Internal Factorisation Systems

Sanjiv Ranchod

Department of Mathematics and Applied Mathematics
University of Cape Town

Supervisor: Professor George Janelidze
Co-supervisor: Dr Tamar Janelidze-Gray

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Plagiarism declaration

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Abstract

We introduce internal factorisation systems for internal categories. We recall the definitions and theory of internal categories and factorisation systems. We develop a diagrammatic calculus of pullbacks for ease of internal calculation. To define an internal factorisation system we define and study the subobjects of isomorphisms, an internalisation of the class of isomorphisms of a category. We provide an abstract example of an internal factorisation system. We then internalise various properties of factorisation systems, such as the two components determining each other, the cancellation properties and the essential uniqueness of factorisations, and show that an internal factorisation system satisfies these internal conditions.

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1 Introduction

1.1 Introduction

An internal category is a generalisation of a (small) category, produced by considering a small category as a diagram in **Set**. The first instances of this notion were given by Ehresmann in [1], and a comprehensive account of the structure may be found in [2]. Their study provides categorical insight into the notion of a category itself. On the other hand, factorisation systems, introduced by Freyd and Kelly [3], appear frequently in categorical algebra and have been studied in various general and concrete contexts (see [4], [5]). Borceaux provides an overview of factorisation systems in [6].

In this work, we introduce the notion of an internal factorisation system. This is done by considering the components of a definition of a factorisation system diagrammatically in a general context, in such a way that an internal factorisation system on an internal category in **Set** is precisely a usual factorisation system on the corresponding small category. We note the presentation herein is self-contained and that material from the first chapters of Mac Lane [7] are sufficient preliminaries for the reader.

We begin by setting up notation for using pullbacks in a way that is convenient for our purposes, paying attention to the universal morphisms induced by pullbacks. One may view this as a *calculus of pullbacks*, in a sense, and it is the primary means of calculation throughout the work.

We next recall and motivate the definition of an internal category and make brief commentary on the associativity condition of the definition. This is followed by the consideration of *subobjects of morphisms* and their properties, an internal generalisation of a class of morphisms of a category. We then define and study the *subobjects of isomorphisms*, a subobject of morphisms containing all isomorphisms of a category, in an internal sense.

We then recall the definitions of orthogonality and factorisations towards the definition of a factorisation system. We further recall well known properties of factorisation systems. Next, we motivate and define internal notions of orthogonality and factorisations and provide the definition of an *internal factorisation system.* We then give a general example of an internal factorisation system.

We finally define internal notions of the properties of factorisation systems expressed earlier. These include the fact that factorisations are unique up to isomorphisms, the cancellation properties and that each component of a factorisation system determines the other. We show that an internal factorisation system satisfies these internal conditions.

In contemporary categorical algebra, the notion of a factorisation is approached via a prefactorisation system, which has proven to be a weaker, yet still powerful structure. However, the generality of a prefactorisation system means that it requires a relatively strong *internal logic* on the ambient category in order to be internalised. We therefore avoid this avenue and define a factorisation system (and indeed an internal factorisation system) directly, without reference to prefactorisation systems. This allows for the internal factorisation system to be defined in the general context of a finitely complete category.

To provide ease of reading, we outline the notation we use in commutative diagrams by way of example. Consider the diagram

$$\begin{array}{ccc} \mathrm{Iso}(C) & \xrightarrow{\pi_2} & \mathrm{Pt}(C) \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1\pi_2} \\ \mathrm{Pt}(C) & \xrightarrow[\pi_2\pi_2]{} & C_1 \end{array} \tag{ISO}$$

This diagram appears in 2.3.3, and represents the definition of the *object of isomorphisms*. It is thus labelled, on the right, with (ISO). When the commutativity of this square is reference in a proceeding diagram, **ISO** will be written in the relevant cell of that diagram, as below:

$$\begin{array}{ccccccc} \mathrm{Iso}(C) & \xrightarrow{\pi_1} & \mathrm{Pt}(C) & \xrightarrow{\pi_2} & C^{\leftarrow\leftarrow} & \xrightarrow{\pi_1} & C_1 \\ {\scriptstyle \langle\pi_2,\pi_1\rangle}\downarrow & \mathbf{PB} \quad \nearrow{\scriptstyle \pi_2} & & \mathbf{ISO} & & \nearrow{\scriptstyle \pi_2} & \\ \mathrm{Iso}(C) & \xrightarrow[\pi_1]{} & \mathrm{Pt}(C) & \xrightarrow[\pi_2]{} & C^{\leftarrow\leftarrow} & & \end{array}$$

We are liberal with our use of these labels. In particular, similar commutative diagrams may have the same label when they are conveying similar results. Additionally, for diagrams that are only to be referenced within a particular proof or remark, lower case letters are used as labels, such as (a). Such labels, when referenced, will appear with brackets, as **(a)**.

1.2 Calculus of pullbacks

We recall here some basic theory of pullbacks, as internal categories are defined in *categories with pullbacks*, and we will make extensive use of them. In particular, we fix notation for the morphisms induced by the universal property of pullbacks, and highlight convenient properties of this notation.

Definition 1.2.1. In a category $\mathbb{C}$, the *pullback* of two morphisms $f : A \to C$ and $g : B \to C$ is the limit of the diagram

$$A \xrightarrow{f} C \xleftarrow{g} B$$

written as

$$\begin{array}{ccc} A \times_C B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ A & \xrightarrow{f} & C \end{array}$$

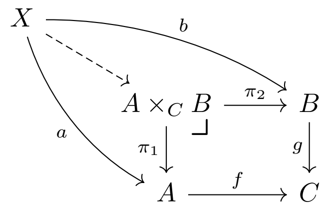
Remark 1.2.2. In general, we will refer to the object $A \times_C B$ as the pullback of f and g. We also overlook the fact that the notation of the pullback omits the morphisms f and g. We will usually write only π_1 and π_2 for the projections of a pullback, even when multiple such morphism appear in the same diagram.

Definition 1.2.3. A category $\mathbb{C}$ *has pullbacks* if the limit of every diagram

$$A \xrightarrow{f} C \xleftarrow{g} B$$

exists.

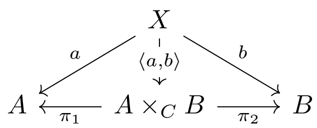
Definition 1.2.4. Let $\mathbb{C}$ be a category with pullbacks. Consider the commutative diagram



The morphism induced by the universal property of the pullback $A \times_C B$ is written as $\langle a, b \rangle : X \to A \times_C B$.

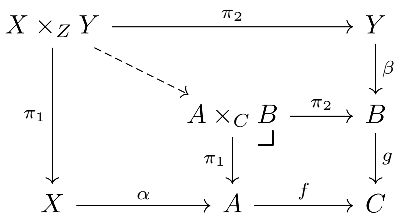
Remark 1.2.5. This notation is clearly well defined, in the sense that if $a = c$ and $b = d$ and $\langle a, b \rangle$ exists, then $\langle a, b \rangle = \langle c, d \rangle$. We will often make use of this fact within commutative diagrams. In such cases, the fact that $a = c$ and $b = d$ will either be explicitly stated, or clear from previous calculation.

Remark 1.2.6. We emphasise that the triangles of this diagram commute:



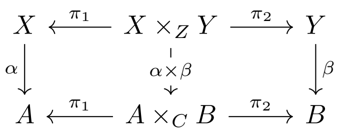
(PB)

Definition 1.2.7. Let $\mathbb{C}$ be a category with pullbacks. Consider the commutative diagram



where $X \times_Z Y$ is some pullback. Then the morphism induced by the universal property of $A \times_C B$ is written as $\alpha \times \beta = \langle \alpha\pi_1, \beta\pi_2 \rangle : X \times_Z Y \to A \times_C B$.

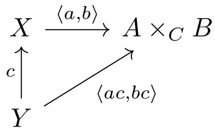
Remark 1.2.8. We, as before, emphasise the commutativity of the squares of this diagram as



(PB)

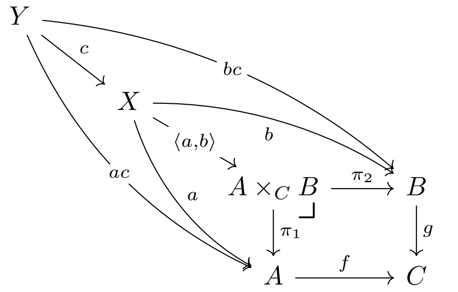
Remark 1.2.9. The notation from the above definition is borrowed from the morphism induced by the universal property of a product. In fact, in the case where $X \times_Z Y$ is the product $X \times Y$, and π_1 and π_2 are product projections instead of pullback projections, the same notation will be used. This convention is also used in the analogous case of $A \times_C B$ being the product $A \times B$.

Proposition 1.2.10. *In a category $\mathbb{C}$ with pullbacks, for some pullback $A \times_C B$ such that $\langle a, b\rangle : X \to A \times_C B$ exists, the following diagram commutes*



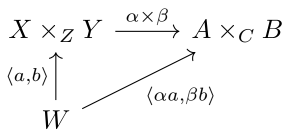
(PB1)

Proof. Observe the following commutative diagram:



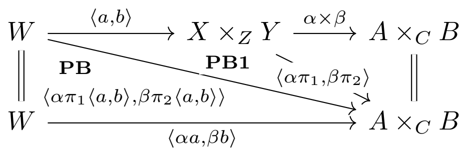
This means that the composition $\langle a, b\rangle c$ satisfies the conditions of the universal morphism induced by the outside of the diagram. By the uniqueness of this universal morphism, we have $\langle a, b\rangle c = \langle ac, bc\rangle$. □

Proposition 1.2.11. *In a category $\mathbb{C}$ with pullback, for two pullbacks $X \times_Z Y$ and $A \times_C B$ such that the morphisms $\langle a, b\rangle : W \to X \times_Z Y$ and $\alpha \times \beta : X \times_Z Y \to A \times_C B$ exist, the following diagram commutes*



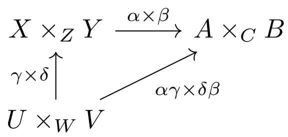
(PB2)

Proof. Noting that by definition, $\alpha \times \beta = \langle \alpha\pi_1, \beta\pi_2\rangle$, we observe that the following diagram commutes.



□

Proposition 1.2.12. *In a category $\mathbb{C}$ with pullbacks, for some pullbacks $U \times_W V$, $X \times_Z Y$ and $A \times_C B$ such the morphisms $\gamma \times \delta : U \times_W V \to X \times_Z Y$ and $\alpha \times \beta : X \times_Z Y \to A \times_C B$ exist, the following diagram commutes*



(PB3)

Proof. Observe that the following diagram commutes

$$\begin{array}{ccccc}
U \times_W V & = & U \times_W V & \xrightarrow{\alpha\delta\times\gamma\beta} & A \times_C B \\
 & \textbf{PB} & & \textbf{PB} & \\
{\scriptstyle \gamma\times\delta}\downarrow & \swarrow{\scriptstyle \langle\gamma\pi_1,\delta\pi_2\rangle} & \downarrow{\scriptstyle \langle\alpha\gamma\pi_1,\beta\delta\pi_2\rangle} & & \| \\
 & \textbf{PB2} & & & \\
X \times_Z Y & \xrightarrow[\alpha\times\beta]{} & A \times_C B & = & A \times_C B
\end{array}$$

□

Proposition 1.2.13. *In a category $\mathbb{C}$ with pullbacks, for all morphisms $a : X \to A \times_C B$ for some pullback $A \times_C B$, we have that $a = \langle \pi_1 a, \pi_2 a\rangle$.* (PB4)

Proof. We note that the following diagram trivially commutes

$$\begin{array}{ccccc}
A \times_C B & & & \xrightarrow{\pi_2 a} & \\
 & \searrow^{a} & & & \\
 & & A \times_C B & \xrightarrow{\pi_2} & B \\
{\scriptstyle \pi_1 a} & & {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow \\
 & & A & \longrightarrow & C
\end{array}$$

so by the uniqueness of universal morphisms, we have that $\langle \pi_1 a, \pi_2 a\rangle = a$. □

Remark 1.2.14. In the above proposition, we may set $a_1 = \pi_1 a$ and $a_2 = \pi_2 a$. Thus, any morphism $a : X \to A \times_C B$ may be written in the form $a = \langle a_1, a_2\rangle$. In such a case, we will implicitly have that $fa_1 = ga_2$. We will use this fact frequently, and reference it as (PB4).

We next recall the well known result on pullbacks, often referred to as the *Pasting Law.*

Proposition 1.2.15. *Let C be a category with pullbacks. Consider the following diagram, where the right hand square is a pullback.*

$$\begin{array}{ccccc}
X & \xrightarrow{a} & A \times_C B & \xrightarrow{\pi_2} & B \\
{\scriptstyle b}\downarrow & & {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
Y & \xrightarrow{c} & A & \xrightarrow{f} & C
\end{array}$$

Then the left hand square is a pullback if and only if the outside rectangle is a pullback.

Remark 1.2.16. In the above, in such a case, we have that $Y \times_A (A \times_C B) \cong X \cong Y \times_C B$, as pullbacks are unique up to isomorphism. We will often identify such objects, omitting the isomorphism between them.

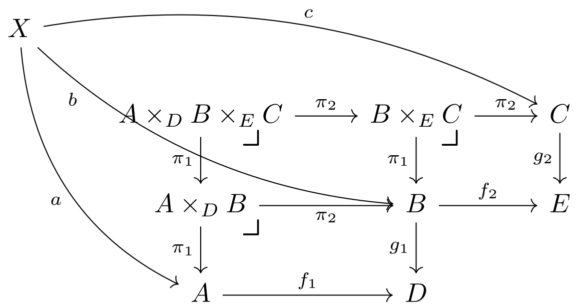
Definition 1.2.17. In a category with pullbacks, a *triple pullback* is a diagram

$$\begin{array}{ccccc}
A \times_D B \times_E C & \xrightarrow{\pi_2} & B \times_E C & \xrightarrow{\pi_2} & C \\
{\scriptstyle \pi_1}\downarrow & \lrcorner & {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle g_2} \\
A \times_D B & \xrightarrow{\pi_2} & B & \xrightarrow{f_2} & E \\
{\scriptstyle \pi_1}\downarrow & \lrcorner & {\scriptstyle g_1}\downarrow & & \\
A & \xrightarrow{f_1} & D & &
\end{array} \qquad \text{(TPB)}$$

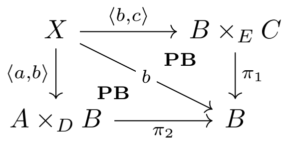
where all the small squares are pullbacks.

Remark 1.2.18. By the above definition, we have that $A \times_D B \times_E C \cong (A \times_D B) \times_B (B \times_E C)$. A direct consequence of the pasting law is that $A \times_D B \times_E C \cong (A \times_D B) \times_E C \cong A \times_D (B \times_E C)$. This fact is often referred to as the *associativity of pullbacks*, and we will again identify these isomorphic objects. We highlight that the commutativity of the top left square of the above diagram means $\pi_1\pi_2 = \pi_2\pi_1$, in this context.

Now, consider a triple pullback $A \times_D B \times_E C$, an object X, and morphisms $a : X \to A$, $b : X \to B$ and $c : X \to C$ such that the following diagram commutes:



Firstly, we have induced universal arrows $\langle a, b\rangle : X \to A \times_D B$ and $\langle b, c\rangle : X \to B \times_E C$. Then, the following diagram commutes



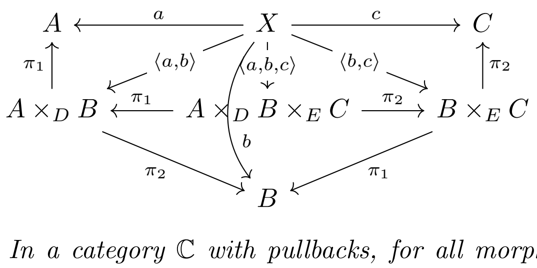
which induces the universal morphism $\langle\langle a, b\rangle, \langle b, c\rangle\rangle : X \to A \times_D B \times_E C$. In line with the notation for a triple pullback, we will write this morphism as below:

Definition 1.2.19. In a category $\mathbb{C}$ with pullbacks, with the setup of the above, we write

$$\langle a, b, c\rangle = \langle\langle a, b\rangle, \langle b, c\rangle\rangle : X \to A \times_D B \times_E C$$

for the universal morphism induced by the triple pullback $A \times_D B \times_E C$.

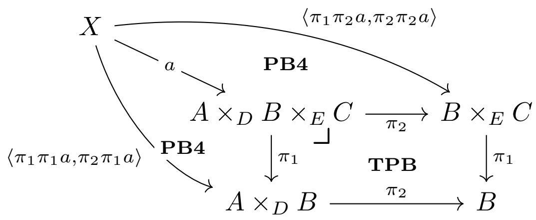
Remark 1.2.20. Note that a, b and c need only satisfy the conditions $f_1 a = g_1 b$ and $f_2 b = g_2 c$ to make the relevant diagram commute, inducing the morphism $\langle a, b, c\rangle : X \to A \times_D B \times_E C$. While this notation has its advantages, it can be somewhat confusing when composing this universal arrow with pullback projections. We thus, as before, emphasise the commutativity of the following diagram.



(TPB)

Proposition 1.2.21. *In a category $\mathbb{C}$ with pullbacks, for all morphisms $a : X \to A \times_D B \times_E C$, for some triple pullback $A \times_D B \times_E C$, we have that $a = \langle \pi_1\pi_1 a, \pi_1\pi_2 a, \pi_2\pi_2 a\rangle$.*

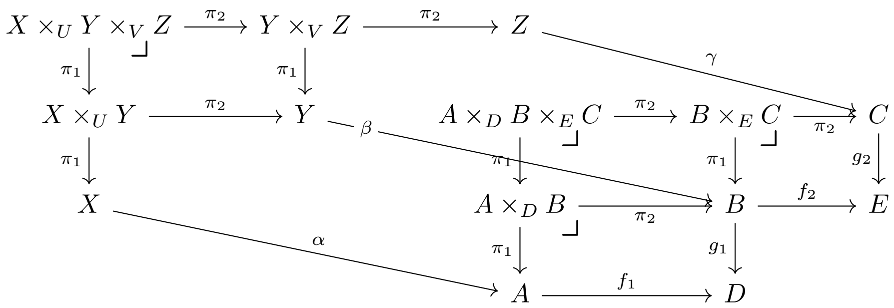
Proof. Observe that the following diagram commutes:



Here, a is a universal morphism for the triple pullback $A \times_D B \times_E C$. Noting that $\pi_2\pi_1 a = \pi_1\pi_2 a$, we have by the uniqueness of the universal arrow, that $a = \langle \pi_1\pi_1 a, \pi_1\pi_2 a, \pi_2\pi_2 a\rangle$. □

Remark 1.2.22. As observed in Remark 1.2.14, we may write any morphism $a : X \to A \times_D B \times_E C$ as $a = \langle a_1, a_2, a_3\rangle$, where $a_1 = \pi_1\pi_1 a$, $a_2 = \pi_1\pi_2 a$ and $a_3 = \pi_2\pi_2 a$.

Definition 1.2.23. Let $\mathbb{C}$ be a category with pullbacks. Consider the commutative diagram



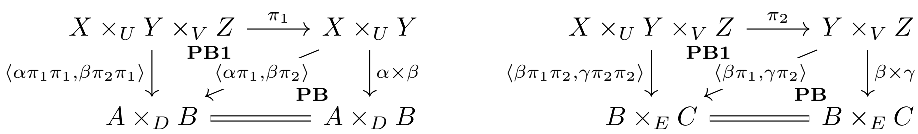
where $X \times_U Y \times_V Z$ is some triple pullback. The morphism induced by the universal property of the triple pullback $A \times_D B \times_E C$ is written

$$\alpha \times \beta \times \gamma = \langle \alpha\pi_1\pi_1, \beta\pi_1\pi_2, \gamma\pi_2\pi_2\rangle : X \times_U Y \times_V Z \to A \times_D B \times_E C$$

Proposition 1.2.24. *In a category $\mathbb{C}$ with pullbacks, in the setup of the above, we have the following equality of morphisms $X \times_U Y \times_V Z \to A \times_D B \times_E C$*

$$\alpha \times \beta \times \gamma = (\alpha \times \beta) \times (\beta \times \gamma)$$

Proof. Firstly note that all the following morphisms exist, and the diagrams commute

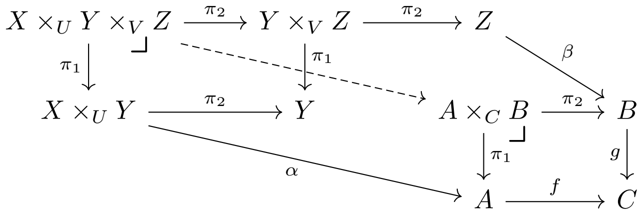


Now, noting that $\pi_1\pi_2 = \pi_2\pi_1$, and by definition $\alpha \times \beta \times \gamma = \langle \pi_1\pi_1\alpha, \pi_1\pi_2\beta, \pi_2\pi_2\gamma\rangle$, we have that the equality of morphisms:

$$\alpha \times \beta \times \gamma = \langle\langle \alpha\pi_1\pi_1, \beta\pi_2\pi_1\rangle, \langle \beta\pi_1\pi_2, \gamma\pi_2\pi_2\rangle\rangle = \langle(\alpha \times \beta)\pi_1, (\beta \times \gamma)\pi_2\rangle = (\alpha \times \beta) \times (\beta \times \gamma)$$

□

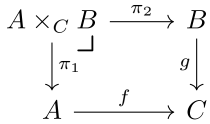
Remark 1.2.25. We now critically consider an important case of universal morphisms between pullbacks and triple pullbacks. Consider the following commutative diagram:



where $X \times_U Y \times_V Z$ is some triple pullback and $A \times_C B$ is a pullback. The universal property of $A \times_C B$ induces the morphism $\langle \alpha\pi_1, \beta\pi_2\pi_2 \rangle : X \times_U Y \times_V Z \to A \times_C B$. Then, by Definition 1.2.7, $\langle \alpha\pi_1, \beta\pi_2\pi_2 \rangle = \alpha \times \beta\pi_2$. We note here that this deviates from conventional notation, for which this universal morphism is written as $\alpha \times \beta$. However, for consistency of calculation, we will go against convention and write $\alpha \times \beta\pi_2$. Similarly, in the same set up, for morphisms $\gamma : X \to A$ and $\delta : Y \times_V Z \to B$, making the expected diagram commute, the universal morphism induced by $A \times_C B$, is written $\gamma\pi_1 \times \delta : X \times_U Y \times_V Z \to A \times_C B$ (instead of simply $\gamma \times \delta$).

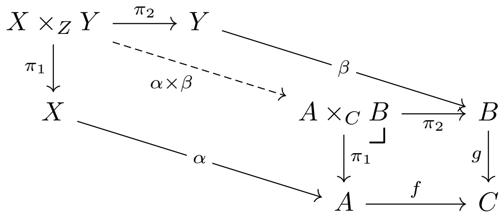
We now note some further well known results on pullbacks.

Proposition 1.2.26. *Let $\mathbb{C}$ be a category will pullbacks, and consider the following pullback in $\mathbb{C}$:*



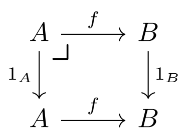
If f is a monomorphism, then π_2 is a monomorphism.

Proposition 1.2.27. *Let $\mathbb{C}$ be a category with pullbacks, and consider the following commutative diagram*



where $A \times_C B$ and $X \times_Y Z$ are two pullbacks, and $\alpha \times \beta$ is induced by the universal property of $A \times_C B$. If α and β are monomorphisms, then $\alpha \times \beta$ is a monomorphism.

Proposition 1.2.28. *Let $f : A \to B$ be a morphism in a category $\mathbb{C}$ with pullbacks. Then the following diagram is a pullback:*



2 Internal Category Theory

We consider internal categories as a means of generalising a (small) category and develop the relevant theory on the structure.

2.1 Basic definitions

The definition of an internal category is motivated by the representation of a small category as a diagram in **Set**, satisfying certain properties. Specifically, let C be a small category, which has C_0 as its set of objects, C_1 as its set of morphisms and $C^{\leftarrow\leftarrow}$ as its set of all pairs of composable morphisms. Then C induces the diagram

$$C_0 \;\underset{c}{\overset{d}{\leftleftarrows}}\!\!\xrightarrow{e}\; C_1 \xleftarrow{\;m\;} C^{\leftarrow\leftarrow} \tag{IC}$$

in **Set**, where the map d assigns a morphism to its domain object, c assigns a morphism to its codomain object, e assigns an object to its identity morphism, and m assigns a pair of composable morphisms to their composition. As C is a category, these functions are going to satisfy certain conditions, namely:

1. The domain and codomain of identity morphisms are appropriate.
2. The domain and codomain of the composition of two morphisms are appropriate.
3. Identity morphisms acts as identity under composition.
4. Composition is associative.

These conditions can be expressed as commutative diagrams in **Set**, which are given in the following definition. We then define an internal category, C, in some category $\mathbb{C}$ as a diagram **IC** in $\mathbb{C}$, satisfying the commutative diagrams of these conditions, such that an internal category in **Set** is exactly a small category. As we use pullbacks to describe this structure in **Set**, we require this of our category $\mathbb{C}$ in the general definition.

Definition 2.1.1. Let $\mathbb{C}$ be a category with pullbacks. An *internal category*, $C = (C_0, C_1, C^{\leftarrow\leftarrow}, d, e, c, m)$, in $\mathbb{C}$ is a diagram

$$C_0 \;\underset{c}{\overset{d}{\leftleftarrows}}\!\!\xrightarrow{e}\; C_1 \xleftarrow{\;m\;} C^{\leftarrow\leftarrow} \tag{IC}$$

where $C^{\leftarrow\leftarrow}$ is defined as the pullback

$$\begin{array}{ccc} C^{\leftarrow\leftarrow} & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle \pi_1}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle c} \\ C_1 & \xrightarrow{d} & C_0 \end{array} \tag{CM}$$

such that the following diagrams commute

$$\begin{array}{ccc} C_0 & \xrightarrow{e} & C_1 \\ {\scriptstyle e}\big\downarrow & \diagdown\!\!\diagdown & \big\downarrow{\scriptstyle d} \\ C_1 & \xrightarrow[c]{} & C_0 \end{array} \qquad \begin{array}{ccccc} C_1 & \xleftarrow{\pi_1} & C^{\leftarrow\leftarrow} & \xrightarrow{\pi_2} & C_1 \\ {\scriptstyle c}\big\downarrow & & \big\downarrow{\scriptstyle m} & & \big\downarrow{\scriptstyle d} \\ C_0 & \xleftarrow{c} & C_1 & \xrightarrow{d} & C_0 \end{array} \tag{IC1, IC2}$$

$$\begin{array}{ccc} C_1 & \xrightarrow{\langle ec,1\rangle} & C^{\leftarrow\leftarrow} \\ {\scriptstyle\langle 1,ed\rangle}\downarrow & \diagdown\!\diagdown & \downarrow{\scriptstyle m} \\ C^{\leftarrow\leftarrow} & \xrightarrow[m]{} & C_1 \end{array} \qquad \begin{array}{ccc} C^{\leftarrow\leftarrow\leftarrow} & \xrightarrow{m\times\pi_2} & C^{\leftarrow\leftarrow} \\ {\scriptstyle\pi_1\times m}\downarrow & & \downarrow{\scriptstyle m} \\ C^{\leftarrow\leftarrow} & \xrightarrow{m} & C_1 \end{array} \tag{IC3, IC4}$$

where $C^{\leftarrow\leftarrow\leftarrow}$ is the triple pullback

$$\begin{array}{ccc} C^{\leftarrow\leftarrow\leftarrow} & \xrightarrow{\pi_1} & C^{\leftarrow\leftarrow} \\ {\scriptstyle\pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle\pi_1} \\ C^{\leftarrow\leftarrow} & \xrightarrow{\pi_2} & C_1 \end{array}$$

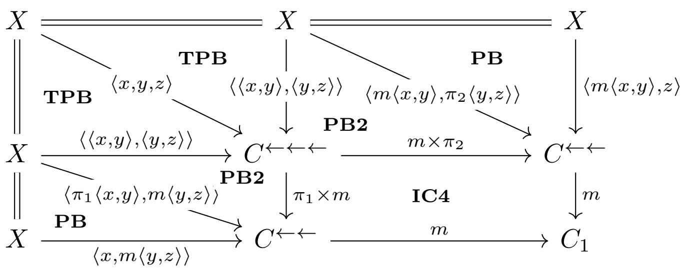
Remark 2.1.2. In the case that $\mathbb{C} = \mathbf{Set}$, **IC1**, **IC2**, **IC3** and **IC4** each correspond to the conditions 1, 2, 3 and 4 on a usual category. C will therefore form a small category in this case, and we will freely refer to it as such.

We will usually use **IC4** in a different form, which we give below.

Proposition 2.1.3. *Let C be an internal category in a category $\mathbb{C}$ with pullbacks. Let X be an object and $\langle x, y, z\rangle : X \to C^{\leftarrow\leftarrow\leftarrow}$ a morphism in $\mathbb{C}$. Then the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\langle m\langle x,y\rangle,z\rangle} & C^{\leftarrow\leftarrow} \\ {\scriptstyle\langle x,m\langle y,z\rangle\rangle}\downarrow & & \downarrow{\scriptstyle m} \\ C^{\leftarrow\leftarrow} & \xrightarrow{m} & C_1 \end{array} \tag{ASC}$$

Proof. We simply observe that the following diagram commutes:



□

Corollary 2.1.4. *Let C be an internal category in a category $\mathbb{C}$ with pullbacks. Let $\alpha\times\beta\times\gamma$: $X \times_U Y \times_V Z \to C^{\leftarrow\leftarrow\leftarrow}$ be some universal arrow induced by the triple pullbacks $C^{\leftarrow\leftarrow\leftarrow}$. Then we have that the following diagram commutes:*

$$\begin{array}{ccc} X \times_U Y \times_V Z & \xrightarrow{m(\alpha\times\beta)\times\gamma\pi_2} & C^{\leftarrow\leftarrow} \\ {\scriptstyle\alpha\pi_1\times m(\beta\times\gamma)}\downarrow & & \downarrow{\scriptstyle m} \\ C^{\leftarrow\leftarrow} & \xrightarrow{m} & C_1 \end{array} \tag{ASC}$$

In fact, this result can be taken further, and we will find the following to be useful. One may view this result as a internal strict coherence condition on the associativity of composition.

Corollary 2.1.5. *Let C be an internal category in a category $\mathbb{C}$ with pullbacks. Let $\langle w, x\rangle$ and $\langle y, z\rangle$ be morphisms $X \to C^{\leftarrow\leftarrow}$ such that the morphism*

$$\langle m\langle w, x\rangle, m\langle y, z\rangle\rangle : X \to C^{\leftarrow\leftarrow}$$

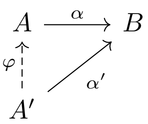
exists. Then we have the following equality of morphisms $X \to C_1$:

$$\begin{aligned} & m\langle w, m\langle m\langle x, y\rangle, z\rangle\rangle \\ =\ & m\langle w, m\langle x, m\langle y, z\rangle\rangle\rangle \\ =\ & m\langle m\langle w, x\rangle, m\langle y, z\rangle\rangle \\ =\ & m\langle m\langle m\langle w, x\rangle, y\rangle, z\rangle \\ =\ & m\langle m\langle w, m\langle x, y\rangle\rangle, z\rangle \end{aligned} \tag{ASC}$$

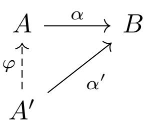
2.2 Subobjects of morphisms

We will frequently need the internal concept of a class of morphisms of a category. In the case of a small category, seen as an internal category C in **Set**, this is a subset of C_1. Thus, if C is an internal category instead in a general category $\mathbb{C}$ with pullbacks, we use a subobject of C_1.

Definition 2.2.1. In a category $\mathbb{C}$, given two monomorphism $\alpha : A \to B$ and $\alpha' : A' \to B$, we say that $\alpha' \leq \alpha$ if there exists a morphism φ such that the following diagram commutes



Definition 2.2.2. In a category $\mathbb{C}$, given two monomorphism $\alpha : A \to B$ and $\alpha' : A' \to B$, we say that α and α' are *equivalent*, and write $\alpha \sim \alpha'$, if there exists an isomorphism φ such that the following diagram commutes



Proposition 2.2.3. *In a category $\mathbb{C}$, given two monomorphism $\alpha : A \to B$ and $\alpha' : A' \to B$, then $\alpha \sim \alpha'$ if and only if $\alpha \leq \alpha'$ and $\alpha' \leq \alpha$.*

Corollary 2.2.4. *In a category $\mathbb{C}$, for a fixed object B of $\mathbb{C}$, $\sim$ forms an equivalence relation on the class of monomorphism with codomain B.*

Definition 2.2.5. In a category $\mathbb{C}$ a *subobject* of some object B of $\mathbb{C}$ is an $\sim$-equivalence class of monomorphisms with codomain B.

Remark 2.2.6. We adopt the common practice of identifying a subobject with some representative monomorphism of the equivalence class. That is, for a monomorphism $\alpha : A \to B$, we will call α a subobject of B. Additionally, a subobject $\alpha : A \to B$ in the category **Set** specifies A as a subset of B. We will thus refer to α as the subset of B in such a context.

Definition 2.2.7. Let C be an internal category in a category $\mathbb{C}$ with pullbacks. A *subobject of morphisms* of C is a subobject of C_1, $\alpha : A \to C_1$.

Of course, if $\mathbb{C} = \mathbf{Set}$ this subobject of morphisms will be a subset of morphisms of the small category C. We now consider a couple of examples that we use frequently.

Example 2.2.8. Note that $1_{C_1} : C_1 \to C_1$ is trivially a monomorphism, and thus a subobject of morphisms. In the case of $\mathbb{C} = \mathbf{Set}$, this is the subset of C_1 containing all morphism. Thus, in general, 1_{C_1} may be considered as the *subobject of all morphisms* of C.

Example 2.2.9. We have that $e : C_0 \to C_1$ is a split monomorphism by **IC1**, and thus a subobject of morphisms. In the cases $\mathbb{C} = \mathbf{Set}$, this is the subset of C_1 containing all identity morphism. We thus will often consider e as the *subobject of identity morphisms* of C.

Definition 2.2.10. Let $\alpha : A \to C_1$ and $\beta : B \to C_1$ be two subobjects of morphisms of an internal category C in a category $\mathbb{C}$ with pullbacks. Then we say that α *contains* β if $\beta \leq \alpha$ as subobjects of C_1. That is, if there exists a morphism $\beta_\alpha : B \to A$ making the following diagram commute

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & C_1 \\ {\scriptstyle \beta_\alpha}\uparrow & \nearrow_{\beta} & \\ B & & \end{array} \tag{C}$$

Remark 2.2.11. In such a case, β_α will be a monomorphism (and thus a subobject of A). Note that by the commutativity of the following diagram, the subobject of all morphisms contains every subobject of morphisms $\alpha : A \to C_1$, with $\alpha_{1_{C_1}} = \alpha$.

$$\begin{array}{ccc} C_1 & \xrightarrow{1_{C_1}} & C_1 \\ {\scriptstyle \alpha}\uparrow & \nearrow_{\alpha} & \\ B & & \end{array}$$

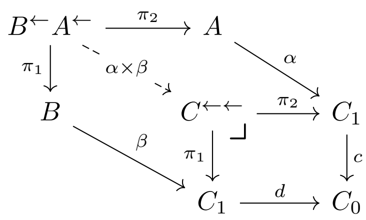
Definition 2.2.12. If a subobject of morphisms $\alpha : A \to C_1$ of an internal category C contains e, then we say that α *contains all identities* of C.

For two classes of morphisms A and B of a category $\mathbb{C}$, we may speak of a class of pairs of composable morphism $f : X \to Y$ and $g : Y \to Z$ in $\mathbb{C}$ such that f is in A and g is in B. We internalise this notion, emphasising the notation used for the pullback.

Definition 2.2.13. Let C be an internal category in a category $\mathbb{C}$ with pullbacks, and let $\alpha : A \to C_1$ and $\beta : B \to C_1$ be two subobjects of morphism of C. Then the *object of pairs of composable morphisms* of α and β is the pullback

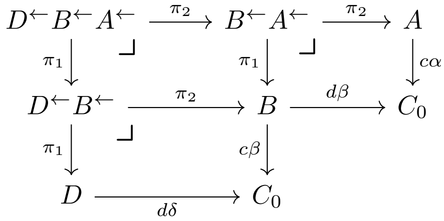
$$\begin{array}{ccc} B{\leftarrow}A{\leftarrow} & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle c\alpha} \\ B & \xrightarrow{d\beta} & C_0 \end{array} \tag{CM}$$

Remark 2.2.14. Note that the following diagram commutes



which induces $\alpha \times \beta$ by the universal property of $C^{\leftarrow\leftarrow}$. Further note that α and β are monomorphism, so by 1.2.27 $\alpha \times \beta$ is a monomorphism, and thus a subobject of $C^{\leftarrow\leftarrow}$.

Remark 2.2.15. In the case were $A = B$ and $\alpha = \beta$, we will write $A^{\leftarrow\leftarrow} = A^{\leftarrow}A^{\leftarrow}$. We extend this notation to triple pullbacks of the form

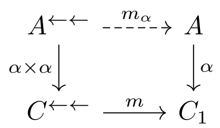


for subobjects of morphisms α, β and δ. Similarly, a subobject $\delta \times \beta \times \alpha : D^{\leftarrow}B^{\leftarrow}A^{\leftarrow} \to C^{\leftarrow\leftarrow\leftarrow}$ is induced by the universal property of $C^{\leftarrow\leftarrow\leftarrow}$. For $D^{\leftarrow}B^{\leftarrow}A^{\leftarrow}$ we write: $B^{\leftarrow\leftarrow}A^{\leftarrow}$ if $\delta = \beta$, $D^{\leftarrow}B^{\leftarrow\leftarrow}$ if $\beta = \alpha$, and $B^{\leftarrow\leftarrow\leftarrow}$ if $\alpha = \beta = \delta$.

Example 2.2.16. Considering the subobject of all morphisms, we have that $C^{\leftarrow\leftarrow} = C_1^{\leftarrow\leftarrow}$ and $C^{\leftarrow\leftarrow\leftarrow} = C_1^{\leftarrow\leftarrow\leftarrow}$.

We will usually need a particular subobject of morphisms to be closed under composition, in an internal sense. Specifically, this notion will be used in the definition of an internal factorisation system.

Definition 2.2.17. Let $\alpha : A \to C_1$ be an subobject of morphisms of an internal category C in a category $\mathbb{C}$ with pullbacks. Then α is *closed under composition* if there exists a morphism $m_\alpha : A^{\leftarrow\leftarrow} \to A$ such that the following diagram commutes:



(CL)

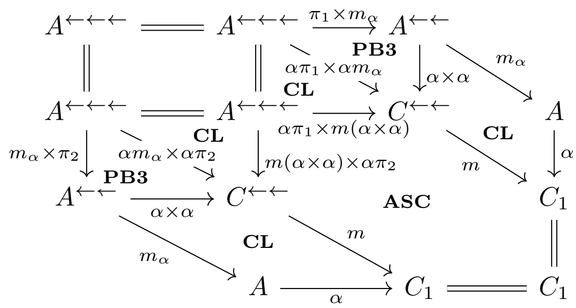
Remark 2.2.18. To see that that this is a reasonable definition, we consider the case when $\mathbb{C} = \mathbf{Set}$. Then, in this definition, C is a small category and A is a subset of morphisms in C, with inclusion α. Then A being closed under composition means that for a pair of composable morphisms (a_1, a_2) in A, $m(\alpha \times \alpha)(a_1, a_2) = \alpha m_A(a_1, a_2)$. As α and $\alpha \times \alpha$ are inclusions, this means that $m_A(a_1, a_2) = m(a_1, a_2) = a_1 a_2$, making the m_A the restriction of m to domain $A^{\leftarrow\leftarrow}$, a subset of $C^{\leftarrow\leftarrow}$. The fact that m_A has codomain A then means that every composition of morphisms in A in again in A. This agrees with the usual definition of closure under composition on a subset of morphisms of a small category.

In this case, m_α, as restriction of composition m, will inherit the associativity of m. We show that this is true in the general internal case.

Proposition 2.2.19. *Let C be an internal category in a category $\mathbb{C}$ with pullbacks, and let $\alpha : A \to C_1$ be a subobject of morphisms that is closed under composition. Then the following diagram commutes:*

$$\begin{array}{ccc} A^{\leftarrow\leftarrow\leftarrow} & \xrightarrow{\pi_1\times m_\alpha} & A^{\leftarrow\leftarrow} \\ {\scriptstyle m_\alpha\times\pi_2}\downarrow & & \downarrow{\scriptstyle m_\alpha} \\ A^{\leftarrow\leftarrow} & \xrightarrow{m_\alpha} & A \end{array} \tag{ASC}$$

Proof. We observe that the following diagram commutes:



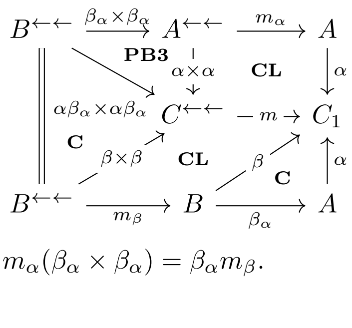
This gives the equality of morphisms $\alpha m_\alpha(\pi_1 \times m_\alpha) = \alpha m_\alpha(m_\alpha \times \pi_2) : A^{\leftarrow\leftarrow\leftarrow} \to C_1$. Then α is a monomorphism, so $m_\alpha(\pi_1 \times m_\alpha) = m_\alpha(m_\alpha \times \pi_2)$. □

We may therefore apply the various forms of **ASC** to any composition morphism m_α for some subobject of morphisms α. We now also note that composition within a subobject of morphisms is closed with respect to being contained in another subobject of morphisms.

Proposition 2.2.20. *Let C be an internal category in a category $\mathbb{C}$ with pullbacks, and let $\alpha : A \to C_1$ and $\beta : B \to C_1$ be two subobjects of morphisms of C, both closed under composition, such that α contains β. Then the following diagram commutes:*

$$\begin{array}{ccc} B^{\leftarrow\leftarrow} & \xrightarrow{m_\beta} & B \\ {\scriptstyle \beta_\alpha\times\beta_\alpha}\downarrow & & \downarrow{\scriptstyle \beta_\alpha} \\ A^{\leftarrow\leftarrow} & \xrightarrow{m_\alpha} & A \end{array} \tag{CL}$$

Proof. The following diagram commutes



and α is a monomorphism, so $m_\alpha(\beta_\alpha \times \beta_\alpha) = \beta_\alpha m_\beta$.

□

2.3 The Subobjects of Isomorphisms

Isomorphisms play a fundamental role in the definition - and many of the properties - of a factorisation system. As we are working internally, we cannot consider individual isomorphisms. Thus, we internalise the notion of the class of all isomorphisms of a category.

Due to their many characterising properties, one may define an isomorphism in many ways. It is convenient, in this context, to view them as morphisms that are both split epimorphisms and split monomorphisms. As such, we first internalise the notion of the class of all split epimorphisms with a specified section, which are called *points* of the category.

Definition 2.3.1. Let C be an internal category in a category $\mathbb{C}$ with pullbacks. The *object of points* of C is the pullback:

$$\begin{array}{ccc} \mathrm{Pt}(C) & \xrightarrow{\pi_2} & C^{\leftarrow\leftarrow} \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle m} \\ C_0 & \xrightarrow{e} & C_1 \end{array} \tag{PT}$$

Remark 2.3.2. As e is a split monomorphism, we have that π_2 is a monomorphism. In particular, π_2 is a subobject of $C^{\leftarrow\leftarrow}$.

To understand this definition, we consider the case $\mathbb{C} = \mathbf{Set}$. Then, $\mathrm{Pt}(C)$ is the subset of pairs of composable morphisms of the small category C, such that their composition is an identity morphism. As such, the images of the composite maps $\pi_1\pi_2 : \mathrm{Pt}(C) \to C_1$ and $\pi_2\pi_2 : \mathrm{Pt}(C) \to C_1$ will contain all the split epimorphisms of C and split monomorphisms of C, respectively. We can thus regard these composites in general as such. Note that they will not be injective maps, and thus in general not monomorphisms, as the section of a split epimorphism (or, dually, the retraction of a split monomorphism) is not necessarily unique.

Nonetheless, as mentioned, to be both a split epimorphism and a split monomorphism is to be an isomorphism, which motivates the following definition.

Definition 2.3.3. Let C be an internal category in a category $\mathbb{C}$ with pullbacks. Then the *object of isomorphisms* of C is the pullback:

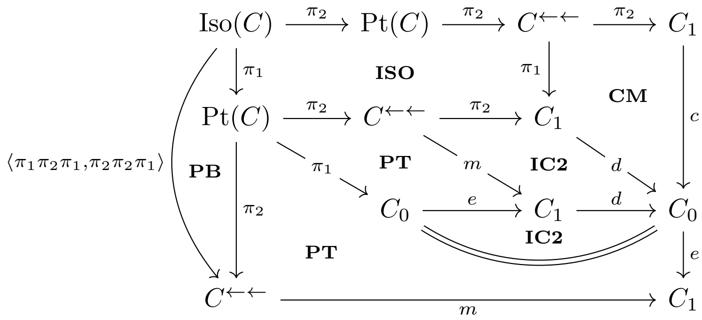
$$\begin{array}{ccc} \mathrm{Iso}(C) & \xrightarrow{\pi_2} & \mathrm{Pt}(C) \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1\pi_2} \\ \mathrm{Pt}(C) & \xrightarrow[\pi_2\pi_2]{} & C_1 \end{array} \tag{ISO}$$

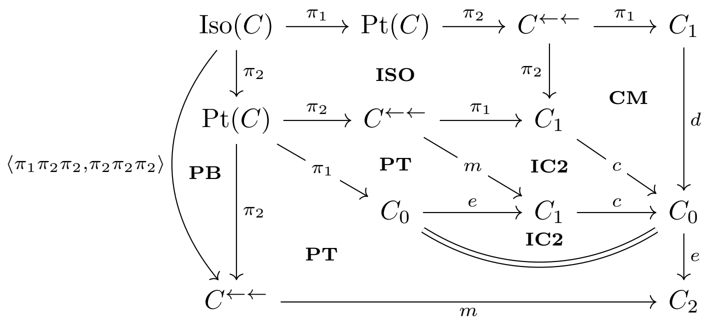
Now, unlike for split epimorphisms, the "section" of an isomorphism - that is, its inverse - is unique. We phrase this as "*there is a correspondence between isomorphisms and their inverses*" and show this property internally. We first, however, consider the following lemma, for its usefulness in calculation.

Lemma 2.3.4. *For an internal category C in a category $\mathbb{C}$ with pullbacks, the following diagram commutes:*

$$\begin{array}{ccccc} \mathrm{Iso}(C) & \xrightarrow{\pi_2\pi_2} & C^{\leftarrow\leftarrow} & \xleftarrow{\pi_2\pi_1} & \mathrm{Iso}(C) \\ {\scriptstyle \langle \pi_1\pi_2\pi_1, \pi_2\pi_2\pi_1 \rangle}\downarrow & & {\scriptstyle ec\pi_2}\downarrow\ \downarrow{\scriptstyle ed\pi_1} & & \downarrow{\scriptstyle \langle \pi_1\pi_2\pi_2, \pi_2\pi_2\pi_2 \rangle} \\ C^{\leftarrow\leftarrow} & \xrightarrow[m]{} & C_1 & \xleftarrow{m} & C^{\leftarrow\leftarrow} \end{array} \tag{IS1}$$

Proof. We observe that the following diagrams commute:





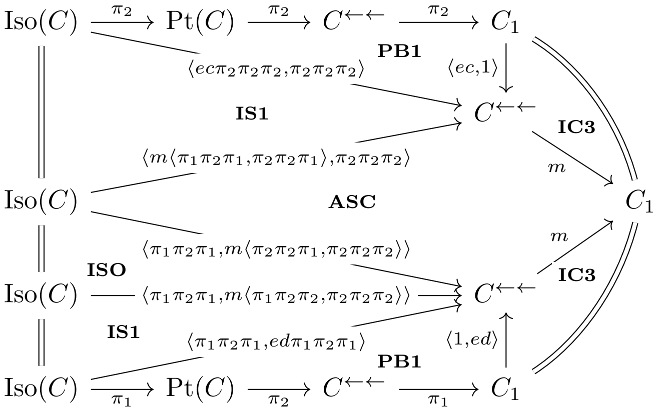
□

Remark 2.3.5. The meaning of the above lemma is seemingly opaque, but it internally describes that the composition of inverse morphisms are identity morphisms. This will become clearer shortly, and we will revisit this lemma once the necessary tools are developed. In the next proposition, the square is the same as that of 2.3.3, with the pullback projections interchanged.

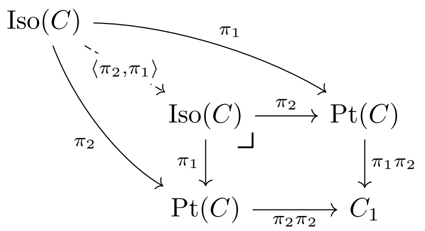
Proposition 2.3.6. *For an internal category C in a category $\mathbb{C}$ with pullbacks, the following diagram is a pullback:*

$$\begin{array}{ccc} \mathrm{Iso}(C) & \xrightarrow{\pi_1} & \mathrm{Pt}(C) \\ {\scriptstyle \pi_2}\downarrow & \lrcorner & \downarrow{\scriptstyle \pi_1\pi_2} \\ \mathrm{Pt}(C) & \xrightarrow[\pi_2\pi_2]{} & C_1 \end{array} \tag{ISO}$$

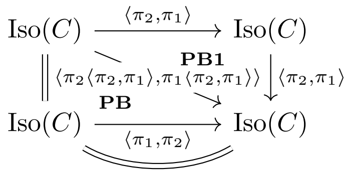
Proof. Firstly, we consider the following commutative diagram, to show that this square commutes:



Then, by the universal property of $\text{Iso}(C)$, we obtain a morphism $\langle \pi_2, \pi_1 \rangle$, as in the following diagram



But then, we have that $\langle \pi_2, \pi_1 \rangle$ is an automorphism by the commutativity of the diagram



□

That the inverse of an isomorphism is unique gives that the *canonical morphism* from $\text{Iso}(C)$ to C_1 should be a monomorphism. The above proposition indicates that there are two such canonical morphism, which we give next.

Proposition 2.3.7. *For an internal category C in a category $\mathbb{C}$ with pullbacks, we have that the following composites are monomorphisms*

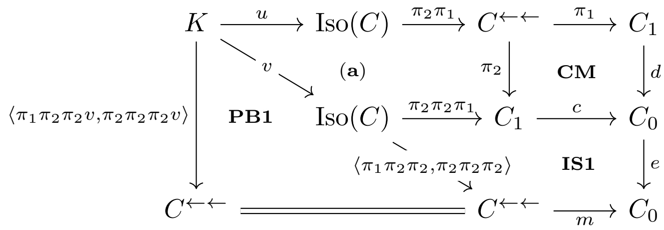
$$\text{Iso}(C) \xrightarrow{\pi_1} \text{Pt}(C) \xrightarrow{\pi_2} C^{\leftarrow\leftarrow} \xrightarrow{\pi_1} C_1$$

$$\text{Iso}(C) \xrightarrow{\pi_1} \text{Pt}(C) \xrightarrow{\pi_2} C^{\leftarrow\leftarrow} \xrightarrow{\pi_2} C_1$$

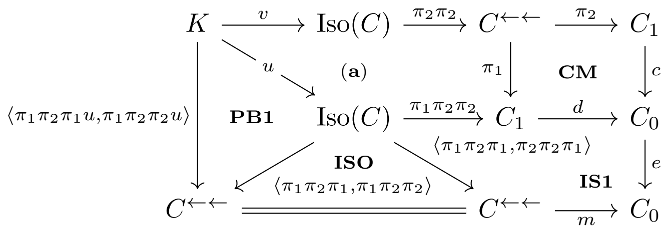
Proof. Consider an object K and morphisms u and v as in the diagram

$$K \overset{u}{\underset{v}{\rightrightarrows}} \text{Iso}(C) \xrightarrow{\pi_2\pi_2\pi_1} C_1 \tag{a}$$

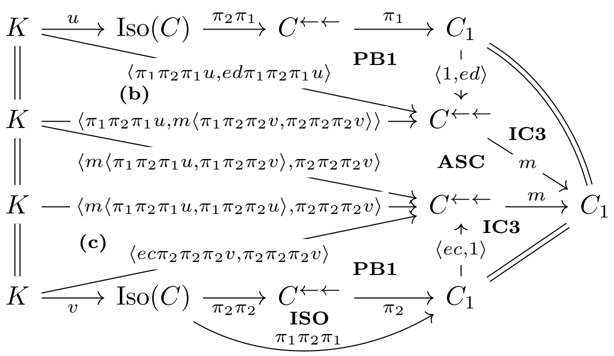
such that $\pi_2\pi_2\pi_1 u = \pi_2\pi_2\pi_1 v$. By **ISO**, we also have that $\pi_1\pi_2\pi_2 u = \pi_1\pi_2\pi_2 v$. Then, observe that the following diagrams commute:



(b)



(c)



meaning that $\pi_1\pi_2\pi_1 u = \pi_1\pi_2\pi_1 v$. By 2.3.6, we also have that $\pi_2\pi_2\pi_2 u = \pi_2\pi_2\pi_2 v$. Now, consider the following four equalities of composites $K \to C_1$:

$$\pi_1\pi_2\pi_1 u = \pi_1\pi_2\pi_1 v \quad \pi_2\pi_2\pi_1 u = \pi_2\pi_2\pi_1 v$$
$$\pi_1\pi_2\pi_2 u = \pi_1\pi_2\pi_2 v \quad \pi_2\pi_2\pi_2 u = \pi_2\pi_2\pi_2 v$$

We note that the leading morphisms π_1 and π_2 are the projections for $C^{\leftarrow\leftarrow}$, and thus are jointly monomorphic. Thus we have the equalities of composites $K \to C^{\leftarrow\leftarrow}$:

$$\pi_2\pi_1 u = \pi_2\pi_1 v \quad \pi_2\pi_2 u = \pi_2\pi_2 v$$

The leading morphism π_2 is a projection of $\mathrm{Pt}(C)$, which by the remark 2.3.2 is a monomorphism, so we have the equalities of morphisms $K \to \mathrm{Pt}(C)$:

$$\pi_1 u = \pi_1 v \quad \pi_2 u = \pi_2 v$$

Here the leading morphisms are the projections of $\mathrm{Iso}(C)$, and are jointly monomorphic. We may thus conclude that $u = v$ and so $\pi_2\pi_2\pi_1$ is a monomorphism. Recalling the automorphism $\langle \pi_2, \pi_1 \rangle$ from 2.3.6, we observe that $\pi_1\pi_2\pi_1$ is isomorphic to $\pi_2\pi_2\pi_1$ by the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Iso}(C) & \xrightarrow{\pi_1} & \mathrm{Pt}(C) & \xrightarrow{\pi_2} & C^{\leftarrow\leftarrow} & \xrightarrow{\pi_1} & C_1 \\
{\scriptstyle \langle \pi_2,\pi_1\rangle}\big\downarrow & \mathbf{PB}\ \nearrow_{\pi_2} & & \mathbf{ISO} & & \nearrow_{\pi_2} & \\
\mathrm{Iso}(C) & \xrightarrow[\pi_1]{} & \mathrm{Pt}(C) & \xrightarrow[\pi_2]{} & C^{\leftarrow\leftarrow} & &
\end{array}$$

and thus $\pi_1\pi_2\pi_1$ is also a monomorphism. □

We now have the necessary results to define the *subobject of isomorphisms*, and determine its properties, allowing us to avoid complicated calculations as in the above. As suggested by 2.3.7, we will speak of two (albeit equivalent) subobjects of isomorphisms.

Definition 2.3.8. Let C be an internal category in a category $\mathbb{C}$ with pullbacks. The *subobjects of isomorphisms* of C are the subobjects of morphisms:

$$\sigma = \pi_2\pi_2\pi_1 = \pi_1\pi_2\pi_2 : \mathrm{Iso}(C) \to C_1 \tag{ISO}$$
$$\sigma' = \pi_1\pi_2\pi_1 = \pi_2\pi_2\pi_2 : \mathrm{Iso}(C) \to C_1$$

Remark 2.3.9. We note that σ and σ' are isomorphic as $\sigma = \sigma'\langle \pi_2, \pi_1 \rangle$ from the proof of 2.3.7 and are thus equivalent as subobjects of C_1. We distinguish these two subobjects for their interactions with each other, and in particular consider σ to be the primary subobject of isomorphisms. For example, we define $\mathrm{Iso}(C)^{\leftarrow\leftarrow}$ as the pullback:

$$\begin{array}{ccc}
\mathrm{Iso}(C)^{\leftarrow\leftarrow} & \xrightarrow{\pi_2} & \mathrm{Iso}(C) \\
{\scriptstyle \pi_1}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle c\sigma} \\
\mathrm{Iso}(C) & \xrightarrow{d\sigma} & C_0
\end{array} \tag{CM}$$

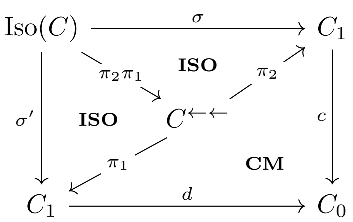
We may now reconsider 2.3.4 under this new notation.

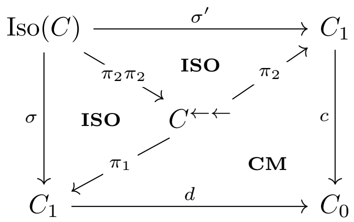
Proposition 2.3.10. *Let C be an internal category in a category $\mathbb{C}$ with pullbacks. Then, the following diagrams commute:*

$$\begin{array}{ccc}
\mathrm{Iso}(C) & \xrightarrow{\sigma} & C_1 \\
{\scriptstyle \sigma'}\big\downarrow & & \big\downarrow{\scriptstyle c} \\
C_1 & \xrightarrow{d} & C_0
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Iso}(C) & \xrightarrow{\sigma'} & C_1 \\
{\scriptstyle \sigma}\big\downarrow & & \big\downarrow{\scriptstyle c} \\
C_1 & \xrightarrow{d} & C_0
\end{array} \tag{IS0}$$

$$\begin{array}{ccccc}
\mathrm{Iso}(C) & \xrightarrow{\sigma} & C_1 & \xrightarrow{d} & C_0 \\
{\scriptstyle \sigma'}\big\downarrow & \searrow^{\langle \sigma',\sigma\rangle} & & & \big\downarrow \\
C_1 & & C^{\leftarrow\leftarrow} & & {\scriptstyle e} \\
{\scriptstyle c}\big\downarrow & & & \searrow^{m} & \big\downarrow \\
C_0 & & \xrightarrow{e} & & C_1
\end{array}
\qquad
\begin{array}{ccccc}
\mathrm{Iso}(C) & \xrightarrow{\sigma'} & C_1 & \xrightarrow{d} & C_0 \\
{\scriptstyle \sigma}\big\downarrow & \searrow^{\langle \sigma,\sigma'\rangle} & & & \big\downarrow \\
C_1 & & C^{\leftarrow\leftarrow} & & {\scriptstyle e} \\
{\scriptstyle c}\big\downarrow & & & \searrow^{m} & \big\downarrow \\
C_0 & & \xrightarrow{e} & & C_1
\end{array} \tag{IS1}$$

Proof. The commutativity of the first two diagrams follows from the commmutative diagrams





This asserts the existence of $\langle \sigma, \sigma' \rangle$ and $\langle \sigma', \sigma \rangle$. Then the commutativity of the second two diagrams is exactly the commutativity of the left and right square in lemma 2.3.4 under definition 2.3.8, respectively. ☐

Remark 2.3.11. Given this proposition, it should be clear that σ and σ' point to inverse isomorphisms in C_1 internally, so that these commutative diagrams says that the composition of inverse isomorphisms is the identity morphism of the suitable domain or codomain object. This, of course, is a characterising property of isomorphisms, which may be internally phrased by the following proposition. Note that we now require our category $\mathbb{C}$ to have products as well as pullbacks, and thus assume it to be finitely complete.

Proposition 2.3.12. *Let C be an internal category in a finitely complete category $\mathbb{C}$. Then the following square is a pullback:*

$$\begin{array}{ccc} \mathrm{Iso}(C) & \xrightarrow{\langle\langle\sigma',\sigma\rangle,\langle\sigma,\sigma'\rangle\rangle} & C^{\leftrightarrows} \\ {\scriptstyle (d,c)\sigma}\downarrow & \lrcorner & \downarrow{\scriptstyle m\times m} \\ C_0 \times C_0 & \xrightarrow{e\times e} & C_1 \times C_1 \end{array} \tag{ISO}$$

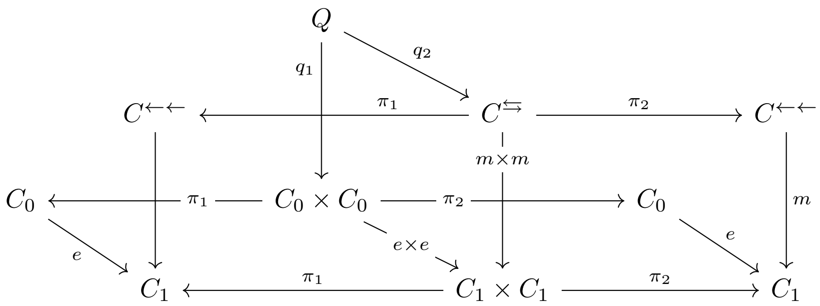
where $C^{\leftrightarrows}$ is defined as the following pullback

$$\begin{array}{ccc} C^{\leftrightarrows} & \xrightarrow{\pi_2} & C^{\leftarrow\leftarrow} \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle (\pi_2,\pi_1)} \\ C^{\leftarrow\leftarrow} & \xrightarrow{(\pi_1,\pi_2)} & C_1 \times C_1 \end{array} \tag{BAF}$$

Proof. Firstly note that the existence of $\langle\langle\sigma',\sigma\rangle,\langle\sigma,\sigma'\rangle\rangle : \mathrm{Iso}(C) \to C^{\leftrightarrows}$ follows from the definition of $C^{\leftrightarrows}$, and the fact that the square commutes follows from 2.3.10. Now, assume that there exists an object Q and morphisms q_1 and q_2 such that the following diagram commutes:

$$\begin{array}{ccc} Q & \xrightarrow{q_2} & C^{\leftrightarrows} \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle m\times m} \\ C_0 \times C_0 & \xrightarrow{e\times e} & C_1 \times C_1 \end{array}$$

We firstly observe that the following diagram commutes by **PB**

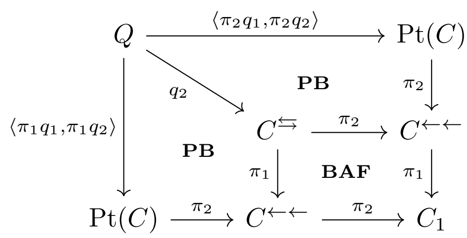


(a)

Which induces, by the universal property of $\mathrm{Pt}(C)$, the two morphisms

$$\langle \pi_1 q_1, \pi_1 q_2\rangle : Q \to \mathrm{Pt}(C)$$
$$\langle \pi_2 q_1, \pi_2 q_2\rangle : Q \to \mathrm{Pt}(C)$$

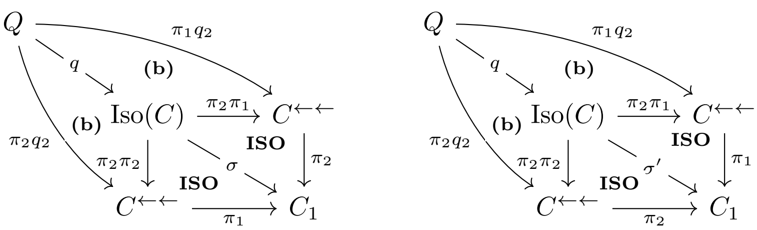
Then the following diagram commutes,



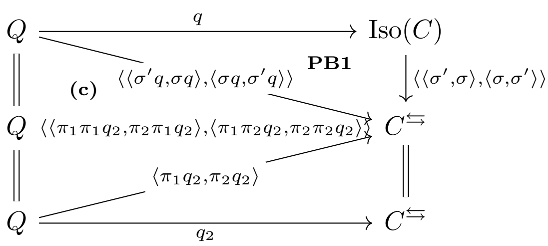
which induces the morphism

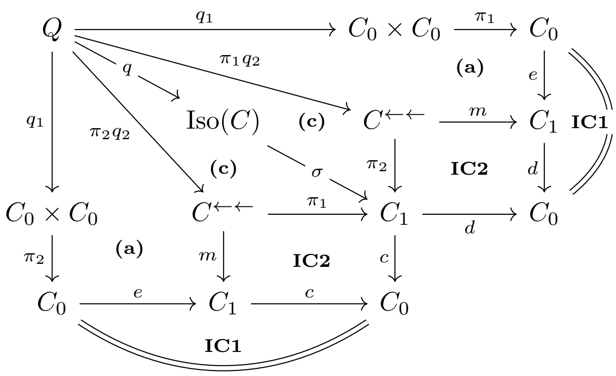
$$q = \langle\langle \pi_1 q_1, \pi_1 q_2\rangle, \langle \pi_2 q_1, \pi_2 q_2\rangle\rangle : Q \to \mathrm{Iso}(C) \tag{b}$$

by the universal property of $\mathrm{Iso}(C)$. We now show that q is the universal arrow making the desired square a pullback by observing the commutativity of the following diagrams:

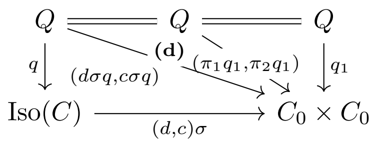


(c)

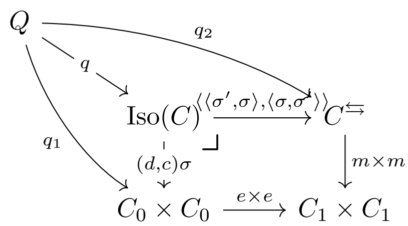




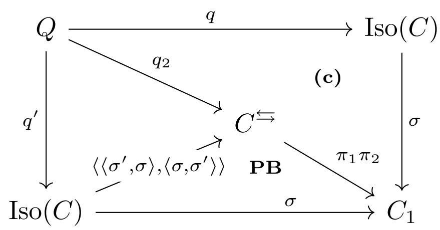
(d)



We therefore have that q is a universal arrow for the diagram

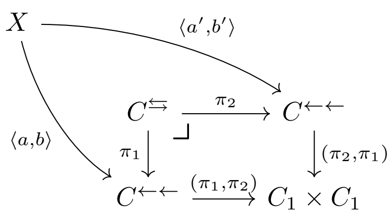


To show that q is unique, assume that there is a morphism $q' : Q \to \mathrm{Iso}(C)$ such that $(d,c)\sigma q' = q_1$ and $\langle\langle\sigma',\sigma\rangle,\langle\sigma,\sigma'\rangle\rangle q' = q_2$. Then the following diagram commutes:

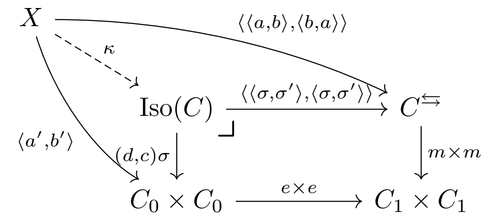


σ is a monomorphism, so $q = q'$. □

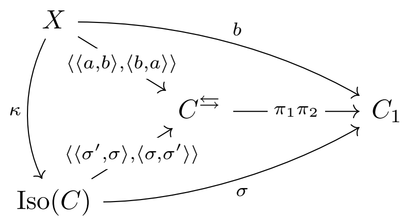
Remark 2.3.13. We note properties of the pullbacks in 2.3.12 for ease of later calculation. Firstly, for some two morphisms $\langle a,b\rangle$, $\langle a',b'\rangle : X \to C^{\leftarrow\leftarrow}$ the diagram



commutes if and only if $a = b'$ and $b = a'$, in which case the universal property of $C^{\leftrightarrows}$ induces a morphism $\langle\langle a, b\rangle, \langle b, a\rangle\rangle : X \to C^{\leftrightarrows}$. Considering 1.2.14, we observe that any morphism $X \to C^{\leftrightarrows}$ is of the form $\langle\langle a, b\rangle, \langle b, a\rangle\rangle$. Next, consider two morphisms $\langle a', b'\rangle : X \to C_0 \times C_0$ and $\langle\langle a, b\rangle, \langle b, a\rangle\rangle : X \to C^{\leftrightarrows}$. Then the outside of the diagram



commutes if and only if $m\langle a, b\rangle = ea'$ and $m\langle b, a\rangle = eb'$, in which case κ is induced by the universal property of the pullback. Then the commutativity of



means that $\sigma\kappa = b$. In the case that b is a monomorphism - and thus a subobject of morphisms of C - we have have that $b \leq \sigma$ and κ is a monomorphism.

Now that we have a developed notion of the subobject of isomorphisms, we make the following definition, which we will use in defining an internal factorisation system.

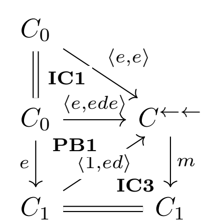
Definition 2.3.14. Let $\alpha : A \to C_1$ be a subobject of morphisms of an internal category C in a category $\mathbb{C}$ with pullbacks. If α contains σ, then we say that α *contains all isomorphisms* of C.

Remark 2.3.15. Note that as σ and σ' are equivalent subobjects of C_1, a subobject of morphisms $\alpha : A \to C_1$ contains all isomorphisms of C if and only if it contains σ'.

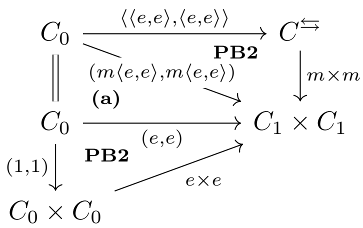
Of course, every identity morphism is an isomorphism. We show this internally.

Proposition 2.3.16. *Let C be an internal category in a category $\mathbb{C}$ with pullbacks. Then σ contains all identities of C.*

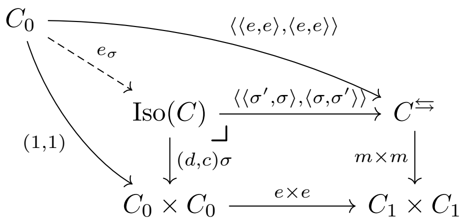
Proof. Firstly note that by **IC1**, the universal property of $C^{\leftarrow\leftarrow}$ induces the morphism $\langle e, e\rangle : C_0 \to C^{\leftarrow\leftarrow}$. Then, the existence of $\langle\langle e, e\rangle, \langle e, e\rangle\rangle : C_0 \to C^{\leftrightarrows}$ follows from remark 2.3.13. Next, observe that the following diagrams commutes:



(a)



This then induces the universal morphism $e_\sigma : C_0 \to \text{Iso}(C)$ making the following diagram commute:



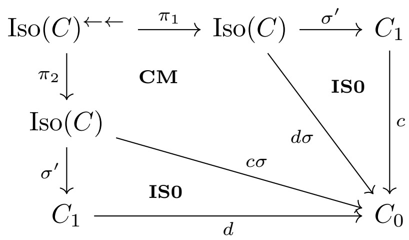
Lastly, we have that $e \leq \sigma$ from remark 2.3.13. $\square$

Corollary 2.3.17. *Let $\alpha : A \to C_1$ be a subobject of morphisms of an internal category C in a finitely complete category $\mathbb{C}$. If α contains all isomorphisms of C then it contains all identities of C. In particular, $e_\alpha = \sigma_\alpha e_\sigma$.*

The composition of isomorphisms is again an isomorphism. Thus, the class of isomorphisms of a category is closed under composition.

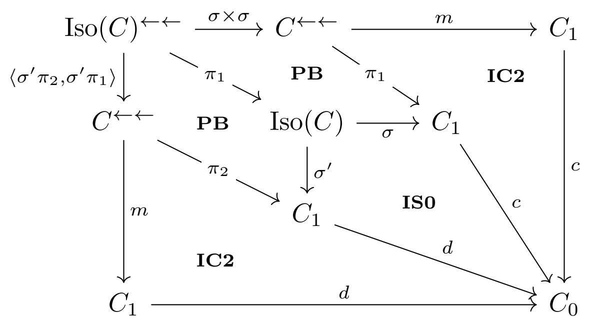
Proposition 2.3.18. *Let C be an internal category in a finitely complete category $\mathbb{C}$. The subobjects of isomorphisms of C are closed under composition.*

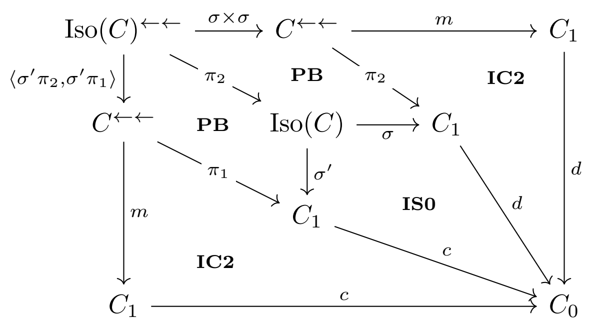
Proof. Given that σ and σ' are equivalent, we need only show that σ is closed under composition. Firstly, note the existence of $\sigma \times \sigma : \text{Iso}(C)^{\leftarrow\leftarrow} \to C^{\leftarrow\leftarrow}$. The commutativity of the diagram



induces the morphism $\langle \sigma'\pi_2, \sigma'\pi_1\rangle : \text{Iso}(C)^{\leftarrow\leftarrow} \to C^{\leftarrow\leftarrow}$ by the universal property of $C^{\leftarrow\leftarrow}$.

Then the commutativity of the diagrams





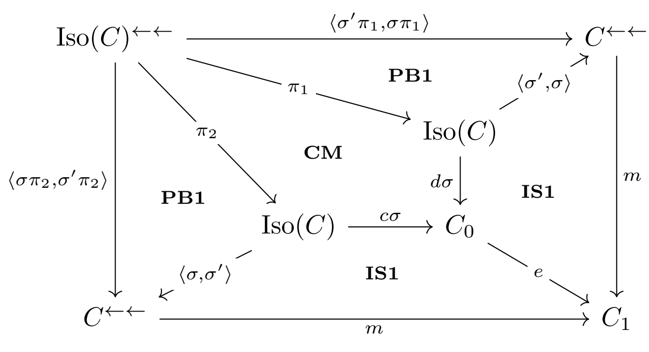
respectively induce the two morphisms

$$\langle m\langle \sigma'\pi_2, \sigma'\pi_1\rangle, m(\sigma \times \sigma)\rangle : \mathrm{Iso}(C)^{\leftarrow\leftarrow} \to C^{\leftarrow\leftarrow}$$
$$\langle m(\sigma \times \sigma), m\langle \sigma'\pi_2, \sigma'\pi_1\rangle\rangle : \mathrm{Iso}(C)^{\leftarrow\leftarrow} \to C^{\leftarrow\leftarrow}$$

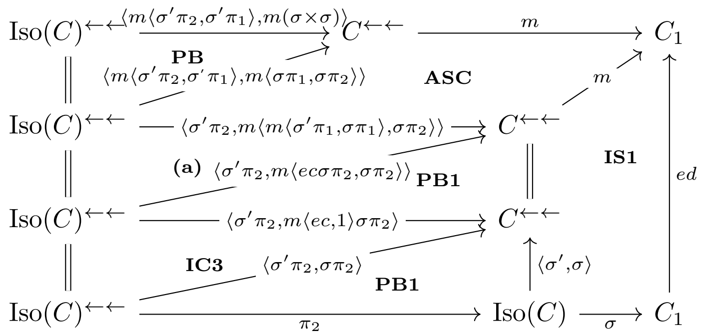
by the universal property of $C^{\leftarrow\leftarrow}$. By remark 2.3.13, we have the morphism

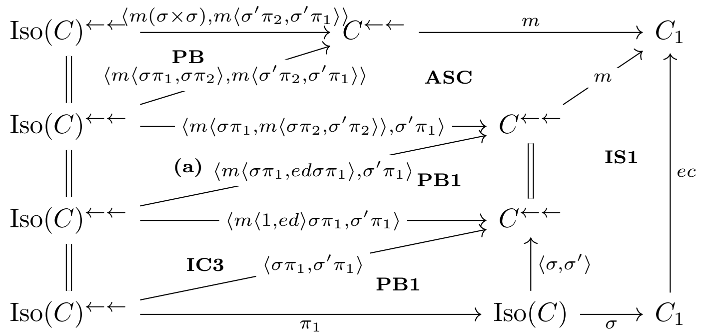
$$\langle\langle m\langle \sigma'\pi_2, \sigma'\pi_1\rangle, m(\sigma \times \sigma)\rangle, \langle m(\sigma \times \sigma), m\langle \sigma'\pi_2, \sigma'\pi_1\rangle\rangle\rangle : \mathrm{Iso}(C)^{\leftarrow\leftarrow} \to C^{\leftrightarrows}$$

Now, we observe the commutative diagrams:

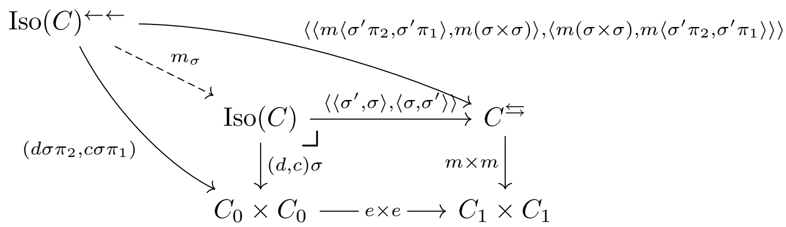


(a)





It then follows from remark 2.3.13 that we obtain a universal morphism $m_\sigma : \mathrm{Iso}(C)^{\leftarrow\leftarrow} \to \mathrm{Iso}(C)$ making the diagram



commute. In particular, we have that $\sigma m_\sigma = m(\sigma\times\sigma)$, so σ is closed under composition. $\square$

3 Internal Factorisation Systems

We begin by recalling the definition of a factorisation system and considering various properties of the structure. We then internalise two conditions of a factorisation system for which we have not yet done so. Finally, we introduce the internal factorisation system and provide a general example.

3.1 Factorisation Systems

To develop the notion of a factorisation system, one must start with the concept of orthogonality of morphisms.

Definition 3.1.1. Let $\mathbb{C}$ be a category, and let $f : X \to Y$ and $g : X' \to Y'$ be two morphisms in $\mathbb{C}$. f is *orthogonal* to g, written $f \downarrow g$, if for all morphisms $u : X \to X'$ and $v : Y \to Y'$ in $\mathbb{C}$ with $vf = gu$, there exists a unique morphism $z : Y \to X'$ such that $u = zf$ and $v = gz$, as in the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & \swarrow{\scriptstyle z} & \downarrow{\scriptstyle v} \\ X' & \xrightarrow[g]{} & Y' \end{array}$$

Remark 3.1.2. Observe that, on the other hand, for morphisms f and g as in the above definition, for any morphism $z' : Y \to X'$, we obtain the following commutative diagram by composition

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle z'f}\downarrow & \swarrow{\scriptstyle z'} & \downarrow{\scriptstyle gz'} \\ X' & \xrightarrow[g]{} & Y' \end{array}$$

In particular, it is easily seen that $z'f$ and gz' are the unique morphisms $X \to X'$ and $Y \to Y'$, respectively, making the above diagram commute. It follows that to say $f \downarrow g$ is precisely to say that there is a correspondence between diagrams in $\mathbb{C}$ of the two forms

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \swarrow & \\ X' & \xrightarrow[g]{} & Y' \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow[g]{} & Y' \end{array}$$

This is key in how orthogonality is defined in an internal setting, and may in fact be phrased as a pullback of Hom-sets of $\mathbb{C}$ in the category **Set**, as observed by Kelly (see e.g. diagram (6.3) of [8]).

Proposition 3.1.3. *Let $f : X \to X'$ and $g : Y \to Y'$ be two morphisms in a category $\mathbb{C}$. Then $f \downarrow g$ if and only if the following diagram is a pullback in the category* **Set***:*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{C}}(Y, X') & \xrightarrow{\mathrm{Hom}_{\mathbb{C}}(f, X')} & \mathrm{Hom}_{\mathbb{C}}(X, X') \\ {\scriptstyle \mathrm{Hom}_{\mathbb{C}}(Y,g)}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathrm{Hom}_{\mathbb{C}}(X,g)} \\ \mathrm{Hom}_{\mathbb{C}}(Y, Y') & \xrightarrow[\mathrm{Hom}_{\mathbb{C}}(f, Y')]{} & \mathrm{Hom}_{\mathbb{C}}(X, Y') \end{array}$$

We make use of the expected definition of classes of morphisms being orthogonal.

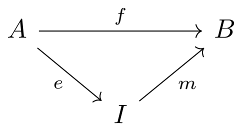
Definition 3.1.4. Let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms of a category $\mathbb{C}$. Then $\mathcal{E}$ is *orthogonal* to $\mathcal{M}$, written $\mathcal{E} \downarrow \mathcal{M}$, if for all morphisms e in $\mathcal{E}$ and m in $\mathcal{M}$, we have that $e \downarrow m$.

We briefly consider some examples to familiarise ourselves with this notion.

Example 3.1.5. The definition of orthogonality 3.1.1 may remind one of the definition of a strong epimorphism. Indeed, it is true that if $\mathrm{StrEpi}(\mathbb{C})$ is the class of strong epimorphisms and $\mathrm{Mono}(\mathbb{C})$ is the class of monomorphisms of some category $\mathbb{C}$, then $\mathrm{StrEpi}(\mathbb{C}) \downarrow \mathrm{Mono}(\mathbb{C})$. In the case that $\mathbb{C}$ is a regular category, regular epimorphism coincide with strong epimorphisms, so if $\mathrm{RegEpi}(\mathbb{C})$ is the class of regular epimorphisms of $\mathbb{C}$, then $\mathrm{RegEpi}(\mathbb{C}) \downarrow \mathrm{Mono}(\mathbb{C})$. We will revisit this latter example shortly.

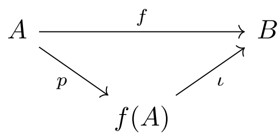
The next consideration in defining a factorisation system is that of a category having a particular type of factorisation.

Definition 3.1.6. Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms of a category $\mathbb{C}$. Then $\mathbb{C}$ *has $(\mathcal{E}, \mathcal{M})$-factorisations* if for all morphisms $f : A \to B$ in $\mathbb{C}$, there exists morphisms $e : A \to I$ in $\mathcal{E}$ and $m : I \to B$ in $\mathcal{M}$ such that $f = me$, as in the following commutative triangle



We call the composite me or the pair (m, e) a *factorisation* of f.

Example 3.1.7. The classical example of such a factorisation occurs in the category **Set**. For every morphism, that is, map, $f : A \to B$ in **Set**, there is a *canonical factorisation* of f



where $f(A)$ is the image of A under f, p is the canonical surjective map into this image, and ι is the inclusion of the image into the codomain B. In **Set**, surjective maps are precisely epimorphisms and inclusions are monomorphism, so **Set** has $(\mathrm{Epi}, \mathrm{Mono})$-factorisations, where Epi and Mono are respectively the class of epimorphism and monomorphisms in **Set**. It is worth noting that this archetypal example inspires the notation used for a factorisation system.

Example 3.1.8. A regular category $\mathbb{C}$ may be defined as a finitely complete category with *pullback stable* $(\mathrm{RegEpi}(\mathbb{C}), \mathrm{Mono}(\mathbb{C}))$-factorisations. The fact that the constituent classes of morphisms are also orthogonal, correctly suggests that they form a factorisation system on $\mathbb{C}$.

Definition 3.1.9. Let $\mathbb{C}$ be a category and let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms of $\mathbb{C}$. Then the pair $(\mathcal{E}, \mathcal{M})$ forms a *factorisation system* on $\mathbb{C}$ if the following four conditions are met:

FS1. $\mathcal{E}$ and $\mathcal{M}$ contain all the isomorphisms of $\mathbb{C}$.

FS2. $\mathcal{E}$ and $\mathcal{M}$ are closed under composition.

FS3. $\mathcal{E} \downarrow \mathcal{M}$.

FS4. $\mathbb{C}$ has $(\mathcal{E}, \mathcal{M})$-factorisations.

Example 3.1.10. It is easily show that RegEpi($\mathbb{C}$) and Mono($\mathbb{C}$) both contain all isomorphisms and are closed under composition for some regular category $\mathbb{C}$. This confirms that $(\text{RegEpi}(\mathbb{C}), \text{Mono}(\mathbb{C}))$ forms a factorisation system on $\mathbb{C}$.

Example 3.1.11. For any category $\mathbb{C}$, the pair (Iso($\mathbb{C}$), $\mathbb{C}_1$) of classes of isomorphisms of $\mathbb{C}$ and all morphisms of $\mathbb{C}$, respectively, forms the *trivial factorisation system* on $\mathbb{C}$. The generality of this example will allow us to provide an analogous notion on an internal category later.

We now consider various well known properties of factorisation systems.

Proposition 3.1.12. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on a category $\mathbb{C}$. Then,*

1. *e in $\mathcal{E}$ if and only if for all m in $\mathcal{M}$, $e \downarrow m$.*
2. *m in $\mathcal{M}$ if and only if for all e in $\mathcal{E}$, $e \downarrow m$.*

What this means on classes may be phrased as follows.

Corollary 3.1.13. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on a category $\mathbb{C}$. Then, for classes of morphisms $\mathcal{E}'$ and $\mathcal{M}'$ of $\mathbb{C}$,*

1. *$\mathcal{E} \downarrow \mathcal{M}'$ if and only if $\mathcal{M}' \subseteq \mathcal{M}$.*
2. *$\mathcal{E}' \downarrow \mathcal{M}$ if and only if $\mathcal{E}' \subseteq \mathcal{E}$.*

In particular, this may be understood as the components of a factorisation system $\mathcal{E}$ and $\mathcal{M}$ determining each other, in the following sense.

Proposition 3.1.14. *Let $\mathbb{C}$ be a category, and let $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{E}', \mathcal{M}')$ be two factorisation systems on $\mathbb{C}$. Then, $\mathcal{E} = \mathcal{E}'$ if and only if $\mathcal{M} = \mathcal{M}'$.*

By the definition of a factorisation system, $\mathcal{E}$ and $\mathcal{M}$ both contain all isomorphisms of the category. It is straight forward to show that this is precisely the subclass of morphism for which they agree.

Proposition 3.1.15. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on a category $\mathbb{C}$. Then $\mathcal{E} \cap \mathcal{M} =$ Iso($\mathbb{C}$), where Iso($\mathbb{C}$) is the class of all isomorphisms of $\mathbb{C}$.*

We now define what it means for a class of morphisms of a category to have the right- or left cancellation property.

Definition 3.1.16. Let $\mathcal{A}$ be a class of morphisms of some category $\mathbb{C}$. Then $\mathcal{A}$ satisfies the *right cancellation property* if for all pairs of composable morphisms of $\mathbb{C}$,

$$A \xrightarrow{f} B \xrightarrow{g} C$$

such that f is in $\mathcal{A}$ and their composition gf is in $\mathcal{A}$, we have that g is in $\mathcal{A}$.

Definition 3.1.17. Let $\mathcal{A}$ be a class of morphisms of some category $\mathbb{C}$. Then $\mathcal{A}$ satisfies the *left cancellation property* if for all pairs of composable morphisms of $\mathbb{C}$,

$$A \xrightarrow{f} B \xrightarrow{g} C$$

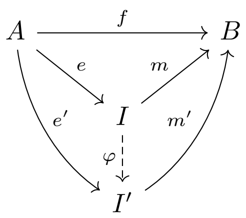
such that g is in $\mathcal{A}$ and their composition gf is in $\mathcal{A}$, we have that f is in $\mathcal{A}$.

These properties are respectively satisfied by each of $\mathcal{E}$ and $\mathcal{M}$ of a factorisation system.

Proposition 3.1.18. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on a category $\mathbb{C}$. Then, $\mathcal{E}$ has the right cancellation property and $\mathcal{M}$ has the left cancellation property.*

We now move on to a central property of a factorisation system, that their factorisations are essentially unique. Note that in the next definition, we do not necessarily require the pair $(\mathcal{E}, \mathcal{M})$ to be a factorisation system.

Definition 3.1.19. Let $\mathbb{C}$ be a category with $(\mathcal{E}, \mathcal{M})$-factorisations. Then these factorisations are *unique up to isomorphism* if for all morphisms $f : A \to B$ in $\mathbb{C}$, and every two factorisations $A \xrightarrow{e} I \xrightarrow{m} B$ and $A \xrightarrow{e'} I' \xrightarrow{m'} B$ of f, there exists a unique isomorphism $\varphi : I \to I'$ making the following diagram commute



Proposition 3.1.20. *Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on a category $\mathbb{C}$. Then $(\mathcal{E}, \mathcal{M})$-factorisations are unique up to isomorphism.*

This proposition directly follows from the fact that in definition 3.1.19, under the assumption that $(\mathcal{E}, \mathcal{M})$ is a factorisation system, $e \downarrow m'$ and $e' \downarrow m$. However, the strength of this property is exhibited by the fact that under the assumptions of FS1, FS2 and FS4, it may replace the orthogonality condition.

Proposition 3.1.21. *Let $\mathcal{E}$ and $\mathcal{M}$ be two classes of morphisms of a category $\mathbb{C}$ such that the pair $(\mathcal{E}, \mathcal{M})$ satisfies* FS1*,* FS2 *and* FS4*. Then the following are equivalent:*

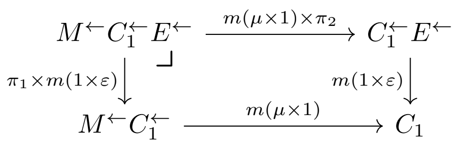
1. *$(\mathcal{E}, \mathcal{M})$ forms a factorisation system on $\mathbb{C}$.*
2. *$(\mathcal{E}, \mathcal{M})$-factorisations are unique up to isomorphism.*

3.2 Internal Orthogonality and Factorisation

In considering the definition of a factorisation system 3.1.9, we already have internal notions of FS1 and FS2. We now aim to internalise orthogonality and $(\mathcal{E}, \mathcal{M})$-factorisation.

As we are working without elements, we utilise definition 3.1.4 on the orthogonality of classes of morphisms. We make particular use of remark 3.1.2 and the correspondence mentioned therein. We first give the definition of internal orthogonality, then provide motivation.

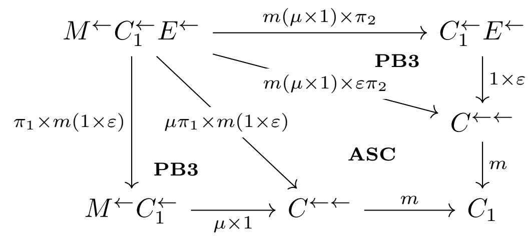
Definition 3.2.1. Let C be an internal category in a category $\mathbb{C}$ with pullbacks. Let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. Then ε is *orthogonal* to μ, written $\varepsilon \downarrow \mu$ if the following diagram is a pullback



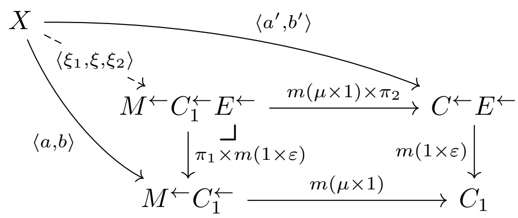
(OTH)

Before we motivate this definition, we consider the nature of the pullback above.

Remark 3.2.2. Firstly, observe that the square always commutes by the following diagram

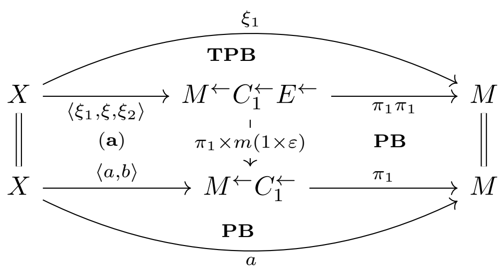


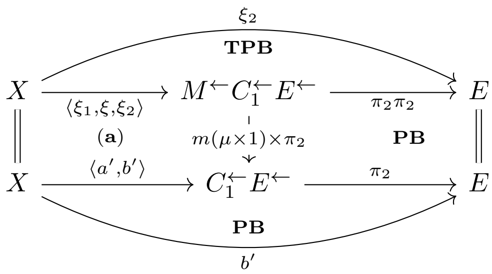
Next, assume that $\varepsilon \downarrow \mu$, and consider morphisms $\langle a, b\rangle : X \to M^{\leftarrow}C_1^{\leftarrow}$ and $\langle a', b'\rangle : X \to C_1^{\leftarrow}E^{\leftarrow}$ making the outside of the diagram below commute. This induces a universal arrow $\langle \xi_1, \xi, \xi_2\rangle : X \to M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow}$ making the whole diagram commute



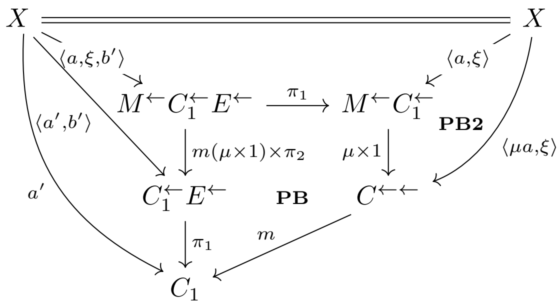
(a)

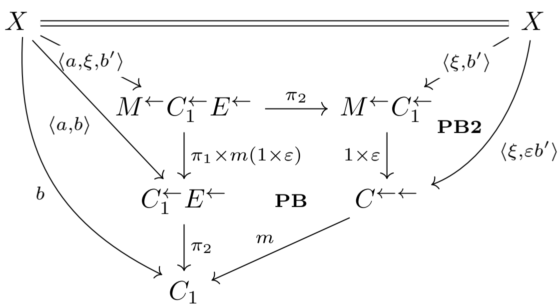
Then, we have that the following two diagrams commute:



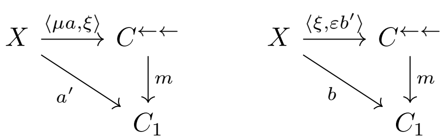


This means that the induced universal arrow is of the form $\langle a, \xi, b' \rangle$. Then, observe that the following two diagrams commute:





Therefore, the universal property of **OTH** induces a unique morphism $\xi : X \to C_1$ which makes the following diagram commute:



(OT1)

On the other hand, any morphism $\xi : X \to C_1$ satisfy these triangles will produce a morphism $\langle a, \xi, b' \rangle : X \to M^{\leftarrow} C_1^{\leftarrow} E^{\leftarrow}$ which satisfies the universal property of **OTH**. We may therefore conclude that the universal property of this pullback is that $\langle a, b \rangle$ and $\langle a', b' \rangle$ induce the unique morphism $\xi : X \to C_1$ satisfying **OT1**.

Additionally, for use in later calculation, note that by remark 1.2.14, a morphism ξ satisfying the above diagrams, will also satisfy the following diagrams:

$$\begin{array}{ccc} X & \xrightarrow{\xi} & C_1 \\ {\scriptstyle \mu a}\downarrow & & \downarrow{\scriptstyle c} \\ C_1 & \xrightarrow{d} & C_0 \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{\varepsilon b'} & C_1 \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle c} \\ C_1 & \xrightarrow[d]{} & C_0 \end{array} \tag{OT2}$$

With the details of the pullback **OTH** established, we now carefully consider the next proposition to motivate the internal definition of orthogonality.

Proposition 3.2.3. *Let C be an internal category in the category* **Set***. Let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. Then $\varepsilon \downarrow \mu$, as in the above definition, if and only if $E \downarrow M$ as subsets of morphisms of C, viewed as a small category, as in definition 3.1.4.*

Proof. Let us firstly recall that C will be a small category and E and M will be subsets of morphisms of C, with respective inclusions ε and μ. Then $M^{\leftarrow}C_1^{\leftarrow}$ is the set of pairs of composable morphisms of C, $X \xrightarrow{u} Y \xrightarrow{f} Z$ with f in M and the morphism $m(\mu \times 1)$ maps such a pair to their composition $X \xrightarrow{fu} Z$. Similarly, $C_1^{\leftarrow}E^{\leftarrow}$ is the set of pairs of composable morphisms $X \xrightarrow{g} Y \xrightarrow{v} Z$ with g in E and the morphism $m(\varepsilon \times 1)$ maps such a pair to their composition $X \xrightarrow{vg} Z$.

Now, let us assume that $E \downarrow M$ as subsets of morphisms of the small category C. Let $\langle a, b\rangle : K \to M^{\leftarrow}C_1^{\leftarrow}$ and $\langle a', b'\rangle : K \to C_1^{\leftarrow}E^{\leftarrow}$ be two morphisms in **Set** which make the following diagram commute:

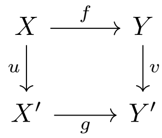
$$\begin{array}{ccc} K & \xrightarrow{\langle a',b'\rangle} & C^{\leftarrow}E^{\leftarrow} \\ & M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow} \xrightarrow{m(\mu\times 1)\times\pi_2} C^{\leftarrow}E^{\leftarrow} & \\ {\scriptstyle \langle a,b\rangle}\searrow & \downarrow{\scriptstyle \pi_1\times m(1\times\varepsilon)} \qquad\qquad {\scriptstyle m(1\times\varepsilon)}\downarrow & \\ & M^{\leftarrow}C_1^{\leftarrow} \xrightarrow{m(\mu\times 1)} C_1 & \end{array}$$

Then, for all $x \in K$, the commutativity of the outside of this diagram means that the following square commutes in C,

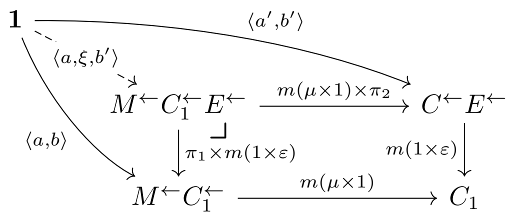
$$\begin{array}{ccc} X & \xrightarrow{b'(x)} & Y \\ {\scriptstyle b(x)}\downarrow & & \downarrow{\scriptstyle a'(x)} \\ X' & \xrightarrow[a(x)]{} & Y' \end{array}$$

where $b'(x)$ is a morphism in E and $a(x)$ is a morphism in M. Then, $E \downarrow M$ implies that $b'(x) \downarrow a(x)$ so there exists a unique morphism $z_x : Y \to X'$ such that $z_x b'(x) = b(x)$ and $a(x)z_x = a'(x)$ in C. We then define the function $\xi : K \to C_1$ in **Set** which maps each x to $\xi(x) = z_x$. It follows from the definition of each z_x that ξ satisfies the equations $\xi(x)b'(x) = b(x)$ and $a(x)\xi(x) = a'(x)$ in C. But this is precisely what it means for ξ to satisfy the commutative triangles of **OT1** in **Set**. The uniqueness of ξ is given by the uniqueness of each z_x so by 3.2.2, ξ is the desired universal morphism, which makes the square **OTH** a pullback. Therefore, $\varepsilon \downarrow \mu$.

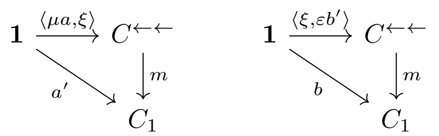
On the other hand, let us now assume that $\varepsilon \downarrow \mu$, so **OTH** is a pullback. Let $f : X \to Y$ in E and $g : X' \to Y'$ in M be two morphisms in C, and consider morphisms $u : X \to X'$ and $v : Y \to Y'$ in C such the following diagram commutes in C:



Noting the singleton set $\mathbf{1} = \{*\}$, this data may be considered as a pair of maps $\langle a, b\rangle : \mathbf{1} \to M^{\leftarrow}C_1^{\leftarrow}$ and $\langle a', b'\rangle : \mathbf{1} \to C_1^{\leftarrow}E^{\leftarrow}$ in **Set**, defined by $a(*) = g$, $b(*) = u$, $a'(*) = v$ and $b'(*) = f$. Then, the commutativity of the above square means that the outside of the following diagram in **Set** commutes, which induces a unique universal morphism $\xi : \mathbf{1} \to C_1$ making the whole diagram commute.



By remark 3.2.2, this implies that the following two triangles of **OT1** commute:



Following these diagrams for the element $*$ of $\mathbf{1}$, we get that $a(*)\xi(*) = a'(*)$ and $\xi(*)b'(*) = b(*)$, which is to say that $g\xi(*) = v$ and $\xi(*)f = u$ in C. Noting that $\xi(*)$ is unique as ξ is unique, we have that $f \downarrow g$ in C and therefore $E \downarrow M$. $\square$

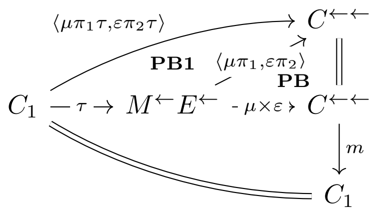
We now move on to the notion of an internal category having factorisations. It is relatively straight forward to internalise definition 3.1.6, with some careful considerations.

Consider a small category C as an internal category in **Set**, and let E and M be two subsets of morphisms of C, with inclusions $\varepsilon : E \to C_1$ and $\mu : M \to C_1$, respectively, into the set of all morphism of C. Then, to say that C has (E, M)-factorisations is to say that there exists a map $\tau : C_1 \to M^{\leftarrow}E^{\leftarrow}$ which assigns each morphism f of C to a pair of composable morphisms e in E and m in M such that me is a factorisation of f. This last statement means that $f = me$, which may be phrased in terms of τ as the condition $m(\mu \times \varepsilon)\tau = 1_{C_1}$, noting that the map $m(\mu \times \varepsilon) : M^{\leftarrow}E^{\leftarrow} \to C_1$ assigns the pair (m, e) to their composition me. We thus arrive at the following definition:

Definition 3.2.4. Let C be an internal category of a category $\mathbb{C}$ with pullbacks. Let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. Then C has (ε, μ)-*factorisations* if there exists a morphism $\tau : C_1 \to M^{\leftarrow}E^{\leftarrow}$ such that $m(\mu \times \varepsilon)\tau = 1_{C_1}$.

Remark 3.2.5. Firstly, note that the above definition is equivalent to requiring the morphism $m(\mu \times \varepsilon)$ to be a split epimorphism. Dropping this back into the context of $\mathbb{C} = \mathbf{Set}$, we observe that the Axiom of Choice says that every epimorphism in **Set** is a split epimorphism. As we model internal categories, and indeed internal factorisation systems, from this context, it is reasonable to consider defining a general internal category C as having (ε, μ)-factorisation if $m(\mu \times \varepsilon)$ is an epimorphism. In fact, one may ask this morphism to be any strength of epimorphism (regular, strong, extremal, etc.). However, as the strongest requirement we impose on $\mathbb{C}$ is that it is finitely complete, it will not in general have a notion of the Axiom of Choice. Furthermore, the explicit existence of τ allows for us to internally refer to factorisations of the morphisms of C. We thus conclude that definition 3.2.4 is suitable.

Remark 3.2.6. We will often use the following form of the fact that an internal category has (ε, μ)-factorisations in calculation:



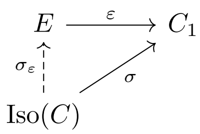
(F1)

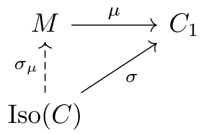
3.3 Internal Factorisation Systems

We are now ready to define an internal factorisation system on an internal category C.

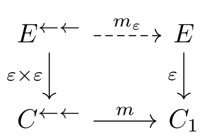
Definition 3.3.1. Let C be an internal category in a finitely complete category $\mathbb{C}$. Let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. Then the pair (ε, μ) forms an *internal factorisation system* on C if the following four conditions are met:

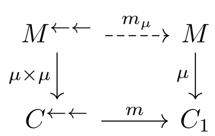
IFS1. ε and μ contain all isomorphisms of C: There exist morphisms σ_ε and σ_μ such the following triangles commute





IFS2. ε and μ are closed under composition: There exist morphism m_ε and m_μ such that the following squares commute:





IFS3. $\varepsilon \downarrow \mu$: The following square is a pullback:

$$\begin{array}{ccc} M^{\leftarrow}C_1^{\leftarrow}E^{\leftarrow} & \xrightarrow{m(\mu\times 1)\times\pi_2} & C^{\leftarrow}E^{\leftarrow} \\ {\scriptstyle \pi_1\times m(1\times\varepsilon)}\downarrow & \lrcorner & \downarrow{\scriptstyle m(1\times\varepsilon)} \\ M^{\leftarrow}C_1^{\leftarrow} & \xrightarrow{m(\mu\times 1)} & C_1 \end{array}$$

IFS4. C has (ε,μ)-factorisations: There exists a morphism τ such that $m(\mu\times\varepsilon)\tau = 1_{C_1}$

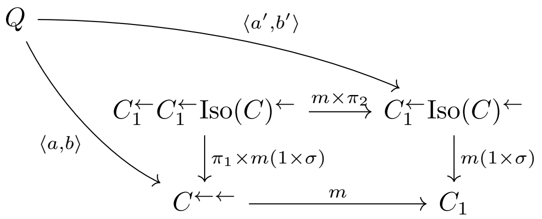
Remark 3.3.2. It is clear from our reasoning throughout, that an internal factorisation system on an internal category in **Set** is precisely a factorisation system on the corresponding small category.

We now proceed to consider an explicit example of an internal factorisation system. In particular, we internalise the example 3.1.11 that $(\text{Iso}(\mathbb{C}),\mathbb{C}_1)$ is a factorisation system on a general category $\mathbb{C}$.

Proposition 3.3.3. *Let C be an internal category of a finitely complete category $\mathbb{C}$. Then the pair $(\sigma, 1_{C_1})$ forms an internal factorisation system on C.*

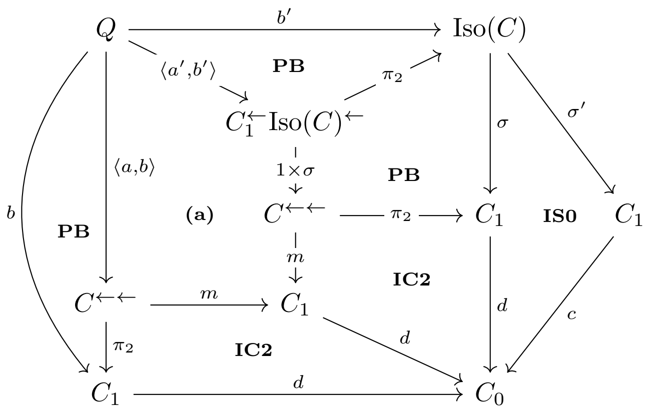
Proof. Firstly note that $\sigma : \text{Iso}(C) \to C_1$ and $1_{C_1} : C_1 \to C_1$ are the subobjects of isomorphism and all morphisms, respectively. We need to show that this pair of subobjects of morphisms satisfies the conditions IFS1 - IFS4. σ trivially contains all isomorphisms of C and 1_{C_1} contains all isomorphisms of C by 2.2.11. Next, σ is closed under composition by 2.3.18 and 1_{C_1} trivially is too.

We now show that $\sigma \downarrow 1_{C_1}$. Consider morphisms $\langle a,b\rangle : Q \to C^{\leftarrow\leftarrow}$ and $\langle a',b'\rangle : Q \to C_1^{\leftarrow}\text{Iso}(C)^{\leftarrow}$ such that the following diagram commutes:



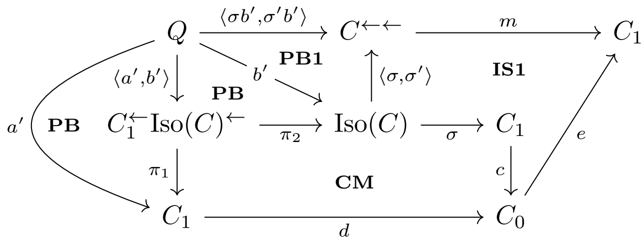
(a)

Observe that the square in the above diagram, which we aim to show is a pullback, commutes by remark 3.2.2. Next consider the commutative diagram

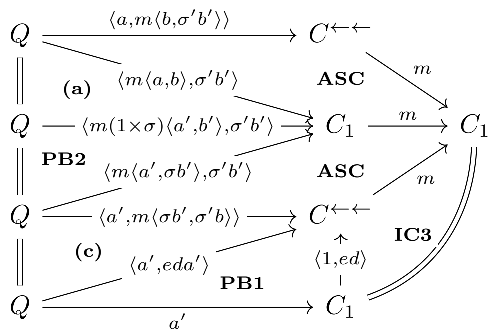


(b)

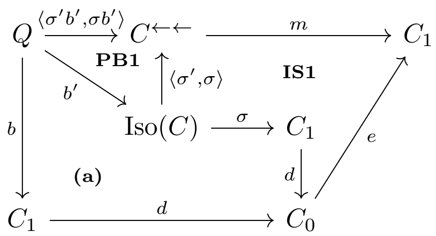
This induces the morphism $\langle b, \sigma' b' \rangle : Q \to C^{\leftarrow\leftarrow}$. We show, via remark 3.2.2, that the morphism $m\langle b, \sigma' b' \rangle : Q \to C_1$ satisfies the universal property of the square of (**a**). The next diagrams commute:



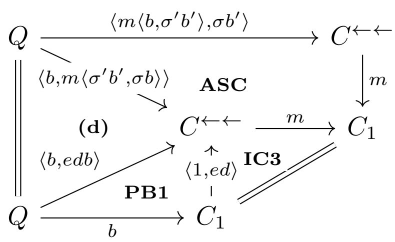
(c)



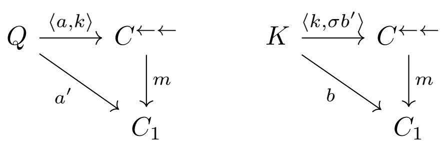
This last diagram means that $m\langle a, m\langle b, \sigma' b' \rangle\rangle = a'$, which is the first condition of **OT1**. We similarly have that the following two diagrams commute:



(d)

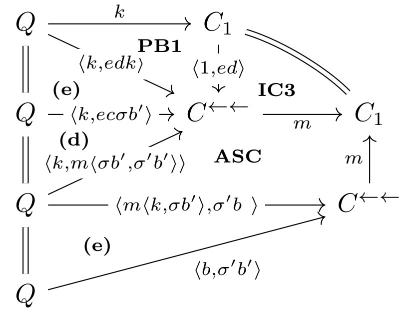


The last of these diagrams is the second condition of **OT1**. We finally need to show that $m\langle b, \sigma' b' \rangle$ is the unique morphism $Q \to C_1$ satisfying this condition. Let $k : Q \to C_1$ be a morphism such that the following diagrams commute:



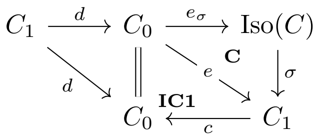
(e)

Note that the existence of $\langle k, \sigma b' \rangle$ implies that $dk = c\sigma b'$. It then follows that the following diagram commutes

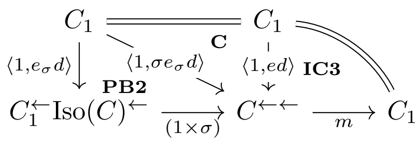


We may thus conclude that $k = m\langle b, \sigma' b' \rangle$, so $m\langle b, \sigma' b' \rangle$ is unique, and thus, by remark 3.2.2, the square of **(a)** is a pullback, so $\sigma \downarrow 1_{C_1}$.

We lastly need to show that C has $(\sigma, 1_{C_1})$-factorisations. That is, that the morphism $m(1 \times \sigma) : C_1^{\leftarrow}\mathrm{Iso}(C)^{\leftarrow} \to C_1$ is a split epimorphism. Observe then, that the following diagram commutes:



which implies the existence of the morphism $\langle 1, e_\sigma d \rangle : C_1 \to C_1^{\leftarrow}\mathrm{Iso}(C)^{\leftarrow}$. Finally, the following diagram commutes:



So by setting $\tau = \langle 1, e_\sigma d \rangle$, we have that $m(1 \times \sigma)$ is a split epimorphism, so C has $(\sigma, 1_{C_1})$-factorisations. $\square$

4 Properties of Internal Factorisation Systems

We may now proceed to consider properties of internal factorisation systems. These properties present as internal forms of the various properties of factorisation systems detailed in 3.1.

4.1 $\varepsilon \cap \mu = \mathrm{Iso}(C)$

Recall that in the category **Set**, if A and B are subsets of some set X, with respective inclusions α and β, then the intersection of A and B may be given by the following pullback in **Set**:

$$\begin{array}{ccc} A \cap B & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & X \end{array}$$

In general, for two subobjects $\alpha : A \to X$ and $\beta : B \to X$ in some category $\mathbb{C}$ with pullbacks, one then defines the intersection of these subobjects by the following similar pullback in $\mathbb{C}$:

$$\begin{array}{ccc} \alpha \cap \beta & \xrightarrow{\pi_2} & B \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\alpha} & X \end{array}$$

In particular, note that as pullbacks preserve monomorphisms, π_1 and π_2 will respectively be subobjects of A and B, as one would expect. With this fact and 3.1.15 in mind, we have the following proposition.

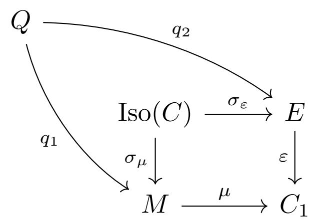
Proposition 4.1.1. *Let C be an internal category in a finitely complete category $\mathbb{C}$. Let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C which satisfy IFS1, IFS2 and IFS3. Then the following square is a pullback:*

$$\begin{array}{ccc} \mathrm{Iso}(C) & \xrightarrow{\sigma_\varepsilon} & E \\ {\scriptstyle \sigma_\mu}\downarrow & \lrcorner & \downarrow{\scriptstyle \varepsilon} \\ M & \xrightarrow{\mu} & C_1 \end{array} \qquad \text{(IEM)}$$

Proof. Firstly, note that by the following diagram, the above square commutes:

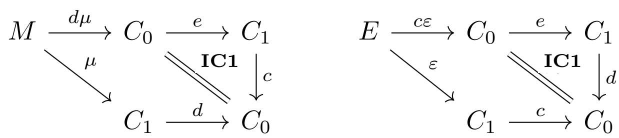
$$\begin{array}{ccc} \mathrm{Iso}(C) & \xrightarrow{\sigma_\varepsilon} & E \\ {\scriptstyle \sigma_\mu}\downarrow & \searrow^{\sigma} \;\; \textbf{IFS1} & \downarrow{\scriptstyle \varepsilon} \\ M & \xrightarrow[\mu]{\textbf{IFS1}} & C_1 \end{array}$$

Next, assume that there exists an object Q in $\mathbb{C}$ and two morphisms $q_1 : Q \to M$ and $q_2 : Q \to E$ such that the following diagram commutes:

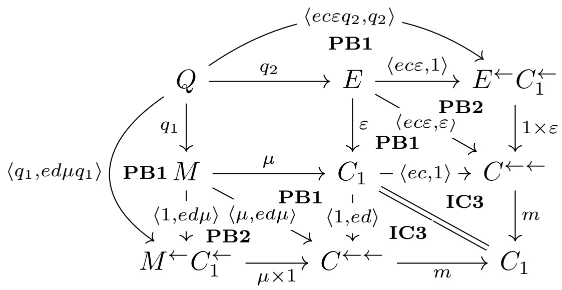


(a)

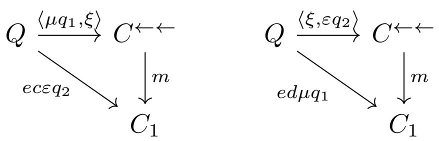
We have that the following diagrams commute:



which induces the morphisms $\langle 1, ed\mu\rangle : M \to M^{\leftarrow}C_1^{\leftarrow}$ and $\langle ec\varepsilon, 1\rangle : E \to C_1^{\leftarrow}E^{\leftarrow}$ by the universal properties of $M^{\leftarrow}C_1^{\leftarrow}$ and $C_1^{\leftarrow}E^{\leftarrow}$ respectively. Next, we observe that the following diagram commutes:



This induces, by the universal property of the pullback **OTH** and remark 3.2.2, a morphism $\xi : Q \to C_1$ which makes the following two diagrams commute:



Now, by assumption, we have that $\mu q_1 = \varepsilon q_2$, so by remark 2.3.13 we have the morphism $\langle\langle \xi, \varepsilon q_2\rangle, \langle \mu q_1, \xi\rangle\rangle : Q \to C^{\leftrightarrows}$. Then, again by remark 2.3.13 and the above commutative triangles we have a universal morphism $\kappa : Q \to \mathrm{Iso}(C)$ such that $\sigma\kappa = \mu q_1 = \varepsilon q_2$. By IFS1, we obtain the following equalities of morphisms:

$$\mu\sigma_\mu\kappa = \mu q_1 \quad \varepsilon\sigma_\varepsilon\kappa = \varepsilon q_2$$

Then, as μ and ε are monomorphisms, we obtain that $\sigma_\mu\kappa = q_1$ and $\sigma_\varepsilon\kappa = q_2$, so that $\kappa : Q \to \mathrm{Iso}(C)$ is a suitable universal arrow for (**a**). We lastly show that κ is the unique such morphism. Let $\kappa' : Q \to \mathrm{Iso}(C)$ be a morphism such that $\sigma_\mu\kappa' = q_1$ and $\sigma_\varepsilon\kappa' = q_2$. Then $\sigma_\mu\kappa' = \sigma_\mu\kappa$, and σ_μ is a monomorphism, so $\kappa = \kappa'$. $\square$

4.2 Factorisation Unique up to Isomorphism

As mentioned in 3.1.21, the strength of the fact that factorisations of a factorisation system are unique up to isomorphisms is expressed in that it may be interchanged with the orthogonality condition in the definition of a factorisation system. We now aim to internalise this property. One may observe that the internal definition of the essential uniqueness of factorisations is similar to that of orthogonality. We use the following motivation to explain this.

For a category $\mathbb{C}$ with an $(\mathcal{E}, \mathcal{M})$ factorisation system (for some classes of morphisms $\mathcal{E}$ and $\mathcal{M}$), for these factorisations to be unique up to isomorphisms is to say that for any four morphisms $e : A \to I$, $e' : A \to A'$ in $\mathcal{E}$ and $m : I \to B$ and $m' : I' \to B$ in $\mathcal{M}$ such that the following square commutes:

$$\begin{array}{ccc} A & \xrightarrow{e} & I \\ {\scriptstyle e'}\downarrow & \swarrow{\scriptstyle z} & \downarrow{\scriptstyle m} \\ I' & \xrightarrow[m']{} & B \end{array}$$

there exists a unique isomorphism $z : I \to I'$ making the whole diagram commute. Let us fix the morphisms e and m', and consider an isomorphism $z' : I \to I'$. Then, by composition, we obtain the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{e} & I \\ {\scriptstyle z'e}\downarrow & \swarrow{\scriptstyle z'} & \downarrow{\scriptstyle m'z'} \\ I' & \xrightarrow[m']{} & B \end{array}$$

Now, by the facts that, as an isomorphism, z' is in $\mathcal{E}$ and $\mathcal{M}$ and that both of these classes are closed under composition, we have that $z'e$ and $m'z'$ are in $\mathcal{E}$ and $\mathcal{M}$, respectively. Furthermore, $z'e$ and $m'z'$ are the unique morphisms $A \to I'$ and $I \to B$ (in these classes) making this diagram commute. We may therefore say that for $(\mathcal{E}, \mathcal{M})$-factorisations to be unique up to isomorphisms is, for a fixed e in $\mathcal{E}$ and m' in $\mathcal{M}$, for there to be a correspondence between the diagrams

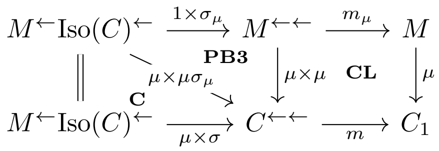
$$\begin{array}{ccc} A & \xrightarrow{e} & I \\ {\scriptstyle e'}\downarrow & & \downarrow{\scriptstyle m} \\ I' & \xrightarrow[m']{} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{e} & I \\ & \swarrow{\scriptstyle z'} & \\ I' & \xrightarrow[m']{} & B \end{array}$$

where e' is in $\mathcal{E}$, m is in $\mathcal{M}$ and z' is an isomorphism.

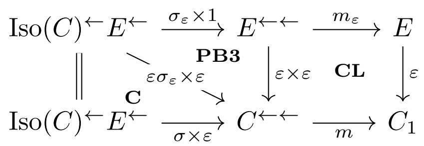
Definition 4.2.1. Let C be an internal category in a finitely complete category $\mathbb{C}$. Let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C which satisfy IFS1 and IFS2. Then (ε, μ)-*factorisations are unique up to isomorphism* if the following square is a pullback:

$$\begin{array}{ccc} M^{\leftarrow}\mathrm{Iso}(C)^{\leftarrow}E^{\leftarrow} & \xrightarrow{m_\mu(\sigma_\mu\times 1)\times\pi_2} & M^{\leftarrow}E^{\leftarrow} \\ {\scriptstyle \pi_1\times m_\varepsilon(\sigma_\varepsilon\times 1)}\downarrow \quad \lrcorner & & \downarrow{\scriptstyle m(\mu\times\varepsilon)} \\ M^{\leftarrow}E^{\leftarrow} & \xrightarrow[m(\mu\times\varepsilon)]{} & C_1 \end{array} \qquad \text{(FUI)}$$

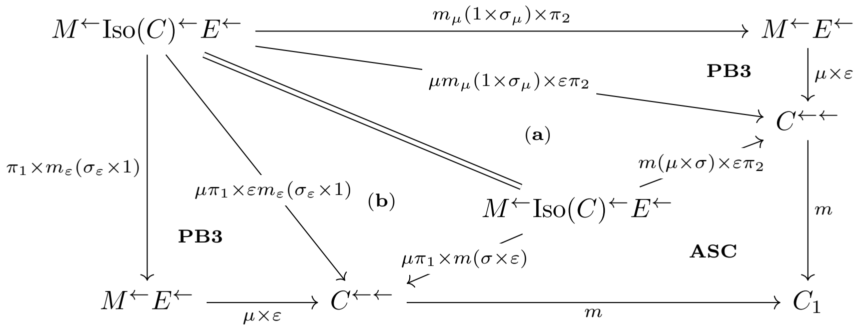
Remark 4.2.2. As was done for **OTH** in 3.2.2, we first consider the nature of this pullback. Firstly, note that in the setup of the above definition, the square will always commute:



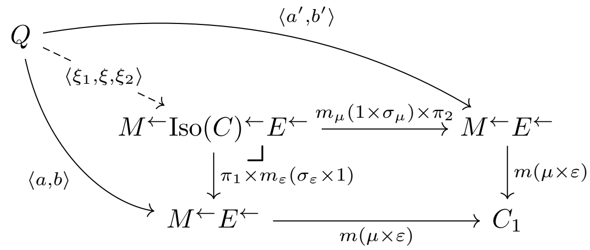
(a)



(b)

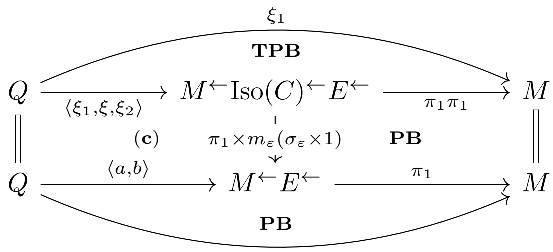


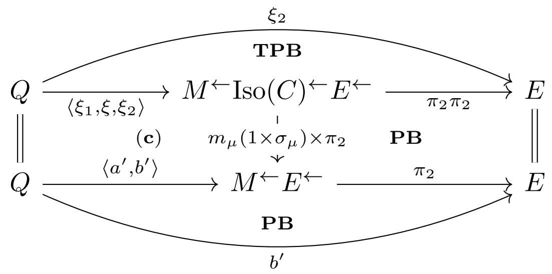
Now, assume that (ε,μ)-factorisations are unique up to isomorphism, and that there exists an object Q in $\mathbb{C}$ and two morphisms $\langle a,b\rangle : Q \to M^{\leftarrow}E^{\leftarrow}$ and $\langle a',b'\rangle : Q \to M^{\leftarrow}E^{\leftarrow\leftarrow}$ making the outside of the next diagram commute. This will induce the universal morphism $\langle \xi_1,\xi,\xi_2\rangle : Q \to M^{\leftarrow}\mathrm{Iso}(C)^{\leftarrow}E^{\leftarrow}$ making the whole diagram commute:



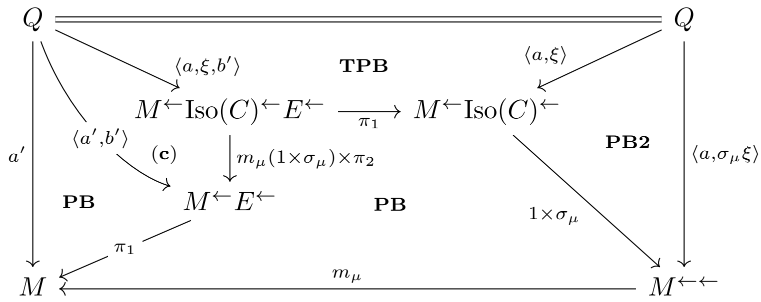
(c)

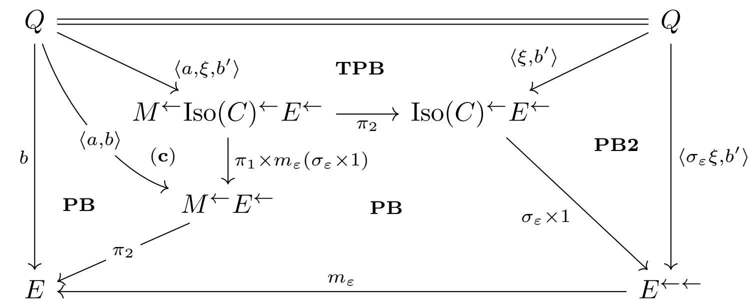
Then, the following diagrams commute:



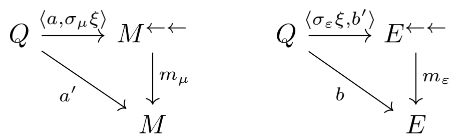


Thus, the universal morphism is of the form $\langle a, \xi, b' \rangle$. Then, we have that the following two diagrams commute:





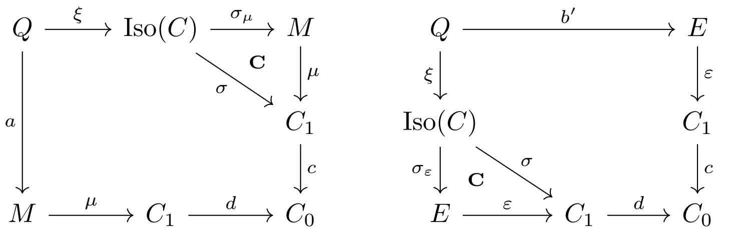
Therefore, given the morphisms $\langle a, b \rangle$ and $\langle a', b' \rangle$,the universal property of **FUI** induces the unique morphism $\xi : Q \to \mathrm{Iso}(C)$ such that the following diagrams commute:



(FU1)

On the other hand, given such a morphism ξ, we may obtain a unique universal morphism $\langle a, \xi, b' \rangle$ for the diagram (**c**). We therefore conclude that the universal property of **FUI** is that $\langle a, b \rangle$ and $\langle a', b' \rangle$ induce a unique morphism $\xi : Q \to \mathrm{Iso}(C)$ making the diagram **FU1** commute.

Note also that a ξ which satisfies the above diagrams will also satisfy the following two diagrams:



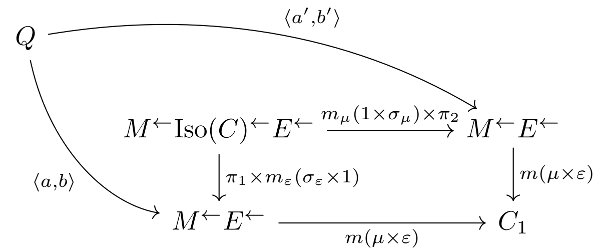
(FU2)

In definition 4.2.1, we do not require that ε and μ satisfy IFS3 or IFS4. This means, that in the context of $\mathbb{C} = \mathbf{Set}$ where C is a small category, we do not require all factorisations to exist (or, indeed, for E to be orthogonal to M). This definition thus means that all factorisations that exist are unique up to isomorphism. To see that the internal definition indeed corresponds to the usual definition, observe that given the correspondence detailed at the start of this section, its similarity the correspondence of 3.1.2 and the similarities of **OTH** and **FUI**, the proof will be similar to that of 3.2.3.

We now show that for an internal factorisation system, factorisations are unique up to isomorphism.

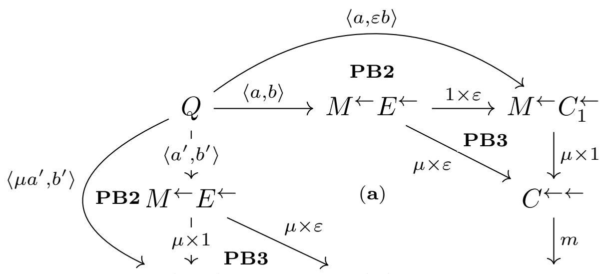
Proposition 4.2.3. *Let C be an internal category in a finitely complete category $\mathbb{C}$. Let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms which satisfy IFS1, IFS2 and IFS3. Then (ε, μ)-factorisations are unique up to isomorphism.*

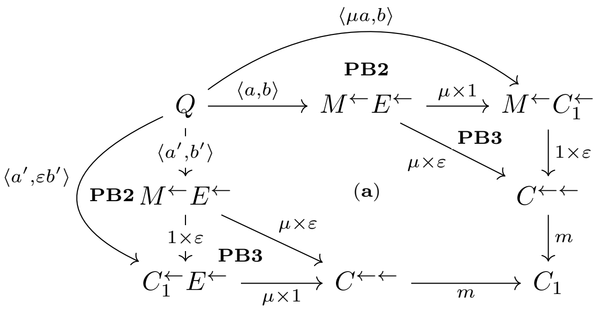
Proof. Let Q be an object and $\langle a, b\rangle : Q \to M^{\leftarrow}E^{\leftarrow}$ and $\langle a', b'\rangle : Q \to M^{\leftarrow}E^{\leftarrow}$ be two morphisms in $\mathbb{C}$ such that the following diagram commutes.



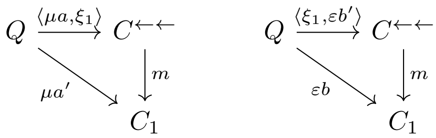
(a)

Then, the following diagrams commute:



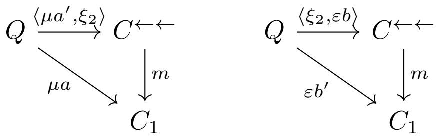


which, by remark 3.2.2 respectively mean that there exists a unique morphisms $\xi_1 : Q \to C_1$ making the following two diagrams commute



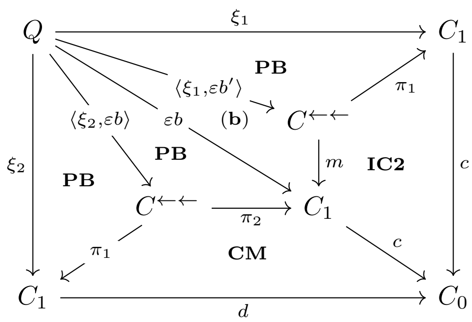
(b)

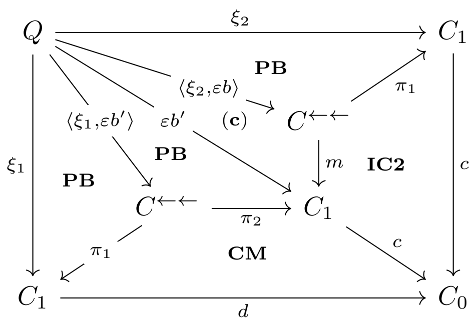
and that there exists a unique morphism $\xi_2 : Q \to C_1$ which makes the following diagrams commute



(c)

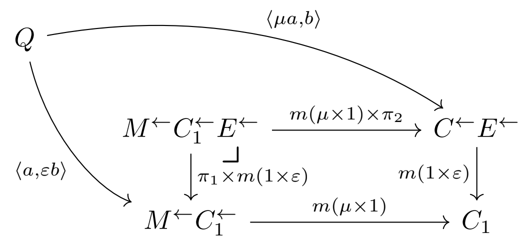
We may then observe that the next two diagrams commute:



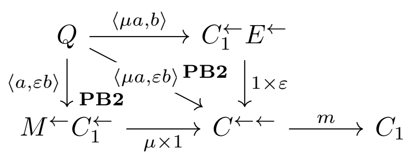


which each, by the universal property of $C^{\leftarrow\leftarrow}$, implies the unique existence of morphisms $\langle \xi_2, \xi_1 \rangle : Q \to C^{\leftarrow\leftarrow}$ and $\langle \xi_1, \xi_2 \rangle : Q \to C^{\leftarrow\leftarrow}$. Then, from remark 2.3.13, we have the morphism $\langle \langle \xi_2, \xi_1 \rangle, \langle \xi_1, \xi_2 \rangle \rangle : Q \to C^{\leftrightarrows}$.

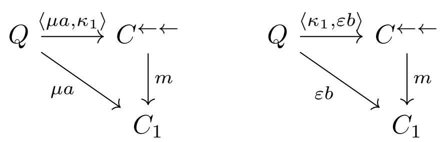
Now, consider the following diagram:



The outside easily commutes by the following diagram:

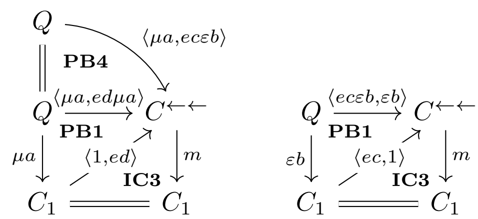


which, by remark 3.2.2 induces a *unique* morphism $\kappa_1 : Q \to C_1$ which makes the following two triangle commute:

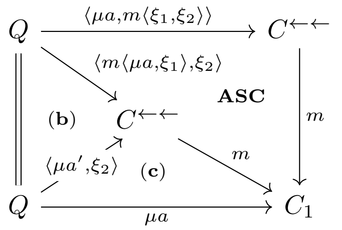


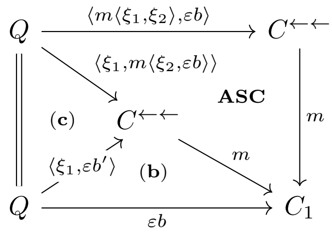
(d)

We prove the equality of the morphisms $ec\varepsilon b$ and $m\langle \xi_1, \xi_2 \rangle$ by showing that they both satisfy this above condition on κ_1. Firstly, recalling 1.2.14 with regard to the morphism $\langle a, b \rangle : Q \to M^{\leftarrow}E^{\leftarrow}$, we have that the following diagrams commute:

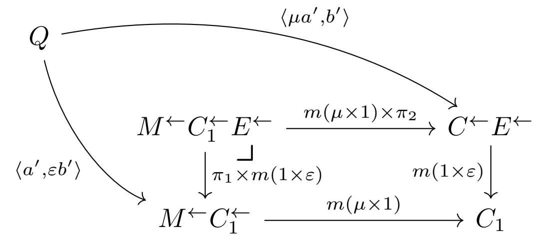


Therefore, by the uniqueness of κ_1, we have that $\kappa_1 = ec\varepsilon b$. Then, the next two diagrams commute:

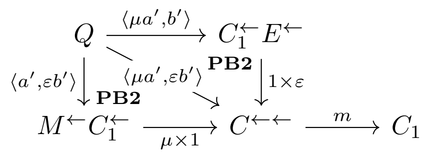




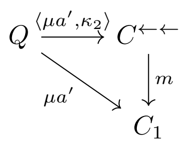
So, by the uniqueness of κ_1, $\kappa_1 = m\langle \xi_1, \xi_2 \rangle$. Thus, $ec\varepsilon b = m\langle \xi_1, \xi_2 \rangle$. We now similarly consider the followings diagram:

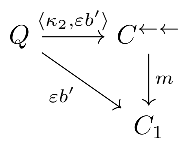


The outside of this diagram commutes by the following:



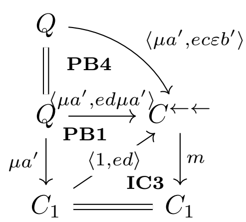
which, by remark 3.2.2 induces a *unique* $\kappa_2 : Q \to C_1$ which makes the following triangles commute:

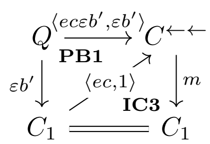




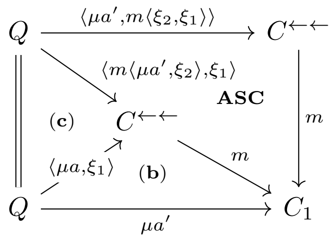
(e)

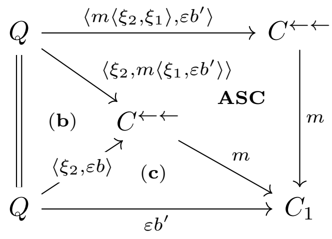
We show that $ec\varepsilon b'$ and $m\langle \xi_2, \xi_1 \rangle$ both satisfy this condition on κ_2, and are thus equal. Again, recall remark 1.2.14 on $\langle a', b' \rangle$ to observe that the following two diagrams commute:





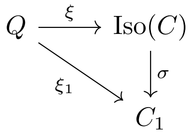
which implies that $\kappa_2 = ec\varepsilon b'$, by the uniqueness of κ_2. Then, the next two diagrams commute:





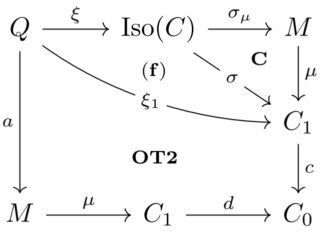
So, $\kappa_2 = m\langle\xi_2, \xi_1\rangle$, and thus $ec\varepsilon b' = m\langle\xi_2, \xi_1\rangle$.

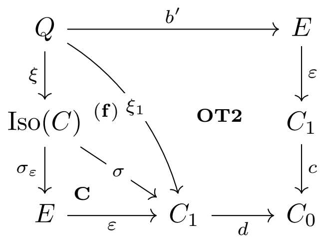
Next, putting together the existence of $\langle\langle\xi_2, \xi_1\rangle, \langle\xi_1, \xi_2\rangle\rangle$ and the facts that $ec\varepsilon b = m\langle\xi_1, \xi_2\rangle$ and $ec\varepsilon b' = m\langle\xi_2, \xi_1\rangle$, in the context of remark 2.3.13, we obtain a morphism $\xi : Q \to \mathrm{Iso}(C)$ which satisfies the following commutative diagram:



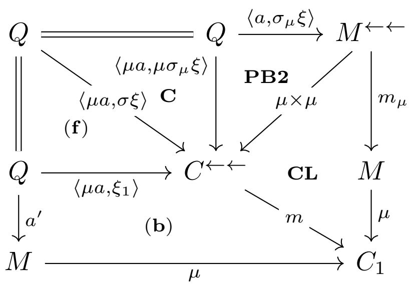
(f)

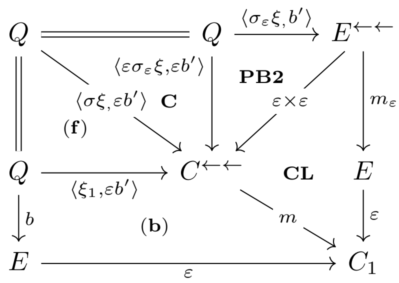
We then have that the following diagrams commute:



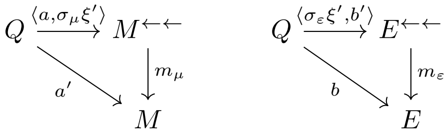


which, by the universal property of $C^{\leftarrow\leftarrow}$, respectively induce the morphisms $\langle a, \sigma_\mu\xi\rangle : Q \to C^{\leftarrow\leftarrow}$ and $\langle\sigma_\varepsilon\xi, b'\rangle : Q \to C^{\leftarrow\leftarrow}$. We then have that the commutativity of the following diagrams



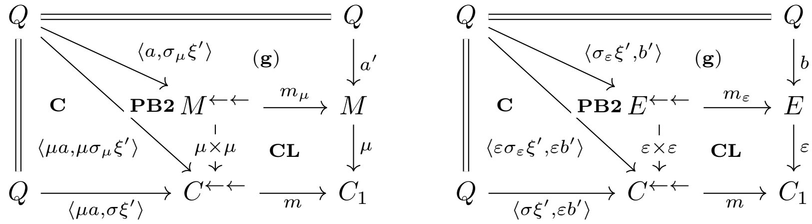


means that $\mu m(\mu\langle a, \sigma)\mu\xi\rangle = \mu a'$ and $\varepsilon m_\varepsilon\langle\sigma_\varepsilon\xi, b'\rangle = \varepsilon b$. μ and ε are monomorphisms, so we may conclude that $m(\mu\langle a, \sigma)\mu\xi\rangle = a'$ and $m_\varepsilon\langle\sigma_\varepsilon\xi, b'\rangle = b$ and so ξ satisfies the universal property of **FUI**, by remark 4.2.2. We finally show that ξ is the unique such morphism. Assume that there exists a morphism $\xi : Q \to \mathrm{Iso}(C)$ such that the following diagrams commute:



(g)

Then, the following diagrams commute:



which means that $\sigma\xi'$ satisfies the universal property of ξ_1 in (**b**). The uniqueness of ξ_1 means that $\xi_1 = \sigma\xi'$. But by (**f**), $\xi_1 = \sigma\xi$, so $\sigma\xi' = \sigma\xi$. Finally, we have that σ is a monomorphism, so $\xi' = \xi$, and thus ξ is unique.

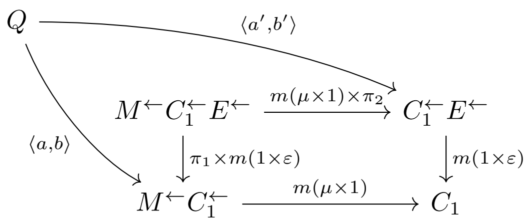
□

We now show, with reference to 3.1.21, that one may substitute the orthogonality in the definition of an internal factorisation system with the essential uniqueness of factorisations.

Proposition 4.2.4. *Let C be an internal category in a finitely complete category $\mathbb{C}$. Let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of C. TFAE:*

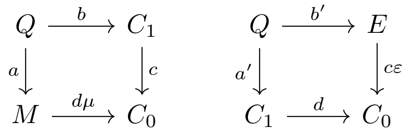
1. *(ε, μ) form an internal factorisation system on C.*
2. *(ε, μ) satisfy* IFS1*,* IFS2*,* IFS4 *and*
   IFS3**: (ε, μ)-factorisations are unique up to isomorphism.*

Proof. By 4.2.3 above, 1 implies 2. Now, assume that 2 holds. We must show that $\varepsilon \downarrow \mu$. Firstly, note that by 3.2.2, the square of **OTH** always commutes. Then, consider an object Q and two morphisms $\langle a, b\rangle : Q \to M^{\leftarrow}C_1^{\leftarrow}$ and $\langle a', b'\rangle : Q \to C_1^{\leftarrow}E^{\leftarrow}$ in $\mathbb{C}$ such that the following diagram commutes:



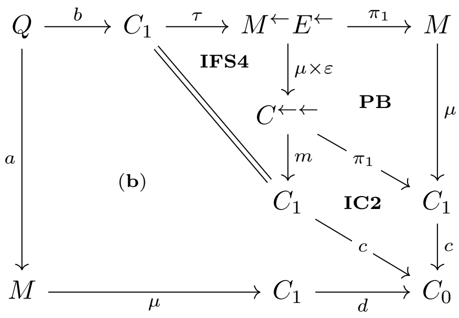
(a)

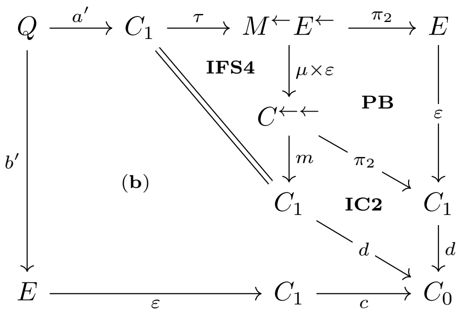
By remark 1.2.14, the existence of $\langle a, b\rangle$ and $\langle a', b'\rangle$ means that the following two diagrams commute:



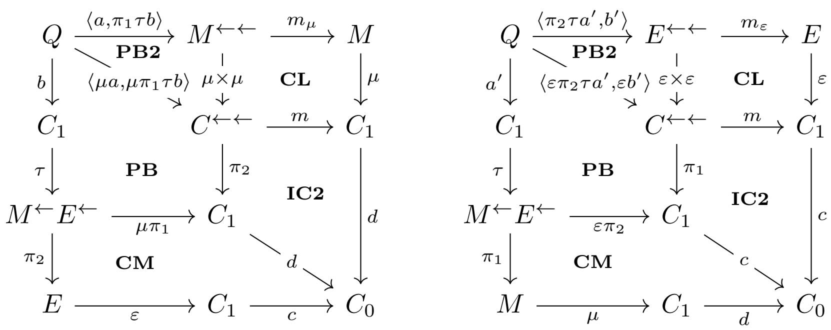
(b)

Now, we observe that the following diagrams commute:



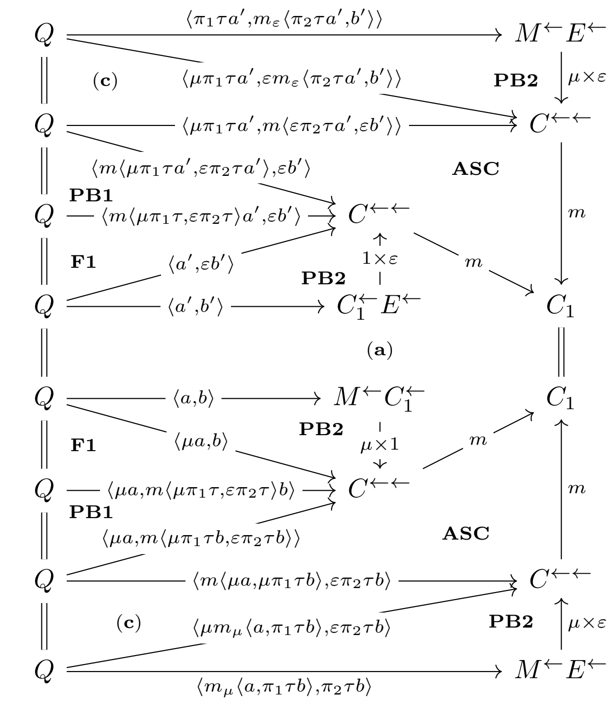


which induce, by the universal properties of $M^{\leftarrow\leftarrow}$ and $E^{\leftarrow\leftarrow}$ respectively, the morphisms $\langle a, \pi_1\tau b\rangle : Q \to M^{\leftarrow\leftarrow}$ and $\langle \pi_2\tau a', b'\rangle : Q \to E^{\leftarrow\leftarrow}$. Then, the following diagrams commute:

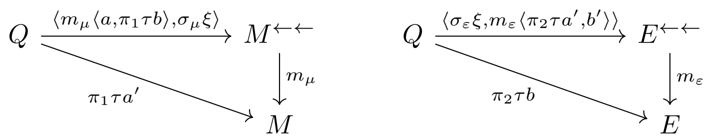


(c)

which respectively induce by the universal property of $M^{\leftarrow}E^{\leftarrow}$ the morphisms $\langle m_\mu\langle a, \pi_1\tau b\rangle, \pi_1\tau b\rangle : Q \to M^{\leftarrow}E^{\leftarrow}$ and $\langle \pi_1\tau a', m_\varepsilon\langle \pi_2\tau a', b'\rangle\rangle : Q \to M^{\leftarrow}E^{\leftarrow}$. Now, the following diagram commutes:

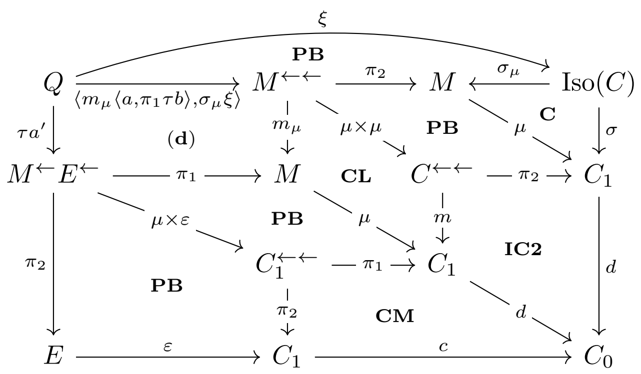


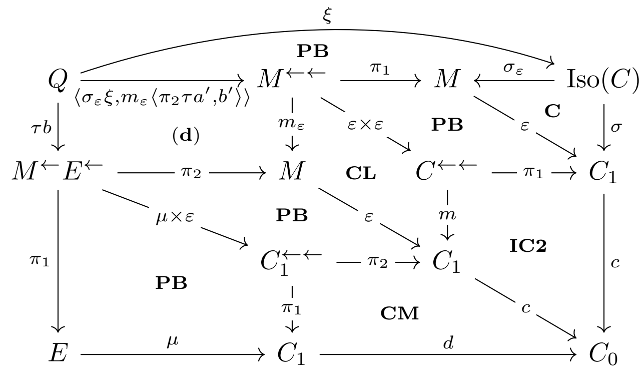
This induces, by 4.2.2, a unique morphism $\xi : Q \to \mathrm{Iso}(C)$ such that the following triangles commute:



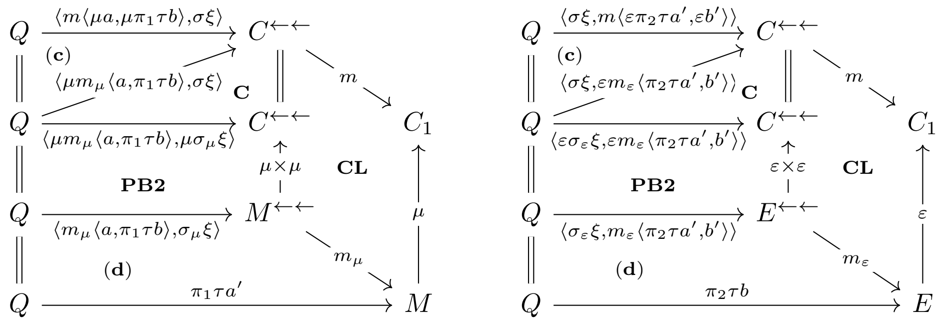
(d)

Then, the next two diagram commute:



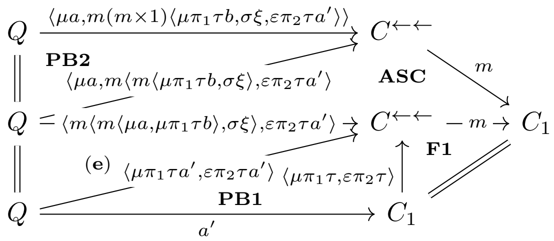


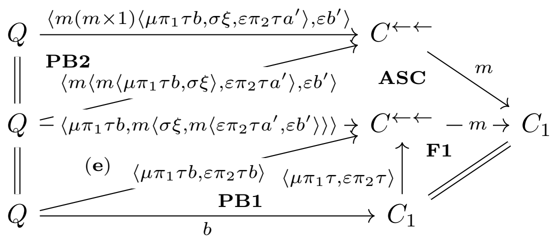
By the definition of the triple pullback $C^{\leftarrow\leftarrow\leftarrow}$, there exists a morphism $\langle \mu\pi_1\tau b, \sigma\xi, \varepsilon\pi_2\tau a'\rangle : Q \to C^{\leftarrow\leftarrow\leftarrow}$. We now show that $m(m\times 1)\langle \mu\pi_1\tau b, \sigma\xi, \varepsilon\pi_2\tau a'\rangle : Q \to C_1$ is the desired universal morphism for (**a**), with reference to remark 3.2.2. Firstly, the following diagrams commute:



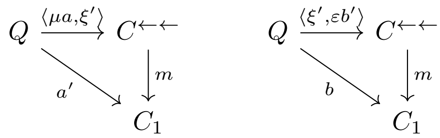
(e)

which makes the next two diagrams commute:



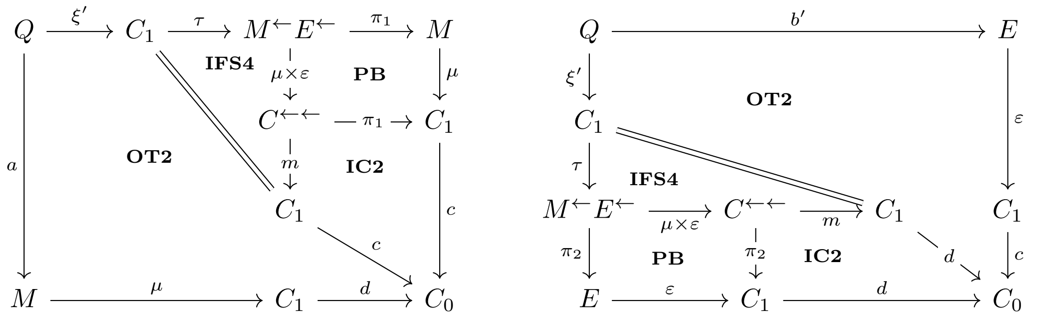


This shows that $m(m\times 1)\langle \mu\pi_1\tau b, \sigma\xi, \varepsilon\pi_2\tau a'\rangle$ is a suitable universal morphism. We lastly must show that this morphism is unique. Assume that there exists some $\xi' : Q \to C_1$ making the following diagrams commute:

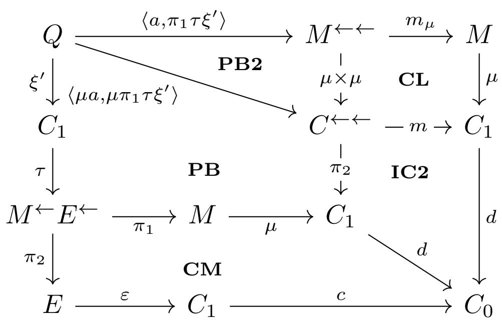


(f)

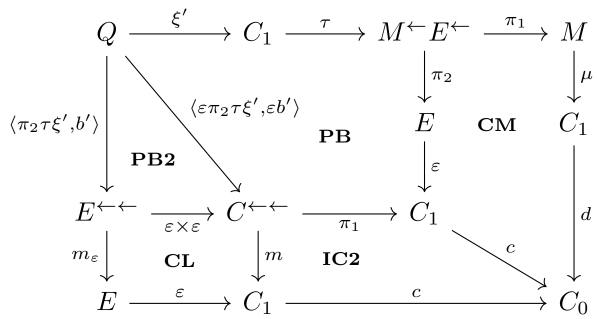
We have the following commute:



These diagrams induce, by the respective universal properties of $M^{\leftarrow\leftarrow}$ and $E^{\leftarrow\leftarrow}$, the morphism $\langle a, \pi_1\tau\xi'\rangle : Q \to M^{\leftarrow\leftarrow}$ and $\langle \pi_2\tau\xi', b'\rangle : Q \to E^{\leftarrow\leftarrow}$. Then, the following diagrams commute:

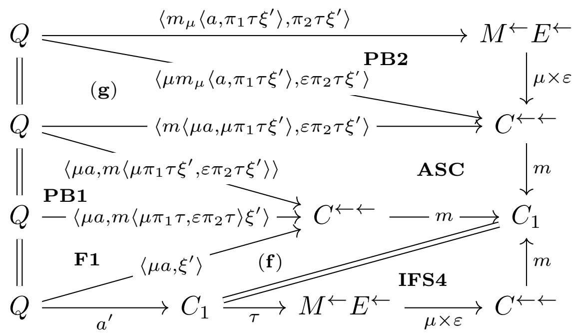


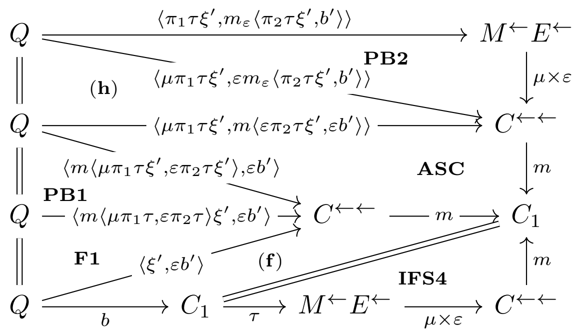
(g)



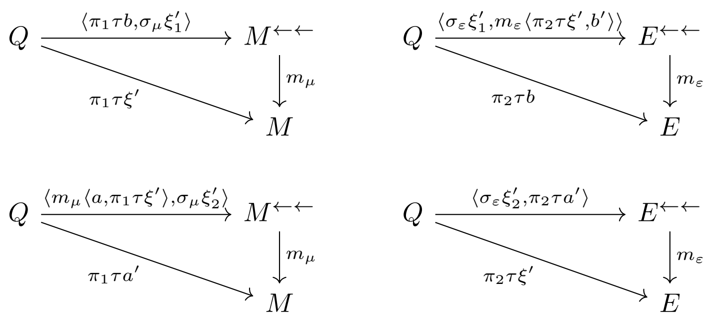
(h)

which, by the universal property of $M^{\leftarrow}E^{\leftarrow}$ respectively induce the morphisms $\langle m_\mu\langle a, \pi_1\tau\xi'\rangle, \pi_2\tau\xi'\rangle :$ $Q \to M^{\leftarrow}E^{\leftarrow}$ and $\langle \pi_1\tau\xi', m_\varepsilon\langle \pi_2\tau\xi', b'\rangle\rangle : Q \to M^{\leftarrow}E^{\leftarrow}$. Then, the next diagrams commute:



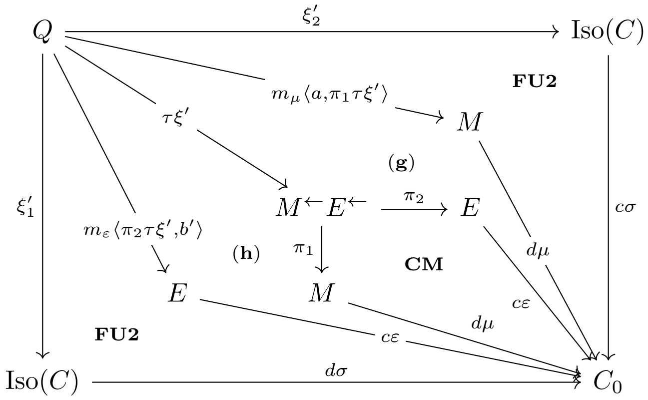


which, by remark 4.2.2 respectively induce the universal morphisms $\xi_2' : Q \to \mathrm{Iso}(C)$ and $\xi_1' : Q \to \mathrm{Iso}(C)$ which satisfy the following diagrams:

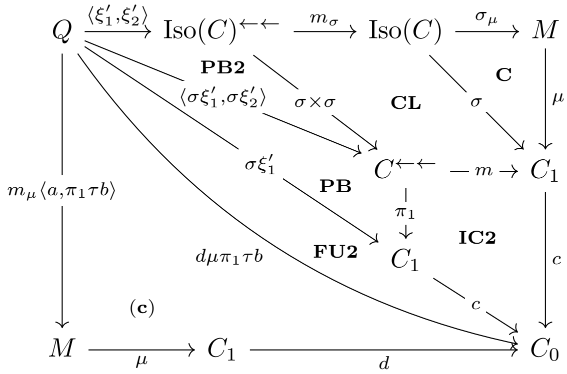


(i)

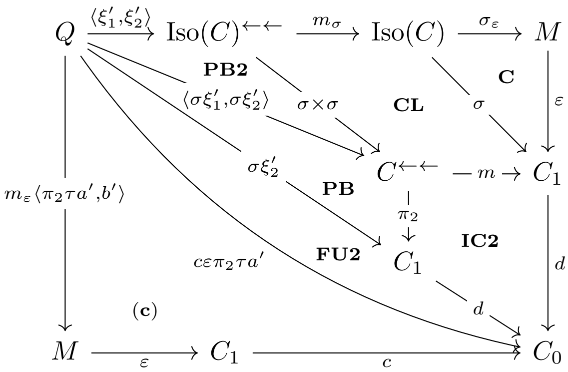
Recalling **FU2** with respect to these morphisms, we have that the following diagram commutes



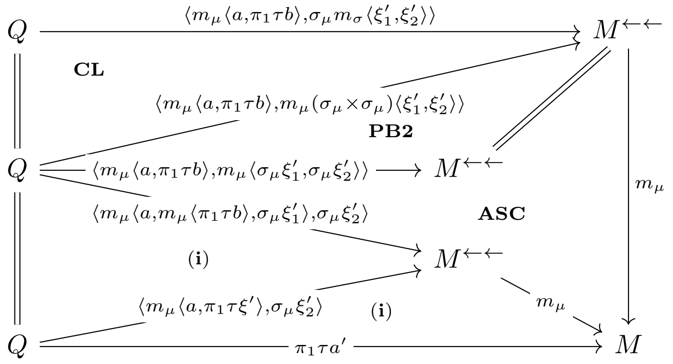
which induces the morphism $\langle\xi_1',\xi_2'\rangle : Q \to \mathrm{Iso}(C)^{\leftarrow\leftarrow}$ by the universal property of $\mathrm{Iso}(C)$. We now aim to show that $m_\sigma\langle\xi_1',\xi_2'\rangle : Q \to \mathrm{Iso}(C)$ satisfies the universal property of ξ, as in (**d**). To this end, we must first show the existence of $\langle m_\mu\langle a,\pi_1\tau b\rangle, \sigma_\mu m_\sigma\langle\xi_1',\xi_2'\rangle\rangle : Q \to M^{\leftarrow\leftarrow}$ and $\langle\sigma_\varepsilon m_\sigma\langle\xi_1',\xi_2'\rangle, m_\varepsilon\langle\pi_2\tau a',b'\rangle\rangle : Q \to E^{\leftarrow\leftarrow}$. This is true by the universal properties of $M^{\leftarrow\leftarrow}$ and $E^{\leftarrow\leftarrow}$, respectively, due to the following commutative diagrams:

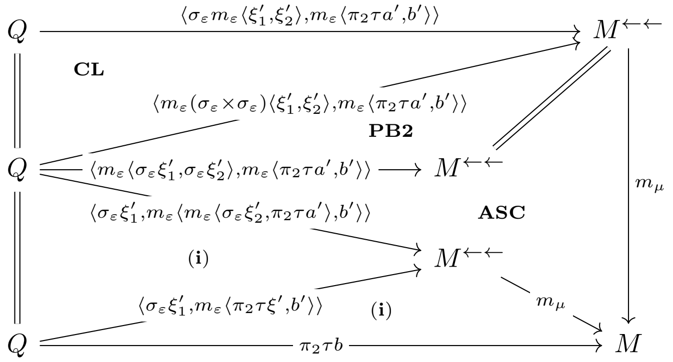


(j)

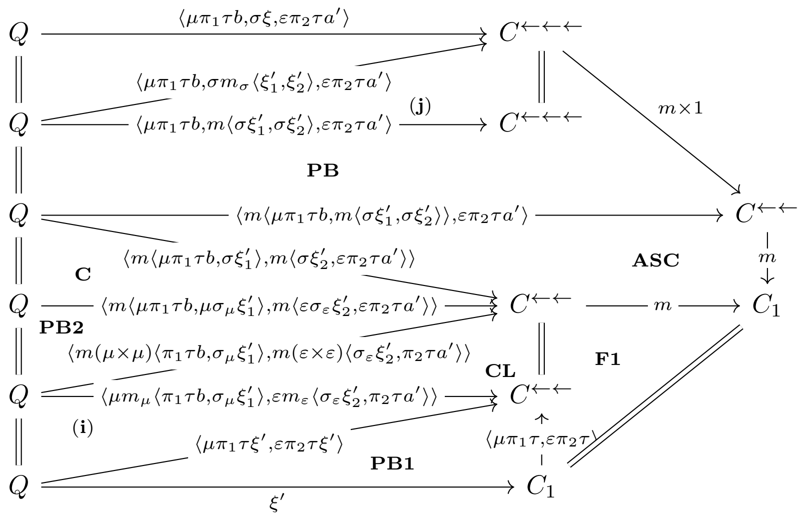


Then, the following diagrams commute:





Therefore, by the uniqueness of the universal morphism $\xi : Q \to \mathrm{Iso}(C)$, we have that $\xi = m_\sigma\langle \xi_1', \xi_2'\rangle$. Finally, we observe that the following diagram commutes:



This means that $m(m \times 1)\langle \mu\pi_1\tau b, \sigma\xi, \varepsilon\pi_2\tau a'\rangle$ is unique, which completes the proof. $\square$

4.3 The Cancellation Properties

We now consider the cancellation properties of 3.1.16 and 3.1.17 with the aim of providing analogous internal definitions. To do this, we make the following straightforward observation on these definitions.

Proposition 4.3.1. *Let* **C** *be a category and let A be a class of morphisms of $\mathbb{C}$ which is closed under composition. TFAE:*

1. *A satisfies the left cancellation property.*
2. *For all pairs of composable morphisms $(f : Y \to Z, g : X \to Y)$ in $\mathbb{C}$:*

$$f \in A \text{ and } fg \in A \iff f \in A \text{ and } g \in A$$

Proof. Under the assumption of 1, the forward of direction of 2 is given by the left cancellation property, and the backwards direction of 2 is given by A being closed under composition. Under the assumption of 2, the left cancellation property is given by the forwards direction of 2. □

We have a similar result for the right cancellation property.

Proposition 4.3.2. *Let* **C** *be a category and let A be a class of morphisms of $\mathbb{C}$ which is closed under composition. TFAE:*

1. *A satisfies the right cancellation property.*
2. *For all pairs of composable morphisms $(f : Y \to Z, g : X \to Y)$ in $\mathbb{C}$:*

$$g \in A \text{ and } fg \in A \iff g \in A \text{ and } f \in A$$

Proof. Dual of the above. □

These observations indicate that under the assumption that a class of morphisms is closed under composition, the class satisfying a cancellation property may be phrased as a correspondence between pairs of composable morphisms for which one morphism and their composition is in the class, and pairs of composable morphisms of the class. The internal definitions are given by this correspondence.

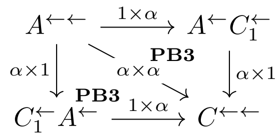
Definition 4.3.3. Let C be an internal category in a category $\mathbb{C}$ with pullbacks, and let $\alpha : A \to C_1$ be a subobject of morphisms of C that is closed under composition. Then α satisfies the *left cancellation property* if the following diagram is a pullback:

$$\begin{array}{ccc} A^{\leftarrow\leftarrow} & \xrightarrow{m_\alpha} & A \\ {\scriptstyle 1\times\alpha}\downarrow & \lrcorner & \downarrow{\scriptstyle \alpha} \\ A^{\leftarrow}C_1^{\leftarrow} & \xrightarrow{m(\alpha\times 1)} & C_1 \end{array} \tag{LCP}$$

Definition 4.3.4. Let C be an internal category in a category $\mathbb{C}$ with pullbacks, and let $\alpha : A \to C_1$ be a subobject of morphisms of C that is closed under composition. Then α satisfies the *right cancellation property* if the following diagram is a pullback:

$$\begin{array}{ccc} A^{\leftarrow\leftarrow} & \xrightarrow{m_\alpha} & A \\ {\scriptstyle \alpha\times 1}\downarrow & \lrcorner & \downarrow{\scriptstyle \alpha} \\ C_1^{\leftarrow}A^{\leftarrow} & \xrightarrow{m(1\times\alpha)} & C_1 \end{array} \tag{RCP}$$

Remark 4.3.5. Notice that the commutativity of the squares of **LCP** and **RCP** comes directly from the definition of a subobject of morphisms being closed under composition, by noting the commutativity of the following diagram:

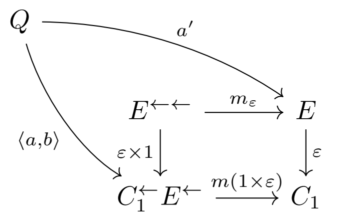


We may therefore consider the cancellation properties to be a strengthening of closure under composition.

We may now show that for an internal factorisation system, (ε, μ), ε has the right cancellation property, and μ has the left cancellation property, an internalisation of 3.1.18.

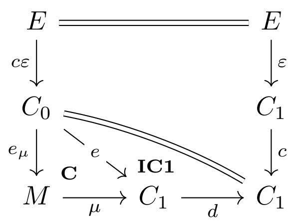
Proposition 4.3.6. *Let C be an internal category in a finitely complete category $\mathbb{C}$, and let (ε, μ) be an internal factorisation system on C. Then ε has the right cancellation property.*

Proof. Firstly, as (ε, μ) is an internal factorisation system, ε is closed under composition. By the above remark, we have that the square **RCP** commutes. Then, assume that there exists an object Q and two morphisms $\langle a, b\rangle : Q \to C_1^{\leftarrow}E^{\leftarrow}$ and $a' : Q \to E$ in $\mathbb{C}$ such that the following diagram commutes:

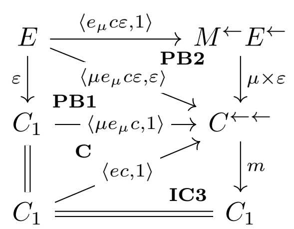


(a)

Recalling that both ε and μ contain all identity morphisms by **IFS1**, observe that the following diagram commutes:

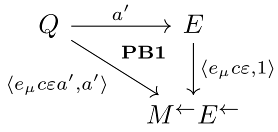


This induces the morphism $\langle e_\mu c\varepsilon, 1\rangle : E \to M^{\leftarrow}E^{\leftarrow}$ by the universal property of $M^{\leftarrow}E^{\leftarrow}$. The commutativity of the next diagram will be useful in later calculation:

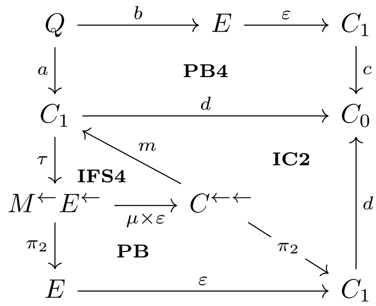


(b)

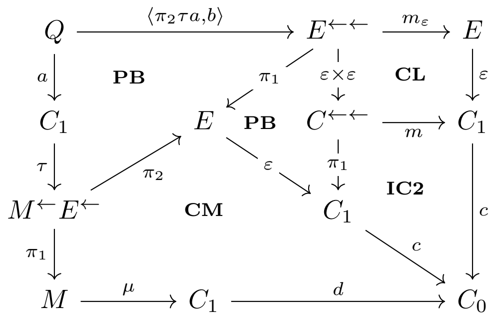
We make use of $\langle e_\mu c\varepsilon a', a'\rangle$, obtained from the following commutative diagram:



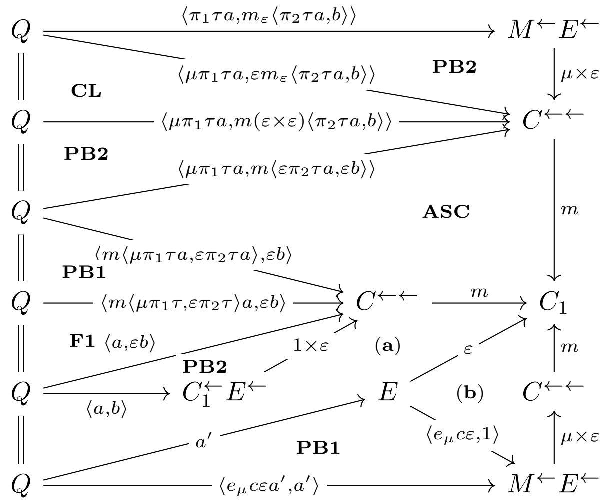
Next, we observe that the following diagram commutes:



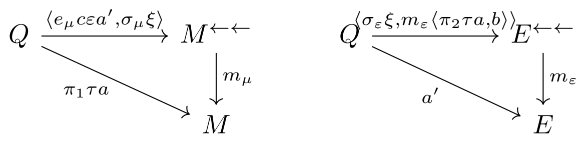
which induces, by the universal property of $E^{\leftarrow\leftarrow}$, the morphism $\langle \pi_2\tau a, b\rangle : Q \to E^{\leftarrow\leftarrow}$. Then the following diagram commutes



which induces the morphism $\langle \pi_1\tau a, m_\varepsilon\langle \pi_2\tau a, b\rangle\rangle : Q \to M^{\leftarrow}E^{\leftarrow}$, by the universal property of $M^{\leftarrow}E^{\leftarrow}$. Now, we have that the following diagram commutes

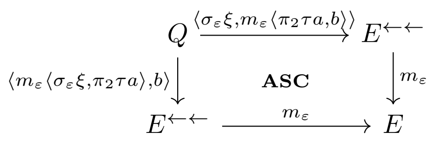


By remark 4.2.2, we have a unique morphism $\xi : Q \to \mathrm{Iso}(C)$ making the following diagrams commute:

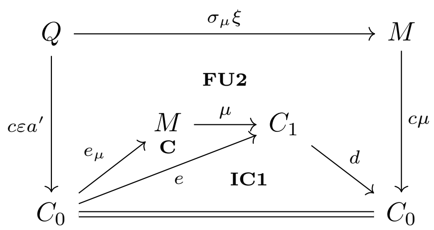


(c)

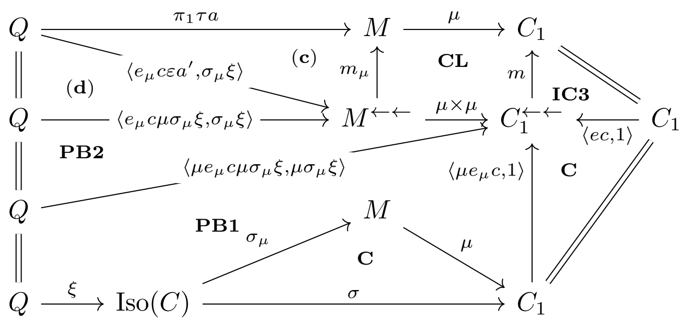
By the associativity of m_ε, we have that the following commutes:



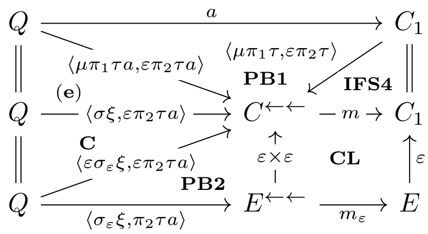
We will show that $\langle m_\varepsilon\langle \sigma_\varepsilon \xi, \pi_2\tau a\rangle, b\rangle : Q \to E^{\leftarrow\leftarrow}$ is the desired universal morphism of (**a**). To do this, we make the following interim calculations:



(d)

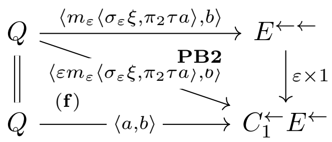


(e)

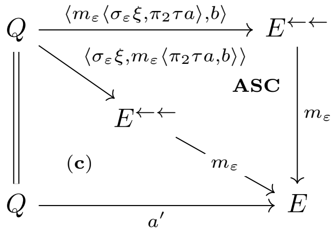


(f)

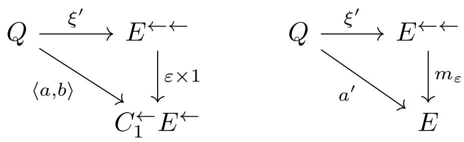
Then the fact that $\langle m_\varepsilon\langle\sigma_\varepsilon\xi, \pi_2\tau a\rangle, b\rangle$ is the desired universal arrow is given by the the following commutative diagrams:



(g)

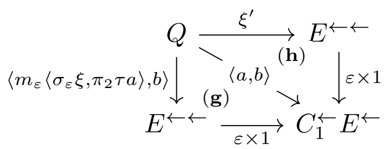


We finally show that this morphism is unique. Assume that there exists a morphism $\xi' : Q \to E^{\leftarrow\leftarrow}$ such that the following diagrams commute:



(h)

Then, the following diagram commutes:

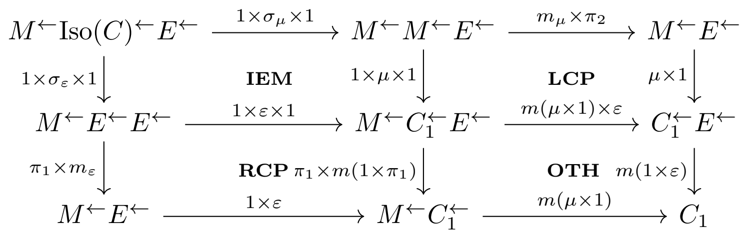


Lastly, ε and 1 are monomorphisms, so by 1.2.27, $\varepsilon \times 1$ is a monomorphism. Therefore, $\xi' = \langle m_\varepsilon \langle \sigma_\varepsilon \xi, \pi_2 \tau a \rangle, b \rangle$. $\square$

The proof of the normally dual result below is similar to that of the of the above, and we thus omit it.

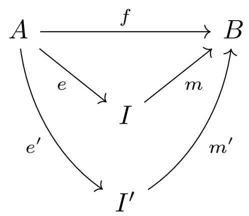
Proposition 4.3.7. *Let C be an internal category in a finitely complete category $\mathbb{C}$, and let (ε, μ) be an internal factorisation system on C. Then μ has the left cancellation property.*

Remark 4.3.8. Given now that we have the cancellation properties for internal factorisation systems, we may make the following observation. Let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms of an internal category C in a finitely complete category $\mathbb{C}$ which satisfy IFS1, IFS2 and IFS3. Consider the diagram:

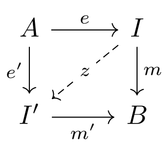


It is straightforward to show that each of the four squares of this diagram is a pullback by the fact that **IEM**, **LCP**, **RCP** and **OTH** are pullbacks, respectively. Therefore, by the pasting law, the whole square is a pullback. The whole square is precisely **FUI**. This result, when considered with $\mathbb{C} = \textbf{Set}$, corresponds to the following proof that factorisations are unique up to isomorphism:

Let $(\mathcal{E}, \mathcal{M})$ be a factorisation system on a (small) category $\mathbb{D}$. Let $f : A \to B$ be a morphism in $\mathbb{D}$ such that $f = me = m'e'$ are two $(\mathcal{E}, \mathcal{M})$-factorisations of f as in the following diagram:



Then the square of the following diagram commutes, and the fact that $e \downarrow m'$ means that there exist a unique morphism $z : I \to I'$ making the whole diagram commute.



Then $\mathcal{E}$ has the right cancellation property so z is in $\mathcal{E}$ and $\mathcal{M}$ has the left cancellation property so z is in $\mathcal{M}$. Finally, if z is in $\mathcal{E}$ and $\mathcal{M}$, then z must be an isomorphism, so the factorisations of f are unique up to isomorphism.

It is important to note that for a usual factorisation system, it is possible to show that each class satisfies the respective cancellation property without using the fact that factorisations are unique up to isomorphism, and thus the above is a valid proof. On the other hand, for an internal factorisation system, we require the essential uniqueness of factorisations to prove the cancellation properties, and thus may not use this internal observation as a proof.

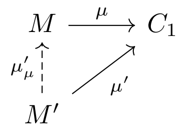
4.4 The order on internal factorisation systems

In the usual sense, the factorisation systems on a category may be endowed with an order which is induced by the inclusion of the class of the components of the factorisation systems, in a particular way. Furthermore, a consequence of this is that each component of a specific factorisation system determines the the other component. We now show that analogous results hold for internal factorisation systems.

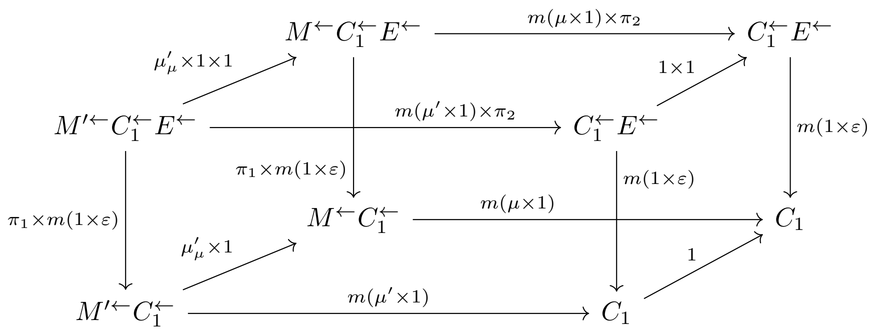
As expressed in 3.1.13, if $(\mathcal{E}, \mathcal{M})$ is a factorisation system, then $\mathcal{E}$ is orthogonal to every subclass of $\mathcal{M}$. We show this internally.

Proposition 4.4.1. *Let C be an internal category in a category $\mathbb{C}$ with pullbacks. Let $\varepsilon : E \to C_1$ and $\mu : m \to C_1$ be two subobjects of morphisms of C such that $\varepsilon \downarrow \mu$. If $\mu' : M' \to C_1$ is a subobject of morphisms of C such that μ contains μ', then $\varepsilon \downarrow \mu'$.*

Proof. Firstly, if μ contains μ', then there exists a monomorphism $\mu'_\mu : M' \to M$ which makes the following triangle commute:



Now, consider the following commutative diagram:

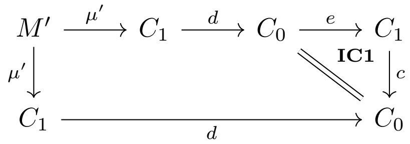


The left and right hand squares are clearly pullbacks, while the back square is a pullback by **OTH** as $\varepsilon \downarrow \mu$. By the pasting law, we have that the front square is a pullback, which means that $\varepsilon \downarrow \mu'$. □

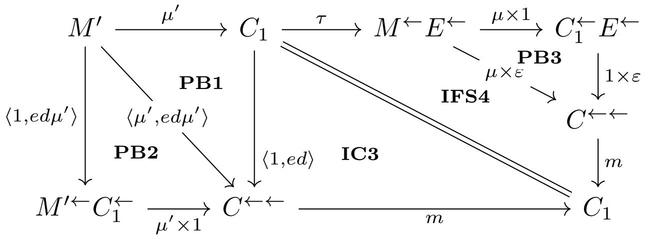
While in the above proposition we only require $\varepsilon \downarrow \mu$, and not that (ε, μ) is a factorisation system, if it is in fact such, we have that the converse of this statement holds, as expressed externally in 3.1.13.

Proposition 4.4.2. *Let C be an internal category in a finitely complete category $\mathbb{C}$ and let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms such that (ε, μ) is an internal factorisation system on C. If $\mu' : M' \to C_1$ is a subobject of morphisms of C then $\varepsilon \downarrow \mu'$ if and only if μ contains μ'.*

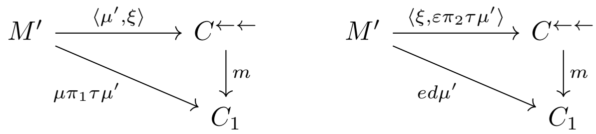
Proof. By **IFS3**, $\varepsilon \downarrow \mu$, and by the previous proposition 4.4.1, if μ contains μ' then $\varepsilon \downarrow \mu'$. We now show the forward direction. Assume that $\varepsilon \downarrow \mu'$ and consider the following commutative diagram:



This induces the universal morphism $\langle 1, d\mu' \rangle : M' \to M'^{\leftarrow}C_1^{\leftarrow}$. Then, the following commutes:

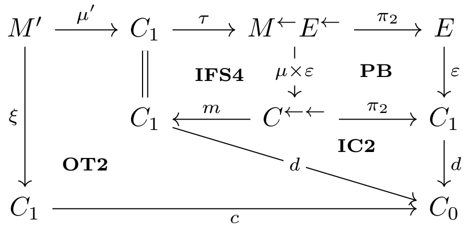


This, given that $\varepsilon \downarrow \mu'$, with reference to 3.2.2, induces a universal morphism $\xi : M' \to C_1$ which makes the following diagrams commute:



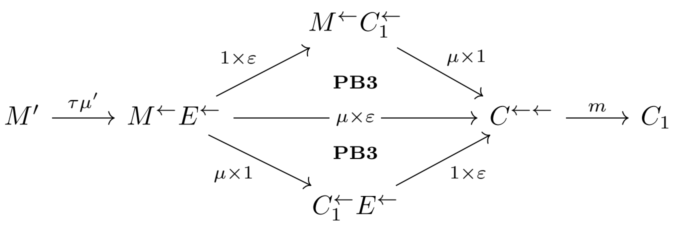
(a)

Next, we observe that the following commutes:

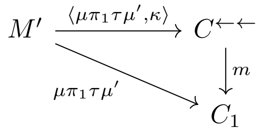


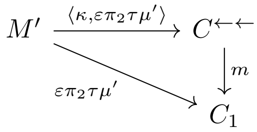
(b)

This induces the morphism $\langle \varepsilon\pi_2\tau\mu', \xi \rangle : M' \to C^{\leftarrow\leftarrow}$. We now aim to show that $m\langle \varepsilon\pi_2\tau\mu', \xi \rangle = ed\mu\pi_1\tau\mu'$. To do this, observe that the following diagram commutes:

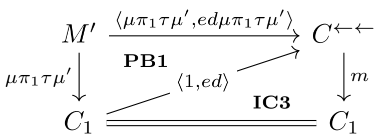


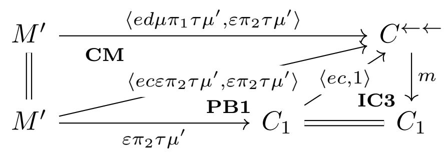
which by remark 3.2.2 and that $\varepsilon \downarrow \mu$ means that there exists a *unique* morphism $\kappa : M \to C^{\leftarrow\leftarrow}$ such that the following commute:



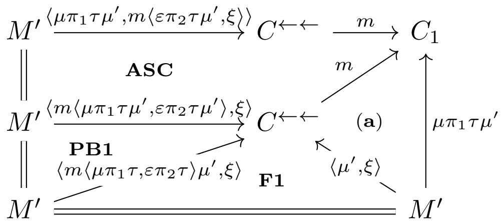


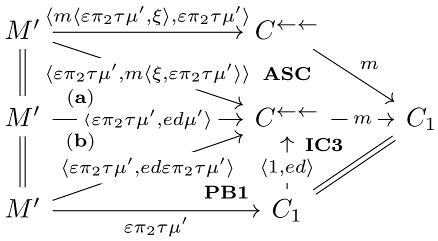
The following commutative diagrams show that $ed\mu\pi_1\tau\mu'$ satisfies this universal property:



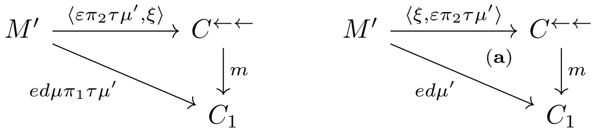


So by the uniqueness of κ, we have that $\kappa = ed\mu\pi_1\tau\mu'$. On the other hand, the next commutative diagrams show that $m\langle \varepsilon\pi_2\tau\mu', \xi\rangle$ satisfies this same universal property:

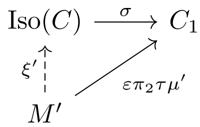




So by the uniqueness of κ, we have that $\kappa = m\langle \varepsilon\pi_2\tau\mu', \xi\rangle$. Therefore, $m\langle \varepsilon\pi_2\tau\mu', \xi\rangle = ed\mu\pi_1\tau\mu'$ as desired. We may now consider the following commutative diagrams in the context of 2.3.13:

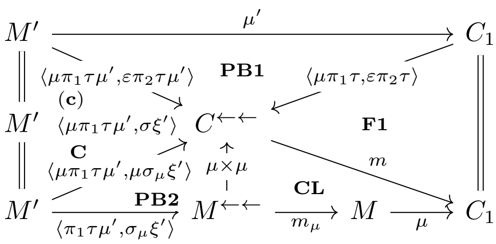


This induces a universal morphism $\xi : M' \to \mathrm{Iso}(C)$ such that the following diagram commutes:



(c)

We finally have that the following diagram commutes:



Which means that μ' factors through μ, and thus μ contains μ'. $\square$

The proof for the following analogous result is similar and thus omitted.

Proposition 4.4.3. *Let C be an internal category in a finitely complete category $\mathbb{C}$ and let $\varepsilon : E \to C_1$ and $\mu : M \to C_1$ be two subobjects of morphisms such that (ε, μ) is an internal factorisation system on C. If $\varepsilon' : E' \to C_1$ is a subobject of morphisms of C then $\varepsilon' \downarrow \mu$ if and only if ε contains ε'.*

We then have the following result, for two internal factorisation systems of a particular category.

Proposition 4.4.4. *Let C be an internal category in a finitely complete category $\mathbb{C}$ such that (ε, μ) and (ε', μ') are two internal factorisation systems on C. Then ε contains ε' if and only if μ' contains μ.*

Proof. ε contains ε' if and only if $\varepsilon' \downarrow \mu$ if and only if μ' contains μ. $\square$

In particular, this means that each component of an internal factorisation system determines the other, as per 3.1.14.

Corollary 4.4.5. *Let C be an internal category in a finitely complete category $\mathbb{C}$, and let (ε, μ) and (ε', μ') be two internal factorisation systems on C. Then $\varepsilon \sim \varepsilon'$ if and only if $\mu \sim \mu'$.*

Proof. If $\varepsilon \sim \varepsilon'$, then ε contains ε' and ε' contains ε. By the above proposition, these statements respectively imply that μ' contains μ and μ contains μ'. So $\mu \sim \mu'$. The reverse direction is dual. □

It then makes sense to make the following definition.

Definition 4.4.6. Let C be an internal category in a finitely complete category $\mathbb{C}$. The *order on internal factorisation systems* of C is defined by

$$(\varepsilon, \mu) \leq (\varepsilon', \mu') \iff \mu \leq \mu'$$

where (ε, μ) and (ε', μ') are internal factorisation systems on C and the order on the right hand side is the order on subobjects of morphisms.

We now note that the internal factorisation system $(\sigma, 1_{C_1})$ is the top element of this order.

Proposition 4.4.7. *Let C be an internal category in a finitely complete category $\mathbb{C}$. Then $(\sigma, 1_{C_1})$ is the top element of the order on internal factorisation systems.*

Proof. Let (ε, μ) be an internal factorisation system on C. Then we have that 1_{C_1} contains μ by 2.2.11. Thus, by definition, $(\varepsilon, \mu) \leq (\sigma, 1_{C_1})$. □

5 References

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