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Existence and Stability of Solutions to the Equations of Fibre Suspension Flows

by

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Abstract

A popular approach to formulating the initial-boundary value problem for fibre suspension flows is that in which fibre orientation is accounted for in an averaged sense, through the introduction of a second-order orientation tensor $\boldsymbol{A}$. This variable, together with the velocity and pressure, then constitutes the set of unknown variables for the problem. The governing equations are balance of linear momentum, the incompressibility condition, an evolution equation for $\boldsymbol{A}$, and a constitutive equation for the stress.

The evolution equation contains a fourth-order orientation tensor $\mathcal{A}$, and it is necessary to approximate $\mathcal{A}$ as a function of $\boldsymbol{A}$, through a closure relation.

The purpose of this these is to examine the well-posedness of the equations governing fibre fibre suspension flows, for various closure relations. It has previously been shown by GP Galdi and BD Reddy that, for the linear closure, the problem is wellposed provided that the particle number, a material constant, is less than a critical value. The work by Galdi and Reddy made of a model in which rotary diffusivity is a function of the flow.

This thesis re-examines these issues in two different ways. First, the second law of thermodynamics is used to establish the constraints that the constitutive equations have to satisfy in order to be compatible with this law. This investigation is carried out for a variety of closure rules. The second contribution of the thesis concerns the existence and uniqueness of solutions to the governing equations, for the linear and quadratic closures; for a model in which the rotary diffusivity is treated as a constant, local and global existence of solutions are established, for sufficiently small data, and in the case of the linear closure, for admissible values of the particle number. The existence theory uses a Schauder fixed point approach.

SUMMARY

It is well-known that the linear closure relation leads to anomalous behavior, in that the rest state of the fluid is unstable [15]. On the other hand, no global solution has been found yet for any suspension flow process. The purpose of this work is to show that, in conformity with results presented elsewhere, the linear closure approximation leads to flows that are stable only for certain ranges of particle number, and furthermore there exists a local and global solution for a specific flow, for both linear and quadratic closure.

The stability is based on the consistency of the constitutive equations with respect to the second law of thermodynamics, and the conditions of monotonic stability in the energetic sense. Four closure are investigated here: linear, quadratic, Hinch&Leal and Smooth orthoropic.

The existence theorem uses the fixed point argument which is decomposed into two linear ones: a Stokes type problem for the velocity and a linear type for the orientation tensor; and one transport problem. The existence of a unique classical solution, local in time, is proven by using a Schauder fixed point theorem, for both linear and quadratic approximation relations.
The global a priori estimates are derived to obtain a unique global classical solution for sufficiently small data and for a constant value of D_r. The solution is found to be stable in absence of the body force.

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Contents

Chapter 1

INTRODUCTION

1.1 Fibre Suspension Process

The flow of fluids with suspensions of fibres (as shown in Figure 1) is an important problem in the processing of fibre composite materials, and has received much attention due to the technological importance of the subject [37]. Indeed, fibre composites are used in a wide variety of applications, ranging from compression-moulded automobile body panels to injection-moulded parts for business machines and consumer goods [2]. Fibre composites are also used to reinforce thermoplastics for injection moulding and extrusion, sheet moulding compounds, and short-fibre GMTs (Glass-mat-reinforced thermoplastics). They are also used to reinforce metallic and ceramic matrices, and polymer-matrix composites [40], and in many other applications.

A new field to be explored is orthopaedic implants, where fibre composites may be used as prostheses. Traditionally, metal prostheses are used in the human body. However, due to their very high stiffness, metals do not transfer stresses to the bone very effectively. Fibre composites are ideal canditates to replace them as they have

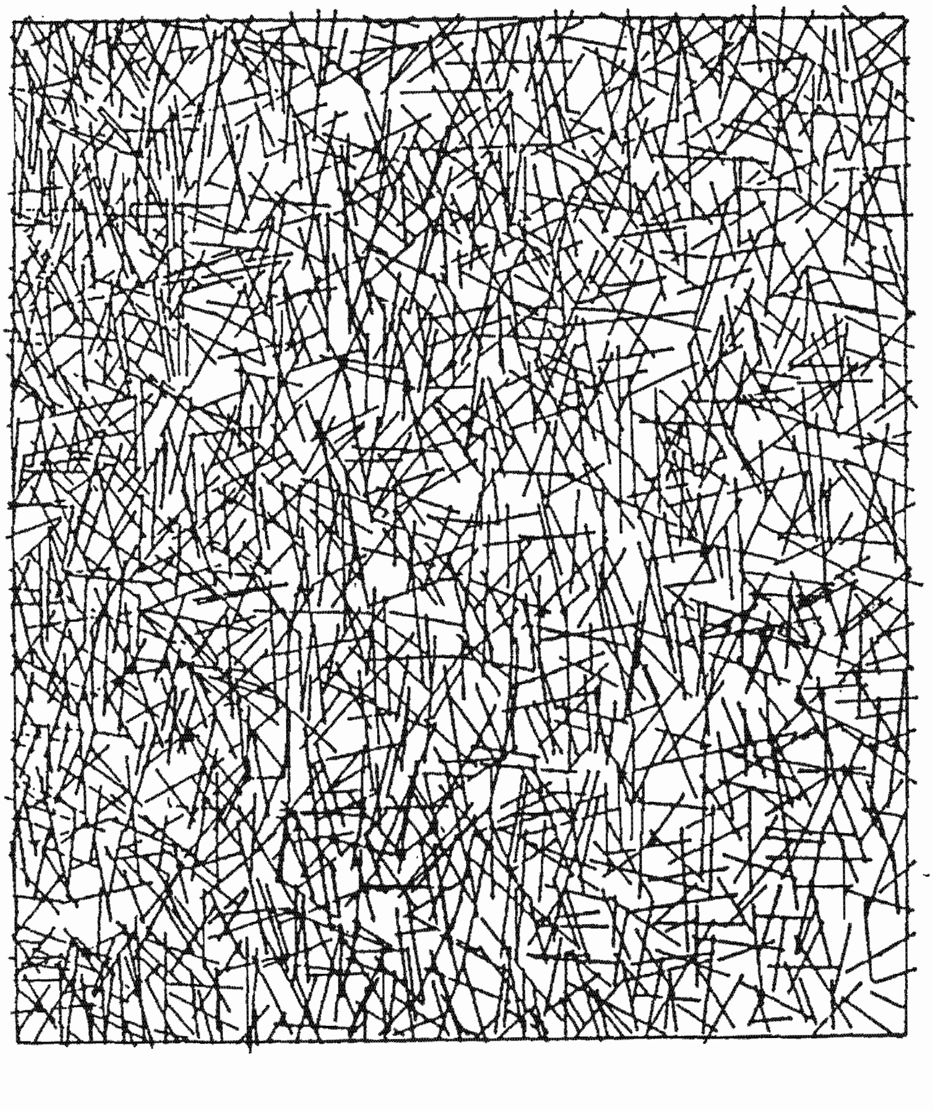

Figure 1: Computer image of fibres digitized from a radiograph of a plaque of sheet moulding compound [31].

moderate stiffness, which can be fine tuned to maximize load transfer to the bone [40].

The use of fibre composite compounds has grown in commercial importance in recent years due to desirable cost and perfomance charateristics, especially in relation to mechanical and thermal properties [24], so the practical interest arises from the economic incentives to use fibre-reinforced composites as replacement for metals. The mechanical properties of fibre composites are not as good as a composite with continuous reinforcing fibre [40]. However, fibre composites can be processed by fast, highly-automated methods such as injection or compression moulding. This gives them an economic advantage over continuous- fibre composites, which are still fabricated by slow, labour-intensive processes such as lay-up followed by autoclave cure [40].

Fibre composites are generally formed by automated methods such as injection moulding, compression moulding or extrusion.

The properties of fibre suspension composite parts depend highly on the way the part is manufactured. If such a material is formed, the flow changes the orientation of the fibres. As the resin or moulding compound deforms to achieve the desired shape, the orientation of fibres is changing. Fibre orientation changes stop when the matrix solidifies, and the orientation pattern becomes part of the microstructure of the finished article [2, 40].

This fibre orientation pattern is the dominant structural feature of a fibre composite. The composite is stiffer and stronger in the direction of greatest orientation, and weaker and more compliant in the direction of least orientation [2].

Therefore, it is very important to know what happens in the fluid state of a fibre composite material. An aspect is that of fibre-fluid and fibre-fibre interactions: it is of considerable fundamental interest to understand how to model such interactions in suspensions, and to be able to describe quantitatively and qualitatively how these interactions affect the rheological properties [10].
The ultimate goal is to learn how to design fibre composite parts effectively, and how to predict the orientation of the fibres, and to control manufacturing processes so as to generate favourable orientation states, and obtain the best possible performance from the fibre composite.

It would be interesting to answer the following questions, no matter which process or material is being considered:

- What parameters describe the orientation of fibres, and how can one measure them?

- How does flow change the orientation of fibres?

Theories exist which can predict the mechanial properties of the composite once the fibre orientation state is known [2].

The choice of the variables used to represent fibre orientation state affects the accuracy and efficiency of the calculation. The use of various parameters to describe the orientation states has a long history. The starting point for most of work on the problem has been JEFFERY's (1922) [22] solution for the motion of a small rigid spheroid in a uniform shear flow of a Newtonian fluid. Jeffery showed that, in the absence of particle body forces or couples, a spheroid will translate with the velocity of the undisturbed fluid at the position of its centre, while its axis of revolution rotates in one of an infinite one-parameter family of possible periodic orbits [21].

Many workers have developed and used orientation parameters to describe material structure and to predict material properties [2]. While these parameters provide a concise way to describe orientation states, their formulation relies on assumptions about the direction of the principal axes of orientation and/or some symmetries in the probability distribution function [2].

One plausible representation is a probability distribution function for fibre orientation. This has been used for some simple problems involving nontrivial flow fields, but only for planar orientation. Unfortunately, the probability distribution function is too unwieldy for numerical calculations of three-dimensional fibre orientation in complex geometries [2].

There have been investigations in the 1980s, especially by ADVANI AND TUCKER [2, 3, 40], and DINH AND ARMSTRONG [10], aimed at deriving quatitative relationships between processing conditions and fibre orientation.

BATCHELOR [4] considered the properties of the bulk stress in a suspension of non-spherical particles, immersed in a Newtonian fluid, and on which a couple (but no force) may be imposed by external means. He defined the bulk stress and bulk velocity gradient in the suspension as averages over an ensemble of realizations, these averages being equal to integrals over a suitably chosen volume of ambient fluid and fibres together, when the suspension is statistically homogeneous. Without restriction on the type or the concentration or the Reynolds number of the motion, Batchelor expressed the contribution to the bulk stress due to the presence of the fibres in terms of integrals involving the stress and velocity over the surface of fibres, together with volume integrals not involving the stress.

DINH AND ARMSTRONG [10] established that in start-up of steady shear or elongational flow the orientation of the fibres and the measurable rheological properties both depend only on the total applied strain. They also obtained the expression of the stress in terms of an integral over a function of the Cauchy strain tensor and the orientation vector for a fibre. Also, they [10] obtained a constitutive equation for a semi-concentrated suspension of rigid fibres in Newtonian fluid. The fibre orientation distribution was calculated using JEFFERY's analysis [24].

ADVANI AND TUCKER [2] have been interested in predictions of flow-induced fibre orientation. In these calculations the orientation state of the fibres plays the role of a structural state variable.

The need of a description of fibre orientation which is both general (like the distribution function) and concise (like the orientation parameters), is therefore evident. One possibility is to use a more compact description of orientation. This requirement is fulfilled by a set of tensors that have been called tensorial order parameters, orientation-moment tensors, conformation tensors, [2], or, as we will refer to them here, *"orientation tensors"*.

Tensors offer considerable advantages for numerical computation because they are a compact description of the fibre orientation state. Such models also have the great advantage that the average behaviour of fibres may be treated in a deterministic manner, and without the need to solve rather complex equations for the probability density function which characterises fibre orientation.

A theory of anisotropic fluids which is rather general in interpretation was formulated by HAND [17]. He introduced the symmetric tensor that described the microscopic structure of a fluid and found the most general expression for the stress tensor as a function of the symmetric tensor and rate of deformation tensor.

It is necessary, in models based on orientation tensors, to make use of a closure approximation in order to express the fourth-order orientation tensor in terms of that of order two. Solutions of the resulting problem depend critically on the choice of closure rule, and it is found that while some rules are particularly well suited to specific flow situations, no single rule can be said to lead to accurate simulations of all flow situations.

ADVANI AND TUCKER [2, 40] have derived the equations of change for the second- and fourth-order tensors, and the fourth-order tensor was estimated by the mean of second tensors through linear, quadratic and hybrid closures. Two different approximations have been introduced by CINTRA AND TUCKER [6], and they were tested against distribution function solutions in a variety of flow fields, both steady and unsteady, by integrating the orientation evolution equations. Further details may be found in Chapter 3.

HINCH AND LEAL [18] derive a number of different closure approximations in their study of suspensions of fibres with Brownian motion. They derive approximations for three closures: for weak (W1) and strong (S1) flows, and a composite rule, commonly referred to as H&L1. Hinch and Leal also developed a second set of approximations, known as second approximations in weak (W2) and strong (S2) flow, and a second (H&L2) composite rule. The H&L rule is studied in Chapter 3.

In a recent study of the well-posedness of the set of equations describing fibre suspensions flows [15], GALDI AND REDDY have shown that there is a close connection between stability, in the Liapounov sense [35], and the particle number, a parameter closely related to fibre concentration, for the case in which a linear closure approximation is used. In particular, the rest state is shown to to be unstable for particle numbers exceeding 35/2. A second result in that work is one on local existence and uniqueness of solutions, for closure rules corresponding to stable rest states.

The inherent instability asserts itself also in the behaviour of the evolution equation, in which concentration is not present as a parameter. It is shown, in a numerical study by REDDY AND MITCHELL [34], that whereas quadratic and hybrid closures converge rapidly to the steady state, the use of the linear closure gives rise to a transient region which is oscillatory for a relatively extended period of time. In this work, Reddy and Mitchell treated the computational problem as on that is coupled, and which was decoupled by adopting a two-stage approach: finite element approximations in space and finite differences in time were used to solve, first, the momentum equations for the velocity and pressure fields, and this was followed by the solution of the evolution equations for the orientation tensors. The evolution equations were solved on an element-by-element basis. The process proceeds iteratively until the differences between solutions in successive iterations is acceptably small.

1.2 The aim of this thesis

This thesis has two main aims:

- first, the thesis works towards a more complete picture of the stability of fibre suspension flows, by investigating thermodynamic consistency and energetic stability for these flows. In particular, it investigates the circumstances under which the constitutive equations are consistent with the second law of thermodynamics, and conditions under which flows are monotonically or exponentially stable, in an energetic sense, for a range of popular closure approximations. Four approximations are studied here: linear, quadratic, the Hinch and Leal first composite, and the smooth orthotropic closure of Cintra and Trucker.

- next, this work establishes conditions for the existence of a unique classical solution, locally and globally in time, for the case in which the linear and quadratic closure are used, and for a constant value of the rotary diffusivity D_r, which is an excellent model for very small fibres, that experience rotary Brownian motion [40]. In the case of both closures, the proof of existence of a local solution uses a fixed point argument. Global a priori estimates are derived to obtain a unique global classical solution for sufficiently small data, and it is proven that that solution is stable in the absence of body forces.

1.3 Notation

We make use of coordinate-free notation wherever convenient, and denote vectors and tensors by bold-face letters.

The scalar product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is denoted by $\boldsymbol{u} \cdot \boldsymbol{v}$, while the corresponding product of two second-order tensors $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted by $\boldsymbol{A}{:}\boldsymbol{B}$. In index form these expressions read $u_i v_i$ and $A_{ij}B_{ij}$, the summation convention on repeated indices being applied at all times. The magnitude of a vector $\boldsymbol{u}$ and a tensor $\boldsymbol{A}$ are then naturally defined by $|\boldsymbol{u}| = (\boldsymbol{u} \cdot \boldsymbol{u})^{1/2}$ and $|\boldsymbol{A}| = (\boldsymbol{A} : \boldsymbol{A})^{1/2}$.

We summarise here some notation that will occur throughout the thesis:

$\boldsymbol{x}$: location of fluid particle

$\boldsymbol{v}$: velocity field of the flow

$\boldsymbol{b}$: body force

p: pressure

μ: dynamic viscosity

ρ: fluid mass density

$\boldsymbol{A}$: second-order orientation tensor.

$\boldsymbol{D}=\frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$ deformation tensor

$\boldsymbol{W}=\frac{1}{2}(\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^T)$ spin tensor

$\boldsymbol{T}$: symmetric Cauchy stress tensor

$\mathcal{A}$: fourth-order orientation tensor

We will also need the following function spaces:

Here Ω denotes a bounded domain of $\mathbb{R}^n$ ($n = 2$ or 3), with boundary Γ. We will assume that Ω is on one side of Γ, and that Γ is at least Lipschitzian or C^1. For details on the regularity of Γ see also Appendix A.2.5.

$\Omega_T = \Omega \times (0,T)$ for $T > 0$

$L^p(\Omega), 1 \leq p \leq \infty$, the Lebesgue spaces with norms defined in the usual way and their norms denoted by $\|\cdot\|_{L^p}$.

$H^k(\Omega)$, $k = 0, 1, ...$ the Sobolev spaces endowed with the inner product $(\cdot,\cdot)_{H^k}$ defined by

$$(\boldsymbol{u},\boldsymbol{v})_{H^k} = \sum_{|\alpha|\le k} \int_\Omega D^\alpha \boldsymbol{u} D^\alpha \boldsymbol{v} dx. \qquad \alpha = (\alpha_1, ..., \alpha_n)$$

$H^k_0(\Omega) = \{f \in H^k(\Omega) : f_{/\Gamma} = 0 \text{ and } D^\alpha f = 0 \text{ on } \Gamma \text{ for } |\alpha| < k\}$

$H^{-k}(\Omega)$ the topological dual space of $H^k_0(\Omega)$.

We denote the space of vector- or tensor valued functions with components in one of the spaces introduced above as follows:
$\mathbb{L}^p(\Omega)$ will denote the space of vector-or tensor valued functions with components in $L^p(\Omega)$.
The same will apply to $\mathbb{H}^k(\Omega)$,...
We will also require the following spaces:

$$\mathbb{H}=\{\boldsymbol{v} : v_i \in L^2(\Omega), \text{div } \boldsymbol{v} = 0 \text{ in } \Omega, \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}$$

$$\mathbb{V}=\{\boldsymbol{v} : v_i \in H^1_0(\Omega), \text{div } \boldsymbol{v} = 0 \text{ in } \Omega\}$$

$$\mathbb{X}=\{\boldsymbol{A} : A_{ij} \in L^2(\Omega), A_{ij} = A_{ji}, \ A_{ii} = 0 \quad \text{a.e in } \Omega\}$$

These spaces are equipped respectively with the norms

$$\|\cdot\|_{\mathbb{H}} \equiv \|\cdot\|_{\mathbb{L}^2} \equiv |\cdot|;$$
$$\|\cdot\|_{\mathbb{V}} \equiv \|\cdot\|_{\mathbb{H}^1} \quad \text{and}$$
$$\|\cdot\|_{\mathbb{X}} \equiv \|\cdot\|_{\mathbb{L}^2}.$$

P denotes the orthogonal projection of $\mathbb{L}^2(\Omega)$ onto $\mathbb{H}$.

$\mathcal{L}(\boldsymbol{v}) = -P\Delta\boldsymbol{v}$

$D(\mathcal{L}) = \mathrm{V} \cap \mathbb{H}^2(\Omega)$

$\|\boldsymbol{v}\|_{D(\mathcal{L})} = \|\mathcal{L}\boldsymbol{v}\|_{L^2}$ is equivalent to the natural $\mathbb{H}^2$-norm.

$\|\boldsymbol{v}\| = |\nabla \boldsymbol{v}|$ is the Dirichlet norm.

We set

$$\|\boldsymbol{v}\|_{\mathbb{L}^p(0,T,\mathbb{H}^k)} = \left(\int_0^T \|\boldsymbol{v}(t)\|^p_{\mathbb{H}^k} dt\right)^{\frac{1}{p}}$$

$\mathbb{L}^p(0,T;\mathbb{H}^k) = \{\boldsymbol{v} : [0,T] \longrightarrow \mathbb{H}^k(\Omega) : \|\boldsymbol{v}\|_{\mathbb{L}^p(0,T;\mathbb{H}^k)} < \infty\}$

For more details on these spaces see Appendix A.2.

1.4 Presentation of the thesis

The rest of this thesis is organised as follow:

Chapter 2 reviews the relevant topics from continuum mechanics. In particular, we review the mechanics of fibre suspension flows, constitutive equations, and closure approximation rules. These lead to a formulation of the initial boundary value problem for fibre suspension flows.

In Chapter 3, we study the thermodynamics and the monotonic stability (in the energetic sense) of fibre suspension flows. The study focusses on four closure rules: the linear, quadratic, Hinch&Leall, and smooth orthotropic closures.

The problem of local existence of solutions is investigated in Chapter 4, for both linear and quadratic closure rules, while the global existence of solutions and the stability of those solutions are the subjects of Chapter 5.

We summarise and review our results in Chapter 6. Finally, some relevant mathematical results are summarised in the Appendix.

Chapter 2

MECHANICS OF FIBRE SUSPENSION FLOWS

2.1 Description of fibre suspensions

2.1.1 Single fibres

Suppose that a fibre is immersed in a viscous incompressible Newtonian fluid. The fibre is assumed to be axisymmetric, like a rigid cylinder or a homogenous ellipsoid, uniform in length and diameter, as shown in Figure 2.
The orientation of a fibre can be described by a unit vector $\boldsymbol{p}$ directed along the fibre axis (Figure 2). With respect to spherical coordinates (θ, ϕ),

$$p_1 = \sin\theta\cos\phi, \quad p_2 = \sin\theta\sin\phi, \quad p_3 = \cos\theta, \tag{2.1}$$

where (p_1, p_2, p_3) are the cartesian components of $\boldsymbol{p}$. The choice of direction for $\boldsymbol{p}$ is arbitrary, since the "head" of the fibre is identical to its "tail". Therefore, any description of the orientation of the fibre must be unchanged if one substitutes

$$-\boldsymbol{p} \quad for \quad \boldsymbol{p} \tag{2.2}$$

or

$$\pi - \theta \ \ for \ \ \theta \ \ and \ \ \phi + \pi \ \ for \ \ \phi. \tag{2.3}$$

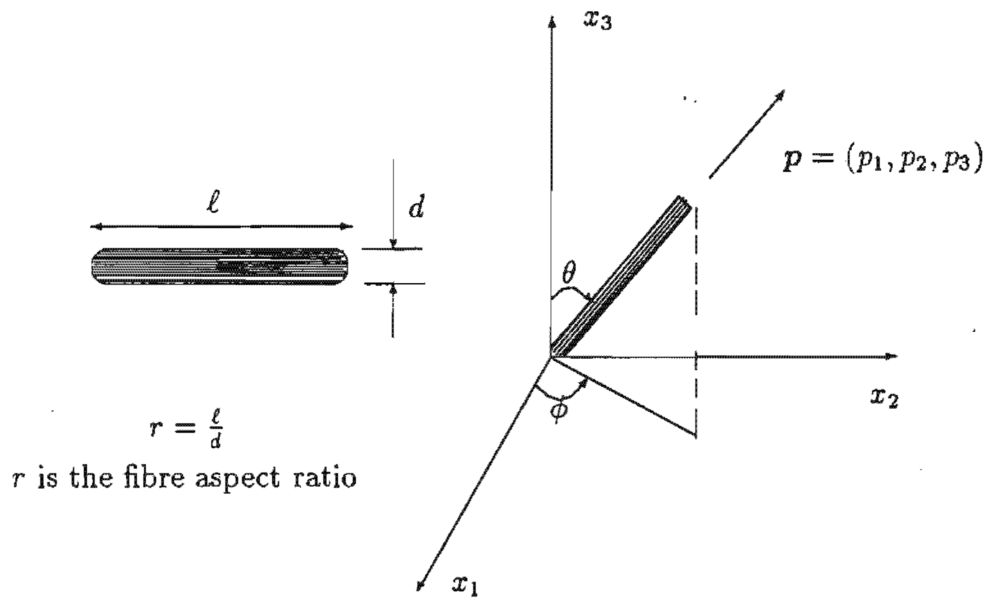

Figure 2: Single fibre orientation

2.1.2 Fibre suspensions: the distribution function

A suspension of uniform cylindrical or ellipsoidal (Figure 2) fibres is characterized by the particle volume fraction h and the fibre aspect ratio r.

A suspension is said to be [40]

$$\left.\begin{array}{r}\text{dilute}\\ \text{semi-dilute}\\ \text{concentated}\end{array}\right\}\text{if}\left\{\begin{array}{l} hr^2 < 1\\ 1 < hr^2 < r\\ r < hr^2\end{array}\right. \tag{2.4}$$

We will be concerned with dilute and semi-dilute suspensions, for which fibres have a low probability of making contact, though the motion of the fibres and the fluid are coupled. We assume also that the fibre concentration is spatially uniform. That is, the number of fibres per unit volume is uniform, though the orientation of the fibres may not be.

Fibres are generally not aligned in the same direction, not even within a very small region; their orientation will differ from point to point, as well as in time, and it is unrealistic to attempt a proper description of this variation in orientation for a single fibre in a suspension. Instead, one approach is to regard the fibres as a sample drawn from an infinite population. This useful approach uses a probability density function $\psi(\boldsymbol{p})$ or $\psi(\theta,\phi)$, whose value for a given orientation gives the probability that a fibre has that particular orientation [15, 40].
This function is defined such that the probability of any given fibre lying in the range between θ_1 and $\theta_1 + d\theta$, and between ϕ_1 and $\phi_1 + d\phi$, is given by [40]

$$\mathbb{P}(\theta_1 \leq \theta \leq \theta_1 + d\theta\ ,\ \phi_1 \leq \phi \leq \phi_1 + d\phi) \;=\; \psi(\theta_1,\phi_1)\sin\theta_1 d\theta d\phi. \tag{2.5}$$

Since every fibre must lie at some angle, the integral of this function over all angles must satisfy

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{2\pi}\psi(\theta,\phi)\sin\theta\ d\theta d\phi \;=\; \int_{S^1}\psi(\boldsymbol{p})\ d\boldsymbol{p} = 1. \tag{2.6}$$

The integral is taken over the unit sphere S^1, that is, over all possible orientations. The property (2.2) or (2.3) implies that the distribution function must be even, in the sense that

$$\psi(\boldsymbol{p}) \;=\; \psi(-\boldsymbol{p}) \tag{2.7}$$

or

$$\psi(\theta,\psi) \;=\; \psi(\pi-\theta,\phi+\pi). \tag{2.8}$$

Various approaches to solutions of the problem have been proposed: **Orientation as a function of deformation** [10] Here we assume that

$$\psi(\boldsymbol{p}) \quad = \quad \frac{1}{4\pi}(\boldsymbol{p}\cdot\boldsymbol{B}\boldsymbol{p})^{-3/2}$$

where $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$ is the right Cauchy-Green deformation tensor, $\boldsymbol{F}=\nabla\phi$ being the deformation gradient, where $\boldsymbol{x} = \phi(\boldsymbol{X}, t)$ is the motion.
This approximation is valid for infinitely slender fibres with no interaction and random initial orientation, and is therefore limited in applicability, in that it cannot be used for models that contain diffusion or interaction terms.

Direct calculation of orientation tensors [2, 39] This approach, which we will use here, allows the probability function to be eliminated in favor of a new field variable.

Consider the motion of fibres in a fluid. JEFFERY's classical analysis [22] treats a single rigid fibre in an infinite body of Newtonian fluid. The unperturbed fluid velocity is assumed to be a linear function of position, and inertia and body forces are assumed to be negligible.
We shall use the Jeffery's solution for the rotational motion of an ellipsoidal particle, which is expressed as the time derivative of the orientation vector $\boldsymbol{p}$ by

$$\dot{p}_i \;= W_{ij}p_j \;+\lambda\;\left[D_{ij}p_j - (D_{kl}p_kp_lp_i)\right]. \tag{2.9}$$

Here

$$\boldsymbol{D} = \frac{1}{2}\left(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T\right)$$

is the deformation rate tensor,

$$\boldsymbol{W} = \frac{1}{2}\left(\nabla\boldsymbol{v} - (\nabla\boldsymbol{v})^T\right)$$

is the spin tensor,

$$\lambda = \frac{r^2 - 1}{r^2 + 1}$$

is a parameter depending on particle aspect ratio, for the sphere $\lambda \longrightarrow 0$, and for the slender rod $\lambda \longrightarrow 1$. Jeffery's equation (2.9) predicts that a single particle in a single shear flow will undergo a periodic rotation [22, 40]. If the flow field is given by

$$v_1 = Kx_2, \qquad v_2 = v_3 = 0, \tag{2.10}$$

where K is the shear rate, then the fibre motion will be given by [40]

$$\tan\theta = \frac{Cr}{\sqrt{cos^2\phi + r^2 sin^2\phi}}, \tag{2.11}$$

$$\cot\phi = rtan\left(\frac{2\pi t}{T} + k\right), \tag{2.12}$$

where T is the period of rotation, that is,

$$T = \frac{2\pi}{K}\left(r + \frac{1}{r}\right), \tag{2.13}$$

and $r = \frac{\ell}{d}$ is the equivalent ellipsoidal axis ratio (see Figure 2).

The orbit constant C and the phase k are determined by the initial position of the fibre. The degree of alignment of fibres is predicted by Jeffery's equation(2.9). If all particles have the same aspect ratio, hence the same period, then the entire distribution of fibre orientation returns to its original state periodically, with period T [40]. However semi-dilute and concentrated suspensions do not display this periodicity. These malalignments are presumably caused by interactions between the fibres.

Current models that includes an interaction effect between fibres are closely related to the theory of rotary Brownian motion [40]. The effects of rotary Brownian motion on fibre orientation are modeled by writing an equation for the time evolution of the probability distribution function ψ, in the form [18, 40]

$$\frac{d\psi}{dt} = -\frac{\partial}{\partial p_i}(\psi \dot{p}_i) + D_r \frac{\partial^2 \psi}{\partial p_i^2} \tag{2.14}$$

The equation (2.9) for the motion of a fibre is likewise modified, and becomes

$$\dot{\boldsymbol{p}} = \boldsymbol{W}\boldsymbol{p} + \lambda[\boldsymbol{D}\boldsymbol{p} - (\boldsymbol{p} \cdot \boldsymbol{D}\boldsymbol{p})\boldsymbol{p}] - \frac{D_r}{\psi}\frac{\partial \psi}{\partial \boldsymbol{p}}. \tag{2.15}$$

Here D_r is the rotary diffusivity due to Browian motion. Equation (2.15) can be written in component form as

$$\dot{p}_i = W_{ij}p_j + \lambda \, [D_{ij}p_j - (D_{kl}p_k p_l p_i)] - \frac{D_r}{\psi}\frac{\partial \psi}{\partial p_i}. \tag{2.16}$$

Folgar and Trucker adapted equation (2.15) by using $\lambda = 1$ in Jeffery's equation and setting

$$D_r = C_I|\boldsymbol{D}| \tag{2.17}$$

where C_I is a constant known as the interaction coefficient, and

$$|\boldsymbol{D}| = (\boldsymbol{D} : \boldsymbol{D})^{1/2}; \tag{2.18}$$

the inner product $\boldsymbol{A} : \boldsymbol{B}$ of two second-order tensors is defined by

$$\boldsymbol{A} : \boldsymbol{B} = A_{ij}B_{ij}. \tag{2.19}$$

All particles in a suspension experience small randomly-oriented forces as they collide with the solvent molecules. In practical composites, the fibres are too large to experience significant Brownian motion. However, when D_r is small equation (2.15) exhibits many of the same qualitative features as non-dilute suspensions. This has led several workers to adopt equation (2.15) as a phenomenological model for fibre orientation in non-dilute suspensions [2, 40].

The case $D_r = 0$ corresponds to that in which rotary diffusion is assumed absent, and the equation (2.15) reduces to that obtained by JEFFERY [22, 40] for an ellipsoid of revolution in a dilute suspension.

The case D_r = constant is an excellent model for very small particles, which experience rotary Brownian motion. Many studies have treated this case over the years [40]. This model will be the subject of Chapters 4 and 5.

The distribution function $\psi(\boldsymbol{p})$ or $\psi(\theta, \phi)$ provides a general description of the orientation state, and it is possible to make direct calculations of ψ for a single point in a suspension. One can discretize the spatial field using any of the usual tools of numerical analysis (finite elements, finite difference, etc.). However there are practical difficulties in using ψ in realistic similations and it is not practical to calculate ψ for hundreds or thousands of points [15, 40]. The orientation tensors provide a way around this difficulty.

2.1.3 Orientation tensors

We first define the averaging operator $< \cdot >$

$$< f > = \int f(\theta, \phi)\psi(\theta, \phi) \sin\theta d\theta d\phi = \int_{S^1} f(\boldsymbol{p})\psi(\boldsymbol{p})d\boldsymbol{p}, \tag{2.20}$$

where f is an arbitrary function.
We observe from (2.6) and (2.20) that

$$< 1 > = 1,$$

so equation (2.6) is frequently referred to as the normalization condition.

The orientation tensor $\mathcal{A}^k$ of order k is obtained by averaging k tensor products of the vector $\boldsymbol{p}$; that is,

$$\mathcal{A}^k = < \underbrace{\boldsymbol{p} \otimes \boldsymbol{p} \otimes \cdots \otimes \boldsymbol{p}}_{k \text{ terms}} >, \tag{2.21}$$

or in component form,

$$\mathcal{A}_{i_1 i_2 \cdots i_k} = < p_{i_1} p_{i_2} \cdots p_{i_k} > . \tag{2.22}$$

The case of a first-order tensor,

$$\mathcal{A}^1 = < \boldsymbol{p} >,$$

is trivial, since from (2.7), and (2.20) we have

$$< -\boldsymbol{p} > = \int_{S^1} (-\boldsymbol{p}) \psi(-\boldsymbol{p}) d\boldsymbol{p} \;=\; -\int_{S^1} \boldsymbol{p} \psi(\boldsymbol{p}) d\boldsymbol{p} \;=\; - < \boldsymbol{p} > \tag{2.23}$$

The condition (2.2) implies that

$$< -\boldsymbol{p} > = < \boldsymbol{p} >,$$

and it follows that $\quad - < \boldsymbol{p} > = < \boldsymbol{p} >$.
Therefore

$$\mathcal{A}^1 = < \boldsymbol{p} > = 0. \tag{2.24}$$

The same applies to any tensor of odd order constructed in the same way. We are thus left with even-ordered tensors. The second-order orientation tensor is given by

$$\mathcal{A}^2 \equiv \boldsymbol{A} \;= < \boldsymbol{p} \otimes \boldsymbol{p} > \quad or \quad A_{ij} \;= < p_i p_j >, \tag{2.25}$$

and the fourth-order tensor by

$$\mathcal{A}^4 \equiv \mathcal{A} \;= < \boldsymbol{p} \otimes \boldsymbol{p} \otimes \boldsymbol{p} \otimes \boldsymbol{p} > \quad or \; \mathcal{A}_{ijkl} \;= < p_i p_j p_k p_l > . \tag{2.26}$$

The tensor $\boldsymbol{A}$ satifies the symmetry and normalization conditions

$$A_{ij} \;=\; A_{ji} \quad and \quad A_{ii} \;=\; 1. \tag{2.27}$$

Indeed,

$$A_{ji} \;= < p_j p_i > = \int_{S^1} p_j p_i \psi(\boldsymbol{p}) d\boldsymbol{p} = \int_{S^1} p_i p_j \psi(\boldsymbol{p}) d\boldsymbol{p} \;=\; A_{ij}, \tag{2.28}$$

and, since $\boldsymbol{p}$ is an unit vector,

$$A_{ii} = < p_i p_i > = < 1 > = 1.$$

There are thus only five independent components in three dimensions and two for plane situations.

The fourth-order tensor $\mathcal{A}$ has the symmetries

$$\mathcal{A}_{ijkl} = \mathcal{A}_{jikl} = \mathcal{A}_{kijl} = \mathcal{A}_{klij} \; etc. \tag{2.29}$$

Futhermore, the higher-order tensors give complete information about their lower order counterparts [15]; for example, from (2.26) and (2.27),

$$\mathcal{A}_{ijkk} = < p_i p_j p_k p_k > = < p_i p_j > = A_{ij}. \tag{2.30}$$

Figure 3 shows some examples of $\boldsymbol{A}$ in $\mathbb{R}^2$.
A complete representation of ψ requires all the terms in the infinite series, involving orientation tensors of all orders. It is too complicated to use ψ in all its generality, in particular if the motion is unsteady and nonhomogeneous, so that $\psi = \psi(\boldsymbol{p}, \boldsymbol{x}, t)$.

The use of only the second-order tensor amounts an approximation of $\psi(\boldsymbol{p})$ by truncating the series. Thus, a tensor representation is always an approximate description.

However, in many theories the second-and fourth order tensors give exactly the information needed to determine many physical properties of composites [40]. Most models use the second-order tensor $\boldsymbol{A}$ to describe and predict orientation.

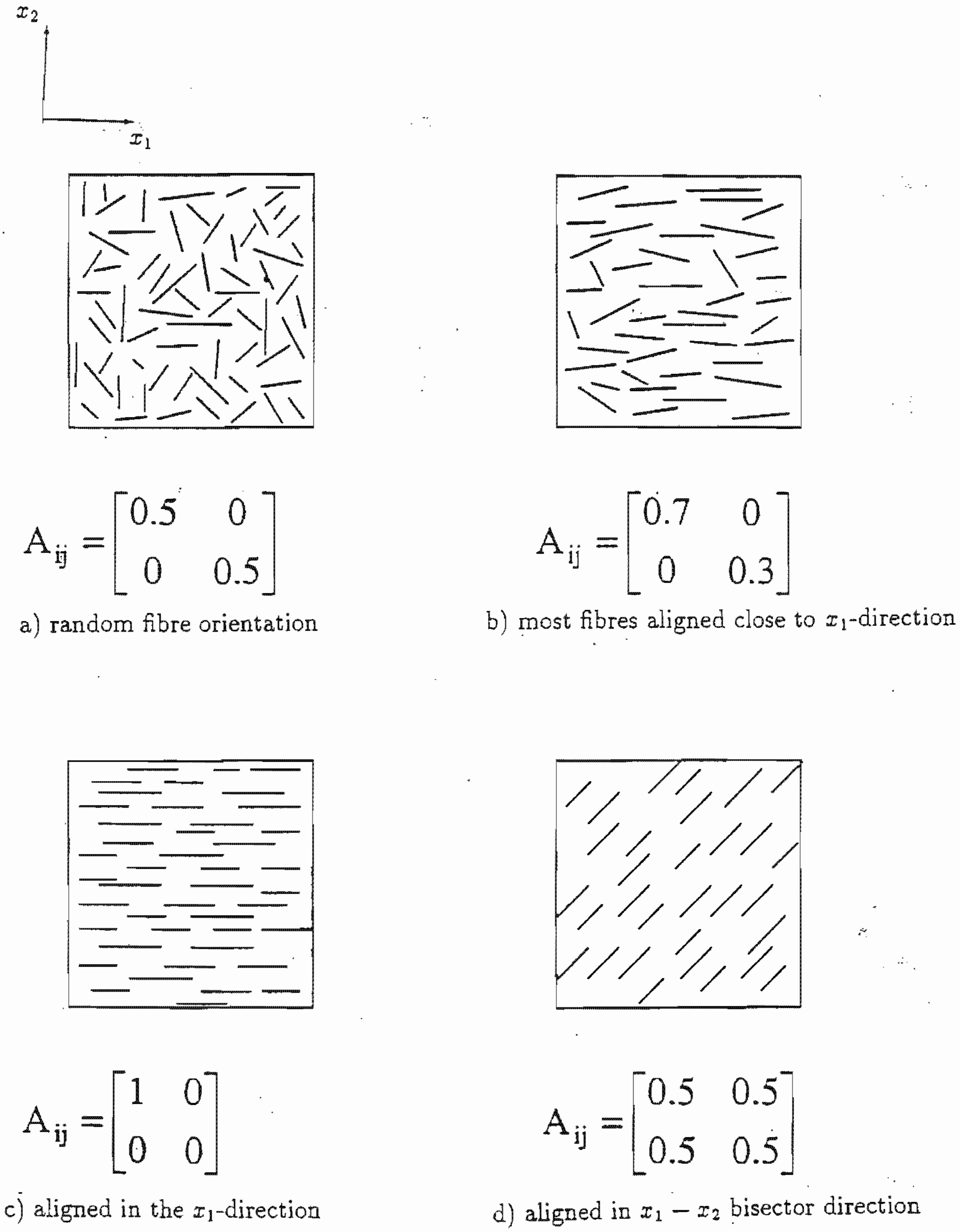

Figure 3: Examples of the orientation tensor in $\mathbb{R}^2$

2.2 Constitutive equations for fibre suspensions

2.2.1 Evolution equation for the orientation tensors [2, 3, 15, 40]

Substituting (2.8) into (2.25) and using (2.16) for $\dot{p}_i$ we obtain

$$\frac{DA_{ij}}{Dt} + (W_{ij}A_{jk} - A_{jk}W_{kj}) - \lambda(D_{ik}A_{kj} + A_{ik}D_{kj} - 2\mathcal{A}_{ijkl}D_{kl}) - D_r(\delta_{ij} - nA_{ij}) = 0, \tag{2.31}$$

$(n = 2 \; or \; 3)$

or

$$\frac{D\boldsymbol{A}}{Dt} + (\boldsymbol{AW} - \boldsymbol{WA}) - \lambda(\boldsymbol{DA} + \boldsymbol{AD} - 2\boldsymbol{\mathcal{A}D}) - D_r(\boldsymbol{I} - n\boldsymbol{A}) = 0, \tag{2.32}$$

where

$$\frac{D\boldsymbol{A}}{Dt} = \frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{A} \tag{2.33}$$

is the material derivative.

Equation (2.31) or (2.32) is known as the *"evolution equation"* for the orientation tensor $\boldsymbol{A}$.

2.2.2 Constitutive equation for stress [2, 3, 15, 40]

We will assume that the fluid is incompressible. The total stress $\boldsymbol{T}$ is then a modification of the constitutive equation for incompressible Newtonian fluids, and is given by

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu\boldsymbol{D} + \boldsymbol{E}, \tag{2.34}$$

where p is the pressure, μ is the solvent viscosity, and $\boldsymbol{D}$ is the deformation tensor.

The extra stress $\boldsymbol{E}$ is found by solving for the stress field around a single massless fibre [3, 15].

A variety of theories exist to predict the viscosity of suspensions of fibers in a Newtonian fluid. All lead to expressions of the form

$$\boldsymbol{E} = 2\mu h \left[K\mathcal{A}\boldsymbol{D} + B(\boldsymbol{DA} + \boldsymbol{AD}) + H\boldsymbol{D} + 2FD_r\boldsymbol{A}\right] \tag{2.35}$$

or, in component form,

$$E_{ij} = 2\mu h \left[K\mathrm{A}_{ijkl}D_{kl} + B(D_{ik}A_{kj} + A_{ik}D_{kj}) + HD_{ij} + 2FD_rA_{ij}\right], \tag{2.36}$$

where h is the particle volume fraction and K, B, H, and F are positive constants. The contribution of D_r to the stress is not significant, and this term is usually negleted [15, 40]. This will be the case henceforth.

Therefore we will express the stress in the form

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu_I\boldsymbol{D} + \boldsymbol{S} \tag{2.37}$$

where

$$\boldsymbol{S} = 2\mu_I\left[N_p\mathcal{A}\boldsymbol{D} + N_s(\boldsymbol{DA} + \boldsymbol{AD})\right] \tag{2.38}$$

and

$$\mu_I = \mu(1 + hH), \qquad N_p = \frac{hK}{1 + hH}, \qquad N_s = \frac{hB}{1 + hH}; \tag{2.39}$$

N_s and N_p are positive constants known as the particle and shear number, respectively.

2.3 Closure Approximations

We wish to develop a model in which the unknowns are $p, \boldsymbol{v}$ and $\boldsymbol{A}$. Unfortunately equations.(2.31),(2.32) and (2.38) contain the tensor $\mathcal{A}$. It is a feature of such evolution equations that the equation for an orientation tensor of a particular rank contains the tensor of the next even rank up [3, 15, 40].

In order to eliminate the tensor $\mathcal{A}$, a closure approximation is used, in which $\mathcal{A}$ is approximated by a function of the second-order tensor $\boldsymbol{A}$. There has been a great deal of work on closures [3, 6, 40]. We focus on four examples which are the subject of the work by ADVANI AND TUCKER [2], HINCH AND LEAL [18], and CINTRA AND TUCKER [6].
In the next section, we will review these closures rules.

2.3.1 Linear closure

The simplest closures are *linear* and *quadratic* approximations. The linear approximation $\mathcal{A}^L$, is defined for problems in $\mathbb{R}^3$ by

$$\begin{aligned} \mathcal{A}^L_{ijkl} &= \frac{1}{7}(A_{ij}\delta_{kl} + A_{ik}\delta_{jl} + A_{il}\delta_{jk} + A_{kl}\delta_{ij} + A_{jl}\delta_{ik} + A_{jk}\delta_{il}) \\ &+ -\frac{1}{35}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \end{aligned} \tag{2.40}$$

or, in tensor form,

$$\mathcal{A}^L\boldsymbol{D} = -\frac{1}{35}\left[(tr\boldsymbol{D})\boldsymbol{I} + 2\boldsymbol{D}\right] + \frac{1}{7}\left[(tr\boldsymbol{D})\boldsymbol{A} + 2\boldsymbol{A}\boldsymbol{D} + 2\boldsymbol{D}\boldsymbol{A} + (\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I}\right]. \tag{2.41}$$

For incompressible fluids ($tr\ \boldsymbol{D}$=0),

$$\mathcal{A}^L\boldsymbol{D} = -\frac{2}{35}\boldsymbol{D} + \frac{1}{7}\left[2\boldsymbol{A}\boldsymbol{D} + 2\boldsymbol{D}\boldsymbol{A} + (\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I}\right]. \tag{2.42}$$

For planar problems $\mathcal{A}^L$ takes the form [3, 40]

$$\begin{aligned} \mathcal{A}^L_{ijkl} &= \frac{1}{6}(A_{ij}\delta_{kl} + A_{ik}\delta_{jl} + A_{il}\delta_{jk} + A_{kl}\delta_{ij} + A_{jl}\delta_{ik} + A_{jk}\delta_{il}) \\ &+ -\frac{1}{24}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \end{aligned} \tag{2.43}$$

One notes that the linear approximation $\mathcal{A}^L$ satisfies the symmetry properties of (2.29). The linear approximation is exact for random distribution of fibres, for which [15]

$$\boldsymbol{A} = \begin{bmatrix} \frac{1}{n} & 0 & \cdots & 0 \\ 0 & \frac{1}{n} & \cdots & 0 \\ \cdots & & & \\ 0 & \cdots & 0 & \frac{1}{n} \end{bmatrix}$$

for a problem in $\mathbb{R}^n$.

2.3.2 Quadratic closure

The quadratic approximation $\mathcal{A}^Q$ is defined by

$$\mathcal{A}^Q = \boldsymbol{A} \otimes \boldsymbol{A}, \tag{2.44}$$

or

$$\mathcal{A}^Q_{ijkl} = A_{ij}A_{kl}. \tag{2.45}$$

This approximation is exact for fully aligned fibres; for example, in the case of fibres aligned parallel to the x_1-axis,

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

One notes that $\mathcal{A}^Q$ does not possess all the symmetries of $\mathcal{A}$ (2.29).
Each of these approximations is suitable for only a range of physical situations, and there are, of course, situations in which neither is a good approximations [3, 15]. We will later examine these two closure rules further, in the context of stability and existence considerations. It is important, though, to expand the set of closure rules being examined, in order to arrive at a broader picture of the range of approximations in use, and their relative merits. For this purpose we add to the linear and quadratic closure two more rules which have received some attention. These two rules, the first Hinch-Leal composite rule and the Cintra-Tucker orthotropic smooth closure, are good candidates for investigation since they are simple in structure, and lead to very good results in certain flow situations.

2.3.3 Hinch-Leal Closure

HINCH AND LEAL [18] have proposed a number of different closure approximations in their study of fibre suspensions exhibiting Brownian motion. One of these, now often referred to as the first Hinch-Leal composite rule, or H&L1, is incorporated into the present study. It has the advantage of being relatively simple, in that it is a quadratic polynomial in the components of $\boldsymbol{A}$. The H&L1 rule is defined by

$$\mathcal{A}\boldsymbol{D} = \frac{1}{5}\left[6\boldsymbol{ADA} + (\boldsymbol{A} : \boldsymbol{D})(2\boldsymbol{I} - \boldsymbol{A}) - 2(\boldsymbol{A}^2 : \boldsymbol{D})\boldsymbol{I}\right]. \qquad (2.46)$$

This closure rule gives accurate results for simple shear flows, and for smaller values of the interaction coefficient. It has the drawback, however, that it gives a nonphysical result in some flows (see Chapter 3).

2.3.4 Orthotropic closure

The final example comes from an extensive study of orthotropic closure rules due to CINTRA AND TUCKER [6]. They take as a point of departure the observation that the tensor $\mathcal{A}$ must be orthotropic, that its principal axes coincide with those of $\boldsymbol{A}$, and that it is a function of only two of the eigenvalues of $\boldsymbol{A}$, the last attribute following from (2.27). These authors develop two rules, referred to as a smooth closure, and a fitted closure. The former has a simple form based on linear interpolation of the eigenvalues of $\boldsymbol{A}$, while the fitted closure rule is based on numerical solutions for the probability density function. We confine attention to the orthotropic smooth closure. In order to describe this rule, it is most convenient to write the nonzero components of $\mathcal{A}$ as a symmetric 6×6 matrix $\overline{\mathcal{A}}$; this matrix has the form

$$\overline{\mathcal{A}} = \begin{pmatrix} \overline{\mathcal{A}}_{11} & \overline{\mathcal{A}}_{12} & \overline{\mathcal{A}}_{13} & 0 & 0 & 0 \\ & \overline{\mathcal{A}}_{22} & \overline{\mathcal{A}}_{23} & 0 & 0 & 0 \\ & & \overline{\mathcal{A}}_{33} & 0 & 0 & 0 \\ & & & \overline{\mathcal{A}}_{23} & 0 & 0 \\ & & \text{SYM} & & \overline{\mathcal{A}}_{13} & 0 \\ & & & & & \overline{\mathcal{A}}_{12} \end{pmatrix}, \tag{2.47}$$

and has six independent components, which are related to $\mathcal{A}_{ijkl}$ according to

$$\begin{aligned} \overline{\mathcal{A}}_{ii} &= \mathcal{A}_{iiii} \quad \text{for } i = 1,2,3 \text{ (no sum on } i), \\ \overline{\mathcal{A}}_{ij} &= \mathcal{A}_{iijj} \quad \text{for } i,j = 1,2,3, \ i \neq j, \text{ (no sum on } i,j). \end{aligned} \tag{2.48}$$

Other nonzero components of $\mathcal{A}$ are obtained by invoking the symmetry properties of this tensor. The elementary identity (2.30) $\mathcal{A}_{ijkk} = A_{ij}$ leads to a further reduction of the number of independent components to three, through the set of equations

$$\begin{aligned} \overline{\mathcal{A}}_{11} + \overline{\mathcal{A}}_{12} + \overline{\mathcal{A}}_{13} &= A_1, \\ \overline{\mathcal{A}}_{12} + \overline{\mathcal{A}}_{22} + \overline{\mathcal{A}}_{23} &= A_2, \\ \overline{\mathcal{A}}_{13} + \overline{\mathcal{A}}_{23} + \overline{\mathcal{A}}_{13} &= A_3, \end{aligned} \tag{2.49}$$

where $(A_1,\ A_2,\ A_3)$ are the eigenvalues of $\boldsymbol{A}$.

The closure rule is then defined by expressing $\overline{\mathcal{A}}_{ii}$ $(i = 1,2,3)$ as a function of two of the eigenvalues A_i (using the fact that $\sum_i A_i = 1$). For the orthotropic smooth closure this expression takes the form

$$\begin{pmatrix} \overline{\mathcal{A}}_{11} \\ \overline{\mathcal{A}}_{22} \\ \overline{\mathcal{A}}_{33} \end{pmatrix} = \begin{pmatrix} -0.15 + 1.15A_1 - 0.10A_2 \\ -0.15 + 0.15A_1 + 0.90A_2 \\ 0.60 - 0.60A_1 - 0.60A_2 \end{pmatrix}, \tag{2.50}$$

and a simple computation, based on (2.49) and (2.50), leads to the expression

$$\begin{pmatrix} \overline{\mathcal{A}}_{12} \\ \overline{\mathcal{A}}_{13} \\ \overline{\mathcal{A}}_{23} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -0.1 + 0.1A_1 + 0.6A_2 \\ 0.4 - 0.4A_1 - 0.40A_2 \\ 0.4 - 0.4A_1 - 0.40A_2 \end{pmatrix} \tag{2.51}$$

for the remaining independent components of $\overline{\mathcal{A}}$ in terms of the eigenvalues of $\boldsymbol{A}$.

2.4 The initial boundary value problem

Now we are in position to state the problem. We assume that the fluid occupies a bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 2or3$) with boundary Γ, and is subjected to the action of a body force $\boldsymbol{b}$ per unit mass. The fluid has mass density ρ. It is required to find the velocity field $\boldsymbol{v}(\boldsymbol{x},\mathrm{t})$, the pressure $p(\boldsymbol{x}, t)$, and the orientation tensor field $\boldsymbol{A}(\boldsymbol{x},\mathrm{t})$ which satisfy the following set of equations:

1. conservation of momentum
$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \mathrm{div}\boldsymbol{T} = \rho\boldsymbol{b}; \tag{2.52}$$

2. conservation of mass (continuity equation)
$$\mathrm{div}\boldsymbol{v} = 0; \tag{2.53}$$

3. constitutive equation for stress
$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu_I[\boldsymbol{D} + N_p\mathcal{A}\boldsymbol{D} + N_S(\boldsymbol{A}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A})]; \tag{2.54}$$

and

4. the evolution equation for the orientation tensor

$$\frac{D\boldsymbol{A}}{Dt} = (\boldsymbol{W}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{W}) + \lambda(\boldsymbol{D}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{D} - 2\mathcal{A}\boldsymbol{D}) + D_r(\boldsymbol{I} - n\boldsymbol{A}). \qquad (2.55)$$

These equations are supplemented by the boundary condition

$$\boldsymbol{v} = \boldsymbol{0} \;\; on \;\; \Gamma \qquad (2.56)$$

and the inital conditions

$$\boldsymbol{v}(\boldsymbol{x},0) = \boldsymbol{v}_0, \quad \boldsymbol{A}(\boldsymbol{x},0) = \boldsymbol{A}_0 \text{ on } \;\Gamma. \qquad (2.57)$$

In (2.54) and (2.55) it is assumed that one of the closure approximations will be used to express $\mathcal{A}$ in terms of $\boldsymbol{A}$.

Chapter 3

ASPECTS OF STABILITY

This Chapter examines the circumstances under which the constitutive equations for fibre suspension flows are consistent with the second law of thermodynamics, and the conditions under which fibre suspension flows are stable, in the energetic sense. The study is based on the formalism of COLEMAN AND GURTIN [8], in which the second law, in the form of the Clausuis- Duheim inequality, is used to determine restrictions on the constitutive equations. These issues are examined in the context of the closure approximations reviewed in Chapter 2, viz: the linear and quadratic closures, the HINCH–LEAL rule, and the smooth orthotropic closure rule of CINTRA AND TUCKER. It is shown that, with the use of the linear closure approximation, the constitutive equations are consistent with the second law, and the flows are monotonically stable, if the particle number does not exceed 35/2. The quadratic closure is consistent, and stable. It is not possible to determine the stability or otherwise of the HINCH-LEAL closure for arbitrary flows, though for biaxial elongation, a case which is known to lead to non-physical results, the closure rule is consistent with the second law of thermodynamics. The smooth orthotropic rule of CINTRA AND TUCKER is shown not to be consistent with the second law for arbitrary flows.

In the present context, the second law will provide a mean of determining restrictions only on the constitutive equation for the stress, and not on the evolution equation.

3.1 Thermodynamic restrictions on the constitutive equations

3.1.1 Preliminary Notions

We assume that the fibre suspension occupies a bounded domain $\Omega \subset \mathbb{R}^n$, (n=2 or 3), with boundary Γ. We assume also that the suspension is mechanically isolated; that is [11],

$$\int_{\Gamma} \boldsymbol{v} \cdot \boldsymbol{T}\boldsymbol{n}\; da + \int_{\Omega} \rho \boldsymbol{v} \cdot \boldsymbol{b}\; dv = 0. \tag{3.1}$$

We suppose, as in [8], that there is no diffusion of mass in the suspension, but that it may deform. We also assume isothermal conditions, so that the purely mechanical theory is relevant.
A mechanical process is described by five functions, defined at $(\boldsymbol{X}, t) \in \Omega \times \mathbb{R}$, where $\boldsymbol{X}$ is the reference position [8, 11]:

1. the motion $\boldsymbol{x} = \phi(\boldsymbol{X}, t)$.

2. the symmetric Cauchy stress tensor $\boldsymbol{T}$=$\boldsymbol{T}(\boldsymbol{X}$,t)

3. the specific body force $\boldsymbol{b}$=$\boldsymbol{b}(\boldsymbol{X}$,t) per unit mass (exerted on the fluid at $\boldsymbol{X}$ by the external world)

4. the specific internal energy $\epsilon = \epsilon(\boldsymbol{X}, t)$ per unit mass

5. the internal state variable (orientation tensor) $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{X}, t)$.

This set of five functions will be called a mechanical process if and only if [8, 11], it is compatible with the law of balance of linear momemtum, (3.2), the law of balance of energy (3.3), and the constitutive equations are consistent with (3.4).

Under sufficient smoothness assumptions and since $\boldsymbol{T}$ is symmetric, these laws are equivalent to the equations [8, 11, 25]

$$\begin{aligned} \rho\ddot{\boldsymbol{x}} - \mathrm{div}\boldsymbol{T} &= \rho\boldsymbol{b} \\ \text{or} & \\ \rho\dot{\boldsymbol{v}} - \mathrm{div}\boldsymbol{T} &= \rho\boldsymbol{b}, \end{aligned} \tag{3.2}$$

and

$$\rho\dot{\epsilon} - \boldsymbol{T} : \boldsymbol{L} = 0, \tag{3.3}$$

where ρ is the mass density and

$$\boldsymbol{L} = \nabla \boldsymbol{v}$$

is the velocity gradient.

3.1.2 The dissipation inequality

A mechanical process is said to be admissible if it is compatible with the above constitutive assumptions. Since we are dealing with a purely mechanical theory, the second law of thermodynamics in this context is the requirement that the energy increase at a rate not exceeding the power expended [9]. If the free energy is denoted by Ψ, then the local form of the second law is the dissipation inequality

$$\rho\dot{\Psi} \leq \boldsymbol{T} : \boldsymbol{D}. \tag{3.4}$$

We will use the methods of COLEMAN, GURTIN AND NOLL [8, 9] to investigate which restrictions, if any, are placed on the constitutive equations by the dissipation inequality. This approach has been used with great effectiveness by DUNN AND FOSDICK [11] in determining the restrictions that apply to the constants appearing in the constitutive equations for fluids of second grade.

The motion of the fluid is described by the function $\boldsymbol{x} = \varphi(\boldsymbol{X}, t)$, in which $\boldsymbol{x}$ and $\boldsymbol{X}$ denote reference and current position, respectively in the fluid. We assume that $\varphi(\boldsymbol{X}, t)$ is smoothly invertible in its arguments, and define the deformation gradient $\boldsymbol{F}$ by

$$\boldsymbol{F} = \nabla_{\mathbf{X}} \varphi. \tag{3.5}$$

If the inverse $\boldsymbol{F}^{-1}$ of $\boldsymbol{F}$ exists, then

$$\boldsymbol{L} = \dot{\boldsymbol{F}} \boldsymbol{F}^{-1}. \tag{3.6}$$

For any sufficiently smooth functions $\boldsymbol{F}$ and $\boldsymbol{A}$,

$$\dot{\Psi}(\boldsymbol{F}, \boldsymbol{A}) = \Psi_F : \dot{\boldsymbol{F}} + \Psi_A : \dot{\boldsymbol{A}}. \tag{3.7}$$

Here Ψ_F and Ψ_A are the derivatives of Ψ with respect to $\boldsymbol{F}$ and $\boldsymbol{A}$.

We now substitute (2.37), (2.38), (3.4) and (3.7) in the dissipation inequality (3.4), and make use also of the incompressibility condition $\operatorname{tr} \boldsymbol{D} = 0$, to obtain

$$\begin{aligned} &\rho \Psi_F \boldsymbol{F}^T : \boldsymbol{L} + \rho \Psi_A : [\boldsymbol{A}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A} + \lambda(\boldsymbol{A}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A} - 2\mathcal{A}\boldsymbol{D}) \\ &\quad + C_I |\boldsymbol{D}|(\boldsymbol{I} - n\boldsymbol{A})] - 2\mu_I[\boldsymbol{D} + N_p \mathcal{A}\boldsymbol{D} + N_s(\boldsymbol{A}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A})] : \boldsymbol{D} \leq 0. \end{aligned} \tag{3.8}$$

Since (3.8) must hold for all incompressible flows, we may replace $\boldsymbol{L}$ by $\alpha \boldsymbol{L}$, where α is any real number: this gives the inequality

$$\alpha[K - (\operatorname{sgn} \alpha) B] - \alpha^2 C \leq 0, \tag{3.9}$$

in which

$$\rho^{-1}K = \Psi_F \boldsymbol{F}^T : \boldsymbol{L} + \Psi_A : [\boldsymbol{AW} - \boldsymbol{WA} + \lambda(\boldsymbol{AD} + \boldsymbol{DA} - 2\mathcal{A}\boldsymbol{D})], \tag{3.10}$$

$$B = n\rho C_I |\boldsymbol{D}| \Psi_A : (\boldsymbol{I} - n\boldsymbol{A}), \tag{3.11}$$

$$C = \boldsymbol{T} : \boldsymbol{D} = 2\mu_I[\boldsymbol{D} + N_p \mathcal{A}\boldsymbol{D} + N_s(\boldsymbol{AD} + \boldsymbol{DA})] : \boldsymbol{D}. \tag{3.12}$$

Lemma 3.1 *Satisfaction of the reduced dissipation inequality for all incompressible flows consistent with equations (2.37)-(2.38) and (2.32), together with the appropriate closure rules, implies that*

$$\begin{aligned} C &\geq 0, \\ B &\geq 0, \\ K &\leq |B|. \end{aligned} \tag{3.13}$$

PROOF.
Suppose first that $\alpha > 0$; then (3.9) can be written in the form

$$(K - B) - \alpha C \leq 0. \tag{3.14}$$

Since α can be chosen arbitrarily small, we must have

$$K - B \leq 0. \tag{3.15}$$

This in turn implies that

$$C \geq 0, \tag{3.16}$$

for otherwise it would be possible to choose α sufficiently large to violate the inequality (3.14).

Suppose next that $\alpha < 0$. Then (3.9) becomes

$$K + B + |\alpha| C \geq 0. \tag{3.17}$$

From this inequality we deduce that

$$K + B \geq 0, \quad \text{and} \quad C \geq 0. \tag{3.18}$$

The inequality $(3.13)_1$ now follows automatically from (3.16) and (3.18), while $(3.13)_2$ may be deduced from (3.15) and $(3.18)_1$. These two inequalities also yield $(3.13)_3$.□

We see from (3.13) and (3.10)–(3.12) that while $(3.13)_1$ and $(3.13)_2$ require a knowledge of the free energy function in order that they may be used to determine restrictions on the constitutive equations, $(3.13)_3$ provides a direct means of determining such restrictions, if any exist. We see also that it is only the constitutive equation for the *stress* that plays a role in the definition of C, so that the second law in the present context will provide a means of determining restrictions only on the stress, and not on the evolution equation (2.32) for $\boldsymbol{A}$. We now examine the restriction $C \geq 0$ for the closures introduced in Chapter 2.

3.1.3 Linear closure

Substitution of (2.42) in (3.12) yields

$$7\left(1 - \frac{2}{35}N_p\right)\boldsymbol{D}:\boldsymbol{D} + N(\boldsymbol{AD} + \boldsymbol{DA}):\boldsymbol{D} \geq 0, \tag{3.19}$$

where

$$N = 2N_p + 7N_s.$$

Theorem 3.1 *The constitutive equation (2.32) with the linear closure approximation (2.42) is compatible with the second law of thermodynamics if*

$$N_p \leq \frac{35}{2}. \tag{3.20}$$

PROOF.
First we note that

$$(\boldsymbol{AD}+\boldsymbol{DA}):\boldsymbol{D}=2\boldsymbol{A}:\boldsymbol{D}^2\geq 0.$$

The last inequality follows from the fact that, if we express $\boldsymbol{A}$ and $\boldsymbol{D}$ in component form relative to the *principal* basis of $\boldsymbol{A}$, then $\boldsymbol{A}:\boldsymbol{D}^2=\sum_{i=1}^{n}A_{ii}D_{ii}^2$. From the definition of $\boldsymbol{A}$ all the terms in this sum are nonnegative.

Thus the inequality (3.19) can be written in the form

$$\left(1-\frac{2N_p}{35}\right)+\frac{2N(\boldsymbol{A}:\boldsymbol{D}^2)}{7|\boldsymbol{D}|^2}\geq 0. \tag{3.21}$$

For $N_p\leq 35/2$ this inequality holds for all flows. □

It is worth observing here that the restriction (3.20) is one that was obtained also by GALDI AND REDDY [15], who showed that the rest state is unstable, in the sense of Liapounov, for particles numbers exceeding the value of 35/2.

3.1.4 Quadratic closure

A repetition of the computations just carried out, this time using the closure rule (2.45), yields

$$|\boldsymbol{D}|^2+N_p(\boldsymbol{A}:\boldsymbol{D})^2+2N_s\boldsymbol{A}:\boldsymbol{D}^2\geq 0. \tag{3.22}$$

This inequality is valid for any flow field and orientation tensor field, and so the second law imposes no restrictions on the equation for the stress.

3.1.5 The Hinch-Leal (H&L1) closure

We return to (3.12) and consider in particular the term $\boldsymbol{AD}:\boldsymbol{D}$. Upon substituting (2.46) in this expression, we find that

$$5\boldsymbol{AD}:\boldsymbol{D}=6\,\mathrm{tr}\,[(\boldsymbol{AD})^2]-(\boldsymbol{A}:\boldsymbol{D})^2. \tag{3.23}$$

The first term is of indeterminate sign, while the second term is negative, and thus it is not possible to determine restrictions for all possible flows.

It is useful, though, to consider the consequences of one particular flow, that of biaxial elongation, for which the evolution equation together with the H&L1 closure is known to give unphysical results [3]. This flow has the form

$$v_1 = \epsilon x_1, \quad v_2 = \epsilon x_2, \quad v_3 = -2\epsilon x_3 \tag{3.24}$$

relative to a set of cartesian coordinates, and the deformation rate is

$$\boldsymbol{D} = \epsilon \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{3.25}$$

A straightforward, if somewhat tedious calculation reveals that, for this flow,

$$\mathrm{tr}\,(\boldsymbol{AD})^2 = \epsilon^2[(A_{11}^2 + 2A_{12}^2 + A_{22}^2) - 4(A_{13}^2 + A_{23}^2 - A_{33}^2)]. \tag{3.26}$$

Now for biaxial elongation flow, we have $A_{22} = A_{11} = A$, say, and so $A_{33} = 1 - 2A$ [3]. In addition, all off-diagonal components are zero. It follows from (3.23) and (3.26) that

$$5\mathcal{A}\boldsymbol{D} : \boldsymbol{D} = 4\epsilon^2[18A^2 - 18A + 5], \tag{3.27}$$

which is easily shown to be nonnegative for $A \leq 1$. The remaining two terms in the expression for C are also nonnegative, and so we see that, while the evolution equation with the HINCH-LEAL law leads to non-physical results for the case of biaxial elongation, as reported in [3], the constitutive equation for the stress is consistent with the second law.

3.1.6 The Smooth Orthotropic Closure Rule

We turn now to the rule defined by (2.50) and (2.51), and substitute these values in (3.12). We focus attention first on the coefficient of N_p, and note that if we arrange the components of $\boldsymbol{D}$ in vector form according to

$$\boldsymbol{d} = [D_{11}\ D_{22}\ D_{33}\ 2D_{23}\ 2D_{13}\ 2D_{12}]^T,$$

then we may write

$$\mathcal{A}\boldsymbol{D} : \boldsymbol{D} = \boldsymbol{d}^T \overline{\mathcal{A}} \boldsymbol{d},$$

where $\overline{\mathcal{A}}$ is defined by (2.47)–(2.51). Now, in order that C be nonnegative for all flows and for all positive values of N_p and N_s, it is necessary that the matrix $\overline{\mathcal{A}}$ be positive semidefinite. We may express $\overline{\mathcal{A}}$ in the form

$$\overline{\mathcal{A}} = \Gamma_0 + \Gamma_1 A_1 + \Gamma_2 A_2, \tag{3.28}$$

in which, from (2.50) and (2.51),

$$\Gamma_0 = \begin{pmatrix} -0.15 & 0.05 & 0.20 \\ 0.05 & -0.15 & 0.20 \\ 0.20 & 0.20 & 0.60 \end{pmatrix}, \Gamma_1 = \begin{pmatrix} 1.15 & 0.05 & -0.20 \\ 0.05 & 0.15 & -0.20 \\ -0.20 & -0.20 & 0.60 \end{pmatrix},$$

and

$$\Gamma_2 = \begin{pmatrix} -0.10 & 0.30 & -0.20 \\ 0.30 & 0.90 & -0.20 \\ 0.30 & -0.20 & -0.60 \end{pmatrix}.$$

Thus, in order that C be nonnegative for all values of A_1 and A_2 in their ranges of physical validity, each of the matrices Γ_i $(i = 0, 1, 2)$ must be positive semidefinite. In other words, the second law implies, under these general conditions, that all of the eigenvalues of the matrices Γ_i are nonnegative.

A routine calculation reveals, however, that the sets $\boldsymbol{\lambda}_i$ of eigenvalues of Γ_i are given by

$$\boldsymbol{\lambda}_0 = [-0.2\ \ 0.2\ \ 0.7],\ \ \boldsymbol{\lambda}_1 = [0.19\ \ 1.18\ \ -0.67],\ \ \boldsymbol{\lambda}_2 = [-0.14\ \ -0.68\ \ 1.02],$$

which contradicts the necessary condition. Thus, while there may well exist admissible flows for which the smooth orthotropic closure is consistent with the second law, it cannot be said to be consistent for all such flows.

3.2 Energetic stability

We begin by establishing an identity involving the kinetic energy of the fluid. We take the scalar product of equation (2.52) with $\boldsymbol{v}$, integrate over Ω and integrate by parts, to obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega}\rho|\boldsymbol{v}|^2\,dx+\int_{\Omega}\boldsymbol{T}:\boldsymbol{L}\,dx=\int_{\Gamma}\boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{n}\,da+\int_{\Omega}\rho\boldsymbol{v}\cdot\boldsymbol{b}\,dx \tag{3.29}$$

where $\boldsymbol{n}$ is the unit outward normal vector to Γ.

We assume that the body is mechanically isolated [9], so that

$$\int_{\Gamma}\boldsymbol{v}\cdot\boldsymbol{T}\boldsymbol{n}\,da+\int_{\Omega}\rho\boldsymbol{v}\cdot\boldsymbol{b}\,dx=0. \tag{3.30}$$

The kinetic energy E of the fluid is defined by

$$E(t)=\frac{1}{2}\int_{\Omega}\rho|\boldsymbol{v}|^2d\boldsymbol{x},$$

and so we have, using (3.12),

$$\begin{aligned}E'(t)&=-\int_{\Omega}\boldsymbol{T}:\boldsymbol{L}\,dx\\&=-\int_{\Omega}C\,dx.\end{aligned} \tag{3.31}$$

We will be interested first in determining the conditions under which fibre suspension flows are monotonically stable, for various closure approximations, in the sense that

$$E'(t)\leq 0\quad\text{for all } t. \tag{3.32}$$

The energetic stability of the suspension, which is a *global* condition, is seen from (3.31) to be intimately connected to the *local* consistency condition $C\geq 0$.

Substitution of (2.37)-(2.38) in (3.31) and use of the incompressibility condition leads to the expression

$$E'(t)=-2\mu_I\|\boldsymbol{D}\|^2-2\mu_I\int_{\Omega}\left(N_s\boldsymbol{A}:\boldsymbol{D}^2+N_p\boldsymbol{A}\boldsymbol{D}:\boldsymbol{D}\right)\,dx; \tag{3.33}$$

here and henceforth we use the notation

$$\|D\| = \left[\int_\Omega D : D \, dx\right]^{1/2}$$

to denote the L^2 norm.

3.2.1 The linear closure approximation

Substitution of (2.42) in (3.33) leads to the expression

$$E'(t) = -\frac{4\mu_I N}{7}\int_\Omega A : D^2 \, dx - 2\mu_I\left(1 - \frac{2}{35}N_p\right)\|D\|^2 \tag{3.34}$$

for the rate of change of kinetic energy. For all $t > 0$, and for all admissible fields v and A, we see from (3.34) and from the positivity of $A : D^2$ that (3.32) holds in the event that $N_p \leq 35/2$. Thus approximations with the linear closure are monotonically stable if $N_p \leq 35/2$.

We can go one step further, by recalling that, from Korn's inequality (see Appendix A.3.3 and the Poincaré Friedrichs inequality (see, for example, [33], and also Appendix A.3.2) there exists a constant $c_1 > 0$ such that

$$\|D\|^2 \geq c_1 E(t); \tag{3.35}$$

thus, from (3.34) we have

$$\begin{aligned} E'(t) + \lambda E(t) &\leq E'(t) + \frac{2}{\rho}\mu_I\left(1 - \frac{2}{35}N_p\right)\|D\|^2 \\ &= -\frac{4}{7}\frac{\mu_I}{\rho}N\int_\Omega A : D^2 \, dx \\ &\leq 0, \end{aligned} \tag{3.36}$$

where $\lambda = \dfrac{2c_1\mu_I}{\rho}\left(1 - \dfrac{2N_p}{35}\right)$. It follows, by integration of (3.36), that

$$E(t) \leq E(0)e^{-\lambda t} \tag{3.37}$$

for all t. In other words, flows approximated by the linear closure are *exponentially stable* if $N_p \leq 35/2$.

3.2.2 The quadratic closure

For this closure, (3.33) takes the form

$$E'(t) + 2\mu_I\|D\|^2 = -2\mu_I \left(N_p \int_\Omega (A : D)^2 \, dx + 2N_s \int_\Omega A : D^2 \, dx\right) \leq 0. \quad (3.38)$$

We immediately deduce the monotonic stability from (3.38), and by application of the inequalities of Korn and Poincaré-Friedrichs as before, find that flows are in addition exponentially stable.

For the Hinch-Leal and smooth orthotropic closure rules, the sign-indeterminacy of $\mathcal{A}D : D$ in (3.33) does not permit any conclusions to be drawn about the stability, or otherwise, of flows approximated by these closures. We summarise the results for linear and quadratic closures in the following theorem.

Theorem 3.2 *(a) If $N_p \leq 35/2$, then fibre suspension flows corresponding to the linear closure approximation are monotonically stable, in the sense that*

$$E'(t) \leq 0. \quad (3.39)$$

Furthermore, such flows are exponentially stable, in the sense that there exists a positive constant λ such that

$$E(t) \leq E(0)e^{-\lambda t} \quad (3.40)$$

for all t.

(b) Fibre suspensions flows corresponding to the quadratic closure approximation are monotonically and exponentially stable.

Chapter 4

Local existence of solutions

As stated in Chapter 2, the case $D_r = \text{constant}$ is an excellent model for very small fibres, which experience rotary Brownian motion. Many studies have treated this case over the years [40]. In this chapter we show, for the cases of the linear and quadratic closures, that the problem (2.52)-(2.57) admits a unique solution locally in time, for the case of constant D_r. Galdi and Reddy [15] have established the existence of a unique solution locally in time for the case in which D_r is given by (2.18), and for the quadratic closure.

The methods used in this Chapter and in the next are based on the work of Guillopé and Saut [16], and also draw on the work of Galdi and Reddy [15].

4.1 Linear closure approximation

4.1.1 Dimensionless and traceless problem

We set

$$\overline{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \quad \overline{\boldsymbol{v}} = \frac{\boldsymbol{v}}{V} \quad \overline{t} = \frac{tV}{L}, \quad \overline{p} = \frac{pL}{\mu V}$$

$$\overline{\boldsymbol{b}} = \rho\frac{\boldsymbol{b}L^2}{\mu V}, \quad Re = \frac{\rho V L}{\mu}, \quad We = \frac{\lambda V}{n D_r L}, \quad \overline{\boldsymbol{A}} = \frac{\boldsymbol{A}L}{V} \tag{4.1}$$

where Re, We are the Reynolds and the Weissenberg numbers respectively, and V and L represent a typical velocity and a typical length of the flow.
It is useful to carry out the following additive decompostion of $\boldsymbol{A}$: we set

$$\boldsymbol{A} = \hat{\boldsymbol{A}} + \boldsymbol{A}^* \tag{4.2}$$

where $\boldsymbol{A}^*$ is a diagonal tensor with trace 1, which implies that $\hat{\boldsymbol{A}}$ is traceless. For convenience, and without any loss in generality, we choose

$$\boldsymbol{A}^* = \frac{1}{n}\boldsymbol{I} \tag{4.3}$$

for a problem in $\mathbb{R}^n$.
We next make use of (4.1)- (4.3) in equation (2.52) and (2.55), and use (2.42) for $\mathcal{A}$, to obtain

$$\left.\begin{aligned} Re(\boldsymbol{v}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}) + \nabla p - \gamma(1 - \tfrac{2}{35}N_p)\Delta\boldsymbol{v} &= \boldsymbol{b} + \mathrm{div}\boldsymbol{S}_L \\ \boldsymbol{S}_L = \tfrac{1}{7}\gamma\left[N(\boldsymbol{AD} + \boldsymbol{DA}) + N_p(\boldsymbol{A} : \boldsymbol{D})\boldsymbol{I}\right] & \end{aligned}\right\} \tag{4.4}$$

and

$$\begin{aligned} \boldsymbol{A} + We\left\{\boldsymbol{A}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + \boldsymbol{AW} - \boldsymbol{WA} - \tfrac{3\lambda}{7}(\boldsymbol{AD} + \boldsymbol{DA})\right\} & \\ = 2\omega\boldsymbol{D} + \tfrac{2\lambda}{7}We(\boldsymbol{A} : \boldsymbol{D})\boldsymbol{I} & \end{aligned} \tag{4.5}$$

For convenience we have denoted $\hat{\boldsymbol{A}}$ by $\boldsymbol{A}$. Here

$$\gamma = \frac{2\mu_I}{\mu}, \qquad N = 2N_p + 7N_s$$

and

$$\omega = \frac{13\lambda}{35 n D_r}. \tag{4.6}$$

Remark.
Equation (4.5) has the same lefthand side as (1.13) in [16], while the second term on the righthand side of (4.5) is an additionnal one.

We will need the following special spaces:

$$\left.\begin{aligned} &\mathbb{H} = \{\boldsymbol{v} : v_i \in L^2(\Omega),\ \ \operatorname{div}\boldsymbol{v} = 0\ \text{ in }\Omega,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0\ \text{ on }\ \Gamma\},\\ &\mathbb{V} = \{\boldsymbol{v} : v_i \in H_0^1(\Omega),\ \ \operatorname{div}\boldsymbol{v} = 0\ \text{ in }\Omega\},\\ &\mathbb{X} = \{\boldsymbol{A} : A_{ij} \in L^2(\Omega),\ \ A_{ij} = A_{ji},\ A_{ii} = 0\ \text{ a.e. in }\Omega\}.\end{aligned}\right\} \tag{4.7}$$

The orthogonal projection of $\mathbb{L}^2(\Omega)$ onto $\mathbb{H}$ is denoted by P, and we define the Stokes operator $\mathcal{L}$ by

$$\mathcal{L}\boldsymbol{v} = -P\Delta\boldsymbol{v}. \tag{4.8}$$

$\mathcal{L}$ has the domain $D(\mathcal{L})=\mathbb{V}\cap\mathbb{H}^2(\Omega)$, and is equipped with the norm $\|\boldsymbol{v}\|_{D(\mathcal{L})} = \|\mathcal{L}\boldsymbol{v}\|_{\mathbb{L}^2}$, which is equivalent to the natural $\mathbb{H}^2$-norm. We introduce the bilinear mapping

$$b(\boldsymbol{v},\boldsymbol{w}) = P(\boldsymbol{v}\cdot\nabla)\boldsymbol{w}. \tag{4.9}$$

In the next section, we implement a fixed point argument, using Schauder's Fixed Point Theorem, to show the existence of a regular solution on a small time interval $(0,T^*)$. This solution satisfies an energy inequality, which implies its uniqueness in that class.

4.1.2 Linearised problems

We study two linearised problems, one for the velocity $\boldsymbol{v}$, and the other for $\boldsymbol{A}$. We first recall, without proof, some well known results for the time-dependent Stokes problem

$$\left.\begin{aligned} &Re\boldsymbol{v}' + \delta\mathcal{L}\boldsymbol{v} = \boldsymbol{F} \\ &\operatorname{div}\boldsymbol{v} = 0 \\ &\boldsymbol{v}(0) = \boldsymbol{v}_0 \end{aligned}\right\} \tag{4.10}$$

where $\boldsymbol{F}$ is a given body force.

Lemma 4.1 *Let* $\delta = \gamma(1 - \frac{2}{35}N_p)$. *Assume that* $\partial\Omega = \Gamma \in C^2$, $\boldsymbol{v}_0 \in \mathbb{V}$ *and* $\boldsymbol{F} \in \mathbb{L}^2(\Omega_T)$. *If* $\delta > 0$, *then the Stokes problem (4.10) admits an unique solution* $\boldsymbol{v} \in \mathbb{L}^2(0,T;D(\mathcal{L})) \cap C([0,T],\mathbb{V})$ *such that* $\boldsymbol{v}' \in \mathbb{L}^2(\Omega_T)$ *and* $p \in L^2(0,T;H^1)$. *Futhermore, there exists a constant* $C_1(Re,\gamma,N_p,\Omega)$ *such that*

$$\begin{aligned} \|\boldsymbol{v}\|^2_{\mathbb{L}^2(0,T;D(\mathcal{L}))\cap\mathbb{L}^\infty(0,T;\mathbb{V})} + \|\boldsymbol{v}'\|^2_{\mathbb{L}^2(\Omega_T)} + \|p\|^2_{\mathbb{L}^2(0,T;H^1)} \\ \leq C_1(\|\boldsymbol{v}_0\|^2_{\mathbb{L}^2(\Omega_T)} + \|\boldsymbol{F}\|^2_{\mathbb{L}^2(\Omega_T)}). \end{aligned} \tag{4.11}$$

Lemma 4.2 *Assume that* $\Gamma \in C^3$, $\boldsymbol{F}' \in \mathbb{L}^2(0,T,\mathbb{H}^{-1})$, $\boldsymbol{v}_0 \in D(\mathcal{L})$. *If* $\delta \geq 0$, *then the unique solution of problem (4.10) satisfies*

$$\begin{aligned} &\boldsymbol{v} \in \mathbb{L}^2(0,T;\mathbb{H}^3) \cap C([0,T];D(\mathcal{L})), \\ &\boldsymbol{v}' \in \mathbb{L}^2(0,T;V) \cap C([0,T];\mathbb{H}), \\ &p \in L^2(0,T;H^2), \end{aligned}$$

and there exists a constant $C_2(Re,\gamma,N_p,N_s,\Omega)$ *such that*

$$\begin{aligned} &\|\boldsymbol{v}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)\cap\mathbb{L}^\infty(0,T;D(\mathcal{L}))} + \|\boldsymbol{v}'\|^2_{\mathbb{L}^2(0,T;\mathbb{V})\cap\mathbb{L}^\infty(0,T;\mathbb{H})} + \|p\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^2)} \\ &\leq C_2\left\{|\mathcal{L}\boldsymbol{v}_0|^2 + \|\boldsymbol{F}\|^2_{\mathbb{L}^1(0,T;\mathbb{H}^1)} + \|\boldsymbol{F}'\|^2_{\mathbb{L}^1(0,T;\mathbb{H}^{-1})} + |\boldsymbol{F}(0)|^2\right\}. \end{aligned} \tag{4.12}$$

Now, we turn to the study of a linearised problem associated with the equation (4.5) for $\boldsymbol{A}$. For a given velocity field $\overline{\boldsymbol{v}}$,

we first show the existence and the uniqueness of a regular solution $\boldsymbol{A}$ to the problem

$$\left.\begin{aligned} We\{\boldsymbol{A}' + (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{A} + \boldsymbol{A}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A} - \tfrac{3\lambda}{7}(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A})\} + \boldsymbol{A} \\ = 2\omega\overline{\boldsymbol{D}} - \tfrac{2\lambda}{7}We(\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I} \\ \boldsymbol{A}(0) = \boldsymbol{A}_0 \qquad a.e \quad in \quad \Omega \end{aligned}\right\} \tag{4.13}$$

where

$$\overline{\boldsymbol{D}} = \frac{1}{2}\left(\nabla\overline{\boldsymbol{v}} + (\nabla\overline{\boldsymbol{v}})^T\right) \quad \text{and} \quad \overline{\boldsymbol{W}} = \frac{1}{2}\left(\nabla\overline{\boldsymbol{v}} - (\nabla\overline{\boldsymbol{v}})^T\right)$$

Lemma 4.3 *Assume that* $\Gamma \in C^1$, $\overline{\boldsymbol{v}} \in \mathbb{L}^1(0,T;\mathbb{H}^3) \cap D(\mathcal{L})$, $\boldsymbol{A}_0 \in \mathbb{H}^2(\Omega)$. *Then the problem (4.13) admits a unique solution* $\boldsymbol{A} \in C([0,T],\mathbb{H}^2)$. *Futhermore, there exists a constant* $C(\Omega,\omega,We)$ *such that*

$$\|\boldsymbol{A}\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^2)} \leq \left(\|\boldsymbol{A}_0\|_{\mathbb{H}^2} + \frac{2\omega}{We}\right)\exp(C\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}). \tag{4.14}$$

In addition, if $\overline{\boldsymbol{v}} \in C([0,T],D(\mathcal{L}))$, *then* $\boldsymbol{A}' \in C([0,T],\mathbb{H}^1)$ *and satisfies*

$$\begin{aligned} &\|\boldsymbol{A}'\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} \leq \\ &C\left(\|\overline{\boldsymbol{v}}_{\mathbb{L}^1(0,T;\mathbb{H}^3)} + \frac{1}{We}\right)\left(\|\boldsymbol{A}_0\|_{\mathbb{L}^1(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right)\exp(C\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}). \end{aligned} \tag{4.15}$$

Proof

The existence of a unique solution follows by the method of characteristics [23], (see also GUILLOPÉ AND SAUT [16]).

Estimate (4.14),

Taking the inner product of (4.13) with $\boldsymbol{A}$, integrating over Ω and using the property

$$((\boldsymbol{v}\cdot\nabla)\boldsymbol{A},\boldsymbol{A})_{\mathbb{L}^2}=0$$

we arrive at the identity

$$\frac{1}{2}\frac{d}{dt}\left(\|\boldsymbol{A}\|_{L^2}^2\right)+\|\boldsymbol{A}\|_{L^2}^2=$$
$$\int_\Omega\left[We\left\{\overline{\boldsymbol{W}}\boldsymbol{A}-\boldsymbol{A}\overline{\boldsymbol{W}}+\frac{3\lambda}{7}(\boldsymbol{A}\overline{\boldsymbol{D}}+\overline{\boldsymbol{D}}\boldsymbol{A})+\frac{2\lambda}{7}(\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I}\right\}+2\omega\overline{\boldsymbol{D}}\right]:\boldsymbol{A}\;dx.$$

Next we apply the operator $\partial/\partial x_l$ to 4.13, take the L^2-inner product of this equation with $\boldsymbol{A}_{,l}$ and simplify, using the property

$$\int_\Omega[(\boldsymbol{v}\nabla)\phi]:\phi\;dx\equiv\int_\Omega v_i\phi_{,i}:\phi=\frac{1}{2}\int_\Omega v_i(\phi:\phi)_{,i}=0$$

to obtain

$$\frac{1}{2}\frac{d}{dt}\,|\boldsymbol{A}|_{H^1}^2+|\boldsymbol{A}|_{H^1}^2=-\int_\Omega\overline{v}_{k,l}\boldsymbol{A}_{,k}:\boldsymbol{A}_{,l}\;dx$$
$$+\int_\Omega\left(\left[We\left\{\overline{\boldsymbol{W}}\boldsymbol{A}-\boldsymbol{A}\overline{\boldsymbol{W}}+\frac{3\lambda}{7}(\boldsymbol{A}\overline{\boldsymbol{D}}+\overline{\boldsymbol{D}}\boldsymbol{A})+\frac{2\lambda}{7}(\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I}\right\}+2\omega\overline{\boldsymbol{D}}\right]_{,l}:\boldsymbol{A}_{,l}\right)\;dx,$$

in which a subscript following a comma denotes partial differentiation with respect to that component. Finally, we apply the operator $\partial^2/\partial x_l\partial x_m$ to (4.13), take the L^2-inner product of this equation with $\boldsymbol{A}_{,lm}$, and simplify in the same way as above, to obtain

$$\frac{1}{2}\frac{d}{dt}\,|\boldsymbol{A}|_{H^2}^2+|\boldsymbol{A}|_{H^2}^2=-\int_\Omega[\overline{v}_{k,lm}\boldsymbol{A}_{,k}:\boldsymbol{A}_{,lm}+2\overline{v}_{k,l}\boldsymbol{A}_{,km}:\boldsymbol{A}_{,lm}]\;dx$$
$$+\int_\Omega\left(\left[We\left\{\overline{\boldsymbol{W}}\boldsymbol{A}-\boldsymbol{A}\overline{\boldsymbol{W}}+\frac{3\lambda}{7}(\boldsymbol{A}\overline{\boldsymbol{D}}+\overline{\boldsymbol{D}}\boldsymbol{A})+\frac{2\lambda}{7}(\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I}\right\}+2\omega\overline{\boldsymbol{D}}\right]_{,lm}:\boldsymbol{A}_{,lm}\right)\;dx,$$

We add all these equations, to obtain

$$\frac{1}{2}\frac{d}{dt}(We\|\boldsymbol{A}\|^2_{\mathbb{H}^2}) + \|\boldsymbol{A}\|^2_{\mathbb{H}^2} = 2\omega(\overline{\boldsymbol{D}},\boldsymbol{A})_{\mathbb{H}^2} + We(\overline{\boldsymbol{W}}\boldsymbol{A} - \boldsymbol{A}\overline{\boldsymbol{W}},\boldsymbol{A})_{\mathbb{H}^2}$$
$$-We\int_\Omega\{\overline{v}_{k,l}\,[\boldsymbol{A}_{,k}:\boldsymbol{A}_{,l} + 2\boldsymbol{A}_{,km}:\boldsymbol{A}_{,lm}] + \overline{v}_{k,lm}\boldsymbol{A}_{,k}:\boldsymbol{A}_{,lm}\}d\boldsymbol{x}$$
$$-2\frac{\lambda We}{7}\left(\left((\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I},\boldsymbol{A}\right)_{\mathbb{H}^2} + \frac{3\lambda}{7}We(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A},\boldsymbol{A})_{\mathbb{H}^2}\right). \tag{4.16}$$

We now estimate the right-hand side of (4.16). To this end, we recall the Sobolev inequality (see Appendix A.3.1)

$$\sup_{x\in\Omega}|u(x)| + \|u\|_{L^4} \leq c\|u\|_{H^2} \quad \text{for all } u \in H^2(\Omega) \tag{4.17}$$

which implies that

$$\left|\int_\Omega \overline{v}_{k,l}\boldsymbol{A}_{,k}:\boldsymbol{A}_{,l}\ dx\right| \leq c\|\overline{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2},$$
$$\left|\int_\Omega \overline{v}_{k,l}\boldsymbol{A}_{,km}:\boldsymbol{A}_{,lm}\ dx\right| \leq c\|\overline{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2}, \tag{4.18}$$
$$\left|\int_\Omega \overline{v}_{k,lm}\boldsymbol{A}_{,k}:\boldsymbol{A}_{,lm}\ dx\right| \leq c\|\overline{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2}.$$

Likewise, from (4.17) we find that

$$\left|((\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I},\boldsymbol{A})_{H^2}\right| \leq c\|\overline{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2},$$
$$\left|((\overline{\boldsymbol{W}}\boldsymbol{A} - \boldsymbol{A}\overline{\boldsymbol{W}},\boldsymbol{A})_{H^2}\right| \leq c\|\overline{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2}, \tag{4.19}$$
$$\left|((\overline{\boldsymbol{D}}\boldsymbol{A} + \boldsymbol{A}\overline{\boldsymbol{D}} + 2\omega\overline{\boldsymbol{D}}),\boldsymbol{A})_{H^2}\right| \leq c\|\overline{\boldsymbol{v}}\|_{H^3}\|\boldsymbol{A}\|_{H^2}(1 + \|\boldsymbol{A}\|_{H^2}).$$

The constant c entering the estimates (4.18)– (4.19) depends only on Ω, and on the material constants. By making use of (4.18)– (4.19) in (4.16) we thus conclude that

$$\frac{1}{2}\frac{d}{dt}(We\|\boldsymbol{A}\|^2_{\mathbb{H}^2}) + \|\boldsymbol{A}\|^2_{\mathbb{H}^2} \leq 2\omega C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|_{\mathbb{H}^2} + 2WeC_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}. \tag{4.20}$$

Inequality (4.20) implies that

$$\frac{d}{dt}(\|\boldsymbol{A}\|_{\mathbb{H}^2}) \leq \frac{2\omega C_0}{We}\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + 2C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|_{\mathbb{H}^2} - \frac{1}{We}\|\boldsymbol{A}\|_{\mathbb{H}^2}, \tag{4.21}$$

now, for a given $\alpha \in \mathbb{R}_+$, we deduce from (4.21) that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[(\|\boldsymbol{A}\|_{\mathbb{H}^2} + \alpha)^2\right) &= (\|\boldsymbol{A}\|_{\mathbb{H}^2} + \alpha)\frac{d}{dt}(\|\boldsymbol{A}\|_{\mathbb{H}^2}) \\ &\leq 2C\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}(\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} + \alpha)\left(\|\boldsymbol{A}\|_{\mathbb{H}^2} + \frac{2\omega}{We}\right) \end{aligned} \tag{4.22}$$

for some positive constant C.

We now set $\alpha = \frac{2\omega}{We}$. It follows from inequality (4.22) that

$$\frac{1}{2}\frac{d}{dt}\left((\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} + \frac{2\omega}{We})^2\right) \leq 2C\|\overline{\boldsymbol{v}}(t)\|_{\mathbb{H}^3}\left(\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} + \frac{2\omega}{We}\right)^2 \tag{4.23}$$

Integration of (4.23) over $(0,t) \subset (0,T)$ gives (4.14). Therefore $\boldsymbol{A}$ belongs to $\mathbb{L}^\infty(0,T,\mathbb{H}^2)$.

Estimate (4.15)

Let us first consider the ordinary differential equation

$$\begin{cases} \frac{d}{dt}\boldsymbol{A}(t,\boldsymbol{x}) = \boldsymbol{G}(t,\boldsymbol{A}) \quad in \quad \Omega_T \\ \boldsymbol{A}(0,\boldsymbol{x}) = \boldsymbol{A}_0 \quad in \quad \Omega \end{cases} \tag{4.24}$$

where

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{A},t) &= \frac{1}{We}\left(2\omega\overline{\boldsymbol{D}} + \frac{2\lambda We}{7}(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I} - \boldsymbol{A}\right) - (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{A} \\ &-(\boldsymbol{A}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A}) + \frac{3\lambda}{7}(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A}). \end{aligned} \tag{4.25}$$

We note that the solution of (4.24) is given by

$$\begin{aligned} \boldsymbol{A}(t) &= \boldsymbol{A}_0 + \int_0^t \left[\frac{1}{We}\left(2\omega\overline{\boldsymbol{D}} + \frac{2\lambda We}{7}(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I} - \boldsymbol{A}\right)\right. \\ &\left. -(\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A}) + \frac{3\lambda}{7}(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A})\right] ds. \qquad (4.26) \end{aligned}$$

If $\boldsymbol{G} \in C([0,T],\mathbb{H}^2)$ then solution (4.26) belongs to $C([0,T],\mathbb{H}^2)$ (at least locally) [13, 30].

If $\overline{\boldsymbol{v}} \in C([0,T],\mathbb{H}^2)$, we have

$$\begin{aligned} \boldsymbol{A}'(t) &= \frac{1}{We}\left(2\omega\overline{\boldsymbol{D}} - \frac{2\lambda We}{7}(\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I} - \boldsymbol{A}\right) - (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{A} \\ &-(\boldsymbol{A}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A}) + \frac{3\lambda}{7}(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A}) \qquad (4.27) \end{aligned}$$

belongs to $C([0,T],\mathbb{H}^1)$. Taking the $\mathbb{H}^1$-norm of both sides of (2.10) we obtain

$$\begin{aligned} \|\boldsymbol{A}'(t)\|_{\mathbb{H}^1} &= C_0\frac{2\omega}{We}\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + \frac{1}{We}C_0\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} + C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} \\ &\leq C_0\left(\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + \frac{1}{We}\right)\left(\|\boldsymbol{A}(t)\|_{\mathbb{H}^2}) + \frac{2\omega}{We}\right), \qquad (4.28) \end{aligned}$$

Equation (4.28) may be written in the form

$$\|\boldsymbol{A}(t)\|_{\mathbb{H}^2} + \frac{2\omega}{We} \geq \frac{\|\boldsymbol{A}'(t)\|_{\mathbb{H}^1}}{C_0\left(\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + \frac{1}{We}\right)}. \qquad (4.29)$$

We integrate (4.23) over $(0,T)$, and deduce that

$$\|\boldsymbol{A}\|_{\mathbb{H}^2} + \frac{2\omega}{We} \leq \left(\|\boldsymbol{A}_0\|_{\mathbb{H}^2} + \frac{2\omega}{We}\right)\exp\{C\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\}. \qquad (4.30)$$

Next, we use (4.30) to deduce from (4.29) that

$$\frac{\|\boldsymbol{A}'|_{\mathbb{H}^1}}{C_0\left(\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + \frac{1}{We}\right)} \leq \left(\|\boldsymbol{A}_0\|_{\mathbb{H}^2} + \frac{2\omega}{We}\right)\exp\{C\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\}, \qquad (4.31)$$

which implies that

$$\|A'(t)\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} \leq$$
$$C\left(\|\overline{v}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)} + \frac{1}{We}\right)\left(\|A_0(t)\|_{\mathbb{L}^1(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right)\exp(C\|\overline{v}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}).$$

Thus (4.15) is proved.□

4.1.3 Local existence of a regular solution

Let us first recall without proof Schauder's theorem [36] for the existence of a fixed point, and the Arzela-Ascoli theorem [5] for the compactness.

Lemma 4.4 SCHAUDER'S FIXED POINT THEOREM

Let M be a non-empty convex subset of a normed space B. Let f be a continuous mapping of M into a compact set $K \subset M$. Then f has a fixed point. □

Lemma 4.5 ARZELA-ASCOLI THEOREM

Let (K,d) be a compact metric space and $\mathcal{H}$ a bounded subset of $C(K,X)$, with X a Banach space. Assume that $\mathcal{H}$ is uniformly equicontinuous, that is,

$$\forall \epsilon > 0 \ \exists \delta > 0 \quad / \quad d(x_1,x_2) < \delta \longrightarrow \left| f(x_1) - f(x_2) \right| < \epsilon \ \ \forall f \in \mathcal{H}.$$

Then $\mathcal{H}$ is relatively compact in $C(K,X)$. □

Let us now consider the following problem:

$$\left.\begin{array}{r} Re\boldsymbol{v}' + \delta\mathcal{L}\boldsymbol{v} = \boldsymbol{b} - Re(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \\ +\mathrm{div}\{\frac{1}{7}\gamma\,[N(\boldsymbol{AD}+\boldsymbol{DA}) + N_p(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I}]\} \\ \\ We\{\boldsymbol{A}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + \boldsymbol{AW} - \boldsymbol{WA} - \frac{3\lambda}{7}(\boldsymbol{AD}+\boldsymbol{DA})\} + \boldsymbol{A} \\ = 2\omega\boldsymbol{D} - \frac{2\lambda}{7}We(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I} \\ \\ \boldsymbol{v}(\cdot,t) \in \mathbb{V}, \quad \boldsymbol{A}(\cdot,t) \in \mathbb{H}^2(\Omega) \cap \mathbb{X} \ a.a.t, \\ \boldsymbol{v}(0) = \boldsymbol{v}_0, \quad \boldsymbol{A}(0) = \boldsymbol{A}_0 \end{array}\right\} \tag{4.32}$$

Theorem 4.1 LOCAL EXISTENCE OF SOLUTION: LINEAR CLOSURE
Assume that $\Gamma \in C^3, \boldsymbol{b} \in \mathbb{L}^2_{loc}(\mathbb{R}_+; \mathbb{H}^1)$, $\boldsymbol{b}' \in \mathbb{L}^2_{loc}(\mathbb{R}_+; \mathbb{H}^{-1})$,
$\boldsymbol{v}_0 \in D(\mathcal{L})$, $\boldsymbol{A}_0 \in \mathbb{H}^2(\Omega)$.
If $\delta > 0$, *then there exist* $T^* > 0$, $\boldsymbol{v} \in \mathbb{L}^2(0, T^*; \mathbb{H}^3) \cap C([0, T^*], D(\mathcal{L}))$, *with* $\boldsymbol{v}' \in \mathbb{L}^2(0, T^*; \mathbb{V}) \cap C([0, T^*], \mathbb{H})$; $p \in L^2(0, T^*; \mathbb{H}^2)$
(p is the associated pressure), and $\boldsymbol{A} \in C([0, T^*], \mathbb{H}^2) \cap \mathbb{X}$, *such that* $(\boldsymbol{v}, \boldsymbol{A}, p)$ *is a solution to the problem (4.32) in* Ω_{T^*}.

Proof.

1. For $T > 0, B_1 > 0$ and $B_2 > 0$, we define the set

$$\begin{aligned} R_T = \{&(\overline{\boldsymbol{v}}, \overline{\boldsymbol{A}}), \overline{\boldsymbol{v}} \in C([0, T], D(\mathcal{L})) \cap \mathbb{L}^2(0, T; \mathbb{H}^3), \\ &\overline{\boldsymbol{v}}' \in C([0, T], \mathbb{H}) \cap \mathbb{L}^2(0, T; \mathbb{V}); \overline{\boldsymbol{A}} \in \mathbb{L}^\infty(0, T; \mathbb{H}^2); \\ &\overline{\boldsymbol{A}}' \in \mathbb{L}^\infty(0, T; \mathbb{H}^1); \overline{\boldsymbol{v}}(0) = \overline{\boldsymbol{v}}_0, \overline{\boldsymbol{A}}(0) = \overline{\boldsymbol{A}}_0 \in \Omega, \\ &\|\overline{\boldsymbol{v}}\|^2_{\mathbb{L}^\infty(0,T;D(\mathcal{L}))\cap\mathbb{L}^2(0,T;\mathbb{H}^3)} + \|\overline{\boldsymbol{v}}'\|^2_{\mathbb{L}^\infty(0,T;\mathbb{H})\cap\mathbb{L}^2(0,T;\mathbb{V})} \leq B_1, \\ &\|\overline{\boldsymbol{A}}\|^2_{\mathbb{L}^\infty(0,T;\mathbb{H}^2)} \leq B_1, \quad \|\overline{\boldsymbol{A}}'\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} \leq B_2\}. \end{aligned} \tag{4.33}$$

   - First we show that, if B_1 is large enough, then $\forall T > 0, R_T \neq \emptyset$.
     Let $\boldsymbol{v}^*$ be the solution of the following problem

$$\begin{aligned} &Re\boldsymbol{v}^{*'} + \delta\mathcal{L}\boldsymbol{v}^* = 0 \qquad a.e\ in\ \mathbb{R}_+ \\ &\boldsymbol{v}^*(t) \in \mathbb{V} \qquad in\ \mathbb{R}_+ \\ &\boldsymbol{v}^*(0) = \boldsymbol{v}_0 \end{aligned}$$

     If $\delta > 0$, then from Lemma 4.2, there exists a constant $D_1(N_p, Re, \Omega)$ such that

$$\|\boldsymbol{v}^*\|_{\mathbb{L}^2(0,T;\mathbb{H}^3)\cap\mathbb{L}^\infty(0,T;D(\mathcal{L}))} + \|\boldsymbol{v}^{*'}\|^2_{\mathbb{L}^\infty(0,T;\mathbb{H})\cap\mathbb{L}^2(0,T;\mathbb{V})} \leq D_1|\mathcal{L}\boldsymbol{v}_0|^2. \tag{4.34}$$

If we set

$$B_1 > D_1|\mathcal{L}v_0|^2 + \|A_0\|_{\mathbb{H}^2}, \tag{4.35}$$

then $(v^*, A_0) \in R_T$, for all $T > 0$.

- R_T is a convex set.
In fact, let $k \in \mathbb{R}$ $0 < k < 1$, $(\overline{v}_1, \overline{A}_1)$, $(\overline{v}_2, \overline{A}_2) \in R_T$.
By definition of R_T: $(\overline{v}_1,\ \overline{v}_2)$ belongs to $C([0,T], D(\mathcal{L})) \cap \mathbb{L}^2(0,T,\mathbb{H}^3)$, and $(\overline{v}_1',\ \overline{v}_2')$ belongs to $C([0,T], \mathbb{H} \cap \mathbb{L}^2(0,T,\mathbb{V})$, then obviously
$(1-k)\overline{v}_1 + k\overline{v}_2$ belongs to $C([0,T], D(\mathcal{L})) \cap \mathbb{L}^2(0,T,\mathbb{H}^3)$, and
$(1-k)\overline{v}_1' + k\overline{v}_2'$ belongs to $C([0,T], \mathbb{H} \cap \mathbb{L}^2(0,T,\mathbb{V})$
Using the Cauchy-Schwarz inequality we deduce that

$$\begin{aligned}
&\|(1-k)\overline{v}_1 + k\overline{v}_2\|^2 + \|(1-k)\overline{v}_1' + k\overline{v}_2'\|^2 \\
&\leq (1-k)^2\|\overline{v}_1\|^2 + 2(1-k)k\|\overline{v}_1\|\|\overline{v}_2\| + k^2\|\overline{v}_2\|^2 \\
&+(1-k)^2\|\overline{v}_1'\|^2 + 2(1-k)k\|\overline{v}_1'\|\|\overline{v}_2'\| + k^2\|\overline{v}_2'\|^2 \\
&\leq (1-k)^2\left(\|\overline{v}_1\|^2 + \|\overline{v}_1'\|^2\right) + k^2\left(\|\overline{v}_2\|^2 + \|\overline{v}_2'\|^2\right) \\
&+(1-k)k\left(\|\overline{v}_1\|^2 + \|\overline{v}_2\|^2 + \|\overline{v}_1'\|^2 + \|\overline{v}_2'\|^2\right) \\
&\leq (1-k)\left(\|\overline{v}_1\|^2 + \|\overline{v}_1'\|^2\right) + k\left(\|\overline{v}_2\|^2 + \|\overline{v}_2'\|^2\right) \\
&\leq (1-k)B_1 + kB_1 = B_1,
\end{aligned}$$

which implies that

$$\|(1-k)\overline{v}_1 + k\overline{v}_2\|^2 + \|(1-k)\overline{v}_1' + k\overline{v}_2'\|^2 \leq B_1$$

where $\|\cdot\|$ denotes one of the following norms $\|\cdot\|_{\mathbb{L}^\infty(0,T;D(\mathcal{L}))\cap\mathbb{L}^2(0,T;\mathbb{H}^3)}$, or $\|\cdot\|_{\mathbb{L}^\infty(0,T;\mathbb{H}))\cap\mathbb{L}^2(0,T;\mathbb{V})}$.
Likewise, we may easily show that
$(1-k)\overline{A}_1 + k\overline{A}_2 \in \mathbb{L}^\infty(0,T;\mathbb{H}^2)$,

$(1-k)\overline{A}_1' + k\overline{A}_2' \in \mathbb{L}^2(0,T,\mathbb{H}^1)$,

and

$$\|(1-k)\overline{A}_1 + k\overline{A}_2\|^2_{\mathbb{L}^\infty(0,T;\mathbb{H}^2)} \leq B_1$$

and

$$\|(1-k)\overline{A}_1' + k\overline{A}_2'\|^2_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} \leq B_2.$$

- Next, we consider the mapping

$$\begin{aligned} \phi \quad &: R_T \longrightarrow X_T = C([0,T];\mathbb{V}) \times C([0,T];\mathbb{H}^1) \\ &(\overline{v},\overline{A}) \longmapsto (v,A), \end{aligned} \tag{4.36}$$

where A and v are the unique solutions of (4.10) and (4.13) respectively, with

$$\begin{aligned} F &= -Re(\overline{v}\cdot\nabla)\overline{v} + b \\ &+\mathrm{div}\left\{\frac{1}{7}\gamma\left[N(\overline{A}\,\overline{D} + \overline{D}\,\overline{A}) + N_p(\overline{A}:\overline{D})I\right]\right\}. \end{aligned} \tag{4.37}$$

Clearly a fixed point of ϕ is a solution of the problem (4.32).

2. For this purpose, we show that ϕ satisfies the conditions of Schauder's Fixed Point Theorem (Lemma 4.4)

   - First we show that there exists T^* such that $\phi(R_{T^*}) \subset R_{T^*}$.
   If $(\overline{v},\overline{A}) \in R_T$, then from (4.37) using (4.18), (4.19) and the Poincaré-Friedrichs inequality to obtain:

$$\begin{aligned} \|F\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^1)} &= \int_0^T (F,F)_{\mathbb{H}^1}\,dt \\ &\leq \int_0^T \left[D_2(\|\overline{v}\|^4_{\mathbb{H}^3} + \|\overline{A}\|^2_{\mathbb{H}^2}\|\overline{v}\|^2_{\mathbb{H}^3}) + \|b\|^2_{\mathbb{H}^1}\right] dt. \end{aligned}$$

   Then

$$\|F\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^1)} \leq D_2 B_1^2 T + \|b\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^1)}, \tag{4.38}$$

where D_2 depends on Ω, N_p, N_s and γ. Also,

$$\begin{aligned}|\boldsymbol{F}(0)|^2 &= (\boldsymbol{F}(0), \boldsymbol{F}(0))_{\mathbb{L}^2}, \\ &\leq D_2\left(|\mathcal{L}\boldsymbol{v}_0|^2 + |\boldsymbol{A}_0|^2 + 1\right)|\mathcal{L}\boldsymbol{v}_0|^2 + |\boldsymbol{b}(0)|^2, \qquad (4.39)\end{aligned}$$

$$\begin{aligned}\|\boldsymbol{F}'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})} &= \int_0^T (\boldsymbol{F}', \boldsymbol{F}')_{\mathbb{H}^{-1}} dt \\ &\leq \int_0^T D_2\left[\|\overline{\boldsymbol{v}}\|^2_{\mathbb{H}^3}\|\overline{\boldsymbol{v}}'\|^2_{\mathbb{H}^2} + \|\overline{\boldsymbol{A}}'\|^2_{\mathbb{H}^1}\|\overline{\boldsymbol{v}}\|^2_{\mathbb{H}^3}\right] dt + \\ &\quad \int_0^T \left[D_2(\|\overline{\boldsymbol{A}}\|^2_{\mathbb{H}^2}\|\overline{\boldsymbol{v}}'\|^2_{\mathbb{H}^2}) + \|\boldsymbol{b}\|^2_{\mathbb{H}^{-1}}\right] dt;\end{aligned}$$

so that

$$\|\boldsymbol{F}'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})} \leq D_2(B_1^2 + B_1B_2^2)T + \|\boldsymbol{b}'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})}. \qquad (4.40)$$

From Lemma 4.2 and 4.3, we use (4.38)–(4.40), to get

$$\begin{aligned}&\|\boldsymbol{v}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)\cap\mathbb{L}^2(0,T;D(\mathcal{L}))} + \|\boldsymbol{v}'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})\cap\mathbb{L}^\infty(0,T;\mathbb{H})} \\ &\leq C_2\left[|\mathcal{L}\boldsymbol{v}_0|^2 + \|\boldsymbol{F}\|^2_{L(0,T;H^1)} + \|\boldsymbol{F}'\|^2_{L(0,T;H^{-1})} + \|\boldsymbol{F}(0)\|^2\right] \\ &\leq C_2\Big[B_1D_2(2B_1 + B_2^2)T + \|\boldsymbol{b}'\|^2_{L(0,T;H^{-1})} + |\boldsymbol{b}(0)|^2 \\ &\quad D_2(|\mathcal{L}\boldsymbol{v}_0|^2 + |\boldsymbol{A}_0| + 1)|\mathcal{L}\boldsymbol{v}_0|^2\Big], \qquad (4.41)\end{aligned}$$

$$\begin{aligned}\|\boldsymbol{A}\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^2)} &\leq \left(\|\boldsymbol{A}_0\|_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right)\exp\left(C_1T\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\right) \\ &\leq \left(\|\boldsymbol{A}_0\|_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right)\exp(C_1TB_1^{1/2}), \qquad (4.42)\end{aligned}$$

and

$$\begin{aligned}\|\boldsymbol{A}'\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} &\leq C_0\left(\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^2(0,T;\mathbb{H}^3)} + \frac{1}{We}\right)\left(\|\boldsymbol{A}_0\|_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right) \\ &\quad \exp\left(C_1T\|\overline{\boldsymbol{v}}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\right) \\ &\leq C_0\left(B_1^{1/2} + \frac{1}{We}\right)\left(\|\boldsymbol{A}_0\|_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right)\exp(C_1TB_1^{1/2}). \\ &\qquad (4.43)\end{aligned}$$

Therefore $(\boldsymbol{v},\boldsymbol{A}) \in R_{T^*}$, if we choose T^* such that the right-hand sides of of (4.41)-(4.43) are bounded respectively by B_1, B_1 and B_2. We make use of (4.35), and we then choose

$$B_1 \geq max\left\{D_1|\mathcal{L}\boldsymbol{v}_0|^2, \left(||\boldsymbol{A}_0|| + \frac{2\omega}{We}\right) e, 2C_2\left[||\boldsymbol{b}'||^2_{\mathbb{L}^1(0,T;\mathbb{H}^1)} + |\boldsymbol{b}(0)|^2 + D_2(|\mathcal{L}\boldsymbol{v}_0|^2 + |\boldsymbol{A}_0|^2 + 1)|\mathcal{L}\boldsymbol{v}_0|^2\right]\right\} \quad (4.44)$$

$$B_2 \geq eC_0\left(B_1^{1/2} + \frac{1}{We}\right)\left(||\boldsymbol{A}_0||_{\mathbb{L}^1(0,T;\mathbb{H}^2)} + \frac{2\omega}{We}\right) \quad (4.45)$$

and

$$T^* \leq T \leq \min\left\{\frac{1}{4C_2D_2(2B_1 + B_2^2)}, \frac{1}{C_1B_1^{1/2}}\right\}. \quad (4.46)$$

Therefore, we have defined constants B_1 and B_2, depending on We, Ω, ω and on the data, and we have defined a time, say T^*, satisfying (4.46), depending on B_1, B_2 and on the data, such that $\phi(R_{T^*}) \subset R_{T^*}$.

- Obviously ϕ is continuous
- By the Arzela-Ascoli Theorem (Lemma 4.5), we deduce that R_{T^*} is compact in
  $X_T = C([0,T];\mathbb{V}) \times C([0,T];\mathbb{H}^1)$.
- Therefore, since R_{T^*} is a non-empty convex subset of X_T, we deduce from the Schauder Fixed Point Theorem (Lemma 4.4) that ϕ has a fixed point, $(\boldsymbol{v},\boldsymbol{A})$ say, which is the solution to the problem(4.32).

3. For the regularity of $(\boldsymbol{v},\boldsymbol{A})$ on Γ see Appendix A.2.5

4. It remains to show that $\boldsymbol{A} \in \mathbb{X}$; that is that $\boldsymbol{A}^T = \boldsymbol{A}$ and tr$\boldsymbol{A}$=0.

   - $\boldsymbol{A}^T = \boldsymbol{A}$.

     We take the transpose of (4.13), to obtain

$$We\left\{\boldsymbol{A}^{T'} + (\overline{\boldsymbol{v}} \cdot \nabla)\boldsymbol{A}^T + \boldsymbol{A}^T\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A}^T - \frac{3\lambda}{7}(\boldsymbol{A}^T\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A}^T)\right\}$$

$$+\boldsymbol{A}^T = 2\omega\overline{\boldsymbol{D}} - \frac{2\lambda}{7}We(\boldsymbol{A}:\overline{\boldsymbol{D}})\boldsymbol{I}. \tag{4.47}$$

Next, we set $\boldsymbol{Q} = \boldsymbol{A}^T - \boldsymbol{A}$, and substract (4.13) from (4.47) to get

$$We\{\boldsymbol{Q}' + (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{Q} + \boldsymbol{Q}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{Q} - \frac{3\lambda}{7}(\boldsymbol{Q}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{Q})\} + \boldsymbol{Q} = 0. \tag{4.48}$$

We now take the $\mathbb{L}^2$-inner product of (4.48) with $\boldsymbol{Q}$, this gives

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{Q}\|^2 + (\boldsymbol{Q}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{Q}, \boldsymbol{Q}) - \frac{3\lambda}{7}(\boldsymbol{Q}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{Q}, \boldsymbol{Q}) + \frac{1}{We}\|\boldsymbol{Q}\|^2 = 0. \tag{4.49}$$

Some terms in (4.49) are now simplified. Using the identity $\boldsymbol{AB}:\boldsymbol{C} = \boldsymbol{B}:\boldsymbol{A}^T\boldsymbol{C}$ and the skew-symmetry tensor of $\overline{\boldsymbol{W}}$, to get

$$(\boldsymbol{Q}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{Q}, \boldsymbol{Q}) = \int_\Omega (\boldsymbol{Q}^T\boldsymbol{Q} : \overline{\boldsymbol{W}} - \boldsymbol{Q}\boldsymbol{Q}^T : \overline{\boldsymbol{W}})dx = 0$$

and

$$(\boldsymbol{Q}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{Q}, \boldsymbol{Q}) \le K\|\boldsymbol{Q}\|^2_{\mathbb{L}^2}\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}.$$

Using the above results, we deduce from (4.49) the inequality

$$\frac{d}{dt}\|\boldsymbol{Q}\|^2_{\mathbb{L}^2} \le \left(2K\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + \frac{1}{We}\right)\|\boldsymbol{Q}\|^2_{\mathbb{L}^2}.$$

Since $\overline{\boldsymbol{v}} \in \mathbb{L}^2(0, T^*; \mathbb{H}^3)$ we may apply Gronwall's Lemma (see Appendix A.3.6) to conclude that $\boldsymbol{Q}=0$; that is $\boldsymbol{A}^T=\boldsymbol{A}$.

- $\mathrm{tr}\boldsymbol{A}=0$

  First, we recall that $\mathrm{tr}\overline{\boldsymbol{A}}=\mathrm{tr}\boldsymbol{A}_0=0$ and we set $Z=\mathrm{tr}\boldsymbol{A}$, and take the trace of both sides of (4.13), to obtain

$$We\{Z' + (\overline{\boldsymbol{v}}\cdot\nabla)Z - \frac{2\lambda}{7}(\boldsymbol{A}:\overline{\boldsymbol{D}})\} + Z = -\frac{2\lambda}{7}We(\boldsymbol{A}:\overline{\boldsymbol{D}}).$$

This implies that

$$Z' + (\overline{\boldsymbol{v}}\cdot\nabla)Z + KZ = 0. \tag{4.50}$$

Clearly, $Z_1(t) = 0$ solves (4.50). Now assume that Z_2 is also a solution of (4.50); then we have

$$\begin{aligned} Z_1' + (\overline{v} \cdot \nabla) Z_1 + K Z_1 &= 0; \\ Z_2' + (\overline{v} \cdot \nabla) Z_2 + K Z_2 &= 0. \end{aligned} \tag{4.51}$$

We set $Z = Z_1 - Z_2$, substract $(4.51)_2$ from $(4.51)_1$, and take the L^2-inner product with Z to obtain

$$\frac{d}{dt}|Z|^2 + K|Z|^2 = 0.$$

We now apply Gronwall's Lemma and conclude that $Z = 0$; that is $Z_1 = Z_2$. This completes the proof of the theorem.□

4.1.4 Uniqueness of the solution

Now, we show that the solution obtained in Theorem 4.1 is the only one in the class of regular solutions.

Theorem 4.2 *Let $T^* > 0$. The problem (4.32) admits at most one solution $(\boldsymbol{v}, \boldsymbol{A})$ in $\mathbb{L}^2(0,T;\mathbb{H}^3) \cap C([0,T]; D(\mathcal{L})) \times C([0,T];\mathbb{H}^2)$.*
The pressure p is unique up to an additive constant in $L^2(0,T;\mathbb{H}^2)$

Proof
As usual, we take the difference of two solutions $(\boldsymbol{v}^1, \boldsymbol{A}^1, p^1)$ and $(\boldsymbol{v}^2, \boldsymbol{A}^2, p^2)$, corresponding to the same data.
Set $\boldsymbol{v}=\boldsymbol{v}^1 - \boldsymbol{v}^2$ and $\boldsymbol{A}=\boldsymbol{A}^1 - \boldsymbol{A}^2$. The vector function $\boldsymbol{v}$ and the tensor function $\boldsymbol{A}$ satisfy the equations

$$\begin{aligned} &Re\{\boldsymbol{v}' + (\boldsymbol{v}^1 \cdot \nabla)\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}^2\} + \gamma\left(1 - \frac{2}{35}N_p\right)\mathcal{L}\boldsymbol{v} = \operatorname{div}(\boldsymbol{S}_1 - \boldsymbol{S}_2), \\ &We\{\boldsymbol{A}' + (\boldsymbol{v} \cdot \nabla)\boldsymbol{A}^1 + (\boldsymbol{v}^2 \cdot \nabla)\boldsymbol{A}\} + \boldsymbol{A} = 2\omega \boldsymbol{D} \\ &-\frac{2\lambda}{7}We(\boldsymbol{A}_1 : \boldsymbol{D}_1 - \boldsymbol{A}_2 : \boldsymbol{D}_2)\boldsymbol{I} - We\left(R(\boldsymbol{A}^1, \boldsymbol{v}) + R(\boldsymbol{A}, \boldsymbol{v}^2)\right), \end{aligned} \tag{4.52}$$

where

$$\begin{aligned} \boldsymbol{S}_i &= \frac{1}{7}\gamma\left[(\boldsymbol{A}_i\boldsymbol{D}_i + \boldsymbol{D}_i\boldsymbol{A}_i) + N_p(\boldsymbol{A}_i : \boldsymbol{D}_i)\boldsymbol{I}\right], \\ R(\boldsymbol{A}, \boldsymbol{v}) &= \boldsymbol{A}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A} - \frac{3\lambda}{7}(\boldsymbol{A}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A}). \end{aligned} \tag{4.53}$$

Now, we take the $\mathbb{L}^2$-inner product of (4.52) with $\boldsymbol{v}$ and $\boldsymbol{A}$ respectively, and integrate over Ω, to obtain

$$\begin{aligned} &Re\left\{\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}(t)\|^2_{\mathbb{L}^2} + \left((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}^2, \boldsymbol{v}\right)\right\} + \delta\|\boldsymbol{v}(t)\|^2_{\mathbb{H}^1} = -\left(\boldsymbol{S}_1 - \boldsymbol{S}_2, \boldsymbol{D}\right), \\ &We\left\{\frac{1}{2}\frac{d}{dt}\|\boldsymbol{A}(t)\|^2_{\mathbb{L}^2} + \left((\boldsymbol{v}\cdot\nabla)\boldsymbol{A}^1, \boldsymbol{A}\right)\right\} + |\boldsymbol{A}(t)|^2_{\mathbb{L}^2} = 2\omega(\boldsymbol{D}, \boldsymbol{A}(t)) \\ &-We\left(R(\boldsymbol{A}^1, \boldsymbol{v}) + R(\boldsymbol{A}, \boldsymbol{v}^2), \boldsymbol{A}(t)\right). \end{aligned} \tag{4.54}$$

Using (4.53), and the Sobolev inequality (see Appendix A.3.1) we obtain the estimates

$$\begin{aligned} |(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}^2| &\le C_0\|\boldsymbol{v}^2\|_{\mathbb{H}^3}|\boldsymbol{v}|, \\ |R(\boldsymbol{A}, \boldsymbol{v}^2)| &\le C_0\|\boldsymbol{v}^2\|_{\mathbb{H}^3}|\boldsymbol{A}|, \\ |(\boldsymbol{v}\cdot\nabla)\boldsymbol{A}^1 + R(\boldsymbol{A}^1, \boldsymbol{v})| &\le C_0\|\boldsymbol{A}^1\|_{\mathbb{H}^2}\|\boldsymbol{v}\|_{\mathbb{H}^3}, \end{aligned} \tag{4.55}$$

also,

$$(\boldsymbol{S}_1 - \boldsymbol{S}_2, \boldsymbol{D}) = \frac{\gamma}{7}N\left(\boldsymbol{A}^1\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A}^1 + \boldsymbol{A}\boldsymbol{D}^2 + \boldsymbol{D}^2\boldsymbol{A}, \boldsymbol{D}\right). \tag{4.56}$$

We make use of (4.55) and (4.56), we add $(4.54)_1$ and $(4.54)_2$, and using Young's Inequality (see Appendix A.3.5) to deduce the energy inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(Re\|\boldsymbol{v}\|^2_{\mathbb{L}^2} + We\|\boldsymbol{A}\|^2_{\mathbb{L}^2}\right) + \gamma\left(1 - \frac{2}{35}N_p\right)\|\boldsymbol{v}\|^2_{\mathbb{H}^2} \\ &\le C_0\|\boldsymbol{A}^1\|_{\mathbb{H}^2}\left(\frac{\epsilon}{2}\|\boldsymbol{v}\|^2_{\mathbb{H}^2} + \frac{1}{2\epsilon}|\boldsymbol{v}|^2\right) + C_0\|\boldsymbol{v}^2\|_{\mathbb{H}^3}\left(\frac{\epsilon}{2}\|\boldsymbol{v}\|^2_{\mathbb{H}^2} + \frac{1}{2\epsilon}|\boldsymbol{A}|^2\right) \\ &+Re\|\boldsymbol{v}^2\|_{\mathbb{H}^3}\|\boldsymbol{v}\|^2_{\mathbb{L}^2} + \|\boldsymbol{A}\|^2_{\mathbb{L}^2} + 2\omega C_0\left(\frac{\epsilon}{2}\|\boldsymbol{v}\|^2_{\mathbb{H}^2} + \frac{1}{2\epsilon}\|\boldsymbol{A}\|^2_{\mathbb{L}^2}\right), \\ &+C_0We\|\boldsymbol{A}^1\|_{\mathbb{H}^2}\left(\frac{\epsilon}{2}\|\boldsymbol{v}\|^2_{\mathbb{H}^2} + \frac{1}{2\epsilon}|\boldsymbol{A}|^2\right) + C_0We\|\boldsymbol{v}^2\|_{\mathbb{H}^3}\|\boldsymbol{A}\|^2_{\mathbb{L}^2}, \end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(Re\|v\|_{\mathbb{L}^2}^2 + We\|A\|^2{}_{\mathbb{L}^2}\right) + \left[\delta - C_0\frac{\epsilon}{2}\|A^1\|_{\mathbb{H}^2}\right]\|v\|_{\mathbb{H}^2}^2 \\
&-C_0\frac{\epsilon}{2}\left[\|v\|_{\mathbb{H}^3}^2 + 2\omega + We\|A^1\|_{\mathbb{H}^2}\right]\|v\|_{\mathbb{H}^2}^2 \\
&\leq \frac{1}{2\epsilon}\left(C_0\|v^2\|_{\mathbb{H}^3} + 2\epsilon + 2\omega C_0 + C_0 We\|A^1\|_{\mathbb{H}^2} + 2\epsilon We\|v\|_{\mathbb{H}^3}^2\right)\|A\|^2{}_{\mathbb{L}^2} \\
&+\frac{1}{2\epsilon}\left(C_0\|A^1\|_{\mathbb{H}^2} + 2\epsilon Re\|v^2\|_{\mathbb{H}^3}\right)\|v\|^2{}_{\mathbb{L}^2} \\
&\leq \left(Re\|v\|^2{}_{\mathbb{L}^2} + We|A|^2\right)\left(C_0\|v^2\|_{\mathbb{H}^3} + k\right), \qquad (4.57)
\end{aligned}$$

where

$$k = \frac{1}{2\epsilon}\left(2\epsilon Re\|v^2\|_{\mathbb{H}^3} + C_0\|A^1\|_{\mathbb{H}^2} + 2\epsilon + 2\omega C_0 + C_0 We + 2\epsilon We\|v^2\|_{\mathbb{H}^3}^2\right). \qquad (4.58)$$

Equation (4.58) is true for any $\epsilon > 0$; we choose ϵ small enough, for example,

$$\epsilon = \frac{\delta}{C_0(\|A^1\|_{\mathbb{H}^2} + \|v^2\|_{\mathbb{H}^3} + 2\omega + We\|A^1\|_{\mathbb{H}^2})},$$

such that the coefficient of $\|v\|_{\mathbb{H}^2}^2$ on the left hand side of (4.57) is positive (this requires also that $N_p < 35/2$).

Then (4.57) reads as follows:

$$\frac{1}{2}\frac{d}{dt}\left(Re\|v\|_{\mathbb{L}^2}^2 + We\|A\|_{\mathbb{L}^2}^2\right) \leq K\left(Re\|v\|_{\mathbb{L}^2}^2 + We\|A\|_{\mathbb{L}^2}^2\right) \qquad (4.59)$$

with

$$K = C_0\|v^2\|_{\mathbb{H}^3} + k.$$

We deduce from (4.59) and Gronwall's Lemma that $Re\|v\|_{\mathbb{L}^2}^2 + We\|A\|_{\mathbb{L}^2}^2 = 0$, and also the pressure p is constant.□

4.2 The quadratic closure approximation

Now we turn to the the quadratic closure approximation for the problem studied in Section 4.1. We prove the local existence of the solution, as well its uniqueness.

4.2.1 Dimensionless and traceless problem

We use the decomposition (4.2), and again set

$$A^* = \frac{1}{n}I,$$

for a problem in $\mathbb{R}^n$. Then, writting A for $\overline{A}$ and using tr$D = 0$ we obtain

$$\begin{aligned} \mathcal{A}D &= (A + A^*) \otimes (A + A^*)D \\ &= (A : D)(A + \frac{1}{n}I). \end{aligned} \tag{4.60}$$

We also set

$$\eta = \frac{1}{nD_r}\frac{V}{L}. \tag{4.61}$$

We substitute the decomposition (4.2) in (2.44) and (2.52), and make use of (4.1), (4.60) and (4.61), to obtain the dimensionless system

$$\begin{aligned} &Re(v' + (v \cdot \nabla)v) + \nabla p - \gamma\left(1 + \frac{N_s}{n}\right)\Delta v = b + \text{div}S_Q, \\ &\eta\left[A' + (v \cdot \nabla)a + (AW - WA) - \lambda(AD + DA)\right] + A \\ &= 2\lambda\eta D - 2\lambda\eta(A : D)(A + \frac{1}{n}I), \end{aligned} \tag{4.62}$$

where

$$S_Q(A, v) = \gamma\left[N_p(A : D)(A + \frac{1}{n}I) + N_s(AD + DA)\right]. \tag{4.63}$$

Remarks

1. Lemmas 4.1 and 4.2 are still valid with $\delta = \gamma(1 + \frac{2N_s}{n})$.

2. We state below a variant of Lemma 4.3.

4.2.2 The linearised problem

For a given admissible velocity field $\overline{v}$ and orientation tensor field $\overline{A}$, let us consider the problem

$$\begin{aligned} &\eta\left[A' + (\overline{v}\cdot\nabla)A + (A\overline{W} - \overline{W}A) - \lambda(A\overline{D} + \overline{D}A)\right] + A \\ &= 2\lambda\eta\overline{D} - 2\lambda\eta(A:\overline{D})(\overline{A} + \frac{1}{n}I), \\ &A(0) = A_0 \quad a.e \ \ in \ \ \Omega. \end{aligned} \tag{4.64}$$

Lemma 4.6 *Assume that* $\Gamma \in C^1$, $\overline{v} \in \mathbb{L}^1(0,T;\mathbb{H}^3) \cap D(\mathcal{L})$, $\overline{A} \in \mathbb{X} \cap C([0,T],\mathbb{H}^1)$, $A_0 \in \mathbb{H}^2(\Omega)$. *Then the problem (4.64) admits a unique solution* $A \in C([0,T],\mathbb{H}^2)$. *Futhermore, there exists a constant* $C(\Omega,\omega,We)$ *such that*

$$\begin{aligned} &\|A\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^2)} \\ &\le (\|A_0\|_{\mathbb{H}^2} + 1)\exp\left[C\left(\|\overline{v}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)}\| + \|\overline{A}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \|\overline{v}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\right)\right]. \end{aligned} \tag{4.65}$$

In addition, if $\overline{v} \in C([0,T],D(\mathcal{L}))$, *then* $A' \in C([0,T],\mathbb{H}^1)$ *and satisfies*

$$\begin{aligned} &\|A'\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} \le \\ &C\left[\|\overline{v}\|_{\mathbb{H}^3}(2\eta\lambda\|\overline{A}\|_{\mathbb{H}^2} + \eta(2\lambda+1)) + 1\right](\|A_0\|_{\mathbb{H}^2} + 1) \\ &\exp\left\{C\left(\|\overline{v}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)}\| + \|\overline{A}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \|\overline{v}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\right)\right\}. \end{aligned} \tag{4.66}$$

Proof

Estimate (4.65)

We take the scalar product of (4.64) with A in $\mathbb{L}^2(\Omega)$, next we apply the operator ∇ to (4.64), and multiply the resulting equation by ∇A, and finally we apply the operator D^2 to (4.64) and multiply the resulting equation by D^2A in $\mathbb{L}^2(\Omega)$. We add the resulting equations, use (4.18) – (4.19) and the Cauchy-Schwarz inequality, to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\eta\|\boldsymbol{A}\|^2_{\mathbb{H}^2}) + \|\boldsymbol{A}\|^2_{\mathbb{H}^2} = 2\eta(\overline{\boldsymbol{D}},\boldsymbol{A})_{\mathbb{H}^2} + \eta(\overline{\boldsymbol{W}}\boldsymbol{A} - \boldsymbol{A}\overline{\boldsymbol{W}},\boldsymbol{A})_{\mathbb{H}^2}\\
&-\eta\int_\Omega\{\overline{v}_{k,l}\,[\boldsymbol{A}_{,k}:\boldsymbol{A}_{,l} + 2\boldsymbol{A}_{,km}:\boldsymbol{A}_{,lm}] + \overline{v}_{k,lm}\boldsymbol{A}_{,k}:\boldsymbol{A}_{,lm}\}d\boldsymbol{x}\\
&-2\eta\lambda\left(\left((\boldsymbol{A}:\overline{\boldsymbol{D}})\overline{\boldsymbol{A}},\boldsymbol{A}\right)_{\mathbb{H}^2} + \eta\lambda(\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A},\boldsymbol{A})_{\mathbb{H}^2}\right)\\
&\le 2\eta C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|_{\mathbb{H}^2} + \eta C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|^2_{\mathbb{H}^2} + 2\eta\lambda C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\overline{\boldsymbol{A}}\|_{\mathbb{H}^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}.
\end{aligned}\tag{4.67}$$

Inequality (4.67) implies that

$$\begin{aligned}
\frac{d}{dt}(\|\boldsymbol{A}\|_{\mathbb{H}^2}) \;\le\;& 2C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3} + 2C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|_{\mathbb{H}^2} + 2\lambda C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\|\overline{\boldsymbol{A}}\|_{\mathbb{H}^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}\\
&-\frac{1}{\eta}\|\boldsymbol{A}\|_{\mathbb{H}^2}.
\end{aligned}\tag{4.68}$$

For a given $\alpha \in \mathbb{R}_+$, $(\alpha \ge 1)$, we deduce from (4.68) that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left((\|\boldsymbol{A}\|_{\mathbb{H}^2} + \alpha)^2\right) \;=\;& (\|\boldsymbol{A}\|_{\mathbb{H}^2} + \alpha)\frac{d}{dt}(\|\boldsymbol{A}\|_{\mathbb{H}^2})\\
\le\;& 2C_0\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}(\lambda\|\overline{\boldsymbol{A}}\|_{\mathbb{H}^2} + 1)(\|\boldsymbol{A}\|_{\mathbb{H}^2} + \alpha)(\|\boldsymbol{A}\|_{\mathbb{H}^2} + \alpha).
\end{aligned}\tag{4.69}$$

We set $\alpha = 1$; it follows from inequality (4.69) that

$$\frac{1}{2}\frac{d}{dt}\left((\|\boldsymbol{A}\|_{\mathbb{H}^2} + 1)^2\right) \le C\|\overline{\boldsymbol{v}}\|_{\mathbb{H}^3}\left(\lambda\|\overline{\boldsymbol{A}}\|_{\mathbb{H}^2} + 1\right)(\|\boldsymbol{A}\|_{\mathbb{H}^2} + 1)^2. \tag{4.70}$$

Integration of (4.69) over $(0,t) \subset (0,T)$ gives (4.65). Therefore $\boldsymbol{A}$ belongs to $\mathbb{L}^\infty(0,T,\mathbb{H}^2)$.

Estimate (4.66)

If $\overline{\boldsymbol{v}} \in C([0,T],\mathbb{H}^2)$, and since $\overline{\boldsymbol{A}} \in C([0,T],\mathbb{H}^1)$, we have

$$\begin{aligned}
\boldsymbol{A}' \;=\;& 2\lambda\overline{\boldsymbol{D}} - 2\lambda(\boldsymbol{A}:\overline{\boldsymbol{D}})(\overline{\boldsymbol{A}} + \frac{1}{n}\boldsymbol{I}) - \frac{1}{\eta}\boldsymbol{A} - (\overline{\boldsymbol{v}}\cdot\nabla)\boldsymbol{A}\\
&-(\boldsymbol{A}\overline{\boldsymbol{W}} - \overline{\boldsymbol{W}}\boldsymbol{A}) + (\boldsymbol{A}\overline{\boldsymbol{D}} + \overline{\boldsymbol{D}}\boldsymbol{A}),
\end{aligned}\tag{4.71}$$

so that A' belongs to $C([0,T],\mathbb{H}^1)$. Taking the $\mathbb{H}^1$-norm of both sides of (4.71), using subadditivity and Poincaré inequalities, we obtain

$$\begin{aligned}\|A'\|_{\mathbb{H}^1} &\leq 2\lambda C_0\|\overline{v}\|_{\mathbb{H}^3} + \frac{1}{\eta}C_0\|A\|_{\mathbb{H}^2} + C_0\|\overline{v}\|_{\mathbb{H}^3}\|A\|_{\mathbb{H}^2} \\ &\leq \frac{C_0}{\eta}\left[\|\overline{v}\|_{\mathbb{H}^3}\left(2\eta\lambda\|\overline{A}\|_{\mathbb{H}^2} + \eta(2\lambda+1)\right) + 1\right]\left(\|A\|_{\mathbb{H}^2}) + 1\right),\end{aligned}$$

and this inequality may be written in the form

$$\|A\|_{\mathbb{H}^2} + 1 \geq \frac{\|A'\|_{\mathbb{H}^1}}{\frac{C_0}{\eta}\left[\|\overline{v}\|_{\mathbb{H}^3}\left(2\eta\lambda\|\overline{A}\|_{\mathbb{H}^2} + \eta(2\lambda+1)\right) + 1\right]}. \tag{4.72}$$

We deduce from (4.70) that

$$\begin{aligned}\|A\|_{\mathbb{H}^2} + 1 &\leq (\|A_0\|_{\mathbb{H}^2} + 1)\exp\left\{C\left(\|\overline{v}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)} + \|\overline{A}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^2)}\right.\right. \\ &\qquad \left.\left. + \|\overline{v}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\right)\right\}.\end{aligned} \tag{4.73}$$

We make use of (4.73) and we deduce from (4.72) that

$$\begin{aligned}&\frac{\|A'\|_{\mathbb{H}^1}}{\frac{C_0}{\eta}\left[\|\overline{v}\|_{\mathbb{H}^3}(\eta\lambda\|\overline{A}\|_{\mathbb{H}^2} + \eta(2\lambda+1)) + 1\right]} \\ &\leq (\|A_0\|_{\mathbb{H}^2} + 1)\exp\left\{C\left(\|\overline{v}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)} + \|\overline{A}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \|\overline{v}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\right)\right\},\end{aligned}$$

which implies that

$$\begin{aligned}\|A'\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} &\leq C\left[\|\overline{v}\|_{\mathbb{H}^3}(2\eta\lambda\|\overline{A}\|_{\mathbb{H}^2} + \eta(2\lambda+1)) + 1\right](\|A_0\|_{\mathbb{H}^2} + 1) \\ &\qquad \exp\left\{C\left(\|\overline{v}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)} + \|\overline{A}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^2)} + \|\overline{v}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\right)\right\},\end{aligned}$$

so that (4.66) is proved.$\blacksquare$

4.2.3 Local existence of regular solution

Let us consider the following problem:

$$\left.\begin{array}{l} Rev' + \delta\mathcal{L}v = \mathrm{div}\gamma\left[N_p(\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A}+\frac{1}{n}\boldsymbol{I}) + N_s(\boldsymbol{AD}+\boldsymbol{DA})\right] \\ \qquad\qquad +\boldsymbol{b} - Re(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \\ \eta\{\boldsymbol{A}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} + (\boldsymbol{AW}-\boldsymbol{WA}) - \lambda(\boldsymbol{AD}+\boldsymbol{DA})\} + \boldsymbol{A} \\ = 2\lambda\eta\boldsymbol{D} - 2\lambda\eta(\boldsymbol{A}:\boldsymbol{D}(\boldsymbol{A}+\frac{1}{n}\boldsymbol{I}) \\ \boldsymbol{v}(\cdot,t)\in\mathbb{V}, \quad \boldsymbol{A}(\cdot,t)\in\mathbb{H}^2(\Omega) \quad and \quad \boldsymbol{A}(\cdot,t)\in\mathbb{X} \quad a.a.t \\ \boldsymbol{v}(0)=\boldsymbol{v}_0, \quad \boldsymbol{A}(0)=\boldsymbol{A}_0. \end{array}\right\} \tag{4.74}$$

Theorem 4.3 Local existence of solution: Quadratic closure
Assume that $\Gamma\in C^3$, $\boldsymbol{b}\in\mathbb{L}^2_{loc}(\mathbb{R}_+,\mathbb{H}^1)$, $\boldsymbol{b}'\in\mathbb{L}^2_{loc}(\mathbb{R}_+,\mathbb{H}^{-1})$, $\boldsymbol{v}_0\in D(\mathcal{L})$, $\boldsymbol{A}_0\in\mathbb{H}^2(\Omega)\cap\mathbb{X}$.
Then there exists $T^*>0$, $\boldsymbol{v}\in\mathbb{L}^2(0,T^*;\mathbb{H}^3)\cap C([0,T^*];D(\mathcal{L}))$, *with* $\boldsymbol{v}'\in\mathbb{L}^2(0,T^*;\mathbb{V})\cap C([0,T^*];\mathbb{H})$; $p\in L^2(0,T^*;\mathbb{H}^2)$ *(p is the associated pressure), and* $\boldsymbol{A}\in C([0,T^*],\mathbb{H}^2)\cap\mathbb{X}$ *such that* $(\boldsymbol{v},\boldsymbol{A},p)$ *is a solution to the problem (4.74) in* Ω_{T^*}.

Proof

1. For $T>0$, $B_1>0$ and $B_2>0$, we define R_T, ϕ as in (4.33) and (4.36) and we set

$$\boldsymbol{F} = -Re(\overline{\boldsymbol{v}}\cdot\nabla)\overline{\boldsymbol{v}} + \boldsymbol{b} + \mathrm{div}\gamma\left[N_p(\overline{\boldsymbol{A}}:\overline{\boldsymbol{D}})(\overline{\boldsymbol{A}}+\frac{1}{n}\boldsymbol{I}) + N_s(\overline{\boldsymbol{A}}\,\overline{\boldsymbol{D}}+\overline{\boldsymbol{D}}\,\overline{\boldsymbol{A}})\right]. \tag{4.75}$$

2. Let us find $T^*>0$ such that $\phi(R_{T^*})\subset R_{T^*}$.
Let $(\overline{\boldsymbol{v}},\overline{\boldsymbol{A}})\in R_T$, as in Section 4.1, using Poincaré- Friedrichs Inequality we

deduce from (4.75) that

$$\begin{aligned}\|F\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^1)} &= \int_0^T \|F(t)\|^2_{\mathbb{H}^1}\,dt \\ &\leq \int_0^T \Big[D_2(\|\overline{v}\|^4_{\mathbb{H}^3} + \|\overline{A}\|^2_{\mathbb{H}^2}\|\overline{v}\|^2_{\mathbb{H}^3}) + \|\overline{A}\|^2_{\mathbb{H}^2}\|\overline{v}\|^4_{\mathbb{H}^3}) \\ &\qquad + \|b\|^2_{\mathbb{H}^1}\Big]\,dt.\end{aligned}$$

Then

$$\|F\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^1)} \leq D_2 B_1^2(B_1+2)T + \|b\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^1)}, \tag{4.76}$$

where D_2 depends on Ω, and on the data, and

$$|F(0)|^2 \;\leq\; D_2\Big(|\mathcal{L}v_0|^2 + |A_0|^4 + \|A_0\|^2\Big)\,|\mathcal{L}v_0|^2 + |b(0)|^2. \tag{4.77}$$

Likewise,

$$\begin{aligned}\|F'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})} &= \int_0^T \|F'\|^2_{\mathbb{H}^{-1}}\,dt \\ &\leq \int_0^T D_2\left[\|\overline{v}\|^2_{\mathbb{H}^3}\|\overline{v}'\|^2_{\mathbb{H}^2} + \|\overline{A}'\|^2_{\mathbb{H}^1}\|\overline{A}\|^2_{\mathbb{H}^2}\|\overline{v}\|^2_{\mathbb{H}^3}\right]dt + \\ &\quad \int_0^T D_2\left[\|\overline{A}\|^4_{\mathbb{H}^2}\|\overline{v}\|^2_{\mathbb{H}^3} + \|\overline{A}\|^2_{\mathbb{H}^2}\|\overline{v}\|^2_{\mathbb{H}^3} + \|b\|^2_{\mathbb{H}^{-1}}\right]dt + \\ &\quad \int_0^T \left[D_2\|\overline{A}\|^2_{\mathbb{H}^2}\|\overline{v}'\|^2_{\mathbb{H}^2} + \|\overline{A}'\|^2_{\mathbb{H}^1}\|\overline{v}\|^2_{\mathbb{H}^3})\right]dt.\end{aligned}$$

Therefore

$$\|F'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})} \leq D_2(B_1^3 + B_1^2B_2^2 + B_1B_2^2 + 3B_1^2)T + \|b'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})}. \tag{4.78}$$

From Lemmas 4.2 and 4.6, and using (4.76)-(4.78), we get

$$\begin{aligned}&\|v\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)\cap\mathbb{L}^2(0,T;D(\mathcal{L}))} + \|v'\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^{-1})\cap\mathbb{L}^\infty(0,T;\mathbb{H})} \\ &\leq C_2\left[|\mathcal{L}v_0|^2 + \|F\|^2_{L(0,T;H^1)} + \|F'\|^2_{L(0,T;H^{-1})} + \|F(0)\|^2\right] \\ &\leq C_2\Big[B_1D_2(2B_1^2 + 4B_1 + B_1B_2^2 + B_2^2)T + \|b'\|^2_{L(0,T;H^{-1})} \\ &\quad + |b(0)|^2 + D_2(|\mathcal{L}v_0|^2 + |A_0|^4 + \|A_0\|^2)|\mathcal{L}v_0|^2\Big],\end{aligned} \tag{4.79}$$

$$\begin{aligned}\|A\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^2)} &\le \left(\|A_0\|_{\mathbb{L}^2(0,T;\mathbb{H}^2)}+1\right)\\ &\quad \exp\left[C\left(\|\overline{v}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)}+\|\overline{A}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^2)}+\|\overline{v}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\right)\right]\\ &\le \left(\|A_0\|_{\mathbb{L}^2(0,T;\mathbb{H}^2)}+1\right)\exp\left(C(2B_1+B_1^{1/2})T\right), \qquad (4.80)\end{aligned}$$

$$\begin{aligned}\|A'\|_{\mathbb{L}^\infty(0,T;\mathbb{H}^1)} &\le C\left[(\|\overline{v}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)}\left(2\eta\lambda(\|\overline{A}\|_{\mathbb{H}^2}+\eta(2\lambda+1))\right)+1\right]\\ &\quad (\|A_0\|_{\mathbb{H}^2}+1)\exp\left\{C(\|\overline{v}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^3)}+\|\overline{A}\|^2_{\mathbb{L}^2(0,T;\mathbb{H}^2)}+\|\overline{v}\|_{\mathbb{L}^1(0,T;\mathbb{H}^3)})\right\}\\ &\le C\left[B_1^{1/2}\left(2\eta\lambda(B_1^{1/2}+\eta(2\lambda+1))\right)+1\right]\\ &\quad (\|A_0\|_{\mathbb{H}^2}+1)\exp\left\{C(2B_1+B_1^{1/2})T\right\}. \qquad (4.81)\end{aligned}$$

Therefore $(v,A)\in R_{T^*}$, if we choose T^* such that the right-hand sides of (4.79)–(4.81) are bounded respectively by B_1, B_1 and B_2. We make use of (4.35), so we choose

$$\begin{aligned}B_1 &\ge \max\Big\{ D_1|\mathcal{L}v_0|^2, (\|A_0\|+1)\,e, 2C_2\Big[\|b'\|^2_{\mathbb{L}^1(0,T;\mathbb{H}^1)}+\\ &\quad |b(0)|^2 + D_2(|\mathcal{L}v_0|^2+\|A_0\|^4+\|A_0\|^2)|\mathcal{L}v_0|^2\Big]\Big\}, \qquad (4.82)\end{aligned}$$

$$B_2 \ge eC\left[B_1^{1/2}\left(2\eta\lambda\left(B_1^{1/2}+\eta(2\lambda+1)\right)\right)+1\right]\left(\|A_0\|_{\mathbb{L}^1(0,T;\mathbb{H}^2)}+1\right), \quad (4.83)$$

and

$$T^* \le T \le \min\left\{\frac{1}{2C_2D_2(2B_1^2+4B_1+B_1B_2^2+B_2^2)}, \frac{1}{C(2B_1+B_1^{1/2})}\right\}. \quad (4.84)$$

Thus with T^* satisfying (4.84), $\phi(R_{T^*})\subset R_{T^*}$.

3. As in the proof of Theorem 4.1, ϕ is continuous. Thus from parts 1) and 2) of this proof, and Schauder's Fixed Foint Theorem, we deduce that ϕ has a fixed point, (v,A) say, which is the solution to the problem (4.74).

4. It remains to show that $A\in\mathbb{X}$. That is $A^T=A$ and tr$A$=0.

- $A^T{=}A$.

  We take the transpose of (4.64), recalling that $\overline{A}^T = \overline{A}$, to get

$$\eta\{A^{T'}+(\overline{v}\cdot\nabla)A^T+A^T\overline{W}-\overline{W}A^T-\lambda(A^T\overline{D}+\overline{D}A^T)\}+A^T = 2\lambda\eta\overline{D}-2\lambda\eta(A:\overline{D})(\overline{A}+\frac{1}{n}I). \qquad (4.85)$$

  We set $Q = A^T - A$, and we substract (4.64) from (4.85): this gives

$$\eta\{Q'+(\overline{v}\cdot\nabla)Q+Q\overline{W}-\overline{W}Q-\lambda(Q\overline{D}+\overline{D}Q)\}+Q=0. \qquad (4.86)$$

  Now, we take the $\mathbb{L}^2$-inner product of (4.86) with Q:

$$\frac{1}{2}\frac{d}{dt}\|Q\|^2+(Q\overline{W}-\overline{W}Q,Q)-\lambda(Q\overline{D}+\overline{D}Q,Q)+\frac{1}{\eta}\|Q\|^2_{\mathbb{L}^2}=0. \qquad (4.87)$$

  As before, we deduce from (4.87) that

$$\frac{d}{dt}\|Q\|^2_{\mathbb{L}^2} \le \left(2K\|\overline{v}\|_{\mathbb{H}^3}+\frac{1}{We}\right)\|Q\|^2_{\mathbb{L}^2}.$$

  Since $\overline{v} \in \mathbb{L}^2(0,T^*;\mathbb{H}^3)$ we may apply Gronwall's Lemma to conclude that $Q{=}0$. So that $A^T{=}A$.

- tr$A{=}0$

  First, we recall that tr$\overline{A}$ = tr$A_0 = 0$ and we set Z = tr$A$, and take now the trace of both sides of (4.64), to obtain

$$\eta\{Z'+(\overline{v}\cdot\nabla)Z-2\lambda(A:\overline{D})\}+Z=-2\lambda\eta(A:\overline{D}),$$

  which implies that

$$Z'+(\overline{v}\cdot\nabla)Z+KZ=0. \qquad (4.88)$$

  Clearly, $Z_1(\mathrm{t}) = 0$ solves (4.88). Now assume that Z_2 is also a solution of (4.88), then we have

$$\begin{aligned} Z_1'+(\overline{v}\cdot\nabla)Z_1+KZ_1&=0\\ Z_2'+(\overline{v}\cdot\nabla)Z_2+KZ_2&=0 \end{aligned} \qquad (4.89)$$

We set $Z = Z_1 - Z_2$, we substract $(4.89)_2$ from $(4.89)_1$, and take the L^2-inner product with Z to obtain

$$\frac{d}{dt}|Z|^2 + K|Z|^2 = 0.$$

We apply Gronwall's Lemma and conclude that $Z = 0$. Therefore $Z_1 = Z_2$. This ends the proof of the theorem.□

4.2.4 Uniqueness of the solution

Now, we show that the solution obtained in the theorem 4.3 is the only one in the class of regular solution.

Theorem 4.4 *Let T>0. The problem (4.32) admits at most one solution* $(\mathbf{v},\mathbf{A})$ *in* $\mathbb{L}^2(0,T;\mathbb{H}^3) \cap C([0,T];D(\mathcal{L})) \times C([0,T];\mathbb{H}^2)$.
The pressure p is unique up to an additive constant in $\mathbb{L}^2(0,T;\mathbb{H}^2)$

See the proof of Theorem 4.2.□

Chapter 5

GLOBAL EXISTENCE OF SOLUTIONS

In this Chapter, we show that the unique solutions obtained in Theorems 4.1 and 4.2 for the linear closure and in Theorem 4.3 and 4.4 for the quadratic closure, are defined for all $t > 0$, for small enough data. These global solutions are proved to be stable in the absence of body forces..

We reiterate here the observation that it is possible, as will be shown in this Chapter, to obtain results on global existence for the model (2.41) in which the rotary diffusivity is assumed constant. GALDI AND REDDY [15] have studied local existence for the case in which D_r is approximated by (2.17 [14], but have indicated that it is not clear how their analysis may be extended to include global existence.

The approach is this Chapter draws substantially on the analysis by GUILLOPÉ AND SAUT in [16].

5.1 Linear closure approximation

In this section, we show that the unique solution obtained in Theorems 4.1 and 4.2 are defined on $\mathbb{R}_+$, if the data are small enough. To this end we first derive some a priori bounds uniform in time, satisfied by that solution.

5.1.1 Some a priori estimates

We recall first that $(\boldsymbol{v},\boldsymbol{A})$ is the unique solution to the problem (4.32). We rewrite (4.20), with $\overline{\boldsymbol{v}} = \boldsymbol{v}$, using Young's Inequality (see Appendix, (A.3.5)), to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(We\|\boldsymbol{A}\|^2_{H^2}) + \|\boldsymbol{A}\|^2_{H^2} \leq\ & 2\omega C_0\|\boldsymbol{v}\|_{H^3}\|\boldsymbol{A}\|_{H^2} + 2WeC_0\|\boldsymbol{v}\|_{H^3}\|\boldsymbol{A}\|^2_{H^2} \\ & \leq 4\omega^2 C_0^2\frac{\epsilon}{2}\|\boldsymbol{v}\|^2_{H^3} + \frac{1}{2\epsilon}\|\boldsymbol{A}\|^2_{H^2} \\ & +4We^2C_0^2\frac{\alpha}{2}\|\boldsymbol{A}\|^4_{H^2} + \frac{1}{2\alpha}\|\boldsymbol{v}\|^2_{H^3}.\end{aligned}$$

We choose $\epsilon = \frac{4}{3}$ and $\alpha = 3/(2\omega^2 C_0^2)$; then the previous inequality reads

$$\frac{d}{dt}(We\|\boldsymbol{A}\|^2_{H^2}) + \frac{1}{2}\|\boldsymbol{A}\|^2_{H^2} \leq 6\omega^2 C_0\|\boldsymbol{v}\|^2_{H^3} + \frac{6We^2}{\omega^2}\|\boldsymbol{A}\|^4_{H^2}. \tag{5.1}$$

We write (4.32)1 in the form

$$\begin{aligned}&(\gamma(1-\frac{2}{35}N_p))\mathcal{L}\boldsymbol{v} = -Re\boldsymbol{v}' + \boldsymbol{b} - Re(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \\ &\text{div}\left\{\frac{1}{7}\gamma\left[N(\boldsymbol{AD}+\boldsymbol{DA}) + N_p(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I}\right)\right\},\end{aligned}$$

we take the H^1-norm of both sides by them self, using Poincaré- Friedrichs and Young's Inequalities to obtain

$$\begin{aligned}\|\mathcal{L}\boldsymbol{v}\|_{H^1} &\leq \frac{c_0}{\gamma^2(1-\frac{2}{35}N_p)^2}\left(Re^2\|\boldsymbol{v}'\|^2_{H^2} + \|\boldsymbol{A}\|^2_{H^2} + \|\boldsymbol{b}\|^2_{H^1} + c_2Re^2\|\boldsymbol{v}\|^2_{H^1}\|\nabla\boldsymbol{v}\|^2_{H^1}\right) \\ &\leq \frac{c_0}{\gamma^2(1-\frac{2}{35}N_p)^2}\left(Re^2\|\boldsymbol{v}'\|^2_{H^2} + \|\boldsymbol{A}\|^2_{H^2} + \|\boldsymbol{b}\|^2_{H^1} + C_2Re^2|\mathcal{L}\boldsymbol{v}|^4\right),\end{aligned}$$

using Korn's Inequality (see Appendix A.3.3), we deduce from the previous inequality an estimate for $\|\boldsymbol{v}\|_{H^3}$, in the form

$$\|\boldsymbol{v}\|_{H^3}^2 \leq C_0 \left[Re^2\|\boldsymbol{v}'\|_{H^2}^2 + \|\boldsymbol{A}\|_{H^2}^2 + \|\boldsymbol{b}\|_{H^1}^2 + C_2 Re^2 |\mathcal{L}\boldsymbol{v}|^4\right]. \tag{5.2}$$

From (5.1) and (5.2), therefore,

$$\begin{aligned} &\frac{d}{dt}(We\|\boldsymbol{A}\|_{H^2}^2) + \left(\frac{1}{2} - 6\omega^2 C_0\right)\|\boldsymbol{A}\|_{H^2}^2 \\ &\leq 6\omega^2 C_0 \left(Re^2\|\boldsymbol{v}'\|_{H^2}^2 + \|\boldsymbol{b}\|_{H^1}^2 + C_2 Re^2|\mathcal{L}\boldsymbol{v}|^4\right) + \frac{3We^2}{\omega^2}\|\boldsymbol{A}\|_{H^2}^4. \end{aligned} \tag{5.3}$$

Remark.

Inequality (5.3) will be used only for values of D_r such that the coefficient of $\|\boldsymbol{A}\|_{H^2}^2$ in the left-hand side is positive. In particular, we will choose D_r such that the coefficient of $\|\boldsymbol{A}\|_{H^2}^2$ is larger than $C_0\omega^2$. That is,

$$\frac{1}{2} - 7\omega^2 C_0 \geq 0. \tag{5.4}$$

We write $(4.32)_1$ in the form

$$\begin{aligned} \mathcal{L}\boldsymbol{v} &= \frac{1}{\delta}\left[-Re^2\boldsymbol{v}' + \boldsymbol{b} + \operatorname{div}\left\{\frac{\gamma}{7}\left[N(\boldsymbol{AD}+\boldsymbol{DA}) + N_p(\boldsymbol{A}:\boldsymbol{D})\boldsymbol{I}\right]\right\}\right] \\ &\quad -Re\frac{1}{\delta}(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \end{aligned} \tag{5.5}$$

where

$$\delta = \gamma\left(1 - \frac{2}{35}N_p\right).$$

We now take the scalar product of (5.5) with $\mathcal{L}\boldsymbol{v}$ in $\mathbb{H}$, using the Cauchy-Schwarz inequality, to obtain

$$\begin{aligned} |\mathcal{L}\boldsymbol{v}| &= \frac{1}{\delta}\left[Re^2|\boldsymbol{v}'||\mathcal{L}\boldsymbol{v}| + |\boldsymbol{b}||\mathcal{L}\boldsymbol{v}| + \frac{\gamma}{7}N\,|(\boldsymbol{AD}+\boldsymbol{DA})|\,\|\boldsymbol{v}\| + \frac{\gamma}{7}N_p|\boldsymbol{A}|\,|\boldsymbol{D}|\,\|\boldsymbol{v}\|\right] \\ &\quad +Re\,|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}|\,|\mathcal{L}\boldsymbol{v}|\,,. \end{aligned}$$

Using the Poincaré-Friedrichs (see Appendix, (A.3.2)) and Young inequalities, we deduce from the previous inequality that there exist constants $C_0 > 0$ and $C_1 > 0$ such that

$$|\mathcal{L}\boldsymbol{v}|^2 \leq \frac{C_0}{\delta}\left[Re^2|\boldsymbol{v}'|^2 + |\boldsymbol{b}|^2 + \|\boldsymbol{A}\|^2_{H^2} + C_2 Re^2\|\boldsymbol{v}\|^4\right] \tag{5.6}$$

where

$$\|\boldsymbol{v}\|^2 = \int_\Omega |\nabla \boldsymbol{v}|^2 d\boldsymbol{x}$$

is the Dirichlet norm.

Now, we take the scalar product in $\mathbb{H}$ of $(4.32)_1$ with $\mathcal{L}\boldsymbol{v}$, and get

$$\frac{d}{dt}(Re\|\boldsymbol{v}\|^2) + \delta|\mathcal{L}\boldsymbol{v}|^2 \leq |\boldsymbol{b}||\mathcal{L}\boldsymbol{v}| + Re|(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}|\|\boldsymbol{v}\| + \frac{2\gamma}{7}N|\boldsymbol{AD}|\|\boldsymbol{v}\| + \frac{\gamma}{7}N_p|\boldsymbol{A}||\boldsymbol{D}|\|\boldsymbol{v}\|.$$

Again using the Young and Poincaré-Friedrichs inequalities, we deduce from the previous inequality that there exists a constant C_3 such that

$$\frac{d}{dt}(Re\|\boldsymbol{v}\|^2) + \delta|\mathcal{L}\boldsymbol{v}|^2 \leq \frac{3}{2\delta}\left(|\boldsymbol{b}|^2 + C_2\|\boldsymbol{A}\|^2_{H^2} + C_3\frac{Re^2}{\delta^2}\|\boldsymbol{v}\|^6\right). \tag{5.7}$$

Next, we differentiate $(4.32)_1$ with respect to t, and take the scalar product in $\mathbb{H}$ of the resulting equation with $\boldsymbol{v}'$, as above, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(Re|\boldsymbol{v}'|^2) + \delta\|\boldsymbol{v}'\|^2 \leq\ & |\boldsymbol{b}'||\mathcal{L}\boldsymbol{v}| + Re\left[|(\boldsymbol{v}'\cdot\nabla)\boldsymbol{v}| + |(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}'|\right]|\mathcal{L}\boldsymbol{v}| \\ & + \frac{2}{7}\gamma N|\boldsymbol{AD}|\|\boldsymbol{v}\| + N_p|\boldsymbol{A}||\boldsymbol{D}|\|\boldsymbol{v}\|.\end{aligned}$$

Using the Cauchy-Schwarz and Young inequalities, we deduce that there exists a constant $C = C(\gamma, N_p, N, \Omega)$ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(Re|\boldsymbol{v}'|^2) + \delta\|\boldsymbol{v}'\|^2 \leq\ & \frac{3C^2}{2\delta}|\boldsymbol{b}'|^2 + \frac{Re^2}{2\epsilon}|\boldsymbol{v}'|^4 + \frac{\epsilon}{2}\|\boldsymbol{v}\|^2_{H^3} \\ & + \frac{C^2}{\delta}\left(\|\boldsymbol{A}'\|^4_{H^1} + \|\boldsymbol{v}\|^4 + \|\boldsymbol{A}\|^4 + \|\boldsymbol{A}'\|^4\right). \end{aligned} \tag{5.8}$$

We also differentiate $(4.32)_2$ with respect to t, and take the scalar product in $\mathbb{H}$ of the resulting equation with $(3\delta/4\omega^2C_0^2)\boldsymbol{A}'$, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\frac{3\delta We}{4\omega^2C_0^2}|\boldsymbol{A}|^2\right)+\frac{3\delta}{8\omega^2C_0^2}|\boldsymbol{A}'|^2 \;\leq\;& \frac{\delta}{2}\|\boldsymbol{v}'\|^2+\frac{We^2}{8\omega^2}\left(\frac{1}{2\epsilon}|\boldsymbol{v}'|^4+\frac{\epsilon}{2}\|\boldsymbol{A}\|_{H^1}^2\right)\\ &+\frac{3We^2\delta}{8\epsilon\omega^2}|\boldsymbol{A}'|^4+\frac{\epsilon}{8}\frac{\delta}{\omega^2C_0^2}\|\boldsymbol{v}\|_{H^3}^2\\ &+\frac{\delta}{8\omega^2}\left(\frac{\epsilon}{2}\|\boldsymbol{A}\|_{H^2}^4+\frac{1}{2\epsilon}\|\boldsymbol{v}'\|^4\right). \qquad (5.9)\end{aligned}$$

Adding (5.8) and (5.9), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{3\delta We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2\right)+\delta\|\boldsymbol{v}'\|^2+\frac{3\delta}{4\omega^2C_0^2}|\boldsymbol{A}'|^2\\ &\leq\frac{3C^2}{\delta}|\boldsymbol{b}'|^2+\frac{1}{\epsilon}\left(\frac{We^2}{8\omega^2}+Re^2\right)|\boldsymbol{v}'|^4+\epsilon\frac{We^2}{8\omega^2}\|\boldsymbol{A}\|_{H^1}^4\\ &+\epsilon\left(1+\frac{\delta}{4\omega^2C_0^2}\right)\|\boldsymbol{v}\|_{H^3}^3+\frac{3\delta We^2}{4\epsilon\omega^2}|\boldsymbol{A}'|^4+\frac{3\delta We^2}{4\epsilon\omega^2}\left(\frac{\epsilon}{2}\|\boldsymbol{A}\|_{H^2}^4+\frac{1}{2\epsilon}\|\boldsymbol{v}'\|^4\right)\\ &+\frac{2C^2}{\delta}\left(\|\boldsymbol{A}'\|_{H^1}^4+\|\boldsymbol{v}\|^4+\|\boldsymbol{A}'\|^4+\|\boldsymbol{v}'\|^4\right). \qquad (5.10)\end{aligned}$$

We multiply inequality (5.3) by $\delta/(12Re^2\omega^2C_0)$ and add the resulting inequality to (5.10), to obtain

$$\begin{aligned}&\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{3\delta We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{3\delta We}{4\omega^2C_0^2}\|\boldsymbol{A}\|_{H^2}^2\right)+\frac{\delta}{2}\|\boldsymbol{v}'\|^2+\frac{3\delta}{4\omega^2C_0^2}|\boldsymbol{A}'|^2\\ &+\frac{\delta}{12Re^2\omega^2C_0^2}\left(\frac{1}{2}-6\omega^2C_0\right)\|\boldsymbol{A}\|_{H^2}^2\\ &\leq\epsilon\left(1+\frac{\delta}{4\omega^2C_0^2}\right)\|\boldsymbol{v}\|_{H^3}^2+\frac{3C^2}{\delta}|\boldsymbol{b}'|^2+\frac{\delta}{2Re^2}\|\boldsymbol{b}\|_{H^1}^2+\frac{C_2\delta}{2}|\mathcal{L}\boldsymbol{v}|^4+\frac{We^2\delta}{4Re^2\omega^4C_0}\|\boldsymbol{A}\|_{H^2}^4\\ &+\epsilon\frac{We^2}{8\omega^2}\|\boldsymbol{A}\|_{H^1}^4+\frac{3\delta We^2}{4\epsilon\omega^2}|\boldsymbol{A}'|^4+\frac{\delta}{4\omega^2}\left(\frac{\epsilon}{2}\|\boldsymbol{A}\|_{H^2}^4+\frac{1}{2\epsilon}\|\boldsymbol{v}'\|^4\right)+\frac{1}{\epsilon}\left(\frac{We^2}{8\omega^2}+Re^2\right)|\boldsymbol{v}'|^4\\ &+\frac{2C^2}{\delta}\left(\|\boldsymbol{A}'\|_{H^1}^4+\|\boldsymbol{v}\|^4+\|\boldsymbol{A}\|^4+\|\boldsymbol{v}'\|^4\right).\end{aligned}$$

Making use of (5.2) to estimate $\|\boldsymbol{v}\|_{H^3}^2$, and using also (5.4), the previous inequality reads

$$\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{3\delta We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{3\delta We}{4\omega^2C_0^2}\|\boldsymbol{A}\|_{H^2}^2\right)+\frac{\delta}{2}\|\boldsymbol{v}'\|^2+\frac{3\delta}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{\delta}{2Re^2}\|\boldsymbol{A}\|_{H^2}^2$$
$$\leq \epsilon C_0\left(1+\frac{\delta}{4\omega^2C_0^2}\right)\left\{Re^2\|\boldsymbol{v}'\|^2+\|\boldsymbol{A}\|_{H^2}^2+\|\boldsymbol{b}\|_{H^1}^2+C_2Re^2|\mathcal{L}\boldsymbol{v}|^4\right\}+\frac{3C^2}{\delta}|\boldsymbol{b}'|^2$$
$$+\frac{\delta}{2Re^2}\|\boldsymbol{b}\|_{H^1}^2+\frac{C_0\delta}{2}|\mathcal{L}\boldsymbol{v}|^4+\frac{We^2\delta}{4Re^2C_0\omega^4}\|\boldsymbol{A}\|_{H^2}^4+\epsilon\frac{We^2}{8\omega^2}\|\boldsymbol{A}\|_{H^1}^4+\frac{3\delta We^2}{4\epsilon\omega^2}|\boldsymbol{A}'|^4$$
$$+\frac{\delta}{4\omega^2}\left(\frac{\epsilon}{2}\|\boldsymbol{A}\|_{H^2}^4+\frac{1}{\epsilon}\|\boldsymbol{v}'\|^4\right)+\frac{2C^2}{\delta}\left(\|\boldsymbol{A}'\|_{H^1}^4+\|\boldsymbol{v}\|^4+\|\boldsymbol{A}\|^4+\|\boldsymbol{v}'\|^4\right)$$
$$+\frac{1}{\epsilon}\left(\frac{We^2}{8\omega^2}+Re^2\right)|\boldsymbol{v}'|^4. \tag{5.11}$$

In order that the coefficient of $\|\boldsymbol{A}\|^2$ and $\|\boldsymbol{v}'\|^2$ be positive we choose
$\epsilon=(\delta\omega^2C_0/(Re^2(4\omega^2C_0^2+\delta))$.
Inequality (5.11) now reads

$$\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{3\delta We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{3\delta We}{4\omega^2C_0^2}\|\boldsymbol{A}\|_{H^2}^2\right)+\frac{\delta)}{2}\|\boldsymbol{v}'\|^2+\frac{3\delta}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{\delta}{4Re^2}\|\boldsymbol{A}\|_{H^2}^2$$
$$\leq\frac{\delta}{4Re^2}\|\boldsymbol{b}\|_{H^1}^2+\frac{3C^2}{\delta}|\boldsymbol{b}'|^2+\frac{\delta}{2Re^2}\|\boldsymbol{b}\|_{H^1}^2+\frac{3C_3\delta)}{4}|\mathcal{L}\boldsymbol{v}|^4+\frac{\delta We^2}{4Re^2C_0\omega^4}\|\boldsymbol{A}\|_{H^2}^4$$
$$+\frac{C_0\delta We^2}{8Re^2(4\omega^2C_0^2+\delta}\|\boldsymbol{A}\|_{H^1}^4+\frac{3We^2Re^2(4\omega^2C_0^2+\delta}{4\omega^4C_0}|\boldsymbol{A}'|^4+\frac{C_0\delta^2}{4Re^2(4\omega^2C_0^2+\delta}\|\boldsymbol{A}\|_{H^2}^4$$
$$+\frac{Re^2(4\omega^2C_0^2+\delta)}{4\omega^4C_0}\|\boldsymbol{v}'\|^4+\frac{Re^2(4\omega^2C_0^2+\delta)}{\delta\omega^2C_0}\left(\frac{We^2}{8\omega^2}+Re^2\right)|\boldsymbol{v}'|^4$$
$$+\frac{2C^2}{\delta}\left(\|\boldsymbol{A}'\|_{H^1}^4+\|\boldsymbol{v}\|^4+\|\boldsymbol{A}\|^4+\|\boldsymbol{v}'\|^4\right).$$

Setting

$$C_5 = \max\left\{\frac{4\omega^2C_0^2+\delta}{4\omega^2C_0},\frac{C_0}{4\omega^2C_0^2+\delta},C_0,\frac{4\omega^2C_0^2+\delta}{\omega^2C_0}\left(\frac{We^2}{8\omega^2}+Re^2\right)\right\},$$

the previous inequality becomes

$$\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{3\delta We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{3\delta We}{4\omega^2C_0^2}\|\boldsymbol{A}\|_{H^2}^2\right)+\frac{\delta}{4}\|\boldsymbol{v}'\|^2+\frac{3\delta}{4\omega^2C_0^2}|\boldsymbol{A}'|^2+\frac{\delta}{4Re^2}\|\boldsymbol{A}\|_{H^2}^2$$

$$
\begin{aligned}
&\leq \frac{3\delta}{4Re^2}\|\boldsymbol{b}\|_{H^1}^2 + \frac{3C^2}{\delta}|\boldsymbol{b}'|^2 + \frac{3C_3\delta}{4}|\mathcal{L}\boldsymbol{v}|^4 + \frac{We^2\delta}{4Re^2C_0\omega^4}\|\boldsymbol{A}\|_{H^2}^4 + \frac{C_5\delta We^2}{8Re^2}\|\boldsymbol{A}\|_{H^1}^4 \\
&+\frac{3C_5We^2Re^2}{4\omega^2}|\boldsymbol{A}'|^4 + \frac{C_5Re^2}{\delta}|\boldsymbol{v}'|^4 + \frac{C_5\delta}{4Re^2}\|\boldsymbol{A}\|_{H^2}^4 + \frac{C_5Re^2}{4\omega^2}\|\boldsymbol{v}'\|^4 \\
&+\frac{2C^2}{\delta}\left(\|\boldsymbol{A}'\|_{H^1}^4 + \|\boldsymbol{v}\|^4 + \|\boldsymbol{A}\|^4 + \|\boldsymbol{v}'\|^4\right). \qquad (5.12)
\end{aligned}
$$

We multiply (5.7) by $\frac{\delta^2}{12C_2Re^2}$, add the resulting inequality to (5.12), and make use of (5.6) to estimate $|\mathcal{L}\boldsymbol{v}|^4$ in (5.12), it follows that, there exists $C_6 > 0$ such that

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\delta^2}{12C_2Re^2}\|\boldsymbol{v}\|^2 + Re|\boldsymbol{v}'|^2 + \frac{3\delta We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2 + \frac{3\delta We}{4\omega^2C_0^2}\|\boldsymbol{A}\|_{H^2}^2\right) + \frac{\delta}{8Re^2}\|\boldsymbol{A}\|_{H^2}^2 \\
&+\frac{\delta}{4}\|\boldsymbol{v}'\|^2 + \frac{3\delta}{4\omega^2C_0^2}|\boldsymbol{A}'|^2 + \frac{\delta^3}{12C_2Re^2}|\mathcal{L}\boldsymbol{v}|^2 \\
&\leq \frac{3\delta}{4Re^2}\|\boldsymbol{b}\|_{H^1}^2 + \frac{\delta}{8C_2Re^2}|\boldsymbol{b}|^2 + \frac{3C_6}{4\delta^2}|\boldsymbol{b}|^4 + \frac{3C^2}{\delta}|\boldsymbol{b}'|^2 + \frac{\delta}{Re^2}\left(\frac{C_5We^2}{8} + \frac{3C_6Re^2}{\delta^2}\right)\|\boldsymbol{A}\|_{H^1}^4 \\
&+\frac{1}{\delta}\left(2C^2 + \frac{C_3}{8C_2}\right)\|\boldsymbol{v}\|^4 + \frac{Re^2}{\delta}\left(3C_6Re^2 + C_5\right)|\boldsymbol{v}'|^4 + \frac{\delta}{Re^2}\left(\frac{We^2}{4C_0\omega^4} + \frac{C_5}{4}\frac{3C_3C_6}{\delta^2}\right)\|\boldsymbol{A}\|_{H^2}^4 \\
&+\frac{Re^2}{\delta}\left(\frac{C_5\delta}{4\omega^4} + \frac{2C^2}{Re^2}\right)\|\boldsymbol{v}'\|^4 + \frac{3C_5We^2Re^2}{4\omega^2}|\boldsymbol{A}'|^4 + + \frac{2C^2}{\delta}\|\boldsymbol{A}\|^4 + \frac{2C^2}{\delta}\|\boldsymbol{A}'\|_{H^1}^4 \\
&+\frac{3C_6C_2^2Re^4}{4\delta}\|\boldsymbol{v}\|^8. \qquad (5.13)
\end{aligned}
$$

We set

$$
\begin{aligned}
C_7 \;=\; \max\Bigg\{&\frac{C_5We^2}{8} + \frac{3C_6Re^2}{\delta^2}, 2C^2 + \frac{C_3}{8C_2}, 3c_6Re^2 + C^5, \frac{We^2}{4C_0\omega^4} + \frac{C_5}{4}\frac{3C_3C_6}{\delta^2}, \\
&\frac{C_5\delta}{4\omega^2} + \frac{2C^2}{Re^2}C_6C_2^2\Bigg\},
\end{aligned}
$$

to obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\delta^2}{12C_2Re^2}\|\boldsymbol{v}\|^2 + Re|\boldsymbol{v}'|^2 + + \frac{3\delta We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2 + \frac{3We\delta}{4\omega^2C_0^2}\|\boldsymbol{A}\|_{H^2}^2\right) \\
&+\frac{\delta}{8Re^2}\|\boldsymbol{A}\|_{H^2}^2 + \frac{\delta}{4}\|\boldsymbol{v}'\|^2 + \frac{3\delta We}{4\omega^2C_0^2}|\boldsymbol{A}'|^2 + \frac{\delta^3}{12dRe^2}\|\boldsymbol{v}\|^2
\end{aligned}
$$

$$\leq \frac{3\delta}{4Re^2}\|\boldsymbol{b}\|_{H^1}^2 + \frac{3\delta}{8C_2Re^2}|\boldsymbol{b}|^2 + \frac{3C^2}{\delta}|\boldsymbol{b}'|^2 + \frac{3C_6}{4\gamma\delta}|\boldsymbol{b}|^4 + \frac{3C_5We^2Re^2}{4\omega^2}|\boldsymbol{A}'|^4$$
$$+C_7\left\{\frac{\delta}{Re^2}\|\boldsymbol{A}\|_{H^1}^4 + \frac{1}{\delta}\|\boldsymbol{v}\|^4 + \frac{Re^2}{\delta}|\boldsymbol{v}'|^4 + \frac{\delta}{Re^2}\|\boldsymbol{A}\|_{H^2}^4 + \frac{Re^2}{\delta}\|\boldsymbol{v}'\|^4\right\}$$
$$+\frac{2C^2}{\delta}\|\boldsymbol{A}\|^4 + \frac{2C^2}{\delta}\|\boldsymbol{A}'\|_{H^1}^4 + \|\boldsymbol{v}\|^6 + C_7\frac{3Re^4}{4\delta}\|\boldsymbol{v}\|^8. \tag{5.14}$$

We now prove an important result which will help us to show the global existence of regular solution.

Lemma 5.1 *[16] Let f be a non negative, absolutely continuous function satisfying the inequality*

$$f' + kf \leq \alpha\left(f^2 + f^3 + f^4 + f^6 + f^{2m}\right) + \beta \tag{5.15}$$

where $m \geq 2$, $k > 0$, $\alpha > 0$ *and* $\beta \geq 0$ *are some constants. Let* $M_0 > 0$, *be the unique solution of* $M^{2m-1} + M^5 + M^3 + M^2 + M - \frac{k}{2\alpha} = 0$, *and* $0 < M < M_0$. *If* $f(0) \leq M$, *and* $\beta \leq \frac{kM}{3}$, *then* $f(t)$ *is bounded by* M *for all* $t > 0$.

Proof

The result is proven by contradiction. Assume that there exists $t > 0$ such that $f(t) > M$ and let

$$t^* = \inf\{t \in \mathbb{R}_+, f(t) > M\}. \tag{5.16}$$

Since f is continuous, we have $f(t^*) = M$, and from (5.15) we see that f is differentiable. Therefore for $t > t^*$,

$$f'(t^*) = f'_+(t^*) = \lim_{t\to t^*_+}\frac{f(t) - f(t^*)}{t - t^*} = \lim_{t\to t^*_+}\frac{f(t) - M}{t - t^*} \geq 0,$$
$$f'(t^*) = f'_+(t^*) \geq 0. \tag{5.17}$$

Since $M_0^{2m-1} + M_0^5 + M^3 + M_0^2 + M_0 - \frac{k}{2\alpha} = 0$, $0 < M < M_0$, $f(0) \leq M$, $f(t^*) = M$, and $\beta \leq \frac{kM}{3}$, from (5.15) we obtain

$$f'(t^*) \leq -kf(t^*) + \alpha\left(f^2(t^*) + f^3(t^*) + f^4(t^*) + f^6(t^*) + f^{2m}(t^*)\right) + \beta$$

$$
\begin{aligned}
&\leq -kM + \alpha\left(M^2 + M^3 + M^4 + M^6 + M^{2m}\right) + \frac{kM}{3} \\
&= -kM + \alpha M\left(M + M^2 + M^3 + M^5 + M^{2m-1}\right) + \frac{kM}{3} \\
&\leq -kM + \frac{kM}{2} + \frac{kM}{3} \\
&= -\frac{kM}{6} < 0.
\end{aligned}
$$

Thus, $f'(t^*) < 0$ which contradicts (5.17). Therefore $f'(t^*) \leq 0 \quad \forall t \geq 0.\square$

Corollary 5.1 *Under the hypotheses of Lemma 5.1, f(t)≤M* $\forall t \geq 0$, *and inequality (5.15) implies that*

$$f' + kf \leq \eta f^2 + \beta, \tag{5.18}$$

where

$$\eta = \alpha\left(1 + M + M^2 + M^4 + M^{2m-1}\right) > 0.$$

5.1.2 Global existence of regular solution: linear closure

Let

$$f(t) = \frac{\delta^2}{12C_2 Re^2}\|\boldsymbol{v}(t)\|^2 + Re|\boldsymbol{v}'(t)| + \frac{3\delta We}{4\omega^2 C_0^2}|\boldsymbol{A}'(t)|^2 + \frac{3\delta We}{4\omega^2 C_0^2}\|\boldsymbol{A}(t)\|^2_{H^2}. \tag{5.19}$$

Then inequality (5.14) takes the form:

$$f'(t) + kf(t) \leq \alpha\left(f^2(t) + f^3(t) + f^4(t)\right) + \beta, \tag{5.20}$$

where $k > 0$, $\alpha > 0$ and $\beta \geq 0$ are constants depending on the data.

Theorem 5.1 GLOBAL EXISTENCE FOR THE LINEAR CLOSURE
Let $\partial\Omega \in C^4$. *If* $\delta = \gamma\left(1 - \frac{2}{35}N_p > 0\right)$, *then there exists* ω_0 *satisfying (5.4) and depending on the data, such that, if* $\omega_0 < \omega$ *and* $\boldsymbol{v}_0 \in D(\mathcal{L})$, $\boldsymbol{A}_0 \in \mathbb{H}^2$, $\boldsymbol{b} \in$

$\mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}^2)$, *and* $\boldsymbol{b}' \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}^1)$ *are small enough in their spaces, then the problem (4.32) admits a unique solution* $(\boldsymbol{v}, \boldsymbol{A})$ *defined for all times* t, *and*

$$\begin{aligned} &\boldsymbol{v} \in C_b(\mathbb{R}_+, D(\mathcal{L})) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{H}^3) \\ &\boldsymbol{v}' \in C_b(\mathbb{R}_+, \mathbb{H}) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{V}) \\ &\boldsymbol{A} \in C_b(\mathbb{R}_+, \mathbb{H}^2) \cap \mathbb{X}; \qquad \boldsymbol{A}' \in C_b(\mathbb{R}_+, \mathbb{H}^1). \end{aligned}$$

Proof.

1. For the regularity of the solution on Γ, see Appendix A.2.5

2. From (5.14), we see that the local solution obtained in Theorem 4.1 satisfies the inequality (5.20) with

$$\beta = \frac{3\delta}{4Re^2}\|\boldsymbol{b}\|^2_{L^\infty(0,T;\mathbb{H}^1)} + \frac{\delta}{8C_2Re^2}|\boldsymbol{b}|^2 + \frac{3C^2}{\delta}|\boldsymbol{b}'|^2 + \frac{3C_6}{4\gamma\delta}|\boldsymbol{b}|^4.$$

   By Lemma 5.1, there exists a constant M_0, depending on the data, such that if $f(0) \le M \le M_0$ and $\beta \le \frac{kM}{3}$, then $f(t)$ is bounded for all $t \in \mathbb{R}_+$. Observe that $f(0) \le M$ if $\boldsymbol{v}_0$, $\boldsymbol{A}_0$ and $\boldsymbol{b}$ are small in their respective spaces. Therefore from the hypotheses, if $\boldsymbol{b} \in \mathbb{L}^\infty(\mathbb{R}_+; \mathbb{H}^2)$, $\boldsymbol{b}' \in \mathbb{L}^2(\mathbb{R}_+, \mathbb{H}^1)$, which are satisfied when $\boldsymbol{v}_0 \in D(\mathcal{L})$, $\boldsymbol{b}(0) \in \mathbb{L}^2$ and $\boldsymbol{A}_0 \in \mathbb{H}^2$. We deduce that

$$\begin{aligned} &\boldsymbol{v} \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{V}) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, D(\mathcal{L})), \\ &\boldsymbol{v}' \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{V}), \\ &\boldsymbol{A} \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}^2), \qquad \boldsymbol{A}' \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{L}^2). \end{aligned} \tag{5.21}$$

3. From inequality (5.6) we deduce that $\boldsymbol{v} \in \mathbb{L}^\infty(\mathbb{R}_+; D(\mathcal{L}))$ and from (5.5), that $\boldsymbol{v} \in \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{H}^3)$.
   In the same way (5.21) implies that $\boldsymbol{v} \in C_b(\mathbb{R}_+; D(\mathcal{L}))$, and $(4.32)_1$ implies

that $v' \in C_b(\mathbb{R}_+; \mathbb{H})$.

We write $(4.32)_2$ in the form

$$We\{A' + (v \cdot \nabla)A + A\} = -We\left\{AW - WA - \frac{3\lambda}{7}(AD + DA)\right\} + 2\omega D - \frac{2\lambda}{7}We(A : D)I. \quad (5.22)$$

The right hand side of (5.22) has its first and third term belonging to $\mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{H}^2)$, so that $A \in C_b(\mathbb{R}_+, \mathbb{H}^3)$. This together with (5.21) implies that the right-hand side of (5.22) belongs to $C_b(\mathbb{R}_+, \mathbb{H}^1)$ and that $A' \in C_b(\mathbb{R}, \mathbb{H}^1)$, because also $A'(0) \in \mathbb{H}^1$ (from the hypotheses of the theorem).□

5.1.3 Stability of the global solution around zero

In this section we show that global solution obtained in Theorem 5.1 is stable in the absence of body force.

Theorem 5.2 STABILITY OF SOLUTION IN THE ABSENCE OF BODY FORCE
Under the hypotheses of Theorem 5.1, we assume also that b*=0. Then the solution* (v,A) *obtained in Theorem 5.1 is exponentially stable.*

Proof.

We choose ω_0 such that it satisfies (5.4). Therefore the solution (v,A) satisfies (5.14), (5.20), (5.15) and consequently (5.18), with $\beta = 0$. Therefore, from (5.18) we deduce that

$$\frac{f'}{f(1 - \frac{\eta}{k}f)} \leq -k,$$

which implies that

$$f(t) \leq \frac{f(0)e^{-kt}}{1 - \frac{\eta}{k}f(0)}, \quad (5.23)$$

with

$$1 - \frac{\eta}{k}f \geq 0.$$

In particular, we use (5.19), which with (5.23) implies that

$$\|v\|_{\mathbb{L}^2}^2 + \|A\|_{\mathbb{L}^2}^2 \leq Ke^{-kt}, \tag{5.24}$$

where K is a positive constant depending on the data.≋

5.2 Quadratic closure approximation

In this Section, we show, as in the previous Section, that the unique solution obtained in Theorems 4.3 and 4.4 are defined on $\mathbb{R}_+$, if the data are small enough. To this end we derive first some a priori bounds uniform in time, satisfied by that solution.

5.2.1 Some a priori estimates

We recall first that the local solution $(\boldsymbol{v},\boldsymbol{A})$ satisfies the conditions of Theorems 4.3 and 4.4. In what follows, $(\boldsymbol{v},\boldsymbol{A})$ is the unique local solution of problem (4.74). Now we take the scalar product of $(4.74)_2$ with $\boldsymbol{A}$ in $\mathbb{L}^2(\Omega)$, apply the operator ∇ to $(4.74)_2$, and multiply the resulting equation equation by $\nabla\boldsymbol{A}$, and finally we apply the operator D^2 to $(4.74)_2$ and we multiply the resulting equation by $D^2\boldsymbol{A}$ in $\mathbb{L}^2(\Omega)$. We add these resulting equations, and as in the previous Section, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\eta\|\boldsymbol{A}\|_{\mathbb{H}^2}^2) + \|\boldsymbol{A}\|_{\mathbb{H}^2}^2 = 2\lambda\eta(\boldsymbol{D},\boldsymbol{A})_{\mathbb{H}^2} + \eta(\boldsymbol{W}\boldsymbol{A} - \boldsymbol{A}\boldsymbol{W}, \boldsymbol{A})_{\mathbb{H}^2}\\ &-2\lambda\eta\int_\Omega\{v_{k,l}\,[\boldsymbol{A}_{,k} : \boldsymbol{A}_{,l} + 2\boldsymbol{A}_{,km} : \boldsymbol{A}_{,lm}] + v_{k,lm}\boldsymbol{A}_{,k} : \boldsymbol{A}_{,lm}\}d\boldsymbol{x}\\ &-\lambda\eta\left[\left((\boldsymbol{A}:\boldsymbol{D})(\boldsymbol{A}+\frac{1}{n}\boldsymbol{I}),\boldsymbol{A}\right)_{\mathbb{H}^2} + (\boldsymbol{A}\boldsymbol{D}+\boldsymbol{D}\boldsymbol{A},\boldsymbol{A})_{\mathbb{H}^2}\right].\end{aligned} \tag{5.25}$$

Using (4.16)–(4.19) and Young's inequality, we deduce from (5.25) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\eta\|\boldsymbol{A}\|_{\mathbb{H}^2}^2\right) + \|\boldsymbol{A}\|_{H^2}^2 \;\leq\;& 2\lambda\eta\|\boldsymbol{v}\|_{H^3}\|\boldsymbol{A}\|_{H_s^2} + C_0\eta\|\boldsymbol{v}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|_{\mathbb{H}^2}^2\\ &+C_0\eta\|\boldsymbol{v}\|_{\mathbb{H}^3}\|\boldsymbol{A}\|_{\mathbb{H}^2}^3\end{aligned}$$

$$\leq \epsilon_1\lambda^2\eta^2\|v\|^2_{\mathbb{H}^3} + \frac{1}{2\epsilon_1}\|A\|^2_{\mathbb{H}^2} + C_0^2\eta^2\frac{\epsilon_2}{2}\|A\|^4_{\mathbb{H}^2}$$
$$+\frac{1}{2\epsilon_2}\|v\|^2_{\mathbb{H}^3} + C_0^2\eta^2\frac{\epsilon_3}{2}\|A\|^6_{\mathbb{H}^2} + \frac{1}{2\epsilon_3}\|v\|^2_{\mathbb{H}^3}.$$

We choose $\epsilon_1 = 2/3$, $\epsilon_2 = \epsilon_3 = 6/(\lambda^2\eta^2)$, and we set $C_1 = 6C_0^2$; the previous inequality now reads

$$\frac{d}{dt}\left(\eta\|A\|^2_{\mathbb{H}^2}\right) + \frac{1}{2}\|A\|^2_{\mathbb{H}^2} \leq 3\lambda^2\eta^2\|v\|^2_{\mathbb{H}^3} + \frac{C_1}{\lambda^2}\|A\|^4_{\mathbb{H}^2} + \frac{C_1}{\lambda^2}\|A\|^6_{\mathbb{H}^2}. \tag{5.26}$$

From $(4.74)_1$, we deduce an estimate for $\|v\|_{\mathbb{H}^3}$, in the form

$$\|v\|^2_{\mathbb{H}^3} \leq C_0\left\{Re^2\|v'\|^2 + \|A\|^2_{\mathbb{H}^2} + \|A\|^4_{\mathbb{H}^2} + \|b\|_{\mathbb{H}^1} + C_2Re^2|\mathcal{L}v|^4\right\}. \tag{5.27}$$

From (5.26) and (5.27) it follows that

$$\frac{d}{dt}\left(\eta\|A\|^2_{\mathbb{H}^2}\right) + \left(\frac{1}{2} - 3C_0\lambda^2\eta^2\right)\|A\|^2_{\mathbb{H}^2} \leq \left(3\lambda^2\eta^2C_0 + \frac{C_1}{\lambda^2}\right)\|A\|^4_{\mathbb{H}^2}$$
$$+\frac{C_1}{\lambda^2}\|A\|^6_{\mathbb{H}^2} + 3C_0\lambda^2\eta^2\left(Re\|v\|^2 + \|b\|^2_{\mathbb{H}^1} + C_2Re^2|\mathcal{L}v|^4\right). \tag{5.28}$$

Remark. Inequality (5.28) will be used only for values of η such that the coefficient of $\|A\|^2_{\mathbb{H}^2}$ in the left-hand side is positive. Specifically, we will choose η such that the coefficient of $\|A\|^2_{\mathbb{H}^2}$ is larger than $C_0\lambda^2\eta^2$, so that

$$\frac{1}{2} - 4C_0\lambda^2\eta^2 \geq 0. \tag{5.29}$$

We write $(4.74)_1$ in the form

$$\mathcal{L}v = \frac{1}{\delta}\left\{-Rev' + b + \text{div}\gamma\left[N_p(A:D)(A + \frac{1}{n}I) + N_s(AD + DA)\right]\right.$$
$$\left. -Re(v\cdot\nabla)v\right\}, \tag{5.30}$$

and deduce from (5.30) the inequality

$$|\mathcal{L}v|^2 \leq \frac{c_0}{\delta}\left\{Re^2|v'|^2 + \|A\|^2_{\mathbb{H}^2} + \|A\|^4_{\mathbb{H}^1} + |b|^2 + c_2Re^2\|v\|^4\right\}. \tag{5.31}$$

Now we take the scalar product of $(4.74)_1$ in $\mathbb{H}$ with $\mathcal{L}\boldsymbol{v}$, and find that there exists a constant C_3 such that

$$\begin{aligned}\frac{d}{dt}\left(Re\|\boldsymbol{v}\|^2\right)+\delta|\mathcal{L}\boldsymbol{v}|^2 \;\le\; & \frac{2}{\delta}\Big(|\boldsymbol{b}|^2+\|\boldsymbol{A}\|^4_{\mathbb{H}^1}+C_2\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\\ & +C_3\frac{Re^2}{\delta^2}\|\boldsymbol{v}\|^4\Big). \qquad (5.32)\end{aligned}$$

Next, we differentiate $(4.74)_1$ with respect to t, and take the scalar product in $\mathbb{H}$ of the resulting equation with $\boldsymbol{v}'$, to obtain

$$\begin{aligned}\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2\right)+\delta\|\boldsymbol{v}'\|^2 \;\le\; & \frac{2C^2}{\delta}|\boldsymbol{b}'|^2+\frac{\delta}{8}\|\boldsymbol{v}'\|^4+\frac{\epsilon}{8}|\boldsymbol{v}'|^4+\frac{Re^2}{2\epsilon}\|\boldsymbol{v}\|^2_{\mathbb{H}^3}\\ & +\frac{C^2}{\delta}\Big(\|\boldsymbol{A}\|^8_{\mathbb{H}^1}+\|\boldsymbol{v}\|^8+\frac{1}{2}\|\boldsymbol{A}'\|^8\\ & +\frac{5}{2}\|\boldsymbol{A}\|^4_{\mathbb{H}^1}+\|\boldsymbol{A}'\|^4+\|\boldsymbol{v}\|^4_{\mathbb{H}^2}+\|\boldsymbol{v}'\|^4\Big). \qquad (5.33)\end{aligned}$$

We also differentiate $(4.74)_2$ with respect to t, and take the scalar product in $\mathbb{H}$ of the resulting equation with $(\delta/80C^2\lambda^2\eta^2)\boldsymbol{A}'$ to have a positive coefficient of $\|\boldsymbol{v}\|^2_{\mathbb{H}^3}$ in (5.35). We thus obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\frac{\delta}{80C^2\lambda^2\eta}|\boldsymbol{A}'|^2\right)+\frac{\delta}{160C^2\lambda^2\eta^2}|\boldsymbol{A}'|^2 \;\le\; & \frac{\delta}{4}\|\boldsymbol{v}'\|^2+\frac{3\epsilon\delta}{160C^2\lambda^2\eta}\|\boldsymbol{v}\|^2_{\mathbb{H}^3}\\ & +\frac{\delta}{16C\lambda^2\eta^2\epsilon}\left(|\boldsymbol{A}|^4+|\boldsymbol{A}'|^4\right)\\ & +\frac{\delta}{1600C\lambda^2\eta^2}\Big(|\boldsymbol{A}|^8+\|\boldsymbol{A}\|^4_{\mathbb{H}^1}\\ & +|\boldsymbol{v}'|^4+\|\boldsymbol{v}'\|^4+|\boldsymbol{A}'|^4\Big). \qquad (5.34)\end{aligned}$$

Adding (5.33) and (5.34), we have

$$\begin{aligned}&\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{\delta}{80C^2\lambda^2\eta}|\boldsymbol{A}'|^2\right)+\frac{\delta}{2}\|\boldsymbol{v}'\|^2+\frac{\delta}{160C^2\lambda^2\eta^2}|\boldsymbol{A}'|^2\\ &\le\frac{4C^2}{\delta}|\boldsymbol{b}'|^2+\frac{3\epsilon\delta}{80C^2\lambda^2\eta}\|\boldsymbol{v}\|^2_{\mathbb{H}^3}+\frac{\delta}{8C\lambda^2\eta^2\epsilon}\left(|\boldsymbol{A}|^4+|\boldsymbol{A}'|^4\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{2C^2}{\delta}\left(\|\boldsymbol{A}\|_{\mathbb{H}^1}^2+\|\boldsymbol{v}\|^8+\frac{1}{2}\|\boldsymbol{A}'\|^8+\frac{5}{8}\|\boldsymbol{A}\|_{\mathbb{H}^1}^4+\|\boldsymbol{A}'\|^4+\|\boldsymbol{v}'\|^4+\|\boldsymbol{A}\|_{\mathbb{H}^2}^4\right)\\
&+\frac{\delta}{800C\lambda^2\eta^2}\left(|\boldsymbol{A}|^8+\|\boldsymbol{A}\|_{\mathbb{H}^1}^4+|\boldsymbol{v}'|^4+\|\boldsymbol{v}'\|^4+|\boldsymbol{A}'|^4\right)\\
&+\frac{\delta}{4}\|\boldsymbol{v}'\|^4+\frac{1}{4\epsilon}|\boldsymbol{v}'|^4+Re^2\epsilon\|\boldsymbol{v}\|_{\mathbb{H}^3}^4. \qquad (5.35)
\end{aligned}
$$

We multiply (5.28) by $\delta/(12C_0\lambda^2\eta^2 Re^2)$, to obtain a positive coefficient of $\|\boldsymbol{v}'\|^2$, and add the resulting inequality to (5.35). We also recall from (5.29) that $\frac{1}{2}-3C_0\lambda^2\eta^2 \geq C_0\lambda^2\eta^2$, hence we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{\delta}{80C^2\lambda^2\eta}|\boldsymbol{A}'|^2+\frac{\delta}{12C_0\lambda^2\eta Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}^2\right)\\
&\frac{\delta}{12Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}^2+\frac{\delta}{2}\|\boldsymbol{v}'\|^2+\frac{\delta}{160C^2\lambda^2\eta^2}|\boldsymbol{A}'|^2\\
&\leq \frac{4C^2}{\delta}|\boldsymbol{b}'|^2+\frac{3\epsilon\delta}{80C^2\lambda^2\eta}\|\boldsymbol{v}\|_{\mathbb{H}^3}^2+\frac{\delta}{4Re^2}\|\boldsymbol{b}\|_{\mathbb{H}^1}^2+\frac{C_2\delta}{4}|\mathcal{L}\boldsymbol{v}|^4+Re^2\epsilon\|\boldsymbol{v}\|_{\mathbb{H}^3}^4\\
&+C_4\left\{\frac{1}{\epsilon}\left[\left(\frac{1}{4}+\frac{\delta}{800C\lambda^2\eta^2}\right)|\boldsymbol{v}'|^4+\delta\left(\frac{\epsilon}{800C\lambda^2\eta^2}+\frac{1}{8C\lambda^2\eta^2}\right)|\boldsymbol{A}'|^4\right]\right.\\
&\left.+\left(\frac{5\epsilon\delta}{4C\lambda^2\eta^2}+\frac{2C^2\epsilon}{\delta}\right)\|\boldsymbol{A}\|_{\mathbb{H}^1}^4\right\}+\frac{\delta}{12C_0\lambda^2\eta^2Re^2}\left(3\lambda^2\eta^2C_0+\frac{C_1}{\lambda^2}\right)\|\boldsymbol{A}\|_{\mathbb{H}^2}^4\\
&+\frac{2C^2}{\delta}\left(\|\boldsymbol{A}\|_{\mathbb{H}^1}^8+\|\boldsymbol{v}\|^8+\frac{1}{2}\|\boldsymbol{A}'\|^8+\|\boldsymbol{A}'\|4+\|\boldsymbol{v}'\|^4+\|\boldsymbol{v}\|_{\mathbb{H}^2}^4\right)\\
&+\frac{C_1\delta}{12C_0\lambda^4\eta^2Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}^6+\frac{\delta}{4}\|\boldsymbol{v}'\|^4. \qquad (5.36)
\end{aligned}
$$

We make use of (5.27) to estimate $\|\boldsymbol{v}\|_{\mathbb{H}^2}^2$ in (5.36), which now reads

$$
\begin{aligned}
&\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{\delta}{80C^2\lambda^2\eta}|\boldsymbol{A}'|^2+\frac{\delta}{12C_0\lambda^2\eta Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}^2\right)\\
&\frac{\delta}{12Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}^2+\frac{\delta}{2}\|\boldsymbol{v}'\|^2+\frac{\delta}{160C^2\lambda^2\eta^2}|\boldsymbol{A}'|^2\\
&\leq \frac{4C^2}{\delta}|\boldsymbol{b}'|^2+\frac{\delta}{4Re^2}\|\boldsymbol{b}\|_{\mathbb{H}^1}^2+\frac{C_2\delta}{4}|\mathcal{L}\boldsymbol{v}|^4\\
&\frac{3\epsilon\delta C_0}{80C^2\lambda\eta}\left(Re^2\|\boldsymbol{v}'\|^2+\|\boldsymbol{A}\|_{\mathbb{H}^2}^2+\|\boldsymbol{A}\|_{\mathbb{H}^2}^4+\|\boldsymbol{b}\|_{\mathbb{H}^1}^2+C_2Re^2|\mathcal{L}\boldsymbol{v}|^4\right)
\end{aligned}
$$

$$
\begin{aligned}
&+C_4\left\{\frac{1}{\epsilon}\left[\left(\frac{1}{4}+\frac{\delta}{800C\lambda^2\eta^2}\right)|\boldsymbol{v}'|^4+\frac{\delta}{4C\lambda^2\eta^2}(\epsilon+1)|\boldsymbol{A}'|^4\right]\right.\\
&\left.+\left(\frac{5\epsilon\delta}{4C\lambda^2\eta^2}+\frac{2C^2\epsilon}{\delta}\right)\|\boldsymbol{A}\|_{\mathbb{H}^1}^4\right\}+\frac{\delta}{12Re^2}\left(3+\frac{C_1}{C_0\lambda^4\eta^2}\right)\|\boldsymbol{A}\|_{\mathbb{H}^2}^4\\
&+\frac{2C^2}{\delta}\left(\|\boldsymbol{A}\|_{\mathbb{H}^1}^8+\|\boldsymbol{v}\|^8+\frac{1}{2}\|\boldsymbol{A}'\|^8+\|\boldsymbol{A}'\|^4+\|\boldsymbol{v}'\|^4+\|\boldsymbol{v}\|^4\right)\\
&+C_4Re^2\epsilon\left\{Re^4\|\boldsymbol{v}'\|^4+\|\boldsymbol{A}\|_{\mathbb{H}^2}^4+\|\boldsymbol{A}\|_{\mathbb{H}^2}^8+\|\boldsymbol{b}\|_{\mathbb{H}^1}^4+Re^4|\mathcal{L}\boldsymbol{v}|^8\right\}\\
&+\frac{C_1\delta}{12C_0\lambda^4\eta^2Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}^6+\frac{\delta}{4}\|\boldsymbol{v}'\|^4. \qquad (5.37)
\end{aligned}
$$

We choose $\epsilon = 10C^2\lambda^2\eta/(9Re^2C_0)$, to obtain a positive coefficient of $\|\boldsymbol{A}\|_{\mathbb{H}^2}^2$; inequality (5.37) now reads

$$
\begin{aligned}
&\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2+\frac{\delta}{80C^2\lambda^2\eta}|\boldsymbol{A}'|^2+\frac{\delta}{12C_0\lambda^2\eta Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}^2\right)\\
&+\frac{\delta}{24Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}^2+\frac{\delta}{24}\|\boldsymbol{v}'\|^2+\frac{\delta}{160C^2\lambda^2\eta^2}|\boldsymbol{A}'|^2\\
&\le\frac{4C^2}{\delta}|\boldsymbol{b}'|^2+\frac{7\delta}{8Re^2}\|\boldsymbol{b}\|_{\mathbb{H}^1}^2+\frac{7C_2\delta}{24}|\mathcal{L}\boldsymbol{v}|^4+\frac{10C^2C_4\lambda^2\eta}{9C_0}\|\boldsymbol{b}\|_{\mathbb{H}^1}^4\\
&+\frac{\delta}{12Re^2}\left(7+\frac{C_1}{C_0\lambda^4\eta^2}+\frac{120C_4C^2\lambda^2\eta Re^2}{9C_0\delta}\right)\|\boldsymbol{A}\|_{\mathbb{H}^2}^4+\frac{C_1\delta}{12C_0\lambda^4\eta^2Re^2}\|\boldsymbol{A}\|_{\mathbb{H}^2}^6\\
&+\frac{C_4}{2C^2\lambda^2\eta}\left\{\frac{9Re^2C_0}{5}\left(\frac{1}{4}+\frac{\delta}{800C\lambda^2\eta^2}\right)|\boldsymbol{v}'|^4+\frac{\delta}{2C\lambda^2\eta^2}\left(\frac{10C^2\lambda^2\eta}{9Re^2C_0}+1\right)|\boldsymbol{A}'|^4\right\}\\
&+\frac{2C^2}{\delta}\left(\|\boldsymbol{A}\|_{\mathbb{H}^1}^8+\|\boldsymbol{v}\|^8+\frac{1}{2}\|\boldsymbol{A}'\|^8+\|\boldsymbol{A}'\|^4+\|\boldsymbol{v}\|^4\right)\\
&+\frac{10CC_4\delta}{9Re^2C_0}\left(\frac{5}{4}+\frac{2C^3\lambda^2\eta}{\delta^2}\right)\|\boldsymbol{A}\|_{\mathbb{H}^1}^4+\frac{\delta}{4}\left(1+\frac{40C_4C^2\lambda^2\eta Re^4}{9C_0\delta}+\frac{8C^2}{\delta^2}\right)\|\boldsymbol{v}'\|^4\\
&+\frac{10C_4C^2\lambda^2\eta}{9C_0}\left(\|\boldsymbol{A}\|^8+Re^4|\mathcal{L}\boldsymbol{v}|^8\right).
\end{aligned}
$$

We set

$$
\begin{aligned}
C_5 = \quad \max \quad &\left\{\frac{10C^2C_4}{9C_0}, 7+\frac{C_1}{C_0\lambda^4\eta^2}+\frac{120C_4C^2\lambda^2\eta Re^2}{9C_0\delta},\right.\\
&\frac{C_4}{2C^2\lambda^2\eta}\left[\left(\frac{1}{4}+\frac{\delta}{800C\lambda^2\eta^2}\right)+\frac{1}{2C}\left(\frac{10C^2\lambda^2\eta}{9Re^2C_0}+1\right)\right],
\end{aligned}
$$

$$1 + \frac{40C_4C^2\lambda^2\eta Re^4}{9C_0\delta} + \frac{8C^2}{\delta^2}, \frac{10C_4C^2}{9C_0}, 10\left(\frac{5}{4} + \frac{2C^3\lambda\eta}{\delta^2}\right)\frac{CC_4}{C_0}\Big\};$$

the previous inequality now reads

$$\begin{aligned}
&\frac{d}{dt}\left(Re|\boldsymbol{v}'|^2 + \frac{\delta}{80C^2\lambda^2\eta}|\boldsymbol{A}'|^2 + \frac{\delta}{12C_0\lambda^2\eta Re^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\right)\\
&\frac{\delta}{24Re^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2} + \frac{\delta}{24}\|\boldsymbol{v}'\|^2 + \frac{\delta}{160C^2\lambda^2\eta^2}|\boldsymbol{A}'|^2\\
&\leq \frac{4C^2}{\delta}|\boldsymbol{b}'|^2 + \frac{7\delta}{8Re^2}\|\boldsymbol{b}\|^2_{\mathbb{H}^1} + \frac{7C_2\delta}{24}|\mathcal{L}\boldsymbol{v}|^4 + C_5\lambda^2\eta\|\boldsymbol{b}\|^4_{\mathbb{H}^1} + C_5\frac{\delta}{12Re^2}\|\boldsymbol{A}\|^4_{\mathbb{H}^2}\\
&+C_5\left\{\frac{9Re^2C_0}{5}|\boldsymbol{v}'|^4 + \frac{\delta}{2\lambda^2\eta^2}|\boldsymbol{A}'|^4\right\} + C_5\frac{\delta}{9Re^2}\|\boldsymbol{A}\|^4_{\mathbb{H}^1} + \frac{C_1\delta}{12C_0\lambda^4\eta^2Re^2}\|\boldsymbol{A}\|^6_{\mathbb{H}^2}\\
&+C_5\frac{\delta}{4}\|\boldsymbol{v}'\|^4 + \frac{2C^2}{\delta}\left(\|\boldsymbol{A}\|^8_{\mathbb{H}^1} + \|\boldsymbol{v}\|^8 + \frac{1}{2}\|\boldsymbol{A}'\|^8 + \|\boldsymbol{A}'\|^4 + \|\boldsymbol{v}\|^4\right)\\
&+C_5\lambda^2\eta\left(\|\boldsymbol{A}\|^8 + Re^4|\mathcal{L}\boldsymbol{v}|^8\right).
\end{aligned} \tag{5.38}$$

We multiply (5.32) by $\delta^2/(96C_2Re^2)$, and add the resulting inequality to (5.38); we also make use of (5.31) to estimate $|\mathcal{L}\boldsymbol{v}|^4$ in (5.38). It follows that there exists C_6 such that

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{\delta^2}{92C_2Re}\|\boldsymbol{v}\|^2 + Re|\boldsymbol{v}'|^2 + \frac{\delta}{80C^2\lambda^2\eta}|\boldsymbol{A}'|^2 + \frac{\delta}{12C_0\lambda^2\eta Re^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\right)\\
&+\frac{\delta}{48Re^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2} + \frac{\delta}{24}\|\boldsymbol{v}'\|^2 + \frac{\delta}{160C^2\lambda^2\eta^2}|\boldsymbol{A}'|^2 + \frac{\delta^3}{96C_2Re^2}|\mathcal{L}\boldsymbol{v}|^2\\
&\leq \frac{4C^2}{\delta}|\boldsymbol{b}'|^2 + \frac{7\delta}{8Re^2}\|\boldsymbol{b}\|^2_{\mathbb{H}^1} + \frac{C_0\delta}{48C_2Re^2}|\boldsymbol{b}|^2 + C_5\lambda^2\eta\|\boldsymbol{b}\|^4_{\mathbb{H}^1} + \frac{\delta}{Re^2}\left(\frac{C_0}{48C_2} + \frac{C_5}{9}\right)\|\boldsymbol{A}\|^4_{\mathbb{H}^1}\\
&+\frac{1}{\delta}\left(\frac{C_0C_3}{48C_2} + 2C^2\right)\|\boldsymbol{v}\|^4 + C_5\lambda^2\eta\|\boldsymbol{A}\|^8 + \frac{Re^2}{\delta}\left(\frac{7C_6Re^2}{24} + \frac{9C_5C_0\delta}{5}\right)|\boldsymbol{v}'|^4\\
&+\frac{Re^2}{\delta}\left(\frac{7C_6Re^2}{24} + \frac{9C_5C_0\delta}{5}\right)|\boldsymbol{v}'|^4 + \frac{1}{\delta}\left(\frac{7C_2C_6Re^4}{24} + 2C^2\right)\|\boldsymbol{v}\|^8\\
&+\frac{\delta}{12Re^2}\left(\frac{7C_6}{2\delta} + C_5\right)\|\boldsymbol{A}\|^4_{\mathbb{H}^2} + \frac{C_6}{24\delta}|\boldsymbol{b}|^4\\
&+\frac{C_1\delta}{12C_0\lambda^4\eta^2Re^2}\|\boldsymbol{A}\|^6_{\mathbb{H}^2} + \frac{1}{\delta}\left(\frac{7C_6}{24} + 2C^2\right)\|\boldsymbol{A}\|^8_{\mathbb{H}^1}
\end{aligned}$$

$$+C_5\frac{\delta}{4}\|\boldsymbol{v}'\|^4 + \frac{2C^2}{\delta}\left(\frac{1}{2}\|\boldsymbol{A}'\|^8 + \|\boldsymbol{A}'\|^4\right) + C_5\frac{\delta}{2\lambda^2\eta^2}|\boldsymbol{A}'|^4$$
$$+Re^4\frac{C_6}{\delta^4}\left\{Re^8|\boldsymbol{v}'|^8 + \|\boldsymbol{A}\|^8_{\mathbb{H}^2} + \|\boldsymbol{A}\|^{16}_{\mathbb{H}^1} + |\boldsymbol{b}|^8 + C_2Re^2\|\boldsymbol{A}\|^{16}\right\}. \quad (5.39)$$

Next, we set

$$C_7 = \max\left\{\frac{C_0}{48C_2} + \frac{C_5}{9}, \frac{C_0C_3}{48C_2} + 2C^2, \frac{7C_6Re^2}{24} + \delta\frac{9C_5C_0}{5}, \frac{7C_2C_6Re^4}{24} + 2C^2, \frac{7C_6}{2\delta} + C_5, \frac{7C_6}{24} + 2C^2, C_6, \frac{C_5\lambda^2\eta\delta^4}{Re^2}\right\}.$$

Since $\|\boldsymbol{v}\|^2 \le d|\mathcal{L}\boldsymbol{v}|^2$, for positive d the inequality (5.39) now reads

$$\frac{d}{dt}\left(\frac{\delta^2}{92C_2Re}\|\boldsymbol{v}\|^2 + Re|\boldsymbol{v}'|^2 + \frac{\delta}{80C^2\lambda^2\eta}|\boldsymbol{A}'|^2 + \frac{\delta}{12C_0\lambda^2\eta Re^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2}\right)$$
$$+\frac{\delta}{48Re^2}\|\boldsymbol{A}\|^2_{\mathbb{H}^2} + \frac{\delta}{24}\|\boldsymbol{v}'\|^2 + \frac{\delta}{160C^2\lambda^2\eta^2}|\boldsymbol{A}'|^2 + \frac{\delta^3}{96C_2Re^2d}\|\boldsymbol{v}\|^2$$
$$\le \frac{4C^2}{\delta}|\boldsymbol{b}'|^2 + \frac{7\delta}{8Re^2}\|\boldsymbol{b}\|^2_{\mathbb{H}^1} + \frac{C_0\delta}{48C_2Re^2}|\boldsymbol{b}|^2 + C_5\lambda^2\eta\|\boldsymbol{b}\|^4_{\mathbb{H}^1} + \frac{7C_6}{24\delta}|\boldsymbol{b}|^4 + \frac{C_6Re^4}{\delta^4}|\boldsymbol{b}|^8$$
$$+\frac{C_7}{\delta}\left(\|\boldsymbol{v}\|^4 + \|\boldsymbol{A}'\|^4 + Re^2|\boldsymbol{v}'|^4\right) + C_7\frac{\delta}{12Re^2}\left(\|\boldsymbol{v}\|^4_{\mathbb{H}^2} + 12\|\boldsymbol{v}\|^4_{\mathbb{H}^1}\right)$$
$$+C_5\frac{\delta}{4}\left(\|\boldsymbol{v}'\|^4 + \frac{2}{\lambda^2\eta^2}|\boldsymbol{A}'|^4\right) + \frac{C_7}{\delta}\left(\|\boldsymbol{A}\|^8_{\mathbb{H}^1} + \|\boldsymbol{A}'\|^8 + \|\boldsymbol{v}\|^8\right)$$
$$+Re^4\frac{C_7}{\delta^4}\left(Re^8|\boldsymbol{v}'|^8 + \|\boldsymbol{A}\|^8_{\mathbb{H}^2} + \|\boldsymbol{A}\|^8\right) + \frac{C_1\delta}{12C_0\lambda^4\eta^2Re^2}\|\boldsymbol{A}\|^6_{\mathbb{H}^2}$$
$$+Re^4\frac{C_7}{\delta^4}\left(\|\boldsymbol{A}\|^{16}_{\mathbb{H}^1} + C_2Re^2\|\boldsymbol{A}\|^{16}\right). \quad (5.40)$$

5.2.2 Global existence of regular solution: quadratic closure

Theorem 5.3 GLOBAL EXISTENCE FOR THE QUADRATIC CLOSURE
Let $\partial\Omega \in C^4$. There exists η_0 satisfying (5.29) and depending on γ, Ω, N_s and on the data, such that, if $\eta_0 < \eta$ and $\boldsymbol{v}_0 \in D(\mathcal{L})$, $\boldsymbol{A}_0 \in \mathbb{H}^2$, $\boldsymbol{b} \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}^2)$, and $\boldsymbol{b}' \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}^1)$ are small enough in their spaces, then the problem (4.74) admits a

unique solution $(\boldsymbol{v},\boldsymbol{A})$ defined for all times t, and

$$\begin{aligned} &\boldsymbol{v} \in C_b(\mathbb{R}_+, D(\mathcal{L})) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{H}^3)\\ &\boldsymbol{v}' \in C_b(\mathbb{R}_+, \mathbb{H}) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{V})\\ &\boldsymbol{A} \in C_b(\mathbb{R}_+, \mathbb{H}^2) \cap \mathbb{X}; \qquad \boldsymbol{A}' \in C_b(\mathbb{R}_+, \mathbb{H}^1)\end{aligned}$$

Proof.

Let

$$f(t) = \frac{\delta^2}{92C_2Re}\|\boldsymbol{v}(t)\|^2 + Re|\boldsymbol{v}'(t)| + \frac{\delta}{80C^2\lambda^2\eta}|\boldsymbol{A}'(t)|^2 + \frac{\delta}{12C_0\lambda^2\eta Re^2}\|\boldsymbol{A}(t)\|^2_{\mathbb{H}^2}. \tag{5.41}$$

Then inequality (5.40) takes the form

$$f'(t) + kf(t) \leq \alpha\Big(f^2(t) + f^3(t) + f^6(t) + f^8(t)\Big) + \beta, \tag{5.42}$$

where $k > 0$, $\alpha > 0$ and $\beta \geq 0$ are some constants depending on the data, and f satisfies the hypotheses of Lemma 5.1, with $n = 4$ and

$$\begin{aligned}\beta \;=\;& \frac{4C^2}{\delta}|\boldsymbol{b}'|^2 + \frac{7\delta}{8Re^2}\|\boldsymbol{b}\|^2_{\mathbb{H}^1} + \frac{C_0\delta}{48C_2Re^2}|\boldsymbol{b}|^2 + C_5\lambda^2\eta\|\boldsymbol{b}\|^4_{\mathbb{H}^1}\\ &+\frac{7C_6}{24\delta}|\boldsymbol{b}|^4 + \frac{C_6Re^4}{\delta^4}|\boldsymbol{b}|^8.\end{aligned}$$

By Lemma 5.1, there exists a constant M_0, depending on $\Omega, N_s, \eta, \gamma$ and Re such that if $f(0) \leq M \leq M_0$ and $\beta \leq \frac{kM}{3}$, then $f(t)$ is bounded for all $t \in \mathbb{R}_+$. Observe that $f(0) \leq M$ if $\boldsymbol{v}_0$, $\boldsymbol{A}_0$ and $\boldsymbol{b}$ are small in their respective spaces. Therefore from the hypotheses, if $\boldsymbol{b} \in \mathrm{L}^\infty(\mathbb{R}_+; \mathbb{H}^2)$, $\boldsymbol{b}' \in \mathbb{L}^2$, which are satisfied when $\boldsymbol{v}_0 \in D(\mathcal{L})$, $\boldsymbol{b}(0) \in \mathbb{L}^2$ and $\boldsymbol{A}_0 \in \mathbb{H}^2$. We deduce that

$$\begin{aligned} &\boldsymbol{v} \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{V}) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, D(\mathcal{L}))\\ &\boldsymbol{v}' \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}) \cap \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{V})\\ &\boldsymbol{A} \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{H}^2); \qquad \boldsymbol{A}' \in \mathbb{L}^\infty(\mathbb{R}_+, \mathbb{L}^2).\end{aligned} \tag{5.43}$$

From inequality (5.31) we deduce that $\boldsymbol{v} \in \mathbb{L}^\infty(\mathbb{R}_+; D(\mathcal{L}))$ and from (5.30) that $\boldsymbol{v} \in \mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{H}^3)$. And inequality (5.43) implies that $\boldsymbol{v} \in C_b(\mathbb{R}_+; D(\mathcal{L}))$, and $(4.74)_1$ implies that $\boldsymbol{v}' \in C_b(\mathbb{R}_+; \mathbb{H})$.

We write $(4.74)_2$ in the form

$$\begin{aligned} \eta\Big(\boldsymbol{A}' + (\boldsymbol{v}\cdot\nabla)\boldsymbol{A}\Big) + \boldsymbol{A} &= -\eta\Big\{\boldsymbol{A}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{A} - \lambda(\boldsymbol{A}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{A})\Big\} \\ &\quad +2\lambda\eta\boldsymbol{D} - 2\lambda\eta(\boldsymbol{A}:\boldsymbol{D})\Big(\boldsymbol{A} + \frac{1}{n}\boldsymbol{I}\Big). \end{aligned} \tag{5.44}$$

The right hand side of (5.44) has its first and third term belonging to $\mathbb{L}^2_{loc}(\mathbb{R}_+, \mathbb{H}^2)$, then $\boldsymbol{A} \in C_b(\mathbb{R}_+, \mathbb{H}^3)$. This together with (5.43) implies that the right-hand side of (5.44) belongs to $C_b(\mathbb{R}_+, \mathbb{H}^1)$ and that $\boldsymbol{A}' \in C_b(\mathbb{R}, \mathbb{H}^1)$, because also $\boldsymbol{A}'(0) \in \mathbb{H}^1$ (from the hypotheses of the theorem).□

5.2.3 Stability of the global solution around zero: quadratic closure

In this section we show that the we show that global solution obtained in Theorem 5.3 is stable with zero body force.

Theorem 5.4 STABILITY OF SOLUTION AROUND ZERO FOR THE QUADRATIC CLOSURE

Under the hypotheses of Theorem 5.3, we assume also that $\boldsymbol{b}=\boldsymbol{0}$. *Then the solution* $(\boldsymbol{v},\boldsymbol{A})$ *obtained in Theorem 5.3 is exponentially stable.*

Proof

We choose η_0 such that it satisfies (5.29). Therefore the solution $(\boldsymbol{v}, \boldsymbol{A})$ satisfies (5.40), (5.41), (5.42) and consequently (5.18), with $\beta = 0$. Therefore, from (5.18) we deduce that

$$\frac{f'}{f(1-\frac{\nu}{k}f)} \le -k$$

and hence, for $f(0) < \frac{k}{\nu}$, we deduce

$$f(t) \leq \frac{f(0)}{1 - \frac{\nu}{k} f(0)} \exp\{-kt\}. \tag{5.45}$$

In particular inequality (5.45) implies

$$\|\boldsymbol{v}\|^2_{\mathbb{L}^2} + \|\boldsymbol{A}\|^2_{\mathbb{L}^2} \leq K \exp\{-kt\} \tag{5.46}$$

where K is some positive constant depending on the data.□

Chapter 6

Discussion and Conclusions

In this Chapter we discuss the main results of the thesis. First we comment on the influence of the choice of closure on thermodynamic consistency and on energetic stability. Next, we return to the the problem of existence and uniqueness of solutions local and global in time.

6.1 Results on Thermodynamic Consistency

Chapter 3 has focussed on the influence of the choice of closure rule on thermodynamic consistency, and on energetic stability. In the form of a dissipation inequality, the second law has been used as a means of determining the restrictions, if any, which must be met by the constitutive equations. In the present context it is restrictions on the constitutive equation for the stress that are relevant. The study has not been exhaustive, in the sense that only a selection of closure rules has been chosen for investigation, and restrictions have been imposed on the constitutive equation for the stress only. Nevertheless, each of the closures studied performs well in selected flow situations. The spectrum of flows which are modelled accurately by at least one of the closures in the study is fairly comprehensive. The techniques and results presented here provide an additional means of understanding the degree of acceptibility of closure rules, over and above existing rheological or numerical assessments. Indeed, the second law of thermodynamics places a simple restriction on the choice of particle number as a requirement for consistency – in the case of the linear closure. On the other hand, the results in respect of the HINCH-LEAL closure are inconclusive, while the smooth orthotropic closure is shown, despite its good performance in selected flow simulations, not to be consistent for all possible flows.

Other closure rules may also be investigated using the methods presented here; the extent to which it is possible to obtain simple restrictions on such closures, for example in the form of bounds on material constants, will depend on the degree of complexity of the closures.

6.2 Results on well-posedness of solutions

In Chapter 4 and 5, we have shown that there exist solutions, local and global in time, for both the linear clsoure when $N_p \leq 35/2$ and for the quadratic closure, and for the case $D_r = \text{constant}$. The constraint on the particle number in the case of the linear closure is one that arises in an investigation of Liapounov stability [35], and also in Chapter 3, in the context of thermodynamic admissibility. To obtain the local solution (and for each closure), we studied two linearised problems, one for the velocity $\boldsymbol{v}$, and the other for $\boldsymbol{A}$. We then implemented a fixed point argument, using Schauder's Fixed Point Theorem, to show that a regular solution exists on a small time interval $(0, T^*)$. This solution satisfies an energy inequality, which implies its uniqueness in its class. For uniqueness, as usual, we took the difference of two solutions which satisfied an energetic inquality; by Gronwall's Lemma, this implied its nullity. For the global solution, we first derived some a priori bounds uniform in time, satisfied by the local solutions obtained in Chapter 4. Then, by the means of Lemma 5.1 and for small enough data, we deduced that there exists a unique solution $(\boldsymbol{v},\boldsymbol{A})$ defined for all time t. In the absence of body force that solutions satisfied Corollary 5.1, which implies that it is exponentially stable.

Our study has investigated two closure approximations with a constant value of D_r; one can investigate the problem of existence using other closure approximations or different tools, with different values of D_r. Ideally this study should be complemented by numerical simulations so as to corroborate the implications of the two closure approximations investigated here. This will be the subject of further study.

Appendix A

RESULTS FROM FUNCTIONAL ANALYSIS AND FUNCTION SPACES

In this appendix we gather in one place a number of results from functional analysis and function spaces that are required in the main body of the thesis.

A.1 Results from Functional Analysis

A.1.1 Normed spaces and Banach spaces

All vector spaces are assumed to be defined over the field of real numbers.
Let V be a vector space. A semi-norm on V is a map $|\cdot|: V \longrightarrow \mathbb{R}^+$ which satisfies

$$|u+v| \leq |u| + |v|, \quad |\alpha u| = |\alpha||u| \quad \forall u, v \in V, \quad \forall \alpha \in \mathbb{R}. \tag{A.1}$$

A norm $\|\cdot\|$ on V is a semi-norm that has the additional property of positive definiteness:

$$\|v\| = 0 \text{ iff } v = 0. \tag{A.2}$$

If $\|\cdot\|$ is a norm on V, then the pair $(V, \|\cdot\|)$ is called a *normed space*. Usually, the norm $\|\cdot\|$ defined over the space V is conventional or is clear from the context, and we simply denote the normed space by V. The notion of norm is a generalization of the absolute value for real numbers. The quantity $\|v\|$ is used to measure the length of a vector $v \in V$, while $\|u - v\|$ is used to measure the distance between two vectors u and v in V.

Two norms $\|\cdot\|$ and $|\!|\!|\cdot|\!|\!|$ on a normed space V are said to be *equivalent* if there are positive constants c_1 and c_2 such that

$$c_1\|v\| \leq |\!|\!|v|\!|\!| \leq c_2\|v\| \quad \forall v \in V. \tag{A.3}$$

A *Cauchy sequence* $\{v_n\}_{n=1}^{\infty}$ in V is a sequence that has the property that for any $\epsilon > 0$ there exists a number $N(\epsilon)$ such that $\|v_n - v_m\| < \epsilon$ for all $n, m > N(\epsilon)$. Certainly, all convergent sequences are Cauchy sequences, though the converse is not true. A subset A of a normed space V is *complete* if and only if every Cauchy sequence in A has a strong limit in A.
A complete normed space is called a *Banach space*.

A.1.2 Inner product and Hilbert spaces

Let V be a vector space. An *inner product* on V is a symmetric bilinear form $(\cdot,\cdot) : V \times V \to \mathbb{R}$ that is also positive definite; that is, $(\cdot,\cdot)$ has the following properties:

$$(u, v) = (v, u) \quad \forall u, v \in V,$$
$$(\alpha u_1 + \beta u_2, v) = \alpha(u_1, v) + \beta(u_2, v) \quad \forall u_1, u_2, v \in V, \ \alpha, \beta \in \mathbb{R},$$

$$(v,v)\geq 0 \quad \forall v\in V, \quad \text{and} \quad (v,v)=0 \Longleftrightarrow v=0.$$

A space V endowed with an inner product $(\cdot,\cdot)$ is called an *inner product space*. When the definition of $(\cdot,\cdot)$ is clear, we will simply denote the inner product space by V.

Every inner product generates a norm according to

$$\|v\| = (v,v)^{1/2},$$

so that every inner product space is a normed space.

A *complete inner product space* is called a *Hilbert space*. Hence a Hilbert space is a Banach space whose norm is induced by an inner product.

Linear operators and linear functionals. Let V and W be vector spaces. A map $L : V \to W$ is also called an *operator*. The operator L is *linear* from V to W if it is additive and homogeneous, that is, if

$$L(u+v) = L(u) + L(v),$$
$$L(\alpha v) = \alpha L(v),$$

for all $u, v \in V$ and $\alpha \in \mathbb{R}$. For a linear operator L, we often write $L(v)$ as Lv. A linear operator is called a *linear functional* if $W = \mathbb{R}$.

The *range* $\mathcal{R}(L)$ and *kernel*, or *null, space* $\mathcal{N}(L)$ of L are subspaces of W and V, defined respectively by

$$\mathcal{R}(L) = \{w \in W : \ w = L(v) \text{ for some } v \in V\},$$
$$\mathcal{N}(L) = \{v \in V : \ L(v) = 0\}.$$

The range $\mathcal{R}(L)$ is the set of the images under the mapping L, while the null space $\mathcal{N}(L)$ consists of the solutions of the equation $L(v) = 0$. Obviously, $0 \in \mathcal{N}(L)$.

A special operator worth mentioning explicitly is the *projection operator* $P : V \to V$, from a vector space V into itself, which is defined to have the property

$$P^2 = P, \quad \text{or} \quad P^2 v = Pv, \ \ \forall\, v \in V.$$

If V and W are normed spaces and L is a map from V into W, then L is said to be *continuous* if $v_n \to v$ in V implies that $L(v_n) \to L(v)$ in W. Furthermore, the map L is said to be *bounded* if for any $r > 0$, there is a constant $R \geq 0$ such that

$$\|L(v)\| \leq R \quad \forall\, v \in V, \ \|v\| \leq r.$$

When L is a linear operator, the boundedness of L is characterized by the existence of a constant $M \geq 0$ such that

$$\|L(v)\| \leq M\|v\| \ \ \forall\, v \in V. \tag{A.4}$$

The properties of continuity and boundedness are equivalent in the case of linear operators: *a linear operator is continuous if and only if it is bounded.*

An operator L from V to W is said to be *Lipschitz continuous* if there exists a constant $c > 0$ such that

$$\|L(v_1) - L(v_2)\| \leq c\,\|v_1 - v_2\| \quad \forall\, v_1, v_2 \in V.$$

Lipschitz continuous operators are continuous, but the converse is not true in general. On the other hand, a linear operator is Lipschitz continuous if and only if it is continuous.

The space $\mathcal{L}(V, W)$; dual space.

Let V and W be normed spaces. We denote by $\mathcal{L}(V, W)$ the space of all bounded linear operators from V to W. For $L \in \mathcal{L}(V, W)$, the quantity

$$\|L\| = \sup_{0 \neq v \in V} \frac{\|Lv\|}{\|v\|} = \sup_{\|v\| \leq 1} \|Lv\| \tag{A.5}$$

is well-defined; furthermore, it can be shown that $\|L\|$ thus defined is a norm on $\mathcal{L}(V, W)$. The space $\mathcal{L}(V, W)$ endowed with the norm (A.5) is a Banach space if W is a Banach space.

The space $\mathcal{L}(V, \mathbb{R})$ of bounded linear functionals on V is known as the *dual space* of V and is denoted by V'. Clearly, then, since $\mathbb{R}$ is complete, V' is a Banach space with the norm

$$\|L\| = \sup_{\|v\| \leq 1} |Lv|. \tag{A.6}$$

Often, we will use ℓ for a bounded linear functional on a normed space V and denote the action of ℓ on a member $v \in V$ by $\langle \ell, v \rangle$ rather than $\ell(v)$. Here, $\langle \cdot, \cdot \rangle$ is the duality pairing between V' and V. In Section 5.2 we will see examples of duality in the context of the function space $L^p(\Omega)$.

Weak convergence. Let V be a normed space and V' its dual. A sequence $\{v_n\}$ in V is said to converge *weakly* in V to v if

$$\lim_{n \to \infty} \langle \ell, v_n \rangle = \langle \ell, v \rangle \quad \forall \, \ell \in V'. \tag{A.7}$$

The notation

$$v_n \rightharpoonup v$$

is used to indicate weak convergence. Strong (norm) convergence implies weak convergence, but the converse does not hold, with the exception of *finite-dimensional spaces*, for which the two forms of convergence coincide.

A.1.3 Embeddings

Embedding results are especially important when we compare Sobolev spaces with different indices; details of Sobolev spaces are given in the next section. Let V and W be normed spaces with $V \subset W$. If there is a constant $c > 0$ such that

$$\|v\|_W \leq c\|v\|_V \quad \forall \, v \in V, \tag{A.8}$$

we say V *is continuously embedded in* W, and write

$$V \hookrightarrow W.$$

This property can be interpreted in various ways; for example, (A.8) states that the identity operator $I : V \to W$ is bounded, or equivalently, continuous. Thus the continuous embedding of V in W implies also that if $v_n \to v$ in V, then $v_n \to v$ in W.

The subspace V is said to be *compactly embedded in* W if

$$v_n \rightharpoonup v \;\text{ in } V \;\text{ implies that } v_n \to v \text{ in } W.$$

This property is expressed in the form

$$V \hookrightarrow\hookrightarrow W,$$

and is equivalent to the statement that the identity operator from V into W is compact.

A.1.4 Dual operators

The generalization to normed spaces of the notion of the transpose of a matrix has many applications in functional analysis. To carry out such a generalization we begin with normed spaces V and W and their duals V' and W'. Let A be a linear operator with domain $\mathcal{D}(A) \subset V$ and range in W. Given $w' \in W'$ we pose the question, under what conditions does there exist $v' \in V'$ such that

$$\langle w', Av\rangle = \langle v', v\rangle \quad \forall\, v \in \mathcal{D}(A)? \tag{A.9}$$

It can be shown that a necessary and sufficient condition for (A.9) to hold is that $\mathcal{D}(A)$ be dense in V; when this is the case, v' is determined uniquely by w'. When

$\mathcal{D}(A)$ is the whole space V, then this procedure defines a linear operator A' from W' to V' such that $A'w' = v'$. The operator A' is called the *dual* operator of A, and we may write

$$\langle w', Av\rangle = \langle A'w', v\rangle \quad \forall\, v \in V,\ w' \in W'.$$

If $\mathcal{D}(A) = V$ and A is bounded, then A' is also bounded, and

$$\|A'\| = \|A\|.$$

A.2 Function Spaces

The function spaces to be discussed include the spaces $C^m(\Omega)$ and $C^m(\overline{\Omega})$ of m-times continuously differentiable functions on Ω and $\overline{\Omega}$, the Lebesgue spaces $L^p(\Omega)$, the Sobolev spaces $W^{m,p}(\Omega)$, and their Hilbert space specializations $H^m(\Omega) = W^{m,2}(\Omega)$. The spaces will be defined on an open bounded domain $\Omega \subset \mathbb{R}^d$ that will be assumed to possess certain prescribed condition of smoothness. In order to give a proper treatment of time-dependent problems, we will later introduce vector-valued function spaces, which permit functions of space and time to be interpreted as maps from a time interval into a Banach or Hilbert space of functions.

A.2.1 The Spaces $C^m(\Omega)$, $C^m(\overline{\Omega})$, and $L^p(\Omega)$

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d \leq 3$ for most applications). Before going on to discuss function spaces, we introduce the useful multi-index notation.

Multi-index notation. Let $\mathbb{Z}^d_+$ denote the set of all ordered d-tuples of nonnegative integers. A member of $\mathbb{Z}^d_+$ will usually be denoted by α or β, where, for example,

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)\,,$$

each component α_i being a nonnegative integer.

We denote by $|\alpha|$ the sum $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$, called the length of α, and by $D^\alpha v$ the partial derivative

$$D^\alpha v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}.$$

Thus if $|\alpha| = m$, then $D^\alpha v$ will denote one of the mth partial derivatives of v. For example, $\alpha = (1, 0, 3)$ belongs to $\mathbb{Z}_+^3$, with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = 1 + 0 + 3 = 4$, and in this case the partial derivative $D^\alpha v$ is the fourth derivative defined by

$$D^\alpha v = \frac{\partial^4 v}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} = \frac{\partial^4 v}{\partial x_1^1 \partial x_2^0 \partial x_3^3} = \frac{\partial^4 v}{\partial x_1 \partial x_3^3}.$$

A.2.2 Spaces of continuous and continuously differentiable functions

We denote by $C(\Omega)$ the space of all real-valued functions that are continuous on Ω. Since Ω is open, a function from the space $C(\Omega)$ is not necessarily bounded; consider, for example, the continuous function $v(x) = \ln x$ on $(0, 1)$. We denote further by $C(\overline{\Omega})$ the space of functions that are *bounded and uniformly continuous* on Ω. The notation $C(\overline{\Omega})$ is consistent with the fact that a bounded and uniformly continuous function on Ω has a unique continuous extension to $\overline{\Omega}$. The space $C(\overline{\Omega})$ is a *Banach space* with the norm

$$\|v\|_{C(\overline{\Omega})} = \sup\{|v(\boldsymbol{x})| : \boldsymbol{x} \in \Omega\} \equiv \max\{|v(\boldsymbol{x})| : \boldsymbol{x} \in \overline{\Omega}\}.$$

For any nonnegative integer m, $C^m(\Omega)$ is defined to be the space of functions that together with their derivatives of order less than or equal to m are continuous; that is,

$$C^m(\Omega) = \{v \in C(\Omega) : D^\alpha v \in C(\Omega) \text{ for } |\alpha| \leq m\}.$$

We likewise set

$$C^m(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : D^\alpha v \in C(\overline{\Omega}) \text{ for } |\alpha| \leq m\}.$$

It is common practice to write $C(\Omega)$ and $C(\overline{\Omega})$ instead of $C^0(\Omega)$ and $C^0(\overline{\Omega})$.

The space $C^m(\overline{\Omega})$ can be endowed with the seminorm

$$|v|_{C^m(\overline{\Omega})} = \sum_{|\alpha|=m} \|D^\alpha v\|_{C(\overline{\Omega})},$$

and it becomes a Banach space when endowed with the norm

$$\|v\|_{C^m(\overline{\Omega})} = \sum_{j=0}^{m} |v|_{C^j(\overline{\Omega})} = \sum_{|\alpha|\leq m} \|D^\alpha v\|_{C(\overline{\Omega})}.$$

And we define

$$C^\infty(\Omega) = \{v \in C(\Omega) : \ v \in C^m(\Omega) \ \ \forall m \in \mathbb{Z}_+\}$$

and

$$C^\infty(\overline{\Omega}) = \{v \in C(\overline{\Omega}) : \ v \in C^m(\overline{\Omega}) \ \ \forall m \in \mathbb{Z}_+\}.$$

These are spaces of infinitely differentiable functions. Finally, we set

$$C_b^k(\Omega) = \left\{u \in C^k(\Omega) : \exists M > 0, |D^\alpha u(x)| < M, \forall x \in \Omega \text{ and } \ |\alpha| \leq k\right\}.$$

This space is equiped with the norm

$$|u| = \sup_{|\alpha|\leq k} \sup_{x\in\Omega} |D_\alpha u(x)|$$

A.2.3 The spaces $L^p(\Omega)$

We denote by $L^p(\Omega)$ the space of (equivalence classes of) measurable functions v for which

$$\int_\Omega |v(\boldsymbol{x})|^p \, dx < \infty,$$

where integration is understood to be in the sense of Lebesgue. The space $L^p(\Omega)$ is a *Banach space* when endowed with the norm $\|\cdot\|_{L^p(\Omega)}$ defined by

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v(\boldsymbol{x})|^p \, dx\right)^{1/p}.$$

When there is no danger of confusion, reference to the domain Ω will be omitted in the symbol for norms. It will also be convenient to write $\|\cdot\|$ for the norm on $L^2(\Omega)$ when this is unlikely to be ambiguous. The quantity $\|\cdot\|_{L^p}$ is a norm only when it is understood that u represents an equivalence class of functions, two functions being equivalent if they are equal almost everywhere (a.e.), that is, equal everywhere except on a subset of Ω of Lebesgue measure zero.

The definition of the spaces $L^p(\Omega)$ can be extended to include the case $p = \infty$ in the following manner. We define the essential supremum (denoted by ess sup) of any measurable function v by

$$\operatorname{ess\,sup}_\Omega v = \inf\{M \in (-\infty, \infty] : \ v(\boldsymbol{x}) \leq M \text{ a.e. in } \Omega\}.$$

Then v is said to be *essentially bounded above* if $\operatorname{ess\,sup}_\Omega v < \infty$. A similar definition of essential infimum may be given, leading to the notion of a function that is essentially bounded below. We say that v is *essentially bounded* if both $\operatorname{ess\,sup}_\Omega v$ and $\operatorname{ess\,inf}_\Omega v$ are finite.

Then we may define

$$L^\infty(\Omega) = \{v : \ v \text{ is essentially bounded on } \Omega\}.$$

This space is a *Banach space* when endowed with the norm

$$\|v\|_{L^\infty(\Omega)} = \operatorname{ess\,sup}_\Omega |v|.$$

Since all continuous functions on a bounded closed set are bounded, we have

$$C(\overline{\Omega}) \hookrightarrow L^\infty(\Omega).$$

The case $p = 2$ is special, in that $L^2(\Omega)$ is an inner product space (indeed, a Hilbert space) when endowed with the inner product

$$(u, v)_{L^2(\Omega)} = \int_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\, dx.$$

This inner product, in turn, generates the norm $\|\cdot\|_{L(\Omega)}$.

Let v be a function defined on Ω. We say that $v \in L^p_{\text{loc}}(\Omega)$ if for any proper subset $\Omega' \subset\subset \Omega$, $v \in L^p(\Omega')$. For any number $p \in [1, \infty)$,

Dual spaces and reflexivity. We define the dual exponent q of $p \in [1, \infty)$ by $1/p + 1/q = 1$ (with the usual convention that $q = \infty$ when $p = 1$). Then the topological dual $[L^p(\Omega)]'$ of $L^p(\Omega)$ may be identified with $L^q(\Omega)$. In particular, $L^2(\Omega)$ may be identified with its dual space. For $1 < p < \infty$ the roles of p and q are symmetric, and so it is clear that

$$L^p(\Omega) = (L^q(\Omega))' = (L^p(\Omega))''.$$

Thus the spaces $L^p(\Omega)$ are reflexive for $1 < p < \infty$.

The spaces $L^1(\Omega)$ and $L^\infty(\Omega)$ are *not* reflexive, though it is possible to identify $L^\infty(\Omega)$ with the dual of $L^1(\Omega)$; this identification is expressed in the form

$$L^\infty(\Omega) = (L^1(\Omega))'.$$

On the other hand, $L^1(\Omega)$ can be identified only with a proper subspace of $(L^\infty(\Omega))'$.

A.2.4 Sobolev Spaces

Assumptions about domains. We introduce a definition that will suffice for most purposes when smoothness assumptions about the boundary of a domain need to be made.

For any point $\boldsymbol{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, set

$$y = x_d \ \text{ and } \ \hat{\boldsymbol{x}} = (x_1, x_2, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}.$$

An open set Ω in $\mathbb{R}^d$ is said to have a *Lipschitz-continuous boundary* Γ if there exist constants $\alpha > 0$ and $\beta > 0$, a finite number of *local* coordinate systems $(\hat{\boldsymbol{x}}^m, y^m)$, and local maps f^m, $m = 1, \ldots, M$, that are Lipschitz-continuous on their respective domains of definition $\{\hat{\boldsymbol{x}}^m : |\hat{\boldsymbol{x}}^m| \leq \alpha\}$ such that

$$\Gamma = \cup_{m=1}^{M} \{(\hat{\boldsymbol{x}}^m, y^m) : \; y^m = f^m(\hat{\boldsymbol{x}}^m), \; |\hat{\boldsymbol{x}}^m| \leq \alpha\},$$

and for $m = 1, \ldots, M$,

$$\{(\hat{\boldsymbol{x}}^m, y^m) : \; f^m(\hat{\boldsymbol{x}}^m) < y^m < f^m(\hat{\boldsymbol{x}}^m) + \beta, \; |\hat{\boldsymbol{x}}^m| \leq \alpha\} \subset \Omega,$$
$$\{(\hat{\boldsymbol{x}}^m, y^m) : \; f^m(\hat{\boldsymbol{x}}^m) - \beta < y^m < f^m(\hat{\boldsymbol{x}}^m), \; |\hat{\boldsymbol{x}}^m| \leq \alpha\} \subset \mathbb{R}^d \backslash \overline{\Omega}.$$

More generally, we say that the boundary is of class X if the functions f^m are of class X, and that it is *smooth* if $X = C^\infty$.

With a slight abuse of terminology, a domain with a Lipschitz boundary is also referred to as a Lipschitz domain, with obvious modifications in nomenclature for boundaries of other classes. *In the following, we always assume that Ω is a Lipschitz domain, unless stated otherwise.* We note, though, that such an assumption is, in fact, not needed for some of the results stated here.

The Sobolev spaces $W^{m,p}(\Omega)$. For any nonnegative integer m and real number $p \geq 1$ or $p = \infty$, we define

$$W^{m,p}(\Omega) = \{v \in L^p(\Omega) : \; D^\alpha v \in L^p(\Omega) \text{ for any } \alpha \in \mathbb{Z}_+^d \text{ with } |\alpha| \leq m\},$$

where derivatives are taken in the distributional sense. Norms in the spaces $W^{m,p}(\Omega)$ are defined by

$$\|v\|_{W^{m,p}(\Omega)} \equiv \|v\|_{m,p,\Omega} = \Big(\sum_{|\alpha| \leq m} \|D^\alpha v\|^p_{L^p(\Omega)} \Big)^{1/p}, \quad 1 \leq p < \infty, \tag{A.10}$$

and

$$\|v\|_{m,\infty,\Omega} = \max_{|\alpha|\le m} \|D^\alpha v\|_{L^\infty(\Omega)}. \tag{A.11}$$

With the norm defined above, the space $W^{m,p}(\Omega)$ becomes a Banach space. We also introduce here the seminorms on the spaces $W^{m,p}(\Omega)$:

$$|v|_{m,p,\Omega} = \Big(\sum_{|\alpha|=m} \|D^\alpha v\|^p_{L^p(\Omega)}\Big)^{1/p}, \quad 1 \le p < \infty,$$
$$|v|_{m,\infty,\Omega} = \max_{|\alpha|=m} \|D^\alpha v\|_{L^\infty(\Omega)}.$$

The space $W^{m,p}(\Omega)$ is *reflexive* if and only if $1 < p < \infty$. We note here that $W^{0,p}(\Omega) = L^p(\Omega)$.

The case $p = 2$ is special, in that $W^{m,2}(\Omega)$ may be assigned an inner product. We set $W^{m,2}(\Omega) \equiv H^m(\Omega)$ and define the inner product on this space by

$$(u,v)_{H^m(\Omega)} = \sum_{|\alpha|\le m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)},$$

where as before, $(\cdot,\cdot)_{L^2(\Omega)}$ denotes the $L^2(\Omega)$ inner product. With this inner product, $H^m(\Omega)$ is a *Hilbert space*. The corresponding norm will be denoted by $\|\cdot\|_{H^m(\Omega)}$ or simply by $\|\cdot\|_{H^m}$, depending on the particular context.

Embeddings of $W^{m,p}(\Omega)$. Some properties regarding embeddings and inclusions are summarized in the following theorem.

Theorem A.1 *The following statements are valid:*

(a) $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow W^{k,p}(\Omega)$ *if* $m > k$.

(b) $\mathcal{D}(\Omega) \hookrightarrow W^{m,p}(\Omega)$.

(c) $C^m(\overline{\Omega}) \hookrightarrow W^{m,p}(\Omega)$.

(d) $C^\infty(\overline{\Omega}) \cap W^{m,p}(\Omega)$ *is dense in* $W^{m,p}(\Omega)$*; in other words, a function in* $W^{m,p}(\Omega)$ *can be approximated by a sequence of functions smooth up to the boundary.*

(e) (Sobolev compact embedding) *If* $k < m - d/p$ *with* $1 \leq p \leq \infty$*, then* $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow C^k(\overline{\Omega})$*; in particular,* $W^{m,p}(\Omega) \hookrightarrow\hookrightarrow C^k(\overline{\Omega})$.

Trace theorems. A uniformly continuous function v on a bounded domain Ω with boundary Γ has a well-defined boundary value, usually denoted by $v|_\Gamma$. This property may be expressed in an alternative manner by the introduction of a map γ called the *trace* operator, which associates with each $v \in C(\overline{\Omega})$ its boundary value $\gamma v = v|_\Gamma$, a function belonging to $C(\Gamma)$.

For a function $v \in W^{m,p}(\Omega)$ the issue of its boundary value is less straightforward: the restriction of v to Γ need not make sense, since Γ is a set of measure zero, and two functions in $W^{m,p}(\Omega)$ are identified if they are equal a.e. Fortunately, it is possible to extend the notion of the trace operator for continuous functions in $C(\overline{\Omega})$ to functions in $W^{m,p}(\Omega)$ for certain ranges of the indices m and p. This result is summarized in the following.

Theorem A.2 *Assume that* $1 \leq p \leq \infty$ *and* $m > 1/p$*. Then there exists a unique bounded linear surjective mapping* $\gamma : W^{m,p}(\Omega) \to W^{m-1/p,p}(\Gamma)$ *such that* $\gamma v = v|_\Gamma$ *when* $v \in W^{m,p}(\Omega) \cap C(\overline{\Omega})$.

In future, when the trace γv of a Sobolev function v on the boundary is defined, we will simply write v for the trace γv.

The trace theorem can be extended to higher-order derivatives on the boundary. In order to avoid complications arising from compatibility conditions we confine attention to higher-order *normal derivatives*, since, for example, the tangential derivative of a function is completely defined if the function itself is known along a boundary.

Let $\boldsymbol{n} = (n_1, \ldots, n_d)^T$ denote the outward unit normal to the boundary Γ of Ω, assumed here to be smooth. The kth normal derivative of a function $v \in C^k(\overline{\Omega})$ is then defined by

$$\frac{\partial^k v}{\partial n^k} \equiv n_{i_1} \cdots n_{i_k} \frac{\partial^k v}{\partial x_{i_1} \cdots \partial x_{i_k}}.$$

The following theorem states the fact that this definition can be extended to functions in certain Sobolev spaces.

Theorem A.3 *(Second Trace Theorem). Assume that Ω is a bounded open set with a $C^{k,1}$ boundary Γ. Assume that $1 \leq p \leq \infty$ and $m > k + 1/p$. Then there exist unique bounded linear and surjective mappings $\gamma_j : W^{m,p}(\Omega) \to W^{m-j-1/p,p}(\Gamma)$ $(j = 0, 1, \ldots, k)$ such that $\gamma_j v = (\partial^j v / \partial n^j)|_\Gamma$ when $v \in W^{m,p}(\Omega) \cap C^{k,1}(\overline{\Omega})$.*

It is important to note that the ranges of the trace operators are proper subsets of $L^p(\Gamma)$. On the other hand, it can be shown that $W^{m-j-1/p,p}(\Gamma)$ is *dense* in $L^p(\Gamma)$, for $j = 0, 1, \ldots, k$.

The space $W_0^{m,p}(\Omega)$. With the definition of traces at our disposal, it is now possible to consider those subspaces of Sobolev spaces characterized by the fact that the functions vanish on the boundary. To this end we define

$$W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : \ \gamma_j v = 0 \text{ for } j < m - 1/p\}.$$

This space may be *equivalently defined by*

$$W_0^{m,p}(\Omega) = \text{the closure of } C_0^\infty(\Omega) \text{ in } W^{m,p}(\Omega).$$

Immediately we see that any function in $W_0^{m,p}(\Omega)$ can be approximated by a sequence of $C_0^\infty(\Omega)$ functions with respect to the norm of $W^{m,p}(\Omega)$.

From the definition and the second trace theorem, $W_0^{m,p}(\Omega)$ is a *closed subspace* of $W^{m,p}(\Omega)$. When $p = 2$, we write $H_0^m(\Omega)$ to replace $W_0^{m,2}(\Omega)$. In particular, we will frequently use the space

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : \ v = 0, \ Dv = 0 \text{ a.e. on } \Gamma\}.$$

Equivalent norms. The following result can be used to generate various equivalent norms (cf. the definition (A.3)) on Sobolev spaces. Recall that over the Sobolev space $W^{k,p}(\Omega)$, $|v|_{k,p,\Omega}$ is the seminorm defined by

$$|v|_{k,p,\Omega} = \Big(\int_\Omega \sum_{|\alpha|=k} |D^\alpha v|^p \, dx\Big)^{1/p}.$$

Theorem A.4 (Equivalent Norm Theorem) *Let Ω be an open, bounded, connected set in $\mathbb{R}^d$ with a Lipschitz boundary, $k \geq 1$, $1 \leq p < \infty$. Assume that $f_j : W^{k,p}(\Omega) \to \mathbb{R}$, $1 \leq j \leq J$, are seminorms on $W^{k,p}(\Omega)$ satisfying two conditions:*

(H_1) $0 \leq f_j(v) \leq c\, \|v\|_{k,p,\Omega} \quad \forall\, v \in W^{k,p}(\Omega)$, $1 \leq j \leq J$.

(H_2) *If v is a polynomial of degree less than or equal to $k-1$ and $f_j(v) = 0$, $1 \leq j \leq J$, then $v = 0$.*

Then, the quantity

$$\|v\| = |v|_{k,p,\Omega} + \sum_{j=1}^J f_j(v)$$

or

$$\|v\| = \Big(|v|_{k,p,\Omega}^p + \sum_{j=1}^J f_j(v)^p\Big)^{1/p}$$

defines a norm on $W^{k,p}(\Omega)$, which is equivalent to the norm $\|v\|_{k,p,\Omega}$.

Proof. We will prove that the quantity

$$\|v\| = |v|_{k,p,\Omega} + \sum_{j=1}^J f_j(v)$$

is a norm on $W^{k,p}(\Omega)$ equivalent to the norm $\|v\|_{k,p,\Omega}$. That

$$\|v\| = \left(|v|^p_{k,p,\Omega} + \sum_{j=1}^{J} f_j(v)^p\right)^{1/p}$$

is also an equivalent norm can be proved similarly.

By the condition (H_1), we see that for some constant $c > 0$,

$$\|v\| \le c\,\|v\|_{k,p,\Omega} \quad \forall\, v \in W^{k,p}(\Omega).$$

So we need only to show that there is another constant $c > 0$ such that

$$\|v\|_{k,p,\Omega} \le c\,\|v\| \quad \forall\, v \in W^{k,p}(\Omega).$$

We argue by contradiction. Suppose that this inequality is false; then we can find a sequence $\{v_l\} \subset W^{k,p}(\Omega)$ with the properties

(a) $\|v_l\|_{k,p,\Omega} = 1$,

(b) $\|v_l\| \le 1/l$

for $l = 1, 2, \ldots$.

From Property (b), we see that as $l \to \infty$,

$$|v_l|_{k,p,\Omega} \to 0$$

and

$$f_j(v_l) \to 0, \quad 1 \le j \le J.$$

Since $\{v_l\}$ is a bounded sequence in $W^{k,p}(\Omega)$, and since

$$W^{k,p}(\Omega) \hookrightarrow\hookrightarrow W^{k-1,p}(\Omega),$$

there is a subsequence of the sequence $\{v_l\}$, still denoted by $\{v_l\}$, and a function $v \in W^{k-1,p}(\Omega)$ such that

$$v_l \to v \quad \text{in } W^{k-1,p}(\Omega), \text{ as } l \to \infty.$$

This property and $|v_l|_{k,p,\Omega} \to 0$ as $l \to \infty$, together with the uniqueness of a limit, imply that

$$v_l \to v \quad \text{in } W^{k,p}(\Omega), \text{ as } l \to \infty$$

and

$$|v|_{k,p,\Omega} = \lim_{l\to\infty} |v_l|_{k,p,\Omega} = 0.$$

We then conclude that v is a polynomial of degree less than or equal to $k-1$. On the other hand, from the continuity of the functionals f_j, $1 \le j \le J$, we find that

$$f_j(v) = \lim_{l\to\infty} f_j(v_l) = 0, \quad 1 \le j \le J.$$

Using the assumption (H_2), we see that $v = 0$, which contradicts the fact that

$$\|v\|_{k,p,\Omega} = \lim_{l\to\infty} \|v_l\|_{k,p,\Omega} = 1.$$

The proof of the result is now completed. □

A.2.5 Regularity of solution on the boundary [27, 29]

We recall here a fundamental result on regularity of the solution in the neighbourhood of the boundary .

Theorem A.5 *Assume that L is an elliptic differential operator of order $2m$, regular enough in a domain Ω. If $u \in H_0^m(\Omega)$ is a solution to the equation $Lu = f$ in Ω, with $f \in H^{k-m}(\Omega)$.*
Then $u \in H^{k+m}(\Omega)$ if $\Omega \in C^{k+m}$

A.3 Some Inequalities

We summarize in this section some useful inequalities, that are often required in this thesis.

A.3.1 Sobolev Inequalities [1, 28]

If Ω is a bounded domain of $\mathbb{R}^n$ with boundary $\Gamma \in C^1$.
From the Imbedding Theorem (Theorem A.1), and for $u \in W^{m,p}(\Omega)$, $m \geq 1$, $1 \leq p < \infty$ we deduce the following

1. if $\frac{1}{p} - \frac{m}{n} = \frac{1}{q} > 0$, $\|u\|_{L^q(\Omega)} \leq C(m,n,p)\|u\|_{W^{m,p}}(\Omega) \quad 1 \leq q < \infty$
   The case $p = 2, \quad m = 1$
   $n = 2, \|u\|_{L^q(\Omega)} \leq C\|u\|_{H_0^1(\Omega)} \quad \forall q, 1 \leq q < \infty$
   $n = 3, \|u\|_{L^6(\Omega)} \leq C\|u\|_{H_0^1(\Omega)}$
   $n = 3, \|u\|_{L^4(\Omega)} \leq C\|u\|_{H_0^1(\Omega)}$
   $n \geq 3, \|u\|_{L^{2n/(n-2)}(\Omega)} \leq C\|u\|_{H_0^1(\Omega)}$

2. if $\frac{1}{p} - \frac{m}{n} = 0$, $\|u\|_{L^q(\Omega)} \leq C(m,p,n,q,\Omega)\|u\|_{W^{m,p}(\Omega)} \quad 1 \leq q < \infty$

3. if $\frac{1}{p} - \frac{m}{n} < 0$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ and for $p \leq q \leq \infty$ we have

$$\sup_{x\in\Omega} |v(x)| \leq C_1(m,p,n,q,\Omega)\|v\|_{W^{m,p}(\Omega)}, \tag{A.12}$$

$$\text{and} \quad \|u\|_{L^q(\Omega)} \leq C_2(m,n,p,\Omega)\|u\|_{W^{m,p}(\Omega)}. \tag{A.13}$$

   The case $p = 2, \quad m = 2$ and $q = 4$, adding (A.12) and (A.13) to obtain

$$\sup_{x\in\Omega} |v(x)| + \|v\|_{L^4(\Omega)} \leq C(\Omega)\|v\|_{H^2(\Omega)}$$

4. $\|u\|_{H^p} \leq C\|u\|_{H^q}$ for $p \leq q$

A.3.2 Poincaré-Friedrichs Inequality

This inequality may be deduced as a consequence of the Equivalent Norm Theorem (Theorem A.4)

$$f_1(v) = \int_{\partial\Omega} |v|\, ds.$$

We take in Theorem A.4 $k = 1$, $p = 2$, $J = 1$, We can then conclude that there exists a constant $c > 0$, depending only on Ω, such that the inequality

$$\|v\|_{1,\Omega} \leq c\,|v|_{1,\Omega} \quad \forall\, v \in H_0^1(\Omega) \tag{A.14}$$

holds. It follows from (A.14) that the seminorm $|\cdot|_1$ is a norm on $H_0^1(\Omega)$, equivalent to the usual $H^1(\Omega)$-norm.

More generally, if Γ_0 is an open, nonempty subset of the boundary Γ, then there is a constant $c > 0$, depending only on Ω, such that

$$\|v\|_{1,\Omega} \leq c\,|v|_{1,\Omega} \quad \forall\, v \in H_{\Gamma_0}^1(\Omega). \tag{A.15}$$

Here,

$$H_{\Gamma_0}^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ a.e. on } \Gamma_0\}.$$

This inequality can be derived by applying Theorem A.4 with $k = 1$, $p = 2$, $J = 1$, and

$$f_1(v) = \int_{\Gamma_0} |v|\, ds.$$

We will often use Poincaré-Friedrichs Inequality in the form

$$\int_\Omega \|\boldsymbol{v}\|^2 dx \leq C \int_\Omega |\nabla \boldsymbol{u}|^2 dx \quad \forall \boldsymbol{u} \in \mathbb{H}_0^1(\Omega)$$

A.3.3 Korn's first inequality [32]

For $\boldsymbol{u} \in [H_0^1(\Omega)]^3$, we define the tensor function

$$\boldsymbol{D}(\boldsymbol{v}) = \frac{1}{2}\left(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T\right).$$

Korn's first inequality states that there exists a constant $c > 0$ depending only on Ω such that

$$\|v\|^2_{[H^1(\Omega)]^3} \leq c \int_\Omega |D(v)|^2 dx \quad \forall v \in [H^1(\Omega)]^3. \tag{A.16}$$

We deduce from Korn's inequality and the Equivalent Norm Theorem (Theorem A.4 that the norms $\|v\|^2_{[H^1(\Omega)]^3}$ and $\int_\Omega |\nabla v| dx = \|v\|$ are equivalent.

A.3.4 Cauchy-Schwarz Inequality

Let V be an inner product space. Then

$$|(u, v)| \leq \|u\| \, \|v\| \quad \forall u, v \in V.$$

A.3.5 Young Inequality

We state here the special case: If a, b are positive numbers. Then

$$ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2 \quad \forall \epsilon > 0.$$

Proof For $a, b \in \mathbb{R}$, for $\epsilon > 0$, we have

$$0 \leq (\epsilon a - b)^2 = a^2 - 2\epsilon ab + \epsilon^2 b^2.$$

So

$$2\epsilon ab \leq \epsilon^2 a^2 + b^2,$$

which implies that

$$ab \leq \frac{\epsilon}{2} a^2 + \frac{b^2}{2\epsilon}$$

A.3.6 Gronwall's Lemma

Theorem A.6 *Let* $u : [0,T] \longrightarrow \mathbb{R}$ *be continuous and nonnegative. Suppose* $C \geq 0, K \geq 0$ *are such that*

$$u(t) \leq C + \int_0^t Ku(s)ds \quad \forall t \in [0,T].$$

Then

$$u(t) \leq Ce^{Kt} \quad \forall t \in [0,T].$$

Proof First, suppose $C > 0$, let

$$U(t) = C + \int_0^t Ku(s)ds > 0;$$

then

$$u(t) \leq U(t).$$

By differentiation of U we find

$$U'(t) = Ku(t);$$

hence

$$\frac{U'(t)}{U(t)} = \frac{Ku(t)}{U(t)} \leq K.$$

So

$$\frac{d}{dt}(\ln U(t)) \leq K,$$

therefore

$$\frac{d}{dt}(\ln U(t)) \leq \ln U(0) + Kt$$

by integration. Since $U(0) = C$, we have by exponentiation

$$U(t) \leq Ce^{Kt},$$

and so

$$u(t) \leq CeKt.$$

If $C = 0$, then apply the above argument for a sequence of positive c_i that tend to 0 as $i \to \infty$.

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