

Pricing Swaptions on Amortising Swaps

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A dissertation submitted to the Faculty of Commerce, University of Cape Town, in partial fulfillment of the requirements for the degree of Master of Philosophy.

September 17, 2018

*MPhil in Mathematical Finance,
University of Cape Town.*



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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy at the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

September 17, 2018

Abstract

In this dissertation, two efficient approaches for pricing European options on amortising swaps are explored. The first approach is to decompose the pricing of a European amortising swaption into a series of discount bond options, with an assumption that the interest rate follows a one-factor affine model. The second approach is using a one-dimensional numerical integral technique to approximate the price of European amortising swaption, with an assumption that the interest rate follows an additive two-factor affine model. The efficacy of the two methods was tested by making a comparison with the prices generated using Monte Carlo methods. Two methods were used to accelerate the convergence rate of the Monte Carlo model, a variance reduction method, namely the control variates technique and a method of using deterministic low-discrepancy sequences (also called quasi-Monte Carlo methods).

Acknowledgements

I would like to thank my supervisor Prof. Thomas McWalter, Ms Cecile Lötter and Greg Mollentz of Old Mutual Specialised Finance for their support and guidance during the writing of this dissertation. I am grateful to individuals in the academic team, in particular, Ralph Rudd, Alex Backwell and Prof. Peter Ouwehand for their enthusiasm, encouragement and their willingness to answer my questions. I also want to extend my special thanks to Prof. David Taylor. He was not only my course convener but a friend and mentor. Last but not least, I am grateful for the patience and support from colleagues, friends and family.

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Chapter 1

Introduction

Many large corporate institutions such as insurance companies and banks have been using interest rate swaps to manage the interest rate exposure that arises from their asset and liability mismatches. These institutions use a management process called asset and liability management (ALM), which is a process of matching assets and cash flows to meet the obligations of the company ([Van Deventer et al., 2013](#)).

Suppose an insurance company is exposed to rising interest rates on the floating rate debt liability side of the balance sheet, one of the approaches to mitigate this risk is by trading an interest rate swap (IRS). An interest rate swap is an agreement between two parties where each takes a position on the direction of the interest rate. For example, a counterparty paying a fixed interest rate for a loan anticipates a decrease in the future floating rate and to protect against the case where the floating rate drops they enter into a swap contract to pay a floating rate and receive a fixed rate over the same given period of the loan. There is no need to exchange the principal amount in a swap contract. To determine a fair value of the swap contract, the interest cash flows from both counterparties must be computed using the same principal amount. The cash flows are paid out on specified payment dates called settlement dates.

As swaps became more useful in hedging interest rate exposures, there has been a rapid growth in the creation of swap related derivatives, in particular, swaptions. A swaption is a financial contract between two counterparties, where one counterparty (the buyer) pays a certain amount (the premium) to the other counterparty (the seller) for an option to enter into an interest rate swap (IRS) at some future date. There are two types of swaptions, namely a payer swaption and a receiver swaption. A payer swaption gives the buyer the right to be a fixed rate payer and a floating rate receiver and a receiver swaption gives the buyer the right to be a floating rate payer and a fixed rate receiver of the cash payments.

Swaptions are over-the-counter (OTC) derivatives. These derivatives are widely used by financial companies to mitigate interest rate exposures. Caps¹ and Swaptions represent the most significant class of fixed-income options in the financial market (Longstaff et al., 2001).

In this dissertation, the objective is to explore practical approaches for modeling the fair market value of an option written on an amortising swap (also called an amortising swaption) using risk neutral pricing. An amortising swap is a swap contract embedded with a decreasing notional profile for the interest rate payments. One of the uses of such instruments is to hedge liabilities whose notional profiles are decreasing. Even though it is hard to hedge this risk correctly, one could still hedge most of the risk. Figure 1.1 below shows three amortising notional profiles for a swap contract with a term of 15 years and a 3-month tenor structure. Example 1 represents a simple linear amortising schedule. Example 2 represents a schedule with three distinct phases, in the first period there is no amortisation, after which small accruals occur and finally a linear amortisation. Example 3 represents an entirely bespoke amortisation schedule.

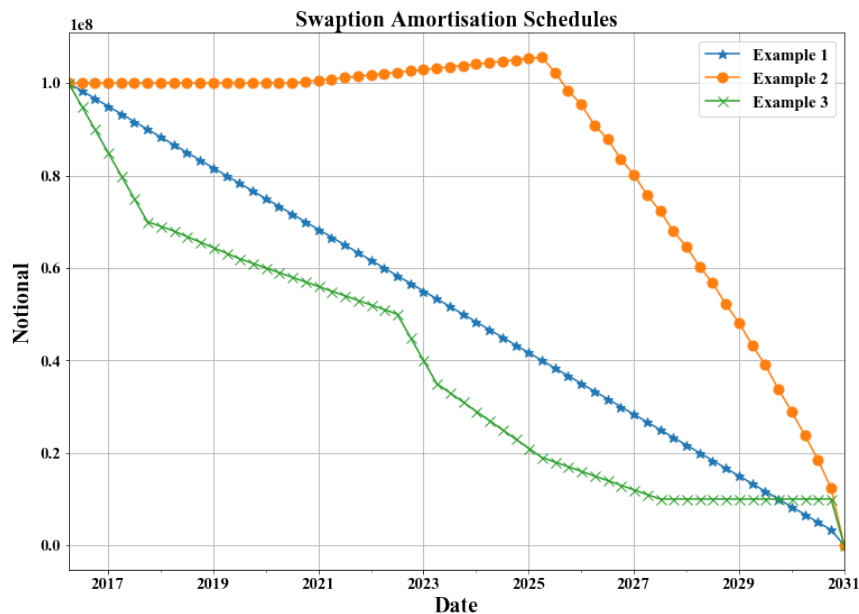


Fig. 1.1: Three different notional profiles.

This dissertation uses two approaches for pricing amortising swaptions, the ap-

¹ Another type of interest rate derivative in which at the end of each period the buyer will receive a payment only if the interest rate is above the strike rate (McInerney and Zastawniak, 2015).

proach of [Jamshidian \(1989\)](#) using single factor models and the approach of [Brigo and Mercurio \(2013\)](#) using double factor models.

This dissertation is organized as follows. Chapter [2](#) provides a brief introduction to Jamshidian's work on pricing coupon-bearing bond options. Chapter [3](#) applies the work of [Jamshidian \(1989\)](#) and ([Brigo and Mercurio, 2013](#), p.158) to pricing swaptions. Chapter [4](#) illustrates the efficacy of these formulae against Monte Carlo based pricing which is taken as the true solution for pricing swaptions. The final chapter provides a summary of the results and points out directions for further research.

Chapter 2

Jamshidian Decomposition

Jamshidian (1989) observed that the price of any derivative with positive cash flows c_i at n time points in the future is a monotonically decreasing function of the short rate. Therefore, one can directly apply the formulas for European options on zero-coupon bonds to European options on coupon-bearing bonds, see Van Deventer et al. (2013) and Jamshidian (1989). When pricing a derivative security that depends directly or indirectly on bonds, the specification of the interest rate process becomes very crucial for the pricing of bonds. This implies that the assumption of a constant or deterministic interest rate process will be too restrictive and therefore a term structure model is required. The difficulty in these models arises with addition to the choice of the number of state variables, the capability of fitting the current term structure and volatility structure. The term structure model is assumed to be determined by the instantaneous short rate and there have been many models proposed describing the dynamics of the short rate. In Jamshidian's (1989) paper, the short rate is a stochastic process $(r_t)_{t \in I}$, a family of random variables indexed by a time interval I , under a risk-neutral measure. The stochastic differential equation (SDE) for the short rate r_t under the risk-neutral measure \mathbb{Q} at $t \in \mathbb{R}_+$ is given as

$$dr_t = \mu dt + \sigma dW_t^{\mathbb{Q}}, \quad (2.1)$$

where $\mu \in \mathbb{R}$ is the drift, $\sigma > 0$ is the volatility and W_t is the Brownian motion. In Jamshidian's framework, equation (2.1) is a one-factor affine term structure model. The yield $\gamma^{(\tau)}$ of a τ -period bond under an affine model is written as

$$\gamma^{(\tau)} = A(\tau) + B(\tau)r, \quad (2.2)$$

with the coefficients $A(\tau)$ and $B(\tau)$ dependent on maturity τ . The main advantage of affine models is that they are tractable with closed-form solutions for bond yields.

Jamshidian (1989) derived an explicit formula for pricing European options on coupon-bearing bonds. The formula derived resembled the Black-Scholes formula.

According to [Jamshidian \(1989\)](#) an option on a portfolio of coupon-bearing bonds can be decomposed into a portfolio of options on discount bonds with appropriate strike prices. The decomposition is known as the Jamshidian trick, see [Henrard \(2009\)](#); [Jamshidian \(1989\)](#) and [Peterson et al. \(2003\)](#).

In a paper by [Hübner \(1997\)](#), the closed-form solution for the standard swaption pricing formula was derived using [Jamshidian's \(1989\)](#) approach. Since the Vasicek model holds the volatility parameter constant, [Hübner \(1997\)](#) showed that the observed term structure and volatility structure can be added into the model by making a slight modification on the variance of the log of the bond price derived by [Jamshidian \(1989\)](#).

[Henrard \(2003\)](#) provided a formula identical to [Jamshidian's \(1989\)](#) for zero-coupon bonds and coupon bearing bonds under the HJM (Heath-Jarrow-Merton) one-factor model. His results were obtained by imposing a condition on the volatility structure of the model. The formula obtained gives satisfactory results for products like bond options and swaptions with settlement dates after the expiry date of the option. The only difficulty with the formula is that one needs to numerically solve a one-dimensional equation involving exponential functions. [Henrard \(2003\)](#) also derived a hedge ratio for hedging the option with the underlying.

Chapter 3

Pricing Methodology

3.1 Jamshidian Trick

This section illustrates how to value an amortising swaption using the Jamshidian trick. To obtain an explicit formula for European swaptions identical to Jamshidian's formula of zero-coupon bonds, one must express the payoff function as an option on a portfolio of coupon-bearing bonds. For simplification, this section starts off by showing how the method is applied to a standard¹ swaption and ends by showing an application to an amortising swaption.

The Jamshidian trick works in a one-factor interest rate model as in Vasicek (1977). Under the Vasicek framework, the interest rate r_t follows an Ornstein-Uhlenbeck process. The model has two drawbacks: First, it allows for negative values of the interest rate. Second, it does not fit the current term structure of interest rates nor the current volatilities exactly. The Ornstein-Uhlenbeck process models mean-reverting behaviour. Let $(r_t)_{t \in T}$, a family of random variables $r_t : \Omega \rightarrow \mathbb{R}$ indexed by set T , be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. The SDE of the short rate as proposed by Vasicek (1977) is given by

$$dr_t = \alpha(b - r_t) dt + \sigma dW_t^{\mathbb{Q}}, \quad \sigma, b, \alpha > 0, \quad (3.1)$$

where W_t is the Brownian motion under \mathbb{Q} , b is the historical average instantaneous rate, α is the speed of mean reversion and σ is the volatility. The process reverts towards the constant arbitrary equilibrium level b and hence $\alpha > 0$. The solution to (3.1) is given, for each $s \leq t$, as

$$r_t = r_s \exp(-\alpha(t-s)) + b(1 - \exp(-\alpha(t-s))) + \sigma \int_s^t \exp(-\alpha(t-u)) dW_u^{\mathbb{Q}}. \quad (3.2)$$

¹ standard means constant notional profile.

Proof. See Appendix A.

It is now clear that r_t conditional on \mathcal{F}_s is normally distributed with mean

$$\mathbb{E}[r_t | \mathcal{F}_s] = r_s \exp(-\alpha(t-s)) + b(1 - \exp(-\alpha(t-s))), \quad (3.3)$$

and variance

$$\text{Var}[r_t | \mathcal{F}_s] = \sigma^2 \int_s^t \exp(-2\alpha(t-u)) du = \frac{\sigma^2}{2\alpha} (1 - \exp(-2\alpha(t-s))). \quad (3.4)$$

3.1.1 Pricing a Standard European Swaption

Discount bond

The price $P(t, T)$ of a discount bond paying 1 at maturity T is given by

$$P(t, T) = \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^T r_s ds \right) \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T. \quad (3.5)$$

The price of a discount bond under the Vasiceck model is

$$P(t, r_t, T) = A(t, T) \exp(-B(t, T)r_t), \quad (3.6)$$

where $A(t, T)$ and $B(t, T)$ are defined implicitly as

$$A(t, T) = \exp \left(\left(b - \frac{\sigma^2}{2\alpha^2} \right) [B(t, T) - (T+t)] - \frac{\sigma^2}{4\alpha} B(t, T)^2 \right)$$

and

$$B(t, T) = \frac{1}{\alpha} [1 - \exp(-\alpha(T-t))].$$

Definition 3.1. A *Coupon-bearing bond* is a contract that ensures the payment at future times T_1, \dots, T_M of the deterministic amounts of currency (coupon payments) $c_j := c_1, \dots, c_M$. Typically, the cash flows are defined as $c_j = N\delta R$ for $j < M$ and $c_j = N(\delta R + 1)$ for $j = M$, where R is a fixed interest rate and N is bond notional value. The last cash flow includes the reimbursement of the notional value of the bond,

$$\text{CB}(t, T_M) = \sum_{j=1}^{M-1} NR\delta P(t, T_j) + N(1 + R\delta) P(t, T_M). \quad (3.7)$$

Interest rate swap

Consider a receiver (forward-start) swap with the first reset date T_0 and payment dates $T_1 < T_2 < \dots < T_M$ (tenor structure) at fixed rate R . The discounted payoff at time $t \leq T_0$ is given by

$$\text{VR}(t) = \sum_{j=1}^M P(t, T_j) \delta N (R - L(t; T_{j-1}, T_j)), \quad (3.8)$$

where $L(t, T_{j-1}, T_j)$ represents the forward Libor rate, which is the floating rate.

Using a no arbitrage argument, the forward Libor rate is given as

$$L(t; T_{j-1}, T_j) = \frac{1}{\delta} \left(\frac{P(t, T_{j-1})}{P(t, T_j)} - 1 \right). \quad (3.9)$$

For the above IRS to be fair, one solves for a unique R , namely the forward swap rate, such that (3.8) equates to zero. The forward swap rate for (3.8) is given by

$$S(t) = \frac{N(P(t, T_0) - P(t, T_M))}{\sum_{j=1}^M N \delta P(t, T_j)}. \quad (3.10)$$

Standard Swaption

Let $H_{T_0}^x \in L^1(\mathbb{Q}, \mathcal{F}_{T_0})$ be the payoff function of the swaption, where $x = \text{rec, pay}$, for receiver and payer swaptions respectively. The price of a receiver swaption at time t with tenor structure $t \leq T_0 < T_1 < \dots < T_M$, a strike rate R and expiry date T_0 is given as the discounted expectation of the payoff as

$$\begin{aligned} \text{RS}(t) &= \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^{T_0} r_s ds \right) H_{T_0}^{\text{rec}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^{T_0} r_s ds \right) N \sum_{j=1}^M \delta P(T_0, T_j) (R - S(T_0))^+ \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T_0, \end{aligned} \quad (3.11)$$

and for the payer swaption as

$$\begin{aligned} \text{PS}(t) &= \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^{T_0} r_s ds \right) H_{T_0}^{\text{pay}} \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^{T_0} r_s ds \right) N \sum_{j=1}^M \delta P(T_0, T_j) (S(T_0) - R)^+ \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T_0, \end{aligned} \quad (3.12)$$

where $\delta = T_j - T_{j-1}$, N is the notional, T_M is the maturity date of the swap, $P(T_0, T_j)$ is the discount bond price and $S(T_0)$ is the forward swap rate at time T_0 . It is important to note that (3.11) and (3.12), which are \mathcal{F}_t -measurable should not be confused with $H_{T_0}^{\text{rec}}$ and $H_{T_0}^{\text{pay}}$, which are \mathcal{F}_{T_0} -measurable.

A payer swaption is a call option on a forward swap rate and a receiver swaption is a put option on a forward swap rate.

Now consider a receiver swaption defined above. Since $P(T_0, T_0) = 1$, the time T_0 value of the receiver swaption expiring at time T_0 is given by

$$\text{RS}(T_0) = \left(\sum_{j=1}^M NR\delta P(T_0, T_j) + NP(T_0, T_M) - N \right)^+. \quad (3.13)$$

Let $P(t, r_t, T)$ be the explicit price function of a discount bond at time t maturing at time T given by (3.6). Equation (3.13) with respect to $P(t, r_t, T)$ is given as

$$\begin{aligned} \text{RS}(T_0) &= \left(\sum_{j=1}^M NR\delta P(T_0, r_{T_0}, T_j) + NP(T_0, r_{T_0}, T_M) - N \right)^+ \\ &= \left(\sum_{j=1}^{M-1} NR\delta P(T_0, r_{T_0}, T_j) + N(1 + R\delta)P(T_0, r_{T_0}, T_M) - N \right)^+. \end{aligned} \quad (3.14)$$

Equation (3.14) is equivalent to a payoff function of a call option written on a coupon-bearing bond with expiry date T_0 and a strike price $K = N$,

$$\text{RS}(T_0) = (\text{CB}(T_0, T_M) - N)^+ = \left(\sum_{j=1}^M c_j P(T_0, r_{T_0}, T_j) - K \right)^+, \quad (3.15)$$

where c_j are the cash flows defined in Definition 3.1. The trick is to convert the positive part of the sum in (3.15) to the sum of positive parts. This trick relies on finding a unique r^* such that

$$\sum_{j=1}^M c_j K_j = N, \quad (3.16)$$

where

$$K_j = P(T_0, r^*, T_j).$$

To achieve the desired decomposition the short rate model must satisfy the following condition:

$$\frac{\partial P(t, r_t, T)}{\partial r_t} < 0, \text{ for all } 0 < t < T.$$

Under the condition above, consider the case where $r_{T_0} < r^*$, we get

$$\sum_{j=1}^M c_j P(T_0, r_{T_0}, T_j) > \sum_{j=1}^M c_j K_j,$$

and so

$$P(T_0, r_{T_0}, T_j) > K_j,$$

which implies the call option expires in the money

$$\begin{aligned} 0 &< \left(\sum_{j=1}^M c_j P(T_0, r_{T_0}, T_j) - N \right)^+ \\ &= \sum_{j=1}^M c_j P(T_0, r_{T_0}, T_j) - N \\ &= \sum_{j=1}^M c_j P(T_0, r_{T_0}, T_j) - \sum_{j=1}^M c_j K_j \\ &= \sum_{j=1}^M c_j (P(T_0, r_{T_0}, T_j) - K_j)^+. \end{aligned} \quad (3.17)$$

Now, consider the case where $r_{T_0} > r^*$, we get

$$\sum_{j=1}^M c_j P(T_0, r_{T_0}, T_j) < \sum_{j=1}^M c_j K_j,$$

and so

$$P(T_0, r_{T_0}, T_j) < K_j,$$

which implies the call option expires out the money

$$\begin{aligned} 0 &= \left(\sum_{j=1}^M c_j P(T_0, r_{T_0}, T_j) - N \right)^+ \\ &= \sum_{j=1}^M c_j P(T_0, r_{T_0}, T_j) - N \\ &= \sum_{j=1}^M c_j P(T_0, r_{T_0}, T_j) - \sum_{j=1}^M c_j K_j \\ &= \sum_{j=1}^M c_j (P(T_0, r_{T_0}, T_j) - K_j)^+. \end{aligned} \quad (3.18)$$

Equations (3.17) and (3.18) show that in either case the receiver swaption value at time T_0 with strike rate N and an expiry date T_0 is given by

$$RS(T_0, R, r_{T_0}, T_0) = \sum_{j=1}^M c_j (P(T_0, r_{T_0}, T_j) - K_j)^+. \quad (3.19)$$

The strikes of each individual discount bond options are adjusted such that all options can be exercised simultaneously.

Taking the time t discounted expectation of equation (3.19) and using the change of numeraire technique in Appendix B yields the following

$$\text{RS}(t, R, r_t, T_0) = \sum_{j=1}^M c_j \text{ZBO}(t, T_0, T_j, K_j, r_t, 1), \quad 0 \leq t \leq T_0, \quad (3.20)$$

Similarly the value of a payer swaption at time t is given as

$$\text{PS}(t, R, r_t, T_0) = \sum_{j=1}^M c_j \text{ZBO}(t, T_0, T_j, K_j, r_t, -1), \quad 0 \leq t \leq T_0, \quad (3.21)$$

where ZBO is the closed form solution of a European bond option given as

$$\text{ZBO}(t, T_0, T_j, K, r_t, \zeta) = \zeta (P(t, r_t, T_j)\Phi(d_1) - KP(t, r_t, T_0)\Phi(d_2)), \quad (3.22)$$

where

$$d_1 = \frac{\log\left(\frac{P(t, r_t, T_j)}{KP(t, r_t, T_0)}\right) + \frac{\bar{\sigma}^2}{2}}{\bar{\sigma}} \quad \text{and} \quad d_2 = d_1 - \bar{\sigma}$$

with

$$\bar{\sigma} = \sigma \sqrt{\frac{1 - \exp(-2\alpha(T_0 - t))}{2\alpha}} B(T_0, T_j),$$

where $\zeta = 1$ (for a call) and -1 (for a put), and Φ is the cumulative normal distribution function. From (3.20) and (3.21), a receiver swaption is decomposed into a sum of call options and a payer swaption is decomposed into a sum of put options.

3.1.2 Pricing an Amortising European Swaption

Amortising Swaption

Consider the case where the notional profile of equation (3.11) and (3.12) is no longer a constant, but an amortising profile. The price of a receiver swaption at time t written on an amortising swap with tenor $T_0 < T_1 < \dots < T_M$, a strike rate R and expiry date T_0 is given as follows

$$\text{RS}_A(t) = \mathbb{E}^{\mathbb{Q}} \left[\exp\left(-\int_t^{T_0} r_s ds\right) \sum_{j=1}^M N_{T_j} \delta P(T_0, T_j) (R - S_A(T_0))^+ \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T_0 \quad (3.23)$$

and for the payer swaption as

$$\text{PS}_A(t) = \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^{T_0} r_s ds \right) \sum_{j=1}^M N_{T_j} \delta P(T_0, T_j) (S_A(T_0) - R)^+ \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T_0, \quad (3.24)$$

where $S_A(t)$ is the amortising forward swap rate. The notional profile in equation (3.23) and (3.24) is defined by N_{T_j} showing that the notional changes at every payment date in the swap. By substituting N_{T_j} into (3.8), the amortising forward swap rate is given by

$$S_A(t) = \frac{\sum_{j=1}^M N_{T_j} (P(t, T_{j-1}) - P(t, T_j))}{\sum_{j=1}^M N_{T_j} \delta P(t, T_j)}. \quad (3.25)$$

To perform the same pricing mechanism as above consider the payoff of an amortizing receiver swaption at the expiry date T_0 ,

$$\text{RS}_A(T_0) = \sum_{j=1}^M N_{T_j} \delta P(T_0, T_j) (R - S_A(T_0))^+. \quad (3.26)$$

By substituting equation (3.25) into equation (3.26), we obtain

$$\text{RS}_A(T_0) = \left(\sum_{j=1}^M N_{T_j} R \delta P(T_0, T_j) - \sum_{j=1}^M N_{T_j} (P(t, T_{j-1}) - P(t, T_j)) \right)^+. \quad (3.27)$$

To apply the Jamshidian method (3.27) should be expressed in the following form

$$\left(\sum_{j=1}^M c_j P(T_0, T_j) - K \right)^+. \quad (3.28)$$

By using elementary algebra we obtain

$$\begin{aligned} \text{RS}_A(T_0) &= \left(\sum_{j=1}^{M-1} (N_{T_j} (R\delta + 1) - N_{T_{j+1}}) P(T_0, T_j) - N_{T_M} (R\delta + 1) P(T_0, T_M) - N_{T_1} \right)^+ \\ &= \left(\sum_{j=1}^M c_j P(T_0, T_j) - N_{T_1} \right)^+, \end{aligned} \quad (3.29)$$

where

$$c_j = \begin{cases} N_{T_j} (\delta R + 1) - N_{T_{j+1}}, & 1 \leq j \leq M - 1, \\ N_{T_j} (\delta R + 1), & j = M. \end{cases}$$

In the case of an amortising notional profile set $N_{T_M} = 0$ such that the cash flow profile becomes

$$c_j = \begin{cases} (N_{T_j}(\delta R + 1) - N_{T_{j+1}}), & 1 \leq j \leq M - 1 \\ 0, & j = M. \end{cases}$$

Following the same steps showed in Section 3.1.2 the time t price of an amortising receiver swaption is given by

$$RS_A(t, R, r_t, T_0) = \sum_{j=1}^M c_j ZBO(t, T_0, T_j, K_j, r_t, 1), \quad 0 \leq t \leq T_0 \quad (3.30)$$

and similarly the time t price of an amortising payer swaption is given by

$$PS_A(t, R, r_t, T_0) = \sum_{j=1}^M c_j ZBO(t, T_0, T_j, K_j, r_t, -1), \quad 0 \leq t \leq T_0, \quad (3.31)$$

where ZBO is given by (3.22). The interpretation of (3.30) and (3.31) is the same as (3.20) and (3.21). Equations (3.30) and (3.31) represent a closed-form solution of an amortising swaption using the Jamshidian trick.

3.2 Numerically Computing a One-Dimensional Integral

This section shows how an amortising swaption can be priced under a two-factor short rate framework by computing a one-dimensional integral numerically. It is pointed out in Brigo and Mercurio (2013) that one-factor short rate models have perfect correlations between interest rates of different maturities which results in a poor model for describing the evolution of the observed yield curve and poor prices for path dependent options. Past literature has explored the curvature of the observed yield curve showing evidence that interest rates of different maturities have some decorrelation between them.

The two-factor affine short rate model, in particular, the additive Gaussian two-factor G2++ model, is one of many models that attempts to capture a more realistic correlation of interest rates of different maturities, see Brigo and Mercurio (2013). The short rate equation r_t under the G2++ model is specified as

$$r_t = x_t + y_t + \varphi_t, \quad \varphi_0 = r_0, \quad (3.32)$$

where $(x_t)_{t \in T}$ and $(y_t)_{t \in T}$ are driven by the SDEs

$$\begin{aligned} dx_t &= -ar_t dt + \sigma dW_t^{1, \mathbb{Q}}, & x_0 &= 0, \\ dy_t &= -br_t dt + \eta dW_t^{2, \mathbb{Q}}, & y_0 &= 0. \end{aligned}$$

Here a, b, σ and η are positive constants, with instantaneously-correlated Brownian motions W^1, W^2 with $dW_t^{1,\mathbb{Q}}dW_t^{2,\mathbb{Q}} = \rho dt$, $-1 \leq \rho \leq 1$. The factors x_t and y_t are correlated Gaussian factors and $\varphi(t)$ is a deterministic function chosen to fit the current observed term structure of discount factors. By solving the SDEs for x_t and y_t above, equation (3.32) for each $s \leq t$ can be rewritten as

$$\begin{aligned} r_t = & x_s \exp(-a(t-s)) + y_s \exp(-b(t-s)) \\ & + \sigma \int_s^t \exp(-a(t-u)) dW_u^{1,\mathbb{Q}} + \eta \int_s^t \exp(-b(t-u)) dW_u^{2,\mathbb{Q}} + \varphi_t. \end{aligned} \quad (3.33)$$

Equation (3.33) is normally distributed with a mean of

$$\mathbb{E}[r_t | \mathcal{F}_s] = x_s \exp(-a(t-s)) + y(s) \exp(-b(t-s)) + \varphi_t$$

and a variance of

$$\begin{aligned} \text{Var}[r_t | \mathcal{F}_s] = & \frac{\sigma^2}{2a} [1 - \exp(-2a(t-s))] \\ & + 2\rho \frac{\sigma\eta}{a+b} [1 - \exp(-(a+b)(t-s))] + \frac{\eta^2}{2b} [1 - \exp(-2b(t-s))]. \end{aligned}$$

A derivation (3.33) is given in [Brigo and Mercurio \(2013\)](#).

Consider the payoff of a receiver amortising swaption with an expiry date T_0 , strike rate R and a swap maturity date T_M . Since the G2++ is an affine model the t price of a receiver amortising swaption in discounted-expected-value form is given as

$$\text{RS}_A(t) = \mathbb{E}^{\mathbb{Q}} \left[\exp \left(- \int_t^{T_0} r_s ds \right) \left(\sum_{j=1}^M c_j P(T_0, r_{T_0}, T_j) - N_{T_1} \right)^+ \middle| \mathcal{F}_t \right], \quad 0 \leq t \leq T_0, \quad (3.34)$$

where c_j is defined in (3.29). Since the discount factor is random, by applying the change of numeraire toolkit in [Appendix B](#), the discount swaption price process in (3.34) can be written under the forward measure \mathbb{Q}_0^T as

$$\text{RS}_A(t) = P(t, r_t, T_0) \mathbb{E}^{\mathbb{Q}^{T_0}} \left[\left(\sum_{j=1}^M c_j P(T_0, r_{T_0}, T_j) - N_{T_1} \right)^+ \right], \quad (3.35)$$

where $P(t, r_t, T_0)$ is the forward numeraire. The price of a discount bond at $t \leq T$, is given by

$$P(t, r_t, T) = A(t, T) \exp(-B(a, t, T)x_t - B(b, t, T)y_t), \quad (3.36)$$

where

$$B(z, t, T) = \frac{1 - \exp(-z(T - t))}{z} \quad (3.37)$$

and

$$A(t, T) = \exp\left(-\int_t^T \varphi_u du - \frac{1}{2}V(t, T)\right), \quad (3.38)$$

with

$$\begin{aligned} V(t, T) = & \frac{\sigma^2}{a^2} \left(T - t + \frac{2}{a} \exp(-a(T - t)) - \frac{1}{2a} \exp(-2a(T - t)) - \frac{3}{2a} \right) \\ & + \frac{\eta^2}{b^2} \left(T - t + \frac{2}{b} \exp(-b(T - t)) - \frac{1}{2b} \exp(-2b(T - t)) - \frac{3}{2b} \right) \\ & + 2\rho \frac{\sigma\eta}{ab} \left(T - t + \frac{\exp(-a(T - t)) - 1}{a} + \frac{\exp(-b(T - t)) - 1}{b} \right. \\ & \left. - \frac{\exp(-(a + b)(T - t)) - 1}{a + b} \right), \end{aligned} \quad (3.39)$$

where the function φ_t is derived by linking the discount factors produced by the model (3.36) with the ones observed in the market. To make things simple, assume that the function φ_t is a constant. Since (3.35) is under the T_0 -forward measure, the SDEs under the T_0 -forward measure are

$$\begin{aligned} dx_t = & \left[-ax_t - \frac{\sigma^2}{a} (1 - \exp(-a(T_0 - t))) - \rho \frac{\sigma\eta}{b} (1 - \exp(-b(T_0 - t))) \right] dt \\ & + \sigma dW_t^{1, \mathbb{Q}^{T_0}} \end{aligned} \quad (3.40)$$

and

$$\begin{aligned} dy_t = & \left[-by_t - \frac{\eta^2}{b} (1 - \exp(-b(T_0 - t))) - \rho \frac{\sigma\eta}{a} (1 - \exp(-a(T_0 - t))) \right] dt \\ & + \eta dW_t^{2, \mathbb{Q}^{T_0}}. \end{aligned} \quad (3.41)$$

The solution to the state variable is

$$\begin{aligned} r_t = & x_s \exp(-a(t - s)) - M_x^{T_0}(s, t) + y_s \exp(-b(t - s)) \\ & - M_y^{T_0}(s, t) + \sigma \int_s^t \exp(-a(t - u)) dW_u^{1, \mathbb{Q}^{T_0}} \\ & + \eta \int_s^t \exp(-b(t - u)) dW_u^{2, \mathbb{Q}^{T_0}} + \varphi_t, \end{aligned} \quad (3.42)$$

where

$$\begin{aligned}
M_x^{T_0}(s, t) &= \left(\frac{\sigma^2}{a^2} + \rho \frac{\sigma\eta}{ab} \right) [1 - \exp(-a(t-s))] \\
&\quad - \frac{\sigma^2}{2a^2} \left[\exp(-a(T_0-t)) - \exp(-a(T_0+t-2s)) \right] \\
&\quad - \frac{\rho\sigma\eta}{b(a+b)} [\exp(-b(T_0-t)) - \exp(-bT_0-at+(a+b)s)], \quad s \leq t, \quad (3.43)
\end{aligned}$$

and

$$\begin{aligned}
M_y^{T_0}(s, t) &= \left(\frac{\eta^2}{b^2} + \rho \frac{\sigma\eta}{ab} \right) [1 - \exp(-b(t-s))] \\
&\quad - \frac{\eta^2}{2b^2} \left[\exp(-b(T_0-t)) - \exp(-b(T_0+t-2s)) \right] \\
&\quad - \frac{\rho\sigma\eta}{a(a+b)} [\exp(-a(T_0-t)) - \exp(-aT_0-bt+(a+b)s)], \quad s \leq t. \quad (3.44)
\end{aligned}$$

Substituting equation (3.36) into equation (3.35) and writing the expression in integral form outputs the following

$$\begin{aligned}
RS_A(t) &= P(t, T_0) \\
&\quad \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left(\sum_{j=1}^M c_j A(T_0, T_j) \exp(-B(a, T_0, T_j)x - B(b, T_0, T_j)y) - N_{T_1} \right) \\
&\quad \times f(x, y) dx dy, \quad (3.45)
\end{aligned}$$

where the function $f(x, y)$ is the joint density function,

$$f(x, y) = \frac{\exp\left(-\frac{1}{2(1-\rho_{xy}^2)} \left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho_x\rho_y \frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 \right]\right)}{2\pi\sigma_x\sigma_y\sqrt{1-\rho_{xy}^2}}. \quad (3.46)$$

Equation (3.45) can be simplified to a single integral by freezing x and evaluating the integral over y , see [Brigo et al. \(2002\)](#), as follows

$$\begin{aligned}
RS_A(t, R, r_t, T_0) &= P(t, T_0) \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{1}{2} \left(\frac{x-\mu_x}{\sigma_x}\right)^2\right)}{\sigma_x\sqrt{2\pi}} \left[\sum_{j=1}^M \lambda_j(x) \exp(\kappa_j(x)) \Phi(h_2(x)) - \right. \\
&\quad \left. N_{T_1} \Phi(h_1(x)) \right] dx \quad (3.47)
\end{aligned}$$

and similarly the price at time $t \leq T_0$ for the amortising payer swaption is given by,

$$\text{PS}_A(t, R, r_t, T_0) = P(t, T_0) \int_{-\infty}^{+\infty} \frac{\exp\left(-\frac{1}{2} \left(\frac{x-\mu_x}{\sigma_x}\right)^2\right)}{\sigma_x \sqrt{2\pi}} \left[N_{T_1} \Phi(-h_1(x)) - \sum_{j=1}^M \lambda_j(x) \exp(\kappa_j(x)) \Phi(-h_2(x)) \right] dx, \quad (3.48)$$

where

$$\begin{aligned} h_1(x) &= \frac{\bar{y}(x) - \mu_y}{\sigma_y \sqrt{1 - \rho_{xy}^2}} - \frac{\rho_{xy}(x - \mu_x)}{\sigma_x \sqrt{1 - \rho_{xy}^2}}, \\ h_2(x) &= h_1(x) + B(b, T_0, T_j) \sigma_y \sqrt{1 - \rho_{xy}^2}, \\ \lambda_j(x) &= c_j A(T_0, T_j) \exp(-B(a, T_0, T_j)x), \\ \kappa_j(x) &= -B(b, T_0, T_j) \left[\mu_y - \frac{1}{2} (1 - \rho_{xy}^2) \sigma_y B(b, T_0, T_j) + \rho_{xy} \sigma_y \frac{x - \mu_x}{\sigma_x} \right], \\ \mu_x &= -M_x^{T_0}(t, T_0), \\ \mu_y &= -M_y^{T_0}(t, T_0), \\ \sigma_x &= \sigma \sqrt{\frac{1 - \exp(-2a(T_0 - t))}{2a}}, \\ \sigma_y &= \eta \sqrt{\frac{1 - \exp(-2b(T_0 - t))}{2b}} \end{aligned}$$

and

$$\rho_{xy} = \frac{\rho \sigma \eta}{(a + b) \sigma_x \sigma_y}.$$

The constant $\bar{y}(x)$ is the unique solution to the equation

$$\sum_{j=1}^M c_j A(T_0, T_j) \exp(-B(a, T_0, T_j)x - B(b, T_0, T_j)\bar{y}(x)) = N_{T_1}. \quad (3.49)$$

To compute the semi-closed form swaption prices given by equation (3.47) and (3.48) one needs to use numerical root finding and integration methods. The integrand in equation (3.47) and (3.48) is a bounded function against a normal distribution.

To compute the integral in (3.47) and (3.48) we adopt the Gauss-Legendre integration technique described by [Acar and Natcheva-Acar \(2009\)](#). The technique computes abscissas $\{x_0, x_2, \dots, x_{99}\}$ and weights $\{w_0, w_1, \dots, w_{99}\}$ so that the formula is

$$\int_l^u f(x) dx \approx \sum_{i=0}^{99} w_i f(x_i). \quad (3.50)$$

The random variable X in (3.47) and (3.48) is normally distributed with mean μ_x and standard deviation σ_x , we write

$$X \sim \mathbb{N}(\mu_x, \sigma_x),$$

where μ_x and σ_x are parameters of the distribution. The boundaries $[l, u]$ of the integral are given as a function of μ_x and σ_x

$$[\mu_x - N\sigma_x, \mu_x + N\sigma_x],$$

see Ferranti (2015). The value $N \in \mathbb{N}$ is chosen using the algorithm given by Acar and Natcheva-Acar (2009) as follows

1. Set $N = 1$.
2. Let $l = -N\sigma_x, u = N\sigma_x$.
3. Compute the integrand at l and u .
 - (a) If $Integrand(l) < 10^{-8}$, set lower bound as l . Else increase N and go to step 2.
 - (b) If $Integrand(u) < 10^{-8}$, set upper bound as u . Else increase N and go to step 2.

To solve for the roots $\bar{y}(x)$ in equation (3.49) for individual values of x in the domain above, we use the `fsolve` function in Python.

Chapter 4

Results

In this chapter, we test the efficacy of the closed form solution under the one-factor affine short rate model and semi-closed form solution under the two-factor affine short rate model. We compare the price derived from the formulas with Monte Carlo based prices which are taken as the true solution of the amortising swaption. Monte Carlo methods are simply described as numerical methods based on random sampling (Niederreiter, 1988). Suppose H is the discounted payoff function of an amortising swaption maturing at time T_0 . Let S_p be the price of the amortising swaption. From Section 3.1, the price of the amortising swaption at $t \leq T_0$ is given by

$$S_p = \mathbb{E}^{\mathbb{Q}} [H], \quad (4.1)$$

The basis of Monte Carlo is that we can estimate the price S_p by simulating an independent and identically distributed (i.i.d) sequence $\{\hat{H}_i, i = 1, 2, \dots, n\}$ where the \hat{H}_i has a mean S_p and variance $\text{Var}(S_p)$. So the price estimate based on n replications is given by the sample mean

$$\hat{S}_p = \frac{1}{n} \sum_{i=1}^n \hat{H}_i, \quad (4.2)$$

By the law of large numbers as $n \rightarrow \infty$ this sample mean is a normally distributed approximation with mean S_p and variance $\text{Var}(S_p)/n$ (Boyle et al., 1997).

Due to the fact that Monte Carlo simulation has an error proportional to $\text{Var}(S_p)/n$ and a slow convergence rate of $\mathcal{O}(1/\sqrt{n})$, the following part will discuss two techniques to increase the speed and the accuracy of the estimates.

Control Variate Method

The underlying principle in this technique is "Use what you know" (Boyle et al., 1997). This technique can only be applied under the standard Monte Carlo ap-

proach¹.

Suppose now we calculate an i.i.d pair $(\hat{H}_i, \hat{H}_{A_i})$, $i = 1, \dots, n$. Where the closed form solution

$$I_A = \mathbb{E}[H_A], \quad (4.3)$$

of the \hat{H}_{A_i} is known. For any fixed β , the control variate estimator is given by

$$\hat{S}_p^\beta = \hat{S}_p + \beta (I_A - \hat{I}_A). \quad (4.4)$$

The purpose of this method is to reduce the variance

$$\text{Var} [\hat{S}_p^\beta] = \text{Var} [\hat{S}_p] + \beta^2 \text{Var} [\hat{I}_A] - 2\beta \text{Cov} [\hat{S}_p, \hat{I}_A], \quad (4.5)$$

such that

$$\text{Var} [\hat{S}_p^\beta] \leq \text{Var} [\hat{S}_p].$$

To find the variance-minimum value of β , β_{\min} , equate the first order derivative of equation (4.5) with respect to β to zero and solve for β which is given as

$$\beta_{\min} = \frac{\text{Cov} [\hat{S}_p, \hat{I}_A]}{\text{Var} [\hat{I}_A]}, \quad (4.6)$$

Note that by using equation (4.6), the inequality above holds. From equation (4.6), a good control variate is one that strongly correlates with the corresponding target integrated. We compute an unbiased estimate $\hat{\beta}_{\min}$ by reserving $n_1 < n$ replications as follows

$$\hat{\beta}_{\min} = \frac{\sum_{i=1}^{n_1} (\hat{H}_i - \hat{S}_p)(\hat{H}_{A_i} - \hat{I}_A)}{\sum_{i=1}^{n_1} (\hat{H}_{A_i} - \hat{I}_A)^2}, \quad (4.7)$$

where H_A is the payoff function of the known quantity. The control variate estimator for an amortising swaption is given by

$$\hat{S}_p^{\hat{\beta}} = \frac{1}{n} \sum_{i=1}^n \hat{H}_i + \hat{\beta} \left(I_A - \frac{1}{n} \sum_{i=1}^n \hat{H}_{A_i} \right). \quad (4.8)$$

Quasi-Random Sequences

¹ In standard Monte Carlo application, the n points generated are described as pseudo-random numbers.

An alternative method used to improve the accuracy of Monte Carlo estimates is the quasi-Monte Carlo (QMC) method. QMCs use low-discrepancy sequences instead of pseudo-random sequences. Low-discrepancy sequences or quasi-random sequences are sequences that are highly equidistributed compared to pseudo-random numbers (Sak and Başoğlu, 2015; Niederreiter, 1978). An advantage of quasi-Monte Carlo methods that is pointed out by Schoenmakers and Heemink (1997) is that QMC sequences can be used as a variance reduction technique due to their speed of convergence. Let the price given by (4.1) be defined on an s dimensional unit cube $[0, 1]^s$,

$$S_p = \int_{[0,1]^s} H(x)dx. \quad (4.9)$$

The quasi-Monte Carlo estimator, based on n replications, is

$$\hat{S}_p = \frac{1}{n} \sum_{i=1}^n \hat{H}(x_i), \quad (4.10)$$

where x_1, x_2, \dots, x_n are uniform random distributed random numbers. Unlike the crude Monte Carlo estimator the variables x_1, x_2, \dots, x_n are not i.i.d samples. Obviously, we want n to be as small as possible since

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \hat{H}(x_i) = \int_{[0,1]^s} g(x)dx. \quad (4.11)$$

The main result on the integration error of quasi-Monte Carlo methods is given by

$$\left| \int_{[0,1]^s} H(x)dx - \frac{1}{n} \sum_{i=1}^n \hat{H}(x_i) \right| \leq V(g)D^s(n, s), \quad (4.12)$$

where $V(g)$ is the bounded variation function in the sense of Hardy and Krause, $D^*(n, s)$ is the star-discrepancy of the set $\{x_1, x_2, \dots, x_n\}$, which measures the uniformity of the distribution of $\{x_1, x_2, \dots, x_n\}$, see Glasserman (2013) and Wang (2001). Equation (4.12) shows that the disadvantage of QMC is that the error depends on the dimension s . In this dissertation, we use Sobol quasi-random sequences, see Sobol (1998) and Glasserman (2013). Sobol numbers have a theoretical error of the order $\mathcal{O}(N^{-1} \log^s n)$. The sequence generated by QMC algorithms is uniformly distributed and since we are interested in generating normally distributed random samples, we use the inverse transform method to perform the conversion from uniform to normal variates.

Consider two short rate models: the Vasicek model and the G2++ model. To test the accuracy in both cases, we compute the difference between the closed form prices and the Monte Carlo prices for different strike prices. We illustrate the numerical results in the next two sections.

4.1 Numerical Results for the Vasicek Model

The prices derived in Section 3.1 are computed under the \mathbb{Q}^{T_0} forward measure, so the short rate SDE under the forward measure is given by

$$dr(t) = [\alpha b - \alpha r(t) - B(t, T_0)\sigma^2] dt + \sigma dW^{\mathbb{Q}^{T_0}}(t), \quad (4.13)$$

and the solution for equation (4.13) is given by

$$r_t = r_s \exp(-\alpha(t-s)) + M_r^{T_0}(s, t) + \sigma \int_s^t \exp(-\alpha(t-u)) dW_t^{\mathbb{Q}^{T_0}}, \quad s \leq t, \quad (4.14)$$

where

$$\begin{aligned} M_r^{T_0}(s, t) &= \left(b - \frac{\sigma^2}{\alpha^2} \right) (1 - \exp(-\alpha(t-s))) \\ &\quad + \frac{\sigma^2}{2\alpha^2} [\exp(-\alpha(T_0-s)) - \exp(-\alpha(T_0+t-2s))], \quad s \leq t. \end{aligned} \quad (4.15)$$

The solution r_t is normally distributed with a mean of

$$\mathbb{E}[r_t | \mathcal{F}_s] = \exp(-\alpha(t-s)) r_s + M_r^{T_0}(s, t)$$

and a variance of

$$\text{Var}[r_t | \mathcal{F}_s] = \frac{\sigma^2}{2\alpha^2} [1 - \exp(-2\alpha(t-s))].$$

We have shown that given r_t for $t \leq T_0$, one can compute T_0 -forward rate

$$r_{T_0} \sim \mathbb{N}(\mu_r(t, T_0), \sigma_r^2(t, T_0)),$$

where

$$\mu_r(t, T_0) = r_t \exp(-\alpha(T_0-t)) + M_r^{T_0}(t, T_0)$$

and

$$\sigma_r^2(t, T_0) = \frac{\sigma^2}{2\alpha^2} [1 - \exp(-2\alpha(T_0-t))].$$

To simulate short rates at time T_0 from $t \leq T_0$ we set

$$r_{T_0} = \mu_r(t, T_0) + \sigma_r(t, T_0) Z_{T_0}, \quad (4.16)$$

where Z_{T_0} is a sequence of n independent random variables drawn from $\mathbb{N}(0, 1)$.

Consider an at-the-money 1 year receiver swaption with a linear amortising notional profile written on a 5 years swap. Both fixed and floating legs occur semiannually or $\delta = 0.5$.

Exhibit 1. Parameters for Vasicek Model

| r_0 | α | b | σ |
|-------|----------|---------|----------|
| 0.07 | 0.1 | 0.04438 | 0.00474 |

Figure 4.1 below shows the evolution of the amortising receiver swaption prices for different Monte Carlo methods over different sample sizes. We compare the closed form solution, equation (3.30), to Monte Carlo estimates. From the 3-standard deviation bounds², one can observe that as the sample size increases, Monte Carlo prices converge to the closed form solution. The Sobol estimates converge much faster than the control variate estimates.

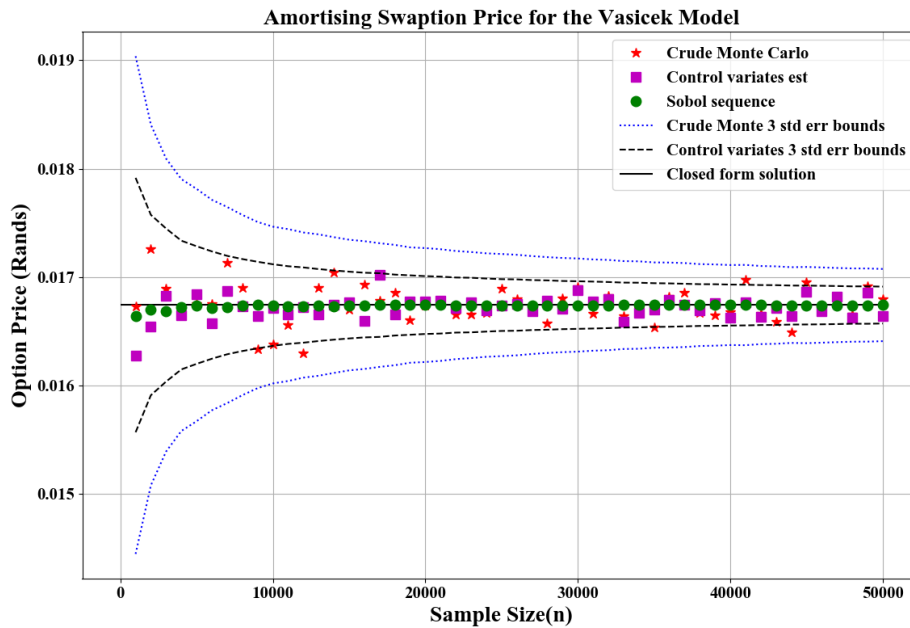


Fig. 4.1: Monte Carlo prices compared to the closed form solution of an amortising receiver swaption.

The price of the option under the different pricing methods shown in Figure 4.1

² This error bound can only be shown for points generated by pseudo-random number generator.

versus term to maturity is shown in Figure 4.2 respectively. The closed form solution and Monte Carlo estimates converge as the term to maturity decreases. This illustrates that the longer term to maturity the option has the higher the probability of it finishing in the money.

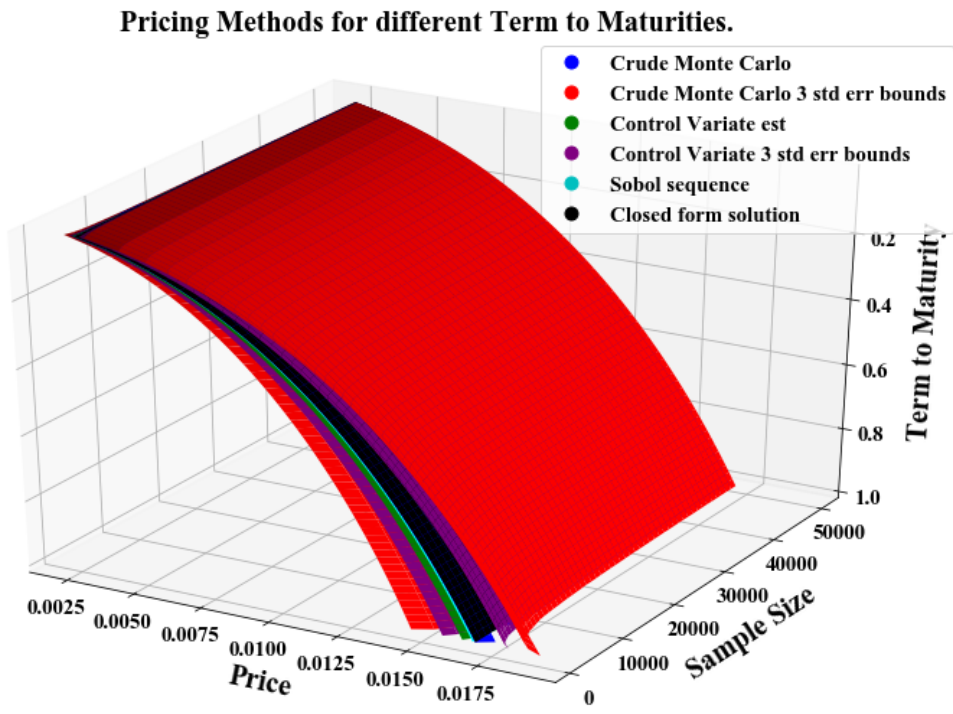


Fig. 4.2: Effects of the term to maturity on convergence of pricing methods.

Figure 4.3 shows individual percentage difference between different Monte Carlo methods and the closed form solution. One can easily observe that the quasi-Monte Carlo estimates have a higher accuracy than the control variate estimates.

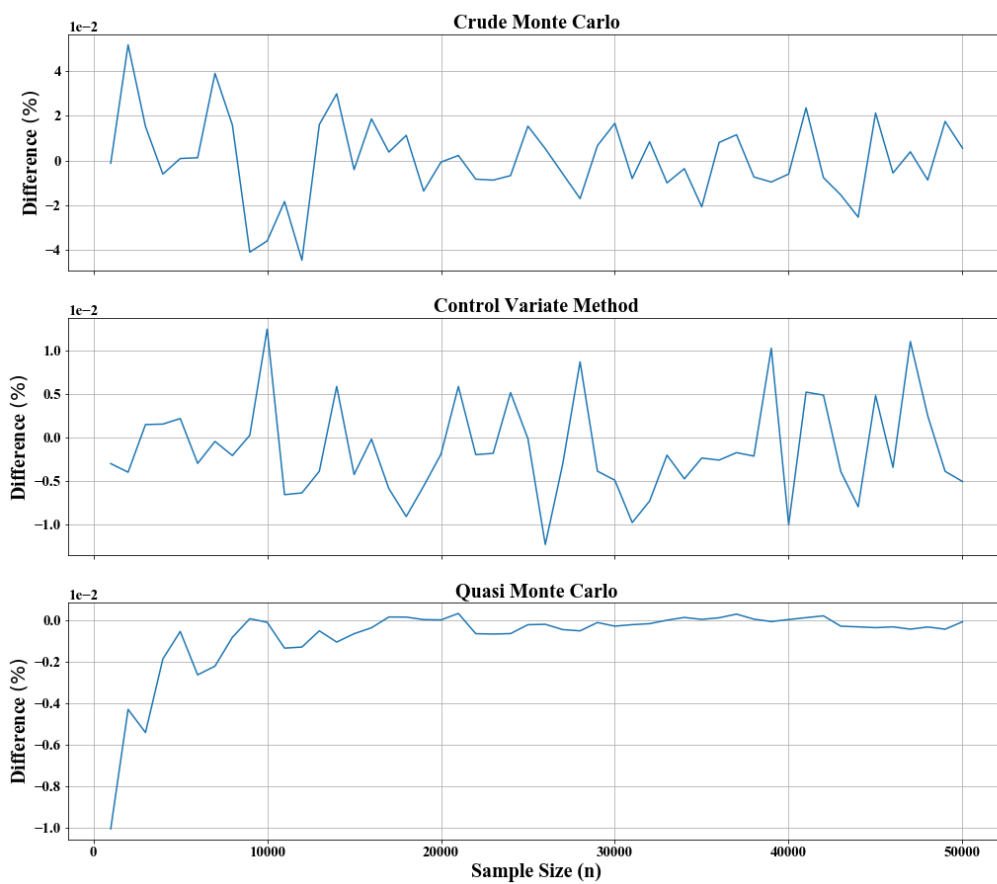


Fig. 4.3: The percentage difference between the closed form solution and Monte Carlo prices for different sample sizes.

Figure 4.4 shows the deviation between the closed form solution and the control variate estimates for different strike prices. The sample size of pseudo-random numbers is 50000. The error obtained relative to the true solution is less than a few parts within 0.0001% and it grows to parts within 0.8% for far in-the-money strike prices.

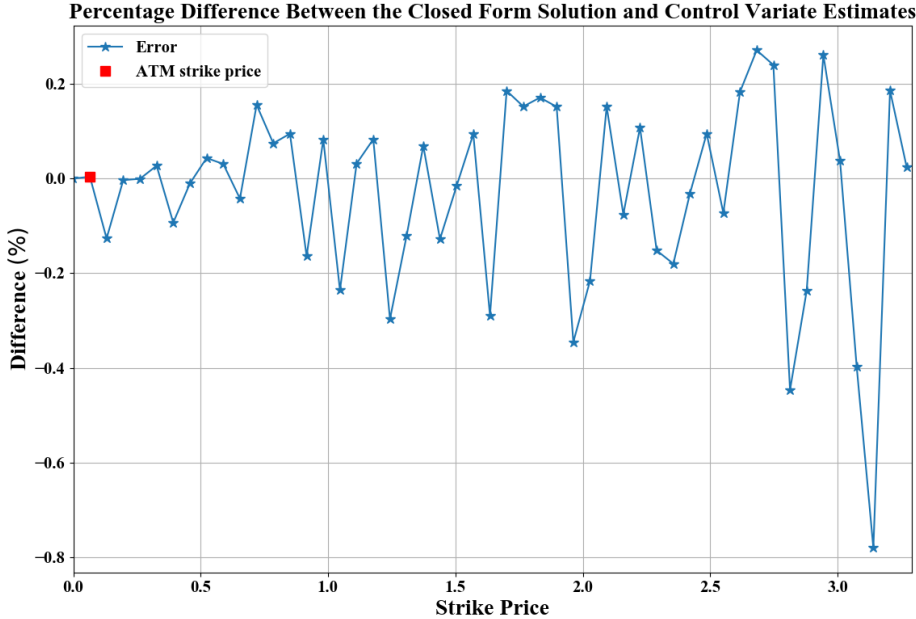


Fig. 4.4: The percentage difference between the closed form solution and control variate estimates for different strike prices.

Figure 4.5 shows the deviation between the closed form solution and Sobol estimates for different strike prices. The sample size of the Sobol random numbers is 50000. The error obtained relative to the closed-form solution was less than a few parts within 0.0001% and it grows to parts within $1.8 \times 10^{-3}\%$ for far in-the-money strike prices.

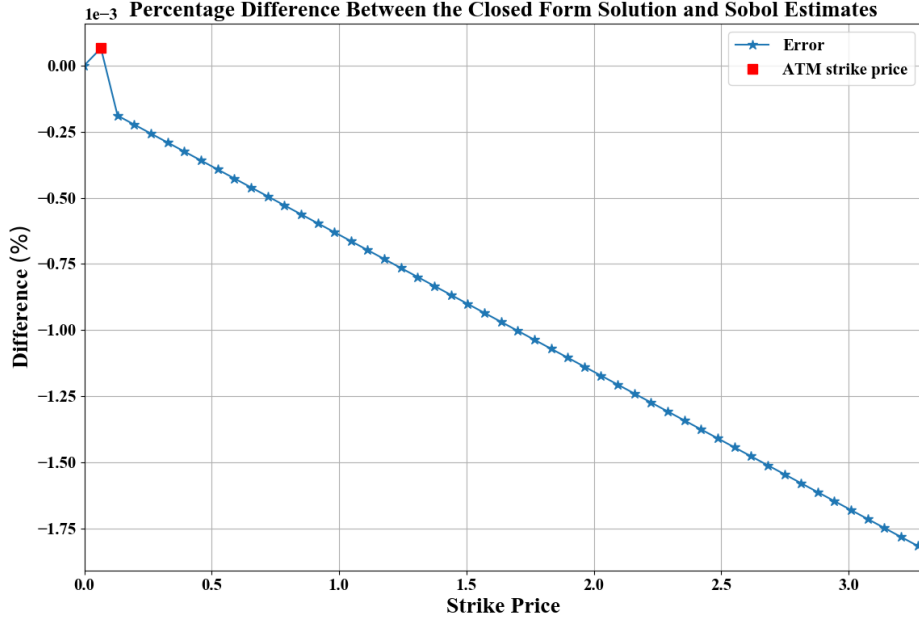


Fig. 4.5: The percentage difference between the closed form solution and Sobol estimates for different strike prices.

4.2 Numerical Results for the G2++ Model

Consider the compact form of equation (3.32) given as follows

$$r_t = \delta + \sum_{i=1}^2 x_t^i, \quad (4.17)$$

with the state variable dynamics given as follows

$$dx_t^i = -a^i x_t^i + \sigma_i dW_t^{i, \mathbb{Q}}, \quad i = 1, 2, \quad (4.18)$$

where $dW_t^i dW_t^j = \rho_{ij}$ with $\rho_{ii} = 1$ and $\varphi_t = \delta$. The bond prices at time $t \leq T$ takes the form

$$P(\tau, x_t) = \exp\left(M(\tau) - \sum_{i=1}^2 B(a^i, \tau) x_t^i\right), \quad \tau = T - t \quad (4.19)$$

where

$$B(a^i, \tau) = \frac{1 - \exp(-a^i \tau)}{a^i}, \quad (4.20)$$

and

$$M(\tau) = -\delta \tau + \sum_{i,j} \frac{\rho_{ij} \sigma^i \sigma^j}{a^i a^j} (\tau - B(a^i, \tau) - B(a^j, \tau) + B(a^i + a^j, \tau)). \quad (4.21)$$

The second term in equation (4.21) is the same as the variance expression given in equation (3.39). The solution to the state variables under the forward measure \mathbb{Q}^{T_0} is given by equation (3.42).

To simulate n replications of the short rate at time T_0 under the measure \mathbb{Q}^{T_0} we set

$$r_{T_0} = \delta + \sum_{i=1}^2 x_{T_0}^i, \quad (4.22)$$

where

$$x_{T_0}^i = x_t^i \exp(-a_i(T_0 - t)) - M_{x^i}^{T_0}(t, T_0) + \sigma_{x^i}(t, T_0)Z_{T_0}, \quad (4.23)$$

and

$$\sigma_{x^i}(t, T_0) = \sigma^i \sqrt{\frac{1 - \exp(-2a^i(T_0 - t))}{2a^i}} \quad (4.24)$$

where $M_{x^i}^{T_0}$ is given by equations (3.43) and (3.44).

Consider the example given in Section 4.1. The parameters for the G2++ are given in Exhibit 2 below.

Exhibit 2. Parameters for the G2++ Model

| x_0^1 | x_0^2 | δ | a^1 | a^2 | σ^1 | σ^2 | ρ |
|---------|---------|----------|---------|---------|------------|------------|--------|
| 0.0 | 0.0 | 0.03 | 0.04050 | 0.04438 | 0.00474 | 0.00301 | 0.85 |

Figure 4.6 below shows the evolution of the amortising receiver swaption prices for different Monte Carlo estimates over different sample sizes. The Monte Carlo estimates are compared to the semi-closed form solution, equation (3.47). One can observe from the 3-standard deviation bounds that as the sample size increases, Monte Carlo estimates converge to the semi-closed form solution.

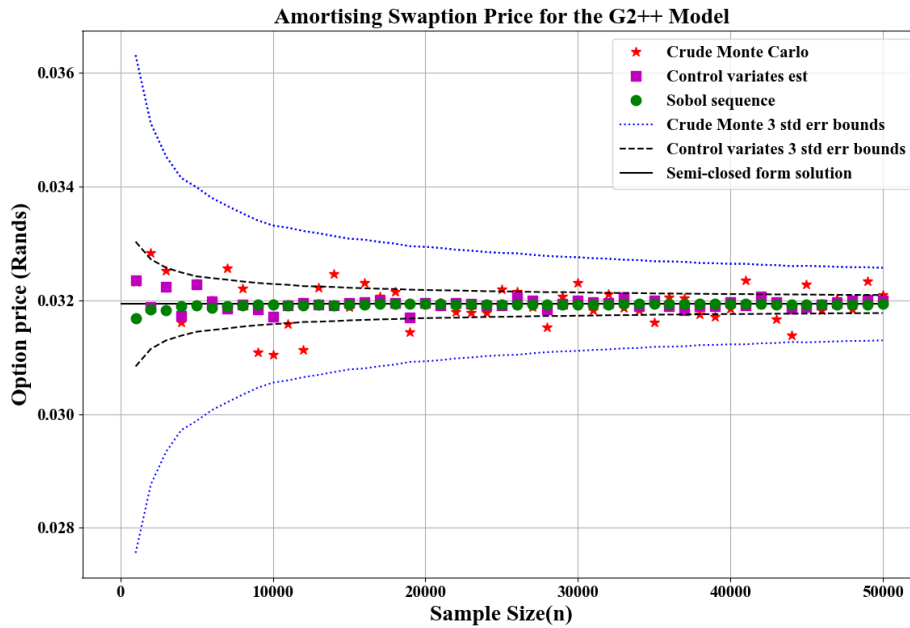


Fig. 4.6: Monte Carlo based prices compared to the semi-closed form solution for an amortising receiver swaption price.

The price of the option under the different pricing methods shown in Figure 4.6 versus term to maturity is shown in Figure 4.7 respectively. The semi-closed form solution and Monte Carlo estimates converge as the term to maturity decreases. This illustrates that the longer term to maturity the option has the higher the probability of it finishing in the money.

Pricing Methods for different Term to Maturities.

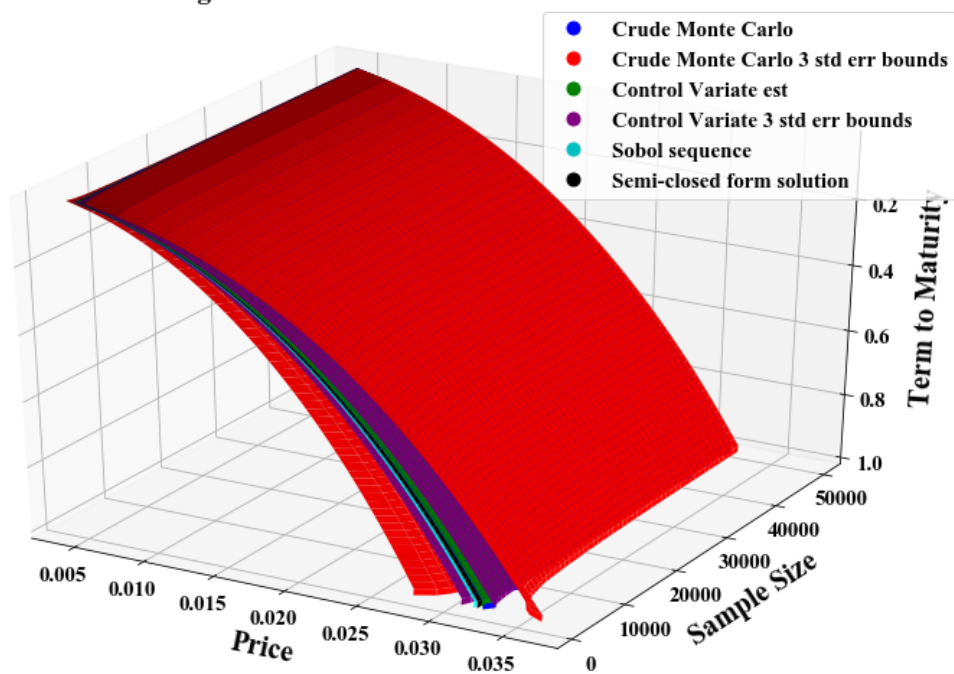


Fig. 4.7: Effects of the term to maturity on convergence of pricing methods.

Figure 4.8 shows individual percentage difference between different Monte Carlo methods and the semi-closed form solution. One can easily observe that the quasi-Monte Carlo estimates are more accurate than control variates estimates.

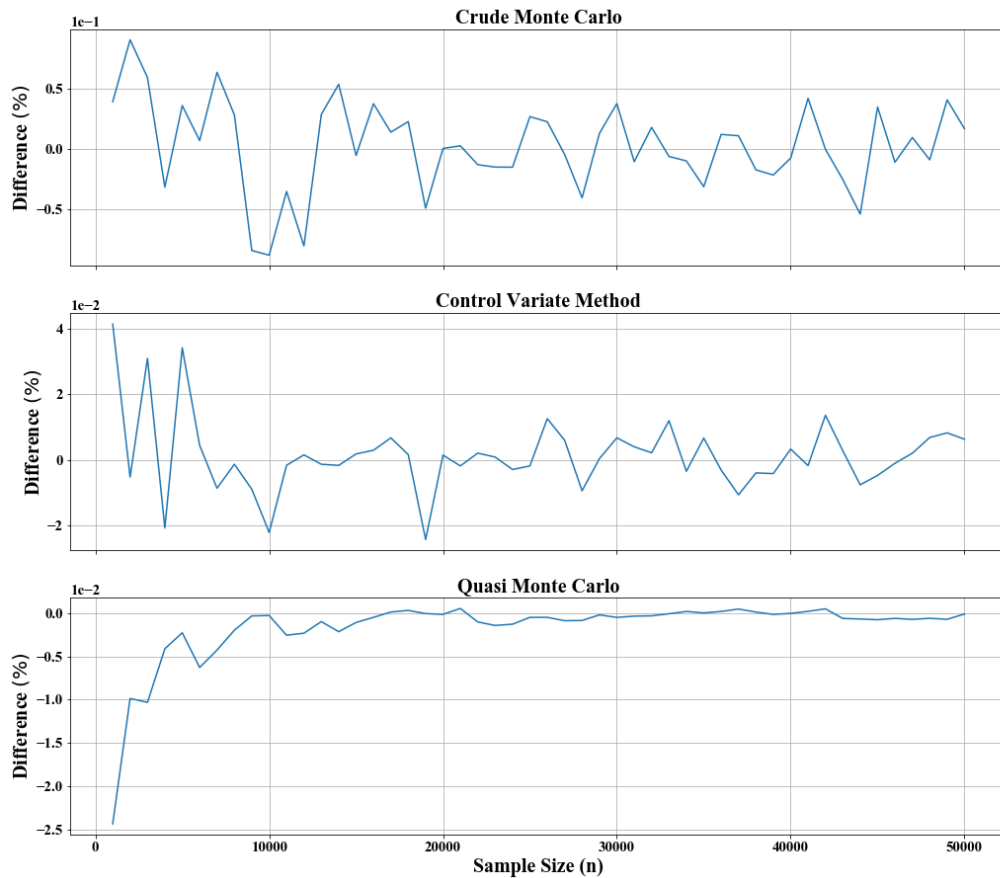


Fig. 4.8: The percentage difference between the semi-closed form solution and different Monte Carlo methods for different sample size.

Figure 4.9 shows the deviation between the semi-closed form prices and the control variate estimates for different strike prices. The sample size of pseudo-random numbers is 50000. The error obtained relative to the true solution is less than a few parts within 0.0001% and it grows to parts within 0.13% for far in-the-money strike prices.

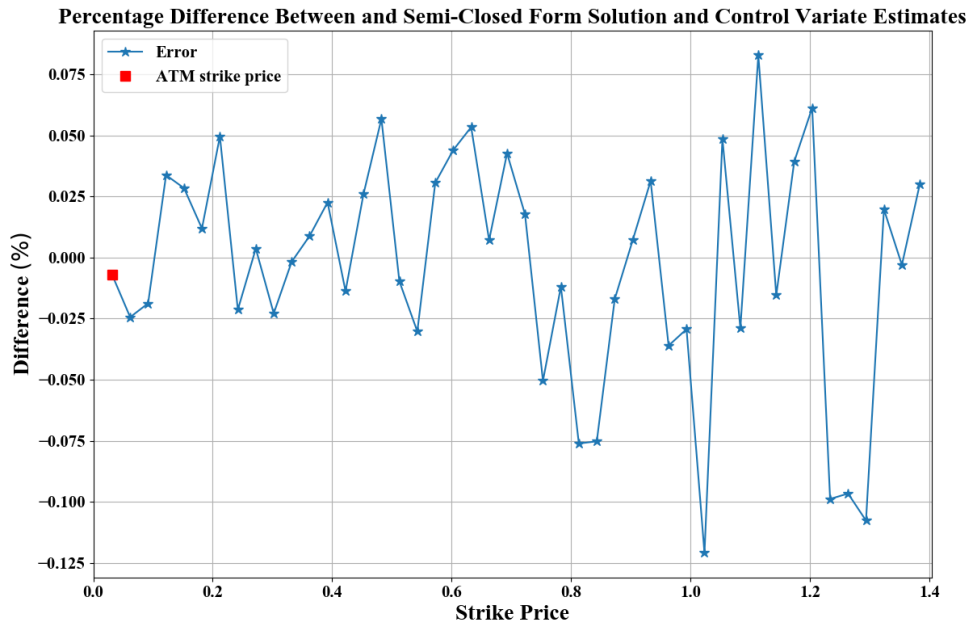


Fig. 4.9: The percentage difference between the semi-closed form solution and control variate estimates for different strike prices.

Figure 4.10 shows the deviation between semi-closed form prices and Sobol estimates for different strike prices. The sample size of quasi-random numbers is 50000. The error obtained relative to the true solution was less than a few parts within 0.0001% and it grows to parts within $1.7 \times 10^{-3}\%$ for far in-the-money strike prices.

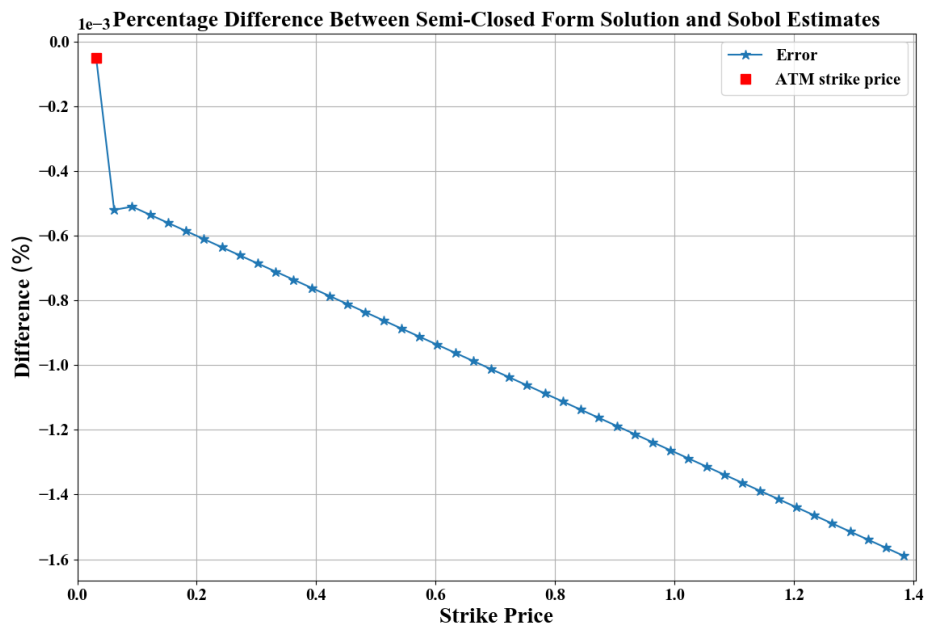


Fig. 4.10: The percentage difference between the semi-closed form solution and Sobol estimates for different strike prices.

Chapter 5

Conclusion

This dissertation explores two methods for computing closed-form prices for amortising swaptions. The first method uses the Jamshidian trick (which converts the price of the swaption into a sum of zero-coupon bond options struck at specific strike prices). The second method computes the price by numerically solving a one-dimensional integral. The short rate model used in both methods is an affine model. The disadvantage of both the Vasicek model and the G2++ models is that they do not capture the dependency of implied volatility on the strike price of the option. In the real-world, swaption prices show the existence of an implied volatility skew. It is necessary to use a better short rate model that can incorporate market volatility structure. This dissertation illustrates that the same methods used for pricing standard swaptions can be applied to price amortising swaption by making a simple modification on the payoff function. In both methods, the price difference between the standard solution and the amortising solution embedded with a flat notional profile is zero. The efficacy of both pricing methods can only be measured for at-the-money strike prices of the swaptions. Monte Carlo simulations perform better when the strike is within the expected volatility. Figures 4.4, 4.5, 4.9 and 4.10 show that for far in-the-money strike prices, the swaption prices from Monte Carlo simulations do not converge since it is highly unlikely for the swap rate to rise so high given the expected return and volatility. It is also shown that quasi-Monte Carlo methods have a significantly better performance than crude Monte Carlo methods. For the examples given in Chapter 4, the Jamshidian decomposition method is ~ 700 times faster than the one-dimensional numerical integration method. A further extension of this dissertation will be to investigate the performance of each method calibrated to market swaption prices and to analyse the price sensitivity for different parameters of interest rate models. The last recommendation for future research on the topic is to improve the speed and precision of pricing under multi-factor term structure models, perhaps applying the method by [Collin-Dufresne and Goldstein \(2002\)](#).

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Appendix A

Stochastic Differential Equation (SDE)

A general SDE is given as

$$dX_t(\omega) = f_t(X_t(\omega))dt + g_t(X_t(\omega))dW_t(\omega), \quad (\text{A.1})$$

where ω denotes that $X_t = X_t(\omega)$ is a random variable and the initial condition $X_0(\omega) = X_0$ with probability one.

In addition, $f_t(X_t(\omega)) \in \mathbb{R}$, $g_t(X_t(\omega)) \in \mathbb{R}$, and $W_t(\omega) \in \mathbb{R}$. We can write (A.1) as an integral equation using Ito's Lemma as follows

$$X_t(\omega) = X_0 + \int_0^t f_v(X_v(\omega))dv + \int_0^t g_v(X_v(\omega))dW_v(\omega). \quad (\text{A.2})$$

Appendix B

The Change of Numeraire Toolkit

This is a brief summary of the change of numeraire toolkit described in chapter 2 of [Brigo and Mercurio \(2013\)](#).

When valuing a contingent claim it is necessary to calculate expectations of future payoffs under the risk-neutral measure discounted to the time of valuing or pricing. Usually the discount factor is given as a function of the interest rate (the short rate). In the one-factor setting, we will require the following definitions:

- The stochastic interest rate process $r_t, 0 \leq t \leq T$,
- The price per asset of a risk-free money market account (MMA) at time t is

$$M_t = \exp\left(\int_0^t r_s ds\right), \quad (\text{B.1})$$

by definition it is given by the dynamics $dM_t = r_t M_t dt$ with $M_0 = 1$. The term M has been used to denote the cash bond instead of B , this is to avoid confusion between the deterministic term in the discount bond price and the cash bond.

- \mathbb{Q} is the risk-neutral measure associated with the MMA,
- Discount factor

$$D(t, T) = \frac{M_t}{M_T} = \exp\left(-\int_t^T r_u du\right). \quad (\text{B.2})$$

Definition B.1. *An arbitrage-free market.* A market is said to be *arbitrage-free* if it is not possible to produce a portfolio with zero investment that has a positive future value with positive probability.

Definition B.2. *A complete market.* A market is said to be *complete* if every attainable claim can be replicated by a self-financing trading strategy.

Definition B.3. *A probability measure \mathbb{Q} on Ω is called a risk-neutral measure if under \mathbb{Q} , the expected return of each risky asset $S^{(i)}$ equals the return r of the riskless asset, that is*

$$\mathbb{E}^{\mathbb{Q}}\left[S_{t+1}^{(i)} | \mathcal{F}_t\right] = (1 + r)S_t^{(i)}, t = 0, 1, 2, \dots, T, \quad (\text{B.3})$$

$i = 0, 1, 2, \dots, d$.

The idea of a unique (equivalent) risk-neutral measure \mathbb{Q} comes from the assumption that the market is arbitrage-free and complete. Let H_T be the value of an attainable claim at time T . The time t value of any attainable claim is given by

$$\pi_t = \mathbb{E}^{\mathbb{Q}} [D(t, T) \cdot H_T | \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}} \left[\frac{M_t}{M_T} \cdot H_T | \mathcal{F}_t \right] = M_t \mathbb{E}^{\mathbb{Q}} \left[\frac{H_T}{M_T} | \mathcal{F}_t \right], \quad (\text{B.4})$$

for any $0 \leq t \leq T$ and π_t is martingale. The time t price π_t with the cash flow normalized by a positive price process of an non-dividend paying asset called a *numeraire* is a martingale.

Let us assume that U_t is another numeraire process given by the dynamics

$$dU_t = r_t U_t dt + \sigma_t^U U_t dW_t^{\mathbb{Q}}, \quad (\text{B.5})$$

where σ_t^U is the volatility and $W^{\mathbb{Q}}$ is a \mathbb{Q} -Brownian motion, such that the cash flow normalised by U_t is martingale. Let $\bar{\mathbb{Q}}$ be the associated equivalent martingale measure of U_t . Applying Radon-Nikodym derivative into equation (B.4) to change between \mathbb{Q} and $\bar{\mathbb{Q}}$ given by

$$L(t) = \frac{d\mathbb{Q}}{d\bar{\mathbb{Q}}} = \frac{M_T}{M_t} \cdot \frac{U_t}{U_T}, \quad (\text{B.6})$$

we get

$$\pi_t = M_t \mathbb{E}^{\bar{\mathbb{Q}}} \left[\frac{H_T}{M_T} \cdot \frac{d\mathbb{Q}}{d\bar{\mathbb{Q}}} | \mathcal{F}_t \right] = U_t \mathbb{E}^{\bar{\mathbb{Q}}} \left[\frac{H_T}{U_T} | \mathcal{F}_t \right]. \quad (\text{B.7})$$

We know that $d\left(\frac{M_t}{U_t}\right)$ is a martingale under the $\bar{\mathbb{Q}}$ measure and therefore

$$d\left(\frac{M_t}{U_t}\right) = -\frac{\sigma_t^U M_t}{U_t} dW_t^{\bar{\mathbb{Q}}}. \quad (\text{B.8})$$

We use Itô's Lemma to evaluate the dynamics of the ratio of B to U . This gives

$$d\left(\frac{M_t}{U_t}\right) = d\left(\frac{1}{U_t}\right) M_t + \left(\frac{1}{U_t}\right) dM_t. \quad (\text{B.9})$$

Let $X_t = \frac{1}{U_t}$ and apply Itô's Lemma to obtain

$$\begin{aligned} dX_t &= \frac{dX_t}{dU_t} dU_t + \frac{1}{2} \frac{d^2 X_t}{dU_t^2} dU_t^2 \\ &= -\frac{1}{U_t^2} (r_t U_t dt + \sigma_t^U dW_t^{\mathbb{Q}}) + \frac{1}{2} \left(\frac{2}{U_t^3}\right) (\sigma_t^U)^2 U_t^2 dt \\ &= -\frac{r_t}{U_t} dt - \frac{\sigma_t^U}{U_t} dW_t^{\mathbb{Q}} + \frac{(\sigma_t^U)^2}{U_t} dt \\ &= \left(\frac{(\sigma_t^U)^2 - r_t}{U_t}\right) dt - \left(\frac{\sigma_t^U}{U_t}\right) dW_t^{\mathbb{Q}}, \end{aligned} \quad (\text{B.10})$$

substituting (B.10) into (B.9) we obtain

$$\begin{aligned}
d\left(\frac{M_t}{U_t}\right) &= M_t dX_t + \left(\frac{r_t M_t}{U_t}\right) dt \\
&= M_t \left[\left(\frac{(\sigma_t^U)^2 - r_t}{U_t}\right) dt - \left(\frac{\sigma_t^U}{U_t}\right) dW_t^{\mathbb{Q}} \right] + \left(\frac{r_t M_t}{U_t}\right) dt \\
&= \frac{(\sigma_t^U)^2 M_t}{U_t} dt - \frac{r_t M_t}{U_t} dt - \frac{\sigma_t^U M_t}{U_t} dW_t^{\mathbb{Q}} + \frac{r_t M_t}{U_t} dt \\
&= \frac{(\sigma_t^U)^2 M_t}{U_t} dt - \frac{\sigma_t^U M_t}{U_t} dW_t^{\mathbb{Q}}.
\end{aligned} \tag{B.11}$$

By comparing equation (B.11) and equation (B.8) we obtain

$$\begin{aligned}
-\frac{\sigma_t^U M_t}{U_t} dW_t^{\bar{\mathbb{Q}}} &= \frac{(\sigma_t^U)^2 M_t}{U_t} dt - \frac{\sigma_t^U M_t}{U_t} dW_t^{\mathbb{Q}} \\
dW_t^{\bar{\mathbb{Q}}} &= dW_t^{\mathbb{Q}} - \sigma_t^U dt.
\end{aligned} \tag{B.12}$$