Asymptotics of the Rough Heston Model

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Joshua J. Hayes
September 15, 2021
Abstract

The recent explosion of work on rough volatility and fractional Brownian motion has led to the development of a new generation of stochastic volatility models. Such models are able to capture a wide range of stylised facts that classical models simply do not. While these models have sound mathematical underpinnings, they are difficult to implement, largely due to the fact that fractional Brownian motion is neither Markovian nor a semimartingale. One idea is to investigate the behaviour of these models as maturities become very small (or very large) and consider asymptotic estimates for quantities of interest. Here we investigate the performance of small-time asymptotic formulae for the cumulant generating function of the Fractional Heston model as presented in Guennoun et al. (2018). These formulae and their effectiveness for small-time pricing are interrogated and compared against the Rough Heston model proposed in El Euch and Rosenbaum (2019).
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Chapter 1

Introduction

The Black-Scholes model serves as a benchmark for option prices. First introduced in Black and Scholes (1973), the model provides closed-form solutions for European call and put option prices given parameters \((S_t, K, r, \sigma, t, T)\) which are specified by some contract. If we fix all the parameters and vary the volatility \(\sigma\), we obtain the time \(t\) price of an option with that volatility. Conversely, given the price of an option, the implied volatility \(\sigma_{BS}\) corresponding to that price can be computed almost instantaneously. This sets up a convenient bijection between option prices and implied volatilities, one result of which is the fact that options are usually quoted in terms of their Black-Scholes implied volatility rather than price.

After collecting market data for options of various strikes \(K\) and maturities \(T\), we may obtain the corresponding implied volatilities \(\sigma_{BS}(K,T)\). Using this data, it is possible to plot the resulting volatility surface. Fixing \(T\) and taking a slice of this surface yields the volatility smile for maturity \(T\). Now, if the Black-Scholes model was a perfect representation of reality, then this volatility smile would be a horizontal line. In reality, what we find in modern markets is that options that are further in-the-money or out-the-money tend to have higher implied volatilities than those at-the-money. The result is a volatility surface that not flat but convex as illustrated in Figure 1.1.

Clearly, the Black-Scholes model requires refinements in order to better account for the behaviour of market data. This is the beginning of the volatility modelling problem, a central area of research in mathematical finance. Now, before a mathematical model qualifies for further consideration, it must at minimum be able to reproduce the market prices of European options without introducing arbitrage opportunities and do so efficiently enough to be practically useful. To this end, it must be possible to calibrate the model parameters to market data; and in the case of volatility modelling, the goal is to fit model implied volatilities to market implied volatilities.

Stochastic volatility models enjoy a central place in the extant literature on
derivatives pricing and perhaps the keystone in this respect is the Heston model introduced in Heston (1993). The (classical) Heston model replaces the constant volatility of Black-Scholes with a stochastic process for the variance of the stock price such that the dynamics of the stock $S_t$ and instantaneous variance $V_t$ are

$$\begin{align*}
dS_t &= S_t \sqrt{V_t} \left( \rho dB_t + \sqrt{1 - \rho^2} dB_t^\perp \right), \quad S_0 > 0, \\
dV_t &= \lambda(\theta - V_t) \, dt + \lambda \nu \sqrt{V_t} \, dB_t, \\
\end{align*}$$

(1.1)

where we assume zero drift for the stochastic process $S_t$ and $V_t$ follows a Cox-Ingersoll-Ross (CIR) process with long run average variance $\theta$, rate of mean reversion $\lambda$, volatility of variance $\lambda \nu$ governed by $\nu$, and correlation $\rho$. The Brownian motions $B$ and $B^\perp$ which drive the processes are independent.

The Heston model is attractive because it allows for very fast pricing and calibration due to the existence of quasi-closed form formulae for European option prices and, by extension, the implied volatility surface. Such methods are detailed in Gatheral (2006). Moreover, the model produces a volatility surface that matches the market quite well when the maturity $T$ is sufficiently large. However, for smaller maturities, the Heston model fits market data much worse as it tends to underestimate the level of volatility as $T \to 0$ and is unable to capture the steepness of the volatility surface without compromising the fit at the long end. In sum, while

![Fig. 1.1: The SPX implied volatility surface at market close on 15 September 2005 plotted against time to expiry and log-strike. Source: Gatheral (2006).](image-url)
the Heston model is an improvement on Black-Scholes and is able to fit the implied volatility surface for large maturities very well, it fails in general to provide a satisfying fit for the short end of the market implied volatility surface. This shortcoming is visually demonstrated in Figure 1.2, and, as Gatheral (2006) explains, is one of the reasons why market practitioners often prefer local volatility models and some academics have proposed the addition of jumps to the stock price process.

![Fig. 1.2: The SPX implied volatility surface compared with the Heston fit at market close on 15 September 2005. The Heston surface has been shifted down by five volatility points for comparison. Source: Gatheral (2006).](image)

We consider neither jump models nor local volatility, but rather a new generation of stochastic volatility models known as rough volatility models. In particular, we investigate a revision of the Heston model dubbed the Rough Heston and introduced in El Euch and Rosenbaum (2019). This model replaces the variance process in the Heston with one that is driven by fractional Brownian motion, resulting in variance sample paths that are much rougher than those produced by the classical Heston model which is driven by ordinary Brownian motion.

Gatheral et al. (2018) argue that models driven by fractional Brownian motion do indeed reflect the behaviour of market data much better than classical models and better capture many stylised facts about markets. In particular, the Rough Heston is able to fit the short end of the volatility surface very well. On the other hand, it presents some serious challenges in terms of tractability as direct simulation of the volatility process is computationally expensive, requiring that the
algorithm store the entire history of the volatility sample path. A major breakthrough in response to this challenge is the derivation of the characteristic function for the Rough Heston model in El Euch and Rosenbaum (2019). The computation of this characteristic function remains quite intensive, but does allow for the recovery of option prices via Fourier techniques in a much more efficient fashion, especially when compared with Monte-Carlo simulation. Nonetheless, that this model remains computationally expensive means that the Rough Heston presents challenges with respect to implementation for pricing and calibration.\footnote{For more detail see Chapter 4.3 for an outline of the algorithm required to compute the Rough Heston characteristic function and Appendix B for some examples of runtimes.}

As a possible response to this challenge, we consider the Fractional Heston model of Guennoun et al. (2018). Like the Rough Heston, this model is an extension of the classical Heston utilising fractional Brownian motion. What is particularly attractive is that Guennoun et al. (2018) derive a closed-form formula for the cumulant generating function (CGF) of the Fractional Heston and prove small-time and large-time asymptotic formulae for the behaviour of this CGF. Not only is this CGF much faster to compute than the characteristic function of the Rough Heston, but the asymptotics are quoted directly in terms of the model parameters.

We concern ourselves chiefly with the small-time behaviour of all models under consideration where the goal is to investigate how well the Fractional Heston and its asymptotic formulae capture the behaviour of the volatility surface produced by the Rough Heston. The asymptotic formulae have the potential to allow for very fast pricing and calibration but, like any approximation, such an approach requires careful consideration to assess both its strengths and limitations. We provide this consideration in the sequel.

The structure of this dissertation is as follows: Chapters 1 and 2 introduce the volatility modelling problem and provide a review of the relevant literature. Chapter 3 reviews the Heston model, its characteristic function, as well as methods for pricing options using this characteristic function. Chapter 4 outlines the Rough Heston model and details both its characteristic function as well as the fractional Adams numerical scheme for solving fractional differential equations. Chapter 5 introduces the Fractional Heston model, the cumulant generating function and its small-time asymptotics. Chapter 6 provides a numerical picture of the models in question and details the strengths and limitations of the Fractional Heston as well as its asymptotics. The conclusion in Chapter 7 summarises our findings and makes some recommendations for further research.

Unless otherwise stated, we work under a risk-neutral measure, and, for the sake of simplicity, assume that both the risk-free rate and dividend yields are zero.
Chapter 2

Literature Review

The existing literature is not without attempts to improve the fit of the Heston model for small maturities whilst recognising the trade-off with respect to efficiency and computational cost.

Local volatility models proceed from the idea that there is a unique risk-neutral density that can be derived from the market prices of European options. First introduced in Dupire (1994) and Derman and Kani (1994), the goal is to price options using an approach that is simpler than those involving stochastic volatility models whilst incorporating the market volatility skew. These models treat volatility as a function $\sigma_L(S_t, t)$ of both the stock price $S_t$ and the time $t$. Since the only source of randomness is the stock price, local volatility models tend to be easy to calibrate. Whilst parametric approaches like that of Brigo and Mercurio (2002) are highly tractable and popular, local volatility models face challenges whenever the volatility of the underlying is not a function of this underlying.

The addition of jumps to the stock price dynamics follows from the idea that such jumps can capture the explosive behaviour of the implied volatility surface at the short end. As time grows large, the effect of the jumps is “smoothed out” with time so as to capture the shape of the volatility surface at the long end. Both Gatheral (2006) and Jacquier et al. (2013) propose an approach that sits within the general framework of affine processes whilst Bates (1996) approaches jumps via independent Lévy processes. However, jumps present serious challenges as the jump part of the process is difficult to hedge, making practical implementation a thorny issue.

Classical stochastic volatility models, such as the Heston model, have been criticised for struggling to accurately capture the risk of price movements on short timescales. This is best demonstrated by the fact that the Heston fails to reproduce the empirically observed explosive behaviour of the at-the-money skew which is
defined by

\[ \psi(\tau) = \left| \frac{\partial}{\partial k} \sigma_{BS}(k, \tau) \right|_{k=0} \]

where \( k \) is log-moneyness and \( \tau = T - t \) is time to expiry. This quantity is best modelled by a power law as shown in Figure 2.1. It is demonstrated in Gatheral (2006) that conventional time-homogeneous stochastic volatility models such as the classical Heston produce an at-the-money skew that is constant for small-time and a sum of decaying exponentials for large-time. At the end of Chapter 4, we show the flat behaviour of the Heston skew in contrast to the Rough Heston which reproduces precisely the desired explosive behaviour as seen in market data.

![Fig. 2.1: Plot of non-parametric estimates of the SPX at-the-money volatility skew as at 20 June 2013 where the red curve is a power law fit \( \psi(\tau) = A\tau^{-0.4} \). Source: Gatheral et al. (2018).](image)

Stochastic volatility models have also been criticised for failing to account for the stylised fact that volatility of returns is a long-memory process. This behaviour has been investigated and modelled in continuous-time by Comte and Renault (1998) and Comte et al. (2012) using fractional Brownian motion with Hurst exponent \( H > 1/2 \). Now, fractional Brownian motion presents its own set of challenges as it is not a semimartingale and non-Markovian unless \( H = 1/2 \) (in which case we have ordinary Brownian motion). This means that we cannot escape the curse of dimensionality as the volatility depends on the entire history of the process. Furthermore, existing Feynman-Kac methods cannot be applied which makes applications challenging.
Despite these challenges, academics working on rough volatility have made significant breakthroughs in recent years. As introduced in El Euch and Rosenbaum (2019) and El Euch et al. (2019), the Rough Heston model has dynamics

\[ dS_t = S_t \sqrt{V_t} \left( \rho dB_t + \sqrt{1 - \rho^2} dB^\perp_t \right), \]

\[ V_t = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(\theta - V_s) \, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda \nu \sqrt{V_t} \, dB_s, \]

(2.1)

where \( \Gamma \) is the standard Gamma function, the parameters \( \lambda, \theta, V_0, \) and \( \nu \) are positive, and \(-1 \leq \rho \leq 1\). The parameter \( \alpha = H + 1/2 \) is restricted to \((1/2, 1)\) and shapes the roughness of the volatility sample paths. Notice that when \( \alpha = 1 \) we obtain the classical Heston model.

This model is non-Markovian and not a semimartingale which means that it is not amenable to many of the standard tools of mathematical finance such as Feynman-Kac techniques. Moreover, the former property means that the volatility is dependent on the entire history of the process and one consequence of this is that the model is difficult to simulate directly, making Monte Carlo pricing of derivatives impractical. Despite these challenges, the Rough Heston has garnered much attention due to several major breakthroughs. The first is detailed in El Euch and Rosenbaum (2018), which shows that, at least theoretically, perfect hedging strategies can be obtained under the Rough Heston model via replicating portfolios made up of the underlying and the forward variance curve. The second is the previously mentioned major achievement of El Euch and Rosenbaum (2019), in the form of the derivation of the characteristic function of the rough Heston model. This means that option prices are accessible via Fourier techniques, and, while still computationally challenging, much easier to estimate than a Monte Carlo approach.

Guennoun et al. (2018) propose a version of the Heston model driven by fractional Brownian motion called the Fractional Heston model. Let \( X_t = \log(S_t) \) be the log-price of the asset. The dynamics are given by

\[ dX_t = -\frac{1}{2} V_t^d \, dt + \sqrt{V_t^d} \, dB_t, \]

\[ dV_t = \lambda(\theta - V_t) \, dt + \lambda \nu \sqrt{V_t} \, dB^\perp_t, \]

\[ V_t^d = \eta + I_{0+}^d V_t, \]

where \( X_0 = 0, V_0 > 0, \) \( d \in (-1/2, 1/2) \), the parameters are such that \( \lambda, \theta, \nu > 0 \), and \( W \) and \( B \) are independent Brownian motions. The operator \( I_{0+}^d \) is the left fractional integral of order \( d \) where

\[ I_{0+}^d V_t = \begin{cases} \frac{1}{\Gamma(d)} \int_0^t (t-s)^{d-1} V_s \, ds, & \text{if } d \in (0, 1/2), \\ \frac{1}{\Gamma(d+1)} \frac{d}{dt} \int_0^t (t-s)^d V_s \, ds, & \text{if } d \in (-1/2, 0). \end{cases} \]
where $\Gamma$ is the standard Gamma function. Of particular interest from Guennoun et al. (2018) are asymptotic formulae for the cumulant generating function

$$m(u, t) = \log(E[e^{uX_t}])$$

as $t \to 0$ and $t \to \infty$. In what follows, we explicate these results and investigate their veracity by implementing both the Rough Heston and Fractional Heston models in full.
Chapter 3

The Heston Model

In this chapter, we outline the celebrated model due to Heston (1993) as well as its characteristic function and methods for pricing European call options.

3.1 Model Description

The Heston model is a time-homogeneous stochastic volatility model where the asset price $S$ and instantaneous variance processes have dynamics

$$
\begin{align*}
\frac{dS_t}{S_t} &= \sqrt{V_t} \left( \rho dB_t + \sqrt{1 - \rho^2} dB_t^H \right), \quad S_0 > 0, \\
\frac{dV_t}{V_t} &= \lambda (\theta - V_t) \, dt + \lambda \nu \sqrt{V_t} \, dB_t
\end{align*}
$$

so that $S_t$ is a stochastic process (where for simplicity we assume zero drift) and $V_t$ follows a CIR process with long-run average variance $\theta$, rate of mean reversion $\lambda$, volatility of variance $\lambda \nu$ governed by $\nu$, and correlation $\rho$.

The Heston model is popular for several reasons: The underlying process is a semimartingale and Markovian, meaning that one can apply Feynman-Kac techniques to compute the characteristic function in closed-form. This allows for very fast pricing and calibration. The Heston also reproduces several stylised facts about market data such as the leverage effect\(^1\), time-varying volatility, and fat tails. Further, the model offers reasonably good estimates of the implied volatility surface especially at larger maturities as exemplified by Figure 1.2.

Additionally, the parameters have an effect on the shape of the volatility surface that is easy to understand: the parameter $\nu$ changes the volatility of variance $\lambda \nu$ which changes the convexity of the smile, the correlation $\rho$ rotates the smile, and the initial volatility $V_0$, mean-reversion level $\theta$, and mean-reversion speed $\lambda$ govern the overall level of the surface over time. To illustrate this behaviour, Figure 3.1 shows how four of the input parameters change the shape of the volatility smile.

\(^1\) The inverse relationship that is empirically observed between the asset price and volatility.
We fixed the parameters
\[ \lambda = 1, \quad \theta = 0.06, \quad V_0 = 0.06, \quad \nu = 0.05, \quad \rho = 0, \quad T = 1/4, \]
and in each of the respective subplots we allowed \( \nu, \rho, \lambda, \) and \( \theta \) to vary.

![Plot of implied volatility against k for different parameters](image)

Fig. 3.1: Effect of input parameters on the classical Heston volatility smile.

### 3.2 Heston’s Characteristic Function and the ‘Little Trap’

In this section we sketch the derivation of the characteristic function under the Heston model. The Itô formula may be applied to the function

\[ L(u, t, V, S) = \mathbb{E} \left[ e^{iu \log(S_T)} \mid \mathcal{F}_t \right], \]

where \( \mathcal{F}_t \) is the natural filtration and \( u \in \mathbb{R} \). Since \( L \) is a martingale, an application of Feynman-Kac yields the following partial differential equation

\[ -\partial_t L(u, t, V, S) = \left( \lambda(\theta - V) \partial_v + \frac{1}{2}(\lambda \nu)^2 V \partial_{vv} + \frac{1}{2} \sigma^2 V \partial_{ss} + \rho \nu \lambda S V \partial_{sv} \right) L(u, t, V, S) \]
with boundary condition \( L(u, T, V, S) = e^{i u \log(S_T)} \). It follows from this PDE that the characteristic function of \( X_t = \log(S_t/S_0) \) is
\[
\mathbb{E} [e^{iuX_t}] = \exp (g(u, t) + V_0 h(u, t)) ,
\]
where \( h \) satisfies the Riccati equation
\[
\partial_t h(u, s) = \frac{1}{2} (-u^2 - iu) + \lambda (iu \rho \nu - 1) h(u, s) + \frac{(\lambda \nu)^2}{2} h^2(u, s), \quad h(u, 0) = 0, \quad (3.2)
\]
and
\[
g(u, t) = \theta \lambda \int_0^t h(u, s) \, ds.
\]
The solution of the Riccati equation (3.2) gives a closed-form formula for the characteristic function. Note that a fractional version of this Riccati equation will occur when outlining the characteristic function of the Rough Heston model in Chapter 4.2.

While it is possible to use the form above for numerics, we employ a more stable numerical specification due to Albrecher et al. (2007). Using the same parameters as in (3.1) and constant risk-free rate \( r \), the so-called ‘Little Trap’ characteristic function of the log-price under the risk-neutral measure is given by
\[
L_T(u) = \exp (A(u, T) + V_0 B(u, T) + i u \log(S_0)) ,
\]
where
\[
A(u, T) = r T i u + \theta \lambda \left( T x_+ - \frac{1}{a} \log \left( \frac{1 - g e^{-T d}}{1 - g} \right) \right) ,
\]
\[
B(u, T) = \frac{1 - e^{-T d}}{1 - g e^{-T d}} x_+ ,
\]
and
\[
a = \frac{(\lambda \nu)^2}{2}, \quad b = \lambda - \rho \lambda \nu i u, \quad c = -\frac{u^2 + i u}{2}, \quad d = \sqrt{b^2 - 4ac},
\]
\[
x_{\pm} = \frac{b \pm d}{2a}, \quad \text{and} \quad g = \frac{x_-}{x_+}.
\]

3.3 Numerics for Call Option Prices

Let \( L_T(u) \) be the characteristic function of the log-price. Assuming no dividends and constant risk-free rate \( r \), we can write the initial value of a European call option as
\[
C(S_0, K, T) = S_0 \Pi_1 - K e^{-r T} \Pi_2 ,
\]
where
\[
\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{e^{-iu \log(K)L_T(u-i)}}{iuL_T(-i)} \right] \, du,
\]
and
\[
\Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re \left[ \frac{e^{-iu \log(K)L_T(u)}}{iu} \right] \, du.
\]

The formulae above follow from the Gil-Pelaez Inversion Theorem which is a standard result in Fourier analysis as detailed in Gil-Pelaez (1951). When implementing this method numerically, simple quadrature suffices to compute the two integrals as follows:

\[
\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^N \Re \left[ \frac{e^{-iu_n \log(K)L_T(u_n-i)}}{iu_nL_T(-i)} \right] \Delta u,
\]
and
\[
\Pi_2 = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^N \Re \left[ \frac{e^{-iu_n \log(K)L_T(u_n)}}{iu_n} \right] \Delta u,
\]

where \(u_n = (n - \frac{1}{2})\Delta u\) is defined by \(\Delta u = \frac{u_{\text{max}}}{N}\) and the limits of integration are \([0, u_{\text{max}}]\). The upper integration limit \(u_{\text{max}}\) is chosen to be sufficiently large such that the error in the estimate is bounded: for sufficiently small \(\varepsilon > 0\), we have

\[
\int_{u_{\text{max}}}^\infty \Re \left[ \frac{e^{-iu \log(K)L_T(u-i)}}{iuL_T(-i)} \right] \, du < \varepsilon,
\]

and

\[
\int_{u_{\text{max}}}^\infty \Re \left[ \frac{e^{-iu \log(K)L_T(u)}}{iu} \right] \, du < \varepsilon.
\]
Chapter 4

The Rough Heston Model

In this chapter, we provide an overview of the Rough Heston model from El Euch and Rosenbaum (2019). We lay out its characteristic function as well as the numerical methods necessary to compute it.

4.1 Model Description

To introduce a rough counterpart of the Heston model, we follow the analysis of El Euch and Rosenbaum (2019) and refer to this model as the Rough Heston. Given a Brownian motion \( B \), a fractional Brownian motion \( B^H \) with Hurst parameter \( H \in (0, 1) \) can be defined using the Mandelbrot-van Ness formula

\[
B^H_t = \frac{1}{\Gamma(H + 1/2)} \int_{-\infty}^{0} \left( (t-s)^{H-1/2} - (-s)^{H-1/2} \right) dB_s + \frac{1}{\Gamma(H + 1/2)} \int_{0}^{t} (t-s)^{H-1/2} dB_s. \tag{4.1}
\]

Since it is the kernel \((t-s)^{H-1/2}\) that is responsible for the rough dynamics in (4.1), to produce rough behaviour in a Heston-like model it makes sense to introduce the kernel \((t-s)^{H-1/2}\) to the instantaneous variance process in (3.1). The resultant Rough Heston model has dynamics

\[
\begin{align*}
\mathrm{d}S_t &= S_t \sqrt{V_t} \left( \rho \mathrm{d}B_t + \sqrt{1-\rho^2} \, \mathrm{d}B^\perp_t \right), \\
V_t &= V_0 + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \lambda (\theta - V_s) \, ds + \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} \lambda \nu \sqrt{V_t} \, dB_s, \quad \tag{4.2}
\end{align*}
\]

where the parameters \( \lambda, \theta, V_0, \) and \( \nu \) are positive and are analogous to those of the Heston model in (3.1), the correlation is \( \rho \), and the parameter \( \alpha \in (1/2, 1) \) governs the roughness of the model. This final parameter is related to the Hurst parameter by \( \alpha = H + 1/2 \). That the model is well-defined and mathematically sound is demonstrated rigorously in El Euch and Rosenbaum (2019). This was first achieved
4.1 Model Description

in El Euch et al. (2018) where Hawkes-type processes are used to build a market microstructure model that in the long run converges to the Rough Heston model.

Before moving on to the characteristic function, note that $\alpha$ shapes the roughness of the volatility sample paths. The smaller the value, the rougher the sample paths. Further, it is easy to see that when $\alpha = 1$ we recover the Heston model. Hence, it is reasonable to think of the Rough Heston as a natural extension of the Heston model and dub it the Rough Heston. An alternative approach due to Guenoun et al. (2018) is provided in Chapter 5.

In Figure 4.1, we show simulations of the sample paths of the volatility and price processes under the Heston and Rough Heston models. The parameters used are

$$\lambda = 2.1, \theta = 0.06, V_0 = 0.03, \nu = 0.1905, \rho = 0, \alpha = 0.6.$$  

It should be clear that the Rough Heston produces volatility sample paths that are much rougher than the Heston model.

![Figure 4.1: Sample paths of the Heston and Rough Heston models.](image-url)
4.2 The Characteristic Function of the Rough Heston Model

In Chapter 3.2 we showed how applying Itô’s formula and Feynman-Kac yields a closed-form formula for the characteristic function of the log-price under the Heston model. In the case of the Rough Heston with $\alpha < 1$, the model is neither Markovian nor a semimartingale. Therefore, we cannot apply the same technique to recover a closed-form formula for the characteristic function.

Fortunately, El Euch and Rosenbaum (2019) offer an alternative approach which allows for a derivation of the characteristic function of the log-price under the Rough Heston model. The surprising result is one which is very similar in spirit to the form of Heston’s original formulae, but is unfortunately computationally intensive to implement. The key difference is that the Riccati equation (3.2) is replaced by a fractional Riccati equation involving fractional derivatives instead of classical derivatives.

Using the same parameters as in (4.2) the characteristic function of the Rough Heston can be written as

$$
E \left[ e^{iu \log(S_T)} \right] = \exp \left( g_1(u, t) + V_0 g_2(u, t) \right),
$$

where

$$
g_1(u, t) = \theta \lambda \int_0^t h(u, s) \, ds, \quad g_2(u, t) = I^{1-\alpha} h(u, t),
$$

such that $h$ is a solution of the fractional Riccati equation

$$
D^\alpha h(u, s) = \frac{1}{2} (-u^2 - iu) + \lambda (iu \rho \nu - 1) h(u, s) + \frac{(\lambda \nu)^2}{2} h^2(u, s), \quad I^{1-\alpha} h(u, 0) = 0,
$$

where $D^\alpha$ and $I^{1-\alpha}$ are the fractional derivative and integral operators defined for any function $f$ as

$$
D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) \, ds,
$$

$$
I^{1-\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f(s) \, ds,
$$

whenever they exist.\(^1\) Notice once again that when $\alpha = 1$ we recover the classical Heston model, but once $\alpha < 1$ the fractional Riccati equations must be solved using a sufficiently sophisticated numerical scheme. This challenge is taken up in the next section.

\(^1\) See Appendix A.2 for a brief derivation of these operators.
4.3 Computing the Characteristic Function

We reproduce the central result from El Euch and Rosenbaum (2019) below for clarity and then sketch the numerical scheme required to solve fractional differential equations.

**Theorem 4.1** (El Euch and Rosenbaum (2019), Thm 4.1). Consider the Rough Heston model as defined in (4.2) with correlation $\rho \in (-1/\sqrt{2}, 1/\sqrt{2}]$. Then for all $t \geq 0$ and fixed $u \in \mathbb{R}$, we have

$$L(u, t) = \exp \left( \theta \lambda I^1 h(u, t) + V_0 I^{1-\alpha} h(u, t) \right),$$

where $h$ is a solution of the fractional Riccati equation

$$D^\alpha h(u, s) = \frac{1}{2}(-u^2 - iu) + \lambda(iu \nu - 1)h(u, s) + \frac{\lambda \nu^2}{2} h^2(u, s),$$

$$I^{1-\alpha} h(u, 0) = 0,$$

which admits a continuous solution.

The characteristic function $L(u, t)$ is defined via the fractional Riccati equation,

$$D^\alpha h(u, t) = F(u, h(u, t)), \quad I^{1-\alpha} h(u, 0) = 0,$$

where

$$F(u, x) = \frac{1}{2}(-u^2 - iu) + \lambda(iu \nu - 1)x + \frac{\lambda \nu^2}{2} x^2.$$  \hfill (4.4)

Several methods exist for solving fractional Riccati equations, most of which stem from the idea that (4.3) implies the Volterra integral equation

$$h(u, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} F(u, h(u, s)) \, ds$$  \hfill (4.5)

whence one develops numerical schemes to solve (4.5). Following El Euch and Rosenbaum (2019), we employ a fractional Adams scheme from Diethelm et al. (2002).

Using the same notation as (4.4), let $g(u, t) = F(u, h(u, t))$ and consider a discrete time grid $(t_k)_{k \geq 0}$ with mesh $\Delta$. We may estimate the integral

$$h(u, t_{k+1}) = \frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} g(u, s) \, ds$$

by

$$\frac{1}{\Gamma(\alpha)} \int_0^{t_{k+1}} (t_{k+1} - s)^{\alpha-1} \hat{g}(u, s) \, ds,$$
where
\[ \hat{g}(u, t) = \frac{t_{j+1} - t_j}{t_{j+1} - t_j} \hat{g}(u, t_j) + \frac{t - t_j}{t_{j+1} - t_j} \hat{g}(u, t_{j+1}), \quad t \in [t_j, t_{j+1}), 0 \leq j \leq k. \]

We obtain a trapezoidal discretization of the fractional integral with the scheme
\[ \hat{h}(u, t_{k+1}) = \sum_{0 \leq j \leq k} a_{j,k+1} F(u, \hat{h}(u, t_j)) + a_{k+1,k+1} F(u, \hat{h}(u, t_{k+1})), \quad (4.6) \]
where the coefficients are defined by
\[ a_{0,k+1} = \frac{\Delta^\alpha}{\Gamma(\alpha + 2)} (k^{\alpha+1} - (k - \alpha)(k+1)^\alpha), \]
\[ a_{j,k+1} = \frac{\Delta^\alpha}{\Gamma(\alpha + 2)} ((k - j + 2)^{\alpha+1} + (k - j)^{\alpha+1} - 2(k - j + 1)^{\alpha+1}), 1 \leq j \leq k, \]
\[ a_{k+1,k+1} = \frac{\Delta^\alpha}{\Gamma(\alpha + 2)}. \quad (4.7) \]

Notice that the scheme (4.6) is implicit as \( \hat{h}(u, t_{k+1}) \) features on both sides of the equation. To construct an explicit scheme, we first compute a pre-estimate of \( \hat{h}(u, t_{k+1}) \) based on a Riemann sum which is then substituted into (4.6). We call this pre-estimate the predictor and denote it as
\[ \hat{h}^P(u, t_{k+1}) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \hat{g}(u, s) \, ds, \]
where \( \hat{g}(u, t) = \hat{g}(u, t_j) \) when \( t \in [t_j, t_{j+1}) \) for \( 0 \leq j \leq k \). From the definitions above, we see that the predictor is given by
\[ \hat{h}^P(u, t_{k+1}) = \sum_{0 \leq j \leq k} b_{j,k+1} F(u, \hat{h}(u, t_j)), \]
where
\[ b_{j,k+1} = \frac{\Delta^\alpha}{\Gamma(\alpha + 1)} ((k - j + 1)^\alpha - (k - j)^\alpha) \text{ for } 0 \leq j \leq k. \]

Therefore, the revised explicit numerical scheme is
\[ \hat{h}(u, t_{k+1}) = \sum_{0 \leq j \leq k} a_{j,k+1} F(u, \hat{h}(u, t_j)) + a_{k+1,k+1} F(u, \hat{h}^P(u, t_{k+1})), \quad \hat{h}(u, 0) = 0, \quad (4.8) \]
where the weights \( a_{j,k+1} \) are identical to (4.7).

That this scheme converges is shown in Li and Tao (2009). It can be shown that for \( t > 0 \) and \( u \in \mathbb{R} \), we have
\[ \max_{t_j \in [0,t]} | \hat{h}(u, t_j) - h(u, t_j) | = o(\Delta), \]
and for any $\varepsilon > 0$, we have
\[
\max_{t_j \in [\varepsilon, t]} \left| \hat{h}(u, t_j) - h(u, t_j) \right| = o(\Delta^{2-\alpha}).
\]

4.4 Comparing the Heston and Rough Heston

An implementation of the fractional Adams scheme from the previous section provides a numerical solution of the fractional Riccati equation (4.3). Having obtained the characteristic function, the Gil-Pelaez inversion formulae allow us to compute call option prices as outlined in Chapter 3.3.

To compare the Heston and the Rough Heston, we use the parameters

\[
\lambda = 2, \quad \theta = 0.04, \quad V_0 = 0.4, \quad \nu = 0.05, \quad \rho = -0.5,
\]

and plot the term-structure of the at-the-money skew defined by

\[
\psi(\tau) = \left| \frac{\partial}{\partial k} \sigma_{BS}(k, \tau) \right|_{k=0}
\]

where $k$ is the log-moneyness, $\tau$ is time to maturity, and $\sigma_{BS}$ is the Black-Scholes implied volatility.

![Fig. 4.2: Term-structure of the at-the-money skew for the Rough Heston ($\alpha = 0.6$) and Heston ($\alpha = 1$).](image-url)
In Figure 4.2, we implement the Rough Heston characteristic function twice: first with $\alpha = 0.6$ for the rough case, and second with $\alpha = 1$ for the classical Heston. We also plot the Heston skew using the ‘Little Trap’ characteristic function from Chapter 3.2 as a sanity check to confirm that the Rough Heston implementation does agree with the classical Heston.

Recalling the empirical example of the at-the-money skew in Figure 2.1, it is clear from Figure 4.2 that the Rough Heston captures the dynamics of the volatility surface much better than the Heston model. This is a massive improvement over non-rough stochastic volatility models as this explosive behaviour is commonly observed in market data and is important for applications as noted in Bayer et al. (2016).
Chapter 5

The Fractional Heston Model

In this chapter, we outline the model introduced in Guennoun et al. (2018) as an alternative approach to introduce roughness to the Heston model. We summarise the model, its cumulant generating function, as well as its the small-time asymptotic formulae.

5.1 Model Description

We shall call the model proposed by Guennoun et al. (2018) the Fractional Heston model. Let $X_t = \log(S_t/S_0)$ be the log-price of the asset. Then the dynamics are given by

$$
\begin{align*}
    dX_t &= -\frac{1}{2} V_t^d \, dt + \sqrt{V_t^d} \, dB_t, \quad X_0 = 0, \\
    dV_t &= \lambda (\theta - V_t) \, dt + \lambda \nu \sqrt{V_t} \, dB^\perp_t, \quad V_0 > 0, \\
    V_t^d &= \eta + I^d_{0+} V_t,
\end{align*}
$$

where $d \in (-1/2, 1/2)$ controls the fractional behaviour, the parameters are such that $\lambda, \theta, \nu > 0$, and the operator $I^d_{0+}$ is the left fractional integral of order $d$ where

$$
I^d_{0+} V_t = \begin{cases} 
    \frac{1}{\Gamma(d)} \int_0^t (t-s)^{d-1} V_s \, ds, & \text{if } d \in (0, 1/2), \\
    \frac{1}{\Gamma(d+1)} \frac{d}{dt} \int_0^t (t-s)^d V_s \, ds, & \text{if } d \in (-1/2, 0).
\end{cases}
$$

Notice that when $d = 0$, we obtain the classical Heston model. This is reminiscent of the Rough Heston in (4.2). The additional parameter $\eta \geq 0$ shifts the sample paths of the instantaneous variance which provides a way to loosen the connection between the mean and variance of $V$.

5.2 The Cumulant Generating Function

For the processes $X$ and $V^d$ in (5.1), the cumulant generating function (CGF) for the log-price and instantaneous variance is $m(u, w, t) = \log (E[e^{uX_t+wV_t^d}])$ where $t \geq 0$.
and \((u, w) \in \tilde{D}_t = \{(u, w) \in \mathbb{R}^2 \mid |m(u, w, t)| < \infty\}\) which is called the effective domain of \(m\). Whilst Guennoun et al. (2018) prove the following theorem in a more general setting, we shall restrict to the case where \(w = 0\), since our analysis of the asymptotic behaviour of the Fractional Heston only requires information about \(X_t\). We shall write \(m(u, w, t) = m(u, t)\) with effective domain \(D_t\) as above.

Theorem 5.1 (Guennoun et al. (2018), Thm 2.2). For any \(t \geq 0\),

\[
\log \mathbb{E} \left[ e^{ux_t} \right] = \frac{u(u - 1)\eta t}{2} - B(t)V_0 + A(t), \tag{5.2}
\]

where \(A\) and \(B\) satisfy

\[
A'(s) + \lambda \theta B(s) = 0, \\
B'(s) + \lambda B(s) + \frac{(\lambda \nu)^2}{2}B(s)^2 + \frac{u(u - 1)}{2\Gamma(d + 1)}s^d = 0, \tag{5.3}
\]

for \(0 \leq s \leq t\), with the initial conditions \(A(0) = B(0) = 0\), and where \(\Gamma\) is the Gamma function.

Fig. 5.1: Comparing the MGF of the Fractional Heston with the MGF of the ‘Little Trap’ Heston.

In Figure 5.1, we implement Theorem 5.1 to plot the moment generating function of the log-price \(X_t\). To this end we fix the parameters

\[
\lambda = 2.1, \quad \theta = 0.05, \quad V_0 = 0.05, \quad \eta = 0.03, \quad \nu = 0.095, \quad \text{and} \quad \rho = 0.
\]

In Figure 5.1, the plot on the left is the fractional case with \(d = -0.3\). For the plot on the right \(d = 0\), and we have set \(V_0 = 0.06\), \(\eta = 0\), and we have also plotted
the moment generating function of the Heston model to confirm that the Fractional Heston reproduces the Heston when $d = 0$.

5.3 Asymptotics of the Fractional Heston Model

For the effective domain $D_t$ of the function $m(u, t)$ we adopt the following conventions: We have $D_{t_{\infty}} = \cap_{s > 0} D_{s \leq t}$ and $D_t = \cup_{s > 0} D_{s \leq t}$. For $\delta > 0$, let $D_{t\delta}^{(\delta)}$ be the effective domain of the pointwise limit of the map $u \mapsto m \left( \frac{u}{t^{1+\delta}}, t \right)$ as $t$ tends to zero.

The following theorem provides small-time asymptotic formulae for the CGF of the Fractional Heston and is proved in Guennoun et al. (2018).

Theorem 5.2 (Guennoun et al. (2018), Prop 3.2). As $t$ tends to zero,

(i) for $d \in (0, 1/2)$, we have $\delta = 1$ whence $D_{0\delta}^{(\delta)} = \mathbb{R}$ and

$$\lim_{t \downarrow 0} t \frac{m \left( \frac{u}{t}, t \right)}{\eta u^2} = \frac{\eta u^2}{2} \text{ for all } u \in D_{0\delta}^{(\delta)},$$

(ii) for $d \in [-1/2, 0)$, we have $\delta = 1 + d$, then $D_{0\delta}^{(\delta)} = \mathbb{R}$ and

$$\lim_{t \downarrow 0} t^{1+d} \frac{m \left( \frac{u}{t^{1+d}}, t \right)}{\frac{V_0 u^2}{2(2 + d)}} = \frac{V_0 u^2}{2(2 + d)} \text{ for all } u \in D_{0\delta}^{(\delta)}.$$

Also proved in Guennoun et al. (2018) is an analogous theorem for large-time asymptotics of the CGF. We state the result below and leave the investigation of the practical usefulness of this theorem to future research.

Theorem 5.3 (Guennoun et al. (2018), Prop 3.3). As $t$ tends to infinity,

(i) for $d \in (0, 1/2)$, we have $D_{\infty} = [0, 1]$ and

$$\lim_{t \uparrow \infty} t^{-1+d/2} m(u, t) = -\frac{2\theta}{\nu(1+d/2)} \sqrt{\frac{u(1-u)}{\Gamma(1+d)}} \text{ for all } u \in D_{\infty},$$

(ii) for $d \in (-1/2, 0)$, there exist $u_- \leq 0, u_+ \geq 1$ such that

$$\lim_{t \uparrow \infty} t^{-1} m(u, t) = \frac{\eta}{2}u(u - 1) \text{ for all } u \in D_{\infty} = [u_-, u_+].$$

One remark worth making is that in Theorem 5.3, the effective domain $D_{\infty}$ is no longer $\mathbb{R}$ but a closed interval, which means that these asymptotics have much more limited scope than the small-time asymptotics. The focus of this dissertation remains on the small-time asymptotic formulae. For this reason, we shall not investigate these large-time asymptotics as far as pricing is concerned.
In Figure 5.2, we have plotted the rescaled CGF of $X_t$ for the small-time case against the asymptotic estimate from Theorem 5.2. We have rewritten the asymptotic equations as
\begin{align*}
    t^{-1} \eta \frac{u^2}{2} \quad \text{and} \quad t^{-(1+d)} \frac{V_0 u^2}{2\Gamma(2+d)}
\end{align*}
and have computed the rescaled cumulant generating functions using the full implementation from Theorem 5.1. In Figure 5.2, the parameters used are
\begin{align*}
    \lambda = 2.1, \quad \theta = 0.05, \quad V_0 = 0.03, \quad \eta = 0.03, \quad \nu = 0.095,
\end{align*}
where $d = -0.3$ in the top row, $d = 0.2$ in the bottom row, $t = 1/10$ on the left, and $t = 1/1000$ on the right. Clearly, at small time scales the asymptotic formulae captures the behaviour of the cumulant generating function remarkably well; especially for the rough case where $d < 0$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.2}
\caption{Rescaled CGF of $X_t$ vs small-time asymptotics from Theorem 5.2.}
\end{figure}

Also proved in Guennoun et al. (2018) are asymptotic formulae for the implied volatility surface under the Fractional Heston model. The small-time formulae are reproduced below. Note that for $(x, t) \in \mathbb{R} \times \mathbb{R}^+$, the implied volatility corresponding to the price of a European option with maturity $t$ and strike $e^x$ is denoted by $\Sigma(x, t)$.
Theorem 5.4 (Guennoun et al. (2018), Thm 3.6). As $t$ tends to zero:

(i) If $d \in (0, 1/2]$, then $\Sigma(x, t)^2$ converges to $\eta$ for any $x \neq 0$.

(ii) If $d \in [-1/2, 0)$, then $t^{-d}\Sigma(x, t)^2$ converges to $V_0/\Gamma(d + 2)$ for any $x \neq 0$.

In what follows we shall investigate how well these formulae capture the prices of European options.
Chapter 6

Numerical Results

In this chapter, we provide a numerical implementation of all the models introduced in this dissertation. We detail the strengths and limitations of the Fractional Heston model, its asymptotic implementation, and note some relevant features of the model when compared with the Rough Heston and Heston models.

6.1 Preparing the Asymptotic Formulae for Pricing

In order to price with the asymptotic formula from Theorem 5.2(ii), we make the following computations. Understanding that this equality holds in the limit, we write

\[ T^{1+d} \mathfrak{m} \left( \frac{u}{T^{1+d}}, T \right) = \frac{V_0 u^2}{2 \Gamma(2+d)}, \]

\[ \mathfrak{m} \left( \frac{u}{T^{1+d}}, T \right) = \frac{V_0 u^2}{2 T^{1+d} \Gamma(2+d)}. \]

We now make the substitution \( u \mapsto uT^{1+d} \) and obtain

\[ \tilde{\mathfrak{m}} (u, T) = \frac{V_0 u^2 T^{2(1+d)}}{2 T^{1+d} \Gamma(2+d)} = \frac{V_0 u^2 T^{1+d}}{2 \Gamma(2+d)}. \]

Finally, we obtain the “implied” asymptotic characteristic function by substituting \( u \mapsto iu \) and taking the exponential so that

\[ L_T(u) = \exp \left( \tilde{\mathfrak{m}} (iu, T) \right) = \exp \left( -\frac{V_0 u^2 T^{1+d}}{2 \Gamma(2+d)} \right). \]  

(6.1)

With this characteristic function now available, it is possible to apply the pricing techniques to (6.1) as outlined in Chapter 3.3.

6.2 Option Prices over Time

In what follows, we use the following parameters for the classical Heston model and the Fractional Heston model (both the full implementation using Theorem 5.1
6.2 Option Prices over Time

and its asymptotic version from Theorem 5.2):

\[ \lambda = 2.1, \ \theta = 0.05, \ V_0 = 0.03, \ \nu = 0.4286, \ \eta = 0, \ \text{and} \ d = -0.3. \]

For the Rough Heston model we select the parameters:

\[ \lambda = 6.3, \ \theta = 0.025, \ V_0 = 0.12, \ \nu = 0.0952, \ \text{and} \ \alpha = 0.6. \]

The reason for the difference between the sets of parameters is twofold: First, though the same Greek letters appear in both models, it would not be appropriate to treat the parameters as if they were identical. Both models reduce to the classical Heston when \( d = 0 \) and \( \alpha = 1 \) (respectively) but their behaviour is not identical. A glance at the two different volatility processes in (4.2) and (5.1) confirms this fact. Secondly, we want to compare the models, so it makes sense to choose parameters that ultimately provide us with similar behaviour as we have done here.

Here we fix three values of log-moneyness\(^1\) for the plots: at-the-money \( k = 0 \), in-the-money \( k = 0.1 \), and out-the-money \( k = -0.1 \). The plots below show option prices as well as their implied volatilities over maturities ranging from \( T = 7/365 \) to \( T = 1 \).

\[ \text{Fig. 6.1: Asymptotic vs quasi-closed-form pricing over time (} k = 0 \text{).} \]

What we are examining here is how closely the asymptotic formula for the Fractional Heston CGF from Theorem 5.2 tracks the option prices and implied volatilities when compared against the closed-form CGF from Theorem 5.1. While it is clear from the proof in Guennoun et al. (2018) that the closed-form CGF converges to the asymptotic formula, we are interested in how well this asymptotic works

\(^1\) Recall that log-moneyness is defined by \( k = \log(S_t/K) \).
As is clear from Figures 6.1, 6.2, and 6.3, the asymptotic characteristic function (6.1) reproduces option prices that are in line with both the Rough Heston and the closed-form formulae of the Fractional Heston on small time scales. This concordance clearly increases as $T \to 0$. Furthermore for options both in- and out-of-the-money, the asymptotic remains close to the full Fractional Heston, as shown in Figures 6.2 and 6.3. This is quite remarkable given that (6.1) has no dependence on the strike and thus one might expect divergent option prices when $k \neq 0$. We
examine this further in the next section. Nonetheless, it is clear from the figures that for very small maturities up to about 1 month the asymptotic characteristic function (6.1) produces reasonable option prices.

### 6.3 Volatility Smiles

The figures in the previous section demonstrate that the asymptotic formula in (6.1) can produce reasonable option prices. We investigate this further by turning our attention to the behaviour of the models at fixed maturities across a range of strikes. Using the same parameters as in the previous section, we compute volatility smiles for small maturities \( T \in \{7/365, 1/12, 1/4\} \).

![Graph](image)

**Fig. 6.4:** Asymptotic vs quasi-closed-form pricing with maturity \( T = 7/365 \).

In Figures 6.4, 6.5, and 6.6, the asymptotic formula for the Fractional Heston CGF returns a flat line for the volatility smile. This is somewhat unsurprising given that there is no dependence on the strike price in the formula (6.1). What is clear is that the volatility surface generated by the asymptotic converges to the asymptotic implied volatility estimate derived in Theorem 5.4(ii); these values are shown as black crosses in the figure. Both these asymptotics have no dependence on the strike price, meaning that the resulting volatility surface is flat and sloping upward as \( T \to 0 \). It is clear from the figure that the asymptotic formula yields a rather good estimate of the at-the-money and near-the-money implied volatility.
Fig. 6.5: Asymptotic vs quasi-closed-form pricing with maturity $T = 1/12$.

Fig. 6.6: Asymptotic vs quasi-closed-form pricing with maturity $T = 1/4$. 
The plots of the volatility smiles provide a clear indication of how the models produce different volatility surfaces and how the asymptotic formula fails to produce a convincing smile. To examine in further detail the differences in prices, we computed the relative differences between the prices produced by the models. This analysis, as detailed in Figures 6.7, 6.8, and 6.9, reveals that all of the models agree quite closely on price for values of log-moneyness such that $-0.25 < k < 0.25$. Remarkably, the asymptotic version of the Fractional Heston maintains rather good agreement with the closed-form Fractional Heston all the way out to maturities of 3 months. This confirms what was observed in Figures 6.1, 6.2, and 6.3.

Fig. 6.7: Relative model price differences for maturity $T = 7/365$. 
6.3 Volatility Smiles

**Fig. 6.8:** Relative model price differences for maturity $T = 1/12$.

**Fig. 6.9:** Relative model price differences for maturity $T = 1/4$. 
6.4 At-the-Money Skew

We compute the at-the-money skew using the same parameters as before with the exception of the parameter $\nu$ governing the volatility of variance. Of particular interest is the fact that the Fractional Heston fails to produce an at-the-money skew with normal parameters, whilst the asymptotic implementation of the Fractional Heston fails to produce any explosion in the skew whatsoever—though the latter is to be expected after noting the plots of the volatility smiles in the previous section. In order to produce any explosive behaviour, the Fractional Heston requires that $\nu$ be increased significantly.

![At-the-money skew term-structure](image)

Fig. 6.10: Term-structure of the at-the-money skew for all models.

In Figure 6.10, the at-the-money skews were generated for the Fractional Heston and Classical Heston. Two cases are considered: the first with $\nu = 0.9524$ and the second with $\nu = 0.0952$. Note that these parameters differ by a factor of 10. What we observe is that the Fractional Heston can reproduce some of the desired behaviour of the at-the-money skew, but fails to capture all of the explosion as $T \to 0$, and is only able to produce this behaviour when the value of $\nu$ is very
large. In addition, this result is not particularly remarkable when compared with the Rough Heston for two reasons: Firstly, the classical Heston produces this same behaviour given the exact same parameters. Secondly, the parameter $\nu = 0.9524$ is rather unrealistic and would lead to serious calibration issues. One clear way to see this is to notice that this skew decays far too slowly and does not demonstrate the power law behaviour characteristic of market data that is exemplified by the Rough Heston.

It seems clear that the Fractional Heston does not provide volatility smiles with the same convexity as those of the Rough Heston as maturities become very small. This is further evidenced by the flat behaviour of the asymptotic Fractional Heston and of the flat limiting result for the implied volatility under the Fractional Heston given by Theorem 5.4.
Chapter 7

Conclusion

We observed that the Fractional Heston is able to reproduce some of the behaviour of the celebrated Rough Heston model. What initially attracted us to this model was the fact that the asymptotics allow for fast closed-form pricing (and calibration) for options with small maturities. We noted that this model does produce prices that agree with the Rough Heston, but is not able to capture the explosive behaviour of the at-the-money skew in the same way. It is clear that the Fractional Heston, in both its full and asymptotic forms, can offer reasonable estimates for option prices when log-moneyness is close to zero. While the asymptotic formula may offer a way to pre-estimate parameters when doing calibration, the full implementation of the Fractional Heston seems most appropriate for robust pricing and calibration. This model still retains a significant advantage over the Rough Heston due to the fact that the CGF only requires that we numerically solve two ordinary differential equations rather than the computationally challenging fractional differential equations appearing in the characteristic function of the Rough Heston.

We recommend further research into this model that would allow for non-zero correlation between the Brownian motions driving the Fractional Heston as defined in (5.1). This seems to be a significant shortcoming of the model as it currently stands and the introduction of correlation may well allow the Fractional Heston the ability to better capture the explosive behaviour of the at-the-money skew. In Guennoun et al. (2018) it is noted that the dynamics of the instantaneous variance of the Fractional Heston when \( \rho \neq 0 \) are given by

\[
\frac{dV_t}{V_t} = \lambda (\theta - V_t) dt + \rho \nu u \sqrt{V_t} \frac{dW_t}{V_t} + \lambda \nu \sqrt{V_t} dW_t.
\]

This process is non-Markovian and so it is not possible to simply modify the approach as current Feynman-Kac methods are not able to deal with such processes. There have been developments which attempt to provide Feynman-Kac tools for fractional Brownian motion as evidenced by Viens and Zhang (2019). We leave such developments and their application as an avenue for future research.
Bibliography


Appendix A

Fractional Brownian Motion

A.1 A Primer on Fractional Brownian Motion

Fractional Brownian motion (fBm) is a generalisation of ordinary Brownian motion which was first introduced in Mandelbrot and Van Ness (1968). The main idea is to relax the condition that the increments be independent.

The idea behind fBm is as follows: Suppose we have a Brownian motion $B_t$ and a choice of Hurst parameter $H \in (0, 1)$ and $T > 0$. A fractional Brownian motion $B^H_t$ with Hurst parameter $H$ is a moving average of the increments $dB_t$ in which past increments of $B_t$ are weighted by $(t - s)^{H - 1/2}$. What this means is that the resulting fBm has some structure of dependence within the process (unless of course $H = 1/2$ in which case we revert to ordinary Brownian motion).

The Mandelbrot-van Ness representation of fBm is given by

$$B^H_t = \frac{1}{\Gamma(H + 1/2)} \int_{-\infty}^{0} \left( (t - s)^{H - \frac{1}{2}} - (-s)^{H - \frac{1}{2}} \right) dW_s + \frac{1}{\Gamma(H + 1/2)} \int_{0}^{t} (t - s)^{H - \frac{1}{2}} dW_s.$$

The value of $H$ has the following effect on the resulting fBm: first, if $H = 1/2$, then we have ordinary Brownian motion; second, if $H > 1/2$, then there is positive correlation between the increments of the process; third, if $H < 1/2$, then there is negative correlation between the increments of the process.

In addition, fBm has several nice properties similar to those of ordinary Brownian motion. We shall summarise them briefly here: First, for $c \in \mathbb{R}$, we have self-similarity

$$B^H_{ct} \sim |c|^H B^H_t.$$

Second, the increments are stationary, that is

$$B^H_t - B^H_s \sim B^H_{t-s}.$$

Third, while the sample paths are almost nowhere differentiable, almost all trajectories are locally Hölder regular of any order strictly less than $H$: for each trajectory, for all $T > 0$ and $\varepsilon > 0$, there is a constant $C$ such that for $0 < s, t < T$, we have

$$|B^H_t - B^H_s| \leq C|t - s|^{H - \varepsilon}.$$
Finally, it is possible to define stochastic integrals with respect to fBm but these integrals are not in general semimartingales like those defined with respect to ordinary Brownian motion.

A.2 A Primer on Fractional Calculus

In this section, we offer an heuristic motivation for the definitions of the fractional derivative and integral operators appearing in Chapter 4.2.

Let $f$ be a function defined for $x > 0$. Recall both the differentiation operator $D$ given by

$$Df(x) = \frac{d}{dx} f(x)$$

and the integration operator $I$ given by

$$If(x) = \int_0^x f(u) du.$$  

These operators can be iteratively applied, leading to a notion of powers of differentiation and integral operators, that is, for a positive integer $n$,

$$I^n f(x) = \left( I \circ I \circ \cdots \circ I \right)_{(n)}(f).$$

Naturally, one asks whether this can be generalised so that $n$ takes on any positive real value.

This can be done as follows: Consider the integral

$$(If)(x) = \int_0^x f(t) dt$$

and apply the operator $I$ again to get

$$(I^2 f)(x) = \int_0^x (If)(t) dt = \int_0^x \left( \int_0^t f(s) ds \right) dt.$$  

For arbitrary $n$, the formula for repeated integration due to Cauchy is

$$(I^n f)(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt.$$  

To extend this formula for non-integer values of $n$, we replace the factorial with the Gamma function

$$(I^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt.$$  

Similarly, the fractional derivative operator can be written

$$(D^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)\alpha} dt.$$
Appendix B

Computational Cost of non-Markovian Models

B.1 The Rough Heston and Fractional Heston

Following Chapter 6.2, we use the parameters below to demonstrate the computational expensiveness of non-Markovian models.

For the Rough Heston we use the parameters:

\[ \lambda = 6.3, \theta = 0.025, V_0 = 0.12, \nu = 0.0952, \text{ and } \alpha = 0.6. \]

For the classical Heston and the Fractional Heston (both the full implementation using Theorem 5.1 and its asymptotic version from Theorem 5.2) we use the parameters:

\[ \lambda = 2.1, \theta = 0.05, V_0 = 0.03, \nu = 0.4286, \eta = 0, \text{ and } d = -0.3. \]

We computed prices of at-the-money options with expiration \( T = 1/12 \) and tabulated these in Table B.1.

<table>
<thead>
<tr>
<th>Model Name</th>
<th>Runtime (secs)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rough Heston</td>
<td>2.2801</td>
</tr>
<tr>
<td>Fractional Heston</td>
<td>1.0532</td>
</tr>
<tr>
<td>Fractional Heston (asymptotic version)</td>
<td>0.0151</td>
</tr>
<tr>
<td>Heston (Rough Heston parameters)</td>
<td>0.0038</td>
</tr>
<tr>
<td>Heston (Fractional Heston parameters)</td>
<td>0.0022</td>
</tr>
</tbody>
</table>

Tab. B.1: Example runtimes for the option pricing models.

It is clear from Table B.1 that the full implementations of non-Markovian models in the first two rows are computationally far more expensive than the asymptotic version of the Fractional Heston model and the classical Heston model. The asymptotic version of the Fractional Heston is much closer to classical Heston in terms of runtime.

The MATLAB code has been optimised as far as possible and the models were run on a Lenovo X1 Carbon running Windows 10 on an Intel i7-8650U quad-core 1.90 GHz (up to 4.20 GHz) CPU with 16 GB RAM.