Quality Control Charts under Random Fuzzy Measurements

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Abstract

We consider statistical process control charts as tools that statistical process control utilizes for monitoring changes; identifying process variations and their causes in industrial processes (manufacturing processes) and which help manufacturers to take the appropriate action, rectify problems or improve manufacturing processes so as to produce good quality products. As an essential tool, researchers have always paid attention to the development of process control charts. Also, the sample sizes required for establishing control charts are often under discussion depending on the field of study. Of late, the problem of Fuzziness and Randomness often brought into modern manufacturing processes by the shortening product life cycles and diversification (in product designs, raw material supply etc) has compelled researchers to invoke quality control methodologies in their search for high customer satisfaction and better market shares (Guo et al 2006).

We herein focus our attention on small sample sizes and focus on the development of quality control charts in terms of the Economic Design of Quality Control Charts; based on credibility measure theory under Random Fuzzy Measurements and Small Sample Asymptotic Distribution Theory. Economic process data will be collected from the study of Duncan (1956) in terms of these new developments as an illustrative example.

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Chapter 1. Introduction

1.1 Background

"The life cycle of products has decreased rapidly and customized short-run manufacturing process becomes quite common for the achievement customer satisfaction"; Pan (2002). We also note from Guo et al (2006) that there exist challenges of diversification that industries confront in product design, raw material supply, and intermediate part manufacturing of final commercial goods involving multi national companies. So, both life cycle and diversification more often carry vagueness into manufacturing process. In this way randomness and fuzziness are both brought into modern manufacturing, compelling researchers to rely on quality control methodologies so as to search for high customer satisfaction and better market shares.

1.2 Statement of the research project

This research report aims to develop Quality Control Charts based on Fuzzy Credibility Measures under Random Fuzzy Measurements and Small Sample Asymptotics. The context of the study is provided by contemporary globalization trends, which have introduced new challenges for quality control charts using:

(a) Small Sample Sizes and

(b) Random Fuzzy Measurements.

1.3 Objectives of the research project

The objectives of this research are to:

- Develop a theoretical understanding of Small Sample Asymptotics, Fuzzy Credibility Measure, and Statistical Control Charts;
- Develop a mathematical understanding of Statistical Control Charts relying on Fuzzy Credibility Measure Theory under Random Fuzzy Measurements and Sparse Sample Availability;
- Develop a practical understanding of Statistical Control Charts for Fuzzy Measurements and evaluate performance of the methods that are based on different measures.

1.4 Development of the research report

* **Chapter 2**: This chapter deals with Statistical process control charts as graphic tools that statistical process control utilizes for monitoring changes, identifying process variations and their causes in an industrial process (manufacturing process). The tools help manufacturers to take appropriate action to rectify problems or improve the process so as to produce good quality products.

Historically, Statistical Process Control (SPC) was developed in 1920s with techniques of sampling inspection and quality control. In 1924, Walter A Shewhart of Bell Telephone Laboratories developed the first sketch of a modern control chart. Many researchers such as H. F. Dodge, H. G. Romig, W. J. Jennett and others worked in the same way; and later the first official use of SPC is recorded during World War II both in the UK and the USA. In the 1950s, a wide use and application of SPC is also recorded in Japanese industry as a strategy of saving money and attracting customers, and then in more current times, SPC has been widely used. Yet, note that right from the beginning the philosophy of researchers was that good quality products sell, and hence consistent good quality leads to greater productivity, and there is no conflict between price and quality. According to G. B. Wetherill et al (1991) SPC is not a magic formula for curing all production ills, but it is a very useful tool to be used in promoting and maintaining the health of a commercial or industrial enterprise. From this we note that any industry or organization producing products or goods, wants them to be of good quality, and the goods are said to be of the good quality when they conform to specific requirements and are fit for use. Quality in quality control is some attribute or characteristic of a process or product for which a standard is established. So, manufacturing industries and service organizations monitor and maintain the quality of their products or services so as to satisfy both themselves providing the business support and the customers who receive the products. Manufacturing industries and services organizations more often rely on various methods during the process production so as to achieve their purposes. Notice that, the process is only monitored when the information of the production process is collected and output examined, leading to verification of whether the required standards have been reached. So, the SPC is used for maintaining after the admitted or the required quality of the products and the technique often used is one of on-line techniques presented under the terms of the preventive method and called Shewart control charts after Dr. Walter A. Shewart (1924); see also Chapter 2 section 2.2 and Wetherill et al (1991). There are some other techniques among which we have: Cumulative Sum (CUSUM) control charts for process variables; Exponentially Weighted moving averages (EWMA) control charts, sampling inspection of input materials and continuous production inspection. We also underline that, most of time the technicians' intention is to have the production process under a state of control but in developing control measures, it is often difficult to know what to plot on the chart, such as a dimension, and what to do when a process-out-of-control signal is given. For that problem we refer to Wetherill et al (1991) to list in the table E2 Appendix E, the stages or methodology of Statistical process control.

The framework of this chapter will start with the concept quality, then follow concepts of quality control charts and then the concepts of Statistical Control Charts. The discussion on Average Run Length (ARL) in Statistical Control charts will also be presented. The Statistical Design of Quality Control charts and the Economic Design of Quality Control Charts will be discussed separately. A particular emphasis in this chapter is on the use of statistical control charts relying on small Random Sample sizes from normal variables but non-normal distributions are also discussed. The chapter will end with a summary

* **Chapter 3** deals with Zadeh's fuzzy mathematics and Fuzzy Credibility Measure Theory. We present here more definitions and theorems on Fuzzy Credibility Measure Theory so as to inform the mathematical study of Fuzzy phenomena.

In fact, for a long time now in mathematics, there has been a difficulty with addressing the problem caused by the study of uncertain (vague) phenomena at the subset or event level, when the case study cannot be treated in probability theory. In order to alleviate such difficulties Zadeh

(1965) initiated Fuzzy mathematics as a foundation to deal with those vague phenomena. In the same vision and so as to measure a fuzzy event, Zadeh (1978) suggested the concept of possibility measure in expecting it to play the role of probability measure in probability theory for Fuzzy phenomena. Since then, many researchers have been relying on possibility theory to sort out problems brought by fuzzy phenomena; see for instance Kaufman and Gupta (1985), Nahmias (1978), Zimmermann (1985), Dubois and Prade (1988), Klir and Yuan (1995), Decooman (1997), and Liu (2002a). Although possibility measure has been widely used, it has no self-duality property as that in probability theory. Since in both theory and practice self-dual measure is absolutely needed, Liu and Liu (2001) have introduced the concept of credibility measure to facilitate working with fuzzy phenomena, in lieu of possibility measure. According to Liu and Liu (2001), credibility theory is a branch of mathematics that studies the behavior of Fuzzy phenomena. As credibility measure possesses self-duality among other properties, it has been admitted to play a similar role to probability measure in probability theory, for Fuzzy phenomena. Note that, in 2004 Liu gave an axiomatic foundation for credibility theory comprising five axioms from which a credibility measure is defined. Also, Liu (2006) stated that the credibility measure theory has two versions, the classical credibility measure and a non-classical one; see also Guo and Zhao (2006). Due to our interest, in this chapter we will be leaning on the non-classical version. Moreover, in credibility theory, according to Liu and Liu (2001), there are concepts that we will not examine all here; such as Fuzzy variable, membership function, credibility distribution, expected value, variance, critical value, entropy, distance, convergence and there are also fundamental theorems such as credibility subadditivity theorem, the credibility extension theorem, the credibility semi-continuity law, the product credibility theorem, and the credibility inversion theorem. We also notice the extensions of credibility theory such as Fuzzy random theory and random Fuzzy theory that we categorize respectively as a branch of mathematics that studies the behavior of Fuzzy Random phenomena and a branch of mathematics that studies the behavior of Random Fuzzy phenomena. By Fuzzy random variable, we mean a function from a probability space to the set of Fuzzy variables (with fuzzy variable a function from Credibility space to the set of real numbers), and by random Fuzzy variable a function from a credibility space to the set of random variables. From Liu (2006) both Fuzzy random variables and random Fuzzy variables are viewed as special cases of a hybrid variable, which is a function from a chance space (product of credibility space and probability space) to the set of real numbers. In this chapter, our interest on Credibility extensions will be on the Random Fuzzy variable denoted by Guo et al (2006) by $\xi = \{X_{\beta(\theta)}, \theta \in \Theta\}$, and defined as a collection of random variables X_β defined on the common probability space (Ω, Δ, P) indexed by a fuzzy variable $\beta(\theta)$ defined on the credibility space $(\Theta, 2^\Theta, C\tilde{r})$. Section 3.1 will be dealing with Zadeh's fuzzy mathematics and section 3.2 the concepts of fuzzy credibility measure theory. In section 3.3 we will present Chance Space and Hybrid Variable so as to illuminate both extensions: random fuzzy variable and fuzzy random variable. A large emphasis on random fuzzy variable theory will be given in section 3.4, and section 3.5 will be dealing with average chance measure of random fuzzy variable and section 3.6 average chance distribution of random fuzzy variable. Section 3.7 will be discussing the particular case of Normal Random Fuzzy Variable (NRFV). An emphasis on a sampling aspect for NRFV will be given in section 3.8 and we will end our chapter with a summary in section 3.9.

* **Chapter 4**: Small Sample Asymptotics distribution theory is herein defined and discussed as the set of asymptotic expansions for approximating the distribution of statistics for very small sample sizes, even down to 1. The improvement of these approximations is then made in terms of Approximate Distribution for Maximum Likelihood Estimates; Empirical Small Sample Asymptotics and Asymptotic Distributions for small sample from Random Fuzzy Variable. The purpose of this chapter is to derive small sample asymptotic approximations, maximum likelihood estimate and empirical small sample asymptotic under random fuzzy environment.

The term Small Sample Asymptotics submerged in asymptotic theory for a long time was for the first time brought to the surface by Frank Hampel (1973) in the basic ideas of the approximation for the mean (See Field and Ronchetti, 1990). In fact, numerous studies undertaken by researchers such as Daniels (1954), Feller (1971), Esscher (1932), Cramer (1938), Khinchin (1949), Christopher and Ronchetti (1990), Ronchetti (1989), Ronchetti and Welsh (1990), and many others during the last few years indicate the importance attached to this approximation tool. So, it emerges that Small Sample Asymptotics are a powerful tool for obtaining asymptotic expansions (for very small sample sizes) for the exact distribution of statistic when it is difficult or "intractable" (see Christopher et al 1990).

The problem of Small Sample Asymptotics can be presented as follows: for a given statistic $T(X_1, X_2, ..., X_n)$ where $X_1, X_2, ..., X_n$, are n independent identically distributed random variables; then it is difficult to calculate analytically the distribution function F_n or the density function f_n of the statistic T_n if we do not know their particular forms. Small Sample Asymptotics arises so as to compute accurate approximations of the distributions of the statistic. Note that the approach found here demands that the underlying distribution be known, for the approach is a functional of the underlying distribution. On the other hand, according to Christopher and Ronchetti (1990), Bhattacharya and Rao (1976), Serfling (1980) and Pollard (1985), many asymptotic normality properties relying on central limit theorem assess the asymptotic theory as a linearization of the statistics T_n, and then admit the linearized statistics as equivalent to T_n for n large so as to make the difference close to zero in probability. The asymptotic distribution found is considered as an approximation to accurate distribution. But, herein we are not going to focus on that approach.

Nevertheless, let us recall that the starting point of the development of the idea of Small Sample Asymptotics was especially the use of the results of the mean by Hampel (1973). From these approximation methods, especially small sample asymptotics, empirical small sample asymptotics is derived. Indeed, from small sample asymptotics, we recall that the approximations found are functionals of the underlying distribution function. It comes out that, the underlying distribution function plays the important role in the approximation process; however, its computation is often difficult in other words "intractable". That difficulty also extends to moment generating functions and cumulant generating functions which are also of importance. Thus, in order to make the computation tractable and non-parametric one derives empirical small sample asymptotic so as to find a very accurate approximation of the small sample asymptotics used. The small sample asymptotics are very accurate approximations to the distribution of the statistic of interest, and due to the computation problem set by underlying distribution, one sorts it out in finding approximation to small sample asymptotics, since any accurate approximations to small sample asymptotics is automatically that of the distribution of the statistic of interest.

So, in order to make the approximation process work, one replaces the underlying distribution function from small sample asymptotics by empirical distribution function that leads to the situation of empirical small sample asymptotic. In the case where the small sample asymptotics considered is functional of underlying cumulant generating function, one can first approximate the cumulant generating function in replacing it by that of sample version; that is, when resampling from the underlying distribution is possible. Unfortunately, the empirical small sample asymptotic found is not a very accurate approximation; so, accuracy is achieved when one considers the cumulant generating function from the sample of the distribution (after resampling) in the place of the sample version. Wherefore, that leads to the new approximation which is very accurate to small sample asymptotics or to the distribution of the statistic of the interest which works well for very small sample sizes, even down to $n = 1$. Usually, one applies the approximation first for $n = 1$, and then substitutes the underlying distribution by the empirical one. The situation obtained leads to an estimator of the underlying density, since the purpose is that of finding the appropriate estimator of the underlying distribution function, which can allow keeping the relative error as small as possible. For more details see Feuerverger (1989), Ronchetti (1989); Field and Ronchetti (1990).

* **Chapter 5**: For decades, in order to achieve customer satisfaction, business environments more often ensure the supervision of the speeds product life cycle, Guo et al (2006). This statement is not so far from the one made by Pan (2002) stating that :"The life cycle of products has decreased rapidly and customized short-run manufacturing processes become quite common for achieving customer satisfaction".

Again according to Guo et al (2006); there exist challenges of diversification that industries face in product design, raw material supply, and intermediate part manufacturing for final commercial goods; involving multi national companies. Both product life cycle and diversification more often carry 'vagueness' into the manufacturing process. Wherefore, the presence of both randomness and fussiness in modern manufacturing compels researchers to lean on quality control methodologies so as to search for high customer satisfaction and better market shares. Statistical Process Control theory and methodology since initiated by Shewhart (1965) have been always used because of their successful industrial applications. They also help statistical processes of control to keep their critical role in modern quality improvement and management. In addition, since introduced by Zadeh (1965) as Fuzzy Mathematics, the presence of Fuzzy phenomena discovered in manufacturing process pushed researchers to bring Zadeh's Fuzzy into Fuzzy quality control developments.

Unfortunately, up to now, quality control charts problems in random fuzzy environments has not y been developed yet. Nevertheless, three fundamental issues are blocking the wider acceptance of fuzzy quality control methodology: the self-duality, the variable-orientation and the membership specification. This chapter combines techniques from the Chapters 2 and 3, and the discussion will be focused on the concept of Average Run Length (ARL) under Random Fuzzy Measurements, and then Economic Design for Normal Random Fuzzy Control Chart which is the key issue of this chapter. We follow the Duncan-Style Economic Design Quality Control chart to design ours under random fuzzy measurements. Simulation issues will also be examined for generating Random Variables. We will also end the chapter with a summary.

* **Chapter 6** Conclusion

This chapter will summarize the whole work.

1.5 Sources of Information

Information on Small Sample Asymptotic, Fuzzy Credibility Measure, and Control Charts is widely available; more often within publications in Journals (See References).

Fuzzy Credibility Theory has been developed by Liu and Liu (2001), and many others researchers have carried their contributions into this field, among them we have: Zhao and Liu (2006), Guo and Zhao (2006).

Data used for the technical analysis have been collected from the study in Economic Design of $\overline{X}$-Charts of Duncan (1956).

Chapter 2. Statistical Quality Control Charts Theory

This chapter deals with the concept of quality in section 2.1, then follows the concept of quality control Charts in section 2.2. Section 2.3 gives us an emphasis on statistical process control charts, in this section we examine the average run length (ARL) of statistical quality control charts; and also statistical design and economic design of the quality control charts are discussed separately, and then we end the chapter by a summary in section 2.4.

2.1 Concept of Quality

The word quality has different meanings subject to the contexts in which people or even a person use it. It is therefore very difficult to find a perfect or precise definition that distinguishes between the bad or high quality of products or services as a single, identifiable characteristic.

In Quality Control the term quality is defined as some attribute or characteristic of a process or product for which a standard is established. The characteristic is usually one of the leading factors on which customers base their choices among competing services and products. Service organisations and manufacturing industries maintain the quality of their services or products so as to satisfy both themselves as providers of the business support and the customers who receive those services or products.

According to Montgomery (1997), the above aims can be reached if both the manufacturers and organisations can achieve three criteria: (a) Design quality goals and manufacturing quality: the quality observed is specified for the process of manufacturing and the manufacturing quality results measure how well manufactured products satisfy the design quality goals; (b) Quality reliability: the product quality is evaluated on the basis of whether or not the product carried out its intended functions; (c) Consumer-oriented quality: the manufacturing industry must be consumer-oriented on its efforts to create and satisfy demand.

We note from Garvin (1987) that it is often necessary to differentiate several dimensions of the word 'quality' and eight of these are:

-Performance, Customers usually evaluate a product and its basic functions to determine if it will perform according to specifications and determine the extent to which it performs them;

- Reliability, sometimes reformulated as 'being free of defectives';

-Durability, a measure of product life, either economically or physically. Economically a product is considered to be durable if its expected cost of repair does not exceed its current value; Physically a product is considered durable if maintenance and repairs are possible. A product that lasts longer is usually viewed as being of higher quality;

- Serviceability, which relates to the time and effort that is needed to repair or routinely maintain a product. The failure of a product is usually viewed as an annoyance, but a prompt repair may relieve part of irritation;

- Aesthetics, a subjective dimension of quality including look (colour, shape and packaging), feel, sound, taste or smell of a product. It is greatly influenced by the preferences of the individual customer. On this dimension of quality it is usually not possible to meet the needs of every customer, but the manufacturer is satisfied if the product meets the needs of a majority of regular, trusted and serious consumers;

- Features adding to the basic functioning of a product, or upgrades, often interpreted as higher quality and performance;

- Perceived quality, a subjective dimension by which customers without full information about a product directly base their quality image on past experiences, the reputation of the manufacturer, the quality of other products from the same manufacturer, or the name of the product;

-Conformity to Standards, as the degree to which a product meets pre-established requirements of the designer, is related to reliability. This dimension of quality is very important in situations where products are used as components in more complex assembly. Specifications on the individual components are usually expressed as a target and a tolerance. If each of the components is just slightly too big or too small, a tight fit is unlikely, and the final product may not perform as intended by the designer, or may wear out early.

It is clear from these eight dimensions, that quality is a multi-dimensional characteristic of a product and can described at various levels during the production process.
Again, according to Garvin (1987) manufacturer should not strive to be the first on all the eight dimensions of quality, but should rather select a number of dimensions on which to compete.
On the other hand, Montgomery (2001) considers the relationship between Quality and Variability, stating that:" Quality is inversely proportional to variability". In other words; the increase of the quality of the product is subject to the decrease of the variability of important characteristics across the product.
The use of statistical process control allows the manufacturer to recognize variation in the process outcomes.
Statistical control methods help to monitor the product and also help to find and eliminate the root causes of product variation, in order to improve the quality in the process that generates the products.

The purpose of developing control charts, according to Shewhart (1924), is to reduce the causes of variation so as to ensure the improvement of quality of the products.

We also note the purpose of a quality control chart is to correct causes of variation by identifying source, then observing and recording measurements from the process, and also detecting changes and the causes of the changes in quality of the products during the process.
During the process, in Statistical Process Control, two sources of variability are considered, namely common cause (of variation) and special cause (of variation). Also, in industrial process quality control two ways are often considered: a product orientation and a process perspective that can be presented below as follows:

a) For a product orientation the interest is on the parts or units after they are manufactured, so that single quality dimension is considered at a time and, the quality of a part is defined subject to the target value with the specified limits for that quality dimension. In manufacturing or engineering we note that range within which the required quality dimensions should fall, is specified.

b) For the case of process perspective the emphasis is on the process and system which produces items. The concern is about monitoring the problems before many "defective" products are produced.

In the process perspective, it is assumed that all variations in a system during the process come from one of two types of causes of variation: natural variation or non-natural variation:

We label causes of natural or random variation as the "common cause" of variation. It will not be of importance to reduce the variation event in unique cases because adjusting the process in response to each deviation from the target value has a high cost.
Also, their essentially momentary or discrete independent presence during the process, allows us to state that the process is operating under control (SPC).

On the other hand, the causes for non-natural or non-random variation called "special causes" or "assignable causes" are not often present during the process; but when they are present, the trouble in the process is evident, signalled by the presence of shifts within the charts or by a direct quality index value out of boundaries or outside of the control area. The important fact is that they can be easily detected, found and eliminated. Once they are eliminated, the process remains under natural or random variation.
We also note that, during the control of the process, when the special cause of variation is eliminated and the common cause of variation is reduced; the system of production is said to have been improved which leads to the expression often used: "improvement of quality".

Moreover, in many cases, improvement of quality by reducing variability is often limited and from the discussion of Snee (1990), on the problem of total quality management in place of SPC, we note his conclusion that the idea of improving quality by reducing variability requires a new way of thinking. He describes this mode as "Statistical thinking", a philosophy adopted by many researchers.

2.2 Quality Control Charts

A control chart is often defined as a plot of quality characteristics as a function of time or sample numbers. It is a statistical powerful tool that detects changes during the process that may affect the quality of the output allows for elimination of the cause of these changes and return the process into its state of control

Control Charts are more frequently used in the analysis and control of manufacturing processes, that is, for producing quality, which is satisfactory, adequate, dependable and economic. See also Irving (1976).

Note that production processes are often monitored for the extent to which products meet specifications. More often, two "enemies" of product quality are considered: (1) deviations from target specifications, and (2) excessive variability around target specifications.

Experimental designs were earlier used to optimize these two quality characteristics for developing the production process and the methods provided in quality control are on-line or in-process quality control procedures to monitor an on-going production process; see Duncan (1974).

Nevertheless, the approach to on-line quality control is simply the extraction of sample of a particular size from the on going production process. Line charts of the variability in those samples are produced and considered their closeness to target specifications.

Thereupon, if a trend emerges in those lines, or if samples fall outside pre-specified limits, then one declares the process to be out-of-control and takes an action to find cause of the problem; (see Shewhart 1931, who is considered to be the first to introduce these methods).

The use of control chart is on-line for monitoring a characteristic of the process and showing whether the process is operating within its control limits of expected variation. Thus they help the manufacturer or user to quickly detect variation in an important quality characteristic. They also facilitate an on-time corrective action to detect undesirable variation.

Note also that the process is said to be out of statistical control when one value falls outside the control limits. According to Ryan (1989) the probability of conducting such a search should be quite small when the process is in state of control.

Indeed, a process is not in statistical control when the variable being measured does not have a stable distribution.

Also, in order to examine if a process is being maintained in a state of statistical control, control charts utilize past data to determine control limits that would apply to future data obtained from that process. These past data are supposed to be relatively recent for they should adequately represent the current process.

Nevertheless, when control charts are being used for the first time, it is then more necessary to determine trial control limits. In using past data, if sample points fall outside the control limits then investigation might be made, since the result can come from the computation of the test control limits. Thereupon test limits should be recomputed only when the cause of any initial points lying outside the limits can be both determined and removed.

If one or more points are outside new control limits and the assignable causes can be both detected and removed then the limits are once again recomputed, and the cycle should be continued until no additional action can be taken.

As soon as the process is in control, the study of process capability is initiated to determine whether or not the process is meeting the specifications.

Recall that the process is said to be either in control or out-of-control, and the objective of control charts is to analyze components of variability so that variability due to assignable causes can be detected and removed.

There exist different benefits for the use of Control Charts that we can present below as follows:

a) Control charts help to distinguish different causes of variation;
b) Control charts are used to estimate the parameters of a production process, and then determine process capability;
c) Control charts furnish useful information for accessing the inherent capability of the process after achieving statistical control;
d) Control charts are used to evaluate the effects of improvement actions, aiming to reduce assignable causes of variability associated with common causes;
e) Control charts ensure the maintenance of the process in its state of statistical control on the basis of continuous monitoring strategies. They also help to keep the process meeting the

requirements over time;

f) A successful control chart program reduces scrap and rework, which are the primary production-killers in any operation. Such reductions lead to stability of the cost of products;

g) In control chart case there are no adjustment measures. In other words, in control charts, we have the philosophy: “if it is not broken, do not fix it”. Guo et al (2006)

2.2.1 Types of Control Charts

There are two types of data in the statistical research milieu: numerical data and attribute data. The nature of control charts matches the type of the product output quality characteristic they report. So, we have statistical process control charts for variables and statistical process control charts for attributes. For their different types of usage, See Table E3 Appendix E, or sub-section 2.2.4 below.

2.2.2 Setting up Control Charts

In this section we follow Porter et al (1992) in setting up control charts for sub-grouped data in the so-called "standard procedure". The same procedures (step 2 to 5 given below) are also presented by Caulcutt (1995).

1. Obtain data;
2. Put the data into sub-groups;
3. Calculate the mean and range of each sub-group;
4. Calculate the overall mean;
5. Estimate the process standard deviation by using $\overline{R}/d_n$ (d_n is Hartley’s constant, depending on the sample size and given in many statistical process control texts);
6. Calculate values for control lines on a mean chart and a range chart by using $(\overline{\overline{x}}),(\overline{R})$ or the estimate $\overline{R}/d_n$, and appropriate constants;
7. Plot the group means onto the mean chart and plot the group ranges onto the range chart;

8. If the control charts indicate that the process is stable or in control, then these charts, or other charts of a more appropriate type based on the same estimates, can be used to monitor future production;
9. If the control charts indicate that the process is unstable, or out-of-control, then the assignable causes should be sought and corrective action taken. The whole procedure should then be repeated

2.2.3 Control Chart plots

The purpose of designing Control charts is to compare an observed sample value to the process expectation in relation to its past performance. In fact, the control of current values is based to the past values.
Also, note that control charts are often time series based. The control of the quality characteristic of the process is easily sought by repeated sampling of the process, and from each sample value a test statistic is computed and plotted on the chart.
Note that the horizontal axis indicates the sample number or time at which a quality characteristic is measured and the vertical axis indicates the value of the quality characteristic.

Three lines are considered as basics for describing a chart; they are inserted on the chart indicating a specific position or level of behaviour of the quality index of the process.
The centerline (CL) represents the specified target level for the quality of the process. The control limits namely the upper control limit (UCL) and the lower control limits (LCL) indicate the boundary within which the computed statistic is expected to fall.

The formulas for computing general forms of control limits can be given below as follows:

$$\begin{aligned} UCL &= E[\text{statistic}] + L\text{x}[\text{standard deviation of statistic}] \\ CL &= E[\text{statistic}] \\ LCL &= E[\text{statistic}] - L\text{x}[\text{standard deviation of statistic}] \end{aligned} \tag{2.1}$$

Nevertheless, most of time, $L = 3$ in other words; control limits are fixed at 3-sigma (3σ) from the centreline, and if it is assumed that the subgroup means have a normal sampling distribution,

the probability that any observed mean will fall within the 6σ bandwidth of $\pm 3\sigma$ is approximated using a normalized variance.
In other words:

$$P(-3 \le z \le 3) = P(z \le 3) - P(z \le -3) = 0.998 - (1 - 0.998) \cong 0.996. \qquad (2.2)$$

In the case where the statistic plotted does not fallow normal distribution, the control limits are always positioned so as to keep the probability for the observed characteristic falling outside the bandwidth very small (less than one percent when the process is in control). See Braverman (1981).

According to Chebyshev's theorem, no matter what the true underlying distribution, about 89% of all observations fall within 3-sigma of the mean or centerline. Here below is the general illustration of the control chart.

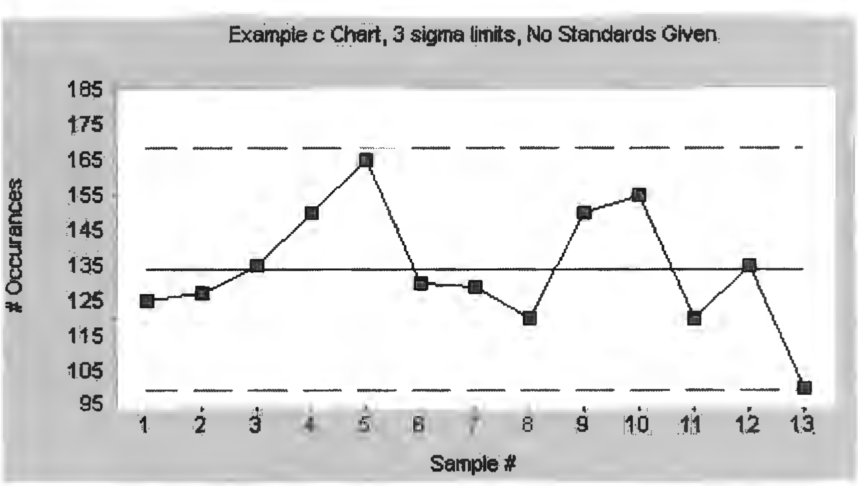

Figure 2.2.1 Control Chart (Six Sigma, 2007, see web references)

More often, one declares the process to be under statistical process control or in a stable state, when the control chart displays a random pattern of variation about the centerline within the upper and lower limits, that is, within the bandwidth.
Note that, when a process is in control, the control chart provides a method for continuously testing a statistical hypothesis.

The "in-control process" is identified as null hypothesis H_0 and the "out-of-control" is identified to the alternative hypothesis H_1. So, as soon as a point value falls outside the boundary, then the null hypothesis is rejected in favour of the alternative.

Thereafter, an action is taken so as to search, find and eliminate assignable causes and return the process back into its state of control.
Some researchers adopt the principle of stopping the process while taking action for eliminating the assignable cause; still others keep the process working until the assignable cause is moved.
Using the first reasoning; when the out of control condition occurs, the process should be stopped and an investigation initiated so as to locate the assignable cause. Once the assignable cause is determined and eliminated, the process can restart and continues.
It should be noted that the control chart is based on the fact that if the process is in- control, the outcomes are predictable, that is, based on previous observations, it is possible for a given set of limits to determine the probability that future observations fall within the bandwidth.

2.2.4 Applications of Control Charts

Recall that the use of Control Charts is based on three fundamental elements: two control limits (upper and lower control limits) set generally at three sigma units on either side of a centerline. According to Dr. Shewhart, these three sigma limits provide the control of the likelihood of committing types I and type II errors.
In fact, these three sigma control limits are set using an average measure of dispersion based on the within subgroup variability. Upper and lower control limits are positioned so that the variation in the sample average, range or standard deviation is only due to common causes.
The construction of control limits utilizes an estimate of the standard deviation for the common cause system that applies to the process when it is operating in the state of statistical control. The standard deviation is computed within each subgroup and then average used to design the control limits.
The design of control chart is based on the methods of rational sampling and rational sub grouping, from which different decisions or questions are taken or answered from data.
According to Ewan (1963), when the quality index computed from data plotted on chart signals that the process is out-of-control; then two types of actions can be taken: Direct action resulting in

the physical alteration of some aspect of the process implies more precise knowledge of cause and effect, and is used when one can put the process at a specific level, or when one knows how to put the process into another way of functioning after the change has been noted. The second indirect action results in attempts to find out why there has been a change in the process. It generally does not imply very specific knowledge about the cause of the change nor how to put immediately the process to the right procedure, nor how to maintain the new better process level.

So, it is often difficult, in Statistical Process Control, during the process to know among all control charts, which is relevant to use for controlling the behavior of data. Researchers often pass by the net examination of data for finding a correct decision on the choice of the chart. In fact, once we have data, an examination is made. After that, if data are numeric or variable, then the next step is to verify whether we have sub group of size 1 or not. If the sub group is of size 1; then X-bar chart or moving average chart is used; otherwise, we use X-bar chart, R-chart or chart for s.

However, if after examination we discover that data are attributes, then it is also important to check if the distribution of the data is Binomial or Poisson.

If data follow Binomial distribution and the samples are of same sizes n; then p-chart is used otherwise we use the np-chart, and when data follow Poisson distribution, the c-chart is used for subgroups with equal sizes and the u-chart for subgroups with unequal sizes .

We present these above statements as summary in the table below (See also table E3 Appendix E)

Table 2.2.1 Application of Control Chart

Control Charts for Variable data	*Control charts for Attributes data*
I- Variable Chart with sub grouping size $\neq 1$	I- Based on Binomial distribution
- X-bar Chart; - R-chart; - S- chart (or S^2-chart)	a) Attributes data with sub group size unequal • np- chart b) Attributes data with sub group size equal • p-chart
II- Variable Chart with sub grouping size $= 1$	II- Based on Poisson distribution

The description of each part of the table above is given in the following sub-sections:

Control Charts for Variables

Variable Data with sub grouping size $\neq 1$

On a numerical scale, the quality characteristic of statistical process quality control charts is referred to as a variable; and the main charts used to plot variable measurements are X-bar chart and R-chart.

We also have other control charts for variables such as s-chart and s^2-chart. The charts show the behaviour of the process, and are quality time based.

- *X-bar Chart*

The X-bar chart monitors between-sample variability. In fact, it controls the average quality level in the process over time; it is then used to control the mean or central tendency of the process. It is also used in common with R-chart for collecting measurements in subgroups.

We can construct the chart using the following formulae

$$\begin{aligned} UCL &= \mu + 3\sigma_{\overline{X}} = \mu + 3\,\sigma/\sqrt{n} \\ Cl &= \mu \\ LCL &= \mu - 3\sigma_{\overline{X}} = \mu - 3\,\sigma/\sqrt{n} \end{aligned} \tag{2.3}$$

where μ is the process mean, σ the process standard deviation, and n the size of the sample.

Since the quality process index μ and σ are unknown, the researchers estimate them from the data assuming the best current estimate of μ is $\bar{x}$. We note average of averages $\bar{\bar{x}}$ computed as follows:

$$\bar{x}_j = \frac{1}{n_j}\sum_{i=1}^{n_j} x_i \tag{2.4}$$

and

$$\bar{\bar{x}} = \frac{1}{m}\sum_{j=1}^{m} \bar{x}_j = \frac{1}{m}\sum_{j=1}^{m}\frac{1}{n_j}\sum_{i=1}^{n_j} x_{ji} = \frac{1}{m}\sum_{j=1}^{m}\sum_{i=1}^{n_j}\frac{x_{ji}}{n_j} \tag{2.5}$$

Recall that $\bar{x}$ $(or\,\bar{x}_j)$ is the average of the j^{th} sample of the size n_j ($n=\sum_{j=1}^{m} n_j$) and m the number of subgroups in a sample.

The required estimate of the process standard deviation σ is a function of average of the sample ranges noted $\bar{R}$ divided by a constant d_2 (function of the sample size).

$$\hat{\sigma}=\frac{\bar{R}}{d_2} \tag{2.6}$$

with $R = x_{\max} - x_{\min}$;

$$\bar{R}=\frac{\sum_{i=1}^{m} R_i}{m} \tag{2.7}$$

and d_2 is the expectation of relative range given by $d_2 = E_w = ER_n / \sigma$, where R_n is a random variable representing the range of a sample size n from a normal distribution of variance σ^2.

We then have the formulae of control limits given by:

$$\begin{aligned} UCL_{\bar{x}} &= \bar{\bar{x}} + 3\bar{R}/(d_2\sqrt{n}) = \bar{\bar{x}} + A_2\bar{R} \\ CL_{\bar{x}} &= \bar{\bar{x}} \\ LCL_{\bar{x}} &= \bar{\bar{x}} - 3\bar{R}/(d_2\sqrt{n}) = \bar{\bar{x}} - A_2\bar{R} \end{aligned} \tag{2.8}$$

where $A_2 = 3/(d_2\sqrt{n})$ (See table C2, appendix C).

We also notice that X-bar chart is used to control data of the current input, so the centreline corresponds to a standard value $\bar{x}"$ obtained from past data and upper and lower control limits computed from estimated standard value of process standard deviation $\sigma"$.

The formulae for control limits are then given by:

$$\begin{aligned} UCL_{\bar{x}} &= \bar{x}" + 3\sigma" = \mu + 3\sigma"/\sqrt{n} \\ CL_{\bar{x}} &= \bar{x}" = \mu \\ LCL_{\bar{x}} &= \bar{x}" - 3\sigma" = \mu - 3\sigma"/\sqrt{n} \end{aligned} \tag{2.9}$$

$$s_n^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{(n-1)} \tag{2.12}$$

Once the standard deviation of each sample is computed and the average of the sample standard deviations calculated, then the control limits are computed as follows:

$$\begin{aligned} UCL_s &= \left(1 + \frac{3}{c_4}\sqrt{1 - c^2{}_4}\right)\bar{s} = B_4\bar{s} \\ CL &= \bar{s} \\ LCL_s &= \left(1 - \frac{3}{c_4}\sqrt{1 - c^2{}_4}\right)\bar{s} = B_3\bar{s} \end{aligned} \tag{2.13}$$

where $c_4 = \sqrt{2/(n-1)}[\Gamma(n/2)/\Gamma(n-1/2)] \sim 1-[1/4(n-1)]$

It is assumed that at the start, the process is in the state of control and the standard deviation is equal to that of standard value σ''.

So, the control limits on the s-chart and s^2-chart can be computed by:

$$\begin{aligned} UCL &= \left(c_4 + 3\sqrt{1 - c^2{}_4}\right)\sigma'' = B_5\bar{s} \\ LCL &= \left(c_4 - 3\sqrt{1 - c^2{}_4}\right)\sigma'' = B_6\bar{s} \end{aligned} \tag{2.14}$$

See Table C2 Appendix C for the value of B_3, B_4, B_5 and B_6

Variable Data with sub grouping size $= 1$

For the purpose of monitoring a process, it is often difficult to get sufficient data quickly enough to be able to divide the data into subgroups. Thereupon, it appears to be necessary to construct control chart using one observation (individual observation or individual measurement). Such charts are called Charts for individual values.

- *R-bar Chart*

Note that, there is no way of estimating standard deviation for deriving the control limits of X-chart. Therefore, moving average (range) and standard deviation methods are used.

- *Moving Range Chart*

The formula of the moving range method is given by:

$$MR_i = |x_i - x_{i-1}| \tag{2.15}$$

which is the absolute difference between two consecutive observations.

So, the average of the moving ranges is the average to obtain an average moving range $\overline{MR}$.

The control limit can be calculated as follows:

$$\begin{aligned} UCL_x &= \bar{x} + 3\overline{MR}/d_2 = \bar{x} + E_2\overline{MR} \\ CL_x &= \bar{x} \\ LCL_x &= \bar{x} - 3\overline{MR}/d_2 = \bar{x} - E_2\overline{MR} \end{aligned} \tag{2.16}$$

where $E_2 = 3/d_2$

The standard deviation is estimated by $\overline{MR}$ and the estimator is made unbiased in dividing $\overline{MR}$ by d_2 (function of sample size n). Again see Table C2 Appendix C for the value of d_2

Also, the standard deviation σ can be estimated by s (standard deviation of all observations); then as another way to derive the moving range method, the upper and lower control limits can be computed as $\bar{x} + 3s$ and $\bar{x} - 3s$ respectively.

Control Charts for Attributes

Attributes Data

The items inspected are inferred to be conforming or non-conforming to the specifications of an attribute or quality characteristic, if the quality characteristics of the process cannot be represented numerically.

- *Binomial Distribution*

The Binomial distribution with parameters n and p is defined by:

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0,1,2,...,n \tag{2.17}$$

The mean and the variance of the Binomial distribution are given by:

$\mu = np$ and $\sigma^2 = np(1-p)$ respectively.

Attributes Data with sub group size unequal

- *p-Chart*

The p-Chart or "Control Chart for fraction non conforming" within a manufacturing process, is the most frequently used attribute control chart for unequal sub group sizes and is based on the Binomial distribution.

When we have a sample which has n units and we are sure that the units produced in a production process operating in a stable manner and the successive units produced are independent and that the probability that any unit will not conform to specifications is p; then the fraction of non-conforming values is given by the ratio of the number of non-conforming items in the sample over the total number of items in the sample.

Let us denote d as the number of non-conforming units in the sample. It is clear that d follows a binomial distribution with parameters n and p given by:

$$p(d = x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0,1,2,...,n \tag{2.18}$$

where the mean and the variance are given by $E(d) = np$ and $V(d) = np(1-p)$ respectively; and the non-conforming sample fraction is defined as:

$$\hat{p} = \frac{d}{n} \tag{2.19}$$

with $E(\hat{p}) = p$ and $V(\hat{p}) = p(1-p)/n$

If we consider the true non-conforming p of the production process or standard value as known, then we can compute the control limits for p-chart as follows:

$$UCL = p + 3\sqrt{\frac{p(1-p)}{n}}$$
$$CL = p \qquad (2.20)$$
$$LCL = p - 3\sqrt{\frac{p(1-p)}{n}}$$

However, if p is unknown, then p is estimated from observed data. According to Montgomery (2001), if d_i represents non-conforming units in sample I, then the fraction non-conforming in the i^{th} sample is given by:

$$\hat{p}_i = \frac{d_i}{n}, \quad i = 1,2,\ldots,n \qquad (2.21)$$

The average of m individual sample fractions non-conforming is then given by:

$$\bar{p} = \frac{\sum_{i=1}^{m} \hat{p}_i}{m} = \frac{\sum_{i=1}^{m} d_i}{m} \qquad (2.22)$$

If p is unknown, then $\bar{p}$ is considered as an estimator of p and then we have the new formulae of control limits for p-chart given by:

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$
$$CL = \bar{p} \qquad (2.23)$$
$$LCL = \bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$$

Attributes Data with sub group size equal

- *np-Chart*

The np-chart is used to plot number of non-conforming when the sample sizes are all equal. If the true fraction non-conforming p in the production process is known or is a standard value specified by management; then the control limits for np-chart are computed by:

$$UCL = np + 3\sqrt{np(1-p)}$$
$$CL = np \qquad (2.24)$$
$$LCL = np - 3\sqrt{np(1-p)}$$

$\widehat{np_i}$ is the charting statistic for each sample.

When the true fraction non-conforming p in the production process is unknown, then the average of m preliminary individual sample fractions non-conforming $\overline{p}$ is used and the control limits for np-chart given by:

$$UCL = n\overline{p} + 3\sqrt{n\overline{p}(1-\overline{p})}$$
$$CL = n\overline{p} \qquad (2.25)$$
$$LCL = n\overline{p} - 3\sqrt{n\overline{p}(1-\overline{p})}$$

$\widehat{np_i}$ is the charting statistic for each sample.

Remark: For a constant sample size, both p-chart and np-charts are equivalent.

Attribute Charts

- *Poisson Distribution*

We use the term Poisson distribution to denote the distribution defined by:

$$P(x) = \frac{e^{-c}c^{x}}{x!}, \quad x = 0,1,2,... \qquad (2.26)$$

where x is the number of non-conformities and $c > 0$ the parameter of the Poisson distribution. Recall that the mean and variance of the Poisson are the parameter c.

Attribute Charts with equal sub group sizes

- *c-Chart*

The c-Chart based upon the Poisson distribution is used when we chart the number of non-conformities for equal sample size. c is considered as the number of non-conformities in each sample but not the fraction of non-conforming.

The control limits for the average number of non-conformities per unit can be given by:

$$UCL = u + 3\sqrt{\frac{u}{n}}$$
$$CL = u \qquad (2.30)$$
$$LCL = u - 3\sqrt{\frac{u}{n}}$$

The true value of u is known or specified by the management.

When the value of u is unknown, then the control limits for u-chart are given by:

$$UCL = \bar{u} + 3\sqrt{\frac{\bar{u}}{n}}$$
$$CL = \bar{u} \qquad (2.31)$$
$$LCL = \bar{u} - 3\sqrt{\frac{\bar{u}}{n}}$$

$\bar{u}$ is the average number of non-conformities per inspection unit from the preliminary sample used as an estimator of u .

Remark: Comparison between Attributes Control Charts and Variable Control Charts.

Table 2.2.2: Advantages of Attributes Control Charts over Variables Control Charts.

Attributes control charts	*Variables control charts*
Used to count the number of non-conforming items or number of non-conformities in a sample	Need actual measurements for the items which are produced
Used to many different non-conformities at the same time	Used for each quality characteristic with separate charts
Inspection is much easier	Inspection is not easy
Need to know if the inspected item meets the specified requirements in classifying them in respect to various quality criteria	
Don't depend on an underlying statistical distribution	
Used for visual inspections for some attributes (correct colour, required cleanliness, etc...)	
Provide more persuasive evidence of quality problems to management	
Need large sample size (more than or equal to 50)	Need small sample size (at least 4 or5)

More sensible	Sensible
	Alert to quality problem before any actual "unacceptable", detected by the attribute chart, will occur.
	Leading indicators of trouble that will; sound an alarm before the number of rejects (scrap) increases in the product process.

2.3 Statistical quality Control Concepts

A process is defined as any combination of machines, tools, methods, materials and/or people employed to obtain specific qualities (or quality characteristics) in a product or service. Also, any change of these constituents results in a new process;

A control in the sense of process control is a feedback loop through which we measure actual performance, compare it with a standard, and act on any marked difference.

Control charts are a powerful statistical tool used in statistical process control (SPC), for monitoring the potential changes in a process and eliminating these changes that may affect the quality of the output in order to maintain the stability of the process.

Statistical quality control is the applications of statistical techniques for measuring and improving the quality of processes for maintaining a steady process and thus generating homogeneity environments for the production.

Statistical process control (SPC) is the applications of statistical techniques for measuring and analyzing the central tendency and variation in processes. In other words SPC is statistical methods used extensively to monitor and improve the quality and productivity of manufacturing processes and service operations (Zachary et al 2000).
According to Wetherill et al (1991), the SPC can be classified into on-line and off-line categories.

- An on-line SPC method is viewed in terms of screening and preventive methods.

Screening SPC methods are a form of sampling inspection. They verify whether the product or goods reach the quality needed. If not the standard items are screened out for recording, for selling at a reduced price or for scrap. Another term of on-line method is the preventive method that is the inspection of the process. It utilizes process control to avoid production of defective items. Among preventive methods we have: (a) Shewart control charts for process variables; (b) Cusum control charts for process variables; (c) Sampling inspection of input material; (d) Continuous production inspection of product.

-An off-line process control is built to design and set up for a product and its production process from the start. The aim of off-line process control chart is to reduce or remove the effect of potential causes of variability by modifying the process, or the product, so as to make it less sensitive to these causes.

Note also that, the principle of applying quality control is the existence of a process that produces a good or service for which some standard of quality can be specified.

2.3.1 Rule of Control Charts

Control charts are powerful tools utilized by statistical process control to monitor production processes in order to decide whether they are in state of control or not. Once the process is in an out-of-control state, action can be taken to find the reasons why the process is in that state. As soon as the assignable cause is detected or found and eliminated, the process is returned back into its state of control. Recall that, the process is said to be in a state of control when it is only subject to common causes.

Again recall that three common lines are said to be basic for all control charts: centreline (CL), upper control limit (UCL) and lower control limit (LCL). These lines constitute the boundaries of control such that when at least one point is found outside the boundary (above the upper control limit or below the lower control limit), then the process is declared to be out-of control.

There are some small changes which may happen inside the boundary that can further lead the process to an out-of-control state. Supplementary criteria are often used to increase the sensibility of the control charts to small changes in the level or variability of the process. Referring to Duncan in quality control and Engineering Statistics, Nelson's Alarm Rule (1984) and other researchers, a number of limits are superimposed on the plot to partition the control charts into three zones A,B and C on each side of the centreline, and each zone is 1-sigma wide. See also Figure E3 Appendix E.

During the process, product records are taken and plotted onto the charts and fall into four zones above the centreline or the four zones below the centreline.
The zone test used for both variable charts and attributes charts helps to detect variation and as soon as a variation is detected, an action is undertaken by the manufacturer to search for the cause, so as to find and eliminate it.

The reasoning is then made using the half of the chart that can be presented from Duncan (1986) as follows:

1) A single point outside three-sigma units away from the centerline signals that there is a cause of variation;
2) Two out of any three consecutive points outside the two-sigma warning limits on one side of the centerline but still inside the control limits, in zone A also signal the presence of a cause;
3) Four out of any five consecutive points outside the one-sigma limits on one side of the centerline in zone B indicate that there is a cause of variation;
4) A run of eight consecutive points on the same side of the centerline, in zone C indicates the presence of cause of variation.

NB: For the study of the sensitivity of the range chart the warning lines have also been imposed on the chart.

2.3.2 Control Charts for unequal sample size

When the samples within a control chart have unequal sizes the way to represent the upper and lower control limits above and below the centerline (target specification) cannot be a straight line. Three ways are considered by researchers to deal with this kind of situation:

1) **Average sample size**: if one desires maintaining the design of a control chart with straight-line control limits, then the average sample size n per sample across all samples has to be computed, and then establish the control limits based on the average sample size. This method is not " exact", but it is advised when sample sizes are reasonably similar to each other.

2) **Variable Control Limits**: Alternatively, different control limits can be calculated for each sample, based on the respective sample sizes. This method will lead to variable control limits, and result in step-chart like control lines in the plot. It ensures that the correct control limits are computed for each sample; but, one loses the simplicity of straight-line control limits.

3) **Stabilized (normalized) Chart**: The best of two worlds (straight line control limits that are accurate) can be accomplished by standardizing the quantity to be controlled, the mean, proportion, etc. according to units of sigma. The control limits can then be expressed in straight lines, while the location of the sample points in the plot depends not only on the characteristic to be controlled, but also on the respective samples n's. The disadvantage of this method is that the values on the vertical (Y) axis are standardized and therefore, those numbers cannot be taken at face value.

2.3.3 Control Charts for non-normal Data

In this section, our attention is focused on the use of Control Charts for non-normal Data (using small sample size), especially the X-bar Chart, since for R, S and S^2 the individual observations are assumed to be normally distributed.

Recall that, the Control limits for the standard X-bar Chart assumed the Sample means to be approximately normally distributed.
Shewhart (1931), in his original work experienced various non-normal distributions for individual observations. He noted that the resulting distribution of means for samples of sizes four has an appropriate standard normal distribution-(based control limits for the means), as long as the underlying distribution of observations are approximately normal.

However, Ryan (1989) pointed out that, when the distribution of observations is highly skewed and the sample sizes are small, then the resulting standard control limits may produce a large number of false alarms (increasing alpha error rate), as well as a larger number of false negatives ("process-is-in-control") or (increased beta-error rate).

So control limits (as well as process capability indices) for $\overline{X}$-Charts can be computed, based on so-called Johnson curves (Johnson 1949), which allow approximating the skewness and kurtosis for a large range of non-normal distributions.

These non-normal $\overline{X}$-Charts are useful when the distribution of the means across the samples is clearly skewed or otherwise non-normal.

Again, according to Ryan (1989), if the samples are of size 4 or 5, then the distribution of $\overline{X}$ will not differ greatly from a normal distribution as long as the distribution of X is reasonably symmetric and bell shaped. Since, from the result of the central limit theorem, we note that the distribution of $\overline{X}$ will be more normal, in general, than the distribution of X.

Nevertheless, when the distribution of X is highly asymmetric such that the distribution of $\overline{X}$ will also be asymmetric for small samples, data can usually be transformed (by log, square root, reciprocal, etc.) so that the transformed data will be approximately normal.

2.3.4 Average Run Length in Statistical Control Charts

In Statistical process control the theory of average run length (ARL) is widely used because of its great importance in analysing a control chart. A run length is defined as a succession of items of the same class. When the output of the process is plotted, for instance the mean, if sample points falls above the mean then they are said to belong to one class, and those falling below the mean are said to belong to an other class, and all points which coincide to the mean are ignored. (See also Guo et al 2006).

So, from these observations, we derive the notions of a run above the mean or a run below the mean, and also "run up", that is, a succession of increasing values, and "run down", a succession of decreasing values.

From Duncan (1986) and others researchers, we note that a run is defined as a sequence of the observations of the same type. Any type of run (run up or down) of eight or more points is a signal of the out-of-control condition. It follows that, the run length provides an important measure of the current effectiveness of a chart that is being used to control the quality of the current output.

One of these performance measures is average run length (ARL), which can be considered in two parts; the in-control and out-of-control background scenarios.

Thus, ARL for the in-control is the average length of a run of in-control points that follows immediately after a change to control has been specified in the process and ARL for the out-of control is the average of a run of out-of-control points that follows immediately after a change away control has been specified in the process.

An ARL curve for a chart used to control the quality of current output can be derived; and if one can calculate that; then any change in a process(for instance increase in the process average) can be detected. A commonly used ARL is the in-control ARL, whose computation is given in the following sub section.

Computation of an Average Run Length

For the computation of an ARL, we consider the general control chart where subgroups of size n are taken from the sample. For each subgroup n_i a statistic T is computed and plotted against the corresponding time. The control limits are computed and inserted into the chart. They are used for calculating the probability that the statistic plots outside the control limits for any subgroup. This is denoted by p_d and given by:

$$P_d = P(T < LCL \quad or \quad T > UCL) \tag{2.32}$$

Consider Y the number of subgroups until the event that the statistic plots outside the control limits occurs. The distribution of Y under the assumption that the subgroups are independent of each other and the process is stationary and computed by:

$$P(Y = j) = (1 - p_d)^{j-1} p_d, j = 1, 2, ..., n \tag{2.33}$$

Thus Y has a geometric distribution with parameter p_d. The average run length is defined as $E[Y]$, and given by:

$$ARL = E[Y] = \sum_{j=1}^{\infty} jP(Y=j) = p_d \sum_{j=1}^{\infty} j(1-p_d)^{j-1} \tag{2.34}$$

The average run length for a given mean shift and subgroup size n for the X-bar chart is calculated by:

$$ARL = \frac{1}{p\left(Z < L - \frac{\Delta\sqrt{n}}{\sigma}\right) + p\left(Z > L + \frac{\Delta\sqrt{n}}{\sigma}\right)} \tag{2.35}$$

where Z is N (0, 1), the size of the mean shift $= \Delta = \mu_{'in-control} - \mu_{'shifted}$, the process variance $= \sigma^2$, 'n-control' mean$= \mu$, control limits $\mu \pm L\sigma / \sqrt{n}$

The sample size for a given ARL, and mean shift is computed as:

$$n = \left(\frac{(Z_{p_d} - L)\sigma}{|\Delta|}\right)^2 \tag{2.36}$$

2.3.5 Statistical Design of Quality Control Charts

The statistical design of quality control charts is the design of quality control charts based on statistical performance for the in-control and out-of control regions in respect to the parameter values. It is viewed as the stage of designing control charts such that every time the process goes into an out-of control condition, it has to be taken back into its previous control state. So, the main objective of the statistical design of quality control charts is to detect shift of the quality index (or the out-of-control condition) during the production process.

In fact, the purpose of statistical design of quality control charts is to take action to find, correct or eliminate the assignable cause of shifts during the production process so as to return the process back into its state of control. It ensures the achievement of the pre-selected levels of the type I and type II errors (Guo et al 2006). It is assumed with the return of the process back into its state of control when it has been troubled by assignable cause.

Also, as a principle, one can utilize the past data of a past stage of control to construct control limits of the current process subject to being maintained under control. That is centreline is plotted at $\overline{X}''$; upper and lower control limits plotted at $\overline{X}'' + k(\sigma''/\sqrt{n})$ and $\overline{X}'' - k(\sigma''/\sqrt{n})$ respectively

The big issue in the statistical design of quality control charts is to detect shifts from target value. Also, due to the fact that control charts do not have the same power of detecting small shifts, some researchers proceed by comparing the power of control charts (or quality procedure) to monitor the quality (index) level. Woodall (1985) offers a discussion of the performance of ARL between Shewhart chart and CUSUM. Page (1961) considers the design of CUSUM charts for a required ARL values at specified shift in the mean assumed important enough to necessitate being quickly discovered.

Some others researchers reconstruct control charts by modifying control limits (ARL) as widely as the previous control limits. Freund (1957) and Duncan (1974) refer to the Shewhart chart with these modifications as an acceptance control chart; and according to Woodall (1985), modified control limits have always been important when shifts in the mean occur without a significant increase in the percentage of non-conforming production.

The last point of our discussion turns to the comparison of Statistical Design and Economic Design of Quality Control Charts (See Economic Design next section).

In fact, in the economic design of control charts, (e.g. for the mean) the small shift is very important; whereas the statistical design of control charts is above all, more appropriate when the time interval between samples is predetermined and any large shift can be easily and quickly detected in respect to its frequency of occurrence.

According to Woodall (1985), the Economic Design model takes into account the total cost of false alarms which is considered proportional to the number of false alarms. That approach often leads to a very low in-control ARL for the cost. We also note that the excessive number of false alarms introduces extra-variability into the process and destroys on site confidence in the control procedure.

Montgomery (1980), on site states that the economic design of control charts is no longer used in practice because of its complexity, and in addition the input variables are often unknown; on the other hand Woodall says that the statistical design of control charts is about designing a control chart to a particular application, even if the resulting procedure is not optimal in the sense of minimizing the total cost.

2.3.6 Economic Design of Quality Control Charts

To begin with, we need to recall that control charts have been used for a long time in statistical process control as a powerful charting method which helps to monitor the process, and allow manufacturers to detect assignable causes during the process and take corrective action. Three approaches are often used to design control charts in statistical process control: the heuristic from Shewhart (1931), the statistically designed control chart, and the third method being the economic design model; see Saniga (1989).

While the statistical design which has two parameters (n, k) indicating the sampling size (n) and the control limit factor (k), the economic design possesses three parameters (n, h, k) indicating sampling size (n), the interval time (h) of withdrawing successive samples during the process, and the third parameter (k) indicates the control limit factor.
Also, recall that statistical design ensures the achievement of the pre-selected levels of the type I and type II errors. The economic design model keeps in control the existing quality levels of the process. In other words; the statistical design of control charts is about returning the process into control when it has been troubled by assignable causes; whereas the economic design keeps the process in the stage of control. See also Guo et al (2006).

The main objective of economic design for quality control chart is the computation of three parameters required for keeping the existing quality levels of the process.

Some researchers have approached economic design by using cost function based especially on general costs such as: (a) sampling cost; (b) cost of process running out of control, (c) cost of process running in-control (d) cost of false alarms; (e) cost of locating an assignable cause when process is out of control and (f) adjustment cost, and some others focused on an income function. The foremost is Duncan (1956) who established a criterion of "optimum design" so as to determine these three unknown parameters (n, h, k), in constructing the net income per hour as a function of chart design variables. Its differentiation in respect to n, h, k, set equal to zero gives three equations which allow specification optimized values of these design variables.

So, according to Goh (1996), these three values obtained help to achieve the balance between acceptable product quality level and acceptable economic level for the concerned production.

Several researchers focused on the Duncan criterion to set up other methods. We can note Saniga (1977) with joint economic design of x-bar and R control charts, Lorenzen and Vance (1986) in determining the economic design of control charts and Collani (1989) with different procedures to determine the economic design of x-bar control charts.

Despite all these variations, the most important issue remains the computation of the required parameters to be used so as to maintain the quality of the production process.

2.4 Summary

Statistical process control chart have been discussed in this chapter as graphic charts used for monitoring specific processes leading to great development and improvement in quality standard. These charts give a graphic appearance of the process which enables any manufacturer or service provider with or without statistical knowledge to know whether the process is in state of control or not help to recognize the types and causes of variation during the process. As soon as the variations occur, an action is taken so as to detect the cause, eliminate it and then quickly remit the process back into its state of control in respect to a target value. So, the process is then said to produce goods of required quality. ARL has been discussed in terms of analyzing charts during the process, and the design of quality control charts has also been discussed statistically in terms of returning the process back into its state of control when it has been affected by assignable causes or economically focusing on the computation of the required parameters, which allow maintaining the process in its state of Control.

Chapter 3. Random Fuzzy Variable Theory

Section 3.1 sets up Zadeh's fuzzy mathematics foundation that connects us to section 3.2 on Liu's Fuzzy Credibility Measure theory which will facilitate a mathematical foundation for quality control charts within random fuzzy environments. Section 3.3 covers chance space and hybrid variable. The random fuzzy variable theory will be examined in section 3.4. Section 3.5 will be dealing normal random fuzzy variable, followed by sampling distributions in section 3.6. Then the end of the chapter will be the summary in section 3.7.

3.1 Zadeh's Fuzzy Mathematics

The problem of uncertainty has long required researchers to find tools that can enable its description and manipulation. The complexity is due to the fact that the uncertainty takes different forms, often in terms of randomness or fuzziness. Randomness had been extensionally handled by researchers, but fuzziness viewed as imprecision, inexactness, vagueness, and complexity has been introduced by Zadeh (1965). He exhibited the difference between both types of uncertainty in introducing Fuzzy Sets Theory *as a means for presenting or dealing with uncertainty.*

We mean by Zadeh's (1965) fuzzy mathematics the intuitive idea of dealing with fuzzy sets. Although fuzzy mathematics aims to develop a methodology for the formulation and solution of problems that are too complex or too ill-defined to be susceptible to analysis by conventional techniques, its theory invokes a functional theory- membership function that is not the regular route. Set to Set class to Set functions of probability space. Consequently, his theory failed to create a true counterpart to probability theory because of the fact that Zadeh's (1978) possibility measure concept (for fuzzy set or fuzzy event) while claimed to play the same role as a probability measure, does not have a particular duality property. The theory leads to a practical set operation approach which is often inconvenient and very complicated.

Since, in both theory and practice a self-duality property is absolutely needed Liu et al (2002) introduced the concept of credibility measure to facilitate working with fuzzy phenomena instead of possibility measure. Credibility measure theory is the field of mathematics that studies the behaviour of fuzzy phenomena and is a theoretical foundation for modeling fuzziness. On the platform of Liu's (2004, 2006) theory, fuzzy variable, random fuzzy variable and fuzzy random variable are all *scalar variables* and characterized by relevant distributions respectively, in much the same way that a random variable is characterized by probability distributions. We no longer need to work with fuzzy variables as set functions and alpha-level cut sets, as required Zadeh's fuzzy mathematics.

Nevertheless, we assume that Zadeh's Extension Principle should be still in force.

3.2 Fuzzy Credibility Measure Theory

Consider Θ a nonempty set and 2^{Θ} its power set (i.e. all subsets of Θ), and assign to each $A \in 2^{\Theta}$ called event, a number $\mathrm{Cr}\{A\}$ which precisely gives the credibility grade that the event A will occur.

Liu (2004) give five axioms listed below for characterising credibility of an event A: $\mathrm{Cr}\{A\}$

Axiom 1 $\mathrm{Cr}\{\Theta\} = 1$

Axiom 2 Cr is increasing, i.e. $\mathrm{Cr}\{A\} \le \mathrm{Cr}\{B\} \quad whenever \quad A \subset B$,

Axiom 3 Cr is self-dual, i.e. $\mathrm{Cr}\{A\} + \mathrm{Cr}\{A^c\} = 1 \quad for \quad any \quad A \in 2^{\Theta}$.

Axiom 4 $\mathrm{Cr}\{\bigcup_i A_i\} \wedge 0.5 = \sup_i \mathrm{Cr}\{A_i\}$, for any $\{A_i\}$ with $\mathrm{Cr}\{A_i\} \le 0.5$,

Axiom 5 Let Θ_k be nonempty sets on which $\mathrm{Cr_k}$ satisfy the first four axioms, $k = 1, 2, \ldots, n.$ respectively, and let $\Theta = \Theta_1 \times \Theta_2 \times \Theta_3 \times \ldots \times \Theta_n$. Then

$\mathrm{Cr}\{(\theta_1, \theta_2, \ldots, \theta_n)\} = \mathrm{Cr}_1\{\theta_1\} \wedge \mathrm{Cr}_2\{\theta_2\} \wedge \ldots \wedge \mathrm{Cr}_n\{\theta_n\}$

for each $(\theta_1, \theta_2, \ldots, \theta_n) \in \Theta$

Notice that $\mathrm{Cr}\{\phi\} = 0$, and, also $0 \le \mathrm{Cr}\{A\} \le 1$, $\forall A \in 2^{\Theta}$

Definition 1 (Liu, 2004), if the set function Cr satisfies the first four axioms then it is called a credibility measure

Theorem 1 (Liu, 2004, Credibility Subadditivity theorem) The credibility measure is sub additive. That is, $\mathrm{Cr}\{A \cup B\} \le \mathrm{Cr}\{A\} + \mathrm{Cr}\{B\} \quad for \quad any \quad A, B \in 2^{\Theta}$

(Proof See Liu, 2004)

Remark 1 If Θ is only a set of two elements, and then the credibility measure is identical with the probability measure.

Definition 2 We define a Credibility space the triplet $(\Theta, 2^{\Theta}, \mathrm{Cr})$ where Θ is a nonempty set, 2^{Θ} the power set of Θ, and Cr the credibility measure; Liu (2006).

Remark 2 The properties of semi continuity, extension, and product are also applicable in credibility theory. We also notice that the credibility measure is a particular case of non-additive measure.

3.2.1 Fuzzy Variable

We describe as a Fuzzy variable ξ a function defined from a credibility space $(\Theta, 2^{\Theta}, \mathrm{Cr})$ to the set of real numbers. As we pointed out in section 3.1, a fuzzy variable is characterized by its credibility distribution which is similar to its counterpart in probability measure theory, where a random variable is characterized by its probability distribution function. Therefore the way of describing fuzzy variable in fuzzy credibility measure theory is different from that in Zahed's fuzzy mathematics.

3.2.2 Credibility Distribution

According to Liu (2002), in contrast to the case of probability distribution function of the random variable; the credibility distribution function $\Lambda : \mathbb{R} \to [0,1]$, of a fuzzy variable ξ is defined by:

$$\Lambda(x) = \mathrm{Cr}\{\theta \in \Theta / \xi(\theta) \le x\} \tag{3.1}$$

Definition 3 According to Liu (2004), the function $\lambda : \mathbb{R} \to [0, +\infty)$ is called credibility density function of a fuzzy variable ξ if and only if we have:

$$\int_{-\infty}^{+\infty} \lambda(y)dy = 1, \quad \Lambda(x) = \int_{-\infty}^{x} \lambda(y)dy, \quad \forall x \in \mathbb{R} \tag{3.2}$$

where Φ is the credibility distribution of the fuzzy variable ξ.

Remark 3

$$\mathrm{Cr}\{a \le \xi \le b\} \ne \int_a^b \lambda(y)dy \tag{3.3}$$

and from Liu (2004) we have

$$\mathrm{Cr}\{\xi \le x\} = \int_{-\infty}^{x} \lambda(y)dy, \quad \mathrm{Cr}\{\xi \ge x\} = \int_{x}^{+\infty} \lambda(y)dy \tag{3.4}$$

Expected value

Due to numerous ways to define expected value operator; we follow Dubois and Prade (1987), Heilpern (1992), Campos and Gonzalez (1989), Gonzalez (1990) and Yager (1981)

Nevertheless, we herein focus on the definition of the expected value applicable to both continuous and discrete fuzzy variables, given by Liu and Liu (2002).

Definition 4 We define the expected value of a fuzzy variable ξ with respect to a distribution Λ, the expression given by:

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\xi \ge r\}dr - \int_{-\infty}^{0} \mathrm{Cr}\{\xi \le r\}dr \tag{3.5}$$

when at least one of the two integrals is finite.

Assume that ξ is a fuzzy variable whose credibility density function λ exists; then from Liu (2002b) we have:

$$E[\xi] = \int_{-\infty}^{+\infty} x\lambda(x)dx \tag{3.6}$$

provided that the Lebesgue integral is finite

If $\lim_{x\to-\infty} \Lambda(x) = 0$ and $\lim_{x\to\infty} \Lambda(x) = 1$, then

$$E[\xi] = \int_{-\infty}^{+\infty} xd\Lambda(x) \tag{3.7}$$

provided that the Lebesgue Stieltjes integral is finite

Remark 4 If ξ and η are independent fuzzy variables with finite expected values, then for any numbers a and b, we have $E[a\xi + b\eta] = aE[\xi] + bE[\eta]$

Variance

The variance of a fuzzy variable indicates the measure of the dispersion of the distribution around its expected value. If the value of variance is small then the fuzzy variable is tightly concentrated around its expected value; and for a large variance value a fuzzy variable has a wide spread around its expected value.

Definition 5 (Liu, 2002) Let ξ be a fuzzy variable with finite expected value e; then the variance of ξ is defined by

$$V[\xi] = E[(\xi - e)^2]. \tag{3.8}$$

Remark 5 If ξ is a fuzzy variable whose variance exists, a and b are real numbers, then $V[a\xi + b] = a^2 V[\xi]$. Also, $V[\xi] = 0$ if and only if ξ is constant.

Suppose that ξ is a fuzzy variable that takes values in [a, b], but whose membership function is otherwise arbitrary. If its expected value is given, say e, then the maximum variance defined by Liu and Liu (2006) in the maximum variance theorem is given by $(e-a)(b-e)$.

3.2.3 Membership Function

Zahed (1965) extended the (set) indicator function into membership function and thus used it to characterize a fuzzy set.

Definition 6 A function is called a set indicator function if

$$\vartheta_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

The indicator function can be considered as the characteristic function that attributes or assigns the value of 0 to the elements which are not members of the set A, and assigns the value 1 to all members of A.

Definition 7 (Zahed, 1965) A membership function is a mapping from set Θ to interval $[0,1]$, denoted as $\mu_A(\theta)$, i.e,

$$\mu_A(\theta):\Theta \rightarrow [0,1] \tag{3.10}$$

The values attributed to different types of elements of the universe indicate the membership grade of the elements related to the set. Such a function is called a membership function and the set defined by it is called a fuzzy set.

The degree of belongingness and strata evolution reflect fuzziness in its membership function. Figure 3.2.1 intuitively demonstrates the fundamental feature of membership function and thus of fuzzy set.

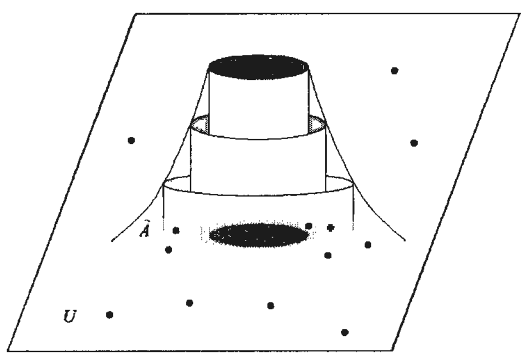

Figure 3.2.1 Membership function

A membership function of a fuzzy variable ξ is now defined as a derived function on the credibility measure. In other words, membership μ is derived from the credibility measure Cr,

$$\mu(x) = (2\,\mathrm{Cr}\{\xi = x\}) \wedge 1, \quad x \in \mathbb{R} \tag{3.11}$$

This function represents the degree of possibility that the fuzzy variable ξ takes a prescribed value.

Remark 6 Strictly speaking, the characterisation of a fuzzy variable does not require membership function on the fuzzy credibility measure theoretical foundation. Credibility distribution is the intrinsic concept associated with a fuzzy variable (membership to possibility measure (grade) does not have the same meaning as that of a credibility distribution grade). Membership function to day plays a role linking Zahed's fuzzy mathematics and Liu's credibility measure theory.

Remark 7 A fuzzy variable ξ has a unique membership function μ, however; a membership function μ may produce several fuzzy variables. See also Liu (2006)

If we know the membership function μ; then the credibility measure Cr can be derived.

Theorem 2 (Credibility Inversion Theorem)

Let ξ be a fuzzy variable with membership function μ. Then for any set B of real numbers, we have

$$\mathrm{Cr}\{\xi \in B\} = \frac{1}{2}(\sup_{x \in B} \mu(x) + 1 - \sup_{x \in B^c} \mu(x)) \tag{3.12}$$

(Proof: see Liu, 2006)

The credibility distribution that the fuzzy variable ξ takes value less than or equal to x (e.g. $\mathrm{Cr}(\xi \leq x)$), if the fuzzy variable ξ is given by a membership function μ, has its credibility distribution function is determined by:

$$\Lambda(x) = \frac{1}{2}(\sup_{y \leq x} \mu(y) + 1 - \sup_{y > x} \mu(y)), \quad \forall x \in \mathbb{R} \tag{3.13}$$

Definition 8 (Liu, 2006) Let ξ be a fuzzy variable with membership μ, then $\forall B \subset \Re$,

$$\mathrm{Cr}(A/B) = \frac{1}{2}[\frac{(2\,\mathrm{Cr}(A \cap B)) \wedge 1}{(2\,\mathrm{Cr}(B) \wedge 1)} + 1 - \frac{(2\,\mathrm{Cr}(A^c \cap B)) \wedge 1}{(2\,\mathrm{Cr}(B) \wedge 1)}] \tag{3.14}$$

provided $\mathrm{Cr}(B) > 0$

3.3 Chance Space and Hybrid Variable

We present an understanding of both the fuzzy random variable and random fuzzy variable using the concepts of chance space and hybrid variable as given by Liu (2006).

Consider a probability space noted by (Ω, A, Pr) , and a credibility space $(\Theta, 2^{\Theta}, \mathrm{Cr})$.

We define a chance space, the product $(\Omega, A, \Pr)\times(\Theta, 2^{\Theta}, \mathrm{Cr})$ of the probability and credibility spaces $\mathbb{R}$.

The hybrid variable $\xi(\omega,\theta)$ is then a function ξ from a chance space $(\Omega, A, \Pr)\times(\Theta, 2^{\Theta}, \mathrm{Cr})$ to the set of real numbers $\mathbb{R}$.

In the case where $\xi(\omega,\theta)$ is a measurable function of ω for each $\theta \in \Theta$, then $\xi(.,\theta)$ is a random variable for each $\theta \in \Theta$. The hybrid variable is a random fuzzy variable, because it is a function from a credibility space $(\Theta, 2^{\Theta}, \mathrm{Cr})$ to the set $\{\xi(.,\theta) : \theta \in \Theta\}$ of random variables.

On the other hand, if $\mathrm{Cr}\{\theta \in \Theta \mid \xi(\omega,\theta) \in B\}$ is measurable function of ω for any set B of real numbers, then the hybrid variable is a fuzzy random variable for it is a measurable function from a probability space (Ω, A, Pr) to the set $\{\xi(\omega,.) : \omega \in \Omega\}$ of fuzzy variables.

From this comment it follows that, fuzzy random variable and random fuzzy variable are special cases of hybrid variables. Nevertheless, it should be noted that our study will be focused on random fuzzy variables.

3.4 Random Fuzzy Variable Theory

We describe as by random fuzzy variable theory, the branch of mathematics that studies the behaviour of Random Fuzzy phenomena.

3.4.1 Random Fuzzy Variable

A random fuzzy variable is a function from the credibility space $(\Theta, 2^{\Theta}, \mathrm{Cr})$ to the set of random variables. Liu (2006)

A random fuzzy variable is also defined as a fuzzy variable taking "random variable" values. Liu (2006)

Moreover, Guo and Zhao (2006), suggested an intuitive explanation of random fuzzy variable in using the concepts of stochastic processes.

Definition 9 (Guo and Zhao, 2006) A random fuzzy variable, denoted as $\xi = \{X_{\beta(\theta)},\ \theta \in \Theta\}$, is a collection of random variables X_β defined on the common probability space $(\Omega, A, \Pr)$ and indexed by a fuzzy variable $\beta(\theta)$ defined on the credibility space $(\Theta, 2^\Theta, \mathrm{Cr})$. In contrast to the interpretation of a stochastic process $X = \{X_t,\ t \in \mathbb{R}^+\}$, a random fuzzy variable is a bivariate mapping from $(\Omega \times \Theta, A \times 2^\Theta)$ to the space $(\mathbb{R}, \beta(\mathbb{R}))$.

In stochastic process theory, the index used is typically time, which is a positive (scalar variable), while in the case of random fuzzy theory the "index" is a fuzzy variable β.

For instance, a random fuzzy variable can be viewed as similar to the stochastic process $X = \{X_{\tau(\omega)},\ \omega \in \Omega\}$ with stopping time $\tau(\omega)$ as its index.

Definition 10 (Liu, 2006) Let ξ and η be random fuzzy variables defined on the Credibility space $(\Theta, 2^\Theta, \mathrm{Cr})$. Then, $\xi = \eta$ if and only if $\xi(\theta) = \eta(\theta)$ for almost all $\theta \in \Theta$

Theorem 3 (Liu, 2006) Let ξ be a random fuzzy variable. If the expected value $E[\xi(\theta)]$ is finite for each θ, then $E[\xi(.)]$ is a fuzzy variable.

(Proof See Liu, 2006)

3.4.2 Random Fuzzy Vector

Definition 11 (Liu, 2006) An n-dimensional random fuzzy vector is a function from the credibility space $(\Theta, 2^\Theta, Cr)$ to the set of n-dimensional random vectors.

Theorem 4 (Liu, 2006)

1) The vector $(\xi, \xi_1, \xi_2, ..., \xi_n)$ is a random fuzzy vector if and only if $\xi_1, \xi_2, ..., \xi_n$ are random fuzzy variables.

2) Let ξ be an n-dimensional random fuzzy vector, and $f : \mathbb{R}^n \to \mathbb{R}$ a measurable function. Then $f(\xi(\theta))$ is a random fuzzy variable

(Proof : See Liu, 2006)

3.4.3 Expected value

Definition 12 (Liu, 2006) Let ξ be a random fuzzy variable. Then the expected value of ξ is defined by:

$$E[\xi] = \int_0^{+\infty} \mathrm{Cr}\{\theta \in \Theta \mid E[\xi(\theta)] \geq r\}dr - \int_{-\infty}^0 \mathrm{Cr}\{\theta \in \Theta \mid E[\xi(\theta)] \leq r\}dr \quad (3.15)$$

provided that at least one of the two integrals is finite

Theorem 5 (Liu, 2006) Assume that ξ and η are independent random variables with finite expected values. Then for any real numbers a and b, we have:

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta] \quad (3.16)$$

(Proof: See Liu, 2006)

3.4.4 Variance

Definition 13 (Liu, 2006) Let ξ be a random fuzzy variable with finite expected value e. Then, the variance of ξ is defined by $V[\xi] = E[(\xi - e)^2]$.

Theorem 6 (Liu, 2006)

1) If ξ is a random fuzzy variable with finite expected value, and a and b are real numbers, then

$$V[a\xi + b] = a^2 V[\xi] \quad (3.17)$$

2) Assume that ξ is a random fuzzy variable whose expected value exists. Then we have

$$V[E[\xi(.)]] \leq V[\xi] \quad (3.18)$$

3) Let ξ be a random fuzzy variable with expected value e. Then

$$V[\xi] = 0 \text{ if and only if } \mathrm{Ch}\{\xi = e\}(1) = 1 \quad (3.19)$$

N.B: For Ch in (3.19) see section 3.6.

(Proof: See Liu, 2006)

3.4.5 Maximum Variance Theorem

Theorem 7 (Liu, 2006)

1) Let f be a convex function on $[a,b]$, and ξ a random fuzzy variable that takes values in $[a,b]$ and has expected value e. Then

$$E[f(\xi(\theta))] \le \frac{b - E[\xi(\theta)]}{b-a} f(a) + \frac{E[\xi(\theta)] - a}{b-a} f(b). \tag{3.20}$$

2) (Maximum Variance Theorem) Let ξ be a random fuzzy variable that takes values in $[a,b]$ and has expected values e, but whose chance distribution (See section 3.6) is otherwise arbitrary. Then

$$V[\xi] \le (e-a)(b-e). \tag{3.21}$$

3) Let ξ be a random fuzzy variable that takes values in $[a,b]$ and has variance υ, but whose chance distribution (See section 3.6) is otherwise arbitrary. Then

$$\frac{a+b-\sqrt{(b-a)^2+4\upsilon}}{2} \le E[\xi] \le \frac{a+b+\sqrt{(b-a)^2+4\upsilon}}{2}. \tag{3.22}$$

(Proof : See Liu, 2006)

3.5 Average Chance Measure of a Random Fuzzy Variable

The chance measure concepts dealt with so far have been mathematically complex. An easier way to deal with them was presented by Guo and Zhao (2006).

In probability theory, the distribution of a random variable ξ on probability space $(\Omega, A, \Pr)$, noted $F_\xi(.)$ is linked to the probability measure of event $\{\omega : \xi(\omega) \le x\}$ by the expression:

$$F_\xi(x) = P(\{\omega : \xi(\omega) \le x\}) \quad \forall \{\omega : \xi(\omega) \le x\} \in A \tag{3.23}$$

For the case of random fuzzy theory, it is believed that the average chance measure plays the same role like that of probability measure P in probability theory.

Definition 14 (Guo and Zhao, 2006) Let ξ be a random fuzzy variable, then the average chance measure denoted by $\mathrm{Ch}(.)$, on a random Fuzzy event $\{\xi \le x\}$, is

$$\mathrm{Ch}\{\xi \le x\} = \int_0^1 \mathrm{Cr}\{\theta \in \Theta \mid \Pr[\xi(\theta) \le x] \ge \alpha\} d\alpha \tag{3.24}$$

From Guo and Zhao (2006), we note six axioms of average chance measure that we can present below as follows:

Let $\mathrm{Ch}(.)$ be an average chance measure on a product measure space $(\Omega \times \Theta, P \times 2^{\Theta})$.

Then we have:

(i) $\mathrm{Ch}\{\phi\} = 0$ and $\mathrm{Ch}\{\Theta\} = 1$;

(ii) (Normality) $\forall A \in 2^{\Theta}, \quad 0 \le \mathrm{Ch}\{A\} \le 1$;

(iii) (Self-Duality) $\forall A \in 2^{\Theta}$ then $\mathrm{Ch}\{A^c\} = 1 - \mathrm{Ch}\{A\}$

(iv) (Weak monotone increasing) for $\forall A \subset B \quad A, B \in 2^{\Theta}, \quad Ch\{A\} \le Ch\{B\}$;

(v) (Semi-Continuity) for $\forall A_n, A \in 2^{\Theta}, \quad n = 1, 2, \ldots, \textit{if} \quad A_n \to A$ then:

$$\lim_{A_n \to A} \mathrm{Ch}\{A_n\} = \mathrm{Ch}\{A\} \tag{3.25}$$

if and only if one of the following conditions holds:

(a) $\mathrm{Cr}(A_n) \le 0.5 \ \& \ A_n \uparrow A$,

(b) $\lim_{n\to\infty} \mathrm{Cr}(A_n) < 0.5 \ \& \ A_n \uparrow A$,

(c) $\lim_{n\to\infty} \mathrm{Cr}(A_n) \ge 0.5 \ \& \ A_n \downarrow A$, and

(d) $\lim_{n\to\infty} \mathrm{Cr}(A_n) > 0.5 \ \& \ A_n \downarrow A$.

(vi) (Sub-additivity) for $\forall A \subset B \quad A, B \in 2^{\Theta}$,

$$\mathrm{Ch}\{A \cup B\} \le \mathrm{Ch}\{A\} + \mathrm{Ch}\{B\} \tag{3.26}$$

3.6 Average Chance Distribution of a Random Fuzzy Variable

We mean by average chance distribution, the function $\Psi(.)$ defined by:

$$\Psi(x) = \mathrm{Ch}\{\xi \le x\} \tag{3.27}$$

The theoretical framework in terms of average chance measure concepts is now established, that is, once the average chance measure for the basic event form $\{\xi \le x\}$ is given, and then the average chance measure for any event A should be established in terms of the basic event $\{\xi \le x\}$. We mean by average chance measure space the triple space given by $(\Omega \times \Theta, P \times 2^{\Theta}, \mathrm{Ch})$.

Let $\Psi(.)$ be average chance distribution of random fuzzy variable ξ on the chance measure space $(\Omega \times \Theta, P \times 2^{\Theta}, \mathrm{Ch})$. Then

(i) $\Psi(-\infty)=0$ and $\Psi(+\infty)=1$;

(ii) For $\forall x \in \mathbb{R}=(-\infty,+\infty)$, $0 \le \Psi(x) \le 1$;

(iii) A non-negative real-valued function $\phi_{\xi}(.)$ is called average chance density for a random fuzzy variable ξ if for $\psi(x) \ge 0$, $x \in \mathbb{R}$ and

$$\Psi(x)=\int_{-\infty}^{x} \psi(\upsilon)d\upsilon \tag{3.28}$$

3.7 Normal Random Fuzzy Variable (NRFV)

Ordinarily, we define a normal random variable X to be a variable whose behaviour is described by a normal distribution denoted by $N(\mu,\sigma^2)$ (i.e. $X \sim N(\mu,\sigma^2)$) where the real numbers μ and σ^2 are respectively mean and variance.

We define a Normal random fuzzy variable X to be a random fuzzy variable having normal distribution $N(\nu,\sigma^2)$ whose mean and the variance do not obey the same rule as that of a normal random variable.

Three cases can be envisaged:

1) The case where the mean ν is fuzzy (i.e. $\nu \sim \mu_{\nu}(.;a_{\nu},b_{\nu},c_{\nu})$), and the variance σ^2 is a fixed non- negative real value;

2) The case where the variance σ^2 is fuzzy (i.e. $\sigma \sim \mu_{\sigma}(.;a_{\sigma},b_{\sigma},c_{\sigma})$), and the mean ν is fixed number;

3) The case where both mean ν and variance σ^2 are fuzzy.

We now examine the two first cases and as a fuzzy variable is described by a membership function, we focus our attention on triangular and trapezoidal membership functions for each case; see also Guo et al (2006).

Normal Random Fuzzy Variable with triangular fuzzy mean and fixed variance

Credibility distribution function for triangular fuzzy mean

From (3.12), we have the triangular credibility distribution function for a fuzzy mean given by:

$$\Lambda(y) = \mathrm{Cr}\{\nu \le y\} = \begin{cases} 0 & y < a_\nu \\ \dfrac{y - a_\nu}{2(b_\nu - a_\nu)} & a_\nu \le y < b_\nu \\ \dfrac{y + c_\nu - 2b_\nu}{2(c_\nu - b_\nu)} & b_\nu \le y < c_\nu \\ 1 & y > c_\nu \end{cases} \tag{3.29}$$

Generally, the fuzzy event for normal random variable $\xi \sim N(\mu, \sigma^2)$ is given by:

$$\begin{gathered} \{\theta \in \Theta : \Pr\{\xi(\theta) \le x\} \ge \alpha\} \\ \{\theta \in \Theta : \Pr\{\xi(\theta) \le x\} \ge \alpha\} \Leftrightarrow \{\nu \in \Theta : \mathbf{x} \ge \nu + \sigma\Phi^{-1}(\alpha)\} \end{gathered} \tag{3.30}$$

Thereafter, the fuzzy event for the case of normal random fuzzy variable with a triangular fuzzy mean can be given by:

$$\{\nu : \Pr\{\xi(\omega, 0) \le x\} \ge \alpha\} \Leftrightarrow \{\nu \in \Theta : \nu \le x - \sigma\Phi^{-1}(\alpha)\} \tag{3.31}$$

The table below gives the range for the integration of the integrand $\mathrm{Cr}\{\nu \in \Theta \mid \Pr\{\xi(\omega, \nu) \le x\} \ge \alpha\}$ with respect to α :

Table 3.7.1 Range with respect to α : (Guo, Zhao and Mi, 2006)

$\nu = g(\alpha) = x - \sigma\Phi^{-1}(\alpha)$	*Range for* α	$\mathrm{Cr}\{\nu \in \Theta \mid \Pr\{\xi(\omega,\nu) \le x\} \ge \alpha\}$
$-\infty < g(\alpha) < a_\nu$	$\Phi\left(\frac{x-a_\nu}{\sigma}\right) < \alpha < 1$	0
$a_\nu \le g(\alpha) < b_\nu$	$\Phi\left(\frac{x-b_\nu}{\sigma}\right) < \alpha < \Phi\left(\frac{x-a_\nu}{\sigma}\right)$	$\frac{x-\sigma\Phi^{-1}(\sigma)-a_\nu}{2(b_\nu-a_\nu)}$
$b_\nu < g(\alpha) < c_\nu$	$\Phi\left(\frac{x-c_\nu}{\sigma}\right) < \alpha < \Phi\left(\frac{x-b_\nu}{\sigma}\right)$	$\frac{x-\sigma\Phi^{-1}(\sigma)+c_\nu-2b_\nu}{2(c_\nu-b_\nu)}$
$g(\alpha) \ge c_\nu$	$0 < \alpha < \Phi\left(\frac{x-c_\nu}{\sigma}\right)$	1

Average chance measure

Consider the event $\{\xi(\omega,\nu) \le x\}$, and using the above table, it follows that the average chance measure can be computed as:

$$\mathrm{ch}\{\xi(\omega,\nu) \le x\} = \int_{\Phi\left(\frac{x-b_\nu}{\sigma}\right)}^{\Phi\left(\frac{x-a_\nu}{\sigma}\right)} \frac{x-\sigma\Phi^{-1}(\sigma)-a_\nu}{2(b_\nu-a_\nu)} d\alpha + \int_{\Phi\left(\frac{x-c_\nu}{\sigma}\right)}^{\Phi\left(\frac{x-b_\nu}{\sigma}\right)} \frac{x-\sigma\Phi^{-1}(\sigma)+c_\nu-2b_\nu}{2(c_\nu-b_\nu)} d\alpha + \int_{0}^{\Phi\left(\frac{x-c_\nu}{\sigma}\right)} 1x d\alpha \tag{3.32}$$

Average chance distribution

By definition the average chance distribution is given by: $\Psi(x) = \mathrm{ch}\{\xi(\omega,\nu) \le x\}$

We find that the expression of the average chance distribution depends on that of the integration $\int_{s_1}^{s_2} \Phi^{-1}(\alpha) d\alpha$ which can be transformed by change of variable, $\alpha = \Phi(u)$, then $d\alpha = \phi(u)du$, where $\phi(.)$ is the probability density function for standard normal variable.

Table 3.7.3 Range with respect to α

$g(\alpha)$	*Range for* α	$Cr\{v \in \Theta : v \le x - \sigma\Phi^{-1}(\alpha)\}$
$-\infty \le g(\alpha) \le a_v$	$\Phi(\frac{x-a_v}{\sigma}) \le \alpha \le 1$	0
$a_v \le g(\alpha) \le b_v$	$\Phi(\frac{x-b_v}{\sigma}) \le \alpha \le \Phi(\frac{x-a_v}{\sigma})$	$\frac{x-\sigma\Phi^{-1}(\alpha)-a_v}{2(b_v-a_v)}$
$b_v \le g(\alpha) \le c_v$	$\Phi(\frac{x-c_v}{\sigma}) \le \alpha \le \Phi(\frac{x-b_v}{\sigma})$	$\frac{1}{2}$
$c_v \le g(\alpha) \le d_v$	$\Phi(\frac{x-c_v}{\sigma}) \le \alpha \le \Phi(\frac{x-d_v}{\sigma})$	$\frac{x-\sigma\Phi^{-1}(\alpha)+d_v-2c_v}{2(d_v-c_v)}$
$g(\alpha) \ge d_v$	$0 \le \alpha \le \Phi(\frac{x-d_v}{\sigma})$	1

Average chance measure

Consider the event $\{\xi(\omega, v) \le x\}$, and using the above table, it follows that the average chance measure can be computed as:

$$\text{ch}\{\xi(\omega,v) \le x\} = \int_{\Phi\left(\frac{x-b_v}{\sigma}\right)}^{\Phi\left(\frac{x-a_v}{\sigma}\right)} \frac{x-\sigma\Phi^{-1}(\alpha)-a_v}{2(b_v-a_v)} d\alpha + \int_{\Phi\left(\frac{x-c_v}{\sigma}\right)}^{\Phi\left(\frac{x-b_v}{\sigma}\right)} \frac{1}{2} \times d\alpha + \int_{\Phi\left(\frac{x-d_v}{\sigma}\right)}^{\Phi\left(\frac{x-c_v}{\sigma}\right)} \frac{x-\sigma\Phi^{-1}(\alpha)+d_v-2c_v}{2(d_v-c_v)} d\alpha + \int_{0}^{\Phi\left(\frac{x-d_v}{\sigma}\right)} 1 \times d\alpha \tag{3.37}$$

Average chance distribution

By definition the average chance distribution is given by $\Psi(x) = \text{ch}\{\xi(\omega,v) \le x\}$

We also find that the expression of the average chance distribution depends on that of the integration $\int_{s_1}^{s_2} \Phi^{-1}(\alpha) d\alpha$ which can be transformed by change of variable $\alpha = \Phi(u)$, so that $d\alpha = \phi(u)du$, where $\phi(.)$ is the probability density function for standard normal variable.

The table below gives various limits for u ;

Table 3.7.4 Limits for u

Limit for α	*Limit for* u
$\Phi\left(\frac{x-a_v}{\sigma}\right)$	$\frac{x-a_v}{\sigma}$
$\Phi\left(\frac{x-b_v}{\sigma}\right)$	$\frac{x-b_v}{\sigma}$
$\Phi\left(\frac{x-c_v}{\sigma}\right)$	$\frac{x-c_v}{\sigma}$
$\Phi\left(\frac{x-d_v}{\sigma}\right)$	$\frac{x-d_v}{\sigma}$

Thus, the expression of the average chance distribution for normal random fuzzy variable with fuzzy trapezoidal fuzzy mean is given by:

$$\begin{aligned}\Psi(x) &= \frac{x-a_v}{2(b_v-a_v)}\left(\Phi\left(\frac{x-a_v}{\sigma}\right)-\Phi\left(\frac{x-b_v}{\sigma}\right)\right)+\\ &+\frac{x+d_v-2c_v}{2(d_v-c_v)}\left(\Phi\left(\frac{x-c_v}{\sigma}\right)-\Phi\left(\frac{x-d_v}{\sigma}\right)\right)+\\ &+\frac{1}{2}\left(\Phi\left(\frac{x-b_v}{\sigma}\right)-\Phi\left(\frac{x-c_v}{\sigma}\right)\right)+\Phi\left(\frac{x-d_v}{\sigma}\right)+\\ &-\frac{\sigma}{2(b_v-a_v)}\int_{\frac{x-b_v}{\sigma}}^{\frac{x-a_v}{\sigma}} u\phi(u)du-\frac{\sigma}{2(d_v-c_v)}\int_{\frac{x-d_v}{\sigma}}^{\frac{x-c_v}{\sigma}} u\phi(u)du\end{aligned} \tag{3.38}$$

Normal Random Fuzzy Variable with triangular standard deviation and fixed mean

Credibility Distribution

The Triangular Credibility Distribution function for fuzzy standard deviation is:

$$\Lambda(z)=\mathrm{Cr}\{\sigma\le z\}=\begin{cases}0 & z<a_\sigma\\ \dfrac{z-a_\sigma}{2(b_\sigma-a_\sigma)} & a_\sigma\le z<b_\sigma\\ \dfrac{z+c_\sigma-2b_\sigma}{2(c_\sigma-b_\sigma)} & b_\sigma\le z<c_\sigma\\ 1 & z>c_\sigma\end{cases} \tag{3.39}$$

The fuzzy event for normal random fuzzy variable with a triangular fuzzy standard deviation can be given by:

$$\{\sigma\in\Theta:\Pr\{\xi(\omega,\sigma)\le x\}\ge\alpha\}\Leftrightarrow\{\sigma\in\Theta:\sigma\le(x-\nu)/\Phi^{-1}(\alpha)\} \tag{3.40}$$

The table below gives the range for the integration of the integrand $\mathrm{Cr}\{\nu\in\Theta\mid\Pr\{\xi(\omega,\nu)\le x\}\ge\alpha\}$ with respect to α :

Table 3.7.5 Range with respect to α : (Guo, Zhao and Mi, 2006)

$\nu=g(\alpha)=(x-\nu)/\Phi^{-1}(\alpha)$	*Range for* α	$\mathrm{Cr}\{\nu\in\Theta\mid\Pr\{\xi(\omega,\nu)\le x\}\ge\alpha\}$
$-\infty<g(\alpha)<a_\sigma$	$\Phi\left(\frac{x-\nu}{a_\sigma}\right)<\alpha<1$	0
$a_\sigma\le g(\alpha)<b_\sigma$	$\Phi\left(\frac{x-\nu}{b_\sigma}\right)<\alpha<\Phi\left(\frac{x-\nu}{a_\sigma}\right)$	$\frac{x-\nu-a_\sigma\Phi^{-1}(\alpha)}{2\Phi^{-1}(\alpha)(b_\sigma-a_\sigma)}$
$b_\sigma<g(\alpha)<c_\sigma$	$\Phi\left(\frac{x-\nu}{c_\sigma}\right)<\alpha<\Phi\left(\frac{x-\nu}{b_\sigma}\right)$	$\frac{x-\nu+(c_\sigma-2b_\sigma)\Phi^{-1}(\alpha)}{2\Phi^{-1}(\alpha)(c_\sigma-b_\sigma)}$
$g(\alpha)\ge c_\nu$	$0<\alpha<\Phi\left(\frac{c_\nu-x}{c_\sigma}\right)$	1

$$\Psi(x)=\frac{a_v}{2(b_v-a_v)}\left(\Phi\left(\frac{x-v}{b_\sigma}\right)-\Phi\left(\frac{x-v}{a_\sigma}\right)\right)+$$

$$+\frac{c_\sigma-2b_\sigma}{2(c_\sigma-b_\sigma)}\left(\Phi\left(\frac{x-v}{b_\sigma}\right)-\Phi\left(\frac{x-v}{c_\sigma}\right)\right)+\Phi\left(\frac{x-v}{c_\sigma}\right)+ \tag{3.42}$$

$$+\frac{x-v}{2(b_\sigma-a_\sigma)}\int_{\frac{x-v}{b_\sigma}}^{\frac{x-v}{a_\sigma}}\frac{1}{u}\phi(u)du+\frac{x-u}{2(c_v-b_v)}\int_{\frac{x-v}{c_\sigma}}^{\frac{x-v}{b_\sigma}}\frac{1}{u}\phi(u)du$$

Trapezoidal credibility distributed standard deviation

The trapezoidal credibility distribution function for fuzzy standard deviation is given by:

$$\Lambda_\sigma(x)=\begin{cases}0 & \text{if } x\le a_\sigma\\ \dfrac{x-a_\sigma}{2(b_\sigma-a_\sigma)} & \text{if } a_\sigma<x\le b_\sigma\\ \dfrac{1}{2} & \text{if } b_\sigma<x\le c_\sigma\\ \dfrac{x+d_\sigma-2c_\sigma}{2(d_\sigma-c_\sigma)} & \text{if } c_\sigma<x\le d_\sigma\\ 1 & \text{if } x>d_\sigma\end{cases} \tag{3.43}$$

Using the same procedure as established above, we obtain the average chance distribution function for trapezoidal fuzzy standard deviation as:

$$\Psi_{V_n}(v)=-\frac{a_v}{2(b_\sigma-a_\sigma)}\left(F_{\chi_n^2}(\frac{v}{a_\sigma^2})-F_{\chi_n^2}(\frac{v}{b_\sigma^2})\right)+$$

$$+\frac{1}{2}\left(F_{\chi_n^2}(\frac{v}{b_\sigma^2})-F_{\chi_n^2}(\frac{v}{c_\sigma^2})\right)+\frac{d_\sigma-2c_\sigma}{2(d_\sigma-c_\sigma)}\left(F_{\chi_n^2}(\frac{v}{c_\sigma^2})-F_{\chi_n^2}(\frac{v}{d_\sigma^2})\right)+$$

$$+F_{\chi_n^2}(\frac{v}{d_\sigma^2})+\frac{\sqrt{n}\Gamma((n-1)/2)}{2^{3/2}(b_\sigma-a_\sigma)\Gamma(n/2)}\left(F_{\chi_{n-1}^2}(\frac{v}{a_\sigma^2})-F_{\chi_{n-1}^2}(\frac{v}{b_\sigma^2})\right)+ \tag{3.44}$$

$$+\frac{\sqrt{n}\Gamma((n-1)/2)}{2^{3/2}(d_\sigma-c_\sigma)\Gamma(n/2)}\left(F_{\chi_{n-1}^2}(\frac{v}{c_\sigma^2})-F_{\chi_{n-1}^2}(\frac{v}{d_\sigma^2})\right)$$

3.8 Sampling Distribution for Normal Random Fuzzy Variable (NRFV)

Sampling for NRFV with triangular fuzzy mean and fixed standard deviation

Consider $\{x_1,\ldots,x_n\}$ i.i.d. simple random sample from normal random fuzzy ξ.

Recall that:

$$\Pr\left(x_i(\omega,\nu)\le x\right)=\Phi\left(\frac{x-\nu}{\sigma_0}\right),\ i=1,2,\ldots,n \tag{3.45}$$

For a fixed mean ν_0; the sample mean is given by:

$$\overline{X}=\frac{1}{n}\sum_{i=1}^{n}x_i \sim N\left(\nu_0,\frac{\sigma_0^2}{n}\right) \tag{3.46}$$

Average Chance Distribution (from Guo et al, 2006)

If the mean is fuzzy parameter $\nu\sim\mu_\nu(\cdot)$, then $\overline{X}$ is a normal random fuzzy variable and the average chance distribution can be obtained by substituting $\sigma_0/\sqrt{n}$ for σ:

$$\Psi_{\overline{X_n}}(x)=\frac{x-a_\nu}{2(b_\nu-a_\nu)}\left(\Phi\left(\frac{x-a_\nu}{\sigma_0/\sqrt{n}}\right)-\Phi\left(\frac{x-b_\nu}{\sigma_0/\sqrt{n}}\right)\right)+$$

$$+\frac{x+c_\nu-2b_\nu}{2(c_\nu-b_\nu)}\left(\Phi\left(\frac{x-b_\nu}{\sigma_0/\sqrt{n}}\right)-\Phi\left(\frac{x-c_\nu}{\sigma_0/\sqrt{n}}\right)\right)+\Phi\left(\frac{x-c_\nu}{\sigma_0/\sqrt{n}}\right)+ \tag{3.47}$$

$$-\frac{\sigma_0/\sqrt{n}}{2(b_\nu-a_\nu)}\int_{\frac{x-b_\nu}{\sigma_0/\sqrt{n}}}^{\frac{x-a_\nu}{\sigma_0/\sqrt{n}}}u\phi(u)du-\frac{\sigma_0/\sqrt{n}}{2(c_\nu-b_\nu)}\int_{\frac{x-c_\nu}{\sigma_0/\sqrt{n}}}^{\frac{x-b_\nu}{\sigma_0/\sqrt{n}}}u\phi(u)du$$

Sampling for NRFV with trapezoidal fuzzy mean and fixed standard deviation

The sampling (probability) distribution theory constitutes the foundation for statistical control charts under random uncertainty. For random fuzzy environments, we face similar requirements in sampling average chance distribution.

For the further developments along quality control charts, we first explore sampling mean statistic

$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$

Average Chance Distribution (from Guo et al, 2006)

We note that for any mean X_0, $\overline{X} \sim N\left(v_0, \sigma^2 / n\right)$ and therefore the average chance distribution for statistic $\overline{X}$ if the mean is a trapezoidal fuzzy variable, is given by:

$$
\begin{aligned}
\Psi(x) = {} & \frac{x-a_v}{2(b_v-a_v)}\left(\Phi\left(\frac{x-a_v}{\sigma^2/n}\right)-\Phi\left(\frac{x-b_v}{\sigma^2/n}\right)\right)+ \\
& +\frac{x+d_v-2c_v}{2(d_v-c_v)}\left(\Phi\left(\frac{x-c_v}{\sigma^2/n}\right)-\Phi\left(\frac{x-d_v}{\sigma^2/n}\right)\right)+ \\
& +\frac{1}{2}\left(\Phi\left(\frac{x-b_v}{\sigma^2/n}\right)-\Phi\left(\frac{x-c_v}{\sigma^2/n}\right)\right)+\Phi\left(\frac{x-d_v}{\sigma^2/n}\right)+ \\
& -\frac{\sigma^2/n}{2(b_v-a_v)}\int_{\frac{x-b_v}{\sigma^2/n}}^{\frac{x-a_v}{\sigma^2/n}} u\phi(u)du-\frac{\sigma^2/n}{2(d_v-c_v)}\int_{\frac{x-d_v}{\sigma^2/n}}^{\frac{x-c_v}{\sigma^2/n}} u\phi(u)du
\end{aligned}
\tag{3.48}
$$

Sampling for NRFV with triangular fuzzy standard deviation and fixed mean

The formula of a sample variance S_n^2 (for a fixed standard deviation σ) follows a Chi-square distribution with degree of freedom $(n-1)$:

$$S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(\frac{x_i - \bar{x}}{\sigma}\right)^2 \sim \chi_{n-1}^2 \tag{3.49}$$

The density distribution function is given by:

$$f_{\chi_{n-1}^2}(y) = \frac{1}{2^{(n-1)/2}} y^{(n-1)/2-1} e^{-y/2}, \quad y > 0 \tag{3.50}$$

The probability distribution function after setting $v = \frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2$ is:

$$\Pr\left\{\chi_{n-1}^2 \le \sigma^2 v\right\} = \frac{1}{2^{(n-1)/2}\Gamma\left((n-1)/2\right)} \int_0^{\sigma^2 v} y^{(n-1)/2-1} e^{-y/2} dy = F_{\chi_{n-1}^2}(\sigma^2 v) \tag{3.51}$$

Average Chance Distribution

From the above statement, a fuzzy event can then be defined by:

$$\left\{\sigma \in \Theta : \Pr\left\{\chi_{n-1}^2 \le \sigma^2 v\right\} \ge \alpha\right\} \tag{3.52}$$

and it follows that:

$$\left\{\sigma \in \Theta : \sigma \ge \sqrt{\frac{1}{v} F_{\chi_{n-1}^2}^{-1}(\alpha)}\right\} \tag{3.53}$$

The limits of α are listed in the table below

Table 3.7.7 Range with respect to α :(Guo, Zhao and Mi, 2006)

$\sigma = g(\alpha) = \sqrt{\frac{1}{v} F_{\chi_{n-1}^2}^{-1}(\alpha)}$	*Range for* α	$Cr\left\{\sigma \in \Theta \mid \Pr\left\{\xi(\omega, v) \le x\right\} \ge \alpha\right\}$
$-\infty < g(\alpha) < a_\sigma$	$0 < \alpha < F_{\chi_{n-1}^2}^{-1}(a_\sigma^2 v)$	1
$a_\sigma \le g(\alpha) < b_\sigma$	$F_{\chi_{n-1}^2}(a_\sigma^2 v) \le \alpha < F_{\chi_{n-1}^2}(b_\sigma^2 v)$	$\dfrac{2b_\sigma - a_\sigma - \sqrt{\frac{1}{v} F_{\chi_{n-1}^2}^{-1}(\alpha)}}{2(b_\sigma - a_\sigma)}$
$b_\sigma < g(\alpha) < c_\sigma$	$F_{\chi_{n-1}^2}(b_\sigma^2 v) \le \alpha \le F_{\chi_{n-1}^2}(c_\sigma^2 v)$	$\dfrac{c_\sigma - \sqrt{\frac{1}{v} F_{\chi_{n-1}^2}^{-1}(\alpha)}}{2(c_\sigma - b_\sigma)}$
$g(\alpha) \ge c_\sigma$	$F_{\chi_{n-1}^2}(c_\sigma^2 v) \le \alpha < 1$	0

Average Chance Measure

$$\text{ch}\{\xi(\omega,v)\le x\} = \int_{F_{\chi^2_{n-1}}(vb_\sigma^2)}^{F_{\chi^2_{n-1}}(vc_\sigma^2)} \frac{c_\sigma - \sqrt{\frac{1}{v}F_{\chi^2_{n-1}}^{-1}(\alpha)}}{2(c_\sigma - b_\sigma)} d\alpha +$$

$$+ \int_{F_{\chi^2_{n-1}}(va_\sigma^2)}^{F_{\chi^2_{n-1}}(vb_\sigma^2)} \frac{2b_\sigma - a_\sigma - \sqrt{\frac{1}{v}F_{\chi^2_{n-1}}^{-1}(\alpha)}}{2(b_\sigma - a_\sigma)} d\alpha + \int_0^{F_{\chi^2_{n-1}}(va_\sigma^2)} 1x d\alpha \qquad (3.54)$$

We transform the expression of average chance measure by change of variable, $\alpha = F_{\chi^2_{n-1}}(u)$ then $d\alpha = f_{\chi^2_{n-1}}(u)du$, where $f_{\chi^2_{n-1}}(.)$ is the Chi-square density function.

Table 3.7.8 Limits for u

Limit for α	Limit for u
$F_{\chi^2_{n-1}}(va_\sigma^2)$	va_σ^2
$F_{\chi^2_{n-1}}(vb_\sigma^2)$	vb_σ^2
$F_{\chi^2_{n-1}}(vc_\sigma^2)$	vc_σ^2

So, the average chance distribution for the sample variance is:

$$\Psi(v) = F_{\chi^2_{n-1}}(a_\sigma^2 v) + \frac{2b_\sigma - a_\sigma}{2(b_\sigma - a_\sigma)}[F_{\chi^2_{n-1}}(b_\sigma^2 v) - F_{\chi^2_{n-1}}(a_\sigma^2 v)] +$$

$$+\frac{c_\sigma}{2(b_\sigma - a_\sigma)}[F_{\chi^2_{n-1}}(c_\sigma^2 v) - F_{\chi^2_{n-1}}(b_\sigma^2 v)] - \frac{1}{2(b_\sigma - a_\sigma)\sqrt{v}} \int_{va_\sigma^2}^{vb_\sigma^2} \sqrt{u}.f_{\chi^2_{n-1}}(u)d\alpha + \qquad (3.55)$$

$$-\frac{1}{2(c_\sigma - b_\sigma)\sqrt{v}} \int_{vb_\sigma^2}^{vc_\sigma^2} \sqrt{u}.f_{\chi^2_{n-1}}(u)d\alpha$$

or

$$\Psi(v) = F_{\chi^2_{n-1}}(a_\sigma^2 v) + \frac{2b_\sigma - a_\sigma}{2(b_\sigma - a_\sigma)}[F_{\chi^2_{n-1}}(b_\sigma^2 v) - F_{\chi^2_{n-1}}(a_\sigma^2 v)] +$$

$$+\frac{c_\sigma}{2(b_\sigma - a_\sigma)}[F_{\chi^2_{n-1}}(c_\sigma^2 v) - F_{\chi^2_{n-1}}(b_\sigma^2 v)] +$$

$$-\frac{1}{2(b_\sigma - a_\sigma)\sqrt{v}}\frac{\sqrt{2}\Gamma(n/2)}{\Gamma((n-1)/2)}[F_{\chi^2_{n-1}}(b_\sigma^2 v) - F_{\chi^2_{n-1}}(a_\sigma^2 v)] + \tag{3.56}$$

$$-\frac{1}{2(c_\sigma - b_\sigma)\sqrt{v}}\frac{\sqrt{2}\Gamma(n/2)}{\Gamma\left((n-1)/2\right)}[F_{\chi^2_{n-1}}(c_\sigma^2 v) - F_{\chi^2_{n-1}}(b_\sigma^2 v)]$$

Sampling for NRFV with trapezoidal fuzzy standard deviation and fixed mean

For a given mean v_0 and a fixed variance σ_0^2, statistic $V_n / \sigma_0^2 = \sum_{i=1}^n \left((X - v_0)/\sigma_0\right)^2 \sim \chi_n^2$ i.e., the statistic V_n / σ_0^2 follows a chi-square distribution with degree of freedom n; we have the following equivalent events

$$\left\{\theta : \Pr\left\{\chi^2_{n-1} \le \frac{V_n(\theta, \omega)}{\sigma^2}\right\} \ge \alpha\right\} \Leftrightarrow \left\{\theta : \sigma(\theta) \le \sqrt{\frac{V_n}{F^{-1}_{\chi^2_n}(\alpha)}}\right\} \tag{3.57}$$

Then, by appropriate partitioning the integration range of α in terms of the trapezoidal standard deviation σ, we obtain the sampling average chance distribution for statistic V_n:

$$\begin{aligned}\Psi(x) = &-\frac{a_\sigma}{2(b_\sigma - a_\sigma)}\left(F_{\chi_n^2}\left(\frac{v}{a_\sigma^2}\right) - F_{\chi_n^2}\left(\frac{v}{b_\sigma^2}\right)\right) + \frac{1}{2}\left(F_{\chi_n^2}\left(\frac{v}{b_\sigma^2}\right) - F_{\chi_n^2}\left(\frac{v}{c_\sigma^2}\right)\right) + \\ &+\frac{d_\sigma - 2c_\sigma}{2(d_\sigma - c_\sigma)}\left(F_{\chi_n^2}\left(\frac{v}{c_\sigma^2}\right) - F_{\chi_n^2}\left(\frac{v}{d_\sigma^2}\right)\right) + F_{\chi_n^2}\left(\frac{v}{d_\sigma^2}\right) + \\ &+\frac{\sqrt{v}\Gamma((n-1)/2)}{2^{3/2}(b_\sigma - a_\sigma)\Gamma(\frac{n}{2})}\left(F_{\chi_n^2}\left(\frac{v}{a_\sigma^2}\right) - F_{\chi_n^2}\left(\frac{v}{b_\sigma^2}\right)\right) + \\ &+\frac{\sqrt{v}\Gamma((n-1)/2)}{2^{3/2}(d_\sigma - c_\sigma)\Gamma(\frac{n}{2})}\left(F_{\chi_n^2}\left(\frac{v}{c_\sigma^2}\right) - F_{\chi_n^2}\left(\frac{v}{d_\sigma^2}\right)\right)\end{aligned} \tag{3.58}$$

3.9 Summary

In this chapter we have discussed the non-classical version of that we call credibility measure theory. For this study of fuzzy phenomena we began by recalling Zadeh's fuzzy mathematics. We used the notion of the fuzzy variable, from which we examined different functions that describe it such as: membership function, credibility distribution function; credibility density function. A substantial emphasis has been placed on one of the two extensions, namely the random fuzzy variable. We defined that concept as a fuzzy variable having random variables values. Its distribution has been examined in terms of both average chance distribution and the average chance density function. We also relied on the particular case of the normal random fuzzy variable whose the distribution we examined under two (2) different cases of its two quality indices (mean and variance). The sampling aspect for the normal random fuzzy variable has been examined for both triangular and trapezoidal fuzzy parameter.

Chapter 4. Small Sample Asymptotics Distribution Theory

In this chapter, we review an approximate distribution for the mean in section 4.1. Section 4.2 will cover the Approximation distribution for maximum likelihood estimates and the case of empirical small sample asymptotic will be discussed in section 4.3, followed in section 4.4 by asymptotic distributions for small samples from random fuzzy variables. The chapter is summarized in section 4.5.

4.1 Approximate Distribution for Mean

4.1.1 One dimension Saddlepoint Approximation

Saddlepoint Approximation

Consider $x_1,...,x_n$ n independent identically distributed (i.i.d) observations with the common distribution function $\mathrm{F}(x)$ and probability density function $\mathrm{f}(x)$.

We also assume that their common moment generating function (CMGF) is:

$$\mathrm{M}(\alpha) = \int_{-\infty}^{+\infty} e^{\alpha x}\, \mathrm{f}(x)dx \tag{4.1}$$

and the cumulant generating function (CGF);

$$\mathrm{K}(\alpha) = \log \mathrm{M}(\alpha) \tag{4.2}$$

where α exists and is a point in the real interval (a,b) containing zero.

The problem is then to approximate the distribution function $\mathrm{F}_n(x)$ or the density function f_n of the arithmetic mean $T_n(x_n,...,x_n) = n^{-1}\sum_{i=1}^{n} x_i$.

According to Ronchetti and Christopher (1990), Wang (1990), Daniels (1954), Field et al (1990) and Reid (1988), the saddlepoint approximation of the distribution of the

statistic $T_n(x_n,...,x_n) = n^{-1}\sum_{i=1}^{n} x_i$ is given by:

$$g_n(x) = [\frac{n}{2\pi K''(\alpha_0)}]^{1/2} \exp\{n[K(\alpha_0) - \alpha_0 t]\} \qquad (4.3)$$

According to Ronchetti and Christopher (1990), this saddlepoint approximation is the leading term of the asymptotic expansion:

$$f_n(t) = [\frac{n}{2\pi K''(\alpha_0)}]^{1/2} \exp\{n[K(\alpha_0) - \alpha_0 t]\}\{1 + \frac{1}{n}[\frac{1}{8}\lambda_4(\alpha_0) - \frac{5}{24}\lambda_3^2(\alpha_0)]\delta^2 + ...\} \qquad (4.4)$$

where the saddlepoint α_0 is the unique root of the equation

$$K'(\alpha_0) = t \qquad (4.5)$$

$$K'(\alpha) = \frac{\int x \exp\{\alpha(t)x\} f(x)dx}{\int \exp\{(\alpha(t)x\} f(x)dx} \qquad (4.6)$$

$$K''(\alpha) = \frac{\int \exp\{(\alpha(t)x\}dx \int x^2 \exp\{\alpha(t)x\} f(x)dx - (\int x \exp\{\alpha(t)x\} f(x)dx)^2}{(\int \exp\{(\alpha(t)x\} f(x)dx)^2} \qquad (4.7)$$

and moreover,

$$\lambda_3(\alpha_0) = K'''(\alpha_0)/[K''(\alpha_0)]^{3/2}$$
$$\lambda_4(\alpha_0) = K^{(iv)}(\alpha_0)/[K''(\alpha_0)]^2 \qquad (4.8)$$

are standardized measures of skewness and kurtosis respectively.

The important property of this approximation is the fact that the error is better than the error given by the Edgeworth expansion (for more details see Ronchetti et al 1990). One can also note that the approximation works well in the tails; in fact, from (4.4)

$$f(t)_n = g_n(x)[1 + 0(1/n)] \qquad (4.9)$$

We can see that $g_n(x) \geq 0$ and that the relative error is of the order n^{-1} uniformly (see also Jensen 1988).

Furthermore, from Daniels (1954), Field (1990), one notes that for a wide class of underlying densities, the coefficient of the term of order n^{-1} does not depend on t. The renormalisation reduces the relative error as much as possible and to bring it down to O($n^{-3/2}$).

According to Lugannani and Rice (1980), one can introduce the saddlepoint formula for the tail probability of the mean $\overline{X}$ as follows:

$$P(\overline{X}) = \exp[n\{k(t) - \alpha_0 t\} + \frac{1}{2}z^2] \times \\ \times [\{1\text{-}\Phi(z)\}(1\text{-}\frac{1}{\sqrt{n}}\frac{\lambda_3}{6}z^3) + \phi(z)\frac{1}{\sqrt{n}}\frac{\lambda_3}{6}(z^2\text{-}1)]\{1\text{+O}(n^{-1})\} \tag{4.10}$$

where α_0 is the solution to $k'(t) = x$, $z = t\{n k''(t)\}^{1/2}$, for $r = 1,\ldots,4$, $\lambda_r = k^{(r)}(t)/\{k''(t)\}^{r/2}$, with $k^{(r)}$ the r^{th} derivative of the cumulant generating function.

Also, note that t is the function of the parameter x as well as any parameter on which the CGF depends; see also Pamela et al (2002)

Another method of deriving saddlepoint approximation is through the conjugate density method:

Conjugate density

The conjugate density method recentres the distribution at the point t of the interest and thereafter, a normal approximation is applied at the centre of the distribution. So, the approximation in the centre is then converted to an approximation for f_n.

According to Feller (1971), Kullback (1960) Ronchetti and Christopher (1990), the conjugate density function is given by:

$$h_t(x) = c(t)\exp\{\alpha(t)(x-t)\}\, f(x) \tag{4.11}$$

where $c(t)$ is given such that h_t is the density and $\alpha(t)$ is chosen such that

$$\int (x-t)\exp\{\alpha(t)(x-t)\}\, f(x)dx = 0 \tag{4.12}$$

i.e. $E_{h_t}X = t$.

and the expression

$$\sigma^2(t) = \int (x-t)^2 \exp\{\alpha(t)(x-t)\} \mathrm{f}(x)dx \tag{4.13}$$

is called the variance of X under the density h_t

The link between the notation of cumulant generating function and that of conjugate density has been given by Ronchetti et al (1991) and at the end of their demonstration they show that both saddlepoint and conjugate methods lead to the same result.

The conjugate density function (4.11) is also considered a local normal approximation. In fact, it has also been obtained by changing the underlying density to the point of interest, using a normal approximation to the mean and also applying centering lemma; see Field et al (1990).

4.1.2 Multivariate Saddlepoint Approximation

Multivariate saddlepoint approximation is explored under two cases: the first derived via multivariate Edgeworth expansion and the second relies on multivariate M-estimators, which will be examined in the next section.

For the first case; we present the result from Reid (1988), and McCullagh (1987).

We consider X a random variable or vector with density $f(x;k)$ where k denote the cumulants of X and the density $\psi(x;k)$.

Then, the one-dimension Edgeworth expansion is given by:

$$\mathrm{f}(x;k) = \psi(x;k)\{1 + \frac{k_3 h_3(x;k)}{6} + \frac{k_4 h_4(x;k)}{24} + \frac{k_3^2 h_6(x;k)}{72} + \ldots\} \tag{4.14}$$

where $h_j(x;k)$ is the Hermite polynomial defined by:

$$h_j(x;k) = (-1)^j \frac{\partial^j \psi(x;k)}{\partial x^j} / \psi(x;k) \tag{4.15}$$

If X is a sum or average of independent identically distributed random variables, then according to Reid (1988), and the standard version of the Edgeworth expansion (Ronchetti et al , 1990) is given at $x = k_1, \quad or \quad z = 0$ by:

$$\frac{1}{\sqrt{2\pi k_2}}\left(1+\frac{3\rho_4}{24}-\frac{15\rho_3^2}{72}+\ldots\right) \tag{4.16}$$

with all the odd powers of $n^{-1/2}$ equals to zero and the expression is in powers of n^{-1} for standard normal variables.

Thus, univariate saddlepoint approximation utilizes this version by an appropriate choice of the parameter of the conjugate density.

For the multivariate version we go from multivariate Edgeworth expansion expressed analogously and the result is again given by Reid (1988):

$$\mathrm{f}(x;k) = \psi(x;k)\{1+\frac{k^{i,j,k}h_{i,j,k}(x;k)}{3!}+\frac{k^{i,j,k,l}h_{i,j,k,l}(x;k)}{4!}+\frac{k^{i,j,k,l,m,n}h_{i,j,k,l,m,n}(x;k)}{6!}+\ldots\} \tag{4.17}$$

where $\psi(x;k)$ is the multivariate normal density of dimension p with vector and covariance matrix matching the mean and covariance X .

The Hermite polynomials are given by:

$$\mathrm{h}_{i,j,k}(x;k) = (-1)^3 \frac{\partial^3 \psi(x;k)}{\partial x^i \partial x^j \partial x^k} / \psi(x;k) \tag{4.18}$$

Using notation from Reid (1988), the random variable is $X = (X^1,\ldots,X^p)$ is denoted by $EX = E(X^1,\ldots,X^p) = (k^1,\ldots,k^p)$; the pxp covariance matrix of X is $(k^{i,j})$, where the moments $k^{i,j}$ is the second cumulant, or covariance of X^i and X^j. Therefore, $k^{i,j,k}$ is the joint third cumulant of X^i, X^j, X^k (in other words that equal to $E[(X^i-k^i)(X^j-k^j)(X^k-k^k)]$), also, $k^{i,j,k,l}$ is the joint fourth cumulant of X^i, X^j, X^k, X^l. Their sizes are respectively $pxpxp$ and $pxpxpxp$ and all indices run from 1 to p.

So the multivariate version of Edgeworth expansion (Ronchetti et al, 1990) above, becomes:

$$(2\pi k_2)^{-p/2}\left|k_{i,j}\right|^{1/2}(1+\frac{3\rho_4-3\rho_{13}^2-2\rho_{23}^2}{24}+...) \tag{4.19}$$

where

$$\rho_4 = k^{i,j,k,l}k_{i,j}k_{k,l},\quad \rho_{13}^2 = k^{i,j,k}k^{l,m,n}k_{i,j}k_{k,l}k_{m,n},\quad \rho_{23}^2 = k^{i,j,k}k^{l,m,n}k_{i,l}k_{j,m}k_{k,n},$$

and (...) in the expression (4.19) is $0(n^{-2})$ under independent identically distributed sampling.

4.2 Approximate Distribution for Maximum likelihood Estimates

4.2.1 Likelihood function

Let X be a random variable, a value in the imposed model $(\mathrm{X},\mathrm{B},\mathrm{P}_\theta,\theta\in\Theta)$. Let $X_i, i=\overline{1,n}$ be n independent random variables having the same law as X, the random vector $\widetilde{X}=(X_1,X_2,...,X_n)$ with value in the model $(\mathrm{X},\mathrm{B},\mathrm{P}_\theta,\theta\in\Theta)^n$ is called a n-sample of X.

The number n is the size of the sample.

Definition 1:

The likelihood function, $\mathrm{L}(x_1,x_2,...,x_n,\theta)$ for $\mathrm{X}^n\times\Theta\to[0,1]$ is defined as:

$$\mathrm{L}(x_1,x_2,...,x_n,\theta)=\prod_{i=1}^{n}\mathrm{f}_\theta(x_i) \tag{4.20}$$

where $\mathrm{f}_\theta(x_i)$ is the law of X_i, $i=1,..,n$ considered as a function of $\mathrm{X}\to[0,1]$.

This function includes the information expressed by the model $(\mathrm{X},\mathrm{B},P_\theta,\theta\in\Theta)^n$.

4.2.2 Maximum Likelihood Estimate

Consider the model $(\mathrm{X},\mathrm{B},P_\theta,\theta\in\Theta)^n$ imposed by a σ-finite measure μ.

Definition 2:

$\hat{\theta}:\mathrm{X}^n\to\Theta$ is said Maximum likelihood Estimator (M.L.E) of θ if: $\forall\tilde{x}\in\mathrm{X}^n,\forall\theta\in\Theta$,

$$\mathrm{L}(\tilde{x},\hat{\theta}(\tilde{x}))\geq\mathrm{L}(\tilde{x},\theta)) \tag{4.21}$$

Example 1:

$X \in U_{[0,\theta]}, \tilde{x} = (x_1,...,x_n)$, $f_\theta(x) = \frac{1}{\theta}.1_{[0.\theta[}(x)$

$$L(\tilde{x},\theta) = \begin{cases} \frac{1}{\theta^n}, & if \quad \tilde{x} \in [0,\theta[^n \\ 0 \quad , & otherwise \end{cases} \tag{4.22}$$

We take $\hat{\theta}(\tilde{x}) = \sup_i x_i <=> \hat{\theta}(\widetilde{X}) = \sup_i X_i \Rightarrow L(\tilde{x}, \sup_i x_i) > L(\tilde{x},\theta) \quad \forall i \quad \sup_i x_i < \theta$.

Then $\hat{\theta}(\widetilde{X})$ is a maximum likelihood estimator

NB: The procedure often used to find the maximum likelihood estimator is to maximize the likelihood function, that is $\partial L(\tilde{x},\theta)/\partial\theta = 0$, and the maximum likelihood estimator is defined to be the unique root of this equation.

4.2.3 Fisher Information

We define the Fisher Information $I_n(\theta)$ carried by a same sample $\widetilde{X}$ on the real parameter θ , the positive quantity:

$$I_n(\theta) = E_\theta[(\frac{\partial \log L(\tilde{x},\theta)}{\partial\theta})^2] \tag{4.23}$$

Theorem 1(Fisher, 1959; Lindsey, 1996):

If the domain of X does not depend of θ, then

$$I_n(\theta) = -E_\theta[\frac{\partial^2 \log L(\tilde{x},\theta)}{\partial\theta^2}] \tag{4.24}$$

Proof

$$\int_{X^n} L(\tilde{x},\theta)dx = 1$$

Using the derivative of log, we get:

$$\frac{\partial L(\tilde{x},\theta)}{\partial\theta} = L(\tilde{x},\theta).\frac{\partial \log L(\tilde{x},\theta)}{\partial\theta}$$

also

$$\frac{\partial}{\partial \theta}\int_{X^n} L(\tilde{x},\theta)dx = 0$$

or

$$\int_{X^n} \frac{\partial L(\tilde{x},\theta)}{\partial \theta} d\tilde{x} = 0$$

since the domain of X does not depend of θ then

$$\int_{X^n} \frac{\partial \log L(\tilde{x},\theta)}{\partial \theta} . L(\tilde{x},\theta)d\tilde{x} = 0$$

or

$$E_\theta[\frac{\partial \log L(\tilde{x},\theta)}{\partial \theta}] = 0$$

then

$$I_n(\theta) = Var(\frac{\partial \log L(\tilde{x},\theta)}{\partial \theta}),$$

therefore $I_n(\theta) > 0$.

Taking the second derivative, it follows that:

$$\frac{\partial^2}{\partial \theta^2}\int_{X^n} L(\tilde{x},\theta)dx = 0$$

or

$$\int_{X^n} \frac{\partial}{\partial \theta}[L(\tilde{x},\theta)\frac{\partial \log L(\tilde{x},\theta)}{\partial \theta}]d\tilde{x} = 0$$

or

$$\int_{X^n} \frac{\partial^2 \log L(\tilde{x},\theta)}{\partial \theta^2} L(\tilde{x},\theta)d\tilde{x} + \int_{X^n} \frac{\partial L(\tilde{x},\theta)}{\partial \theta} \times \frac{\partial \log L(\tilde{x},\theta)}{\partial \theta} d\tilde{x} = 0 .$$

In other words

$$E_\theta[\frac{\partial^2 \log L(\tilde{x},\theta)}{\partial \theta^2}] + E_\theta[(\frac{\partial \log L(\tilde{x},\theta)}{\partial \theta})^2]$$

or

$$E_\theta[\frac{\partial^2 \log L(\tilde{x},\theta)}{\partial \theta^2}] + I_n(\theta) = 0$$

Thus

$$I_n(\theta) = -E_\theta[\frac{\partial^2 \log L(\tilde{x},\theta)}{\partial \theta^2}]$$

4.2.4 Approximate Distribution formula

We recall here that for the case of a large sample, the central limit theorem can always be applied. Nevertheless, if a variable does seem approximately normal (e.g. $\hat{\theta} \sim N\{\theta, I_n(\hat{\theta})^{-1}\}$), then according to Efron (1998), and Durbin (1980), the approximate density of maximum likelihood estimate is given by:

$$p_\theta(\hat{\theta}) \approx (2\pi)^{-1/2} \left|I_n(\hat{\theta})\right|^{1/2} \exp\left\{-\frac{I_n(\hat{\theta})}{2}(\hat{\theta}-\theta)^2\right\} \tag{4.25}$$

N.B: The above result helps us to work with small samples.

The following quadratic approximation has also been shown

$$\log\frac{L(\theta)}{L(\hat{\theta})} \approx -\frac{I_n(\hat{\theta})}{2}(\hat{\theta}-\theta)^2 \tag{4.26}$$

Another form of the approximation can be given by:

$$p_\theta(\hat{\theta}) \approx (2\pi)^{-1/2} \left|I_n(\hat{\theta})\right|^{1/2} \frac{L(\theta)}{L(\hat{\theta})} \tag{4.27}$$

Example 2: (Efron et al, 1998)

1. Let $x_1, x_2, ..., x_n$ be an i.i.d. sample from normal distribution (e.g. $N\{\theta, \sigma^2\}$) with σ^2 known. From the central limit theorem, it is known that $\hat{\theta} = \overline{X}$ is $N\{\theta, \sigma^2 / n\}$. So with the above formula:

$$\log L(\theta) = -\frac{1}{2\sigma^2}\left\{\sum_i (x_i - \bar{x}) + n(\bar{x} - \theta)^2\right\}$$

$$\log L(\theta) = -\frac{1}{2\sigma^2}\sum_i (x_i - \bar{x})^2 \tag{4.28}$$

$$I_n(\hat{\theta}) = \sigma^2 / n$$

Thus,

$$p_\theta(\hat{\theta}) \approx (2\pi)^{-1/2} \left|\sigma^2 / n\right|^{1/2} \exp\left\{-\frac{n}{2\sigma^2}(\bar{x}-\theta)^2\right\} \tag{4.29}$$

As we can see, this is the normal distribution $N(\theta, \sigma^2 / n)$

2. Suppose that y follows the Poisson distribution with mean θ, the MLE of θ is $\hat{\theta} = y$, and the Fisher information is $\mathrm{I}(\hat{\theta}) = 1/\hat{\theta} = 1/y$.

It comes that the formula of approximation is given by:

$$\mathrm{p}_\theta(y) \approx (2\pi)^{-1/2}(1/y)^{1/2} \frac{e^{-\theta}\theta^y / y!}{e^{-y}y^y y!} \tag{4.30}$$

$$\mathrm{p}_\theta(y) \approx (2\pi)^{-1/2}(1/y)^{1/2} \frac{e^{-\theta}\theta^y / y!}{e^{-y}y^y y!}$$

$$\Rightarrow \mathrm{p}_\theta(y) \approx \frac{e^{-\theta}\theta^y / y!}{(2\pi y)^{1/2} e^{-y} y^y y!}$$

Nelder and Pregibon (1987) suggested an approximation of the denominator as $(2\pi(y+1/6)^{1/2}e^{-y}y^y)$ for all y>0, then the approximate formula of Barndorff-Nielsen (1983) is given by:

$$\mathrm{p}^*_\theta(\hat{\theta}) = c(\theta)(2\pi)^{-1/2}\left|I_n(\hat{\theta})\right|^{1/2} \frac{L(\theta)}{L(\hat{\theta})} \tag{4.31}$$

This formula is often o a magic formula; see Efron (1998).

Also, according to Durbin (1980), the exponential family formula (3.29) is exactly the saddlepoint approximation given by:

$$\mathrm{p}_\theta(\hat{\theta}) = c(\theta)(2\pi)^{-1/2}\left|\mathrm{I}_n(\hat{\theta})\right|^{1/2} \frac{\mathrm{L}(\theta)}{\mathrm{L}(\hat{\theta})}\left\{1+O(n^{-3/2})\right\} \tag{4.32}$$

The error is $O(n^{-3/2}) = bn^{-3/2}$, with b bounded.

Notice that, the approximation distribution for the maximum likelihood estimate is not unique, in fact it depends on the nature of the underlying distribution from which the maximum likelihood estimate has been constructed. The examples examined above show us different types of approximation.

Another type of maximum likelihood estimator is the M-Estimator that we discuss in the following sub-section.

4.2.5 One dimension M-Estimators

According to Huber (1964, 1967), M-estimators are the statistics T_n defined as solution of the equation:

$$\sum_{i=1}^{n} \psi(x_i, t) = 0 \tag{4.33}$$

where $x_1, x_2, \ldots, x_n$ denote n independent observations with common density f ; (i.e. $\mathrm{f}(x,\theta)$).
If $\psi(x;\theta) = \partial \mathrm{f}(x,\theta) / \partial\theta$, then the statistics T_n is called the maximum likelihood estimate of θ.

Let $\mathrm{f}_n(t)$ be the density of the statistic T_n; then, the approximation of $\mathrm{f}_n(t)$ can be accomplished in writing T_n as a mean up to a specific order and then using the saddlepoint approximation to the mean as treated above in the case of conjugate density. For M-estimates, according to Ronchetti and Christopher (1990) the conjugate density for a fixed value of t is given by:

$$\mathrm{h}_t(x,\theta) = c(t)\exp\{\alpha(t)\psi(x,t)\}\,\mathrm{f}(x,\theta) \tag{4.34}$$

this conjugate density is used to center T_n at t such that $E_{h_t}(T_n) = t$ up to first order, and the approximation of f_n at t is made by a low-order Edgeworth Expansion. α in (4.34) is the solution of the following equation:

$$\int \psi(x,t)\exp\{\alpha(t)\psi(x,t)\}\,\mathrm{f}(x,\theta)dx = 0 \tag{4.35}$$

4.3 Empirical Small Sample Asymptotics

4.3.1 Empirical Saddlepoint Approximations for the Mean

Let $X_1,...,X_n$ be n independent identically distributed random variables with the common density function $f(x)$ and the finite moment generating function defined by:

$$M(t) = \int_{-\infty}^{+\infty} e^{tx} f(x)dx \tag{4.36}$$

and the cumulant generating function given by:

$$K(t) = \log M(t) \tag{4.37}$$

We recall from Daniels (1954), Lugannani and Rice (1988), Christopher and Ronchetti (1990) that the saddleppoint approximation for density of $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$ is given by:

$$f_n(x) = [\frac{n}{2\pi K''(t)}]^{1/2} \exp\{n[K(t) - tx]\} \tag{4.38}$$

where $t = t(x)$ is the unique real root of the equation $K'(t) = x$.

Nevertheless note that, the computation of the exact analytic forms of $M(t)$ and $K(t)$ given above, often poses problems. One may estimate empirically these transforms by $M_n(t)$ and $K_n(t)$ when sampling from $f(x)$ is possible. $K_n(t)$ denotes the sample version of $K(t)$, defined by:

$$K_n(t) = \log M_n(t) \tag{4.39}$$

with

$$M_n(t) = \frac{1}{n}\sum_{i=1}^{n} e^{tx} \tag{4.40}$$

That leads to the situation of empirical saddlepoint approximation, with new approximation:

$$\hat{f}_n(x) = [\frac{n}{2\pi K_n''(\hat{t})}]^{1/2} \exp\{n[K_n(\hat{t}) - \hat{t}x]\} \tag{4.41}$$

where $\hat{t}$ replacing t in (4.58) is defined such that $K'_n(\hat{t}) = x$ and x is in the interval centered in 0. See also Feuerverger (1989)

It was demonstrated by Feuerverger (1989) that the relative error subject to this new approximation is not good enough for by considering the ratio:

$$\frac{\hat{f}_n(x)}{f_n(x)} = [\frac{K_n''(\hat{t})}{K''(\hat{t})}]^{1/2} \exp\{n[K_n(\hat{t}) - \hat{t}K'_n(\hat{t})] - n[K(t) - tK'(t)]\}, \qquad (4.42)$$

he showed that $\hat{f}_n(x)/f_n(x) \to 1$ as the exponent term in (4.41) tends to zero and $K_n''(\hat{t}) \to K''(\hat{t})$.

To sort out this error problem, one replaces sample version K_n by K_m based on a sample size m with $m > n$. This step leads to the new estimator function defined by:

$$\hat{f}_{m,n}(x) = [\frac{n}{2\pi K_m''(\hat{t})}]^{1/2} \exp\{n[K_m(\hat{t}) - \hat{t}x]\} \qquad (4.43)$$

As constructed, the new $\hat{f}_n$ so found may be a good estimator for f_n.

We also note that $\hat{f}_n(x)$ is defined on the region of x-values where $K'_n(t) = x$ has a solution and the interval of the interest is $(x^{(1)}, x^{(n)})$ from the smallest to largest order statistics.

Theorem 2:

1) Suppose the conditions of the assumption given by Feuerverger (1989) are met, and let $\hat{f}_{m,n}(x)$ denote the saddlepoint approximation $f_n(x)$ but based now on the sample cumulant function $K_m(t)$ from a sample of size m. Then

$$\frac{\hat{f}_{m,n}(x)}{f_n(x)} = 1 + o(m^{\frac{-1}{2}}n), \qquad (4.44)$$

where the error term is uniform over any interval of x-values corresponding to an interval of t-values interior to the domain $I/2$ of the normal convergence.
(Proof see Feuerverger, 1989)

2) Let $g_n(x)$ denote the saddlepoint approximation for the normalized variable $n^{\frac{1}{2}}(\overline{X} - \mu)$, where $\mu = E(X)$ and let $\hat{g}_n(x)$ denote the corresponding normalized empirical saddlepoint approximation, but centred now at $\overline{X}$ instead of μ; then it follows that:

$$\frac{\hat{g}_n(x)}{g_n(x)} = 1 + o(n^{\frac{-1}{2}}), \tag{4.45}$$

(Proof see Feuerverger, 1989)

As the saddlepoint approximation $f_n(x)$ is very close to the distribution of $\overline{X}$, even for very small values of n the use of empirical versions of the $n=1$ saddlepoint approximation $\hat{f}_{m,1}(x)$ appears to be a non-parametric density estimator for $f(x)$.

From:

$$\hat{f}_{m,1}(x) - f(x) = \{\hat{f}_{m,1}(x) - f_1(x)\} + \{f_1(x) - f(x)\} \tag{4.46}$$

we can see that, the error consists of two components, the first represents sampling variability, and the second bias.

A numerical study made by Feuerverger (1989) on these two components of error (4.45) showed that $\hat{f}_{m,1}$ was almost a reliable estimator to $f_1(x)$, whereas $f_1(x)$ was sometimes very close to $f(x)$. His suggestion is that this bias might be reduced through prior adjustment of the sample or by the inclusion of higher terms in the saddlepoint approximation. See also Ronchetti et al (1990)

4.3.2 Empirical Saddlepoint Approximations for One Dimension M-Estimators

Let $x_1, x_2, ..., x_n$ be n independent observations with common probability density function f, and thestatistical function T_n an M-estimator unique solution of the equation:

$$\sum_{i=1}^{n} \psi(x_i, T_n) = 0 \tag{4.47}$$

According to conditions 1 to 5 from Ronchetti and Welsh (1993), the saddlepoint approximation to the probability density function $f_n(t)$ of the statistic T_n made at a fixed point t_0, in a particular compact interval containing zero is given by:

$$f_n(t_0) = (n/2\pi)^{1/2} c^{-n}(t_0)\, A(t_0)/\sigma(t_0)[1 + 0(1/n)] \tag{4.48}$$

where $\alpha(t_0)$ is the solution of the equation:

$$\int \psi(x,t_0)\exp\{\alpha\psi(x,t_0)\}\,\mathrm{f}(x)dx = 0 \tag{4.49}$$

and

$$c^{-1}(t_0) = \int \exp\{\alpha\psi(x,t_0)\}\,\mathrm{f}(x)dx \tag{4.50}$$

$$\sigma^2(t_0) = \mathrm{E}_{t_0}\,\psi^2(x,t_0), \quad \mathrm{A}(t_0) = \mathrm{E}_{t_0}[D\psi(x,t_0)] \tag{4.51}$$

E_{t_0} is the expectation with respect to the conjugate density:

$$\mathrm{h}_{t_0}(x) = c(t_0)\exp\{\alpha(t_0)\psi(x,t_0)\}\,\mathrm{f}(x) \tag{4.52}$$

In setting $n=1$ from (21) one obtains the small sample (or saddlepoint) approximation denoted $\mathrm{g}_1(t)$ at the point t, equivalent to $\mathrm{A}(t)/c(t)\sigma(t)$, where A, c and σ are defined as in (4.51).

So, it appears that $\mathrm{g}_1(t)$ obtained, depends on ψ which defines the M-estimator.

Ronchetti et al (1990) demonstrated by illustrations using the approximation for the mean and four specific underlying distributions, that if g_1 is used as a density estimator, and then two problems occur.

The first problem involves different choices of ψ, since each of them defines a different estimator.

The second problem is that of the underlying density to be estimated enters in the computation of g_1 through A, c, and σ.

In order to sort out the second problem, the underlying distribution in A, c, and σ is replaced by empirical distribution function. One obtains a non-parametric density estimator as follows:

$$\widehat{\mathrm{g}_1}(t) = \widehat{D}_1\widehat{\mathrm{A}}(t)/(\hat{c}(t)\widehat{\sigma}(t)) \tag{4.53}$$

where $\widehat{\alpha}(t)$ is given by the equation:

$$\sum_{i=1}^{n}\psi(x_i,t)\exp\{\widehat{\alpha}(t)\psi(x_i,t)\} = 0 \tag{4.54}$$

and

$$\hat{c}(t) = \left(\frac{1}{n}\sum_{i=1}^{n}\exp\{\widehat{\alpha}(t)\psi(x_i,t)\}\right)^{-1} \tag{4.55}$$

$$\widehat{\mathrm{A}}(t) = \hat{c}(t)\frac{1}{n}\sum_{i=1}^{n}\frac{\partial\psi(x_i,t)}{\partial t}\exp\{\widehat{\alpha}(t)\psi(x_i,t)\} \tag{4.56}$$

$$\hat{\sigma}^2(t) = \hat{c}(t)\frac{1}{n}\sum_{i=1}^{n}\psi^2(x_i,t)\exp\{\hat{\alpha}(t)\psi(x_i,t)\} \tag{4.57}$$

$$\widehat{D}_1 = \left(\int A(t)/(\hat{c}(t)\hat{\sigma}(t))dx\right)^{-1} \tag{4.58}$$

One can also assess the quality of the estimator using the difference $\hat{g}_1(t) - f(t)$ such that:

$$\hat{g}_1(t) - f(t) = [\hat{g}_1(t) - g_1(t)] + [g_1(t) - f(t)] \tag{4.59}$$

Note that the first term (variability) decreases as n increases, and the second one (bias) remains fixed. See also Ronchetti et al (1990).

4.4 Asymptotic Distributions for Small Sample from Random Fuzzy Variable

We now discuss the approximate distribution of random fuzzy variable applicable even when random fuzzy variable is not normally distributed.

We follow different types of approximations reviewed above to design small sample average chance distribution and empirical small sample average chance distribution of random fuzzy variable. We also examine the approximation using maximum likelihood for the case of random fuzzy variable.

4.4.1 Saddlepoint Approximation

Consider $\xi_1, \xi_2, \ldots, \xi_n$ to be n independent identically distributed (i.i.d) random fuzzy variables with the common average chance distribution function $\Psi_\xi(x)$ and the average chance density function $\psi_\xi(x)$ such that:

$$\Psi(x) = \int_{-\infty}^{x}\psi_\xi(x)dv \tag{4.60}$$

We also note from Liu et al (2006) that the characteristic function of a random fuzzy variable ξ with chance distribution $\Psi_\xi(x;\alpha)$ is given by:

$$\varphi(t;\alpha) = \int_{-\infty}^{+\infty} e^{itx} d\Psi_{\xi}(x;\alpha), \quad \forall t \in \Re, \alpha \in (0,1] \tag{4.61}$$

provided that the Lebesgue-Stieltjes integral exists, where

$e^{itx} = \cos tx + i \sin tx$ and $i = \sqrt{-1}$.

From this statement we can extend the definition of characteristic function of random fuzzy variable to the case of average chance distribution as follows:

$$\varphi_{\xi}(t) = \int_{-\infty}^{+\infty} e^{itx} d\Psi_{\xi}(x), \quad \forall t \in \Re \tag{4.62}$$

or

$$\varphi_{\xi}(t) = \int_{-\infty}^{+\infty} e^{itx} \psi_{\xi}(x) dx, \quad \forall t \in \Re \tag{4.63}$$

where $\psi_{\xi}(x)$ is the average chance density as given above,

provided that the Lebesgue-Stieltjes integral exists, where

$e^{itx} = \cos tx + i \sin tx$ and $i = \sqrt{-1}$.

It is well known for non-fuzzy case that the above expression is also called Fourrier transform of the signal and has a maximum at the origin $t = 0$, as $\psi_{\xi}(x) \geq 0$ where $\psi_{\xi}(x)$ is one dimensional continuous positive real-valued random variable. Then for a continuous distribution in replacing it by α one gets the integral:

$$\mathrm{M}_{\xi}(\alpha) = \int_{-\infty}^{+\infty} e^{\alpha x} \psi_{\xi}(x) dx \tag{4.64}$$

which is called moment generating function used to characterize the distribution of an ergodic signal.

The cumulant generating function (CGF) is given by:

$$\mathrm{K}_{\xi}(\alpha) = \log \mathrm{M}_{\xi}(\alpha) \tag{4.65}$$

where α exists and is a point in the real interval (a, b) containing zero.

We can still apply the moment generating function and cumulant generating function when the environment is random fuzzy in replacing the distribution of random variable by that of random fuzzy variable.

Coming back to our need the problem remains that of approximating the average chance distribution function $\Psi_{\xi}(x)$ or the average chance density function $\psi_{\xi}(x)$ of the arithmetic fuzzy sample mean $\tilde{\xi}_n = n^{-1}\sum_{i=1}^{n}\xi_i$.

Following Ronchetti and Christopher (1990), Wang (1990), Daniels (1954), Field et al (1990) and Reid (1988), the approximation or the saddlepoint approximation of the average chance density function of the statistic $\tilde{\xi}_n = n^{-1}\sum_{i=1}^{n}\xi_i$ can be given by:

$$g_n(x) = [\frac{n}{2\pi K_{\xi}''(\alpha_0)}]^{1/2} \exp\{n[K_{\xi}(\alpha_0) - \alpha_0 t]\} \tag{4.66}$$

Recall from Ronchetti and Christopher (1990), this saddlepoint approximation emerges as the leading term of the expansion:

$$f_n(t) = [\frac{n}{2\pi K_{\xi}''(\alpha_0)}]^{1/2} \exp\{n[K_{\xi}(\alpha_0) - \alpha_0 t]\}\{1 + \frac{1}{n}[\frac{1}{8}\lambda_4(\alpha_0) - \frac{5}{24}\lambda_3^2(\alpha_0)]\delta^2 + \ldots\} \tag{4.67}$$

where the saddlepoint α_0 is the unique root of the equation

$$K_{\xi}'(\alpha_0) = t \tag{4.68}$$

$$K_{\xi}'(\alpha) = \frac{\int x\exp\{\alpha(t)x\}\psi_{\xi}(x)dx}{\int \exp\{(\alpha(t)x\}\psi_{\xi}(x)dx} \tag{4.69}$$

$$K_{\xi}''(\alpha) = \frac{\int \exp\{(\alpha(t)x\}dx \int x^2 \exp\{\alpha(t)x\}\psi_{\xi}(x)dx - (\int x\exp\{\alpha(t)x\}\psi_{\xi}(x)dx)^2}{(\int \exp\{(\alpha(t)x\}\psi_{\xi}(x)dx)^2} \tag{4.70}$$

and moreover,

$$\lambda_3(\alpha_0) = K_{\xi}'''(\alpha_0)/[K_{\xi}''(\alpha_0)]^{3/2} \tag{4.71}$$

and

$$\lambda_4(\alpha_0) = K_{\xi}^{(iv)}(\alpha_0)/[K_{\xi}''(\alpha_0)]^2 \tag{4.72}$$

are standardized measures of skewness and kurtosis respectively.

4.4.2 Conjugate density

Consider $\xi_1, \xi_2, ..., \xi_n$ n independent identically distributed (i.i.d) random fuzzy variables with the common average chance distribution function $\Psi_\xi(x)$ and the average chance density function $\psi_\xi(x)$.

Referring to Khinchin (1960), Feller (1971), Kullback (1960) Ronchetti and Christopher (1990), the average chance conjugate density function can be given by:

$$\mathrm{h}_t(x) = c(t)\exp\{\alpha(t)(x-t)\}\psi_\xi(x) \tag{4.73}$$

where $c(t)$ is given such that h_t is the density and $\alpha(t)$ is chosen such that:

$$\int (x-t)\exp\{\alpha(t)(x-t)\}\psi_\xi(x)dx = 0 \tag{4.74}$$

and the expression

$$\sigma^2(t) = \int (x-t)^2 \exp\{\alpha(t)(x-t)\}\psi_\xi(x)dx \tag{4.75}$$

is also called the variance of X under the density h_t

Recall from Ronchetti et al (1991) that both saddlepoint approximation and conjugate density function lead to the same result; and the approximation (4.63) can also be considered as a local normal approximation; see Field et al (1990) for the non-fuzzy case.

4.4.3 Approximate Distribution for Maximum likelihood

Estimates small sample from Random Fuzzy Variable

Likelihood function

Let ξ be a random fuzzy variable, a value in the imposed model $(\mathrm{X}, \mathrm{B}, \mathrm{Cr}_\alpha, \alpha \in \mathrm{H})$. Let ξ_i, $i = 1, ..., n$ be n independent random fuzzy variables having the same law as ξ, the random fuzzy vector $\tilde{\xi} = (\xi_1, \xi_2, ..., \xi_n)$ with value in the model $(\mathrm{X}, \mathrm{B}, \mathrm{Cr}_\alpha, \alpha \in \mathrm{H})^n$, then $\tilde{\xi}$ is called a n-fuzzy sample of ξ and n is the size of the fuzzy sample.

Definition 3:

We define the likelihood function for a random fuzzy variable as the function $L(\xi_1(\theta), \xi_2(\theta), ..., \xi_n(\theta), \alpha)$ where $X^n \times H \to [0,1]$:

$$L(\xi_1(\theta), \xi_2(\theta), ..., \xi_n(\theta), \alpha) = \prod_{i=1}^{n} \psi_\alpha(x_i) \quad (4.76)$$

or

$$L(x_1, x_2, ..., x_n, \alpha) = \prod_{i=1}^{n} \psi_\alpha(x_i) \quad (4.77)$$

where $\xi_i(\theta) = x_i, \forall\ i = 1, ..., n;\ \theta \in \Theta,\ \psi_\alpha(x_i)$ is the law of $\xi_i,\ i = \overline{1, n}$ considered as a function defined from X to $[0,1]$.

The function $L(x_1, x_2, ..., x_n, \alpha)$ includes the information expressed by the model $(X, B, Cr_\alpha, \alpha \in H)^n$.

Maximum Likelihood Estimate

Consider the model $(X, B, Cr_\alpha, \alpha \in H)^n$ dominated by a σ-finite measure μ.

Definition 4:

$\hat{\alpha} : X^n \to H$ is called the maximum likelihood estimator (MLE) of α for a random fuzzy variable if: $\forall \tilde{x} \in X^n, \forall \alpha \in H$:

$$L(\tilde{x}, \hat{\alpha}(\tilde{x})) \geq L(\tilde{x}, \alpha)) \quad (4.78)$$

NB: Another way used to find the maximum likelihood estimator from random fuzzy variable is to maximize the likelihood function, that is $\partial L(\tilde{x}, \alpha) / \partial \alpha = 0$, and the maximum likelihood estimator is considered as the unique root of this equation.

Fisher Information

We can also define fisher information $I_n(\alpha)$ carried by the fuzzy sample $\tilde{\xi}$ on the real parameter α, as the positive quantity:

$$I_n(\alpha) = E_\alpha[(\frac{\partial \log L(\tilde{x}, \alpha)}{\partial \alpha})^2] \quad (4.79)$$

Theorem 3:

If the domain of X does not depend of α, then

$$I_n(\alpha) = -E_\alpha\left[\frac{\partial^2 \log L(\tilde{x},\alpha)}{\partial \alpha^2}\right] \tag{4.80}$$

(Proof, see the non-fuzzy case studied above)

Approximate Distribution formula

We also assume that in the case of a large sample, the central limit theorem can always be applied. Nevertheless, if a variable does seem approximately normal (e.g. $\hat{\theta} \sim N\{\theta, I_n(\hat{\theta})^{-1}\}$), then we can still focus on the results from Efron (1998) and Durbin (1980) to give the approximate density of maximum likelihood estimate (for random fuzzy case) as:

$$\gamma_\alpha(\hat{\alpha}) \approx (2\pi)^{-1/2}\left|\mathrm{I}_n(\hat{\alpha})\right|^{1/2} \exp\left\{-\frac{\mathrm{I}_n(\hat{\alpha})}{2}(\hat{\alpha}-\alpha)^2\right\} \tag{4.81}$$

The following quadratic approximation can also be applied

$$\log\frac{\mathrm{L}(\alpha)}{\mathrm{L}(\hat{\alpha})} \approx -\frac{\mathrm{I}_n(\hat{\alpha})}{2}(\hat{\alpha}-\alpha)^2 \tag{4.82}$$

Another form of the approximation can be given by:

$$\gamma_\alpha(\hat{\alpha}) \approx (2\pi)^{-1/2}\left|\mathrm{I}_n(\hat{\alpha})\right|^{1/2}\frac{\mathrm{L}(\alpha)}{\mathrm{L}(\hat{\alpha})} \tag{4.83}$$

For this case of random fuzzy variable, we can also use the result from Durbin (1980), which states that the exponential family formula (4.80) is exactly the saddlepoint approximation given by:

$$\gamma_\alpha(\hat{\alpha}) = c(\alpha)(2\pi)^{-1/2}\left|\mathrm{I}_n(\hat{\alpha})\right|^{1/2}\frac{\mathrm{L}(\alpha)}{\mathrm{L}(\hat{\alpha})}\{1+O(n^{-3/2})\} \tag{4.84}$$

Note that, as for the non fuzzy case, here too the approximation distribution for the maximum likelihood estimate is not unique; it depends on the nature of the underlying distribution from which the maximum likelihood estimate has been constructed.

4.4.4 Empirical Small Sample Asymptotics

Empirical Saddlepoint Approximations for the Mean

We assume $\xi_1,...,\xi_n$, to be n independent identically distributed random fuzzy variables with the common average chance density function $\psi_\xi(x)$ and the finite moment generating function defined by:

$$M_\xi(t) = \int_{-\infty}^{+\infty} e^{tx}\psi_\xi(x)dx \tag{4.85}$$

and the cumulant generating function given by:

$$K_\xi(t) = \log M_\xi(t) \tag{4.86}$$

From the previous result we recall that the saddleppoint approximation for density of $\bar{\xi} = n^{-1}\sum_{i=1}^{n}\xi_i$ is given by:

$$\psi_n(x) = [\frac{n}{2\pi K_\xi''(t)}]^{1/2} \exp\{n[K_\xi(t) - tx]\} \tag{4.87}$$

where $t = t(x)$ is the unique real root of the equation $K_\xi'(t) = x$.

Nevertheless note that, the computation of the exact analytic forms of $M_\xi(t)$ and $K_\xi(t)$ is often intractable. To alleviate the computation problem we may estimate empirically both $M_\xi(t)$ and $K_\xi(t)$ by $M_n(t)$ and $K_n(t)$ respectively; when sampling from $\psi(x)$ is possible.

$K_n(t)$ denotes the sample version of $K(t)$, defined by:

$$K_n(t) = \log M_n(t) \tag{4.88}$$

with

$$M_n(t) = \frac{1}{n}\sum_{i=1}^{n} e^{tx} \tag{4.89}$$

That leads to the situation of empirical saddlepoint approximation, with new approximation:

$$\hat{\psi}_n(x) = [\frac{n}{2\pi K_n''(\hat{t})}]^{1/2} \exp\{n[K_n(\hat{t}) - \hat{t}x]\} \tag{4.90}$$

where $\hat{t}$ replacing t in (4.76) is defined such that $K'_n(\hat{t}) = x$ and x is in the interval centred at 0.

We can still follow the reasoning of the non fuzzy case to evaluate the efficacy of the method by examining the approximation error.

4.5 Summary

In this chapter, we examined the approximate distribution for mean, especially the one-dimension and multivariate saddlepoint approximation. Conjugate density was also discussed as another way of dealing with saddlepoint approximation. The approximate distribution for maximum likelihood estimates was also examined, as well as the particular case of M-Estimators. Further, empirical small sample asymptotics were discussed as an improvement of approximation methods from small sample asymptotics. The final emphasis of this chapter was on asymptotic distributions for small samples from the Random Fuzzy Variable which is an improvement on small sample asymptotics and empirical small sample asymptotics under a random fuzzy environment. The main purpose of this chapter was to familiarise ourselves with ways of finding asymptotic expansions or approximations of the distribution of the statistics in cases where the computation of the exact distribution is very difficult, that is, for very small sample size n, even down to 1. There is a need to extend the approximation methods and use almost the same procedures as are usually applied in fuzzy environments.

Chapter 5. Random Fuzzy Quality Control Charts

In section 5.1, the concept of average run length (ARL) under random fuzzy measurements is discussed. Then section 5.2 focuses on Duncan-style economic design for normal random fuzzy control charts. A discussion of the efficiency of the method will be offered, and a simulation will be examined. A chapter summary follows in section 5.3.

5.1 The Concept of ARL in Random Fuzzy Quality Control Charts

The ARL under random fuzzy environments is similar to ARL for the ordinary case in chapter 2. The same role is phrased under two cases: in-control and out of control.

Note that, the in-control ARL involves the fuzzy sample points that follow immediately after a change has been specified in the process; and the out-of control ARL is the average length of a run of out-of-control fuzzy sample points that follow immediately after a change has been specified in the process. Also, an ARL curve for a chart used to control the quality of current output can be derived. If one can calculate that indicator then for any given change in a process, for instance; an increase in the process average can be detected.

In this case of fuzzy measurements, a commonly used ARL is the in-control ARL as for the ordinary study.

For the computation of ARL under fuzzy measurements one can consider the general control chart technique where subgroups of size n are taken from the sample; and for each subgroup a statistic is computed and plotted against the corresponding time.

The control limits are computed as indicated below and then inserted into the chart. They are then used for calculating the average chance grade that the statistic plotted, falls outside the control limits for any subgroup.

As illustration, we examine below two cases: ARL for normal random fuzzy measurements with triangular mean and ARL for normal random fuzzy measurements having trapezoidal mean.

5.1.1 ARL under Normal Random Fuzzy Variable with Triangular Mean and Fixed Variance

For the case of an $\overline{X}$-chart under the assumption that the underlying distribution function is that of normal random fuzzy variable, and the fuzzy mean has a triangular membership $\mu_{(a_v, b_v, c_v)}(.)$, the control limits are given by:

$$
\begin{aligned}
&\text{Upper Control Limit} : (a_v + 2b_v + c_v)/4 + 3\sigma_T/\sqrt{n} \\
&\text{Centre-line} \quad : \quad (a_v + 2b_v + c_v)/4 \qquad (5.1) \\
&\text{Lower Control Limit} : (a_v + 2b_v + c_v)/4 + 3\sigma_T/\sqrt{n}
\end{aligned}
$$

where σ_T is random fuzzy standard deviation given by:

$$\sigma_T = \sqrt{\sigma^2 + V} \qquad (5.2)$$

σ in (5.2) is the standard deviation from random variable and V the variance from fuzzy variable defined by:

$$
\begin{aligned}
V(\overline{X}_n \le x) = V &= E\left[\left(\overline{X}_n - E\left[\overline{X}_n\right]\right)^2\right] \\
&= \int_{-\infty}^{+\infty}\left(x - \frac{a_v + 2b_v + c_v}{4}\right)^2 \lambda(\mathrm{x})\mathrm{dx}\,, \qquad (5.3)
\end{aligned}
$$

and $\lambda(\mathrm{x})$ in (5.3) is the credibility density function (See Liu, 2002b for the expectation formula in terms of credibility density function). So (5.3) becomes:

$$
\begin{aligned}
V = &\frac{1}{384(b_v - a_v)}\left[2b_v - 3a_v + c_v\right]^3 + \\
&+ \frac{1}{384}\left(\frac{1}{(b_v - a_v)} - \frac{1}{(c_v - b_v)}\right)\left[2b_v - a_v - c_v\right]^3 + \qquad (5.4) \\
&+ \frac{1}{384(c_v - b_v)}\left[2c_v - a_v - 2b_v\right]^3
\end{aligned}
$$

$a_v < b_v < c_v$

and the average chance distribution function is given by:

$$\Psi_{\overline{X}_n}(x) = \frac{x-a_v}{2(b_v-a_v)}\left(\Phi\left(\frac{x-a_v}{\sqrt{\sigma^2+V}}\right)-\Phi\left(\frac{x-b_v}{\sqrt{\sigma^2+V}}\right)\right)+$$

$$+\frac{x+c_v-2b_v}{2(c_v-b_v)}\left(\Phi\left(\frac{x-b_v}{\sqrt{\sigma^2+V}}\right)-\Phi\left(\frac{x-c_v}{\sqrt{\sigma^2+V}}\right)\right)+ \tag{5.5}$$

$$+\Phi\left(\frac{x-c_v}{\sqrt{(\sigma^2+V)}}\right)-\frac{\sqrt{\sigma^2+V}}{2(b_v-a_v)}\int_{\frac{x-b_v}{\sqrt{\sigma^2+V}}}^{\frac{x-a_v}{\sqrt{\sigma^2+V}}} u\phi(u)du-\frac{\sqrt{\sigma^2+V}}{2(c_v-b_v)}\int_{\frac{x-c_v}{\sqrt{\sigma^2+V}}}^{\frac{x-b_v}{\sqrt{\sigma^2+V}}} u\phi(u)du$$

According to Guo et al (2006), if ϖ denotes the average chance grade when the process is out-of-control, then we have:

$$\varpi = \int_{UCL}^{\infty}\psi(x)dx+\int_{-\infty}^{LCL}\psi(x)dx \tag{5.6}$$

We know that:

$$\int_{UCL}^{\infty}\psi(x)dx = 1-\int_{-\infty}^{UCL}\psi(x)dx = 1-\Psi_{\overline{X}_n}(UCL) \tag{5.7}$$

and

$$\int_{-\infty}^{LCL}\psi(x)dx = \Psi_{\overline{X}_n}(LCL) \tag{5.8}$$

with $\psi(x)$ the average chance density defined by $\psi(x) = d\Psi_{\overline{X}_n}(x)/dx$ where

$\Psi_{\overline{X}_n}(x)$ is average chance distribution function for normal random fuzzy variable with fuzzy mean having a triangular membership function previously defined.

Thus, the average chance grade when the process is out-of-control can be given by:

$$\varpi = 1-\Psi_{\overline{X}_n}[(a_v+2b_v+c_v)/4+3\sqrt{\sigma^2+V}/\sqrt{n}]+\Psi_{\overline{X}_n}[(a_v+2b_v+c_v)/4-3\sqrt{\sigma^2+V}/\sqrt{n}] \tag{5.9}$$

The ARL will be the expected value of a geometric distribution with parameter ϖ (see chapter 2), in other words:

$$ARL = 1/\varpi \tag{5.10}$$

The value of ARL can be approximated by first guessing the values of a_v, b_v, and c_v which specify the position of the triangular membership function of the fuzzy mean. We can assume the variance (σ^2) to be a small fixed real value corresponding to control chart parameter n. The R Program in Appendix A addresses computation burden.

Results and Comments

Table 5.1.1 ARL for Triangular Fuzzy Mean

a_v	b_v	c_v	n	k	σ	V	ϖ	ARL
0	1	2	8	3	2	0.33	0.307	3.26
0	1	2	5	3	2	0.33	0.196	5.10
0	1	2	4	3	2	0.33	0.148	6.76
9	10	11	8	3	2	0.33	0.307	3.26
89	119	131	8	3	2	210.42	0.430	2.33

From the above result, we remind that ARL remains a powerful performance measure for analyzing control chart during the process and here we can see that it can also be derived under random fuzzy measurements; nevertheless, assumptions are necessary for the computation of the ARL.

For in-control ARL; the objective is then to large values of ALR. Accurate assumptions of the form of triangular membership function (position of a_v, b_v, c_v) are desirable so as to avoid high values of the probability for the quality index plotted to fall outside the control zone. For small sample sizes, control limits width and standard deviation can also be selected so that the probability of finding the sample point outside the control zone is as small as possible.

5.1.2 ARL for Normal Random Fuzzy Variable with Trapezoidal Mean and Fixed Variance

An $\overline{X}$-chart for normal random fuzzy variable with trapezoidal fuzzy mean ($\mu_{(a_v,b_v,c_v,d_v)}(.)$) control limits are given as follows:

$$\text{Upper Control Limit}: \frac{a_v+b_v+c_v+d_v}{4}+3\sigma_T/\sqrt{n}$$

$$\text{Centre-line} \quad : \frac{a_v+b_v+c_v+d_v}{4} \tag{5.11}$$

$$\text{Lower Control Limit}: \frac{a_v + b_v + c_v + d_v}{4} - 3\sigma_T / \sqrt{n}$$

where σ_T is the random fuzzy standard deviation given by:

$$\sigma_T = \sqrt{\sigma^2 + V} \tag{5.12}$$

σ is the standard deviation from random variable and V the variance from fuzzy variable defined by:

$$\begin{aligned} V(\overline{X}_n \le x) = V &= E\left[\left(\overline{X}_n - E\left[\overline{X}_n\right]\right)^2\right] \\ &= \int_{-\infty}^{+\infty} \left(x - \frac{a_v + b_v + c_v + d_v}{4}\right)^2 \lambda(\mathrm{x})\mathrm{dx}, \end{aligned} \tag{5.13}$$

where $\lambda(\mathrm{x})$ is the credibility density function (See Liu, 2002b). The variance in (5.13) can be written as:

$$\begin{aligned} V &= \frac{1}{384(b_v - a_v)}\left\{[3b_v - a_v - c_v - d_v]^3 - [3a_v - b_v - c_v - d_v]^3\right\} + \\ &+ \frac{1}{384(d_v - c_v)}\left\{[3d_v - a_v - b_v - c_v]^3 - [3c_v - a_v - b_v - d_v]^3\right\} \end{aligned} \tag{5.14}$$

$a_v < b_v < c_v < d_v$

and the average chance distribution function is given by:

$$\begin{aligned} \Psi_{\overline{X}_n}(x) &= \frac{x - a_v}{2(b_v - a_v)}\left(\Phi\left(\frac{x - a_v}{\sqrt{\sigma^2 + V}}\right) - \Phi\left(\frac{x - b_v}{\sqrt{\sigma^2 + V}}\right)\right) + \\ &+ \frac{x + d_v - 2c_v}{2(d_v - c_v)}\left(\Phi\left(\frac{x - c_v}{\sqrt{\sigma^2 + V}}\right) - \Phi\left(\frac{x - d_v}{\sqrt{\sigma^2 + V}}\right)\right) + \\ &+ \frac{1}{2}\left(\Phi\left(\frac{x - b_v}{\sqrt{\sigma^2 + V}}\right) - \Phi\left(\frac{x - c_v}{\sqrt{\sigma^2 + V}}\right)\right) + \Phi\left(\frac{x - d_v}{\sqrt{\sigma^2 + V}}\right) + \\ &- \frac{\sqrt{\sigma^2 + V}}{2(b_v - a_v)} \int_{\frac{x - b_v}{\sqrt{\sigma^2 + V}}}^{\frac{x - a_v}{\sqrt{\sigma^2 + V}}} u\phi(u)du - \frac{\sqrt{\sigma^2 + V}}{2(d_v - c_v)} \int_{\frac{x - d_v}{\sqrt{\sigma^2 + V}}}^{\frac{x - c_v}{\sqrt{\sigma^2 + V}}} u\phi(u)du \end{aligned} \tag{5.15}$$

So, we have:

The chart of interest is $\overline{X}$-chart and the underlying distribution function of the process performance is led by that of normal random fuzzy variable. Sub-section 5.2.1 deals with characteristics of the process, which are presented in tables (parameters of the process, process factors, time factors, elements in income). In sub-section 5.2.2 we utilize average chance density to derive the probability P that the fuzzy sample point will fall outside the control limits and the probability α of a fuzzy sample point falling outside the control limits when the process is in control.

We also utilize numerical procedures to compute the parameters.

5.2.1 Outline

The principle of the economic design of quality control charts under random fuzzy measurements is the same as that of the ordinary case (in chapter 2). In other words the computation of the three parameters of the process indicating the sample size (n), the interval time (h) of withdrawing successive samples during the process, and the control limits factor (k) remains the same here. However the change is that now we use average chance distribution function for the computation of these control chart parameters.

Also, recall that compared to the objective of the statistical design of control charts, which ensure the achievement of pre-selected levels of type I error and type II errors or returning the process to a state of control when it has been in an out-of-control state, the economic design under random fuzzy environment also aims to keep in-control the existing quality levels of the process or to keep the process in the state of control.

During the economic design of control charts, some researchers construct a cost function and another income function, with the purpose of determining the required control chart parameters. In addition, during the stage of correcting or eliminating assignable causes; some researchers shut down the process until the problem brought by assignable causes is sorted out, but others leave the process working while they sort out the problem.

We follow the Duncan-style economic design control charts by constructing net income per hour which is a function of chart design variables whose differences with respect to n, h, k are set equal to zero, yielding three equations which lead to optimize values of these design variables.
So, according to Goh (1996), these three values help to achieve the balance between the acceptable product quality level and the acceptable economic level for the concerned production.
Also, in Duncan's approach the process is not shut down while sorting out the problem brought into the production process by an assignable cause. The process continues to work.

Referring to Liu and Liu (2002) and Guo et al (2006), we can compute the mean of average chance distribution for $\overline{X}$-chart for normal random fuzzy variable as follows:

$$E[\xi] = \int_{0}^{+\infty} \mathrm{Cr}\left\{\nu \in \Theta : \mathrm{E}_{\infty}\left[\xi(\omega,\nu)\right] \geq r\right\} dr - \int_{-\infty}^{0} \mathrm{Cr}\left\{\nu \in \Theta : \mathrm{E}_{\infty}\left[\xi(\omega,\nu)\right] \leq r\right\} dr \quad (5.19)$$

We also have the variance given by:

$$V[\xi] = E\left[\left(\xi(\omega,\nu) - E[\xi(\omega,\nu)]\right)^2\right] \quad (5.20)$$

We here also examine both ARL cases: normal random fuzzy sample mean with triangular membership function and normal random fuzzy sample mean with trapezoidal membership function.

Duncan-Style Economic Design under Normal Random Fuzzy Variable with triangular mean

For a particular case of normal random fuzzy sample mean with triangular membership function (again see Liu and Liu, 2002; Guo et Zhao, 2006), we have the expectation of the mean $\overline{X}_n$ given by:

$$E\left[\overline{X}_n\right] = \frac{a_\nu + 2b_\nu + c_\nu}{4} \quad (5.21)$$

The variance of $\overline{X}_n$ is also given by:

$$V\left[\overline{X}_n\right] = \sigma_T^{\,2} / n \quad (5.22)$$

and the standard deviation:

$$\sigma_{\overline{X}_n} = \sigma_T / \sqrt{n} \quad (5.23)$$

where σ_T is given in (5.2) and $\Psi_{\bar{X}_n}(x)$ given in (5.5)

To the above statements we can add that the centreline, upper and lower control limits for controlling the current process, can be established using the past fuzzy mean and standard deviation under state of control. We can present these lines as follows:

$$\text{Upper Control Limit}: \frac{a_v"+2b_v"+c_v"}{4}+k\sigma_T"/\sqrt{n}$$

$$\text{Centerline}: \frac{a_v"+2b_v"+c_v"}{4} \tag{5.24}$$

$$\text{Lower control limit}: \frac{a_v"+2b_v"+c_v"}{4}-k\sigma_T"/\sqrt{n}$$

where $(a_v"+2b_v"+c_v")/4$, and $\sigma_T"$ are random fuzzy standard measurements taken from the past state of control. Note that the process starts with the state of control when the random fuzzy process mean $(a_v'+2b_v'+c_v')/4$ is equal to $(a_v"+2b_v"+c_v")/4$ and the random fuzzy process standard deviation σ_T'equal to $\sigma_T"$

As soon as a shift in $(a_v'+2b_v'+c_v')/4$ occurs when the current fuzzy process mean varies from $(a_v"+2b_v"+c_v")/4$ to $(a_v"+2b_v"+c_v")/4+\delta\sigma_T"$ or from $(a_v"+2b_v"+c_v")/4$ to $(a_v"+2b_v"+c_v")/4-\delta\sigma_T"$ then the opportunity of finding an assignable cause can be noted and an inspection has to be made so as to detect quickly the cause, take corrective action and return the process back into its state of control. However, if the fuzzy sample point $(a_v'+2b_v'+c_v')/4$ falls outside the control limits, then the presence of an assignable cause is effective and the need remains (as for ordinary case) to eliminate the assignable cause and return the process back into its state of control.

We follow Duncan (1956) to redefine the net income function per hour under the random fuzzy environment.

Process Description

Recall that in this section, we follow Duncan-style economic design, to reconstruct the net income function per hour for the case of normal random fuzzy variable with fuzzy mean having triangular membership function. Our aim here is the determination of the sample size (n) to be employed; the interval time (h) between successive samples; and the multiplier (k) of sigma to be used so as to determine control limits for maximal income.

Table 5.2.1 Process Parameters

Notation	*Explanation*
n	Sample size
h	Interval between samples
k	Control limit factor

Table 5.2.2 Process Factors

Notation	*Explanation*
Q	Probability that the assignable cause will go undetected, or probability that a sample point will fall within control limits when there has been change in the process mean, or probability of acceptance of sample (point)
P	Probability that the assignable cause will be detected, or the probability that a sample point will fall out-side the control limits, or the probability of rejection of the sample (point)
α	Probability of a sample point falling outside the control limits when the process is in control
$Q^{r-1}P$	Probability that after assignable cause has been occurred it will be found on the r^{th} sample taken after the shift
$1/p$	The mean number of samples taken before the shift in the process is detected

Table 5.2.3 Time factors

Notation	*Explanation*
$\left(h/2-\lambda h^2/12\right)$	The average time for the assignable cause to occur in the interval between samples
$1/\lambda$	Average time for the assignable cause to occur
$(h/p)-\left(h/2-\lambda h^2/12\right)$	Average time the process will be outside of the control limits
en	The time to take and inspect a sample and to compute the results
D	Average time taken to find an assignable cause after a point falls outside the control limits

$\beta = \dfrac{1/\lambda}{1/\lambda + (1/p - 1/2 + \lambda h/12)h + en + D}$	The proportion time a process will be in control
$\lambda = \dfrac{(1/p - 1/2 + \lambda h/12)h + en + D}{1/\lambda + (1/p - 1/2 + \lambda h/12)h + en + D}$	The proportion time a process will be out of control
$\dfrac{\alpha}{\lambda h}$	The expected number of false alarms before the process goes out of control
$\dfrac{\beta\alpha}{h}$	The expected number of false alarms per hour of operations
$1/\lambda + (1/p - 1/2 + \lambda h/12)h + en + D$	Average cycle until next assignable cause occurs
$\varepsilon = \dfrac{1}{1/\lambda + (1/p - 1/2 + \lambda h/12)h + en + D}$	Average number of times per hour that the process actually goes out of control

Table 5.2.4 Elements of Income

Notation	*Explanation*
V_0	Average income per hour of the process under control conditions at standard centerline level
V_1	Average income per hour at the new levels (upper or lower control limits)
T	Cost of seeking an assignable cause when none exists
$\dfrac{\beta\alpha T}{h}$	Expected loss per hour between cause of false alarms
W	Average cost of finding the assignable cause when it occurs
εW	Average cost per hour of finding an assignable cause when it occurs
$(b/h) + (cn/h)$	Hourly cost of maintaining the control chart related to its design
b	The cost per sample of sampling and charting independent of sample size
c	The cost per unit of measurement of an item of product and other control chart operations directly to the size of the sample
$I = \beta V_0 + \gamma V_1 - \beta\dfrac{\alpha T}{h} - \varepsilon W - \dfrac{b}{h} - \dfrac{cn}{h}$ or $I = V_0 - \dfrac{\lambda MB + \alpha T/h + \lambda W}{1 + \lambda B} - \dfrac{b}{h} - \dfrac{cn}{h}$	The net income per hour
$L = \dfrac{\lambda MB + \alpha T/h + \lambda W}{1 + \lambda B} + \dfrac{b}{h} + \dfrac{cn}{h}$	Loss-cost

where $M = V_0 - V_1$ with assumption that the process was originally centered between the specification limits.

$$\chi+\gamma=1,\ B=(ah+en+D),\ a=(1/p-1/2+\lambda h/12) \tag{5.25}$$

5.2.2 Optimum Design Aspect

It is assumed by Duncan (1956) that optimum values of the process parameters have a relationship since the partial derivatives of L with respect to n, h, and k set equal to zero give three differential equations.

After transformation by first nominating the values of α with given $\lambda, M, e, d, T, W, B, c$ so as to eliminate the neglected quantities; the system of 3 equations can be written as follows:

$$\frac{-\lambda h^2 M(\partial p/\partial n)}{p^2}+c=0 \quad (1)$$

$$\lambda h^2 M(1/p-1/2)-\alpha T-b-cn=0 \quad (2) \tag{5.26}$$

$$\frac{-\lambda h^2 M(\partial p/\partial k)}{p^2}+T\,\partial\alpha/\partial k=0 \quad (3)$$

Note that, for non fuzzy case, according to Lorenzen et al (1986), the in-control time is exponentially distributed with mean $1/\lambda$ and during the search state, if the process continues, the average time for occurrence of the assignable cause is $1/\lambda$.

From these statements, under random fuzzy measurements, especially the case of normal random fuzzy, we can state that the in-control time is a random fuzzy variable, exponentially distributed as $\lambda(\theta)e^{-\lambda(\theta)t}, \theta\in\Theta$ with the same formula for the mean or average time for occurrence of the assignable cause as for the ordinary case.

However, the difference is that, the parameter λ of the exponential distribution is a fuzzy variable defined on the credibility space $(\Theta, 2^{\Theta}, Cr)$ having membership function μ_λ indicating the grade of possibility that the fuzzy variable $\lambda(\theta)$ takes some prescribed value.

To be specific, we consider a fuzzy variable $\lambda(\theta)$ having triangular membership function define by:

$$\mu_v(x) = \begin{cases} \dfrac{x-a_v}{b_v-a_v}, & if \quad a_v \le x \le b_v \\ \dfrac{x-b_v}{c_v-b_v}, & if \quad b_v \le x \le c_v \\ 0 \ , & \text{otherwise} \end{cases} \tag{5.27}$$

and the credibility distribution defined by:

$$\begin{aligned} &\Lambda_v : \mathrm{R} \to [0;1] \\ &x \to \Lambda_v(x) = \frac{1}{2}\left(\sup_{y \le x} \mu_v(y) + 1 - \sup_{y > x} \mu_v(y) \right) \end{aligned} \tag{5.28}$$

or

$$\Lambda_v(x) = \begin{cases} 0, & if \quad x \le a_v \\ \dfrac{x-a_v}{2(b_v-a_v)}, & if \quad a_v \le x \le b_v \\ \dfrac{x-b_v}{2(c_v-b_v)}, & if \quad b_v \le x \le c_v \\ 1, & if \quad x \ge c_v \end{cases} \tag{5.29}$$

The credibility density function is given by:

$$\lambda_v(x) = \begin{cases} \dfrac{1}{2(b_v-a_v)}, & if \quad a_v \le x \le b_v \\ \dfrac{1}{2(c_v-b_v)}, & if \quad b_v \le x \le c_v \\ 0, & \text{otherwise} \end{cases} \tag{5.30}$$

From these above statements, under a random fuzzy environment, especially the case of the normal random fuzzy, we can intuitively re-write the probability P that the assignable cause will be detected, or the probability that a fuzzy sample point will fall outside the control limits, by:

$$P = \int_{-\infty}^{-k-\delta\sqrt{n}} \psi_{\overline{X}_n}(z)dz + \int_{k-\delta\sqrt{n}}^{\infty} \psi_{\overline{X}_n}(z)dz \tag{5.31}$$

$$=1+(\Psi_{\overline{X}_n}(-k-\delta\sqrt{n})-\Psi_{\overline{X}_n}(k-\delta\sqrt{n}))$$

Also, the probability α of a fuzzy sample point falling outside the control limit when the process is in control can be given as:

$$\alpha=\int_{-\infty}^{-k}\psi_{\overline{X}_n}(z)dz+\int_{k}^{\infty}\psi_{\overline{X}_n}(z)dz \qquad (5.32)$$

$$=1+(\Psi_{\overline{X}_n}(-k-\delta\sqrt{n})-\Psi_{\overline{X}_n}(k-\delta\sqrt{n}))$$

where $\psi_{\overline{X}_n}(x)=d\Psi_{\overline{X}_n}(x)/dx$ is the average chance density of the normal random fuzzy variable with triangular fuzzy mean. From Guo et al (2006) we have the average chance distribution function of the normal random fuzzy variable:

$$\Psi_{\overline{X}_n}(x)=\frac{x-a_v}{2(b_v-a_v)}\left(\Phi\left(\frac{x-a_v}{\sigma_T}\right)-\Phi\left(\frac{x-b_v}{\sigma_T}\right)\right)+$$

$$+\frac{x+c_v-2b_v}{2(c_v-b_v)}\left(\Phi\left(\frac{x-b_v}{\sigma_T}\right)-\Phi\left(\frac{x-c_v}{\sigma_T}\right)\right)+ \qquad (5.33)$$

$$+\Phi\left(\frac{x-c_v}{\sigma_T}\right)-\frac{\sigma_T}{2(b_v-a_v)}\int_{\frac{x-b_v}{\sigma}}^{\frac{x-a_v}{\sigma}}u\phi(u)du-\frac{\sigma_T}{2(c_v-b_v)}\int_{\frac{x-c_v}{\sigma}}^{\frac{x-b_v}{\sigma}}u\phi(u)du$$

where σ_T is defined in (5.5), $\Phi(x)$ is standard normal distribution function and $\phi(x)$ density function.

So, in applying the formula of derivative of the integral for P with respect to k and n, and for α with respect to k, we get:

$$\frac{\partial P}{\partial k}=-\psi_{\overline{X}_n}(-k-\delta\sqrt{n})-\psi_{\overline{X}_n}(k-\delta\sqrt{n}) \qquad (5.34)$$

and

$$\frac{\partial P}{\partial n}=-\frac{\delta}{2\sqrt{n}}\psi_{\overline{X}_n}(-k-\delta\sqrt{n})+\frac{\delta}{2\sqrt{n}}\psi_{\overline{X}_n}(k-\delta\sqrt{n}) \qquad (5.35)$$

also

$$\frac{\partial \alpha}{\partial k}=-\psi_{\overline{X}_n}(-k)-\psi_{\overline{X}_n}(k) \qquad (5.36)$$

Thus, from the system of equations given above we can rewrite Duncan's results under random fuzzy measurements as follows:

$$h = \sqrt{\frac{\alpha T + b + cn}{\lambda M(1/p - 1/2)}} \quad (1)'$$

$$-n + \frac{p^2(1/p - 1/2)}{\partial P / \partial n} = \frac{\alpha T + b}{c} \quad (2)' \qquad (5.37)$$

$$\frac{[\psi_{\overline{X}_n}(-k) + \psi_{\overline{X}_n}(k)][\psi_{\overline{X}_n}(-k - \delta\sqrt{n}) - \psi_{\overline{X}_n}(k - \delta\sqrt{n})]}{\psi_{\overline{X}_n}(-k - \delta\sqrt{n}) + \psi_{\overline{X}_n}(k - \delta\sqrt{n})} = \frac{2c\sqrt{n}}{\delta T} \quad (3)'$$

where (1)' was obtained from (2), and (2)' was obtained by substituting (1)' in (1), and finally (3)' was obtained in combining (1) and (3) and using the transformation of the derivatives $\partial P / \partial n$, $\partial P / \partial k$, and $\partial \alpha / \partial k$.

Procedure for Computation of Process Parameters: n, h and k

The procedure to find the approximated optimum values of n, h and k for fixed values of $\delta, \lambda, M, e, D, T, b,$ and c begins from the equations (1)', (2)', (3)'.

Recall that, the reasoning adopted by Duncan in this case study leant much on the standard normal distribution function, which is symmetric and positive.

Unfortunately, in our case, average chance density distribution function as found and used in integration for the computation of the probabilities P and α is neither symmetric nor positive.

Therefore the transformation of these probabilities cannot be made in the same way as used by Duncan.

In order to simplify the computation issues we proceed first by fixing the value of the third parameter k=3; then our discussion and assumption will be made with fixed value of control limit width; afterwards, we will compute numerical values of α and P using R program.

Nevertheless, we make the assumptions as follows: first, the standard deviation from random variable is supposed to be small ($\sigma = 1$ or $\sigma = 2$ for example), and the position of the triangular (membership function) has to be considered such that the values of a_v, b_v, c_v be close to that of control limit width k. In other words; the deviation of the values of a_v, b_v, c_v has to be small in expecting that α (the chance for the fuzzy sample point to fall outside control zone while the process is in its state of control) will have a small value as well. Once α is computed, we utilize economic data (from Duncan, 1956) for $\delta = 2$ to compute the quantity $(\alpha T + b)/c$, which corresponds to $-n + [p^2(1/p - 1/2)]/(\partial p/\partial n)$, and then we determine the value of n in using Duncan's graph, (see Table 5.2.5 or Figure 5.2.1)

Note that as long as the underlying distribution remains a standard normal distribution, we can utilize Duncan's graph or Duncan's table to find the value of n which helps to compute P and finally h.

The R program in Appendix A was used for the computation of α and P and the results are listed in the table 5.2.6 below.

Table 5.2.5 Economic design of $\overline{X}$ charts. (from Duncan, 1956 p.235)

Selected value of $[p^2(1/p-1/2)/(\partial p/\partial n)]-n=(\alpha T+b)/c$ for $\delta=2$

n	$k=2$	$k=2.5$	$k=3$
2	0.4	-------	--------
3	3.3	0.4	--------
4	14.5	3.7	0.0
5	54.5	14.6	3.3
6	202.3	48.4	12.6
7	750.6	156.4	38.9
8		507.5	111.9
9			328.8

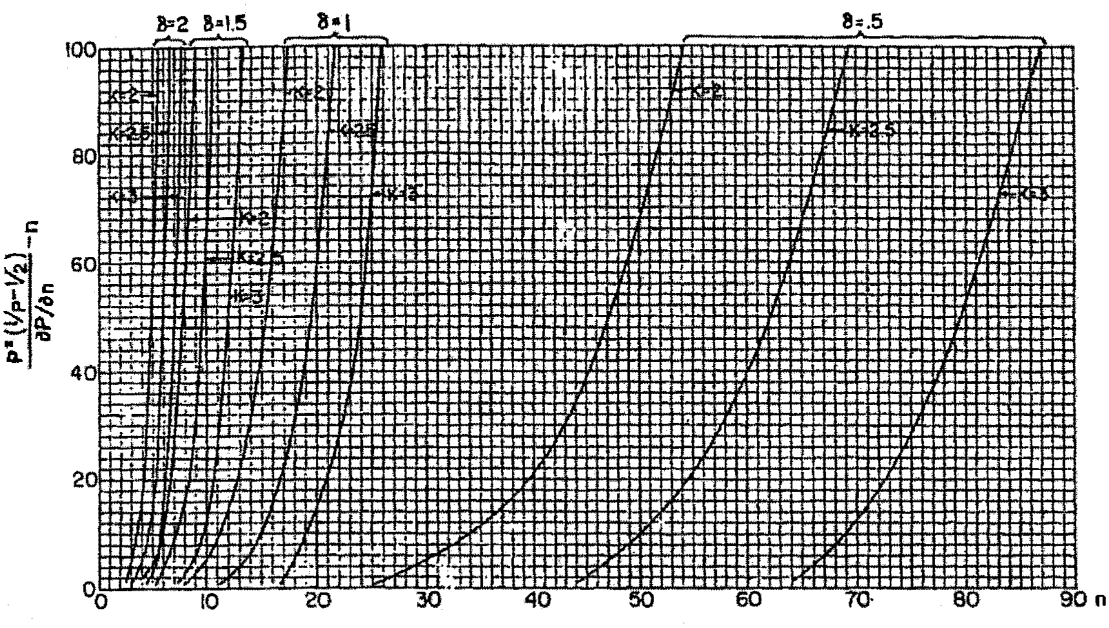

FIG. I

GRAPH OF $\frac{P^2\,(1/P-1/2)}{\partial P/\partial n}-n$ AS A FUNCTION OF n

Figure 5.2.1: Economic design of $\overline{X}$-charts. (Duncan, 1956, p.235)

Example: Computation of the process parameters n, h and k for the case of triangular fuzzy mean and fixed variance using Table 5.2.5 or Figure 5.2.1

$\delta = 2, M = \$100, e = .05D = 2, T = \$50, w = \$25, b = \$.50, c = \$.10,$
$\lambda = 0.01, av = 0, bv = 1, cv = 2$

We assume for this case that: $\lambda = \lambda(\theta_0) = 0.01$ with $\theta_0 \in \Theta$

We start by fixing control limit width $k = 3$, the standard deviation from random variable $\sigma = 2$ and variance from fuzzy variable $V = 0.33$. With these values of k and σ we go to program R to compute the value of the probability $\alpha = 0.209$. Then we use this value of α and economic data given above to calculate the expression:

$[p^2(1/p - 1/2)]/(\partial p/\partial n) - n = (\alpha T + b)/c = (0.209 \times 50 + 0.5)/0.10 \approx 214$ and refer to Duncan table or Duncan graph to find the corresponding value of n.

The use of the table can be explained as follows:

Once we have found the value of $(\alpha T + b)/c$ as computed above Duncan's table provides the corresponding value of n for the approximate value of $(\alpha T + b)/c \approx 214$, $\delta = 2$ and $k = 3$. In the third column of the table we have the approximate value of $(\alpha T + b)/c \approx 111.9$ which corresponds to the value of n on the same line than $(\alpha T + b)/c$ and situated in the first column $n \approx 8$.

We can also use the Duncan's graph to find the same value of $n \approx 8$. In fact, using the graph, we have the vertical axis represented by:

$[p^2(1/p - 1/2)]/(\partial p/\partial n) - n = (\alpha T + b)/c = (0.209 \times 50 + 0.5)/0.10 \approx 214$; and the horizontal representing the values of n. On the top of the graph we have values of δ, for this case $\delta = 2$; and in the graph we have boundaries or zones specifying different values of δ and in each zone there are lines, which indicate the values of control limit width k.

So, after calculating the value $(\alpha T + b)/c \approx 214$ we draw an imaginary horizontal line going from the vertical line and crossing the zone of $\delta = 2$ (See fig.5.2.1); then in the zone we take the point of intersection between that line drawn and the one that specifies the value of $k = 3$. This point of intersection is projected on the horizontal line, which falls on $n \approx 8$.

We return to the R program to compute the probability $p \approx 0.955$ and finally we calculate the value of the time of withdrawing samples $h = \{(\alpha T + b + cn)/[\lambda M(1/p - 1/2)]\}^{1/2} \approx 4.63$.

We apply this process of computation to find all values of our process parameters. The results are given in the table below:

Table 5.2.6.a. Economic Design of $\overline{X}$ -Chart

		___Assumed Cost and Risk Factors(from Duncan) ______								------------------ Ours ----------------------					Approximate ------ Optimum Chart		
Example Number	Degree of shift- δ	λ	M($)	e	D	T($)	W($)	b($)	c($)	a_ν	b_ν	c_ν	σ	V	n	h	K (fixed)
1	2	.01	100	.05	2	50	25	.50	.10	0	1	2	2	.33	8	4.63	3
2	2	.02	100	.05	2	50	25	.50	.10	0	1	2	2	.33	8	3.28	3
3	2	.03	100	.05	2	50	25	.50	.10	0	1	2	2	.33	8	2.68	3
4	2	.02	50	.05	2	50	25	.50	.10	0	1	2	2	.33	8	4.63	3
5	2	.01	1,000	.05	2	50	25	.50	.10	0	1	2	2	.33	8	1.47	3
6	2	.01	10,000	.05	2	50	25	.50	.10	0	1	2	2	.33	8	0.46	3
7	2	.01	100	.05	2	50	25	.50	.10	0	1	2	2	.33	8	4.63	3
8	2	.01	100	.05	20	50	25	.50	.10	0	1	2	2	.33	8	4.63	3
9	2	.01	100	.05	2	5	2.5	.50	.10	0	1	2	2	.33	8	2.07	3
10	2	.01	100	.05	2	500	250	.50	.10	0	1	2	2	.33	8	13.91	3
11	2	.01	100	.50	2	50	25	.50	.10	0	1	2	1	.33	6	2.64	3
12	2	.01	100	.50	2	50	25	.50	.10	0	1	2	.5	.33	6	1.97	3
13	2	.01	100	.05	2	50	25	5.00	.10	0	1	2	2	.33	8	5.45	3
14	2	.01	100	.05	2	50	25	.50	1.00	0	1	2	2	.33	6	5.32	3
15	2	.01	100	.05	2	50	25	.50	10.00	0	1	2	2	.33	4	8.44	3
16	2	.01	1,000	.05	2	50	25	.50	1.00	0	1	2	2	.33	5	1.57	3
17	1	.01	12.87	.05	2	50	25	.50	.10	0	1	2	2	.33	25	13.32	3
18	1	.01	128.70	.05	2	50	25	.50	.10	0	1	2	2	.33	25	4.21	3
19	1	.01	12.87	.05	2	500	250	.50	.10	0	1	2	2	.33	26	37.67	3
20	1	.01	12.87	.05	2	50	25	.50	1.00	0	1	2	2	.33	18	18.24	3
21	2	.07	100	.05	2	50	25	.50	.10	0	1	2	2	.33	8	1.75	3
22	2	.07	50	.05	2	50	25	.50	.10	0	1	2	2	.33	8	2.48	3
23	2	.01	100	.05	2	50	25	.50	.10	0	1	2	2	.33	8	4.63	3
24	2	.14	100	.05	2	50	25	.50	10.00	0	1	2	2	.33	4	2.26	3
25	2	.15	100	.05	2	500	250	.50	.10	0	1	2	2	.33	8	3.51	3
26	2	.01	100	.05	2	50	25	.50	.10	9	10	11	12	.33	8	10.12	3
27	2	.01	100	.05	2	50	25	.50	.10	89	119	131	161	210.42	8	10.13	3

Table 5.2.6.b. Economic Design of $\overline{X}$-Chart

	Approximate Optimum chart (from Duncan)			Exact Optimum Chart (from Duncan)			------------ ------ Loss-cost for 100 hours of operation--------------------				
							(ours)	(from Duncan)	(from Duncan)	(ours)	(from Duncan)
Example Number	n	h	k	n	h	k	Approximate Optimum($)	Approximate Optimum($)	Exact Optimum($)	n=5,h=1 k=3	n=5,h=1 k=3
1	5	1.3	3.2	5	1..3	3..2	159.54	402.13	401.35	1,411.42	1,345.74
2	5	0.9	3.2				314.40	696.55		1,663.95	1,599.10
3	5	0.8	3.2				562.90	962.39		1,903.64	1,893.63
4	5	1.3	3.2				157.85	416.49		1,400.64	1,336.82
5	5	0.3	3.2	4	0.4	3.0	2,658.46	2,756.67	2,698.09	3,844.54	3,769.07
6	5	0.1	3.2				61,637.97	23,682.73		28,175.72	28,002.35
7	5	1.3	3.2	2	0..9	2.7	159.54	608.86	540.35	1,411.42	1,345.74
8	5	1.3	3.2	6	1.8	3.2	453.27	1,840.78	1,836.55	2,706.31	2,650.54
9	4	1.3	2.4				220.05	362.14		474.45	466.91
10	6	1.3	3.8	6	1.4	3.7	132.38	659.37	636.93	10,781.08	10,134.06
11	5	1.3	3.2				300.15	402.13		893.17	1,345.74
13	7	3.3	3.2	6	3.6	2.4	151.37	589.86	586.85	1,871.25	1,795.74
14	2	1.6	2.6	3	2.6	2.4	160.57	594.13	563.62	1,871.25	1,795.74
15	2	7.0	1.4				289.22	1,010.74		6,371.25	6,295.74
16	2	0.5	2.6	3	0.8	2.4	2,662.01	3,241.65	3,176.19	4,403.44	4,219.07
17	17	5.6	2.8	14	5.4	2.7	576.82	142.82	141.78	1,175.87	1,111.13
18	17	1.9	2.8	12	1.6	2.6	872.65	647.50	626.66	1,489.06	1,423.02
19	22	6.0	3.5	20	6.1	3.4	4,379.49	365.83	364.79	10,545.49	9,889.45
20	8	10.4	2.0	8	12.0	1.9	1,204.90	245.05	243.34	1,625.86	1,561.13
21	5	0.8	3.2				1,491.45	962.39		2,801.58	1,893.63
22	5	1.3	3.2				634.13	416.49		1,957.86	1,336.82
23	5	0.3	3.2	4	0.4	3.0	340.80	2,756.67	2,698.09	1,421.25	3,769.07
24	5	0.1	3.2				2,582.33	23,682.73		8,933.35	28,002.35
25	5	1.3	3.2	2	0..9	2.7	2,776.20	608.86	540.35	6,475.35	1,345.74
26	5	1.3	3.2	5	1..3	3..2	118.24	402.13	401.35	5,253.32	1,345.74
27	5	0.9	3.2				118.29	696.55		5,258.19	1,599.10

Discussion

From the table 5.2.6 above, it emerges that several assumptions have to be established so as to obtain the required results of the method.

The first assumption is to establish the relationship between the membership function of fuzzy mean (fuzzy variable) and control limits width k (here k=3).

We have to be sure that without another factor that assumption alone can ensure a small probability α for the sample point plotted to fall outside control zone when the process is under control.

Another assumption pertains to the time interval between withdrawing samples during the production process.

From the table, we can see that the probability α and the interval time h are proportional. In fact, when the interval time h is small we have a small probability α, and when the interval time h is large, the probability α is also large. This means that the chance of producing non-conforming products of undesirable quantity is obvious when h is too large.

We also need to take note of the deviation from the target value, that is, the standard deviation, which is supposed to be small as well. Large standard deviation leads to the large probability α, which we hush to avoid when we are expecting good quality of products.

The table below also gives the comparison between the loss-cost of the approximate optimum design and that of an arbitrary design $\overline{X}$ -chart at n=5, h=1 and k=3.

Further, we note that the variation of the sample size is inversely linked to the variation of δ. A small value of δ (for instance less or equal to 0.5) requires a large sample size (more or equal to 40); whereas a large value of δ requires small sample size.

Hence, we can claim that $\overline{X}$ chart is used to detect large shifts but not small shifts.

On the other hand, we remark that the value of h varies conversely with the loss rate M.

The variations of both cost of seeking trouble when none exists (T) and average cost of seeking trouble when it does exist (W) modify slightly the value of n. As shown by Duncan for the ordinary case, these costs also have an effect upon the optimum value of k for random fuzzy measurements; variations in the unit cost of inspection and charting (c) and the delay factor (e) also have an effect on n, h and k.

The variation of the cost of visiting the process to take a sample (b) affects n and h. We also notice that, the approximate optimum loss-cost obtained for this case of random fuzzy measurements is a slightly lower than for the ordinary case when the sample size is not too small and the interval time h is also quite large (at least 3 hours). Otherwise as we can see for the arbitrary case, n=5, h=1, k=3, the loss-cost for fuzzy measurements is generally higher than that of the corresponding non-fuzzy case.

Duncan-style Economic Design under normal random fuzzy variable with trapezoidal mean

For the case of normal random fuzzy sample mean with trapezoidal membership function (again see Liu and Liu, 2002), we have the expectation of the mean $\overline{X}_n$ given by:

$$E\left[\overline{X}_n\right] = \frac{a_v + b_v + c_v + d_v}{4} \tag{5.38}$$

The variance of $\overline{X}_n$ is also given by:

$$V\left[\overline{X}_n\right] = \sigma_T^{\,2} / n \tag{5.39}$$

and the standard deviation

$$\sigma_{\overline{X}_n} = \sigma_T / \sqrt{n} \tag{5.40}$$

where σ_T and $\Psi_{\overline{X}_n}(x)$ defined in ARL part for the case of trapezoidal fuzzy mean.

Thus we can claim (with emphasis on statistical design under normal random fuzzy measurements) that the centreline, upper and lower control limits for controlling the current process, can be established in using the past fuzzy mean and fuzzy standard deviation during a state of control. We can present these lines as follows:

$$\begin{aligned} &\text{Upper control limit}: \frac{a_v'' + b_v'' + c_v'' + d_v''}{4} + k\sigma_T'' / \sqrt{n} \\ &\text{Centerline}: \frac{a_v'' + b_v'' + c_v'' + d_v''}{4} \\ &\text{Lower control limit}: \frac{a_v'' + b_v'' + c_v'' + d_v''}{4} - k\sigma_T'' / \sqrt{n} \end{aligned} \tag{5.41}$$

where $(a_v'' + b_v'' + c_v'' + d_v'')/4$, and σ_T'' are fuzzy standard measurements

Note that the process starts with the state of control when the random fuzzy process mean $(a_v' + b_v' + c_v' + d_v')/4$ is equal to $(a_v'' + b_v'' + c_v'' + d_v'')/4$ and the fuzzy process standard deviation σ_T' equal to σ_T''

As soon as a shift in $(a_v'+b_v'+c_v'+d_v')/4$ occurs when the current random fuzzy process mean varies from $(a_v''+b_v''+c_v''+d_v'')/4$ to $(a_v''+b_v''+c_v''+d_v'')/4+\delta\sigma_T''$ or from $(a_v''+b_v''+c_v''+d_v'')/4$ to $(a_v''+b_v''+c_v''+d_v'')/4-\delta\sigma_T''$, and the opportunity of finding an assignable cause is evident. An inspection is made so as to detect quickly the cause, correct the action and return the process back into its state of control. However, if the fuzzy sample point $(a_v'+b_v'+c_v'+d_v')/4$ falls outside the control limits, then the presence of assignable cause is effective and the need is to eliminate the assignable cause and return the process back into its state of control.

Procedure for Computation of Process Parameters: n, h and k

As in the case of triangular fuzzy mean, we also proceed here by first fixing the value of the third parameter k=3. Then our procedure will be established with fixed value of control limits width k, and we will compute numerical values of α and p using the R program. Nevertheless, we also make assumptions as follows: first, the standard deviation is supposed to be small ($\sigma=1$ or $\sigma=2$ for example), and the position of the trapezium has to be considered such that, the values of a_v, b_v, c_v, and d_v be close to the value of control limit width k. In other words; the deviation of the values of a_v, b_v, c_v, and d_v from k has to be small to warrant a small value of α as well. Once α is computed, we utilize economic data (from Duncan, 1956) for $\delta=2$ (or $\delta=1$); to calculate the quantity $(\alpha T+b)/c$, which corresponds to $-n+[p^2(1/p-1/2)]/(\partial p/\partial n)$, and then we determine the value of n using Duncan's graph or Duncan's table, (See Table 5.2.5 or Figure 5.2.1).

We recall that as long as the underlying distribution function remains the standard normal distribution, we can utilize the Duncan's graph; to find n, and compute p, then finally h.

The average chance distribution function $\Psi_{\overline{X}_n}(.)$ of the normal random fuzzy variable with trapezoidal fuzzy mean has been previously defined. $\Phi(.)$ indicates the standard normal distribution function with density $\phi(.)$ as specified above in the previous section. R Program is used in Appendix A for the computation of α and P; the results are listed in the following table:

Table 5.2.7.a. Economic Design of $\overline{X}$ -Chat

	Assumed Cost and Risk Factors (from Duncan)									Ours						Approximate Optimum Chart		
Example Number	Degree of shift- δ	λ	M($)	e	D	T($)	W($)	b($)	c($)	a_ν	b_ν	c_ν	d_ν	σ	V	n	h	K (fixed)
1	2	.01	100	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	8	5.60	3
2	2	.02	100	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	8	3.24	3
3	2	.03	100	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	8	3.24	3
4	2	.02	50	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	8	5.61	3
5	2	.01	1,000	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	8	1.77	3
6	2	.01	10,000	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	8	0.56	3
7	2	.01	100	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	8	5.61	3
8	2	.01	100	.05	20	50	25	.50	.10	0	1	2	3	2	1.08	8	5.61	3
9	2	.01	100	.05	2	5	2.5	.50	.10	0	1	2	3	2	1.08	8	2.31	3
10	2	.01	100	.05	2	500	250	.50	.10	0	1	2	3	2	1.08	8	17.10	3
11	2	.01	100	.50	2	50	25	.50	.10	0	1	2	3	1	1.08	6	4.79	3
12	2	.01	100	.50	2	50	25	.50	.10	0	1	2	3	.5	1.08	6	4.47	3
13	2	.01	100	.05	2	50	25	5.00	.10	0	1	2	3	2	1.08	8	6.31	3
14	2	.01	100	.05	2	50	25	.50	1.00	0	1	2	3	2	1.08	6	6.42	3
15	2	.01	100	.05	2	50	25	.50	10.00	0	1	2	3	2	1.08	4	9.23	3
16	2	.01	1,000	.05	2	50	25	.50	1.00	0	1	2	3	2	1.08	5	1.86	3
17	1	.01	12.87	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	25	14.48	3
18	1	.01	128.70	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	25	4.58	3
19	1	.01	12.87	.05	2	500	250	.50	.10	0	1	2	3	2	1.08	26	40.99	3
20	1	.01	12.87	.05	2	50	25	.50	1.00	0	1	2	3	2	1.08	18	43.81	3
21	2	.07	100	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	8	2.12	3
22	2	.07	50	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	8	3.00	3
23	2	.01	100	.05	2	50	25	.50	.10	0	1	2	3	2	1.08	8	5.61	3
24	2	.14	100	.05	2	50	25	.50	10.00	0	1	2	3	2	1.08	4	2.47	3
25	1	.15	100	.05	2	500	250	.50	.10	0	1	2	3	2	1.08	8	4.00	3
26	2	.01	100	.05	2	50	25	.50	.10	9	10	11	12	2	1.08	8	10.12	3
27	2	.01	100	.05	2	50	25	.50	.10	89	119	131	161	2	516.00	8	10.13	3

Table 5.2.7.b Economic Design of $\overline{X}$ -Chat

	Approximate Optimum chart (from Duncan))			Exact Optimum Chart (from Duncan)			Loss-cost for 100 hours of operation (ours)	(from Duncan)	(from Duncan)	(ours)	(from Duncan)
Example Number	n	h	k	n	h	k	Approximate Optimum($)	Approximate Optimum($)	Exact Optimum($)	n=5,h=1 k=3	n=5,h=1 k=3
1	5	1.3	3.2	5	1..3	3..2	146.79	402.13	401.35	1,904.06	1,748.01
2	5	0.9	3.2				320.27	696.55		2,149.25	1,996.97
3	5	0.8	3.2				505.80	962.39		2,381.65	2,232.00
4	5	1.3	3.2				143..94	416.49		1,879.02	1,727.28
5	5	0.3	3.2	4	0.4	3.0	23,61.604	2,756.67	2,698.09	4,402.92	4,241.71
6	5	0.1	3.2				52,181.17	23,682.73		29,391.55	29,178.68
7	5	1.3	3.2	2	0..9	2.7	146.52	608.86	540.35	1,904.06	1,748.81
8	5	1.3	3.2	6	1.8	3.2	385.73	1,840.78	1,836.55	3,124.75	2,992.00
9	4	1.3	2.4				205.22	362.14		530.30	514.17
10	6	1.3	3.8	6	1.4	3.7	122.33	659.37	636.93	15,641.74	14,086.40
11	5	1.3	3.2				196.78	402.13		1,415.19	1,748.01
13	7	3.3	3.2	6	3.6	2.4	189.79	589.86	586.85	2,354.06	2,198.01
14	2	1.6	2.6	3	2.6	2.4	178.48	594.13	563.62	2,354.06	2,198.01
15	2	7.0	1.4				166.09	1,010.74		6,854.06	6,698.01
16	2	0.5	2.6	3	0.8	2.4	1,024.16	3,241.65	3,176.19	4,852.92	4,691.71
17	17	5.6	2.8	14	5.4	2.7	22.93	142.82	141.78	1,662.38	1,956.592
18	17	1.9	2.8	12	1.6	2.6	466.17	647.50	626.66	1,974.10	1,506.63
19	22	6.0	3.5	20	6.1	3.4	529.48	365.83	364.79	15,412.12	13,844.98
20	8	10.4	2.0	8	12.0	1.9	10.35	245.05	243.34	2,112.38	1,956.592
21	5	0.8	3.2				118.24	962.39		3,201.47	2,232.00
22	5	1.3	3.2				118.17	416.49		2,367.25	1,727.28
23	5	0.3	3.2	4	0.4	3.0	1,310.46	2,756.67	2,698.09	1,904.06	4,241.71
24	5	0.1	3.2				561.89	23,682.73		9,266.21	29,178.68
25	5	1.3	3.2	2	0..9	2.7	146.52	608.86	540.35	16,219.56	1,748.81
26	5	1.3	3.2	5	1..3	3..2	2,347.35	402.13	401.35	5,282.83	1,748.01
27	5	0.9	3.2				3,801.07	696.55		5,258.17	1,996.97

Comments

From the above table, it comes out that the technique also works as stated in the previous study. Nevertheless, some defects can also be noted: The technique is generally costly for very small values of control charts parameters (for instance n=5, h=1, k=3), and also a certain number of assumptions have to be made while dealing with this technique.

We remind that the membership function of the fuzzy variable has an important influence over control limit width. For this case we used trapezoidal membership function with the values a_v, b_v, c_v, and d_v (which specify the position of the trapezium) near to that of control limits width k. However, if the variation is large enough, we obtain high probability α (of finding a fuzzy sample point outside control zone). And such value of probability indicates the opportunity to produce undesirable quantity of non-conforming products (or items). Also, the fuzzy variable (λ) from in-control time has to be well specified since the variation of λ can cause high cost.

We also underlined several economic data to influence the value of control charts parameters, which lead to high cost. The value of economic data can also be well selected so as to get the required control charts parameters

5.3 Simulation

We recall that, the normal random fuzzy variable is given by: $X_\nu \sim> N(\nu, 2)$ where ν here is a fuzzy mean having triangular membership function defined by:

$$\mu_\nu(x) = \begin{cases} \dfrac{x-a_\nu}{b_\nu - a_\nu}, & if \quad a_\nu \le x \le b_\nu \\ \dfrac{x-b_\nu}{c_\nu - b_\nu}, & if \quad b_\nu \le x \le c_\nu \\ 0 \;, & \text{otherwise} \end{cases} \tag{5.42}$$

We also recall the definition of the credibility distribution function of the fuzzy mean ν as: $R \to [0,1]$

$\Lambda_\nu(x) = \dfrac{1}{2}\left(\sup_{y\le x} \mu_\nu(y) + 1 - \sup_{y>x} \mu_\nu(y)\right) \quad \forall x \in R$, in other words:

$$\Lambda_\nu(x) = \begin{cases} 0, & if \quad x \le a_\nu \\ \dfrac{x-a_\nu}{2(b_\nu - a_\nu)}, & if \quad a_\nu \le x \le b_\nu \\ \dfrac{x-b_\nu}{2(c_\nu - b_\nu)}, & if \quad b_\nu \le x \le c_\nu \\ 1, & if \quad x \ge c_\nu \end{cases} \tag{5.43}$$

and the credibility density function:

$$\lambda_\nu(x) = \begin{cases} \dfrac{1}{2(b_\nu - a_\nu)}, & if \quad a_\nu \le x \le b_\nu \\ \dfrac{1}{2(c_\nu - b_\nu)}, & if \quad b_\nu \le x \le c_\nu \\ 0, & \text{otherwise} \end{cases} \tag{5.44}$$

1) if $a_\nu \le x \le b_\nu$, then we have:

$$\Lambda_\nu(x) = y = \frac{x-a_\nu}{2(b_\nu - a_\nu)} \Rightarrow x = 2(b_\nu - a_\nu)y + a_\nu \text{ and } 0 \le y \le \frac{1}{2} \tag{5.45}$$

2) if $b_v \le x \le c_v$, then we have:

$$\Lambda_v(x) = y = \frac{x - b_v}{2(c_v - b_v)} \Rightarrow x = 2(c_v - b_v)y - c_v + 2b_v \text{ and } \frac{1}{2} \le y \le 1 \qquad (5.46)$$

From the probability theory and statistics that, we note that, the uniform distribution is a family of probability distributions such that for each member of the family, all intervals of the same length on the distribution's support are equally probable.

From this statement, we acquire the random value of uniform distribution, then we generate fuzzy mean from the triangular distribution function, and then we utilize these results for the mean to generate the normal random fuzzy variable from the normal distribution.
We use excel to design this work. See Appendix A for illustration.

5.3 Summary

In this chapter, we have first discussed the ARL under random fuzzy measurements that we said to play the same role as that of non-fuzzy cases examined earlier. It analyses charts during the production process (see chapter 2 for discussion on the non-fuzzy case). The main emphasis was to develop the economic-design quality control chart using the Duncan-style especially $\overline{X}$ charts. At the end of this study we note that the method also works well under random fuzzy measurements as for ordinary cases (non-fuzzy case). However, a simple remark that needs to be made is that the approximate loss-costs are higher than those of the ordinary case, especially when n=5, h=1, and k=3.

We utilized economic data from Duncan on ordinary study (non-fuzzy case) to design a control chart focusing on random fuzzy measurements. The computation of the three control chart parameters allows the maintenance of the process in a state of control. The simulation issue has been examined and numerical examples given as illustrations. Further studies may need to work on different values of k and examine its variation regarding different costs.

Chapter 6. Conclusion

In conclusion to this thesis, we first need to recall the aim of the study, which was stated as: "The development of statistical quality control charts based on credibility measure theory under random fuzzy measurements and small sample asymptotics".

Although several researchers have worked for a long time in the area of statistical process control using different techniques, for example, cumulative sum charts (CUSUM), the exponential weighted moving average method (EWMA), and so on, however the development of quality control charts under random fuzzy measurements has not been treated yet. Wherefore, this work has endeavoured to address that gap. In this study, we particularly focused on the economic design of quality control charts under random fuzzy measurements.

In order to realise our aim, we first reviewed in chapter 2 relevant statistical process techniques used in statistical quality control, and then came chapter 3 where we examined random fuzzy theory and made ourselves familiar with random fuzzy measurements. The small sample asymptotic method discussed in chapter 4 ensured the approximation of the distributions of the statistics of interest for a very small sample size; even down to 1. We also improved the approximations of these distributions in terms of empirical small sample asymptotics methods, the maximum likelihood estimator and the small sample asymptotic random fuzzy approach. We then combined the study of chapter 2 and that of chapter 3 to design chapter 5: Random Fuzzy Quality Control Charts, and the core of the latter chapter was the economic design of quality control charts under random fuzzy measurements; that is the design of quality control chart parameters (n: sample size; h: interval between sample, k: control limits width) under random fuzzy measurements.

The chart of the interest is the X-bar chart and the technique followed is that of the economic design of control charts developed by Duncan (1956). Economic data were obtained from Duncan (1956) and adapted to the economic design of quality control charts under random fuzzy measurements in establishing the control chart parameters; and from the discussion made; it emerged that the technique worked as it would in ordinary cases.

Nevertheless, some defects have been notified. The technique is generally costly for very small values of control chart parameters (for instance $n=5$, $h=1$, k=3). Another defect we can highlight is the number of assumptions, which have to be made while designing this technique.

In the example we showed that the credibility distribution function of the fuzzy variable has an important influence over control limit width. For both cases triangular and trapezoidal membership functions, we used the values of a_v, b_v, c_v (which specify the position of the triangle) and av, bv, cv, dv (specifying the trapezium) near to that of k (the control limits width). However, if the variation is large enough, that leads to high probability α of finding a fuzzy sample point outside the control zone. And such value of probability; indicates the opportunity of producing undesirable quantities of non-conforming products (or items). Another fuzzy variable that has to be looked after is $\lambda(\theta)$ from exponential distribution. In fact, $\lambda(\theta)$ has an influence on the value of h and on the cost: the high values of $\lambda(\theta)$ lead to small values of h and small costs.

Several economic data have also been noted to influence the value of control chart parameters, leading to high costs. Therefore, the value of economic data might also be well selected so as to get the required control chart parameters.

Recommendations

The economic design of statistical quality control charts under random fuzzy measurements could be used as a useful technique for designing control chart parameters (*n*: sample size; *h*: interval between sample; and *k*: control limits width), which facilitate the maintenance of existing levels of production processes in a state of control.

An accurate relationship between credibility distribution function and control limit width could also serve to avoid high values of probability α of finding a random fuzzy sample point to fall outside control zone. Since the high value of α during the production process can lead to produce undesirable or non-conforming products.

Economic data could be selected such that the computation of control chart parameters conforms to producers' needs. An assumption can be made on the behaviour of the fuzzy variable $\lambda(\theta)$ so as to deal with conforming values that suit our needs.

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Appendix A: R Code

A1. R Program for the computation of ARL under normal random fuzzy variable having triangular fuzzy mean and fixed variance.

```
# Program R for ϖ

F=function (av, bv, cv, n, sigma, V)

{expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}

u1= ((av+ (2*bv) +cv)/4) + 3*(((sigma^2)+V)^.5)/ (n^.5)
u2= ((av+ (2*bv) +cv)/4) - 3*(((sigma^2)+V)^.5)/ (n^.5)
a1= (u1-av)/ (2*(bv-av))
a2= (u1-av)/(((sigma^2)+V)^.5)
a3= (u1-bv)/(((sigma^2)+V)^.5)
a4= (u1+cv-(2*bv))/ (2*(cv-bv))
a5= (u1-cv)/(((sigma^2)+V)^.5)
a6=-(((sigma^2)+V)^.5)/ (2*(bv-av))
a7=-(((sigma^2)+V)^.5)/ (2*(cv-bv))

b1= (u2-av)/ (2*(bv-av))
b2= (u2-av)/(((sigma^2)+V)^.5)
b3= (u2-bv)/(((sigma^2)+V)^.5)
b4= (u2+cv-(2*bv))/ (2*(cv-bv))
b5= (u2-cv)/(((sigma^2)+V)^.5)

int1=integrate(expN,lower=a3,upper=a2)$value
int2=integrate(expN,lower=a5,upper=a3)$value
int3=integrate(expN,lower=b3,upper=b2)$value
int4=integrate(expN,lower=b5,upper=b3)$value

cum1=pnorm(q=a2, mean=0,sd=1,lower.tail=TRUE, log.p=FALSE)
cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum4=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum5=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum6=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
sum2=a1*(cum1-cum2) +a4*(cum2-cum3) +cum3+a6*int1+a7*int2
sum1=b1*(cum4-cum5) +b4*(cum5-cum6) +cum6+a6*int3+a7*int4

p=1+ (sum1-sum2)
p2=1-sum2
```

```
  p1=sum1
  list (sum2=sum2, sum1=sum1, p=p, p1=p1, p2=p2)

 }

 F
```

Example A1

> # Program R for the computation of ARL under normal random fuzzy variable having triangular fuzzy mean.

> # The average Run Length for triangular case

```
> F=function (av, bv, cv, n, sigma, V)
+
+ {expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}
+  u1= ((av+ (2*bv) +cv)/4) + 3*(((sigma^2)+V)^.5)/ (n^.5)
+  u2= ((av+ (2*bv) +cv)/4) - 3*(((sigma^2)+V)^.5)/ (n^.5)
+  a1= (u1-av)/ (2*(bv-av))
+  a2= (u1-av)/(((sigma^2)+V)^.5)
+  a3= (u1-bv)/(((sigma^2)+V)^.5)
+
+  a4= (u1+cv-(2*bv))/ (2*(cv-bv))
+  a5= (u1-cv)/(((sigma^2)+V)^.5)
+  a6=-(((sigma^2)+V)^.5)/ (2*(bv-av))
+  a7=-(((sigma^2)+V)^.5)/ (2*(cv-bv))
+  b1= (u2-av)/ (2*(bv-av))
+  b2= (u2-av)/(((sigma^2)+V)^.5)
+  b3= (u2-bv)/(((sigma^2)+V)^.5)
+  b4= (u2+cv-(2*bv))/ (2*(cv-bv))
+  b5= (u2-cv)/(((sigma^2)+V)^.5)
+
+  int1=integrate(expN,lower=a3,upper=a2)$value
+  int2=integrate(expN,lower=a5,upper=a3)$value
+  int3=integrate(expN,lower=b3,upper=b2)$value
+  int4=integrate(expN,lower=b5,upper=b3)$value
+
+  cum1=pnorm(q=a2, mean=0,sd=1,lower.tail=TRUE, log.p=FALSE)
+  cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum4=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum5=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum6=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  sum2=a1*(cum1-cum2) +a4*(cum2-cum3) +cum3+a6*int1+a7*int2
+  sum1=b1*(cum4-cum5) +b4*(cum5-cum6) +cum6+a6*int3+a7*int4
+  p=1+ (sum1-sum2)
```

```
+   p2=1-sum2
+   p1=sum1
+   list (sum2=sum2, sum1=sum1, p=p, p1=p1, p2=p2)
+   }
>  F(0,1,2,5,2,.33)
$sum2
[1] 0.9018835

$sum1
[1] 0.09811645

$p
[1] 0.1962329

$p1
[1] 0.09811645

$p2
[1] 0.09811645
```

$$\varpi = 1 - \Psi_{\overline{X}_n}((a_v + 2b_v + c_v)/4 + 3\sqrt{\sigma^2 + V}/\sqrt{n}) + \Psi_{\overline{X}_n}((a_v + 2b_v + c_v)/4 - 3\sqrt{\sigma^2 + V}/\sqrt{n}) \approx 0.196$$

$$ARL = 1/\varpi \approx 5.10$$

A2. R Program for the computation of ARL under normal random fuzzy variable having trapezoidal fuzzy mean with fixed variance.

Program R for ϖ

```
F=function (av, bv, cv,dv, n, sigma, V)

{expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}
  u1= ((av+ bv +cv+dv)/4) + 3*(((sigma^2)+V)^.5)/ (n^.5)
  u2= ((av+ bv +cv+dv)/4) - 3*(((sigma^2)+V)^.5)/ (n^.5)
  a1= (u1-av)/ (2*(bv-av))
  a2= (u1-av)/(((sigma^2)+V)^.5)
  a3= (u1-bv)/(((sigma^2)+V)^.5)
  a4= (u1+dv-(2*cv))/ (2*(dv-cv))
  a5= (u1-cv)/(((sigma^2)+V)^.5)
  a6=-(((sigma^2)+V)^.5)/ (2*(bv-av))
  a7=-(((sigma^2)+V)^.5)/ (2*(cv-bv))

  b1= (u2-av)/ (2*(bv-av))
```

```
b2= (u2-av)/(((sigma^2)+V)^.5)
b3= (u2-bv)/(((sigma^2)+V)^.5)
b4= (u2+dv-(2*cv))/ (2*(cv-bv))
b5= (u2-cv)/(((sigma^2)+V)^.5)

int1=integrate(expN,lower=a3,upper=a2)$value
int2=integrate(expN,lower=a5,upper=a3)$value
int3=integrate(expN,lower=b3,upper=b2)$value
int4=integrate(expN,lower=b5,upper=b3)$value

cum1=pnorm(q=a2, mean=0,sd=1,lower.tail=TRUE, log.p=FALSE)
cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum4=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum5=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum6=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
sum2=a1*(cum1-cum2) +a4*(cum2-cum3) +cum3+a6*int1+a7*int2
sum1=b1*(cum4-cum5) +b4*(cum5-cum6) +cum6+a6*int3+a7*int4

p=1+ (sum1-sum2)
p2=1-sum2
p1=sum1

list (sum2=sum2, sum1=sum1, p=p, p1=p1, p2=p2)
}
F
```

Example A2

```
> # R Program of ARL under normal random fuzzy variable having trapezoidal fuzzy mean


> # The average run length for trapezoidal case)
> F=function (av, bv, cv,dv, n, sigma, V)
+
+ {expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}
+   u1= ((av+ bv +cv+dv)/4) + 3*(((sigma^2)+V)^.5)/ (n^.5)
+   u2= ((av+ bv +cv+dv)/4) - 3*(((sigma^2)+V)^.5)/ (n^.5)
+   a1= (u1-av)/ (2*(bv-av))
+   a2= (u1-av)/(((sigma^2)+V)^.5)
+   a3= (u1-bv)/(((sigma^2)+V)^.5)
+
+   a4= (u1+dv-(2*cv))/ (2*(dv-cv))
+   a5= (u1-cv)/(((sigma^2)+V)^.5)
+   a6=-(((sigma^2)+V)^.5)/ (2*(bv-av))
+   a7=-(((sigma^2)+V)^.5)/ (2*(cv-bv))
+   b1= (u2-av)/ (2*(bv-av))
+   b2= (u2-av)/(((sigma^2)+V)^.5)
```

```
+   b3= (u2-bv)/(((sigma^2)+V)^.5)
+   b4= (u2+dv-(2*cv))/ (2*(cv-bv))
+   b5= (u2-cv)/(((sigma^2)+V)^.5)
+
+   int1=integrate(expN,lower=a3,upper=a2)$value
+   int2=integrate(expN,lower=a5,upper=a3)$value
+   int3=integrate(expN,lower=b3,upper=b2)$value
+   int4=integrate(expN,lower=b5,upper=b3)$value
+
+  cum1=pnorm(q=a2, mean=0,sd=1,lower.tail=TRUE, log.p=FALSE)
+  cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum4=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum5=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum6=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  sum2=a1*(cum1-cum2) +a4*(cum2-cum3) +cum3+a6*int1+a7*int2
+  sum1=b1*(cum4-cum5) +b4*(cum5-cum6) +cum6+a6*int3+a7*int4
+   p=1+ (sum1-sum2)
+   p2=1-sum2
+   p1=sum1
+   list (sum2=sum2, sum1=sum1, p=p, p1=p1, p2=p2)
+   }
>  F(0,1,2,3,5,2,1.083)
$sum2
[1] 0.8988276

$sum1
[1] 0.1028734

$p
[1] 0.2040458

$p1
[1] 0.1028734

$p2
[1] 0.1011724
```

=> $\varpi \approx 0.204$

ARL= $1/\varpi \approx 4.90$

A3. R program used for the computation problems on Duncan-style economic design for normal random fuzzy variable quality control charts.

```
# R Program for alpha (the case of triangular fuzzy mean)

F=function(av,bv,cv,k,sigma,V)

{expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}

a1=(k-av)/(2*(bv-av))
a2=(k-av)/ (((sigma^2)+V)^.5)
a3=(k-bv)/ (((sigma^2)+V)^.5)
a4=(k+cv-(2*bv))/(2*(cv-bv))
a5=(k-cv)/ (((sigma^2)+V)^.5)
a6=-(((sigma^2)+V)^.5)/(2*(bv-av))
a7=-(((sigma^2)+V)^.5)/(2*(cv-bv))

b1=-(k+av)/(2*(bv-av))
b2=-(k+av)/ (((sigma^2)+V)^.5)
b3=-(k+bv)/ (((sigma^2)+V)^.5)
b4=(-k+cv-(2*bv))/(2*(cv-bv))
b5=-(k+cv)/ (((sigma^2)+V)^.5)

int1=integrate(expN,lower=a3,upper=a2)$value
int2=integrate(expN,lower=a5,upper=a3)$value
int3=integrate(expN,lower=b3,upper=b2)$value
int4=integrate(expN,lower=b5,upper=b3)$value

cum1=pnorm(q=a2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum4=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum5=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum6=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)

sum2=a1*(cum1-cum2)+a4*(cum2-cum3)+cum3+a6*int1+a7*int2

sum1=b1*(cum4-cum5)+b4*(cum5-cum6)+cum6+a6*int3+a7*int4

sum=1+(sum1-sum2)

list(sum1=sum1,sum2=sum2,sum=sum)

}
F
```

```
# R Program for P (the case of triangular fuzzy mean)
 F=function(av,bv,cv,n,k,sigma,teta,V)

 {expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}
 a1=(k-(teta*(n^.5))-av)/(2*(bv-av))
 a2=(k-(teta*(n^.5))-av)/ (((sigma^2)+V)^.5)
 a3=(k-(teta*(n^.5))-bv)/ (((sigma^2)+V)^.5)
 a4=(k-(teta*(n^.5))+cv-(2*bv))/(2*(cv-bv))
 a5=(k-(teta*(n^.5))-cv)/ (((sigma^2)+V)^.5)
 a6=-(((sigma^2)+V)^.5)/(2*(bv-av))
 a7=-(((sigma^2)+V)^.5)/(2*(cv-bv))

 b1=-(k+(teta*(n^.5))+av)/(2*(bv-av))
 b2=-(k+(teta*(n^.5))+av)/ (((sigma^2)+V)^.5)
 b3=-(k+(teta*(n^.5))+bv)/ (((sigma^2)+V)^.5)
 b4=(-k-(teta*(n^.5))+cv-(2*bv))/(2*(cv-bv))
 b5=-(k+(teta*(n^.5))+cv)/ (((sigma^2)+V)^.5)

 int1=integrate(expN,lower=a3,upper=a2)$value
 int2=integrate(expN,lower=a5,upper=a3)$value
 int3=integrate(expN,lower=b3,upper=b2)$value
 int4=integrate(expN,lower=b5,upper=b3)$value

 cum1=pnorm(q=a2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
 cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
 cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
 cum4=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
 cum5=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
 cum6=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)

 sum2=a1*(cum1-cum2)+a4*(cum2-cum3)+cum3+a6*int1+a7*int2

 sum1=b1*(cum4-cum5)+b4*(cum5-cum6)+cum6+a6*int3+a7*int4

 p=1+(sum1-sum2)

 p2=1-sum2

 p1=sum1

list(sum2=sum2,sum1=sum1,p=p,p1=p1,p2=p2)

   }
   F
```

Example A3

```
> # R Program for alpha (triangular fuzzy mean)
> F=function(av,bv,cv,k,sigma,V)
+
+  {expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}
+
+  a1=(k-av)/(2*(bv-av))
+  a2=(k-av)/ (((sigma^2)+V)^.5)
+  a3=(k-bv)/ (((sigma^2)+V)^.5)
+  a4=(k+cv-(2*bv))/(2*(cv-bv))
+  a5=(k-cv)/ (((sigma^2)+V)^.5)
+  a6=-(((sigma^2)+V)^.5)/(2*(bv-av))
+  a7=-(((sigma^2)+V)^.5)/(2*(cv-bv))
+
+  b1=-(k+av)/(2*(bv-av))
+  b2=-(k+av)/ (((sigma^2)+V)^.5)
+  b3=-(k+bv)/ (((sigma^2)+V)^.5)
+  b4=(-k+cv-(2*bv))/(2*(cv-bv))
+  b5=-(k+cv)/ (((sigma^2)+V)^.5)
+
+  int1=integrate(expN,lower=a3,upper=a2)$value
+  int2=integrate(expN,lower=a5,upper=a3)$value
+  int3=integrate(expN,lower=b3,upper=b2)$value
+  int4=integrate(expN,lower=b5,upper=b3)$value
+
+  cum1=pnorm(q=a2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum4=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum5=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum6=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+
+  sum2=a1*(cum1-cum2)+a4*(cum2-cum3)+cum3+a6*int1+a7*int2
+
+  sum1=b1*(cum4-cum5)+b4*(cum5-cum6)+cum6+a6*int3+a7*int4
+
+  sum=1+(sum1-sum2)
+
+  list(sum1=sum1,sum2=sum2,sum=sum)
+
+  }
>  F(0,1,2,3,2,.33)
$sum1
[1] 0.03197297
```

```
$sum2
[1] 0.8226784

$sum
[1] 0.2092946


> # R Program for P ( triangular fuzzy mean)
> F=function(av,bv,cv,n,k,sigma,teta,V)
+
+ {expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}
+ a1=(k-(teta*(n^.5))-av)/(2*(bv-av))
+ a2=(k-(teta*(n^.5))-av)/ (((sigma^2)+V)^.5)
+ a3=(k-(teta*(n^.5))-bv)/ (((sigma^2)+V)^.5)
+ a4=(k-(teta*(n^.5))+cv-(2*bv))/(2*(cv-bv))
+ a5=(k-(teta*(n^.5))-cv)/ (((sigma^2)+V)^.5)
+ a6=-(((sigma^2)+V)^.5)/(2*(bv-av))
+ a7=-(((sigma^2)+V)^.5)/(2*(cv-bv))
+
+ b1=-(k+(teta*(n^.5))+av)/(2*(bv-av))
+ b2=-(k+(teta*(n^.5))+av)/ (((sigma^2)+V)^.5)
+ b3=-(k+(teta*(n^.5))+bv)/ (((sigma^2)+V)^.5)
+ b4=(-k-(teta*(n^.5))+cv-(2*bv))/(2*(cv-bv))
+ b5=-(k+(teta*(n^.5))+cv)/ (((sigma^2)+V)^.5)
+
+ int1=integrate(expN,lower=a3,upper=a2)$value
+ int2=integrate(expN,lower=a5,upper=a3)$value
+ int3=integrate(expN,lower=b3,upper=b2)$value
+ int4=integrate(expN,lower=b5,upper=b3)$value
+
+ cum1=pnorm(q=a2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+ cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+ cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+ cum4=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+ cum5=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+ cum6=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+
+ sum2=a1*(cum1-cum2)+a4*(cum2-cum3)+cum3+a6*int1+a7*int2
+
+ sum1=b1*(cum4-cum5)+b4*(cum5-cum6)+cum6+a6*int3+a7*int4
+
+ p=1+(sum1-sum2)
+
+ p2=1-sum2
+
+ p1=sum1
```

```
+
+  list(sum2=sum2,sum1=sum1,p=p,p1=p1,p2=p2)
+
+  }
>  F(0,1,2,5,3,2,2,.33)
$sum2
[1] 0.1262544

$sum1
[1] 4.166975e-05

$p
[1] 0.8737873

$p1
[1] 4.166975e-05

$p2
[1] 0.8737456
```

A4. R program used for the computation problems on Duncan-style economic design for normal random fuzzy variable quality control charts.

```
# R Program for alpha (the case of trapezoidal fuzzy mean)
 F=function(av,bv,cv,dv,k,sigma,V)

  { expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}

  a1=(k-av)/(2*(bv-av))
  a2=(k-av)/ (((sigma^2)+V)^.5)
  a3=(k-bv)/ (((sigma^2)+V)^.5)
  a4=(k+dv-(2*cv))/(2*(dv-cv))
  a5=(k-cv)/ (((sigma^2)+V)^.5)
  a6=(k-dv)/ (((sigma^2)+V)^.5)
  a7=-(((sigma^2)+V)^.5)/(2*(bv-av))
  a8=-(((sigma^2)+V)^.5)/(2*(dv-cv))

  b1=-(k+av)/(2*(bv-av))
  b2=-(k+av)/ (((sigma^2)+V)^.5)
  b3=-(k+bv)/ (((sigma^2)+V)^.5)
  b4=(-k+dv-(2*cv))/(2*(dv-cv))
  b5=-(k+cv)/ (((sigma^2)+V)^.5)
  b6=-(k+dv)/ (((sigma^2)+V)^.5)

  int1=integrate(expN,lower=a3,upper=a2)$value
```

```
int2=integrate(expN,lower=a6,upper=a5)$value
int3=integrate(expN,lower=b3,upper=b2)$value
int4=integrate(expN,lower=b6,upper=b5)$value

cum1=pnorm(q=a2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum4=pnorm(q=a6,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum5=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum6=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum7=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum8=pnorm(q=b6,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)

sum2=a1*(cum1-cum2)+a4*(cum3-cum4)+.5*(cum2-cum3)+cum4+a7*int1+a8*int2
sum1=b1*(cum5-cum6)+b4*(cum7-cum8)+.5*(cum6-cum7)+cum8+a7*int3+a8*int4
sum=1+(sum1-sum2)

list(sum1=sum1,sum2=sum2,sum=sum)

}

F
```

R Program for the computation of P (the case of trapezoidal fuzzy mean)

```
F=function(av,bv,cv,dv,n,k,sigma,teta,V)

{expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}

a1=(k-(teta*(n^.5))-av)/(2*(bv-av))
a2=(k-(teta*(n^.5))-av)/sigma
a3=(k-(teta*(n^.5))-bv)/sigma
a4=(k-(teta*(n^.5))+dv-(2*cv))/(2*(dv-cv))
a5=(k-(teta*(n^.5))-cv)/sigma
a6=(k-(teta*(n^.5))-dv)/sigma
a7=-sigma/(2*(bv-av))
a8=-sigma/(2*(dv-cv))

b1=-(k+(teta*(n^.5))+av)/(2*(bv-av))
b2=-(k+(teta*(n^.5))+av)/sigma
b3=-(k+(teta*(n^.5))+bv)/sigma
b4=(-k-(teta*(n^.5))+dv-(2*cv))/(2*(dv-cv))
b5=-(k+(teta*(n^.5))+cv)/sigma
b6=-(k+(teta*(n^.5))+dv)/sigma

int1=integrate(expN,lower=a3,upper=a2)$value
```

```
int2=integrate(expN,lower=a6,upper=a5)$value
int3=integrate(expN,lower=b3,upper=b2)$value
int4=integrate(expN,lower=b6,upper=b5)$value

cum1=pnorm(q=a2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum4=pnorm(q=a6,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum5=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum6=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum7=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
cum8=pnorm(q=b6,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)

sum2=a1*(cum1-cum2)+a4*(cum3-cum4)+.5*(cum2-cum3)+cum4+a7*int1+a8*int2
sum1=b1*(cum5-cum6)+b4*(cum7-cum8)+.5*(cum6-cum7)+cum8+a7*int3+a8*int4

p=1+(sum1-sum2)

p2=1-sum2

p1=sum1

list(sum2=sum2,sum1=sum1,p=p,p1=p1,p2=p2)

}

 F
```

Example A4

```
> # R Program for alpha (trapezoidal fuzzy mean)
>  F=function(av,bv,cv,dv,k,sigma,V)
+
+   { expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}
+
+   a1=(k-av)/(2*(bv-av))
+   a2=(k-av)/ (((sigma^2)+V)^.5)
+   a3=(k-bv)/ (((sigma^2)+V)^.5)
+   a4=(k+dv-(2*cv))/(2*(dv-cv))
+   a5=(k-cv)/ (((sigma^2)+V)^.5)
+   a6=(k-dv)/ (((sigma^2)+V)^.5)
+   a7=-(((sigma^2)+V)^.5)/(2*(bv-av))
+   a8=-(((sigma^2)+V)^.5)/(2*(dv-cv))
+
+   b1=-(k+av)/(2*(bv-av))
```

```
+  b2=-(k+av)/ (((sigma^2)+V)^.5)
+  b3=-(k+bv)/ (((sigma^2)+V)^.5)
+  b4=(-k+dv-(2*cv))/(2*(dv-cv))
+  b5=-(k+cv)/ (((sigma^2)+V)^.5)
+  b6=-(k+dv)/ (((sigma^2)+V)^.5)
+
+  int1=integrate(expN,lower=a3,upper=a2)$value
+  int2=integrate(expN,lower=a6,upper=a5)$value
+  int3=integrate(expN,lower=b3,upper=b2)$value
+  int4=integrate(expN,lower=b6,upper=b5)$value
+
+  cum1=pnorm(q=a2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum4=pnorm(q=a6,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum5=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum6=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum7=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+  cum8=pnorm(q=b6,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+
+  sum2=a1*(cum1-cum2)+a4*(cum3-cum4)+.5*(cum2-cum3)+cum4+a7*int1+a8*int2
+  sum1=b1*(cum5-cum6)+b4*(cum7-cum8)+.5*(cum6-cum7)+cum8+a7*int3+a8*int4
+  sum=1+(sum1-sum2)
+
+  list(sum1=sum1,sum2=sum2,sum=sum)
+
+  }
> F(0,1,2,3,3,2,1.083)
$sum1
[1] 0.0347821

$sum2
[1] 0.7256783

$sum
[1] 0.3091038

> # R Program for computation of P (trapezoidal fuzzy mean)
> F=function(av,bv,cv,dv,n,k,sigma,teta,V)
+
+  {expN=function(u){u*exp(-.5*(u^2))/sqrt(2*pi)}
+
+  a1=(k-(teta*(n^.5))-av)/(2*(bv-av))
+  a2=(k-(teta*(n^.5))-av)/sigma
+  a3=(k-(teta*(n^.5))-bv)/sigma
+  a4=(k-(teta*(n^.5))+dv-(2*cv))/(2*(dv-cv))
+  a5=(k-(teta*(n^.5))-cv)/sigma
```

```
+   a6=(k-(teta*(n^.5))-dv)/sigma
+   a7=-sigma/(2*(bv-av))
+   a8=-sigma/(2*(dv-cv))
+
+   b1=-(k+(teta*(n^.5))+av)/(2*(bv-av))
+   b2=-(k+(teta*(n^.5))+av)/sigma
+   b3=-(k+(teta*(n^.5))+bv)/sigma
+   b4=(-k-(teta*(n^.5))+dv-(2*cv))/(2*(dv-cv))
+   b5=-(k+(teta*(n^.5))+cv)/sigma
+   b6=-(k+(teta*(n^.5))+dv)/sigma
+
+  int1=integrate(expN,lower=a3,upper=a2)$value
+   int2=integrate(expN,lower=a6,upper=a5)$value
+  int3=integrate(expN,lower=b3,upper=b2)$value
+   int4=integrate(expN,lower=b6,upper=b5)$value
+
+   cum1=pnorm(q=a2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+   cum2=pnorm(q=a3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+   cum3=pnorm(q=a5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+   cum4=pnorm(q=a6,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+   cum5=pnorm(q=b2,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+   cum6=pnorm(q=b3,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+   cum7=pnorm(q=b5,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+   cum8=pnorm(q=b6,mean=0,sd=1,lower.tail=TRUE,log.p=FALSE)
+
+   sum2=a1*(cum1-cum2)+a4*(cum3-cum4)+.5*(cum2-cum3)+cum4+a7*int1+a8*int2
+   sum1=b1*(cum5-cum6)+b4*(cum7-cum8)+.5*(cum6-cum7)+cum8+a7*int3+a8*int4
+
+   p=1+(sum1-sum2)
+
+   p2=1-sum2
+
+   p1=sum1
+
+   list(sum2=sum2,sum1=sum1,p=p,p1=p1,p2=p2)
+
+   }
>    F(0,1,2,3,5,3,2,2,1.08)
$sum2
[1] 0.09460934
$sum1
[1] 2.004982e-05
$p
[1] 0.9054107
$p1
[1] 2.004982e-05
$p2
[1] 0.9053907
```

A5. Simulation

1) Col A: UNIFORM(0,1),
2) A2: =RAND()
3) Col B: GEN MEAN FROM TRIANGULAR DIST
4) B2: =IF(A2<0.5,(2*(E2-D2)*A2)+D2,(2*(F2-E2)*A2)-F2+(2*E2))
5) Col C: GEN RANDOM FV FROM NORM DIST WITH MEAN IN COL B
6) C2: =NORMINV(RAND(),B2,G2)
7) Col D: SUGGEST THE VALUE a
8) Col E : SUGGEST THE VALUE b
9) Col F: SUGGEST THE VALUE c
10) Col G: SUGGEST THE VALUE STANDARD DEVIATION

Example A5

1) Col A: UNIFORM (0, 1)
2) A3:=0.896669911
3) Col B: GEN MEAN FROM TRIANGULAR DIST
4) B3:=0.793339822
5) Col C: GEN RAND FV FROM NORM DIST WITH MEAN IN COL B
6) C3:= 1.282812457
7) Col D: a= -1
8) Col E: b= 0
9) Col F: c= 1
10) Col G: $\sigma = 2$

So, random variables can be generated as follow:

$$\nu_1 = 1.282812457 \Rightarrow \mathrm{N}(\nu_1, 2)$$

$$\nu_2 = 0.245425524 \Rightarrow \mathrm{N}(\nu_2, 2)$$

$$\nu_3 = 2.947701763 \Rightarrow \mathrm{N}(\nu_3, 2)$$

Appendix B: Notations

Table B1. Notations and formulas for designing charts

CL	Center line
UCL	Upper control limit
LCL	Lower control limit
n	Sample size
m	Sample number
$\overline{X}$	Average measurements
$\overline{\overline{X}}$	Average of average
R	Range
$\overline{R}$	Average of range
σ	Process standard deviation (sigma)
D_i	Nonconforming units in sample i
$\hat{p}$	The fraction nonconforming
$\overline{p}$	The average of the individual sample fractions nonconforming
ARL	Average run length

Table B2. Notations for credibility measure theory

Θ	Fuzzy universe
2^{Θ}	Power set of Θ
$\mu(.)$	Membership function
$Cr(.)$	Credibility measure
$\Lambda(.)$	Credibility distribution function
$\lambda(.)$	Credibility density function
$\Psi(.)$	Average chance distribution function
$\psi(.)$	Average chance density function

Table B3. Notations in small sample asymptotics and empirical small sample asymptotics

T_n	Statistic
M_i	Moments
K_i	Cumulant

Table B4. Notations in random fuzzy quality control charts

$\mu_{(a_v,b_v,c_v)}(.)$	Triangular membership function
$\overline{\omega}$	Average chance grade
P	Probability that a sample point will fall out-side the control limits
α	Probability of a sample point falling outside the control limits when the process is in control

Appendix C: Formulas for Designing Charts

Table C1. Control limits for various variable control charts

$\bar{x}$-chart	User factor(s)	CL	UCL	LCL
σ assumed known	A	$\bar{\bar{x}}$	$\bar{\bar{x}}+A\sigma$	$\bar{\bar{x}}-A\sigma$
$\bar{R}$ for estimating σ	A_2	$\bar{\bar{x}}$	$\bar{\bar{x}}+A_2\sigma$	$\bar{\bar{x}}-A_2\bar{R}$
$\hat{\sigma}$ for estimating σ	A_1	$\bar{\bar{x}}$	$\bar{\bar{x}}+A_1\hat{\sigma}$	$\bar{\bar{x}}-A_1\hat{\sigma}$

R-chart	User factor (s)	CL	UCL	LCL
σ assumed known	D_1 and D_2	$\bar{R}=d_2\sigma$	$D_2\sigma$	$D_1\sigma$
$\bar{R}$ for estimating σ	D_3 and D_4	$\bar{R}$	$D_4\bar{R}$	$D_3\bar{R}$

Table C2. Definitions for charts[1]

$\bar{x}$	$\frac{1}{n}\sum_{i=1}^{n} x_i$
$\bar{\bar{x}}$	$\frac{1}{m}\sum_{i=1}^{m} \bar{x}_i$
R	$x_{\max}-x_{\min}$
$\bar{R}$	$\frac{1}{m}\sum_{i=1}^{m} R_i$
A	$\frac{3}{\sqrt{n}}$
A_2	$\frac{3}{d_2\sqrt{n}}$
A_3	$\frac{3}{c_4\sqrt{n}}$
B_3	$1-\frac{k}{c_4}$
B_4	$1+\frac{k}{c_4}$
B_5	$c_3-3\sqrt{1-c_4^2}$

B_6	$c_3+3\sqrt{1-c_4^2}$
D_1	d_2-3d_3
D_2	d_2+3d_3
D_3	$1-3\frac{d_3}{d_2}$
D_4	$1+3\frac{d_3}{d_2}$
c_4	$\sqrt{\frac{2}{n-1}}\frac{\Gamma(n/2)}{\Gamma(n-1/2)}\cong 1-\frac{1}{4(n-1)}$
k	$3\sqrt{1-c_4^2}$

Table C3. Control charts formulas for attribute data

	CL	UCL	LCL	Notes
P (fraction)	$\bar{p}$	$\bar{p}+3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$	$\bar{p}-3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$	If varies use $\bar{n}$ or individual n_i
np(number of non-conforming)	$n\bar{p}$	$n\bar{p}+3\sqrt{n\bar{p}(1-\bar{p})}$	$n\bar{p}-3\sqrt{n\bar{p}(1-\bar{p})}$	n must be a constant
C (count of nonconformance)	$\bar{c}$	$\bar{c}+3\sqrt{\bar{c}}$	$\bar{c}-3\sqrt{\bar{c}}$	n must be a constant
U (count of nonconformance/unit)	$\bar{u}$	$\bar{u}+3\sqrt{\frac{\bar{u}}{n}}$	$\bar{u}-3\sqrt{\frac{\bar{u}}{n}}$	If n varies use $\bar{n}$ or individual n_i

Table C4. Definitions for charts[2]

$\widehat{p}$	$\frac{D_i}{n}, i = 1, 2, \ldots, m$
$\overline{p}$	$\frac{\sum_{i=1}^{m} D_i}{mn} = \frac{\sum_{i=1}^{m} \widehat{p}_i}{m}$
u	$\frac{c}{n}$
$\overline{u}$	$\frac{\sum_{i=1}^{n} u_i}{n}$

Appendix D: Factors for constructing Variables Control Charts

Table D1. Factors for constructing variables control charts

Observation In	Chart for Average			Chart for Standard Deviations					
	Factors for Control Limits			Factors for Center Line		Factors for Control Limits			
Sample, n	A	A2	A3	C4	1/C4	B3	B4	B5	B6
2	2.121	1.880	2.659	0.7979	1.2533	0	3.267	0	2.606
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276
4	1.500	0729	1.628	0.9213	1.0854	0	2.266	0	2.088
5	1.342	0.577	1.427	0.9400	1.0638	0	2.089	0	1.964
6	1.225	0.483	1.287	0.9515	1.0510	0.030	1.970	0.029	1.874
7	1.134	0.419	1.182	0.9594	1.0423	0.118	1.882	0.113	1.806
8	1.061	0.373	1.099	0.9650	1.0363	0.185	1.815	0.179	1.751
9	1.000	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707
10	0.949	0.308	0.975	0.9727	1.0281	0.284	1.716	0.276	1.669
11	0.905	0.285	0.927	0.9754	1.0252	0.321	1.679	0.313	1.637
12	0.866	0266	0.886	0.9776	1.0229	0.354	1.646	0.346	1.610
13	0.832	0.249	0.850	0.9794	1.0210	0.382	1.618	0.374	1.585
14	0.802	0.235	0.717	0.9810	1.0194	0.406	1.594	0.399	1.563
15	0.775	0.223	0.789	0.9823	1.0180	0.428	1.572	0.421	1.544
16	0.750	0.212	0.763	0.9835	1.0168	0.448	1.552	0.440	1.526
17	0.728	0.203	0.718	0.9845	1.0157	0.466	1.534	0.458	1.511
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496
19	0.688	0.187	0.698	0.9862	1.0140	0.497	1.503	0.490	1.483
20	0.671	0.180	0.680	0.9869	1.0133	0.510	1.490	0.504	1.470
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459
22	0.640	0.167	0.647	0.9882	1.0119	0.534	1.466	0.528	1.448
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429
25	0.600	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.420

Observations In	Chart for Range						
	Factors for Center Line			Factors for Control Limits			
Sample,n	d 2	1/d2	d 3	D1	D2	D3	D4
2	1.128	0.8865	0.853	0	3.686	0	3.267
3	1.693	0.5907	0.888	0	4.358	0	2.575
4	2.059	0.4857	0.880	0	4.698	0	2.282
5	2.326	0.4299	0.864	0	4.918	0	2.115
6	2.534	0.3946	0.848	0	5.078	0	2.004
7	2.704	0.3698	0.833	0.204	5.204	0.076	1.924
8	2.847	0.3512	0.820	0.388	5.306	0.136	1.864
9	2.970	0.3367	0.808	0.547	5.393	0.184	1.816
10	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	3.173	0.3152	0.787	0.811	5.535	0.256	1.744
12	3.258	0.3069	0.778	0.922	5.594	0.283	1.717
13	3.336	0.2998	0.770	1.025	5.647	0.307	1.693
14	3.407	0.2935	0.763	1.118	5.696	0.328	1.672
15	3.472	0.2880	0.756	1.203	5.741	0.347	1.653
16	3.532	0.2831	0.750	1.282	5.782	0.363	1.637
17	3.588	0.2787	0.744	1.356	5.820	0.378	1.622
18	3.640	0.2747	0.739	1.424	5.856	0.391	1.608
19	3.689	0.2711	0.734	1.487	5.891	0.403	1.597
20	3.735	0.2677	0.729	1.549	5.921	0.415	1.585
21	3.778	0.2647	0.724	1.605	5.951	0.425	1.575
22	3.819	0.2618	0.720	1.659	5.979	0.434	1.566
23	3.858	0.2592	0.716	1.710	6.006	0.443	1.557
24	3.895	0.2567	0.712	1.759	6.031	0.451	1.548
25	3.931	0.2544	0.708	1.806	6.056	0.459	1.541

Appendix E: Statistical Control Concepts

Table E1. Statistical notions

Sample	A part, or subgroup, or subset of a population.
Random sample or simple random sample in sampling theory	A sample selected in such a manner that every single element of the population from which the sample is taken has an equal chance of being included in the sample and each selection is independent of every other selection.
Sample size	The number of units used in the sample
Small random sample	Random sample having small size; the required sample sizes are less than or equal to 30 (in certain cases less or equal to 36) with desired characteristics the reduction of cost, time loss and sampling error.
Bias in the sample	Particular elements in the population are more likely to be included in the sample than some other elements

Table E2 (Wetherill and Brown, 1991): Statistical process methodology

Stage 1. Process flow:	
1.1	Draw a schematic diagram of the flow of the process, and note the stages or phases in the process
1.2	Study the flow of data from the process. Note where and when this data is stored, communication links, etc
Stage 2, Determine the problem:	
2.1	Collect people's opinions about the problem, including the customer
2.2	Determine the important product variables, whether or not they are measured
2.3	Collect and analyze data on these variables using moving average, CuSum, and process capability studies
2.4	Calculate the costs of non-conforming product
2.5	Interpret the data using process log books, and by consultation with process engineers and operators
Stage 3. Explore the process	
3.1	Collect information about the process: (a) Known from technical sources and reports

	(b) Relationships or material believed, sometimes strongly to be relevant (c) Conjectures and opinions
3.2	Break the process down into modules, if possible, and decide on any extra data necessary to achieve this modularization
3.3	Collect data available from quality control or other routine operations. Decide on extra data required and collect the data
3.4	Analyze and interpret the data required
3.5	Design and carry out experiments on the plant in order to test and establish empirical or theoretical models
3.6	Choose the types of SPC charts to use and decide where to put them
3.7	Implement SPC: This stage will often involve training, and some sort of 'public relations' exercise with staff

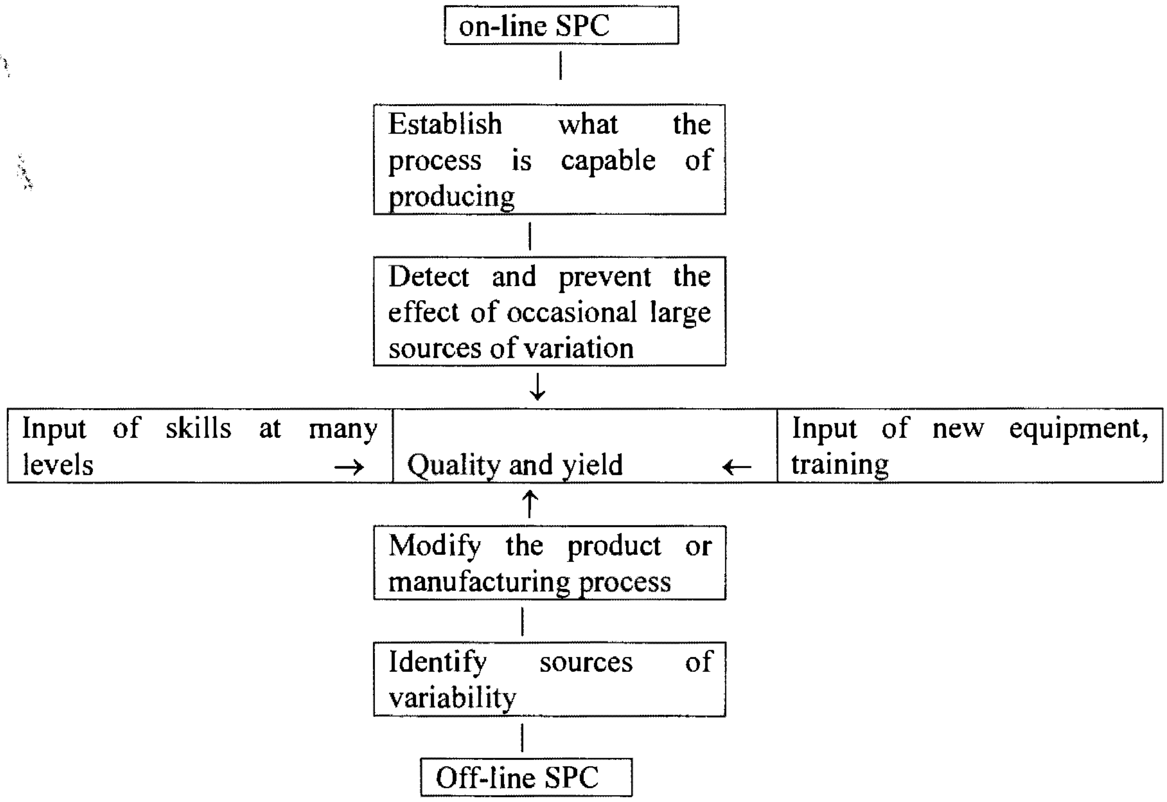

Figure E1. Factors affecting the success of SPC (Wetherill and Brown, 1991)

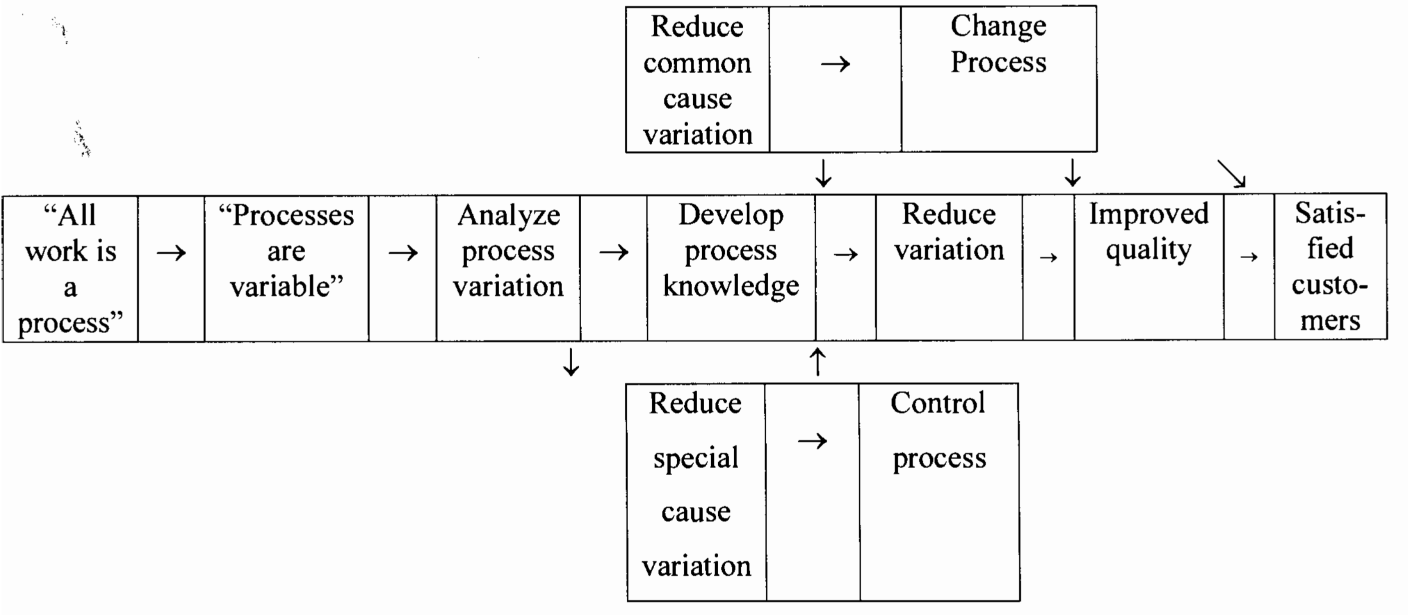

Figure E2. Statistical thinking and quality improvement (Snee, 1990)

Table E3. (Ryan, 1989) The use of control charts

A- For measurement data with sub grouping (one variable)	
A.1 For controlling process variability	
	i R-chart 1.Without probability limits 2. With probability limits
	ii S-chart 1 Without probability limits 2 With probability limits
	iii. S^2-chart
A.2 For controlling a process mean	
	1. $\overline{X}$-chart
	2. CUSUM procedures i. Geometric moving average chart ii. Acceptance chart iii Modified limits iv Difference charts 1. For paired data 2. For independent data
B. For measurement data without Sub grouping (one variable)	
B.1 For controlling process variability	1. Moving range chart 2. CUSUM for process variability
B.2 For controlling a process mean	1. Individual observations chart (X-chart) 2. CUSUM for individual observations 3. Moving average chart
C. For attribute data	
C.1 For nonconforming units	1. NP-chart 2. NP-chart with arcsin transformation 3. P chart 4. Modified P and NP charts 5. CUSUM procedures
C.2 For nonconformities	1. C-chart 2. C-chart with square root transformation 3. U-chart 4. KU-chart 5. D-chart 6. CUSUM procedures
D. Multivariate charts for measurement data	
D.1 With subgrouping	
D.2 Without subgrouping	

Table E4: Alarm rules (Duncan's Quality Control and Industrial Statistics)

a)	A single point outside 3-sigma control limits;
b)	A run of 7 consecutive point up, down or on one side of the centerline;
c)	2 consecutive points outside the 2-sigma warning limits on one side of the centerline but still inside the control limits;
d)	4 consecutive points outside the 1-sigma limits on one side of the centerline;
e)	"Obvious" cycles up and down

Table E5. Nelson's Alarm Rules (Journal of Quality Technology, 1984)

a)	A single point outside 3-sigma control limits;
b)	9 consecutive points on one side of centerline;
c)	6 consecutive points increasing or decreasing;
d)	14 consecutive points alternating up and down;
e)	2 out of any 3 consecutive points outside the 2-sigma warning limits on one side of the centerline but still inside the control limits;
f)	4 out of 5 consecutive points outside the 1-sigma limits on one side of the centerline;
g)	15 consecutive points inside 1-sigma limits
h)	8 consecutive point with none inside 1-sigma limits

	Single point above upper control limit	UCL
	Zone A $\bar{x}+2\sigma$ *to* $\bar{x}+3\sigma$ 2 out of 3 points in zone A or above	
	Zone B $\bar{x}+\sigma$ *to* $\bar{x}+2\sigma$ 4 out of 5 points in zone B or above	
Sample Quality Characteristic	Zone C $\bar{x}+\sigma$ 8 in a row in zone C or above	
		CL
	Zone C $\bar{x}-\sigma$ 8 in a row in zone C or below	
	Zone B $\bar{x}-\sigma$ *to* $\bar{x}-2\sigma$ 4 out of 5 points in zone A or below	
	Zone A $\bar{x}-2\sigma$ *to* $\bar{x}-3\sigma$ 2 out of 3 points in zone A or below	
	Single point below lower control limit	LCL

Sample Number

Figure E3. Alarm Rules (Duncan's Quality Control and Industrial Statistics)