Functorial Transitive Quasi-uniformities and their Bicompletions

by

Zaid Kimmie

Thesis presented for the Degree of

Doctor of Philosophy

in the Department of Mathematics and Applied Mathematics

under the supervision of

Prof. G.C.L. Brümmer

August 1995

Acknowledgements

I wish to thank my supervisor Professor Guillaume Brümmer for his constant support and encouragement throughout the course of my graduate studies. His questions and criticisms have contributed greatly to my mathematical education and he has provided an inspiring example to me. In particular I would like to thank Prof. Brümmer for kindly providing the financial support without which I would not have been able to fully pursue my interest in mathematics.

I wish to express my gratitude to the Categorical Topology Research Group, through its chairman Prof. Brümmer, for providing a stimulating and inspirational research environment. I also wish to thank the Categorical Topology Research Group for allowing me the use of its facilities and for providing me with employment opportunities.

I wish to thank the Department of Mathematics and Applied Mathematics at the University of Cape Town, through its Head, Prof. C. Brink, for employment opportunities granted to me during the course of my studies.

I acknowledge financial support received from the FRD during 1992 and 1993.

To my parents: Thank you. Your sacrifice, your support and your example have made all of this possible for me.

Finally, to Erica, my closest friend and companion: Thank you for always being there, thank you for **everything**.

Contents

Introduction

The study of quasi-uniform spaces arises from the removal of the symmetry axiom in the definition of a uniform space. Quasi-uniformities were first investigated by Nachbin [1948] and the term quasi-uniformity was first introduced by Császár [1960]. The roots of the theory date back even further, because asymmetric distance functions such as quasi-metrics naturally induce quasi-uniformities and such a function was already exhibited by Hausdorff [1935]. The work by Pervin [1962, 1963] and the monograph by Murdeshwar and Naimpally [1966] provided a basis for further research in the field of quasi-uniformities.

Many of the initial results about quasi-uniform spaces are closely related to the corresponding results about uniform spaces but the greater generality of quasi-uniformities became clear with the result, due to Császár [1960] and Pervin [1962], that every topological space has a compatible quasi-uniformity. This can be contrasted with the situation in uniform spaces, where a topological space has a compatible uniformity if and only if it is completely regular.

It is in the study of the intriguing relationship between quasi-uniform spaces and topological spaces that categorical methods play an important role. Brümmer [1969] first considered the class of all functorial quasi-uniformities and in particular described the coarsest and finest functorial admissible quasi-uniformities.

Transitive quasi-uniform spaces form an important subcategory of the category of quasi-uniform spaces and they play a role almost as general as that of quasi-uniform spaces in the study of topological properties. In fact almost all of the canonically induced quasi-uniformities that have been studied in the literature are transitive. It is well known that when one adds a transitivity axiom to the uniform space axioms the corresponding topological spaces one obtains from these uniform spaces are the zero-dimensional spaces. However, in the realm of quasi-uniform spaces the addition of a transitivity axiom leads to no such loss of generality amongst the corresponding topological spaces; *every* topological space admits a transitive quasi-uniformity e.g. the Császár-Pervin quasi-uniformity due to Császár [1960] and Pervin [1962].

Much of the initial work on transitive quasi-uniform spaces is due to Fletcher [1970, 1971a, 1971b] and Fletcher and Lindgren [1972a, 1972b]. Fletcher used interior-preserving open covers on a topological space (called Q-covers in [1970, 1971b]) to construct compatible transitive quasi-uniformities and Fletcher and Lindgren [1972a] generalised this construction to account for all compatible transitive quasi-uniformities. The first indication of the functorial nature of the Fletcher method appeared in [Fletcher 1972] when "continuous classes" of coverings were introduced. Fletcher and Lindgren [1972a] also showed that a well known canonical quasi-uniformity, viz. the semi-continuous quasi-uniformity, was transitive. At this stage the existence of a number of canonical transitive quasi-uniformities had already been established e.g. the Pervin quasi-uniformity, the point finite quasi-uniformity [Fletcher 1971a], the locally finite quasi-uniformity [Fletcher 1971b] and the well-monotone quasi-uniformity [Junnila 1978]. Halpin [1974] was only partially successful in his attempt to demonstrate that all functorial transitive quasi-uniformities could be accounted for by the Fletcher construction. The question was settled when Brümmer [1984], by tidying up Halpin's work, showed that indeed all functorial transitive quasi-uniformities could be obtained by the Fletcher construction.

In Chapter 1 we revisit the relationship between the use of interior-preserving open covers and Brümmer's spanning construction. This allows us to tighten up the description of some functorial transitive quasi-uniformities (including the semi-continuous, the point-finite, the well-monotone and the fine transitive) in terms of their spanning classes. In addition we produce a partial answer to a question posed in [Brümmer 1977, p.81] by showing that the locally finite, the point-finite and the well-monotone covering quasi-uniformities cannot be spanned by single spaces. We also prove some results about the functorial transitive quasi-uniformities that will be used in later chapters.

The notion of bicompletion, first introduced by Császár [1960] in a more general context and developed for quasi-uniform spaces by Salbany [1970], is the natural analogue of the notion of completion in uniform spaces (see also [Brümmer 1982]). At first the terms doubly-complete [Császár 1963], s-complete [Brümmer 1977, 1978], complete [Salbany 1970], [Brümmer 1982], pair-complete [Fletcher and Lindgren 1978] and [Fletcher and Hunsaker 1992] were used. However, the use of the term *bicomplete* has become widely accepted, e.g. [Fletcher and Lindgren 1982], [Künzi and Brümmer 1987] and [Künzi and Ferrario 1992]. The initial work on the bicompletion of quasi-uniform spaces considered mainly the categorical analogues of the uniform completion [Salbany 1970], [Brümmer 1977, 1978, 1982]. The question of characterising the conditions on the induced topology of a quasi-uniform space that would effectively determine whether the quasi-uniform space is bicomplete was first raised in [Künzi and Brümmer 1987]. The particular situation investigated in that paper was that of totally bounded quasi-uniform spaces. The paper by Künzi and Ferrario [1992] solved the question for two of the most important canonical quasi-uniformities viz. the well-monotone and the fine transitive quasi-uniformities. That paper also gave substantial partial answers to the question for the cases of the fine quasi-uniformity and the semi-continuous quasi-uniformity.

In Chapter 2 we focus on the bicompletion of functorial transitive quasi-uniformities. By introducing a characterisation of the bicompletion in terms of prime open filters we extend the work of Künzi and Ferrario to encompass general functorial transitive quasi-uniformities. We characterise the topological spaces on which a given functorial transitive quasi-uniformity is bicomplete. Amongst the canonical transitive quasi-uniformities which we consider are the point-finite and the locally-finite covering quasi-uniformities. In their 1990 paper Künzi and Ferrario asked whether the fine transitive quasi-uniformity was bicomplete whenever the fine quasi-uniformity was bicomplete. We are able to answer the analogous question in the setting of transitive quasi-uniformities, where we are able to find a functorial transitive quasi-uniformity which is strictly coarser than the fine transitive quasi-uniformity but which is bicomplete whenever the fine transitive quasi-uniformity is bicomplete. We end the chapter with another application of our characterisation of the bicompletion, finding a transitive functorial quasi-uniformity which is bicompletion-idempotent but fails to be lower completion-true (in the sense of Brümmer [1992]).

Chapter 3 continues to consider the bicompleteness concept by studying the monads on the category of topological spaces which are induced by the bicompletion of certain functorial transitive quasi-uniformities. The fact that the bicompletion of certain functorial quasi-uniformities gives rise to monads on the category of topological spaces was first discussed by Brümmer [1979]. That paper also asked for a characterisation of the category of algebras of the monad induced by the bicompletion of the Császár-Pervin quasi-uniformity. This question was solved by Simmons [1982] and equivalently by Wyler [1984], K.H. Hofmann [1984], Salbany [1984] and Banaschewski and Brümmer [1988]. We extend the results of Simmons and are able to characterise the category of algebras of monads arising from the bicompletion of a certain class of transitive functorial quasi-uniformities.

After Reilly [1972] introduced the notion of zero-dimensionality in bitopological spaces, Halpin [1974] and Bîrsan [1974] showed that the zero-dimensional bitopological spaces were precisely those bispaces which admitted transitive quasi-uniformities. This corresponds to the characterisation of zero-dimensional topological spaces as those spaces which admit transitive uniformities. In an analogous fashion strongly zero-dimensional bitopological spaces are defined in [Banaschewski and Brümmer 1990] as those spaces whose Stone-Čech compactification (cf. [Salbany 1970] and [Császár 1972]) is zero-dimensional. In our final chapter we briefly explore some of the relationships between transitive quasi-uniformities and bitopological spaces. In particular we consider the manner in which canonical (transitive) quasi-uniformities on topological spaces can be extended to canonical quasi-uniformities on bitopological spaces, building on the work of Brümmer [1977, 1982]. We are able to give a partial answer to a question posed by Brümmer [1977, 1968], by showing that there exists a proper class of functorial quasi-uniformities for which this extension process is not unique. While many questions concerning the interactions between functorial transitive quasi-uniformities and strongly zero-dimensional bitopological spaces remain unanswered, we are able to make some progress on identifying those extensions which are transitive-valued on the strongly zero-dimensional bitopological spaces.

Chapter 0

Preliminaries

In this chapter we will provide an overview of notation, terminology and fundamental results which we will use in this thesis.

Set-theoretic and topological notions

Our set-theoretic assumptions and terminology about sets and classes are taken from [Herrlich and Strecker 1973].

For a set X, $\mathcal{P}(X)$ is the power set of X.
We denote the real numbers by $\mathbf{R}$, the closed interval $[0,1]$ by $\mathbf{I}$, the natural numbers by $\mathbf{N}$, the integers by $\mathbf{Z}$ and the rationals by $\mathbf{Q}$. The space $\mathbf{R}_d$ is the set of real numbers equipped with the discrete topology, and for $A \subseteq \mathbf{R}$, A_d is the appropriate subspace of $\mathbf{R}_d$.
If X is a topological space then $\mathcal{O}X$ denotes the topology on X. A collection $\mathcal{A} \subseteq \mathcal{O}X$ is called:

(i) an *open spectrum* if $\mathcal{A} = \{A_n\}_{n\in\mathbf{Z}}$ and $A_n \subseteq A_{n+1}$ (all $n \in \mathbf{Z}$) and $\bigcap \mathcal{A} = \emptyset$ and $\bigcup \mathcal{A} = X$;

(ii) *well-monotone* if $\mathcal{A}$ is well-ordered by inclusion;

(iii) *point-finite* if for every $x \in X$, $\{A \mid x \in A \in \mathcal{A}\}$ is finite;

(iv) *locally finite* if for every $x \in X$, there exists an $O \in \mathcal{O}X$ such that $x \in O$ and $\{A \mid A \cap O \neq \emptyset\}$ is finite.

Categorical Notions

We will use the terminology of [Adámek, Herrlich and Strecker 1990] for categorical notions.

The class of objects of a category $\mathbf{X}$ is written Ob$\mathbf{X}$, and that of morphisms Mor$\mathbf{X}$ (though we may write $X \in \mathbf{X}$ for $X \in \text{Ob}\mathbf{X}$ if no confusion can occur). If X and Y are objects of $\mathbf{X}$, the set of morphisms from X to Y is denoted by $\mathbf{X}(X, Y)$.

Let $X, Y \in \mathbf{X}$ and let $W: \mathbf{X} \to \mathbf{Set}$ be a forgetful functor. We say that X is *W-coarser* than Y, or that Y is *W-finer* than X, written $X \leq_W Y$, if there is a morphism $h: Y \to X$ in $\mathbf{X}$ satisfying $Wh = 1$ (where 1 is the appropriate identity in $\mathbf{Set}$). For functors $F, G: \mathbf{Y} \to \mathbf{X}$ we write $F \leq_W G$ if $FY \leq_W GY$ for every $Y \in \mathbf{Y}$ and $F <_W G$ if $F \leq_W G$ but $FY \neq GY$ for some $Y \in \mathbf{Y}$. We will write $X \leq Y$ and $F \leq G$ where no confusion can arise.

Suppose there is a forgetful functor $W: \mathbf{X} \to \mathbf{Set}$. A subcategory $\mathbf{Y}$ of $\mathbf{X}$ is called *initially dense* in $\mathbf{X}$ if for each $X \in \mathbf{X}$ the source $\mathbf{X}(X, Y)$, with Y ranging through the objects of $\mathbf{Y}$, is initial with respect to W.

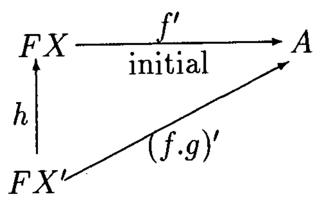
The spanning construction

Hušek [1967] and Brümmer [1969, 1971] introduced a method, termed the spanning construction [Brümmer, 1979], for constructing sections of forgetful functors. We will apply the spanning construction to the category of quasi-uniform spaces and the forgetful functors T_1 and $\overline{T}$, but we give a more general description of the construction here.

Let $\mathbf{X}$ and $\mathbf{Y}$ be categories, $U: \mathbf{Y} \to \mathbf{X}$ and $E: \mathbf{X} \to \mathbf{Set}$ be amnestic faithful functors such that $W = EU$ is topological and U preserves initial sources. Let $\mathbf{A}$ be any class of objects of $\mathbf{Y}$. We can then define a functor $F: \mathbf{X} \to \mathbf{Y}$ as follows:

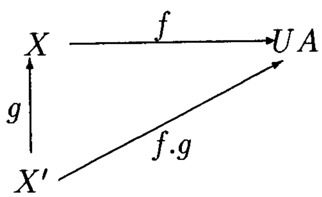
For any $X \in \mathbf{X}$ there exists a unique W-initial source $(f': B \to A)$ satisfying $Wf' = Ef$, where f ranges through $\mathbf{X}(X, UA)$ and A through $\mathbf{A}$. Let $FX = B$.
For any morphism $g: X' \to X$ in $\mathbf{X}$, consider the following diagrams:

In $\mathbf{Y}$,



all $A \in \mathbf{A}$

In $\mathbf{X}$,



all $f \in \mathbf{X}(X, UA)$, all $A \in \mathbf{A}$

The lower diagram commutes for all the given f. The source $(f': FX \to A)$ is the one given by the definition of FX. The source $((f.g)': FX' \to A)$ consists of the maps $f.g: X' \to UA$ regarded as maps from FX' to A. They are morphisms in $\mathbf{Y}$ by the definition of FX'. Now the initiality of $(f': FX \to A)$ gives us a unique morphism $h: FX' \to FX$ satisfying $Wh = Eg$. We let $Fg = h$.
It is easily seen that F is a functor; we call it the *functor spanned by* $\mathbf{A}$ and write $F = \langle \mathbf{A} \rangle_U$.

A functor F is a *U-section* when F is a section (i.e. right inverse) of U, i.e. $UF = 1_X$.

0.0.1 **Proposition** [Brümmer 1971,1976]

Let $\mathbf{A}$ *be a class of objects of* $\mathbf{Y}$ *and let* $F = \langle \mathbf{A} \rangle_U$. *Then* F *is a* U*-section iff* $U\mathbf{A}$ *is initially dense in* $\mathbf{X}$.

0.0.2 **Proposition** [Brümmer,1982]

Any U*-section* F *is spanned by its range* $\mathcal{R}_F$, *i.e.* $F = \langle \mathcal{R}_F \rangle_U$ *where* $\mathcal{R}_F = \{FX \mid X \in \mathbf{X}\}$.

Quasi-Uniform Spaces

Our basic reference on quasi-uniform spaces is [Fletcher and Lindgren 1982].

A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ such that

(i) $\{(x,x) \mid x \in X\} \subseteq U$ for every $U \in \mathcal{U}$;

(ii) $(\forall U \in \mathcal{U})$ $(\exists V \in \mathcal{U})$ such that $V \circ V \subseteq U$.

The pair $(X,\mathcal{U})$ is then called a quasi-uniform space. We shall often just denote a quasi-uniform space by X. Then $\operatorname{ent} X$ will denote the set $\mathcal{U}$ of entourages of X.
Let $(X,\mathcal{U})$ and $(Y,\mathcal{H})$ be quasi-uniform spaces. The map $f\colon X \to Y$ is *quasi-uniformly continuous* if for every $H \in \mathcal{H}$, $(f \times f)^{-1}H \in \mathcal{U}$. We shall denote the category of quasi-uniform spaces and quasi-uniformly continuous maps by **Qu**. We shall write **Top** for the category of topological spaces and continuous mappings. Then $\mathbf{Top_0}$ denotes its full subcategory of T_0-spaces.
If $(X,\mathcal{U})$ is a quasi-uniform space, we denote by $\tau(\mathcal{U})$ the topology on X having the neighbourhood system $\{U(x) \mid U \in \mathcal{U}\}$ for each $x \in X$, where $U(x) = \{y \in X \mid$

$(x,y) \in U\}$. We define the forgetful functors $T_1\colon \mathbf{Qu} \to \mathbf{Top}$ and $\overline{T}\colon \mathbf{Qu} \to \mathbf{Bitop}$ by $T_1(X,\mathcal{U}) = (X,\tau(\mathcal{U}))$ and $\overline{T}(X,\mathcal{U}) = (X,\tau(\mathcal{U}),\tau(\mathcal{U}^{-1}))$.

The quasi-uniform space $\mathbf{R}_q$ has $\mathbf{R}$ as underlying set and $\{\{(x,y) \mid y < x+\epsilon\} \mid \epsilon > 0\}$ as basis for its set of entourages. For any subset A of $\mathbf{R}$, A_q is a quasi-uniform subspace of $\mathbf{R}_q$ and A_u is the topological space obtained by applying the functor T_1 to A_q. In particular for $\mathbf{D} = \{0,1\}$, $\mathbf{D}_q$ is the Sierpinski quasi-uniform space.

Let C_1^* be the coarsest T_1-section and let ϕ be the finest T_1-section. Brümmer [1971] has shown that $C_1^* = \langle \mathbf{D}_q \rangle_{T_1}$ and $\phi = \langle \mathbf{Qu} \rangle_{T_1}$.

Bicomplete Quasi-Uniform spaces

If $\mathcal{U}$ is a quasi-uniformity on a set X, then $\mathcal{U}^*$ will denote the coarsest uniformity which is finer than both $\mathcal{U}$ and $\mathcal{U}^{-1}$. The uniformity $\mathcal{U}^*$ has as subbase for its entourages $\{U \cap U^{-1} \mid U \in \mathcal{U}\}$. A filter $\mathcal{F}$ on X is a *$\mathcal{U}^*$-Cauchy* filter if for each $U \in \mathcal{U}$ there exists $F \in \mathcal{F}$ such that $F \times F \subseteq U$. $\mathcal{F}$ is a *minimal $\mathcal{U}^*$-Cauchy filter* if $\mathcal{F}$ is $\mathcal{U}^*$-Cauchy and there exists no $\mathcal{U}^*$-Cauchy filter which is strictly coarser than $\mathcal{F}$. A quasi-uniform space $(X,\mathcal{U})$ is *bicomplete* iff $(X,\mathcal{U}^*)$ is a complete uniform space, i.e. if every $\mathcal{U}^*$-Cauchy filter converges in $(X,\tau(\mathcal{U}^*))$. The bicompletion of a quasi-uniform space $(X,\mathcal{U})$ is a space $(\tilde{X},\tilde{\mathcal{U}})$, where $\tilde{X}$ consists of the minimal $\mathcal{U}^*$-Cauchy filters and $\tilde{\mathcal{U}}$ has basis $\{\tilde{U} \mid U \in \mathcal{U}\}$ where $\tilde{U} := \{(\mathcal{F},\mathcal{G}) \mid \exists F \in \mathcal{F}\ \exists G \in \mathcal{G}$ such that $F \times G \subseteq U\}$. We denote the reflector to the bicomplete spaces in $\mathbf{Qu}$ by K and the unit of the adjunction by k.

Transitive Quasi-Uniform spaces

A quasi-uniform space $(X,\mathcal{U})$ is *transitive* if $\mathcal{U}$ has a base consisting of transitive entourages. An open cover $\mathcal{A}$ of X is an *interior-preserving open cover* if for each $\mathcal{B} \subseteq \mathcal{A}$, $\bigcap \mathcal{B}$ is open. For each $X \in \mathbf{Top}$ let Φ_X be a collection of interior-preserving open covers on X. For each $\mathcal{A} \in \Phi_X$ and for each $x \in X$, let $\mathcal{A}_x := \{A \mid x \in A \in \mathcal{A}\}$. For each $\mathcal{A} \in \Phi_X$, $U_\mathcal{A} := \bigcup_{x \in X}\{x\} \times \bigcap \mathcal{A}_x$. It is easy to see that $U_\mathcal{A}$ is a transitive entourage. Fletcher [1971a] showed that if $\bigcup \Phi_X$ is a base for $\mathcal{O}X$ then $\{U_\mathcal{A} \mid \mathcal{A} \in \Phi_X\}$ forms a subbase for a transitive admissible quasi-uniformity on X and also that every transitive admissible quasi-uniformity on X can be constructed from a suitable collection of interior-preserving open covers in this manner.

An indexed class $\Phi = (\Phi_X)_{X \in \mathbf{Top}}$ will be termed a *natural kind of covers* [Brümmer 1984] if Φ_X is a collection of covers for each $X \in \mathbf{Top}$ and whenever $f: X \to Y$ is a continuous map and $\mathcal{A} \in \Phi_Y$ then $f^{-1}[\mathcal{A}] \in \Phi_X$. $(\Phi_X)_{X \in \mathbf{Top}}$ will be called an *adequate* [Brümmer 1984] collection of covers if for each $X \in \mathbf{Top}$, $\bigcup \Phi_X$ forms a basis for $\mathcal{O}X$. If Φ is an adequate natural kind of interior-preserving open covers then Φ induces a T_1-section F, say, where for each $X \in \mathbf{Top}$, FX is the quasi-uniform space with subbase $\{U_\mathcal{A} \mid \mathcal{A} \in \Phi_X\}$ [Brümmer 1984].

We shall often use the identity $U_\mathcal{A} \cap U_\mathcal{B} = U_{\mathcal{A} \wedge \mathcal{B}}$ where $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A} \text{ and } B \in \mathcal{B}\}$ valid when $\mathcal{A}$ and $\mathcal{B}$ are covers of X [Brümmer 1984].

For notions about quasi-uniform spaces which we have left undefined we refer the reader to [Fletcher and Lindgren 1982]. More information about transitive-valued T_1-sections may be found in [Brümmer 1984], and about general T_1-sections and $\overline{T}$-sections in [Brümmer 1977, 1982].

A list of functors

K: $\mathbf{Qu} \to \mathbf{Qu}$ is the bicompletion functor.

The following are T_1-sections:

$\mathcal{C}_1^* = \langle \mathbf{D}_q \rangle_{T_1}$ - the Császár-Pervin functor;

$\mathcal{C}_1 = \langle \mathbf{R}_q \rangle_{T_1}$ - the semi-continuous functor;

ϕ_t - the finest transitive valued T_1-section;

ϕ - the finest T_1-section;

$\mathcal{C}_h$ is induced by $\{\mathcal{A} \cup \{X\} \mid \mathcal{A}$ is an open-spectrum on some $O \in \mathcal{O}X\}$;

D is induced by the open spectra which have $\emptyset$ as a member;

D_h is induced by $\{\mathcal{A} \cup \{X\} \mid \mathcal{A}$ is an open-spectrum containing $\emptyset$ on some $O \in \mathcal{O}X\}$;

B is induced by the open spectra which have X as a member;

W - the well-monotone functor, induced by all well-monotone open covers;

W_κ is induced by all well-monotone open covers with cardinality $\leq \kappa$;

W_u is induced by all well-decreasing interior-preserving open covers;

Q_{pf} - the point-finite functor, induced by all point-finite open covers;

Q_{lf} - the locally finite functor, induced by all locally finite open covers;

Q_{hlf} - the hereditary locally finite functor, induced by $\{\mathcal{A} \cup \{X\} \mid \mathcal{A}$ is a locally finite open cover of some $O \in \mathcal{O}X\}$.

Chapter 1

Constructing transitive T-sections

1.1 Introduction

There are two well-known canonical methods of assigning transitive admissible quasi-uniformities to topological spaces, viz. the spanning construction due to Hušek [1964] and Brümmer [1969] and the use of interior-preserving open covers due to Fletcher [1970, 1971a, 1971b] and Fletcher and Lindgren [1972a, 1972b, 1982].

Following on the work of Fletcher and Lindgren, Halpin [1974] and Brümmer [1984] showed that every *transitive* T_1-section can be induced by an adequate natural kind of interior-preserving open covers. Brümmer [1984] also described the largest such kind of covers that induces a given transitive T_1-section. It is clear that a T_1-section F is transitive if and only if it is spanned by a collection of transitive quasi-uniform spaces.

The most striking aspect of transitive quasi-uniform spaces is that they can be constructed by considering the interior-preserving open covers of their induced topological spaces. We can, to a certain extent, mimic this construction (due to Fletcher) to investigate, in more detail, the transitive analogue to Brümmer's spanning construction.

1.2 A spanning construction

Throughout this section $\mathcal{S}$ will denote an initially dense (in **Top**) collection of topological spaces. For each $X \in \mathcal{S}$ let Φ_X be an adequate collection of interior preserving open covers. For each $Y \in \mathbf{Top}$ set

$$\Gamma_Y := \{f^{-1}\mathcal{A} \mid f\colon Y \to X \text{ continuous}, X \in \mathcal{S}, \mathcal{A} \in \Phi_X\}.$$

1.2.1 **Lemma**

(i) $(\Gamma_Y)_{Y \in \mathbf{Top}}$ *is a natural kind of covers.*

(ii) For each $Y \in \mathbf{Top}$, Γ_Y *is an adequate collection of interior preserving open covers.*

Proof. (i) Let $\mathcal{A} \in \Gamma_Y$ and suppose that $g\colon Z \to Y$ is a continuous map. By definition $\mathcal{A} = f^{-1}\mathcal{B}$ where $f\colon Y \to X$ is a continuous map, $X \in \mathcal{S}$ and $\mathcal{B} \in \Phi_X$. Then $g^{-1}\mathcal{A} = (fg)^{-1}\mathcal{B} \in \Gamma_Z$ since $fg\colon Z \to X$ is continuous.
(ii) The fact that $\bigcup \Gamma_Y$ is a subbase for $\mathcal{O}Y$ follows immediately since $\mathcal{S}$ is initially dense.

1.2.2 **Definition**

$(\Phi_X)_{X \in \mathcal{S}}$ *determines* a transitive T_1-section, F, where FY is the quasi-uniform space with subbase $\{U_{\mathcal{A}} \mid \mathcal{A} \in \Gamma_Y\}$ (Γ_Y as above).

The relationship between the spanning construction and the construction outlined above is given by the following proposition.

1.2.3 **Proposition**

Let $(\Phi_X)_{X\in\mathcal{S}}$ *determine a transitive* T_1*-section* F *as above. For each* $X \in \mathcal{S}$ *let* SX *be the quasi-uniform space with subbase* $\{U_{\mathcal{A}} \mid \mathcal{A} \in \Phi_X\}$. *Then* $F = \langle\{SX \mid X \in \mathcal{S}\}\rangle_{T_1}$.

1.2.4 **Definition**

(i) For each topological space X, we will denote by $\mathcal{I}(X)$ the set of all interior-preserving open covers on X.

(ii) A class, $\mathcal{S}$, of topological spaces *generates* a transitive T_1-section, F, if $(\mathcal{I}(X))_{X\in\mathcal{S}}$ determines F.

1.2.5 **Proposition**

For a transitive T_1*-section* F *and an initially dense (in* **Top***) class of topological spaces,* $\mathcal{S}$*, the following are equivalent:*

(i) F *is generated by* $\mathcal{S}$*;*

(ii) $F = \langle\phi_t X \mid X \in \mathcal{S}\rangle_{T_1}$ *;*

(iii) F *is the coarsest section which coincides with* ϕ_t *on* $\mathcal{S}$.

1.2.6 **Definition**

(i) A topological space (X,τ) is *Alexandroff-discrete* if τ is closed under arbitrary intersections.

(ii) In an Alexandroff-discrete space X an *Alexandroff subbase* is a family $\mathcal{S}$ of open sets such that every open set of X is a union of intersections of subfamilies of $\mathcal{S}$.

We will denote the full subcategory of **Top** which consists of the Alexandroff-discrete spaces by **Alex**.

Given an interior-preserving open cover $\mathcal{A}$ of a topological space X, we will denote by $X_{\mathcal{A}}$ the topological space which has the same underlying set as X and which has $\mathcal{A}$ as Alexandroff subbase. Furthermore, since $\mathcal{A}$ is an interior-preserving open cover, the identity map $i_{\mathcal{A}}: X \to X_{\mathcal{A}}$ is continuous.

1.2.7 Theorem

The fine transitive functor, ϕ_t, is generated by **Alex**.

Proof. Let F be the functor generated by **Alex**. Clearly $F \leq \phi_t$ so we only need to show that $\phi_t \leq F$. For any $X \in$ **Top**, we have the initial source

$$(i_{\mathcal{A}}: \phi_t X \to \phi_t X_{\mathcal{A}})_{\mathcal{A} \in \mathcal{I}(X)}$$

because the subbasic entourage $U_{\mathcal{A}}$ of $\phi_t X$ coincides with $(i_{\mathcal{A}} \times i_{\mathcal{A}})^{-1} U_{\mathcal{A}}$ in which $U_{\mathcal{A}}$ happens also to be an entourage of $\phi_t X_{\mathcal{A}}$. The result follows.

Denote by **Pow** the full subcategory of **Top** consisting of spaces homeomorphic to the complete atomic boolean algebras with the upper sets (in the given partial order) as open sets. Clearly **Pow** $\subseteq$ **Alex**.

1.2.8 Lemma

$\mathcal{P}(A)$ with the topology of upper sets has basic open sets of the form $\uparrow\{B\}$, $B \subseteq A$.

Proof. Let $\mathcal{U}$ be an upper set in $\mathcal{P}(A)$, that is, if $U \in \mathcal{U}$ and $U \subseteq V \in \mathcal{P}(A)$, then $V \in \mathcal{U}$. The claim to be proved is that $\mathcal{U}$ is a union of sets of the form $\uparrow\{B\}$, more precisely:

$$\mathcal{U} = \bigcup \{ \uparrow\{B\} \mid B \in \mathcal{U}\}.$$

Clearly $\mathcal{U} \subseteq \bigcup\{ \uparrow\{B\} \mid B \in \mathcal{U}\}$. To see the reverse inclusion let $V \in \bigcup\{ \uparrow\{B\} \mid B \in \mathcal{U}\}$; so $V \in \uparrow\{B\}$ for some $B \in \mathcal{U}$; i.e. $B \subseteq V \subseteq A$. Since $\mathcal{U}$ is an upper set and $B \in \mathcal{U}$, it follows that $V \in \mathcal{U}$; and hence $\bigcup\{ \uparrow\{B\} \mid B \in \mathcal{U}\} \subseteq \mathcal{U}$.

1.2.9 **Lemma**

Each upper set of $\mathcal{P}(A)$ is a union of intersections of sets of the form $\uparrow\{a\}$, $a \in A$.

Proof. The above lemma gives the basic open sets of the form $\uparrow\{B\}$, $B \subseteq A$. Now $\uparrow\{B\} = \bigcap\{ \uparrow\{a\} \mid a \in B\}$, because $V \in \uparrow\{B\} \iff B \subseteq V \iff (\forall a \in B)$ $(a \in V) \Leftrightarrow (\forall a \in B)\ (V \in \uparrow\{a\}) \Leftrightarrow V \in \bigcap\{ \uparrow\{a\} \mid a \in B\}$.

We shall use the above lemmas in the following situation: $\mathcal{A}$ is a family of subsets of a given set X; $\mathcal{P}(\mathcal{A})$ equipped with the topology of upper sets is then a power space and in particular an Alexandroff space, and has an Alexandroff subbase consisting of all sets of the form $\uparrow\{A\}$, $A \in \mathcal{A}$.

1.2.10 **Proposition**

The fine transitive functor, ϕ_t, is generated by **Pow**.

Proof. Let X be a topological space and let $\mathcal{A} \in \mathcal{I}(X)$. Define $f_{\mathcal{A}}: X \to \mathcal{P}(\mathcal{A})$ by $f_{\mathcal{A}}(x) = \{A \in \mathcal{A} \mid x \in A\}$.

Then, considering $\uparrow\{A\} \subseteq \mathcal{P}(\mathcal{A})$, we have: (for $A \in \mathcal{A}$)

$$\begin{aligned} f_{\mathcal{A}}^{-1} \uparrow\{A\} &= \{x \in X \mid f_{\mathcal{A}}(x) \in \uparrow\{A\}\} \\ &= \{x \in X \mid \{A\} \subseteq f_{\mathcal{A}}(x)\} \\ &= \{x \in X \mid A \in f_{\mathcal{A}}(x)\} \\ &= \{x \in X \mid x \in A\} \\ &= A. \end{aligned}$$

Since $\{ \uparrow\{A\} \mid A \in \mathcal{A}\}$ is an Alexandroff subbase for $\mathcal{P}(\mathcal{A})$, and $A \in \mathcal{O}X$, it follows that $f_{\mathcal{A}}: X \to \mathcal{P}(\mathcal{A})$ is continuous.

Consider the functor $F = \langle\{\phi_t Y \mid Y \in \mathbf{Pow}\}\rangle_{T_1}$. From its definition it follows that the continuous map $f_{\mathcal{A}}: X \to \mathcal{P}(\mathcal{A})$ is lifted to a map $f'_{\mathcal{A}}: FX \to \phi_t\mathcal{P}(\mathcal{A})$ in **Qu**. Consider also the collection $\Sigma = \{ \uparrow\{A\} \mid A \in \mathcal{A}\}$ which is an interior-preserving open cover of $\mathcal{P}(\mathcal{A})$. The calculation above shows that $f_{\mathcal{A}}^{-1}\Sigma = \mathcal{A}$. One has $U_\Sigma \in \operatorname{ent}\phi_t\mathcal{P}(\mathcal{A})$ and $(f \times f)^{-1}U_\Sigma = U_{f_{\mathcal{A}}^{-1}\Sigma} = U_{\mathcal{A}}$ and the entourages $U_{\mathcal{A}}$ form a subbase for the quasi-uniformity of $\phi_t X$. Hence $(\phi_t f_{\mathcal{A}}: \phi_t X \to \phi_t\mathcal{P}(\mathcal{A}))_{\mathcal{A}\in\mathcal{I}(X)}$ is an initial source. Now $f'_{\mathcal{A}} = \phi_t f_{\mathcal{A}}.1_X$ for each $\mathcal{A} \in \mathcal{I}(X)$ and thus $1_X: FX \to \phi_t X$ is a **Qu**-morphism, i.e. $\phi_t X \leq FX$. Since $F \leq \phi_t$ always we have $F = \phi_t$.

1.2.11 Definition

For each transitive T_1-section F,

(i) $\check{F}$ is the transitive T_1-section spanned by $\{FX \mid X \in \mathbf{Top}, FX = \phi_t X\}$;

(ii) $\bar{F}$ is the transitive T_1-section spanned by $\{\phi_t X_{\mathcal{A}} \mid X \in \mathbf{Top}$ and $\mathcal{A} \in \mathcal{I}(X)$ and $U_{\mathcal{A}} \in \operatorname{ent} FX\}$.

To see that $\check{F}$ and $\bar{F}$ are T_1-sections, note that the given spanning classes contain the Sierpinski object.

If $F = \bar{F}$ then we will say that F is *closed.*

1.2.12 Proposition

For every transitive T_1-section F, $\check{F} \leq F \leq \bar{F}$.

Proof. $\check{F} \leq F$ since the given spanning class for $\check{F}$ is contained in a known spanning class for F, viz. $\{FX \mid X \in \mathbf{Top}\}$. To see that $\bar{F} \geq F$, let $X \in \mathbf{Top}$

and let $\mathcal{A} \in \mathcal{I}(X)$ with $U_{\mathcal{A}} \in \text{ent}\, FX$. Now $1_{\mathcal{A}}: X \to X_{\mathcal{A}}$ is continuous, and by the definition of $\bar{F}$, $1_{\mathcal{A}}: \bar{F}X \to \phi_t X_{\mathcal{A}}$ is a **Qu**-morphism. Since $U_{\mathcal{A}} \in \text{ent}\, \phi_t X_{\mathcal{A}}$, $(1_{\mathcal{A}} \times 1_{\mathcal{A}})^{-1} U_{\mathcal{A}} = U_{1_{\mathcal{A}}^{-1}\mathcal{A}} = U_{\mathcal{A}} \in \text{ent}\, \bar{F}X$ and hence $\bar{F} \geq F$.

It is easy to see that $\check{F}$ is the finest ϕ_t-generated functor coarser than F. We will give an example to show that $F, \check{F}, \bar{F}$ do not always coincide.

1.2.13 **Definition**

(i) Let $m \in \mathbf{N}$ and let $x, y \in \mathbf{N}^m$. Then $x \succ y$ provided that there exists an $i \in \{1, \ldots, m\}$ such that $\pi_i(x) > \pi_i(y)$;

(ii) A (finite or infinite) sequence $(x_k)_{k \in J}$, where $J \subseteq \mathbf{N}$ with each $x_k \in \mathbf{N}^m$ is *h-decreasing* if for all $j, k \in \mathbf{N}$, $x_j \succ x_k$ whenever $k > j$.

1.2.14 **Remark**

Note that $\succ$ is not a transitive relation, e.g. (in $\mathbf{N}^2$) $(2,2) \succ (4,1) \succ (3,3)$.

1.2.15 **Proposition**

Let $k \in \mathbf{N}$. Then there are no infinite h-decreasing sequences in $\mathbf{N}^k$.

Proof. The assertion clearly holds for $k = 1$ since $\mathbf{N}$ is well-ordered. Assume that the result is true for all $k < p$ but that there does exist an infinite h-decreasing sequence (x_n) in $\mathbf{N}^p$. We can choose an infinite subsequence (z_m) of (x_n) such that $(\pi_1(z_m))$ is an infinite increasing (but not necessarily strictly increasing) sequence in $\mathbf{N}$. Notice that (z_m) is again an infinite h-decreasing sequence in $\mathbf{N}^p$. Now consider the sequence (y_m) in $\mathbf{N}^{p-1}$ given by $\pi_i(y_m) = \pi_{i+1}(z_m)$ $(i = 1, \ldots, p-1)$. We claim that (y_m) is also an infinite h-decreasing sequence: Let $j, k \in \mathbf{N}$ with $k > j$. Then

$z_j \succ z_k$, i.e. $\exists i \in \{1,\ldots,p\}$ such that $\pi_i(z_j) > \pi_i(z_k)$. Since $\pi_1(z_j) \leq \pi_1(z_k)$, it follows that $\exists i \in \{1,\ldots,p-1\}$ such that $\pi_{i+1}(z_j) > \pi_{i+1}(z_k)$, i.e. $\exists i \in \{1,\ldots,p-1\}$ such that $\pi_i(y_j) > \pi_i(y_k)$.

This contradicts our assumption that $\mathbf{N}^{p-1}$ contains no infinite h-decreasing sequences and hence $\mathbf{N}^p$ can contain no infinite h-decreasing sequences.

Recall that we use $\mathbf{N}$ to denote the discrete topological space on the natural numbers, D_q for the Sierpinski quasi-uniform space and that W denotes the well-monotone functor.

1.2.16 **Proposition**

Let F be the transitive T_1-section spanned by $\{W\mathbf{N}, D_q\}$. Then $W\mathbf{N} = F\mathbf{N} < \phi_t\mathbf{N}$.

Proof. $W\mathbf{N}$ is in the spanning set of F so $W\mathbf{N} \leq FT_1(W\mathbf{N}) = F\mathbf{N}$. Since W is spanned by the class $\{WX \mid X \in \mathbf{Top}\}$ which contains $\{W\mathbf{N}, D_q\}$, we have $F \leq W$. Thus $W\mathbf{N} = F\mathbf{N}$. Since $F\mathbf{N} \leq \phi_t\mathbf{N}$, we suppose that $\phi_t\mathbf{N} \leq F\mathbf{N}$, i.e. that $\phi_t\mathbf{N} \leq W\mathbf{N}$. Since $\mathbf{N}$ is a discrete space, $\phi_t\mathbf{N}$ is discrete with basis consisting of a single entourage $U_{\mathcal{D}}$ where $\mathcal{D} := \{\{n\} \mid n \in \mathbf{N}\}$. Hence there exist well-monotone open covers $\mathcal{A}^1,\ldots,\mathcal{A}^m$ of $\mathbf{N}$ with $U := U_{\mathcal{A}^1} \cap \ldots U_{\mathcal{A}^m} \subseteq U_{\mathcal{D}}$. Because $\mathbf{N}$ is countable each $\mathcal{A}^i$ is countable and can without loss of generality be regarded as infinite. Let $\mathcal{A}^i = \{A^i_n \mid n \in \omega\}$ where $A^i_0 \subseteq A^i_1 \subseteq \ldots$. For $p \in \mathbf{N}$, let

$$n(p,i) = \min\{n \in \omega \mid p \in A^i_n\}.$$

For $p \in \mathbf{N}$, define $x_p \in \mathbf{N}^m$ by:

$$x_p = (n(p,1), n(p,2), \ldots, n(p,m)).$$

By Proposition 1.2.15 the sequence $(x_p)_{p\in\mathbf{N}}$ is not an h-decreasing sequence, so that there exists $j,k \in \mathbf{N}$ with $j > k$ such that for every $i \in \{1,\ldots,m\}$ we have

$\pi_i(x_k) \leq \pi_i(x_j)$, i.e. $n(k,i) \leq n(j,i)$, i.e. $A^i_{n(k,i)} \subseteq A^i_{n(j,i)}$. Now $U_{\mathcal{A}^i}(k) = \bigcap\{A \mid k \in A \in \mathcal{A}^i\} = A^i_{n(k,i)}$. Now $U(k) = A^1_{n(k,i)} \cap \ldots \cap A^m_{n(k,m)}$ and $U(j) = A^1_{n(j,i)} \cap \ldots \cap A^m_{n(j,m)}$, and since $A^i_{n(k,i)} \subseteq A^i_{n(j,i)}$, $U(k) \subseteq U(j)$. However since $U \subseteq U_{\mathcal{D}}$, $U(k) = \{k\}$ and $U(j) = \{j\}$, contradicting $j \neq k$.

1.2.17 **Proposition**

$W\mathbf{N} < \langle\{\phi_t\mathbf{N}, D_q\}\rangle_{T_1}\ \mathbf{N}$.

Proof. Let $G := \langle\{\phi_t\mathbf{N}, D_q\}\rangle_{T_1}$. Since $\phi_t\mathbf{N}$ belongs to the spanning class of G, $\phi_t\mathbf{N} \leq G(T_1\phi_t\mathbf{N}) = G\mathbf{N}$. From Proposition 1.2.16 it then follows that $W\mathbf{N} < G\mathbf{N}$.

The full strength of the following Lemmas will be needed only later but we state them in their general form here.

1.2.18 **Lemma**

*Let $\mathcal{A}, \mathcal{B}$ be interior preserving open covers on $X \in$ **Top**. If $U_{\mathcal{A}} \subseteq U_{\mathcal{B}}$, then for each $B \in \mathcal{B}$, $B = \bigcup\{\bigcap \mathcal{A}_x \mid x \in B\}$, where $\mathcal{A}_x := \{A \mid x \in A \in \mathcal{A}\}$.*

Proof. For each $x \in B$, since $U_{\mathcal{A}} \subseteq U_{\mathcal{B}}$, we have $U_{\mathcal{A}}(x) \subseteq U_{\mathcal{B}}(x)$, i.e. $\bigcap \mathcal{A}_x \subseteq \bigcap \mathcal{B}_x$, and so $\bigcup\{\bigcap \mathcal{A}_x \mid x \in B\} \subseteq B$. Equality then follows since $x \in \bigcap \mathcal{A}_x$.

1.2.19 **Lemma**

*Let $\mathcal{A}, \mathcal{B}$ be interior preserving open covers on $X \in$ **Top**. If $U_{\mathcal{A}} \subseteq U_{\mathcal{B}}$, then $|\mathcal{B}| \leq 2^{2^{|\mathcal{A}|}}$.*

Proof. The mapping given by $B \mapsto \{\mathcal{A}_x \mid x \in B\}$ from $\mathcal{B}$ to $\mathcal{P}(\mathcal{P}(\mathcal{A}))$ is injective by Lemma 1.2.18.

1.2.20 **Proposition**

Let F be the section spanned by $\{W\mathbf{N}, D_q\}$. Then $\breve{F} < F$.

Proof. First we need to show that $F > C_1^*$. Since Proposition 1.2.16 gives $W\mathbf{N} = F\mathbf{N}$ we need only show that $W\mathbf{N} > C_1^*\mathbf{N}$. If this does not hold, then $W\mathbf{N} = C_1^*\mathbf{N}$. Let $\mathcal{A} = \{A_n \mid n \in \mathbf{N}\}$ where $A_n = \{1, 2, \ldots, n\}$. Now $U_{\mathcal{A}} \in \operatorname{ent} W\mathbf{N}$ so we can find finite open covers $\mathcal{D}_1, \ldots, \mathcal{D}_n$ with $U_{\mathcal{D}_1} \cap \ldots \cap U_{\mathcal{D}_n} = U_{\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n} \subseteq U_{\mathcal{A}}$. Since $\mathcal{D}_1 \wedge \ldots \wedge \mathcal{D}_n = \mathcal{D}$ is again a finite open cover we have, by Lemma 1.2.19, that $|\mathcal{A}| < 2^{2^{|\mathcal{D}|}}$, i.e. $\mathcal{A}$ is finite. This contradicts our choice of $\mathcal{A}$ and hence $W\mathbf{N} > C_1^*\mathbf{N}$. Now suppose that $\breve{F} = F$. Since $F = \breve{F} = \langle\{FX \mid X \in \mathbf{Top} \text{ and } \phi_t X = FX\}\rangle_{T_1}$, and since $F > C_1^*$, not every FX in the spanning class of $\breve{F}$ is in the range of C_1^*, and therefore there is some $Y \in \mathbf{Top}$ with $FY = \phi_t Y > C_1^* Y$. From the definition of $F = \langle\{W\mathbf{N}, D_q\}\rangle_{T_1}$ it is clear that if every continuous function from Y to $\mathbf{N}$ were bounded, then one would have $FY = C_1^* Y$. Since the latter is not the case, we have at least one unbounded continuous $f\colon Y \to \mathbf{N}$. Now $f[Y]$ is homeomorphic to $\mathbf{N}$, so we can arrange for f to be surjective. Consider the open cover $\mathcal{D} = \{\{n\} \mid n \in \mathbf{N}\}$ of $\mathbf{N}$. Since f is continuous, $f^{-1}\mathcal{D} = \{f^{-1}\{n\} \mid n \in \mathbf{N}\}$ is an interior-preserving open cover of Y. In Proposition 1.2.16 we saw that $\phi_t\mathbf{N}$ is discrete. Now $U_{\mathcal{D}} \in \operatorname{ent} \phi_t\mathbf{N}$, and since $f\colon Y \to \mathbf{N}$ is continuous, we have $\phi_t f\colon \phi_t Y \to \phi_t\mathbf{N}$ is quasi-uniformly continuous, and so $(f \times f)^{-1} U_{\mathcal{D}} = U_{f^{-1}\mathcal{D}} \in \operatorname{ent} \phi_t Y = \operatorname{ent} FY$. Being surjective, $f\colon Y \to \mathbf{N}$ has at least one (continuous) section $g\colon \mathbf{N} \to Y$ so $(g \times g)^{-1} U_{f^{-1}\mathcal{D}} = U_{g^{-1}f^{-1}\mathcal{D}} \in \operatorname{ent} F\mathbf{N}$. Since $g^{-1}f^{-1}\mathcal{D} = \mathcal{D}$, we have $U_{\mathcal{D}} \in \operatorname{ent} F\mathbf{N}$. This makes $F\mathbf{N}$ the discrete quasi-uniformity, hence $F\mathbf{N} = \phi_t\mathbf{N}$. This contradicts Proposition 1.2.16, so $\breve{F} < F$.

1.2.21 **Lemma**

Let F be a transitive T_1-section. Then $F = \bar{F} \implies F = \breve{F}$.

Proof. If $F = \bar{F}$ then F is spanned by

$$\{\phi_t X_{\mathcal{A}} \mid X \in \mathbf{Top}, \mathcal{A} \in \mathcal{I}(X), U_{\mathcal{A}} \in \operatorname{ent} FX\},$$

which class is contained in $\{FX \mid X \in \mathbf{Top}$ and $FX = \phi_t X\}$. The latter class spans $\check{F}$, so that $F \leq \check{F}$ and hence $F = \check{F}$.

1.2.22 **Proposition**

Let F be the section spanned by $\{W\mathbf{N}, D_q\}$. Then $F < \bar{F}$.

Proof. Lemma 1.2.21 and Proposition 1.2.20.

1.3 The point-finite functor

We can apply the techniques used to find the generating class of ϕ_t to other well-known transitive T_1-sections. Given a set B, we will use $\mathcal{P}(B)$ to denote the partially ordered space on the power set of B with the usual inclusion as order as well as the corresponding Alexandroff-discrete space.

1.3.1 **Definition**

For each topological space X and each $\mathcal{A} \in \mathcal{I}(X)$ let $\mathcal{F}(\mathcal{A})$ be the subspace of $\mathcal{P}(\mathcal{A})$ which consists of all finite subsets of $\mathcal{A}$.

1.3.2 **Lemma**

Let $\mathcal{X}$ be either of $\mathcal{P}(\mathcal{A})$ or $\mathcal{F}(\mathcal{A})$. For $A \in \mathcal{A}$ let $\mathcal{O}_A := \{\mathcal{K} \mid A \in \mathcal{K} \in \mathcal{X}\}$. Then

(i) $\{\mathcal{O}_A \mid A \in \mathcal{A}\} \cup \{\mathcal{X}\}$ is an interior-preserving open cover of $\mathcal{X}$;

(ii) for $\Psi := \{\mathcal{O}_A \mid A \in \mathcal{A}\} \cup \{\mathcal{X}\}$, $(\mathcal{K}_1, \mathcal{K}_2) \in U_\Psi \Rightarrow \mathcal{K}_1 \subseteq \mathcal{K}_2$;

(iii) if Φ *is any other open cover of* $\mathcal{X}$*,* $U_\Psi \subseteq U_\Phi$.

Proof. (i) We need only show that $\mathcal{O}_A$ is open in $\mathcal{X}$ since $\mathcal{X}$ is an Alexandroff-discrete space. Note that $A \in \mathcal{K}$ means $\{A\} \subseteq \mathcal{K}$ and also that if $A \in \mathcal{A}$ then $\{A\} \subseteq \mathcal{A}$ which gives $\{A\} \in \mathcal{P}(\mathcal{A})$. Hence $\mathcal{O}_A$ is $\uparrow(\{A\})$, (where the order is taken as the inclusion order on $\mathcal{X}$, i.e. taking the upset in $\mathcal{P}(\mathcal{A})$ and then restricting to $\mathcal{X}$). Thus $\mathcal{O}_A$ is open in $\mathcal{X}$.
(ii) Suppose that $(\mathcal{K}_1, \mathcal{K}_2) \in U_\Psi$ and that $A \in \mathcal{K}_1$. Note that $A \in \mathcal{K}_1 \Rightarrow \mathcal{K}_1 \in \mathcal{O}_A$. Now $\mathcal{K}_2 \in U_\Psi(\mathcal{K}_1) = \bigcap\{\mathcal{M} \mid \mathcal{K}_1 \in \mathcal{M} \in \Psi\}$. Thus $\mathcal{K}_1 \in \mathcal{O}_A \in \Psi$ gives $\mathcal{K}_2 \in \mathcal{O}_A$, and hence $A \in \mathcal{K}_2$.
(iii) Let $(\mathcal{K}_1, \mathcal{K}_2) \in U_\Psi$. Consider any $\mathcal{O}$ with $\mathcal{K}_1 \in \mathcal{O} \in \Phi$. Since $\mathcal{O}$ is an upper set and $\mathcal{K}_1 \subseteq \mathcal{K}_2$, we have $\mathcal{K}_2 \in \mathcal{O}$. We then have $\mathcal{K}_2 \in \bigcap\{\mathcal{O} \mid \mathcal{K}_1 \in \mathcal{O} \in \Phi\} = U_\Phi(\mathcal{K}_1)$, i.e. $(\mathcal{K}_1, \mathcal{K}_2) \in U_\Phi$.

1.3.3 **Proposition**

Q_{pf} *is generated by the class* $\Lambda := \{\mathcal{F}(\mathcal{A}) \mid X \in \textbf{Top}, \mathcal{A} \in \mathcal{I}(X)\}$.

Proof. Let F be the functor which is generated by Λ. Let Y be a topological space and let $\mathcal{A}$ be a point-finite open cover of Y. Define $f\colon Y \to \mathcal{F}(\mathcal{A})$ by $f(x) = \mathcal{A}^x := \{A \mid x \in A \in \mathcal{A}\}$. Now f is well defined since $\mathcal{A}^x$ is finite for each x. For each $A \in \mathcal{A}$ it is clear that $f^{-1}\mathcal{O}_A = A$, and therefore (in the notation of Lemma 1.3.2) $f^{-1}\{\mathcal{O}_A \mid A \in \mathcal{A}\} = \mathcal{A}$. Thus every point-finite open cover of Y is the preimage under a continuous map of some open cover of a space in Λ. Hence $Q_{pf} \leq F$.
Now let $\mathcal{F}(\mathcal{A}) \in \Lambda$, $Y \in \textbf{Top}$ and let $g\colon Y \to \mathcal{F}(\mathcal{A})$ be a continuous map. Since Ψ, as defined in Lemma 1.3.2, is a point-finite open cover of $\mathcal{F}(\mathcal{A})$, $g^{-1}\Psi$ is a point-finite open cover of Y. Now let Φ be any other (interior-preserving) open cover of $\mathcal{F}(\mathcal{A})$. By lemma 1.3.2(iii) $U_\Psi \subseteq U_\Phi$ and hence $U_{g^{-1}\Psi} \subseteq U_{g^{-1}\Phi}$. Thus $F \leq Q_{pf}$.

1.4 Generating sections via ordinals

Let α be an ordinal. There are two natural topologies on α that we will consider and which we will refer to as the upper and lower topologies.

1.4.1 Definition

(i) α_u is the topological space (α, τ_u) where $O \in \tau_u$ if and only if O is an upper set;

(ii) α_l is the topological space (α, τ_l) where $O \in \tau_l$ if and only if O is a lower set.

Let **Ord** be the category of all ordinals, $\mathbf{Ord}_u$ the category of all ordinals with the upper topology and $\mathbf{Ord}_l$ the category of all ordinals with the lower topology. $\mathbf{Ord}_u$ and $\mathbf{Ord}_l$ are subcategories of **Alex**.

1.4.2 Proposition

(i) $\mathcal{C}_1$ is generated by $\mathbf{Z}_l$ (the integers with lower sets as open sets).

(ii) D is generated by ω_l.

(iii) B is generated by ω_u.

Proof. (*i*) Let $G := \langle\{\phi_t \mathbf{Z}_l\}\rangle_{T_1}$. We claim that $\mathcal{C}_1 = G$. Let $\mathcal{B}$ be any interior-preserving open cover of $\mathbf{Z}_l$. If $\mathcal{B}$ cannot yet be indexed as an open spectrum on $\mathbf{Z}_l$, then it can be by adding $\emptyset$ to it to produce the open spectrum $\mathcal{B}^*$. Note that $U_{\mathcal{B}} = U_{\mathcal{B}^*}$. For $X \in \mathbf{Top}$, we have a subbase of ent GX consisting of sets of the form $(f \times f)^{-1} U_{\mathcal{B}} = U_{f^{-1}\mathcal{B}} = U_{f^{-1}\mathcal{B}^*}$, with continuous $f: X \to \mathbf{Z}_l$ and $\mathcal{B}$ as above. Since $f^{-1}\mathcal{B}^*$ is an open spectrum on X, we have shown that $GX \leq \mathcal{C}_1 X$.

Now consider any subbasic member $U_{\mathcal{A}}$ of $\operatorname{ent} \mathcal{C}_1 X$, where $\mathcal{A} = \{A_n \mid n \in \mathbf{Z}\}$ is an open spectrum on X. Construct $f: X \to \mathbf{Z}_l$ by $f(x) = n$ if $x \in A_n \backslash A_{n-1}$. Then f is continuous since $f^{-1}(-\infty, m] = A_m$. Letting $\mathcal{D} = \{(-\infty, n] \mid n \in \mathbf{Z}\}$, we have $f^{-1}\mathcal{D} = \mathcal{A}$. Now $U_{\mathcal{A}} = U_{f^{-1}\mathcal{D}} = (f \times f)^{-1} U_{\mathcal{D}} \in \operatorname{ent} GX$, and since sets of the form $U_{\mathcal{A}}$ form a subbase for $\operatorname{ent} \mathcal{C}_1 X$, we see that $\mathcal{C}_1 X \leq GX$. Hence $\mathcal{C}_1 = G$.
(*ii*) and (*iii*) follow easily from (*i*).

1.4.3 **Corollary**

$\mathcal{C}_1 \mathbf{Z}_l = \phi_t \mathbf{Z}_l$.

1.4.4 **Proposition**

$\mathcal{C}_1$ is generated by $\{\omega_u, \omega_l\}$.

Proof. Let F be the T_1-section generated by $\{\omega_u, \omega_l\}$. Since $\phi_t \omega_l$ and $\phi_t \omega_u$ have bases consisting of the single entourage $U_{\mathcal{O}\omega_l}$ and $U_{\mathcal{O}\omega_u}$ respectively, and both $\mathcal{O}\omega_l$ and $\mathcal{O}\omega_u$ are open spectra, it follows that $\phi_t \omega_l = \mathcal{C}_1 \omega_l$ and $\phi_t \omega_u = \mathcal{C}_1 \omega_u$. Since F is spanned by $\{\phi_t \omega_l, \phi_t \omega_u\} = \{\mathcal{C}_1 \omega_l, \mathcal{C}_1 \omega_u\}$ it follows that $F \leq \mathcal{C}_1$.
Now consider an open spectrum on $\mathcal{A}$ on any $X \in \mathbf{Top}$. Let $\mathcal{A}' = \{A_n \mid A_n \in \mathcal{A} \text{ and } n \leq 0\} \cup \{X\}$ and let $\mathcal{A}'' = \{A_n \mid A_n \in \mathcal{A} \text{ and } n \geq 0\} \cup \{\emptyset\}$. We proceed as in the proof of Proposition 1.4.2. Consider $\mathcal{B} = \{[n, \infty) \mid n \in \omega\}$ which is an open base and an open spectrum on ω_u. For $h: X \to \omega_u$, $h^{-1}\mathcal{B}$ is then an open spectrum on X. Define $f: X \to \omega_u$ by $f(x) = n$ if $x \in A_{-n} \setminus A_{-(n-1)}$; this f is continuous since $f^{-1}[n, \infty) = A_{-n}; n \in \omega$. Moreover $f^{-1}\mathcal{B} \cup \{X\} = \mathcal{A}'$ so that $U_{\mathcal{A}'} \in \operatorname{ent} FX$. Now consider the open base and open spectrum $\mathcal{D} = \{[0, n) \mid n \in \omega\} \cup \{\emptyset\}$ on ω_l. For any $k: X \to \omega_l$, $k^{-1}\mathcal{D}$ is an open spectrum on X. Define $g: X \to \omega_l$ by $g(x) = n$ if $x \in A_n \setminus A_{n-1}$. This g is continuous since $g^{-1}[0, n) = A_n$. Moreover $g^{-1}\mathcal{D} = \mathcal{A}''$ so

that $U_{\mathcal{A}''} \in \operatorname{ent} FX$. Now $U_{\mathcal{A}'} \cap U_{\mathcal{A}''} \subseteq U_{\mathcal{A}}$, so $U_{\mathcal{A}} \in \operatorname{ent} FX$, and thus $\mathcal{C}_1 X \leq FX$. Thus $F = \mathcal{C}_1$.

1.4.5 **Proposition**

W, the well-monotone functor, is generated by $\mathbf{Ord}_l$.

Proof. Let F be the functor generated by $\mathbf{Ord}_l$. Clearly $F \leq W$: Let $\alpha_l \in \mathbf{Ord}_l$. Then $\mathcal{O}\alpha_l$ is a well-monotone open cover (since α is a well-ordered set) and thus $f^{-1}\mathcal{O}\alpha_l$ is a well-monotone open cover on X for any continuous $f\colon X \to \alpha_l$ and $X \in \mathbf{Top}$.

Conversely, let $X \in \mathbf{Top}$ and let $\mathcal{M}$ be any well-monotone open cover of X. Then there is an indexing $\mathcal{M} = \{M_\alpha \mid \alpha < \tau\}$ with τ an ordinal and $M_\alpha \subseteq M_\beta$ whenever $\alpha \leq \beta$. Define $f\colon X \to \tau$ by letting $f(\alpha)$ be the least ordinal α for which $x \in M_\alpha$. Clearly then $f^{-1}(\downarrow \beta) = M_\beta$ for every $\beta \in \tau$. Thus $f^{-1}\{\downarrow \alpha \mid \alpha \in \tau\} = \mathcal{M}$, and therefore $WX \leq FX$. Thus $W = F$.

1.4.6 **Remark**

1. No set-indexed subcollection of $\mathbf{Ord}_l$ can generate W. In fact for each cardinal κ we have a T_1-section $W_\kappa < W$, where W_κ is induced by all well-monotone open covers of cardinality $\leq \kappa$. The proof of this fact is a simple application of Lemma 1.2.19.

2. The above proof can be generalised to show that W_κ is generated by $\{\alpha_l \mid \alpha \in \mathbf{Ord}$ and $|\alpha| \leq \kappa\}$.

3. Every subcollection of $\mathbf{Ord}_l \cup \mathbf{Ord}_u$ generates a transitive T_1-section, provided that the collection contains at least one space of more than one point. In particular we have the functor W_u which is generated by $\mathbf{Ord}_u$.

4. We have shown that ϕ_t is generated by **Alex** and also by **Pow**. It was remarked earlier that this is equivalent to the statement that ϕ_t is spanned by $\{\phi_t X \mid X \in \mathbf{Alex}\}$ or $\{\phi_t X \mid X \in \mathbf{Pow}\}$. The nature of these categories makes it easy to describe the action of ϕ_t on these spaces. Let $X \in \mathbf{Alex}$. Then $\mathcal{O}X$ is an interior preserving open cover with $U_{\mathcal{O}X} \subseteq U_{\mathcal{A}}$ for any other interior preserving open cover $\mathcal{A}$ of X. Thus the fine transitive quasi-uniformity on X has a base consisting of a single entourage, namely $U_{\mathcal{O}X}$, from which it easily follows that $\phi_t X = \phi X$, i.e. X is a transitive topological space, cf. [Fletcher and Lindgren 1982]. Hence all the functors which we have discussed above can be generated by a class of transitive topological spaces.

1.4.7 **Definition** [Brümmer, 1976, p.128]

A (transitive) T_1-section is *simple* if it is spanned by a set of quasi-uniform spaces.

1.4.8 **Remark**

Brümmer [1976] showed that the fine functor ϕ is not simple and by a similar proof Halpin's thesis [1974] shows that ϕ_t is not simple.

1.4.9 **Proposition**

*Let F be a simple transitive T_1-section. Then sup$\{\ |\mathcal{A}|\ \ |U_{\mathcal{A}} \in \mathrm{ent}\, FX, \mathcal{A} \in \mathcal{I}(X)$ and $X \in$ **Top**$\}$ exists.*

Proof. Since F is spanned by a set of quasi-uniform spaces it is also spanned by their product, and then we have a single topological space S such that $F = \langle\{FS\}\rangle_{T_1}$. Now let $\kappa = \sup\{\ |\mathcal{A}|\ \ |\mathcal{A} \in \mathcal{I}(S)$ and $U_{\mathcal{A}} \in \mathrm{ent}\, FS\}$. Consider any $X \in$ **Top** and let $\mathcal{B}$ be an interior preserving open cover on X with $U_{\mathcal{B}} \in \mathrm{ent}\, FX$.

Since $F = \langle\{FS\}\rangle_{T_1}$ and since $\{U_{\mathcal{A}} \mid \mathcal{A} \in \mathcal{I}(S) \text{ and } U_{\mathcal{A}} \in \operatorname{ent} FS\}$ is a base for $\operatorname{ent} FS$ [Brümmer 1984, Sch. 2.8], there exist finitely many continuous maps $f_i\colon X \to S$ and $\mathcal{A}_i \in \mathcal{I}(S)$ with $U_{\mathcal{A}_i} \in \operatorname{ent} FS$ $(i = 1, \ldots, n)$ such that

$$\bigcap_{i=1}^{n} (f_i \times f_i)^{-1} U_{\mathcal{A}_i} \subseteq U_{\mathcal{B}}$$

and then by [Brümmer 1984, Lemma 2.5]

$$U_{\bigwedge_{i=1}^{n} f_i^{-1}\mathcal{A}_i} \subseteq U_{\mathcal{B}}.$$

We note that $|f_i^{-1}\mathcal{A}_i| = |\mathcal{A}_i| \leq \kappa$ and then $|\bigwedge_{i=1}^{n} f_i^{-1}\mathcal{A}_i| \leq \aleph_0\kappa$. Then by Lemma 1.2.19 $|\mathcal{B}| \leq 2^{2^{\aleph_0\kappa}}$.

1.4.10 Corollary

Each of the T_1-sections $\phi_t, Q_{pf}, Q_{lf}, W$ is non-simple.

Proof. If F is any one of $\phi_t, Q_{pf}, Q_{lf}, W$ and κ is any cardinal number then one can find a topological space X and an interior-preserving open cover $\mathcal{A}$ of X with $U_{\mathcal{A}} \in \operatorname{ent} FX$ and $|\mathcal{A}| > \kappa$ as follows:

(i) If $F \in \{\phi_t, Q_{lf}, Q_{pf}\}$: Take X discrete with $|X| = 2^{\kappa}$ and let $\mathcal{A}$ be the cover of X by singletons;

(ii) If $F = W$: Let $X = \alpha_l$ where α is the initial ordinal of cardinality 2^{κ} and let $\mathcal{A} = \mathcal{O}\alpha_l$.

1.4.11 Remark

Corollary 1.4.10 partly answers the question in [Brümmer 1977, p. 81].

We now prove a somewhat surprising lemma about the topology of power spaces. It is needed for the proof of the one of the main results of this section.

1.4.12 **Lemma**

Let X be a power space and a an atom of X. Let $\{A_i \mid i \in I\}$ be a collection of open sets in X, with $a \in A_i$ for each $i \in I$. Then $\bigcap_I A_i = \uparrow a$ if and only if $A_i = \uparrow a$ for some $i \in I$.

Proof. $\Rightarrow$: Suppose that for each $i \in I$, $A_i \neq \uparrow a$. Then, for every $i \in I$, $\exists y_i \in A_i \backslash \uparrow a$. Now $\bigvee_I y_i \notin \uparrow a$ but $\bigvee_i y_i \in \bigcap_I A_i$, since each A_i is an upper set. Hence $\bigcap_I A_i \neq \uparrow a$.

$\Leftarrow$: Immediate.

For a power space X and a T_1-section F, $FX = \phi_t X$ if and only if $U_{\mathcal{O}X} \in \operatorname{ent} FX$. We note that X is homeomorphic to $\mathcal{P}(\alpha)$ where $\alpha \in \mathbf{Ord}$ and $|\alpha|=|\mathrm{At}X|$, $\mathrm{At}X$ meaning the set of atoms of X.

1.4.13 **Lemma**

Let F be a T_1-section with $F \not\geq \phi_t$. Then there exists a cardinal κ such that $FX \neq \phi_t X$ whenever $|X| \geq 2^\kappa$ and $X \in$ **Pow**.

Proof. Suppose that $F\mathcal{P}(\alpha) = \phi_t\mathcal{P}(\alpha)$ for each ordinal α. By Proposition 1.2.10 $\phi_t = \langle\{\phi_t\mathcal{P}(\alpha) \mid \alpha \in \mathbf{Ord}\}\rangle_{T_1}$ and since $F = \langle\{FX \mid X \in \mathbf{Top}\}\rangle_{T_1}$, the spanning class for ϕ_t is contained in the spanning class of F. Thus $\phi_t \leq F$, contradicting our assumption that $F \not\geq \phi_t$. Hence we can find an ordinal α such that $F\mathcal{P}(\alpha) \neq \phi_t\mathcal{P}(\alpha)$. Let α be the first ordinal with this property. Consider any ordinal μ with $|\mu| \geq |\alpha|$ and $F\mathcal{P}(\mu) = \phi_t\mathcal{P}(\mu)$. Then $U_{\mathcal{O}\mathcal{P}(\mu)} \in \operatorname{ent} F\mathcal{P}(\mu)$. Since $|\alpha| \leq |\mu|$, there is a natural embedding $e\colon \mathcal{P}(\alpha) \longrightarrow \mathcal{P}(\mu)$ and so $e^{-1}\mathcal{O}\mathcal{P}(\mu) = \mathcal{O}\mathcal{P}(\alpha)$. Since $Fe\colon F\mathcal{P}(\alpha) \rightarrow F\mathcal{P}(\mu)$ is quasi-uniformly continuous, $(e \times e)^{-1}U_{\mathcal{O}\mathcal{P}(\mu)} \in \operatorname{ent} F\mathcal{P}(\alpha)$ and thus $U_{\mathcal{O}\mathcal{P}(\alpha)} = U_{e^{-1}\mathcal{O}\mathcal{P}(\mu)} \in \operatorname{ent} F\mathcal{P}(\alpha)$. Now $U_{\mathcal{O}\mathcal{P}(\alpha)} \in \operatorname{ent} F\mathcal{P}(\alpha)$ implies

$F\mathcal{P}(\alpha) = \phi_t\mathcal{P}(\alpha)$, a contradiction. The above argument shows that whenever $|\mu| \geq |\alpha|$, $F\mathcal{P}(\mu) \neq \phi_t\mathcal{P}(\mu)$. Let $\kappa = |\alpha|$. Then for every $X \in \mathbf{Pow}$, $|X| \geq 2^\kappa \Rightarrow FX \neq \phi_t X$.

Let X be an infinite power space and let $\mathcal{A} \in \mathcal{I}(X)$. We will denote the set $\{a \in \mathrm{At}(X) \mid U_\mathcal{A}(a) = \uparrow a\}$ by $\mathrm{At}_\mathcal{A}(X)$. By Lemma 1.4.12 $\mathrm{At}_\mathcal{A}(X) = \{a \in \mathrm{At}(X) \mid \uparrow a \in \mathcal{A}\}$.

1.4.14 **Lemma**

Let X be a an infinite power space and let $\mathcal{A}_1, \ldots, \mathcal{A}_n \in \mathcal{I}(X)$ be such that $U_{\mathcal{A}_i} \neq U_{\mathcal{O}X}$ for each $i \in \{1, \ldots, n\}$. Then $U_{\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_n} \neq U_{\mathcal{O}X}$ if $|At_{\mathcal{A}_i}(X)| < |At(X)|$ for each $i \in \{1, \ldots, n\}$.

Proof. Let $\mathcal{A} = \mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_n$. Observe that, for $\mathcal{A} \in \mathcal{I}(X)$, $U_\mathcal{A} = U_{\mathcal{O}X}$ implies $U_\mathcal{A}(a) = U_{\mathcal{O}X}(a) = \uparrow a$ for every $a \in \mathrm{At}(X)$, so that $\mathrm{At}_\mathcal{A}(X) = At(X)$. Note that $U_\mathcal{A}(a) = \bigcap_{i=1}^n U_{\mathcal{A}_i}(a)$. If $U_\mathcal{A}(a) = \uparrow a$, then $\bigcap_{i=1}^n U_{\mathcal{A}_i}(a) = \uparrow a$, and then by Lemma 1.4.12, $U_{\mathcal{A}_i}(a) = \uparrow a$ for some $i = 1, \ldots, n$. Thus $a \in \mathrm{At}_\mathcal{A}(X)$ if and only if $a \in \mathrm{At}_{\mathcal{A}_i}(X)$ for some $1 \leq i \leq n$, and so

$$\mathrm{At}_\mathcal{A}(X) = \bigcup_{1 \leq i \leq n} \mathrm{At}_{\mathcal{A}_i}(X)$$

so that $|\mathrm{At}_\mathcal{A}(X)| = \max_{1 \leq i \leq n} |\mathrm{At}_{\mathcal{A}_i}(X)| < |\mathrm{At}(X)|$. Hence $U_\mathcal{A} \neq U_{\mathcal{O}X}$.

1.4.15 **Lemma**

Let $F \neq \phi_t$ be a transitive T_1-section and let $\mathcal{P}(\alpha)$ be a power space with $F\mathcal{P}(\alpha) \neq \phi_t\mathcal{P}(\alpha)$. Then $|At_\mathcal{A}(\mathcal{P}(\alpha))| < |\alpha|$ whenever $U_\mathcal{A} \in ent\, F\mathcal{P}(\alpha)$.

Proof. Let $U_\mathcal{A} \in \mathrm{ent}\, F\mathcal{P}(\alpha)$ and let $Y = \{\beta \in \alpha \mid \{\beta\} \in \mathrm{At}_\mathcal{A}(\mathcal{P}(\alpha))\} = \{\beta \in \alpha \mid \uparrow\{\beta\} \in \mathcal{A}\}$. Since $Y \subseteq \alpha$ we have a natural embedding $i : \mathcal{P}(Y) \to \mathcal{P}(\alpha)$. Let $\mathcal{D} = i^{-1}\mathcal{A}$. For each $y \in Y$, $i(\{y\}) \in \mathrm{At}_\mathcal{A}\mathcal{P}(\alpha)$, i.e. $\uparrow i(\{y\}) \in \mathcal{A}$, and thus

$i^{-1}\uparrow(i\{y\}) \in \mathcal{D}$. Also note that

$$\begin{aligned} X \in i^{-1}\uparrow i(\{y\}) &\Leftrightarrow i(X) \in \uparrow i(\{y\}) \\ &\Leftrightarrow i(\{y\}) \subseteq i(X) \\ &\Leftrightarrow \{y\} \subseteq X \\ &\Leftrightarrow X \in \uparrow\{y\} \end{aligned}$$

and hence, for all $y \in Y$: $\uparrow\{y\} = i^{-1}\uparrow i(\{y\}) \in \mathcal{D}$. Thus $\mathcal{B} := \{\uparrow\{y\} \mid y \in Y\} \subseteq \mathcal{D}$, and since $\mathcal{B}$ is a base for $\mathcal{OP}(Y)$ we have $U_{\mathcal{B}} = U_{\mathcal{OP}(Y)}$ and hence $U_{\mathcal{D}} \subseteq U_{\mathcal{OP}(Y)}$. Now $U_{\mathcal{D}} = U_{i^{-1}\mathcal{A}} = (i \times i)^{-1}U_{\mathcal{A}} \in \operatorname{ent} F\mathcal{P}(Y)$, so $U_{\mathcal{OP}(Y)} \in \operatorname{ent} F\mathcal{P}(Y)$, and so $F\mathcal{P}(Y) = \phi_t\mathcal{P}(Y)$. If $|Y| = |\alpha|$ then $\mathcal{P}(Y)$ is homeomorphic to $\mathcal{P}(\alpha)$ and then $F\mathcal{P}(\alpha) = \phi_t\mathcal{P}(\alpha)$, contradicting our data. Thus $|Y| \neq |\alpha|$. Since $Y \subseteq \alpha$ we must have $|Y| < |\alpha|$.

1.4.16 **Theorem**

Let $\{F_i\}_I$ be a set of transitive T_1-sections with the property that $F_i \neq \phi_t$ for each $i \in I$. Then

$$F := \bigvee_I F_i \neq \phi_t.$$

Proof. By Lemma 1.4.13 we can find a cardinal κ_i for each $i \in I$ such that $F_iX \neq \phi_tX$ whenever X is a power space and $|X| \geq 2^{\kappa_i}$. Now consider the ordinal $\alpha = \sup\{\kappa_i \mid i \in I\}$ and the quasi-uniform space $F\mathcal{P}(\alpha)$. For each $i \in I$, $F_i\mathcal{P}(\alpha) < \phi_t\mathcal{P}(\alpha)$. Now $F\mathcal{P}(\alpha)$ has quasi-uniform subbasis

$$\mathcal{S} = \{U_{\mathcal{A}} \mid \mathcal{A} \in \mathcal{I}(\mathcal{P}(\alpha)) \text{ and } U_{\mathcal{A}} \in \operatorname{ent} F_i\mathcal{P}(\alpha) \text{ for some } i \in I\}.$$

Let $U_{\mathcal{A}} := \mathcal{U}_{\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_n}$ be a basic entourage of $F\mathcal{P}(\alpha)$, i.e. $U_{\mathcal{A}_j} \in \mathcal{S}$ for $j = 1, \ldots, n$. If $F\mathcal{P}(\alpha) = \phi_t\mathcal{P}(\alpha)$, then there exists a basic entourage $U_{\mathcal{A}}$ of $F\mathcal{P}(\alpha)$ such that

$U_{\mathcal{A}} \subseteq U_{\mathcal{OP}(\alpha)}$ (since $U_{\mathcal{OP}(\alpha)} \in \text{ent}\, \phi_t \mathcal{P}(\alpha)$). Since the reverse inclusion always holds, we have $U_{\mathcal{A}} = U_{\mathcal{OP}(\alpha)}$. Now by Lemma 1.4.15, for each j,

$$|\text{At}_{\mathcal{A}_j}(\mathcal{P}(\alpha))| < |\text{At}(\mathcal{P}(\alpha))| = |\alpha|$$

and thus by Lemma 1.4.14 $U_{\mathcal{A}} \neq U_{\mathcal{OP}(\alpha)}$. Hence $F\mathcal{P}(\alpha) \neq \phi_t \mathcal{P}(\alpha)$, and thus $F \neq \phi_t$.

Chapter 2

Bicompletions of transitive quasi-uniformities

2.1 Introduction

A quasi-uniform space $(X,\mathcal{U})$ is *bicomplete* if every minimal $\mathcal{U}^*$-Cauchy filter converges in the topology $\tau(\mathcal{U}^*)$. We recall that $\mathcal{F}$ is a $\mathcal{U}^*$-Cauchy filter iff for each $U \in \mathcal{U}$ there exists an $x \in X$ such that $U(x) \cap U^{-1}(x) \in \mathcal{F}$ [Fletcher and Lindgren 1982]. The bicompletion of a quasi-uniform space $(X,\mathcal{U})$ is a space $(\tilde{X},\tilde{\mathcal{U}})$, where $\tilde{X}$ consists of the minimal $\mathcal{U}^*$-Cauchy filters and $\tilde{\mathcal{U}}$ has basis $\{\tilde{U} \mid U \in \mathcal{U}\}$ where $\tilde{U} := \{(\mathcal{F},\mathcal{G}) \in \tilde{X} \times \tilde{X} \mid \exists F \in \mathcal{F}\ \exists G \in \mathcal{G} \text{ such that } F \times G \subseteq U\}$.
$K\colon \mathbf{Qu} \longrightarrow \mathbf{Qu}$ will denote the functor which assigns to each space its bicompletion. We will denote the unit of this bicompletion by $k_X\colon X \longrightarrow KX$. We note that k_X is an initial mapping and is an embedding if and only if X is a T_0 space. For any quasi-uniform space X, KX is a T_0-quasi-uniform space.

2.2 Minimal $\mathcal{U}^*$-Cauchy filters

In this section we present an alternative characterisation of the bicompletion of a transitive quasi-uniform space by extending some constructions due to Brümmer [1979] and Salbany [1984].
Let X be a topological space. A filter, $\mathcal{F}$, on X is *zero-dimensional* if there exists a collection $\mathcal{M}$ of open sets and a collection $\mathcal{N}$ of closed sets such that $\mathcal{M} \cup \mathcal{N}$ is a

subbase for $\mathcal{F}$. The filter $\mathcal{F}$ is *compressed* [Császár 1963] if for every $O \in \mathcal{O}X, O \in \mathcal{F}$ or $X \backslash O \in \mathcal{F}$.

For a given filter $\mathcal{F}$ on a topological space X we will write $\mathcal{F}_o$ ($\mathcal{F}_c$) for the restriction of $\mathcal{F}$ to the open (closed) sets of X.

2.2.1 **Lemma** [Salbany 1984]

Let $\mathcal{F}$ be a compressed filter. Then

(i) $\mathcal{F}_o$ is a prime open filter;

(ii) $\mathcal{F}_c$ is a prime closed filter;

(iii) $\mathcal{F}_c = \{A \mid X\backslash A \in \mathcal{O}X \;\wedge\; X\backslash A \notin \mathcal{F}_o\}$.

2.2.2 **Proposition**

Let $\mathcal{F}$ be a zero-dimensional filter. Then $\mathcal{F}_o \cup \mathcal{F}_c$ is a subbase for $\mathcal{F}$.

2.2.3 **Lemma** [Brümmer 1979]

Let $\mathcal{U}$ be a quasi-uniformity finer than the Pervin quasi-uniformity on a topological space X. Let $\mathcal{F}$ be a $\mathcal{U}^*$-Cauchy filter. Then $\mathcal{F}$ is a compressed filter.

For the rest of this chapter T_1 will denote the forgetful functor from Qu to Top, F will be used to denote a *transitive* T_1-section and $(\Phi_X)_{X \in \mathbf{Top}}$ will denote an adequate natural kind of interior preserving open covers which induces F.

We shall repeatedly and without explicit notice use the fact that F is finer than the Pervin functor $\mathcal{C}_1^*$ [Brümmer 1969].

Our first aim is to develop a characterisation of the bicompletion, KFX, of FX in terms of prime open filters and interior preserving open covers. This will then enable us to examine the topological spaces which arise from the bicompletion. It will become clear that we can replace prime open filters with compressed zero-dimensional filters or prime closed filters.

2.2.4 **Definition**

An interior-preserving open cover, $\mathcal{A}$, of a topological space X *does not separate* $B \subset X$ if for every $A \in \mathcal{A}$, $A \cap B \neq \emptyset \;\Rightarrow\; B \subseteq A$.

2.2.5 **Lemma**

Let $\mathcal{A}$ be an interior preserving open cover. Then $U^*_{\mathcal{A}} := U_{\mathcal{A}} \cap U^{-1}_{\mathcal{A}}$ consists of all pairs (x,y) such that $\mathcal{A}$ does not separate $\{x,y\}$.

Proof.

$$\begin{aligned}
(x,y) \in U_{\mathcal{A}} \cap U^{-1}_{\mathcal{A}} &\iff (x,y) \in U_{\mathcal{A}} \text{ and } (x,y) \in U^{-1}_{\mathcal{A}} \\
&\iff y \in \mathcal{A}_x \text{ and } x \in \mathcal{A}_y \\
&\iff (\forall A \in \mathcal{A})(x \in A \Leftrightarrow y \in A) \\
&\iff \mathcal{A} \text{ does not separate } \{x,y\}
\end{aligned}$$

2.2.6 **Corollary**

Let $F \subseteq X$ and $\mathcal{A} \in \mathcal{I}(X)$. Then F is not separated by $\mathcal{A}$ iff $F \times F \subseteq U^*_{\mathcal{A}}$.

Proof. Suppose that F is not separated by $\mathcal{A}$ and let $(x,y) \in F \times F$. Clearly $\mathcal{A}$ does not separate $\{x,y\}$ since $\{x,y\} \subseteq F$. Hence $(x,y) \in U^*_{\mathcal{A}}$. Conversely suppose that $F \times F \subseteq U^*_{\mathcal{A}}$ and that $\mathcal{A}$ separates F. Then there exists $A \in \mathcal{A}$ and $x,y \in F$ such that $x \in A$ and $y \notin A$. But $(x,y) \in F \times F \subseteq U^*_{\mathcal{A}}$, so by Lemma 2.2.5 $\mathcal{A}$ does not separate $\{x, y\}$. This contradicts our assumption that $x \in A$ and $y \notin A$. Hence $\mathcal{A}$ does not separate F.

We will write that $\mathcal{G}$ is an $(FX)^*$-Cauchy filter when we wish to indicate that $\mathcal{G}$ is a $\mathcal{U}^*$-Cauchy filter, where $\mathcal{U}$ is the quasi-uniformity of FX.

2.2.7 Proposition

$\mathcal{G}$ is an $(FX)^*$-Cauchy filter iff $(\forall \mathcal{A} \in \Phi_X)$ $(\exists G \in \mathcal{G})$ *such that $\mathcal{A}$ does not separate G.*

Proof. $\Rightarrow$: Let $\mathcal{G}$ be an $(FX)^*$-Cauchy filter. For each $\mathcal{A} \in \Phi_X$ there exists a $G \in \mathcal{G}$ such that $G \times G \subset U_{\mathcal{A}} \cap U_{\mathcal{A}}^{-1}$. By Corollary 2.2.6 above, $\mathcal{A}$ does not separate G.

$\Leftarrow$: Let $\mathcal{A} \in \Phi_X$. If there is a $G \in \mathcal{G}$ such that $\mathcal{A}$ does not separate G then by Corollary 2.2.6 $G \times G \subset U_{\mathcal{A}}$, and $\mathcal{G}$ is an $(FX)^*$-Cauchy filter.

2.2.8 Notation

Given an open cover $\mathcal{A}$ on a topological space X and a filter $\mathcal{G}$ on X, set $\mathcal{A}_{\mathcal{G}} := \mathcal{A} \cap \mathcal{G} (= \{A \mid A \in \mathcal{A} \text{ and } A \in \mathcal{G}\})$ and set $\mathcal{A}_{\mathcal{G}^c} := \{A \mid A \in \mathcal{A} \ \wedge \ A \notin \mathcal{G}\}$.

2.2.9 Definition

Consider a topological space X. A filter $\mathcal{F}$ on X is said to be a Φ_X*-filter* if $\mathcal{F}$ is a prime open filter and $\forall \mathcal{A} \in \Phi_X$:

(i) $\bigcap \mathcal{A}_{\mathcal{F}} \in \mathcal{F}$;

(ii) $\bigcup \mathcal{A}_{\mathcal{F}^c} \notin \mathcal{F}$.

2.2.10 **Remark**

After Proposition 2.2.19 it will become clear that the Φ_X-filters depend just on the space X and the functor F and are independent of the particular natural kind Φ which served to induce F.

2.2.11 **Proposition**

Let $\mathcal{F}$ be an $(FX)^*$-Cauchy filter. Then $\mathcal{F}_o$ is a Φ_X-filter.

Proof. Let $\mathcal{A} \in \Phi_X$. Since $\mathcal{F}$ is an $(FX)^*$-Cauchy filter, we can (by Proposition 2.2.7) find $F \in \mathcal{F}$ such that $\mathcal{A}$ does not separate F. Let $A \in \mathcal{A}_{\mathcal{F}}$. Then $F \cap A \neq \emptyset$, so $F \subseteq A$ by Definition 2.2.4, giving $F \subseteq \bigcap \mathcal{A}_{\mathcal{F}}$. We thus have that $F \subseteq \bigcap \mathcal{A}_{\mathcal{F}}$, so that $\bigcap \mathcal{A}_{\mathcal{F}} \in \mathcal{F}$, and hence $\bigcap \mathcal{A}_{\mathcal{F}} \in \mathcal{F}_o$.
For each $A \in \mathcal{A}_{\mathcal{F}^c}$, $A \cap F = \emptyset$, hence $(\bigcup \mathcal{A}_{\mathcal{F}^c}) \cap F = \emptyset$ and thus $\bigcup \mathcal{A}_{\mathcal{F}^c} \notin \mathcal{F}$, so that $\bigcup \mathcal{A}_{\mathcal{F}^c} \notin \mathcal{F}_o$.
Finally $\mathcal{F}_o$ is a prime open filter by Lemmas 2.2.1 and 2.2.3.

2.2.12 **Remark**

From the above Proposition and Lemma 2.2.3 we know that given an $(FX)^*$-Cauchy filter, $\mathcal{F}$, and $\mathcal{A} \in \Phi_X$, $\bigcap \mathcal{A}_{\mathcal{F}} \setminus \bigcup \mathcal{A}_{\mathcal{F}^c} \in \mathcal{F}$. Denote this set by $[\mathcal{F}\mathcal{A}]$.

2.2.13 **Proposition**

Let $\mathcal{F}$ be an $(FX)^*$-Cauchy filter and let $\mathcal{A} \in \Phi_X$. Then $[\mathcal{F}\mathcal{A}]$ is the largest element of $\mathcal{F}$ which is not separated by $\mathcal{A}$.

Proof. Let F be an element of $\mathcal{F}$ which is not separated by $\mathcal{A}$. Then $F \subseteq \bigcap \mathcal{A}_{\mathcal{F}}$ since $F \cap A \neq \emptyset$ for each $A \in \mathcal{A}_{\mathcal{F}}$. Suppose that $x \in F$ and $x \in \bigcup \mathcal{A}_{\mathcal{F}^c}$. Then there exists an $A \in \mathcal{A}_{\mathcal{F}^c}$ such that $x \in A$. But then $F \subseteq A$ so that $A \in \mathcal{A}_{\mathcal{F}}$, a contradiction. Hence $F \subseteq \bigcap \mathcal{A}_{\mathcal{F}} \backslash \bigcup \mathcal{A}_{\mathcal{F}^c}$.

To see that $[\mathcal{F}\mathcal{A}]$ is not separated by $\mathcal{A}$, suppose that $[\mathcal{F}\mathcal{A}] \not\subseteq A \in \mathcal{A}$. Then there is an $x \in [\mathcal{F}\mathcal{A}]$ such that $x \notin A$. Now since $[\mathcal{F}\mathcal{A}] \subseteq \bigcap \mathcal{A}_{\mathcal{F}}$ it follows that $A \notin \mathcal{F}$. Hence $A \in \mathcal{A}_{\mathcal{F}^c}$, i.e. $A \cap [\mathcal{F}\mathcal{A}] = \emptyset$.

We will now show that on a given topological space, X, there is a bijective correspondence between the set of Φ_X-filters and the set of minimal $(FX)^*$-Cauchy filters.

2.2.14 **Notation**

Given a prime open filter $\mathcal{F}$, let $\mathcal{F}_r = \{F \mid X \backslash F \notin \mathcal{F} \text{ and } X \backslash F \in \mathcal{O}X\}$. It is easy to see that $\mathcal{F}_r$ is a prime closed filter. (This is similar to a proof in [Salbany 1984, p. 496]). We will refer to $\mathcal{F}_r$ as *the complementary filter of* $\mathcal{F}$.

2.2.15 **Lemma**

Let $\mathcal{F}$ be a prime open filter. Then $\mathcal{F} \cup \mathcal{F}_r$ is a subbase for a filter.

Proof. Suppose that $F \cap A = \emptyset$ where $F \in \mathcal{F}$ and $A \in \mathcal{F}_r$. Then $F \subseteq X \backslash A$, so $X \backslash A \in \mathcal{F}$ and hence $A \notin \mathcal{F}_r$, a contradiction. Thus $A \cap F \neq \emptyset$.

2.2.16 **Notation**

We will denote by $\hat{\mathcal{F}}$ the filter for which $\mathcal{F} \cup \mathcal{F}_r$ is a subbase.

2.2.17 Lemma

Let $\mathcal{F}$ be a prime open filter. Then,

(i) $(\hat{\mathcal{F}})_o = \mathcal{F}$;

(ii) $(\hat{\mathcal{F}})_c = \mathcal{F}_r$;

(iii) $(\hat{\mathcal{F}})_o \subseteq \mathcal{F}$.

Proof. (i) Clearly $\mathcal{F} \subseteq (\hat{\mathcal{F}})_o$ so let $O \in (\hat{\mathcal{F}})_o$. Then $O \in \hat{\mathcal{F}}$. Now if $O \notin \mathcal{F}$ then $X \backslash O \in \mathcal{F}_r$ and hence $X \backslash O \in \hat{\mathcal{F}}$, contradicting $O \in \hat{\mathcal{F}}$. Thus $O \in \mathcal{F}$.
(ii),(iii) Similar.

2.2.18 Lemma

Let X be a topological space and let $\mathcal{F}$ be a Φ_X-filter. Then $\hat{\mathcal{F}}$ is a minimal $(FX)^*$-Cauchy filter.

Proof. We firstly show that $\hat{\mathcal{F}}$ is an $(FX)^*$-Cauchy filter. Let $\mathcal{A} \in \Phi_X$. By Lemma 2.2.17 $\mathcal{A}_{\mathcal{F}} = \mathcal{A}_{\hat{\mathcal{F}}}$ and $\mathcal{A}_{\mathcal{F}^c} = \mathcal{A}_{\hat{\mathcal{F}}^c}$ so that by Definition 2.2.9 $\bigcap \mathcal{A}_{\hat{\mathcal{F}}} \in \hat{\mathcal{F}}$ and $\bigcup \mathcal{A}_{\hat{\mathcal{F}}^c} \notin \hat{\mathcal{F}}$. Thus $X \setminus \bigcup \mathcal{A}_{\hat{\mathcal{F}}^c} \in \mathcal{F}_r$, which gives $X \setminus \bigcup \mathcal{A}_{\hat{\mathcal{F}}^c} \in \hat{\mathcal{F}}$. Now $\bigcap \mathcal{A}_{\hat{\mathcal{F}}} \cap (X \setminus \bigcup \mathcal{A}_{\hat{\mathcal{F}}^c})$ is an element of $\hat{\mathcal{F}}$ which is, as in the proof of Lemma 2.2.13, not separated by $\mathcal{A}$. By Proposition 2.2.7 $\hat{\mathcal{F}}$ is an $(FX)^*$-Cauchy filter.
To see that $\hat{\mathcal{F}}$ is minimal suppose that $\mathcal{G} \subseteq \hat{\mathcal{F}}$ and that $\mathcal{G}$ is an $(FX)^*$-Cauchy filter. If $\mathcal{G} \neq \hat{\mathcal{F}}$, we can find $F \in \mathcal{F} \cup \mathcal{F}_r$ such that $F \notin \mathcal{G}$. Hence by Lemma 2.2.3 $X \backslash F \in \mathcal{G}$. But $X \backslash F \notin \mathcal{F}_r$ contradicting $\mathcal{G} \subseteq \hat{\mathcal{F}}$. Hence $\hat{\mathcal{F}}$ is a minimal $(FX)^*$-Cauchy filter.

2.2.19 **Proposition**

There is a bijection between the set of minimal $(FX)^*$-Cauchy filters and the set of Φ_X-filters.

Proof. Given a topological space X we denote the set of minimal $(FX)^*$-Cauchy filters by $\tilde{X}$ and the set of Φ_X-filters by X_F. Let $f: \tilde{X} \to X_F$ assign to each element, $\mathcal{F}$ of $\tilde{X}$ its open set restriction, $\mathcal{F}_o$. Proposition 2.2.11 ensures that $f(\mathcal{F}) \in X_F$. Suppose that $f(\mathcal{F}) = f(\mathcal{G})$. Now $(\mathcal{F}_o)\hat{}$ has subbase $\mathcal{F}_o \cup \mathcal{F}_{or}$ and since $\mathcal{F}_{or} \subseteq \mathcal{F}$, $\widehat{f(\mathcal{F})} \subseteq \mathcal{F}$. Since $\widehat{f(\mathcal{F})}$ is a minimal $(FX)^*$-Cauchy filter, it follows that $\mathcal{F} = \widehat{f(\mathcal{F})}$. Similarly $\mathcal{G} = \widehat{f(\mathcal{G})}$. But now $\widehat{f(\mathcal{F})} = \widehat{f(\mathcal{G})}$ since $f(\mathcal{F}) = f(\mathcal{G})$. Hence $\mathcal{F} = \mathcal{G}$ and f is $1-1$. For a given Φ_X-filter $\mathcal{F}$, $\hat{\mathcal{F}}$ is a minimal $(FX)^*$-Caucy filter and by Lemma 2.2.17 $f(\hat{\mathcal{F}}) = \mathcal{F}$.

2.2.20 **Remark**

The unique minimal $(FX)^*$-Cauchy filter coarser than a given $(FX)^*$-Cauchy filter, $\mathcal{F}$, is the filter generated by all open and all closed sets in $\mathcal{F}$. To see this, note that $\mathcal{F}_c = (\mathcal{F}_o)_r$, so that then $(\mathcal{F}_o)\hat{}$ is generated by $\mathcal{F}_o \cup \mathcal{F}_c$, and is then a minimal Cauchy filter by Lemma 2.2.18, clearly coarser than $\mathcal{F}$. It is interesting to compare this construction to the one needed in the case of a general quasi-uniform space, as in [Fletcher and Lindgren 1982, Proposition 3.30].

We now describe a natural topology on the set of all Φ_X-filters and show that the bijective correspondence described in Proposition 2.2.19 becomes a homeomorphism between our newly constructed space and T_1KFX.

2.2.21 **Definition**

Let $\mathcal{D}$ be a collection of prime open filters on a space X. For each $O \in \mathcal{O}X$ let $O^* := \{\mathcal{F} \in \mathcal{D} \mid O \in \mathcal{F}\}$.

2.2.22 **Lemma**

Let $O, O_1, O_2 \in \mathcal{O}X$. Then

(i) $O_1^* \cup O_2^* = (O_1 \cup O_2)^*$

(ii) $O_1^* \cap O_2^* = (O_1 \cap O_2)^*$.

Proof. (i)

$$\begin{aligned} \mathcal{F} \in (O_1 \cup O_2)^* &\iff O_1 \cup O_2 \in \mathcal{F} \\ &\iff O_1 \in \mathcal{F} \text{ or } O_2 \in \mathcal{F} \\ &\iff \mathcal{F} \in O_1^* \text{ or } \mathcal{F} \in O_2^* \\ &\iff \mathcal{F} \in O_1^* \cup O_2^*. \end{aligned}$$

(ii)

$$\begin{aligned} \mathcal{F} \in (O_1 \cap O_2)^* &\iff O_1 \cap O_2 \in \mathcal{F} \\ &\iff O_1 \in \mathcal{F} \text{ and } O_2 \in \mathcal{F} \\ &\iff \mathcal{F} \in O_1^* \text{ and } \mathcal{F} \in O_2^* \\ &\iff \mathcal{F} \in O_1^* \cap O_2^*. \end{aligned}$$

Let X_F be the topological space on the set of all Φ_X-filters which has $\{O^* \mid O \in \mathcal{O}X\}$ as a base for its topology. However, we first need to make some observations about the space T_1KFX. It is clear that $\{\tilde{U_{\mathcal{A}}} \mid \mathcal{A} \in \Phi_X\}$ is a subbase for the quasi-uniformity of KFX. It then follows that T_1KFX has $\{\tilde{U_{\mathcal{A}}}(\mathcal{F}) \mid \mathcal{F} \in T_1KFX, \mathcal{A} \in \Phi_X\}$ as a subbase for its open sets.

2.2.23 **Lemma**

Let $\mathcal{F}, \mathcal{G} \in KFX$ and let $\mathcal{A} \in \Phi_X$. Then $\mathcal{G} \in \tilde{U_{\mathcal{A}}}(\mathcal{F}) \iff \bigcap \mathcal{A}_{\mathcal{F}} \in \mathcal{G}$.

Proof. Let $\mathcal{G} \in \tilde{U}_{\mathcal{A}}(\mathcal{F})$ and $A \in \mathcal{A}_{\mathcal{F}}$ and let $\mathcal{B} := \{A, X\}$. Then $\tilde{U_{\mathcal{A}}} \subseteq \tilde{U_{\mathcal{B}}}$, so $\mathcal{G} \in \tilde{U_{\mathcal{B}}}(\mathcal{F})$. Since $A \in \mathcal{F}$ and $U_{\mathcal{B}} = (A \times A) \cup ((X \setminus A) \times X)$, we can see that $G \subseteq A$, so that $A \in \mathcal{G}$ and hence $\bigcap \mathcal{A}_{\mathcal{G}} \subseteq \bigcap \mathcal{A}_{\mathcal{F}}$. But $\bigcap \mathcal{A}_{\mathcal{G}} \in \mathcal{G}$ by the proof of Proposition 2.2.11.

Conversely, suppose that $\bigcap \mathcal{A}_{\mathcal{F}} \in \mathcal{G}$. By Remark 2.2.12 $[\mathcal{F}\mathcal{A}] \in \mathcal{F}$, and it is easily seen that $[\mathcal{F}\mathcal{A}] \times \bigcap \mathcal{A}_{\mathcal{F}} \subseteq U_{\mathcal{A}}$. Thus $(\mathcal{F}, \mathcal{G}) \in \tilde{U_{\mathcal{A}}}$.

2.2.24 **Theorem**

T_1KFX is homeomorphic to X_F.

Proof. Let f be the bijection given in the proof of Proposition 2.2.19. We show that f is continuous and open. Let $O \in \mathcal{O}X$. Then $O^* \in \mathcal{O}X_F$ is a basic open set. Whenever $\mathcal{F} \in O^*$ it follows that $\hat{\mathcal{F}} \in f^{-1}[O^*]$. Set $\mathcal{A} := \{O, X\}$. Then $U_{\mathcal{A}} \in \operatorname{ent} FX$ and hence $\tilde{U_{\mathcal{A}}} \in \operatorname{ent} KFX$. We now show that $\tilde{U_{\mathcal{A}}}(\hat{\mathcal{F}}) \subseteq f^{-1}[O^*]$, which ensures the continuity of f. Let $\hat{\mathcal{G}} \in \tilde{U_{\mathcal{A}}}(\hat{\mathcal{F}})$. Then $(\hat{\mathcal{F}}, \hat{\mathcal{G}}) \in \tilde{U_{\mathcal{A}}}$ and hence $O \in \hat{\mathcal{G}}$, i.e. $\hat{\mathcal{G}} \in f^{-1}[O^*]$.

We now show that f is open: Let $\tilde{U_{\mathcal{A}}}(\mathcal{F})$ be a basic open set in T_1KFX, where $\mathcal{A} \in \Phi_X$. Let $O := \bigcap \mathcal{A}_{\mathcal{F}}$. Then by Lemma 2.2.23 $f(\mathcal{G}) \in O^* \Leftrightarrow O \in \mathcal{G} \Leftrightarrow \mathcal{G} \in \tilde{U_{\mathcal{A}}}(\mathcal{F}) \Leftrightarrow f(\mathcal{G}) \in f[\tilde{U_{\mathcal{A}}}(\mathcal{F})]$, and hence $f[\tilde{U_{\mathcal{A}}}(\mathcal{F})]$ is open.

We now describe the Φ_X-filters, i.e. the prime open filters which occur as elements of T_1KFX, for some well known transitive T_1-sections.

2.2.25 **Definition** [Frolík 1963]

Let Γ be a collection of covers on a topological space X. A filter $\mathcal{F}$ is Γ-Cauchy if for each $\mathcal{A} \in \Gamma$ there exists $A \in \mathcal{A}$ such that $A \in \mathcal{F}$.

2.2.26 **Definition**

Let Γ be a collection of open families on a topological space, X. An open filter is Γ-prime if whenever $\mathcal{A} \in \Gamma$ and $A \notin \mathcal{F}$ for all $A \in \mathcal{A}$ then $\bigcup \mathcal{A} \notin \mathcal{F}$.

2.2.27 **Remark**

We will use Definitions 2.2.25 and 2.2.26 in the case where Γ consists of, amongst others, the countable open covers, the point-finite open covers, the well-monotone open covers. In particular we will be using the terms *countable-open-Cauchy*, *point-finite prime* and *well-monotone-prime.*

For a given transitive T_1-section, F, Φ^F will denote an adequate natural kind of interior preserving open covers which induces F.

2.2.28 **Proposition**

$T_1KC_1^*X$ consists of all prime open filters.

Proof. Let $X \in$ **Top** and the Φ_X be the collection of all finite open covers of X. Then it is clear from Definition 2.2.9 that every prime open filter is a Φ_X-filter.

2.2.29 **Proposition**

T_1KC_1X consists of the prime open filters which are countable-open-Cauchy and have the countable intersection property.

Proof. We take $\Phi_X^{\mathcal{C}_1}$ to be the set of all open spectra on X. Let $\mathcal{F}$ be a $\Phi_X^{\mathcal{C}_1}$-filter. Suppose that $\mathcal{A} = \{A_n \mid n \in \mathbf{N}\}$ is a countable collection of open sets with $\mathcal{A} \subseteq \mathcal{F}$ and $\bigcap \mathcal{A} = \emptyset$. We construct an open spectrum $\mathcal{B} = \{B_n \mid n \in \mathbf{N}\}$ by setting $B_n = \bigcap_{m \leq n} A_m$. Now $\mathcal{B} \subseteq \mathcal{F}$ since each element of $\mathcal{B}$ is a finite intersection of elements of $\mathcal{F}$. But $\bigcap \mathcal{B} = \emptyset$ contradicting our assumption that $\mathcal{F}$ is a $\Phi_X^{\mathcal{C}_1}$-filter. Hence $\mathcal{F}$ has the countable intersection property. To see that $\mathcal{F}$ is countable-open Cauchy suppose that $\mathcal{A} = \{A_n \mid n \in \mathbf{N}\}$ is a countable open collection with $\bigcup \mathcal{A} = X$. We construct an open spectrum $\mathcal{B} = \{B_n \mid n \in \mathbf{N}\}$ with $B_n = \bigcup_{m \leq n} A_m$. Now $\bigcup \mathcal{B} = X$ so there must be a $B \in \mathcal{B} \cap \mathcal{F}$. Since each element of $\mathcal{B}$ is a finite union of elements of $\mathcal{A}$ and $\mathcal{F}$ is prime, we can find an element of $\mathcal{A}$ which is also an element of $\mathcal{F}$.

Conversely, let $\mathcal{F}$ be a prime open filter which is countable-open Cauchy and has the countable intersection property. We verify conditions (i) and (ii) of Definition 2.2.9. Let $\mathcal{A}$ be an open spectrum. Clearly $\mathcal{A} \not\subseteq \mathcal{F}$ since $\bigcap \mathcal{A} = \emptyset$ and $\mathcal{F}$ has cip. Thus there is a smallest element of $\mathcal{A} \cap \mathcal{F}$, and so $\bigcap \mathcal{A}_{\mathcal{F}} \in \mathcal{F}$. Now consider $\bigcup \mathcal{A}_{\mathcal{F}^c}$. Clearly $\mathcal{A}_{\mathcal{F}^c} \neq \mathcal{A}$ since $\bigcup \mathcal{A} = X$ and $\mathcal{F}$ is countable-open Cauchy. We can now find a largest element of $\mathcal{A}_{\mathcal{F}^c}$, so that $\bigcup \mathcal{A}_{\mathcal{F}^c} \notin \mathcal{F}$.

Recall that $\mathcal{C}_h$: **Top** $\to$ **Qu** assigns to each topological space X the quasi-uniformity induced by $\{\mathcal{A} \cup \{X\} \mid \mathcal{A}$ is an open-spectrum on some $O \in \mathcal{O}X\}$.

2.2.30 Proposition

T_1KC_hX consists of the prime open filters which are countably-prime and have the countable intersection property.

Proof. We prove what is not covered by the above proposition. Let $\mathcal{F}$ be a $\Phi_X^{C_h}$-filter and let $\mathcal{A} = \{A_n \mid n \in \mathbf{N}\}$ be a collection of open sets with $\bigcup \mathcal{A} \in \mathcal{F}$. As in the above proposition we construct an open spectrum, $\mathcal{B}$, on the subspace of X given by $\bigcup \mathcal{A}$. Then $\mathcal{B} \cup \{X\} \in \Phi_X^{C_h}$. Since $\mathcal{F}$ is a $\Phi_X^{C_h}$-filter we can find a $B \in \mathcal{B} \cap \mathcal{F}$. As in the above proposition we conclude that $\mathcal{F}$ is countably prime.

Recall the following functorial quasi-uniformities on a given topological space, X:

(i) D is induced by the open spectra which have $\emptyset$ as a member;

(ii) D_h is induced by the hereditary open spectra;

(iii) B is induced by the open spectra which have X as a member.

The following results follow immediately from the above propositions 2.2.29 and 2.2.30.

2.2.31 Proposition

T_1KDX consists of the prime open filters which are countable-open Cauchy.

2.2.32 Proposition

T_1KD_hX consists of the prime open filters which are countably prime.

2.2.33 Proposition

T_1KBX consists of the prime open filters which have the countable intersection property.

We are not able to find such a simple characterisation in the case of the locally-finite functor but if one considers the hereditary locally finite functor we have:

2.2.34 Proposition

$T_1KQ_{hlf}X$ consists of the prime open filters which are locally-finite prime.

Proof. Let $\mathcal{F} \in T_1KQ_{hlf}X$. Let $\mathcal{A}$ be a locally finite collection of open sets. If $\mathcal{A} \subseteq \mathcal{F}$, then $\bigcap \mathcal{A} \in \mathcal{F}$, so that $\bigcap \mathcal{A} \neq \emptyset$ and hence $\mathcal{A}$ is finite. If $\mathcal{A} \cap \mathcal{F} = \emptyset$, then $\bigcup \mathcal{A} \notin \mathcal{F}$ by part (ii) of Definition 2.2.9.

The converse follows easily.

2.2.35 Proposition

$T_1KQ_{pf}X$ consists of the prime open filters, $\mathcal{F}$, for which every point finite collection in $\mathcal{F}$ is finite and $\mathcal{F}$ is point-finite prime.

Proof. Similar to the proof above.

The elements of the bicompletion of the well-monotone quasi-uniformity will be of particular importance later in this chapter.

2.2.36 Lemma

$\mathcal{F} \in T_1KWX$ if and only if $\mathcal{F}$ is a prime open filter and $\mathcal{F}$ is well-monotone prime.

Proof. Let $\mathcal{F} \in T_1KWX$ and suppose that $\mathcal{A}$ is a well-monotone collection of open sets in X. Then $\mathcal{A} \cup \{X\}$ is a well-monotone open cover of X. If $\mathcal{A} \cap \mathcal{F} = \emptyset$ then $\mathcal{A}_{\mathcal{F}^c} = \mathcal{A}$ and by Definition 2.2.9 $\bigcup \mathcal{A}_{\mathcal{F}^c} \notin \mathcal{F}$.

Conversely, suppose that $\mathcal{A}$ is a well-monotone open cover. Clearly $\mathcal{A}_{\mathcal{F}^c}$ is a well-monotone collection and hence $\bigcup \mathcal{A}_{\mathcal{F}^c} \notin \mathcal{F}$. It is always true that $\bigcap \mathcal{A}_{\mathcal{F}} \in \mathcal{F}$ whenever $\mathcal{A}$ is a well-monotone open cover.

2.2.37 **Proposition**

$\mathcal{F} \in T_1KWX$ if and only if $\mathcal{F}$ is a completely prime open filter.

Proof. Let $\mathcal{U} \subseteq \mathcal{O}X$ be an infinite collection of open sets. We well-order $\mathcal{U}$ so that $\mathcal{U} = \{O_\alpha\}_{\alpha<\beta}$ and β is a limit ordinal. For each $\tau < \beta$, define

$$A_\tau = \bigcup_{\alpha \leq \tau} O_\alpha.$$

Then

$$A_\tau = \bigcup_{\alpha<\tau} A_\alpha \cup O_\tau.$$

Assume that $\bigcup \mathcal{U} \in \mathcal{F}$ and $\forall \alpha < \beta$, $O_\alpha \notin \mathcal{F}$. Now $\bigcup \mathcal{U} = \bigcup_{\tau<\beta} A_\tau \in \mathcal{F}$ and $\{A_\tau \mid \tau < \beta\}$ is a well-monotone open collection, so by Lemma 2.2.36 there exists a τ such that $A_\tau \in \mathcal{F}$. Take τ_o to be the first such τ. Now

$$A_{\tau_o} = \bigcup_{\alpha<\tau_o} A_\alpha \cup O_{\tau_o}$$

and since $O_{\tau_o} \notin \mathcal{F}$ and $\mathcal{F}$ is prime, $\bigcup_{\alpha<\tau_o} A_\alpha \in \mathcal{F}$. Again by Lemma 2.2.36 we can find $\alpha < \tau_o$ such that $A_\alpha \in \mathcal{F}$, contradicting our assumption that τ_o is the first such τ.

We now prove another important property of the well-monotone functor: it is the coarsest transitive T_1-section, G, such that for every space X, the elements of T_1KGX are completely prime open filters. The proof needs the following two lemmas:

2.2.38 **Lemma**

If G is a T_1-section with $G \not\geq W$, then there is an ordinal α such that

$$G\beta_l < W\beta_l = \phi_t\beta_l$$

for every ordinal $\beta \geq \alpha$.

Proof. Recall that W is generated by $\mathbf{Ord}_l$, i.e. $W = \langle\{\phi_t\alpha_l \mid \alpha \in \mathbf{Ord}\}\rangle$, so if for each $\alpha \in \mathbf{Ord}$, $W\alpha_l = G\alpha_l$, then (since $W\alpha_l = \phi_t\alpha_l$) $\{\phi_t\alpha_l \mid \alpha \in \mathbf{Ord}\} \subseteq \mathrm{range}G$, and then $W \leq G$, contradicting that $G \not\geq W$. So we have a least ordinal α for which $G\alpha_l \neq W\alpha_l$. We now show that the same occurs at every ordinal $\beta \geq \alpha$. Assume that $G\beta_l = W\beta_l$, with $\beta \geq \alpha$. There is a natural embedding $i: \alpha_l \to \beta_l$, and then $i^{-1}\mathcal{O}\beta_l = \mathcal{O}\alpha_l$. Since $U_{\mathcal{O}\beta_l} \in \mathrm{ent}\,\phi_t\beta_l = \mathrm{ent}\,W\beta_l = \mathrm{ent}\,G\beta_l$, we then have $(i \times i)^{-1}U_{\mathcal{O}\beta_l} \in \mathrm{ent}\,G\alpha_l$, i.e. $U_{\mathcal{O}\alpha_l} \in \mathrm{ent}\,G\alpha_l$, so that $G\alpha_l = \phi_t\alpha_l = W\alpha_l$, a contradiction.

2.2.39 **Remark**

An ordinal α as in the above lemma is necessarily infinite. This is because a finite ordinal admits a unique quasi-uniformity.

2.2.40 **Lemma**

Let G be a T_1-section with $G \not\geq W$, and let α be an ordinal having the property stated in the above lemma. Then for every ordinal β and for every well-monotone open cover M of β_l:

$$U_{\mathcal{M}} \in \mathit{ent}\,G\beta_l \Longrightarrow |\mathcal{M}| \leq |\alpha|\,.$$

Proof. Let α be as in the above Lemma 2.2.38. Suppose that we have an ordinal β and a well-monotone open cover M of β_l with $U_{\mathcal{M}} \in \mathrm{ent}\,G\beta_l$ but $|\mathcal{M}| > |\alpha|$. Now

$|\alpha|<|\mathcal{M}|\leq|\mathcal{O}\beta_l|=|\beta|$, so that $\alpha<\beta$. Now take an indexing $\mathcal{M}=\{M_\psi \mid \psi<\tau\}$ with τ an ordinal and $M_\psi \subset M_\eta$ whenever $\psi<\eta$. Define $f\colon\alpha\to\beta$ by:

$$(\forall\eta\in\alpha)(f(\eta) \text{ is the least element of } M_{\eta+1}\backslash M_\eta).$$

Recall that $M_{\eta+1}\backslash M_\eta \neq \emptyset$. So for $\psi,\eta\in\alpha$ with $\psi<\eta$, $f(\psi)<f(\eta)$, and hence $f\colon\alpha_l\to\beta_l$ is continuous. For $\eta\in\alpha$,

$$f^{-1}[M_{\eta+1}]=\downarrow\eta$$

since $f(\psi)\in M_{\eta+1}\Rightarrow M_\psi\subset M_{\eta+1}\Rightarrow\psi<\eta+1\Rightarrow\psi\in\downarrow\eta$, and clearly $\psi\in\downarrow\eta\Rightarrow f(\psi)\in M_{\eta+1}$. Consequently $f^{-1}\mathcal{M}=\mathcal{O}\alpha_l$, so that $U_{\mathcal{O}\alpha_l}=U_{f^{-1}\mathcal{M}}=(f\times f)^{-1}U_{\mathcal{M}}\in \operatorname{ent} G\alpha_l$ (since $U_{\mathcal{M}}\in G\beta_l$) and hence $G\alpha_l=\phi_t\alpha_l=W\alpha_l$ contradicting the choice of α.

2.2.41 **Proposition**

W is the coarsest transitive T_1-section, G, such that for every topological space, X, the elements of T_1KGX are completely prime open filters of X.

Proof. Let G be a any transitive T_1-section for which the elements of T_1KGX are completely prime open filters. Suppose that $G\not\geq W$. By Lemma 2.2.40 above there exists an ordinal α such that $G\beta_l<W\beta_l$ whenever $\beta\geq\alpha$. Let λ be a regular cardinal strictly larger than α. Consider the prime open filter, $\mathcal{F}$, on λ_l which has λ as its only element. Clearly $\mathcal{F}$ is not completely prime. We show that $\mathcal{F}\in T_1KG\lambda_l$: Let $\mathcal{A}$ be any (well-monotone) open cover of λ_l for which $U_{\mathcal{A}}\in\operatorname{ent} G\lambda_l$. Since $\lambda>|\alpha|$, $|\mathcal{A}|\leq|\alpha_l|<\lambda$. If $A\neq\lambda$ is any other element of $\mathcal{A}$, then $|A|<\lambda$. Since also $\mathcal{A}<\lambda$ and λ is a regular cardinal and $\bigcup\mathcal{A}=\lambda$, we have $\lambda\in\mathcal{A}$. Consequently $\bigcap\mathcal{A}_{\mathcal{F}}=\lambda$, i.e.

$\bigcap \mathcal{A}_{\mathcal{F}} \in \mathcal{F}$. Furthermore $\mathcal{A}_{\mathcal{F}^c} = \mathcal{A}\backslash\{\lambda\}$, so that $\bigcup \mathcal{A}_{\mathcal{F}^c} \neq \lambda$, i.e. $\bigcup \mathcal{A}_{\mathcal{F}^c} \notin \mathcal{F}$. Thus $\mathcal{F} \in T_1 K G \lambda_l$, contradicting our initial assumption on G.

The following example shows that W is not the only transitive T_1-section for which the bicompletion consists of exactly the completely prime open filters.

2.2.42 **Example**

Let $G = W \vee Q_{lf}$. For every topological space X, T_1KGX consists exactly of the completely prime open filters of X.

2.2.43 **Remark**

We have seen that Lemma 2.2.40 holds for all spaces of the form β_l. But we cannot replace β_l by arbitrary topological spaces as the following example shows. Let X be an infinite discrete topological space, and consider $G = Q_{lf}$. Then $G \not\geq W$, but $GX = Q_{lf}X = \phi_t X$, so that $U_{\mathcal{M}} \in \operatorname{ent} GX$ for every well-monotone open cover $\mathcal{M}$ of X, and there is no upper bound for $|\mathcal{M}|$ as X varies.

2.3 Bicompletions of functorial transitive quasi-uniformities

In this section we will consider T_1KFX as an extension of X via prime open filters, specifically the Φ_X-filters. Using this characterisation we will determine necessary and sufficient conditions on a topological space X in order that FX be bicomplete. It is well known that a T_o space X can be embedded in T_1KFX via the map T_1k_{FX}, which under the characterisation which we are using assigns to each $x \in X$ the open neighbourhood filter at x. We will reserve the letter j_{FX} for the map T_1k_{FX}, and will merely write j if the context is clear.

2.3.1 **Proposition**

Let G and H be transitive T_1-sections with $G \geq H$. Then T_1KGX is a subspace of T_1KHX for every topological space X.

Proof. Let Φ, Γ be the largest adequate natural kind of interior preserving open covers inducing F and H respectively. We need only show that for a given topological space X, every Φ_X-filter is a Γ_X-filter. This follows easily since $\Gamma_X \subseteq \Phi_X$. The one extension is then a subspace of the other, since both are filter extensions over the same X.

For the rest of this section cl **will denote the closure in** T_1KFX.

2.3.2 **Definition**

(i) For each transitive T_1-section F let $\mathbf{U}_F := \{X \in \mathbf{Top} \mid KFX \geq FT_1KFX\}$.

(ii) For each $\mathcal{F} \in KFX$ let $\lambda(\mathcal{F}) := \{x \in X \mid \forall O \in \mathcal{O}X,\ x \in O \Rightarrow O \in \mathcal{F}\}$, i.e. the set of limit points of $\mathcal{F}$ in X.

2.3.3 **Lemma**

For $\mathcal{F} \in T_1KFX$, $cl\{\mathcal{F}\} = \{\mathcal{G} \in T_1KFX \mid \mathcal{G} \subseteq \mathcal{F}\}$.

Proof. For $O \in \mathcal{O}X$, $O \in \mathcal{G} \Rightarrow O \in \mathcal{F}$ if and only if $\mathcal{G} \in O^* \Rightarrow \mathcal{F} \in O^*$.

We caution the reader that $\mathcal{G} \subseteq \mathcal{F}$ need not imply $\mathcal{G} = \mathcal{F}$, because the filters are not minimal Cauchy. So cl$\{\mathcal{F}\}$ can differ from $\{\mathcal{F}\}$.

2.3.4 **Lemma**

Let $\mathcal{F} \in KFX$. Then $\lambda(\mathcal{F}) = j_{FX}^{-1}[cl\{\mathcal{F}\}] = \bigcap \mathcal{F}_r$.

Proof. Let $x \in \lambda(\mathcal{F})$. Then $x \in j_{FX}^{-1}[\mathrm{cl}\{\mathcal{F}\}]$ is immediate from Lemma 2.3.3.
Let $x \in j_{FX}^{-1}[\mathrm{cl}\{\mathcal{F}\}]$ and let A be such that $X \setminus A \in \mathcal{O}X$ and $X \setminus A \notin \mathcal{F}$. Now $j(x) \in \mathrm{cl}\{\mathcal{F}\}$ so if $x \in X \setminus A$ then by Lemma 2.3.3 we have $X \setminus A \in \mathcal{F}$, a contradiction. Hence $x \in A$, and so $x \in \bigcap \mathcal{F}_r$.
Let $x \in \bigcap \mathcal{F}_r$ and suppose that there exists $O \in \mathcal{O}X$ such that $x \in O$ but $O \notin \mathcal{F}$. Then $x \in X \setminus O$, a contradiction.

We recall that the b-topology of a topological space X has as base the collection $\{O \mid O \in \mathcal{O}X \text{ or } X \setminus O \in \mathcal{O}X\}$. A mapping $f: X \to Y$ is b-dense if the closure, in the b-topology of Y, of $f[X]$ is Y.

2.3.5 **Proposition**

For a T_1-section F and a topological space X the following are equivalent:

(i) $X \in \mathbf{U}_F$;

(ii) $KFX \geq C_1^* T_1 KFX$;

(iii) $j: X \to T_1 KFX$ *is b-dense.*

Proof. $(i) \Rightarrow (ii)$: trivial.
$(ii) \Rightarrow (iii)$: Condition (ii) implies that $T_1(KFX)^*$ is the b-topology of $T_1 KFX$, and since j is $T_1(KFX)^*$-dense the result follows.
(iii) $\Rightarrow$ (i): [Brümmer, 1992] The epimorphisms in $\mathbf{Top}_o$ are known to be the b-dense maps; more generally, a map $f: X \to Y$ with $Y \in \mathbf{Top}_o$ and $X \in \mathbf{Top}$ is

right-cancellable if and only if f is b-dense. With this in mind the result is given by applying the proof of [Brümmer 1992, 4.2] to the single space X.

2.3.6 Remark

We can thus see that the elements of $\mathbf{U}_F$ are precisely those topological spaces, X, for which KFX is finer than the associated Pervin structure on T_1KFX. The following result provides further characterisations of $\mathbf{U}_F$ in terms of the elements of KFX.

2.3.7 Definition

Let $X \in \mathbf{Top}$ and let $\mathcal{F}$ be a prime open filter on X. $\mathcal{F}$ is *irreducible* [Hoffmann 1979] if $X \backslash \lambda(\mathcal{F}) \notin \mathcal{F}$.

2.3.8 Proposition

For a transitive T_1-section F and a topological space X, the following are equivalent:

(i) $j: X \to T_1KFX$ is b-dense;

(ii) every $\mathcal{F} \in T_1KFX$ is completely prime;

(iii) every $\mathcal{F} \in T_1KFX$ is irreducible;

(iv) $T_1KFX \subseteq T_1KWX$.

Proof. $(i) \Rightarrow (ii)$: Let $\mathcal{F} \in T_1KFX$ and let j be b-dense. Suppose that $\{U_i\}_I \subseteq \mathcal{O}X$ is a collection with the property that $\bigcup_I U_i \in \mathcal{F} \in KFX$. Recalling Definition 2.2.21 we then see that $\mathcal{F} \in (\bigcup_I U_i)^* \in \Omega T_1KFX$. By the b-density of j we have that $j^{-1}[\mathrm{cl}\{\mathcal{F}\} \cap (\bigcup_I U_i)^*] \neq \emptyset$, i.e. there exists an $x \in X$ such that

$j(x) \in \mathrm{cl}\{\mathcal{F}\}$ and $j(x) \in (\bigcup_I U_i)^*$. Then $x \in \bigcup_I U_i$ and also by Lemma 2.3.2 whenever $x \in O \in \mathcal{O}X$, then $O \in \mathcal{F}$. Since $x \in U_i$ for some $i \in I$ it now follows that $U_i \in \mathcal{F}$.

$(ii) \Rightarrow (iii)$: Let $\mathcal{F} \in KFX$ be completely prime. Then $X \setminus \lambda(\mathcal{F}) = X \setminus \bigcap \mathcal{F}_r = \bigcup\{O \in \mathcal{O}X \mid O \notin \mathcal{F}\}$. Since $X \setminus \lambda(\mathcal{F})$ is a union of open sets which are not elements of $\mathcal{F}$ it then follows that $X \setminus \lambda(\mathcal{F}) \notin \mathcal{F}$, and thus $\mathcal{F}$ is irreducible.

$(iii) \Rightarrow (i)$: Let $\mathcal{F}$ be an element of T_1KFX and let O^* be a basic open neighbourhood of $\mathcal{F}$. Then $j^{-1}[\mathrm{cl}\{\mathcal{F}\} \cap O^*] = \lambda(\mathcal{F}) \cap O$ by Lemma 2.3.4 since $j^{-1}[O^*] = O$. Since $X \setminus \lambda(\mathcal{F}) \notin \mathcal{F}$ and $\lambda(\mathcal{F})$ is closed by Lemma 2.3.4, we have $\lambda(\mathcal{F}) \in \mathcal{F}_r$. By Lemma 2.2.15 $\mathcal{F} \cup \mathcal{F}_r$ is a filter subbase and $O, \lambda(\mathcal{F}) \in \mathcal{F} \cup \mathcal{F}_r$ so that $\lambda(\mathcal{F}) \cap O \neq \emptyset$. Thus j is b-dense.

$(ii) \Rightarrow (iv)$: Suppose that every $\mathcal{F} \in T_1KFX$ is completely prime. By Proposition 2.2.41 T_1KWX consists of exactly the completely prime open filters, and hence $T_1KFX \subseteq T_1KWX$.

$(iv) \Rightarrow (ii)$: Suppose that $T_1KFX \subseteq T_1KWX$. Then every element of T_1KFX must be completely prime since every element of T_1KWX is completely prime, by Proposition 2.2.41.

2.3.9 Remark

1. There are similarities between the above proof and the results contained in [Künzi and Ferrario 1991, Proposition 1].

2. If in the above proposition we require that conditions (i) - (iv) hold for every topological space, then by Proposition 2.2.41 the stated conditions are equivalent to the condition $W \leq F$.

2.3.10 Lemma

$x \in cl_X\{y\} \Longleftrightarrow j(x) \in cl\{j(y)\}$.

Proof. Clear, since j is an initial map.

2.3.11 Lemma

Let $X \in \mathbf{U}_F$ and let B be a closed irreducible set in T_1KFX. Then $j^{-1}[B]$ is a closed irreducible set in X.

Proof. We need only show that $j^{-1}[B]$ is irreducible since it is clearly closed. Suppose that $O_1 \cap j^{-1}[B] \neq \emptyset \neq O_2 \cap j^{-1}[B]$ for $O_1, O_2 \in \mathcal{O}X$. Now $O_i = j^{-1}[O_i^*]$ for $i = 1, 2$. Now $O_1^* \cap B \neq \emptyset \neq O_2^* \cap B$ so $O_1^* \cap O_2^* \cap B \neq \emptyset$. By the b-density of j we have $j^{-1}[O_1^*] \cap j^{-1}[O_2^*] \cap j^{-1}[B] \neq \emptyset$ and thus $j^{-1}[B]$ is irreducible.

2.3.12 Proposition

Let F be a transitive T_1-section and let $X \in \mathbf{U}_F$. If $\mathcal{G} \in T_1KFX \setminus j[X]$, then $j^{-1}[cl\{\mathcal{G}\}]$ is a non-trivial closed irreducible set in X.

Proof. Let $B := \mathrm{cl}\{\mathcal{G}\}$. By Lemma 2.3.11 $j^{-1}[B]$ is a closed irreducible set in X. Suppose that $j^{-1}[B] = \mathrm{cl}_X\{x\}$ for some $x \in X$. Then $j(x) \in \mathrm{cl}\{\mathcal{G}\}$. Since T_1KFX is a T_0-space and $j(x) \neq \mathcal{G}$, $\mathcal{G} \notin \mathrm{cl}\{j(x)\}$. Furthermore $\mathrm{cl}\{j(x)\} \subseteq B$ since B is closed. Now $\mathcal{G} \in B \cap T_1KFX \backslash \mathrm{cl}\{j(x)\}$. We now show that $j^{-1}[B \cap T_1KFX \backslash \mathrm{cl}\{j(x)\}] = \emptyset$, which contradicts the b-density of j: Suppose that there is a $y \in X$ such that $j(y) \in B \cap T_1KFX \backslash \mathrm{cl}\{j(x)\}$. Then $y \in j^{-1}[B] = \mathrm{cl}_X\{x\}$. By Lemma 2.3.10 $j(y) \in \mathrm{cl}\{j(x)\}$, which contradicts $j(y) \in T_1KFX \backslash \mathrm{cl}\{j(x)\}$. Hence no such y exists and $j^{-1}[B \cap T_1KFX \backslash \mathrm{cl}\{j(x)\}] = \emptyset$, allowing us to conclude that B is non-trivial.

2.3.13 Notation

For every irreducible set B, let $\mathcal{O}_B := \{O \in \mathcal{O}X \mid B \cap O \neq \emptyset\}$.

The irreducibility of B ensures that $\mathcal{O}_B$ is a completely prime open filter.

2.3.14 Definition

Let $\Phi = (\Phi_X)_{X \in \mathbf{Top}}$ be an adequate natural kind of interior-preserving open covers inducing a T_1-section, F. Let X be a topological space. Then:

(i) An irreducible set, B, is said to be Φ_X*-irreducible* if for each $\mathcal{A} \in \Phi_X$, $\bigcap\{A \in \mathcal{A} \mid A \cap B \neq \emptyset\} \cap B \neq \emptyset$;

(ii) X is a Φ*-space* iff every closed Φ-irreducible set is a point closure.

2.3.15 Remark

1. In case all finite open covers belong to Φ_X, the assumption of irreducibility in the definition of a Φ_X-irreducible set is redundant. But the assumption is necessary for the general case: Consider the example

$$\Phi_X := \{\{O, X\} \mid O \in \mathcal{O}X\}$$

which defines an adequate natural kind of open covers. Then every subset of any space X satisfies the second part of Definition 2.3.14(i).

2. We recall that a T_0-space is sober if every closed irreducible set is a point closure, and that a topological space is quasi-sober if every closed irreducible set is the closure of a (not necessarily unique) point. It then follows that every quasi-sober space is a Φ-space.

3. The next result shows that if Φ and Ψ induce the same T_1-section F, the Φ_X-irreducible sets are the same as the Ψ_X-irreducible sets, and hence the Φ-spaces coincide with the Ψ-spaces.

2.3.16 **Lemma**

Let $(\Phi_X)_{X\in\mathbf{Top}}$ be an adequate natural kind of interior-preserving open covers inducing a T_1-section F. For each topological space, X, and irreducible set $B \subseteq X$, B is Φ_X-irreducible if and only if $\mathcal{O}_B \in T_1KFX$.

Proof. $\Rightarrow$: Let B be Φ_X-irreducible. We check that $\mathcal{O}_B$ satisfies Definition 2.2.9. By our assumption $\bigcap\{A \in \mathcal{A} \mid A \in \mathcal{O}_B\} \in \mathcal{O}_B$ for each $\mathcal{A} \in \Phi_X$. Thus we need only check that, for each $\mathcal{A} \in \Phi_X$, $\bigcup\{A \in \mathcal{A} \mid A \notin \mathcal{O}_B\} \notin \mathcal{O}_B$. This is clear since $A \notin \mathcal{O}_B \Leftrightarrow A \subseteq X\backslash B$.

$\Leftarrow$: Immediate.

2.3.17 **Definition**

A subset B of X has the *FCI property* if B is irreducible and the open filter $\mathcal{O}_B$ has the countable intersection property.

2.3.18 **Remark**

In [Nel and Wilson 1972] the FCI property was defined for not necessarily irreducible sets. But we only use it for irreducible sets, and then the definition above is equivalent to, and simpler than that by Nel and Wilson.

2.3.19 **Proposition**

Let $\Phi_X = \{all\ open\ spectra\ on\ X\}$ and let B be a closed irreducible set in X. Then B has the FCI property if and only if B is Φ_X-irreducible.

Proof. Let B have the FCI property. Then $\mathcal{O}_B$ has the countable intersection property. Since $\mathcal{O}_B$ is completely prime it is countable-open-Cauchy and hence, by Proposition 2.2.29, $\mathcal{O}_B \in T_1KC_1X$. By Lemma 2.3.16 B is Φ_X-irreducible.
Conversely, suppose that B is Φ_X-irreducible. By Lemma 2.3.16 $\mathcal{O}_B \in T_1KC_1X$ and thus $\mathcal{O}_B$ has the countable intersection property.

2.3.20 **Definition** [Nel and Wilson 1972]

A topological space is a *quasi-fc-space* if every closed irreducible set with the FCI property is a point closure.

2.3.21 **Remark**

Nel and Wilson defined the concept of an fc-space, which is a T_0 quasi-fc-space. They showed that the category of all fc-spaces, which we will denote by **FC** is the epireflective hull in T_0 of the real line with the upper topology, equivalently the natural numbers with the upper topology.

The following proposition is now obvious.

2.3.22 **Proposition**

A topological space, X, is a quasi-fc-space $\Longleftrightarrow$ X is an open-spectrum-space $\Longleftrightarrow$ X is a point-finite-open-spectrum-space.

Let X be a topological space. If Φ_X consists of any one of the following collections:

(i) all finite open covers;

(ii) all well-monotone open covers;

(iii) all locally finite open covers;

(iv) all open spectra containing $\emptyset$,

then X is a Φ-space if and only if X is quasi-sober. We give the proof for the locally-finite case as an example.

2.3.23 **Proposition**

X is a (locally finite)-space iff X is quasi-sober.

Proof. Let B be an irreducible set of a topological space X and let $\mathcal{A}$ be a locally finite open cover of X. Consider the collection $\mathcal{B} = \{A \mid A \in \mathcal{A} \text{ and } A \cap B \neq \emptyset\}$. $\mathcal{B}$ is finite since every pair of elements in $\mathcal{B}$ has non empty intersection and $\mathcal{A}$ is locally finite. Thus $\bigcap \mathcal{B} \cap B \neq \emptyset$ and B is locally-finite-irreducible. Clearly every locally-finite-irreducible set is irreducible and hence X is a (locally-finite)-space iff X is quasi-sober.

The following result is somewhat unexpected.

2.3.24 **Proposition**

Let X be a topological space and let Φ_X be the collection of all point-finite open covers on X. Then X is a Φ-space if and only if X is a quasi-fc-space.

Proof. It is clear that every quasi-fc-space is a Φ-space since every point-finite open spectrum is a point-finite open cover. Conversely suppose that B is an irreducible set with the FCI property and that $\mathcal{A}$ is a point-finite open cover. If

$\bigcap\{A \mid A \in \mathcal{A}$ and $A \cap B \neq \emptyset\} \cap B = \emptyset$, then since $\{A \cap B \mid A \in \mathcal{A}\}$ is a point-finite collection there must be a countable collection such that $\{A_n \cap B \mid n \in \mathbf{N}\} = \emptyset$. This contradicts our assumption that B is a set with the FCI property and hence B is Φ_X-irreducible.

2.3.25 **Lemma**

X is a Φ-space whenever FX is bicomplete.

Proof. Suppose that FX is bicomplete. Let B be a closed Φ-irreducible set in X. By Lemma 2.3.16 $\mathcal{O}_B$ is a Φ_X-filter. Since FX is bicomplete, $\mathcal{O}_B$ is the open neighbourhood filter of some $x \in X$. We show that $B = \mathrm{cl}_X\{x\}$: Let $y \in B \cap O$ where $O \in \mathcal{O}X$. Then $O \in \mathcal{O}_B$ and hence $x \in O$. Thus $y \in \mathrm{cl}\{x\}$ and hence $B \subseteq \mathrm{cl}_X\{x\}$. X is thus a Φ-space since every closed Φ_X-irreducible set is a point closure.

2.3.26 **Theorem**

Let F be a transitive T_1-section and let $(\Phi_X)_{X \in \mathbf{Top}}$ be an adequate natural kind of interior preserving open covers inducing F. Let X be a topological space. Then FX is bicomplete if and only if every Φ_X-filter is completely prime and X is a Φ-space.

Proof. $\Rightarrow$: Suppose that FX is bicomplete. By Lemma 2.3.25 X is a Φ-space. The map $j: X \to T_1KFX$ is surjective, and is hence b-dense. By Proposition 2.3.8 every Φ_X-filter is completely prime.

$\Leftarrow$: Suppose that every element of T_1KFX is completely prime, that X is a Φ-space and that FX is not bicomplete. Let $\mathcal{F} \in T_1KFX \setminus j[X]$ and let $B := j^{-1}[\mathrm{cl}\{\mathcal{F}\}]$. By Propositions 2.3.5, 2.3.8 and 2.3.12 B is irreducible. We show that B is Φ_X-irreducible: Let $\mathcal{A} \in \Phi_X$ and let $A \in \mathcal{A} \cap \mathcal{O}_B$. Note that such an A must exist since $\mathcal{A}$ is a cover of X. Let $x \in A \cap B$. Then $j(x) \in \text{cl}\{\mathcal{F}\}$ and $j(x) \in A^*$. Hence $\mathcal{F} \in A^*$, and thus $A \in \mathcal{F}$. Now $\bigcap\{A \in \mathcal{A} \mid A \in \mathcal{O}_B\} \supseteq \bigcap\{A \in \mathcal{A} \mid A \in \mathcal{F}\} \in \mathcal{F}$. In addition, since $\mathcal{F}$ is completely prime, we have by Lemma 2.3.4, Definition 2.3.7 and Proposition 2.3.8 that $X \setminus B \notin \mathcal{F}$. Hence $\bigcap\{A \in \mathcal{A} \mid A \in \mathcal{O}_B\} \not\subseteq X \setminus B$, i.e. $\bigcap\{A \in \mathcal{A} \mid A \in \mathcal{O}_B\} \cap B \neq \emptyset$. We have shown that B is Φ-irreducible. By Proposition 2.3.12 B is not a point closure. This contradicts our assumption that X is a Φ-space. Hence we can conclude that FX is bicomplete.

2.3.27 **Corollary**

Let F be a transitive T_1-section induced by an adequate natural kind of interior-preserving open covers Φ. Then the following are equivalent for a topological space X:

(i) FX is bicomplete;

(ii) X is a Φ-space and every Φ_X-filter is a completely prime open filter;

(iii) X is a Φ-space and $KFX \geq FTKFX$;

(iv) X is a Φ-space and $j\colon X \to T_1KFX$ is b-dense.

(v) X is a Φ-space and $T_1KFX \subseteq T_1KWX$.

Proof. Theorem 2.3.26 with Propositions 2.3.5, 2.3.8 and Definition 2.3.2(i).

2.3.28 **Corollary**

Let F be a transitive T_1-section and let X be a Φ-space with $FX \geq WX$. Then FX is bicomplete.

Proof. Proposition 2.3.1 with Corollary 2.3.27 ((v) $\Rightarrow$ (i)).

2.4 Examples and applications

We now investigate applications of Theorem 2.3.26 to a few well-known transitive sections.

We shall denote by $\mathbf{Alex}_0$ ($\mathbf{Pow}_0$) the subcategories of T_0 spaces in **Alex** (**Pow**). Recall that ϕ_t: **Top** $\to$ **Qu** is the fine-transitive functor.

2.4.1 **Proposition**

For $X \in \mathbf{Top}_0$, $\phi_t X$ is bicomplete $\iff$ X is in the epireflective hull in $\mathbf{Top}_0$ of $\mathbf{Alex}_0$ $\iff$ X is in the epireflective hull in $\mathbf{Top}_0$ of $\mathbf{Pow}_0$.

Proof. If $X \in$ **Alex**, $\phi_t X$ is bicomplete by Corollary 2.3.28 since $\phi_t > W$ and X is an $\mathcal{I}(X)$-space in the sense of Definition 2.3.14. Now $\phi_t = \langle\{\phi_t X \mid X \in \mathbf{Alex}_0\}\rangle_{T_1} = \langle\{\phi_t X \mid X \in \mathbf{Pow}_0\}\rangle_{T_1}$ (see Theorem 1.2.7 and Proposition1.2.10) and then the result follows from [Brümmer 1992, Theorem 14].

Recall that, for a cardinal κ, W_κ is the transitive T_1-section induced by all well-monotone open covers of cardinality $\leq \kappa$. An open filter, $\mathcal{F}$, is *κ-prime* if it is prime and, given a well-monotone open collection $\mathcal{M}$, with $|\mathcal{M}| \leq \kappa$ and $\bigcup \mathcal{M} \in \mathcal{F}$, $\mathcal{M} \cap \mathcal{F} \neq \emptyset$. For a topological space X, the elements of $T_1 K W_\kappa X$ are the κ-prime open filters on X.

2.4.2 **Definition** [Vaughan 1974]

Let a, b be infinite cardinals. A topological space X is *$[a, b]$-compact* if every open cover $\mathcal{U}$ with $|\mathcal{U}| \leq b$ admits a subcover $\mathcal{U}'$ with $|\mathcal{U}'| < a$. X is $[a, \infty]$-compact if it is $[a, b]$-compact for every $b \geq a$.

2.4.3 **Proposition**

For a topological space X

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$$

where

(i) $W_\kappa X \geq WX$.

(ii) Every well-monotone open collection, $\mathcal{M}$, *on* X *satisfies* $|\mathcal{M}| \leq \kappa$;

(iii) X *is hereditarily* $[\kappa^+, \infty]$*-compact;*

(iv) Every κ*-prime open filter on* X *is completely prime;*

Proof. $(i) \Rightarrow (ii)$: Suppose that $W_\kappa X \geq WX$ and let $\mathcal{M}'$ be a well-monotone open collection in X. Let $\mathcal{M} = \mathcal{M}' \cup \{X\}$. Then $\mathcal{M}$ is a well-monotone open cover. By assumption there exists well-monotone open covers $\mathcal{A}_1, \ldots, \mathcal{A}_n$ with $|\mathcal{A}_i| \leq \kappa$ for each $i \in \{1, \ldots, n\}$ and $U := U_{\mathcal{A}_1} \cap \ldots \cap U_{\mathcal{A}_n} \subseteq U_{\mathcal{M}}$. For each $x \in X$ and for $i \in \{1, \ldots, n\}$ let A_{ix} be the least element of $\mathcal{A}_i$ containing x. Then $|\{U(x) \mid x \in X\}| \leq \kappa$ since $U(x) = A_{1x} \cap \ldots \cap A_{nx}$ and $|\mathcal{A}_i| \leq \kappa$ for each i. Since $U \subseteq U_{\mathcal{M}}$, $|\{U_{\mathcal{M}}(x) \mid x \in X\}| \leq |\{U(x) \mid x \in X\}|$, and since $|\{U_{\mathcal{M}}(x) \mid x \in X\}| = |\mathcal{M}|$, we have $|\mathcal{M}| \leq \kappa$.

$(ii) \Rightarrow (iii)$: Let $B \subseteq X$ and let $B \subseteq \bigcup \mathcal{U}$ where $\mathcal{U} \subseteq \mathcal{O}X$ and $\mathcal{U} = \{O_\alpha \mid \alpha \in \tau\}$. We pick a subsequence $\{\alpha_\mu \mid \mu \in \nu\}$ of τ by letting $\alpha_0 = 0$ and letting α_μ be the least element η of τ for which

$$O_\eta \cup \bigcup_{\sigma<\mu} O_{\alpha_\sigma} \neq \bigcup_{\sigma<\mu} O_{\alpha_\sigma}.$$

Now let $\mathcal{M} = \{M_\mu \mid \mu \in \nu\}$ where $M_\mu = \bigcup_{\beta \leq \mu} O_{\alpha_\beta}$. Clearly $\mathcal{M}$ is a well-monotone open collection and by condition (i), $|\mathcal{M}| = |\nu| \leq \kappa$. Now $B \subseteq \bigcup \mathcal{U} = \bigcup \mathcal{M} = \bigcup_{\mu<\nu} O_{\alpha_\mu}$, so that $\mathcal{U}$ has a subcover of B of cardinality $< \kappa + 1$.

$(iii) \Rightarrow (iv)$: Suppose that X is hereditarily $[\kappa^+, \infty]$-compact and let $\mathcal{F}$ be a κ-prime open filter. Let $\bigcup \mathcal{U} \in \mathcal{F}$ where $\mathcal{U} \subseteq \mathcal{O}X$. By assumption there exists a subcover $\mathcal{A} = \{A_\alpha \mid \alpha \in \tau\}$ of $\mathcal{U}$ with $|\tau| \leq \kappa$. Now let $\mathcal{M} = \{M_\alpha \mid \alpha \in \tau\}$ where $M_\alpha = \bigcup_{\beta \leq \alpha} A_\alpha$. By assumption on $\mathcal{F}$ there exists an $\alpha \in \tau$ such that $M_\alpha \in \mathcal{F}$. Let β be the least such member of τ. Then $M_\beta = \bigcup_{\alpha < \beta} M_\alpha \cup A_\beta$. If $A_\beta \in \mathcal{F}$ we are done so assume that $A_\beta \notin \mathcal{F}$. Then $\bigcup_{\alpha < \beta} M_\alpha \in \mathcal{F}$ and since $|\beta| \leq \kappa$ there is a $\mu < \beta$ such that $M_\mu \in \mathcal{F}$. This contradicts our assumption on the minimality of β and hence $A_\beta \in \mathcal{F}$ and $\mathcal{F}$ is completely prime.

2.4.4 **Proposition**

Let X be a topological space and suppose that we have point-finite open spectra $\mathcal{A}_1, \ldots, \mathcal{A}_m$ and a well-monotone open cover $\mathcal{M}$ with $U_{\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_m} \subseteq U_{\mathcal{M}}$. Then $\mathcal{M}$ is finite.

Proof. Suppose that there did exist point-finite open spectra $\mathcal{A}_1, \ldots, \mathcal{A}_m$ and an infinite well-monotone open cover $\mathcal{M}$ on a topological space X with $U_{\mathcal{A}_1} \cap \ldots \cap U_{\mathcal{A}_m} \subseteq U_{\mathcal{M}}$. By defining a (continuous) map f from the countable discrete space $\mathbf{N}$ to X such that $f^{-1}\mathcal{M} = \{\{0, \ldots, n\} \mid n \in \mathbf{N}\}$ one obtains point-finite open spectra $f^{-1}\mathcal{A}_1, \ldots, f^{-1}\mathcal{A}_m$ and an infinite well-monotone open cover $f^{-1}\mathcal{M}$ with $U_{f^{-1}\mathcal{A}_1} \cap \ldots \cap U_{f^{-1}\mathcal{A}_m} \subseteq f^{-1}\mathcal{M}$. Thus to prove the stated result it is enough to show that it holds on the countable discrete space $\mathbf{N}$ with $\mathcal{M} = \{\{0, \ldots, n\} \mid n \in \mathbf{N}\}$. If $\mathcal{A}$ is a point-finite open spectrum, we will assume that $\mathcal{A} = \{A_n \mid n \in \omega\}$, where $A_0 = X \supset A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots$ and $\bigcap_{n \in \omega} A_n = \emptyset$. Now let $\mathcal{A}_1, \ldots, \mathcal{A}_m$ be point finite open spectra on $\mathbf{N}$ and let $\mathcal{M}$ be as just stated. Suppose that $U := U_{\mathcal{A}_1 \wedge \ldots \wedge \mathcal{A}_m} \subseteq U_{\mathcal{M}}$. For each $i \in \{1, \ldots, m\}$ and $p \in \mathbf{N}$ let $A_{n(i,p)}$ be the smallest member of $\mathcal{A}_i$ which contains p. Define a sequence $(x_p)_{p\in\omega}$ in $\mathbf{N}^m$ by:

$$x_p = (n(1,p), \ldots, n(m,p)).$$

We now see that (x_p) is an infinite h-decreasing sequence in $\mathbf{N}^m$: Let $p, q \in \omega$ with $p > q$. Then $p \notin U(q)$ since $U(q) \subseteq U_{\mathcal{M}}(q) = \{0, \ldots, q\}$. Now suppose that $\pi_i(x_p) \geq \pi_i(x_q)$ for each i, i.e. that $n(i,p) \geq n(i,q)$ for each i. Then, for each i, $A_{n(i,p)} \subseteq A_{n(i,q)}$ so that

$$p \in U(p) = A_{n(1,p)} \cap \ldots \cap A_{n(m,p)} \subseteq A_{n(1,q)} \cap \ldots \cap A_{n(m,q)} = U(q).$$

But $p \notin U(q)$ so $x_q \succ x_p$. We have already seen that $\mathbf{N}^m$ cannot contain any infinite h-decreasing sequences, hence our original assumption, that $U \subseteq U_{\mathcal{M}}$ and $\mathcal{M}$ is infinite, must be false.

2.4.5 **Proposition**

Let X be a hereditarily compact quasi-fc space. Then BX is bicomplete.

Proof. Suppose that X is hereditarily compact. Then every well-monotone open cover must be finite and hence $BX \geq WX = \mathcal{C}_1^* X$. If we further assume that X is a quasi-fc space, then by Proposition 2.3.22 and Corollary 2.3.28 BX is bicomplete.

2.4.6 **Proposition**

Let X be a topological space. The implications

$$(i) \Rightarrow (ii) \Rightarrow (iii)$$

hold, where $(i), (ii)$ and (iii) are given by

(i) $Q_{pf}X \geq WX$;

(ii) every well-monotone open collection on X is refined by a point finite open collection;

(iii) every element of $T_1 K Q_{pf} X$ is completely prime.

Proof.

$(i) \Rightarrow (ii)$: Let $\mathcal{M}'$ be a well-monotone open collection. Then $\mathcal{M} = \mathcal{M}' \cup \{X\}$ is a well-monotone open cover. By assumption there exists a point-finite open cover $\mathcal{A}$ with $U_{\mathcal{A}} \subseteq U_{\mathcal{M}}$. Now $\{U_{\mathcal{A}}(x) \mid x \in X\}$ is point finite and for each $M \in \mathcal{M}'$ we can find $x \in X$ such that $U_{\mathcal{A}}(x) \subseteq M$. (Since $\{U_{\mathcal{M}}(x) \mid x \in X\} = \mathcal{M}$.)

$(ii) \Rightarrow (iii)$: Let $\mathcal{F} \in T_1 K Q_{pf} X$ and let $\mathcal{U} = \{O_\alpha \mid \alpha \in \tau\}$ be a collection of open sets with $\bigcup \mathcal{U} \in \mathcal{F}$. Consider the well-monotone collection $\mathcal{M} = \{M_\alpha \mid \alpha \in \tau\}$, where $M_\alpha = \bigcup_{\beta \leq \alpha} O_\beta$. Let $\mathcal{A}$ be a point-finite refinement of $\mathcal{M}$. By the definition of $\mathcal{F}$ we can find $A \in \mathcal{A}$ such that $A \in \mathcal{F}$. Hence there is an element M_α of $\mathcal{M}$ with $M_\alpha \in \mathcal{F}$. Now let β be the least ordinal α for which $M_\alpha \in \mathcal{F}$. Since $M_\beta = (\bigcup_{\alpha<\beta} M_\alpha) \cup O_\beta$ and $\mathcal{F}$ is a prime open filter, $\bigcup_{\alpha<\beta} M_\alpha \in \mathcal{F}$ or $O_\beta \in \mathcal{F}$. If $O_\beta \in \mathcal{F}$ then $\mathcal{F}$ is completely prime and we are done, so suppose that $\bigcup_{\alpha<\beta} M_\alpha \in \mathcal{F}$. Again we can find a point-finite open refinement $\mathcal{B}$ of $\{M_\alpha \mid \alpha < \beta\}$. Then there is $\alpha < \beta$ such that $M_\alpha \in \mathcal{F}$. This contradicts our assumption on the minimality of β and hence $O_\beta \in \mathcal{F}$.

2.4.7 **Proposition**

Let X be a topological space. The implications

$$(i) \Rightarrow (ii) \Rightarrow (iii)$$

hold, where (i), (ii) and (iii) are given by

(i) $Q_{hlf} X \geq WX$;

(ii) every well-monotone open collection on X is refined by a locally finite open collection;

(iii) every element of $T_1 K Q_{hlf} X$ is completely prime.

Proof. The proof is identical to the proof of Proposition 2.4.6 above.

2.4.8 **Proposition**

Let X be a topological space. Then

(i) $Q_{pf}X$ is bicomplete if X is a quasi-fc-space and every well-monotone open collection has an open point-finite refinement;

(ii) $Q_{hlf}X$ is bicomplete if X is quasi-sober and every well-monotone open collection has an open locally-finite refinement;

(iii) BX is bicomplete if X is a quasi-fc-space and is hereditarily compact ;

(iv) $W_\kappa X$ is bicomplete if X is a quasi-sober space and is hereditarily $[\kappa+1,\infty]$-compact.

Proof. Corollary 2.3.28 and Propositions 2.4.3, 2.4.4, 2.4.5 and 2.4.6.

Recall that for a topological space, X, $\mathcal{I}(X)$ consists of all interior-preserving open covers of X.

2.4.9 **Definition**

(i) A topological space, X, is an *$\mathcal{I}$-space* if every closed $\mathcal{I}(X)$-irreducible set is a point closure.

(ii) An interior-preserving open collection $\mathcal{A}$ is *well-decreasing* if it can be indexed as $\mathcal{A} := \{A_\alpha \mid \alpha \in \tau\}$ for some ordinal τ so that $A_\alpha \subseteq A_\beta$ whenever $\alpha > \beta$.

Let W_u be the T_1-section generated by $\mathbf{Ord}_u$. It is easy to see that W_u is induced by all well-decreasing interior-preserving open covers of X and that W_u preserves subspaces.

2.4.10 **Proposition**

*Let $X \in$ **Top**, let Φ_X be the collection of all well-decreasing interior preserving open covers of X, and let $B \subseteq X$ be a closed, irreducible set. Then B is Φ_X-irreducible if and only if B is $\mathcal{I}_X$-irreducible.*

Proof. Suppose that $B \subseteq X$ is Φ_X-irreducible but that B is not $\mathcal{I}(X)$-irreducible. Then there is $\mathcal{A} \in \mathcal{I}(X)$ such that

$$\bigcap\{A \mid A \in \mathcal{A} \text{ and } A \cap B \neq \emptyset\} \cap B = \emptyset.$$

Let $\{A_\alpha \mid \alpha \in \tau\}$ be a well ordering of $\mathcal{B} := \{A \mid A \in \mathcal{A} \text{ and } A \cap B \neq \emptyset\}$. Consider the well-decreasing interior-preserving collection $\mathcal{W} = \{W_\alpha \mid \alpha \in \tau\}$ where

$$W_\alpha = \bigcap_{\beta \leq \alpha} A_\beta = (\bigcap_{\beta < \alpha} A_\beta) \cap A_\alpha = (\bigcap_{\beta < \alpha} W_\beta) \cap A_\alpha.$$

Now $\bigcap \mathcal{W} \cap B = \bigcap \mathcal{B} \cap B = \emptyset$, and since $\mathcal{W}$ is a well-decreasing interior preserving collection there must exist $\alpha \in \tau$ such that $W_\alpha \cap B = \emptyset$. Let α_o be the least element of τ satisfying this condition, then

$$W_{\alpha_o} \cap B = \bigcap_{\beta < \alpha_o} W_\beta \cap A_{\alpha_o} \cap B = \emptyset.$$

Since B is irreducible and $A_{\alpha_o} \cap B \neq \emptyset$, it follows that $\bigcap_{\beta<\alpha_o} W_\beta \cap B = \emptyset$. But this is impossible since $\{W_\beta \mid \beta < \alpha_o\}$ is a well-decreasing collection and $W_\beta \cap B \neq \emptyset$ for $\beta < \alpha_o$. Hence our initial assumption is false and B is $\mathcal{I}(X)$-irreducible.

The converse is immediate.

In their 1991 paper Künzi and Ferrario asked whether for every space, X, $\phi_t X$ was bicomplete if and only if ϕX was bicomplete. A more general question is of course whether there is a T_1-section strictly coarser than ϕ which is bicomplete whenever ϕ is bicomplete. In the realm of transitive T_1-sections one could ask a similar question: Does there exist a transitive T_1-section $F < \phi_t$ such that for every topological space X, FX is bicomplete if and only if $\phi_t X$ is bicomplete? The above proposition provides the basis for a positive answer.

2.4.11 **Proposition**

Let $F = W \vee W_u$. Then $F < \phi_t$ and FX is bicomplete whenever $\phi_t X$ is bicomplete.

Proof. By Theorem 2.3.26 and Proposition 2.4.10 FX is bicomplete if and only if X is an $\mathcal{I}$-space if and only if $\phi_t X$ is bicomplete.

2.4.12 **Remark**

We now have a range of different characterisations of the category $\mathcal{E}_{\phi_t} = \{X \mid X \in \mathbf{Top}_0$ and $\phi_t X$ bicomplete $\}$:

$\mathcal{E}_{\phi_t} = \mathrm{ERH}_{T_0}\{\mathbf{Alex}_0\} = \mathrm{ERH}_{T_0}\{\mathbf{Pow}_0\} = \mathrm{ERH}_{T_0}\{\mathbf{Ord}_u\}$, where $\mathrm{ERH}_{T_0}\{\mathbf{A}\}$ denotes the epireflective hull of the subcategory $\mathbf{A}$ in the category of T_0-topological spaces.

2.5 A bicompletion-idempotent T_1-section which is not lower-bicompletion-true

Brümmer [1992] calls a T_1-section F *lower K-true* (*upper K-true*) if $KF \leq FT_1KF$ ($KF \geq FT_1KF$), and *K-true* if it both lower and upper K-true. (His definitions were in fact given for a general categorical setting.) Brümmer remarks that if F is K-true, then T_1KF is idempotent, and mentions the open problem of finding a T_1-section F (in any setting) such that T_1KF is idempotent but F fails to be K-true. In what follows we provide a solution to this problem.

2.5.1 **Lemma**

Let $\mathcal{A} \in \Phi_X$ and let $\mathcal{F}, \mathcal{G}$ be minimal $(FX)^*$-Cauchy filters. Then $(\mathcal{F},\mathcal{G}) \in \widetilde{U_\mathcal{A}} \Leftrightarrow \mathcal{A}_\mathcal{F} \subseteq \mathcal{A}_\mathcal{G}$.

Proof. $\Rightarrow$: Let F be in $\mathcal{A}_\mathcal{F}$, i.e. $F \in \mathcal{A}$ and $F \in \mathcal{F}$. Since $F \in \mathcal{A}$, the Fletcher entourage $U_\mathcal{A}$ is contained in the Pervin entourage $S_F := (F \times F) \cup X \setminus F \times X$. Then $\widetilde{U_\mathcal{A}} \subseteq \widetilde{S_F}$, and so by hypothesis the pair $(\mathcal{F},\mathcal{G}) \in \widetilde{S_F}$. Thus we have $H \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $H \times G \subseteq S_F$. Clearly then $(H \cap F) \times G$ is contained in S_F, and hence G is contained in F. (This uses that $H \cap F$ is non-void because H and F are in the same filter.) Thus $F \in \mathcal{A}_\mathcal{G}$, as required.

$\Leftarrow$: By Proposition 2.2.11 $\bigcap \mathcal{A}_\mathcal{F} \in \mathcal{F}$, and hence is in $\mathcal{G}$. By Remark 2.2.12 the set $[\mathcal{F}\mathcal{A}]$ is in $\mathcal{F}$. From this it follows that $[\mathcal{F}\mathcal{A}] \times \bigcap \mathcal{A}_\mathcal{F} \subseteq U_\mathcal{A}$ and then we have $(\mathcal{F},\mathcal{G}) \in \widetilde{U_\mathcal{A}}$, as required.

From the above lemma we can deduce the structure of the quasi-uniform space KFX where the points of KFX are the Φ_X-filters: ent KFX has basis $\{\widetilde{U_\mathcal{A}} \mid U_\mathcal{A} \in \text{ent}\, FX \text{ and } \mathcal{A} \in \mathcal{I}(X)\}$ where $(\mathcal{F},\mathcal{G}) \in \widetilde{U_\mathcal{A}} \Leftrightarrow \mathcal{A}_\mathcal{F} \subseteq \mathcal{A}_\mathcal{G}$.

2.5.2 Lemma

Let $\mathcal{A} \in \Phi_X$. Then $\widetilde{U_\mathcal{A}} = U_{\mathcal{A}^*}$.

Proof. Let $\mathcal{F}, \mathcal{G} \in KFX$, then

$$\begin{aligned}(\mathcal{F},\mathcal{G}) \in U_{\mathcal{A}^*} &\iff (\forall A \in \mathcal{A})\ \mathcal{F} \in A^* \Rightarrow \mathcal{G} \in A^* \\ &\iff ((\forall A \in \mathcal{A})\ A \in \mathcal{F} \Rightarrow A \in \mathcal{G} \\ &\iff \mathcal{A}_\mathcal{F} \subseteq \mathcal{A}_\mathcal{G} \\ &\iff (\mathcal{F},\mathcal{G}) \in \widetilde{U_\mathcal{A}} \text{ by Lemma 2.5.1.}\end{aligned}$$

2.5.3 Lemma

Let F be a transitive T_1-section and let X be a topological space with $FX \geq WX$. Then $\mathcal{O}T_1KFX = \{O^* \mid O \in \mathcal{O}X\}$.

Proof. Recall that $\{O^* \mid O \in \mathcal{O}X\}$ forms a basis for $\mathcal{O}T_1KFX$ so consider $\bigcup_I O_i^* \in \mathcal{O}T_1KFX$. Since $FX \geq WX$ every element of T_1KFX is completely prime. Now

$$\begin{aligned}\mathcal{F} \in \bigcup_I O_i^* &\iff \mathcal{F} \in O_i^* \text{ for some } i \in I \\ &\iff O_i \in \mathcal{F} \text{ for some } i \in I \\ &\iff \bigcup_I O_i \in \mathcal{F} \\ &\iff \mathcal{F} \in (\bigcup_I O_i)^*.\end{aligned}$$

Hence $\bigcup_I O_i^* = (\bigcup_I O_i)^*$ and the result is proved.

2.5.4 Lemma

Let X be a topological space and let $\mathcal{A}$ be an infinite well-decreasing interior-preserving open cover of X. Then $U_\mathcal{A} \notin \mathrm{ent}\, WX$.

Proof. Suppose that $U_{\mathcal{A}} \in \operatorname{ent} WX$. We can define a continuous map $f: \mathbf{N} \to X$ by setting $f(n) = x_n$ where $x_n \in A_n \backslash A_{n+1}$. Clearly $f^{-1}\mathcal{A} = \{\{n, n+1, \ldots\} \mid n \in \mathbf{N}\}$ and $U_{f^{-1}\mathcal{A}} \in \operatorname{ent} W\mathbf{N}$. Let $\mathcal{B} = \{\{0, \ldots, n\} \mid n \in \mathbf{N}\}$. Since $\mathcal{B}$ is a well-monotone open cover $U_{\mathcal{B}} \in \operatorname{ent} W\mathbf{N}$ and hence $U_{\mathcal{B}} \cap U_{f^{-1}\mathcal{A}} \in \operatorname{ent} W\mathbf{N}$. But $U_{\mathcal{B}}(n) \cap U_{f^{-1}\mathcal{A}}(n) = \{n\}$ for every $n \in \mathbf{N}$ and thus $W\mathbf{N} = \phi_t\mathbf{N}$. This cannot hold, for by Proposition 1.2.16 $W\mathbf{N} < \phi_t\mathbf{N}$. Hence $U_{\mathcal{A}} \notin \operatorname{ent} WX$.

A topological space, X, is *superrigid* [van Douwen 1990] if every continuous map $f: X \to X$ is the identity of X or is constant. A topological space X is *irreducible* if every two non-empty open sets in X intersect. For the rest of this section, let Z be the countable T_1 superrigid space described in [van Douwen 1990]. By some relabelling of van Douwen's construction one easily sees that Z has an infinite sequence of infinite subsets Z_n with the following properties:

1. $Z = \bigcup_{n \in \omega} Z_n$ and $\bigcup_{n<m} Z_n \neq Z$ for any $m < \omega$.

2. If $n \neq m$ then $Z_n \cap Z_m$ is finite.

3. Z_n is a closed irreducible subset of Z.

2.5.5 Lemma

There exists a set $B = \{z_n \mid n \in \omega\} \subseteq Z$ with $B \cap Z_n = \{z_n\}$ for every $n \in \omega$.

Proof. Take $z_0 \in Z_0$ and for every $n > 0$ let

$$z_n \in \bigcup_{m \leq n} Z_m \backslash \bigcup_{m<n} Z_m = Z_n \backslash \bigcup_{m<n} Z_m.$$

Let $B = \{z_n \mid n \in \omega\}$ and let $n \in \omega$. Then $B \cap Z_n = \{z_n\}$.

2.5.6 Remark

B is a closed discrete subspace of Z since sets which have finite intersection with each Z_n are closed.

Let $\mathcal{D} = \{D_n \mid n \in \omega\}$ be the interior-preserving open cover of Z given by $D_n = Z \setminus \{z_n\}$. Let $\mathcal{U}$ be the quasi-uniformity on Z with subbase

$$\{U_{\mathcal{A}} \mid \mathcal{A} \text{ is a well-monotone open cover of } Z \text{ or } \mathcal{A} = \mathcal{D}\}.$$

Let F be the transitive T_1-section spanned by $\{\phi_t \alpha_l \mid \alpha \in \mathbf{Ord}\} \cup \{(Z, \mathcal{U})\}$. In view of Proposition 1.4.5, for every $X \in \mathbf{Top}$, ent FX has $\{U_{\mathcal{A}} \mid \mathcal{A}$ is a well-monotone open cover of $X\} \cup \{U_{f^{-1}\mathcal{D}} \mid f\colon X \to Z$ continuous $\}$ as subbase. In particular, since Z is a superrigid space, ent FZ has $\{U_{\mathcal{A}} \mid \mathcal{A}$ is a well-monotone open cover of $Z\} \cup \{\mathcal{D}\}$ as subbase. Thus $FZ = (Z, \mathcal{U})$.

For every topological space X let Φ_X be the collection of all well-monotone open covers of X together with all open covers of the form $f^{-1}\mathcal{D}$ where $f\colon X \to Z$ is a continuous map. Then $\Phi = (\Phi_X)_{X \in \mathbf{Top}}$ is a natural adequate kind of interior preserving open covers inducing F.

2.5.7 Lemma

FZ is not bicomplete.

Proof. We show that Z is not a Φ-space (see Definition 2.3.14), i.e. Z contains a non-trivial closed Φ_Z-irreducible subset. We have already noted that Z_0 is a closed irreducible subset of Z. Now let $\mathcal{A} \in \Phi_Z$. If $\mathcal{A}$ is a well-monotone open cover then $\bigcap\{A \mid A \in \mathcal{A} \text{ and } A \cap Z_0 \neq \emptyset\} \cap Z_0 \neq \emptyset$ is immediate so consider $\bigcap\{D \mid D \in \mathcal{D} \text{ and } D \cap Z_0 \neq \emptyset\}$. Every element of $\mathcal{D}$ meets Z_0 since Z_0 is infinite. Now $\bigcap_{n\in\omega} D_n = \bigcap_{n\in\omega}(Z \setminus \{z_n\}) = Z \setminus \bigcup_{n\in\omega}\{z_n\}$. We also have that $Z_0 \cap Z \setminus \bigcup_{n\in\omega}\{z_n\} =$

$Z_0 \backslash \{z_0\}$ since $Z_0 \cap \bigcup_{n\in\omega}\{z_n\} = \{z_0\}$. Hence Z_0 is Φ_Z-irreducible. In addition, Z_0 is non-trivial since Z is a T_1-space and Z_0 is infinite.

2.5.8 **Lemma**

Let $f: T_1KFZ \to Z$ be continuous. Then f is constant.

Proof. Let $f: T_1KFZ \to Z$ be continuous. If $|f.j[Z]| = 1$, then since $j[Z]$ is dense in T_1KFZ and Z is a T_1-space, $|f[T_1KFZ]| = 1$. So suppose that $|f.j[Z]| > 1$. Then $f.j: Z \to Z$ is not constant and then since Z is superrigid $f.j = 1_X$. Since Z_0 is Φ_Z-irreducible (see proof of Lemma 2.5.7), we have by Lemma 2.3.16 that $\mathcal{F} := \mathcal{O}_{Z_0} = \{O \mid O \cap Z_0 \neq \emptyset\} \in T_1KFZ$. It is easy to check that $\lambda(\mathcal{F}) = Z_0$ and then by Lemma 2.3.4 $j^{-1}[\text{cl}\{\mathcal{F}\}] = Z_0$, i.e. for every $z \in Z_0, j(z) \in \text{cl}\{\mathcal{F}\}$. Since Z is a T_1-space and $|Z_0| > 1$, there exists a $z \in Z_0 \backslash \text{cl}\{f(\mathcal{F})\} = Z_0 \backslash \{f(\mathcal{F})\}$. Now $Z \backslash \{f(\mathcal{F})\}$ is an open set which contains z and so $f^{-1}[Z \backslash \{f(\mathcal{F})\}]$ is an open set in T_1KFZ containing $j(z)$ (since $j(z) \in f^{-1}(z)$) but which does not contain $\mathcal{F}$, i.e. $j(z) \notin \text{cl}\{\mathcal{F}\}$, contradicting our choice of z. Thus f is constant.

2.5.9 **Lemma**

Let $\mathcal{D}$ be the open cover of Z described after Remark 2.5.6 above.

Then $U_{\mathcal{D}^*} \notin \text{ent}\, FT_1KFZ$.

Proof. Recalling the construction of the functor F, we note that $\text{ent}\, F(T_1KFZ)$ has as subbase $\{U_{\mathcal{A}} \mid \mathcal{A}$ is a well-monotone open cover of $T_1KFZ\} \cup \{U_{f^{-1}\mathcal{D}} \mid f: T_1KFZ \to Z$ is continuous $\}$. But by Lemma 2.5.8 every continuous map from T_1KFZ to Z is constant and hence $F(T_1KFZ) = W(T_1KFZ)$.

Now suppose that $U_{\mathcal{D}^*} \in \text{ent}\, WT_1KFZ$. Let $B = \{z_n \mid n \in \omega\}$ be as in Lemma 2.5.5. Define the (continuous) map $f: \mathbf{N} \to Z$ by $f(n) = z_n$. Now $j.f: \mathbf{N} \to T_1KFZ$ is continuous and thus $U_{(j.f)^{-1}\mathcal{D}^*} = U_{f^{-1}j^{-1}\mathcal{D}^*} \in \text{ent}\, W\mathbf{N}$. Now $j^{-1}\mathcal{D}^* = \mathcal{D}$ and $f^{-1}[D_n] = f^{-1}[Z \setminus \{z_n\}] = \mathbf{N} \setminus f^{-1}(z_n) = \mathbf{N} \setminus \{n\}$, and hence $f^{-1}j^{-1}D^* = \{\mathbf{N} \setminus \{n\} \mid n \in \mathbf{N}\}$. For every $m, n \in \mathbf{N}$ with $m \neq n$, $n \in \mathbf{N} \setminus \{m\} = f^{-1}j^{-1}[D_m^*]$ but $m \notin \mathbf{N} \setminus \{m\}$. Since $U_{f^{-1}j^{-1}\mathcal{D}^*}(n)$ is the intersection of all members of $f^{-1}j^{-1}\mathcal{D}^*$ which have n as a member, it follows that $U_{f^{-1}j^{-1}\mathcal{D}^*}(n) = \{n\}$, i.e. $U_{f^{-1}j^{-1}\mathcal{D}^*}$ is the discrete entourage on $\mathbf{N}$. But this cannot hold since by Proposition 1.2.16 $W\mathbf{N} < \phi_t\mathbf{N}$. Hence $U_{\mathcal{D}(n)^*} \notin \text{ent}\, WT_1KFZ$.

2.5.10 **Proposition**

F is not lower-K-true, but is upper-K-true.

Proof. By Lemma 2.5.2 KFZ has basis $\{U_{\mathcal{A}^*} \mid U_{\mathcal{A}} \in \text{ent}\, FZ,\, \mathcal{A} \in \mathcal{I}(Z)\}$. By Lemma 2.5.9 there exists an $\mathcal{A} \in \mathcal{I}(Z)$ with $U_{\mathcal{A}} \in \text{ent}\, FZ$ but $U_{\mathcal{A}^*} \notin \text{ent}\, FT_1KFZ$, i.e. $KFZ \not\leq FT_1KFZ$. Since $F \geq W$, F is upper-K-true with $KFZ > FT_1KFZ$.

2.5.11 **Lemma**

*Let $X \in$ **Top** and let $f: X \to Z$ be continuous. For every completely prime open filter $\mathcal{F}$ on X, $\bigcap\{f^{-1}D_n \mid f^{-1}D_n \in \mathcal{F}$ and $D_n \in \mathcal{D}\} \in \mathcal{F}$.*

Proof. Recall that for a completely prime open filter $\mathcal{F}$, $\lambda(\mathcal{F})$ is irreducible and for every $O \in \mathcal{O}X$, $O \in \mathcal{F} \Leftrightarrow O \cap \lambda(\mathcal{F}) \neq \emptyset$. Now $f^{-1}D_n = f^{-1}[Z \setminus \{z_n\}] = X \setminus f^{-1}(z_n)$, and hence $f^{-1}D_n \in \mathcal{F} \Leftrightarrow X \setminus f^{-1}(z_n) \cap \lambda(\mathcal{F}) \neq \emptyset \Leftrightarrow \lambda(\mathcal{F}) \not\subseteq f^{-1}(z_n)$. Let $\{n_i \mid i \in I\} \subseteq \mathbf{N}$ be such that $f^{-1}D_{n_i} \in \mathcal{F}$ for each $i \in I$. Now suppose that $\bigcap\{f^{-1}D_{n_i} \mid i \in I\} \notin \mathcal{F}$, i.e. $\bigcap\{X \setminus \{f^{-1}(z_{n_i})\} \mid i \in I\} \notin \mathcal{F}$. Then $(X \setminus \bigcup_{i \in I} f^{-1}(z_{n_i})) \cap \lambda(\mathcal{F}) = \emptyset$, i.e. $\lambda(\mathcal{F}) \subseteq \bigcup_{i \in I} f^{-1}(z_{n_i})$. This cannot hold since $\{z_{n_i} \mid i \in I\}$ is a closed discrete subspace of Z and $\lambda(\mathcal{F})$ is irreducible. Hence $\bigcap_{i \in I} f^{-1} D_{n_i} \in \mathcal{F}$.

2.5.12 **Proposition**

Let $X \in \mathbf{Top}$. *Then* $T_1 KFX = T_1 KWX$.

Proof. Since $F \geq W$, T_1KFX is a subspace of T_1KWX. Now let $\mathcal{F}$ be a completely prime open filter. Then by the lemma above $\mathcal{F} \in T_1KFX$, i.e. $T_1KWX \subseteq T_1KFX$. Hence $T_1KFX = T_1KWX$.

2.5.13 **Corollary**

T_1KF *is idempotent.*

Proof. By [Künzi and Ferrario 1991] we know that T_1KW is idempotent.

Chapter 3

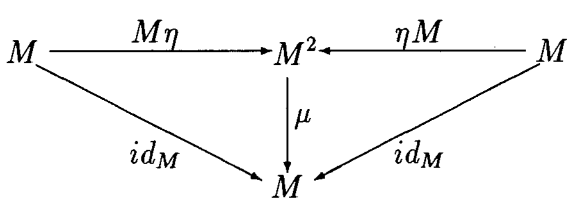
Monads induced by bicompletions

3.1 The transitive bicompletion as a monad

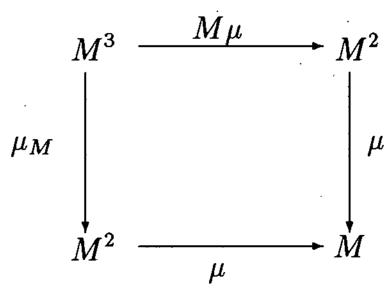
3.1.1 **Definition** [MacLane 1971]

A *monad* on a category, $\mathbf{X}$, is a triple (M, η, μ) where $M: \mathbf{X} \longrightarrow \mathbf{X}$ is a functor and $\eta: 1_{\mathbf{X}} \longrightarrow M$ and $\mu: M^2 \longrightarrow M$ are natural transformations making the following diagrams commute:

a)



b)



Recall that for every transitive T_1-section F we can characterise the elements of T_1KFX where $X \in \mathbf{Top}$, as the Φ_X-filters.

For the rest of this paper let F be a fixed lower K-true transitive T_1-section and let $M := T_1KF$.

Elements of MX will be denoted by $\underline{u}$ and $\underline{v}$ while $\mathcal{F}$ and $\mathcal{G}$ will be used to denote elements of M^2X.

3.1.2 Proposition

*Let $X, Y \in$ **Top** and let $f: X \to Y$ be continuous. Then $KFf: KFX \to KFY$ is given by, for every $\underline{u} \in KFX$,*

$$KFf(\underline{u}) = \{O \in \mathcal{O}Y \mid f^{-1}[O] \in \underline{u}\}$$

Proof. Let $g: KFX \to KFY$ be given by $g(\underline{u}) = \{O \in \mathcal{O}Y \mid f^{-1}[O] \in \underline{u}\}$ for every $\underline{u} \in KFX$. We check that for $\underline{u} \in KFX$, $g(\underline{u}) := \underline{v} \in KFY$. It is clear that $\underline{v}$ is an open filter and it is prime since whenever $O_1 \cup O_2 \in \underline{v}$ it follows that $f^{-1}[O_1] \in \underline{u}$ or $f^{-1}[O_2] \in \underline{u}$, and thus $O_1 \in \underline{v}$ or $O_2 \in \underline{v}$. We now need to show that $\underline{v}$ is a Φ_Y-filter: Note that for each $\mathcal{A} \in \Phi_Y$, $f^{-1}\mathcal{A} \in \Phi_X$. Let $\mathcal{B} = \mathcal{A} \cap \underline{v} = \{A \in \mathcal{A} \mid A \in \underline{v}\}$. Then $f^{-1}\mathcal{B} \subseteq \underline{u}$ and since $\underline{u}$ is a Φ_X-filter, $\bigcap f^{-1}\mathcal{B} \in \underline{u}$. Hence $f^{-1}\bigcap\mathcal{B} \in \underline{u}$ and thus $\bigcap\mathcal{B} \in \underline{v}$. Let $\mathcal{D} = \{A \in \mathcal{A} \mid A \notin \underline{v}\}$. Recall that $(f^{-1}\mathcal{A})_{\underline{u}^c} = \{f^{-1}[A] \in f^{-1}\mathcal{A} \mid f^{-1}[A] \notin \underline{u}\}$. For each $D \in \mathcal{D}$, $f^{-1}[D] \notin \underline{u}$, so that $f^{-1}\mathcal{D} \subseteq (f^{-1}\mathcal{A})_{\underline{u}^c}$, and thus $\bigcup f^{-1}\mathcal{D} \subseteq \bigcup(f^{-1}\mathcal{A})_{\underline{u}^c}$. Since $\underline{u}$ is a Φ_X-filter, $\bigcup(f^{-1}\mathcal{A})_{\underline{u}^c} \notin \underline{u}$, and hence $\bigcup f^{-1}\mathcal{D} \notin \underline{u}$. Thus $\bigcup\mathcal{D} \notin \underline{v}$ and $\underline{v}$ is a Φ_Y-filter.

We now show that g is quasi-uniformly continuous. Recall that for each $\mathcal{A} \in \Phi_Y$, $\mathcal{A}^* = \{A^* \mid A \in \mathcal{A}\}$ and $U_{\mathcal{A}^*}$ is a basic entourage of KFY. Now let $\mathcal{A} \in \Phi_Y$. Then $(g \times g)^{-1}U_{\mathcal{A}^*} = U_{g^{-1}\mathcal{A}^*}$. For each $A \in \mathcal{A}$:

$$\begin{aligned} \underline{u} \in g^{-1}[A^*] &\iff g(\underline{u}) \in A^* \\ &\iff A \in g(\underline{u}) \end{aligned}$$

$$\iff f^{-1}[A] \in \underline{u}$$

$$\iff \underline{u} \in (f^{-1}[A])^*,$$

and so $\{g^{-1}[A^*] \mid A \in \mathcal{A}\} = \{(f^{-1}[A])^* \mid A \in \mathcal{A}\}$. Since $(f^{-1}\mathcal{A}) \in \Phi_X$, $U_{(f^{-1}\mathcal{A})^*} = U_{g^{-1}\mathcal{A}^*} \in \text{ent}\, KFX$ and thus g is quasi-uniformly continuous.

Let $x \in FX$. Then

$$\begin{aligned} O \in k_{FY}.Ff(x) &\iff Ff(x) \in O \\ &\iff f(x) \in O \\ &\iff x \in f^{-1}[O] \\ &\iff f^{-1}[O] \in k_{FX}(x) \\ &\iff O \in g(k_{FX}(x)). \end{aligned}$$

Thus $k_{FY}.Ff = g.k_{FX}$. By naturality $k_{FY}.Ff = KFf.k_{FX}$, and thus $g.k_{FX} = KFf.k_{FX}$. Since k_{FX} is an epimorphism relative to T_0 quasi-uniform spaces [Fletcher and Lindgren 1982, Theorem 3.33] it follows that $g = KFf$.

3.1.3 Corollary

*Let $X, Y \in$ **Top** and let $f: X \to Y$ be continuous. Then for each $\underline{u} \in MX$, $Mf(\underline{u}) = \{O \in \mathcal{O}Y \mid f^{-1}[O] \in \underline{u}\}$.*

3.1.4 Lemma

Let $O \in \mathcal{O}X$. Then:

$(Mf)^{-1}[O^*] = f^{-1}[O]^*$.

Proof.

$$\underline{u} \in (Mf)^{-1}[O^*] \iff Mf(\underline{u}) \in O^*$$

$$\begin{aligned} &\iff O \in Mf(\underline{u}) \\ &\iff f^{-1}[O] \in \underline{u} \\ &\iff \underline{u} \in f^{-1}[O]^* \end{aligned}$$

If $\mathcal{A} \subseteq \mathcal{O}X$ then $\mathcal{A}^* := \{A^* \mid A \in \mathcal{A}\}$.

3.1.5 Lemma

Let $\mathcal{A} \in \Phi_X$. Then for any $\mathcal{B} \subseteq \mathcal{A}$:

(i) $\bigcap \mathcal{B}^* = (\bigcap \mathcal{B})^*$;

(ii) $\bigcup \mathcal{B}^* = (\bigcup \mathcal{B})^*$.

Hence, if $\mathcal{A}$ is an interior preserving open cover on X then $\mathcal{A}^*$ is an interior preserving open cover on MX.

Proof. (i):

$$\begin{aligned} \underline{v} \in \bigcap \mathcal{B}^* &\iff \underline{v} \in B^* \ \forall B \in \mathcal{B} \\ &\iff B \in \underline{v} \ \forall B \in \mathcal{B} \\ &\iff \bigcap \mathcal{B} \in \underline{v} \quad \text{because } \underline{v} \text{ is a } \Phi_X\text{-filter} \\ &\iff \underline{v} \in \left(\bigcap \mathcal{B}\right)^* \end{aligned}$$

(ii): Let $\underline{u} \in \bigcup \mathcal{B}^*$. Then there exists a $B \in \mathcal{B}$ such that $\underline{u} \in B^*$, so that $B \in \underline{u}$, and then $\bigcup \mathcal{B} \in \underline{u}$, which gives $\underline{u} \in (\bigcup \mathcal{B})^*$. Conversely let $\underline{u} \in (\bigcup \mathcal{B})^*$. Then $\bigcup \mathcal{B} \in \underline{u}$. Suppose that $B \notin \underline{u}$ for every $B \in \mathcal{B}$. Then $\bigcup \mathcal{B} \subseteq \bigcup \mathcal{A}_{\underline{u}^c} \notin \underline{u}$ since $\underline{u}$ is a Φ_X-filter. Thus $\bigcup \mathcal{B} \notin \underline{u}$, a contradiction. So there exists a $B \in \mathcal{B}$ with $B \in \underline{u}$, i.e. $\underline{u} \in B^*$, so $\underline{u} \in \bigcup \mathcal{B}^*$.

3.1.6 Proposition

Let $\mu_X : M^2X \longrightarrow MX$ *be given by* $\mu_X(\mathcal{F}) = \mu\mathcal{F} := \{O \in \mathcal{O}X \mid O^* \in \mathcal{F}\}$. *Then* μ_X *is continuous.*

Proof. We first need to show that for each $\mathcal{F} \in M^2X$, $\mu\mathcal{F} \in MX$, i.e. that $\mu\mathcal{F}$ is a Φ_X-filter. Let $\mathcal{A} \in \Phi_X$. Then $U_\mathcal{A} \in \text{ent}\, FX$ and by Lemma 2.5.2 $U^*_\mathcal{A} \in \text{ent}\, KFX$. Since F is lower-K-true $KFX \leq FTKFX$, and so $U^*_\mathcal{A} \in \text{ent}\, FTKFX = \text{ent}\, FMX$. Thus $\mathcal{A}^* \in \Gamma_{MX}$.

Now

$$\begin{aligned} A \in \mathcal{A}_{\mu\mathcal{F}} &\iff A \in \mu\mathcal{F} \text{ and } A \in \mathcal{A} \\ &\iff A^* \in \mathcal{F} \text{ and } A^* \in \mathcal{A}^*, \end{aligned}$$

so $(\mathcal{A}_{\mu\mathcal{F}})^* = (\mathcal{A}^*)_\mathcal{F}$.

We have $\mathcal{F} \in M^2X$, i.e. $\mathcal{F}$ is a Γ_{MX}-filter, and since $\mathcal{A}^* \in \Gamma_{MX}$, $\bigcap(\mathcal{A}^*)_\mathcal{F} \in \mathcal{F}$, i.e. $\bigcap(\mathcal{A}_{\mu\mathcal{F}})^* \in \mathcal{F}$.

By Lemma 3.1.5, $\bigcap(\mathcal{A}_{\mu\mathcal{F}})^* = (\bigcap \mathcal{A}_{\mu\mathcal{F}})^*$, so $(\bigcap \mathcal{A}_{\mu\mathcal{F}})^* \in \mathcal{F}$ i.e. $\bigcap \mathcal{A}_{\mu\mathcal{F}} \in \mu\mathcal{F}$ which is the first condition for $\mu\mathcal{F}$ to be a Φ_X-filter.

To prove the second condition we note that:

$$\begin{aligned} A \in \mathcal{A}_{(\mu\mathcal{F})^c} &\iff A \in \mathcal{A} \text{ and } A \notin \mu\mathcal{F} \\ &\iff A^* \in \mathcal{A}^* \text{ and } A^* \notin \mathcal{F}. \end{aligned}$$

Thus $(\mathcal{A}_{(\mu\mathcal{F})^c})^* = (\mathcal{A}^*)_{\mathcal{F}^c}$. Again since $\mathcal{F}$ is a Γ_{MX}-filter and $\mathcal{A}^* \in \Gamma_{MX}$, $\bigcup(\mathcal{A}^*)_{\mathcal{F}^c} \notin \mathcal{F}$, i.e. $\bigcup(\mathcal{A}_{(\mu\mathcal{F})^c})^* \notin \mathcal{F}$. By Lemma 3.1.5 (ii), then $(\bigcup \mathcal{A}_{(\mu\mathcal{F})^c})^* \notin \mathcal{F}$, i.e. $\bigcup \mathcal{A}_{(\mu\mathcal{F})^c} \notin \mu\mathcal{F}$, which concludes the proof that $\mu\mathcal{F} \in MX$.

To show that μ_X is continuous we need only check that the inverse image of each basic open is again an open set. Now,

$$\begin{aligned}
\mathcal{F} \in {\mu_X}^{-1}[O^*] &\iff \mu_X(\mathcal{F}) \in O^* \\
&\iff O \in \mu_X(\mathcal{F}) \\
&\iff O^* \in \mathcal{F} \\
&\iff \mathcal{F} \in (O^*)^*
\end{aligned}$$

which shows that for every $O \in \mathcal{O}X$, $\mu_X^{-1}[O^*] = (O^*)^*$.

3.1.7 Definition

The map $\mu_X\colon M^2X \to MX$ and the filter $\mu\mathcal{F}$ will be as given in the above Proposition. The map $\eta_X\colon X \to MX$ is given by $\eta_X(x) = \{O \in \mathcal{O}X \mid x \in O\}$.

3.1.8 Proposition

$\eta\colon 1 \to M$ and $\mu\colon M^2 \to M$ are natural transformations.

Proof. Let X and Y be topological spaces and let $f : X \longrightarrow Y$ be a continuous map. For each $x \in X$:

$$\begin{aligned}
O \in \eta_Y(f(x)) &\iff f(x) \in O \\
&\iff x \in f^{-1}[O] \\
&\iff f^{-1}[O] \in \eta_X(x) \\
&\iff O \in Mf(\eta_X(x))
\end{aligned}$$

Hence $\eta_Y.f = Mf.\eta_X$ and η is natural.

Let $\mathcal{F} \in M^2X$, then:

$$\begin{aligned}
O \in \mu_Y(M^2f(\mathcal{F})) &\iff O^* \in M^2f(\mathcal{F}) \\
&\iff (Mf)^{-1}[O^*] \in \mathcal{F} \text{ by Proposition 3.1.2} \\
&\iff (f^{-1}[O])^* \in \mathcal{F} \text{ by Lemma 3.1.4}
\end{aligned}$$

3.2 A classification of interior-preserving open covers

3.2.1 Definition

(i) Let $\mathcal{A}$ be an interior-preserving open cover. A subcollection $\mathcal{B}$ of $\mathcal{A}$ is said to be of type

I1 if $\bigcap \mathcal{B} = \emptyset$ or $\mathcal{B}$ is finite;

I2 if $\bigcap \mathcal{B} = \emptyset$ or $\bigcap \mathcal{B} \in \mathcal{B}$;

(ii) An open cover is said to be of type In, for $n = 1, 2$ if every subcollection is of type In.

3.2.2 Examples

1. Let Φ_X be the collection of either the point finite, the locally finite or the hereditary locally finite covers of a topological space X. Then each $\mathcal{A} \in \Phi_X$ is of type I1.

2. Let Φ_X be the collection of either the well monotone open covers, the open spectra or the hereditary open spectra. Then each $\mathcal{A} \in \Phi_X$ is of type I2.

3. On $(\omega + 1)_u$ the collection of all complements of finite ordinals forms a cover which is neither of type I1 nor of type I2.

3.3 The Eilenberg-Moore category of (M, η, μ)

In this section we will characterise the category of algebras (the Eilenberg-Moore category) of the monads which are generated by functors for which Φ consists of open covers of type I1 or I2. As was mentioned above, almost all of the known examples of transitive T_1-sections fall into this category. The restriction to functors of type I1 or type I2 plays a role only in Lemma 3.3.17, where it would appear to be crucial.

This problem was solved for the case $F = \mathcal{C}_1^*$ by Simmons [1982] and the methods that will be used here are outlined in [Simmons 1982].

3.3.1 **Definition**

Let (M, η, μ) be a monad on a category $\mathcal{X}$. An M-algebra is a pair (X, α) where $X \in \mathcal{X}$ and $\alpha : MX \longrightarrow X$ is a continuous map making the following diagrams commute:

(i)

$$\begin{array}{ccc} X & \xrightarrow{\eta X} & MX \\ & \searrow^{1_X} & \downarrow{\scriptstyle \alpha} \\ & & X \end{array}$$

(ii)

$$\begin{array}{ccc} M^2X & \xrightarrow{\mu X} & MX \\ {\scriptstyle M\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ MX & \xrightarrow[\alpha]{} & X \end{array}$$

An M-algebra morphism from (X, α) to (Y, β) is an $\mathcal{X}$-morphism $f : X \longrightarrow Y$ which makes the following diagram commute:

(iii)

$$\begin{array}{ccc} MX & \xrightarrow{Mf} & MY \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \beta} \\ X & \xrightarrow[f]{} & Y \end{array}$$

The category of M-algebras and their morphisms will be denoted by **M-alg**.

3.3.2 Remark

Recall that given an element $\underline{u}$, of KFX we denote its convergence set by $\lambda\underline{u}$.

3.3.3 Definition

Let $X \in \mathbf{Top}$ and let F be a transitive T_1-section.

(1) X is *F-compact* if $\lambda\underline{u} \neq \emptyset$ for every $\underline{u} \in KFX$.

(2) Let $O, V \in \mathcal{O}X$. Then O is *F-compact* in V (written $O \ll_F V$) iff for every element $\underline{u}$ of KFX, $\lambda\underline{u} \cap V \neq \emptyset$ whenever $O \in \underline{u}$, i.e. if each Φ_X-filter containing O converges to some point in V.

(3) A closed set B in X is *Φ_X-irreducible* if for each $\mathcal{A} \in \Phi_X$, $\bigcap\{A \in \mathcal{A} \mid A \cap B \neq \emptyset\} \cap B \neq \emptyset$.

(4) X is a *Φ-space* if the only Φ_X-irreducible sets are the point closures.

(5) X is *core-F-compact* if for every $O \in \mathcal{O}X$, $O = \bigcup\{U \in \mathcal{O}X \mid U \ll_F O\}$.

(6) X is *F-stable* if whenever $V_1 \ll_F V_2$ and $O_1 \ll_F O_2$ then $V_1 \cap O_1 \ll_F V_2 \cap O_2$.

(7) A continuous map $f : X \longrightarrow Y$ is said to be *stably-F-compacted* if whenever $U \ll_F O$ in Y then $f^{-1}[U] \ll_F f^{-1}[O]$ in X.

3.3.4 Remark

1. F-compactness provides a generalisation of compactness and it is well known that C_1^*-compactness is equivalent to compactness, since C_1^*-compactness is equivalent to the condition that every prime open filter converges.

2. It was noted earlier that Definition 3.3.3(4) is a generalisation of soberness and that X is an open-spectrum space iff X is a quasi-fc-space.

3. Definition 3.3.3(2) generalises the notion of relative compactness. Simmons [1982] has shown that, for $O, U \in \mathcal{O}X$, O is relatively compact in U iff $O \ll_{\mathcal{C}_1^*} O$. Definition 3.3.3(5) is then the analogue of core-compactness.

3.3.5 Lemma

If $\bigcup_I O_\alpha^* = \bigcup_J V_j^*$, *then* $(\bigcup_I O_\alpha)^* = (\bigcup_J V_i)^*$.

Proof. Suppose that $\bigcup_I O_\alpha^* = \bigcup_J V_j^*$. Then $\eta^{-1}[\bigcup_I O_\alpha^*] = \eta^{-1}[\bigcup_J V_j^*]$, and thus $\bigcup_I O_\alpha = \bigcup_J V_i$.

3.3.6 Notation

If $A = \bigcup_I O_1^* \in \Omega MX$, then $\hat{A}$ will denote $(\bigcup_I O_i)^*$. Note that $\eta_X^{-1}\hat{A} = \eta_X^{-1}A$.
For $\underline{u} \in MX$, $\mathcal{F}_{\underline{u}} := \{B \subseteq MX \mid \eta_X^{-1}B \in \underline{u}\}$.

3.3.7 Proposition

$\mathcal{F}_{\underline{u}}$ *is an element of* M^2X.

Proof. It is easy to see that $\mathcal{F}_{\underline{u}}$ is a prime open filter, so we need to show that $\mathcal{F}_{\underline{u}}$ is a Φ_{MX}-filter, i.e. that for every $\mathcal{B} \in \Phi_{MX}$ conditions (i) and (ii) of Definition 2.2.9 hold. Since $\eta_X\colon X \to MX$ is continuous, it follows that for every $\mathcal{B} \in \Phi_{MX}$, $\eta_X^{-1}\mathcal{B} \in \Phi_X$. Let $\mathcal{B} \in \Phi_{MX}$ and let $B \in \mathcal{B} \cap \mathcal{F}_{\underline{u}}$. Then $\eta_X^{-1}B \in \eta_X^{-1}\mathcal{B} \cap \underline{u}$. Now,

$$\begin{aligned} & \{\eta_X^{-1}B \mid B \in \mathcal{B} \text{ and } B \in \mathcal{F}_{\underline{u}}\} \\ = \; & \{\eta_X^{-1}B \mid B \in \mathcal{B} \text{ and } \eta_X^{-1}B \in \underline{u}\} \\ = \; & \eta_X^{-1}\{B \mid B \in \mathcal{B} \text{ and } B \in \mathcal{F}_{\underline{u}}\} \end{aligned}$$

and

$$\begin{aligned} & \{\eta_X^{-1}B \mid B \in \mathcal{B} \text{ and } \eta_X^{-1}B \notin \underline{u}\} \\ = & \{\eta_X^{-1}B \mid B \in \mathcal{B} \text{ and } B \notin \mathcal{F}_{\underline{u}}\} \\ = & \eta_X^{-1}\{B \mid B \in \mathcal{B} \text{ and } B \notin \mathcal{F}_{\underline{u}}\}. \end{aligned}$$

Since $\underline{u}$ is a Φ_X-filter, $\bigcap\{\eta_X^{-1}B \mid B \in \mathcal{B} \text{ and } \eta_X^{-1}B \in \underline{u}\} \in \underline{u}$ and $\bigcup\{\eta_X^{-1}B \mid B \in \mathcal{B} \text{ and } \eta_X^{-1}B \notin \underline{u}\} \notin \underline{u}$. By the above calculation, $\eta_X^{-1}\bigcap\{B \mid B \in \mathcal{B} \text{ and } B \in \mathcal{F}_{\underline{u}}\} \in \underline{u}$ and $\eta_X^{-1}\bigcup\{B \mid B \in \mathcal{B} \text{ and } B \notin \mathcal{F}_{\underline{u}}\} \notin \underline{u}$, i.e. $\bigcap\{B \mid B \in \mathcal{B} \text{ and } B \in \mathcal{F}_{\underline{u}}\} \in \mathcal{F}_{\underline{u}}$ and $\bigcup\{B \mid B \in \mathcal{B} \text{ and } B \notin \mathcal{F}_{\underline{u}}\} \notin \mathcal{F}_{\underline{u}}$.

3.3.8 Proposition

For each $\underline{u} \in MX$, $\mu_X(\mathcal{F}_{\underline{u}}) = \underline{u}$.

Proof.

$$\begin{aligned} O \in \mu_X(\mathcal{F}_{\underline{u}}) & \iff O^* \in \mathcal{F}_{\underline{u}} \\ & \iff \eta_X^{-1}O^* \in \underline{u} \\ & \iff O \in \underline{u}. \end{aligned}$$

Let the functor F be fixed. We will write $\ll$ for $\ll_F$ where no confusion can occur.

3.3.9 Proposition

Let $A, B \in \Omega MX$. Then $A \ll B$ if and only if $A \in \mathcal{F} \Rightarrow \mu\mathcal{F} \in B$ for every $\mathcal{F} \in M^2X$.

Proof. $\Longrightarrow$: Let $\mathcal{F} \in M^2X$ and let $A \in \Omega MX$ be such that $A \in \mathcal{F}$. Since $A \ll B$, there exists a $\underline{v} \in \lambda(\mathcal{F})$ such that $\underline{v} \in B$. Since $\{O^* \mid O \in \mathcal{O}X\}$ is a base for the topology of MX, we can find an $O \in \mathcal{O}X$ with $\underline{v} \in O^* \subseteq B$. It then follows that $O^* \in \mathcal{F}$, and by the definition of $\mu\mathcal{F}$, $O \in \mu\mathcal{F}$. Hence $\mu\mathcal{F} \in O^*$ and we can conclude that $\mu\mathcal{F} \in B$.

$\Leftarrow$: We show that $\mu\mathcal{F} \in \lambda(\mathcal{F})$, from which the desired result follows immediately. Let O^* be a basic open set in MX and suppose that $\mu\mathcal{F} \in O^*$. By the definition of $\mu\mathcal{F}$, $O^* \in \mathcal{F}$, i.e. $\mu\mathcal{F}$ is a limit point of $\mathcal{F}$.

3.3.10 **Corollary**

Let $A, B \in \Omega MX$. Then $A \ll B$ if and only if $\hat{A} \subseteq B$.

Proof. $\Longrightarrow$: Suppose that $A \ll B$ and let $\underline{u} \in \hat{A}$. Since $\eta_X^{-1}\hat{A} = \eta_X^{-1}A \in \underline{u}$ we have, using the definition of $\mathcal{F}_{\underline{u}}$, that $A \in \mathcal{F}_{\underline{u}}$. By Proposition 3.3.8 $\underline{u} = \mu(\mathcal{F}_{\underline{u}})$ and hence by Proposition 3.3.9 $\underline{u} = \mu(\mathcal{F}_{\underline{u}}) \in B$.

$\Leftarrow$: Let $A, B \in \Omega MX$ with $\hat{A} \subseteq B$. Let $\mathcal{F} \in M^2X$ with $A \in \mathcal{F}$. We show that $\mu\mathcal{F} \in B$. Since $A \subseteq \hat{A}$, $\hat{A} \in \mathcal{F}$. Now $\hat{A} = O^*$ for some $O \in \mathcal{O}X$, and so $O = \eta_X^{-1}\hat{A} \in \mu\mathcal{F}$. Hence $\mu\mathcal{F} \in O^* = \hat{A} \subseteq B$.

3.3.11 **Corollary**

Let $A, B \in \Omega MX$. Then $A \ll B$ if and only if $\exists O \in \mathcal{O}X$ such that $A \subseteq O^* \subseteq B$.

3.3.12 **Corollary**

Let $A \in \Omega MX$. Then $\hat{A} = \{\mu\mathcal{F} \mid A \in \mathcal{F} \in M^2X\}$.

3.3.13 **Definition** cf [Simmons 1982]

A T_0 topological space X is said to be *stably-F-compact* if it is an F-compact, core-F-compact, F-stable Φ-space.

We will denote the category of stably-F-compact spaces and stably-F-compacted maps by **SFC**.

3.3.14 Proposition

*Let $X, Y \in$ **Top**. Then MX is a stably-F-compact space and $Mf: MX \to MY$ is a stably-F-compacted map whenever $f: X \to Y$ is continuous.*

Proof. We show that the conditions of Definition 3.3.13 are satisfied. MX is F-compact: Let $\mathcal{F} \in M^2X$. Then $\lambda(\mathcal{F}) \neq \emptyset$ since $\mu\mathcal{F} \in \lambda(\mathcal{F})$. MX is a Φ-space: Let $B \subseteq MX$ be a closed Φ_X-irreducible set. By Proposition 2.3.16 $\mathcal{F}_B = \{F \mid F \cap B \neq \emptyset \text{ and } F \in \Omega MX\} \in M^2X$ and it is clear that $\mu\mathcal{F}_B \in \lambda(\mathcal{F}_B) = B$. We now show that $B = \mathrm{cl}_{MX}\{\mu\mathcal{F}_B\}$, from which it follows that MX is a Φ-space. Let $\underline{u} \in B$ and suppose that $\underline{u} \in O^*$. Then $O^* \in \mathcal{F}_B$ and hence $O \in \mu\mathcal{F}_B$, i.e. $\mu\mathcal{F}_B \in O^*$. MX is core-F-compact: Let $A \in \Omega MX$ and let $\underline{u} \in A$. There exists an $O \in \mathcal{O}X$ such that $\underline{u} \in O^* \subseteq A$. By Corollary 3.3.11 $O^* \ll A$ and since $A = \bigcup\{O^* \mid O^* \subseteq A\}$ it follows that MX is core-F-compact. It remains to show that MX is F-stable so let $A_1, A_2, B_1, B_2 \in \Omega MX$ with $A_1 \ll B_1$ and $A_2 \ll B_2$. By Corollary 3.3.11 there exist $O_1, O_2 \in \mathcal{O}X$ such that $A_i \subseteq O_i^* \subseteq B_i$ for $i = 1, 2$. Thus $A_1 \cap A_2 \subseteq O_1^* \cap O_2^* = (O_1 \cap O_2)^* \subseteq B_1 \cap B_2$, and by Corollary 3.3.11 $A_1 \cap A_2 \ll B_1 \cap B_2$. We have shown that MX is an F-compact, core-F-compact, F-stable, Φ-space and thus MX is a stably-F-compacted space. Finally we show that $Mf: MX \to MY$ is a stably-F-compacted map whenever $f: X \to Y$ is continuous. Suppose that $f: X \to Y$ is continuous and that $A \ll B$ where $A, B \in \mathcal{O}MY$. By Corollary 3.3.11 there exists an $O \in \mathcal{O}Y$ such that $A \subseteq O^* \subseteq B$. Now $Mf^{-1}A \subseteq Mf^{-1}O^* = (f^{-1}O)^* \subseteq Mf^{-1}B$ and by Corollary 3.3.11 $Mf^{-1}A \ll Mf^{-1}B$.

Φ_X-irreducible. Let $\mathcal{A} \in \Phi_X$ and let $\mathcal{B} := \{A \in \mathcal{A} \mid A \cap \lambda\underline{u} \neq \emptyset\}$. Now if $A \in \mathcal{B}$, then $A \in \underline{u}$ and hence $\bigcap \mathcal{B} \neq \emptyset$. Since F is a functor of type I1 or type I2, it follows that $\mathcal{B}$ is either finite or well ordered. In the first case $\bigcap \mathcal{B} \cap \lambda\underline{u} \neq \emptyset$ by the irreducibility of $\lambda\underline{u}$, and in the second case $\bigcap \mathcal{B} \cap \lambda\underline{u} \neq \emptyset$ since $\bigcap \mathcal{B} \in \mathcal{B}$.

3.3.18 **Proposition**

Let X be a stably-F-compact space. Then for each $\underline{u} \in MX$, $\lambda\underline{u}$ is the closure of a unique point.

Proof. By Lemma 3.3.17 $\lambda\underline{u}$ is a closed, irreducible Φ-irreducible set. Since X is a Φ-space, $\lambda\underline{u}$ is a point closure. Uniqueness follows since X is a T_0 space.

3.3.19 **Definition**

Let $X \in \mathbf{SFC}$. Define $\alpha_X : MX \longrightarrow X$ by $\alpha_X(\underline{u}) = x$ such that $\mathrm{cl}_X\{x\} = \lambda\underline{u}$. (Where no confusion can occur we will drop the subscript from α_X.)

3.3.20 **Lemma**

*Let $X \in$ **SFC** and let $V, O \in \mathcal{O}X$ with $V \ll O$. Then $V^* \subseteq \alpha_X^{-1}[O]$.*

Proof. Let $\underline{u} \in V^*$. Since $X \in \mathbf{SFC}$, $\lambda\underline{u} = \mathrm{cl}_X\{\alpha_X(x)\}$ for some $x \in X$. Now $V \ll O$ and thus $\lambda\underline{u} \cap O \neq \emptyset$. Hence $\alpha_X(\underline{u}) \in O$, i.e. $\underline{u} \in \alpha^{-1}O$.

3.3.21 **Lemma**

*Let $X \in$ **SFC**. Then α_X is continuous, stably-F-compacted and $\alpha_X.\eta = 1_X$.*

Proof. We first show that α is continuous. Let $O \in \mathcal{O}X$ and let $\underline{u} \in \alpha^{-1}[O]$. Then $\alpha(\underline{u}) \in O$ and since X is core-F-compact there exists a $V \in \mathcal{O}X$ with $\alpha(\underline{u}) \in V \ll O$.

By Lemma 3.3.20 $V^* \subseteq \alpha^{-1}O$. Now $\underline{u} \in V^* \subseteq \alpha^{-1}O$ and hence $\alpha^{-1}O$ is open. This shows that α is continuous.

We now show that α is stably-F-compacted. Let $V, O \in \mathcal{O}X$ with $V \ll O$. For each $\underline{u} \in \alpha^{-1}V$, $\lambda\underline{u} \cap V \neq \emptyset$ and thus $V \in \underline{u}$ so that $\underline{u} \in V^*$. Thus $\alpha^{-1}V \subseteq V^*$. By Lemma 3.3.20 $V^* \subseteq \alpha^{-1}O$, so that $\alpha^{-1}V \subseteq V^* \subseteq \alpha^{-1}O$. Now by Corollary 3.3.11 $\alpha^{-1}V \ll \alpha^{-1}O$.

Lastly we show that $\alpha.\eta_X = 1_x$. Let $x \in X$. Then $\eta_X(x) \in MX$ is the open neighbourhood filter of x in X and since X is a T_0-space, $\lambda(\eta_X(x)) = \{x\}$. Thus $\alpha(\eta_X(x)) = x$.

3.3.22 Lemma

*Let $X, Y \in$ **SFC** and let $f, g : MX \longrightarrow Y$ be stably-F-compacted maps such that $f\eta_X = g\eta_X$. Then $f = g$.*

Proof. Let $\underline{u} \in MX$. Note that $f.\eta_X : X \to Y$ so that $M(f.\eta_X) : MX \to MY$. Let $\mathcal{F} := M(f.\eta_X)(\underline{u})$. We show that $\lambda(\mathcal{F}) = \mathrm{cl}_Y\{f(\underline{u})\}$. Recall that $M(f.\eta_X)(\underline{u}) = \{O \in \mathcal{O}Y \mid \eta_X^{-1}f^{-1}O \in \underline{u}\}$.

$$\begin{aligned} \text{Now } f(\underline{u}) \in O \in \mathcal{O}Y &\Rightarrow \underline{u} \in f^{-1}O \\ &\Rightarrow \eta_X^{-1}f^{-1}O \in \underline{u} \\ &\Rightarrow O \in M(f.\eta_X)(\underline{u}) = \mathcal{F} \end{aligned}$$

so that $f(\underline{u}) \in \lambda(\mathcal{F})$. Now suppose that $O \in \mathcal{O}Y$ and $O \cap \lambda(\mathcal{F}) \neq \emptyset$. Since Y is core-F-compact we can find a $V \in \mathcal{O}Y$ with $V \in \mathcal{F}$ and $V \ll O$. Since f is a stably-F-compacted map, $f^{-1}V \ll f^{-1}O$ in MX. By Corollary 3.3.11 there exists a $W \in \mathcal{O}X$ such that $f^{-1}V \subseteq W^* \subseteq f^{-1}O$ and thus $\eta_X^{-1}f^{-1}V \subseteq \eta_X^{-1}W^* = W$. Since $V \in \mathcal{F}$, $\eta_X^{-1}f^{-1}V \in \underline{u}$ and then $W \in \underline{u}$, i.e. $\underline{u} \in W^*$.Then $\underline{u} \in f^{-1}O$, and hence $f(\underline{u}) \in O$. We have shown that for $O \in \mathcal{O}Y$, $O \cap \lambda(\mathcal{F}) \neq \emptyset \Rightarrow f(\underline{u}) \in O$, i.e. that $\lambda(\mathcal{F}) = \mathrm{cl}_Y\{f(\underline{u})\}$. Now suppose that $f.\eta_X = g.\eta_X$. Then, for every $\underline{u} \in MX$, $M(f.\eta_X)(\underline{u}) = M(g.\eta_X)(\underline{u})$, so that $\lambda(M(f.\eta_X)(\underline{u})) = \lambda(M(g.\eta_X)(\underline{u})) = \mathrm{cl}_Y\{f(\underline{u})\} = \mathrm{cl}_Y\{g(\underline{u})\}$. Since Y is a T_0 space, $f(\underline{u}) = g(\underline{u})$.

3.3.23 Proposition

*For each $X \in$ **SFC**, (X, α_X) is an M-algebra.*

Proof. Let $X \in \mathbf{SFC}$ and let $\alpha_X : MX \longrightarrow X$ be as given in Definition 3.3.19. We verify the conditions of Definition 3.3.1. By Lemma 3.3.21 condition (i) of Definition 3.3.1 is already fulfilled so we need only check condition (ii), i.e. that $\alpha_X.\mu_X = \alpha_X.(M\alpha_X)$. Consider the following diagram:

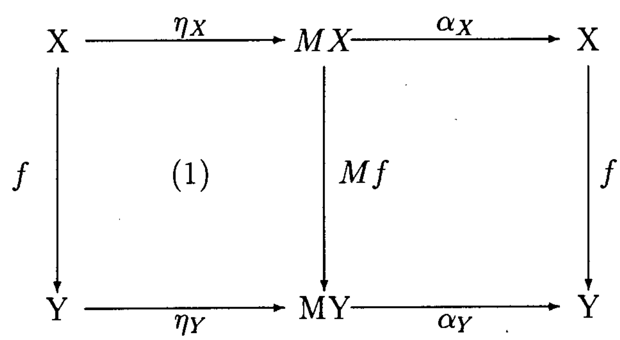
$$\begin{array}{ccccc} MX & \xrightarrow{\eta_{MX}} & M^2X & \xrightarrow{\mu_X} & MX \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle M\alpha_X} & & \downarrow{\scriptstyle \alpha_X} \\ X & \xrightarrow{\eta_X} & MX & \xrightarrow{\alpha_X} & X \end{array}$$

By naturality $M\alpha_X.\eta_{MX} = \eta_X.\alpha_X$, so that $\alpha_X.M\alpha_X.\eta_{MX} = \alpha_X.\eta_X.\alpha_X$. By Lemma 3.3.21 and Definition 3.1.1, $\alpha_X.\eta_X.\alpha_X = 1_X.\alpha_X = \alpha_X = \alpha_X.\mu_X.\eta_{MX}$. Hence $\alpha_X.M\alpha_X.\eta_{MX} = \alpha_X.\mu_X.\eta_{MX}$. By Proposition 3.3.14 and Proposition 3.3.15 $\alpha_X.M\alpha_X$ and $\alpha_X.\mu_X$ are stably-F-compacted and by Lemma 3.3.22 η_{MX} is an **SFC**-epimorphism and thus $\alpha_X.M\alpha_X = \alpha_X.\mu_X$ as required.

3.3.24 Proposition

*Let $X, Y \in$ **SFC** and suppose that $f: X \to Y$ is stably-F-compacted. Then $f: (X, \alpha_X) \to (Y, \alpha_Y)$ is an* **M-alg**-*morphism.*

Proof. We need to show that $f.\alpha_X = \alpha_Y.Mf.$. Consider the following diagram:



By naturality (1) commutes, i.e. $\eta_Y.f = Mf.\eta_X$, and then $\alpha_Y.\eta_Y.f = \alpha_Y.Mf\eta_X$. By Lemma 3.3.21, Proposition 3.3.14 and Proposition 3.3.17 $\alpha_Y.\eta_Y f$ and $f.\alpha_X$ are stably-F-compacted maps and hence by Proposition 3.3.22 $\alpha_Y.Mf = f.\alpha_X$.

We now have a functor $\mathcal{K}$ from **SFC** to **M-alg** which assigns to each stably-F-compacted space X the pair (X, α_X) and which leaves stably-F-compacted maps unchanged. To show that $\mathcal{K}$ is an isomorphism of categories we need to check that it is surjective on objects and morphisms.

3.3.25 **Lemma**

Let $O \in \mathcal{O}X$. Then $MX \backslash O^* = cl_{KX}\{(MX \backslash O^*) \cap \eta_X X\}$.

Proof. Clearly $\text{cl}_{KX}\{(MX \backslash O^*) \cap \eta_X X\} \subseteq MX \backslash O^*$ so let $\underline{u} \in MX \backslash O^*$. Let $V \in \mathcal{O}X$ with $V \in \underline{u}$. Since $V \nsubseteq O$ there exists $x \in V \backslash O$, i.e. $\eta_X(x) \in V^* \backslash O^*$. Now $\eta_X(x) \in MX \backslash O^*$, so $\eta_X(x) \in \eta_X X \cap MX \backslash O^* \cap V^*$, i.e. $V^* \cap (\eta_X X \cap MX \backslash O^*) \neq \emptyset$. Thus $\underline{u} \in \text{cl}_{MX}\{MX \backslash O^* \cap \eta_X X\}$.

3.3.26 **Proposition**

Let (X, θ) be an M-algebra. Then X is stably-F-compacted and $\theta = \alpha_X$.

Proof. We verify the conditions of Definition 3.3.13. X is a T_0-space: Since $\theta.\eta_X = 1_X$, η_X is injective and hence X is a T_0-space.

X is F-compact: Let $\underline{u} \in MX$ and recall that $\lambda\underline{u} = \bigcap\{X \backslash O \mid O \in \mathcal{O}X \text{ and } O \notin \underline{u}\}$. Suppose that $O \notin \underline{u}$. We show that $\theta(\underline{u}) \in X \backslash O$, from which it follows that $\theta(\underline{u}) \in \lambda\underline{u}$ and hence that X is F-compact. First we note that for $x \in X$, $\eta_X(x) \in O^* \Leftrightarrow x \in O \Leftrightarrow \theta(\eta_X(x)) \in O \Leftrightarrow \eta_X(x) \in \theta^{-1}O$ and thus $(MX \setminus O^*) \cap \eta_X X \subseteq MX \setminus \theta^{-1}O$. By Lemma 3.3.25 $MX \setminus O^* = \mathrm{cl}_{MX}\{(MX \setminus O^*) \cap \eta_X X\} \subseteq MX \setminus \theta^{-1}O$ and thus $\theta^{-1}O \subseteq O^*$. By our assumption $\underline{u} \notin O^*$, and hence $\underline{u} \notin \theta^{-1}O$, and thus $\theta(\underline{u}) \notin O$ as required.

X is a Φ-space: Let B be a closed Φ_X-irreducible set in X. By Lemma 2.3.16 $\underline{u} := \{O \in \mathcal{O}X \mid O \cap B \neq \emptyset\} \in MX$. Then $\lambda\underline{u} = B$ so that $\theta(\underline{u}) \in B$. We show that $B = \mathrm{cl}_X\{\theta(\underline{u})\}$. Let $x \in B$. Then $O \in \eta_X(x) \Rightarrow x \in O \Rightarrow O \in \underline{u}$, and thus $\eta_X(x) \in \mathrm{cl}_{MX}\{\underline{u}\}$. Since θ is continuous, $\theta(\eta_X(x)) \in \mathrm{cl}_X\{\theta(\underline{u})\}$, i.e. $x \in \mathrm{cl}_X\{\theta(\underline{u})\}$. Thus $B = \mathrm{cl}_X\{\theta(\underline{u})\}$ and X is a Φ-space.

X is core-F-compact: Let $x \in O \in \mathcal{O}X$. Since $\theta.\eta_X = 1_X$, $\eta_X(x) \in \theta^{-1}O$. We can find a $V \in \mathcal{O}X$ with $\eta_X(x) \in V^* \subseteq \theta^{-1}O$ so that $x \in V \subseteq O$. To finish we need to show that $V \ll O$, so suppose that $V \in \underline{u} \in MX$. Then $\underline{u} \in V^* \subseteq \theta^{-1}O$, i.e. $\theta(\underline{u}) \in O$. Since $\theta(\underline{u}) \in \lambda\underline{u}$, $\lambda\underline{u} \cap O \neq \emptyset$ and $V \ll O$.

X is F-stable: Let $O_1, O_2, V_1, V_2 \in \mathcal{O}X$ with $V_i \ll O_i$ for $i = 1, 2$. Now

$$\begin{aligned} V_1 \cap V_2 \in \underline{u} &\Longrightarrow V_1 \in \underline{u} \ \wedge\ V_2 \in \underline{u} \\ &\Longrightarrow \lambda\underline{u} \cap O_1 \neq \emptyset \ \wedge\ \lambda\underline{u} \cap O_2 \neq \emptyset \\ &\Longrightarrow \theta(\underline{u}) \in O_1 \ \wedge\ \theta(\underline{u}) \in O_2 \end{aligned}$$

(since $\lambda\underline{u} = \mathrm{cl}_X\{\theta(\underline{u})\}$)

$$\begin{aligned} &\Longrightarrow \theta(\underline{u}) \in O_1 \cap O_2 \\ &\Longrightarrow \lambda\underline{u} \cap O_1 \cap O_2 \neq \emptyset \end{aligned}$$

Hence $V_1 \cap V_2 \ll O_1 \cap O_2$.

Finally, to see that $\theta = \alpha_X$ we show that $\lambda\underline{u} = \mathrm{cl}_X\{\theta(\underline{u})\}$. Let $x \in \lambda\underline{u}$. For every $O \in \mathcal{O}X$, $x \in O \Longleftrightarrow \eta_X(x) \in O^* \Rightarrow \underline{u} \in O^*$, and hence $\eta_X(x) \in \mathrm{cl}_X\{\underline{u}\}$. Now $\theta(\eta_X(x)) \in \mathrm{cl}_X\{\theta(\underline{u})\}$, and thus $x \in \mathrm{cl}_X\{\theta(\underline{u})\}$.

3.3.27 **Proposition**

Let $f\colon (X, \alpha_X) \to (Y, \alpha_Y)$ be a morphism in **M-alg**. *Then $f\colon X \to Y$ is a stably-F-compacted map.*

Proof. By Proposition 3.3.26 X and Y are stably-F-compacted spaces and α_X and α_Y are as given in Definition 3.3.1. Let $V, O \in \mathcal{O}Y$ with $V \ll O$. We need to show that $f^{-1}V \ll f^{-1}O$ so suppose that $f^{-1}V \in \underline{u} \in MX$. Then $V \in Mf(\underline{u})$ and since $V \ll O$, $\alpha_Y(Mf(\underline{u})) \in O$. Since $\alpha_Y.Mf = f.\alpha_X$ (see Definition 3.1.1), $f(\alpha_X(\underline{u})) \in O$ and thus $\alpha_X(\underline{u}) \in f^{-1}O$, which gives $f^{-1}O \cap \lambda\underline{u} \neq \emptyset$. This shows that $f^{-1}V \ll f^{-1}O$ and hence f is stably-F-compacted.

3.3.28 **Theorem**

Let F be a lower K-true T_1-section induced by an adequate natural kind of interior-preserving open covers of type I1 or I2. Then **SFC** *is isomorphic to* **M-alg**.

Chapter 4

Strongly zero-dimensional bispaces

4.1 Introduction

Recall that a *bitopological space* (*bispace*) (X, τ_1, τ_2) is a set X endowed with two topologies, τ_1 and τ_2. Given bispaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) one says that a map $f : X \longrightarrow Y$ is *bicontinuous* if both $f : (X, \tau_1) \longrightarrow (X, \sigma_1)$ and $f : (X, \tau_2) \longrightarrow (X, \sigma_2)$ are continuous. **Bitop** is the category which has the bispaces as objects and the bicontinuous maps as morphisms. A bispace, (X, τ_1, τ_2), is *completely regular* if there is a quasi-uniform space $(X, \mathcal{U})$ with $(X, \tau_1, \tau_2) = (X, \tau_{\mathcal{U}}, \tau_{\mathcal{U}^{-1}})$. **Cr2Top** is the full subcategory of **Bitop** which consists of the completely regular bispaces.

The forgetful functors T_1 : **Qu** $\longrightarrow$ **Top** and $\overline{T}$: **Qu** $\longrightarrow$ **Cr2Top** assign to each quasi-uniform space $(X, \mathcal{U})$ the topological space $(X, \tau_{\mathcal{U}})$ and the bispace $(X, \tau_{\mathcal{U}}, \tau_{\mathcal{U}^{-1}})$ respectively.

K_1 : **Cr2Top** $\longrightarrow$ **Top** is the functor which forgets the second topology of a bispace.

Q_1 : **Top** $\longrightarrow$ **Cr2Top** is the unique right inverse to K_1, where $Q_1(X, \tau) = (X, \tau, \tau^*)$ and τ^* has base $\{X \backslash O \mid O \in \tau\}$.

Let F be a $\overline{T}$- (respectively T_1-) section. Then $\mathcal{M}_F := \{X \in \mathbf{Qu} \mid X \leq F\overline{T}X\}$ (respectively $\{X \in \mathbf{Qu} \mid X \leq FT_1X\}$). If $F : \mathcal{X} \longrightarrow \mathcal{Y}$ is any functor then $\mathcal{R}_F := \{FX \mid X \in \mathcal{X}\}$.

4.2 Extending T_1-sections to $\overline{T}$-sections

For basic results about the functors $T_1, \overline{T}, K_1$ and Q_1 we refer the reader to Brümmer [1977].

4.2.1 **Definition** [Brümmer 1982]

A $\overline{T}$-section G is a *$\overline{T}$-section extension* of a T_1-section F if $GQ_1 = F$.

4.2.2 **Remark**

Brümmer [1977] has observed that if a T_1-section F factors through a $\overline{T}$-section G, then necessarily $F = GQ_1$.

4.2.3 **Proposition** [Brümmer 1977, 1982]

Let F be a T_1-section. Then

(i) $\langle \mathcal{R}_F \cup \{\mathbf{I}_q\}\rangle_{\overline{T}} =: \tilde{F}$ *is the coarsest $\overline{T}$-section extension of F;*

(ii) $\langle \mathcal{M}_F\rangle_{\overline{T}} =: \hat{F}$ *is the finest $\overline{T}$-section extension of F;*

(iii) $\mathcal{B}$ *spans a $\overline{T}$-section extension of F whenever* $\mathcal{R}_F \cup \{\mathbf{I}_q\} \subseteq \mathcal{B} \subseteq \mathcal{M}_{\mathcal{F}}$*;*

(iv) $\tilde{F} = \langle\{FT_1A \mid A \in \mathcal{A}\} \cup \{\mathbf{I}_q\}\rangle_{\overline{T}}$ *whenever* $F = \langle\mathcal{A}\rangle_{T_1}$ *;*

(v) the $\overline{T}$-section extensions of F form a large-complete lattice.

4.2.4 **Proposition**

Let $F = \langle\mathcal{A}\rangle_{T_1}$ *be a T_1-section. Then* $G := \langle\mathcal{A} \cup \{\mathbf{I}_q\}\rangle_{\overline{T}}$ *is a $\overline{T}$-section extension of F.*

Proof. Let $X \in \mathbf{Top}$ and let $f : X \longrightarrow T_1A$ be continuous where $A \in \mathcal{A}$. Then $Q_1f : Q_1X \longrightarrow Q_1T_1A$ is bicontinuous, and since $Q_1T_1A \geq \overline{T}A$ it follows that $Q_1f : Q_1X \longrightarrow \overline{T}A$ is bicontinuous. Then, noting that $T_1(FX) = T_1(GQ_1X) = X$, we see that $FX \leq GQ_1X$. Conversely if $g : Q_1X \longrightarrow \overline{T}A$ is bicontinuous, where $A \in \mathcal{A} \cup \{\mathbf{I}_q\}$, then $K_1g : K_1Q_1X \longrightarrow K_1\overline{T}A$ is continuous, i.e. $g : X \longrightarrow T_1A$ is continuous. Noting also that $\overline{T}(GQ_1X) = \overline{T}(FX) = X$, we see that $GQ_1X \leq FX$. The result then follows.

4.2.5 **Remark**

(i) $\mathcal{C}_1^*$ extends uniquely to the coarsest $\overline{T}$-section $\mathcal{C}_b^*$.

(ii) Brümmer [Br82] has shown that $\mathcal{C}_1$ does not extend to a unique $\overline{T}$-section.

(iii) It is still not known whether Brümmer's conjecture [Br77] that every T_1-section other than $\mathcal{C}_1^*$ extends to a proper class of $\overline{T}$-section extensions is true.

Denote the real line by $\mathbf{R}$, the unit interval by $\mathbf{I}$, the integers by $\mathbf{Z}$ and the natural numbers by $\mathbf{N}$. $\mathbf{R}_q$ is the quasi-uniform space with $\mathbf{R}$ as underlying set and $\{\{(x,y) \mid y < x + \epsilon\} \mid \epsilon > 0\}$ as basis. For any subset A of $\mathbf{R}$, A_q is the corresponding quasi-uniform subspace of $\mathbf{R}_q$. $T_1\mathbf{R}_q := \mathbf{R}_u$ is the real line with the upper topology and $\overline{T}\mathbf{R}_q := \mathbf{R}_b$ is the real line with the upper and lower topologies.
Recall from Proposition 1.4.2 that $\mathcal{C}_1$ is spanned by $\mathbf{Z}_q$ and that $\phi_t\mathbf{Z}_u = \mathcal{C}_1\mathbf{Z}_u$. By Proposition 4.2.3 and 4.2.4 $\tilde{\mathcal{C}}_1 = \langle\{\mathbf{Z}_q, \mathbf{I}_q\}\rangle_{\overline{T}}$ is the coarsest $\overline{T}$-section extension of $\mathcal{C}_1$. Let $\mathcal{C}_b$ be the $\overline{T}$-section extension of $\mathcal{C}_1$ which is spanned by $\mathbf{R}_q$. Brümmer [1982] has shown that $\tilde{\mathcal{C}}_1 < \mathcal{C}_b$. We give a different proof of this fact here.

4.2.6 **Proposition**

Let G be any $\overline{T}$-section extension of a transitive T_1-section F where $F \geq \mathcal{C}_1$. If G is spanned by $\mathcal{A} \cup \{\mathbf{I}_q\}$ where $\mathcal{A}$ is a class of transitive quasi-uniform spaces, then $G < \hat{F}$.

Proof. We show that $G\mathbf{R}_b = \mathcal{C}_b^*\mathbf{R}_b$ from which it easily follows that $G\mathbf{R}_b < \mathcal{C}_b\mathbf{R}_b \leq \hat{F}\mathbf{R}_b$. Now the bispace $\mathbf{R}_b$ has no non trivial u-open l-closed subsets and hence $G\mathbf{R}_b$ can contain no non-trivial transitive entourages. Since G is spanned by $\mathcal{A} \cup \{\mathbf{I}_q\}$ and each $A \in \mathcal{A}$ is a transitive quasi-uniform space, it follows that $G\mathbf{R}_b = \mathcal{C}_b^*\mathbf{R}_b$.

4.2.7 **Corollary**

If F is a transitive T_1-section and $F \geq \mathcal{C}_1$, then $\tilde{F} < \hat{F}$.

Proof. By Proposition 4.2.3 the coarsest $\overline{T}$-section extension of F is spanned by $\mathcal{R}_F \cup \{\mathbf{I}_q\}$.

4.2.8 **Corollary**

$\tilde{\mathcal{C}_1} < \mathcal{C}_b$.

4.2.9 **Remark**

The above result shows that there is a proper class of T_1-sections which have non-unique $\overline{T}$-section extensions. This gives a partial answer to a question posed in [Brümmer 1977].

4.3 Strongly zero-dimensional bispaces

4.3.1 Definition

A bispace (X, τ_1, τ_2) is *zero-dimensional* if τ_1 has a base of τ_2-closed sets and τ_2 has a base of τ_1-closed sets.

4.3.2 Remark

Halpin [1974] and Bîrsan [1974] showed that the zero-dimensionality of (X, τ_1, τ_2) is equivalent to (X, τ_1, τ_2) admitting a transitive quasi-uniformity.

We will denote the Stone-Čech compactification functor for bispaces (introduced by Császár [1972] and Salbany [1970]) by $\overline{\beta}$. Recall that a topological space is said to be *strongly zero-dimensional* if its Stone-Čech compactification is zero-dimensional. In an analogous fashion Banaschewski and Brümmer have defined strongly zero-dimensional bispaces.

4.3.3 Definition

A completely regular bispace (X, τ_1, τ_2) is *strongly zero-dimensional* if $\overline{\beta}(X, \tau_1, \tau_2)$ is zero-dimensional.

4.3.4 Remark

For every topological space X, Q_1X is strongly zero-dimensional. Banaschewski and Brümmer [1990] have shown that not every strongly zero-dimensional bispace is in the range of Q_1.

Banaschewski and Brümmer [1990] have shown that a completely regular topological space X is strongly zero-dimensional if and only if every functorial admissible uniformity on X is transitive. In the same paper they then investigated the analogous situation for bispaces. The following result is crucial in this regard. Let $\overline{p}$ denote the totally bounded reflector in **Qu**, cf. [Brümmer 1977].

4.3.5 **Lemma** [Fletcher and Lindgren 1982]

The totally bounded reflector $\overline{p}$ preserves transitivity.

Proof. [Fletcher and Lindgren 1982, Lemma 6.3]

4.3.6 **Proposition** [BaBr90]

*Let $X \in$ **Cr2Top**. Then X is strongly zero-dimensional iff $C_b^* X$ is transitive.*

4.3.7 **Remark**

We can thus characterise the strongly zero-dimensional bispaces as those bispaces which admit at least one functorial transitive quasi-uniformity since if GX is transitive, where G is a $\overline{T}$-section, then $\overline{p}GX = C_b^* X$ is transitive and hence X is strongly zero-dimensional. However the bispace version of the result mentioned after remark 4.3.4 is still outstanding, viz. the characterization of those bispaces for which every functorial admissible quasi-uniformity is transitive.

We now turn our attention to characterising those $\overline{T}$-sections which are transitive-valued on precisely the strongly zero-dimensional bispaces. To this end we make the following definition.

4.3.8 **Definition**

A $\overline{T}$-section is *transitive-fitting* if it is transitive-valued precisely on the strongly zero-dimensional bispaces.

4.3.9 **Remark**

We could omit the word 'precisely' in the above definition, since if GX is transitive, so is $\overline{p}GX = C_b^*X$, and then X is strongly zero-dimensional.

We will denote the full subcategory of **Cr2Top** consisting of the strongly zero-dimensional bispaces by **SZ2Top**.

4.3.10 **Proposition**

Let G be a transitive-fitting $\overline{T}$-section. Then G is an extension of a transitive T_1-section.

Proof. For every $X \in$ **Top**, Q_1X is strongly zero-dimensional [Banaschewski and Brümmer 1990]. Since G is transitive-fitting, GQ_1X is transitive.

4.3.11 **Remark**

(i) Not every $\overline{T}$-section is transitive-fitting since there are non-transitive T_1-sections. In particular ϕ_b, the finest $\overline{T}$-section is not transitive-fitting [Banaschewski and Brümmer 1990].

(ii) There are non-transitive-fitting $\overline{T}$-sections coarser than transitive-fitting $\overline{T}$-sections. [Banaschewski and Brümmer 1990, Prop. 1.4] exhibits two transitive-fitting $\overline{T}$-sections, C_b^* and G, and a $\overline{T}$-section $\overline{S}$ which is not transitive-fitting, such that $C_b^* < \overline{S} < G$.

(iii) Künzi [1992b] solved a question of Banaschewski and Brümmer [1990] by showing that the functor $\mathcal{C}_b$ is transitive-fitting.

Recall that for a topological space, X, we denote the set of interior-preserving open covers of X by $\mathcal{I}(X)$. Given a collection $\mathcal{A}$ of subsets of a set X, $\mathcal{A}^c := \{X \backslash A \mid A \in \mathcal{A}\}$.

4.3.12 **Definition**

For a bispace (X, τ_1, τ_2), $\mathcal{I}(X, \tau_1, \tau_2)$ is the set of all $\mathcal{A} \in \mathcal{I}(X, \tau_1)$ such that $\mathcal{A}^c \in \mathcal{I}(X, \tau_2)$.

Recall that for $A \subseteq X$, $S_A := (A \times A) \cup (A^c \times X)$.

4.3.13 **Proposition**

Let (X, τ_1, τ_2) be a completely regular bispace. Then the following are equivalent:

(i) (X, τ_1, τ_2) is strongly zero-dimensional;

(ii) $\mathcal{C}_b^*(X, \tau_1, \tau_2)$ is transitive;

(iii) $ent\mathcal{C}_b^*(X, \tau_1, \tau_2)$ has base $\{U_\mathcal{A} \mid \mathcal{A} \in \mathcal{I}(X, \tau_1, \tau_2) \ \wedge \ U_\mathcal{A} \in ent\mathcal{C}_1^*(X, \tau_1)\}$;

(iv) $ent\mathcal{C}_b^*(X, \tau_1, \tau_2)$ has base $\{U_\mathcal{A} \mid \mathcal{A}$ is a finite cover of X and $\mathcal{A} \subseteq \tau_1$ and $\mathcal{A}^c \subseteq \tau_2\}$;

(v) $ent\mathcal{C}_b^*(X, \tau_1, \tau_2)$ has subbase $\{S_A \mid A \in \tau_1$ and $A^c \in \tau_2\}$.

Proof. (i) $\Leftrightarrow$ (ii): See Proposition 4.3.6.

(v) $\Rightarrow$ (ii) is clear.

(iii) $\Rightarrow$ (iv): One only needs to note that for an interior-preserving open cover $\mathcal{A}$ of $X \in \mathbf{Top}$, $U_{\mathcal{A}} \in \operatorname{ent} \mathcal{C}_1^* X \Leftrightarrow \mathcal{A}$ is finite cf. Lemma 1.2.19.

(iv) $\Leftrightarrow$ (v): clear.

It only remains to show that $(ii) \Rightarrow (iii)$: Consider any transitive entourage U of $\mathcal{C}_b^*(X, \tau_1, \tau_2)$. Then $U = U_{\mathcal{A}}$ for some (finite) $\mathcal{A} \in \mathcal{I}(X, \tau_1)$ (see [Fletcher and Lindgren 1982, Theorem 2.6]). Since $(X, \tau_1, \tau_2) \leq Q_1(X, \tau_1)$ we have $\mathcal{C}_b^*(X, \tau_1, \tau_2) \leq \mathcal{C}_b^* Q_1(X, \tau_1) = \mathcal{C}_1^*(X, \tau_1)$ and hence $U_{\mathcal{A}} \in \operatorname{ent} \mathcal{C}_1^*(X, \tau_1)$. Clearly $U_{\mathcal{A}}^{-1} = U_{\mathcal{A}^c}$ and since $U_{\mathcal{A}}^{-1}$ belongs to the inverse quasi-uniformity of $\mathcal{C}_b^*(X, \tau_1, \tau_2)$, the same result referred to above makes $\mathcal{A}^c \in \mathcal{I}(X, \tau_2)$. Hence $\mathcal{A} \in \mathcal{I}(X, \tau_1, \tau_2)$.

4.3.14 **Proposition**

Let $\mathcal{A}$ be any class of transitive quasi-uniform spaces spanning a T_1-section F. Then $G := \langle \mathcal{A} \cup \{\mathbf{I}_q\} \rangle_{\overline{T}}$ is a transitive-fitting $\overline{T}$-section extension of F.

Proof. By Proposition 4.2.4, $GQ_1 = F$. Furthermore $G = \langle \mathcal{A} \rangle_{\overline{T}} \vee \langle \mathbf{I}_q \rangle_{\overline{T}} = \langle \mathcal{A} \rangle_{\overline{T}} \vee \mathcal{C}_b^*$. Since $\langle \mathcal{A} \rangle_{\overline{T}}$ is transitive-valued, and $\mathcal{C}_b^*$ is transitive-fitting [Banaschewski and Brümmer 1990], so is G.

4.3.15 **Proposition**

A transitive T_1-section F has at least one transitive-fitting $\overline{T}$-section extension. In particular $\tilde{F}$ is transitive-fitting.

Proof. By proposition 4.3.14 $\tilde{F}$ is transitive-fitting since it is spanned by $\langle \mathcal{R}_F \cup \{\mathbf{I}_q\} \rangle_{\overline{T}}$ and $\mathcal{R}_F$ is a collection of transitive quasi-uniform spaces.

Recall that the transitive T_1-section F is *closed* if F is generated by $\{X_{\mathcal{A}} \mid \mathcal{A} \in \mathcal{I}(X), U_{\mathcal{A}} \in \operatorname{ent} FX$ and $X \in \mathbf{Top}\}$. Denote by $\underline{\mathcal{A}}$ the collection consisting of all unions of intersections of elements of $\mathcal{A}$, i.e. the topology of $X_{\mathcal{A}}$. It is easy to check that $U_{\mathcal{A}} = U_{\underline{\mathcal{A}}}$.

4.3.16 **Lemma**

Let F be a transitive closed T_1-section and let G be a transitive-fitting $\overline{T}$-section extension of F. Then $\operatorname{ent} G(X, \tau_1, \tau_2)$ has $\mathcal{S} := \{U_{\mathcal{A}} \mid \mathcal{A} \in \mathcal{I}(X, \tau_1, \tau_2) \wedge U_{\mathcal{A}} \in \operatorname{ent} F(X, \tau_1)\}$ as base whenever (X, τ_1, τ_2) is strongly zero-dimensional.

Proof. Let (X, τ_1, τ_2) be a strongly zero-dimensional bispace. Since G is transitive-fitting $G(X, \tau_1, \tau_2)$ is transitive. As in the proof of Proposition 4.3.13 we can consider a transitive entourage $U = U_{\mathcal{A}} \in \operatorname{ent} G(X, \tau_1, \tau_2)$ where $\mathcal{A} \in \mathcal{I}(X, \tau_1)$. Since $(X, \tau_1, \tau_2) \leq Q_1(X, \tau_1)$ we have $G(X, \tau_1, \tau_2) \leq GQ_1(X, \tau_1) = F(X, \tau_1)$ and hence $U_{\mathcal{A}} \in \operatorname{ent} F(X, \tau_1)$. Clearly $U_{\mathcal{A}}^{-1} = U_{\mathcal{A}^c}$ and since $U_{\mathcal{A}}^{-1}$ belongs to the inverse quasi-uniformity of $G(X, \tau_1, \tau_2)$ we have $\mathcal{A}^c \in \mathcal{I}(X, \tau_2)$. Hence $\mathcal{A} \in \mathcal{I}(X, \tau_1, \tau_2)$.
Conversely, let $\mathcal{A} \in \mathcal{I}(X, \tau_1, \tau_2)$ and let $U_{\mathcal{A}} \in \operatorname{ent} F(X, \tau_1)$. We have to show that $U_{\mathcal{A}} \in \operatorname{ent} G(X, \tau_1, \tau_2)$. Consider the Alexandroff-discrete space $X_{\mathcal{A}} = (X, \underline{\mathcal{A}})$. Since F is closed, $FX_{\mathcal{A}} = \phi_t X_{\mathcal{A}}$ which has the singleton base $\{U_{\mathcal{A}}\}$ for its entourages. Now $\underline{\mathcal{A}} \subseteq \tau_1$ since $\mathcal{A} \subseteq \tau_1$ and $\mathcal{A}$ is interior preserving and $\underline{\mathcal{A}}^c \subseteq \tau_2$ since $\mathcal{A}^c \subseteq \tau_2$ and $\mathcal{A}^c$ is τ_2-interior preserving. We note that $Q_1(X, \underline{\mathcal{A}}) = (X, \underline{\mathcal{A}}, \underline{\mathcal{A}}^c) \leq (X, \tau_1, \tau_2)$ so that $F(X, \underline{\mathcal{A}}) = GQ_1(X, \underline{\mathcal{A}}) \leq G(X, \tau_1, \tau_2)$. Since $U_{\mathcal{A}} \in \operatorname{ent} F(X, \underline{\mathcal{A}})$, we have $U_{\mathcal{A}} \in \operatorname{ent} G(X, \tau_1, \tau_2)$.

4.3.17 **Remark**

The set $\mathcal{S}$ in Lemma 4.3.16 is exactly the set of all transitive entourages of $G(X, \tau_1, \tau_2)$.

4.3.18 **Theorem**

Let F be a closed transitive T_1-section. Then G is a transitive-fitting $\overline{T}$-section extension of F $\iff$ G is a $\overline{T}$-section and $GX = \tilde{F}X$ for every strongly zero-dimensional bispace X.

Proof. "$\Rightarrow$": Let $X \in$ **SZ2Top**. Then both ent GX and ent $\tilde{F}X$ have the same base given in Lemma 4.3.16. Thus $GX = \tilde{F}X$.

"$\Leftarrow$": For $Y \in$ **Top**, $Q_1 Y$ is strongly zero-dimensional(Remark 4.3.4) so that $GQ_1Y = \tilde{F}Q_1Y = FY$ and thus $GQ_1 = F$. By Proposition 4.3.15 it is clear that G is transitive-fitting.

4.3.19 **Corollary**

Let F be closed. Then, every $\overline{T}$-section extension of F is transitive-fitting if and only if $\hat{F}$ is transitive-fitting.

Proof. This is immediate from Theorem 4.3.18 together with the inequality $\tilde{F} \leq G \leq \hat{F}$ for every $\overline{T}$-section extension G of F.

4.3.20 **Questions**

(i) Banaschewski and Brümmer [1990, Question 1.8(i)] asked whether the finest $\overline{T}$-section extension of ϕ_t - i.e. our $\hat{\phi}_t$ - is transitive-fitting. It is easy to see that ϕ_t is a closed T_1-section. Thus by Corollary 4.3.19 the question is equivalently whether every $\overline{T}$-section extension of ϕ_t is transitive-fitting. We note that $\tilde{\phi}_t$ is transitive-fitting and that there is a finest transitive-fitting $\overline{T}$-section extension of ϕ_t, namely the functor G

of [Banaschewski and Brümmer 1990] which is the finest of all transitive-fitting $\overline{T}$-sections.

Another equivalent version of the Banaschewski and Brümmer question, by virtue of Theorem 4.3.18 is whether all $\overline{T}$-section extensions of ϕ_t coincide on **SZ2Top**. We observe that ϕ_t has more than one $\overline{T}$-section extension because of Corollary 4.2.7.

(ii) The only examples known to us of $\overline{T}$-sections which are not transitive-fitting fail to be extensions of transitive T_1-sections. Is every $\overline{T}$-section extension of a transitive T_1-section transitive-fitting? In particular it would be nice to know this for the case of the semi-continuous T_1-section $\mathcal{C}_1$ (of which we do not know whether it is closed).

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