INTRODUCTION TO LATTICE GAUGE THEORIES

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INTRODUCTION

It is generally accepted today that QCD is the most likely candidate to describe the strong interactions between hadrons. Although it is possible to investigate the theory in the short distance region using conventional perturbation theory, non-perturbative methods are needed to describe the theory in the physically interesting long distance (confinement) region. A prerequisite of any such method is that, starting from basic principles, it should provide a manner of obtaining quantitative physically meaningful results.

It was with this in mind that Wilson [1] introduced the concept of a lattice structure in an attempt to extract information from QCD (e.g. mass spectra and confinement). In this regularized theory, Quantum Field Theory is defined as the limit of a theory with a short distance (ultra violet) cut off a and a volume (infrared) cut off L .

It was one of the first achievements of the newly established theory to show the existence of confinement on the lattice in the strong coupling limit (see Chapter 6). The analogy that exists between the strong coupling expansion of the lattice theory (i.e. a perturbative expansion in terms of the inverse coupling constant), and the relativistic string model, made it possible to relate the (colour) force between a quark-antiquark pair to the string tension between them. A linearly rising potential as a function of the interquark distance R would signal confinement.

It was also apparent from the outset that a strong similarity existed between the strong coupling (SC) regime of the theory and the high temperarture phase in Statistical mechanics. This made it posssible to implement several well-known methods to Lattice Gauge Theories (LGT) that had long been applied to statistical systems, e.g. mean field (MF) methods and Monte Carlo (MC) simulations.

Mean field theory has turned out to be a very useful analytical method to investigate LGT. Applied naively, this implies replacing all the links in the partition function except one by their average (mean field) value M, which is taken to be proportional to the unit matrix (see Chapter 7). It turned out, howevever, that this could not be reconciled with Elitzur's theorem, which states that the (vacuum) expectation value

of a non-gauge invariant quantity (here the link $U_{x,\mu}$) should vanish identically. Although this resulted in the MF approximation being suspect for some time, a solution to the problem was provided by the saddle point approximation, where higher order corrections to the lowest order MF result were included. By summing over all the degenerate saddle point solutions, a vanishing mean field value is obtained in agreement with Elitzur's theorem (see Chapter 7).

The drawback to both mean field and strong coupling methods is that their use as qualitative methods to investigate the theory is limited. For SC methods in QCD this inplies that only the SC phase can be investigated, while little or nothing can be said about the physically interesting weak coupling phase (corresponding to the continuum limit). Near the expected transition points of the various gauge theories all results become suspect due to the singular nature of the phase transition. Mean field methods on the other hand essentially coincide with the SC expansion in the SC region, while in the weak coupling regime it is a rearranged weak coupling expansion.

In the investigation of finite temperature systems MC techniques have been widely applied (see Part II). A thorough understanding of QCD at finite temperature (and density) is vital for many physical applications that have become particularly relevant during the last few years with the construction of accelerators large enough to investigate the possible formation of a quark-gluon plasma. To study the phase transition between the confined hadronic phase and the deconfined (asymptotically free) quark-gluon phase it is important to investigate physical observables such as the energy density over the entire possible temperature range. In this particular case a phase transition will exhibit itself in the form of a sudden change in the behaviour of the physical observable. For the pure gauge theory one investigates the phase structure using as order parameter the Polyakov (thermal Wilson) loop $L(\vec{x})$. This is based on the observation that the pure (SU(N)) Gauge theory is invariant under a global Z(N) transformation, while $L(\vec{x})$ is not. The Polyakov loop can therefore serve as order parameter for confinement by virtue of its identification with the free energy of a quark source: $\langle L \rangle = e^{-\beta F_q}$, $\beta = T^{-1}$. If $\langle L \rangle = 0$, the theory is in a confining phase (corresponding to $F_q = \infty$) whereas if $\langle L \rangle \neq 0$ it is in a non-confining phase. For the full theory (i.e. where the effects of quarks are also included) this identification is not possible as the fermion action explicitly breaks the

global Z(N) symmetry, which results in <L> being non-zero for both phases.

Concerning the calculations itself, it has been established with reasonable certainty that the deconfinement transition in the pure gauge theory with SU(2) and SU(3) as gauge groups is second and first order respectively. Incorporating dynamical fermions into the theory has turned out to be more difficult, and as yet still remains an open problem. The difficulty arises as the Grassmann variables (necessary to describe the anti-commutation properties of the fermion fields) cannot be simulated on a computer and therefore have to be integrated out first, giving an effective fermion determinant. The resulting determinant is highly non-local, so that some kind of approximation is unavoidable. Several approximation schemes have been proposed to deal with this problem, e.g. the hopping parameter expansion (i.e. an expansion in the inverse mass) and the pseudofermion approximation. With all this is mind it is clear that the critical parameters of the phase transition for full QCD are not yet completely resolved. This also applies to the chiral symmetry restoring phase transition, which for quarks in the fundamental represenation occurs virtually simultaneously with the deconfinement transition.

Finally, the results from lattice calculations can only be physically meaningful once the appropriate continuum limit value has been obtained. Alternatively, lattice results (e.g. the deconfinement temperature T_c and string tension) can only be taken seriously if they are near the continuum limit. To ensure that this is indeed the case, use is made of the renormalization group equation for LGT, which provides a prescription of how the coupling constant must change if the lattice spacing is taken to zero.

The thesis is organized as follows.

Part I is a general introduction to LGT. The theory is discussed from first principles, so that for the interested reader no previous knowledge is required, although it is assumed that he/she will be familiar with the rudiments of relativistic quantum mechanics.

Part II is a review of QCD on the lattice at finite temperature and density. Monte Carlo results and analytical methods are discussed. An attempt has been made to include most relevant data up to the end of 1987, and to update some earlier reviews existing on the subject.

To facilitate an understanding of the techniques used in LGT, provision has been made in the form of a separate Chapter on Group Theory and Integration, as well as four Appendices, one of which deals with Grassmann variables and integration.

CHAPTER 1: LATTICE GAUGE THEORY BASICS

1. Lattice Formalisms

To obtain a suitable regularized theory, two different (but equivalent) approaches have been proposed in the literature, namely the Lagrangian and the Hamiltonian formalisms.

The Lagrangian formalism was originally put forward by Wilson [1]. In this formulation, gauge invariance is an exact local symmetry (see Chapter 2) of the action. The first step after the discretization of continuum space-time is to perform a Wick rotation from Minkowski to Euclidian space. All calculations are done using the Euclidian metric.

An advantage of this formalism is the possibility of making an anology between the field theory defined on the lattice and Statistical mechanics, for which several well known calculational methods exist (see Chapters 6 and 7). The field action can be identified with the energy of a configuration, while the vacuum functional integral becomes the partition function. It should, however, be kept in mind that Statistical mechanics is defined on (say) d-dimensional space, while the corresponding QFT problem is defined in d-dimensional space-time.

An alternative approach is the Hamiltonian formalism proposed by Kogut and Susskind [4]. In this formalism, time is treated as a continuous variable, i.e. only space is discretized. A Hamiltonian is defined which describes the quark and gauge degrees of freedom on the spatial lattice. Creutz [5] has shown that the Kogut-Susskind Hamiltonian for LGT can be derived from Wilson's Lagrangian formulation using the transfer matrix formalism [10] in a special gauge (i.e. $A_0 = 0$).

One of the advantages that the Lagrangian formalism has over the Hamiltonian one is that because of the gauge invariant nature of the former, gauge fixing need not be applied. Gauge invariance, however, allows gauge fixing to be implemented without changing the "physics" of the gauge invariant quantities.

It must also be kept in mind that the introduction of a lattice structure destroys the Lorentz (i.e. Euclidian) and rotational invariance of

the original continuum theory. It is expected that these will be restored in the continuum limit ($a \to 0$).

Quantization of the theory is carried out using the (Feynman)path integral formalism (see section 4). We will only consider the Lagrangian formalism which has received widest attention in the literature.

2. Lattice Phenomenology

The lattice Λ is introduced [1;3a;10;11] by defining a set of points $x_\mu \in \Lambda$, $x_\mu = (x_0 , x_1 , \ldots, x_{d-1})$ where $x_i = an_i$ and $n_i = 0, \pm 1, \pm 2 \ldots$ to form the vertices of a hypercubical lattice with spacing a. Although we will only consider lattices with a cubical structure, many alternatives have been discussed in the literature, e.g. the random lattice [6].

The matter fields (e.g. scalar and fermion) are defined on the lattice points, while the gauge fields are defined on the nearest neighbour (n.n.) links and play a crucial role in the context of local gauge invariance (see Chapter 2).

The links $U_{x,\mu}$ take their values in the Gauge group:

$$U_{x,\mu} \equiv U_\mu(x) = e^{iagA_\mu(x)} \tag{1}$$

with $A_\mu(x) \equiv T^a A^a_\mu(x)$, $A^a_\mu(x)$ being the gauge potential. (1a)

The Gauge groups that will be considered will always be **compact** Lie groups, which offer advantages for integration over the group space (see Chapter 5).

The link variables have a definite direction determined by $\mu = 0,1,2, \ldots,d-1$ however, reversing the direction of the link between two nearest neighbour sites will not produce a new degree of freedom but rather requires that U becomes its inverse in the group, i.e. if

$$\underset{x \qquad x+\mu}{\longrightarrow} \quad U_{x,\mu} \equiv U_{x,x+\mu} ,$$

then $$\underset{x \qquad x+\mu}{\longleftarrow} \equiv \underset{x+\mu \qquad x}{\longrightarrow} \quad U^\dagger_{x,\mu} \equiv U_{x+\mu,x} = U_{x+\mu,-\mu} . \tag{2}$$

For a d dimensional cubic lattice with linear (finite) dimension L (i.e. $L = \sum_i x_i$), the number of lattice sites is equal to

$$N \equiv L^d. \tag{3}$$

The corresponding number of links is Nd. Also, in general most lattices have periodic boundary conditions, i.e. with $x_{i+L} = x_i$. To get a feeling for the way that physical quantities are defined on the lattice, we consider any function f(x) defined at the lattice points.

To go to momentum space we introduce the Fourier transform

$$g(k) = a^4 \sum_{x=-\infty}^{\infty} e^{i(k.x)a} f(x) \ . \tag{4}$$

To get the inverse relation we use

$$\int_{-\pi/a}^{\pi/a} e^{ik.(x-y)a} \, dk = \delta_{x,y} \left(\frac{2\pi}{a}\right)^4 \tag{5}$$

where the boundaries of the integral are fixed by the fact that g(k) is periodic over the interval

$$|k_\mu| \leq \frac{\pi}{a} \quad (k \equiv k_\mu) \ , \tag{6}$$

i.e. the momentum values are constrained within the first Brillouin zone. The inverse transform therefore reads

$$f(x) = \frac{1}{(2\pi)^4} \int_{-\pi/a}^{\pi/a} d^4k \, g(k) \, e^{-i(k.x)a} \tag{7}$$

It follows that the lattice regularization has introduced a natural cut-off in momentum space, with cut-off value $k_\mu = \pm \frac{\pi}{a}$, or

$$\Lambda_{\text{cut-off}} \sim \frac{1}{a}$$

(hence in the continuum limit ($a \to 0$), $\Lambda \to \infty$, i.e. Λ is an ultra-violet cut-off).

Given a continuum (Euclidian) Lagrangian we can define the corresponding lattice theory as follows:

- scalar fields (spin 0) are defined on the lattice points
- vector fields (spin 1) are defined on the links
- spinor fields are also defined on the lattice parts, however these are plagued by a degeneracy problem which has to be removed or suitably side-stepped (see Chapter 4).

Just as hypercubical lattices are not the only possibility that can be considered, the internal structure of the lattice may also be altered. If the lattice spacing in the different directions (usually space and time) are not the same, we have an asymmetric lattice which is characterized by the asymmetry parameter

$$\xi \equiv \frac{a_\sigma}{a_\tau} , \tag{8}$$

where a_σ is the lattice spacing in the spatial directions
a_τ is the lattice spacing in the time direction.

It follows that a symmetric lattice would correspond to the choice

$$\xi = 1 \qquad (\text{i.e. } a_\sigma = a_\tau = a). \tag{9}$$

Unless stated otherwise, we will consider lattices with a symmetric structure in the following. Asymmetric lattices are of interest for the theory at finite temperature (see Part II).

3. Free Scalar (spin 0) Field on a Lattice

To show how the discretization of continuum space-time is implemented, we consider as an example the complex scalar field [2;8] with a ϕ^4 interaction term.

In order to go to a discrete space-time formalism, the following substitutions are made:

(a) the derivative of the scalar field is replaced by a difference between fields on neighbouring lattice sites. The simplest choice is to use the nearest neighbour (n.n.) difference

$$\partial_\mu \phi(x) \to \frac{1}{a}[\phi(x+\hat{\mu}) - \phi(x)] \tag{10}$$

with (i) $\phi(x+\hat{\mu}) \equiv \phi(x+a\hat{\mu})$ where $\hat{\mu}$ is the unit vector in the direction μ.

(ii) and x denotes the four vector $x = (x_0, \vec{x})$.

Note that if we choose a next-to-nearest neighbour difference, we can write the derivatve in a more **symmetric** form:

$$\partial_\mu \phi(x) \to \frac{1}{2a}[\phi(x+\hat{\mu}) - \phi(x-\hat{\mu})] \tag{11}$$

(b) The integral of the continuum theory is approximated by the sum

$$\int d^4x \to a^4 \sum_x . \tag{12}$$

In the Euclidian continuum formalism the action for a <u>scalar</u> field is given by [11] (d=4)

$$S = \int d^4x \, [\tfrac{1}{2}|\partial_\mu \phi|^2 + V(\phi)] \tag{13}$$

with $V(\phi) = \tfrac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4$ for the specific ϕ^4 interaction. (13a)

The corresponding lattice expression for the free theory is obtained by using eqs.(11) and (12) (setting $\lambda = 0$)

$$S(\phi) = \sum_x \tfrac{1}{2} a^4 \left\{ \sum_{\mu=0}^{3} \left[\frac{\phi(x+\hat{\mu}) - \phi(x)}{a}\right]^2 + m^2\phi^2(x) \right\}$$

$$= \sum_x \left\{ \frac{a^2}{2} \sum_{\mu=0}^{3} \left[\phi(x+\hat{\mu}) - \phi(x)\right]^2 + a^4 \frac{m^2}{2} \phi^2(x) \right\}. \tag{14}$$

To find the spectrum of the free field theory we go to momentum space using the Fourier transform given in eq.(7)

For the first term in the action (11) this gives

$$I = \sum_x \left[\phi(x+\hat{\mu}) - \phi(x)\right]^2 = \sum_x \left\{ \int \frac{d^4k}{(2\pi)^4} e^{i(k.x+\hat{\mu}a)}\phi(k) - e^{ik.x}\phi(k) \right\}^2$$

$$= \sum_x \int \frac{d^4k'}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \left[e^{i(k.x+\hat{\mu}a)} \phi(k) - e^{ik.x} \right] \times$$

$$\left[e^{-ik'.(x+\hat{\mu}a)} \phi^*(k') - e^{ik'.x} \phi^*(k') \right]$$

$$= \sum_x \int \frac{d^4k'}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \phi(k)\phi^*(k') \left[e^{ik.(x+\hat{\mu}a)} e^{-ik'.(x+\hat{\mu}a)} \right.$$

$$\left. - e^{ik.x} e^{-ik'.(x+\hat{\mu}a)} - e^{ik.(x+\hat{\mu}a)} e^{-ik'.x} + e^{ik.x} e^{-ik'.x} \right]$$

$$= \sum_x \int \frac{d^4k'}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \phi(k)\, \phi^*(k')\, e^{i(k-k').x} \left[1 + e^{ik.\hat{\mu}a} e^{-ik'.\hat{\mu}a} \right.$$

$$\left. - e^{-ik.\hat{\mu}a} - e^{+ik.\hat{\mu}a} \right] . \qquad (15)$$

Using the relation

$$\sum_x e^{i(k'-k).x} = (2\pi)^4 \delta^{(4)} (k-k') \qquad (16)$$

we have

$$I = \int \frac{d^4k}{(2\pi)^4} \phi(k)\, \phi^*(k) \left[e^{ik.\hat{\mu}a} - 1 \right] \left[e^{-ik.\hat{\mu}a} - 1 \right]$$

$$= 4 \int \frac{d^4k}{(2\pi)^4} \phi(k)\phi^*(k) \sin^2 \left(\frac{k_\mu a}{2} \right) . \qquad (17)$$

The action for the free field is therefore given by

$$S(\phi) = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \left[\sum_\mu \frac{4}{a^2} \sin^2 \left(\frac{k_\mu a}{2} \right) + m^2 \right] |\phi(k)|^2 . \qquad (18)$$

In momentum space each mode now contributes to the action a quantity $S(k) \equiv m^2 + (4/a^2) \sum_\mu \sin^2(\frac{1}{2}ak_\mu)$. In the limit $a \to 0$, the standard continuum form $m^2 + k^2$ is recovered. In the limit where m goes to zero, but finite a, S(k) is zero if all k_μ are zero or a multiple of $\frac{2\pi}{a}$. These latter values are, however, outside the region of integration and therefore S(k) has only one pole and only one particle is described. This situation does not exist for spin ½ particles, leading to additional complications when fermions are involved in the theory (see Chapter 4).

4. Quantization of Lattice Gauge Theories [11]

As remarked earlier, quantization of the theory is carried out using the path integral formalism. Using this formalism we can define the Euclidian Green's function for the scalar field by

$$G(\phi_{x_1}, \ldots, \phi_{x_n}) \equiv \langle 0|\phi_{x_1}\phi_{x_2} \ldots \phi_{x_n}|0\rangle$$

$$= \frac{1}{Z}\int_{-\infty}^{\infty} (\prod_x d\phi_x)\ \phi_{x_1}, \ldots, \phi_{x_n}\ e^{-S(\phi)} \qquad (19)$$

where (i) $S(\phi)$ is given by eq.(14), and

$$\text{(ii)}\quad Z = \int_{-\infty}^{\infty} (\prod_x d\phi_x)\ e^{-S(\phi)}$$

$$\equiv \int_{-\infty}^{\infty} [d\phi]\ e^{-S(\phi)} \qquad (20)$$

is the path integral of the scalar theory which corresponds to the partition function of a d-dimensional Statistical system.

The expectation value of any physical quantity $O(\phi)$ can also be defined by

$$\langle O\rangle = Z^{-1}\int [d\phi]\ O\ e^{-S(\phi)}. \qquad (20)$$

with Z given by eq.(20).

CHAPTER 2: GAUGE INVARIANCE IN LATTICE GAUGE THEORIES

1. Review of Gauge Invariance in the Continuum Theory

Consider a matter field multiplet $\phi_i(x)$ belonging to an irreducible representation i of the gauge group G. In this representation we will denote the generators by T^a and the gauge potentials by $A^a_\mu(x)$ with, for $G \equiv SU(N)$, $a = 1,\ldots, N^2-1$ (see Chapter 5).

Consider the SU(N) local gauge transformation

$$\phi_i(x) \to V_{ij}(x)\phi_j(x)$$

$$\phi^*_i(x) \to \phi^*_j(x)V^\dagger_{ji}(x) \qquad (1)$$

where

(i) $V(x) \equiv V(g(x)) \qquad g(x) \in SU(N)$
(ii) V is unitary .

Under the transformation above the field $\phi(x)$ and its covariant derivative

$$[D_\mu\phi(x)]_i \equiv [\partial_\mu\delta_{ij} + ig(T^a)_{ij}A^a_\mu(x)]\phi_j(x) \qquad (2)$$

transforms identically,

while the gauge potentials $A^a_\mu(x)$ that build up the adjoint multiplet transform inhomogeneously:

$$A^a_\mu(x)T^a \equiv A_\mu(x) \to V(x)A_\mu(x)V^\dagger(x) - (i/g)V(x)\partial_\mu V^\dagger(x) \ . \qquad (3)$$

The second term in eq.(2) ensures that the action is gauge invariant under (1).

2. Implication of Local Gauge Invariance for LGT

If the procedure for constructing a gauge invariant action described above is naively applied to lattice theories - i.e. by associating

vector gauge potentials with the links and replacing the derivatives by finite differences - the resulting discretized covariant derivative will only have the correct transformation properties if the lattice spacing a becomes very small:

$$
\begin{aligned}
D_\mu\phi(x) &= [\partial_\mu + igA_\mu(x)]\phi(x) \\
&\to a^{-1}[\phi(x+\mu) - \phi(x)] + igA_\mu(x)\phi(x) \\
&= a^{-1}\phi(x+\mu) - a^{-1}[1 - igaA_\mu(x)]\phi(x) \\
&\approx a^{-1}[\phi(x+\mu) - e^{-iagA_\mu(x)}\phi(x)] \qquad \text{(small a)} \\
&= a^{-d/2}[\phi_{x+\mu} - U^\dagger_{x,\mu}\phi_x] \qquad (4)
\end{aligned}
$$

where we defined $U_{x,\mu} \equiv e^{iagA_\mu(x)}$, (4a)

and introduced the dimensionless variables ϕ_x with

$$\phi_x = a^{(d-2/2)}\phi(x). \qquad (5)$$

In the action, the lattice covariant derivative appears in the form

$$
\begin{aligned}
\int d^dx\ |D_\mu\phi|^2 &\to a^d \sum_{x,\mu} a^{-d}|\phi_{x+\mu} - U^\dagger_{x,\mu}\phi_x|^2 \\
&= \sum_{x,\mu} |\phi_{x+\mu}|^2 - 2\mathrm{Re}\sum_{x,\mu} \phi^*_{x+\mu}\, U^\dagger_{x,\mu}\phi_x + \sum_{x,\mu} |U^\dagger_{x,\mu}|^2\, |\phi_x|^2
\end{aligned}
\qquad (6)
$$

Using $|U^\dagger_{x,\mu}|^2 = 1$, and $\sum_{x,\mu} |\phi_{x+\mu}|^2 = \sum_{x,\mu} |\phi_x|^2$ (if the sum goes over all lattice points) we can rewrite eq.(6) as

$$\int d^dx\ |D_\mu\phi|^2 = 2d \sum_{x,\mu} |\phi_x|^2 - 2\mathrm{Re}\sum_{x,\mu} \phi^*_{x+\mu}\, U^\dagger_{x,\mu}\phi_x. \qquad (7)$$

Equation (7) now provides another possibility for constructing a gauge invariant theory if we note that the factor $U^\dagger_{x,\mu}$ (which is associated with the link (x+μ, x) (see later)) appears in the combination $\phi^*_{x+\mu}\, U^\dagger_{x,\mu}\phi_x$.

3. Application to the Spin 0 Field

As an example, we consider in detail the implication of gauge invariance for the Spin 0 field on the lattice.

From Chapter 1 we have the following expression for the spin 0 field

$$S(\phi) = \sum_x \left\{ \tfrac{1}{2}a^2 \sum_\mu [\phi(x+\hat{\mu}) - \phi(x)]^2 + \tfrac{1}{2}a^4 m^2 |\phi(x)|^2 \right\}$$

$$= \sum_x \left\{ \tfrac{1}{2}a^2 \sum_\mu [|\phi(x+\hat{\mu})|^2 + |\phi(x)|^2 - \phi(x+\hat{\mu})\phi^\dagger(x) \right.$$

$$\left. - \phi^\dagger(x+\hat{\mu})\phi(x) + \tfrac{1}{2}a^4 m^2 |\phi(x)|^2 \right\}. \qquad (8)$$

The mass term, as well as the first two quadratic terms are clearly invariant under the transformation (1).

To make the mixed terms gauge invariant we implement our earlier suggestion and insert a factor $U_{x,\mu}$, such that

$$\phi^\dagger(x)\phi(x+\hat{\mu}) \rightarrow \phi^\dagger(x)U_{x,\mu}\phi(x+\hat{\mu}) \qquad (9)$$

where $U_{x,\mu}$ is an element of the gauge group [with definition (4a)] and interpreted as the link ($x \rightarrow x+\hat{\mu}$) connecting the matter fields at the sites x and $x+\hat{\mu}$ with $\hat{\mu}$ the unit vector giving the specific direction.

Under a local gauge transformation (1), $U_{x,\mu}$ transforms as

$$U_{x,\mu} \rightarrow V(x)U_{x,\mu}V^\dagger(x+\hat{\mu}), \qquad (10)$$

which ensures gauge invariance of expressions like eq.(9), as

$$\phi^\dagger(x)U_{x,\mu}\phi(x+\hat{\mu}) \rightarrow \phi^\dagger(x)V^\dagger(x)V(x)U_{x,\mu}V^\dagger(x+\hat{\mu})V(x+\hat{\mu})\phi(x+\hat{\mu})$$

$$= \phi^\dagger(x)U_{x,\mu}\phi(x+\hat{\mu}). \qquad (11)$$

Hence, in order to obtain a gauge invariant Spin 0 theory, the group elements $U_{x,\mu}$ had to be introduced (instead of the gauge potentials $A_\mu(x)$ needed in the case of the continuum theory). This result also applies to other (matter) fields defined on the lattice, e.g. the Spin ½ fields (see Chapter 4).

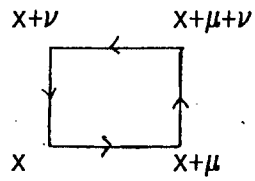
CHAPTER 3: PURE GAUGE SECTOR (SPIN 1 FIELDS) ON THE LATTICE

1. Basic Construction

In constructing an Euclidian action for the pure gauge sector the following requirements should be kept in mind:

(a) the action should be defined at each space-time point of the lattice,
(b) it should, as far as the lattice structure allows, keep the original global symmetries of the continuum Yang-Mills action (such as parity and charge conservation),
(c) it should have a local gauge invariance,
(d) it should give the correct form of the Yang-Mills action in the continuum limit($a \to 0$).

Consider the **closed** loop made up of four nearest neighbours (n.n.) links with origin at the point x:



This is called an **elementary plaquette** U_p [1] (elementary because we only consider n.n. links) where

$$U_p = U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x) \tag{1}$$

$$= U_\mu(x)U_\nu(x+\mu)U_\mu^{-1}(x+\nu)U_\nu^{-1}(x) \tag{1a}$$

Note that the the hermitian conjugate of the above (i.e. $U_p^\dagger$), gives the link variables in the reverse order.

We will now show that the trace of this elementary plaquette forms the simplest gauge invariant candidate for the gauge sector action.
Consider the local gauge transformation (see Chapter 2)

$$U_\mu(x) \to V(x)U_\mu(x)V^\dagger(x+\mu)\ . \tag{2}$$

Under this transformation the elementary plaquette transforms as

$$U_p = U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)$$

$$\to \quad V(x)U_\mu(x)V^\dagger(x+\mu)V(x+\mu)U_\nu(x+\mu)V^\dagger(x+\mu+\nu)V(x+\mu+\nu)U_\mu^\dagger(x+\nu)V^\dagger(x+\nu) \times$$

$$V(x+\nu)U_\nu^\dagger(x)V^\dagger(x)$$

$$= V(x)U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)V^\dagger(x). \qquad (3)$$

If we now take the trace and use its cyclic property we get

$$\text{tr } U_p = \text{tr } U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)$$

$$\to \text{tr}[V(x)U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)V^\dagger(x)]$$

$$= \text{tr } U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x). \qquad (4)$$

The action for the pure gauge sector can therefore be written as

$$S(U) = c \sum_p (\text{tr } U_p + \text{tr } U_p^\dagger)$$

$$= c \sum_p \text{Re tr } U_p, \qquad (5)$$

where (i) the sum runs over all possible elementary plaquettes;
(ii) the constant c must be chosen in order to give the correct continuum Yang-Mills action in the classical continuum limit.

Determining the constant factor in eq.(5)

Writing out the explicit form of the the link variables that form a plaquette, we have

$$U_p = U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x)$$

$$= \exp[iagA_\mu(x)].\exp[iagA_\nu(x+\mu)].\exp[-iagA_\mu(x+\nu)].\exp[-iagA_\nu(x)].$$

From $\partial_\mu A_\nu(x) = \frac{1}{a}[A_\nu(x+\mu) - A_\nu(x)]$

$$\equiv \frac{1}{a}[A_{x+\mu,\nu} - A_{x,\nu}],$$

it follows that

$$A_{x+\mu} \approx a\partial_\mu A_{x,\nu} + A_{x,\nu} \ . \tag{6}$$

Therefore

$$U_p = \exp[iagA_{x,\mu}].\exp[iag(a\partial_\mu A_{x,\nu} + A_{x,\nu})].\exp[-iag(a\partial_\nu A_{x,\mu} + A_{x,\mu})] \times$$

$$\exp[-iagA_{x,\nu}]. \tag{7a}$$

Using the Baker-Hausdorff formula

$$e^X e^Y = e^{X + Y + \frac{1}{2}[X,Y] + \ldots} , \tag{7b}$$

we have

$$U_p = \exp\Big[ia^2 g\{ \partial_\mu A_{x,\nu} - \partial_\nu A_{x,\mu} - g[A_{x,\mu},A_{x,\nu}]\}$$

$$- \frac{1}{2} a^3 g^2 \{[A_{x,\mu},\partial_\mu A_{x,\nu}] - [A_{x,\nu},\partial_\nu A_{x,\mu}]\} + \text{higher order terms} \Big]$$

$$= 1 + ia^2 gF_{\mu\nu} - \frac{1}{2} a^3 g^2 \{[A_{x,\mu},\partial_\mu A_{x,\nu}] - [A_{x,\nu},\partial_\nu A_{x,\mu}]\} +$$

$$+ \frac{1}{2} (ia^2 gF_{\mu\nu})^2 + \ldots ,$$

$$= 1 + ia^2 gF_{\mu\nu} - \frac{1}{2} a^4 g^2 F_{\mu\nu}^2 + O(a^6) + \ldots , \tag{8}$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - g[A_\mu,A_\nu]$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - igf_{abc} A_\mu^a A_\nu^b T^c \tag{9}$$

and $A_\mu = A_\mu(x) = T^a A_\mu^a(x)$. ($T^a$ are the SU(N) generators)

Therefore $\operatorname{tr} U_p = \operatorname{tr}(1 + ia^2 gF_{\mu\nu} - \frac{1}{2} a^4 g^2 F_{\mu\nu}^2 + \ldots)$

$$= \operatorname{tr} 1 + ia^2 g \operatorname{tr}(F_{\mu\nu}) - \frac{1}{2} a^4 g^2 \operatorname{tr}(F_{\mu\nu}^2) + \ldots$$

$$= \operatorname{tr} 1 - \frac{1}{4} a^4 g^2 F_{\mu\nu}^a F_{\mu\nu}^a + \ldots . \tag{10}$$

For the opposite directed plaquette $U_p^\dagger$ we get the identical result, hence

$$\sum_p (\mathrm{tr}U_p + \mathrm{tr}U_p^\dagger) = \sum_p 2(\mathrm{tr}\ 1 - \frac{1}{4} a^4 g^2 F^a_{\mu\nu} F^a_{\mu\nu})$$

$$= 2\sum_p \mathrm{tr}\ 1 + 2g^2 \sum_p (-\ a^4\ \frac{1}{4}\ F^a_{\mu\nu} F^a_{\mu\nu}).$$

To go to the continuum limit ($a \to 0$) we make the substitution

$$a^4 \sum_x \ \to \ \int d^4x,$$

so that $$\sum_p = \sum_x \sum_{\mu\nu} \ \to \ \int d^4x\ (2a^4)^{-1} \sum_{\mu\nu}, \tag{11}$$

where the factor $\frac{1}{2}$ was obtained from the symmetry under μ,ν exchange.

Therefore

$$\sum_p \mathrm{tr}(U_p + U_p^\dagger)$$

$$\to\ 2g^2 \int d^4x\ (2a^4)^{-1}\ [-\ \frac{1}{4}\ a^4 F^a_{\mu\nu} F^a_{\mu\nu}] + \ 2\sum_p \mathrm{tr}\ 1$$

$$=\ -g^2 \int d^4x\ \frac{1}{4}\ F^a_{\mu\nu} F^a_{\mu\nu}\ + \text{constant}\ , \tag{12}$$

where summation over the indices μ,ν is implied.

The pure gauge action therefore gives the correct form of the Euclidian continuum Yang-Mills action provided the constant factor is chosen to be $-g^{-2}$, i.e.

$$S = -g^{-2} \sum_p \mathrm{tr}(U_p + U_p^\dagger) \tag{13}$$

$$= -2g^{-2} \sum_p \mathrm{Re}\ \mathrm{tr}U_p. \tag{13a}$$

Remarks

(i) The action above is called the Wilson action. It is not unique - the requirements listed at the beginning only ensure that the correct continuum limit action is obtained.

(ii) By summing over all possible orientated plaquettes we obtain an action which is **real** (eq. (13a)), with the sum now running over all unorientated plaquettes.

(iii) For colour SU(N) the trace in eq. (13a) only runs over the colour indices.

2. Alternative form of the Wilson Action

By incorporating the constant factor $2\sum_p \mathrm{tr}\ 1$ (which was dropped in eq. (13a)) we obtain the following form of the Wilson action:

$$\sum_p \mathrm{tr}(U_p + U_p^\dagger) = 2\sum_p [\mathrm{tr}\ 1 - \tfrac{1}{4} a^4 g^2 F_{\mu\nu}F_{\mu\nu}]$$

$$\Rightarrow \sum_p \mathrm{Re}\ \mathrm{tr}(U_p) = \sum_p [\mathrm{tr}\ 1 - \tfrac{1}{4} a^4 g^2 F_{\mu\nu}F_{\mu\nu}]$$

$$\overset{a}{\underset{0}{\doteq}} \sum_p \mathrm{tr}\ 1 - \tfrac{1}{2} g^2 \int d^4x\ \tfrac{1}{4} F^a_{\mu\nu}F^a_{\mu\nu}\ .$$

Therefore

$$S = \int d^4x\ \tfrac{1}{4} F^a_{\mu\nu}F^a_{\mu\nu} = (2/g^2)\sum_p [\mathrm{tr}\ 1 - \mathrm{Re}\ \mathrm{tr}\ U_p]$$

$$= (2N/g^2)\sum_p [1 - \tfrac{1}{N} \mathrm{Re}\ \mathrm{tr}\ U_p],$$

$$\Rightarrow S \equiv \beta \sum_x \sum_{\mu\nu} (1 - P^{\mu\nu}_x), \qquad (14)$$

where (i) $\beta = 2N/g^2$ for SU(N)

(ii) $P^{\mu\nu}_x = \frac{1}{N} \mathrm{Re}\ \mathrm{tr}\ U_p$

$$= \tfrac{1}{N} \mathrm{Re}\ \mathrm{tr}\ U^\mu_x U^\nu_{x+\mu} U^{\mu\dagger}_{x+\nu} U^{\nu\dagger}_x$$

(iii) $\sum_p = \sum_x \sum_{\mu\nu}$ $(\equiv \sum_x \sum_{\mu\neq\nu})$

For an asymmetric lattice ($\xi \equiv a_\sigma/a_\tau \neq 1$), eq.(14) reads (see e.g.[22])

$$S = (2N/g_\sigma^2)(a_\tau/a_\sigma) \sum_x \sum_{\mu<\nu<4} P_x^{\mu\nu} + (2N/g_\tau^2)(a_\sigma/a_\tau) \sum_x \sum_{\mu<4} P_x^{\mu 4}, \quad (15)$$

where the sum in the first term runs over all plaquettes containing four space-like links while the sum in the second term runs over plaquettes with two time-like and two space-like links.

3. Generalization of the Gauge Theory Action - other candidates

As we have noted previously, the Wilson action is not unique. It is therefore possible to generalize the action to improve some specific features of the model as long as the correct continuum limit remains unaltered.

We first rewrite the Wilson action (13) in the form

$$S = -(2/g^2)\sum_p \text{Re tr}(U_p)$$

$$= -(\beta/N)\sum_p S_p , \quad (16a)$$

where β is given by eq.(14a) and S_p is a function of U_p only, with

$$S_p = \text{Re tr}(U_p). \quad (16b)$$

The most frequently used generalization is to replace the trace operation in eq.(16) by another real function, namely the character.

Using the results from Group theory (see Chapter 5), we can rewrite the gauge invariant action in terms of a character expansion of all the irreducible representations of the group [12;15];

$$S = -(\beta/N)\sum_p \sum_r (\alpha_r/d_r) \text{ Re } \chi_r(U_p) \quad (17)$$

with $$S_p = \sum_r (\alpha_r/d_r) \text{Re } \chi_r(U_p), \quad (18)$$

and d_r the dimension of the representation r. The sum $\sum_p$ runs over all unorientated plaquettes; if we summed over orientated plaquettes the substitution

$$\text{Re}\,\chi_r(U_p) \to \chi_r(U_p) + \chi_r(U_p^\dagger)$$

$$= \chi_r(U_p) + \chi_{\bar{r}}(U_p) \qquad (19)$$

must be made (with $\bar{r}$ denoting the conjugate representation of r). Note that unless stated otherwise, ß will always stand for $2N/g^2$.

Examples

(1) The "fundamental-adjoint" mixed action

This action is defined by (see [15] and reference therein)

$$S = -(\beta/N)\sum_p [(\alpha_f/d_f)\,\text{Re}\chi_f(U_p) + (\alpha_a/d_a)\,\text{Re}\chi_a(U_p)]$$

$$\equiv -N^{-1}\sum_p [(\beta_f/d_f)\,\text{Re}\chi_f(U_p) + (\beta_a/d_a)\,\text{Re}\chi_a(U_p)], \qquad (20)$$

where (i) $\beta\alpha_f = \beta_f$
$\beta\alpha_a = \beta_a$; (20a)

(ii) for SU(N) $d_f = \chi_f(1) = N$
$d_a = \chi_a(1) = N^2 - 1.$ (20b)

Specifically, for SU(2) we have

$$S = -2^{-1}\sum_p [(\beta_f/2)\,\text{Re}\,\chi_f(U_p) + (\beta_a/3)\,\chi_a(U_p)]. \qquad (21)$$

Note that if we use the alternative normalization of eq.(14), the mixed SU(2) action has the form

$$S = -\sum_p [\beta(1 - \tfrac{1}{2}\,\text{Re tr}\,U_p) + \beta_a(1 - \tfrac{1}{3}\,\text{tr}_a U_p)], \qquad (21a)$$

where the trace without a subscript is taken in the two dimensional fundamental representation and tr_a is taken in the three dimensional adjoint representation.

The following remarks are in order.

(i) If $\beta_a = 0$, the model reduces to the usual Wilson formulation.

(ii) If β ($\equiv \beta_f$) = 0, the action will only depend on the adjoint representation, which gives the form of the Wilson action for the orthogonal groups G/Z [e.g. SU(2)/Z(2) ≈ SO(3)].
More explicitly, the Wilson action for the gauge group SU(N)/Z(N) is given by

$$\begin{aligned} S &= -\beta \sum_p (\alpha_a/d_a)\, \chi_a(U_p) \\ &= -\sum_p (\beta_a/d_a)\, [\chi_f^2(U_p) - 1] \\ &= \sum_p (\beta_a/d_a)\, [1 - \mathrm{tr}(U_p)\mathrm{tr}(U_p^\dagger)], \end{aligned} \tag{22}$$

where we used the following identity for the character of the adjoint representation of SU(N) (which is also a faithful representation of SU(N)/Z(N)),

$$\chi_a(U) = \mathrm{tr}_a(U) = \mathrm{tr}(U)\mathrm{tr}(U^\dagger) - 1. \tag{22a}$$

(2) Another form of the gauge field action was proposed by Manton [13,14] which uses the shortest geodesic distance $d(U_p, I)$ from U_p to the unit element I with respect to the invariant metric on the group manifold:

$$\begin{aligned} S_M &\equiv (1/g_m^2) \sum_p [d(U_p, I)] \\ &= (1/g_m^2) \sum_p \mathrm{tr}(X_p^2), \end{aligned} \tag{23}$$

where X_p is related to the element U_p of the Lie group by

$$U_p = \exp[iX_p]. \tag{23a}$$

E.g., for SU(2) we have (see Chapter5)

$$U_p = \exp[i\theta_p \vec{\tau}\cdot\vec{n}\,], \tag{24}$$

which gives

$$S_M = (2/g_m^2) \sum_p \theta_p^2$$

$$= \tfrac{1}{2}\beta_m \sum_p \theta_p^2 . \qquad (\beta_m = 4/g_m^2) \tag{24a}$$

(3) One of the shortcomings of Manton's action is that the periodicity in the angle variables for compact groups makes the action multivalued and singular for "conjugate" points on the group manifold.
The action proposed by Villain gives the action in terms of the "heat kernel" over the gauge group, i.e. in terms of the matrix elements of $\exp[(1/2)g^2\Delta]$, where Δ satisfies the diffusion equation on the Lie group (or generally the group manifold):

$$\Delta(g,t) = \frac{\partial f(g,t)}{\partial t} ,$$

where $g \in G$, $t = N/\beta$; $f = \exp[-S_V]$. (25a)

The solution of this equation gives the heat kernel action [14;15;16],

$$\exp[-S_V] = \prod_p \sum_r d_r \chi_r(U_p) \exp[-C_r^{(2)}/N\beta], \tag{26}$$

where $C_r^{(2)}$ is the quadratic Casimir invariant for the representation r.

For SU(N) the heat kernel action is given by

$$\exp[-S_V] = \prod_p \left(\prod_{\ell=0}^{\infty} [(\ell+1) \frac{\sin(\ell+1)\theta_p}{\sin(\theta_p)} \exp[-\ell(\ell+2)(g_V^2/8)] \right), \tag{27a}$$

or in terms of indices running over half-integer values,

$$\exp[-S_V] = \prod_p \left(\prod_{\ell=\frac{1}{2}}^{\infty} [(2\ell+1) \frac{\sin(\ell+1)\theta_p}{\sin(\theta_p)} \exp[-\ell(\ell+1)(g_V^2/4)] \right) . \tag{27b}$$

4. Gauge Field Partition Function

The gauge field theory with action defined in section 1 can be quantized by integrating $\exp[-S(U)]$ over all possible values of U_x^μ on the lattice, i.e.

$$Z(g^2) = \int \prod_{x,\mu} (dU_{x,\mu}) \exp[-S(U)]$$

$$\equiv \int [dU] \exp[-S(U)] \,, \tag{28}$$

where $[dU] = \prod_{x,\mu} (dU_{x,\mu})$ (28a)

$\prod_{x,\mu}$ → product over all links of the lattice.

$Z(g^2)$ is called the partition function of the gauge field sector.

Remarks

(1) For an integration measure we use the normalized Haar measure which has the advantage that the specific form of the measure need not be known, while the integrals over the group elements can be calculated using the properties of the Haar measure (see Chapter 5).

(2) The partition function is well defined; the integrals over the group elements U_x^μ are finite because only compact Lie groups are considered.

This is in sharp contrast to the situation in continuum field theory where the functional integrals are plagued by divergences because of overcounting of possible paths due to the gauge invariant nature of the measure. As the measure is not compact, this leads to infinite volume factors that have to be factored out, usually by restricting the measure using the Fadeev-Popov Ansatz.

(3) The local gauge invariance of the action, together with the invariance properties of the normalized Haar measure, ensure that the (vacuum) expectation value of a physical quantity O(U) will also have a local gauge invariance under the transformation in eq.(2), where

$$\langle O(U) \rangle = Z^{-1} \int [dU] \, O(U) \exp[-S(U)]. \tag{29}$$

(4) A four dimensional hypercubic lattice with linear dimension L has $4L^4$ links. The corresponding partition function over the links is therefore defined by $4L^4$ integrals over the group space.

5. Universality

As we have noted previously, there is considerable freedom in the choice of lattice action for the pure gauge theory. The requirement that all actions lead to the same continuum result, however, necessitates the introduction of a universality principle, namely that all physical quantities calculated on the lattice must be independent of the specific choice of action used .

To compare the three most often used gauge theory actions, we first rewrite the Wilson action in (16a) in the form [SU(2)]

$$S = -(\beta/N) \sum_p \text{Re tr } U_p$$

$$= -(\beta/N) \sum_p \cos\theta_p \qquad [\beta = 2N/g_W^2] \tag{30}$$

where we used the following form for the SU(2) parametrization (see Chapter 5)

$$U_p = \cos\theta_p + i\vec{\sigma}\cdot\hat{n} \sin\theta_p. \tag{30a}$$

For the Manton and Villain actions we use the expressions in (24a) and (27a) respectively.

Although the three actions are required to have the same physical content in the (relevent) continuum limit, it is obvious that each will have its own scale parameter. This in turn will influence the values of quantities like the string tension and critical coupling that are usually expressed in terms of the lattice (or continuum) scale parameter (see Part II).

The following ratios for SU(2) have been calculated using the relation between the different lattice and continuum scale parameters [14a;b]:

$$\Lambda_L^{(M)} = 3{,}07\ \Lambda_L^{(W)}$$

$$\Lambda_L^{(M)} = 2{,}45\ \Lambda_L^{(V)}. \tag{31}$$

Using Monte Carlo calculations, the following non-perturbative results for the string tension have been obtained [14b]

$$\sigma = 83 \pm 14\ \Lambda_L^{(W)*}$$

$$16{,}2 \pm 0{,}5\ \Lambda_L^{(M)}$$

$$48{,}5 \pm 2{,}6\ \Lambda_L^{(V)} \qquad (32)$$

(* from $[10a;b]_{II}$ - see Part II).

If universality is indeed valid, then the string tension should be action independent. These predictions, along with theoretical ones (obtained in the Weak Coupling limit $g^2 \to 0$ [14a;b] and including higher order corrections [14c]) are given in table 3.1 .

Table 3.1

Ref.	[14a;b]	[14c]	from σ
$\Lambda_L^{(M)}/\Lambda_L^{(W)}$	3,07	3,33	5,14(87)
$\Lambda_L^{(M)}/\Lambda_L^{(V)}$	2,45	2,92	2,99(19)
$\Lambda_L^{(V)}/\Lambda_L^{(W)*}$	1,25	1,14	1,71(18)

* obtained from the previous two ratios .

Although there seems to be a relative (order of magnitude) agreement in the values obtained above, a more conclusive argument would be if signs of universality is found in the ratio of σ with another physical quantity (calculated for the different actions) .

Such a quantity is the critical temperature, which has been calculated for the pure gauge sector using the different actions. The Monte Carlo results for T_c are [14d]

$$T_c = 42{,}8\ \Lambda_L^{(W)}$$

$$= 10{,}5\ \Lambda_L^{(M)}$$

$$= 27{,}3\ \Lambda_L^{(V)} \ . \qquad (33)$$

Combining the results in (32) and (33), the ratio $\sqrt{\sigma}/T_c$ should be the same for all three actions. Numerically this gives

$$\begin{aligned} \sqrt{\sigma}/T_c &= 1{,}94 \pm 0{,}33 \qquad \text{[Wilson])} \\ &= 1{,}54 \pm 0{,}05 \qquad \text{[Manton]} \\ &= 1{,}78 \pm 0{,}10 \qquad \text{[Villain]} \end{aligned} \qquad (34)$$

which is consistent with universality within two standard deviations.

CHAPTER 4: FERMIONS ON THE LATTICE

1. Continuum Euclidian Action

The Dirac Lagrangian in continuum Minkowski space is given by [17]

$$\mathcal{L} = \bar{\psi}(i\gamma_\mu \partial^\mu - m)\psi \qquad [\psi \equiv \psi(x) = \psi(t,\vec{x})]$$

$$= \bar{\psi} i\gamma_0 \partial^0 \psi + \psi i\vec{\gamma}\cdot\vec{\partial}\psi - m\bar{\psi}\psi \ . \tag{1}$$

Performing a Wick rotation ($t = -i\tau$) we can go to Euclidian space (see Appendices A and B), which gives

$$\mathcal{L} \rightarrow \mathcal{L} = i\bar{\psi}\,\gamma_0(-i)\left(\frac{\partial\psi}{\partial\tau}\right) + i\bar{\psi}\,\vec{\gamma}\cdot\vec{\partial}\psi - m\bar{\psi}\psi \ . \tag{2}$$

Using the hermitian choice for the Euclidian γ matrices $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$ (see App. B) implies

$$\mathcal{L} = -\bar{\psi}\gamma_0\left(\frac{\partial\psi}{\partial\tau}\right) - \bar{\psi}\,\vec{\gamma}\cdot\vec{\partial}\psi - m\bar{\psi}\psi$$

$$= -\bar{\psi}(\gamma_\mu \partial^\mu + m)\psi \ . \tag{3}$$

The corresponding Euclidian action is given by

$$S = \int d^4x\ (-\mathcal{L}) = \int d^4x\ \bar{\psi}(\gamma_\mu \partial^\mu + m)\psi \ , \tag{4}$$

where $d^4x \equiv d\tau\, d^3x = dx_4\, d^3x$. (4a)

2. Defining Fermions Naively on an Infinite Lattice

Using the Euclidian continuum action of a free Dirac particle (3), we can introduce the lattice regularization "naively" by replacing the derivatives by finite differences and the integral by a summation over all lattice points. We here essentially repeat the procedure that was used for the scalar field (Chapter 1):

$$\text{(i)} \quad \int d^4x \rightarrow a^4 \sum_x \tag{5}$$

(ii) $\partial_\mu \psi(x) \to \frac{1}{a}[\psi(x+\mu) - \psi(x)]$ (n.n.)

$$\to \frac{1}{2a}[\psi(x+\mu) - \psi(x-\mu)] \tag{6a}$$

where $\mu\ (\equiv \hat{\mu}) \equiv a\hat{e}_\mu$. (6b)

Using the replacements above in eq.(3), we obtain the following expression for the naive fermion lattice action (S_N) in 4 dimensional Euclidian space:

$$S_N = \sum_x a^4 \Big\{ \sum_{\mu=1}^{4} [\bar{\psi}(x)\gamma_\mu \psi(x+\mu) - \bar{\psi}(x)\gamma_\mu \psi(x-\mu)](2a)^{-1} + m\bar{\psi}(x)\psi(x) \Big\}, \tag{7}$$

where the quark fields are denoted by $\psi(x) \equiv \psi_\alpha^{ij}$, and $i = 1,\ldots,N$; $j = 1,\ldots,4$; $\alpha = u,d,s\ldots$ are the colour, Dirac and flavour indices respectively.

The following remarks are in order.

(a) As we sum over all lattice points we can make the replacement

$$\sum_x \bar{\psi}(x)\gamma_\mu \psi(x-\mu) \to \sum_x \bar{\psi}(x+\mu)\gamma_\mu \psi(x). \tag{8a}$$

(b) The lattice Dirac fields can be rescaled (following Wilson [1])

$$\psi_x \equiv m^{-\frac{1}{2}} \psi(x). \tag{8b}$$

(c) The action in eq.(7) can be extended to any (finite) dimension d.

(d) For $m = 0$ the action contains a (hidden) global $U(N_f) \otimes U(N_f)$ symmetry, where N_f is the number of independent continuum flavours. This continuous symmetry will be made explicit in sec. 7 by "spin-diagonalization".

Using (8a) we can rewrite eq.(7) as

$$S_N = \sum_x a^4 \Big\{ \sum_{\mu=1}^{4} [\bar{\psi}(x)\gamma_\mu \psi(x+\mu) - \bar{\psi}(x+\mu)\gamma_\mu \psi(x)](2a)^{-1} + m\bar{\psi}(x)\psi(x) \Big\} \tag{9}$$

which, after rescaling the fields (using (8b)), gives

$$S_N = \sum_x a^4 \Big\{ \sum_{\mu=1}^{4} [\bar{\psi}_x \gamma_\mu \psi_{x+\mu} - \bar{\psi}_{x+\mu} \gamma_\mu \psi_x](2am)^{-1} + \bar{\psi}_x \psi_x \Big\} . \qquad (10)$$

This action is not unique by the same reasoning that applied to the spin 1 case.

We must also keep in mind that the action in eq.(10) applies to the free theory and is therefore is not gauge invariant. To obtain an invariant action we have to insert the (colour) gauge fields $U_{x,\mu}$ (see Chapter 2).
This leads to the following gauge invariant form of the naive quark - gauge field interaction:

$$S_N = \sum_x a^4 \Big\{ \sum_{\mu=1}^{4} [\bar{\psi}_x \gamma_\mu U_{x,\mu} \psi_{x+\mu} - \bar{\psi}_{x+\mu} \gamma_\mu U^{\dagger}_{x,\mu} \psi_x](2am)^{-1} + \bar{\psi}_x \psi_x \Big\} . \qquad (11)$$

3. Naive Fermions on a Finite Lattice

The concept of a finite lattice (at least in one direction) will play an important role in the finite temperature formalism that will be discussed in Part II.

In a finite temperature theory the lattice will always be finite in the time direction, which in this case is identified with the temperature of the system,

$$\beta \equiv T^{-1} = N_\tau a_\tau ,$$

where N_τ is the number of lattice points in the time/temperature direction. However, as β is also used in the literature to denote the gauge coupling, care must be taken not tot confuse these two definitions. Hence in Part II, the use of β will be resticted to denote the coupling, i.e. $\beta = 2N/g^2$ (see Chapter 3).

As we are working with fermions , the quark fields are represented by Grassmann variables (see Appendix C) which satisfy the **anti-periodic** boundary conditions

$$\psi(\vec{x},\tau) = -\psi(\vec{x},\tau+\beta)$$

$$\bar{\psi}(\vec{x},\tau) = -\bar{\psi}(\vec{x},\tau+\beta) . \qquad (12)$$

The continuum Dirac equation for a free particle at finite temperature is given by [23;24]

$$S = \int_0^{\beta} d\tau \int_V d^3x\ \bar{\psi}(\gamma_\mu \partial^\mu + m)\psi\ , \tag{13}$$

where $\beta = T^{-1}$ and $\psi \equiv \psi(\vec{x},\tau)$. (13a)

On a finite asymmetric lattice this becomes

$$S_N = \sum_x a_\tau a_\sigma^3 \left\{ \sum_{\mu=1}^{4} [\bar{\psi}(x)\gamma_\mu \psi(x+\mu) - \bar{\psi}(x+\mu)\gamma_\mu \psi(x)](2a)^{-1} + m\bar{\psi}(x)\psi(x) \right\} \tag{14}$$

which is equivalent to the naive action on an infinite asymmetric lattice.

In terms of the rescaled fields (8(b)), eq.(14) now reads

$$S = \sum_x a_\tau a_\sigma^3 \left\{ \sum_{\mu=1}^{4} [\bar{\psi}_x \gamma_\mu U_{x,\mu} \psi_{x+\mu} - \bar{\psi}_{x+\mu}\ \gamma_\mu U^{\dagger}_{x,\mu} \psi_x](2am)^{-1} + \bar{\psi}_x \psi_x \right\}. \tag{15}$$

4. Propagator for Naive Lattice Fermions

4.1 Infinite Lattice

To investigate the lattice fermion propagator we first go to momentum space by intoducing a Fourier transform of the function f(x) defined on the lattice points. For a four dimensional infinite lattice the Fourier transform of f(x) is given by

$$g(p) = \sum_{x=-\infty}^{\infty} a^4 \exp[-i(p.x)a]\ f(x). \tag{16a}$$

Using the relation

$$\int_{-\pi/a}^{\pi/a} d^4p\ \exp[ip.(m-n)a] = \delta_{m,n}(2\pi/a)^4, \tag{16b}$$

we obtain the inverse function

$$f(x) = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \exp[i(p.x)a]\, g(p). \qquad (16c)$$

Making a Fourier transform of the Dirac fields ψ_x hence gives

$$\psi_x = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \exp[i(p.x)a]\, \phi_p \ , \qquad (17)$$

where the boundaries on the integral are given by the endpoints of the first Brillouin zone on the lattice.

The action (10) can now be rewritten (using (17)) as

$$S = \sum_x a^4 \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \cdot \int_{-\pi/a}^{\pi/a} \frac{d^4p'}{(2\pi)^4} \times$$

$$\Big[(2ma)^{-1} \sum_\mu \{\bar{\phi}_p, \exp[-ip'.xa]\gamma_\mu \exp[ip.(x+\mu)a]\phi_p$$

$$- \bar{\phi}_p, \exp[-ip'.(x+\mu)a]\gamma_\mu \exp[i(p.x)a]\, \phi_p + \exp[i(p-p').xa]\, \bar{\phi}_p, \phi_p \Big] \qquad (18)$$

$$= \sum_x a^4 \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4} \int_{-\pi/a}^{\pi/a} \frac{d^4p'}{(2\pi)^4} \exp[i(p-p').xa] \times$$

$$\Big[(2ma)^{-1} \sum_\mu [\bar{\phi}_p, \gamma_\mu \phi_p \exp(ip_\mu a) - \bar{\phi}_p, \gamma_\mu \phi_p \exp(-ip'_\mu a)] + \bar{\phi}_p, \phi_p \Big], \qquad (18)$$

where we introduced the notation $(p.\mu) \equiv p_\mu a$. (18a)

Using the relation

$$\sum_x \exp[ix.(p-p')a] = (2\pi/a)^4\, \delta(p-p') \qquad (18b)$$

in eq.(18) gives

$$S = \int_{-\pi/a}^{\pi/a} \frac{d^4p}{(2\pi)^4}\, \bar{\phi}_p \Big[\frac{1}{2ma} \sum_\mu (e^{ip_\mu a} - e^{-ip_\mu a})\gamma_\mu + 1 \Big] \phi_p \ . \qquad (19)$$

The naive fermion propagator is therefore given by

$$G(p) = \left[\frac{1}{2ma} \sum_{\mu} \left(e^{ip_\mu a} - e^{-ip_\mu a} \right)\gamma_\mu + 1 \right]^{-1} . \qquad (20)$$

It is interesting to note that in the limit $a \to 0$ the naive lattice prapagator reduces to the correct continuum expression:

$$G^{-1}(p) \overset{a}{\underset{0}{\to}} i\gamma_\mu (ma)^{-1} p_\mu a + 1$$

$$= i\gamma_\mu p_\mu + m , \qquad (21)$$

where we used $\sin(p_\mu a) \approx p_\mu a + \ldots .$ (a small) (21a)

Determining the poles of the propagator [2;20]

To determine the poles we first rewrite the propagator in the form

$$G(p) = \left[1 + \sum_{\mu} i\gamma_\mu \frac{\sin(p_\mu a)}{ma} \right]^{-1}$$

$$= \left[1 - \sum_{\mu} i\gamma_\mu \frac{\sin(p_\mu a)}{ma} \right] \left[1 + \sum_{\mu} \sum_{\nu} (ma)^{-2} \gamma_\mu\gamma_\nu \sin(p_\mu a) \sin(p_\nu a) \right]^{-1}$$

Now, $\gamma_\mu\gamma_\nu \sin(p_\mu a) \sin(p_\nu a) = \frac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu)\sin(p_\mu a) \sin(p_\nu a)$

$$= \sin^2(p_\mu a), \qquad (22a)$$

where we used the relation $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$. (see App. B)

Hence

$$G(p) = \left[1 - \sum_{\mu} \gamma_\mu \frac{\sin p_\mu a}{ma}\right] / \left[1 + \sum_{\mu} \left(\frac{\sin p_\mu a}{ma} \right)^2\right] . \qquad (23)$$

The propagator will therefore have poles for

$$\left[1 + \sum_{\mu} \left(\frac{\sin p_\mu a}{ma} \right)^2\right] = 0 . \qquad (24)$$

In order to go back to Minkowski space we make the transformation $p_4 \to ip_0$ ($\equiv iE$), which gives

$$\sin^2(p_4 a) \to -\sinh^2(p_0 a). \tag{24a}$$

Eq.(24) now reads

$$(ma)^{-2}\sinh^2(p_0 a) - (ma)^{-2}\sum_{i=1}^{3}\sinh^2(p_0 a) - 1 = 0$$

$$\Rightarrow \quad \sinh^2(p_0 a) = \sum_{i=1}^{3}\sinh^2(p_0 a) + (ma)^2. \tag{25}$$

Keeping in mind that $-\pi/a \leq p \leq \pi/a$, the propagator therefore has 16 poles at the points $p = (0,0,0,0), (\pi/a,0,0,0),\ldots.,$ $(\pi/a,\pi/a,\pi/a,\pi/a)$ instead of the expected single pole at the origin. To illustrate the fermion species doubling more explicitly, we consider the one dimensional case with $p_i = \pi/a$. Equation (25) now reads

$$\sinh^2(p_0 a) = (ma)^2$$

$$\Rightarrow \quad E \equiv p_0 = \pm \text{ arc } \sinh(ma), \tag{26}$$

where the $E > 0$ and $E < 0$ solutions correspond to the particle and anti-particle ones respectively.

However, taking for p_i the values $\pm \pi/a$ would give the same result. A fermion with momentum $\pm \pi/a$ (i.e. of the order of the cut-off) therefore has the same energy as a fermion at rest ($p = 0$).

The state with momentum $p = \pm \pi/a$ is however a distinct one from the $p = 0$ case. This follows from the Pauli principle which states that the fermion wave function must have alternate signs at alternate lattice sites. In one dimension we therefore have a doubling of fermion species, which in the four dimensional case will lead to a 16 ($= 2^4$) degenaracy.

This unexpected property of the lattice fermion action is a general one and not a consequence of the specific form of the action that was used.

Species doubling is closely related to chiral symmetry: it turns out that in order to have the correct fermion action in the Weak Coupling phase (which corresponds to the continuum limit), any lattice fermion formulation must either have a multiplicity of states or no explicit continuous chiral invariance [20].

The axial anomaly that occurs in continuum field theory should also not be present in a regularized theory - the extra fermions are needed to cancel this anomaly. It is interesting to note that if an axial charge is attributed to each of the fermions it turns out that the number of charges necessary to cancel the anomaly is sixteen, eight with charge 1 and eight with charge -1 [20].
Also, the so-called No-Go theorems [18] state that no entirely satisfactory choice for a fermion action is possible on the lattice.

The only way to avoid species doubling completely is to use a non-local lattice derivative, e.g. the so-called SLAC fermions [19]. These theories unfortunately encounter severe problems in the continuum limit and as a result have not been used very often in the literature.

In order to overcome the species doubling problem (and still use a local action) two methods have been introduced in the the literature.
The one method put forward by Wilson results in a lifting of the degenaracy by adding an irrelevant (in the continuum limit) factor to the naive action, while the second method (proposed by Kogut et al.) thins out the degenaracy by so-called spin-diagonalization. These two methods will be discussed in greater detail in sections 5 and 7.

4.2 Naive propagator on a finite symmetric lattice

The problems of species doubling is also apparent for the the finite lattice as the size of the lattice never played a role in the analysis of the propagator in section 4.1.
What will be different however are the momentum space quantities, as we need to introduce a new set of Fourier transforms.

On a finite lattice the Fourier transform of a quantity f(x) defined on the lattice points is given by [32]

$$g(p) = \beta^{-1} \sum_{j=1}^{N} a \exp[-i\pi(\frac{2j+1}{\beta})a\tau] \sum_{\vec{n}=-\infty}^{\infty} \exp[-i(\vec{p}\cdot\vec{x})a]\ f(\vec{x},\tau), \qquad (28)$$

where

(i) the first exponential factor ensures the anti-periodicity of the theory in the time/temperature direction,

(ii) we used a version of the identity

$$N^{-1} \sum_{k=1}^{N} \exp[2\pi i k(m-n/N)] = \delta_{m,n}$$

which follows trivially from the geometrical representation of complex numbers.

By inverting eq.(28) we obtain the following expression for f(x)

$$f(x) = \beta^{-1} \sum_{j=1}^{N} \int_{-\pi/a}^{\pi/a} \frac{d^3p}{(2\pi)^3} \exp[i\pi(\frac{2j+1}{\beta})a\tau] \exp[i(\vec{p}\cdot\vec{x})a] \, f(p). \qquad (29)$$

Applying the expression above to the quark fields gives

$$\psi(x) = \beta^{-1} \sum_{j=1}^{N} \int_{-\pi/a}^{\pi/a} \frac{d^3p}{(2\pi)^3} \exp[i\pi(\frac{2j+1}{N})\tau] \exp[i(\vec{p}\cdot\vec{x})a] \, \phi_p$$

$$= \beta^{-1} \sum_{j=1}^{N} \int_{-\pi/a}^{\pi/a} \frac{d^3p}{(2\pi)^3} \exp[i(p.x)a] \, \phi_p, \qquad (30)$$

with (i) $\beta = a_\tau N_\tau \equiv aN$

(ii) $p \equiv \left(\vec{p} \, , (\frac{2j+1}{Na})\pi \right)$. (30a)

Using the above, we can rewrite the action (14) (with $a_\sigma = a_\tau$) as

$$S = \sum_{\vec{x}} a^4 \sum_{j} \sum_{j'} \int_{-\pi/a}^{\pi/a} \frac{d^3p}{(2\pi)^3} \int_{-\pi/a}^{\pi/a} \frac{d^3p'}{(2\pi)^3} \exp[i(p-p').xa] \times$$

$$\left((2ma)^{-1} [\bar{\phi}_{p'} \gamma_\mu \phi_p \exp(ip'_\mu a) - \bar{\phi}_{p'} \gamma_\mu \phi_p \exp(-ip'_\mu a) + \bar{\phi}_{p'} \phi_p \right)$$

$$= \beta \sum_{j=1}^{N} \int_{-\pi/a}^{\pi/a} \frac{d^3p}{(2\pi)^3} \bar{\phi}_p \left[(2ma)^{-1} \sum_{\mu} (e^{ip_\mu a} - e^{-ip_\mu a}) \gamma_\mu + 1 \right] \phi_p \, , \qquad (31)$$

where we also used

$$\text{(i)} \quad \sum_{k=1}^{N} \exp[ik \tfrac{\beta}{N} (p_0 - p_0')] = N\delta_{p_0,\, p_0'} \tag{31a}$$

$$\text{(ii)} \quad \sum_{\vec{x}=-\infty}^{\infty} \exp[i(\vec{p}-\vec{p}')\cdot\vec{x}a] = (2\pi)^3 \delta^3(a(p-p'))$$

$$= (2\pi/a)^3 \delta^3(p-p') \; . \tag{31b}$$

Equation (31) now clearly shows that the propagator for the finite lattice is equivalent to the one deduced earlier for the infinite lattice (19) (up to a redefinition of the momentum tensor).

5. Wilson's Fermions

The Wilson action is formed by adding terms that leave the continuum limit unaltered while at the same time removing the degenaracy in the fermion species [2;20;21].

The general expression for Wilson's action for free fermions ($N_f = 1$) on a four dimensional symmetrical infinite lattice is given by

$$S_W = \sum_x a^4 \Big\{ -\sum_{\mu=1}^{4} [\bar{\psi}(x)(r-\gamma_\mu)\psi(x+\mu) + \bar{\psi}(x+\mu)(r+\gamma_\mu)\psi(x)]\tfrac{1}{2a} + \bar{\psi}(x)\psi(x) \Big\} \tag{32}$$

which corresponds to adding a term

$$ra^4 \sum_x \sum_\mu (1/2a)[\bar{\psi}(x+\mu) - \bar{\psi}(x)][\psi(x+\mu) - \psi(x)]$$

$$= ra^4 \sum_x \sum_\mu (1/2a)[-\bar{\psi}(x+\mu)\psi(x) - \bar{\psi}(x)\psi(x+\mu) + 2\bar{\psi}(x)\psi(x)] \tag{32a}$$

where $0 \leq r \leq 1$. The choice $r = 1$ corresponds to Wilson fermions. It is easy to show that this term goes to zero in the formal classical continuum limit.

The naive massless action has a classical symmetry

$$SU_V(N_f) \otimes SU_A(N_f) \otimes U_V(1) \otimes U_A(1) \tag{33}$$

where N_f stands for the number of (continuum) flavours.

The extra term (32a) in the Wilson action explicitly breakes the continuous and discrete γ_5 invariance, and hence $SU_A(N_f) \otimes U_A(1)$. The Wilson action therefore has a residual $SU_V(N_f) \otimes U_V(1)$ symmetry.
However, since the symmetry breaking terms are proportional to the lattice spacing a, chiral symmetry should be recovered in the continuum limit (where the chiral $SU(N_f)$ symmetry should be broken spontaneously in order to produce the needed Goldstone Bosons).

Introducing a new set of rescalled fields

$$\psi(x) \rightarrow \psi_x = [2(m_i a + 4)]^{-\frac{1}{2}} \psi(x), \tag{34a}$$

we can rewrite the Wilson action (32) in the form (showing the flavour indices explicitly)

$$S_W = \sum_x \sum_{i=1}^{N_f} a^3 \left(-\sum_{\mu=1}^{4} [\bar{\psi}_x(1-\gamma_\mu)\psi^i_{x+\mu} + \bar{\psi}^i_{x+\mu}(1+\gamma_\mu)\psi^i_x] + (8+2m_i a)\bar{\psi}^i_x\psi^i_x \right)$$

$$= \sum_x \sum_{i=1}^{N_f} a^3 \left(-\sum_{\mu=1}^{4} K_i [\bar{\psi}_x(1-\gamma_\mu)\psi^i_{x+\mu} + \bar{\psi}^i_{x+\mu}(1+\gamma_\mu)\psi^i_x] + \bar{\psi}^i_x\psi^i_x \right) \tag{34b}$$

with $K^i = (8 + 2m_i a)^{-1}$. (34c)

K^i is called the hopping parameter for the different quark flavours and is proportional to the amplitude of moving a quark by one lattice spacing.

For the free theory discussed above ($U_{x,\mu} = 1$) and a symmetric lattice, the relationship between K^i and the quark masses m_i is given by

$$\tfrac{1}{2}\left[(K^i)^{-1} - (K_c)^{-1} \right] = \exp(m_i a) - 1, \tag{34d}$$

with $K_c = 1/8$.

To obtain the expression for the gauge invariant interacting theory we again insert the gauge fields, which gives

$$S_W = \sum_x \sum_{i=1}^{N_f} a^3 \left(-\sum_{\mu=1}^{4} K^i [\bar{\psi}_x(1-\gamma_\mu)U^\mu_x\psi^i_{x+\mu} + \bar{\psi}^i_{x+\mu}(1+\gamma_\mu)U^{\mu\dagger}_x\psi^i_x] + \bar{\psi}^i_x\psi^i_x \right). \tag{35}$$

6. Propagator for the Wilson Action

Repeating the procedure in section 4 to find the naive action propagator, we obtain the following expression for the Wilson action propagator [2]

$$G_W(p) = \left[1 - K^i \sum_\mu [(1 - \gamma_\mu)e^{ip_\mu a} + (1 + \gamma_\mu)\, e^{-ip_\mu a}]\right]^{-1}$$

$$= \left[1 - 2K^i \sum_\mu [\cos(p_\mu a) - i\gamma_\mu \sin(p_\mu a)\right]^{-1}. \qquad (36)$$

The propagator in eq.(36) has only one pole at p = (0,0,0,0); the flavour degenaracy has therefore been lifted.

What has effectively taken place is that the added terms have given large masses [of the order of the cut-off ($\approx a^{-1}$)] to the fifteen unwanted fermions (which are still present), so that they will disappear from the theory as the limit $a \to 0$ is taken, leaving only the desired zero momentum solution. In this limit, the Wilson propagator reduces to

$$G(p) \to [1 - 2K^i(4 - i\sum_\mu \gamma_\mu p_\mu a)]^{-1}$$

$$= [2K^i a(\frac{1-8K^i}{2K^i a} + i\gamma_\mu p_\mu)]^{-1}, \qquad (37)$$

which is equivalent to the Dirac continuum propagator (Euclidian space) if the identification

$$m_i = \frac{1-8K^i}{2K^i a} \qquad (37a)$$

is made [2]. To find the physical energy spectrum we go back to Minkowski space, which gives

$$G(p) \to G(p) = [1 - 8K^i + 2K^i\gamma_0 p_0 a + 2iK^i \vec{\gamma}\cdot\vec{p}\,]^{-1}$$

$$= -(2K^i a)^{-1}[\gamma_0 E - \vec{\gamma}\cdot\vec{p} - m]^{-1}, \qquad (38)$$

which, if we again make the identification (37a), is the usual continuum Dirac propagator in Minkowski space (up to a normalization factor).

7. Staggered Fermions

The idea originally put forward by Kogut and Susskind [29] (using the Hamiltonian formalism) was to reduce the fermion degenaracy by using a suitable transformation to diagonalize the spin matrices and to reinterprete the result as differrent flavours.

The same idea can be applied to the Lagrangian formalism, using the construction put forward by Kawamoto and Smit [30].

Consider the following transformation of the quark fields (satisfying the naive lattice action)

$$\psi_\alpha^{if}(x) = T_{\alpha\beta}(x)\chi_\beta^{if}(x)$$

$$\bar{\psi}_\alpha^{if}(x) = \bar{\chi}_\beta^{if}(x)T_{\beta\alpha}^\dagger(x), \tag{39}$$

where T is a unitary operator defined by

$$T(x) = (\gamma_1)^{x_1}\,(\gamma_2)^{x_2}\,(\gamma_3)^{x_3}\,(\gamma_4)^{x_4} \tag{40a}$$

and $T^\dagger$ has the form (because the γ matrices are Euclidian)

$$T^\dagger(x) = (\gamma_4)^{x_4}\,(\gamma_3)^{x_3}\,(\gamma_2)^{x_2}\,(\gamma_1)^{x_1}\,. \tag{40b}$$

In (39) both ψ and χ are four component fields, with χ for example of the form

$$\chi = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{40c}$$

We can now consider the effects of the transformations (39) on the terms in the naive action:

(i) the mass term is invariant under (39),

(ii) the kinetic term transforms as

$$\bar{\psi}(x)\gamma_\mu\psi(x+\hat{\mu}) \rightarrow \bar{\chi}(x)T^\dagger(x)\gamma_\mu T(x+\hat{\mu})\chi(x+\hat{\mu}) \ . \tag{41}$$

To see how the factor $T^\dagger(x)\gamma_\mu T(x+\hat{\mu})$ looks like we take as example the case if $\hat{\mu} \equiv a\hat{e}_2 = 2$;

$$T^\dagger(x)\gamma_2 T(x+2)$$

$$= (\gamma_4)^{x_4}(\gamma_3)^{x_3}(\gamma_2)^{x_2}(\gamma_1)^{x_1}\gamma_2(\gamma_1)^{x_1}(\gamma_2)^{x_2+1}(\gamma_3)^{x_3}(\gamma_4)^{x_4}$$

$$= (-1)^{x_1} T^\dagger(x)T(x) = (-1)^{x_1}, \tag{42}$$

where we used the anti-commutation relations of the hermitian γ matrices and the fact that $(\gamma_2)^2 = 1$ (see App. B).

Hence, in general

$$T^\dagger(x)\gamma_1 T(x+a\hat{e}_1) = 1 \equiv \eta_1(x)$$

$$T^\dagger(x)\gamma_2 T(x+a\hat{e}_2) = (-1)^{x_1} \equiv \eta_2(x)$$

$$T^\dagger(x)\gamma_3 T(x+a\hat{e}_3) = (-1)^{x_1+x_2} \equiv \eta_3(x) \quad \text{etc.}, \tag{43}$$

which can be extended to a d-dimensional lattice by defining

$$\eta_\mu(x) = (-1)^{x_1 + x_2 + \ldots + x_{\mu-1}}, \qquad \mu = 1,\ldots,d \ . \tag{43a}$$

We also note that

$$\eta_\mu(x) = T^\dagger(x)\gamma_\mu T(x+\hat{\mu})$$

$$= \eta_\mu^\dagger(x) = T^\dagger(x+\hat{\mu})\gamma_\mu T(x). \tag{43b}$$

Using eq.(39), we can rewrite the naive action (d=4 symmetric lattice)

$$S_s = \sum_x a^4 \left[\sum_{\mu=1}^{4} \eta_\mu(x)[\bar{\chi}_x\chi_{x+\mu} - \bar{\chi}_{x+\mu}\chi_x](2a)^{-1} + m\,\bar{\chi}_x\chi_x \right]$$

$$= \sum_x \left[a^3 \sum_{\mu=1}^{4} \tfrac{1}{2}\eta_\mu(x)[\bar{\chi}_x\chi_{x+\mu} - \bar{\chi}_{x+\mu}\chi_x] + a^4 m\,\bar{\chi}_x\chi_x \right] . \tag{44}$$

This action is called the "Staggered" or "Kogut-Susskind" action.

The following remarks are in order.

(i) The action is diagonal in the spin index, i.e. χ as defined in (40c) describes four single component complex fields. The method used is therefore called **spin-diagonalization.**

(ii) For d = 4 the action (44) is a sum of four identical actions, one for each value of the Dirac index. Using the fact that the naive action has a degenaracy of 16f (f = number of single fermion or naive flavours), each of the four actions above will describe 4f Dirac particles.

(iii) Usually only one component of χ is kept (which reduces the number of flavours by a quarter), and the remaining 4f flavours are then interpreted as physical flavour degrees of freedom.

(iv) If we take n to denote the number of non-colour indices of the theory (i.e. the number of Dirac-flavour degrees of freedom), then $n = 4N_f$, where N_f is the number of continuum flavours.

The naive action describes 16f particles in the continuum limit, hence

$$n = 4N_f = 16f \tag{45}$$

For a d-dimensional space the number of Dirac components of a spinor is $2^{[d/2]}$, which implies

$$f = N_f/2^{[d/2]}. \tag{45a}$$

The choice of f = 1 (i.e. $N_f = 1$) is usually associated with Euclidian Staggered fermions. Showing the number of flavours explicitly, the action for free Staggered fermions on a d-dimensional lattice reads

$$S = \sum_x \sum_{\alpha=1}^{f} \Big\{ a^{d-1} \sum_{\mu=1}^{d} \tfrac{1}{2} \eta_\mu(x) [\bar{\chi}_\alpha(x)\chi_\alpha(x+\hat{\mu}) - \bar{\chi}_\alpha(x+\hat{\mu})\chi_\alpha(x)] + a^d m \bar{\chi}_\alpha(x)\chi_\alpha(x) \Big\} . \tag{46}$$

For the interacting theory the action reads

$$S_s = \sum_x \sum_{\alpha=1}^{f} \Big[a^{d-1} \sum_{\mu=1}^{d} \frac{1}{2} \eta_\mu(x) [\bar{\chi}_\alpha(x) U_{x,\mu} \chi_\alpha(x+\hat{\mu}) - \bar{\chi}_\alpha(x+\hat{\mu}) U^\dagger_{x,\mu} \chi_\alpha(x)]$$

$$+ a^d m \bar{\chi}_\alpha(x) \chi_\alpha(x) \Big] . \qquad (47)$$

(v) For the Staggered fermion action only a global $U_V(f) \otimes U_A(f)$ (chiral) symmetry survives (where the naive action had a $U_V(4f) \otimes U_A(4f)$ symmetry) [30].

This symmetry has its origin in the fact that χ's at odd (even) sites are only coupled to χ's at even (odd) sites. [By odd (even) it is meant that $\sum_{\mu=1}^{d} x_\mu$ is an odd (even) integer].

The transformations of this global symmetry are given by

x_μ even:

$$\chi_x \to e^{i\alpha} \chi_x$$

$$\bar{\chi}_x \to e^{-i\beta} \bar{\chi}_x$$

x_μ odd:

$$\chi_x \to e^{i\beta} \chi_x$$

$$\bar{\chi}_x \to e^{-i\alpha} \bar{\chi}_x \qquad (48)$$

where α and β are two independent phases such that $V \equiv e^{i\beta}$ and $W \equiv e^{i\alpha}$ are independent unitary matrices acting on the space of non-colour indices.

This symmetry is only present for the massless theory; the mass term (which couples two fermions at the same site) explicitly breakes the $U(f) \otimes U(f)$ symmetry down to its diagonal $U(f)$ subgroup (i.e. for which $V = W$).

A consequence of this symmetry is that mass counterterms do not occur in renormalization (therefore m(bare) = 0 → m(renormalized) = 0).

(vi) If the Clifford algebra is generated by the d γ-matrices satisfying

$$\{\gamma_\mu,\gamma_\nu\} = -2\delta_{\mu\nu} \qquad \text{(see App. B)}$$

i.e. the γ matrices are chosen to be anti-hermitian, then the relation (43b) is no longer satisfied,

$$\eta_\mu^\dagger(x) = T^\dagger(x+\hat{\mu})\gamma_\mu^\dagger T(x)$$

$$= -T^\dagger(x+\hat{\mu})\gamma_\mu T(x). \tag{49}$$

The Staggered action now reads [26;31]

$$S_s = \sum_x \sum_{\alpha=1}^{f} \Big\{ -a^{d-1} \sum_{\mu=1}^{d} \frac{1}{2}\eta_\mu(x)[\bar{\chi}_\alpha(x)U_{x,\mu}\,\chi_\alpha(x+\hat{\mu}) + \bar{\chi}_\alpha(x+\hat{\mu})U_{x,\mu}^\dagger\,\chi_\alpha(x)]$$

$$+ a^d m\bar{\chi}_\alpha(x)\chi_\alpha(x)\Big\}\,. \tag{50}$$

To conclude this section we briefly discuss the construction of (four component) quark fields out of the one component χ fields (for simplicity we restrict ourselves to the free theory - the interacting theory follows analogously).

Using the fact that the Lagrangian is periodic under a double lattice shift, we can define a four dimensional hypercube H(y) [31] with origin at the site 2y and corners

$$r = 2y + \eta \qquad \eta = 0 \text{ or } 1\,. \tag{51}$$

As η is varied at fixed y, r spans the corners of a four dimensional hypercube. Hence, the set of all hypercubes forms a lattice with spacing 2a. We can now define four flavoured quarks at the level of each hypercube by

$$q_\mu^\alpha(y) = \frac{1}{8}\sum_\eta \Gamma_\eta^{\mu\alpha}\,\chi_\eta(y)$$

$$\bar{q}_\mu^\alpha(y) = \frac{1}{8}\sum_\eta \bar{\chi}_\eta(y)\Gamma_\eta^{\mu\alpha\dagger} \tag{52}$$

where $\eta \equiv (\eta_1,\eta_2,\eta_3,\eta_4) = (0,0,0,0),\ldots,(1,1,1,1)$ runs over the sixteen corners of the hypercube. The indices μ and α denote the Dirac and flavour indices respectively, and the sixteen matrices

$$\Gamma_\eta = \gamma_1^{\eta_1}\gamma_2^{\eta_2}\gamma_3^{\eta_3}\gamma_4^{\eta_4} \tag{53}$$

form a basis for the four dimensional Clifford algebra. If the (Euclidian) γ matrices are chosen to be hermitian, then all the Γ_η will be unitary.

In terms of the $q\bar{q}$ fields the action (44) becomes [31]

$$S_S = (2a^4)\sum_h \Big\{ \sum_\mu [\bar{q}(y)(\gamma_\mu\otimes 1)\Delta_\mu q(y) + a\bar{q}(y)(\gamma_5\otimes\gamma_\mu^*\gamma_5^*)\delta_\mu q(y)] + m\bar{q}(y)(1\otimes 1)q(y)\Big\}, \tag{54}$$

where $\sum_h$ denotes the sum over all hypercubes.

In each tensor product in (54), the first (second) matrix acts in Dirac (flavour) space. For any function f(y)

$$\Delta_\mu f(y) = \frac{1}{4a}[f(x+\mu) - f(x-\mu)] \overset{a}{\underset{o}{\to}} \partial_\mu f(y)$$

$$\delta_\mu f(y) = \frac{1}{4a^2}[f(y+\mu) - f(y-\mu) - 2f(y)] \overset{a}{\underset{o}{\to}} \partial_\mu^2 f(y). \tag{55}$$

The first term in eq.(54) corresponds to the naive action for 4 ($\equiv 2^{d/2}$) free (massless) Dirac fermions on a lattice with spacing 2a, while the second term (involving second order lattice derivatives) is responsible for lifting the fermion degeneracy.

8. Partition Function for Euclidian Lattice Fermions

The partition function for Euclidian lattice fermions is given by

$$Z_F = \int \Big(\prod_{x,\mu} dU_x^\mu\Big)\Big(\prod_x d\psi_x d\bar{\psi}_x\Big)\exp[-S_F(U;\psi,\bar{\psi})]$$

$$\equiv \int [dU][d\psi d\bar{\psi}]\exp(-S_F), \tag{56}$$

where S_F is the fermion lattice action and the quark fields ψ_x and $\bar{\psi}_x$ are regarded as independent Grassmann variables (see App. C).

In order to fascilitate integration over the Grassmann variables we first rewrite the Wilson action on a 4 dimensional symmetric lattice in the form

$$S_F^W \equiv \sum_x \sum_{i=1}^{N_f} \bar{\psi}_x^i Q_{x,y}^i \psi_y$$

$$= \sum_x \sum_{i=1}^{N_f} \bar{\psi}_x^i [\delta_{x,y} - K^i \sum_{\mu=1}^{4} (M_\mu)_{x,y}] \psi_y^i, \qquad (57)$$

with $(M_\mu)_{x,y} = (1-\gamma_\mu) U_x^\mu \delta_{x,y-\mu} + (1+\gamma_\mu) U_x^{\mu\dagger} \delta_{x,y+\mu}$. (57a)

(In the matrix M the colour and spin indices have been surpressed to simplify the notation)

For Staggered fermions the partition function reads

$$Z_F = \int (\prod_{x,\mu} dU_x^\mu) (\prod_x d\chi_x d\bar{\chi}_x) \exp[-S_F^S(U;\chi,\bar{\chi})]$$

$$\equiv \int [dU][d\chi d\bar{\chi}] \exp(-S_F^S) , \qquad (58)$$

where S_F^S is given by (47), and χ and $\bar{\chi}$ are independent one component Grassmann variables.

For integration purposes the action can again be rewritten in the form (d=4)

$$S_F^S \equiv \sum_x \sum_{\alpha=1}^{f} \bar{\chi}_\alpha(x) Q_{x,y}^\alpha \chi_\alpha(y)$$

$$= \sum_x \sum_{\alpha=1}^{f} \bar{\chi}_\alpha(x) [\frac{1}{a} \sum_{\mu=1}^{4} D_{x,y}^{(\mu)} + m_\alpha \delta_{x,y}] \chi_\alpha(y) , \qquad (59)$$

with $D_{x,y}^{(\mu)} = \frac{1}{2} \eta_\mu(x) [U_x^\mu \delta_{x,y-\mu} - U_x^{\mu\dagger} \delta_{x,y+\mu}]$. (59a)

CHAPTER 5: GROUP THEORY AND INTEGRATION

1. Basic Definitions of Group Theory [36-41]

For completeness, we give a few basic definitions of Group Theory that will be used in the sections that follow.

(a) Conjugate elements and classes

An element $g(i)$ of the group G is said to be conjugate to an element $g(j)$ of the same group if there exists an element $g(n) \in G$ such that

$$g(i) = g(n)g(j)g^{-1}(n). \qquad (1)$$

If $g(k)$ and $g(j)$ are both conjugate to $g(i)$, then it follows trivially that $g(k)$ and $g(j)$ are also conjugate to each other. All elements of a group that are conjugate belong to the same class. Furthermore, no element of the group may belong to more than one class.

(b) The Group Rearrangement Theorem (discrete groups)

If $g(i)$ is any element of the group G, and $g(j)$ is allowed to run over all elements of G, then the product element $g(k) = g(j)g(i)$ also runs over all elements of G, each element appearing only once.

This value of $g(k)$ is unique: assume that $g(k)$ can be written as a product of two different group elements $g(j)$ and $g'_j(j)$, i.e. $g(k) = g(j)g(i) = g'(j)g(i)$. Multiplying through with $g^{-1}(i)$ this implies that $g(j) = g'(j)$.

(c) Continuous Groups and Lie Groups

A group G is defined to be a r-parameter continuous group if all its elements can be labelled by r real continuously varying parameters.

Let the group elements be denoted by $g(a_1,\ldots,a_r) \equiv g(a)$. Since a group must posses an identity element, there must exist some set of parameters $a^0 = (a_1^0,\ldots,a_r^0)$ such that $g(a^0)$ behaves like the identity,

$$g(a^0)g(a) = g(a)g(a^0) = g(a). \qquad (2)$$

The identity parameter (a^0) is usually taken to be 0, so that the identity element corresponds to

$$g(0,\ldots,0) = g(0) = e. \tag{3}$$

The existence of an inverse requires that for any parameter a there exists an a′ such that

$$g(a)g(a') = g(a')g(a) = e$$

$$\text{or} \quad g(a') = [g(a)]^{-1}. \tag{4}$$

Also, the closure property requires that if g(a) and g(b) ε G, then g(a)g(b) = g(c) must also be an element of G. (5)
In order for this to be satisfied the parameter c must be a real function of the real parameters a and b;

$$c_i = \phi_i(a_1,\ldots,a_r;b_1,\ldots,b_r)$$

$$\equiv \phi_i(a;b) \qquad i = 1,2,\ldots,r \tag{6}$$

or just $c = \phi(a;b)$ for short.

If we require additionally that the parameters of a product be analytic functions of the parameters of the factors (i.e. the function in eq. (6) will have derivatives of all orders with respect to both arguments) and that the a′ in (4) be analytic functions of the parameter a, then G is called a **r parameter Lie group.**

An r parameter Lie group of transformations can similarly be defined by

$$x_i' = f_i(x_1,\ldots,x_n;a_1,\ldots,a_r) \qquad i = 1,\ldots,n$$

$$\text{or just} \quad x' = f(x,a). \tag{7}$$

(d) Group representations

If there exists a set of linear transformations T in the vector space S which is homomorphic to the group G [i.e. in the sense that $T(g)T(g') = T(gg')$; $T(e) = 1$], this is called a **representation** of the group G.
If the dimension of the vector space S is n, then the representation is said to be n-dimensional. S is called the representation space of T. Also, the representation T(g) is said to be faithfull if there exists an isomorphism (1-1 relationship) between the operations T(g) and the group elements g.

(e) Matrix representation

If all elements of a group G are mapped onto n x n matrices such that the group multiplication law (which is replaced by matrix multiplication) is satisfied, then this set of matrices form a n-dimensional representation of the group. Hence, let there be a homomorphism of G onto D, with

$$g(0) \rightarrow D(0), \qquad g(a) \rightarrow D(a),$$

such that

$$\text{if } g(a)g(b) = g(c), \text{ then } D(a)D(b) = D(c). \tag{8}$$

(f) Two representations M(a) and D(a) are **equivalent** if there exists a similarity transformation such that

$$M(a) = AD(a)A, \tag{9}$$

where A is any fixed matrix for all group elements.

(g) Let $D^n = \{\ldots, D^n(a), \ldots\}$ and $D^m = \{\ldots, D^m(a), \ldots\}$ be two irreducible representations of the group $G = \{\ldots, g_a, \ldots\}$ and let N be the order of the group and d_m be the dimensionality of the representation D^m.

Then

$$\sum_{a=1}^{N} D^{(m)}_{ik}(a)[D^{(n)}_{jl}(a)]^{-1} = \frac{N}{d_m}\, \delta_{mn}\, \delta_{il}\, \delta_{kj}. \tag{10}$$

For unitary representations we have the equivalent result

$$\sum_{a=1}^{N} D^{(m)}_{ik}(a)[D^{(n)}_{lj}(a)]^* = \frac{N}{d_m}\,\delta_{mn}\,\delta_{il}\,\delta_{kj}. \qquad (11)$$

(h) Representations of compact Lie groups

A Lie group is said to be **compact** if every infinite sequence of its elements has a limit in the group (i.e. every infinite sequence converges to a value that is also an element of the group).

The importance of compact groups lie in the following theorems:

(a) All representations of a compact group are unitary (up to equivalence).
(b) All representations of a compact group are discrete.
(c) All irreducible representations of a compact group are finite dimensional.

The theorems above imply that all representations of a compact group are either finite dimensional irreducible, or if not, direct sums of finite dimensional representations.
Also, by implication the compact groups have a finite volume.

2. The Special Unitary Groups

2.1 SU(N)

A representation isomorphic to the abstract group SU(N) is given by the set of N x N special unitary matrices U (special in the sense that the detU = 1).
The matrices of the representations of SU(N) have N^2-1 independent parameters. This follows from the fact that a general N x N complex matrix has $2N^2$ arbitrary real parameters, while the requirments $UU^\dagger = 1$ and det U = 1 imposes N^2 and 1 restriction respectively.
Correspondingly, SU(N) has N^2-1 generators T_a which satisfy the closed algebra

$$[T_a, T_b] = if_{abc}T_c \qquad (12)$$

$$\mathrm{Tr}\ T_a = 0, \tag{13}$$

where: (i) $a,b,c = 1,\ldots, N^2-1 \equiv$ order of the group

(ii) f_{abc} are the real, totally anti-symmetric structure constants of the group normalized such that

$$f_{abc}f_{dbc} = N\delta_{ab}. \tag{14}$$

These N^2-1 linear independent matrices therefore form a basis which is chosen to satisfy the normalization

$$\mathrm{Tr}(T_aT_b) = \tfrac{1}{2}\,\delta_{ab}. \tag{15}$$

The N^2-1 generators of the infinitesimally small transformations are related to the group elements U by

$$U = \exp[i\Sigma_a\, \alpha_a T_a], \qquad a = 1,\ldots,N^2-1 \tag{16}$$

where α_a are arbitrary real parameters.

Furthermore, unitarity implies that the generators must be hermitian matrices, of which $N-1 \equiv r$ can be chosen to be diagonal with r the rank of the group.

These r generators, or linear combinations there of, can be chosen so that they commute with each other - they are then called **charge operators** and denoted by H_i. These operators have the additional property of satisfying the Cartan subalgebra of the group ($[H_i,H_j] = 0$ $i,j = 1,\ldots,r$). The Abelian subgroup generated by the charge operators is called the **Cartan subgroup.**

The smallest non-trivial irreducible representation is called the fundamental representation ($T_a = \frac{1}{2}\lambda_a$) and is by definition of dimension N.

The adjoint representation is generated by the structure constants of the group $\{(T_a)_{bc} = -if_{abc}\}$. The dimensionality of this representation is therefore equal to the number of generators ($= N^2-1$).

2.2 SU(2)

For SU(2) we have $T_a = \frac{1}{2}\sigma_a \qquad a = 1,2,3$ (17)

where σ_a are the Pauli spin matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{18}$$

2.3 SU(3)

For SU(3) the group parameters are given by $T_a = \frac{1}{2}\lambda_a \qquad a = 1,\ldots,8$
where λ_a are the Gell-Man matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda_8 = [3]^{-\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{19}$$

which satisfies eqs. (13) and (15).
The rank of SU(3) is two; the usual choice of the Cartan subalgebra is $H_1 = T_3$ and $H_2 = T_8$.

3. Invariant Group Integration for Lie Groups

3.1 Preliminaries [36;40]

For the case of finite dicrete groups we had the result (section I) that any function f(g) defined on the group manifold satisfies the relation

$$\sum_j f(g_j) = \sum_j f(g_j g_i) = \sum_k f(g_k) \,. \qquad (\text{where } g_j g_i = g_k,\ g_j \equiv g(j)) \tag{20}$$

Also $$\sum_j f(g_j) = \sum_j f(g_i^{-1} g_j), \tag{21}$$

which implies that the weight attached to an arbitrary group element g_j is the same as the one attached to the element $g_i^{-1} g_j$, which is obtained by left translation with the element g_i^{-1}.

For a continuous group G with real parameters $a_1, a_2, \ldots, a_r \in M$ we define the integral over the parameter space M as

$$\int_M d\mu(g). \tag{22}$$

As $g \equiv g(a) = g(a_1, a_2, \ldots, a_r)$,

we have that $d\mu(g) = d\mu(a_1, a_2, \ldots, a_r) \equiv d\mu(a)$. (22a)

For Lie groups we now need a similar property in order to construct a <u>left invariant measure</u>, i.e. a measure such that if f(g) is any function defined on the group, then

$$\int_M d\mu(g)\ f(g) = \int_M d\mu(g)\ f(g'g) = \int_M d\mu(g'g)\ f(g'g). \tag{23}$$

This is again a reasonable requirement because if the integral over g involves each g exactly once, then the integral over g'g involves each group operation also exactly once, but now in a different order.

Since f is an arbitrary function, the measure in (23) must satisfy

$$d\mu_L(g) = d\mu_L(g'g) \qquad \text{for all } g' \in G \tag{24}$$

i.e. it should be left invariant.

Measures satisfying the condition

$$d\mu_R(g) = d\mu_R(gg') \quad \text{(i.e. right invariance)} \tag{25}$$

can also be defined in a similar way.

3.2 Construction of a left invariant measure

If G is any Lie group, then the measure defined over its parameter space (eq.(22)) factors into [36;37;40]

$$d\mu(a) = \rho(a)d(a), \tag{27}$$

where (i) $\rho(a)$ is the density (weight) function of the group G;
(ii) d(a) is the Euclidian volume element .

Note that from now on a will always denote the set $(a_1,\ldots,a_r)$, $b = (b_1,\ldots,b_r)$ etc. .

From Section 1 we have

$$g_r(a)g_s(b) = g_t(c),$$

with $c_i = \phi_i(a;b)$. The measure can therefore be written in the form

$$\begin{aligned} d\mu(g(a)g(b)) &= d\mu(g(c)) = d\mu(c) \\ &= \rho(c)dc \\ &= \rho(\phi(a;b))\ J(a;b)da, \end{aligned} \tag{27}$$

with $\phi(a;b) = \{\phi_1(a;b),\ldots,\phi_r(a;b)\}$, and J(a;b) the Jacobian,

$$J(a;b) = \begin{vmatrix} \dfrac{\partial\phi_1(a;b)}{\partial a_1} & \dfrac{\partial\phi_2(a;b)}{\partial a_1} & \cdots & \dfrac{\partial\phi_r(a;b)}{\partial a_1} \\ \vdots & & & \vdots \\ \dfrac{\partial\phi_1(a;b)}{\partial a_r} & \dfrac{\partial\phi_2(a;b)}{\partial a_r} & \cdots & \dfrac{\partial\phi_r(a;b)}{\partial a_r} \end{vmatrix}. \tag{28}$$

From eq.(24) it follows that left invariance of the measure implies that the following condition must hold

$$d\mu(a) = \rho(a)da = \rho[\phi(a;b)]J(a;b)da. \tag{29}$$

Therefore $\rho(a) = \rho[\phi(a;b)]J(a;b)$ (30)

or $\rho[(\phi(a;b)] = \rho(a)/J(a;b)$. (30a)

The density function $\rho(\phi(a,b))$ can now be calculated by choosing any convenient value for a.
Choosing $a = 0$ means that $g(a) = e$ (i.e. the identity element of the group), which implies that

$$\phi_i(0;b) = b_i = c_i, \qquad \text{(from (6))} \tag{31}$$

so that

$$\rho[\phi(0;b)] = \rho(b) = \rho(0)/J(a;b)\big|_{a=0}. \tag{32}$$

In the neighbourhood of the identity (which we are considering) the value of $\rho(0)$ can be arbitrarily fixed (usually equal to one).

Substituting the above, the measure (29) becomes

$$d\mu(a) = [\rho(0)/J(a;b)\big|_{a=0}]\, da \tag{33}$$

$$= J^{-1}(a;b)\big|_{a=0}\, da. \tag{33a}$$

The invariant measure defined above is called the Haar measure of the group, with the property

$$N \int d\mu(a) = V, \tag{34}$$

where V is the group volume. It is usually normalized to unity

$$\int [d\mu(a)] = 1 \tag{34a}$$

where, in future, [ ] will always denote the normalized measure.

For unitary groups we introduce the notation [dU] for the normalized Haar measure, with $U \equiv g \in SU(N)$ [or U(N)].

Examples

(i) As an elementary example we calculate the density function $\rho(c)$ for the continuous two parameter non-abelian group of transformations in R_1 [37]

$$f(\alpha_1\alpha_2;x) = \alpha_1 x + \alpha_2 .$$

Operations of this group are defined on the coordinate x(m), where m is some fixed point. The group operation (α_1,α_2) is defined by

$$x' = (\alpha_1, \alpha_2)x = \alpha_1 x + \alpha_2 \qquad (\alpha_1 \neq 0)$$

From the group multiplication law we therefore have the result

$$x'' = f(\beta_1\beta_2;x') \qquad x' = f(\alpha_1 \alpha_2;x)$$

Therefore $x'' = \beta_1 x' + \beta_2 = \beta_1(\alpha_1 x + \alpha_2) + \beta_2$

$$= f(\beta_1\alpha_1, \beta_1\alpha_2 + \beta_2;x),$$

which gives $\phi(\beta_1,\beta_2; \alpha_1,\alpha_2) = (\beta_1\alpha_1, \beta_1\alpha_2 + \beta_2)$

or $c_1 = \phi_1(\beta;\alpha) = \beta_1\alpha_1$

$$c_2 = \phi_2(\beta;\alpha) = \beta_1\alpha_2 + \beta_2 .$$

Now, $\dfrac{\partial\phi_1}{\partial\alpha_1} = \beta_1$; $\dfrac{\partial\phi_2}{\partial\alpha_2} = \beta_1$; $\dfrac{\partial\phi_2}{\partial\alpha_1} = \dfrac{\partial\phi_1}{\partial\alpha_2} = 0.$

Thus, $J(\alpha;\beta)\big|_{\alpha=0} = (\beta_1)^2$

and $\rho(c) = \rho(0)/\ (\beta_1)^2 .$ $\qquad [\rho(0) = 1]$

The measure is therefore given by

$$d\alpha\ (\beta_1)^{-2} = d\alpha_1 d\alpha_2\ (\beta_1)^{-2}. \qquad (35)$$

(ii) A unitary representation of the group **SU(2)** in terms of the Euler angles α,β and γ is given by [40]

$$U(\alpha,\beta,\gamma) = \begin{pmatrix} e^{i(\gamma+\alpha)/2}\cos(\beta/2) & e^{i(\gamma-\alpha)/2}\sin(\beta/2) \\ -e^{-i(\gamma-\alpha)/2}\sin(\beta/2) & e^{-i(\gamma+\alpha)/2}\cos(\beta/2) \end{pmatrix} \tag{36}$$

where $0 \le \alpha,\beta,\gamma \le 2\pi$ (36a)

which is of the form

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}$$

with $z_1 z_1^* + z_2 z_2^* = 1$.

Writing $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$

$$z_2 = x_2 + iy_2 = r_2 e^{i\theta_2}, \tag{37}$$

we have that the product of two group elements gives

$$\begin{pmatrix} x_1' + iy_1' & x_2' + iy_2' \\ -x_2' + iy_2' & x_1' - iy_1' \end{pmatrix} \begin{pmatrix} x_1 + iy_1 & x_2 + iy_2 \\ -x_2 + iy_2 & x_1 - iy_1 \end{pmatrix}$$

$$\approx \begin{pmatrix} X_1 + iY_1 & X_2 + iY_2 \\ -X_2 + iY_2 & X_1 - iY_1 \end{pmatrix} \tag{38}$$

where, in general $X_i = X_i(x_1',x_2',y_1',y_2',x_1,x_2,y_1,y_2)$, and similarly for Y_i, with $i = 1,2$.

Calculating the Jacobian near the identity we have

$$d\mu(x_1x_2;y_1,y_2) = c[(x_1 + y_1)^2 + (x_2 + y_2)^2 + 2(x_1y_2 - x_2y_1)^2 + 2(x_1x_2 + y_1y_2)^2\ ]^{-1}\ dx_1dx_2dy_1dy_2 \ . \tag{39}$$

Using (37) this gives

$$d\mu(r_1,r_2,\theta_1,\theta_2) = cr_1r_2/(r_1^2 + r_2^2)^2\ \ dr_1dr_2d\theta_1d\theta_2 \tag{40}$$

which, after making the transformation (with $r_1^2 + r_2^2 = 1$)

$$\psi = r_1^2 + r_2^2$$

$$\phi = 2r_1 r_2 \ , \tag{41}$$

implies

$$d\mu(\psi,\phi,\theta_1,\theta_2) = (c\phi/2\psi^2)[4(\psi^2 - \phi^2)^{\frac{1}{2}}]^{-1}\ d\psi d\phi d\theta_1 d\theta_2 \ . \tag{42}$$

Restricting the integration to the hypersurface $\psi = 1$ we have

$$d\mu(\phi,\theta_1,\theta_2) = (c\phi/8)(1 - \phi^2)^{-\frac{1}{2}}\ d\phi d\theta_1 d\theta_2 \ . \tag{43}$$

This hypersurface can now be parametrized in terms of the Euler angles

$$r_1 = \cos(\beta/2)$$

$$r_2 = \sin(\beta/2)$$

which, together with (41), results in

$$d\mu(\beta,\theta_1,\theta_2) = (c\sin\beta/8)\ d\beta d\theta_1 d\theta_2 \ . \tag{44}$$

From (37) we have the identification

$$\theta_1 = (\alpha + \gamma)/2$$

$$\theta_2 = (\alpha - \gamma)/2$$

which, substituted into (44), finally gives

$$d\mu(\alpha,\beta,\gamma) = (c/16)\sin\beta\ d\alpha d\beta d\gamma \ . \tag{45}$$

Using the conventional normalization in (34a), the constant c is given by

$$c = 2/\pi \ ,$$

so that $[d\mu(\alpha,\beta,\gamma)] = (1/8\pi)\sin\beta\ d\alpha d\beta d\gamma \ ,$ (46)

with α,β and γ as in (36a).

It can be shown that if the group in question is compact, the density functions $\rho_L(a)$ and $\rho_R(a)$ which gives the left and right measures, are equal [37].
Thus, for compact groups the left and right invariant measures will also be equal and the group integrals will converge.
When the group is noncompact the measures may be unequal, but the integrals will diverge.

In addition to the property of invariance, the group integrals must satisfy the usual properties of ordinary c-number integrals, namely

<u>linearity</u>: $$\int dg[af(g) + bh(g)] = a \int dg\ f(g) + b \int dg\ h(g) \qquad (47)$$

<u>positivity</u>: $$\int dg\ f(g) > 0\ . \qquad (48)$$

3.3 <u>Calculation of the Haar Measures for the Special Unitary Groups</u>

Although the invariant measure for SU(2) has already been derived in the previous section, we will now calculate the Haar measures in a different manner which is easier (especially for the higher order groups) and explicitly uses the properties of the unitary groups .

(a) <u>SU(2)</u>

Any group element of the matrix representation of SU(2) can be written as

$$U = \exp[i\theta\ \vec{\tau}\cdot\hat{n}], \qquad (49)$$

where (i) $\vec{\tau} = \frac{1}{2}\vec{\sigma}$, with $\vec{\sigma}$ the Pauli matrices defined in (18);

(ii) $\hat{n}$ is a unit three dimensional vector;

(iii) $0 \leq \theta \leq 4\pi$.

The representation (49) can be written in the linear form

$$U = \exp[i\theta\ \vec{\tau}\cdot\hat{n}]$$

$$1 + i\theta\ \vec{\tau}\cdot\hat{n}\ - (1/2!)[\vec{\tau}\cdot\hat{n}]^2 - (i/3!)[\vec{\tau}\cdot\hat{n}]^3 + \ldots$$

$$= 1 - (1/2)(\theta/2)^2 + (1/4!)(\theta/2)^4 + \ldots$$

$$+ i\vec{\sigma}\cdot\hat{n}\ [(\theta/2) - (1/3!)(\theta/2)^3 + \ldots\ ]$$

$$= \cos(\theta/2) + i\vec{\sigma}\cdot\hat{n}\ \sin(\theta/2)$$

$$\equiv \cos(\phi) + i\vec{\sigma}\cdot\hat{n}\ \sin(\phi), \qquad (50)$$

with $\phi \equiv \theta/2$, $0 \leq \phi \leq 2\pi$.

Hence equation (50) is of the form

$$U = a_0 + i\vec{a}\cdot\vec{\sigma}\ , \qquad (51)$$

where $a_0 = \cos(\phi) = \cos(\theta/2)$

$$\vec{a} = \hat{n}\ \sin(\phi)$$

with the condition $a_0^2 + |\vec{a}|^2 = 1.$ (51a)

From the above, and the choice of normalization of the invariant measure (34a) it follows that

$$\int d^4a\ \delta(a^2-1) \approx \int dU\ . \qquad (51b)$$

Calculating the normalization factor we have

$$N \int d^4a\ \delta(a^2-1) = N \int da_0 \int d^3a\ \delta(a_0^2 + |\vec{a}|^2 - 1)$$

$$= N \int da_0\ |\vec{a}|^2\ d(|\vec{a}|)\ d\Omega\ \delta(a_0^2 + |\vec{a}|^2 - 1)$$

$$= \tfrac{1}{2}N \int_{-1}^{1} da \int d\Omega\ (1 - a_0^2)^{\frac{1}{2}}$$

$$= 2\pi N \int_0^{\pi} \sin^2\theta \, d\theta = \pi^2 N.$$

If eq.(51b) is normalized to unity the above implies that $N^2 = \pi^{-2}$, hence

$$[dU] = \pi^{-2} \, da_0 \, d^3a \, \delta(a^2-1)$$

$$= \pi^{-2} \, d^4a \, \delta(a^2-1). \qquad (52)$$

Alternatively, if the parametrization of eq.(50) is chosen the normalized Haar measure reads

$$[dU] = (1/2\pi^2) \sin^2\phi \, d\phi \, d^2n \, , \qquad (0 \leq \phi \leq 2\pi) \qquad (53)$$

as $$\int [dU] = (1/2\pi^2) \int d^2n \, [\tfrac{1}{2} \int_0^{2\pi} d\phi \, (1 - \cos 2\phi)]$$

$$= \pi(1/2\pi^2) \int d^2n = 1. \qquad (53a)$$

(b) <u>SU(3)</u>

The SU(3) group manifold can be parametrized in terms of two normalized complex three dimensional vectors $\vec{u}$ and $\vec{v}$.

Each element U of SU(3) can therefore be written in matrix form

$$U = \begin{pmatrix} u_1^* & u_2^* & u_3^* \\ v_1^* & v_2^* & v_3^* \\ w_1 & w_2 & w_3 \end{pmatrix} \qquad (54)$$

with $\vec{w} = \vec{u} \times \vec{v}$

$$w_i = \varepsilon_{ijk} u_j v_k \qquad (55a)$$

and the constraints (so that $U^{\dagger}U = 1$):

$$\vec{u}^* \cdot \vec{v} = 0$$

$$|u_1|^2 + |u_2|^2 + |u_3|^2 = 1$$

$$|v_1|^2 + |v_2|^2 + |v_3|^2 = 1$$

$$\sum_i w_i w_i^* = \sum_i \varepsilon_{ijk}\varepsilon_{ilm}\, u_j v_k u_l^* v_m^*$$

$$= \sum_i |u_i|^2 |v_i|^2 - (\vec{u}\cdot\vec{v}^*)(\vec{v}\cdot\vec{u}) = 1$$

$$\sum_i w_i v_i = 0 = \sum_i w_i u_i . \qquad (55b)$$

Note also that the SU(3) constraint (det U = 1) is also satisfied,

$$\det U = \varepsilon_{ijk}\, u_i^* v_j^* w_k$$

$$= \varepsilon_{ijk}\,\varepsilon_{klm}\; u_i^* v_j^* u_l v_m \qquad \text{(substituting } w_k\text{)}$$

$$= (\sum_i |u_i|^2)(\sum_j |v_j|^2) - (\vec{u}^*\cdot\vec{v})(\vec{u}\cdot\vec{v}^*) = 1. \qquad (55c)$$

The total number of independent variables is eight. This follows because u_i and v_i are complex (thus contributing twelve variables), while there are four constraints in order to ensure that detU = 1 and $UU^\dagger = 1$.

The group integral over the Haar measure can therfore be written as

$$\int [dU] = N' \int_{-\infty}^{\infty} d^6u \int_{-\infty}^{\infty} d^6v\; \delta(|u|^2-1)\;\delta(|v|^2-1)\delta(u^*.v) \qquad (56)$$

where N′ is chosen in order for the measure to satisfy the normalization (34a) and by $\delta(u.v^*)$ it is understood that the real and imaginary parts are seperately equal to zero, i.e.

$$\delta(u.v^*) = \delta\Big[\mathrm{Re}(u.v^*)\Big]\;\delta\Big[\mathrm{Im}(u.v^*)\Big] . \qquad (56a)$$

(c) <u>SU(N)</u>

For SU(N) we can generalize our previous results for SU(3) by writing any group element in the form of an N x N matrix

$$\begin{pmatrix} u_1^* & u_2^* & \cdots & u_N^* \\ v_1^* & v_2^* & \cdots & v_N^* \\ \vdots & & & \\ t_1^* & t_2^* & \cdots & t_N^* \\ w_1 & w_2 & \cdots & w_N \end{pmatrix} \tag{57}$$

with $w_i = (-1)^{N-1} \varepsilon_{ii_1 \ldots i_{N-1}} u_{i_1} \ldots u_{i_{N-1}}$ (57a)

(where the $(-1)^{N-1}$ factor is needed to ensure that detU = 1 [note that for N = 3 the result above reduces to the one in (55a)])

and the constraints

$$\sum_{i=1}^{N} |u_i|^2 = 1$$

$$\sum_{i=1}^{N} |v_i|^2 = 1 \quad \text{etc.}$$

$$\sum_{i=1}^{N} u_i^* v_i = 0 \text{ etc.} \tag{57b}$$

(which ensures that $UU^\dagger = 1$).

The normalized Haar measure is therefore given by

$$\int [dU] = N' \int_{-\infty}^{\infty} d^{2N}u \ldots \int_{-\infty}^{\infty} d^{2N} t \; \delta(|u|^2-1) \ldots \delta(|t|^2-1) \times$$

$$\delta(u^*.v) \ldots \delta(u^*.t) \; , \tag{58}$$

where $\delta(u^*.v)$ again implies that the real and imaginary parts are seperately equal to zero, i.e.

$$\delta(u^*.v) \equiv \delta\left(\sum_{i=1}^{N} \mathrm{Re}(u_i^* v_i) \right) \delta\left(\sum_{i=1}^{N} \mathrm{Im}(u_i^* v_i) \right) . \tag{58a}$$

3.4 General Results of SU(N) Group Integrals

In many cases the explicit form of the Haar measure need not be known, and instead the specific properties of the Haar measure can be used.

For the SU(N) invariant Haar measure we have the following results:

$$\int [dU] = 1 \tag{59}$$

$$\int [dU]\, U_{ij} U^{\dagger}_{kl} = \frac{1}{N} \delta_{il}\delta_{jk} \tag{60}$$

$$\int [dU]\, U = \int [dU]\, U^{\dagger} = 0 \tag{61}$$

$$\int [dU]\, U_{i_1 j_1} \cdots U_{i_N j_N} = \frac{1}{N!}\, \varepsilon_{i_1 \cdots i_N}\, \varepsilon_{j_1 \cdots j_N} \tag{62}$$

$$\begin{aligned} &\int [dU]\, U_{i_1 j_1}\, U^{\dagger}_{k_1 l_1}\, U_{i_2 j_2}\, U^{\dagger}_{k_2 l_2} \\ &= (N^2-1)^{-1} [\delta_{i_1 l_1}\delta_{i_2 l_2}\, \delta_{j_1 k_1}\delta_{j_2 k_2} + \delta_{i_1 l_2}\delta_{i_2 l_1}\, \delta_{j_1 k_2}\delta_{j_2 k_1}] \\ &\quad - \{N(N^2-1)\}^{-1} [\delta_{i_1 l_1}\delta_{i_2 l_2}\, \delta_{j_1 k_2}\delta_{j_2 k_1} + \delta_{i_1 l_2}\delta_{i_2 l_1}\, \delta_{j_1 k_1}\delta_{j_2 k_2}] \end{aligned} \tag{63}$$

$\int [dU]\, U_{ij}U_{kl} = \int [dU]\, U^{\dagger}_{ij}U^{\dagger}_{kl} = 0$ except for SU(2) in which case

$$\int [dU]\, U_{ij}U_{kl} = \tfrac{1}{2}\delta_{il}\delta_{jk} \tag{64}$$

$$\int [dU]\, \mathrm{tr}(AU^{\dagger})\, \mathrm{tr}(UB) = N^{-1}\mathrm{tr}(AB) \tag{65}$$

The proofs of these results rely on the invariance properties of the measure. For illustration we will now give a few below [with [dU] always defined by (58)].

(i) Consider the integral $\int [dU]\, U$. The invariance of the measure implies that $\int [dU]\, U = \int [dU]\, W_1 U W_2^{\dagger}$. This however can only be true if $\int [dU]\, U = 0$.

We note that the Haar measure (58) is invariant under the substitution

$$U \to -U ,$$

i.e
$$\begin{aligned} u_i &\to -u_i \\ v_i &\to -v_i \\ &\cdot \quad\quad \cdot \\ &\cdot \quad\quad \cdot \\ &\cdot \quad\quad \cdot \\ t_i &\to -t_i \end{aligned} \tag{66}$$

for any i and $1 \leq i \leq N$.

(ii) Consider $\int [dU]\, U_{ij} U^{\dagger}_{kl}$. This integral vanishes identically if U_{ij} and $U^{\dagger}_{kl}$ are unrelated because of the invariance of the measure under the transformation $U \to -U$. However, if $j = k$ and $i = l$, then

$$U_{ij} U^{\dagger}_{kl} = U_{ij} U^{\dagger}_{ji}$$
$$= |u_1|^2 + \ldots + |u_N|^2 + |v_1|^2 + \ldots + |t_N|^2,$$

where summation over the indices i and j was assumed.

Using the fact that

$$|u_1|^2 + \ldots + |u_N|^2 = 1$$

$$|v_1|^2 + \ldots + |v_N|^2 = 1 \quad \text{etc.}, \text{ we thus have}$$

$$\int [dU]\, U_{ij} U^{\dagger}_{kl} = \frac{1}{N} \delta_{il} \delta_{jk}.$$

(iii) The proof of (65) follows trivially if the traces are written out:

$$I = \int [dU]\, \mathrm{tr}(AU^{\dagger})\, \mathrm{tr}(UB)$$
$$= \sum_{ijkl} \int [dU]\, A_{ij} U^{\dagger}_{ji}\, U_{kl} B_{lk}$$
$$= N^{-1} \sum_{ijkl} A_{ij}\, B_{lk}\, \delta_{jl} \delta_{ik} \qquad \text{(using (60))}$$
$$= N^{-1}\, \mathrm{tr}(AB).$$

4. Group Characters

4.1 General Definitions

Using a similarity transformation (i.e. change of basis) an infinite number of equivalent (matrix) representations can be constructed from any given representation. The trace of any representation however is always independent of a similarity transformation.

We now use this property and define the character of the group element g(a) in the representation $D^{(r)}$ by

$$\chi_r(a) = \text{tr } D^{(r)}(a)$$

$$= \sum_i D^{(r)}_{ii}(a). \tag{67}$$

To show that the character function is invariant under a similarity transformation (i.e. that equivalent representations have the same set of characters) we proceed as follows.

Under a similarity transformation the representation $D^{(r)}(a)$ transforms as

$$D^{(r')} = AD^{(r)}A^{-1}. \tag{68}$$

Applying the trace defined in (67) gives

$$\chi^{(r')} = \sum_i D^{(r')}_{ii}(a)$$

$$= \sum_{ijk} A_{ij} D^{(r)}_{jk}(a) A^{-1}_{ki}$$

$$= \sum_{jk} D^{(r)}_{jk}(a)(A^{-1}A)_{kj}$$

$$= \sum_j D^{(r)}_{jj}(a) \equiv \chi^{(r)}(a). \tag{69}$$

Also, all elements of a group G that are in the same class will have the same character. For example, let g(m) and g(j) be two group elements which are in the same class. Then for any representation D,

$$\chi^{(r)}(m) = \sum_i D^{(r)}_{ii}(njn^{-1})$$

$$= \sum_{ijk} D^{(r)}_{ij}(n)\, D^{(r)}_{jk}(j)\, D^{(r)}_{ki}(n^{-1})$$

$$= \sum_j D^{(r)}_{jj}(m). \qquad (70)$$

Note also that if $D^{(r)}$ is self conjugate, then all the characters in $D^{(r)}$ are real. Also, in any representation of any group (discrete or continuous) the character of the unit element g(e) is d_r, where d_r is the dimension of the representation $D^{(r)}$, i.e.

$$\dim(D^{(r)}) \equiv tr(1)$$

$$= \chi_r(0,0,\ldots,0) = d_r. \qquad (71)$$

From the definition it follows that characters satisfy similar orthogonality relations to the representations in (11) and (12).

Setting i = k and j = l in (11) we have

$$\sum_a D^{(m)}_{ii}(a)\, D^{(n)-1}_{jj}(a) = (N/d_m)\, \delta_{mn}\delta_{ij}.$$

which, after performing the summation over all i and j, gives

$$\sum_a \chi^{(m)}(a)\, \chi^{(n)-1}(a) = N\delta_{mn}. \qquad (72)$$

For unitary representations this gives

$$\sum_a \chi^{(m)}(a)\, \chi^{(n)*}(a) = N\delta_{mn}. \qquad (73)$$

4.2. Characters of the Irreducible Representations of the Unitary Groups

For SU(N) we define the group character using the "charge operators" H_j of the group (section II) with j = 1,...r where r is the rank of the group. The character for a group element of SU(N) corresponding to the r'th irreducible representation is defined by

$$\chi_r(\alpha_1,..,\alpha_{N-1}) = \mathrm{tr}_r \exp[i \sum_{j=1}^{N-1} \alpha_j H_j] \tag{74}$$

where $\alpha_1,..,\alpha_{N-1}$ are simple numbers that are used to parametrize the group elements for a specific (irreducible) representation r. It is important to note that the characters are defined only on the N-1 diagonal operators.

In anology to the case of discrete groups, representations of continuous groups also satisfy orthogonality relations, namely

$$\int_G D^{(m)}_{ik}(a)\, D^{(n)-1}_{jl}(a) d\mu(a) = (V/d_m)\, \delta_{mn}\delta_{il}\delta_{kj} \tag{75}$$

and for unitary representations (e.g. Lie groups like U(N) and SU(N))

$$\int_G D^{(m)}_{ik}(U)\, D^{(n)*}_{lj}(U) d\mu(U) = (V/d_m)\, \delta_{mn}\delta_{il}\delta_{kj}. \tag{76}$$

The corresponding relations for the characters therefore read

$$\int_G \chi^{(m)}(U)\, \chi^{(n)*}(U) d\mu(U) \equiv \int_G \chi^{(m)}(U)\, \chi^{(n)}(U^*) d\mu(U)$$

$$= V\delta_{mn}. \tag{77}$$

For Lie groups the group volume is finite, so that the normalized Haar measure can thus be used, which gives

$$\int_G [dU]\, \chi^{(m)}(U)\, \chi^{(n)}(U^*) = \delta_{mn}. \tag{78}$$

Also, in general if

$$D^{(r)}(a) \otimes D^{(r')}(a) = D^{(m)}(a) + D^{(n)}(a)$$

then $$\chi^{(r)}(a) \otimes \chi^{(r')}(a) = \chi^{(m)}(a) + \chi^{(n)}(a). \tag{79}$$

E.g. the SU(3) decomposition $3 \otimes 3 = 6 + \bar{3}$ will corresponds to

$$(\chi_3)^2 = \chi_6 + \chi_3^* . \tag{79a}$$

By definition it also follows that (compare eq.(71))

$$\chi_r(1) = d_r, \tag{80}$$

which for SU(N) implies

$$\begin{aligned} \chi_f(1) &= d_f = N \\ \chi_o(1) &= d_o = 1 \\ \chi_a(1) &= d_a = N^2-1. \end{aligned} \tag{81}$$

(where f denotes the fundamental reperentation, o the trivial one and a the adjoint representation)

4.3 Properties of Unitary Group Characters

Using (in most cases) only the orthogonality property (78), it is easy to show that the character functions satisfy the following relations:

(i) $$\int [dU]\, \chi_r^*(U)\chi_s(TU) = d_r^{-1}\, \delta_{rs}\chi_r(T) \tag{82}$$

If T = I, the above implies

$$\int [dU]\, \chi_r^*(U)\chi_s(IU) = d_r^{-1}\, \delta_{rs}\chi_r(I) = \delta_{rs}$$

which is just the relation (78).

(ii) $$\int [dU]\, \chi_r(UW)\chi_s^*(T^\dagger U)$$

$$= \int [dU]\, \chi_r(UW)\chi_s(U^\dagger T) = d_r^{-1}\, \delta_{rs}\chi_r(WT). \tag{83}$$

Also

$$\int [dU]\, \chi_r(UW)\chi_s^*(TU) = d_r^{-1}\, \delta_{rs}\chi_r(WT^\dagger) \tag{83a}$$

(iii) $$\int [dU]\, \chi_r(U) = \delta_{ro} \tag{84}$$

As dU = dUT for any T ε G, this implies

$$\int [dU]\, \chi_r(UT) = \delta_{ro}. \tag{84a}$$

(iv) For SU(2), $\chi_r(U^\dagger) = \chi_r^*(U) = \chi_r(U)$ (85)

(v) $\int [dU]\, \chi_r(U)\chi_s(U) = n_s$, (86)

where n_s is the number of singlets in $r \otimes s$ $[D^{(r)} \otimes D^{(s)}]$.

(vi) Any class function defined on the group elements can be expanded as a linear combination of characters

$$f(U) = f(VUV^{-1}) = \sum_{r=1}^{\infty} c_r \chi_r(U). \tag{87}$$

To determine c_r we use (78), which gives the result

$$c_r = \int [dU]\, \chi_r^*(U) f(U) \tag{88}$$

In the expresssions above T and W may also stand for any combination of group elements (links), e.g.

$$T = \prod_{\ell'} U_{\ell'}, \qquad (\ell \neq \ell') .$$

As an example we calculate the characters for SU(2) which has only one charge operator $H_1 \equiv J_3$ (so called because of the identification of SU(2) with isospin and/or angular momentum).

J_3 can take on 2J + 1 possible values

$$|J_3| \leq J \quad \text{or} \quad J_3 = -J, -J+1, \ldots, J \tag{89}$$

The character of SU(2) (corresponding to the spin representation) is therefore given by

$$\chi_J(\alpha) = \mathrm{tr}_J \exp[i\alpha J_3] = \sum_{J_3=-J}^{J} e^{i\alpha J_3}$$

$$= \sin\alpha(J+\tfrac{1}{2})/\sin(\alpha/2), \tag{90}$$

where $J = 0, \tfrac{1}{2}, 1, \ldots$ and α is a single continuous variable with $0 \leq \alpha \leq 2\pi$.

5. Calculation of One-link Integrals

In Lattice Gauge theories one-link integrals occur in wide variety of forms. We will consider a few of these.

(a) The generating functional form for the integral in LGT is

$$W[J,K] = \int [dU] \exp[\mathrm{tr}(JU + U^{\dagger}K)] \tag{91}$$

where K and J are arbirary N x N matrices that act as sources and [dU] is the SU(N) Haar measure.

In [35c] equation (1) was evaluated with K = 0, i.e.

$$W(J) = W[J,K=0]$$

$$= \int [dU] \exp[\mathrm{tr}JU]. \tag{92}$$

Using the invariance property of the Haar measure, W(J) can be written in terms of a power series

$$W(J) = \sum_{i=0}^{\infty} a_i (\det J)^i, \tag{92a}$$

with $$\det J = \frac{1}{N!}\, \varepsilon_{i_1 \ldots i_N} \varepsilon_{j_1 \ldots j_N} J_{i_1 j_1} \cdots J_{i_N j_N}. \tag{92b}$$

Using a recursion relation the coefficients a_i ($i \neq 0$) can be calculated (from the normalization of the measure it follows that $a_0 = 1$), which results in (92a) reading (SU(N)):

$$W(J) = \sum_{i=0}^{\infty} \frac{2! \ldots (N-1)!}{i! \ldots (i+N-1)!} (\det J)^i. \tag{92c}$$

(b) In the absence of (external) sources the single link integral has the general form

$$Z_N(A) = \int [dU] \exp[\beta N \,\mathrm{tr}(UA + A^{\dagger}U^{\dagger})], \tag{93}$$

where $$\beta \equiv (g^2 N)^{-1} \tag{93a}$$

and $A = \sum_i U_i$ is the sum of unitary matrices that complete the neighbouring plaquettes that couple to the link U.

This integral has been calculated in [35d] by approximate means using the $N \to \infty$ limit and assuming that it is allowable to exchange this limit with the SCE of the exponent in the integrand. [Note that in the calculations the value of ß (which is small) is kept fixed as $N \to \infty$].

This approximation gives (U(N))

$$Z_N(A) \approx \exp[ß^2 N \, tr(AA^\dagger)]$$

$$= \exp[(Ng^4)^{-1} \, tr(AA^\dagger)]. \qquad (94)$$

Using the Weyl parametrization of the U(N) measure (see Chapter 7) the integral in (93) can be evaluated exactly if A is proportional to the unit matrix (i.e. A = aI):

$$Z_N(aßN) = \det I_{i-j}(2aßN), \qquad (95)$$

where I_n is the modified Bessel function (see App. D).

To obtain the corresponding equation for SU(N) the restrictions $\sum_{i=1}^{N} \phi_i = 0 (\mathrm{mod} 2\pi)$ must be added to the original parametrization of the U(N) measure, which leads to

$$Z_N(aßN) = \sum_{m=-\infty}^{\infty} \det I_{m+j-i}(2aßN). \qquad (96)$$

[A similar expression was derived in Chapter 7, sec. 5B]

(c) Another method of evaluating the single [U(N)] link integral with external sources (as defined in (91)) is to make use of a character expansion [35e].

Consider the generating functional

$$Z_N(J,J^\dagger) = \int [dU] \exp[ßN \, tr(JU + U^\dagger J^\dagger)], \qquad (97)$$

where J and $J^\dagger$ are arbirtrary sources and ß is again given by (93a).

Using the expansion

$$\exp[\beta N \, \mathrm{tr}(JU)] = \sum_r c_r(\beta N)\, \chi_r(JU) \tag{98}$$

we obtain

$$Z_N(J,J^\dagger) = \sum_{r,r'} c_r c_{r'} \int [dU] \chi_r(JU) \chi_{r'}(U^\dagger J^\dagger). \tag{99}$$

Using the orthogonality of the characters (section IV) gives

$$Z_N(JJ^\dagger) = \sum_r c_r^2 \, (d_r)^{-1} \chi_r(JJ^\dagger), \tag{100}$$

where d_r is the dimension of the representation r.

The coefficients c_r are given by the inverse relation

$$c_r(\beta N) = \int [dU] \chi_r(U^\dagger)\, e^{\beta N \, \mathrm{tr} U}. \tag{101}$$

Again using the Weyl parametrization of the Haar measure, and noting that any representation of U(N) can be labelled by N integers $r = (n_1,\ldots,n_N)$ with $n_1 \geq n_2 \geq \ldots \geq n_N \geq 0$, where n_i corresponds to the number of boxes in the Young tableau, we have the result [35e]

$$\begin{aligned} c_r &= \frac{1}{N!}\, \varepsilon_{i_1,\ldots,i_N} \varepsilon_{j_1,\ldots,j_N} \times \\ &\qquad \prod_{k=1}^{N} \int_0^{2\pi} (d\phi_k/2\pi)\, \exp[-i\phi_k(n_{i_k} + j_k - i_k)] \exp[\beta N e^{i\phi_k}] \\ &= \frac{1}{N!}\, \varepsilon_{i_1,\ldots,i_N} \varepsilon_{j_1,\ldots,j_N} \times \\ &\qquad \prod_{k=1}^{N} \left(\int_0^{2\pi} (d\phi_k/2\pi)\, \exp[-i\phi_k(n_{i_k} + j_k - i_k)] \exp[\beta N \cos\phi_k] \right) \\ &= \det I_{|n+j-i|}(\beta N). \end{aligned} \tag{102}$$

Using the series expansion of the Bessel function $I_n(x)$, this gives

$$c_r = \det\left(\frac{1}{(n_i+j-i)\pi} \right) (\beta N)^{n_1+\ldots+n_N}$$

$$= \left(n_1\pi \ldots n_N\pi \right)^{-1} \prod_{1\le i\le j\le N} [1- (n_j/n_i+j-i)]. \qquad (103)$$

Substituting the above into eq.(100) gives

$$Z_N(JJ^\dagger) = \sum_{n_1\ge\ldots\ge n_N\ge 0} \Big((\beta N)^{2(n_1+\ldots+n_N)}/[n_1! \ \ldots \ n_N!]^2 \ \times$$

$$\prod_{1\le i\le j\le N} [1- \{n_j/(n_i+j-i)\}]^2 \ \ (d_r)^{-1}\chi_r(JJ^\dagger)\Big) \qquad (104)$$

with $r \equiv (n_1,n_2,\ldots,n_N)$.

Equation (104) can now be evaluated for the different characters corresponding to the representation r.

Examples of the U(N) characters are

$$\chi_{(1,0,\ldots,0)} = tr(JJ^\dagger) \qquad [\text{with } d_{(1,0,\ldots,0)} = N]$$

$$\chi_{(2,0,\ldots,0)} = (2\pi)^{-1}[tr^2 JJ^\dagger) + tr\{(JJ^\dagger)^2\}] \ , \text{ with}$$

$$d_{(2,0,\ldots,0)} = \tfrac{1}{2}N(N+1) \ . \qquad (105)$$

(d) The one-link integral with sources $J^\dagger$ and J has been calculated for U(N) and SU(N) with N = 2 and 3 in [35a].
We will now consider the case of N = 2 in more detail.

Using the parametrization

$$U = \begin{pmatrix} u_1^* & u_2^* \\ w_1 & w_2 \end{pmatrix} \tag{106}$$

and noting that $w_1 = -\varepsilon_{12}u_2 = -u_2$

$$w_2 = -\varepsilon_{21}u_1 = u_1 ,$$

we obtain the same prametrization as in (51),i.e

$$\int [dU] \equiv \pi^{-2} \int_{-\infty}^{\infty} d^4u \; \delta(|u|^2-1). \tag{107}$$

Using the following integral representation of the Dirac delta function

$$\delta(|u|^2-1) = (2\pi)^{-1} \int_{-\infty}^{\infty} ds \; e^{is(|u|^2-1)} , \tag{108}$$

we therefore have

$$\int [dU] \equiv (2\pi^3)^{-1} \int_{-\infty}^{\infty} d^4u \int_{-\infty}^{\infty} ds \; e^{is(|u|^2-1)}. \tag{109}$$

Consider the integral in (97) with

$$A \equiv \beta NJ = 2\beta J$$

$$B \equiv \beta NJ^\dagger = 2\beta J^\dagger$$

Hence,

$$Z_{N=2}(A,B) = \int [dU] \exp[tr(UA + U^\dagger B)] \tag{110}$$

with integrand

$$\mathrm{tr}(UA + U^{\dagger}B)$$

$$= \mathrm{tr}\left[\begin{pmatrix} u_1^* & u_2^* \\ -u_2 & u_1 \end{pmatrix} A + \begin{pmatrix} u_1 & -u_2^* \\ u_2 & u_1 \end{pmatrix} B \right]$$

$$= \mathrm{tr}\left[\begin{pmatrix} u_1^* & u_2^* \\ -u_2 & u_1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + \begin{pmatrix} u_1 & -u_2^* \\ u_2 & u_1 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \right]$$

$$= u_1^* a_{11} + u_2^* a_{21} - u_2 a_{12} + u_1 a_{22} + u_1 b_{11} - u_2^* b_{21} + u_2 b_{12} + u_1^* b_{22} . \tag{111}$$

Defining

$$u_1 = v_1 + iv_2$$

$$u_2 = v_3 + iv_4 , \tag{112}$$

we can rewrite eq.(111) as

$$\mathrm{tr}(UA + U^{\dagger}B)$$

$$= v_1(a_{11} + a_{22} + b_{11} + b_{22}) + iv_2(-a_{11} + a_{22} + b_{11} - b_{22})$$

$$= v_3(a_{21} - a_{12} - b_{21} + b_{12}) + iv_4(-a_{21} - a_{12} + b_{21} + b_{12})$$

$$\equiv \sum_{i=1}^{4} v_i A_i . \tag{113}$$

Therefore

$$Z_2(A,B) = \int [dU] \exp[\mathrm{tr}(UA + U^{\dagger}B)]$$

$$= (2\pi^3)^{-1} \int_{-\infty}^{\infty} ds\, e^{-is} \int_{-\infty}^{\infty} d^4v\, e^{isv^2} \exp[\sum_{i=1}^{4} v_i A_i]$$

$$= (2\pi^3)^{-1} \int_{-\infty}^{\infty} ds\, e^{-is} \prod_{i=1}^{4} \left(\int_{-\infty}^{\infty} dv_i \exp[isv_i^2] \exp[v_i A_i] \right) . \tag{114}$$

Completing the square in the integrand gives

$$\int_{-\infty}^{\infty} dv_i \exp[isv_i^2] \exp[v_i A_i]$$

$$= \int_{-\infty}^{\infty} dv_i \exp\left[is[v_i^2 + 2v_i(A_i/2is) - (A_i^2/4s^2)] + is(A_i^2/4s^2)\right]$$

$$= \sqrt{(\pi/-is)}\, \exp[i(A_i^2/4s)]. \qquad (115)$$

so that eq.(114) now reads

$$Z_2(A,B) = (2\pi^3)^{-1} \int_{-\infty}^{\infty} ds\, e^{-is}\, (-\pi^2/s^2)\, \exp[(i/4s) \sum_i A_i^2]. \qquad (116)$$

Using the series expansion of the exponential function we can write

$$Z_2(A,B) = -(2\pi)^{-1} \sum_{m=0}^{\infty} \frac{i^m}{m!} \int_{-\infty}^{\infty} ds\, s^{-(m+2)} \left[\frac{1}{4} \sum_{i=1}^{4} A_i^2\right]^m. \qquad (117)$$

The integral can now be evaluated by partial integration, which gives

$$Z_2(A,B) = -(2\pi)^{-1} \sum_{m=0}^{\infty} \frac{i^m}{m!} \frac{(-i)^{m+1}}{(m+1)!} (-2\pi i) \left[\frac{1}{4} \sum_{i=1}^{4} A_i^2\right]^m$$

$$= \sum_{m=0}^{\infty} [m!(m+1)!]^{-1} \left[\frac{1}{4} \sum_{i=1}^{4} A_i^2\right]^m. \qquad (118)$$

Noting that

$$A_1^2 + A_2^2 = 4(b_{11} + a_{22})(a_{11} + b_{22})$$

$$A_3^2 + A_4^2 = 4(-a_{12} + b_{12})(a_{21} - b_{21})$$

and

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

$$\mathrm{tr}AB = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}\ ,$$

we have that

$$\frac{1}{4}\sum_{i=1}^{4} A_i^2$$

$$= \det A + \det B + \mathrm{tr}AB. \tag{119}$$

Hence

$$\int [dU]\, \exp[\mathrm{tr}(UA + U^{\dagger}B)] = \sum_{m=o}^{\infty} [m!(m+1)!]^{-1}(\det A + \det B + \mathrm{tr}AB)^m. \tag{121}$$

Comparing the result above with the series expansion of the modified Bessel functions

$$I_1(z) = \tfrac{1}{2}z \sum_{k=o}^{\infty} (z^2/4)^k/[k!(k+1)!] \tag{121}$$

we obtain the following expression

$$\int [dU]\, \exp[\mathrm{tr}(UA + U^{\dagger}B)]$$

$$= 2I_1(z)/z\ , \tag{122}$$

where $z^2/4 = \det A + \det B + \mathrm{tr}AB.$ (122a)

CHAPTER 6: STRONG COUPLING METHODS

1. Preliminaries

One of the reasons for introducing a lattice structure to describe QCD is that it provides a way of solving the theory exactly when the coupling constant g becomes very large.

In this limit the theory exhibits confinement (as will be shown below). This so-called Strong Coupling Expansion (SCE) is similar to the high temperature limit in Statistical Physics [1,3c,7,11,15] (this analogy is only possible because we are working in an Euclidian framework).

This method is well-defined because it has been rigorously proven [7] that high temperature (Strong Coupling) expansions have a finite radius of convergence (see also [15]). This implies that confinement holds for all powers of ß as long as ß remains sufficiently small.

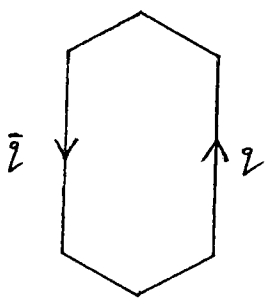
2. The Wilson Loop as Order Parameter

We want to show that the Wilson loop expectation value describes the creation, propagation and annihilation of a static quark-antiquark source, and that its behaviour determines the confinement property of QCD.

Consider now a $q\bar{q}$ pair with infinite mass (i.e. all kinetic degrees of freedom are "frozen" and no virtual $q\bar{q}$ production takes place). These $q\bar{q}$ pairs now serve as the static sources of colour charges which are characteristic for an $SU(3)_c$ triplet.

First of all the $q\bar{q}$ pair must be separated to a relative distance R from each other. This configuration is then held for a "time" $T \gg R$. Finally, the $q\bar{q}$ pair is allowed to annihilate ($R \to 0$).

Diagramatically



The propagation of the $q\bar{q}$ pair is given by the matrix element

$$\langle f|e^{-HT}|i\rangle \tag{1}$$

where (i) $|i\rangle$ and $|f\rangle$ are the initial and final states respectively
(ii) e^{-HT} is the (time) evolution operator.

This matrix can be written in terms of path integrals,

$$\langle f|e^{-HT}|i\rangle = \frac{1}{Z}\int [d\phi][dA]\ \exp[-\int d^4x\ \mathcal{L} + ig\int d^4x\ A_\mu J_\mu]$$

$$= \frac{1}{Z}\int [d\phi][dA]\ \exp[-S(\phi) + S_I] \tag{2}$$

with $Z = \int [d\phi]\ e^{-S(\phi)}$ (2a)

and $S_I = ig\int d^4x\ A_\mu J_\mu$, (2b)

which arises from the interaction of the quark charges with the fluctuating background gluon fields.

If we consider a closed contour C (i.e. $|i\rangle = |f\rangle$), the interaction term S_I reduces to

$$S_I = ig\oint_C d^4x\ A_\mu(x) \tag{3}$$

where $A_\mu(x) = A^a_\mu(x)\ T^a$. (3a)

This follows because the system now has a singlet character, i.e. the source $J_\mu(x)$ can be chosen such that

$$J^a_\mu(x) = T^a\ \delta_{\mu o}\ \delta^3(\vec{x} - \vec{R}) - T^a\ \delta_{\mu o}\ \delta^3(\vec{x} - \vec{R}'). \tag{3b}$$

For a closed contour the LHS of (2) gives

$$\langle f|e^{-HT}|i\rangle = e^{-V(R)T}\ \langle f|i\rangle$$

$$= e^{-V(R)T} \tag{4}$$

(because the sources are static, the energy is purely potential).

Therefore

$$\langle f|e^{-HT}|i\rangle = e^{-V(R)T}$$

$$= \frac{1}{Z}\int [d\phi]\, e^{-S + ig\int_C d^4x\, A_\mu(x)}$$

$$= \langle \mathrm{tr}\, e^{ig\int_C d^4x\, A_\mu(x)}\rangle$$

$$= \langle A(C) \equiv W(C). \qquad (5)$$

A(C) is called the Wilson loop. We note that because of the trace (which is needed to sum over all possible colour combinations), the Wilson loop has a local gauge invariance under the transformations defined in Chapter 2.

This is in agreement with Elitzur's theorem [44], which requires that the average value of any non-gauge invariant quantity vanishes identically, irrespective of the coupling.

The Wilson loop is a non-local order parameter which distinguishes (for pure gauge theories) between the different phases of the theory, and will next be linked to the question of confinement.

3. The Wilson Loop and Confinement

Consider the same model as in section 2, with a quark at y = (t,0) and an anti-quark at x = $(t,\vec{R})$ (the quark sources must again be heavy in order to prevent virtual $q\bar{q}$ pairs from forming.

The $q\bar{q}$ state at a time t can be represented in terms of a gauge invariant operator M(x,y) [56], which creates a widely separated $q\bar{q}$ pair:

$$|q(t,\vec{0})\ \bar{q}\ (t,\vec{R})\rangle = \sum_C f(C)\ M[(t,\vec{R}),(t,\vec{0});C]|0\rangle \qquad (6)$$

where $M(x,y;C) = \bar{q}(x)U(P,x,y)q(y)$ can be interpreted as a meson creation operator, $|0\rangle$ is the vacuum state and f(C) is the amplitude of the state with path C.

Also $U(P,x,y) = P \exp[ig \int_y^x A_\mu \, dx^\mu]$. (6b)

(where P denotes that the path is directed).

For t = 0 we choose the path to be a straight line with $x_0 = y_0 = 0$. The $q\bar{q}$ pair can be destroyed with $M^\dagger$.

The system is now allowed to evolve for a time t = T, after which the overlap between the $q\bar{q}$ states at t = 0 and t = T is measured.

$$\Omega(T,R) = \langle 0|M^\dagger[(T,\vec{0}),(T,\vec{x})]\, M[(0,\vec{0}),(0,\vec{x})]|0\rangle \, . \tag{7}$$

Inserting a complete set of energy eigenstates, we obtain for the Euclidian formalism

$$\Omega(T,R) = \sum_n |\langle n|M[(0,\vec{0});(0,\vec{x})]|0\rangle|^2 \, e^{-E_n t} \, . \tag{8}$$

The asymptotic behaviour for large T is dominated by the least energetic state which couples to M. This smallest energy eigenvalue corresponds to the potential energy of the static $q\bar{q}$ system separated by a distance R.

Hence

$$\lim_{T\to\infty} \Omega(T,R) \sim e^{-E(R)T} \tag{9}$$

From (7) we have

$$\Omega(T,R) = \langle 0|M^\dagger[(T,\vec{0}),(T,\vec{x})]\, M[(0,\vec{0}),(0,\vec{x})]|0\rangle$$

$$= \langle 0|\bar{q}(T,\vec{x})U[(T,\vec{x}),(T,\vec{0});C]q(T,\vec{0})\bar{q}(0,\vec{0})U[(0,\vec{0}),(0,\vec{x});C]q(0,\vec{x})|0\rangle \, .$$

If the quarks act as external sources, the propagator can be written in the form

$$\langle 0|q(T,\vec{0})\bar{q}(0,\vec{0})|0\rangle$$

$$= \left\langle \exp[i\int_0^T A_0(\tau,\vec{x})d\tau] \right\rangle \langle 0|q(T,\vec{0})\bar{q}(0,\vec{0})|0\rangle_{free}$$

$$\approx U[(T,\vec{0}),(0,\vec{0});C]\ e^{-mT}.$$

Similarly

$$\langle 0|q(T,\vec{x})\bar{q}(0,\vec{x})|0\rangle$$

$$\approx\ U[(0,\vec{x}),(T,\vec{x});C]\ e^{-mT},$$

hence

$$\Omega(T,R) \approx e^{-2mT}\ W(C)\ , \tag{10}$$

with $$W(C) = \langle 0|\mathrm{tr}U[x,x;C]|0\rangle\ . \tag{10a}$$

Comparison of eqs.(9) and (10) gives

$$\lim_{T\to\infty} W(C) \approx e^{-T[E(R)-2m]}\ . \tag{11a}$$

In a confining theory we expect $V(R)$ [$\equiv E(R)$] to increase without bound as R increases, i.e.

$$V(R) \to \infty \quad \text{as} \quad R \to \infty\ .$$

If one considers the model where a colour electric flux tube (i.e. a string with finite width) is formed between the sources, then one expects

$$V(R) \underset{\infty}{\overset{R}{\to}} \sigma R\ + \text{correction} \tag{12}$$

where σ is the so-called **string tension** of the flux tube (the specific form of the correction to the linear term is given in Part II, sec.3).

We note that eq.(12) is dimensionless - to restore the right physical dimensions the lattice spacing a has to be used.
For eq.(12) this implies that $\frac{1}{a}V(R)$ is the physical potential energy at a distance $r = Ra$, so that

$$\tfrac{1}{a}V(R) \underset{\infty}{\overset{Ra}{\longrightarrow}} \frac{\sigma}{a^2}\,Ra, \tag{12a}$$

where σ/a^2 is the physical string tension and Ra is the physical string distance.

For a non-confining theory (like QED) we expect

$$V(R) \underset{\infty}{\overset{R}{\to}} \text{constant} = 2m. \qquad (m = m_q) \tag{12b}$$

For large loops we should therefore have a different behaviour for confining and non-confining theories

$$W(C) \equiv \langle 0|A(C)|0\rangle\Big|_{\text{Eucl.}} \to e^{-\sigma RT} = e^{-\sigma.(\text{Area of loop})} : \text{confinement}$$

$$\langle 0|A(C)|0\rangle\Big|_{\text{Eucl.}} \to e^{-2mT} = e^{-m.(\text{Perimeter of loop})} : \text{no confinement} \tag{13}$$

4. The Wilson Loop in Lattice Gauge Theories

In the lattice formalism, a Wilson loop is given by a product of links that form a closed loop (C) [11], i.e.

$$A(C) = \frac{1}{N} \operatorname{tr}\left(\prod_{\ell \varepsilon C} U_\ell\right)$$

$$\equiv \frac{1}{N} \operatorname{tr}\left(\prod_{C} U\right), \tag{14}$$

where the $\frac{1}{N}$ factor is added to ensure that A(C) is normalized to one if all the links in the loop are set to unity.

For the expectation value of the Wilson loop we thus have

$$W(C) \equiv \langle A(C)\rangle = \frac{1}{Z} \int [dU] \, \frac{1}{N} \operatorname{tr}\left(\prod_{C} U\right) e^{-S} \tag{15}$$

where

(i) [dU] is the normalized SU(N) Haar measure

(ii) $Z = \int [dU] \, e^{-S}$ (15a)

(iii) S is the Wilson action defined in Chapter 3.

The Wilson loop (eq.(14)), because of its gauge invariant nature, can now be used as order parameter to investigate confinement on the lattice.

We note that W(C) is a function of the bare coupling g and the order of the gauge group, i.e.

$$W(C) \equiv W_C(g^2, N)$$

5. Calculation of the Leading Order Contribution to <A(C)>

In analogy to the high temperature expansions for thermodynamical systems, Strong Coupling expansions are performed by expanding the "Boltzmann factor" in powers of $\frac{1}{g^2}$ ($\equiv \beta/2N$) [11], i.e.

$$e^{-S(U)} = \prod_p e^{-S_p}$$

$$= \prod_p \exp[\frac{\beta}{2N} (\mathrm{tr}\, U_p + \mathrm{tr}\, U_p^\dagger)]$$

$$= \prod_p [1 + \frac{\beta}{2N} (\mathrm{tr}\, U_p + \mathrm{tr}\, U_p^\dagger) + \frac{1}{2!} (\frac{\beta}{2N})^2 (\mathrm{tr}\, U_p + \mathrm{tr}\, U_p^\dagger)^2 + \ldots]. \tag{16}$$

Hence, to leading order ($O(\beta)$)

$$\langle A(C)\rangle = \frac{1}{Z} \int [dU]\, \frac{1}{N}\, \mathrm{tr}\, (\prod_{\ell \varepsilon C} U_\ell) \prod_p [1 + \frac{\beta}{2N} (\mathrm{tr}\, U_p + \mathrm{tr}\, U_p^\dagger) + \ldots]. \tag{17}$$

Again choosing the Wilson loop to be a $n \times n$ rectangular one, we can write

$$\frac{1}{N}\, \mathrm{tr}\, (\prod_{\ell \varepsilon C} U_\ell) = \frac{1}{N} \sum_{j_1 \ldots j_n} (U_{\ell_1})_{j_1 j_2} (U_{\ell_2})_{j_2 j_3} \ldots (U_{\ell_n})_{j_n j_1} \tag{18}$$

so that

$$\langle A(C)\rangle = \frac{1}{Z} \int [dU]\, \frac{1}{N} \sum_{j_1 \ldots j_n} (U_{\ell_1})_{j_1 j_2} \ldots (U_{\ell_n})_{j_n j_1} \times$$

$$\prod_p [1 + \frac{\beta}{2N} (\mathrm{tr}\, U_p + \mathrm{tr}\, U_p^\dagger) + \ldots]. \tag{19}$$

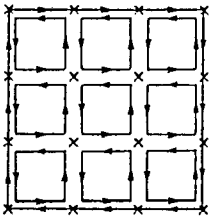
Using the following group integration results for SU(N) (see Chapter 5),

$$\int [dU] = 1$$

$$\int [dU]\, U_{ij}\, U^{+}_{kl} = \frac{1}{N}\, \delta_{il}\, \delta_{jk} \quad , \tag{20}$$

the first non-vanishing contribution to eq.(19) will be the term in the expansion giving the correct number of U and $U^{\dagger}$'s, i.e. all links must occur at least quadratically in the form tr $UU^{\dagger}$.

This corresponds to "tiling" the surface of the rectangular Wilson loop with elementary plaquettes,



The lowest order contribution will therefore correspond to

$$W(C) \sim \frac{1}{N} \left(\frac{\beta}{2N}\right)^{N_p} \left(\frac{1}{N}\right)^{N_L} (N)^{N_S}, \tag{21}$$

with:

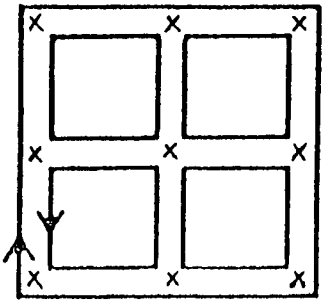
N_p = Number of plaquettes needed to tile the surface

N_L = Number of link pairs ↑↓

N_S = Number of lattice sites on the loop

where each pair of links contributed a factor $\frac{1}{N}$, while a factor N was obtained from each site on the surface (N = tr 1).

As an example we consider a $\underline{2 \times 2}$ Wilson loop:



with $N_p = 4$; $N_L = 12$; $N_S = 9$.

Eq.(21) therefore reads

$$W(C) \underset{0}{\beta} \frac{1}{N} \left(\frac{\beta}{2N}\right)^4 \left(\frac{1}{N}\right)^{12} (N)^9$$

$$= \left(\frac{\beta}{2N^2}\right)^4 = \left(\frac{1}{g^2 N}\right)^4 . \qquad (22)$$

Remarks

(i) The result obtained above can be extended to larger loops in an obvious way as long as the loops are planar. In general the result (to lowest order) will be given by

$$W(C) \underset{0}{\beta} \left(\frac{\beta}{2N^2}\right)^{N_p}$$

$$= \left(\frac{1}{g^2 N}\right)^{N_p} \qquad (23)$$

where N_p is the minimum number of plaquettes needed to tile the area included by the loop.

(ii) Eq.(23) is only valid for $N \geq 3$. For $N = 2$ the direction of the plaquette does not matter since

$$\int dU \, U_{ij} \, U_{kl} = \frac{1}{2} \delta_{ik} \delta_{jl} . \qquad (24)$$

Both possible orientations of the plaquette variable (U_p and $U_p^\dagger$) now contribute (which is not the case for $N \geq 3$ as the integral (eq.(24)) vanishes).

The contributions from U_p and $U_p^\dagger$ are the same; we have a factor $\frac{\beta}{N^2} = \frac{\beta}{4}$ for each plaquette and therefore an effective factor 2^{N_p} in W(C),

$$W(C) \underset{0}{\beta} \left(\frac{\beta}{4}\right)^{N_p} \qquad (SU(2)) \qquad (25)$$

6. Correspondence between SCE and Confinement

Since all the elementary plaquettes have the same area ($\sim a^2$), there exists a one-to-one correspondence between the number of plaquettes (N_p) and the minimum area (A) enclosed by a (planar) loop; for a rectangular loop with spatial and temporal dimensions R and T respectively this would imply

$$a^2 N_p = A = TR. \tag{26}$$

The previous results can now be rewritten as ($N \geq 3$)

$$W(C) \underset{o}{\overset{\beta}{\to}} (\frac{\beta}{2N^2})^{A/a^2} [1 + \ldots]$$

$$\equiv e^{-\sigma A}, \tag{27}$$

where $$\sigma = -\frac{1}{a^2} \ln (\frac{\beta}{2N^2}) + \ldots \tag{27a}$$

is the string tension, while for $N = 2$

$$W(C) \underset{o}{\overset{\beta}{\to}} (\frac{\beta}{N^2})^{A/a^2} [1 + \ldots]$$

$$= e^{-\sigma A}, \tag{28}$$

with $$\sigma = -\frac{1}{a^2} \ln (\frac{\beta}{4}) . \tag{28a}$$

The dots (...) emphasize the fact that we are only considering lowest order results. The following remarks are in order:

(i) Eqs.(27) and (28) correspond to an <u>area law</u> behaviour, which is valid for arbitrary shaped loops. If the loop C lies in a plane the leading contribution if $\beta \to 0$ will always follow the tiling of the minimal surface bounded by the loop. The gauge invariant Wilson loop expectation value can thus be used as order parameter for confinement.

(ii) Analogously to the continuum theory [eq.(13)] we have the following results for a confining and non-confining theory:

$$\langle W(C)\rangle = \langle W(R,T)\rangle \to \begin{matrix}\exp[-(\text{Area of loop})] \to \text{confinement} \\ \exp[-(\text{perimeter of loop})] \to \text{non-confinement}\end{matrix} \qquad (29)$$

(iii) The area law is lost in the following cases:

(a) When quarks are introduced as dynamical variables (and thus no longer act as mere static colour sources). This situation leads to a favourable environment for the (widely separated) static sources to create virtual $q\bar{q}$ pairs.

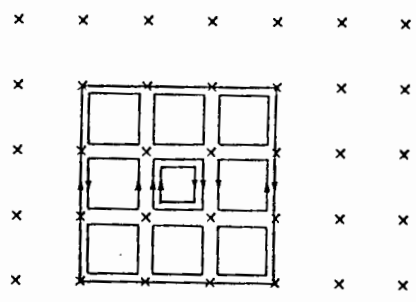
(b) When the centre of the gauge group is not non-trivial, i.e. singlets appear in the direct product of the fundamental representation with any number a adjoint representations. In physical terms this implies that if a finite number of gluons (gauge fields) can neutralize a static source in the adjoint representation by screening it, then a gauge invariant object is formed when the gauge fields surround the edge of the (large) Wilson loop to give it a <u>perimeter</u> law behaviour .

7. <u>Higher Order Corrections to W(C)</u>

To extrapolate the Strong Coupling Expansions to larger ß values requires that the expansions be carried out to higher orders. These terms are also needed to obtain accurate estimates for quantities like the string tension etc.. For higher order terms in ß, the surface of the Wilson loop is no longer minimally covered, and W(C) is of the form

$$W(C) = \frac{\beta}{2N^2}\,[1 + O(\beta) + O(\beta^2) + \ldots]. \qquad (N \geq 3) \qquad (30)$$

E.g., the $O(\beta)$ correction is obtained by tiling any plaquettes of the surface twice with the same orientation:



This gives the result for N=3 [11]

$$\langle A(C)\rangle = \left(\frac{\beta}{18}\right)^{N_p}\left[1 + \frac{1}{2!}\,N_p\left(\frac{\beta}{6}\right) + \ldots\right] \tag{31a}$$

$$\approx \left(\frac{\beta}{18}\right)^{A/a^2} e^{\frac{1}{2}(\beta/6)A/a^2}$$

$$\equiv e^{-\sigma A}, \tag{31b}$$

$$\text{with}\quad \sigma a^2 = -\ln\left(\frac{\beta}{18}\right) - \frac{\beta}{12} \tag{31c}$$

where the factor N_p was obtained because any plaquette may be tiled twice, while the factor $\frac{\beta}{6}$ came from the action.

Although it is possible in principle to calculate higher order corrections to W(C) using the SCE defined in eq.(17), it becomes difficult due to the fact that any given plaquette may be tiled many times, which makes integration over the link variables cumbersome .

In order to simplify matters it is convenient to replace the SCE by a character expansion.

8. Character Expansion Techniques [9;15;11]

The Boltzman weight can be expanded (see Chapter 5, Section V)

$$e^{-S} = \prod_p e^{-S_p}$$

$$\text{with}\quad e^{-S_p} = \exp\left[\frac{\beta}{N}\,\mathrm{Re\ tr}\,U_p\right] = \exp\left[\frac{\beta}{N}\,\chi\,(U_p)\right]$$

$$= \sum_r c_r\,\chi_r\,(U_p) \tag{32}$$

$$\text{and}\quad c_r = \int [dU]\,\chi_r^*(U)\,e^{\beta/N\,\chi(U)}. \tag{32a}$$

As an example we calculate the expansion coefficients c_r for **SU(2)**.

The group characters for any irreducible representation r is given by (see Chapter 5)

$$\chi_r(U) = \frac{\sin(r+\frac{1}{2})\theta}{\sin(\theta/2)} \ . \qquad (33)$$

Choosing the character associated with the spin ½ representation ($r=\frac{1}{2}$), we have

$$\chi_{\frac{1}{2}}(U) = 2\cos(\theta/2) \ . \qquad (33a)$$

The LHS of (32) is now given by

$$e^{-S_p} = e^{\beta/2\ \chi_{\frac{1}{2}}(U_p)}$$

$$= e^{\cos(\theta/2)}, \qquad (34)$$

so that

$$c_r = \int [dU]\ \chi_r^*(U)\ e^{\beta\cos(\theta/2)}$$

$$= \int_0^{4\pi} \int \sin^2(\theta/2)\ \frac{d\theta}{2\pi}\ \frac{d^2\hat{n}}{4\pi}\ \frac{\sin(r+\frac{1}{2})\theta}{\sin(\theta/2)}\ e^{\beta\cos(\theta/2)}$$

$$= \frac{1}{\pi}\int_0^{2\pi} d(\tfrac{\theta}{2})\sin(\theta/2)\ \sin(r+\tfrac{1}{2})\theta\ e^{\beta\cos(\theta/2)}$$

$$\equiv \frac{1}{2\pi}\int_0^{2\pi} d\phi\ [\cos(2r\phi) - \cos[2(r+1)\phi]\ e^{\beta\cos\phi}$$

$$= I_{2r}(\beta) - I_{2(r+1)}(\beta)$$

$$= \frac{2(2r+1)}{\beta}\ I_{2r+1}(\beta) \ . \qquad \text{(see App. D)} \qquad (35)$$

Eq.(32) is therefore given by

$$e^{-S_p} = e^{\beta\cos(\theta/2)}$$

$$= \sum_r 2(2r+1)\ I_{2r+1}(\beta)/\beta\ \chi_r(U)\ , \qquad r = 0, \tfrac{1}{2}, \ldots \ . \qquad (36)$$

Defining

$$d_r = 2r + 1; \qquad [SU(2)]$$

$$\beta_r(\beta) = 2I_{2r+1}(\beta)/\beta \ , \tag{37}$$

we can write

$$e^{\frac{1}{2}\beta \cos(\theta/2)} = \sum_{r=0}^{\infty} d_r \, \beta_r(\beta) \, \chi_r(U). \tag{37a}$$

Factorizing out the term corresponding to the trivial representation (r=0) we have

$$e^{\frac{1}{2}\beta \cos(\theta/2)} = \frac{2I_1(\beta)}{\beta} + \sum_{r\neq 0} d_r \beta_r(\beta) \, \chi_r(U)$$

$$= \frac{2I_1(\beta)}{\beta} \left[1 + \sum_{r\neq 0} d_r z_r(\beta) \, \chi_r(U)\right] \tag{38}$$

where (i) $z_r \equiv \dfrac{I_{2r+1}(\beta)}{I_1(\beta)}$

(ii) $\chi_0(U) = 1.$ (38a)

The SU(2) result in eq.(37a) can be generalized to any gauge group G

$$e^{-S_p} = e^{\beta/N \, \chi(U_p)}$$

$$\Rightarrow \ e^{-S_p} = \sum_{r=0}^{\infty} d_r \beta_r(\beta) \, \chi_r(U_p)$$

$$\equiv \sum_{r=0}^{\infty} c_r \, \chi_r(U_p). \tag{39}$$

The trivial coefficient c_0 is given by

$$c_0 = \beta_0 = \int [dU] \, \chi_0^*(U) \, \exp[\tfrac{\beta}{N} \chi(U)]$$

$$= \int [dU] \, \exp[\tfrac{\beta}{N} \chi(U)] \ , \tag{40a}$$

so that $e^{\beta/N \, \chi(U)} = \beta_0 \left[1 + \sum_{r\neq 0} b_r \chi_r(U_p)\right]$ (40b)

where we used $c_r/c_0 = c_r/\beta_0 \equiv b_r.$ (40c)

Character Expansion for SU(3)

For SU(3) the Boltzmann factor is given by

$$e^{-S_p} = \exp\left[\frac{\beta}{6}\left(\chi_3(U_p) + \chi_{\bar{3}}(U_p)\right)\right]$$

$$= 1 + \frac{\beta}{6}[\chi_3(U_p) + \chi_{\bar{3}}(U_p)] + \frac{\beta^2}{72}[\quad]^2 + \ldots , \qquad (41)$$

where $\chi_{\bar{3}}(U_p) = \chi_3(U_p^\dagger)$. (41a)

From eq.(40a) we have

$$c_0 = \int [dU]\, e^{-S_p(U_p)}$$

$$= \int [dU]\, [1 + \frac{\beta}{6}\left(\chi_3(U_p) + \chi_{\bar{3}}(U_p)\right) + \frac{\beta^2}{72}\left(\chi_3 + \chi_{\bar{3}}\right)^2 + \ldots]$$

$$= 1 + 2(\frac{\beta^2}{72}) + \ldots$$

$$= 1 + \frac{\beta^2}{6} + \ldots , \qquad (42)$$

where we used

(i) $\int [dU] = 1$

(ii) $\int [dU]\, \chi_3(U_p) = 0 = \int [dU]\, \chi_{\bar{3}}(U_p)$

(iii) $\int [dU]\, \chi_{\bar{3}}(U_p)\chi_3(U_p) = 1 \qquad (3 \otimes \bar{3} = 8 + 1)$

$\int [dU]\, \chi_3(U_p)\chi_3(U_p) = 0 = \int [dU]\, \chi_{\bar{3}}(U_p)\chi_{\bar{3}}(U_p) \qquad (3 \otimes 3 = 6 + \bar{3})$. (42a)

The expansion coefficients for SU(3) are given by

$$c_r = \int [dU]\, \chi_r^*(U_p)\, e^{-S_p(U)}$$

$$= \int [dU]\, \chi_{\underline{r}}(U_p)\, \exp[\tfrac{\beta}{6}\,(\chi_3(U_p) + \chi_{\bar{3}}(U_p))]. \qquad (43)$$

Therefore

$$b_3 = \frac{1}{\beta_0} \int [dU]\, \chi_3^*(U_p)\, [1 + \tfrac{\beta}{6}\,\{\chi_3(U_p) + \chi_{\bar{3}}(U_p)\} + \ldots]$$

$$\text{and}\quad b_{\bar{3}} = \frac{1}{\beta_0} \int [dU]\, \chi_3(U_p)\, [1 + \tfrac{\beta}{6}\,\{\chi_3(U_p) + \chi_{\bar{3}}(U_p)\} + \ldots]\ . \qquad (44)$$

Therefore, using the results in (42a) we obtain

$$b_3 = b_{\bar{3}} = \beta/6 + O(\beta^2). \qquad (44a)$$

Similarly, for the higher representations we find [11]

$$b_6 = b_{\bar{6}} = (\beta^2/72) + O(\beta^3)$$

$$= (b_3^2/2) + O(\beta^3) \qquad (45a)$$

and

$$b_8 = \frac{\beta^2}{36} + O(\beta^3)$$

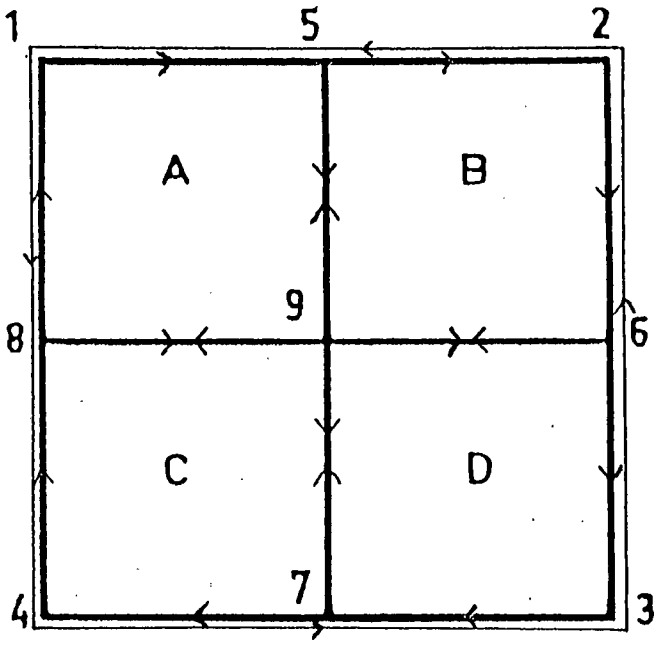
$$= b_3^2 + O(\beta^3)\ . \qquad (45b)$$

9. Character Expansions applied to the Wilson Loop

As before, the lowest order contribution is obtained by tiling the minimum area with plaquettes.

For simplicity we consider a 2 x 2 Wilson loop as example. The results can then be generalized to larger loops.

Let the 2 x 2 loop by indiced by [8]



The expectation value of the Wilson loop is now given by

$$W(C) = \langle \chi_s\left(\prod_{\ell \varepsilon C} U_\ell\right)\rangle$$

$$\equiv \frac{1}{Z}\int [dU]\, \chi_s\left(\prod_{\ell \varepsilon C} U_\ell\right) \exp\left[-\sum_p S_p\right] \tag{46}$$

with $e^{-S_p} = e^{(\beta/3)\chi(U_p)}$

$$= \sum_{r=0}^{\infty} c_r\, \chi_r(U_p) \tag{46a}$$

Consider the integral

$$I = \int [dU]\, \chi_s\left(\prod_{\ell \varepsilon C} U_\ell\right) \prod_p e^{-S_p}$$

$$= \int [dU]\, \chi_s\left(\prod_{\ell \varepsilon C} U_\ell\right) \prod_p \left(\sum_{r=0}^{\infty} c_r\, \chi_r(U_p)\right)$$

$$\equiv \int [dU']\, \chi_s\left(\prod_{\ell \varepsilon C} U_\ell\right) \int [dU_{int}] \prod_p \sum_{r=0}^{\infty} c_r(U_p), \tag{47}$$

where $[dU] = [dU'] [dU_{int}]$

with $[dU_{int}] = dU_{59}\, dU_{69}\, dU_{79}\, dU_{89}$ (47a)

and $\prod_p \Rightarrow pl(A).pl(B).pl(C).pl(D)$

Now

$$I_2 \equiv \int [dU_{int}] \prod_p [\sum_r c_r \chi_r(U_p)]$$

$$= \sum_{r_A r_B r_C r_D} \sum \sum \sum \int dU_{59}\, dU_{69}\, dU_{79}\, dU_{89}\, c_{r_A}\, c_{r_B}\, c_{r_C}\, c_{r_D} \times$$

$$\chi_{r_A}(U_{15}U_{59}U_{98}U_{81})\, \chi_{r_B}(U_{52}U_{26}U_{69}U_{95})\, \chi_{r_C}(U_{63}U_{37}U_{79}U_{96})\, \chi_{r_D}(U_{74}U_{48}U_{89}U_{97}) \quad (48a)$$

Consider the intergrations over the links separately .

A: $\int dU_{59}\, \chi_{r_A}(U_{15}U_{59}U_{98}U_{81})\, \chi_{r_B}(U_{52}U_{26}U_{69}U_{95}) \qquad (U_{95} = U_{59}^{\dagger})$ _

$$= \int dU_{59}\, \chi_{r_A}(U_{59}U_{98}U_{81}U_{15})\, \chi_{r_B}(U_{59}^{\dagger}U_{52}U_{26}U_{69})$$

$$= d_{r_A}^{-1}\, \delta_{r_A}\, \chi_{r_A}(U_{98}U_{81}U_{15}U_{52}U_{26}U_{69})$$

B: $\int dU_{89}\, \chi_{r_D}(U_{74}U_{48}U_{89}U_{97})\, \chi_{r_A}(U_{98}U_{81}U_{15}U_{52}U_{26}U_{69})$

$$= \int dU_{89}\, \chi_{r_D}(U_{89}U_{97}U_{74}U_{48})\, \chi_{r_A}(U_{89}^{\dagger}U_{81}U_{15}U_{52}U_{26}U_{69})$$

$$= d_{r_D}^{-1}\, \delta_{r_A, r_D}\, \chi_{r_D}(U_{97}U_{74}U_{48}U_{81}U_{15}U_{52}U_{26}U_{69})$$

$$\equiv d_{r_D}^{-1}\, \delta_{r_A, r_D}\, \chi_{r_D}(VU_{69}) \qquad [V \equiv U_{97}U_{74}U_{48}U_{81}U_{15}U_{52}U_{26}]$$

C: $\int dU_{69}\, \chi_{r_D}(U_{69}V)\, \chi_{r_C}(U_{69}^{\dagger}W) \qquad [W \equiv U_{63}U_{37}U_{79}U_{96}]$

$$= d_{r_D}^{-1}\, \delta_{r_D, r_C}\, \chi_{r_D}(VW)$$

$$= d_{r_D}^{-1}\, \delta_{r_D, r_C} \chi_{r_D}(U_{97}U_{74}U_{48}U_{81}U_{15}U_{52}U_{26}U_{63}U_{37}U_{97}^{\dagger})$$

D: $$\int dU_{79}\, \chi_{r_D}\, (U_{97}U_{74}U_{48}U_{81}U_{15}U_{52}U_{26}U_{63}U_{37}U_{97}^{\dagger})$$

$$= \chi_{r_D}\, (U_{15}U_{52}U_{26}U_{63}U_{37}U_{74}U_{48}U_{81})$$

Combining A, B, C, D gives

$$I_2 = \sum_r (c_r^4/d_r^3)\chi_r(U_{15}U_{52}U_{26}U_{63}U_{37}U_{74}U_{48}U_{81})$$

$$= \sum_r (c_r^4/d_r^3)\chi_r(U'_{12}U'_{23}U'_{34}U'_{41}) \qquad \text{where } U'_{12} = U_{15}U_{52}\text{ , etc.}$$

Using the above we finally obtain

$$I = \left(\frac{c_s}{d_s}\right)^4 , \tag{49}$$

with

$$W(C) = \left(\frac{b_s}{d_s}\right)^4 \tag{50}$$

(where the factor β_0 cancelled out in the denominator).

For SU(3) we choose s to represent the fundamental representation (i.e. $s = f \equiv 3$), hence

$$W(C) = (b_3/3)^4 \tag{51}$$

(where we used $d_f = \chi_3(1) = 3$). (51a)

This result is conistent with the one obtained in (22) if we use eq.(44a).

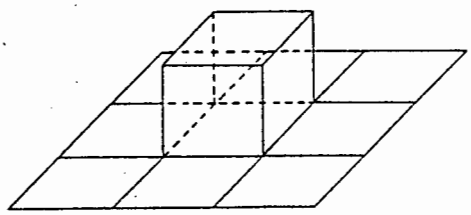
Introducing the variable $t = \frac{1}{3} b_3$, (52)

we can rewrite the string tension (27a) for SU(3) in the form

$$a^2\sigma = -\ln t \,. \tag{53}$$

First order correction ($O(t^4)$) [11]

The $O(t^4)$ correction is due to a cubical "bump" on the tiled surface (as shown below)



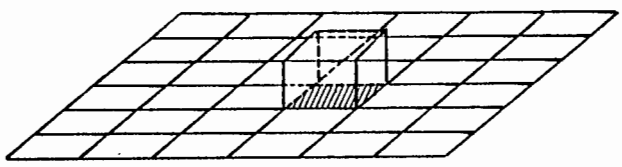
This contribution adds four new plaquettes to the surface, corresponding to the four possible orientations of the bump [the surface can fluctuate either above or below the plane, or in the two transverse directions corresponding to the two remaining dimensions of 4-dimensional space-time].

Also the bump can occur on any plaquette of the surface*

$$\langle A(C)\rangle \equiv W(C) = (t)^{\text{Area}/a^2}\,[1 + {}^{*}4(\text{Area}/a^2)t^4 + O(t^5)] \tag{54}$$

so that $$a^2\sigma = \ln t - 4t^4 + O(t)^5. \tag{55}$$

To order t^5 there is an additional contribution from a bump with a "floor", which arises from having a non-trivial (i.e. with representation $\bar{3}$) base for the cube.



After performing all the internal link integrals (which open up all the plaquettes except the floor one), we are left with the following contribution to σ:

$$(a^2\sigma)_{O(t^5)} \sim 4t^4\,(3t)\int dU_1\,dU_2\,dU_3\,dU_4\,\chi_3(U_p)\,\chi_3(U_p)\,\chi_3(U_p)$$

$$\equiv -12t^5\int dU_p\,\chi_3(U_p)\,\chi_3(U_p)\,\chi_3(U_p)$$

$$= -12t^5 n_s = -12t^5$$

where n_s = # singlets in $3 \otimes 3 \otimes 3$ $(= 1)$.

The only difference between the $O(t^5)$ contribution above and the $O(t^4)$ one (i.e. bump without a floor), is therefore a factor $b_3 = 3t$;

$$a^2\sigma = -\ln t - 4t^4 - 12t^5 + O(t^6). \tag{56}$$

The results above (with d=4) can be written in the general form [15]

$$a^2\sigma = -\ln t - \sum_n K_n t^n, \tag{57}$$

where the values of the coefficients K_n (to order n = 14) are given in table 6.1 for the different gauge groups (see [15] and references therein).

Table 6.1 (from [15])

Coefficients K_n for the strong coupling expansion of the string tension in four dimensions. $K = -\ln t - \Sigma K_n t^n$

	n										
G	4	5	6	7	8	9	10	11	12	13	14
Z_2	4	0	4	0	56	0	144	0	$\frac{3616}{3}$	0	4276
U(1)	4	0	2	0	$\frac{170}{3}$	0	$\frac{2125}{24}$	0	$\frac{862619}{720}$	0	$\frac{5754751}{2160}$
SU(2)	4	0	0	0	$\frac{176}{3}$	0	$\frac{10936}{405}$	0	$\frac{1532044}{1215}$	0	$\frac{3596102}{5103}$
SU(3)	4	12	−10	−36	$\frac{391}{2}$	$\frac{1131}{10}$	$\frac{2550837}{5120}$	$\frac{-5218287}{2048}$	$\frac{285551579}{61440}$		
SU(∞)	4	0	8	0	56	0	344	0	$\frac{4588}{3}$	0	11688

10. String Tension from SCE

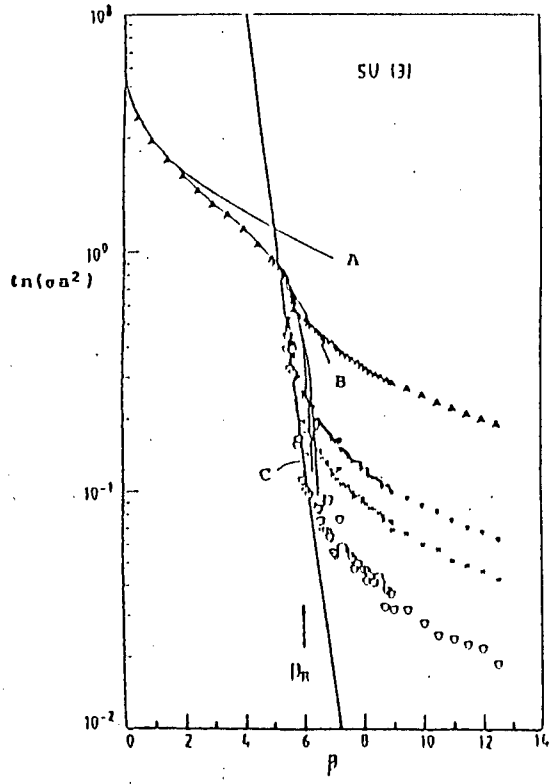


FIGURE 6.1 : SU(3) string tension as a function of the coupling β. The points A-D denote the zeroth, 11th, 12th and 10th order. Also indicated is the roughening coupling β_R .

We will now calculate the ratio σ/Λ_L^2 , where Λ_L is the dimensionless (Euclidian) scale parameter [see Part II], from Strong Coupling results.

This calculation is based on the observation that Strong Coupling results will have the correct slope predicted by asymptotic freedom provided they lie in a narrow range of possible β values [15]

$$\sigma a^2 |_{O(n)} \approx K_n \exp[-6\pi^2(\beta - \beta^{(n)})/11] \quad (\text{for } \beta \approx \beta^{(n)}) \qquad (58)$$

and K_n given in table 6.1 .

Dividing the equation above by the Renormalization Group Equation (RGE) [see Part II]

$$\Lambda_L a^2 = (6\pi\beta/11)^{121/102} \exp[-6\pi^2\beta/11] \qquad (59)$$

we obtain

$$\sigma/\Lambda_L^2 |_{O(n)} \approx K_n (6\pi^2\beta^{(n)}/11)^{-121/102} \exp[6\pi^2\beta^{(n)}/11] \; . \qquad (60)$$

It must however be emphasized that eq.(60) only provides an estimate for σ/Λ_L^2 (this will become clear later). We will now discuss the case of SU(3) in greater detail (along with the Monte Carlo data [Part II]).

In **fig. 6.1** the results for the SU(3) string tension using a SCE for σ to order t^8, t^{10} and t^{12} are given. The expansion seems to converge well up to $\beta \approx 5$. For $\beta \approx 6$ higher order terms start to play a more important role. For each order the narrow range in β can be seen for which the slope of the curve agrees with the one predicted by the RGE.

In order to obtain a rough estimate for the (continuum) value of σ, a straight line is fitted to the slope tangent to the curve of the SCE to a given order, and in so doing the value of $\Lambda_L/\sqrt{\sigma}$ is extracted.

For the highest order results for the SCE this gives [15]

$$\text{10 th order:} \quad \Lambda_L/\sqrt{\sigma} = 3{,}7 \times 10^{-3};$$

$$\text{12 th order:} \quad \Lambda_L/\sqrt{\sigma} = 3{,}5 \times 10^{-3}. \qquad (61)$$

The results obtained above unfortunately do not agree well with MC results computed on large lattices, which give the value

$$\Lambda_L/\sqrt{\sigma} = 9{,}5 \times 10^{-3} \qquad (62)$$

which is more than twice as large [see Part II].

11. Roughening

One of the questions that may be asked is whether Strong Coupling calculations can be systematically improved by going to higher orders. Unfortunately it appears that calculations incorporating higher order terms are plagued by a variety of problems.

One serious limitation becomes evident when non-planar (off-axis) Wilson loops are considered. This is because the minimal area bounded by the loop must be composed of elementary plaquettes, which implies that the potential energy of the quark sources at (0,0) and (x,y) is

$$V(x,y) = \sigma(|x| + |y|)$$

and not

$$V(x,y) = \sigma(\sqrt{x^2+y^2}).$$

As a result rotational invariance (which should be restored as $g \to 0$) is badly broken by the leading order SCE .

Moreover, the minimal surface contributing to an off-axis loop is highly degenerate due to the transverse quantum fluctuations of the surface whith amplitudes comparable to the radius of the loop.
It is important to note that the restoration of rotational (Lorentz) invariance implies that the surface bounded by an on-axis Wilson loop* must fluctuate wildly. This in turn means that higher order terms in the SCE of $\langle A(C)\rangle$ become important [* for the calculation of σ, the Wilson loops are usually chosen to be on the coordinate axis].

Another problem is roughening, which concerns the long-wavelength vibrations of strings and which results in the inroduction of non-analycities in the way of Strong Coupling Expansions, especially for quantities like the string tension. This phenomenon makes it difficult to extract the continuum limit (with $g \to 0$) of such quantities via analytic means, as the unbounded transverse fluctuations of the string surface may generate a singularity in $\sigma(\beta)$ at some critical value of β [$\beta = \beta_R$]. This is called the roughening transition [42a-c].

Drouffe and Zuber [15] have conjectured that the singularity might be an essential one, i.e. of "infinite order", which blocks the continuation of the Strong Coupling series to the Weak Coupling region.

However, it has been shown [42d] that for any non-zero temperature (and finite coupling) the roughening transition is washed out by the (thermal) transverse fluctuations of the electric flux tube connecting the $q\bar{q}$ pairs. This is due to the fact that for any non-zero temperature the string wave function diverges like $\sqrt{L}$, where L is the length of the string. This implies that the string will be rough for any $T > 0$.

Another method of evading the roughening singularity is to derive the SCE using off-axis quarks, i.e. with non-planar Wilson loops. The surface will again be rough to lowest order which prevents a phase transition from occuring [42e].

It is also interesting to note that the large transverse vibrations (massless modes) generate a universal (and σ independent) correction to the static potential [42b], with

$$\Delta V(r) = -\frac{(d-2)}{r}\frac{\pi}{24}\,, \tag{64}$$

so that

$$V(r) = \sigma r - \frac{(d-2)}{r}\frac{\pi}{24} + \text{constant} + O(r^{-2})\,. \tag{65}$$

We will refer to this point again in Part II.

CHAPTER 7: MEAN FIELD TECHNIQUES

1. Introduction

Mean Field (MF) theory, like the Strong Coupling methods discussed in Chapter 6, provides us with a way of solving Lattice Gauge theories analytically. This is particularly important to verify computer calculations, which are often plaqued by purely computational restrictions (e.g. lattice size) which in turn heavily influences the results thus obtained.

MF theory is a well established method in Statistical physics to obtain the critical points for phase transitions. Unfortunately, the direct application of MF techniques to LGT violate Elitzur's theorem [44] to lowest order (see also Chapter 6). It can, however, be shown that the inclusion of higher order terms in the approximation corrects this violation (see Sec.4).

2. MF applied to the Ising (Spin) Model: "Intuitive Method" [11]

Consider an Ising model with s_i belonging to the discrete Abelian group $Z_2 = \{1,-1\}$. If we place a spin s_i from the set Z_2 on each site of a d-dimensional hypercubic lattice, we have as partition function

$$Z = \sum_{\{s\}} \exp(\beta \sum_{\{ij\}} s_i s_j). \qquad (1)$$

A given spin s_i will now interact with 2d nearest neighbours, denoted by $\sum_{j(i)} s_j$, with the average $\frac{1}{2d} \sum_{j(i)} s_j$ behaving like some mean field when d goes to infinity.

For the Ising model the mean field corresponds to the magnetization M, defined as the expectation value for any given spin

$$M = \langle s_i \rangle. \qquad (2)$$

Consider now a particular site i, and replace the spins on all the neighbouring sites with their average value M.

The Boltzmann probability for the spin on site i to have the value s_i is therefore given by

$$P(s_i) = \frac{e^{\beta \sum_{ij} s_i s_j}}{e^{\beta \sum_{j(i)} s_j} + e^{-\beta \sum_{j(i)} s_j}}. \qquad (3)$$

Making the replacements: $s_j \rightarrow M = \langle s_j \rangle$

$$\sum_{j(i)} \rightarrow 2d$$

we have

$$P(s_i) = \frac{e^{2d\beta M s_i}}{2 \cosh (2d\beta M)} . \qquad (4)$$

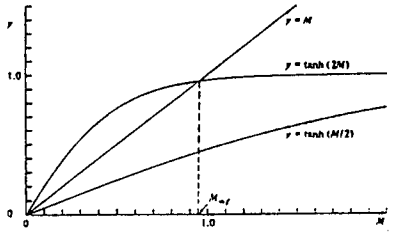
Requiring that the average value of s_i also be M , we obtain the self-consistency relation:

$$M = \sum_{\{s\}} s_i P(s_i)$$

$$= \frac{1 \cdot e^{2d\beta M}}{2 \cosh(2d\beta M)} + \frac{-1 \cdot e^{2d\beta M}}{2 \cosh(2d\beta M)}$$

$$= \tanh(2d\beta M). \qquad (5)$$

For small ß (i.e. large T, $\beta \approx T^{-1}$), eq.(5) has the unique solution M = 0. The MF approximation therefore correctly predicts that the magnetization vanishes at high temperatures. Graphically we have the following solution for eq.(5)



From the asymptotic behaviour we see that $\beta_{MF} = \frac{1}{2d}$ while for $\beta \geq \beta_{MF} = \frac{1}{2d}$, eq.(5) also has a favoured non-trivial solution with M > 0 (as well as a symmetric one at M < 0).

This non-trivial solution corresponds to the spontaneous breaking of the discrete Z_2 symmetry of the original system. MF theory therefore predicts a **phase transition** at β_{MF}.

3. Variational MF Approximation [11;15;3a;43]

3.1 Basics

One possible way of deriving a MF approximation is to formulate the technique in variational form.

In this approximation non-coupled independent fields are considered in an external (mean) field h , where h in general represents the global effect of other fields interacting with a given variable U_ℓ.

This method is based on the **convexity** property of the exponential function:

given any function f over some set X = {x} for which there exists a normalized measure $\rho(x)$, then, because the exponential function is convex,

$$\langle e^f \rangle \geq e^{\langle f \rangle} \tag{6}$$

where $$\langle f \rangle_\rho = \int_X f(x)\, \rho(x)\, dx. \tag{6a}$$

3.2 Formalism

Consider the form of the partition function:

$$Z = \int [d\phi]\, \exp[S(\phi) - S_h(\phi) + S_h(\phi)]. \tag{7}$$

An appropriate measure will give for any function g the expectation value:

$$\langle g \rangle_h = \int [d\phi]\, \exp[S_h(\phi)] g(\phi) / \int [d\phi]\, \exp[S_h(\phi)], \tag{8}$$

so that

$$Z \equiv \langle \exp[S(\phi) - S_h(\phi)] \rangle_h \, Z_h. \tag{9}$$

Using the convexity property, (eq.(6)), we have

$$\frac{Z}{Z_h} = \langle \exp[S(\phi) - S_h(\phi)] \rangle_h \geq \exp\langle S(\phi) - S_h(\phi) \rangle_h$$

$$\Rightarrow \quad \frac{Z}{Z_h} \geq \exp\langle S(\phi) - S_h(\phi) \rangle_h$$

$$\ln Z \geq \langle S(\phi) - S_h(\phi) \rangle_h + \ln Z_h, \tag{10}$$

which implies

$$F \geq \langle S(\phi) - S_h(\phi) \rangle_h + F_h, \tag{10a}$$

where $F \equiv \ln Z; \quad F_h \equiv \ln Z_h.$ (10b)

The variational estimate of F is obtained by minimizing the RHS with respect to the parameter h, which optimizes the bound and gives the consistency condition

$$F_{var} = \underset{h}{\text{Max}}[F_h + \langle S(\phi) - S_h(\phi) \rangle_h] \tag{11}$$

where $\underset{h}{\text{Max}}[\ ]$ denotes $\frac{\partial}{\partial h}[\ ] = 0.$

3.3 Application to the d-dimensional Ising Model [11;15]

First add and subtract a "source term" in the partition function:

$$Z = \sum_{\{s\}} \exp[\beta \sum_{ij} s_i s_j + h \sum_i s_i - h \sum_i s_i], \tag{12}$$

where h will later become the variational parameter.

A suitable (discrete) measure will give the following expectation value for a function f

$$\langle f\rangle = \sum_s [\exp(h \sum_i s_i)\, f(s)] / \sum_s \exp(h \sum_i s_i). \tag{13}$$

Using eq.(13) we can rewrite Z in the form

$$Z = \langle \exp[\beta \sum_{i,j} s_i\, s_j - h \sum_i s_i] \rangle \sum_s \exp(h \sum_i s_i). \tag{14}$$

Applying the convexity property (eq.(6)) we obtain the inequality

$$Z / \sum_s \exp(\sum_i s_i) = \langle \exp[\beta \sum_{i,j} s_i s_j - h \sum_i s_i] \rangle$$

$$\geq \exp\left[\langle \beta \sum_{i,j} s_i s_j - h \sum_i s_i \rangle \right]. \tag{15}$$

We will now discuss eq.(15) in more detail. Consider the right hand side first

(a) the second term gives (using eq.(13)):

$$\langle -h \sum_i s_i \rangle = -\sum_s \left[(h \sum_i s_i) \exp(h \sum_i s_i) \right] / \sum_S \exp(h \sum_i s_i). \tag{16}$$

Now: $\sum_s \exp(h \sum_i s_i)$

$$= \left(\sum_{s_1=\pm 1} e^{h s_1} \right) \left(\sum_{s_2=\pm 1} e^{h s_2} \right) (\ldots) = [2\cosh h]^{\Lambda}, \tag{17}$$

where $\Lambda \equiv$ number of lattice sites on a d-dimensional hypercubical lattice with $\Lambda = L^d$ (L = # lattice sites in one linear direction. (18)

$$\sum_s (h \sum_i s_i) \exp(h \sum_i s_i) = h \frac{d}{dh} [2\cosh(h)]^{\Lambda}$$

$$= h\Lambda\, [2\cosh(h)]^{\Lambda - 1}\, 2\sinh(h). \tag{19}$$

Using eq.(19) and eq.(17) in eq.(16) give:

$$\langle -h \sum_i s_i \rangle = - \frac{h\Lambda [2\cosh(h)]^{\Lambda - 1}\, 2\sinh(h)}{[2\cosh(h)]^{\Lambda}}$$

$$= - \Lambda h \tanh(h). \qquad (20)$$

(b) The first term in the RHS of eq.(15) can be written (using eq.(13))

$$\langle \beta \sum_{i,j} s_i s_j \rangle = \Big(\sum_s (\beta \sum_{i,j} s_i s_j) \exp(h \sum_i s_i) \Big) / \exp(h \sum_i s_i). \qquad (21)$$

The numerator now gives

$$\sum_s [\sum_{i,j} \beta\, s_i s_j \exp(h \sum_i s_i)]$$

$$= \sum_s [\beta\, s_1 s_2 \exp(hs_1 + hs_2)][\exp(hs_3 + hs_4 + \ldots] + \sum_s (\ldots)$$

$$= 4\Lambda d\beta \sinh^2(h)[2\cosh(h)]^{\Lambda-2}, \qquad (22)$$

so that

$$\langle \beta \sum_{i,j} s_i s_j \rangle = 4\Lambda d\beta \sinh^2(h)[2\cosh(h)]^{\Lambda-2}/[2\cosh(h)]^{\Lambda}$$

$$= \Lambda \beta d \tanh^2(h). \qquad (23)$$

Hence eq.(15) finally reads

$$Z/\sum_s \exp(\sum_i s_i) = \langle \exp[\, \beta \sum_{i,j} s_i s_j - h \sum_i s_i] \rangle$$

$$\geq \exp[-\Lambda h \tanh(h) + \Lambda \beta d \tanh^2(h)], \qquad (24)$$

so that

$$Z \geq \exp\Big[-\Lambda h \tanh(h) + \Lambda \beta d \tanh^2(h) + \log[2\cosh(h)]^{\Lambda}\Big]. \qquad (24a)$$

Using as definition for the (Helmholtz) free energy F

$$Z = e^{-\beta F},$$

we obtain the following bound for the free energy per site

$$\beta F = -\Lambda^{-1} \ln Z \leq \beta F_{MF}$$

$$\equiv -d\beta \tanh^2(h) + h \tanh(h) - \log[2\cosh(h)]. \quad (25)$$

Minimizing the RHS of eq.(25) with respect to the parameter h gives a lower bound on the free energy:

$$\frac{d}{dh}(\beta F_{MF}) = 0 = \frac{d}{dh}\left[-d\beta \tanh^2(h) + h \tanh(h) - \log[2\cosh(h)]\right]$$

$$= \operatorname{sech}^2(h)[h - 2d\beta \tanh(h)]. \quad (26)$$

Remark: Eq.(26) is equivalent to the earlier result for the Ising model [eq.(4)] if we make the identification $h = 2d\beta M$, because then

$$M = \frac{\tanh(2d\beta M)\ \operatorname{sech}^2(2d\beta M)}{\operatorname{sech}^2(2d\beta M)}$$

$$= \tanh(2d\beta M) \equiv \text{eq.(4)}$$

3.4 Variational MF Approximation applied to Pure SU(N) LGT [11;43]

Consider the pure SU(N) LGT partition function (see Chapter 3),

$$Z = \int [dU]\ e^{-S(U)}$$

$$= \int [dU] \exp\left[\frac{\beta}{N} \sum_p \text{Re tr } U_p\right] \quad (27)$$

(i.e. where S(U) is the Wilson action) and introduce a one-parameter constant external field h which is linearly coupled to the gauge fields:

$$S_h(U_\ell) = \frac{h}{N} \sum_\ell \text{Re tr } (U_\ell) \quad (28)$$

and $$Z_h = \int [dU] \exp[S_h(U_\ell)]. \quad (29)$$

We can now define the expectation value

$$\langle O \rangle_h = \frac{1}{Z_h} \int [dU]\ O \exp[S_h(U_\ell)], \quad (30)$$

so that

$$\frac{Z}{Z_h} = \langle \exp[S(U) - S_h(U_\ell)] \rangle_h . \tag{31}$$

Using eq.(7) gives

$$\frac{Z}{Z_h} \geq \exp[\langle S(U) - S_h(U_\ell) \rangle_h], \tag{32}$$

or equivalently

$$\begin{aligned} \ln Z &\geq \ln Z_h + \langle S(U) - S_h(U_\ell) \rangle_h \\ &= \ln Z_h + \langle S(U) \rangle_h - \langle S_h(U_\ell) \rangle_h . \end{aligned} \tag{32a}$$

As we are using an independent link formulation, eq.(29) can be written in the form

$$\begin{aligned} Z_h &= \int \left(\prod_\ell dU_\ell \right) \exp\left[\frac{h}{N} \sum_\ell \mathrm{Re\ tr}\ U_\ell \right] \\ &= \prod_\ell \left(\int dU_\ell \exp\left[\frac{h}{N} \sum_\ell \mathrm{Re\ tr}\ U_\ell \right] \right) \\ &\equiv \prod_\ell \left[z(h_\ell) \right] = \left[z(h_\ell) \right]^{\Lambda d}, \end{aligned} \tag{33}$$

where $\Lambda d = L^d d$ = # of links. (33a)

Define

$$\begin{aligned} W(h) &= \ln Z_h \\ &= \ln \left[\prod_\ell z(h)_\ell \right] = \sum_\ell \ln z(h)_\ell \\ &\equiv \sum_\ell w(h_\ell) . \end{aligned} \tag{34}$$

It follows that

$$\frac{d}{dh} W(h) = \frac{d}{dh} [\ln Z_h] = \Lambda d\, w'(h_\ell), \tag{35}$$

where $w'(h_\ell) \equiv \frac{d}{dh} w(h_\ell) = \frac{1}{z(h_\ell)} \frac{d}{dh} z(h_\ell) = \langle U_\ell \rangle_h .$ (35a)

The expectation values in eq.(32a) can therefore be written as

$$\langle S_h(U_\ell)\rangle_h = \frac{1}{Z_h}\int [dU]\; S_h(U_\ell)\; \exp[\tfrac{h}{N}\sum_\ell \mathrm{Re\ tr}\ U_\ell]$$

$$= h\, W'(h) = \Lambda dh\, w'(h); \tag{36}$$

$$\langle S_w(U)\rangle_h = \frac{1}{Z_h}\int [dU]\; (\sum_p \tfrac{\beta}{N}\,\mathrm{Re\ tr}\ U_p)\; \exp[\tfrac{h}{N}\sum_\ell \mathrm{Re\ tr}\ U_\ell]$$

$$= \beta \sum_p \prod_\ell \left(\frac{1}{z(h_\ell)}\int dU_\ell\, \tfrac{1}{N}\,\mathrm{Re\ tr}\ U_p\, \exp[\tfrac{h}{N}\,\mathrm{Re\ tr}\ U_\ell]\right)$$

$$\equiv \tfrac{1}{2}d(d-1)\Lambda\beta\,[w'(h)]^4, \tag{37}$$

where we used

$$\sum_p \Rightarrow \tfrac{1}{2}d(d-1)\Lambda \quad (= \#\text{ of plaquettes}). \tag{37a}$$

Substituting eqs.(36) and (37) into eq.(32a) gives:

$$\ln Z \geq \ln Z_h + \tfrac{1}{2}d(d-1)\,\Lambda\beta[w'(h)]^4 - \Lambda dhw'(h)$$

$$= \Lambda d\left(w(h) + \tfrac{1}{2}d(d-1)\beta[w'(h)]^4 - hw'(h)\right). \tag{38}$$

Defining [15]

$$F = \frac{1}{N}\ln Z \quad \text{we have}$$

$$F/d = \frac{\ln Z}{Nd} \geq \frac{F_{MF}}{d} = \max_h\{w(h) - hw'(h) + \beta^*\,\frac{1}{4}\,w'(h)^4\}, \tag{39}$$

where $\beta^* = 2\beta(d-1)$ and F/d is the free energy per link. (39a)

The following remarks are in order.

(i) All calculations were done using $e^{S_h(U_\ell)}$ as weight, i.e. using an independent-link approximation.

(ii) From eq.(35a) it is apparent that Elitzur's theorem [44] is violated in this approximation.

(iii) The following consistency equation is obtained by maximizing F_{MF} with respect to h:

$$h = 2(d-1)\beta\,[w'(h)]^3.$$

If $h = 0$, the function $w'(h)$ vanishes. This value is therefore always a local minimum. For high temperatures this is the only solution, while for low temperatures (large β) there also exists non-zero solutions.

The result above is true for all gauge groups. MF theory therefore predicts a **first order** phase transition seperating the Strong Coupling region (where the $h = 0$ solution dominates) and the Weak Coupling region (with solutions $h \neq 0$).

(iv) Calculations can also be performed using the temporal axial gauge, i.e. setting all the temporal links equal to one so that they no longer act as dynamical variables.

Only spatial links are now present in the one-parameter action, and as a result the quantity

$$z = \beta^*[w'(h)]^3$$

$$= 2(d-1)\beta[w'(h)]^3$$

in equation (39) must be replaced by [45]

$$z = 2(d-2)\,\beta[w'(h)]^3 + 2\beta w'(h)$$

$$= \frac{\beta^*(d-2)}{(d-1)}\,[w'(h)]^3 + \frac{\beta^*}{(d-1)}\,w'(h). \qquad (40)$$

The free energy per link is now given by

$$\frac{F}{(d-1)} \geq \frac{F_{MF}}{(d-1)} = \max_h\Big(w(h) - hw'(h) + \frac{\beta^*(d-2)}{4(d-1)}\,[w'(h)]^4 + \frac{\beta^*}{(d-1)}\,\frac{1}{4}\,[w'(h)]^2\Big). \qquad (41)$$

For large d eq.(41) is equivalent to eq.(39). Although the problem of the violation of Elitzur's theorem is avoided by fixing the gauge, the

MF Ansatz will now violate the 90^0 rotation symmetry of the original lattice theory in the lowest order approximation. For a detailed discussion see [15].

(v) There are certain disadvantages in using the variational approximation:

(a) the one parameter independent link action (eq.(28)) cannot be used to calculate corrections and also does not comply with the requirement of gauge invariance.

(b) it is only accurate if the fluctuations around the self-consistently determined MF are small and can be neglected. This is the case if d is large (i.e. a given field (link) interacts with a large number of nearest neighbours) or in the Weak Coupling (large ß) regime when almost all the degrees of freedom are "frozen" in an ordered state.

4. Saddle Point Approximation

In order to calculate corrections the lowest order MF result, the saddle point approximation [43;46;15;48] is used.

This method is based upon the fact that any interacting system (consisting of spins or links) of a gauge theory problem is equivalent to a problem of independent degrees of freedom in a random external field. In this approximation the lowest order MF equations are obtained as the stationary conditions describing a saddle point.

Consider the partition function

$$Z[J] = \int [dU]\, e^{-S(U) + J.U}, \tag{42}$$

where $[dU] = \prod_x dU_x$,

and we introduced a source term

$$J.U \equiv \sum_x J_x U_x. \tag{42a}$$

Using the Fourier representation of the Dirac δ function

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iq.x}\, dq$$

$$\equiv \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{q.x}\, dq, \tag{43}$$

we can write

$$\delta(U-V) = \int_{-i\infty}^{i\infty} d\left(\frac{h}{2\pi i}\right) e^{(u-v).h} \tag{43a}$$

or

$$\delta(U_x-V_x) = \int \frac{dh_x}{2\pi i}\, e^{(u_x-v_x)h_x}, \tag{43b}$$

where we used $$d\left(\frac{h}{2\pi i}\right) \equiv \prod_x \left(\frac{dh_x}{2\pi i}\right) . \tag{43c}$$

The integrand in (42) can now be written as

$$e^{-S(U) + J.U} = \int [dV]\, \delta(U-V)\, e^{-S(V) + J.U}, \tag{44}$$

which gives for the partition function

$$Z[J] = \iint [dU][dV]\, \delta(U-V)\, e^{-S(V) + J.U}$$

$$= \iiint [dU][dV]\, d\left(\frac{h}{2\pi i}\right) \exp[-S(V) + J.U + (U-V).h]$$

$$= \iiint \prod_x \left(dU_x dV_x \frac{dh_x}{2\pi i}\right) \exp\left(-\sum_x [(J_x + h_x)U_x - V_x h_x] - S(V)\right). \tag{45}$$

Introducing $\omega(J) = \ln \int [dU]\, e^{J.U}$

$$= \ln\left(\prod_x \int dU_x\, e^{J_x U_x}\right), \tag{46}$$

we can write

$$Z[J] = \int [dV]\, d\left(\frac{h}{2\pi i}\right) \exp[\omega(J + h) - V.h - S(V)]$$

$$= \int [dV]\, d\left(\frac{h}{2\pi i}\right) \exp[\sum_x \omega(J_x + h_x) - \sum_x V_x h_x - S(V)]. \qquad (47)$$

The MF result is now given by the saddle-point approximation of the integral above in both variables h and V.

Determining the saddle-points

We first rewrite the integral in the form

$$Z[J] = \int [dV]\, d\left(\frac{h}{2\pi i}\right) \exp[-A(V,h)], \qquad (48)$$

where $A(V,h) = -\omega(J + h) + V.h + S(V).$ (48a)

The saddle point is now given by the stationary equations

$$\text{(i)} \quad \frac{\partial A(V,h)}{\partial V_x} = 0 = \frac{\partial S}{\partial V_x} + h_x$$

$$\rightarrow \quad \left.\frac{\partial S(V)}{\partial V_x}\right|_{V=V^*} = -h_x^*;$$

$$\text{(ii)} \quad \frac{\partial A(V,h)}{\partial h_x} = 0 = -\frac{\partial \omega(J_x+h_x)}{\partial h_x} + V_x$$

$$\rightarrow \quad \left.\frac{\partial \omega(J_x+h_x^*)}{\partial h_x}\right|_{h=h^*} = V_x^*. \qquad (49)$$

(hence (V_x^*, h_x^*) is the saddle-point solution).

The lowest order contribution to

$$F[J] \equiv \log Z[J] \qquad (50)$$

from a single saddle-point is

$$F^*[J] = -S(V^*) - V^*.h^* + \omega(J + h^*). \qquad (51)$$

We note, in passing, that because we introduced a source term we can regard F[J] as a generating functional with $F^*[J] \equiv W_{MF}[J]$.

In order to calculate corrections to the MF result above we rewrite the variables h and V as

$$h_x = h_x^* + i\eta_x$$

$$V_x = V_x^* + \phi_x \tag{52}$$

so that

$$\Delta h_x = i\eta_x$$

$$\Delta V_x = \phi_x. \tag{52a}$$

Expanding in fluctuations ($\Delta V, \Delta h$) around the MF solution gives:

$$Z[J] = \exp[-S(V^*) - V^*\cdot h^* + \omega(J+h^*)] \quad \times$$

$$\int_{-\infty}^{\infty} d[\Delta V] \int_{-i\infty}^{i\infty} d\left[\frac{\Delta h}{2\pi i}\right] \exp\Big[-\tfrac{1}{2} S''(V^*)(\Delta V)^2 - (\Delta V)\cdot(\Delta h) +$$

$$\tfrac{1}{2}\omega''(J + h^*)(\Delta h)^2\Big] \quad \times$$

$$\exp\Big\{\sum_{n=3}^{\infty} \frac{1}{n!}\Big[-S^{[n]}(V^*)(\Delta V)^n + \omega^{[n]}(J + h^*)(\Delta h)^n\Big]\Big\}, \tag{53}$$

with $Z_{sp} \equiv \exp[-S(V^*) - V^*\cdot h^* + \omega(J+h^*)]$ (53a)

and where [n] denotes the n^{th} derivative (with respect to h).

Using (55a) we thus have for the <u>first</u> correction

$$Z[J] = Z_{sp} \iint [d\phi]\, d\left[\frac{\eta}{2\pi}\right] \exp\Big[-\tfrac{1}{2} S''(V^*)\,\phi_x^2$$

$$- i\phi.\eta - \tfrac{1}{2}\omega''(J + h^*)(\eta_x)^2\Big]. \tag{54}$$

The integral is of Gaussian form, which can be readily evaluated to give [15]

$$Z[J] = Z_{sp}\, \det{}^{-\frac{1}{2}}[1 - S''(V^*)\,\omega''(V + h^*)]$$

$$= Z_{sp} \exp[-\tfrac{1}{2} \operatorname{tr} \ln\{(1 - S''(V^*)\,\omega''(V + h^*))\}]. \tag{55}$$

The one-loop contribution from the Gaussian fluctuations to F = log Z is therefore given by

$$F_1 = -\tfrac{1}{2}\mathrm{tr}\,\ln[1 - S''(V^*)\,w''(V + h^*)]. \tag{56}$$

The following remarks are in order

(i) For the saddle-point to be stable, the matrix $1 - S''(V^*)\,\omega''(V + h^*)$ must be positive definite.

(ii) If the saddle-point solutions do not share the same symmetries of the original theory, then the stationary conditions (49) will be degenerate. This is especially true for gauge theories.

For continuously degenerate saddle-points, gauge fixing in the Weak Coupling phase must be introduced from the start to avoid the matrix above developing zero modes (corresponding to zero mass excitations) [47]. The effect of gauge fixing would then be to render the quadratic terms of the Gaussian fluctuations around the mean field positive semi-definite.

For discrete groups gauge fixing is not necessary as no zero modes occur which prevent the calculation of higher order corrections in the fluctuation expansion around the saddle-point [47]. The usual choice for fixing the gauge is the axial gauge [47,55,15], although the choice of a covariant gauge [54] has also been advocated.

Brézin and Drouffe [43a] have shown that the problem associated with the violation of Elitzur's theorem (sec. 3.4) can be suitably remedied by using a more general expression for the MF trial action, namely

$$S(U_\ell;\{h\}) = N^{-1} \sum_{<ij>} \mathrm{Re}\ \mathrm{tr}(h_{ij}U_{ji}), \tag{57}$$

where, in contrast to eq.(28), a mean field h_{ij} is now associated with each link (with <ij> denoting the lattice links). Choosing $h_{ij} = h^\dagger_{ji}$, they used the following ansatz for the Saddle Point solutions given in eq.(49)

$$h_{ij} = V_i h V_j^{-1} \qquad h \in R,\ V_i \in G \tag{58}$$

which is dependent on the real constant h and a set of local gauge transformations $\{V_i\}$ [with $V_i \equiv V_i(x)$; $V_j^{-1} \equiv V_j^{-1}(x+\mu)$].

Hence, because eq.(58) is degenerate with respect to V_i, if h_{ij} is a solution, then so is h'_{ij}. By averaging U over all possible Saddle Point configurations, one obtains the result

$$\langle U_{ij} \rangle = \int dV_i dV_j^{-1} \, w'(h) \, V_i V_j^{-1} \equiv 0 \tag{59}$$

so that Elitzur's theorem is satisfied (compare eq.(35a)).

5. General Applications of MF Approximations

5.1 Gauge Models [45;50]

Using the Wilson action in the fundamental representation the partition function for SU(N) reads (see Chapter 3)

$$Z = \int \left(\prod_\ell dU_\ell\right) \exp\left(\beta \sum_p \frac{\mathrm{tr}U_p + \mathrm{tr}U_p^\dagger}{2\mathrm{tr}\ 1}\right)$$

$$= \int [dU] \exp\left(\frac{\beta}{2N} \sum_p \mathrm{tr}(U_p + U_p^\dagger)\right). \qquad \left(\beta = \frac{2N}{g^2}\right) \tag{60}$$

The MF equations are obtained by replacing all the link variables $U_{x,\mu} \equiv U_\ell$ by their average values $\langle U_\ell \rangle$, except one (U_L) ("Weiss MF" method).

Hence

$$Z \to Z_{MF} = \int dU_L \exp\left(z(\mathrm{tr}U_L + \mathrm{tr}U_L^\dagger)/(2\ \mathrm{tr}\ 1)\right), \tag{61}$$

where

(i) $z = 2(d-1)\beta m^3$ (61a)

(d is the number of dimensions and (2d-1) is the number of plaquettes).

(ii) m is the solution of the MF self-consistency equation.

$$m1 = \langle U_L \rangle$$

$$= \frac{d}{dz} \ln \left[\int dU_L \exp\left(z(\mathrm{tr} U_L + U_L^\dagger)/2\mathrm{tr}\ 1 \right) \right]_{z=2(d-1)\beta m}$$

$$= \frac{d}{dz} \ln Z_{MF} \Big|_{z=2(d-1)\beta m^3} \equiv Q(z). \qquad (62)$$

We will now calculate eq.(62) for a few gauge groups.

(a) $G \equiv U(1)$ [45]

The abelian group U(1) can be parameterized by

$$U(1) = \left\{ e^{i\phi} : 0 < \phi \leq 2\pi \right\} \qquad (63)$$

with measure

$$\int dg\ f(g) = \frac{1}{2\pi} \int_0^{2\pi} d\phi\ f(e^{i\phi}). \qquad (64)$$

Hence, if $U_L \in U(1)$, then $U_L = e^{i\phi}$ and

$$z\ \mathrm{tr}\left[U_L + U_L^\dagger \right] = 2z \cos\phi\ ; \quad 2\mathrm{tr}\ 1 = 2 \qquad (65)$$

so that eq.(62) now reads

$$m = \frac{\int_0^{2\pi} d\phi \cos\phi\ e^{z\cos\phi}}{\int_0^{2\pi} d\phi\ e^{z\cos\phi}} = \frac{I_1(z)}{I_0(z)}\ , \qquad (66)$$

where $I_n(z)$ is the modified Bessel function of order n (see Appendix D) and z is defined in eq.(61a).

(b) $G \equiv SU(2)$ [45;47;49;51]

One possible parameterization for a group element of SU(2) is (see Chapter 5)

$$U = 1 \cos\phi + i\hat{n}\cdot\vec{\sigma} \sin\phi, \qquad (67)$$

where $0 \leq \phi \leq 2\pi$, and $\hat{n}$ is a unit three dimensional vector.

The corresponding normalized Haar measure over the group manifold is given by (Chapter 5)

$$[dU] = \frac{\sin^2\phi \, d\phi \, d^2n}{2\pi^2}. \tag{68}$$

For χ we choose the character associated with the spin ½ representation, i.e.

$$\chi_{\frac{1}{2}}(u) = \mathrm{tr}_{\frac{1}{2}} U$$

$$= 2\cos\phi. \tag{69}$$

Hence $\mathrm{tr}_{\frac{1}{2}}(U_L + U_L^\dagger) = 2\,\mathrm{tr}_{\frac{1}{2}}(1\cos\phi)$

$$= 4\cos\phi$$

and $\quad 2\mathrm{tr}\ 1 = 4, \tag{70}$

which gives

$$Z_{MF} = \int dU_L \exp[z\,\mathrm{tr}_{\frac{1}{2}}(U_L + U_L^\dagger)/2\mathrm{tr}\ 1]$$

$$= \frac{1}{2\pi^2}\int d^2n \int_0^{2\pi} \sin^2\phi \, e^{z\cos\phi}\, d\phi$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sin^2\phi \, e^{z\cos\phi}\, d\phi \ . \tag{71}$$

Rewriting

$$e^{z\cos\phi} = \frac{1}{z}\frac{d}{d(\cos\phi)} e^{z\cos\phi} \tag{71a}$$

we have

$$Z_{MF} = \frac{1}{\pi}\int_0^{2\pi} \sin^2\phi \, d\phi \left[\frac{1}{z}\frac{d}{d(\cos\phi)} e^{z\cos\phi}\right]$$

$$= -\frac{1}{\pi}\int_0^{2\pi} \sin\phi \, d(\cos\phi) \left[\frac{1}{z}\frac{d}{d(\cos\phi)} e^{z\cos\phi}\right]. \tag{71b}$$

Using partial integration this gives

$$Z_{MF} = \frac{1}{z}\frac{1}{\pi}\int_0^{2\pi} \cos\phi \; e^{z\cos\phi}\, d\phi$$

$$= \frac{I_1(z)}{z} \; . \tag{72}$$

Similarly it follows that

$$\int dU_L \frac{\mathrm{tr}(U_L + U_L^\dagger)}{2\ \mathrm{tr}\ 1} \exp[z\ \mathrm{tr}(U_L+U_L^\dagger)/2\ \mathrm{tr}\ 1]$$

$$= \frac{1}{\pi}\int_0^{2\pi} \sin^2\phi\, \cos\phi \left[\frac{1}{z}\frac{d}{d(\cos\phi)} e^{z\cos\phi}\right] d\phi$$

$$= \frac{1}{z\pi}\int_0^{2\pi} \cos 2\phi\; e^{z\cos\phi}\, d\phi \;\; = \frac{2I_2(z)}{z} \; . \tag{73}$$

From eq.(62) it thus follows that (using eqs.(72) and (73))

$$m = \frac{I_2(z)}{I_1(z)} \; , \tag{74}$$

which, using (D.3) gives

$$m = \frac{I_2(z)}{I_1(z)} \;\; = \;\; \tfrac{1}{2}\, \frac{I_3(z) - I_1(z)}{I_2(z) - I_0(z)} \; . \tag{74a}$$

(c) Fundamental-adjoint mixed action [51,52]

Using the mixed SU(N) fundamental-adjoint action

$$S = -\sum_p \beta_F \left[\frac{\chi_F(U_p)}{\chi_F(1)} + \beta_A \frac{\chi_A(U_p)}{\chi_F(1)}\right] \; , \tag{75}$$

two paramters have to be introduced in order to describe the MF, i.e.

$$\langle U_F \rangle = m_F 1_F$$

$$\langle U_A \rangle = m_A 1_A \; . \tag{76}$$

Hence

$$S \to S_{MF} = -(2d-1)\left[\frac{\beta_F}{2N} m_F^3 \, \mathrm{tr}(U_L + U_L^\dagger) + \frac{\beta_A}{N^2-1} m_A^3 (|\mathrm{tr}\, U|^2 - 1)\right] \tag{77}$$

where the number of plaquettes in d dimensions is again (2d-1), while the rest of the notation used is discussed in Chapter 3.

Since U is unitary it can be diagonalized (see Chapter 5) with eigenvalues $e^{i\phi_j}$, j = 1, ... N. The SU(N) invariant Haar measure factorizes into [see discussion leading up to eq.(105)]

$$dU = \prod_{j=1}^{N} \frac{d\phi_j}{2\pi} |\Delta(\phi)|^2 \, 2\pi \, \delta_p(\Sigma_j \phi_j), \tag{78}$$

where $\Delta(\phi)$ is the Vandermonde determinant and $\delta_p(\Sigma_j \phi_j)$ a periodic δ function inserted to enforce det U = 1.

The partition function now reads

$$Z = \int \prod_{j=1}^{N} d\phi_j \prod_{j'<j} \sin^2\left[\frac{\phi_{j'} - \phi_j}{2}\right] \delta_p(\Sigma_j \phi_j) e^{-S(\phi)} \tag{79}$$

Eq.(79) can now be solved numerically [51].

Several physical quantities can now be calculated, e.g. the plaquette energy

$$E_p = \langle S_p \rangle = \left\langle \frac{\mathrm{tr}\,(U_p + U_p^\dagger)}{2\mathrm{tr}\ 1} \right\rangle$$

$$= \frac{d}{d\beta'} \ln \left[\int \left(\prod_\ell dU_\ell\right) \exp\left(\beta' S_p + \beta \sum_{p' \neq p} S_{p'}\right) \right]_{\beta=\beta'}. \tag{80}$$

In the lowest order MF approximation (i.e. replacing all the links except one with their average values), this gives [45;49]

$$\langle S_p \rangle = \frac{\int dU_L \left[\frac{m^3 \ tr(U_L + U_L^\dagger)}{2tr\ 1}\right] exp\left[z \ \frac{tr(U_L + U_L^\dagger)}{2tr\ 1}\right]}{\int dU_L exp\left[z \frac{tr(U_L + U_L^\dagger)}{2tr\ 1}\right]}$$

$$= m^4. \tag{81}$$

Using an improved MF approximation, namely to replace U by $\langle U \rangle = m$ everywhere except on all four links of the plaquette p gives

$$\langle S_p \rangle = \frac{d}{d\beta'} \ln \left[\int \prod_{\ell \varepsilon p} dU_\ell \ exp\left\{\beta' S_p + \beta(2d-3)m^3 \sum_{\ell \varepsilon p} \frac{tr(U_\ell + U_\ell^\dagger)}{2tr\ 1}\right\}\right]\Bigg|_{\beta=\beta'} \tag{82}$$

where the number of nearest neighbour plaquettes in d dimensions is (2d-3).

Eq.(77) can now be evaluated using character expansion techniques [45,49] (see also Chapter 5).

For SU(2), the character expansion of e^{-S} is given by

$$e^{\frac{1}{2}\alpha \chi_{\frac{1}{2}}(u)} = \sum_{r=0}^{\infty} d_r b_r(\alpha) \chi_r(U) \tag{83}$$

where (i) $d_r = 2r + 1$ (83a)

(ii) and the expansion coefficients are given by

$$b_r(\alpha) = \frac{2 I_{r+1}(\alpha)}{\alpha} \tag{83b}$$

Using the orthogonality properties of the characters it is straightforward to show that [49]

$$\langle S_p \rangle = \langle \tfrac{1}{2} \chi_{\frac{1}{2}}(U_p) \rangle$$

$$= \frac{\sum_r d_r^2 \ b_r^4(z) \ \frac{d}{dz'} b_r(z')}{\sum_r d_r^2 \ b_r^4(z) \ b_r(z')}\Bigg|_{\substack{z'=\beta \\ \beta'=\beta}} \tag{84}$$

where $z = (2d-3)\beta m^3$. (84a)

5.2 Chiral Models [53;52]

Consider the SU(N) ⊗ SU(N) chiral partition function

$$Z = \int [dU] \exp\left(\tfrac{1}{2}\beta \sum_{<ij>} \mathrm{tr}\, U_i^\dagger U_j + \mathrm{h.c.}\right), \tag{85}$$

where i labels the lattice sites and <ij> labels links.

If we now make the Ansatz of replacing all the links except the ith one by a fixed matrix K, we have

$$Z = \int [dU] \exp\left(\tfrac{1}{2}\beta z\, \mathrm{tr}(UK^\dagger + KU^\dagger)\right), \tag{86}$$

where z = 2d is the number of nearest neighbour links.

Writing K = J/z the partition function in eq.(86) reduces to the single site problem

$$\begin{aligned} Z_{ss} &\equiv \exp[-F_{ss}(J,J^\dagger)] \\ &= \int [dU] \exp[\tfrac{1}{2}\beta\, \mathrm{tr}(U_{ji}J_{ij}^\dagger + J_{ji}U_{ij}^\dagger)], \end{aligned} \tag{87}$$

with the MF self-consistency equation given by

$$<U> = K = J/z \tag{88}$$

(and the expectation value taken with respect to eq.(85)).

Now, $F_{ss}(J,J^\dagger) = -\ln Z_{ss}$

therefore

$$\begin{aligned} \frac{\partial F_{ss}}{\partial J_{ss}^\dagger} &= -\frac{1}{Z_{ss}} \int [dU] \exp[\tfrac{1}{2} \frac{\beta}{2} U_{ji} \exp[\tfrac{1}{2} \beta\, \mathrm{tr}\, (UJ^\dagger + JU^\dagger)] \\ &= -\frac{\beta}{2} <U_{ji}>. \end{aligned} \tag{89}$$

Equation (87) can therefore be written in the form

$$-\frac{2}{\beta}\frac{\partial F_{ss}}{\partial J^{\dagger}_{ij}} = \frac{1}{z} J_{ji}. \tag{89a}$$

Defining the MF free energy

$$F_{MF}(J,J^{\dagger}) \equiv \frac{\beta}{2z}\,\mathrm{tr}(J^{\dagger}J) + F_{ss}(J,J^{\dagger}) \tag{90}$$

we have

$$\frac{\partial}{\partial J^{\dagger}_{ij}} F_{MF} = 0 \tag{91}$$

which can be used to determine the value of the mean field J.

Using a global Ansatz

$$J = \alpha 1 \ , \tag{92}$$

equation (87) becomes

$$Z_{ss} = e^{-F_{ss}}$$

$$= \int [dU]\, \exp[\tfrac{1}{2}\alpha\beta\ \mathrm{tr}(U + U^{\dagger})]. \tag{93}$$

Expanding the integrand in eq.(87) in powers of β and using the properties of Haar measure (Chapter 5) we have

$$\int [dU]\, \exp[\tfrac{1}{2}\beta \mathrm{tr}(UJ^{\dagger} + JU^{\dagger})]$$

$$= \int [dU] \left\{1 + \tfrac{1}{2}\beta(\mathrm{tr}\ UJ^{\dagger} + \mathrm{tr}\ JU^{\dagger}) + \frac{1}{2!}(\beta/2)^2 \Big[\quad\Big]^2 + \ldots\right\}$$

$$= \int [dU] \left\{\frac{1}{2!}(\beta/2)^2 [2\mathrm{tr}(UJ^{\dagger})\ \mathrm{tr}(JU^{\dagger})]\right.$$

$$+ \frac{6}{4!}(\beta/2)^4 (\mathrm{tr}\ UJ^{\dagger})^2 (\mathrm{tr}\ JU^{\dagger})^2]$$

$$\left. + \frac{1}{N!}(\beta/2)^N [(\mathrm{tr}\ UJ^{\dagger})^N + \mathrm{tr}\ (JU^{\dagger})^N] + \ldots\right\}$$

$$= \frac{1}{2^2 N}\beta^2\,\mathrm{tr}(J^\dagger J) + \frac{1}{(N^2-1)}\frac{\beta^4}{2^5}\mathrm{tr}^2(J^\dagger J) - \frac{\beta^4}{N(N^2-1)}\frac{1}{2^5}\mathrm{tr}(J^\dagger JJJ^\dagger)$$

$$+ \frac{\beta^N}{2^N N!}\left[\det(J) + \det(J^\dagger)\right] + \ldots \quad . \tag{94}$$

For general J we thus find [53]

$$F_{MF} = \frac{\beta}{2z}\mathrm{tr}(J^\dagger J) - \frac{\beta^2}{2^2 N}\mathrm{tr}(J^\dagger J) - \frac{\beta^4}{2^5 N^2(N^2-1)}\mathrm{tr}^2(J^\dagger J)$$

$$+ \frac{\beta^4}{2^5 N(N^2-1)}\mathrm{tr}(J^\dagger JJJ^\dagger) - \frac{\beta^N}{2^N N!}[\det(J) + \det(J^\dagger)] + \ldots \quad . \tag{95}$$

Using the global Ansatz (eq.(92)), equation (95) reduces to

$$F_{MF} = \frac{\beta}{2z}\mathrm{tr}(J^\dagger J) - \frac{\beta^2}{2^2 N}\mathrm{tr}(J^\dagger J) - \frac{\beta^N}{2^N N!}[\det J + \det J^\dagger] + \ldots \tag{95a}$$

(i.e. the two fourth order terms cancel).

For SU(2), eq.(95a) gives (to lowest order)

$$F_{MF} = \frac{\beta}{2z}\mathrm{tr}(J^\dagger J) - \frac{\beta^2}{2^2 2}\mathrm{tr}(J^\dagger J) - \frac{\beta^2}{2^2 2!}[\det J + \det J^\dagger] + \ldots$$

$$= \frac{\beta}{2z}\alpha^2\mathrm{tr}1 - \frac{\beta^2}{2^2 .2}\alpha^2\mathrm{tr}1 - \frac{\beta^2}{2^2 2!}[2\alpha^2 \det 1] + \ldots$$

$$= \frac{\beta}{z}\alpha^2 - \frac{\beta^2\alpha^2}{2} + \ldots \qquad (z = 2d = 6)$$

$$= \tfrac{1}{2}\beta(\tfrac{1}{3} - \beta)\alpha^2 + \ldots \quad . \tag{96}$$

The MF approximation therefore predicts a continuous phase transition transaction at $\beta_c = \frac{1}{3}$.

For SU(3) the determinantal interaction enters in lowest powers as a cubic, which results in the phase transition being first order [53]

$$F_{MF} = \frac{1}{4}\beta(1-\beta)\alpha^2 - \frac{1}{24}\beta^3\alpha^3 + \ldots \quad . \tag{97}$$

In order to determine the phase transition points more accurately, eq.(93) can be evaluated exactly using SU(N) integration techniques.

We shall first solve the U(N) integral and then afterwards add the SU(N) constraints.

As U is a unitary, it can be diagonalized by a unitary matrix D,

$$U = D\begin{pmatrix} e^{i\phi_1} & & & \\ & e^{i\phi_2} & & \\ & & \ddots & \\ & & & e^{i\phi_N} \end{pmatrix} D^\dagger \equiv DU_D D^\dagger \tag{98}$$

The integrand of eq.(93) can hence be written as

$$\exp[\tfrac{1}{2}\alpha\beta(\mathrm{tr}U + \mathrm{tr}U^\dagger)] = \exp[\tfrac{1}{2}\alpha\beta\ \mathrm{tr}(U_D + U_D^\dagger)]$$

$$= \exp[\alpha\beta(\cos\phi_1 + \ldots + \cos\phi_N)], \tag{99}$$

where we used the cyclic property of the trace

$$\mathrm{tr}\ U = \mathrm{tr}(DU_D D^\dagger) = \mathrm{tr}\ U_D. \tag{99a}$$

Using the Weyl parameterization we can write for the Haar measure in eq.(93),

$$[dU] = d\mu(\phi)\ [dD], \tag{100}$$

$$\text{where } d\mu(\phi) = \left(\prod_{i=1}^{N} \frac{d\phi_i}{2\pi}\right) |\Delta(\phi)|^2 \qquad (0 \le \phi_i \le 2\pi) \tag{100a}$$

and $\Delta(\phi)$ is the Vandermonde determinant

$$\Delta(\phi) = \frac{1}{\sqrt{N!}}\, \varepsilon_{j_1 \cdots j_N}\, e^{i\phi_1(N-j_1)} \ldots e^{i\phi_N(N-j_N)} \tag{100b}$$

$$= \det[e^{i\phi(N-j)}]\ . \tag{100c}$$

As the integrand [eq.(99)] is independent of D, the integration over [dD] can be normalized to unity. With this in mind we can write for eq.(93)

$$Z_{ss} = \int d\mu(\phi)\ \exp[2\beta(\cos\phi_1 + \ldots + \cos\phi_N)]$$

$$= \left(\prod_{i=1}^{N} \frac{d\phi_i}{2\pi}\right) |\Delta(\phi)|^2 \exp[\alpha\beta \prod_{i=1}^{N} \cos\phi_i] \ . \qquad (101)$$

Now $|\Delta\phi|^2 = \Delta(\phi)\cdot\bar{\Delta}(\phi)$

$$= \frac{1}{N!}\ \varepsilon_{i_1\ldots j_N}\ \varepsilon_{j_1\ldots j_N}\ e^{i\phi_1(j_1-i_1)} \ \ldots\ e^{i\phi_N(j_N-i_N)}$$

$$= \frac{1}{N!}\ \varepsilon_{i_1\ldots i_N}\ \varepsilon_{j_1\ldots j_N} \prod_{r=1}^{N} e^{i\phi_r(j_r-i_r)} \ . \qquad (101a)$$

Thus

$$Z_{ss} = \int \left(\prod_{r=1}^{N} \frac{d\phi_r}{2\pi}\right) |\Delta(\phi)|^2 \prod_{r=1}^{N} \exp[\alpha\beta\ \cos\phi_r]$$

$$= \frac{1}{N!}\ \varepsilon_{i_1\ldots i_N}\ \varepsilon_{j_1\ldots j_N} \prod_{r=1}^{N} \left\{\int_0^{2\pi} \frac{d\phi_i}{2\pi}\ e^{i\phi_r(j_r-i_r)}\ e^{\alpha\beta\ \cos\phi_r}\right\}$$

$$\equiv \det I_{|j-i|}(\alpha\beta), \qquad [U(N)] \qquad (102)$$

where $$I_{|j-i|}(\alpha\beta) = \int_0^{2\pi} \frac{d\phi}{2\pi}\ e^{\alpha\beta\ \cos\phi\ \pm\ i(n-m)\phi} \ . \qquad (102a)$$

For <u>SU(N)</u> the restrictions

$$\sum_{i=1}^{N} \phi_i = 0(\mathrm{mod}2\pi) \qquad (103)$$

must be added to the U(N) parameterization (eq.(100a)) in order to enforce the condition det U = 1.

To this end we insert a periodic delta function into the measure (eq.(100a)), namely [53]

$$2\pi\ \delta_p\left(\sum_{r=1}^{N} \phi_r\right) \equiv \sum_{\ell=-\infty}^{\infty} 2\pi\ \delta\left(\sum_{r=1}^{N} \phi_r\ -\ 2\pi\ell\right)$$

$$= \sum_{m=-\infty}^{\infty} \exp[im(\sum_r \phi_r)] \qquad (104)$$

so that

$$[dU] = d\mu(\phi)\ 2\pi\delta_p(\sum_r \phi_r)$$

$$= \sum_{m=-\infty}^{\infty} d\mu(\phi)\ \exp[im(\sum_{r=1}^{N} \phi_r)], \tag{105}$$

where we took into account that the integral will again be independent of D.

The single-site integral now reads

$$Z_{ss} = \sum_{m=-\infty}^{\infty} \int d\mu(\phi)\ \exp[im(\sum_{r=1}^{N} \phi_r)]\ \exp[\alpha\beta \prod_{r=1}^{N} \cos\phi_r]$$

$$= \sum_{m=-\infty}^{\infty} \frac{1}{N!}\ \varepsilon_{i_1\ldots i_N}\ \varepsilon_{j_1\ldots j_N} \prod_{r=1}^{N} \left(\int_0^{2\pi} \frac{d\phi_r}{2\pi}\ \exp[i\phi_r(m+j_r-i_r)+\alpha\beta\cos\phi_r]\right)$$

$$= \sum_{m=-\infty}^{\infty} \det I_{m+j-i}(\alpha\beta). \tag{106}$$

The exact expression for the free MF energy is therefore

$$F_{MF} = \frac{N}{2z}\ \beta\alpha^2 - \ln \sum_{r=-\infty}^{\infty} \det\ I_{r+j-i}(\alpha\beta). \tag{107}$$

If the Bessel functions are now expanded in powers of $\alpha\beta$, the epressions obtained earlier in (96) and (97) are recovered for N =2 and 3 respectively. For futher results, see [53].

PART II

1. Field Theories at Finite Temperature

To introduce the concept of temperature in QFT (which is necessary to describe a wide variety of physical applications e.g: heavy-ion colisions) we use as Ansatz the Statistical partition function

$$Z = \mathrm{tr}\, e^{-\beta H}$$

$$= \sum_{\phi} \langle\phi| e^{-\beta H} |\phi\rangle \tag{1.1}$$

where H is the Hamilton operator and

$$\beta = T^{-1} \qquad (k_B = 1) \tag{1.1a}$$

Interpreting the partition function as a summation over all possible independent eigenstates we can use the Feynman functional formalism [1] which gives

$$\sum_{\phi} \langle\phi| e^{-\beta H} |\phi\rangle = N' \int (d\phi)\, e^{-S(\phi)} \tag{1.2}$$

where N' is a temperature dependent normalzation factor, and we introduced the variable

$$\tau \equiv it.$$

To find the partition function Z, the integration in (1.2) is restricted to those fields satisfying the following periodic boundary conditions

$$\phi(\vec{x},0) = \phi(\vec{x},\beta). \tag{1.2a}$$

We thus have the result

$$Z = N' \int (d\phi)\, e^{-S(\phi)}$$

$$= N' \int (d\phi)\, \exp\Big[\int_0^{\beta} d\tau \int d^3x\, \mathcal{L} \Big] \tag{1.3}$$

with $\mathcal{L}$ ($\equiv \mathcal{L}(\phi, \partial^e_\mu \phi)$) the Euclidian Lagrangian and $\partial^e_\mu \equiv (i\frac{\partial}{\partial\tau}, \vec{\nabla})$.

Eq. (3) is now a "typical" path integral, i.e. it consists of only classical fields.

Introducing a suitable normalized measure we finally have

$$Z = \int [d\phi]\, e^{-S(\phi)}. \tag{1.4}$$

In applying the formalism above to gauge fields, special care must be taken because there are now only two independent degrees of freedom for a massless vector field, while the Lagrangian of the theory has typicaly four degrees of freedom for most renormalizable gauges [2].

Choosing a suitable "physical" gauge (in particular the axial gauge), however, results in each gauge field having only two degrees of freedom. This particular choice also has the advantage that Fadeev-Popov ghost fields need not be intoduced to get rid of the unwanted degrees of freedom. For other gauges the Fadeev-Popov Ansatz has to be included.

In general the partition function for a gauge theory will have the form [1;2;5e]

$$Z = (N')^m \int [dA]\, \exp\Big(\int_0^\beta d\tau \int d^3x\, \mathcal{L}(A) \Big)\, \det M_f\, \delta[f_a(A_\mu)], \tag{1.5}$$

where $\det M_f\, \delta[f_a(A_\mu)] = \det\left|\frac{\partial f_a}{\partial \theta}\right|\, \delta[f_a(A_\mu)]$ is the Fadeev-Popov Ansatz which restricts the functional measure and thus get rid of the unwanted redundancy in the quantization procedure. The set of functions $\theta(x)$ parametrize the (infinitesmal) gauge transformations of $\mathcal{L}$, while m is the total number of physical polarization states (degrees of freedom).

Also, the Lagrangian of the Yang-Mills theory is given by

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu}\, F^{a\mu\nu} \tag{1.6}$$

where $$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu \tag{1.6a}$$

with f_{abc} the SU(N) structure constants (see Chapter 5), and g the coupling constant of the theory.

Again introducing a normalized measure we have the following form for the continuum Yang-Mills partition function

$$Z = \int_p [dA]\, e^{-S} \tag{1.7}$$

where p denotes the periodicity of the measure [i.e. $A(\vec{x},0) = A(\vec{x},\beta)$].

Thermal averages of physical variables are defined by

$$\langle O \rangle = \frac{1}{Z} \mathrm{tr}[O\, e^{-\beta H}] \,, \tag{1.8}$$

where Z is the partition function defined in (1.1).

In terms of the formalism discussed above, this implies

$$\langle O \rangle = \frac{1}{Z} \int_p [dA]\, O\, e^{-S}, \tag{1.9}$$

with Z defined in (1.7).

2. Lattice Gauge Theories at Finite Temperature

2.1 General Formalism

The motivation behind the introduction of the lattice regularization in zero temperature Field Theory (Chapter 1) is also valid for the finite temperature formalism.

The only difference as far as the lattice structure self is concerned is that the lattice must now be finite in at least one direction (the so-called time or temporal direction). As the Euclidian framework is used, this choice is arbitrary.

For a lattice finite in all directions the number of sites in the spatial and temporal directions will be denoted respectively by N_σ and N_τ [3]. In general the lattice also need not be symmetric, in which case the lattice spacings in the spatial and temporal directions are denoted by a_σ and a_τ.

The asymmetry parameter is defined by

$$\xi = a_\sigma / a_\tau \ . \tag{2.1}$$

A symmetric lattice will therefore correspond to the choice $\xi = 1$ (but with $N_\sigma \neq N_\tau$ for finite temperature systems).

To ensure the required periodicity of the gauge fields in τ, the lattice should be periodic in at least the temporal direction, i.e.

$$\hat{ae}_1 = \hat{ae}_{1+N_\tau} . \tag{2.1}$$

The volume and temperature on the lattice are thus defined by

$$V = (N_\sigma a_\sigma)^3$$

$$T^{-1} = N_\tau a_\tau \ . \tag{2.3}$$

It is important to note that for physical quantities the thermodynamic limit corresponds to taking $N_\sigma \to \infty$, whith the lattice spacing a_σ kept fixed. Also, the continuum limit of the theory corresponds to the limits N_σ , $N_\tau \to \infty$ with a_σ and $a_\tau \to 0$ while V and T^{-1} are kept fixed.

For lattice finite temperature systems, the gauge field action used most often is the Wilson one defined on an asymmetric lattice (see also Chpt.3):

$$S_G(U) = (2N/g_\sigma^2)(a_\tau/a_\sigma) \sum_x \sum_{\mu<\nu<4} P_x^{\mu\nu} + (2N/g_\tau^2)(a_\sigma/a_\tau) \sum_x \sum_{\mu<4} P_x^{\mu 4} \qquad (2.4)$$

where $P_x^{\mu\nu} \equiv 1 - \frac{1}{N} \text{Re tr } U_x^\mu U_{x+\mu}^\nu U_{x+\nu}^{\mu\dagger} U_x^{\nu\dagger}$

$$\equiv 1 - \frac{1}{N} \text{Re tr } U_x^{\mu\nu}, \qquad (2.4a)$$

and (i) $\sum_x \sum_{\mu<\nu<4} \equiv \sum_{\{P_\sigma\}}$ implies summation over all space-like plaquettes (i.e with all four links space-like) ;

(ii) $\sum_x \sum_{\mu<4} \equiv \sum_{\{P_\tau\}}$ implies summation over all time-like plaquettes (i.e with two spatial and two time-like links).

[The notation $U_{x,\mu}$ and U_x^μ will both be used to denote a link]

Defining

$$K_\sigma \equiv (1/g_\sigma^2)(a_\tau/a_\sigma) = (1/g_\sigma^2)\, \xi^{-1};$$

$$K_\tau \equiv (1/g_\tau^2)(a_\sigma/a_\tau) = (1/g_\tau^2)\, \xi \qquad (2.5)$$

we can rewrite eq.(2.3) in the form

$$S(U) = 2NK_\sigma \sum_{\{P_\sigma\}} P_x^{\mu\nu} + 2NK_\tau \sum_{\{P_\tau\}} P_x^{\mu 4}. \qquad (2.6)$$

The partition function is given by

$$Z_E(T,V) = \int \left(\prod_{x,\mu} dU \right) e^{-S(U)}$$

$$\equiv \int [dU]\, e^{-S(U)}, \qquad (2.7)$$

while the thermodynamic average is defined (analogously to (1.8)) for any operator O(U) by

$$\langle O(U) \rangle = Z^{-1} \int [dU]\, O(U)\, e^{-S(U)}. \qquad (2.8)$$

The following remarks are in order.

(i) As a consequence of the asymmetry of the lattice there are now two coupling constants $g_\sigma(a,\xi)$ and $g_\tau(a,\xi)$. This is to ensure that physical quantities in the continuum limit will be regularization scheme independent, as an independent variation in a_σ and a_τ can be compensated by suitably adjusting the two couplings.

For a symmetric lattice ($\xi = 1$) both couplings are equal (Chpt. 3):

$$g_\sigma(a,1) = g_\tau(a,1) \equiv g(a). \tag{2.9}$$

(ii) In the Weak Coupling limit the couplings g_σ and g_τ can be expanded in terms of the bare coupling constant to give the following finite lattice size corrected values [4]

$$g_\sigma^{-2}(a,\xi) = g^{-2}(a) + c_\sigma(\xi) + O(g^2)$$

$$g_\tau^{-2}(a,\xi) = g^{-2}(a) + c_\tau(\xi) + O(g^2) \tag{2.10}$$

where the functions $c_\sigma(\xi)$ and $c_\tau(\xi)$ are numerically known for SU(N) [4], and eq.(2.9) implies that $c_\sigma(1) = 0 = c_\tau(1)$.
In later calculations, derivatives with respect to T and V will be replaced by derivatives with respect to a_σ and ξ (section 2.4).
In anticipation we define

$$(\partial\, g_\sigma^{-2}(a,\xi)/\partial\xi)_{\xi=1} = (\partial c_\sigma(\xi)/\partial\xi)_{\xi=1} \equiv c'_\sigma$$

$$(\partial\, g_\tau^{-2}(a,\xi)/\partial\xi)_{\xi=1} = (\partial c_\tau(\xi)/\partial\xi)_{\xi=1} \equiv c'_\tau\ . \tag{2.11}$$

(iii) Using (2.3) and (2.5) the lattice couplings can be written as

$$K_\sigma = g_\sigma^{-2}\ (N_\tau T a_\sigma)^{-1};$$

$$K_\tau = g_\tau^{-2}\ N_\tau T a_\sigma\ . \tag{2.12}$$

2.2 Order Parameters and symmetries

One possibility to distinguish between the phases of a gauge theory is to use the free energies of static configurations of massive quarks and anti-quarks as order parameter. This can be done as follows [5].

First intoduce the operators $\psi_a^\dagger(\vec{x}_i,\tau)$ and $\psi_a(\vec{x}_i,\tau)$ which create and annihilate static quarks with colour a at position $(\vec{x}_i,\tau)$, along with their charge conjugates $\psi_a^{\dagger c}$ and ψ_a^c for anti-quarks [5c] at $(\vec{x}_i,0)$.
These fields therefore act as creation and annihilation operators, which satisfy the equal time anti-commutation relations

$$\{\psi_a(\vec{x}_i,\tau),\ \psi_b^\dagger(\vec{x}_j,\tau)\} = \delta_{ij}\delta_{ab}, \tag{2.13}$$

and simalarly for ψ^c.

The quark fields satisfy the static time-evolution relation

$$\left(\frac{1}{i}\frac{\partial}{\partial\tau} - A_0(\vec{x},\tau)\right)\psi(\vec{x}_i,\tau) = 0.$$

Integrating this equation gives

$$\psi(\vec{x}_i,\tau) = \mathrm{Texp}\left[\, i\int_0^\tau d\tau'\ A_0(\vec{x},\tau')\right]\ \psi(\vec{x}_i,0) \tag{2.14}$$

where T denotes a time ordered exponential.

The free energy of a configuration of N_q quarks and $N_{\bar{q}}$ anti-quarks relative to the vacuum is now defined by

$$\exp[-\beta F(\vec{x}_1,\ldots,\vec{x}_{N_q};\vec{y}_1,\ldots,\vec{y}_{N_{\bar{q}}})]$$

$$= (N^{N_q+N_{\bar{q}}})^{-1} \sum_{|s>} \langle s|\ e^{-\beta H}\ |s\rangle, \tag{2.15}$$

where the summation goes over all states $|s\rangle$ with heavy quarks at $\vec{x}_1,\ldots,\vec{x}_{N_q}$ and anti-quarks at $\vec{y}_1,\ldots,\vec{y}_{N_{\bar{q}}}$.

Using the quark operators we thus have

$$|s> = \sum_{\{a,b\}} \psi^{\dagger}_{a_1}(\vec{x}_1,0) \ldots \psi^{\dagger}_{a_{N_q}}(\vec{x}_{N_q},0)\psi^{\dagger c}_{b_1}(\vec{y}_1,0) \ldots \psi^{\dagger c}_{b_{N_{\bar{q}}}}(\vec{y}_{N_{\bar{q}}},0) \; |s'> \quad (2.16)$$

where the state $|s'>$ contains no heavy quarks. Using the above, eq.(2.15) now reads

$$\exp[-\beta F_{N_q N_{\bar{q}}}] = (N^{N_q+N_{\bar{q}}})^{-1} \times$$

$$\sum_{|s'>} <s'| \sum_{\{a,b\}} \psi_{a_1}(\vec{x}_1,0) \ldots \psi_{a_{N_q}}(\vec{x}_{N_q},0)\psi^{c}_{b_1}(\vec{y}_1,0) \ldots \psi^{c}_{b_{N_{\bar{q}}}}(\vec{y}_{N_{\bar{q}}},0) \times$$

$$e^{-\beta H} \psi^{\dagger}_{a_1}(\vec{x}_1,0) \ldots \psi^{\dagger}_{a_{N_q}}(\vec{x}_{N_q},0)\psi^{\dagger c}_{b_1}(\vec{y}_1,0) \ldots \psi^{\dagger c}_{b_{N_{\bar{q}}}}(\vec{y}_{N_{\bar{q}}},0) \; |s'> \quad (2.17)$$

Since $e^{-\beta H}$ is the generator for Euclidian time translations ($e^{-\beta H} = e^{-\tau H} = e^{itH}$), i.e. $e^{\beta H} O(t) e^{-\beta H} = O(\tau+\beta)$ for any operator $O(\tau)$, equation (2.17) becomes (inserting factors $e^{-\beta H}e^{\beta H}$)

$$\exp[-\beta F_{N_q N_{\bar{q}}}] = (N^{N_q+N_{\bar{q}}})^{-1} \times$$

$$\sum_{|s'>} <s'| e^{-\beta H} \psi_{a_1}(\vec{x}_1,\beta) \psi^{\dagger}_{a_1}(\vec{x}_1,0) \ldots \psi_{a_{N_q}}(\vec{x}_{N_q},\beta) \psi^{\dagger}_{a_{N_q}}(\vec{x}_{N_q},0) \times$$

$$\psi^{c}_{b_1}(\vec{y}_1,\beta) \psi^{\dagger c}_{b_1}(\vec{y}_1,0) \ldots \psi^{c}_{b_{N_{\bar{q}}}}(\vec{y}_{N_{\bar{q}}},\beta) \psi^{\dagger c}_{b_{N_{\bar{q}}}}(\vec{y}_{N_{\bar{q}}},0) \; |s'> \quad (2.18)$$

Defining the (thermal) Wilson line

$$L(\vec{x}) \equiv \frac{1}{N} \mathrm{tr}\, \mathrm{Texp}\, [i \int_0^{\beta} d\tau \, A(\vec{x},\tau)] \quad (2.19)$$

and using eq.(2.14) we can rewrite (2.18) as

$$\exp[-\beta F_{N_q N_{\bar{q}}}] = \mathrm{tr}[e^{-\beta H} L(\vec{x}_1)\ldots L(\vec{x}_{N_q}) L^{\dagger}(\vec{y}_1)\ldots L^{\dagger}(\vec{y}_{N_{\bar{q}}})] \; . \quad (2.20)$$

As we are only considering the pure gluon theory, the thermodynamic average (2.8) can be used, which after deviding out the vacuum expression (corresponding to the state with no quarks and anti-quarks) gives

$$\exp[-\beta F_{N_q N_{\bar{q}}}] = < L(\vec{x}_1)\ldots L(\vec{x}_{N_q}) L^{\dagger}(\vec{y}_1)\ldots L^{\dagger}(\vec{y}_{N_{\bar{q}}})>, \quad (2.21)$$

where $F_{N_q N_{\bar{q}}}$ now stands for the vacuum corrected value.

For a single quark we have

$$e^{-\beta F_q} = \langle L(\vec{x}) \rangle, \tag{2.22a}$$

while the correlation function of two Wilson lines defines the free energy of a $q\bar{q}$ pair

$$e^{-\beta F_{q\bar{q}}(\vec{x})} = \langle L(0) L(\vec{x}) \rangle \tag{2.22b}$$

Providing the so-called cluster decomposition holds, we have for $|\vec{x}| \to \infty$

$$\langle L(0) L(\vec{x}) \rangle \to \langle L(0) \rangle \langle L^{\dagger}(\vec{x}) \rangle$$

$$= |\langle L(0) \rangle|^2 \tag{2.23}$$

so that $F_{q\bar{q}} \to 2F_q$. (2.24)

The value of F_q (i.e. whether it is finite or not) will therefore be a test for confinement. On the lattice the corresponding expression for the Polyakov loop (thermal Wilson loop) is given by

$$L(\vec{x}) = \mathrm{tr} \prod_{\tau=1}^{N_\tau} U_{(\vec{x},\tau),4} \tag{2.25}$$

which consists of the product of all the links in the temporal direction taken at given spatial site $\vec{x}$ (This follows by virtue of the relation between the gauge potential $A_\mu(x)$ and the corresponding element of the gauge group U; $U = \exp[iagA_\mu(x)]$). The periodicity in the gauge fields $A_\mu(x)$ and the fact that the links form a closed loop ensure that $L(\vec{x})$ is gauge invariant.

The Wilson action (2.4) is invariant under the gauge transformations

$$U_{x,\mu} \to V_x U^{\mu}_x V^{\dagger}_{x+\mu} \tag{2.26}$$

for arbitrary $V_x \in G$ (Chapter 3). However, if the centre (C) of the Gauge group is non-trivial, then the lattice action is also invariant under transformations which rotate (flip) all the time-like links at a fixed time $\tau = \tau_0$, i.e.

$$U^0_{(x,\tau_0)} \rightarrow zU^0_{(x,\tau_0)} \tag{2.27}$$

where $z = \exp[2r\pi i/N], \quad 0 \leq r \leq N-1 \quad (SU(N))$ (2.27a)

and $C = \{g_i \in G \mid g_i g_j g_i^{-1} = g_j \text{ for all } g_j \in G\}$. (2.27b)

These transformations form a global symmetry called the **centre symmetry** [5d].

To show that the action is indeed invariant under (2.27) we note that in (2.4) only the time-like plaquettes contain time-like links. A plaquette containing no flipped links is clearly invariant, while a flipped link must necessarily appear twice:

$$\begin{aligned} U^{\mu o}_x &\rightarrow U^{\mu}_x z U^{o}_{x+\mu} U^{\mu\dagger}_{x+o} U^{o\dagger}_x z^{\dagger} \\ &= U^{\mu o}_x , \end{aligned} \tag{2.28}$$

where the last step followed since z commutes with all $U \in G$.

The Polyakov loop (2.25) is clearly not invariant under (2.27) and transforms as

$$L(\vec{x}) \rightarrow zL(\vec{x}). \tag{2.29}$$

The expectation value $\langle L \rangle$ is consequently an order parameter for the centre symmetry which vanishes if the symmetry is unbroken. This will signal confinement as

$$\langle L \rangle = 0 \text{ corresponds to } F_q \rightarrow \infty .$$

Explicitly, in the confining (low temperature) phase, the static $q\bar{q}$ potential will rise linearly, i.e.

$$F_{q\bar{q}}(\vec{x}) \approx \sigma(|\vec{x}|), \tag{2.30}$$

which in turn will result in the correlation function $P(\vec{x})$ decaying exponentially, where

$$\Gamma(\vec{x}) \equiv \langle L(0)L^{\dagger}(\vec{x})\rangle$$

$$\approx \exp[-\sigma|\vec{x}|\beta\ ]. \qquad (2.30a)$$

If the global symmetry is spontaneously broken, then $\langle L(\vec{x})\rangle$ will be different from zero, and F_q finite. This can be seen by noting that $\exp[-\beta F_{N_q N_{\bar{q}}}]$ transforms as

$$\exp[-\beta F_{N_q N_{\bar{q}}}] \to \exp[-2\pi i r(N_q - N_{\bar{q}})/N].\exp[-\beta F_{N_q N_{\bar{q}}}] \qquad (2.31)$$

which, unless $(N_q - N_{\bar{q}}) = pN$ (p some integer) (2.32)

implies that $\exp[-\beta F_{N_q N_{\bar{q}}}] = 0.$ (2.33)

Eq. (2.33) corresponds to divergent free energy, while (2.32) describes the situation where the symmetry is spontaneously broken, i.e. with an N-ality of non-singlet configurations of quarks which may have a finite free energy.

For SU(N) the degeneracy in the possible broken symmetry ground states (corresponding to the N possible distinct expectation values of L,

$$\langle L\rangle = e^{2\pi i r/N} L_0 \quad , \quad r = 0,1,\ldots,N-1 \quad \text{[see (2.27a)]}),$$

implies that $\langle L\rangle$ may be interpreted as an order parameter similar to the magnetization in a Z(N) spin model. This point will be discussed in more detail later. Also, for gauge groups with a dicrete centre symmetry (e.g. SU(N)) the correlation function $\Gamma(\vec{x})$ for the high temperature phase will decay exponentially (with additional power law corrections)[5d]

$$\Gamma(\vec{x})\big|_{x\to\infty} \sim |\langle L\rangle|^2[1 + O(e^{-\mu|\vec{x}|})] \qquad (2.34)$$

with $F_{q\bar{q}} \sim O(e^{-\mu|\vec{x}|})$ (2.34a)

which has the form of the Debye screening (characteristic of an electric plasma).

For the quenched theory, confinement is the result of the anti-screening property of the gluon vacuum polarization which leads to the formation of the colour electric string.
It must, however, be kept in mind that quarks by themselves cannot produce confinement; in fact the quark vacuum polarization screens the colour charge which leads to a weakening of the confining force produced by the gluons (see e.g. [2]). This feature will be of particular importance in the full theory where the effect of dynamical fermions must also be taken into account (see section 5).

For Lattice Gauge theories it has been rigorously shown [6] that systems with spatial dimensions $d \geq 3$ and gauge groups SU(N) [or U(N)] undergo a deconfining phase transition at high temperature, i.e. from a confining phase (at strong coupling) to a phase with free (infintely heavy) quarks and gluons at weak coupling (the case of SU(2) has been explicitly dealt with in [6c]).

Specifically, we have shown that for a pure SU(N) gauge theory the deconfinement transition can be characterized by the spontaneous breaking of the global centre Z(N) symmetry. At this point it is important to note that the formalism described above is only valid for a theory with no dynamical quarks, because in the presence of dynamical fermions the Z(N) symmetry is explicitly broken and hence $\langle L \rangle \neq 0$ for all temperatures. This point will be discussed in section 5.

2.3 Renormalizability and Scaling

One of the prerequisites of Lattice Gauge theories is that all results obtained should be independent of the lattice regularization scheme used, i.e. all physical quantities must become independent of the cut-off (a^{-1}), if the cut-off becomes large ($a \to 0$). This requirement of renormalizability uniquely fixes the coupling dependence of any function f(a,g) representing a physical quantity.

Cut-off independence implies

$$a\frac{d}{da} f(a,g)\Big|_{a\to 0} = 0,$$

i.e. $$\left(a \frac{\partial}{\partial a} - \beta(g)\frac{\partial}{\partial g}\right) f(a,g)\Big|_{a\to 0} = 0 \qquad (2.35)$$

where $\beta(g) = -\, a \frac{\partial g}{\partial a}$ (2.35a)

is the so-called Callan-Symanzik equation which relates the lattice spacings at different couplings (or vice versa), with solution

$$\frac{a(g)}{a(g_0)} = \exp\left[-\int_{g_0}^{g} dg'/\beta(g') \right]. \tag{2.36}$$

The leading order behaviour of $\beta(g)$ can be determined using perturbation theory (i.e. small g) [7], which gives

$$\beta(g) = -\beta_0 g^3 - \beta_1 g^5 + O(g^7). \tag{2.37}$$

The first two coefficients are renormalization scheme independent. For the pure SU(N) gauge theory the value of the coefficients are given by [7]

$$\beta_0 = (4\pi)^{-2}\left[\frac{11}{3} N \right];$$

$$\beta_1 = (4\pi)^{-2}\left[\frac{34}{3} N^2 \right]. \tag{2.38}$$

Defining $\Lambda_L \equiv a^{-1} \exp\left[-\int_{g_0}^{g} dg'/\beta(g') \right]$ (2.39)

and using (2.38), we have to lowest order the solution

$$a\Lambda_L = \exp\{-(2\beta_0 g^2)^{-1} + (\beta_1/2\beta_0^2)\ln(\beta_0 g^2)\}.[1 + O(g^2)] \tag{2.40}$$

where Λ_L is an (arbitrary) cutoff-independent mass parameter which sets the scale for QCD. Equation (2.40) is usually called the Renormalization Group Equation (RGE) for Lattice Gauge theories. The following remarks are in order.

(i) The coupling g should be regarded as the bare or unrenormalized coupling constant. From (2.40) it follows that the continuum limit $a \to 0$ is formally recovered for $g \to 0$.
Of importance for numerical calculations is the fact that the temperature can be increased by increasing the inverse coupling squared $\beta = 6/g^2$.
This follows because a variation in the value of the coupling g would imply a corresponding variation in the lattice spacing by virtue of the RGE (2.40). This in turn provides a way of adjusting the temperature via (2.3).

(ii) In LGT, a mass prediction is of the form

$$m = a^{-1} f(g), \tag{2.41}$$

where g is the (bare) coupling and the correct dimension is provided by the lattice spacing a. In order to obtain a finite quantity when $a \to 0$, the coupling g should approach some critical value g_c, i.e.

$$f(g)\big|_{a\to 0} = f(g_c) = 0. \tag{2.42}$$

Scaling

Physical quantities calculated on the lattice e.g. hadron masses (m_i) and string tension (σ), are all dimensionless with their continuum behaviour determined by (2.36), e.g. for the string tension we have

$$\sigma_i a = \text{const. } \exp\Big[-\int_{g_0}^{g} dg'/\beta(g') \Big] \tag{2.43}$$

for a universal $\beta(g)$.

This implies

$$\sigma_i a/\sigma_j a = \text{constant} = \sigma_i/\sigma_j \tag{2.44}$$

which is called **scaling**. It is important to note that this form of the cutoff-independence generally includes higher order contributions to the β function. For sufficiently small couplings $\beta(g)$ should gradually approach the 2 loop expression given in (2.37), so that

$$\sigma a = \text{const. } \exp\Big[-(2\beta_0 g^2)^{-1} + (\beta_1/2\beta_0^2)\, \ln(\beta_0 g^2) \Big]. \tag{2.45}$$

The approximation where the $O(g^2)$ corrections in eq.(2.40) are neglected is called **asymptotic scaling**.

2.4 (Euclidian) Thermodynamics of pure Yang-Mills LGT

Using the relations (2.3), the following expressions for thermodynamic quantities can be derived by replacing the usual derivatives (with respect to β $(=T^{-1})$ and V) by derivatives with respect to a_σ and ξ with N_σ and N_τ fixed [8]. In the following analysis, the lattice action will always be given by (2.4).

The energy density is defined by

$$\varepsilon_G^E = -\frac{1}{V}\left(\frac{\partial}{\partial\beta}\ln Z_E\right)_V$$

$$= (\xi^2/N_\sigma^3 N_\tau a^4)\left(\frac{\partial}{\partial\xi}\ln Z_E\right)_a \qquad (2.46)$$

where we used $\left.\frac{\partial}{\partial\beta}\right|_V = -(\xi^2/N_\tau a_\sigma)\left(\frac{\partial}{\partial\xi}\right)_{a_\sigma}$.

Now $\left(\frac{\partial}{\partial\xi}\ln Z_E\right)_a$

$$= Z^{-1}\int [dU]\left(\frac{\partial}{\partial\xi}\, e^{-S_G(U)}\right)_a$$

$$= -Z^{-1}\int [dU]\, e^{-S_G(U)}\left(\frac{\partial}{\partial\xi}\, S_G(U)\right)_a$$

$$= -Z^{-1}\int [dU]\, e^{-S_G(U)}\Big[(\partial K_\sigma/\partial\xi)\sum_{\{P_\sigma\}}[1 - N^{-1}\mathrm{Re}\ \mathrm{tr}UUU^\dagger U^\dagger]$$

$$+ (\partial K_\tau/\partial\xi)\sum_{\{P_\tau\}}[1 - N^{-1}\mathrm{Re}\ \mathrm{tr}UUU^\dagger U^\dagger]\Big]$$

$$= -2N(3N_\sigma^3 N_\tau)[(\partial K_\sigma/\partial\xi)\langle P_\sigma\rangle + (\partial K_\tau/\partial\xi)\langle P_\tau\rangle] \qquad (2.47)$$

where $\langle P_\sigma\rangle \equiv \bar{P}_\sigma$

$$= (3N_\sigma^3 N_\tau Z_E)^{-1}\int [dU]\, e^{-S_G(U)}\det{}^n Q \sum_{\{P_\sigma\}}[1 - N^{-1}\mathrm{Re}\ \mathrm{tr}UUU^\dagger U^\dagger] \qquad (2.48)$$

is the thermodynamic average (2.8) of the space-like plaquettes (and similarly for $\bar{P}_\tau$).

The gluon energy density for SU(N) is therefore given by

$$\varepsilon_G^E a^4 = -\xi^2 6N[(\partial K_\sigma/\partial\xi)_a \langle P_\sigma\rangle + (\partial K_\tau/\partial\xi)_a \langle P_\tau\rangle]. \qquad (2.49)$$

The derivatives above can now be evaluated using the Weak Coupling expressions of the coupling constants (2.10):

$$\begin{aligned}(\partial K_\sigma/\partial\xi) &= \frac{\partial}{\partial\xi}\left[\, g_\sigma^{-2}(a_\sigma,\xi)\xi^{-1}\,\right]_a \\ &= -\xi^{-2}\, g_\sigma^{-2}(a_\sigma,\xi) + \xi^{-1}c'_\sigma(\xi); \\ (\partial K_\tau/\partial\xi) &= \frac{\partial}{\partial\xi}\left[\, g_\tau^{-2}(a_\sigma,\xi)\xi\,\right]_a \\ &= g_\sigma^{-2}(a_\sigma,\xi) + \xi c'_\tau(\xi). \end{aligned} \qquad (2.50)$$

Substituting the above into (2.46) we finally have

$$\varepsilon_G^E a^4 = 6N[g^{-2}(\langle P_\sigma\rangle - \langle P_\tau\rangle) - (c'_\sigma\langle P_\sigma\rangle + c'_\tau\langle P_\tau\rangle)]. \qquad (2.51)$$

To obtain the expression for the physical gluon energy density, we use the fact that for the finite temperature formalism (using an asymmetric lattice) the vacuum contribution can be approximated by subtracting from any given quantity the corresponding expresssion calculated on a (sufficiently large) symmetric lattice (with $N_\sigma = N_\tau$) [8].

Using this approximation the physical gluon energy density is given by

$$\varepsilon_G = \varepsilon_G^E - \varepsilon_G^{vac}$$

$$\text{with} \quad \varepsilon_G a^4 = 6N[g^{-2}(\bar{P}_\sigma - \bar{P}_\tau) - c'_\sigma(\bar{P} - \bar{P}_\sigma) + c'_\tau(\bar{P} - \bar{P}_\tau)] \qquad (2.52)$$

where we used the notation $\langle P\rangle \equiv \bar{P}$. For a symmetric lattice $\bar{P}_\sigma = \bar{P}_\tau = \bar{P}$. Getting rid of the lattice units the SU(N) physical gluon energy is given by

$$\varepsilon_G/T^4 = 6NN_\tau^4[g^{-2}(\bar{P}_\sigma - \bar{P}_\tau) + c'_\sigma(\bar{P} - \bar{P}_\sigma) + c'_\tau(\bar{P} - \bar{P}_\tau)]. \qquad (2.53)$$

To avoid any possible ambiguity, the notation ß will in future be used explicitly to denote the SU(N) gauge coupling, i.e. $ß \equiv 2N/g^2$.

3. String Tension

The calculation of the string tension is important because the string tension between a quark-antiquark pair is directly linked to confinement in the flux tube (or string) model (see Chapter 5).

For confinement there must exist a long range term in the static interquark potential. In Chapter 5 we have shown that for a confining theory, the (planar) Wilson loop expectation values behave like

$$W[R,T] = \text{const.}\exp[-TV(R) - bL + c] \tag{3.1}$$

where $$V(R) = \lim_{T\to\infty} V_T(R) \tag{3.2}$$

is the static $q\bar{q}$ potential with upper bound

$$V_T(R) = -\ln\left(\frac{W[R,T]}{W[R,T-1]}\right) \tag{3.2a}$$

and $L = 2(R+T)$ is the perimeter of the loop. This term is due to the self energy of the $q\bar{q}$ pair.

Several methods exist in the literature to extract the static potential from W[R,T]. Essentially they only differ in the way that the self energy contribution is eliminated. The string tension is obtained from the asymptotic behaviour of the potential V(R) for large R. For Monte Carlo (MC) calculations the value of R is limited by the size of the lattices that are used so that some extrapolation method must be applied.

In most calculations using planar Wilson loops the long distance part of the potential (i.e. without the self energy term) is fitted with the form

$$V(R) \underset{\infty}{\overset{R}{\to}} \sigma R + \alpha/R \tag{3.4}$$

where the Coulomb-like correction is predicted by the flux-tube (string) model [12], which for d = 4 gives

$$\alpha = -\pi/12 \ . \tag{3.5}$$

The most general form for the Wilson loop (in d dimensions) is given by [36]

$$W[R,T] = \exp\Big[-\sigma RT - m(R+T) + c$$

$$(d-2)\Big(\frac{\pi}{24}\frac{T}{R} + \ln\big(\frac{R}{4}\big) + \tfrac{1}{2}\sum_{n=1}^{\infty} \ln[1 - \exp(2n\pi T/R)]\Big)\Big] \quad . \quad (3.6)$$

The last term is only dependent on the number of transverse dimensions [=(d-2)]. In the earliest calcultation Creutz [10a;b] proposed the ratio ("Creutz ratio")

$$\chi(R,T) \equiv -\ln\Big[\frac{W[R,T]W[R-1,T-1]}{W[R-1,T]W[R,T-1]} \Big] \quad (3.7)$$

to eliminate the self energy contribution. The string tension is then given by

$$\sigma = \lim_{R,T\to\infty} \chi(R,T). \quad (3.8)$$

This follows if we only consider the first three terms in (3.6) and set c = 0, which gives

$$\chi(R,T) = \sigma. \quad (3.9)$$

More recent calculations have incorporated the Coulomb-like correction in the static potential. Getting rid of the self energy terms, the static potential is fitted to the form of V(R) in (3.4) (see e.g. [11e]).

Various other fitting procedures have been used to obtain the long range potential from W[R,T] (see individual papers in [10], [11]).A recent proposal has been to use the Polyakov loop (thermal Wilson loop) correlation function defined by

$$\Gamma(s) = \sum_{xyz} \langle P(x,y,z)P^{\dagger}(x,y,z+s)\rangle. \quad (3.10)$$

The potential (3.4) is obtained by fitting (see e.g. [11f])

$$\Gamma(s) = \text{const.}\Big[\exp[-N_T V(s)] + \exp[-N_T V(N_Z - s)]\Big] \quad (3.11)$$

where s is assumed to be large and the second term is due to the periodicity in the chosen direction (here z). For large s, V(s) is expected to behave like

$$V(s) \approx \sigma s \qquad (3.12)$$

with σ the required string tension.

Assuming that the MC data are in the scaling region (which turns out to be a yet unresolved problem), the following perturbative estimate for the physical string tension is obtained (after restoring the physical dimensions)

$$a^2\sigma\Big|_{g^2\to 0} = \sigma/\Lambda_L^2 \exp[(-48\pi^2/11Ng^2) - (102/121)\ln(11Ng^2/48\pi^2)] \; . \qquad (3.13)$$

3.1 SU(2) string tension

Billoire and Marinari [10d] and Berg and Billoire [10e] used the correlations between Polyakov loops (with $\vec{p} = 0$ and with [10e] using the Polyakov loops in the adjoint representation) to obtain estimates for the string tension. Their results showed remarkable similarity and, more interesting, both estimates were nearly two times smaller than the earlier ones obtained by Creutz [10a;b] and Bhanot and Rebbi [10c] (see table 3.1).

Measurements of planar Wilson loops by Gutbrod [10g] showed deviations from asymptotic scaling for all Creutz ratios in the interval (with $\beta = 4/g^2$) $2{,}4 \leq \beta \leq 2{,}6$, with the magnitude of the deviations dependent on the geometrical size of the ratios (with ratios formed out of large Wilson loops showing larger deviations than those formed from smaller ones). Calculating Creutz ratios on a large lattice (in order to minimize finite lattice effects) he obtained an even smaller estimate for the string tension than in [10d] and [10e], and also about 50% smaller than the value given in [10f].

In [10h] and [10i] the analysis was repeated for the interval $2{,}5 \leq \beta \leq 2{,}8$ where a similar behaviour was found regarding asymptotic scaling than for the interval in [10g].

In sharp contrast is the result obtained by Koibuchi [10j] using a Langevin simulation, where the value for the string tension was found to be consistent with the earlier estimates in [10a-c], namely

$$\Lambda_L/\sqrt{\sigma} = 0{,}013(1).$$

To obtain the value for Λ_L in table 3.1, the value for the string tension obtained from the Regge slope estimate for α' is used, where

$$\sigma(0) = (2\pi\alpha')^{-1}, \tag{3.14}$$

with $\alpha' = (1\text{GeV})^{-2}$, (3.14a)

so that $\sigma(0) = 0{,}16\ \text{GeV}^2 = \left(400\ \text{MeV}\right)^2$. (3.14b)

Except for the result of [10j], the value for the string tension is much smaller than initially thought. This seems to indicate that either the coupling range used is not yet in the scaling regime, and/or the quantities used (e.g. Creutz ratios) still show too large deviations from asymptotic scaling than expected, even at high ß values [10i].

Table 3.1

Ref.	lattice	$\Lambda_L/\sqrt{\sigma}$	$\sqrt{\sigma}/\Lambda_L$	Λ_L (MeV)
[10a;b]	10^4	0,013(2)	77(10)	5,2(8)
[10c]	16^4	0,011(2)	91(12)	4,4(8)
[10d]*	8^4	0,020(1)	50(4)	8,0(4)
[10e]*	$4^3 \times 32$	0,0185	54(5)	7,4(6)
[10f]	8^4	0,018(1)	56(3)	7,1(4)
[10g]	24^4	0,027(3)	37(3)	10,8(8)
[10h]	24^4	0,0318	31	12,9
[10j]	$8^3 \times 16$	0,013(1)	77(6)	5,2(4)

* results obtained from the zero momentum Polyakov correlation function [see (3.10)]

3.2 SU(3) string tension

The same problems that plague the SU(2) string tension calculations are also present for the case of SU(3). In addition, the complexity of the SU(3) parametrization means that even larger lattices are needed for the calculation of quantities like the Wilson loops [for SU(2) the Pauli matrices provide a relative simple parametrization]. Keeping in mind that the largest lattices that have been used up till now are of the order $12^3 \times 16$ - $24^3 \times 48$, this imposes serious limitations on the sizes of the Wilson loops that can be calculated.

In one of the first calculations, Creutz and Moriarty [11a] obtained a value for σ consistent with the first estimate given in [10b], namely

$$\sqrt{\sigma}/\Lambda_L = 167(24). \qquad (3.15)$$

[the value quoted in [10b] was $\sqrt{\sigma}/\Lambda_L = 200(46)$]

This is shown in **fig. 3.1**, together with the estimates from different order SC expansions (see Chapter 6).

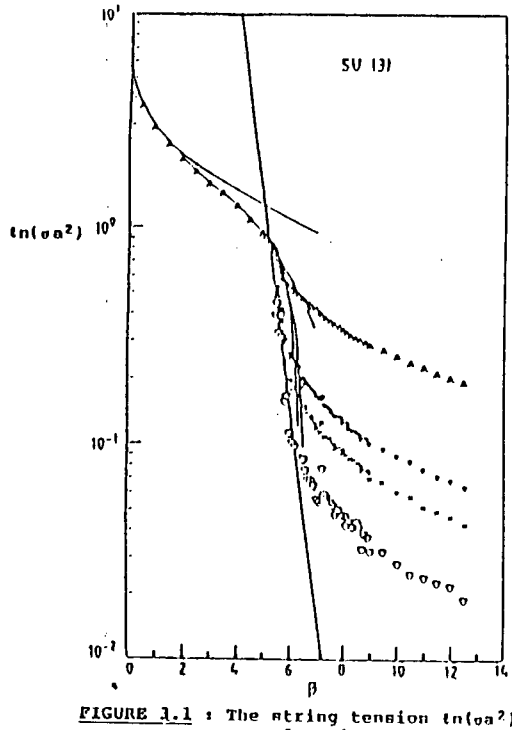


FIGURE 3.1 : The string tension $\ln(\sigma a^2)$ as a function of the coupling β. Earlier data from [11a] as well as from SC expansions to different orders.

More recent calculations have however given estimates for σ which are far lower than the values given above, suggesting that the earlier

More recent calculations have however given estimates for σ which are far lower than the values given above, suggesting that the earlier results suffered heavily from finite size effects. All results are shown in table 3.2.

Parisi et al. [11b] used the connected two-part (zero momentum) Polyakov loop correlation function (3.10), but with the loops in one of the spatial directions (which is chosen to be periodic, i.e. P forms a closed loop).

De Forcrand et al. [11f] used a "source method" to supplement the Polyakov loop correlations: by measuring the loop-loop correlations, σ is determined at a finite physical temperature (the string tension should decrease as the temperature of the system is increased).

Essentially all other methods are based on the calcultions of conventional Wilson loops, with a linear-plus-Coulomb form for V(R)[as in (3.4)]. The $q\bar{q}$ potential is then extracted e.g. by either fitting it directly to the form in (3.4) [11e], or by fitting the Creutz ratio to

$$\chi(R,T) = F(\sqrt{R(R-1)}) + b(R)/T(T-1) \qquad (3.16)$$

where the first term is the potential from which the self energy contribution has been eliminated (and interpreted as the static force between the $q\bar{q}$ pair) while the second term represents the Coulomb-like behaviour [11d]. The force thus obtained is then fitted to the form of the potential in (3.4).

Table 3.2

Ref.	$\beta=6/g^2$	$\sqrt{\sigma}a$	$a^2\sigma$	lattice	$a\Lambda_L(10^{-3})$	$\sqrt{\sigma}/\Lambda_L$
[11a]				6^4		167(24)
[11b]	6,0	0,197(10)	0,039(4)†	$10^3 \times 20$	2,346	84(4)
[11c]	5,40	0,755(13)	0,57(2)	16^4	4,600	164(3)
	5,70	0,469(20)	0,22(4)		3,287	150(15)
[11d]	5,60	0,530(8)	0,281(9)	$8^3 \times 12$	3,679	
	5,80	0,332(3)	0,111(3)		2,938	
	6,00	0,246(3)	0,061(2)		2,346	
	6,20	0,189(2)	0,036(2)		1,872	
	6,40	0,155	0,024		1,480	104*
[11e]	6,00	0,250(8)**		$12^3 \times 16$	2,346	104(2)
[11f]	5,50	0,583(13)†	0,340(15)	$6^3 \times 12$	4,112	142(3)
	5,70	0,367(7)	0,135(5)	$8^3 \times 16$	3,287	112(2)
	5,90	0,253(9)	0,064(5)	$10^3 \times 20$	2,626	96(4)
	6,00	0,205(8)	0,042(3)	$12^3 \times 24$	2,346	87(4)
[11g]	6,00	0,22(2)			2,346	94(9)
	6,30	0,15(2)			1,673	90(12)
[11h]	6,00	0,214	0,046	16^4	2,346	91
	6,30	0,132	0,0173	$24^3 \times 48$	1,673	79
[11i]	6,00	0,230	0,053***	$10^3 \times 20$	2,346	98
[11k]	5,90			$10^3 \times 32$	2,626	93(1)
[11l]	6,00	0,209(3)	0,0439(11)	$16^3 \times 10$	2,346	89(1)

* average for β in the interval 6,0 - 6,4

** obtained from fit in $\beta \in (6,0;6,4)$

*** obtained from a linear fit of the results in [11b]

† obtained from the zero momentum Polyakov correlation function [see (3.10)]

Remarks

(i) The results quoted here for $\sqrt{\sigma}$ of [11d] were obtained from [11f] as no explicit results are given there .

(ii) From the values for $\sqrt{\sigma}/\Lambda_L$ we conclude that asymptotic scaling only sets in for β (at least) $> 6,0$.

However, results from Monte Carlo Renormalization Group calculations suggest that scaling might set in for $\beta \geq 5,7$ (see [11g] and references therein).

(iii) The result in [11i] was obtained from the data of [11b] using a different fitting procedure .

(iv) The "best" current estimate for the SU(3) string tension therefore seems to be

$$\sqrt{\sigma}/\Lambda_L = 95 \pm 10 \ . \qquad (3.17)$$

4. Pure Yang-Mills Deconfinement Phase Transition (Quenched Theory)

In pure LGT the deconfinement phase transition (d.p.t.) can be studied in two ways:

(i) by looking for a discontinuity (sudden change) in the behaviour of thermodynamical quantities like the energy density as a function of the coupling or temperature;

(ii) by using a specific order parameter to distinguish between the two phases, e.g. $\langle L \rangle$.

It must be kept in mind,however, that only L has a direct relation to confinement as ε_G is only the internal energy of the gluon gas.

Although in the analysis below the Wilson form of the pure gauge sector action has been used, it is important to keep in mind that deconfinement has been shown to be independent of the choice of action (see Chapter 3, Section 5).

4.2 SU(2) Quenched Theory

Calculations of the SU(2) theory provided the first numerical evidence of a d.p.t. in Lattice Gauge theories [5a;b].

Investigation of the behaviour of the order parameter $\langle L \rangle$ at the transition point showed a continuous change characteristic of a second order transition. In [5b] this analysis was repeated with the interesting result that for $N_\tau = 2$ and 3 the approximants to β_c were more or less the same, which suggested that the scaling region had been reached (i.e. the continuum limit is well represented by $N_\tau = 3$).

In [13a;b] and [8a] the energy density εa^4 [(2.52)] was calculated on a symmetric lattice with $N_\sigma = 10$, $N_\tau = 2$-4.
Perturbation theory in the high temperature limit gives the following result for the energy density

$$\varepsilon = (N_g \pi^2/15) T^4 [1 - \alpha_s (5N_c/\pi) + \alpha_s^{3/2} (80/\sqrt{3}) (N_c/\pi)^{3/2} \pm \ldots] \quad (4.1)$$

where $N_g = N_c^2 - 1$ and $N_c(SU(2)) = 2$.

For sufficiently high temperatures the SU(2) energy density should therefore attain the Stefan-Boltzmann limit since the coupling constant is a decreasing function of T

$$\varepsilon_{SB} = \frac{1}{5}\pi^2 T^4 . \qquad (4.1a)$$

To be consistent, the lattice results should also share this behaviour. This was indeed found to be the case . In **fig. 4.1** the normalized energy density $(\varepsilon/\varepsilon_{SB})$ is shown as a function of the temperature T, evaluated on a $10^3 \times 3$ lattice.

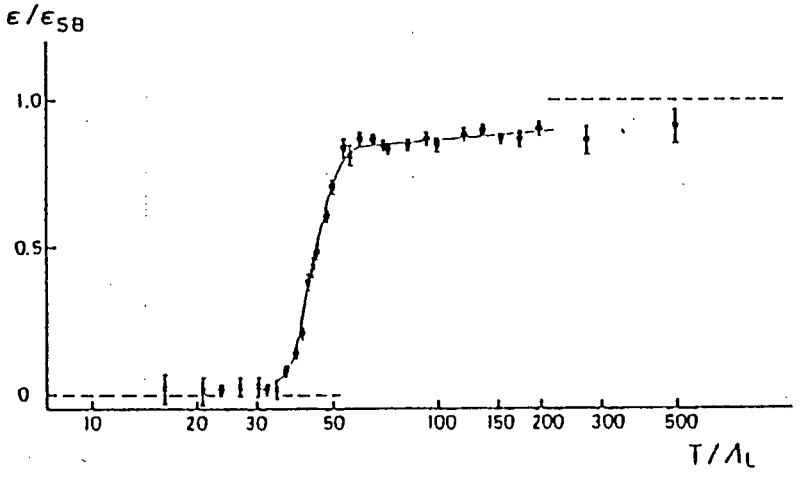


FIGURE 4.1 : The ratio $\varepsilon/\varepsilon_{SB}$ for a $10^3 \times 3$ lattice versus temperature. (From [8a])

The following remarks are in order.

(i) For high temperatures ($T/\Lambda_L \geq 100$) the MC results correspond well with the anticipated Stefan-Boltzmann limit which on the lattice reads

$$\varepsilon_{SB} a^4 = \pi^2/(5N_T^4) \qquad (4.2)$$

i.e. the deconfined gluon gas behaves like a gas of non-confining particles.

(ii) At around $T = 40\Lambda_L$ the energy density shows a rapid change over a relatively small temperature interval. The shape of the transition is characteristic of a second order transition (i.e rapid change but not a singularity-like behaviour). This is in agreement with theoretical predictions [5d], where it was shown that the SU(2) system is in the

same universality class as the 3-dimensional Ising model which is known to have a second order phase transition .

To determine the critical temperature the behaviour of the quantity

$$\partial(\varepsilon/\varepsilon_{SB})/\partial(4/g^2)$$

is considered which coincides with the specific heat c_V in the thermodynamic limit.

The location of the critical temperature is deduced from the peak in $\partial(\varepsilon/\varepsilon_{SB})/\partial(4/g^2)$. For a $10^3 \times 4$ lattice, c_V showed a peak at $\beta_c = 2{,}28$. Assuming asymptotic scaling holds, we have $T_c = 40(2)\Lambda_L$. Using the lower estimate for Λ_L [10a-c;j], namely $\Lambda_L = 5{,}2(8)$, we obtain

$$T_c = 208(10)\ \text{MeV}, \qquad (4.3a)$$

while the higher estimate [10d-h] $\Lambda_L = 7{,}1(4)$ gives

$$T_c = 284(10). \qquad (4.3b)$$

Further results are summerized in table 4.1.

Table 4.1

Ref.	lattice (N_τ, N_σ)	$\beta_c = 4/g_c^2$	T_c^* (MeV)
[5b]	(1,5)	0,75	310
	(2,5)	1,8	230
	(3,6)	2,15	200
[13a;8b]	(2,10)	1,90	161
	(3,10)	2,19	224
	(4,10)	2,28	208(10)

* Using $\Lambda_L = 5{,}2(8)$ MeV

4.2 Quenched SU(3) Deconfinement Phase Transition

For the SU(3) Yang-Mills theory, MC calculations [14] have established that the deconfinement phase transition is of first order.

This is in agreement with theoretical considerations: the pure SU(3) gauge system resembles a 3-state Potts (3-dimensional spin) model, which is known to have a first order phase transition [5d;e].

As the quenched theory has been reviewed in the literature [3b;c],[14e] we will only mention a few results and rather concentrate on the issue of perturbative scaling.

In most cases, a high statistics MC evaluation was carried out to ensure that the d.p.t. is independent of the spatial dimensions of the lattice. The SU(3) gauge theory in the confined phase can be in one of three physically equivalent Z_3 modes (2.27a)

$$\bar{L}/|\bar{L}| = 1;\ \exp[2\pi i/3]\ ;\ \exp[4\pi i/3] \tag{4.4}$$

where $\bar{L}$ is the lattice average over all spatial sites $\vec{x}$, i.e.

$$\langle L\rangle \approx \bar{L} = \lim \frac{1}{I} \sum_{i=1}^{I} L(i) \tag{4.5}$$

and i denotes a given configuration of links $\{U_{x,\mu}\}$.

One of the characteristics of a first order transition is that the system can coexist in both phases at the transition point. This characteristic was indeed found to be present in the behaviour of the order parameter <L>, calculated as a function of the number of iterations, starting in each case (corresponding to a specific value of the coupling $\beta = 6/g^2$) from a completely ordered (i.e. $U_{x,\mu} = 1$, all x) and completely random (i.e all $U_{x,\mu}$ different) configuration [14].

For a $8^3 \times 3$ lattice this shown in **fig. 4.2.1** [14e]. At $\beta = 5{,}5531$ ($T/\Lambda_L = 86$) there is a clear two state signal, i.e. $\beta_c = 5{,}5531$.
In **fig. 4.2.2** the energy denisty ε/T^4 as a function of the coupling β is given for an $8^3 \times 3$ lattice. This clearly shows that the transition is of first order (i.e a very abrupt change in ε/T^4) at the same value of β that was obtained from the investigation of the order parameter in

fig. 4.2.1. As was the case for SU(2), the energy density approaches the Stefan-Boltzmann limit.

$$\varepsilon/T^4 = (N^2-1)/15 = (8/15)\pi^2 \qquad (4.6)$$

from below.

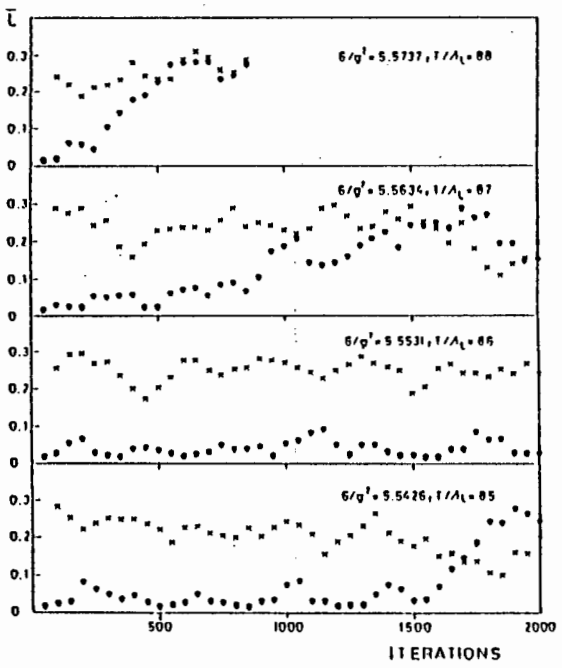


FIGURE 4.2.1 : Polynkov loop average $\bar{L}$ as a function of the number of iterations after an ordered (crosses) and random (circles) start (from [14d])

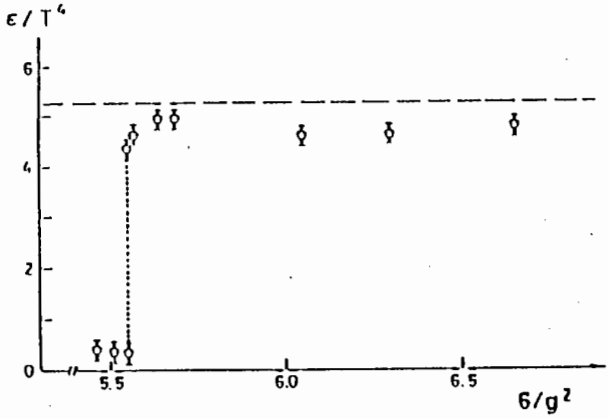


FIGURE 4.2.2 : Energy density (ε/T^4) as a function of ($6/g^2$) evaluated on an $8^3 \times 3$ lattice (from [14e]).

The problem of locating the onset of the scaling regime in SU(3) is of fundamental importance since it is only in this limit that continuum physics can be extracted from lattice calculations.

For SU(3) the asymptotic dependence between the lattice spacing a and the coupling g is given by the RGE

$$\Lambda_L a = \exp[-(8\pi^2/11g^2) - \frac{51}{121}\ln(11g^2/16\pi^2)] \qquad (4.7)$$

or in terms of ß

$$\Lambda_L a = \exp[-(\frac{4\pi}{33})^2 ß + \frac{51}{121}\ln(\frac{8}{33}\pi^2 ß)]. \qquad (4.8)$$

For sufficiently weak coupling the functional dependence between the critical temperature and coupling should be given by

$$T_c/\Lambda_L = N_\tau^{-1} \exp[-(\frac{4\pi}{33})^2 ß - \frac{51}{121}\ln(\frac{8}{33}\pi^2 ß)]. \qquad (4.9)$$

The requirement that the physical deconfinement temperature T_c be independent of the lattice spacing in the scaling regime implies that

$$T_c/\Lambda_L = \text{constant} \quad \text{(i.e independent of } \beta\text{)} .$$

In table 4.2 the most recent results for the critical couplings and their approximants are sumed up.

Table 4.2

Ref.	N_σ	N_τ	β_c	T_c/Λ_L
[14e]	8;10;12	2	5,11(1)	78(1)
	8;10	3	5,55(1)	86(1)
	8;10	4	5,70(1)	76(1)
[14g]	12	5	5,79 - 5,82	68,5±1
	16	6	5,92 - 5,94	65,5±1
[14h]	∞	2	5,097(1)*	77,5
	∞	4	5,696(4)	76
	∞	6	5,877(6)	62
	∞	8	6,00(2)	53(1)
	11	10	6,09(3)	47(1)
[14i]	19;13	8	6,02(2)	54(1)
	17	10	6,15(3)	50(1)
	17;19	12	6.32(3)	51(2)
	19	14	6,47(3)	52(2)
[14j]	16	10	6,065(27)	46(1)
	16	12	6,261(20)	48(1)
	16	14	6,355(26)	45(1)
[14l]	8	4	5,67(1)	73
	10	5	5,79(1)	68

* value of β_c extrapolated to a lattice of infinite spatial dimension

In [14g] a pronounced non-scaling behaviour was found for $\beta \leq 6,1$, while [14i] reported asymptotic scaling only for $6,15 < \beta_c < 6,50$. This result was (more or less) confirmed by [14j] where asymptotic scaling was observed for $\beta > 6,07$.

In **fig. 4.2.3** the results for the critical temperature T_c as a function of β_c are plotted. This clearly shows that scaling can only be considered to have set in for $\beta > 6$ (i.e. $g^{-2} > 1$), which corresponds to $N_\tau \geq 8$.

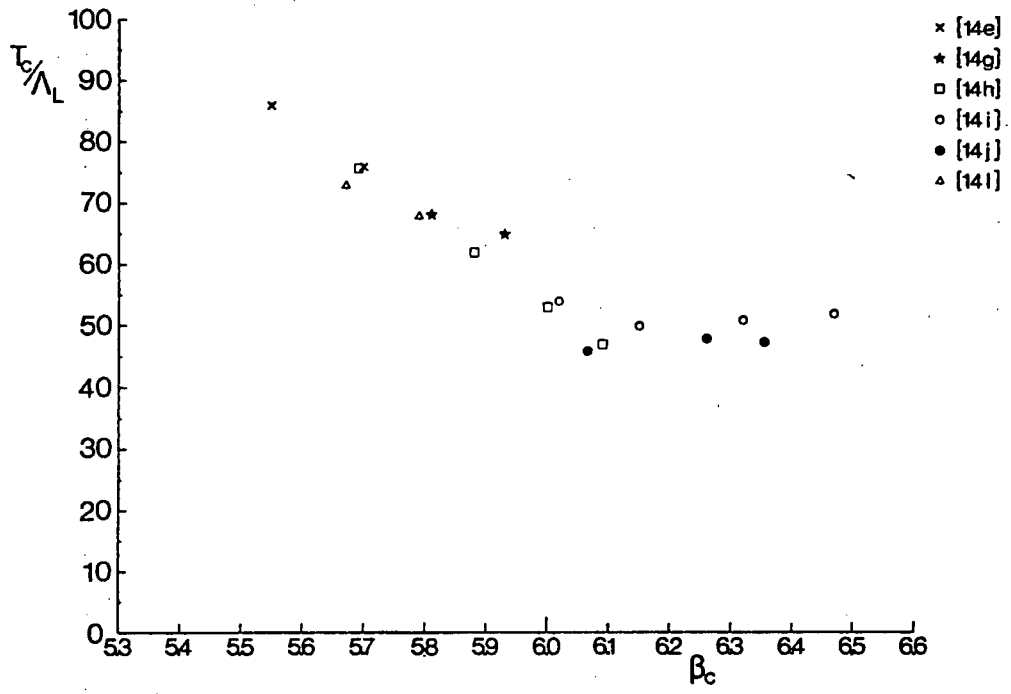


fig. 4.2.3: Schematic diagram showing the possible onset of scaling in the values of the critical temperature.

A lower bound for the onset of asymptotic scaling is thus given by [14k]:

$$\beta \geq 6,2. \qquad (N_\tau \geq 12) \qquad (4.10)$$

The violation of asymptotic (2-loop perturbative) scaling in the interval $5,7 < \beta_c < 6,2$ seems to be independent of asymmetry [14k], i.e. the violations are not due to the particular lattice regularization chosen. This suggests that the coupling range $5,7 < \beta_c < 6,2$ is a regime of non-perturbative but universal scaling.

Comparison of the above with the results obtained earlier for the string tension shows a correspondence in the general behaviour as a function of

the coupling g. It must however be kept in mind that the question of scaling has not yet been resolved (as the results of [14h;i] clearly show).

To obtain a rough estimate for the relation between the critical temperature and the string tension we use the ß = 6,3 results from [11g;h] and [14i;j] which gives

$$T_c/\sqrt{\sigma} \approx 0{,}56. \tag{4.11}$$

If we now make the approximation of using the SU(2) phenemological value for the string tension (3.16), i.e.

$$\sqrt{\sigma} \approx 400 \text{ MeV},$$

then the critical temperature for the quenched SU(3) Yang-Mills system is given by

$$T_c \approx 224 \text{ MeV}, \tag{4.12}$$

which is in qualitive agreement with the quenched SU(2) result.

It is therefore likely that the critical temperature will be in the interval

$$200 \leq T_c \leq 240. \tag{4.13}$$

5. Including Dynamical Quarks - Full QCD

5.1 Introduction

At large distances (i.e. the confinement regime) the flux tubes describing the $q\bar{q}$ potential are broken by virtual $q\bar{q}$ pairs. In addition, large Wilson loops will now tend to follow a perimeter law. For calculational purposes the pure gauge theory has to be adapted to provide for the inclusion of virtual quarks, which gives rise to the following problems.

(i) The presence of the quark term in the action breaks the global invariance of the original pure Yang-Mills action (described in section 2.2) under the centre of the gauge group. This is due to the fact that the anti-periodic boundary conditions of the fermion fields explicitly breaks the Z(N) symmetry of the original action. As we have seen earlier, the d.p.t. in the quenched approximation is associated with the spontaneous breakdown of the global Z(N) symmetry, with order parameter given by the Polyakov loop L.

Because Z(N) is no longer a "good" symmetry, L cannot serve as order parameter for deconfinement. Production of virtual $q\bar{q}$ pairs leads to the situation where $\langle L \rangle$ need not be zero (even at low temperatures) due to their effect of effectively screening the colour field. $\langle L \rangle$ is therefore not a suitable order parameter to distinguish between the two phases, so that the deconfinement transition can only be investigated by looking for sudden changes in the behaviour of physical observables, e.g. the energy density.

(ii) To evaluate the fermion matrices large lattices are needed due to the non-local nature of the fermion determinant. As the lattice sizes used in calculations are relatively small (due to computer limitations), the influence of the finite lattice size effects on the results obtained must always be kept in mind.

5.2 Formalism

In continuum field theory the QCD Lagrangian for massless quarks (one flavour) is given by

$$\mathcal{L}(\psi,A) = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \bar{\psi} (i\gamma_{\mu}\partial^{\mu} - gA_{a}\lambda^{a})\psi \qquad (5.1)$$

with $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f_{abc} A^b_\mu A^c_\nu$. (5.1a)

The corresponding (Euclidian) action is

$$S(\psi,A) = -\int_0^\beta d\tau \int d^3x \, \mathcal{L}(\psi,A) \tag{5.2}$$

with periodic (anti-periodic) boundary conditions for the gauge (fermion) fields respectively.

The full SU(N) QCD partition function is given by

$$\begin{aligned} Z_E &\equiv Z_E(N_\sigma, N_\tau, g) \\ &= \int \Big(\prod_{x,\mu} dU_{x,\mu}\Big)\Big(\prod_x d\psi_x d\bar{\psi}_x\Big) \exp[-S(U,\psi,\bar{\psi};g,m,\mu;\xi)] \\ &\equiv \int [dU][d\psi\, d\bar{\psi}]\, e^{-S} \end{aligned} \tag{5.3}$$

with $S = S_G + S_F$ (5.3a)

where S_G is the gluon action defined in (2.4) and S_F is the fermion action (Chapter 4) which may describe Wilson or Staggered fermions.

Wilson fermions on an asymmetric lattice are defined by

$$\begin{aligned} S_F^W &\equiv \sum_{i=1}^{N_f} \sum_{x,x'} \bar{\psi}^i_x Q^i_{x,x'} \psi^i_{x'} \\ &= \sum_{i=1}^{N_f} \sum_{x,x'} \bar{\psi}^i_x [\delta_{x,x'} - K^i_\tau (M^{(4)})_{x,x'} - K^i_\sigma (\sum_{j=1}^{3} M^{(j)})_{x,x'}] \psi^i_{x'} \end{aligned} \tag{5.4}$$

where $(M_\mu)_{x,x'} = U^\mu_x (1-\gamma_\mu)\delta_{x,x'-\mu} + U^{\mu\dagger}_{x-\mu}(1+\gamma_\mu)\delta_{x,x'+\mu}$. (5.4a)

For an asymmetric lattice the couplings K^i_τ and K^i_σ are defined by [8b]

$$\begin{aligned} K^i_\tau &= \tfrac{1}{2}(m a_\tau + 3\xi^{-1} + 1)^{-1} \\ &\equiv \tfrac{1}{2} k_\tau(g^2,\xi) \frac{1}{3\xi^{-1} + 1} ; \end{aligned} \tag{5.5a}$$

$$K_\sigma^i = \tfrac{1}{2}(ma_\tau + 3\xi + \xi)^{-1}$$

$$\equiv \tfrac{1}{2}k_\sigma(g^2,\xi)\,\frac{1}{3+\xi}\,. \qquad (5.5b)$$

Note that for a symmetric lattice ($\xi = 1$) the above reduces to

$$K_\tau^i = \frac{1}{8}\,k_\tau(g^2,\xi)$$

$$= \frac{1}{8}\,k_\sigma(g^2,\xi) = K_\sigma^i \equiv K^i, \qquad (5.5a)$$

where K^i is the "usual" hopping parameter defined in Chapter 4. Also, for non-interacting quarks, $k_\sigma(g^2,\xi)$ and $k_\tau(g^2,\xi)$ are equal to unity [8b].

For sufficiently small g^2, the ξ dependence of $k(g^2,\xi)$ is small [15]:

$$k(g^2,\xi)\Big|_{g^2\to 0} = 1 + f(\xi)g^2 + O(g^4). \qquad (5.6)$$

Hence, for small g^2 the hopping parameter $K_{\sigma,\tau}^i$ can be approximated by

$$K\big|_{\xi=1} \approx \frac{1}{8}\left(1 + 0{,}11g^2 + O(g^4)\right) \qquad (5.7)$$

which follows from $f(1) \approx 0{,}11$ [15].

Staggered fermions on an asymmetric lattice are defined by

$$S_F^S \equiv \sum_{f=1}^{n} \sum_{x,x'} \bar{\chi}_f(x) Q^f_{x,x'} \chi_f(x')$$

$$= \sum_{f=1}^{n} \sum_{x,x'} \bar{\chi}_f(x)\left[a_\sigma^{-1} \sum_{i=1}^{3} D^{(i)}_{x,x'} + a_\tau^{-1} D^{(4)}_{x,x'} + m_f\delta_{x,x'}\right] \chi_f(x') \qquad (5.8)$$

where $D^{(\mu)}_{x,x'} = \tfrac{1}{2}\eta_\mu(x)\left[U^\mu_x\,\delta_{x,x'-\mu} - U^{\mu\dagger}_{x'}\,\delta_{x,x'+\mu}\right]$

and $D \equiv \sum_{\mu=0}^{3} D^{(\mu)}_{x,x'}$. $\qquad (5.8a)$

For calculation purposes it is convenient to express the partition function (5.3) in terms of purely bosonic variables by integrating out the fermion fields (see App. C).

This leads to the following form of the partition function

$$Z_E = \int [dU]\, e^{-S_{eff}} \tag{5.9}$$

where $e^{-S_{eff}} = e^{-S_G} \det{}^n Q$ (5.9a)

with $n = N_f$ (Wilson fermions)

$n = N_f/4$. (Staggered fermions)

For positive definite Q the effective action can be written as

$$S_{eff} = S_G(U) - n\mathrm{Tr}\, \ln[Q(U)], \tag{5.10}$$

where Tr denotes the trace over colour, space and spin indices.

For Wilson fermions we note that detQ(U)/detQ(1) is positive definite, while for Staggered fermions more care must be taken as Q = D + m is not positive definite: using the shorthand notation

$$S_F^S = \bar{\chi}(D + m)\chi = \bar{\chi}Q\chi$$

we note that $D^\dagger = -D$, so that

$$Q^2 \equiv Q^\dagger Q = (D + m)(-D + m), \tag{5.11}$$

i.e. Q^2 is positive definite.

Hence $n\mathrm{Tr}\, \ln Q = \frac{1}{2} n\mathrm{Tr}\, \ln Q^2$

$$= \tfrac{1}{2} n\mathrm{Tr}\, \ln[-D^2 + m^2]. \tag{5.12}$$

5.3 Physical Observables

For the full theory, the analogue of the thermal expectation value of a physical observable O in (2.8) is given by

$$\langle O \rangle = Z^{-1} \int [dU][d\psi\, d\bar{\psi}]\, O(U,\psi,\bar{\psi})\, e^{-S} \tag{5.13}$$

with $Z = \int [dU][d\psi\, d\bar{\psi}]\, e^{-S}$. (5.13a)

Carrying out the gaussian integration over the fermionic variables gives

$$\langle O\rangle = Z_E^{-1} \int [dU]\, \langle O(U,\psi,\bar{\psi})\rangle_U\, e^{-S_{eff}} \tag{5.14}$$

where $Z_E = \int [dU]\, e^{-S_{eff}}$ (5.14a)

with S_{eff} defined as in (5.9) or (5.10).

$\langle O\rangle_U$ now is the expectation value of O in the presence of a fixed background gauge field $U_{x,\mu}$,

$$\langle O\rangle_U = \int [d\psi\, d\bar{\psi}]\, O\, e^{-S_F} / \int [d\psi\, d\bar{\psi}]\, e^{-S_F}. \tag{5.14b}$$

The Euclidian energy density for the full theory is defined analogously to (2.46), with the difference that Z_E represents the full partition function, i.e.

$$\varepsilon^E = -\frac{1}{V} \left(\frac{\partial}{\partial \beta} \ln Z_E\right)_V$$

$$= (\xi^2/N_\sigma^3 N_\tau a^4) \left(\frac{\partial}{\partial \xi} \ln Z_E\right)_a \tag{5.15}$$

with $\varepsilon^E = \varepsilon_G^E + \varepsilon_F^E$. (non-interacting theory) (5.16)

(a) Wilson fermions

For the gluon energy in the full theory we obtain the same result as for the quenched theory (Sec. 2.4), except that $\bar{P}_\sigma$ is now defined by

$$\bar{P}_\sigma = (3N_\sigma^3 N_\tau Z_E)^{-1} \int [dU]\, e^{-S_G(U)} \det{}^n Q \sum_{\{P_\sigma\}} [1 - N^{-1} \mathrm{Re}\ \mathrm{tr} UUU^\dagger U^\dagger], \tag{5.17}$$

i.e. the averaging is carried out with both the Boltzmann factor and fermion determinant as weights (and similarly for $\bar{P}_\tau$). For $\xi = 1$ the SU(N) physical gluon energy is given by

$$\varepsilon_G/T^4 = 6NN_\tau^4 [g^{-2}(\bar{P}_\sigma - \bar{P}_\tau) + c_\sigma'(\bar{P} - \bar{P}_\sigma) + c_\tau'(\bar{P} - \bar{P}_\tau)]. \tag{5.18}$$

The quark-gluon energy density is defined by (N_f=1)

$$\varepsilon_F^E = -\frac{1}{V}\left(\frac{\partial \ln Z_E}{\partial \beta}\right)_V$$

$$= -(\xi^2/N_\sigma^3 N_\tau a^4 Z_E)\left(\frac{\partial Z_E}{\partial \xi}\right)_a . \qquad (5.19)$$

Using the identity $\det Q = e^{Tr \ln Q}$ we have

$$\left(\frac{\partial}{\partial \xi} Tr \ln Q\right)_a = Tr\left(Q^{-1}\left(\frac{\partial Q}{\partial \xi}\right)_a\right)$$

$$= (\partial K_\tau^i/d\xi)_a Tr(Q^{-1}M^{(4)}) + (\partial K_\sigma^i/d\xi)_a Tr\left(\sum_{j=1}^{3} M^{(j)}Q^{-1}\right). \qquad (5.20)$$

Hence $\varepsilon_F^E a^4 = -(\xi^2/N_\sigma^3 N_\tau Z_E)\int [dU]\ e^{-S_G}\ \det Q \left[(\partial K_\tau^i/d\xi)_a Tr(Q^{-1}M^{(4)})\right.$

$$\left. + (\partial K_\sigma^i/d\xi)_a Tr\left(\sum_{j=1}^{3} M^{(j)}Q^{-1}\right)\right]. \qquad (5.21)$$

For $\xi = 1$ this implies

$$\varepsilon_F^E a^4 = (N_\sigma^3 N_\tau Z_E)^{-1}\int [dU]\ e^{-S_{eff}}\left[(3K^i/4)Tr(M^{(4)}Q^{-1}) + \frac{K^i}{4} Tr\left(\sum_{j=1}^{3} M^{(j)}Q^{-1}\right)\right]$$

(b) Staggered fermions [3c]

For Staggered fermions we use

$$\varepsilon_E \equiv -V^{-1}\left(\frac{\partial \ln Z_E}{\partial \beta}\right)_V$$

$$= -V^{-1}\langle \partial S/\partial(T^{-1})\rangle_V$$

$$= (N_\sigma^3 N_\tau a^3)^{-1}\langle \partial S/\partial a_\tau\rangle \qquad (5.22)$$

where for a non-interacting theory we again have

$\varepsilon^E = \varepsilon_G^E + \varepsilon_G^E$, with Z_E the full partition function [(5.9)].

For an asymmetric lattice the above implies

$$\varepsilon_G^E = (2N/N_\sigma^3 N_\tau a_\sigma^3 a_\tau)\Big[<(a_\tau/a_\sigma g_\sigma^2) \sum_x \sum_{\mu<\nu<4} P_x^{\mu\nu} - (a_\sigma/a_\tau g_\tau^2) \sum_x \sum_{\mu<4} P_x^{\mu 4}>$$

$$+ (a_\tau^2/a_\sigma)(\partial g_\sigma^{-2}/da_\tau)<\sum_x \sum_{\mu<\nu<4} P_x^{\mu\nu}> + a_\sigma(\partial g_\tau^{-2}/da_\tau)<\sum_x \sum_{\mu<4} P_x^{\mu 4}>\Big] \quad (5.23)$$

Using the relations

$$(\partial g_\sigma^{-2}/da_\tau)\big|_{\xi=1} = -a^{-1}c'_\sigma$$

$$(\partial g_\tau^{-2}/da_\tau)\big|_{\xi=1} = -a^{-1}c'_\tau \quad (5.23a)$$

(which are numerically known for SU(N) and different N_f [16]), we have for a symmetric lattice

$$\varepsilon_G^E a^4 = (2N/N_\sigma^3 N_\tau)\Big[g_\sigma^{-2} <\sum_x \sum_{\mu<\nu<4} P_x^{\mu\nu} - \sum_x \sum_{\mu<4} P_x^{\mu 4}>$$

$$- c'_\sigma <\sum_x \sum_{\mu<\nu<4} P_x^{\mu\nu}> - c'_\tau <\sum_x \sum_{\mu<4} P_x^{\mu 4}>\Big] . \quad (5.24)$$

In order to obtain the physical gluon energy density (ε_G) the vacuum contribution must be subtracted . This can be approximated by using the energy density evaluated on a large enough symmetrical lattice with $N_\sigma = N_\tau$ (see section 2.4) , which gives

$$\varepsilon_G^{vac} a^4 = -(N/N_\sigma^4)[c'_\sigma + c'_\tau] \sum_x \sum_{\mu<\nu} P_x^{\mu\nu} \quad (5.25)$$

with $\varepsilon_G \equiv \varepsilon_G^E - \varepsilon_G^{vac}$. (5.26)

The quark-gluon energy density is given by

$$\varepsilon_F^E = (N_\sigma^3 N_\tau a^3)^{-1} <\partial S_F/\partial a_\tau>$$

$$= (N_f/N_\sigma^3 N_\tau a_\sigma^3 a_\tau 4) <\mathrm{Tr}\{a_\tau^{-1} D^{(4)} (a_\sigma^{-1} \sum_{j=1}^{3} D^{(j)} + a_\tau^{-1} D^{(4)} + m)^{-1}\}> \quad (5.28)$$

where we used

$$(\partial S_F/\partial a_\tau) = -n\ \mathrm{Tr}[\ (D+m)^{-1} \frac{\partial}{\partial a_\tau} D]$$

$$= -(N_f/4)\ \mathrm{Tr}[(D+m)^{-1}\ (-a_\tau^{-2}\ D^{(4)})].$$

The physical quark-gluon energy density for a symmetric lattice is given by [3c]

$$\varepsilon_F = \varepsilon_F^E - \varepsilon_F^{vac}$$

$$= (N_f/N_\sigma^3 N_\tau a^4 4)\langle \mathrm{Tr}\, D^{(4)} (\sum_{j=1}^{3} D^{(j)} + m)^{-1} \rangle - 3N_f(16a^4)^{-1}. \qquad (5.29)$$

5.4 Renormalization Group Equation (RGE) for Lattice QCD

For SU(N) QCD with N_f (massless) flavours the RGE is given by (2.40), with universal constants

$$\beta_0 = (11N - 2N_f)/48\pi^2$$

$$\beta_1 = \left[17N^2 - 5NN_f - \frac{3}{2}[(N^2-1)/N]\right] (384\pi^4)^{-1}. \qquad (5.30)$$

Explicitly for SU(3), in terms of the coupling ß, this gives

$$a\Lambda_L = \exp\left[-[4\pi^2\beta/(33-2N_f)] + [(459-57N_f)/(33-2N_f)^2]\log[8\pi^2\beta/(33-2N_f)]\right]. \qquad (5.31)$$

In order to convert estimates from MC calculations from lattice into physical units, we use the following perturbative results for Wilson and Staggered fermions with $N_f \neq 0$.

Table 5.1: Wilson fermions

Ref.	N_f	Λ_{MOM}/Λ_L	$\Lambda_{\overline{MS}}$
[17a]	0	83,5	10,8
	1	89,4	
	2	96,7	
	3	105,8	19,2
	4	117,5	24,7
[17b]	0	83,5	
	3	105,7	
	4	117,0	

Table 5.2: Staggered fermions

Ref.	N_f	Λ_L/Λ_{MIN}	Λ_{MIN}/Λ_L
[17c]	0	0,0922	10,85
	2	0,0605	16,52
	3	0,0339	29,50
	4	0,0347	28,78

Ref.	N_f	$\Lambda_L/\Lambda_{\overline{MS}}$	$\Lambda_{\overline{MS}}/\Lambda_L$
[20f]	0	0,0347	28,8
	2	0,0332	30,1
	3	0,0176	56,8
	4	0,0131	76,3

6. The Deconfinement Phase Transition in the full Theory - Monte Carlo (MC) results

The inclusion of fermions in the quenched theory leads to an action that is highly non-local. In addition to very large lattices needed for the calculations it is of course impossible to simulate Grassmann variables by MC methods. The following algorithms have been used to approximate the full theory:

(a) Hopping parameter expansion (HPE) [18d;e]
(b) Pseudofermions (PF) [18j]
(c) Molecular Dynamics (MD) [18f-i]
(d) Exact algorithm (EA) [18a-c]
(e) Langevin algorithm (LA) [18k-o]
(f) Hybrid algorithm (HA) [18p-t]

(the references above refer to technical discussions of the various methods).

In addition to the deconfinement transition, the restoration of chiral symmetry will also be investigated - we will treat the two cases seperately at first, although in most cases (i.e. for quarks in the fundamental representation), the chiral and deconfinement transitions occur almost coincidentally (see later).

The results for Wilson and Staggered fermions are summarized in tables 6.1 and 6.2 respectively.

6.1 Hopping Parameter Expansion (HPE)

The use of Wilson fermions in calculations has certain advantages and disadvantages. Their virtue is that they include no spurious unphysical states in the zero mass limit and give the correct form of the Dirac Lagrangian in the continuum limit. However, Wilson fermions explicitly break the chiral symmetry (even for zero mass fermions) except in the continuum limit (see Chapter 4). We will now briefly review the HPE and some of the quantities used in this approximation.

The HPE is based on the observation that for Wilson fermions detQ

[= det(1 - KM)] is strictly positive for $0 \leq K \leq 1/8$, i.e. $KM < 1$. (Note for MC purposes the lattice is usually chosen to be symmetric).

An expansion in powers of K (or equivalently m^{-1} as $K \approx m^{-1}$) [18d;e] can therefore be made:

$$N_f \ln \det Q = N_f \mathrm{Tr} \ln(1-KM)$$

$$= -N_f \mathrm{Tr} \sum_{p=1}^{\infty} \frac{K^p}{p} M^p \tag{6.1}$$

where $M \equiv \sum_{\mu=1}^{4} M^{(\mu)}$. (6.1a)

This expansion corresponds to a sum of Wilson loops by virtue of the trace (i.e. only closed loops will contribute) .

For $N_\tau = 2, 3$ the lowest order term in K corresponds to a Polyakov loop, i.e.

$$S_{eff} = S_G(U) + \text{constant}.K^{N_\tau} N_f \sum_{\vec{x}} \mathrm{Re}\, L(\vec{x}) + O(K^4) . \tag{6.2}$$

More explicitly, to fourth order in K we have [18d]

$$S_{eff} \equiv S_G(U) + S_F(L)$$

$$= S_G(U) - 2N_f(2K)^{N_\tau} \sum_{\vec{x}} (L(\vec{x}) + \text{c.c.})$$

$$- 8N_f K^4 \sum_{P_\sigma, P_\tau} \mathrm{tr}(UUU^\dagger U^\dagger) + \text{c.c.}) + O(K^5)$$

$$= S_G(U) - 4N_f(2K)^{N_\tau} \sum_{\vec{x}} \mathrm{Re}\ L(\vec{x})$$

$$- 16N_f K^4 \sum_{P_\sigma, P_\tau} \mathrm{Re\,tr}(UUU^\dagger U^\dagger) + O(K^5) \tag{6.3}$$

with $L(\vec{x}) = N^{-1} \mathrm{tr} \prod_{\tau=1}^{N_\tau} U_{(\vec{x},\tau),4}$. (6.4)

It is interesting to note that the effect of the third term [$O(K^4)$] in the effective action is only to shift the coupling by

$$\beta \to \beta + 16.3\, N_f K^4 \ . \tag{6.5}$$

The effect of the fermions is therefore mainly present in the term containing the Polyakov loop. For small values of the hopping parameter (i.e. large m_q) the quark propagator may be expanded

$$Q^{-1}_{ij}(U) = [1 - K^i M]_{ij}$$

$$= \sum_{p=o}^{\infty} K^p [M_{ij}]^p \tag{6.6}$$

with M again as in (6.1a).

For free Wilson fermions on a symmetric lattice the relation between the hopping parameter K and the corresponding quark masses (m_i) is [3c]

$$\tfrac{1}{2}(K_i^{-1} - K_c^{-1}) = \exp(m_i a) - 1$$

$$\text{with } K_c = 1/8 \ . \tag{6.7}$$

Note that the above is not valid for the interacting theory (i.e. for $U \neq 1$), for which K_c is not known explicitly.

Substituting (6.6) in (5.20) we obtain the following expression for the quark-gluon energy density in the HPE ($\xi = 1$):

$$\varepsilon_F a^4 = \frac{3}{4} (N_\sigma^3 N_\tau)^{-1} \sum_{p=o}^{\infty} K^{p+1} Z_E^{-1} \int [dU]\, e^{-S_{eff}} \Big[\mathrm{Tr}(M^{(4)} M^p) - \frac{1}{3} \sum_{j=1}^{3} \mathrm{Tr}(M^{(j)} M^p) \Big] \ . \tag{6.8}$$

Using the property that only closed loops contribute we obtain the following expression [19d]

$$\varepsilon_F a^4 = N_f [3(2K)^{N_\tau} \mathrm{Re}\bar{L} + 144 K^4 (\bar{P}_\sigma - \bar{P}_\tau) + O(K^5)] \tag{6.9}$$

or in terms of the physical temperature

$$\varepsilon_F/T^4 = N_\tau^4 N_f[3(2K)^{N_\tau}\mathrm{Re}\bar{L} + 144K^4(\bar{P}_\sigma - \bar{P}_\tau) + O(K^5)]. \qquad (6.9a)$$

The gluon energy density is given by (5.18), with $c'_{\sigma,\tau}$ numerically known from [16].

To compensate for finite lattice-size effects we normalize the energy densities to their respective Stefan-Boltzmann limits calculated on lattices of similar sizes.

For the quark-gluon energy density the limiting expression is obtained for the free theory which has $\bar{P}_{\tau,\sigma} = 0$ (because U = 1) and K = 1/8 so that

$$\varepsilon^{SB}a^4 = 3N_f(1/4)^{N_\tau} . \qquad (6.10)$$

For other physical variables in the HPE (e.g. pressure etc.) the reader is refered to [19e].

Monte Carlo results using HPE

In [19b] the normalized gluon energy density (5.18) for SU(3) was calculated on a $8^3 \times 3$ lattice as a function of the coupling ß ($\equiv 6/g^2$) using a fourth order HPE. This is shown in fig. 6.1 (where the result for the quenched theory is also included for comparison).

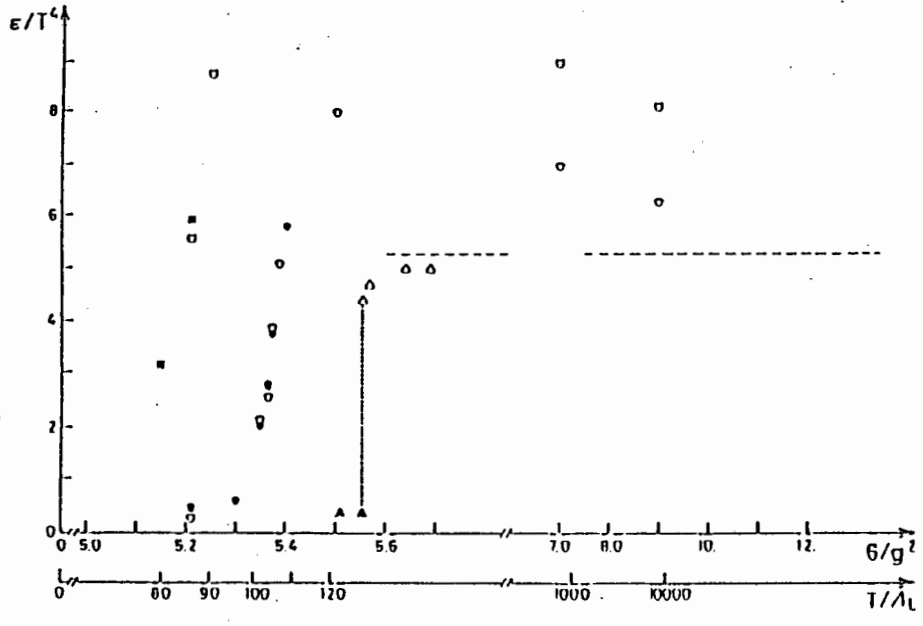


FIGURE 6.1 : Energy density ϵ_G as a function of $6/g^2$ for K = 0.2 (squares) 0.15 (circles), 0 (triangles). Open and solid symbols denote ordered or disordered starts, respectively. The dashed line gives the corresponding ideal gas value (from [19b]).

What clearly emerges is that the deconfinement phase transition now occurs at a lower value of ß (for K = 0,15 , $ß_c \approx 5,38$ while for K = 0,20 $ß_c \approx 5,24$). The RGE (5.31) with $N_f = 1$ gives the following estimates for the critical temperature:

$$K = 0,15 \rightarrow T_c/\Lambda_L = 89,$$

$$K = 0,20 \rightarrow T_c/\Lambda_L = 105. \qquad (6.11)$$

Comparing the above with the result for the pure gauge theory (calculated on a lattice with the same size), i.e. $T_c/\Lambda_L = 86$, one sees that T_c seems not to have been effected much by the inclusion of dynamical fermions into the theory. From the form of ε one also sees that the phase transition now seems to be of second order (i.e. ε shows a continuous change for the interval around $ß_c$).

Using the Strong Coupling value for K_c, i.e. $K_c = 0,25$, the following values for the quark masses are obtained using (6.7):

$$m_q a \approx 0,840 \quad (K = 0,15)$$
$$m_q a \approx 0,40 \quad (K = 0,20) . \qquad (6.12)$$

For a $N_\tau = 3$ lattice this gives

$$m_q/T_c = 2,5$$
$$m_q/T_c = 1,2 \qquad (6.13)$$

respectively.

The effect of higher order terms in the HPE can be seen in **fig. 6.2** where the total (normalized) energy density $\varepsilon/\varepsilon_{SB}$ is plotted for different values of the coupling using a fourth and fifth order HPE on a $8^3 \times 3$ lattice [19d].

The effect of the higher order terms is to shift $ß_c$ to lower values - the fourth order results should therefore be considered as an upper bound. Further results are given in table 6.1.

Hasenfratz et al. [19c] studied the order parameter L on a $8^3 \times 2$ lattice with $N_f = 3$. They found that the first order transition of the quenched theory gradually weakened as K was increased, and for K > 0,055

(ma < 2,1) it disappeared completely. Although this result has been verified using analytical techniques [29] (see Sec. 9), it must be kept in mind that only a lowest order HPE was used.

Table 6.1 (from [19d])

N_τ	β_c	O(K)	T_c/Λ_L (N_f=2)
3	5,300(50)	4	152(10)
4	5,575(25)	4	162(5)
5	5,725(25)	4	157(5)
3	5,250(50)	5	142(10)

All other approximations use Staggered fermions, as they are well suited for the study of chiral symmetry. The chiral limit is at m = 0 with m the bare quark mass. For Wilson fermions the value of the hopping parameter for which the pion mass is zero must be found first before the chiral limit can be investigated. When dealing with Staggered fermions it is also important to keep in mind that the use of N_f to represent the number of flavours is only valid close to the continuum limit.

6.2 Pseudofermion Stochastic Algorithm

The Pseudofermion (PF) method provides an approximation for detQ in (5.9) (with $n = N_f/4$). This method uses the property that if Q is a positive operator, then it satisfies the relation [18j]

$$Q^{-1}_{ji} = \langle \bar{\phi}_i \phi_j \rangle$$

$$= \int [d\bar{\phi}][d\phi]\ \bar{\phi}_i \phi_j \exp[-\sum_{i,j} \bar{\phi}_i Q_{ij} \phi_j] / \int [d\bar{\phi}][d\phi] \exp[-\sum_{i,j} \bar{\phi}_i Q_{ij} \phi_j]$$

where $\bar{\phi}$ and ϕ are complex bosonic fields called pseudofermions (which have the properties of fermions but are ordinary numbers).

For Staggered fermions we have seen that the operator Q is not positive definite, while Q^2 is. For calculational purposes the form

$$S_{eff} = S(U) - N_f/8\ \mathrm{Trln}[-D^2 + m^2] \qquad (6.15)$$

is therefore used.

If the Metropolis algorithm is used for example, the whole determinant need not be calculated, only the change if

$$U_{x,\mu} \rightarrow U_{x,\mu} + \delta U_{x,\mu} \equiv \bar{U}_{x,\mu} , \tag{6.16}$$

i.e. if the links are suitably updated .

Under the update (6.16) the matrix Q changes, $Q \rightarrow Q + \delta Q$, hence the ratio $\frac{\det(Q + \delta Q)}{\det Q}$ is needed, which for small enough δU gives

$$\frac{\det(Q + \delta Q)}{\det Q} = \det(1 + Q^{-1}\delta Q)$$

$$= 1 + \mathrm{Tr}\, Q^{-1}\delta Q + O(\delta U^2). \tag{6.18}$$

The PF method can now be used to find Q^{-1}_{xy}

$$Q^{-1}_{xy} = (Q^\dagger Q)_{xz} Q^\dagger_{zy} = \langle \bar{\phi}_z \phi_x \rangle Q^\dagger_{zy} \tag{6.19}$$

where we used

$$(Q^\dagger Q)^{-1}_{xz} = Z_\phi^{-1} \int (\prod_x d\bar{\phi}_x d\phi)\, \bar{\phi}_z \phi_x \exp[- \sum_{i,j} \bar{\phi}_i (Q^\dagger Q)^{-1}_{ij} \phi_j]. \tag{6.19a}$$

The PF method proposed by Fucito et al. [18j] uses the observation that if the Metropolis algorithm is used, then in calculating the change (6.16) one only needs

$$\delta S_{eff} = S_{eff}(\bar{U}) - S_{eff}(U). \tag{6.20}$$

If higher order corrections $[O(\delta U^2)]$ can be neglected, then δS_{eff} can be approximated by [18j;20a]

$$\delta S_{eff} = \delta S_G - \frac{1}{8} N_f \sum_{x,\mu} \frac{\partial}{\partial U} \mathrm{Tr}[\ln(D+m)(-D+m)] \tag{6.21}$$

with $\delta S_{eff} = S_{eff}(\bar{U}) - S_{eff}(U)$ etc. . (6.21a)

Now $\frac{\partial}{\partial U} \mathrm{Tr}[\ln(D+m)(-D+m)]$

$$= \mathrm{Tr}\left(\tfrac{1}{2}\eta_\mu(x) \left[\frac{1}{D+m} - \frac{1}{-D+m}\right]\right)_{x+\mu i, xj}$$

$$= \mathrm{Tr}\left(\tfrac{1}{2}\eta_\mu(x) \left[\frac{1}{D+m} - \frac{1}{D^\dagger+m}\right]\right)_{x+\mu i, xj}$$

$$\equiv \mathrm{Tr}(J_{x,\mu})_{ij}. \qquad (6.21b)$$

Hence

$$\delta S_{eff} = \delta S_G - \frac{1}{8} N_f \sum_{x,\mu} \mathrm{Tr}(J_{x,\mu} + J^\dagger_{x,\mu})\delta U^\mu_x + O(\delta U^2)$$

$$= \delta S_G - \frac{1}{4} N_f \sum_{x,\mu} \mathrm{Re}\ \mathrm{Tr} J_{x,\mu}\ \delta U^\mu_x + O(\delta U^2)\ . \qquad (6.22)$$

Note that $J^\dagger_{x,\mu} = -J_{x,\mu}$. (6.22a)

The "current" $J_{x,\mu}$ can now be calculated using the PF method.

The deconfining phase transition has been studied by Fucito et al. [20d;e] using the PF method described above on 6^3, $8^3 \times 4$ lattices.

Using the real part of the Polyakov loop (2.25) as order parameter, metastable states (indicative of a first order transition) were found at $\beta_c = 5{,}3 \pm 1^*$ for $N_f = 3$ and $m_q a = 0{,}1$ (* improving the statistics of their method around β_c, they obtained the more accurate estimate $\beta_c = 5{,}2455$ [20i]).

Comparison with quenched data (using a lattice of the same size) showed that β_c had shifted to a lower value while the transition remained first order. This was confirmed by the $ma = 0{,}2$ data, which gave $\beta_c = 5.35$.
If asymptotic scaling is assumed to hold, the following value for the critical coupling is obtained

$$T_c/\Lambda_L = 171 \pm 78. \qquad (6.23)$$

However, for the same number of flavours and quark mass, Gavai and Karsch [20g] found that the Polyakov loop and energy density showed a continuous change over a small coupling range ($\Delta\beta \approx 0{,}1$) reminescant of a second order transition. This confirmed some of the earlier results of [20b;c], where it was found that the behaviour of the energy density and

Polyakov loop indicated a second order transition for $N_f = 2$ and $m_q a = 0,1$; 0,5 and 0,2.

A recent investigation by Attig et al. [20j] showed that for $N_f = 8$ and a fixed quark mass $ma = 0,1$ the phase transition calculated on a $6^3 \times 2$ lattice is no longer of first order (which was the case for the quenched theory corresponding to $N_f = 0$).
Instead the order parameter <ReL> showed a smooth continuous change around β_c suggesting a second order transition. However, for a $6^3 \times 4$ lattice the calculations showed that the first order transition of the quenched theory was still present for $N_f = 8$ and $ma = 0,1$ ($\beta_c = 4,81$). This is in agreement with the result of Gavai [20h] for a $8^3 \times 4$ lattice, where a first order transition was obtained with $\beta_c = 4,78$. This also reflects well on the small effect of the finite volume which leads to $\Delta\beta \approx 0,03$. The results of [20j] and [20h] for $N_\tau = 4$ are shown in fig. 6.3.

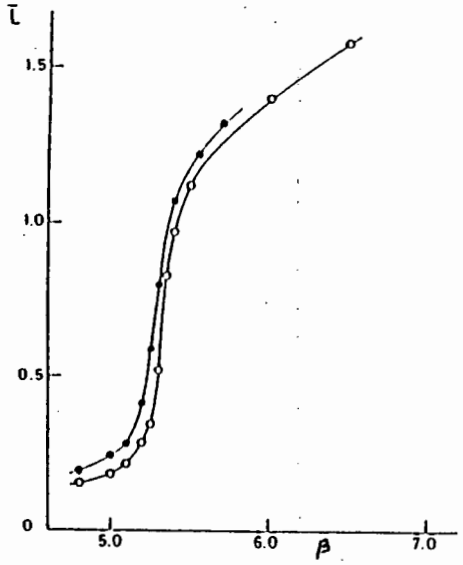


FIGURE 6.2 : Comparison of results in the fourth (open circles and fifth (filled circles) HPE on an $8^3 \times 3$ lattice for the Polyakov loop average $\bar{L}$ (from [19d]).

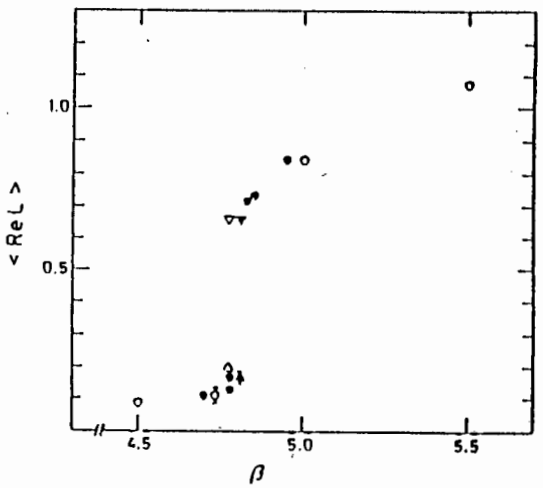


FIGURE 6.3 : < Re L > as a function of β for $N_f = 8$ and $ma = 0.1$ evaluated on a $6^3 \times$ lattice [20j] (filled circles and triangles) compared with an $8^3 \times 4$ lattice [20h] (open circles and triangles), (from [20j])

For the larger lattice ($6^3 \times 6$) the previous results for $N_\tau = 4$ also applied with $\beta_c = 4{,}99$. To test for possible scaling the critical coupling was evaluated using a different algorithm (but still the basic PF method) on a 4^4 lattice with $ma = 0{,}15$ and $N_f = 8$. This was compared with the results obtained on the $6^3 \times 6$ lattice . With $\beta_c(N_\tau=4) \approx 4{,}825$ this gives (using the RGE)

$$T_c(N_\tau=4)/T_c(N_\tau=6) = 1{,}02(10) \qquad (6.24)$$

which is consistent with scaling.

6.3 Other methods

Using the Microcanonical method [18f-i] to incorporate the full fermion determinant into the quenched theory, Polonyi et al. [21a] calculated the energy density (see section 5.3) for $N_f = 4$, $ma = 0{,}10$ and $0{,}08$ on a $8^3 \times 4$ lattice. They found a very abrupt deconfining transition at $\beta \approx 5{,}1$ ($ma = 0{,}08$) similar to the one obtained from the quenched theory on a lattice of similar size.

A strong first order transition was also obtained by Kogut et al. [21b] using the same method for $N_f = 8$ and $ma = 0{,}1$ on a $8^3 \times 4$ lattice. The results are summed up in table 6.2 below.

The first results using an exact algorithm were obtained by Fishler and Roskies [25a] on a 4^4 asymmetrical lattice. For four flavours and massless quarks they found no evidence for a phase transition.
Although in principle a method using an exact algorithm should be preferable, no qualitative estimates (e.g. for T_c, T_{ch}) can be made. This is due to the fact that the lattice size is restricted by the complicated nature of the algorithm, and the fact that $N_\tau = N_\sigma$, which is not suited for finite temperature calculations as the finite lattice-size effects will be comparable with the finite temperature effects that one wants to measure.

Table 6.2

Ref.	lattice	N_f	ma	β_c
[20d;e]	$6^3, 8^3 \times 4$	3	0,1	5,30(10)
		3	0,2	5,35(10)
[20i]	$6^3, 8^3 \times 4$	3	0,1	5,2455
[20g]	$8^3 \times 4$	3	0,1	≈5.25
[20h]	$8^3 \times 4$	8	0,1	4,78
[20j]	$6^3 \times 2$	8	0,1	
	$6^3 \times 4$	8	0,1	4,81
	$6^3 \times 6$	8	0,1	4,99
[21a]	$8^3 \times 4$	4	0,08	≈ 5,1
[21b]	$8^3 \times 4$	8	0,1	4,67(1)

7 Chiral Symmetry on the Lattice

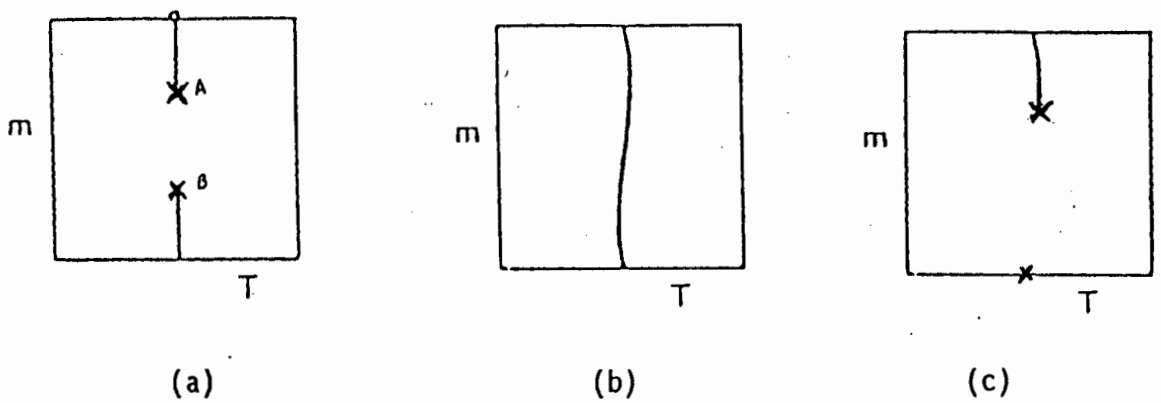
7.1 Preliminaries

Chiral symmetry in physics is based on the observation that the u and d quark masses are very ssmall. However, since the masses are non-zero this is only an approximate symmetry (i.e chiral symmetry is only exact for $m_q = 0$).

Chiral symmetry is **spontaneously** broken at T = 0 even if the u and d quarks are massless. This is due to the effect of instantons ($\vec{E}\cdot\vec{B}$ fluctuations) which do not respect handedness in the QCD cacuum.

A suitable order parameter in this limit is the fermion condensate $\langle\bar{\psi}\psi\rangle$ - a non-zero value would give the amplitude for a left-handed quark to move in a closed loop and end up as a right handed. If the chiral symmetry phase transition (c.s.p.t.) is strong first order, then the transition should persist for small but finite quark mass.

Possible forms for the generic phase diagram are given in **fig. 7.1**



(a) (b) (c)

fig. 7.1

If both transitions are first order, the phase diagram may look like the one in fig. 7.1(a), where A is the critical point (i.e. continuous transition) at the end of a line of first order deconfinement transitions (with the circle (o) denoting the result for the quenched theory) while B is the critical point of the line of first order chiral symmetry phase transitions. For a specific range of m values the high and low temperature regimes will be continuously connected, with no phase transition present if T is raised.

Another possibility is that the deconfined and chiral transitions are linked by a phase boundary fig. 7.1(b). A phase transition would then be inevitable. If the chiral transition is second order, the phase diagram will have the form of fig. 7.1(c). This specific scenario has been suggested by $N_f = 2$ results (See Section 7.3.1).

In [22b] a study of effective σ models predicts a first order c.s.p.t. at finite temperature for the SU(N) flavour group with $N_f \geq 3$ (massless flavours). For $N_f = 2$ the transition is predicted to be first order if and only if the effects of the U(1) axial anomaly are much weaker at finite than at zero temperature. The transition is second order if and only if the anomaly is temperature independent.

At high enough temperatures and/or densities it is expected that chiral symmetry will be restored. For Lattice Gauge theories this has been rigorously shown to be the case [22c].

We will also see (section 7.2) that for quarks in the fundamental representation of SU(N) there is a simultaneous deconfinement transition. This is not true for other representations e.g. the adjoint one.

Questions that have been tackled in the literature include the possibility of a deconfined chirally symmetric state and the nature of the transition (i.e. is it discontinuous, continuous or just a cross-over.

Order parameter

As $\bar{\psi}\psi$ is not invariant under the chiral transformation $\psi \to \gamma_5\psi$, it can be used as order parameter for the c.s.p.t. . The usual criterion for chiral symmetry breaking is that the chiral condensate $\langle\bar{\psi}\,\psi(m)\rangle$ should vanish as $m \to 0$. For a lattice with finite volume (i.e. N_σ finite) this has to be modified as a finite system will (in a large enough time interval) rotate through all its degenerate minima of the effective potential [23b;c]. This implies that $\langle\bar{\psi}\ \psi(m)\rangle = 0$ even if the symmetry is dynamically broken. The correct expression is given by [23b]

$$\lim_{m\to 0}\lim_{N_s\to\infty} \langle\bar{\psi}\,\psi(m)\rangle = \text{constant} \neq 0 \qquad (7.1)$$

In terms of the one component fermion fields (used for Staggered fermions) the expression above takes on the form

$$\lim_{m\to 0}\lim_{N_s\to\infty} \langle\bar{\psi}\,\psi(m)\rangle = \lim_{m\to 0}\lim_{N_s\to\infty} (N_s^{-1}\langle\bar{\chi}_{x,i}\chi_{x,i}\rangle) \qquad (7.2)$$

where i is the colour index and N_s is the number of sites . The order of the limits above is important. We also note that the condensate $\langle\bar{\psi}\,\psi(m)\rangle$ is the quark propagator at zero spacing (averaged over gauge fields), i.e.

$$\langle\bar{\psi}\,\psi(m)\rangle = \frac{3}{N}\langle \mathrm{tr} Q^{-1}\rangle. \qquad (7.3)$$

7.2 Quenched theory

It has been shown in [23a] that for quenched SU(2)

$$T_{ch}/T_c = 1{,}60(20) \qquad (7.4)$$

for quarks in the fundamental representation.

For QCD [SU(3)] the deconfinement temperature and chiral symmetry restoration temperature are almost coincident [14b]:

$$1{,}0 \leq T_{ch}/T_c \leq 1{,}05\ . \qquad (7.5)$$

For fermions in the adjoint representation the two transitions take place at different temperatures (see [23a;d]). In fig. 7.2 the results for the order parameter $\langle\bar{\chi}\chi\rangle$ is shown for quenched SU(2) and SU(3).

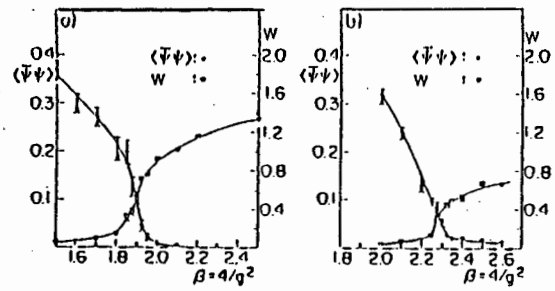


FIG. 7.2a : $\langle\bar{\psi}\psi\rangle$ and W vs $\beta = 4/g^2$ for SU(2) gauge theory on (a) 2×8^3 and (b) 4×8^3 lattices. The curves are meant to guide the eye; they are not precise fits. (From [14b]).

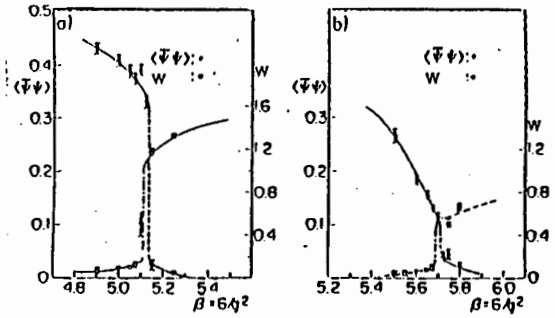


FIG. 7.2b : $\langle\bar{\psi}\psi\rangle$ and W vs $\beta = 6/g^2$ for SU(3) gauge theory on (a) 2×8^3 and (b) 4×8^3 lattices. (From [14b]).

7.3 Full theory

As we have noted previously, Staggered fermions have a remnant continuous chiral symmetry which results in $m_q = 0$ being the chiral limit. For QCD (with colour group SU(3)) the parameters for the analysis of the chiral symmetry is the quark mass and the number of fermions. We will now discuss the results for different numbers of quark flavours. [References with the same roman numbers use more or less the same MC method, e.g. all references denoted by [26] uses the Hybrid Stochastic method]. All results will be tabulated seperately after each section.

7.3.1 Results for 2 Staggerd flavours

Gavai et al. [20b;d;e] investigated the chiral phase transition on a $6^3 \times 2$ lattice for $m_q a = 0{,}1$; $0{,}15$ and $0{,}20$. Compared to the quenched results, the order parameter $\langle\bar{\psi}\psi\rangle$ (extapolated linearly to $m \to 0$) showed a smoother, more continuous behaviour (fig. 7.3.1) which suggested a possible second order transition. This behaviour was also present for <L> and the energy density.

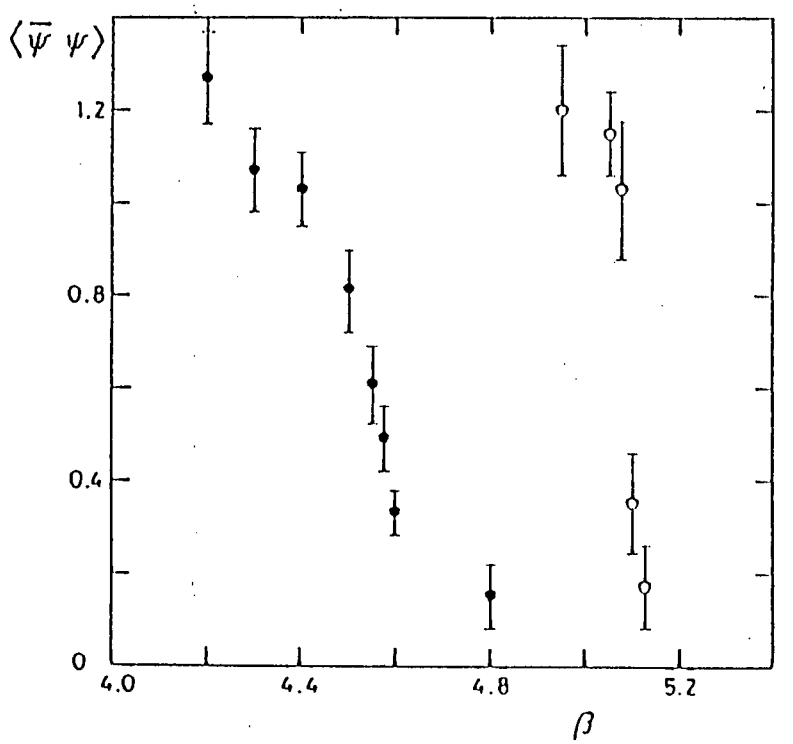


FIGURE 7.3.1 : The chiral condensate as a function of β on a 6^3 x 2 lattice. The filled circles represent two flavour QCD while the open circles are for the quenched theory (from [20c]).

Fukugita et al. [24b] however claimed that the chiral transition strengthened as $m_q a \to 0$, changing from a continuous cross-over at $m_q a = 0{,}2$ to a first order one (as soon as) $m_q a = 0{,}1$.

This is again in contrast to the results of Gottlieb et al. [26e], where an increasing rapid crossover in $\langle\bar{\psi}\psi\rangle$ was found as the mass was lowered, but no conclusive evidence* for a first order phase transition existed even at ma = 0,025 [* the metastability signals in <ReL> were not unambiguous, although $\langle\bar{\psi}\psi\rangle$ did change in a narrow interval of ß for ma = 0,025].

Kogut and Sinclair [26g] found a first order transition for ma = 0,0125 (with clear evidence for coexisting states in the behaviour of <L>), but none for ma = 0,025. This seems to be in agreement with [26e].

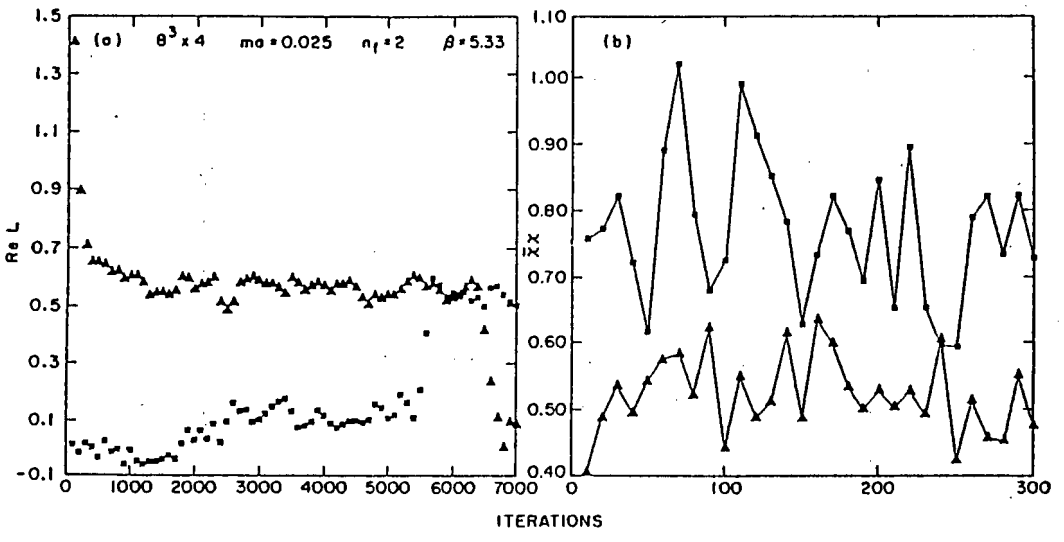


FIGURE 7.3.2 : Metastability signals for two-flavour QCD on an $8^3 \times 4$ lattice with ma = 0.025. Squares and triangles describe runs from disordered and ordered runs respectively. (From [24d])

Gavai et al. [24d], however, did find clear metastability signals for the Polyakov loop and chiral order parameter for ma = 0,025 (fig. 7.3.2) which strongly suggests that the transition is indeed first order. For higher masses no metastability signals were obtained in agreement with the previous results of [24b] and [26e].

Also, Gavai et al. [24e] have found that the transition is first order for an interpolation of an isodoublet of quarks with ma = 0,025 and a third whose mass can be varied between 0,025 and ∞.

In a recent calculation by Gupta et al. [25c] using an exact algorithm on a 4^4 lattice, evidence was found that the transition persists for larger quark masses (ma = 0,20). This is consistent with the transition seen at ma = 0,1 in [24b].

The results above seem to suggest that the line of first order transitions should extend at least to ma = 0,0125 in the phase diagram (fig. 7.1). In addition to the problem of the uncertainty in the critical mass m_{ch} [where the line of first order transitions end (point B on the phase diagram 7.1)], there is also a lack of theoretical predictions on the order of the transition for $N_f = 2$.

Table 7.1 ($N_f = 2$)

Ref.	lattice	ma	β_{ch}
[20b;c]	$6^3 \times 2$	$\to 0$	$\approx 4,6$
[24b]	$8^3 \times 4$	0,10	5,37 - 5,39
		$\to 0$	5,29 - 5,32
[24d]	$8^3 \times 4$	0,10	5,37 - 5,38
		0,025	$\approx 5,33$
[24e]	$8^3 \times 4$	0,025*	
[26g]	$8^3 \times 4$	0,0125	5,2875
	4^4	0,025	5,25
		0,0125	5,225
[25c]	4^4	$\approx 0,2$	

* mass for an isodoublet of qaurks (see discussion)

7.3.2 Results for 3 Staggered flavours

For $N_f = 3$ theoretical considerations [22b] predict a first order transition.

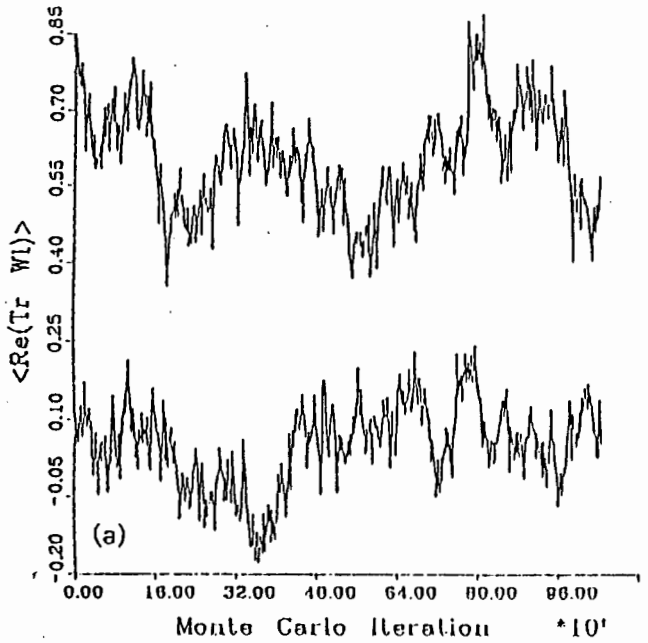


FIGURE 7.3.3 : Polyakov loop expectation value as a function of the number of iterations at $\beta_c = 5.3$ (m = 0.1) (from |20e|).

Fucito et al. [20e;f] have confirmed this; the value of the quark condensate undergoes a discontinuous change, while clear signs of metastability was found in the behaviour of <ReL> in the neighbourhood of the transition with $\beta_{ch} = 5{,}25(10)$ (see fig. 7.3.3). What was also interesting is that for quark masses as high as 0,10 and 0,20 the deconfinement and chiral restoration phase transitions coincided.

Gavai and Karsch [20g] found however that although $\langle\bar{\psi}\psi\rangle_{m\to 0}$ showed a rapid change in behaviour as a function of the coupling, no signs of metastability was found around $\beta_{ch} \approx 5{,}3$. Hence, although a strong first order transition was unlikely, they could not rule out a weak (fluctuation driven) first order transition.

However, the more recent results of Gavai et al. [24c] seem to support the notion of a first order transition. Metastabilities in the behaviour of <ReL> was found for ma = 0,025 using a 4^4 and $8^3 \times 4$ lattice, with the larger lattice having a stronger signal which suggests a strong first order transition (fig. 7.3.4). This result has been confirmed by Kogut et al. [26g] using the Hybrid MC method.

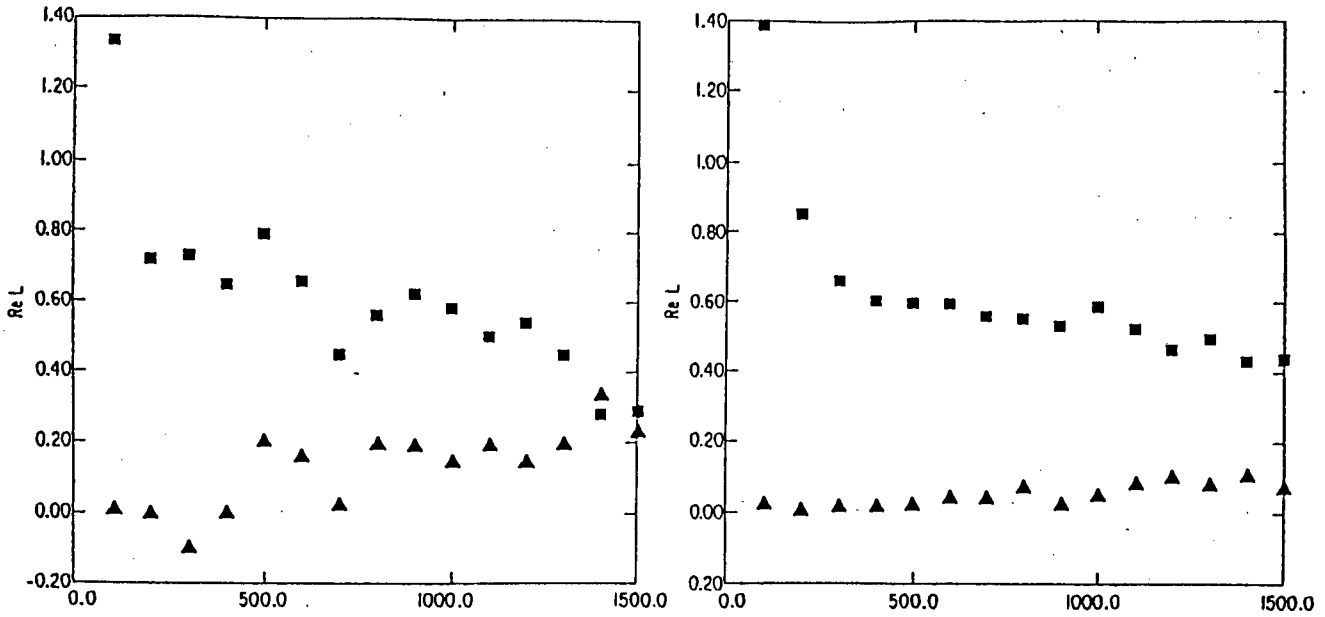


FIGURE 7.3.4 : Metastability signals for the Polyakov loop in three flavour QCD on a 4^4 and $8^3 \times 4$ lattice (left and right respectively) with ma 0 0.025 (from [24c])

Table 7.2 (N_f= 3)

Ref.		lattice	ma
[20e;f]	$6^3, 8^3 \times 4$	0,1;0,2 → 0	5,25(1)
[20g]	$8^3 \times 3$	0,1; 0,074 → 0	≈ 5,25
[24c]	$8^3 \times 4$	≈ 5,1	
[26g]	$8^3 \times 4$	0,025	≈ 5,1

7.3.3 Results for 4 Staggered flavours

There seems to be concensus that for $N_\tau = 4$ and 4 continuous degenerate flavours the chiral symmetry phase transition for small quark masses (i.e ma $\leq$ 0,025) is of strong first order.

On 4^4 lattices using an exact algorithm this transition has been shown to survive up to ma = 0,2 [25c], although the ma = 0,20 data did not show signs of metastability (see fig. 7.3.5).

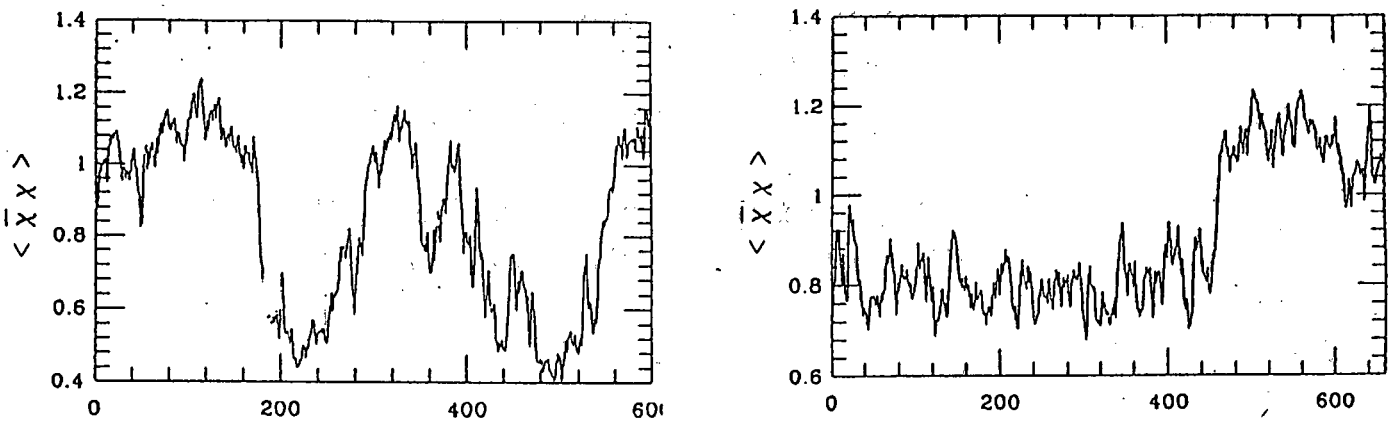


FIGURE 7.3.5 : Time evolution of the chiral condensate for four flavour QCD on a 4^4 lattice for ma = 0.10 and 0.20 (left and right respectively) (from [25c]).

However, results calculated on a $6^3 \times 4$ lattice using the same algorithm seem to suggest that the problem of finite volume effects has not yet been resolved, especially for large masses (i.e ma $\approx$ 0,2).

For larger lattices (usually $8^3 \times 4$) approximate methods are used. Fukugita and Ukawa [24a] obtained metastability for ma = 0,1 indicating a first order transition, while for ma $\approx$ 0,2 - 0,3 the phase transition was continuous. Their results therefore supports the scenario that the first order deconfining transition of the quenched theory (at $m \to \infty$) smoothly continues into the region of small m_q. They also found that the two transitions appeared simultaneously.

Most of the approximate results for $N_f = 4$ have been obtained using the Hybrid Stochastic method. Kogut and Sinclair [26a;b] found a rapid crossover for ma = 0,1 and 0,5, but no metastability, while for ma = 0,250 there was a definite first order transition. The deconfinement transition for large m_q seemed therefore to have been weakened by the inclusion of finite intermediate masses.

Karsch et al. [26d] and Gottlieb et al. [26e] only found a clear signal of metatsability for ma = 0,025 (in agreement with [25b]).

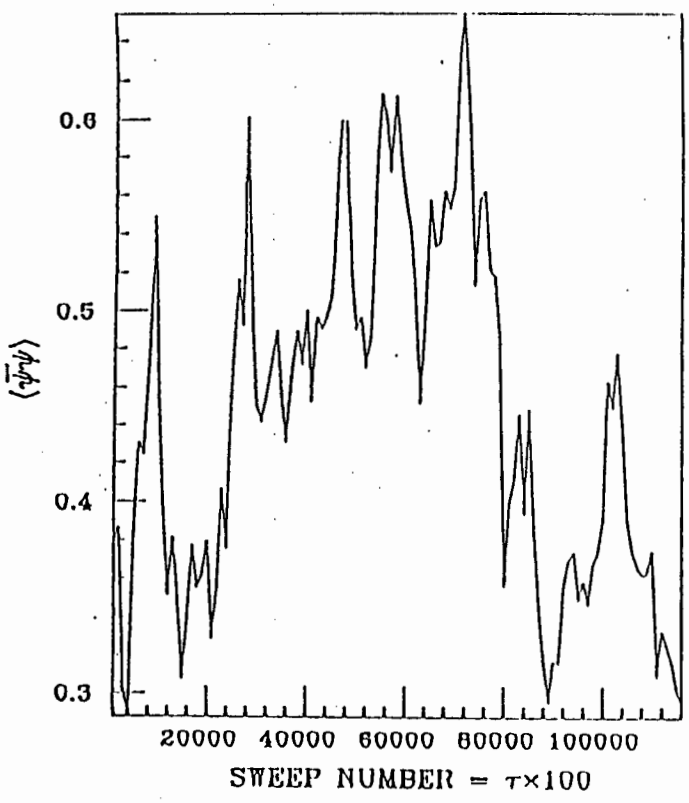


FIGURE 7.3.6 :

Evidence of coexisting states in the time evolution of the chiral condensate at $\beta = 5.125$ (corresponding to ma =0.025) evaluated on a $10^3 \times 6$ lattice (from [26c]).

On a $10^3 \times 6$ lattice Kovacs et al. [26c] found a first order transition at ma = 0,025 (fig. 7.3.6), but not for ma = 0,05 (in agreement with [26b]). If asymptotic scaling is assumed to hold, the zero mass extapolated data in [26b;c] suggests that $T_c/\Lambda_{\overline{MS}} = 2,14(10)$ (while [26d] obtained $T_c/\Lambda_{\overline{MS}} = 2,77(15)$).

The most recent results of Kogut et al. [26f] using a $8^3 \times 4$ lattice show that the transition is first order for ma = 0,0375; 0,0250 and 0,0125 (with metastable states found in all cases). On a $10^3 \times 6$ lattice long runs with ma = 0,0250 also favoured a first order transition. with $\beta_{ch} = 5{,}125$ (see fig. 7.3.7). This result is significant, as it has been shown by Kogut [26a] that (within statistical and systematic uncertainties) asymptotic scaling starts around $\beta \approx 5{,}10$ .The results above can therefore be regarded as being characteristic of the continuum limit. This is not the case for the smaller lattice, where the critical couplings are typically near $\beta_{ch} = 5{,}00$, i.e. in the Strong Coupling region. For the $10^3 \times 6$ lattice the critical coupling was found to be $T_{ch}/\Lambda_{\overline{MS}} = 2{,}14(10)$. It is also interesting to note that the discontinuities in the observables (e.g. $\Delta\langle L\rangle$, $\Delta\langle\bar{\psi}\psi\rangle$) are larger for ma = 0,0125 than for ma = 0,0250 [26d], which implies that the chiral transition is strenghening as $m \to 0$ (as expected [22b]).

Using the Langevin algorithm, Gavai et al. [24c] confirmed earlier results of a first order transition for ma = 0,0250 on a $8^3 \times 4$ lattice.

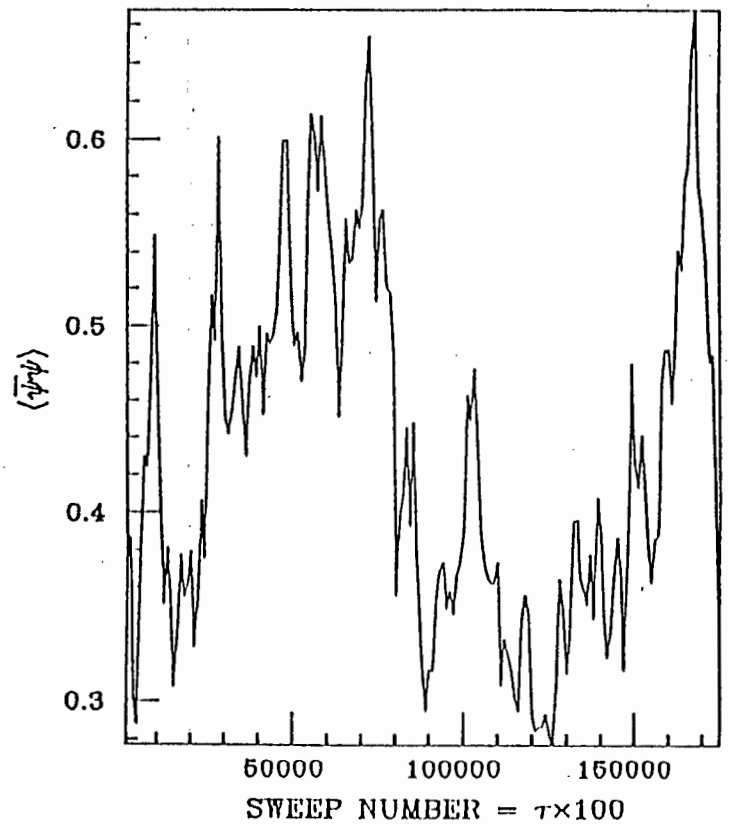


FIGURE 7.3.7 : Time evolution of the chiral condensate on a $10^3 \times 6$ lattice at m = 0.025, β = 5.125 for four-flavour QCD (from [26f]).

Table 7.3 ($N_f = 4$)

Ref.	lattice	ma	β_{ch}
[24c]	$8^3 \times 4$	0,0250	$\approx$ 5,03
[25b]	4^4	0,0250	$\approx$ 4,9
[25c]	4^4	0,20	5,14
		0,10	5,04(3)
		0,050	4,95
		0,0250	4,910(3)
[26a;b]	$10^3 \times 6$	0,250	5,509
		0,100	5,325
		0,050	5.175
		$\rightarrow$ 0,00	5,010(25)
[26c]	$10^3 \times 6$	0,0250	5,125(25)
[26d]	4^4	0.0250	4.94(4)
	$8^3 \times 4$	0,0250	4,96(3)
		$\rightarrow$ 0,00	4,91(3)
[26e]	$8^3 \times 4$	0,500	5,45
		0,400	5,42
		0,100	5,13
		0,050	4.96(3)
		0,0250	4,96(3)
		$\rightarrow$0,00	4,90(3)
[20f]	$8^3 \times 4$	0,0375	4,99(1)
		0,0250	4,94(1)
		0,0125	4,920(6)
		$\rightarrow$ 0,00	4,90(1)
	$10^3 \times 6$	0,0250	5,125(25)

7.3.4 Results for many Staggered flavours

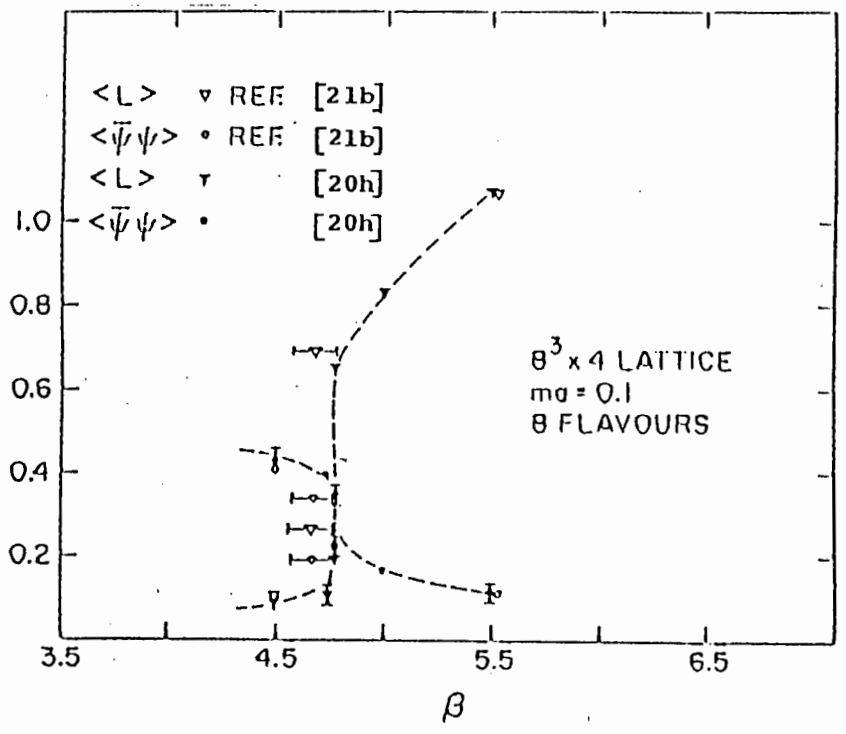


FIGURE 7.3.8 : (From [20h])

Kogut et al. [21b] have found a strong first order phase transition at $\beta_{sh} = 4,67(10)$ with evidence for metastable states for $N_f = 8$ using a $8^3 \times 4$ lattice with ma = 0,10. This was confirmed by Gavai [20h], who found a two-state signal at $\beta = 4,78$. Their combined results are shown in fig. 7.3.8.

More recently, Kogut and Sinclair [26h] have investigated $N_f = 8$ and 12 using lattices of sizes 4^4, 6^4 and 8^4 with fixed quark mass ma = 0,1. They obtained first order transitions in almost all cases, but the results were strongly influenced by finite lattice size effects.
Also, the fact that lattices with $N_\sigma = N_\tau$ were used, make it difficult to distinguish between a possible finite temperature transition or a finite lattice size induced one.

8. QCD at Finite Density

To obtain a realistic explanation of most physical applications the concept of a system at finite baryon density (i.e non-zero chemical potential) must be introduced. It is generally believed that at suffiently high densities and zero temperature, QCD will have a deconfining phase transition analogous to the one at $\mu = 0$ for finite temperature systems.

Starting from first principles, chemical potential is introduced via the partition function

$$Z(\beta,\mu) = \text{tr } e^{-\beta(H-\mu\hat{N})} \tag{8.1}$$

where $\hat{N}$ is the particle number operator with

$$N = N_q - N_{\bar{q}} . \tag{8.1a}$$

For the spin ½ continuum theory the operator N is given by

$$N = \int d^3x\ \bar{\psi}(x)\gamma_0\psi(x) \tag{8.2}$$

which is the fourth component of the (conserved) Noether current

$$J_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x). \tag{8.3}$$

The Dirac action for the continuum theory is therefore given by

$$S = \int_0^\beta d\tau \int d^3x\ \bar{\psi}(x)(\gamma_\mu\partial_\mu + m - \mu\gamma_0)\psi(x) \tag{8.4}$$

with corresponding partition function

$$Z = \int [d\psi d\bar{\psi}]\ e^{-S} \tag{8.5}$$

where the chemical potential was introduced via the source term N.

8.1 Chemical potential on the lattice

8.1.1 Naive formalism

To illustrate how chemical potential can be incorporated into the formalism we first consider the Naive lattice action [27;28a]. For the continuum theory the conserved current was obtained by

$$\bar{\psi}\frac{\delta S}{\delta\bar{\psi}} - \frac{\delta S}{\delta\psi}\psi = 0,$$

which gives $$\frac{\partial}{\partial x_\mu}\left[\bar{\psi}(x)\gamma_\mu\psi(x)\right] = 0 . \tag{8.6}$$

The naive action is defined by (see Chapter 4)

$$S = \sum_x a^4\left\{\frac{1}{2a}\sum_\mu\left[\bar{\psi}(x)\gamma_\mu\psi(x+\hat{\mu}) - \bar{\psi}(x+\hat{\mu})\gamma_\mu\psi(x)\right] + m\bar{\psi}(x)\psi(x)\right\}. \tag{8.7}$$

Therefore

$$a^{-4}\left\{\bar{\psi}\frac{\delta S}{\delta\bar{\psi}} - \frac{\delta S}{\delta\psi}\psi\right\}$$

$$= \frac{1}{2a}\sum_\mu\left\{\bar{\psi}(x)\gamma_\mu\psi(x+\hat{\mu}) - \bar{\psi}(x)\gamma_\mu\psi(x-\hat{\mu}) - \bar{\psi}(x-\hat{\mu})\gamma_\mu\psi(x) + \bar{\psi}(x+\hat{\mu})\gamma_\mu\psi(x)\right\}$$

$$= a^{-1}\sum_\mu\left\{\tfrac{1}{2}\left[\bar{\psi}(x)\gamma_\mu\psi(x+\hat{\mu}) + \bar{\psi}(x+\hat{\mu})\gamma_\mu\psi(x)\right]\right.$$

$$\left. - \tfrac{1}{2}\left[\bar{\psi}(x-\hat{\mu})\gamma_\mu\psi(x) + \bar{\psi}(x)\gamma_\mu\psi(x-\hat{\mu})\right]\right\}$$

$$= 0,$$

with the conserved current given by

$$J_\mu = \tfrac{1}{2}\left[\bar{\psi}(x)\gamma_\mu\psi(x+\hat{\mu}) + \bar{\psi}(x+\hat{\mu})\gamma_\mu\psi(x)\right]. \tag{8.10}$$

The corresponding expression for the particle number operator is

$$N = \Sigma^* J_0$$

$$= \Sigma^* a^3\tfrac{1}{2}\left[\bar{\psi}(x)\gamma_0\psi(x+\hat{0}) + \bar{\psi}(x+\hat{0})\gamma_0\psi(x)\right] \tag{8.11}$$

where Σ^* denotes the sum for $\vec{x}$ ε plane and τ constant.

Proceeding analogously to the continuum case we have the following result for the naive action

$$S = \sum_x a^4 \left\{ \frac{1}{2a} [\bar{\psi}(x)\gamma_0(1-a\mu)\psi(x+\hat{0}) - \bar{\psi}(x+\hat{0})\gamma_0(1+a\mu)\psi(x)] \right.$$

$$\left. + (2a)^{-1} \sum_{j=1}^{3} [\bar{\psi}(x)\gamma_j\psi(x+\hat{j}) - \bar{\psi}(x+\hat{j})\gamma_j\psi(x)] + m\bar{\psi}(x)\psi(x) \right\} . \quad (8.12)$$

In order to obtain a finite expression, counterterms must be added [27a;b]. One choice [27a;28a] is to replace

$$1-a\mu \to e^{-\mu a}$$

$$1+a\mu \to e^{+\mu a} \quad (8.13)$$

which making the change $\mu \leftrightarrow -\mu$ * gives

$$S = \sum_x a^4 \left\{ \frac{1}{2a} [\bar{\psi}(x)\gamma_0 e^{a\mu}\psi(x+\hat{0}) - \bar{\psi}(x+\hat{0})\gamma_0 e^{-a\mu}\psi(x)] \right.$$

$$\left. + (2a)^{-1} \sum_{j=1}^{3} [\bar{\psi}(x)\gamma_j\psi(x+\hat{j}) - \bar{\psi}(x+\hat{j})\gamma_j\psi(x)] + m\bar{\psi}(x)\psi(x) \right\} \quad (8.14)$$

(* in the literature cited, the chemical potential is introduced via $-\mu N$, and not μN as is the case here).

Another choice that has been proposed is to replace [27b]

$$1+\mu a \to (1+\mu a)/(1-\mu^2 a^2)^{\frac{1}{2}}$$

$$1-\mu a \to (1-\mu a)/(1-\mu^2 a^2)^{\frac{1}{2}} . \quad (8.15)$$

It is important to note that these counterterms do not influence the continuum limit.

8.1.2 Staggered fermions

The Staggered fermion action is given by

$$S = a^4 \sum_x \left\{ a^{-1} \sum_\mu \tfrac{1}{2}\eta_\mu(x)[\bar{\chi}(x)\chi(x+\hat{\mu}) - \bar{\chi}(x+\hat{\mu})\chi(x)] + m\bar{\psi}(x)\chi(x) \right\} \quad (8.16)$$

Now

$$a^{-4}\left[\bar{\chi}\frac{\delta S}{\delta\bar{\chi}} - \frac{\delta S}{\delta\chi}\chi\right]$$

$$= \frac{1}{2a}\sum_{\mu}\Big[\eta_\mu(x)\bar{\chi}(x)\chi(x+\hat{\mu}) + \eta_\mu(x)\,\bar{\chi}(x+\hat{\mu})\chi(x)$$
$$-\eta_\mu(x-\hat{\mu})\bar{\chi}(x-\hat{\mu})\chi(x) - \eta_\mu(x-\hat{\mu})\bar{\chi}(x)\chi(x-\hat{\mu})\Big]$$

$$= 0\ ,\ \text{so that}$$

$$J_\mu = \tfrac{1}{2}\eta_\mu(x)[\bar{\chi}(x)\chi(x+\hat{\mu}) + \bar{\chi}(x+\hat{\mu})\chi(x)] \qquad (8.17)$$

and

$$N = \sum^{*}\tfrac{1}{2}a^3\eta_0(x)[\bar{\chi}(x)\chi(x+\hat{0}) + \bar{\chi}(x+\hat{0})\chi(x)]. \qquad (8.18)$$

The appropriate form of the Staggered action is therefore [using (8.13)]

$$S = \sum_x a^4\Big[\frac{1}{2a}\eta_0(x)[\bar{\chi}(x)e^{a\mu}\chi(x+\hat{0}) - \bar{\chi}(x+\hat{0})e^{-a\mu}\chi(x)]$$

$$+ (2a)^{-1}\sum_{j=1}^{3}[\bar{\chi}(x)\chi(x+\hat{j}) - \bar{\chi}(x+\hat{j})\chi(x)] + m\bar{\chi}(x)\chi(x)\Big]\ . \qquad (8.19)$$

8.1.3 Wilson fermions

The most general form of the lattice fermion action containing no species doubling effects is

$$S = \sum_x a^4\Big[-\frac{1}{2a}\sum_\mu[\bar{\psi}(x)(r-\gamma_\mu)\psi(x+\hat{\mu}) + \bar{\psi}(x+\hat{\mu})(r+\gamma_\mu)\psi(x)] + m\bar{\psi}(x)\psi(x)\Big] \qquad (8.20)$$

with $r = 1$ corresponding to Wilson fermions.

Now

$$a^{-4}\left[\bar{\psi}\frac{\delta S}{\delta\bar{\psi}} - \frac{\delta S}{\delta\bar{\psi}}\psi\right]$$

$$= a^{-1}\sum_\mu\Big[\tfrac{1}{2}\sum_\mu[\bar{\psi}(x-\hat{\mu})(r-\gamma_\mu)\psi(x) - \bar{\psi}(x)(r+\gamma_\mu)\psi(x-\hat{\mu})]$$

$$- \tfrac{1}{2}[\bar{\psi}(x)(r-\gamma_\mu)\psi(x+\hat{\mu}) - \bar{\psi}(x+\hat{\mu})(r+\gamma_\mu)\psi(x)\Big]$$

$$= 0$$

which implies that [27b]

$$J_\mu = \tfrac{1}{2}[\bar{\psi}(x-\hat{\mu})(r-\gamma_\mu)\psi(x) - \bar{\psi}(x)(r+\gamma_\mu)\psi(x-\hat{\mu})]. \qquad (8.21)$$

Putting r = 0 (naive case) we again obtain the earlier result (8.10).

With a little manipulation the following expression for the Wilson fermion action on a asymmetric lattice is obtained [27b]

$$S \equiv \sum_{x,x'} \bar{\psi}(x) Q_{x,x'} \psi(x')$$

$$\text{with } Q_{x,x'} = \delta_{x,x'} - K_\sigma \sum_{i=1}^{3} [(r-\gamma_i)\delta_{x,x'-i} + (r+\gamma_i)\delta_{x,x'+i}]$$

$$- K_\tau [f_1(\mu a)(r-\gamma_0)\delta_{x,x'-0} + f_2(\mu a)(r+\gamma_0)\delta_{x,x'+0}] \qquad (8.22)$$

and

$$K_\tau = (a_\sigma^3/2)[ma^3 a_\tau + 3ra_\tau a_\sigma^2 + ra_\sigma^3]^{-1} ,$$

$$K_\sigma = \xi^{-1} K_\tau = \tfrac{1}{2}[ma_\tau + 3r + \xi r]^{-1} \qquad (8.22a)$$

In eq.(8.22), $f_1(\mu a) \equiv 1+\mu a$ and $f_2(\mu a) \equiv 1-\mu a$, which can now be replaced by either (8.13) or (8.15).

The corresponding interacting theory at finite densities is obtained by making the following substitutions:

$$U_{x,4} \to e^{\mu a} U_{x,4}$$

$$U^\dagger_{x,4} \to e^{-\mu a} U^\dagger_{x,4} \qquad \text{or} \qquad (8.23a)$$

$$U_{x,4} \to [(1+\mu a)/(1-\mu^2 a^2)^{\frac{1}{2}}] U_{x,4}$$

$$U^\dagger_{x,4} \to [(1-\mu a)/(1-\mu^2 a^2)^{\frac{1}{2}}] U^\dagger_{x,4}. \qquad (8.23b)$$

The above apply to both Staggered and Wilson fermions.

The connection between chemical potential and the number of quarks present can be interpreted by using (8.23a).
The replacements $U \to e^{\mu a}U$ and $U^\dagger \to e^{-\mu a}U^\dagger$ describe the propagation of quarks in the positive and negative time directions respectively. Only those fermions that complete a full trajectory in the finite temporal direction will pick up factors of $e^{\pm\mu a}$, e.g. for n complete trajectories, the factors will be $e^{\pm n\mu a}$. Those who tumble back will acquire exponential factors that cancel.

8.2 MC results of finite density QCD

One seemingly insurmountable problem of finite density MC calculations is that the fermion determinant is complex. This is due to the fact that for $\mu \neq 0$ the U and $U^\dagger$ terms in Q are no longer hermitian conjugates.
[It is however important to note that due to the nature of integration over the Haar measure, the partition function remains real].

8.2.1 Quenched approximation

The first calculations for the quenched theory at finite densities were done by Kogut et al.[28a] using Staggered fermions to calculte $\langle J_0 \rangle$, which is defined by

$$\begin{aligned} \langle J_0 \rangle &= (\beta V)^{-1} \frac{\partial \ln Z}{\partial \mu} \\ &= (N_\sigma^3 N_\tau a_\sigma^3)^{-1} \frac{\partial \ln Z}{\partial (\mu a_\tau)} \end{aligned} \tag{8.24}$$

with J_0 as defined in (8.17).

Similarly, the chiral order parameter is obtained from

$$\begin{aligned} \langle \bar{\psi}\psi \rangle &= (\beta V)^{-1} \frac{\partial \ln Z}{\partial m} \\ &= (N_\sigma^3 N_\tau a_\sigma^3)^{-1} \frac{\partial \ln Z}{\partial (m a_\tau)} \cdot \end{aligned} \tag{8.25}$$

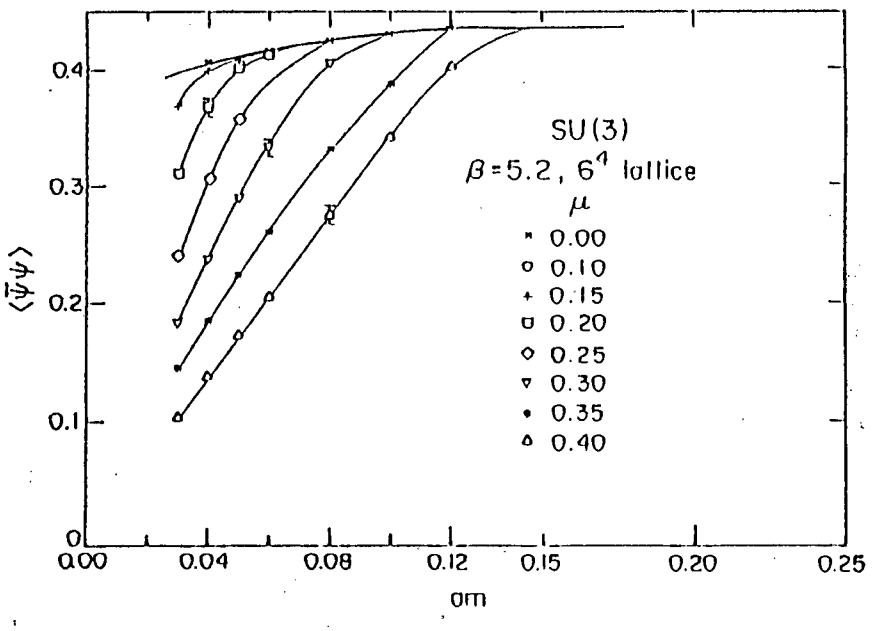


FIGURE 8.1 : The chiral order parameter $\langle\bar{\psi}\psi\rangle$ versus m for various values of μ with β = 5.2 evaluated on a 6^4 lattice (from [28a]).

Although the results for SU(3) suffered from finite size effects (a 6^4 lattice was used) they did find that chiral symmetry was abruptly* restored for the extrapolated data ($m \to 0$) at $\mu_{ch}a \approx 0{,}3$ (* although the transition was abrupt, there was not enough evidence for a first order transition). The results for the order parameter are shown in fig. 8.1. It is interesting to note that the MC results compared well with the Mean Field predictions.

8.2.2 Full theory

The behaviour of quarks and gluons at finite temperature and density was first investigated by Nakamura [28b] for SU(2) (using Wilson fermions). Although no definite predictions for T_{ch} and β_{ch} were obtained, it was found that the gluon energy density was influenced strongly by an increase in μ: for large values of the chemical potential the gluon gas exhibited a limiting behaviour far removed from the ideal gas one of the quenched theory. The quark energy density also showed more rapid increase compared to the $\mu = 0$ case.

The SU(3) deconfinement transition at finite baryon density has been investigated by Engels and Satz [28c] and Berg et al. [28d] using the hopping parameter expansion for Wilson fermions. Before discussing their results, we first give a brief review of the formalism used (which is only a slight modification of the results given in sec. 6.1.1).

From (6.3) it follows that a suitable modification of the effective action to include a finite chemical potential will have the form (to leading order):

$$S_F(L) = -2N_f(2K)^{N_\tau} \sum_{\vec{x}} [L(\vec{x})\, e^{N_\tau \mu a} + L^*(\vec{x})\, e^{-N_\tau \mu a}] \tag{8.26}$$

where $L(\vec{x})$ is the Polyakov loop defined in (6.4).

Using $L(\vec{x}) = \mathrm{Re}L(\vec{x}) + i\, \mathrm{Im}L(\vec{x})$, $S_f(L)$ will have the form

$$S_F(L) = -4N_f(2K)^{N_\tau} \sum_{\vec{x}} [\, \mathrm{Re}L(\vec{x})\cosh(N_\tau \mu a) + i\mathrm{Im}L(\vec{x})\sinh(N_\tau \mu a)]. \tag{8.27}$$

$$\text{with full partition function } Z = \int [dU]\, e^{-S_G(U) - S_F(L)}. \tag{8.28}$$

It is apparent from the above that $S_F(L)$ is only real if $\mu = 0$. Also, exchanging variables $U \to U^\dagger$ we see that

$$\mathrm{Im}Z = \int [dU]\, e^{-S_G(U) - \mathrm{Re}S_F(L)} \sin[-4N_f(2K)^{N_\tau} \sum_{\vec{x}} \mathrm{Im}\, L(\vec{x})\sinh(N_\tau \mu a)] \tag{8.29}$$

vanishes, which implies that Z is real.

It can now be shown that [28d]

$$\langle \mathrm{Im}L(x) \rangle = 0$$

$$\langle \mathrm{Im}\, S_F(L) \rangle \sim \langle \sum_{\vec{x}} L(\vec{x}) \rangle = 0 \tag{8.30}$$

which serves as motivation for making a "partial quenching" approximation in which $\mathrm{Im}S_F(L) = 0$ everywhere.

Monte Carlo calculations can now be performed using only the real part of the action together with $S_G(U)$ - explicit μ dependence of the physical quantities are guaranteed by the presence of the term $\mathrm{Re}\, S_F(L)$.

We also note here that we can write

$$\text{Re } S_F(L) = -4N_f(2K)^{N_\tau} \sum_{\vec{x}} \text{Re } L(\vec{x})\cosh(N_\tau \mu a)$$

$$\equiv -h \sum_{\vec{x}} \text{Re } L(\vec{x}) \qquad (8.31)$$

with $h = 4N_f(2K)^{N_\tau}\cosh(N_\tau \mu a)$. (8.32)

This form of the effective action will be discused in section 9 where analytical methods will be considered.

Calculations were performed on a $8^3 \times 3$ lattice with $N_f = 2$, using the lowest order (K^4) HPE. The results of [28d] are given in table 8.1. In fig. 8.2 the behaviour of <ReL> as a function of ß for different values of the chemical potential is plotted.

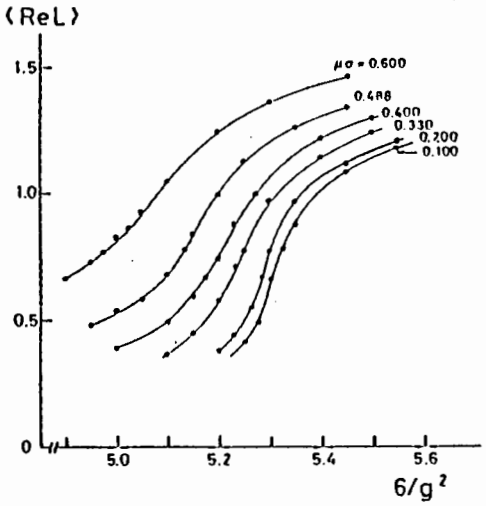


FIGURE 8.2 : <Re L> vs $6/g^2$ for two flavour QCD evaluated for different μa. Curves are drawn to guide the eye (from [28d]).

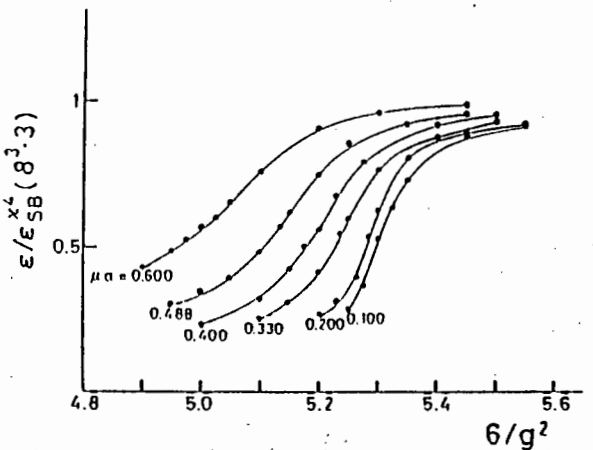


FIGURE 8.3 : Normalized energy density as a function of $6/g^2$ evaluated on a $8^3 \times 3$ lattice (from [28d]).

The results showed that the critical coupling had shifted to lower values (i.e. lower temperatures) as the chemical potential was increased but also that the transition became less abrubpt for higher values of μ.

The same behaviour was apparent for the total energy density, where

$$\varepsilon = V^{-1}[T^2(\frac{\partial \ln Z}{\partial T})_{\mu,V} + \mu T(\frac{\partial \ln Z}{\partial \mu})_{T,V}] \ . \qquad (8.33)$$

In fig. 8.3 the normalized* energy density is plotted as a function of the coupling (* the energy density is normalized to the value $\varepsilon_{SB}^{K^4}$ for an ideal gas of quarks and gluons calculated on a $8^3 \times 3$ lattice using a fourth order HPE).

The transition parameters can now be obtained from either ε or <ReL> for different values of μ. They are given in table 8.1, along with the critical temperatures using the (N_f= 2) RGE [(5.31)].

For the highest value of the chemical potential considered ($\mu/\Lambda_L \approx 200$), the critical temperature compared to the $\mu = 0$ case dropped by ± 25%:

$$T_c(\mu=0)/T_c(\mu/\Lambda_L \approx 200) \approx 0{,}74. \qquad (8.34)$$

Table 8.1 (from [28d])

μa	μ/Λ_L	β_c (ε)	$T_c/\Lambda_L(\varepsilon)$	β_c (ReL)	T_c/Λ_L(ReL)
0,10	45	5,299	151	5,288	149
0,20	89	5,282	148	5,294	150
0,33	139	5,239	180	5,239	140
0,40	161	5,194	132	5,214	136
0,488	184	5,151	125	5,128	126
0,60	203	5,063	112	5,077	114

To conclude this section, we note that all attempts so far to include the full fermionic determinant have met with little success [28e]. The results obtained from existing calculations are in general inconsistent with conventional ideas of chiral symmetry breaking. It seems therefore that the only reliable way to study systems at finite chemical potential is to use analytical approximation methods. These methods will be discussed in the next section.

9. Analytical methods in Finite Temperature LGT

Analytical methods are important to verify whether the results from MC calculations (sections 6 - 8) are physically meaningfull and not for example due to the specific approximation or the size of the lattice used. This is because in most cases, analytical methods can be used to evaluate the theory exactly. Also, as we have seen in section 8, MC calculations for systems at finite densities run into difficulties due to the complex determinant [28e].

9.1 Mean Field Techniques

In [29a] an anology was made between the full SU(3) theory (including non-zero chemical potential) and the Z(3) spin model in the high temperature - strong coupling limit*, i.e. the SU(3) QCD partition function can be approximated by the Z(3) partition function

$$Z = \sum_{z_x} \exp[\, \beta(a_\sigma/a_\tau) \sum_{x,i} \tfrac{1}{2} (z_x z^*_{x+i} + c.c) + h(K,\mu) \sum_x \tfrac{1}{2}(z_x + c.c)] \quad (9.1)$$

where $h(K,\mu)$ denotes the external magnetic field dependent on both the hopping parameter and chemical potential. In general h can have a real as well as imaginary part [29a].
[* the Strong Coupling limit here implies that $a_\tau/a_\sigma << 1$, so that only the time-like part of $S_G(U)$ and S_F are considered].

Using a MC and MF analysis of the Z(3) theory it was found that the first order transition of the quenched theory [with $\mu = 0$ (corresponding to $h = 0$)] was weakened by the external magnetic term and disappeared for suffiently large K (i.e. suffiently small m_q). Hence, for $h > h_{cr}$ the transition will be entirely absent, while at
$h = h_{cr}$ the transition will be second order. For SU(3) this was a confirmation of the MC results obtained in [19c]. It must however be kept in mind that the existence and value of h_{cr} is theory dependent. The Z(3) results therefore do not bear directly on the phase structure of the SU(3) theory.

An of SU(N) LGT at finite temperature and Strong Coupling was done by Green and Karsch [29d]. For the pure gauge theory they confirmed numerical calculations in obtaining a second order transition for N = 2, while

for $N \geq 3$ the phase transition was of first order. Using Wilson fermions, they utilized a HPE to incorporate the effect of quarks into the quenched theory. As their method can be used in a wide variety of forms we will discuss it briefly.

Consider the full QCD partition function defined in section 5:

$$Z = \int [dU] \det{}^{n} Q \; e^{-S_G(U)}, \tag{9.2}$$

where for Wilson fermions $n = N_f$, $S_G(U)$ is the Wilson pure gauge action and for a symmetric lattice we have that

$$Q = 1 - K^{i} M, \text{ with}$$

$$M = U^{\mu}_{x}(1-\gamma_{\mu})\delta_{x,x'-\mu} + U^{\mu\dagger}_{x'}(1+\gamma_{\mu})\delta_{x,x'+\mu} \; . \tag{9.2a}$$

Considering first the pure gauge sector (i.e. with $K = 0$), an approximation whereby the spacelike plaquettes are neglected for sufficiently small β is implemented [29d]. This effectively means that the partition function is evaluated under the constraint $U_{P_\sigma} = 1$ (where P_σ denotes the space-like plaquettes). We first rerwrite the pure gauge (Wilson) action in terms of a character expansion (see Chapter 6)

$$Z = \int [dU] \prod_{P_\tau} \Big[\sum_{r} d_r z_r \, \chi_r(U_{P_\tau}) \Big], \tag{9.3}$$

where the product is over all timelike plaquettes (P_τ) only. As all time-like plaquettes contain two spacelike links, these must now be integrated out (see fig. 9.1).

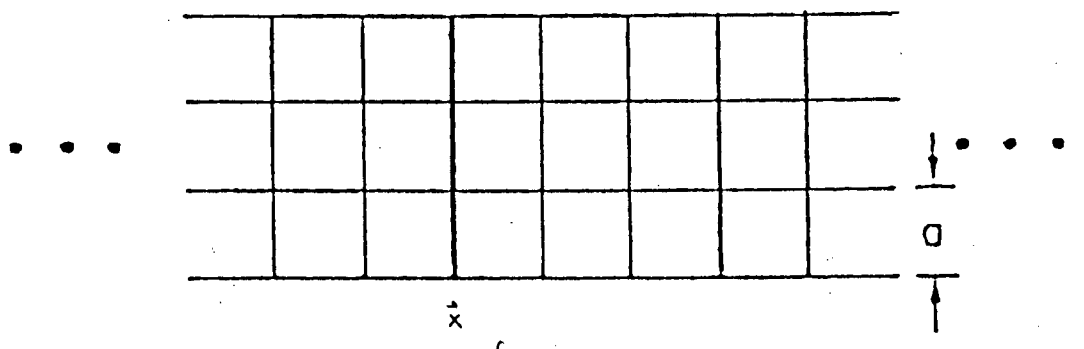


fig. 9.1

Consider one such ladder with $N_\tau = 4$ as example (fig. 9.2).

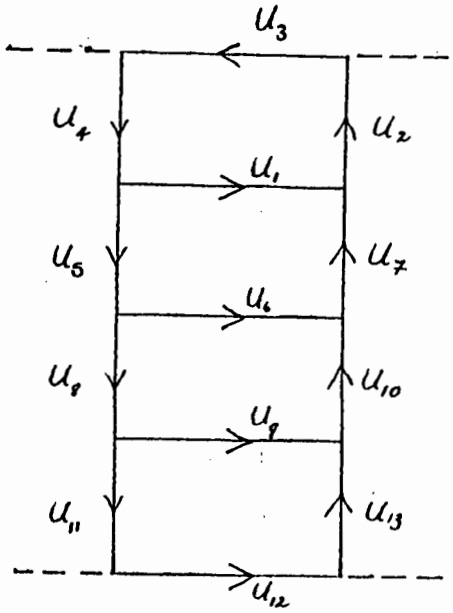


fig. 9.2

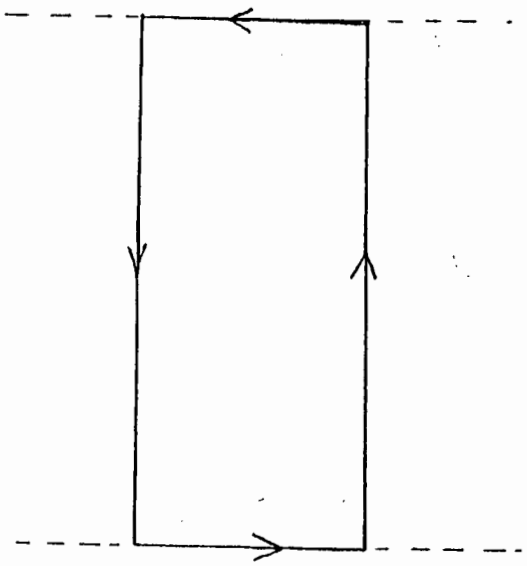


fig. 9.3

[Note that because the lattice is finite in the temporal direction (N_τ = 4) and the gauge fields satisfy periodic boundary conditions, the links U_3 and U_{12} will be the same (except for having different directions)].

The spacelike links (U_1, U_6, U_9 and U_{12}) can now be integrated out using the properties of group integrations (see Chapter 5, sec. IV), which gives the following effective partition function (up to a constant factor that includes the trivial character coefficient)

$$Z = \int \Big(\prod_{\vec{x}} dW_{\vec{x}} \Big) \prod_{\vec{x},\ell} \Big[1 + \sum_r z_r(\beta)^{N_\tau} \chi_r(W(\vec{x}))\, \chi_r(W^\dagger(\vec{x}+\hat{\ell})) \Big], \qquad (9.4)$$

where $W(\vec{x})$ is similar to the Polyakov loop at a spatial site $\vec{x}$:

$$W(\vec{x}) = \prod_{\tau=1}^{N_\tau} U_{(\vec{x},\tau),4} \qquad (9.5)$$

and with the first and second products in (9.4) running over all spatial sites and links of a three dimensional lattice respectively (see fig. 9.3).

A second approximation, namely that ß is small enough that only the fundamental term in (9.4) need be considered, gives

$$Z_{eff} \approx \int \left(\prod_{\vec{x}} dW_{\vec{x}} \right) \exp\left[\beta' \left(\sum_{\vec{x},\ell} \operatorname{tr} W(\vec{x}) \operatorname{tr} W^{\dagger}(\vec{x}+\hat{\ell}) + \text{c.c.} \right) \right]$$

$$= \int \left(\prod_{\vec{x}} dW_{\vec{x}} \right) \exp\left[2\beta' \left(\sum_{\vec{x},\ell} \operatorname{Re} \operatorname{tr} W(\vec{x}) \operatorname{tr} W^{\dagger}(\vec{x}+\hat{\ell}) \right) \right], \qquad (9.6)$$

where $\beta' = (z_f)^{N_\tau} \equiv (z_3)^{N_\tau}$ [SU(3)] . (9.7)

[In terms of the notation introduced in Chapter 6,

$$z_3 \equiv b_3/3 \quad (= t), \qquad (9.8)$$

where in general $b_r = c_r/c_0$, with c_r the character expansion coefficient (I.6.43)

$$c_r = \int [dU]\, \chi_r^*(U) \exp\left[\frac{\beta}{6} \left(\chi_3(U_p) + \chi_{\bar{3}}(U_p) \right) \right].$$

Explicitly, we have [29d] (see also Chapter 6)

$$z_3 = \beta + \frac{3}{2}\beta^2 - \frac{135}{24}\beta^4 + O(\beta^5), \qquad (9.9)$$

where $\beta = (3g^2)^{-1}$ (9.9a)

so that, to lowest order ($N_\tau = 2$)

$$\beta' = \beta^2 + 3\beta^3 + \frac{9}{4}\beta^4 + \ldots \quad . \qquad (9.10)$$

Equation (9.6) describes an effective (3 dimensional) spin model [29b;d] with the Hamiltonian formed by nearest neighbour interactions between the spins $\operatorname{tr} W(\vec{x})$.

Introducing fermions via the Wilson formalism, we have the following form of the effective action for a system with $\mu \neq 0$ (see sec. 8.2.2) [30a;b]

$$Z = \int \left(\prod_{\vec{x}} dW_{\vec{x}} \right) \exp \Big[2\beta' \Big(\sum_{\vec{x},\ell} \mathrm{Re\ tr}\, W(\vec{x})\ \mathrm{tr}\, W^{\dagger}(\vec{x}+\hat{\ell}) + 4N_f (2K)^{N_\tau} \sum_{\vec{x}} [\mathrm{Re\ tr}\, W(\vec{x}) \cosh(N_\tau \mu a) + i\,\mathrm{Im\ tr}\, W(\vec{x}) \sinh(N_\tau \mu a)] \Big]$$

$$\equiv \int [dW] \exp[-S_G(U) - S_F(W)]. \qquad (9.11)$$

The factor $h \equiv 4N_f(2K)^{N_\tau}$ may be considered as an external field: a non-zero h will imply that the global Z(N) symmetry of the pure gauge theory is broken. For $h = 0$, a MF analysis of eq.(9.11) will give the same self-consitent equations as obtained for the chiral model in [29b] (see Chapter 7, sec. 5B):

$$Z_{MF} \equiv Z_{ss} = \int dW \exp[2\beta'(2d)\ \mathrm{Re\ tr}\, W(x)\ \mathrm{tr}\, m - S_F(W)]$$

$$= \int dW \exp[4d\beta'\ \mathrm{Re\ tr}\, W - S_F(W)], \qquad (9.12)$$

where the number of nearest neighbour spatial links is given by

$$\sum_{\vec{x},\ell} \Rightarrow 2d. \qquad (9.12a)$$

The self-consistency equation is therefore given by

$$\langle \mathrm{Re\ tr} W \rangle = m, \qquad (9.13)$$

with $\langle \mathrm{Re\ tr} W \rangle = (12\beta' m)^{-1} \frac{\partial}{\partial m} Z_{ss}.$ (9.14)

Hence $F_{MF} = 6\beta' m^2 + F_{ss}$ (9.15)

with $F_{ss} = -\ln Z_{ss}$, (9.16)

so that $\frac{\partial}{\partial m} F_{MF} = 0$ (9.17)

where F_{MF} is the MF free energy.

The single site integral in eq.(9.12) can be rewritten in the form [30a;b]

$$Z_{ss} = \int dW \exp[(4d\beta' + f_1)\mathrm{Re\ tr}\ W + if_2 \mathrm{Im\ tr}\ W], \tag{9.18}$$

where $f_1 = 4N_f(2K)^{N_\tau} \cosh(n_\tau \mu a)$;

$$f_2 = 4N_f(2K)^{N_\tau} \sinh(n_\tau \mu a). \tag{9.19}$$

There is a remarkable similarity between the (MF) single site integral above and the full partition function of eq.(8.28) used for the MC calculations. This is also true if $W(\vec{x}) \in Z(3)$, in which case eq.(9.18) resembles the partition function defined earlier for the Z(3) spin model (eq.(9.1)).

For small μ ($\mu \approx 0$), the imaginary part of the external field h is small and can hence be neglected. This would correspond to the "partial quenching" approximation where $\mathrm{Im}S_F = 0$ was taken [28d]. For small μ one therefore obtains the following minimum value for f_1:

$$f_1 \rightarrow 4N_f(2K)^{N_\tau}, \tag{9.20}$$

while the partition function has the form of a single link integral (see discusssion in Chapter 5).

If μ is large enough however, both f_1 and f_2 will contribute. The single site partition function can be evaluated analogously to the chiral single site integral in [29b] (see Chapter 7), which gives [30a;b]

$$Z_{ss} = \sum_{\ell=-\infty}^{\infty} \exp(-3\ell\phi)\ \mathrm{det} I_{\ell+j-i}(\alpha) \tag{9.21}$$

where $\alpha = [(2d\beta' + f_1)^2 - f_2^2\ ]^{\frac{1}{2}}$ (9.22)

and $\phi = \cosh^{-1}[(2d\beta' + f_1)/\alpha]$. (9.23)

From eq.(9.15) it follows that the MF energy is given by

$$F_{MF} = 2d\beta' m^2 - \ln[\sum_{\ell=-\infty}^{\infty} \exp(-3\ell\phi)\ \mathrm{det} I_{\ell+j-i}(\alpha)]\ . \tag{9.24}$$

Results

A. Pure gauge theory ($K = \mu = 0$)

Celik et al. [30a;b] have evaluated the single site integral [eq.(9.21)] using a series expansion of the modified Bessel functions (up tp 30 th order in α). In fig. 9.4 the MF free energy (eq.(9.22) is plotted as a function of $\langle L(\vec{x})\rangle$ (= $\langle trW\rangle$ = m) for different values of $2d\beta'$ and $N_\tau = 2$.

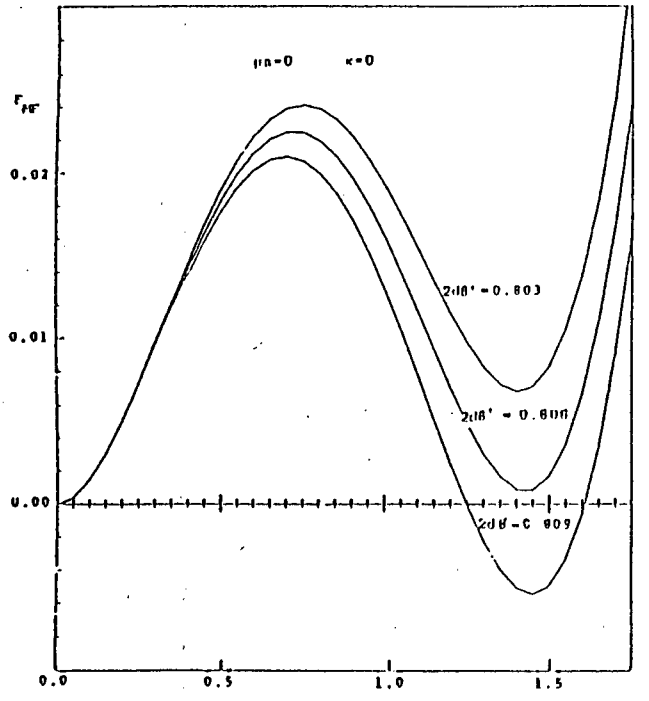


FIGURE 9.4 : Mean field energy F_{mF} versus $\langle L(x)\rangle$ for several values of $2d\beta'$ with $\mu a = 0 = K$ (from [30a;b]).

For $2d\beta' = 0{,}806$ (i.e. $d = 3 \Rightarrow \beta' = 0{,}134$), there are two degenerate minima which confirms MC results that the phase transition is first order.

B. Full theory ($\mu = 0$)

In [19b;d], MC results showed a second order transition at β_c $(=6/g_c^2) \approx 5{,}35$ (see table 6.1) for $N_f = 2$, $N_\tau = 3$ and $K = 0{,}12$ which corresponds to and external field value

$$h = 2N_f(2K)^{N_\tau} = 0{,}055. \qquad (9.25)$$

For $N_\tau = 3$ and $h \equiv f_1 = 0{,}059$ (value for which the line of first order transitions end [30b]), the value obtained for $\beta_c = 6/g_c^2$ was 5.94, with the transition now of second order and $\Delta(6/g_c^2) = 0{,}59$. There is therefore a remarkable correspondence between the MC and MF results.

C. Full theory ($\mu \neq 0$)

In [30a] the MF energy was evaluated for K =0,05 (which fixes the value of m_q by virtue of the relation (6.7) with the (SC) value for K_{crit}) and $\mu a = 0{,}1$. This is shown in fig. 9.5. For $2d\beta' = 0{,}7495$, two degenerate minima characteristic of a first order phase transition were obtained. It must however be kept in mind that the quark mass that was used is very high, so that no qualitative predictions regarding T_c etc. can be made.

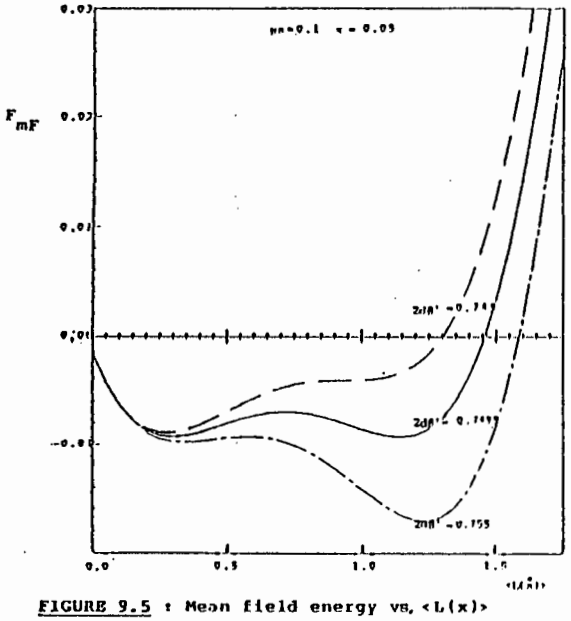


FIGURE 9.5 : Mean field energy vs. $\langle L(x)\rangle$ calculated at fixed values $\mu a = 0.1$ and $K = 0.05$ (from [30a]).

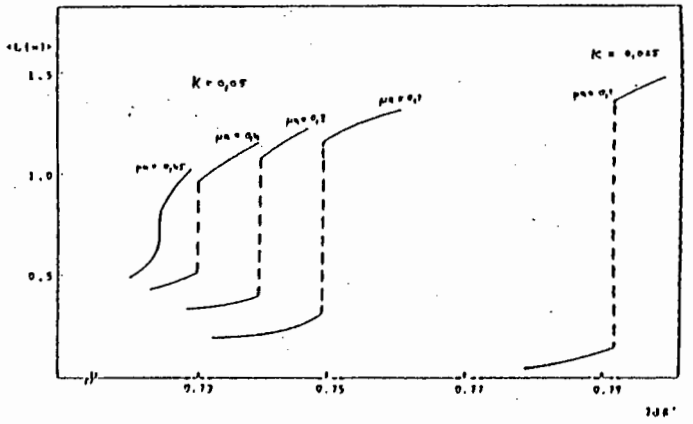


FIGURE 9.6 : Value of the mean field vs. $2d\beta'$ for two values of the hopping parameter (from [30a]).

To obtain the critical curve in the T-μ plane, the MF was evaluated for different values of β' with different sets of values for μ and K. This is shown in fig. 9.6. For K =0,5 and $\mu \approx 0{,}45$, the sharp change in the value of $\langle L(\vec{x})\rangle$ disappeared, which implies that the two degenerate minima of F_{MF} coincides, i.e. the transition is now second order.

9.2 Extreme Strong Coupling Limit

Another possible analytical approach to LGT is to use an effective action in the extreme SC limit (where the contribution from the pure gauge theory is totally neglected, i.e. the limiting value $g^2 = \infty$ is taken). The advantage in considering the SC limit is that the effective action is linear in the link variables $U_{x,\mu}$. If, in addition, gauge fixing is also applied, then the integration over the link variables can be greatly facilitated. The resulting effective action is a functional of only the fermionic variables χ and $\bar{\chi}$, which can be integrated out using the standard rules of Grassmann integration (see App. C). It is important to note that in this ultra SC limit there can be no deconfinement, so that only the chiral transition is generally investigated.

We will now discuss the SC method proposed by Damgaard et al. [32a;b], altough other similar techniques have been applied (see [31;32c-f]).

Let us first consider QCD at finite temperature, with colour gauge group U(N) in d+1 dimensions. A finite chemical potential will be considered later. The choice of U(N) (which has no non-trivial centre) implies that only meson states will be present. In addition, to obtain a consistent continuum flavour interpretation for the Staggered fermions, N_τ must be even (see Chapter 4). The lattice action in this limit is given by (N_f= 1)

$$S = \sum_{\tau=1}^{\beta} \sum_{\vec{x}} \left[-m\bar{\chi}_x\chi_x - \tfrac{1}{2} \sum_{\mu=0}^{d} \eta_\mu(x)[\bar{\chi}_x U_{x,\mu}\chi_{x+\mu} - \bar{\chi}_{x+\mu}U^\dagger_{x,\mu}\chi_x] \right], \qquad (9.26)$$

with corresponding partition function

$$Z = \int [dU][d\chi d\bar{\chi}]\, e^S , \qquad (9.27)$$

where $[dU] = [dU_0][dU_j]$ (with U_0 and U_j describing the temporal and spacelike links respectively and $U_0 \equiv U_0(\vec{x},\tau)$ etc.), $[d\chi] = \prod_x d\chi_x$ etc.. (also note that β now stands for $\beta = N_\tau a_\tau = T^{-1}$)

As the gauge variables satisfy periodic boundary conditions (2.1), a gauge can be found in which the gauge variables U_0 are time-independent and diagonal [29c] (see also Chapter 7)

$$U_0(\vec{x},\tau) = D(\vec{x}) = \mathrm{diag}\left(e^{i\phi_1(\vec{x})/\beta},\ldots,e^{i\phi_N(\vec{x})/\beta}\right) . \qquad (9.28)$$

The corresponding expression for the Haar measure is

$$[dU_0][dU_j] = \prod_{b=1}^{N}\left(\frac{d\theta_b}{2\pi}\right) |\Delta(e^{i\theta_1},\ldots,e^{i\theta_N})|^2 \, [dU_j]$$

$$\equiv \prod_{b=1}^{N}\left(\frac{d\theta_b}{2\pi}\right) \Delta[D] \, [dU_j]. \qquad (9.29)$$

The action now reads

$$S = \sum_{\tau=1}^{\beta} \sum_{\vec{x}} \Big[-m\bar{\chi}_x\chi_x - \tfrac{1}{2}\sum_{j=o}^{d}\eta_j(x)[\bar{\chi}_x U_{x,j}\chi_{x+j} - \bar{\chi}_{x+j}U^{\dagger}_{x,j}\chi_x]$$

$$-\tfrac{1}{2}[\bar{\chi}_x D(\vec{x})\chi_{x+o} - \bar{\chi}_{x+o}D^{*}(\vec{x})\chi_x]\Big] . \qquad (9.30)$$

As the fermionic action is linear in $U_{x,j}$, the integration can be done explicitly. Considering only the term containing the space-like links we have

$$I_1 \equiv \prod_{\vec{x}}\prod_{j=1}^{d}\int dU_{x,j}\, \exp\left[\tfrac{1}{2}\eta_j(x)[\bar{\chi}^a_x U^{ab}_{x,j}\chi^b_{x+j} - \bar{\chi}^c_{x+j}U^{\dagger cd}_{x,j}\chi^d_x]\right]$$

$$= \prod_{\vec{x}}\prod_{j=1}^{d}\left[1 + \frac{1}{2!}\frac{1}{4}\int dU_{x,j}\,(-2)\bar{\chi}^a_x\chi^b_{x+j}\bar{\chi}^c_{x+j}\chi^d_x\, U^{ab}_{x,j}\, U^{\dagger cd}_{x,j} + \ldots\right]$$

$$= \exp\left[-\frac{1}{4N}\sum_{\vec{x}}\sum_{j=1}^{d}\bar{\chi}^a_x\chi^b_{x+j}\bar{\chi}^b_{x+j}\chi^a_x + \text{higher orders}\right], \qquad (9.31)$$

where the colour indices are explicitly shown. The four fermion (meson) term corresponds to the leading term in a $\frac{1}{d}$ expansion, which has been shown to offer a good approximation to the full effective action [31a;b].

The action (9.30) now has the form

$$S = \sum_{\tau=1}^{\beta} \sum_{\vec{x}} \Big(-m\bar{\chi}_x\chi_x - \frac{1}{4N} \sum_{j=1}^{d} \bar{\chi}_x^a \chi_{x+j}^b \bar{\chi}_{x+j}^b \chi_x^a$$

$$-\tfrac{1}{2}[\bar{\chi}_x D(\vec{x})\chi_{x+o} - \bar{\chi}_{x+o} D^*(\vec{x})\chi_x]\Big). \qquad (9.32)$$

This integration has therefore yielded an effective (fermion) action involving only interactions between Fermi fields at equal times τ at nearest sites in space. We note in passing that if we define [32b]

$$J^\dagger_{x,j} = \tfrac{1}{2}\eta_j(x)\bar{\chi}_x^b \chi_{x+j}^a$$

$$J_{x,j} = -\tfrac{1}{2}\eta_j(x)\bar{\chi}_{x+j}^b \chi_x^a \,, \qquad (9.33a)$$

then we can use the results of the one-link integrals given in Chapter 5 as all the U_j links will have the form

$$\int dU_j \exp[\mathrm{tr}(U_j^\dagger J_j + J_j^\dagger U_j)]. \qquad (9.33b)$$

In order to simplify integration over the Grassmann fields, the meson-meson factor in the four fermion term is decoupled by introducing an auxilliary field σ_x by means of the following identity (see also [32c;d])

$$\exp\Big[\frac{1}{4N}\chi_x^a \bar{\chi}_x^a \chi_{x+j}^b \bar{\chi}_{x+j}^b\Big]$$

$$= \int [d\sigma]\exp\Big[-\frac{1}{4}N\sigma_x\sigma_{x+j} + \frac{1}{4}\sigma_x\chi_{x+j}^a\bar{\chi}_{x+j}^a + \frac{1}{4}\sigma_x\chi_x^a\bar{\chi}_x^a\Big], \qquad (9.34)$$

where we have the following correspondence

$$\sigma_x \approx \frac{1}{N}\chi_x^a\bar{\chi}_x^a \qquad (9.35a)$$

which can be made precise by using the (vacuum) expectation values, i.e.

$$\langle\sigma\rangle = \frac{1}{N}\langle\chi\bar{\chi}\rangle. \qquad (9.35b)$$

Making the approximation that σ_x is a classical field (i.e. $\sigma_x \approx \sigma$ or in other words expanding σ_x around its constant value and ignoring the corrections) we have the following expression for the action (9.32) (using also the identity (9.34))

$$S = \sum_{\tau=1}^{\beta} \sum_{\vec{x}} \Big(-m\bar{\chi}_x\chi_x - \frac{1}{4} Nd\sigma^2 - \tfrac{1}{2}\sigma d\bar{\chi}_x\chi_x$$

$$-\tfrac{1}{2}[\bar{\chi}_x D(\vec{x})\chi_{x+o} - \bar{\chi}_{x+o} D^*(\vec{x})\chi_x]\Big). \qquad (9.36)$$

Introducing a Fourier transform (to accomodate the antiperiodicity constraint of the fermion fields) we have

$$\chi(\vec{x},\tau) = (1/\sqrt{\beta}) \sum_{m=1}^{\beta} \exp[2\pi i(m+\tfrac{1}{2})\tau\beta^{-1}]\ \chi_m(\vec{x}) \qquad (9.37)$$

and similarly for $\bar{\chi}(\vec{x},\tau)$.

Substituting the above into the action (and performing the $\sum_{\tau}$ summation) we obtain

$$S = \sum_{\vec{x}} \Big(-\frac{1}{4} Nd\beta\sigma^2 - \sum_{m=1}^{\beta} \sum_{b=1}^{N} \bar{\chi}_m(\vec{x})\{(m+\tfrac{1}{2}\sigma d)$$

$$+ i \sin([2\pi(m+\tfrac{1}{2}) + \theta_b]/\beta)\}\chi_m(\vec{x})\Big). \qquad (9.38)$$

We note in passing that for systems at zero temperature (including the possibility of a non-zero chemical potential), the requirement of introducing a Fourier transform like the one in (9.37) is not necessary, and the integrations over the time-like links can be done similarly to the space-like links in (9.31) (see [32c;d],[31b;c])

Integrating out the Grassmann variables (see App. C) and omitting the mass term (which may be reinstated by a shift $\sigma \to \sigma + 2m/d$) (9.39) gives the determinant

$$\prod_{b=1}^{N} \prod_{m=1}^{\beta} \Big[\tfrac{1}{2}d\sigma + i\sin([2\pi(m+\tfrac{1}{2}) + \theta_b]/\beta)\Big], \qquad (9.40)$$

so that the action (9.38) now reads

$$S = \sum_{\vec{x}} \Big(-\frac{1}{4} Nd\beta\sigma^2 - \sum_{m=1}^{\beta} \sum_{b=1}^{N} \ln[\tfrac{1}{2}d\sigma + i\sin([2\pi(m+\tfrac{1}{2}) + \theta_b]/\beta)\Big) \qquad (9.41)$$

and only the interpretation over the auxilliary field σ and the time like links remaining, i.e.

$$Z = \int [d\sigma][dU_o]e^S \ . \tag{9.42}$$

The fact that ß is even (which has already been used in the integration over the Grassmann variables) can be used to give the following identity (obtained in [30a;b])

$$\prod_{m=1}^{ß} [\sinh(s) - i\cosh(\tfrac{2\pi m}{ß} + \theta)] = 2^{1-ß}[\cosh(ßs) - i^{ß}\cos(ß\theta)]. \tag{9.43}$$

Making the identification

$$\tfrac{1}{2}d\sigma \equiv \sinh(s), \tag{9.44}$$

we can rewrite the expression in (9.40) (and in the process get rid of the product $\prod_m$)

$$I_2 \equiv \prod_b \prod_m \left[\tfrac{1}{2}d\sigma + i\sin([2\pi(m+\tfrac{1}{2}) + \theta_b]/ß)\right]$$

$$= \prod_b \prod_m \left[\sinh(s) - i\cos(\tfrac{2\pi m}{ß} + \theta' + \tfrac{\pi}{2})\right] \qquad [\theta' \equiv (\pi+\theta_b)/ß]$$

$$= \prod_b \left[2^{1-ß}[\cosh(ßs) - i^{ß}\cos(ß\theta)]\right] \qquad [\theta \equiv \theta'+\tfrac{\pi}{2}]$$

Using $\cos(ß\theta) = (-1)^{ß/2 + 1}\cos(\theta_b)$ we therefore have

$$I_2 = \prod_b \left[2^{1-ß}[\cosh(ßs) + \cos(\theta_b)]\right] \tag{9.45}$$

$$= 2^{-ßN} \prod_b [2\cosh(ßs) + 2\cos(\theta_b)]. \tag{9.45a}$$

The partition function (suitably nomalized) now reads

$$Z = \int [d\sigma][dU_o] \exp\left[- \tfrac{1}{4} Ndß \sum_{\vec{x}} \sigma^2\right] \prod_{\vec{x}} \prod_b W(\theta_b), \tag{9.46}$$

where $W(\theta_b) = 2\cosh(ßs) + 2\cos(\theta_b)$. (9.46a)

Considering first the integration over the time-like links, we can use the expression for the Vandermonde determinant given in Chapter 7, which gives the general result (see also [33])

$$\int [dU_o] \prod_b W(\theta_b) = \prod_b \left(\int_{-\pi}^{\pi} \frac{d\theta_b}{2\pi} W(\theta_b) \Delta[D] \right)$$

$$= N! \det(I_{|j-i|}), \tag{9.47}$$

where $I_n \equiv \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} W(\theta_b) e^{i\theta}$ (9.47a)

and the factor N! was obtained because the particular form of $W(\theta)$ implies that $I_n = I_{-n}$.

From (9.47a) it follows that

$$I_o = 2\cosh(\beta s) \; ; \; I_1 = 1 \; ; \; I_n = 0 \quad (n \geq 2). \tag{9.48}$$

The determinant $C_N \equiv \det(I_{|n-m|})$ (9.49)

can now be evaluated using the following recursion relation:

$$C_N = I_o C_{N-1} - I_1^2 C_{N-2}, \tag{9.50}$$

which was obtained exactly from the orthogonal polynomial method [33].

Further evaluation of (9.50) now gives

$$\begin{pmatrix} C_N \\ C_{N-1} \end{pmatrix} = \begin{pmatrix} I_o & -I_1^2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_{N-1} \\ C_{N-2} \end{pmatrix}$$

$$= \begin{pmatrix} I_o & -I_1^2 \\ 1 & 0 \end{pmatrix}^{N-2} \begin{pmatrix} C_2 \\ C_1 \end{pmatrix}, \tag{9.51}$$

which follows as the first matrix on the RHS is independent of N.

Now $c_2 = \frac{1}{2!} \varepsilon_{i_1 i_2} \varepsilon_{j_1 j_2} I_{|j_1 - i_1|} I_{|j_2 - i_2|}$

$$= 4\cosh^2(\beta s) - 1;$$

and $$c_1 = I_0 = 2\cosh(\beta s), \tag{9.52}$$

so that

$$\begin{pmatrix} C_N \\ C_{N-1} \end{pmatrix} = \begin{pmatrix} I_0 & -I_1^2 \\ 1 & 0 \end{pmatrix}^{N-2} \begin{pmatrix} 4\cosh^2(\beta s) - 1 \\ 2\cosh(\beta s) \end{pmatrix}. \tag{9.53}$$

C_N can now be written in terms of the eigenvectors $\lambda_\pm$ (see $[32]_I$)

$$C_N = \frac{\lambda_+^{N+1} - \lambda_-^{N+1}}{\lambda_+ - \lambda_-}, \tag{9.54}$$

where $$\lambda_+ + \lambda_- = I_0 = 2\cosh(\beta s)$$

$$\lambda_+\lambda_- = I_1^2 = 1. \tag{9.55}$$

Using the expressions in (9.55), the value of λ_+ and λ_- can be obtained, which finally gives

$$C_N = \frac{\sinh[(N+1)\beta s]}{\sinh(\beta s)}. \tag{9.56}$$

The final form of the partition function (9.46) is therefore

$$Z = \int [d\sigma] \prod_{\vec{x}} \left(\exp[-\tfrac{1}{4} N d\beta\sigma^2] \, C_N \right)$$

$$\equiv \int [d\sigma] \exp[-\sum_{\vec{x}} V_{eff}], \tag{9.57}$$

where $$V_{eff}(\sigma) = \tfrac{1}{4} N d\beta\sigma^2 - \ln\left(\frac{\sinh[(N+1)\beta s]}{\sinh(\beta s)}\right). \tag{9.58}$$

For sufficiently small σ the effective potential is of the form [30a;b]

$$V_{eff} = \tfrac{1}{4} N\beta d(1 - \tfrac{1}{6}(N+2)\beta d)\sigma^2 + O(\sigma^4). \tag{9.59}$$

From $$(\partial^2 V_{eff}/\partial\sigma^2)\big|_{\sigma=0} = 0 \tag{9.60}$$

it follows that $$\tfrac{1}{6}(N+2)\beta d = 1, \tag{9.60a}$$

which, after restoring the appropriate lattice spacings a_τ and a_σ show a clear second order phase transition at the critical temperature $T_{ch} = (a_\tau \beta_{ch})^{-1}$ determined by

$$\tilde{\beta} = (a_\sigma/a_\tau)^2 \beta_{ch} = \frac{6}{(N+2)d} \cdot \tag{9.61}$$

For $T < T_{ch}$ the system will be in a chiral symmetric phase. The effect of the bare quark mass can be incorporated by the shift (9.39). The resulting effective potential will contain a mass term with negative coefficient which is linearly coupled to σ, so that the phase transition is completely washed out. For a futher discussion see [30b].

The next logical step would be to include a finite chemical potential. For $\mu \neq 0$ and $T = 0$ this has been discussed by Damgaard et al. [32c]. Working with colour gauge group SU(N), they considered the effects of the baryon terms (in addition to the meson terms), which have the form

$$B_x = d^{N/4} \frac{1}{N!} \varepsilon_{i_1 \ldots i_N} \chi_x^{i_1} \cdots \chi_x^{i_N} \cdot \tag{9.62}$$

After integrating out both the spatial and timelike links, the following form of the effective action (to leading order) was obtained

$$S = \sum_{\vec{x}} \Big[\sum_j \Big(\frac{1}{4N} M_x^a M_{x+j}^b + \frac{1}{N!} (\det A_j + \det \bar{A}_j) + \frac{1}{4N} M_x^a M_{x+o}^b + \frac{1}{N!} (e^{N\mu} \det A_o + e^{-N\mu} \det \bar{A}_o) \Big], \tag{9.63}$$

where $M_x^a = \chi_x^a \bar{\chi}_x^a$;

$$\det A_j = \frac{1}{N!} (-1)^N (\tfrac{1}{2})^N [\eta_j(x)]^N \varepsilon_{i_1 \ldots i_N} \varepsilon_{j_1 \ldots j_N} \bar{\chi}_x^{i_1} \chi_{x+j}^{j_1} \cdots \bar{\chi}_x^{i_N} \chi_{x+j}^{j_N};$$

$$\det \bar{A}_j = \frac{1}{N!} (\tfrac{1}{2})^N [\eta_j(x)]^N \varepsilon_{i_1 \ldots i_N} \varepsilon_{j_1 \ldots j_N} \bar{\chi}_{x+j}^{i_1} \chi_x^{j_1} \cdots \bar{\chi}_{x+j}^{i_N} \chi_x^{j_N}. \tag{9.64}$$

Collecting the baryon terms in a baryon potential (V_B) and again introducing an auxilliary field σ for the meson terms, the following form for the partition function is obtained

$$Z = \int [d\sigma] \exp\Big[- \sum_x \frac{Nd}{4} \sigma^2 + \sum_x \ln[\sigma^N + (\tfrac{2}{d})(-1)^{N(N-1)/2} V_B] \Big], \tag{9.65}$$

which is valid for N even and odd, and where V_B contains the chemical potential factors $e^{\pm N\mu}$ as the meson fields carry zero net quark number.

For large values of μ, eq.(9.65) would correspond to a one parameter action

$$Z = \int [d\sigma] \exp[- \sum_x V_{eff}]$$

$$\text{with} \quad V_{eff} = \frac{1}{4} Nd\sigma^2 - \ln(\sigma^N + d^{-N} e^{N\mu}). \tag{9.66}$$

An analysis of V_{eff} for the different gauge groups in four dimensions gave the following predictions:

for $N = 2$ the transition was second order, while for $N \geq 3$ all transitions were found to be of first order (only the cases of $N = 2,3,4$ were explicitly calculated).

If the baryon terms are neglected, the effective potential reduces to

$$V_{eff} \approx \frac{1}{4} Nd\sigma^2 - \ln\sigma \ , \tag{9.67}$$

which shows remarkable correspondence to the earlier result in (9.58).

For SU(2) the effect of the presence of the baryon mass terms manifests itself in baryonic condensate <B> which is non-zero at finite μ [32d] (while for most other gauge groups (i.e. $N \geq 3$) the baryon condensate is either zero or very small [32e;f]). Dagotto et al. have claimed that the nonvanishing baryon condensate is responsible for the breaking of chiral symmetry, so that no transition occured at $g^2 = \infty$, even though the mesonic condensate vanished for all non-zero chemical potentials (and T=0).

10. Concluding Remarks

The significance of MC results lies in the fact that they provide the first numerical investigation of QCD properties such as the phase transitions signalling confinement and chiral symmetry restoration. In the preceding, I gave a review of QCD on the lattice at finite temperature and density. An attempt was made to include most relevant data up to the end of 1987, and to update existing reviews on the subject.

For the pure (quenched) theory results seem to be unanimous in predicting a first order transition. However, for the full theory where the effect of quarks are also included, results at present are still not conclusive. From the latest results it is apparent that some kind of phase transition (be it strong or fluctuation driven first order or even second order) is indeed present, which for $N_f \geq 3$ is in fair agreement with theoretical predictions based on the investigation of the sigma model [22b]. The uncertainties in the values of the critical quantities (e.g. m_{ch}, T_{ch}) are to a large extent due to the inadequate size of the lattices used in the calculations (in turn a result of present day limits on computer technology) and the algorithms that are used to approximate the full theory.

Regarding systems at finite baryon density, the situation at present is even more uncertain [28e]. Calculations have been done using the so-called partial quenching approximation (sec. 8.2), but in order to obtain qualitative estimates for the critical quantities the full (complex) determinant must be considered. Although several methods have been proposed (see e.g. [35]), the results obtained up till now are not combatible with the generally assumed behaviour of systems at finite densities.

Analytical methods have led to several interesting results. Some of the most important of these and their results have been discussed in section 9. The SC method is based on the assumption that the effects of the gauge plaquettes can be ignored to lowest order. This greatly facilitates the integration over the link variables (usually one of the biggest problems facing analytical techniques) as these are now only present in the quark-gluon interaction terms of the fermion action. Except for the case of SU(2), SC methods, combined with a mean field analysis, applied to lattice SU(N) at finite density have predicted a

first order chiral transition for $N \geq 3$ [32c;d;e] in agreement with (quenched) MC results.

As a further probe of the phase diagram, calculations have been done for finite T at zero baryon density, using a SC analysis for the U(N) theory. The transition was found to be of second order [32b]. A more general method that can be applied to both U(N) and SU(N) systems has been presented recently [36]. The necessary condition for the occurence of a phase transition was determined, but no explicit numerical calculation was given. The possibility exists of extending this method to include baryon terms, which in turn could be used to include a finite chemical potential in the theory.
A pure MF analysis of SU(N) has shown that the line of first order transitions end in a critical point (corresponding to a critical value of an external "magnetic" field) [30a;b].

The investigation of the intermediate region of the T-μ phase diagram (i.e. with $T,\mu \neq 0$) is still in a preliminary stage. Predictions have been made that for SU(N), with $N > 3$ (N even), the theory in the SC limit has a chiral symmetry restoration phase transition. For low T and high μ the transition is of first order, while for low μ and high T it is of second order [33f].

To conclude, the possibilities for future research using analytical methods are still far from being exhausted. In addition, it is not unreasonable to expect that MC methods will provide more conclusive results in the near future.

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APPENDIX A: METRIC NOTATIONS

Minkowski Space

Using the covariant notation we define the metric tensor

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} \tag{A.1}$$

such that, if $x^{\mu} = (x^0, x^1, x^2, x^3)$

$$\equiv (x^0, -\vec{x}) \tag{A.2}$$

then $x_{\mu} = g_{\mu\nu} x^{\nu} = (x_0, x_1, x_2, x_3)$

$$= (x^0, -\vec{x}) \tag{A.3}$$

where $x^0 = x_0 = t.$ $(c = 1)$ (A.4)

The metric of the Minkowski space is therefore given by

$$s^2 = g_{\mu\nu}\, x^{\mu} x^{\nu} = x_{\mu} x^{\mu}$$

$$= x_0^2 - \vec{x} \cdot \vec{x} \,. \tag{A.5}$$

The derivatives are defined by

$$\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \tag{A.6}$$

$$\partial^{\mu} \equiv \frac{\partial}{\partial x_{\mu}} = \left(\frac{\partial}{\partial t}, -\vec{\nabla}\right) \tag{A.7}$$

where $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ (A.8)

such that $\partial_{\mu}\partial^{\mu} = \frac{\partial^2}{\partial x_0^2} - \vec{\nabla}.\vec{\nabla}$ (A.9)

The momentum vector is given by

$$p^{\mu} = (E\ ,\ \vec{p}) \tag{A.10}$$

Euclidian Space

For Euclidian space we define the positive metric

$$\delta^{\mu\nu} = \delta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{A.11}$$

The Wick rotation implies that

$$x_4 = ix_0$$
$$(\text{or } \tau = it); \tag{A.12}$$

$$p_4 = ip_0 \, . \tag{A.13}$$

Therefore

$$s^2 = x_\mu x^\mu = \left(- \vec{x} \cdot \vec{x} - x_4^2\right)$$

$$= - \left(\vec{x} \cdot \vec{x} + x_4^2\right) \, . \tag{A.14}$$

APPENDIX B : DIRAC MATRICES

Minkowski Space

The γ matrices satisfy the anti-commutation relations

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$$

$$= 2g^{\mu\nu} \tag{B.1}$$

where $\gamma^\mu = (\gamma^0,\vec{\gamma})$ (B.2)

$$\gamma_\mu = g_{\mu\nu}\,\gamma^\nu = (\gamma^0,-\vec{\gamma}) \tag{B.3}$$

with γ^0 the hermitian and γ^i the anti-hermitian,

$$(\gamma^0)^\dagger = \gamma^0$$

$$(\gamma^i)^\dagger = -\gamma^i \tag{B.4}$$

Also $\gamma^5 = \gamma_5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3$

$$= -i\,\gamma_0\gamma_1\gamma_2\gamma_3 \tag{B.5}$$

$$(\gamma^5)^2 = I \tag{B.6}$$

$$\{\gamma_5,\gamma^\mu\} = 0 \tag{B.7}$$

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu,\gamma_\nu] \tag{B.8}$$

$$\gamma_\mu\partial^\mu \equiv \gamma_0\partial^0 + \gamma_i\partial^i = \gamma_0\frac{\partial}{\partial x_0} + (-\vec{\gamma})\cdot(-\vec{\nabla}) = \gamma_0\frac{\partial}{\partial t} + \vec{\gamma}\cdot\vec{\nabla}$$

$$= \gamma^\mu\partial_\mu\ . \tag{B.9}$$

Representations for γ matrices

The most frequently used representation which satisfies (B.1) is the Dirac representation with

$$\gamma^0 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \tag{B.10}$$

where σ^i are the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{B.11}$$

Euclidian Space

For the Euclidian space definitions of the γ matrices we can make use of one of two possible choices.

1. Hermitian choice

Here γ_0 is left unaltered while γ^i is changed

$$(\gamma_0)_E = (\gamma_0)_D = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{B.12}$$

$$\begin{aligned} (\vec{\gamma})_E &= -i(\vec{\gamma})_D \\ &= -i \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -i\vec{\sigma} \\ i\vec{\sigma} & 0 \end{pmatrix} . \end{aligned} \tag{B.13}$$

Note that

$$(\gamma_i)_E = +(\gamma^i)_E \quad \text{because } \gamma_\mu = \delta_{\mu\nu}\, \gamma^\nu \tag{B.14}$$

$$(\gamma_5)_E = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} . \tag{B.15}$$

This choice of γ matrices satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2\, \delta^{\mu\nu} \tag{B.16}$$

with $$(\gamma_0)^2 = (\gamma_i)^2 = I \tag{B.17}$$

Also $$\begin{aligned} \gamma_\mu \partial^\mu &\equiv \gamma_0 \partial^0 + \partial_i \partial^i \\ &= \gamma_0 \partial^0 + \vec{\gamma} \cdot \vec{\nabla} \end{aligned} \tag{B.18}$$

2. Antihermitian choice

We can define Euclidian γ matrices which satisfy

$$\{\gamma_\mu,\gamma_\nu\} = -2\delta_{\mu\nu} \ . \tag{B.19}$$

This would correspond to the choice

$$\begin{aligned} (\gamma_o)_E &= -i(\gamma_o)_D \\ (\vec{\gamma})_E &= (\vec{\gamma})_D \end{aligned} \tag{B.20}$$

with

$$\begin{aligned} (\gamma_o)_E^\dagger &= -(\gamma_o)_E \\ (\vec{\gamma})_E^\dagger &= -(\vec{\gamma})_E \end{aligned} \tag{B.21}$$

APPENDIX C: GRASSMANN VARIABLES

1. Motivation

For boson fields (e.g. spin 0) the vacuum expectation value of a time ordered product of fields is represented in the path integral formalism by a functional integral (i.e. partition function in Euclidian space) over ordinary c-number functions (classical scalar fields ϕ).

To extend functional methods to the case of fermion fields however, demands that the classical fields act as anti-commutating c-numbers which will ensure that the anti-commutation property of the fermion fields is preserved.

Therefore, to extend the path integral formalism to include fermions it is necessary to introduce a representation of the fermion Hilbert space as monomials of generators $x_1, x_2, \ldots$ of a Grassmann algebra [27;28]. (Note that all references quoted refer to those in Part I)

2. Finite Dimensional Grassmann Algebra

If the generators $x_1, \ldots, x_n$ of an algebra satisfy the relations

$$\{x_i, x_k\} \equiv x_i x_k + x_k x_i$$
$$= 0, \qquad \text{(C.1)}$$

then it is called a Grassmann algebra (G_n) with n generators.

In particular (C.1) implies

$$x_i^2 = 0 . \qquad i = 1, \ldots, n \qquad \text{(C.2)}$$

The dimension of the algebra is n while the linear space formed by the n generators and linear combinations thereof (also denoted by G_n) is 2^n dimensional,

e.g. if $n = 2$ (G_2), then the algebra is formed by x_1, x_2 and the linear space has the basis $1, x_1, x_2, x_1 x_2$.

In general a basis for the linear space G_n is given by the monomials

$$1,\ x_1,\ldots,x_n,\ x_1x_2,x_3x_4,\ldots,x_{n-1}x_n,\ x_1x_2x_3,\ldots,$$

$$x_{n-2}x_{n-1}x_n,\ldots,\ x_1\ldots,x_n \qquad (C.3)$$

Every element $f(x)$ of the algebra G_n can be written in general as a linear combination of monomials

$$f(x) = f_0 + \sum_{r=1}^{n} f_1(r)x_r + \sum_{r_i}^{n} f_2(r_1r_2)x_{r_1}x_{r_2} + \ldots$$

$$+ \sum_{r_i}^{n} f_n(r_1,\ldots,r_n)\, x_{r_1}\ldots x_{r_n}. \qquad (C.4)$$

This expansion is not unique. However, this can be brought about by choosing the coefficient functions $f_i(r_1,\ldots,r_i)$ (which are usually real or complex numbers) to satisfy the condition

$$f_i(r_1,\ldots,r_i) = 0 \quad \text{if} \quad r_i \geq r_j$$

for at least one pair of indices $i<j$.

The resulting expression is unique,

$$f(x) = f_0 + \sum_{r=1}^{n} f_1(r)x_r + \sum_{r_1<r_2} f_2(r_1r_2)x_{r_1}x_{r_2} + \ldots$$

$$+ \sum_{r_1<\ldots<r_n} f_n(r_1,\ldots,r_n)\, x_{r_1}\ldots x_{r_n}. \qquad (C.5)$$

The following remarks are in order.

(i) In general the expansion will always be finite because of the property (C.1).

(ii) The simplest example of the expansion (C.5) is the one-dimensional algebra

$$f = f_0 + f_1\, x_1 \qquad (C.6)$$

where, if f is a c-number, then f_0 is also one while f_1 must be a Grassmann variable.

(iii) Instead of having the coefficients f as ordinary c-numbers they can be chosen to be anti-symmetric functions of the indices $r_1,\ldots,r_n$, i.e. $f_i(r_1,\ldots,r_n)$ will change sign under the permutation of any pair of arguments. It can be shown [27] that this specific choice will ensure that the general expansion of (C.4) is unique.

3. Derivatives of Grassmann Variables

Because of the property (C.1) we need to define both a left and right derivative $(\frac{\partial}{\partial x_i} f)$ and $(f \frac{\partial}{\partial x_i})$ of an element f(x) of the algebra G_n.

As both derivatives are linear operators in G_n they can be defined on the basis (C.3) of G_n. The derivatives are then given by

$$\begin{aligned}\frac{\partial}{\partial x_i} x_{r_1}\ldots x_{r_n} &= (\frac{\partial}{\partial x_i} x_{r_1})\, x_{r_2}\ldots x_{r_n} - x_{r_1}(\frac{\partial}{\partial x_i} x_{r_2})\, x_{r_1}\ldots x_{r_n} \\ &\quad + \ldots + (-1)^{n-1} x_{r_1}\ldots x_{r_{n-1}}(\frac{\partial}{\partial x_i} x_{r_n}) \\ &= \delta_{ir_1} x_{r_2}\ldots x_{r_n} - \delta_{ir_2} x_{r_1} x_{r_3}\ldots x_{r_n} \\ &\quad + \ldots + (-1)^{n-1} x_{r_1}\ldots x_{r_{n-1}} \qquad \text{(C.7a)}\end{aligned}$$

$$\begin{aligned}x_{r_1}\ldots x_{r_n}\frac{\partial}{\partial x_i} &= x_{r_1}\ldots x_{r_{n-1}}(x_{r_n}\frac{\partial}{\partial x_i}) - x_{r_1}\ldots x_{r_{n-2}} x_{r_n}(x_{r_{n-1}}\frac{\partial}{\partial x_i}) \\ &\quad + \ldots + (-1)^{n-1} x_{r_2}\ldots x_{r_n}(x_{r_1}\frac{\partial}{\partial x_i}) \\ &= \delta_{ir_n} x_{r_1}\ldots x_{r_{n-1}} - \delta_{ir_{n-1}} x_{r_1}\ldots x_{r_{n-2}} x_{r_n} \\ &\quad + \ldots + (-1)^{n-1}\delta_{ir_1} x_{r_2}\ldots x_{r_n} \qquad \text{(C.7b)}\end{aligned}$$

To calculate the derivative of the product $x_{r_1} \ldots x_{r_n}$ with respect to x_i, the element x_p must be permutated [using (C.1)] to the first position in the product for the left derivative to be calculated, while for the right derivative x_p has to be permutated to the last position.

The properties of the derivatives can be summed up by the relations

$$\{\frac{\partial}{\partial x_i}, x_j\} = \delta_{ij} = \{x_j, \frac{\overleftarrow{\partial}}{\partial x_i}\}, \tag{C.8}$$

where we introduced the notation $\frac{\overleftarrow{\partial}}{\partial x}$ to denote the right derivative.

As an example we calculate the derivatives of the two dimensional algebra G_2 with

$$f = a + cx_1 + ex_1x_2 \quad .$$

Now

$$\frac{\partial}{\partial x_1} f(x) = c + ex_2 \qquad \frac{\partial}{\partial x_2} f(x) = - ex_1$$

$$f(x) \frac{\overleftarrow{\partial}}{\partial x_1} = c - ex_2 \qquad f(x) \frac{\overleftarrow{\partial}}{\partial x_1} = -ex_2 \tag{C.9}$$

For differentiation with respect to Grassmann variables the chain differentiation rule also holds. This follows immediately from the definition of the derivatives in (C.7).

For example, let $x_k = \sum_r c_{kr} y_r$, and hence $f(x) = f[x(y)]$.

Then

$$\frac{\partial}{\partial y_r} f[x(y)] = \frac{\partial}{\partial x_k} f[x(y)] \frac{\partial x_k}{\partial y_r}$$

$$= \frac{\partial}{\partial x_k} [f(x)]_{x=x(y)} c_{kr} \; ; \tag{C.10a}$$

$$f[x(y)] \frac{\overleftarrow{\partial}}{\partial y_r} = f[x(y)] \frac{\overleftarrow{\partial}}{\partial x_k} \frac{\partial x_k}{\partial y_r}$$

$$= [f(x) \frac{\overleftarrow{\partial}}{\partial x_k}]_{x=x(y)} c_{kr} \tag{C.10b}$$

where for both cases summation over the index k is implied.

For the differentiation of a product of arbitrary elements f_1 and f_2 of the Grassmann algebra we have the result [27]

$$\frac{\partial}{\partial x_i}(f_1 f_2) = (\frac{\partial}{\partial x_i} f_1) f_2 - f_2 (\frac{\partial}{\partial x_i} f_2)$$

$$(f_1 f_2) \frac{\overleftarrow{\partial}}{\partial x_i} = f_2 (f_1 \frac{\overleftarrow{\partial}}{\partial x_i}) - (f_2 \frac{\overleftarrow{\partial}}{\partial x_i}) f_1 \qquad \text{(C.11)}$$

Using the relation $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\} = 0$ (C.12)

(which follows trivially from the definition of the derivatives), we can write for the second derivatives

$$\frac{\partial}{\partial x_1}(\frac{\partial}{\partial x_2} f) = - \frac{\partial}{\partial x_2}(\frac{\partial}{\partial x_1} f)$$

$$(f \frac{\overleftarrow{\partial}}{\partial x_1}) \frac{\overleftarrow{\partial}}{\partial x_2} = - (f \frac{\overleftarrow{\partial}}{\partial x_2}) \frac{\overleftarrow{\partial}}{\partial x_1}$$

$$(\frac{\partial}{\partial x_1} f) \frac{\overleftarrow{\partial}}{\partial x_2} = \frac{\partial}{\partial x_1} (f \frac{\overleftarrow{\partial}}{\partial x_2}) \qquad \text{(C.13)}$$

4. Integration

Consider the one dimensional Grassmann algebra with

$$f(x) = a + bx \; . \qquad \text{(C.14)}$$

This means that $\frac{\partial^2}{\partial x^2} f(x) = 0$.

In the usual Riemann-Lebesgue formalism integration is usually defined as the inverse of differentiation. However, as the example above clearly shows, such an inverse for Grassmann variables is ill defined.

In order to construct a suitable integral we therefore have to resort to a formal definition of the integration operation which preserves some of the general properties of integration over c-numbers. The requirement used is that the integral over the Grassmann variables should be invariant under a translation of the integration variable by a constant factor, i.e.

$$\int dx\ f(x) = \int dx\ f(x+a). \qquad (C.16)$$

This requires that

$$\int dx = 0$$

$$\int dx\ x = 1\ , \qquad (C.18)$$

which for the n-dimensional case would correspond to the following definition for the single integrals

$$\int dx_i = 0$$

$$\int dx_i x_j = \delta_{ij}\ . \qquad i,j = 1,\ldots,n \qquad C.18)$$

Examples

(a) For f defined in (C.14) we note that

$$\int dx\ f(x) = \int dx\ (a+bx) = b$$

$$= \frac{\partial}{\partial x}\ f(x). \qquad (C.19)$$

For the one-dimensional case the definition of integration and (left) differentiation therefore lead to the same result.

(b) Let x and $\bar{x}$ be independent Grassmann variables such that

$$\int dx = \int d\bar{x} = 0$$

$$\int dx\ x = \int d\bar{x}\ \bar{x} = 1\ .$$

Then, because $x^2 = \bar{x}^2 = 0$ we have that

$$e^{-x\bar{x}} = 1 - x\bar{x}\ ,$$

and hence

$$\int d\bar{x}dx\ e^{-x\bar{x}} = \int d\bar{x}dx - \int d\bar{x}dx\ \bar{x}x$$

$$= 0 + \int d\bar{x}dx\ x\bar{x}$$

$$= 1. \qquad (C.20)$$

For the n-dimensional Grassmann algebra we define the set

$dx_1, dx_2, \ldots, dx_n$ satisfying the conditions

$$\{dx_i, dx_k\} = \{x_k, x_i\} = 0 \ . \qquad (C.21)$$

Using the above, multiple integrals can be constructed by iteration of the single integrals defined in (C.18).

The integral defined by the formulas (C.18) and (C.21) is called the integral on the Grassmann algebra by the generators $x_1, \ldots, x_n$, and is given for any element f(x) of the algebra by [27]

$$\int dx_n \ldots dx_1\ f(x). \qquad (C.21a)$$

5. Properties of Grassmann Integrals

5.1 Change of variables

Consider a linear change of variables

$$x_i = \sum_k c_{ik} y_k \ . \qquad i,k = 1, \ldots, n \qquad (C.22)$$

E.g. for n = 2 we have

$$x_1 x_2 = (c_{11}y_1 + c_{12}y_2)(c_{21}y_1 + c_{22}y_2)$$

$$= c_{11}c_{21}y_1y_1 + c_{12}c_{21}y_2y_1 + c_{11}c_{22}y_1y_2 + c_{12}c_{22}y_2y_2$$

$$= (c_{11}c_{22} - c_{12}c_{21})\ y_1y_2$$

$$= \det C\ y_1y_2 \ ,$$

with $C = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}$. (C.23)

In general, for i = n we have

$$x_1 x_2 \ldots x_n = \varepsilon_{i_1, \ldots, i_n} c_{1 i_1} \cdots c_{n i_n} y_1 \cdots y_n$$

$$= (\det C)\, y_1 \ldots y_n \,. \qquad (C.24)$$

To ensure that the normalization conditions of (C.18) are still satisfied we must have that

$$dx_n \ldots dx_1 = (\det C)^{-1}\, dy_n \ldots dy_1$$

hence

$$\int dx_n \ldots dx_1\, f(x) = \det{}^{-1} C \int dy_n \ldots dy_1 f[y(x)] \,. \qquad (C.26)$$

This result can be compared to the one obtained using the (usual) Riemann integration where the transformation determinant is given by detC (and not $\det^{-1} C$).

5.2 Calculation of Gaussian-type integrals with Grassmann variables

Consider the integral

$$I(A) = \int dx_n \ldots dx_1 \exp[\tfrac{1}{2}(x^T A x)]$$

$$= \int dx_n \ldots dx_1 \exp[\tfrac{1}{2} \sum_{i,k}^{n} x_i (A)_{ik} x_k] \qquad (C.27)$$

with A a real n x n anti-symmetric matrix.

(a) For illustration we calculate the integral for n = 2.

Then $A = \begin{vmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{vmatrix}$

and $I(A) = \int dx_2 dx_1 \exp[\tfrac{1}{2} \sum_{i,k=1}^{2} x_i (A)_{ik} x_k]$

$$= \int dx_2 dx_1 \exp[\, a_{12} x_1 x_2]$$

$$= \int dx_2 dx_1 \, [1 + a_{12} x_1 x_2]$$

$$= a_{12}$$

$$= \det{}^{\frac{1}{2}} A. \qquad (C.28)$$

(b) We now want to calculate the integral for the n-dimensional case (C.27) [28].
Noting that A is real and anti-symmetric, we consider the matrix $iA \equiv A_h$ (which is hermitian). A_h can now be diagonalized using a unitary transformation

$$V(iA)V^\dagger = VA_hV^\dagger$$

$$= A_d, \qquad (C.29)$$

where A_d is real and diagonal.

The elements of A_d satisfy the eigenvalue equation

$$\det| A_h - \lambda I| = 0. \qquad (C.30)$$

Since $A^T = -A$,

$$\det| A_h - \lambda I|^T = \det|-A_h - \lambda I| = 0.$$

Thus, if λ is a solution then so is $-\lambda$, and A_d is of the form

$$A_d = \begin{pmatrix} a & & & & \\ & -a & & & 0 \\ & & b & & \\ & & & -b & \\ & 0 & & & \ddots \end{pmatrix} \qquad (C.31)$$

Using a proper orthogonal transformation the matrix A_d can be brought into the form A_s, where

$$A_s = \begin{pmatrix} \lambda_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & & 0 \\ & \lambda_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \end{pmatrix} \qquad (C.32)$$

This can easily be verified using a 2 x 2 unitary matrix

$$s = 2^{-\frac{1}{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix} \qquad (C.32a)$$

which has the property

$$s(-i) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} s^\dagger = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (C.32b)$$

Thus
$$\begin{aligned} SV(A)V^\dagger S^\dagger &\equiv UAU^\dagger \\ &= S(-iA_d)S^\dagger \\ &\equiv A_s \end{aligned} \qquad (C.33)$$

where (i) $$S = \begin{pmatrix} s & & & & \\ & s & & & 0 \\ & & s & & \\ & & & s & \\ & 0 & & & \ddots \end{pmatrix} \qquad (C.33a)$$

(ii) $U \equiv SV$ (C.33b)

(iii) A_s is given by (C.32).

Carrying out a linear substitution

$$x_i = \sum_k u_{ik} y_k,$$

and using (C.26), together with detU = det(SV) = (detS)(detV) = 1, give

$$I_n = \int dy_n \ldots dy_1 \exp[\tfrac{1}{2} \sum_{i,j}^{n} y_i (A_s)_{ij} y_j]. \qquad (C.34)$$

For n even this gives

$$I_n = \int dy_n \ldots dy_1 \exp\left(\tfrac{1}{2} (2)[\lambda_1 y_1 y_2 + \lambda_2 y_3 y_4 + \ldots + \lambda_{\frac{1}{2}n} y_{n-1} y_n] \right)$$

$$= \int dy_n \ldots dy_1 \left(1 + [\lambda_1 y_1 y_2 + \ldots] + \frac{1}{2!}[\lambda_1 y_1 y_2 + \ldots]^2 + \ldots \right.$$

$$\left. + \frac{1}{(n/2)!}[\lambda_1 y_1 y_2 + \ldots + \lambda_{\frac{1}{2}n} y_{n-1} y_n]^{\frac{1}{2}n} + \ldots \right) . \qquad \text{(C.35a)}$$

The only non-zero contribution to (C.35a) is given by

$$I_n = \int dy_n \ldots dy_1 \frac{1}{(n/2)!}[\lambda_1 y_1 y_2 + \ldots + \lambda_{\frac{1}{2}n} y_{n-1} y_n]^{\frac{1}{2}n}$$

$$= \frac{1}{(n/2)!} \sum_P \lambda_1 \lambda_2 \ldots \lambda_{\frac{1}{2}n}$$

$$= \lambda_1 \lambda_2 \ldots \lambda_{\frac{1}{2}n}$$

$$= \det{}^{\frac{1}{2}} A_n \qquad \text{(C.35b)}$$

where the summation index P above imlplies that the sum over all possible combinations of $\{1, \ldots, \frac{1}{2}n\}$ must be taken.

For n odd, $I_n = 0$ as the series expansion of the exponent will always result in one dx_i having an integrand that is equal to one .

Remarks

The following results are a direct consequence of the above (n even):

(i) $$I_n = \int dy_n \ldots dy_1 \exp[\tfrac{1}{2} y^T A y]$$

$$= \det{}^{\frac{1}{2}} A_n \quad [\equiv \text{(C.35b)}] \quad (n \text{ even}) \qquad \text{(C.35c)}$$

(ii) $$I_n = \int dy_1 \ldots dy_n \exp[-\tfrac{1}{2} y^T A y]$$

$$= \det{}^{\frac{1}{2}} A_n \qquad \text{(C.35d)}$$

(iii) $$I_n = \int dx_n \ldots dx_1 \exp[x^T A x]$$

$$= 2^{\frac{1}{2}n} \lambda_1 \lambda_2 \ldots \lambda_{\frac{1}{2}n}$$

$$= \det{}^{\frac{1}{2}} 2A . \qquad \text{(C.36)}$$

The following extension of the integrand is often encountered:

$$I_n = \int dx_n \ldots dx_1 \exp[-\tfrac{1}{2}x^T A x + \theta^T x] \qquad (C.37)$$

where the θ_i are Grassmann variables satisfying the additional constraints

$$\{\theta_i, \theta_j\} = 0$$

$$\{\theta_i, x_j\} = 0 \ . \qquad (C.37a)$$

Completing the square in the exponent and shifting the integration variable we have

$$x^T A x - 2\theta^T x = (x + A^{-1}\theta)A(x + A^{-1}\theta) + \theta^T A^{-1}\theta$$

$$\equiv y^T A y + \theta^T A^{-1}\theta$$

with $y = x + A^{-1}\theta$. Note that we made explicit use of the anti-symmetry of A, i.e. $A^T = -A$.

Hence

$$I_n = \int dy_n \ldots dy_1 \exp[-\tfrac{1}{2}y^T A y + \tfrac{1}{2}\theta^T A^{-1}\theta \ ]$$

$$= \det{}^{\frac{1}{2}} A_n \exp[\tfrac{1}{2}\theta^T A^{-1}\theta]$$

$$= \exp[\tfrac{1}{2}\ \mathrm{tr}(\ln A)]\exp[\ \tfrac{1}{2}\theta^T A^{-1}\theta\ ]. \qquad (C.37c)$$

6. Extension of Grassmann Integration to Complex Matrices

In order to extend Grassmann variables to represent complex fermion fields the concept of a Grassamnn algebra with involution is introduced as follows [27].

Let G be a Grassmann algebra with inner product in which there is defined a one to one mapping of the any element onto itself,

$$f \leftrightarrow f^*$$

such that the following conditions are satisfied:

(i) $(f^*)^* = f$

(ii) $(f_1 f_2)^* = f_2^* f_1^*$

(iii) $(\alpha f)^* = \bar{\alpha} f^*$ (α complex)

(iv) If the inner product is defined for f and g, then it is also defined for f^* and g^* and $f^*.g^* = g.f$

(v) Each of the topological subspaces E^1 of the Grassmann algebra can be decomposed into the direct sum of subspaces

$$E^1 = F + F^*$$

$F \bigwedge H^1$ is orthogonal to $F^* \bigwedge H^1$ where H^1 is the appropriate Hilbert space [33].

The mapping satisfying these conditions is called an **involution in G** (i.e an involution of the algebra on itself) .

The elements of the Grassmann algebra with involution is given by $f_1, \ldots, f_n, \bar{f}_1, \ldots, \bar{f}_n$ (and we introduced the notation $\bar{f} \equiv f^*$). It follows from the definition that they obey the anti-commutation relations (C.1).

If G is now an algebra with involution, we can define the integral

$$I(f) = \int (\prod_i d\bar{x}_i dx) \; f(x,\bar{x}) \tag{C.38}$$

where $\prod_{i=1}^{n} d\bar{x}_i dx = d\bar{x}_n dx_n \ldots d\bar{x}_1 dx_1$ (C.38a)

and $f(x,\bar{x})$ is usually of the form $e^{\pm \bar{x} M x}$.

Example

$$I(f) = \int (\prod_i d\bar{x}_i dx_i) \; e^{-\bar{x} M x}$$

$$= \int (\prod_i d\bar{x}_i dx_i) \; \exp[-\sum_{ij} \bar{x}_i (M)_{ij} x_j]$$

$$= \int (\prod_i d\bar{x}_i dx_i)[1 - \sum_j \bar{x}_i (M)_{ij} x_j + \ldots + \frac{(-1)^n}{n!} (\sum_{ij} \bar{x}_i (M)_{ij} x_j)^n + \ldots]$$

$$= \int (\prod_i d\bar{x}_i dx_i) \frac{(-1)^n}{n!} \sum_{P(ij)} (\bar{x}_{i_1} (M)_{i_1 j_1} x_{j_1})(\ldots)(\bar{x}_{i_n} (M)_{i_n j_n} x_{j_n})$$

$$= \sum_{P(i,j)} \frac{(-1)^n}{n!} \int d\bar{x}_n dx_n \ldots d\bar{x}_1 dx_1 \, \bar{x}_{i_1} x_{j_1} \ldots \bar{x}_{i_n} x_{j_n} \, M_{i_1 j_1} \ldots M_{i_n j_n}$$

$$= \frac{(-1)^n}{n!} (-1)^n \sum_{P(i,j)} \varepsilon_{i_1 \ldots i_n} \varepsilon_{j_1 \ldots j_n} \, M_{i_1 j_1} \ldots M_{i_n j_n}$$

$$= \det M_n \, , \qquad \text{(C.39)}$$

where $\sum_{P(i,j)}$ denotes the sum over all possible permutations of i and j and the ε factors were introduced to ensure that the property (C.1) is satisfied.

The following results are a direct consequence of the above:

$$\text{(i)} \quad \int (\prod_i dx_i d\bar{x}_i) \, e^{-\bar{x} M x} = \int dx_n d\bar{x}_n \ldots dx_1 d\bar{x}_1 \, e^{-\bar{x} M x}$$

$$= (-1)^n \det M_n ; \qquad \text{(C.40a)}$$

$$\text{(ii)} \quad \int (\prod_i dx_i d\bar{x}_i) \, e^{\bar{x} M x} = \det M_n . \qquad \text{(C.40b)}$$

It is important to note that for the integrals above n may assume even or odd values as the combination will always result in an even number of Grassmann variables.

7. Infinite Dimensional Grassmann Algebras

The linear space E is called the direct sum of the spaces E_i if it can formally be written as

$$f = f_0 + f_1 + \ldots + f_i + \ldots \qquad \text{(C.41a)}$$

where $f \in E$.

An infinite dimensional Grassmann algebra is an algebra with an infinite number of generators, with the algebra constructed by the direct sum of topological linear spaces E_i.

It can be shown [27] that any arbirtary element f of the algebra G can uniquely be written in the form

$$f(\alpha) = \sum_{n=0}^{\infty} \int dx_1 \ldots dx_n \, \phi(x_1,\ldots,x_n)\, \alpha(x_1)\ldots\alpha(x_n)$$

$$\equiv \sum_{n=0}^{\infty} \int d^n x \, \phi(x_1,\ldots,x_n)\, \alpha(x_1)\ldots\alpha(x_n), \qquad (C.42)$$

where (i) $\phi(x_1,\ldots,x_n)$ is an anti-symmetric function of the n variables (usually a generalized function) ,

(ii) $\alpha(x_i)$ are the generators of the algebra G which for arbitrary algebras play the same role as the generators for the finite dimensional algebras,

$$\alpha(x) = \sum_{n=0}^{\infty} b_n(x)\alpha_n(x), \qquad (C.42a)$$

with $\{b_n\}$ a suitable basis in the Hilbert space (i.e. $\{b_n\}$ forms an orthonormal complete basis [33]).

Let M be some set with measure dx and G be a Grasmann algebra with inner product and a set of generators $\alpha(x)$, $x \in M$.

We can now define left and right derivatives . We first define the derivatives with respect to the basis elements $\alpha(x)$ of the algebra:

$$\underline{\text{left derivative}} \quad \left[\frac{\delta}{\delta\alpha(x)}\, \alpha(x_1)\ldots\alpha(x_n)\right] = \delta(x - x_1)\alpha(x_2)\ldots\alpha(x_n)$$

$$- \delta(x - x_2)\alpha(x_1)\alpha(x_3)\ldots\alpha(x_n) + \ldots +$$

$$+ (-1)^{n-1}\, \delta(x - x_n)\alpha(x_1)\ldots\alpha(x_{n-1})$$

$$\underline{\text{right derivative}} \quad \left[\alpha(x_1)\ldots\alpha(x_n)\, \frac{\delta}{\delta\alpha(x)}\right] = \delta(x - x_n)\alpha(x_1)\ldots\alpha(x_{n-1})$$

$$- \delta(x - x_{n-1})\alpha(x_1)\ldots\alpha(x_{n-2})\alpha(x_n) + \ldots +$$

$$+ \ (-1)^{n-1} \ \delta(x - x_1)\alpha(x_2)\ldots\alpha(x_n) \qquad (C.43)$$

where $\delta(x-x_i)$ is the Dirac delta function defined on M :

$$\int \delta(x-y)f(y)\, dy = f(x). \qquad (C.44)$$

The left and right derivatives of an element $f(y) \in G$, where

$$f = \int d^n x \ \phi(x_1,\ldots,x_n) \ \alpha(x_1)\ldots\alpha(x_n) \ , \qquad (C.45)$$

follows immediately [using (C.43)] ;

$$\frac{\delta}{\delta\alpha(x)} f = n \int d^{n-1}x \ \phi(x,x_2,\ldots,x_n) \ \alpha(x_2)\ldots\alpha(x_n)$$

$$f \frac{\delta}{\delta\alpha(x)} = n \int d^{n-1}x \ \phi(x_1,\ldots x_{n-1},x) \ \alpha(x_1)\ldots\alpha(x_{n-1}) \qquad (C.46)$$

where the fact that $\phi(x_1,\ldots,x_n)$ was chosen to be anti-symmetric was also used .

8. Integrals on an Infinite Dimensional Grassmann Algebra (Continual Integrals)

Let G be a Grassmann algebra with involution and corresponding decomposition of the defining space E^1 , $E^1 = F + F^*$.

Consider the subalgebra G_{L_n} of G which is generated by the subspace $L_n = F_n + F_n^*$, and let $\{\alpha_i\}$ and $\{\bar{\alpha}_i\}$ be an orthonormal basis for F_n and F_n^* repectively .

Consider the integral

$$I_n(f_n) = \int (\prod_i d\bar{\alpha}_i d\alpha_i) \ f_n(\bar{\alpha},\alpha \ ) \qquad (C.47)$$

where $f_n(\bar{\alpha},\alpha)$ is the value of f on G_{L_n} .

It can be shown [27] that the integral in (C.47) is independent of the choice of the orthonormal basis in F_n.

Definition: If the limit $I(f) = \lim_{n\to\infty} I_n(f_n)$ exists independent of a specific choice of the system of subspaces F_n (and the corresponding adjoint spaces F_n^*), then this limit is called the **continual integral** of the functional f on the algebra G ,

$$I(f) = \int [d\bar{\alpha}\, d\alpha]\, f(\bar{\alpha},\alpha) \tag{C.48}$$

where $$[d\bar{\alpha}\, d\alpha] = \lim_{n\to\infty} d\bar{\alpha}_n d\alpha_n \ldots d\bar{\alpha}_1 d\alpha_1 \tag{C.48a}$$

and $$[d\alpha] \equiv [d\alpha(x)]. \tag{C.48b}$$

Examples

Consider the integral

$$I(f) \equiv \int [d\alpha]\, \exp[\tfrac{1}{2} \int dxdy\ \alpha(x)\ A(x,y)\alpha(y)] \tag{C.49}$$

where $$[d\alpha] = [d\alpha(x)] = \lim_{n\to\infty} d\alpha_n \ldots d\alpha_1 \tag{C.49a}$$

and $$A(x,y) = -\, A(y,x). \tag{C.49b}$$

The matrix element A_{pq} of the operator A in a suitable orthonormal basis can be written formally as

$$A_{pq} = \int dxdy\ b_p(x)A(x,y)b_q(y) \tag{C.50}$$

where the symbols $b_i(x)$, $b_j(y)$ satisfy the relations

$$\{b_i(x),\ b_j(y)\} = 0\ . \tag{C.50a}$$

This ensures that

$$A_{pq} = -\, A_{qp}\ . \tag{C.50b}$$

Hence

$$I = \lim_{n\to\infty} \int d\alpha_n \ldots d\alpha_1\ \exp[\tfrac{1}{2} \sum_{p,q}^{n} \alpha_p A_{pq}\ \alpha_q]. \tag{C.51}$$

Noting the correspondence with the result obtained in (C.35) we can immediately write

$$I = \lim_{n\to\infty} [\det A_n]^{\frac{1}{2}}, \tag{C.52}$$

where A_n is again a real n x n anti-symmetric matrix.

The final result is

$$I = \int [d\alpha] \exp[\tfrac{1}{2} \int dxdy\ \alpha(x)\ A(x,y)\alpha(y)]$$

$$= \det{}^{\frac{1}{2}} A \tag{C.53}$$

with $A(x,y) = - A(y,x)$.

We note that this result is independent of the basis $\{b_i(x)\}$ (due to the linear substitution that was carried out in the evaluation).

Also,

$$\int [d\alpha] \exp[\int dxdy\ \alpha(x)\ A(x,y)\alpha(y)]$$

$$= \det{}^{\frac{1}{2}}(2A). \tag{C.54}$$

For an algebra with involution, and measure as defined in (C.48a), we can repeat the analysis above by noting that the generalization of (C.38) gives

$$\int [d\bar{\alpha}\ d\alpha] \exp[-\int dxdy\ \bar{\alpha}(x)\ M(x,y)\alpha(y)]$$

$$= \det M$$

$$= \exp[\mathrm{tr}\ \ln M], \tag{C.55}$$

where $M(x,y)$ is arbitrary.

APPENDIX D: MODIFIED BESSEL FUNCTIONS

The modified Bessel function $I_n(z)$ is defined as

$$I_n(z) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \cos(n\phi)\, e^{z\cos\phi} \tag{D.1}$$

$$= \frac{1}{\pi} \int_0^{\pi} d\phi \cos(n\phi)\, e^{z\cos\phi} \,. \tag{D.2}$$

Using symmetry arguments we can also write the above in the form

$$I_n(z) = \frac{1}{2\pi} \int_0^{2\pi} d\phi\, e^{z\cos\phi \pm in\phi} \,. \tag{D.3}$$

The modified Bessel functions satisfy the following equality

$$I_{2j}(z) - I_{2(j+1)}(z) = \frac{2(2j+1)}{z} I_{2j+1}(z). \tag{D.4}$$

Proof: Using the notation of (D.3) we can write the LHS as

$$I_{2j}(z) - I_{2j+2}(z) = \frac{1}{2\pi} \int_0^{2\pi} d\phi\, e^{z\cos\phi} \left[e^{i(2j\phi)} - e^{i(2j+2)\phi}\right]$$

$$= -\frac{i}{\pi} \int_0^{2\pi} d\phi\, e^{z\cos\phi}\, e^{i(2j+1)\phi} \sin\phi$$

$$= \frac{i}{\pi} \int_0^{2\pi} d(\cos\phi)\, \frac{1}{z} \left(\frac{d}{d(\cos\phi)} e^{z\cos\phi}\right) e^{i(2j+1)\phi}$$

$$= \frac{2(2j+1)}{2\pi z} \int_0^{2\pi} d\phi\, e^{z\cos\phi}\, e^{i(2j+1)\phi}$$

$$= \frac{2(2j+1)}{z} I_{2j+1}(z) \,.$$

Also $$\frac{d}{dz} I_n(z) = \tfrac{1}{2}[I_{n+1}(z) + I_{n-1}(z)] \tag{D.5}$$

which follows from

$$\frac{d}{dz} I_n(z) = \frac{1}{2\pi} \frac{d}{dz} \left(\int_0^{2\pi} d\phi \cos(n\phi)\, e^{z \cos\phi} \right)$$

$$= \frac{1}{4\pi} \left(\int_0^{2\pi} d\phi \cos(n+1)\phi\, e^{z \cos\phi} + \int_0^{2\pi} d\phi \cos(n+1)\phi\, e^{z \cos\phi} \right) .$$

Note that $$I_0'(z) = I_1(z) \tag{D.6}$$

If the argument is small we have the following power-series expansion

$$I_n(z) = (\tfrac{1}{2}z)^n \sum_{k=0}^{\infty} (\tfrac{1}{2}z^2)^k / [k!\Gamma(n+k+1)] . \tag{D.7}$$

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