Extensive categories, commutative semirings and Galois theory

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Firstly, to my supervisor, Professor George Janelidze, thank you for all your years of wonderful support, encouragement and patience. You have impressed upon me a vision of mathematics I could never have dreamed existed, and left me standing in awe of the beauty of categories.

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Abstract

We describe a Galois theory of commutative semirings as a Boolean Galois theory in the sense of Carboni and Janelidze as presented in [4]. Such a Galois structure then naturally suggests an extension to commutative semirings of the classical theory of quadratic equations over commutative rings, itself presented in [17]. We show, however, that this proposed generalization is impossible for connected commutative semirings which are not rings, leading to the conclusion that for the theory of quadratic equations, "minus is needed". Finally, by considering semirings B which have no non-trivial additive inverses and no non-trivial zero divisors, we present an example of a normal extension of commutative semirings which has an underlying B-semimodule structure isomorphic to $B \times B$.

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Introduction

One immediate investigation one pursues after *calculating* a new concrete Galois theory from the abstract categorical one developed by G. Janelidze, in say [11], is to examine the "simplest" normal extensions of such a theory. In many such theories, even "simple" extensions may require significant work to describe; however for the *Boolean Galois theories* [4], particularly the classical algebraic theories, one can often find examples more readily by virtue of their "geometric" character.

For example, one notable feature of the Galois theory of commutative rings, originally developed by A. R. Magid in [22], is the complete formulation of what we call the *theory of quadratic equations over commutative rings*. This is the statement that for each monic separable quadratic polynomial f over a connected commutative ring B in which the discriminant of f is invertible, the rank-two extension $B[x]/f(x)B[x]$ of B by f is always a normal extension of B. See [17, Section 2].

We say "complete" here because although it certainly has been well known for a long time that, in particular, for each such quadratic polynomial f over a *field* B, the extension of B by f is a two-dimensional B-algebra which splits as the product $B \times B$ of B-algebras if and only if f has a root in B, its complete categorical reformulation offers a much more general viewpoint, and is arguably cleaner.

This new-and-improved categorical approach now allows us to entertain the thought that perhaps this theory of quadratic equations might not "necessitate" the structure of a ring. In other words, since we have at hand the Boolean Galois theories, of which Magid's theory is a special case, we very likely have a *Boolean Galois theory of commutative semirings*, which generalizes Magid's theory and which now allows us to propose a *theory of quadratic equations over commutative semirings*.

Outline

Chapter 1

Chapter 1 introduces the concepts and various equivalent formulations of extensive categories, as well as detailing their important "geometric" properties as may be found in [5]. We introduce the category of commutative semirings and show that it is indeed a lextensive category, as is the category Stone of profinite topological spaces. These two categories will be central points of focus throughout the entire thesis.

Since the categorical Galois theory uses the descent data of descent morphisms $p : E \longrightarrow B$ in a category and traces the connection between internal category actions with respect to the kernel pair $\mathsf{Eq}(p)$ of such morphisms, and the algebras over the monad induced by them, we naturally introduce monoidal categories and monoids over them, leading to the special example of monads and their associated algebras. The descent theory we describe is really the monadic descent theory of [15] and all of our focus is directed at the split monadicity Theorem 1.52 which is sufficient to arrive at the important Beck monadicity criterion. We also make a minor typing correction to [16, Lemma 2.5], the corrected version appearing in Theorem 1.53. No new results appear in this chapter.

Chapter 2

The short Chapter 2 provides the important ingredients of the categorical Galois theory, namely Galois structures and the concept of admissibility. We present a brief development leading to the fundamental theorem of Galois theory in Theorem 2.11 which of course places into context all the work of this thesis, even though it is not specifically necessary for any of the results of the thesis. No new results appear in this chapter.

Chapter 3

Chapter 3 considers the (Boolean) Galois theory of commutative semirings as a special case of the general theory in [4]. The result essentially hinges on Proposition 3.23 which shows that the B-semialgebra B^n for each natural number n is finitely presentable for any connected commutative semiring B. Although we may derive this Galois structure directly from the finite presentability of B^n and Proposition 3.4 while avoiding our slight digression into considerations of the inductive completion of a category, we feel the current presentation is more informative. Moreover, this digression affords us the opportunity to provide an accurate construction of the inductive completion of a category not found in many (English) texts.

The result that finite products of connected objects are admissible with respect to this Galois structure appears in [4, Theorem 3.6], but we provide a proof of this fact in Corollary 3.29.

Finally, having at hand the split monadicity theorem of Chapter 1, we show, in Corollary 3.33, that homomorphisms of semirings $f : B \longrightarrow E$ which are split monomorphisms of B-semimodules are effective for descent, considered as morphisms in $\mathsf{CSemiRing}^{\mathrm{op}}$. Although this corollary may be deduced from general theory in [16] by constructing monads over the category of commutative monoids which are induced by commutative semirings, and examining the appropriate induced morphisms of monads, we present a direct proof which does not require detailed calculations involving these categorical structures. Furthermore, we make another small typing correction to [16, Theorem 4.1], the corrected version being found now in Theorem 3.30.

Chapter 4

Chapter 4 contains several interesting additional remarks building on the Galois theory of commutative semirings developed in Chapter 3. We begin by developing the classical theory of quadratic equations over commutative rings, showing that very little knowledge of the theory of separable algebras over commutative rings–in, for example, [17] or [23]–is needed for its development, as long as we restrict ourselves to the case where our "base" connected commutative ring B is a field. Focusing almost entirely on simple considerations of the complementary idempotents of a commutative ring and other elementary categorical concepts we are able with our approach to recover the classical result of Corollary 4.8.

The benefit of this presentation is that we are now able to propose, in Conjecture 4.11, a natural generalization to commutative semirings of this classical theory. We show that almost all the ingredients, including the descent theory from Chapter 3, are in place to deduce a theory of quadratic equations over commutative semirings.

Despite the promise of success, our Theorem 4.16 shows that a generalization of this kind to connected commutative semirings which are not rings is out of reach. As unfortunate as this result proves to be, it does at least inform us of the importance of the structure of the ring in the theory of quadratic equations.

At this point, still seeking a "simple" normal extension of commutative semirings, we note that we may construct a more general B-semialgebra E whose underlying B-semimodule structure is the canonical B-semimodule structure of $B \times B$, and whose multiplication includes that of the B-semialgebra arising from a *separable quadratic equation over B* (see Section 4.3) as a special case. When B has no non-trivial zero divisors and no non-trivial additive inverses, we show in Proposition 4.23 that each pair (e_1, e_2) of (multiplicatively) invertible elements of

B uniquely determines a B-semialgebra structure on the canonical B-semimodule $B \times B$ whose identity element is (e_1, e_2) and which has non-trivial complementary idempotents. Finally, we conclude, again using only simple features of complementary idempotents, elementary categorical concepts, and the descent theory from Chapter 3, that each such B-semialgebra so determined is a normal extension of commutative semirings, the first known of its kind.

The well known *Burnside rig* of a distributive category is the (large) commutative semiring that has as elements isomorphism classes of objects of the category, and its addition and multiplication are given by sums and products in the category. We note, therefore, that the class of those (large) commutative semirings arising as Burnside rigs over lextensive categories seem to be important candidates for the semiring B in Proposition 4.23 because such semirings necessarily have no non-trivial additive inverses.

Chapter 1

Algebraic and categorical structures

In this chapter and the next, we detail all necessary structures and results needed for the development of a Galois theory of semirings. In doing so, we present the most important features of the categorical Galois theory as developed by G. Janelidze and his coauthors. Good general references for this categorical theory include [2], [4], [11] and [12].

1.1 Extensive categories

Much of the success of category theory comes from the clarification that arises from the organization into special-purpose families of "similar" categories. Examples include abelian categories, regular and exact categories, toposes, and accessible categories, to name only but a few. Here, we consider another such organization.

Categories such as Sets, Top of topological spaces and its full subcategories OCCTop of topological spaces with open connected components and LCTop of locally connected topological spaces, Cat of categories and Preord of preorders, and many, many others, each have evident similar properties of a specific geometric character. For example, each of these categories has pullbacks and finite coproducts, in which (finite) coproduct inclusion morphisms are monomorphisms with the factors of coproducts being "disjoint" from one another. In each of these categories each object determines a set of *complemented subobjects* (see Chapter 3) which forms a Boolean algebra, and so each of these categories admits a reflection in the (opposite) category Bool of Boolean algebras. We think of the objects of Bool as "indexing" the decomposition of each object into its subobjects. These categories are examples of *extensive categories*, an organization which explains the existence of these (very specific) geometric phenomena. As stated in [5] extensive categories have coproducts which are well-behaved, in a way we shall make precise here. The basic theory of these categories and some aspects of their history is developed in [5]. We shall add another important example to our list of extensive categories: the opposite category of commutative semirings $\mathsf{CSemiRing}^{\mathrm{op}}$.

Definition 1.1. A category $\mathbb{C}$ with finite coproducts is said to be extensive, if, for every A and B in $\mathbb{C}$, the coproduct functor

$$S : (\mathbb{C} \downarrow A) \times (\mathbb{C} \downarrow B) \longrightarrow (\mathbb{C} \downarrow A+B) \tag{1.1}$$

is an equivalence of categories.

Example 1.2. The category Top of topological spaces is extensive. For, consider a continuous map $\gamma : Z \to A+B$ from a topological space Z into the coproduct $A+B$ of spaces A and B. Since $A+B$ is the disjoint union of A and B which are closed-and-open subspaces of it, it is the case that γ splits Z as the coproduct $\gamma^{-1}(A)+\gamma^{-1}(B)$. By continuity of γ, each of these factors is closed-and-open in Z. Furthermore, there are continuous maps $\gamma_1 : \gamma^{-1}(A) \longrightarrow A$ and $\gamma_2 : \gamma^{-1}(B) \longrightarrow B$ given by the restriction of γ to the corresponding inverse images, so that clearly $\gamma = \gamma_1 + \gamma_2$. Sending (Z,γ) to the pair $\big((\gamma^{-1}(A),\gamma_1),(\gamma^{-1}(B),\gamma_2)\big)$ therefore determines a functor making the functor $S : (\mathsf{Top} \downarrow A) \times (\mathsf{Top} \downarrow B) \longrightarrow (\mathsf{Top} \downarrow A+B)$ an equivalence.

Example 1.3. The category Sets of sets is extensive.

We now formally define the principal algebraic structure of this thesis.

Definition 1.4. A commutative semiring is a system $S = (S,0,1,+,\cdot)$ in which:

1. $(S,0,+)$ and $(S,1,\cdot)$ are commutative monoids;
2. the multiplication $\cdot$ in S distributes over the addition $+$ in S ; that is, for all s,t,u in S, $s\cdot(t+u) = s\cdot t + s\cdot u$. The symbol $\cdot$ will be dropped in calculations when there can be no confusion about the multiplication involved;
3. the additive identity 0 is a multiplicative zero; that is, $s0 = 0$ for every s in S.

Semirings were formally introduced in [27] and are frequently defined not necessarily assuming the third condition above. However, in this thesis all semirings will be of the above type.

Example 1.5. The system $\mathbf{N} = (\mathbf{N},0,1,+,\cdot)$ of natural numbers together with the familiar operations of addition and multiplication is a commutative semiring.

Definition 1.6. A category $\mathscr{T}$ with a denumerable set $\{T^0, T^1, \ldots, T^n, \ldots\}$ of distinct objects, each object T^n being the n-th power of the object T^1, is called an algebraic theory. The collection of all finite-product preserving functors $F : \mathscr{T} \longrightarrow \mathsf{Set}$ and the collection all natural transformations between such functors form a category $\mathsf{Mod}_{\mathscr{T}}$, the category of $\mathscr{T}$-models.

The collection of all commutative semirings and their semiring homomorphisms form a category and, as seen earlier, is denoted CSemiRing. Moreover, CSemiRing is of course a variety of algebras and, therefore, is obviously equivalent to the category of models of an algebraic theory.

For a survey on the background, basic and more advanced theory, and historical developments, as well as further references of semirings, the reader is urged to consult [7], [9] and [10].

Let S be a commutative semiring, and e a (multiplicative) idempotent in it. The subset $eS = \{es \in S \mid s \in S\} = \{s \in S \mid es = s\}$ of S forms a semiring $eS = (eS, 0, e, +, \cdot)$ in which $+$, 0, and $\cdot$ are calculated as in S. This makes eS a subsemiring of S if and only if it coincides with S (that is, when $e = 1$ in S).

Since CSemiRing is a variety of algebras, it has products created by the forgetful functor $U : \mathsf{CSemiRing} \longrightarrow \mathsf{Sets}$; that is to say, CSemiRing has products whose underlying set is the Cartesian product of sets and product projections are projection maps. Finite products in CSemiRing have a special property.

Proposition 1.7. *In a commutative semiring S, suppose two elements e_1 and e_2 satisfy*

$$e_1 + e_2 = 1 \tag{1.2}$$

$$e_1 e_2 = 0 \tag{1.3}$$

Then e_1 and e_2 are idempotents in S, and the maps $\sigma : e_1S \times e_2S \longrightarrow S$ and $\theta : S \longrightarrow e_1S \times e_2S$ defined by $\sigma(a,b) = a + b$ and $\theta(s) = (e_1s, e_2s)$, respectively, are isomorphisms in CSemiRing *inverse to each other.*

Proof. Straightforward calculations show that e_1 and e_2 are idempotents in S, and both σ and θ are homomorphisms of semirings.

Next, σ is the inverse of θ since

$$\sigma\big(\theta(s)\big) = e_1s + e_2s = (e_1 + e_2)s = s$$

and, letting $a = e_1 s_1$ and $b = e_2 s_2$,

$$\begin{aligned}\theta\big(\sigma(a,b)\big) = \theta\big(\sigma(e_1s_1, e_2s_2)\big) &= \big(e_1(e_1s_1 + e_2s_2), e_2(e_1s_1 + e_2s_2)\big)\\ &= (e_1e_1s_1 + e_1e_2s_2, e_2e_1s_1 + e_2e_2s_2)\\ &= (e_1s_1 + 0, 0 + e_2s_2)\\ &= (a,b)\end{aligned}$$

□

Definition 1.8. For a commutative semiring S, elements e_1 and e_2 in S satisfying equalities (1.2) and (1.3) above are called complementary idempotents. If neither e_1 nor e_2 is 0, then e_1 and e_2 are said to be non-trivial complementary idempotents.

Example 1.9. The category $\mathsf{CSemiRing}^{\mathrm{op}}$ is extensive. Given a homomorphism of semirings $\gamma : A \times B \longrightarrow Z$ from the product $A \times B$ of semirings A and B into a semiring Z, the elements $d_1 = (1,0)$ and $d_2 = (0,1)$ in $A \times B$ obviously satisfy equalities (1.2) and (1.3) above, and hence so do the elements $e_1 = \gamma(d_1)$ and $e_2 = \gamma(d_2)$. By Proposition 1.7, the map $\theta : Z \longrightarrow e_1Z \times e_2Z$ defined by $z \longmapsto (e_1z, e_2z)$ is an isomorphism of semirings. Then, defining maps $\gamma_1 : A \longrightarrow e_1Z$ and $\gamma_2 : B \longrightarrow e_2Z$ by $a \longmapsto \gamma(a,0)$ and $b \longmapsto \gamma(0,b)$, respectively, we have γ_1 and γ_2 are homomorphisms of semirings and $\gamma_1 \times \gamma_2 = \theta\gamma$. Therefore, we may define a functor $R : (A \times B \downarrow \mathsf{CSemiRing}) \longrightarrow (A \downarrow \mathsf{CSemiRing}) \times (B \downarrow \mathsf{CSemiRing})$ by $(Z,\gamma) \longmapsto \big((e_1Z, \gamma_1), (e_2Z, \gamma_2)\big)$ which is inverse to the product functor $\times : (A \downarrow \mathsf{CSemiRing}) \times (B \downarrow \mathsf{CSemiRing}) \longrightarrow (A \times B \downarrow \mathsf{CSemiRing})$.

Definition 1.10. Suppose B is a commutative semiring. The coslice category $(B \downarrow \mathsf{CSemiRing})$ is called the category of commutative B-semialgebras over the semiring B.

We may also describe B-semialgebras from the perspective of universal algebra. Firstly, a *B-semimodule* over a commutative semiring $B = (B, 0, 1, +, \cdot)$ is a system $A = (A, 0, +, \omega)$ where $(A, 0, +)$ is a commutative monoid, and $\omega = \big(\omega_b : A \longrightarrow A\big)_{b \in B}$ is a B-indexed family of functions satisfying (writing $\omega_b(a) = ba$) $1a = a$, $(bb')a = b(b'a)$, $b0 = 0 = 0a$, $b(a + a') = ba + ba'$, and $(b + b')a = ba + b'a$ for all b and b' in B, and a and a' in A. Then, a commutative B-semialgebra over a commutative semiring $B = (B, 0, 1, +, \cdot)$ is a system $A = (A, 0, 1, +, \cdot, \omega)$ where $(A, 0, 1, +, \cdot)$ is a commutative semiring and $(A, 0, +, \omega)$ is a B-semimodule with $b(aa') = (ba)a'$ for all b in B, and a and a' in A.

It is not hard to show that the commutative semiring $B[x]$ of polynomials over a commutative semiring B, with the canonical inclusion homomorphism of semirings $i : B \longrightarrow B[x]$, satisfies an identical universal property to that of the commutative ring $R[x]$ of polynomials over a commutative ring R. Specifically, for any commutative semiring S, any homomorphism of semirings $g : B \longrightarrow S$ and any element $s \in S$, there is a unique homomorphism of semirings $h : B[x] \longrightarrow S$ sending x to s and satisfying $hi = g$. The proof is identical to the commutative ring case (see, for example, [20, Section III.7]). Thus we have shown

Example 1.11. The canonical inclusion homomorphism of semirings $i : B \longrightarrow B[x]$ above is a free object in $(B \downarrow \mathsf{CSemiRing})$ on the single-element set.

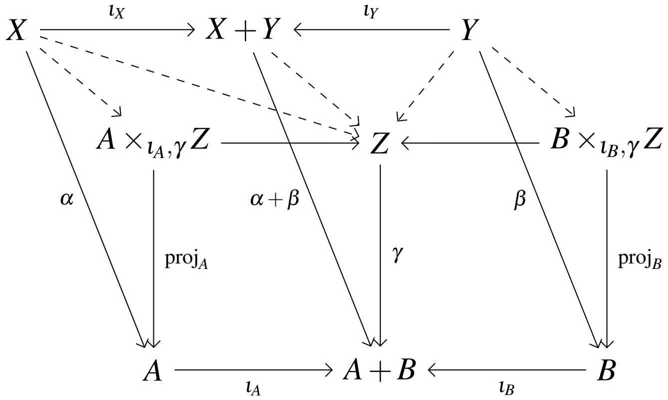
Top, Sets and $\mathsf{CSemiRing}^{\mathrm{op}}$, in addition to being extensive, have pullbacks along coproduct injections. These two conditions conspire to produce some wonderfully simple and powerful results, which have the benefit of allowing us to reformulate extensivity as an "internal" property; that is to say, a property involving only a fixed number of objects and morphisms between them in a category, instead of constructions of morphisms across the whole category.

Let us note first that, for a category $\mathbb{C}$ with finite coproducts and pullbacks along coproduct injections, we may define the functor $R : (\mathbb{C} \downarrow A+B) \longrightarrow (\mathbb{C} \downarrow A) \times (\mathbb{C} \downarrow B)$ for any objects A and B by $(Z,\gamma) \longmapsto \Big(\big(A \times_{\iota_A,\gamma} Z, \mathrm{proj}_A\big), \big(B \times_{\iota_B,\gamma} Z, \mathrm{proj}_B\big) \Big)$ as pullback squares in the diagram

$$\begin{array}{ccccc} A \times_{\iota_A,\gamma} Z & \longrightarrow & Z & \longleftarrow & B \times_{\iota_B,\gamma} Z \\ \Big\downarrow{\scriptstyle \mathrm{proj}_A} & & \Big\downarrow{\scriptstyle \gamma} & & \Big\downarrow{\scriptstyle \mathrm{proj}_B} \\ A & \xrightarrow[\iota_A]{} & A+B & \xleftarrow[\iota_B]{} & B \end{array} \tag{1.4}$$

Proposition 1.12. *In a category $\mathbb{C}$ with finite coproducts and pullbacks along coproduct injections, the functor $R : (\mathbb{C} \downarrow A+B) \longrightarrow (\mathbb{C} \downarrow A) \times (\mathbb{C} \downarrow B)$ above is right adjoint to the coproduct functor of (1.1).*

Proof. Consider the diagram



(1.5)

in $\mathbb{C}$ in which the bottom row is a coproduct diagram, the (front) squares are pullback squares, and the back top row is a coproduct diagram. By the universal properties of coproducts and pullbacks, the dashed arrows are associated to one another via the following bijections natural in the respective arguments

$$\begin{aligned}
&\hom_{(\mathbb{C}\downarrow A+B)}\big((X+Y,\alpha+\beta),(Z,\gamma)\big) \\
&\cong \hom_{(\mathbb{C}\downarrow A+B)}\big((X,\iota_A\alpha),(Z,\gamma)\big)\times\hom_{(\mathbb{C}\downarrow A+B)}\big((Y,\iota_B\beta),(Z,\gamma)\big) \\
&\cong \hom_{(\mathbb{C}\downarrow A)}\Big(\big(X,\alpha\big),\big(A\times_{\iota_A,\gamma}Z,\mathrm{proj}_A\big)\Big)\times\hom_{(\mathbb{C}\downarrow B)}\Big(\big(Y,\beta\big),\big(B\times_{\iota_B,\gamma}Z,\mathrm{proj}_B\big)\Big) \\
&\cong \hom_{(\mathbb{C}\downarrow A)\times(\mathbb{C}\downarrow B)}\bigg(\Big(\big(X,\alpha\big),\big(Y,\beta\big)\Big),\Big(\big(A\times_{\iota_A,\gamma}Z,\mathrm{proj}_A\big),\big(B\times_{\iota_B,\gamma}Z,\mathrm{proj}_B\big)\Big)\bigg)
\end{aligned} \tag{1.6}$$

□

Definition 1.13. A category $\mathbb{C}$ with finite coproducts is said to be lextensive if it is extensive and admits pullbacks along coproduct injections.

Example 1.14. Top, Sets and $\mathsf{CSemiRing}^{\mathrm{op}}$ are lextensive categories.

Here now is our internal version of lextensivity, guided by Proposition 1.12.

Theorem 1.15. *A category $\mathbb{C}$ with finite coproducts and pullbacks along coproduct injections is lextensive if and only if for every commutative diagram of the form*

$$\begin{array}{ccccc}
X & \longrightarrow & Z & \longleftarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
A & \longrightarrow & A+B & \longleftarrow & B
\end{array} \tag{1.7}$$

where the bottom row is a coproduct diagram, the following are equivalent:

1. *The top row is a coproduct diagram.*
2. *The squares are pullbacks.*

Proof. For each $\big((X,\alpha),(Y,\beta)\big)\in(\mathbb{C}\downarrow A)\times(\mathbb{C}\downarrow B)$, the component

$$\eta_{\big((X,\alpha),(Y,\beta)\big)} : \big((X,\alpha),(Y,\beta)\big)\longrightarrow RS\big((X,\alpha),(Y,\beta)\big)$$

of the unit $\eta : 1 \longrightarrow RS$ of the adjunction in Proposition 1.12 is the morphism corresponding to $1_{X+Y} : X+Y \longrightarrow X+Y$ in the sequence of natural bijections of (1.6), taking $(Z,\gamma) = (X+Y,\alpha+\beta)$. Considering diagram (1.5) in such a situation then, having each such component (of the unit) being an isomorphism is the same as having the squares being pullbacks in diagram (1.7) whenever the top row is a coproduct.

By a similar argument for the counit $\varepsilon : SR \longrightarrow 1$, having each component of the counit being an isomorphism is the same as having the top row being a coproduct in diagram (1.7) whenever the squares are pullbacks.

In conclusion then $\mathbb{C}$ is lextensive if and only if $(S,R,\eta,\varepsilon) : (\mathbb{C}\downarrow A)\times(\mathbb{C}\downarrow B) \longrightarrow (\mathbb{C}\downarrow A+B)$ is an adjoint equivalence if and only if condition (1) $\Longleftrightarrow$ condition (2) in diagram (1.7). □

Corollary 1.16. *Any subcategory $\mathbb{S}$ of a lextensive category $\mathbb{C}$, closed under coproducts and pullbacks along coproduct injections in $\mathbb{C}$, is lextensive.*

Proof. Under the assumptions on $\mathbb{S}$, one easily constructs a functor $R_{\mathbb{S}} : (\mathbb{S}\downarrow A+B) \longrightarrow (\mathbb{S}\downarrow A)\times(\mathbb{S}\downarrow B)$ via the diagram (1.4), which is inverse to the coproduct functor $+ : (\mathbb{S}\downarrow A)\times(\mathbb{S}\downarrow B) \longrightarrow (\mathbb{S}\downarrow A+B)$ by Theorem 1.15 and the constructions of its proof. □

Example 1.17. The category Stone of totally disconnected compact Hausdorff spaces (profinite spaces) is lextensive. Indeed, since Stone is a reflective subcategory in Top, which itself is complete, Stone is closed under finite limits in Top. Furthermore, Stone is easily seen to be closed under finite coproducts in Top. Therefore, since Top is lextensive Stone is too, by Corollary 1.16.

Example 1.18. Since the category CRing of commutative rings is obviously closed under finite products in CSemiRing and is a coreflective subcategory in CSemiRing, $\mathsf{CRing}^{\mathrm{op}}$ is lextensive by Corollary 1.16.

Example 1.19. The category FinSets of finite sets is lextensive, again by Corollary 1.16.

If (A,α) is any object of the slice category $(\mathbb{C}\downarrow C)$ for some $C \in \mathbb{C}$, then

$$\big((\mathbb{C}\downarrow C)\downarrow(A,\alpha)\big) \approx \big(\mathbb{C}\downarrow A\big)$$

Therefore, the following result is immediate once we know how to form limits and coproducts in the slice category $(\mathbb{C}\downarrow C)$.

Proposition 1.20. *[5, Proposition 4.8] If a category $\mathbb{C}$ is lextensive, then any slice category $(\mathbb{C}\downarrow C)$ is too, for every $C \in \mathbb{C}$.*

Example 1.21. The categories $(\mathsf{Top} \downarrow X)$,$(B \downarrow \mathsf{CSemiRing})^{\mathrm{op}}$ and $(R \downarrow \mathsf{CRing})^{\mathrm{op}}$ are lextensive, for each topological space X, each commutative semiring B and each commutative ring R.

Definition 1.22. A category $\mathbb{C}$ with finite coproducts and pullbacks along coproduct injections is said to have:

1. universal coproducts, if in diagram (1.7), condition (2) implies condition (1);
2. disjoint coproducts, if for any A, B in $\mathbb{C}$, $0 \cong A \times_{\iota_A,\iota_B} B$, where 0 (called *zero*) is (any) initial object in $\mathbb{C}$ and $\iota_A : A \longrightarrow A+B$ and $\iota_B : B \longrightarrow A+B$ are the canonical coproduct injections;
3. a strict zero if every morphism in $\mathbb{C}$ into its zero is an isomorphism.

As seen in Top, Sets and $\mathsf{CSemiRing}^{\mathrm{op}}$, zeroes are strict and coproduct injection morphisms are inclusion maps. In fact, these facts hold more generally in any lextensive category $\mathbb{C}$.

Lemma 1.23. *If* $0 \longrightarrow Z \longleftarrow Y$ *is a coproduct diagram, then* $Y \longrightarrow Z$ *is an isomorphism.*

Proof. Since the diagram

$$0 \longrightarrow Y \overset{1_Y}{=\!=\!=\!=} Y$$

is always a coproduct diagram, if $0 \longrightarrow Z \longleftarrow Y$ is a coproduct diagram, it follows $Y \longrightarrow Z$ is an isomorphism. □

Proposition 1.24. *A category* $\mathbb{C}$ *with finite coproducts and pullbacks along coproduct injections has disjoint coproducts if, in diagram (1.7), condition (1) implies condition (2).*

Proof. For a coproduct diagram $A \longrightarrow A+B \longleftarrow B$, consider the diagram

$$\begin{array}{ccccc}
A & \overset{1_A}{=\!=\!=\!=} & A & \longleftarrow & 0 \\
{\scriptstyle 1_A}\Big\| & & \Big\downarrow & & \Big\downarrow \\
A & \longrightarrow & A+B & \longleftarrow & B
\end{array}$$

Since the top row is a coproduct diagram, the squares are pullbacks. Since the right hand square is a pullback, the coproduct $A \longrightarrow A+B \longleftarrow B$ is disjoint. □

Proposition 1.25. *In a lextensive category $\mathbb{C}$, every coproduct injection is a monomorphism.*

Proof. Since $\mathbb{C}$ has universal coproducts, the top row in the diagram

$$\begin{array}{ccccc} A \times_{\iota_A,\iota_A} A & \xrightarrow{\mathrm{proj}_2} & A & \longleftarrow & A \times_{\iota_A,\iota_B} B \\ {\scriptstyle \mathrm{proj}_1}\downarrow & & \downarrow{\scriptstyle \iota_A} & & \downarrow \\ A & \xrightarrow[\iota_A]{} & A+B & \xleftarrow[\iota_B]{} & B \end{array}$$

is a coproduct diagram. Since $\mathbb{C}$ has disjoint coproducts by Proposition 1.24, $A \times_{\iota_A,\iota_B} B \cong 0$. Therefore, by Lemma 1.23, $\mathrm{proj}_2 : A \times_{\iota_A,\iota_A} A \longrightarrow A$ is isomorphism. But then, since the squares are pullbacks, $\mathrm{proj}_2 = \mathrm{proj}_1$ and also $\iota_A : A \longrightarrow A+B$ is a monomorphism. □

Proposition 1.26. *If $\mathbb{C}$ has universal coproducts, then it has a strict zero.*

Proof. Given a morphism $X \longrightarrow 0$ and considering the commutative diagram

$$\begin{array}{ccccc} X & \overset{\iota_1 = 1_X}{=\!=\!=\!=} & X & \overset{\iota_2 = 1_X}{=\!=\!=\!=} & X \\ \downarrow & & \downarrow & & \downarrow \\ 0 & =\!=\!=\!= & 0 & =\!=\!=\!= & 0 \end{array}$$

we conclude that its top row is a coproduct diagram since its squares are obviously pullbacks. Therefore, the canonical morphism $\theta : \hom_{\mathbb{C}}(X,-) \longrightarrow \hom_{\mathbb{C}}(X,-) \times \hom_{\mathbb{C}}(X,-)$, induced by composition with the coproduct injection morphisms, is an isomorphism. Since X has a morphism into 0, it also has morphisms into all other objects; that is, $\hom_{\mathbb{C}}(X,A)$ is non-empty for each $A \in \mathbb{C}$.

For any $A \in \mathbb{C}$, take f and g in $\hom_{\mathbb{C}}(X,A)$. Then we have the morphisms $h_1 = \theta_A^{-1}(f,f) : X \longrightarrow A$ and $h_2 = \theta_A^{-1}(g,f) : X \longrightarrow A$. However, $h_1 = h_1\iota_2 = f = h_2\iota_2 = h_2$, and so $f = g$. □

Definition 1.27. Let $p : E \longrightarrow B$ be a chosen morphism of some category $\mathbb{C}$ which has pullbacks. The pullback functor p^* determined by p is the functor

$$p^* : (\mathbb{C} \downarrow B) \longrightarrow (\mathbb{C} \downarrow E)$$

defined by $p^*(A,\alpha) = (E \times_{p,\alpha} A, \mathrm{proj}_1)$ as in the pullback diagram

$$\begin{array}{ccc} E \times_{p,\alpha} A & \xrightarrow{\mathrm{proj}_2} & A \\ {\scriptstyle \mathrm{proj}_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ E & \xrightarrow[p]{} & B \end{array}$$

Definition 1.28. A category $\mathbb{C}$ with finite coproducts and pullbacks is said to have pullbacks distributive with respect to (finite) coproducts, if for every morphism $p : E \longrightarrow B$, the pullback functor $p^* : (\mathbb{C} \downarrow B) \longrightarrow (\mathbb{C} \downarrow E)$ preserves finite coproducts.

Proposition 1.29. *A category $\mathbb{C}$ with finite coproducts and pullbacks has pullbacks distributive with respect to finite coproducts if it has universal coproducts.*

Proof. Suppose $p : E \longrightarrow B$, $t : X \longrightarrow B$ and $s : Y \longrightarrow B$ are morphisms in $\mathbb{C}$. Consider the diagram

$$\big(E \times_{p,t} X, \mathrm{proj}_E\big) \overset{1_E \times \iota_X}{\longrightarrow} \big(E \times_{p,[t,s]} (X+Y), \mathrm{proj}_E\big) \overset{1_E \times \iota_Y}{\longleftarrow} \big(E \times_{p,s} Y, \mathrm{proj}_E\big) \tag{1.8}$$

in $(\mathbb{C} \downarrow E)$ which is the image of the coproduct diagram

$$\big(X,t\big) \xrightarrow{\iota_X} \big(X+Y, [t,s]\big) \xleftarrow{\iota_Y} \big(Y,s\big)$$

in $(\mathbb{C} \downarrow B)$ under the pullback functor p^*, and where $[t,s] : X+Y \longrightarrow B$ is the unique morphism such that $[t,s]\iota_X = t$ and $[t,s]\iota_Y = s$. We show that diagram (1.8) is a coproduct diagram.

A routine calculation shows that each of the squares in the diagram

$$\begin{array}{ccccc} E \times_{p,t} X & \xrightarrow{1_E \times \iota_X} & E \times_{p,[t,s]} (X+Y) & \xleftarrow{1_E \times \iota_Y} & E \times_{p,s} Y \\ {\scriptstyle \mathrm{proj}_X}\downarrow & & \downarrow{\scriptstyle \mathrm{proj}_{X+Y}} & & \downarrow{\scriptstyle \mathrm{proj}_Y} \\ X & \xrightarrow[\iota_X]{} & X+Y & \xleftarrow[\iota_Y]{} & Y \end{array}$$

is a pullback. Therefore, since $\mathbb{C}$ has universal coproducts, the top row is a coproduct diagram. This implies the desired distributivity. □

Example 1.30. Since $\mathsf{CSemiRing}^{\mathrm{op}}$ is lextensive, it has universal coproducts and, therefore, has pullbacks distributive with respect to finite coproducts. Moreover, because coproducts in $\mathsf{CSemiRing}^{\mathrm{op}}$ are products in $\mathsf{CSemiRing}$, and pullbacks in $\mathsf{CSemiRing}^{\mathrm{op}}$ are calculated as tensor products in $\mathsf{CSemiRing}$,

$$E \otimes_B (X \times Y) \cong (E \otimes_B X) \times (E \otimes_B Y)$$

for commutative B-semialgebras E, X and Y.

As expected, there is another important reformulation of lextensivity, one which connects it to earlier work in geometry, topology and topos theory developed by A. Grothendieck and his followers.

Theorem 1.31. *[5, Proposition 2.14] A category $\mathbb{C}$ with finite coproducts and pullbacks along coproduct injections is lextensive if and only if it has universal coproducts and disjoint coproducts.*

In a category $\mathbb{C}$ with coproducts, there are at least two formulations of what we might call a *connected object*. As we shall see below, these may be thought of as internal and external definitions of connectedness. Usefully, the notion of lextensive category is powerful enough to make those two and several other notions of connectedness equivalent in any lextensive category.

Theorem 1.32. *Suppose $\mathbb{C}$ is a lextensive category. For any object $C \in \mathbb{C}$, the following conditions are equivalent:*

1. *C is not an initial object, and if $X \longrightarrow C \longleftarrow Y$ is a coproduct diagram, then either X or Y is an initial object.*

2. *C is not an initial object, and if $X \longrightarrow C \longleftarrow Y$ is a coproduct diagram, then either $X \longrightarrow C$ or $Y \longrightarrow C$ is an isomorphism.*

3. *C is not an initial object, and any morphism from C to a coproduct $X+Y$ factors through one of the two coproduct injections $X \longrightarrow X+Y$ and $Y \longrightarrow X+Y$.*

4. *Any morphism from C to a coproduct $X+Y$ factors through exactly one of the two coproduct injections $X \longrightarrow X+Y$ and $Y \longrightarrow X+Y$, and such a factorization is unique.*

5. *The functor $hom_{\mathbb{C}}(C,-) : \mathbb{C} \longrightarrow \mathsf{Sets}$ preserves binary coproducts.*

6. *The functor hom*$_{\mathbb{C}}(C,-) : \mathbb{C} \longrightarrow \mathsf{Sets}$ *preserves finite coproducts.*

Proof. Obviously (6) $\Longrightarrow$ (5) $\Longleftrightarrow$ (4) $\Longrightarrow$ (3). We show (3) $\Longrightarrow$ (2), (2) $\Longrightarrow$ (1), (1) $\Longrightarrow$ (5) and (5) $\Longrightarrow$ (6).

(3) $\Longrightarrow$ (2): For a coproduct diagram $X \longrightarrow C \longleftarrow Y$, without loss of generality we can assume that $1_C : C \longrightarrow C$ factors through $X \longrightarrow C$. This gives a commutative diagram of the form

$$\begin{array}{ccccc} C & = & C & \longleftarrow & 0 \\ \downarrow & & \| & & \downarrow \\ X & \longrightarrow & C & \longleftarrow & Y \end{array}$$

whose both squares are pullbacks because the rows are coproduct diagrams. Since the square on the left is a pullback, $X \longrightarrow C$ is an isomorphism.

(2) $\Longrightarrow$ (1) involves a similar construction and is equally easy to prove.

(1) $\Longrightarrow$ (5): For a given morphism $C \longrightarrow A+B$, consider the commutative diagram

$$\begin{array}{ccccc} A\times_{A+B}C & \longrightarrow & C & \longleftarrow & B\times_{A+B}C \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A+B & \longleftarrow & B \end{array}$$

whose squares are pullbacks. Since the top row is then a coproduct diagram, one has, without loss of generality, that $B\times_{A+B}C \cong 0$. By Lemma 1.23, $A\times_{A+B}C \longrightarrow C$ is an isomorphism. This shows that the canonical map $\hom_{\mathbb{C}}(C,A)+\hom_{\mathbb{C}}(C,B) \longrightarrow \hom_{\mathbb{C}}(C,A+B)$ induced by composition with the coproduct injection morphisms $A \longrightarrow A+B$ and $B \longrightarrow A+B$, is a surjection. Finally, since the squares are pullbacks, C is not initial, and $\mathbb{C}$ has a strict zero, the map $\hom_{\mathbb{C}}(C,A)+\hom_{\mathbb{C}}(C,B) \longrightarrow \hom_{\mathbb{C}}(C,A+B)$ is a bijection. Therefore, the functor $\hom_{\mathbb{C}}(C,-) : \mathbb{C} \longrightarrow \mathsf{Sets}$ preserves binary coproducts.

(5) $\Longrightarrow$ (6): We need only show that $\hom_{\mathbb{C}}(C,0)$ is the empty set. Since C is not initial by, say (5) $\Longrightarrow$ (3), and $\mathbb{C}$ has a strict zero, the result follows. □

Definition 1.33. An object C in a lextensive category $\mathbb{C}$ is called connected if it satisfies the equivalent conditions of Theorem 1.32.

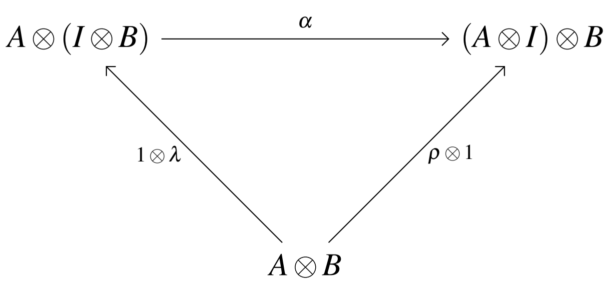
Example 1.34. A non-trivial commutative B-semialgebra (A,α) is connected exactly when it has no non-trivial complementary idempotents. For, suppose such a commutative B-semialgebra A has non-trivial complementary idempotents a_1 and a_2. Then, $A \cong a_1A \times a_2A$ where both of a_1A and a_2A are non-trivial commutative B-semialgebras. Therefore, A is not connected. On the other hand, if $\phi : A_1 \times A_2 \longrightarrow A$ is an isomorphism of B-semialgebras where both of A_1 and A_2 are non-trivial commutative B-semialgebras, then A has the non-trivial complementary idempotents $a_1 = \phi(1,0)$ and $a_2 = \phi(0,1)$.

1.2 Monoidal categories, monads and algebras

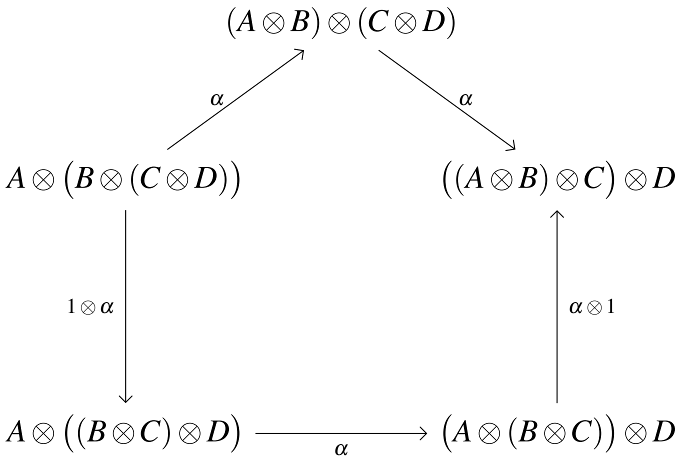
Monoidal categories are the context in which we consider the theory of monads and their algebras. As we shall see, monoidal categories are "just the right environment" to make precise the concept of a monoid object in a category and actions of such monoids on objects of a second category.

Definition 1.35. A monoidal category is a system $(\mathbb{C},I,\otimes,\alpha,\lambda,\rho)$ in which:

- $\mathbb{C}$ is a category;
- I is an object of $\mathbb{C}$;
- $\otimes : \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$ is a functor, written as $\otimes(A,B) = A\otimes B$;
- $\alpha = \big(\alpha_{A,B,C} : A\otimes(B\otimes C) \longrightarrow (A\otimes B)\otimes C\big)_{A,B,C\in\mathbb{C}}$, $\lambda = \big(\lambda_A : A \longrightarrow (I\otimes A)\big)_{A\in\mathbb{C}}$ and $\rho = \big(\rho_A : A \longrightarrow (A\otimes I)\big)_{A\in\mathbb{C}}$ are families are isomorphisms in $\mathbb{C}$, natural in each of their arguments, and such that the diagram



and the pentagonal associativity diagram



commute.

As in the diagrams, we shall just use α, λ and ρ without the subscripts. We will also often write $\mathbb{C} = (\mathbb{C}, I, \otimes) = (\mathbb{C}, I, \otimes, \alpha, \lambda, \rho)$.

For a justification of these definitions, as well as examples showing their necessity, the reader is urged to consult [19, Section VII.1].

A monoidal category $(\mathbb{C}, I, \otimes, \alpha, \lambda, \rho)$ is said to be *strict* if: $A \otimes (B \otimes C) = (A \otimes B) \otimes C$ for all A, B, C in $\mathbb{C}$; $A \otimes I = A = I \otimes A$ for all A in $\mathbb{C}$; and α, λ and ρ are the identity morphisms.

Example 1.36. A category $\mathbb{C}$ with finite products admits a natural monoidal structure by taking $\otimes = \times$ the (chosen) binary product functor, $I = 1$ the terminal object of $\mathbb{C}$ and α, λ, and ρ the canonical isomorphisms $A \times (B \times C) \cong (A \times B) \times C$ and $A \times 1 \cong A \cong 1 \times A$ arising from the universality of $\times$.

Since the category Cat of categories has finite products, it is itself a monoidal category, with $\mathsf{Cat} = (\mathsf{Cat}, \times, \mathbb{1})$. Therefore, α, λ and ρ in Definition 1.35 above are in fact natural isomorphisms between appropriate functors.

The next example is especially important.

Example 1.37. Each category $\mathbb{X}$ determines a strict monoidal category

$$\mathsf{End}(\mathbb{X}) = \big(\mathsf{End}(\mathbb{X}), 1_{\mathbb{X}}, \bullet\big)$$

of endofunctors $F : \mathbb{X} \longrightarrow \mathbb{X}$, where $1_{\mathbb{X}}$ is the identity functor on $\mathbb{X}$ and $\bullet$ is composition of functors.

Definition 1.38. Given a monoidal category $\mathbb{C}$, a monoid in $\mathbb{C}$ is a triple $M = (M, m, e)$, in which M is an object in $\mathbb{C}$ and $m : M \otimes M \longrightarrow M$ and $e : I \longrightarrow M$ are morphisms in $\mathbb{C}$ making the diagram

$$\begin{array}{ccccccc}
M \otimes (M \otimes M) & \xrightarrow{\alpha} & (M \otimes M) \otimes M & \xrightarrow{m \otimes 1} & M \otimes M & \xleftarrow{(e \otimes 1)\lambda} & M \\
\downarrow{\scriptstyle 1 \otimes m} & & & & \downarrow{\scriptstyle m} & {\large //} & \downarrow{\scriptstyle (1 \otimes e)\rho} \\
M \otimes M & & \xrightarrow{\quad m \quad} & & M & \xleftarrow{m} & M \otimes M
\end{array} \tag{1.9}$$

commute.

It turns out that the monoids in the strict monoidal category $\mathsf{End}(\mathbb{X})$ of Example 1.37, called *monads*, are closely related to the theory of adjunctions, the canonical examples of this relationship coming from universal algebra. The precise connection, known as *monadicity*, relates, in some sense, two fundamental ideas of category theory with each other: the algebra-like theory of monoidal categories, with its monoids and monoidal actions on the one hand and the theory of adjunctions on the other.

Definition 1.39. A monad on a category $\mathbb{X}$ is a monoid in the monoidal category $\mathsf{End}(\mathbb{X})$. In detail, a monad on $\mathbb{X}$ is a triple $T = (T, \mu, \eta)$, in which $T : \mathbb{X} \longrightarrow \mathbb{X}$ is a functor and $\mu : T^2 \longrightarrow T$ and $\eta : 1_{\mathbb{X}} \longrightarrow T$ are natural transformations such that the diagram

$$\begin{array}{ccccc}
T^3 & \xrightarrow{\mu T} & T^2 & \xleftarrow{\eta T} & T \\
\downarrow{\scriptstyle T\mu} & & \downarrow{\scriptstyle \mu} & {\large //} & \downarrow{\scriptstyle T\eta} \\
T^2 & \xrightarrow{\mu} & T & \xleftarrow{\mu} & T^2
\end{array}$$

commutes.

Definition 1.40. Let $T=(T,\mu,\eta)$ be a monad on a category $\mathbb{X}$. A T-algebra is a pair (X,ξ) in which X is an object in $\mathbb{X}$ and $\xi : T(X) \longrightarrow X$ is a morphism in $\mathbb{X}$ such that the diagram

$$\begin{array}{ccccc} T^2(X) & \xrightarrow{\mu_X} & T(X) & \xleftarrow{\eta_X} & X \\ {\scriptstyle T(\xi)}\downarrow & & \downarrow{\scriptstyle \xi} & \swarrow\!\!= & \\ T(X) & \xrightarrow[\xi]{} & X & & \end{array}$$

commutes.

A morphism $h:(X,\xi)\longrightarrow(Y,\zeta)$ of T-algebras is a morphism $h: X\longrightarrow Y$ of $\mathbb{X}$ such that the diagram

$$\begin{array}{ccc} T(X) & \xrightarrow{T(h)} & T(Y) \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \zeta} \\ X & \xrightarrow[h]{} & Y \end{array}$$

commutes.

The class of all T-algebras and the class of all their morphisms form a category, the category $\mathbb{X}^T$ of T-algebras.

Given a monad T on $\mathbb{X}$, there is an obvious forgetful functor $U^T:\mathbb{X}^T\longrightarrow\mathbb{X}$ defined by $U^T(X,\xi)=X$. In fact, this forgetful functor has a left adjoint.

Theorem 1.41. *[19, Section VI.2, Theorem 1] For a monad $T=(T,\mu,\eta)$ on $\mathbb{X}$, the functor $F^T:\mathbb{X}\longrightarrow\mathbb{X}^T$ defined by $F^T(X)=\big(T(X),\mu_X\big)$ is a left adjoint of U^T. The unit of the adjunction is the natural transformation $\eta^T=\eta:1_{\mathbb{X}}\longrightarrow T=U^TF^T$ of the monad T. The counit is the natural transformation $\varepsilon^T:F^TU^T\longrightarrow 1_{\mathbb{X}^T}$ defined by $\big(\varepsilon^T\big)_{(X,\xi)}=\xi$.*

1.3 Monadicity

It turns out that, for a given monad $T=(T,\mu,\eta)$ on $\mathbb{X}$, we may "recover" T from the adjunction $(F^T,U^T,\eta^T,\varepsilon^T):\mathbb{X}\longrightarrow\mathbb{X}^T$ of Theorem 1.41 of the previous section. Let us first generalize precisely how an arbitrary adjunction $(F,U,\eta,\varepsilon):\mathbb{X}\longrightarrow\mathbb{A}$ determines a monad on $\mathbb{X}$.

Theorem 1.42. *[19, Section VI.1] For every adjunction* $(F,U,\eta,\varepsilon):\mathbb{X}\longrightarrow\mathbb{A}$*, the triple* $T=(T,\mu,\eta)$ *defined by:*

- $T=UF$*;*
- η *of* (T,η,μ) *is the same as* η *of* (F,U,η,ε)*;*
- $\mu=U\varepsilon F$*,*

is a monad on $\mathbb{X}$*.*

Corollary 1.43. *For a monad* $T=(T,\mu,\eta)$ *on* $\mathbb{X}$*, the monad* T' *determined via Theorem 1.42 by the adjunction* $(F^T,U^T,\eta^T,\varepsilon^T):\mathbb{X}\longrightarrow\mathbb{X}^T$ *of Theorem 1.41 is the monad* T*; that is,* $T=T'$*.*

It is natural then at this point to ask whether, conversely, each adjunction $(F,U,\eta,\varepsilon):\mathbb{X}\longrightarrow\mathbb{A}$ arises from the monad it determines. In fact, there is much more to this story which may be briefly understood with the following additional remarks.

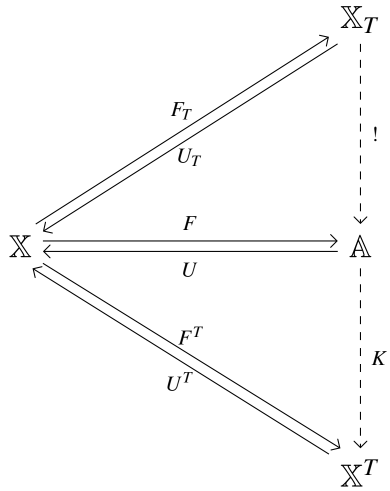
For adjunctions $(F,U,\eta,\varepsilon):\mathbb{X}\longrightarrow\mathbb{A}$ and $(F',U',\eta',\varepsilon'):\mathbb{X}'\longrightarrow\mathbb{A}'$, a *map of adjunctions* $(L_1,L_2):(F,U,\eta,\varepsilon)\longrightarrow(F',U',\eta',\varepsilon')$ is a pair of functors $L_1:\mathbb{X}\longrightarrow\mathbb{X}'$ and $L_2:\mathbb{A}\longrightarrow\mathbb{A}'$ such that $F'L_1=L_2F$ and $L_1U=U'L_2$.

Fixing $\mathbb{X}$, let us denote by $\mathsf{Adj}(\mathbb{X})$ the category whose objects are adjunctions $(F,U,\eta,\varepsilon):\mathbb{X}\longrightarrow\mathbb{A}$ for some category $\mathbb{A}$, and whose morphisms are maps of adjunctions $(L_1,L_2):(F,U,\eta,\varepsilon)\longrightarrow(F',U',\eta',\varepsilon')$ where $L_1=1_{\mathbb{X}}$, the identity functor on $\mathbb{X}$. Note that this is *not* the 2-category of categories, adjunctions and conjugate pairs of natural transformations. See, for example, [19, Chapter IV] for details.

For a monad T on $\mathbb{X}$, consider $\mathsf{Adj}(\mathbb{X},T)$, the full subcategory of $\mathsf{Adj}(\mathbb{X})$ whose object adjunctions determine (via Theorem 1.42) the monad T. $\mathsf{Adj}(\mathbb{X},T)$ has a terminal object.

Theorem 1.44. *[19, Section VI.3, Theorem 1] Let* $(F,U,\eta,\varepsilon):\mathbb{X}\longrightarrow\mathbb{A}$ *and* $T=(T,\eta,\mu)$ *be as in Theorem 1.42. Then there exists a unique functor* $K:\mathbb{A}\longrightarrow\mathbb{X}^T$*, defined by* $K(A)=(U(A),U(\varepsilon_A))$*, with* $U^TK=U$ *and* $KF=F^T$*.*

The category $\mathbb{X}^T$ is called the *Eilenberg-Moore category* of T. It turns out that $\mathsf{Adj}(\mathbb{X},T)$ also has an initial object. This category $\mathbb{X}_T$, called the *Kleisli category* of T, is the full subcategory of $\mathbb{X}^T$ with objects all free T-algebras $F^T(X)=(T(X),\mu_X)$ for each $X\in\mathbb{X}$. Again, we have the adjunction $(F_T,U_T):\mathbb{X}\longrightarrow\mathbb{X}_T$ where the functors F_T and U_T are the restrictions of F^T and U^T to $\mathbb{X}_T$, respectively. A convenient display is:



(1.10)

Definition 1.45. Let $(F,U,\eta,\varepsilon) : \mathbb{X} \longrightarrow \mathbb{A}$ and $T = (T,\eta,\mu)$ be as in Theorems 1.42 and 1.44. Then:

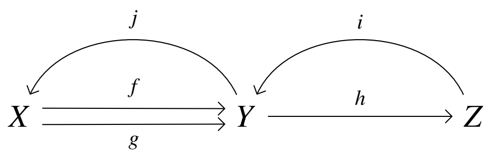
1. the functor $K : \mathbb{A} \longrightarrow \mathbb{X}^T$ as in Theorem 1.44 and diagram (1.10) above is called the comparison functor;

2. the functor $U : \mathbb{A} \longrightarrow \mathbb{X}$ is said to be monadic if the comparison functor K is a category equivalence.

Our original question now reduces to asking when the functor U of a given adjunction $(F,U,\eta,\varepsilon) : \mathbb{X} \longrightarrow \mathbb{A}$ is monadic. To answer this, we make the following definition.

Definition 1.46.

1. A diagram of the form



(1.11)

in a category $\mathbb{C}$ in which $hf = hg$, $hi = 1_Z$, $fj = 1_Y$ and $gj = ih$ is called a split fork.

2. The pair (f,g) in diagram (1.11) is said to be contractible if there exists a morphism $k : Y \longrightarrow X$ with $fk = 1_Y$ and $gkf = gkg$.

A triple (f,g,h) of morphisms in $\mathbb{C}$ where $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ are parallel morphisms and $h : Y \longrightarrow Z$ is a morphism satisfying $hf = hg$ is also called a *fork*.

Remark 1.47. Since the conditions specified in Definitions 1.46(1) and 1.46(2) are purely equational, for any functor $F : \mathbb{C} \longrightarrow \mathbb{D}$, the image of a split fork in $\mathbb{C}$ under F is again a split fork in $\mathbb{D}$, and the image of a contractible pair in $\mathbb{C}$ under F is again contractible in $\mathbb{D}$.

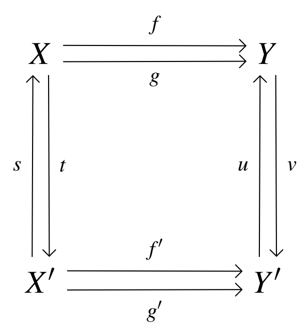
Proposition 1.48. *Suppose the parallel morphisms* $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ *and the morphism* $h : Y \longrightarrow Z$ *form a fork in* $\mathbb{C}$*. There exists a pair* (i, j) *of morphisms* $i : Z \longrightarrow Y$ *and* $j : Y \longrightarrow X$ *which altogether form a split fork if and only if the fork* (f, g, h) *is a coequalizer and the pair* (f, g) *is contractible.*

Proof. "Only if": For a fork (f, g, h) in $\mathbb{C}$, suppose there exists a pair (i, j) of morphisms $i : Z \longrightarrow Y$ and $j : Y \longrightarrow X$ such that $hf = hg$, $hi = 1_Z$, $fj = 1_Y$ and $gj = ih$. We first show that the fork (f, g, h) is a coequalizer diagram. For, given an object M and a morphism $m : Y \longrightarrow M$ of $\mathbb{C}$ such that $mf = mg$, suppose there is a morphism $t : Z \longrightarrow M$ satisfying $th = m$. Then, $t = thi = mi$, and so such a morphism t is uniquely determined. Therefore, (f, g, h) is coequalizer diagram. Next, an easy check shows that, with $k = j$, the pair (f, g) is contractible.

"If": Suppose, for the coequalizer (f, g, h), the fork (f, g) is contractible. Then, there exists a morphism $k : Y \longrightarrow X$ with $fk = 1_Y$ and $gkf = gkg$. Consider the morphism $gk : Y \longrightarrow Y$. Since (f, g) is contractible, $(gk)f = (gk)g$, and so, by the universal property of the morphism $h : Y \longrightarrow Z$, there exists a unique morphism $i : Z \longrightarrow Y$ such that $gk = ih$. Finally, for the morphism $hi : Z \longrightarrow Z$, $(hi)h = h(ih) = h(gk) = (hg)k = (hf)k = h(fk) = h$. Therefore, again by the universal property of the morphism h, $hi = 1_Z$. Therefore, the pair (i, k) together with the fork (f, g, h) form a split fork. □

The following results are taken from [16].

Proposition 1.49. *In the diagram*



in which $vf = f't$, $vg = g't$, $fs = uf'$, $gs = ug'$, $ts = 1_{X'}$ *and* $vu = 1_{Y'}$*, if* (f, g) *is contractible then* (f', g') *is too.*

Proof. Given $j : Y \longrightarrow X$ satisfying $fj = 1_Y$ and $gjf = gjg$, taking $j' = tju : Y' \longrightarrow X'$ is easily seen to make the pair (f', g') contractible. □

Corollary 1.50. *Let* $\mathbb{C} \overset{F}{\underset{G}{\rightrightarrows}} \mathbb{D}$ *be parallel functors between categories* $\mathbb{C}$ *and* $\mathbb{D}$*, and* $\tau : F \longrightarrow G$ *a split epimorphism of functors. If, for a parallel pair of morphisms* (f, g) *in* $\mathbb{C}$*,* $\big(F(f), F(g)\big)$ *is contractible, then* $\big(G(f), G(g)\big)$ *is too.*

The following theorem, called the *Beck monadicity criterion* formally now answers our question of precisely when we may recover an adjunction $(F, U, \eta, \varepsilon) : \mathbb{X} \longrightarrow \mathbb{A}$ from the monad it determines. There are several equivalent formulations of this criterion; we present the one most suitable to our needs (see, for example, [19, Section VI.7, Theorem 1] and [19, Section VI.7, Exercise 6]).

Theorem 1.51. *Let* $\mathbb{A}$ *and* $\mathbb{X}$ *be categories with coequalizers. For an adjunction* $(F, U, \eta, \varepsilon) : \mathbb{X} \longrightarrow \mathbb{A}$*, the following conditions are equivalent:*

1. *The functor* $U : \mathbb{A} \longrightarrow \mathbb{X}$ *is monadic (c.f. Definition 1.45).*
2. *The functor* U *reflects isomorphisms, and* U *preserves coequalizers of those pairs* (f, g) *for which* $\big(U(f), U(g)\big)$ *is contractible.*

We have the following useful sufficient condition called the *split monadicity theorem in the presence of coequalizers*.

Theorem 1.52. *Let* $\mathbb{A}$ *and* $\mathbb{X}$ *be categories with coequalizers. For an adjunction* $(F, U, \eta, \varepsilon) : \mathbb{X} \longrightarrow \mathbb{A}$*, the functor* $U : \mathbb{A} \longrightarrow \mathbb{X}$ *is monadic if the counit* $\varepsilon : FU \longrightarrow 1_{\mathbb{A}}$ *is a split epimorphism.*

Proof. Since ε is a split epimorphism, there exists a natural transformation $\zeta : 1_{\mathbb{A}} \longrightarrow FU$ such that $\varepsilon\zeta = 1_{\mathbb{A}}$. For any $f : A \longrightarrow A'$ in $\mathbb{A}$, if $U(f)$ is invertible, then f is invertible with inverse $f^{-1} = \varepsilon_A FU(f)^{-1}\zeta_{A'}$. Therefore, U reflects isomorphisms. U also *reflects contractiblity*. For, suppose (f, g) is a parallel pair of morphisms in $\mathbb{A}$ such that $\big(U(f), U(g)\big)$ is contractible. Then $\big(FU(f), FU(g)\big)$ is contractible too, and so by Corollary 1.50, (f, g) is contractible. Calling (f, g, h) the coequalizer diagram of (f, g), the fork (f, g, h) is a split fork in $\mathbb{A}$ by Remark 1.47. Then $\big(U(f), U(g), U(h)\big)$ is a split fork in $\mathbb{X}$, and so, by Remark 1.47 again, the coequalizer of (f, g) is preserved by U. Therefore, U satisfies the criteria of Theorem 1.51, and it is monadic. □

The following "induced" monadicity theorem describes another useful condition for the monadicity of a functor $U : \mathbb{A} \longrightarrow \mathbb{X}$ which will be useful for our purposes in Chapter 3.

Theorem 1.53. *[16, Lemma 2.5, with misprint corrected] Let $\mathbb{A}$ and $\mathbb{X}$ be categories with coequalizers. For an adjunction $(F,U,\eta,\varepsilon) : \mathbb{X} \longrightarrow \mathbb{A}$, the functor $U : \mathbb{A} \longrightarrow \mathbb{X}$ is monadic if and only if there exists a commutative diagram*

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\quad U \quad} & \mathbb{X} \\ {\scriptstyle H_1}\downarrow & & \downarrow{\scriptstyle H_2} \\ \mathbb{A}' & \xrightarrow[\quad U' \quad]{} & \mathbb{X}' \end{array} \tag{1.12}$$

of functors such that:

1. *U' is monadic (or at least reflects isomorphisms and preserves coequalizers of those pairs (f,g) for which $\big(U'(f),U'(g)\big)$ is contractible);*
2. *H_1 preserves all coequalizers and reflects isomorphisms;*
3. *H_2 reflects isomorphisms.*

Proof. “If”: We show that condition (2) of Theorem 1.51 is satisfied. Since diagram (1.12) is commutative, and since both U' and H_1 reflect isomorphisms, so too does U.

Next, suppose $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ is a pair of parallel morphisms in $\mathbb{A}$ such that $\big(U(f),U(g)\big)$ is contractible. Let the (chosen) coequalizer diagram of (f,g) be the fork (f,g,h), with $h : Y \longrightarrow Z$, and let the (chosen) coequalizer diagram of $\big(U(f),U(g)\big)$ be the fork $\big(U(f),U(g),h'\big)$, with $h' : U(Y) \longrightarrow Z'$. Then:

1. $\big(H_2U(f),H_2U(g)\big)$ is contractible and hence so is $\big(U'H_1(f),U'H_1(g)\big)$.
2. Therefore, since H_1 preserves all coequalizers, U' (being monadic) preserves the coequalizer of $\big(H_1(f),H_1(g)\big)$. Therefore, $U'H_1$ preserves the coequalizer of (f,g), and hence so does H_2U.
3. By Remark 1.47, $\big(U(f),U(g)\big)$ being contractible makes the fork $\big(U(f),U(g),h'\big)$ a split fork, and hence $\big(H_2U(f),H_2U(g),H_2(h')\big)$ is a split fork (and a coequalizer diagram) too. That is, H_2 preserves the coequalizer of $\big(U(f),U(g)\big)$.
4. Since $\big(U(f),U(g),h'\big)$ is a coequalizer diagram in $\mathbb{X}$, there exists a unique morphism $t : Z' \longrightarrow U(Z)$ such that $th' = U(h)$. Since H_2 preserves the coequalizer of $\big(U(f),U(g)\big)$ by point 3, in $\mathbb{X}'$ we have $H_2(t) : H_2(Z') \longrightarrow H_2U(Z)$ is the unique morphism such

that $H_2(t)H_2(h') = H_2U(h)$, as in the diagram below. Because H_2U preserves the coequalizer of (f,g) by point 2, $H_2(t)$ is an isomorphism. Finally, since H_2 reflects isomorphisms, t is an isomorphism, $\big(U(f),U(g),U(h)\big)$ is a coequalizer diagram in $\mathbb{X}$, and so U preserves the coequalizer of (f,g).

$$\begin{array}{ccccc} H_2U(X) & \underset{H_2U(g)}{\overset{H_2U(f)}{\rightrightarrows}} & H_2U(Y) & \xrightarrow{H_2(h')} & H_2(Z') \\ & & & \searrow^{H_2U(h)} & \Big\downarrow {\scriptstyle H_2(t)} \\ & & & & H_2U(Z) \end{array}$$

"Only if": Simply take $H_1 = 1_{\mathbb{A}}$, $H_2 = 1_{\mathbb{X}}$ and $U' = U$. $\square$

Chapter 2

Galois theory in general categories

2.1 Galois structures and admissibility

Galois structures and the concept of admissibility are the fundamental building blocks of the purely categorical Galois theory.

Definition 2.1. A Galois structure is a system $(\mathbb{C},\mathbb{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{G})$ in which

$$(I,H,\eta,\varepsilon):\mathbb{C}\longrightarrow\mathbb{X}$$

is an adjunction and $\mathbf{F}$ and $\mathbf{G}$ are classes of morphisms in $\mathbb{C}$ and $\mathbb{X}$ respectively, satisfying the following conditions:

1. $I(\mathbf{F})\subseteq\mathbf{G}$ and $H(\mathbf{F})\subseteq\mathbf{G}$.

2. The category $\mathbb{C}$ admits pullbacks along morphisms from $\mathbf{F}$, and the class $\mathbf{F}$ is pullback stable; similarly, the category $\mathbb{X}$ admits pullbacks along morphisms from $\mathbf{G}$, and the class $\mathbf{G}$ is pullback stable. Furthermore, the classes $\mathbf{F}$ and $\mathbf{G}$ contain all isomorphisms of $\mathbb{C}$ and $\mathbb{X}$, respectively.

3. $\mathbf{F}$ and $\mathbf{G}$ are closed under composition.

Example 2.2. Each adjunction $(I,H,\eta,\varepsilon):\mathbb{C}\longrightarrow\mathbb{X}$ trivially determines a Galois structure $\Gamma=(\mathbb{C},\mathbb{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{G})$ in which $\mathbf{F}$ and $\mathbf{G}$ are the classes $\mathbb{C}_1$ and $\mathbb{X}_1$ of all morphisms of $\mathbb{C}$ and $\mathbb{X}$, respectively. In this case, we will simply write $(\mathbb{C},\mathbb{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{G})=(\mathbb{C},\mathbb{X},I,H)$.

Let us briefly provide some background remarks here. The *classical sheaf condition* is the equalizer diagram

$$C(U) \longrightarrow \prod_{i \in I} C(U_i) \rightrightarrows \prod_{i,j} C(U_i \cap U_j)$$

in Sets, for continuous real-valued functions $C(U)$ on an open subspace U (with open cover $(U_i)_{i\in I}$) of a topological space X (see, for example, [21, Section II.1]).

As we shall see quite briefly in this chapter, one may think of categorical Galois theory as a particularly important application of *monadic descent theory*, which is developed in [15]. In that paper, the authors show how a special case of this monadic descent theory, called *topological descent theory*, which is motivated by the classical sheaf condition above and includes it as a special case, may be introduced. Specifically, one first selects an open cover $(U_i)_{i\in I}$ of a topological space B together with the continuous function $p : E = \bigsqcup_{i\in I} U_i \longrightarrow B$, which is the inclusion map on each summand, and a class $\mathbf{E}$ of continuous functions which is closed under composition with homeomorphisms. Then for the full subcategory $\mathbf{E}(E)$ of $(\mathsf{Top} \downarrow E)$ of $\mathbf{E}$-bundles over E, the so-called *descent data* for an $\mathbf{E}$-bundle (C,γ) over E with respect to p yields exactly the cocycle condition which appears in the original descent theory for sheaves.

The monadic descent theory is the observation that these generalizations themselves have purely categorical counterparts using the theory of monads, and that such an observation, remarkably, requires very little knowledge of the original descent theory of sheaves. Specifically, one may replace Top with an arbitrary category $\mathbb{C}$ with pullbacks and analogously select a class $\mathbf{E}$ of morphisms of $\mathbb{C}$ which are closed under composition with isomorphisms. Fixing a morphism $p : E \longrightarrow B$ in $\mathbb{C}$ and requiring that $\mathbf{E}$ be stable with respect to pullbacks along p, we may construct the pullback functor $p^* : \mathbf{E}(B) \longrightarrow \mathbf{E}(E)$. It is shown then in [15, Section 2.1] that when $\mathbf{E}$ is additionally closed under composition with p, p^* has a left adjoint $p_!$ and the descent data becomes precisely the Eilenberg-Moore category (see Definition 1.40) over the monad T^p on $\mathbf{E}(E)$ induced by adjunction $p_! \dashv p^*$.

Returning to Definition 2.1 above, the specification of classes $\mathbf{F}$ and $\mathbf{G}$ of the kind seen in the definition of a Galois structure Γ can therefore be thought of as enforcing minimal conditions which still allow for the use of the theory of monads, as presented in the previous chapter. As we shall soon see precisely, the preceding remarks about the monadic descent theory call for a slight variation of the adjunction $(I,H,\eta,\varepsilon) : \mathbb{C} \longrightarrow \mathbb{X}$ from a Galois structure Γ. Fortunately, since $\mathbf{F}$ has the appropriate pullbacks, we may construct exactly what we need.

Definition 2.3. Given a Galois structure $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{G})$ and an object $B \in \mathbb{C}$, the adjunction induced by Γ is the adjunction

$$\big(I^B, H^B, \eta^B, \varepsilon^B\big) : \mathbf{F}\big(B\big) \longrightarrow \mathbf{G}\big(I(B)\big)$$

in which:

1. $\mathbf{F}(B)$ is the full subcategory in $(\mathbb{C} \downarrow B)$ with objects all pairs (A, α) with $\alpha : A \longrightarrow B$ a morphism of $\mathbf{F}$; similarly, $\mathbf{G}\big(I(B)\big)$ is the full subcategory in $\big(\mathbb{X} \downarrow I(B)\big)$ with objects all pairs (X, ϕ) with $\phi : X \longrightarrow I(B)$ a morphism of $\mathbf{G}$.
2. $I^B\big(A, \alpha\big) = \big(I(A), I(\alpha)\big)$.
3. $H^B(X, \phi) = (B \times_{HI(B)} H(X), \mathrm{proj}_1)$ as defined by the pullback

$$\begin{array}{ccc} B \times_{HI(B)} H(X) & \xrightarrow{\mathrm{proj}_2} & H(X) \\ \Big\downarrow{\scriptstyle \mathrm{proj}_1} & & \Big\downarrow{\scriptstyle H(\phi)} \\ B & \xrightarrow[\eta_B]{} & HI(B) \end{array} \tag{2.1}$$

4. The (component of the) unit ${\eta^B}_{(A,\alpha)} = \langle \alpha, \eta_A \rangle : A \longrightarrow B \times_{HI(B)} HI(A)$.
5. The (component of the) counit ${\varepsilon^B}_{(X,\phi)}$ is the composite

$$I\big(B \times_{HI(B)} H(X)\big) \xrightarrow{I(\mathrm{proj}_2)} IH(X) \xrightarrow{\varepsilon_X} X$$

   where proj_2 is as in diagram (2.1).

The admissibility condition is at the very heart of the categorical Galois theory.

Definition 2.4. An object B of $\mathbb{C}$ is said to be admissible (with respect to the Galois structure Γ) if the counit $\varepsilon^B : I^B H^B \longrightarrow 1_{\mathbf{G}(I(B))}$ is an isomorphism.

The proposition below gives the crucial, yet obvious, property of objects admissible with respect to Γ.

Proposition 2.5. *For a Galois structure* $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{G})$ *and an object* B *in* $\mathbb{C}$*, the following are equivalent:*

1. *B is admissible.*
2. *The functor* $H^B : \mathbf{G}\big(I(B)\big) \longrightarrow \mathbf{F}\big(B\big)$ *is fully faithful.*

The important point is this: if B is admissible with respect to Γ, then the full subcategory of all those objects $(A, \alpha) \in \mathbf{F}\big(B\big)$ for which the component of the unit ${\eta^B}_{(A,\alpha)}$ is an isomorphism (a category which we shall define in the next section) can be identified with the category $\mathbf{G}\big(I(B)\big)$.

2.2 Monadic extensions and coverings

Recall that the pullback functor p^* determined by a morphism $p : E \longrightarrow B$ in a category $\mathbb{C}$ which has pullbacks is defined by $p^*(A, \alpha) = (E \times_{p,\alpha} A, \mathrm{proj}_E)$. Note that p^* has the left adjoint $p_!$

$$p_! : (\mathbb{C} \downarrow E) \longrightarrow (\mathbb{C} \downarrow B)$$

defined by $p_!(D, \delta) = (D, p\delta)$.

For a Galois structure $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{G})$, let $p : E \longrightarrow B$ be a morphism of $\mathbf{F}$. Then, since the class $\mathbf{F}$ is pullback stable, the pullback functor p^* induces another functor

$$\mathbf{F}(B) \longrightarrow \mathbf{F}(E)$$

which we will also label as p^*.

Since $\mathbf{F}$ is closed under composition, (D, δ) in $\mathbf{F}(E)$ implies $(D, p\delta)$ is in $\mathbf{F}(B)$; that is, the pullback functor $p^* : \mathbf{F}(B) \longrightarrow \mathbf{G}(E)$ has the left adjoint $p_! : \mathbf{F}(E) \longrightarrow \mathbf{F}(B)$.

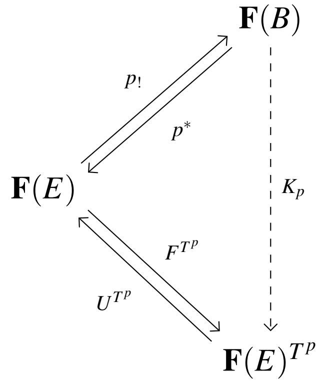
Therefore, for a morphism $p : E \longrightarrow B$ of $\mathbf{F}$, we denote by T^p the corresponding monad on $\mathbf{F}(E)$ determined by the adjunction $(p_!, p^*) : \mathbf{F}(E) \longrightarrow \mathbf{F}(B)$ as in Theorem 1.42.

Definition 2.6. Let $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{G})$ be a Galois structure and suppose $p : E \longrightarrow B$ is a morphism of $\mathbf{F}$. The pair (E, p) (or simply the morphism p) is said to be a monadic extension of B (or an *effective descent morphism*) if p^* is monadic.

Note that, one often drops condition (3) in Definition 2.1 of a Galois structure Γ and rather includes in the definition of a monadic extension the requirement that (D, δ) is in $\mathbf{F}(E)$ implies $(D, p\delta)$ is in $\mathbf{F}(B)$. This of course places fewer restrictions on the classes $\mathbf{F}$ and $\mathbf{G}$

and yet, under these conditions, one will still achieve the fundamental theorem of categorical Galois theory (Theorem 2.11 below). However, for the sake of brevity in this thesis, we shall continue with the extra requirement on $\mathbf{F}$ and $\mathbf{G}$.

We know that when $p : E \longrightarrow B$ is a monadic extension of B, the comparison functor K_p in the diagram



is an equivalence of categories.

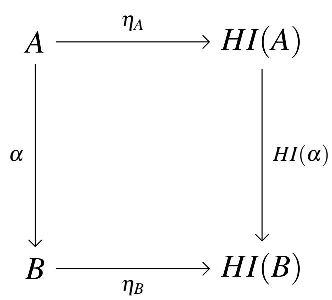
Definition 2.7. Let $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{G})$ be a Galois structure and B an object in $\mathbb{C}$.

1. An object (A,α) of $\mathbf{F}(B)$ is called a trivial covering of B if the component of the unit ${\eta^B}_{(A,\alpha)} : (A,\alpha) \longrightarrow H^B I^B(A,\alpha)$ of the induced adjunction $\big(I^B, H^B, \eta^B, \varepsilon^B\big) : \mathbf{F}\big(B\big) \longrightarrow \mathbf{G}\big(I(B)\big)$ is an isomorphism. The full subcategory in $\mathbf{F}(B)$ with objects all the trivial coverings of B is denoted $\mathsf{TrivCov}(B)$;

2. Suppose for a morphism $p : E \longrightarrow B$ in $\mathbf{F}$, the pair (E, p) is a monadic extension of B. An object (A,α) of $\mathbf{F}(B)$ is said to be split over the monadic extension (E, p) if $p^*(A,\alpha)$ is a trivial covering of E. Furthermore, (E, p) is called an I-normal extension (or simply a normal extension) of B if it is split over itself. The full subcategory in $\mathbf{F}(B)$ of all objects split over (E, p) is denoted $\mathsf{Spl}(E, p)$.

3. An object (A,α) of $\mathbf{F}(B)$ is said to be a covering of B if it belongs to $\mathsf{Spl}(E, p)$ for some monadic extension (E, p) of B.

Note that $(A,\alpha) \in \mathsf{TrivCov}(B)$ means exactly that the diagram



is a pullback.

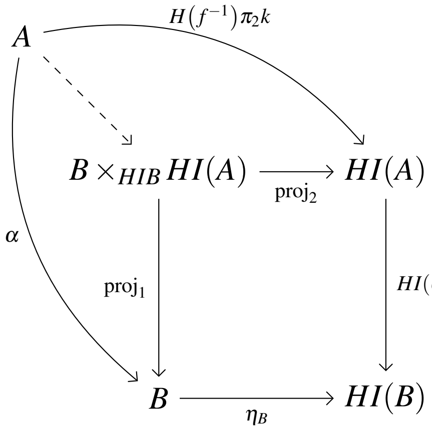
Proposition 2.8. *Let* $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{G})$ *be a Galois structure and* B *an object in* $\mathbb{C}$ *which is admissible with respect to* Γ. *An object* $(A,\alpha) \in \mathbf{F}(B)$ *is a trivial covering of* B *if and only if* $(A,\alpha) \cong H^B(X,\phi)$ *for some* $(X,\phi) \in \mathbf{G}(I(B))$

Proof. Obviously, if $(A,\alpha) \in \mathsf{TrivCov}(B)$ then the component of the unit $\eta^B{}_{(A,\alpha)} : (A,\alpha) \longrightarrow H^B I^B(A,\alpha)$ of the induced adjunction $(I^B, H^B, \eta^B, \varepsilon^B) : \mathbf{F}(B) \longrightarrow \mathbf{G}(I(B))$ is an isomorphism, and so taking $I^B(A,\alpha) = (X,\phi) \in \mathbf{G}(I(B))$ gives $(A,\alpha) \cong H^B(X,\phi)$.

On the other hand, suppose $k : (A,\alpha) \longrightarrow H^B(X,\phi)$ is an isomorphism. Since B is admissible, the component of the counit $\varepsilon^B{}_{(X,\phi)} : I(B \times_{HI(B)} H(X)) \longrightarrow X$ is an isomorphism, and we have the following sequence of composable isomorphisms in $\mathbf{F}(B)$:

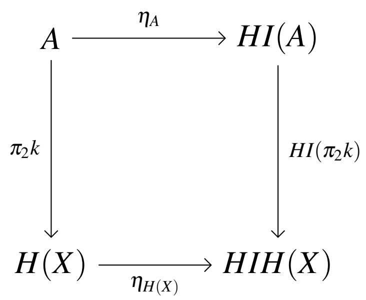
$$(A,\alpha) \xrightarrow{\ k\ } H^B(X,\phi) \xrightarrow{H^B\left(\varepsilon^B_{(X,\phi)}\right)^{-1}} H^B I^B H^B(X,\phi) \xrightarrow{H^B I^B(k^{-1})} H^B I^B(A,\alpha)$$

A routine calculation shows that its composite is the unique morphism in the following diagram



(2.2)

where $f = \varepsilon^B_{(X,\phi)} I(k) = \varepsilon_X I(\pi_2 k)$ and π_2 is the canonical projection morphism $B \times_{HI(B)} H(X) \longrightarrow H(X)$. We show that $H(f^{-1})\pi_2 k = \eta_A$. Indeed, since the diagram



commutes, we obtain

$$\begin{aligned}\pi_2 k &= H(\varepsilon_X)\eta_{H(X)}\pi_2 k \\ &= H(\varepsilon_X)HI(\pi_2 k)\eta_A \\ &= H\big(\varepsilon_X I(\pi_2 k)\big)\eta_A \\ &= H(f)\eta_A\end{aligned}$$

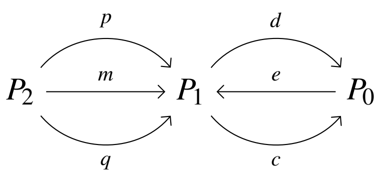
Therefore, the outer square in diagram (2.2) is a pullback and so (A,α) is a trivial covering of B. □

The "important point" we made after Proposition 2.5 can now be made precise: whenever B is admissible with respect to a Galois structure $\Gamma = (\mathbb{C},\mathbb{X},I,H,\eta,\varepsilon,\mathbf{F},\mathbf{G})$, Proposition 2.8 shows that the functor H^B of the induced adjunction $\big(I^B,H^B,\eta^B,\varepsilon^B\big) : \mathbf{F}\big(B\big) \longrightarrow \mathbf{G}\big(I(B)\big)$ in turn induces a functor $\mathbf{G}\big(I(B)\big) \longrightarrow \mathsf{TrivCov}(B)$ which is fully faithful and essentially surjective on objects. Therefore, $\mathsf{TrivCov}(B) \approx \mathbf{G}\big(I(B)\big)$.

2.3 The fundamental theorem

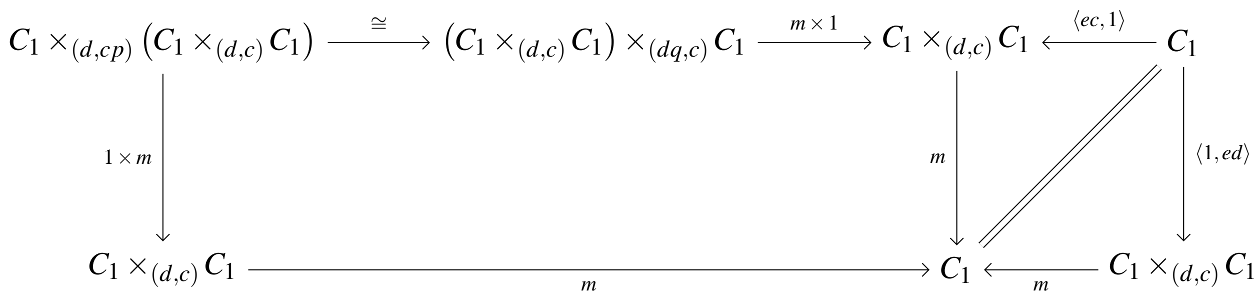
In this section we give a very brief survey of the fundamental theorem of categorical Galois theory due to G. Janelidze. For a more detailed account, the reader is urged to consult [2], [11] and [12].

An *internal precategory* in a category $\mathbb{C}$ is a diagram $P = (P_0,P_1,P_2,d,c,p,q,e,m) =$



in $\mathbb{C}$, in which $de = 1 = ce$, $dp = cq$, $dm = dq$ and $cm = cp$. An internal precategory in Sets is simply called a precategory.

An *internal category* in a category $\mathbb{C}$ with pullbacks is an internal precategory C in $\mathbb{C}$, in which the diagram formed by d, c, p, q is a pullback (giving $C_2 = C_1 \times_{(d,c)} C_1$) and the diagram

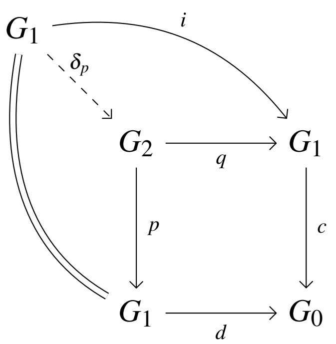


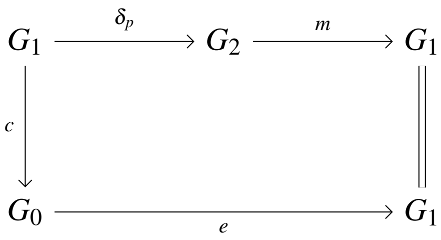
commutes.

An *internal pregroupoid* $G = (G, i)$ in a category $\mathbb{C}$ is an internal precategory G in $\mathbb{C}$ with a morphism $i : G_1 \longrightarrow G_1$ satisfying $di = c$, $ci = d$, $ie = e$ and $i^2 = 1_{G_1}$.

An *internal groupoid* G in a category $\mathbb{C}$ with pullbacks is an internal pregroupoid $G = (G, i)$ which is also an internal category in $\mathbb{C}$, and in which the morphism i satisfies the following:

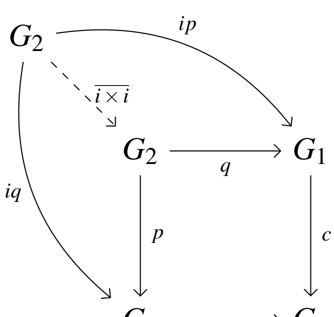
1. For the unique morphism $\delta_p = \langle 1_{G_1}, i \rangle : G_1 \longrightarrow G_2$ in the diagram on the left

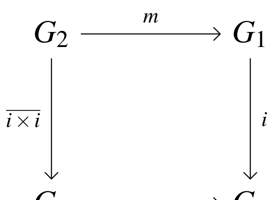




   the diagram on the right commutes, and similarly for a similar diagram involving δ_q;

2. For the unique morphism $\overline{i \times i} = \langle iq, ip \rangle : G_2 \longrightarrow G_2$ in the diagram on the left



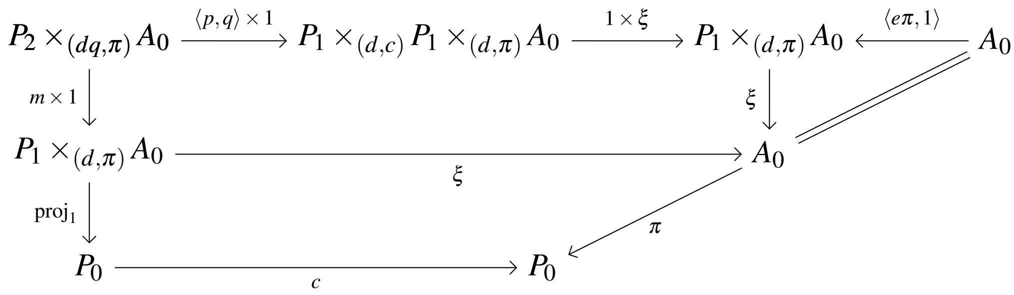


the diagram on the right commutes.

For an internal precategory $P = (P_0, P_1, P_2, d, c, e, m)$ in a category $\mathbb{C}$ with pullbacks, an *internal category action of* P, or simply a P-*action* is a diagram $A = (A_0, \pi, \xi) =$

$$P_1 \times_{(d,\pi)} A_0 \xrightarrow{\ \xi\ } A_0 \xrightarrow{\ \pi\ } P_0$$

such that the diagram

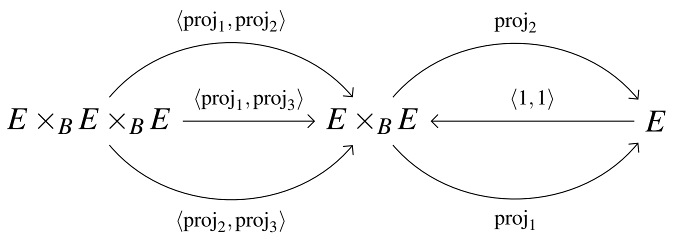


commutes, where

$$\begin{aligned} P_1 \times_{(d,c)} P_1 \times_{(d,\pi)} A_0 &\cong \big(P_1 \times_{(d,c)} P_1\big) \times_{(d\mathrm{proj}_2,\pi)} A_0 \\ &\cong P_1 \times_{(d,c\mathrm{proj}_1)} \big(P_1 \times_{(d,\pi)} A_0\big) \end{aligned}$$

The category of P-actions will be denoted by $\mathbb{C}^P$. When P is an internal category, the top horizontal morphism $\langle p,q\rangle \times 1$ above is an isomorphism.

Consider a morphism $p : E \longrightarrow B$ in $\mathbb{C}$ with pullbacks. The kernel pair of p, denoted by $\mathsf{Eq}(p)$ is the internal groupoid



in which

$$\begin{aligned} E \times_B E \times_B E &\cong \big(E \times_{(p,p)} E\big) \times_{(p\mathrm{proj}_1, p\mathrm{proj}_2)} \big(E \times_{(p,p)} E\big) \\ &\cong \big(E \times_{(p,p)} E\big) \times_{(p\mathrm{proj}_2, p)} E \\ &\cong E \times_{(p, p\mathrm{proj}_1)} \big(E \times_{(p,p)} E\big) \end{aligned}$$

and proj_1, proj_2 and proj_3 the appropriate projection morphisms.

$\mathsf{Eq}(p)$ is in fact an *internal preorder* (an internal category whose domain morphism and codomain morphism are jointly monic) and, furthermore, is an *internal equivalence relation* (hence the notation $\mathsf{Eq}(p)$) in $\mathbb{C}$.

For a Galois structure $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{G})$ and an internal precategory $P = (P_0, P_1, P_2, d, c, e, m)$ in $\mathbb{C}$, the full subcategory in $\mathbb{C}^P$ with objects all $A = (A_0, \pi, \xi) \in \mathbb{C}^P$ in which π is in $\mathbf{F}$, is denoted $\mathbb{C}^P \cap \mathbf{F}$. Furthermore, the full subcategory in $\mathbb{C}^P \cap \mathbf{F}$ in which (A_0, π) is a trivial covering (of P_0) is denoted $\mathsf{Triv}\big(\mathbb{C}^P\big)$.

Suppose Γ is a Galois structure. In the previous section we mentioned that the class $\mathbf{F}$ is defined such that the category of descent data $\mathsf{Des}_{\mathbf{F}}(p)$ (see [15]) for p over $\mathbf{F}$ is precisely the Eilenberg-Moore category $\mathbf{F}(E)^{T^p}$ of the monad T^p induced by the adjunction $p_! \dashv p^*$. Concretely,

Theorem 2.9. *[15, Proposition 2.2] For a Galois structure* $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{G})$ *and an object* (D, δ) *in* $\mathbf{F}(E)$*, the morphism*

$$\overline{\delta} = \langle proj_1, proj_2 \rangle : \big(E \times_{(p,p)} E\big) \times_{(pproj_2, \delta)} D \longrightarrow E \times_{(p, p\delta)} D$$

is an isomorphism and hence determines a category isomorphism

$$\mathbf{F}(E)^{T^p} \cong \mathbb{C}^{\mathsf{Eq}(p)} \cap \mathbf{F}$$

Consider an internal precategory $P = (P_0, P_1, P_2, d, c, e, m)$ in $\mathbb{C}$, in which P_0, P_1 and P_2 are admissible with respect to a Galois structure $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{G})$. Recall from Definition 2.3 that this means precisely that the composite

$$I\big(P_i \times_{HI(P_i)} H(X)\big) \xrightarrow{I(\text{proj}_2)} IH(X) \xrightarrow{\varepsilon_X} X$$

is an isomorphism for each $i = 0, 1, 2$ and $\big(X, \phi\big) \in \mathbf{G}(I(B))$. Therefore, [12, Section 2] shows that the result

$$B \text{ admissible} \quad \Longrightarrow \quad \mathsf{TrivCov}\big(B\big) \approx \mathbf{G}\big(I(B)\big)$$

can be expanded in the following way.

Theorem 2.10. *For a Galois structure* $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{G})$ *and an internal precategory* $P = (P_0, P_1, P_2, d, c, e, m)$ *in* $\mathbb{C}$*, in which* P_0*,* P_1 *and* P_2 *are admissible with respect to* Γ*, there is an equivalence of categories*

$$\mathsf{Triv}\big(\mathbb{C}^P\big) \approx \mathbb{X}^{I(P)} \cap \mathbf{G}$$

where, analogously, $\mathbb{X}^{I(P)} \cap \mathbf{G}$ *is the full subcategory in* $\mathbb{X}^{I(P)}$ *with objects all* $X = (X_0, \pi, \xi) \in \mathbb{X}^{I(P)}$ *in which* π *is in* $\mathbf{G}$*.*

If $I : \mathbb{C} \longrightarrow \mathbb{X}$ is a functor between categories $\mathbb{C}$ and $\mathbb{X}$ with pullbacks, then the image $I(P)$ under I of an internal precategory P in $\mathbb{C}$ is certainly an internal precategory in $\mathbb{X}$, as is $I(G)$ an internal pregroupoid in $\mathbb{X}$ whenever G is an internal pregroupoid in $\mathbb{C}$. Since I does not necessarily preserve pullbacks, we cannot conclude further that $I(C)$ is an internal category in $\mathbb{X}$ whenever C is an internal category in $\mathbb{C}$. However, when the following canonical morphisms

$$I\big(C_1 \times_{(d,c)} C_1\big) \longrightarrow I(C_1) \times_{(I(d),I(c))} I(C_1) \tag{2.3}$$
$$I\big(C_1 \times_{(d,c)} C_1 \times_{(d,c)} C_1\big) \longrightarrow I(C_1) \times_{(I(d),I(c))} I(C_1) \times_{(I(d),I(c))} I(C_1) \tag{2.4}$$

are isomorphisms in $\mathbb{X}$, then $I(C) = \big(I(C_0), I(C_1), I(C_2), I(d), I(c), I(e), I(m)\big)$ is an internal category (respectively, internal groupoid) in $\mathbb{X}$ whenever $C = (C_0, C_1, C_2, d, c, e, m)$ is an internal category (respectively, internal groupoid) in $\mathbb{C}$.

The image $\mathsf{Gal}(E, p) = I\big(\mathsf{Eq}(p)\big)$ is called the *Galois pregroupoid* for the morphism $p : E \longrightarrow B$ in $\mathbb{C}$. When p is a normal extension of B, then the above morphisms in (2.3) and (2.4) are in fact isomorphisms, and hence in this case $\mathsf{Gal}(E, p)$ is an internal groupoid, called the *Galois groupoid*. If, additionally, E is *I-connected*–that is, $I(E)$ is a terminal object in $\mathbb{X}$–then $\mathsf{Gal}(E, p)$ is a group object internal to $\mathbb{X}$.

Putting all these pieces together we get our most important result in this section.

Theorem 2.11. *[12, Theorem 6.8] Let* $\Gamma = (\mathbb{C}, \mathbb{X}, I, H, \eta, \varepsilon, \mathbf{F}, \mathbf{G})$ *be a fixed Galois structure. Suppose* (E, p) *is a monadic extension of* B *in* $\mathbb{C}$*, and* E*,* $E \times_B E$ *and* $E \times_B E \times_B E$ *are admissible with respect* Γ*. Then, the functor*

$$\mathsf{Spl}(E, p) \longrightarrow \mathbb{X}^{\mathsf{Gal}(E,p)} \cap \mathbf{G}$$

determined by the diagram

$$
\begin{array}{ccccc}
\mathbf{F}(B) & \xrightarrow{K_p} & \mathbf{F}(E)^{T^p} \cong \mathbb{C}^{\mathsf{Eq}(p)} \cap \mathbf{F} & & \\
\uparrow \subseteq & & \uparrow \subseteq & & \\
\mathsf{Spl}(E,p) & \dashrightarrow & \mathsf{Triv}\big(\mathbb{C}^{\mathsf{Eq}(p)}\big) & \xrightarrow{\approx} & \mathbb{X}^{\mathsf{Gal}(E,p)} \cap \mathbf{G}
\end{array}
$$

in which the vertical arrows are the inclusion functors and K_p is the comparison functor previously described, is a category equivalence.

Chapter 3

The Galois theory of commutative semirings

Since the development of a purely categorical Galois theory, as presented very briefly in Chapter 2, the classical examples have been extended substantially, with specializations to the theory of central extensions [13], the theory of covering spaces in algebraic topology and even as far as Tannaka Duality in quantum algebras [14]. Much of the utility of the purely categorical Galois theory comes from the fact that these specializations come out of *calculation* from purely categorical constructions.

It turns out that the principal motivating example of the categorical Galois theory, the separable Galois theory of commutative rings developed by Magid in [22], itself is a member of a specialization of the purely categorical theory, known as *Boolean Galois theory*.

The origin of Boolean Galois theory is the fact that for a lextensive category $\mathbb{C}$, the *complemented subobjects* (see Definition 3.25 below) of any object form a Boolean algebra. Knowing this, a functor $I : \mathbb{C} \longrightarrow \mathsf{Stone}$ can be constructed via the Stone Duality by sending each object to its Boolean algebra of complemented subobjects. When $\mathbb{C}$ is the opposite category of commutative rings, such I is isomorphic to the Pierce spectrum functor used in Magid's work.

As a result, in [4], A. Carboni and G. Janelidze develop a general environment which captures the classical Galois theories of A. Grothendieck, as in, for example, [8], and Magid's generalization of it to commutative rings, and presents them as Boolean Galois theories.

This environment is formalized around the notion of a *geometric category*: a lextensive category $\mathbb{C}$ containing a *profinite subcategory* (see Section 3.2 below) and an adjunction $(I, H, \eta, \varepsilon) : \mathbb{C} \longrightarrow \mathsf{Stone}$ in which H is fully faithful, and which admits "enough" admissible objects.

In this chapter, we similarly apply these ideas to the opposite category of commutative semirings to develop a *(Boolean) Galois theory of commutative semirings*. To this end, one may indeed show that $\mathsf{CSemiRing}^{\mathrm{op}}$ is geometric. However, in this chapter we follow a slightly more direct route.

3.1 Filtered colimits

Is it possible to recover an infinite-dimensional vector space V over some field using only its finite dimensional subspaces? As a first guess, the answer seems that it is possible, and that we should be able to describe V (in a way made precise in this section) as the "limit" of an increasing sequence $V_1 \subseteq V_2 \subseteq V_3 \subseteq \dots$ of finite dimensional subspaces of V. Since there are many similar such limits in, for example, *algebraic categories* (see, for example, [1, Section 3.9]), we introduce some ideas to discuss them generally.

Definition 3.1. A non-empty small category J is said to be filtered when:

1. for every pair of objects j, j' in J, there is some object k and morphisms $u : j \longrightarrow k$ and $v : j' \longrightarrow k$ in J;
2. for each pair of parallel morphisms $i \overset{w_2}{\underset{w_1}{\rightrightarrows}} j$, there is some object k and morphism $s : j \longrightarrow k$ in J such that $sw_1 = sw_2$.

Dually, a category K said to be *cofiltered* when K^{op} is filtered.

We would like now to construct the *free filtered-colimit completion* of a *locally small* category $\mathbb{A}$; that is, one with small hom-sets. As we shall show, this is possible because, for each such category $\mathbb{A}$ and class $\mathscr{K}$ of small categories, we may form the *closure of* $\mathbb{A}$ *in* $\mathsf{Sets}^{\mathbb{A}^{op}}$ *under* $\mathscr{K}$*-colimits*.

Definition 3.2. Let $\mathbb{A}$ be a category with small hom-sets. For $\mathscr{K}$ a class of small categories, the closure of $\mathbb{A}$ under $\mathscr{K}$-colimits, denoted $\mathbb{A}_{\mathscr{K}}$, is the smallest subcategory of $\mathsf{Sets}^{\mathbb{A}^{\mathrm{op}}}$ containing the representables $\hom_{\mathbb{A}}(-,A)$ for every $A \in \mathbb{A}$, and closed under $\mathscr{K}$-colimits. $Y_{\mathbb{A},\mathscr{K}} : \mathbb{A} \longrightarrow \mathbb{A}_{\mathscr{K}}$ denotes the functor induced from the Yoneda embedding functor.

For a detailed construction of $\mathbb{A}_{\mathscr{K}}$ in the 2-category $\mathscr{V}$-CAT of $\mathscr{V}$-categories, for $\mathscr{V}$ a general (complete and cocomplete) symmetric monoidal closed category $\mathscr{V}$, the reader is encouraged to consult [18, Section 3.5].

Each pair $\left(\mathbb{A}_{\mathscr{K}}, Y_{\mathbb{A},\mathscr{K}}\right)$ satisfies the universal property described by the *generalized Yoneda embedding theorem*:

Proposition 3.3. *[4, Proposition 1.1] For every category $\mathbb{C}$ with $\mathscr{K}$-colimits, composition with $Y_{\mathbb{A},\mathscr{K}}$ determines an equivalence of categories*

$$hom_{\mathscr{K}\text{-}\mathsf{CAT}}\left(\mathbb{A}_{\mathscr{K}},\mathbb{C}\right) \approx hom_{\mathsf{CAT}}(\mathbb{A},\mathbb{C})$$

where $\mathscr{K}$-CAT is the 2-category of $\mathscr{K}$-colimit complete categories and CAT is the 2-category of (locally small) categories.

Proposition 3.3 exhibits $\mathbb{A}_{\mathscr{K}}$ as the free $\mathscr{K}$-colimit completion of $\mathbb{A}$ (see, for example, [18, Sections 5.7 - 5.10]).

Consider the category equivalence $\mathrm{hom}_{\mathsf{CAT}}(\mathbb{A},\mathbb{C}) \longrightarrow \mathrm{hom}_{\mathscr{K}\text{-}\mathsf{CAT}}\left(\mathbb{A}_{\mathscr{K}},\mathbb{C}\right)$ of Proposition 3.3 above. Given a functor $T : \mathbb{A} \longrightarrow \mathbb{C}$, the $\mathscr{K}$-colimit preserving functor, denoted $\overline{T} : \mathbb{A}_{\mathscr{K}} \longrightarrow \mathbb{C}$, which is the image of T under this category equivalence, is said to be *essentially unique* since any other $\mathscr{K}$-colimit preserving functor $\overline{S} : \mathbb{A}_{\mathscr{K}} \longrightarrow \mathbb{C}$ such that $\overline{S} \cdot Y_{\mathbb{A},\mathscr{K}} = T$ is necessarily isomorphic to $\overline{T}$.

Proposition 3.4. *[4, Proposition 1.2] Under the assumptions of Proposition 3.3, the essentially unique functor $\overline{T} : \mathbb{A}_{\mathscr{K}} \longrightarrow \mathbb{C}$ determined by some $T : \mathbb{A} \longrightarrow \mathbb{C}$ is fully faithful if:*

1. *T is fully faithful;*
2. *each hom-functor $hom_{\mathbb{C}}(T(A),-) : \mathbb{C} \longrightarrow \mathsf{Sets}$ preserves $\mathscr{K}$-colimits.*

Because of the essential uniqueness of the $\mathscr{K}$-colimit preserving functors $\overline{T} : \mathbb{A}_{\mathscr{K}} \longrightarrow \mathbb{C}$, one quickly has:

Corollary 3.5. *Each functor $F : \mathbb{A} \longrightarrow \mathbb{B}$ induces a $\mathscr{K}$-colimit preserving functor $F_{\mathscr{K}} : \mathbb{A}_{\mathscr{K}} \longrightarrow \mathbb{B}_{\mathscr{K}}$, which is an equivalence whenever F is.*

Informally, the construction of $\mathbb{A}_{\mathscr{K}}$ proceeds by transfinite induction, starting with $\mathbb{A}$ and adding at each step thereafter, all $\mathscr{K}$-colimits in $\mathsf{Sets}^{\mathbb{A}^{\mathrm{op}}}$ of objects from the previous step. For certain classes of small categories $\mathscr{K}$, the construction terminates after the first step because adding all $\mathscr{K}$-colimits in $\mathsf{Sets}^{\mathbb{A}^{\mathrm{op}}}$ of objects of $\mathbb{A}$ produces a full subcategory of $\mathsf{Sets}^{\mathbb{A}^{\mathrm{op}}}$ closed under $\mathscr{K}$-colimits. In these cases, we may, therefore, give a precise description of $\mathbb{A}_{\mathscr{K}}$. In particular, this is possible when we take $\mathscr{K}$ to be the class of small filtered categories. In other words, we may now give a construction of a category $\mathsf{Ind}(\mathbb{A})$ and a fully faithful functor $\mathbb{A} \longrightarrow \mathsf{Ind}(\mathbb{A})$ such that $\mathsf{Ind}(\mathbb{A})$ has filtered colimits and each object $F \in \mathsf{Ind}(\mathbb{A})$ is a filtered colimit of objects of $\mathbb{A}$. Moreover, the constructed functor $\mathbb{A} \longrightarrow \mathsf{Ind}(\mathbb{A})$ will satisfy a universal property similar to Proposition 3.3.

To begin with, we often think of diagrams $F : J \longrightarrow \mathbb{A}$, where J is a (small) filtered category, as being "placeholders" for the colimits over them (possibly non-existent in $\mathbb{A}$). That is, in the example of vector spaces over a field K, we may think of an infinite-dimensional vector space V *as being* a diagram $F : J \longrightarrow \mathsf{Vect}_K{}^*$, where J is an infinite filtered category and $\mathsf{Vect}_K{}^*$ is the category of finite-dimensional vector spaces over K.

Next, for small filtered categories J and K, and functors $F : J \longrightarrow \mathbb{A}$ and $G : K \longrightarrow \mathbb{A}$, consider the set

$$\mathrm{Lim}_J \mathrm{Colim}_K \mathrm{hom}_{\mathbb{A}}\big(F(j), G(k)\big)$$

which can of course be presented as

$$\mathrm{Lim}_J \mathrm{Colim}_K \mathrm{hom}_{\mathbb{A}}\big(F(j), G(k)\big) \cong \prod_{j\in J}{}^{\dagger} \left(\bigsqcup_{k\in K} \mathrm{hom}_{\mathbb{A}}\big(F(j), G(k)\big) \Big/ \sim \right) \tag{3.1}$$

The equivalence relation $\sim$ here is defined as

$$\big(k, \phi : F(j) \longrightarrow G(k)\big) \sim \big(k', \psi : F(j) \longrightarrow G(k')\big) \tag{3.2}$$

if and only if there is a $k'' \in K$ and morphisms $u : k \longrightarrow k''$ and $v : k' \longrightarrow k''$ such that $G(u)\phi = G(v)\psi$. For clarity, we shall denote the equivalence class of an element (k, ϕ) in $\bigsqcup_{k\in K} \mathrm{hom}_{\mathbb{A}}\big(F(j), G(k)\big)$ by $\overline{(k, \phi)}$.

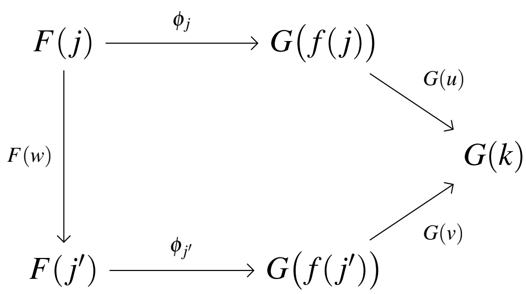
Furthermore, the limit $\prod_{j\in J}{}^{\dagger}$ is the subset of the product $\prod_{j\in J}$ such that for every j, j' and $w : j \longrightarrow j'$ the diagram commutes

$$\begin{array}{ccc}
 & \prod_{i\in J}{}^{\dagger} \left(\bigsqcup_{k\in K} \mathrm{hom}_{\mathbb{A}}\big(F(j), G(k)\big) \Big/ \sim \right) & \\
 & \Big\downarrow \subseteq & \\
 & \prod_{i\in J} \left(\bigsqcup_{k\in K} \mathrm{hom}_{\mathbb{A}}\big(F(j), G(k)\big) \Big/ \sim \right) & \\
 \swarrow \pi_{j'} & & \searrow \pi_j \\
 \bigsqcup_{k\in K} \mathrm{hom}_{\mathbb{A}}\big(F(j'), G(k)\big) \Big/ \sim & \xrightarrow{F(w)^*} & \bigsqcup_{k\in K} \mathrm{hom}_{\mathbb{A}}\big(F(j), G(k)\big) \Big/ \sim
\end{array}$$

Therefore, an element of the right hand side of (3.1) is a pair (f,ϕ) where $f : J_0 \longrightarrow K_0$ is a map from the object set of J to the object set of K, and

$$\phi = \overline{\big(f(j),\phi_j\big)}_{j\in J}$$

a family of equivalence classes (with respect to the equivalence $\sim$ in (3.2)) of morphisms in $\mathbb{A}$ satisfying the following: for every $j, j' \in J$ and every $w : j \longrightarrow j'$, $\big(f(j),\phi_j\big) \sim \big(f(j'),\phi_{j'}F(w)\big)$. This condition is obviously equivalent to the existence of some $k \in K$ and morphisms $u : f(j) \longrightarrow k$, $v : f(j') \longrightarrow k$ in K such that the diagram



commutes.

Note, importantly, that two "parallel" elements (f,ϕ) and (g,ψ) in (3.1) are, of course, the same element if and only if $\big(f(j),\phi_j\big) \sim \big(g(j),\psi_j\big)$, for each $j \in J$.

These discussions lead us to the next definition. From now on, we shall assume that all categories $\mathbb{A}$ are locally small, and all filtered categories J, K and L are small.

Definition 3.6. The inductive completion (also called the free filtered-cocompletion) of a category $\mathbb{A}$ is the category $\mathsf{Ind}(\mathbb{A})$ with:

- objects all functors $F : J \longrightarrow \mathbb{A}$ where J is a filtered category;
- morphisms $(F : J \longrightarrow \mathbb{A}) \longrightarrow (G : K \longrightarrow \mathbb{A})$ all elements of

$$\prod_{j\in J}{}^{\dagger} \left(\coprod_{k\in K} \hom_{\mathbb{A}}\big(F(j),G(k)\big) \Big/ \sim \right)$$

- the composite of morphisms $(f,\phi) : F \longrightarrow G$ and $(g,\psi) : G \longrightarrow H$ the pair $(h,\theta) : F \longrightarrow H$ where $h = gf$ and $\theta_j = \overline{\big(gf(j),\psi_{f(j)}\phi_j\big)}$ the equivalence class of the pair $\Big(h(j),\psi_{f(j)}\phi_j : F(j) \longrightarrow H\big(h(j)\big)\Big)$.

Remark 3.7. The composition of morphisms in Definition 3.6 is well-defined because, for $(f,\phi) : (F : J \longrightarrow \mathbb{A}) \longrightarrow (G : K \longrightarrow \mathbb{A})$ and $(g,\psi) : (G : K \longrightarrow \mathbb{A}) \longrightarrow (H : L \longrightarrow \mathbb{A})$:

1. if $(x,\lambda) \in \overline{(f(s),\phi_s)}$, where $s \in J$ and $x \in K$, then $\big(g(x),\psi_x\lambda\big) \in \overline{\big(g\big(f(s)\big),\psi_{f(s)}\phi_s\big)}$;
2. if $(y,\tau) \in \overline{(g(x),\psi_x)}$, where $x \in K$ and $y \in L$, then $(y,\tau\lambda) \in \overline{(g(x),\psi_x\lambda)}$.

As promised, we have the following series of results which justify the name and notation of $\mathsf{Ind}(\mathbb{A})$ (see Theorem A.1 in Appendix A for the proof).

Theorem 3.8. *In* $\mathsf{Ind}(\mathbb{A})$:

1. *Each object $F : J \longrightarrow \mathbb{A}$ is the filtered colimit (over J) of the objects $F(j) \in \mathbb{A}$ considered as functors $\mathbb{1} \longrightarrow \mathbb{A}$.*
2. *Each $H : I \longrightarrow \mathsf{Ind}(\mathbb{A})$ for I filtered has a colimit in $\mathsf{Ind}(\mathbb{A})$.*
3. *For each object $A \in \mathbb{A}$ considered as a functor $\mathbb{1} \longrightarrow \mathbb{A}$, the functor*
$$hom_{\mathsf{Ind}(\mathbb{A})}\big(A,-\big) : \mathsf{Ind}(\mathbb{A}) \longrightarrow \mathsf{Sets}$$
preserves filtered colimits.
4. *The canonical inclusion functor $\mathbb{A} \longrightarrow \mathsf{Ind}(\mathbb{A})$ is fully faithful.*

Corollary 3.9. *In* $\mathsf{Ind}(\mathbb{A})$:
$$hom_{\mathsf{Ind}(\mathbb{A})}(F,G) \cong Lim_J Colim_K hom_{\mathbb{A}}\big(F(j),G(k)\big)$$
for $F : J \longrightarrow \mathbb{A}$ and $G : K \longrightarrow \mathbb{A}$ with J and K filtered.

Corollary 3.10. *Let $\mathscr{K}$ be the category of all small filtered categories, and $\mathbb{A}_{\mathscr{K}}$ the closure of $\mathbb{A}$ under $\mathscr{K}$-colimits. Then $\mathbb{A}_{\mathscr{K}} \approx \mathsf{Ind}(\mathbb{A})$.*

Proof. By Proposition 3.4 there is an essentially unique fully faithful functor $\overline{T} : \mathbb{A}_{\mathscr{K}} \longrightarrow \mathsf{Ind}(\mathbb{A})$. Moreover, since $\mathbb{A}_{\mathscr{K}}$ has all $\mathscr{K}$-colimits and $\overline{T}$ preserves them, we have the desired result. □

Example 3.11. The category of vector spaces Vect_K over a field K is the inductive completion of the category of finite-dimensional vector spaces $\mathsf{Vect}_K{}^*$ over K; that is, $\mathsf{Vect}_K \approx \mathsf{Ind}\big(\mathsf{Vect}_K{}^*\big)$.

Example 3.12. The category of Boolean algebras Bool is the inductive completion of the category of finite Boolean algebras $\mathsf{FinBool}$; that is, $\mathsf{Bool} \approx \mathsf{Ind}(\mathsf{FinBool})$. In fact, this equivalence of categories holds precisely because each *finitely presentable* Boolean algebra is finite.

Definition 3.13. Suppose $\mathscr{T}$ is an algebraic theory. A $\mathscr{T}$-model M is said to be finitely presentable when there are morphisms u, v and c in $\mathsf{Mod}_{\mathscr{T}}$ making the diagram

$$F(X) \underset{v}{\overset{u}{\rightrightarrows}} F(Y) \xrightarrow{\quad c \quad} M$$

a coequalizer diagram, for finite sets X and Y, and $F : \mathsf{Sets} \longrightarrow \mathsf{Mod}_{\mathscr{T}}$ the free functor to $\mathsf{Mod}_{\mathscr{T}}$.

Remark 3.14. The terminology of "finitely presentable" in Definition 3.13 is justified since if M is a finitely presentable $\mathscr{T}$-model, then the morphisms u and v correspond by adjunction to maps $X \underset{\overline{v}}{\overset{\overline{u}}{\rightrightarrows}} UF(Y)$, so that M is the quotient of $F(Y)$ by the congruence generated by the pairs $\big(\overline{u}(x), \overline{v}(x)\big)$ for each $x \in X$. That is, M is finitely presentable exactly when it is obtained from finitely many generators Y and finitely many relations, $\overline{u}(x) = \overline{v}(x)$, one for each $x \in X$, between terms constructed from those generators.

Proposition 3.15. *[1, Proposition 3.8.12] The category* $\mathsf{Mod}_{\mathscr{T}}$ *of* $\mathscr{T}$*-models for an algebraic theory* $\mathscr{T}$ *is the inductive completion of the full subcategory* $\mathsf{FinPresMod}_{\mathscr{T}}$ *of finitely presentable ones; that is,* $\mathsf{Mod}_{\mathscr{T}} \approx \mathsf{Ind}\big(\mathsf{FinPresMod}_{\mathscr{T}}\big)$.

We note that, over a field K, the finitely presentable vector spaces, the *finitely generated* vector spaces, and the finite dimensional vector spaces obviously all coincide. Therefore, not only is each such vector space a colimit $V \cong \mathrm{Colim}_J V_j$ of finite dimensional vector spaces $V_j \cong K^{n_j}$ over a filtered category J, but, in fact, V is a *filtered union* of its finite dimensional subspaces, as remarked above. See [1, Section 3] for details.

Dually to $\mathsf{Ind}(\mathbb{A})$, we have:

Definition 3.16. The projective completion $\mathsf{Pro}(\mathbb{A})$ (also called the free cofiltered-completion) of a category $\mathbb{A}$ is the opposite category $\mathsf{Pro}(\mathbb{A}) = \mathsf{Ind}\big(\mathbb{A}^{\mathrm{op}}\big)^{\mathrm{op}}$ of the inductive completion of $\mathbb{A}^{\mathrm{op}}$.

Example 3.17. Since $\mathsf{FinSets} \cong \mathsf{FinBool}^{\mathrm{op}}$, we have that $\mathsf{Pro}\big(\mathsf{FinSets}\big) \approx \mathsf{Stone}$ via the Stone Duality.

3.2 Profinite subcategories

Taking seriously the work of [22] as guiding example, as well as our comment about assigning a Boolean algebra of complemented subobjects to each object C in a general lextensive category $\mathbb{C}$, a natural idea is then to determine how to construct a *profinite subcategory* of $\mathbb{C}$; that is, a full subcategory $\mathbb{P} \approx \mathsf{Stone}$ of $\mathbb{C}$. In this section we consider a definition of a *profinite object* in $\mathbb{C}$, made precise by Definition 2 below, and then describe minimal conditions on $\mathbb{C}$ such that the full subcategory $\mathbb{P}$ of all such profinite objects in $\mathbb{C}$ is a profinite subcategory of $\mathbb{C}$.

The results in this section are taken from [4, pp. 649 & 650]. *For the rest of this chapter, $\mathbb{C}$ denotes, more generally, a fixed (not necessarily lextensive) category with a terminal object* 1.

Definition 3.18.

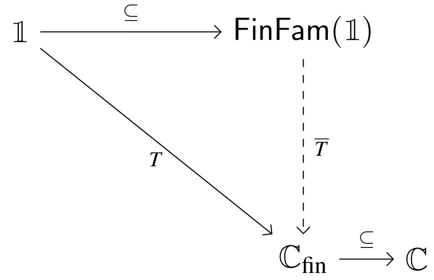
1. An object $C \in \mathbb{C}$ is said to be finite if there is a natural number n, a coproduct $n \cdot 1$ of the terminal object 1 in $\mathbb{C}$ n times and an isomorphism $n \cdot 1 \cong C$. The full subcategory in $\mathbb{C}$ of all such objects is denoted $\mathbb{C}_{\text{fin}}$.

2. An object $C \in \mathbb{C}$ is said to be profinite if there is a (small) cofiltered category I and a functor $H : I \longrightarrow \mathbb{C}$ sending all objects and morphisms into $\mathbb{C}_{\text{fin}}$ such that the limit $\text{Lim}_I H$ exists in $\mathbb{C}$ and there is an isomorphism $\text{Lim}_I H \cong C$. The full subcategory in $\mathbb{C}$ of all such objects is denoted $\mathbb{C}_{\text{profin}}$.

Lemma 3.19. *Suppose that the functor hom*$_{\mathbb{C}_{\text{fin}}}(1,-) : \mathbb{C}_{\text{fin}} \longrightarrow \mathsf{Sets}$ *preserves finite coproducts. Then* $\mathsf{FinSets} \approx \mathbb{C}_{\text{fin}}$.

Proof. Let $\mathscr{K}$ be the class of all finite discrete categories. Then for any category $\mathbb{A}$, $\mathbb{A}_{\mathscr{K}} \approx \mathsf{FinFam}(\mathbb{A})$, the category of finite families of objects from $\mathbb{A}$.

Note that $\mathbb{C}_{\text{fin}}$ obviously has finite coproducts. Since $1 \in \mathbb{C}_{\text{fin}}$, the unique functor $\mathbb{C}_{\text{fin}} \longrightarrow \mathbb{1}$ has a right adjoint $T : \mathbb{1} \longrightarrow \mathbb{C}_{\text{fin}}$, which is fully faithful. Furthermore, since $\text{hom}_{\mathbb{C}_{\text{fin}}}(1,-) : \mathbb{C}_{\text{fin}} \longrightarrow \mathsf{Sets}$ preserves finite coproducts, the essentially unique (finite) coproduct-preserving functor $\overline{T}$ in the diagram



is fully faithful by Proposition 3.4, and is also obviously essentially surjective on objects. Therefore, $\mathbb{C}_{\text{fin}} \approx \mathsf{FinFam}(\mathbb{1})$.

An analogous argument shows $\mathsf{FinSets} \approx \mathsf{FinFam}(\mathbb{1})$. Therefore, there is an equivalence of categories $S : \mathsf{FinSets} \longrightarrow \mathbb{C}_{\text{fin}}$ defined by $S(X) = n \cdot 1$ where X has n elements. □

Lemma 3.20. *The following conditions are equivalent in the category* $\mathbb{C}$:

1. $\mathbb{C}$ *has cofiltered limits of finite objects and* $hom_{\mathbb{C}^{\text{op}}}(C,-) : \mathbb{C}^{\text{op}} \longrightarrow \mathsf{Sets}$ *preserves them, for each* $C \in \mathbb{C}_{\text{fin}}$.
2. $\mathbb{C}_{\text{profin}}$ *has cofiltered limits and* ${\mathbb{C}_{\text{fin}}}^{\text{op}} \subseteq \mathsf{FinPres}({\mathbb{C}_{\text{profin}}}^{\text{op}})$*, where* $\mathsf{FinPres}({\mathbb{C}_{\text{profin}}}^{\text{op}})$ *is the full subcategory of objects* $C \in {\mathbb{C}_{\text{profin}}}^{\text{op}}$ *such that* $hom_{{\mathbb{C}_{\text{profin}}}^{\text{op}}}(C,-) : {\mathbb{C}_{\text{profin}}}^{\text{op}} \longrightarrow \mathsf{Sets}$ *preserves filtered colimits.*

Proof. We shall, rather informally, refer to a functor $J \longrightarrow {\mathbb{C}_{\text{fin}}}^{\text{op}}$, for J a filtered category, and its composite with the inclusion functor ${\mathbb{C}_{\text{fin}}}^{\text{op}} \longrightarrow \mathbb{C}^{\text{op}}$, by the same notation F. By Definition 3.6, sending each $F \in \mathsf{Ind}({\mathbb{C}_{\text{fin}}}^{\text{op}})$ to its colimit in $\mathbb{C}^{\text{op}}$ determines a functor $\mathsf{Ind}({\mathbb{C}_{\text{fin}}}^{\text{op}}) \longrightarrow {\mathbb{C}_{\text{profin}}}^{\text{op}}$ which is essentially surjective on objects. If $F, G \in \mathsf{Ind}({\mathbb{C}_{\text{fin}}}^{\text{op}})$, then by Theorem 3.8, $F : J \longrightarrow {\mathbb{C}_{\text{fin}}}^{\text{op}}$ and $G : K \longrightarrow {\mathbb{C}_{\text{fin}}}^{\text{op}}$ are, respectively, the filtered colimits over J and K of the objects $F(j) \in {\mathbb{C}_{\text{fin}}}^{\text{op}}$ and $G(k) \in {\mathbb{C}_{\text{fin}}}^{\text{op}}$ considered as functors $\mathbb{1} \longrightarrow {\mathbb{C}_{\text{fin}}}^{\text{op}}$. Supposing $\text{hom}_{\mathbb{C}^{\text{op}}}(C,-) : \mathbb{C}^{\text{op}} \longrightarrow \mathsf{Sets}$ preserves filtered colimits of finite objects of $\mathbb{C}$ for each $C \in \mathbb{C}_{\text{fin}}$,

$$\begin{aligned}
\text{hom}_{\mathbb{C}^{\text{op}}}(\text{Colim}_J F, \text{Colim}_K G) &\cong \text{Lim}_J \text{hom}_{\mathbb{C}^{\text{op}}}\big(F(j), \text{Colim}_K G\big) \\
&\cong \text{Lim}_J \text{Colim}_K \text{hom}_{\mathbb{C}^{\text{op}}}\big(F(j), G(k)\big) \\
&\cong \text{Lim}_J \text{Colim}_K \text{hom}_{{\mathbb{C}_{\text{fin}}}^{\text{op}}}\big(F(j), G(k)\big) \\
&\cong \text{Lim}_J \text{Colim}_K \text{hom}_{\mathsf{Ind}({\mathbb{C}_{\text{fin}}}^{\text{op}})}\big(F(j), G(k)\big) \\
&\cong \text{hom}_{\mathsf{Ind}({\mathbb{C}_{\text{fin}}}^{\text{op}})}(F, G)
\end{aligned}$$

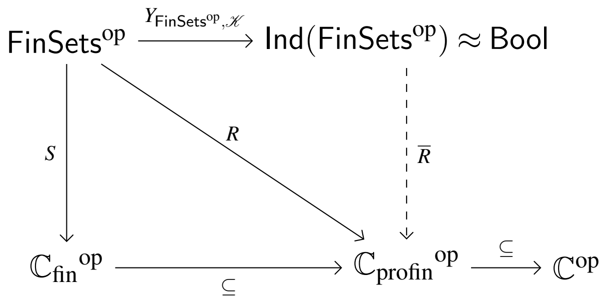
where the penultimate and last step follow by Theorem 3.8 and Corollary 3.9, respectively. Thus the functor $\mathsf{Ind}({\mathbb{C}_{\text{fin}}}^{\text{op}}) \longrightarrow {\mathbb{C}_{\text{profin}}}^{\text{op}}$ described above is additionally fully faithful and so $\mathsf{Ind}({\mathbb{C}_{\text{fin}}}^{\text{op}}) \approx {\mathbb{C}_{\text{profin}}}^{\text{op}}$. Therefore, $\mathbb{C}_{\text{profin}}$ has cofiltered limits and ${\mathbb{C}_{\text{fin}}}^{\text{op}} \subseteq \mathsf{FinPres}({\mathbb{C}_{\text{profin}}}^{\text{op}})$, again by Theorem 3.8.

On the other hand, $\mathbb{C}_{\text{profin}}$ having cofiltered limits and ${\mathbb{C}_{\text{fin}}}^{\text{op}} \subseteq \mathsf{FinPres}({\mathbb{C}_{\text{profin}}}^{\text{op}})$ trivially gives $\text{hom}_{\mathbb{C}^{\text{op}}}(C,-)$ preserving filtered colimits of finite objects for each $C \in \mathbb{C}_{\text{fin}}$. □

Note that of course $\mathsf{Ind}({\mathbb{C}_{\text{fin}}}^{\text{op}}) \approx {\mathbb{C}_{\text{profin}}}^{\text{op}}$ is the same thing as $\mathsf{Pro}(\mathbb{C}_{\text{fin}}) \approx \mathbb{C}_{\text{profin}}$.

Proposition 3.21. *Suppose $\mathbb{C}$ satisfies the assumptions of Lemma 3.19 and the equivalent conditions of Lemma 3.20. Then $\mathsf{Bool} \approx {\mathbb{C}_{\text{profin}}}^{\text{op}}$, and there is a fully faithful functor $H : \mathsf{Stone} \longrightarrow \mathbb{C}$ which is completely determined (up to isomorphism) by the following preservation properties: H preserves 1 and its finite copowers, and H preserves cofiltered limits.*

Proof. With the functor S (considered by the opposite categories) from Lemma 3.19 and considering now the case $\mathscr{K}$ equal to the class of all filtered categories, we have that $\mathbb{A}_{\mathscr{K}} \approx \mathsf{Ind}(\mathbb{A})$ for each category $\mathbb{A}$. In the diagram



R, the resulting composite with S, extends to a filtered-colimit preserving functor $\overline{R} : \mathsf{Bool} \longrightarrow {\mathbb{C}_{\text{profin}}}^{\text{op}}$. Since $\mathsf{Ind}({\mathbb{C}_{\text{fin}}}^{\text{op}}) \approx {\mathbb{C}_{\text{profin}}}^{\text{op}}$, and S is an equivalence, $\overline{R}$ is too. Letting H, considered as a functor between opposite categories, be the composite of the functor $\overline{R}$ and the inclusion functor ${\mathbb{C}_{\text{profin}}}^{\text{op}} \longrightarrow \mathbb{C}^{\text{op}}$, we have the desired result. □

3.3 Finitely presentable semirings

For any algebraic theory $\mathscr{T}$, its category of models $\mathsf{Mod}_{\mathscr{T}}$, being a variety of algebras, has all filtered colimits. It turns out that, in much the same way that there exist internal and external definitions of connected objects which coincide in a lextensive category (see Theorem 1.32), there is a connection between finitely presentable $\mathscr{T}$-models M and filtered-colimit preserving hom-functors over any algebraic theory $\mathscr{T}$.

Proposition 3.22. *[1, Proposition 3.8.14] Let $\mathscr{T}$ be an algebraic theory. A $\mathscr{T}$-model M is finitely presentable if and only if the functor*

$$hom_{\mathsf{Mod}_{\mathscr{T}}}(M, -) : \mathsf{Mod}_{\mathscr{T}} \longrightarrow \mathsf{Sets}$$

preserves filtered colimits.

Note that this proposition can be seen as a justification of the notation $\mathsf{FinPres}$ in Lemma 3.20.

Consider the (algebraic) category $(B \downarrow \mathsf{CSemiRing})$ for B a commutative semiring. The free functor $F : \mathsf{Sets} \longrightarrow (B \downarrow \mathsf{CSemiRing})$ restricted to *finite sets* is given by $F(\{x_1, x_2, \ldots, x_n\}) = B[x_1, x_2, \ldots, x_n]$, the B-semialgebra of polynomials in n indeterminates (see Example 1.11). Of course, when $B = \mathbf{N}$, $(B \downarrow \mathsf{CSemiRing}) \cong \mathsf{CSemiRing}$.

Proposition 3.23. *For each $n \in \mathbf{N}$ and any commutative semiring B, the B-semialgebra B^n can be obtained via the coequalizer diagram*

$$B[x_1, x_2, \ldots, x_m] \underset{proj_2}{\overset{proj_1}{\rightrightarrows}} B[y_1, y_2, \ldots, y_n] \xrightarrow{\ c\ } B^n$$

Here $m = \frac{n}{2}(n-1)+1$; $proj_1$ is the homomorphism of B-semialgebras induced by the injective map $\mu : \{x_1, x_2, \ldots, x_m\} \longrightarrow B[y_1, y_2, \ldots, y_n]$ with $\mu(x_1) = y_1y_2$, $\mu(x_2) = y_1y_3$, $\ldots$, $\mu(x_{n-1}) = y_1y_n$, $\mu(x_n) = y_2y_3$, $\mu(x_{n+1}) = y_2y_4$, $\ldots$, $\mu(x_{m-1}) = y_{n-1}y_n$ and $\mu(x_m) = \sum_{i=1}^n y_i$; $proj_2$ is the homomorphism of B-semialgebras induced by the map $\lambda : \{x_1, x_2, \ldots, x_m\} \longrightarrow B[y_1, y_2, \ldots, y_n]$ with

$$\lambda(x_i) = \begin{cases} 0 & \text{if } i < m \\ 1 & \text{if } i = m \end{cases}$$

c is the homomorphism of B-semialgebras induced by the injective map $\kappa : \{y_1, y_2, \ldots, y_n\} \longrightarrow B^n$ with $\kappa(y_i) = \delta_i$ where $\delta_i \in B^n$ is given by

$$\pi_j \delta_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Proof. Certainly by definition of proj_1, proj_2, c and δ_i, $c\mathrm{proj}_1 = c\mathrm{proj}_2$ on $\{x_1, x_2, \ldots, x_m\}$, and hence on $B[x_1, x_2, \ldots, x_m]$. Next, if S is a commutative B-semialgebra and $f : B[y_1, y_2, \ldots, y_n] \longrightarrow S$ a homomorphism of B-semialgebras such that $f\mathrm{proj}_1 = f\mathrm{proj}_2$, then there are $s_1 = f(y_1), s_2 = f(y_2), \ldots, s_n = f(y_n) \in S$ (possibly non-distinct) with $\sum_{i=1}^n s_i = 1_S$ and $s_is_j = 0$ if $i \neq j$. Therefore, if there is a homomorphism of B-semialgebras $h : B^n \longrightarrow S$ such that the diagram

$$\begin{array}{ccc} B[y_1, y_2, \ldots, y_n] & \xrightarrow{\ c\ } & B^n \\ & \searrow^{f} & \downarrow{h} \\ & & S \end{array}$$

commutes, then, since for any $\underline{b} \in B^n$, $\underline{b} = \sum_{i=1}^n b_i \delta_i = \sum_{i=1}^n b_i c(y_i)$ where $b_i \in B$, it is the case that $h(\underline{b}) = \sum_{i=1}^n b_i h\big(c(y_i)\big) = \sum_{i=1}^n b_i s_i$. Therefore, h is uniquely determined.

Finally, since $\sum_{i=1}^n s_i = 1_S$ and $s_i s_j = 0$ if $i \neq j$ implies each s_i is an idempotent in S, it is a routine calculation to check that the map $h(\underline{b}) = \sum_{i=1}^n b_i s_i$ is a homomorphism of B-semialgebras and makes the above diagram commute. $\square$

We are now in a position to construct a fully faithful functor $H : \mathsf{Stone} \longrightarrow \mathsf{CSemiRing}^{\mathrm{op}}$. Recall from Example 1.9 that $\mathsf{CSemiRing}^{\mathrm{op}}$ is lextensive. Since $\mathbf{N}$ has no non-trivial idempotents, the terminal object $1 = \mathbf{N}$ in $\mathsf{CSemiRing}^{\mathrm{op}}$ is connected. Therefore, $\mathsf{CSemiRing}^{\mathrm{op}}$ satisfies the hypothesis of Lemma 3.19.

Furthermore, Propositions 3.22 and 3.23 in fact show that for each $n \in \mathbf{N}$, $\hom_{\mathsf{CSemiRing}}(\mathbf{N}^n, -) : \mathsf{CSemiRing} \longrightarrow \mathsf{Sets}$ preserves *all* filtered colimits in $\mathsf{CSemiRing}$, and hence filtered colimits of finite objects. Therefore, the hypothesis of Proposition 3.21 is also satisfied. We have proved:

Theorem 3.24. *There exists a unique (up to isomorphism) fully faithful functor $H : \mathsf{Stone} \longrightarrow \mathsf{CSemiRing}^{op}$ satisfying $H(X) = \mathbf{N}^X$ for each finite set X.*

3.4 The Boolean algebra of complemented subobjects

Let us detail slightly more accurately the comments made at the beginning of this chapter. In a category $\mathbb{C}$ which satisfies the hypothesis of Lemma 3.19, consider a fixed coproduct diagram

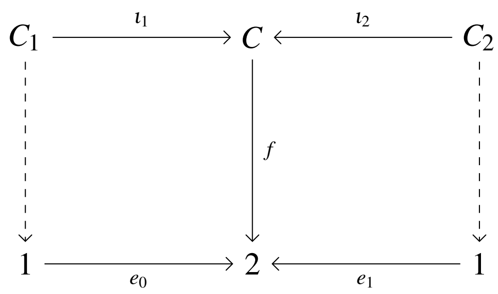
$$1 \xrightarrow{\quad e_0 \quad} 2 \xleftarrow{\quad e_1 \quad} 1 \tag{3.3}$$

Since $\mathsf{FinSets} \approx \mathbb{C}_{\mathrm{fin}}$, diagram (3.3) is the image of a coproduct diagram $\mathbf{1} \longrightarrow \mathbf{2} \longleftarrow \mathbf{1}$ in $\mathsf{FinSets}$, where $\mathbf{1}$ and $\mathbf{2}$ are one- and two-element sets, respectively. Furthermore, given any coproduct diagram $\mathbf{1} \longrightarrow \mathbf{2} \longleftarrow \mathbf{1}$ in $\mathsf{FinSets}$, there are unique maps $\neg : \mathbf{2} \longrightarrow \mathbf{2}$, $\wedge : \mathbf{2} \times \mathbf{2} \longrightarrow \mathbf{2}$ and $\vee : \mathbf{2} \times \mathbf{2} \longrightarrow \mathbf{2}$ such that $\mathbf{2}$ has an internal Boolean algebra structure in $\mathsf{FinSets}$. Therefore, by the equivalence of categories $\mathsf{FinSets} \approx \mathbb{C}_{\mathrm{fin}}$, diagram (3.3) uniquely determines on 2 an internal Boolean algebra structure in $\mathbb{C}_{\mathrm{fin}}$.

Next, by application of the Yoneda embedding $Y : \mathbb{C} \longrightarrow \mathsf{Sets}^{\mathbb{C}^{\mathrm{op}}}$, each diagram (3.3) determines on $\hom_{\mathbb{C}}(-, 2)$ an internal Boolean algebra structure in $\mathsf{Sets}^{\mathbb{C}^{\mathrm{op}}}$. Therefore, we obtain a functor

$$I = \hom_{\mathbb{C}}(-, 2) : \mathbb{C}^{\mathrm{op}} \longrightarrow \mathsf{Bool} \tag{3.4}$$

When $\mathbb{C}$ is in fact a lextensive category, we may clarify these constructions slightly further. Note that, in this case, for each object $C \in \mathbb{C}$, and any morphism $f : C \longrightarrow 2$, we may form the diagram



in which the squares are pullbacks and hence, by Theorem 1.15, the top row is a coproduct diagram in $\mathbb{C}$. By Proposition 1.25, ι_1 and ι_2 are monomorphisms. Since $\mathbb{C}$ has disjoint coproducts, the pullback of ι_1 along ι_2 is an initial object. These observations motivate the following definition:

Definition 3.25. For an object C in a lextensive category $\mathbb{C}$ with a connected terminal object 1, let P denote the set of all pairs $(\iota_1 : C_1 \longrightarrow C, \iota_2 : C_2 \longrightarrow C)$ of monomorphisms determined by pulling f back, in turn, along the coproduct injection morphisms e_0 and e_1 of a given diagram (3.3), for each $f \in \hom_{\mathbb{C}}(C,2)$. Let $\sim$ denote the equivalence relation on P defined by

$$(\iota_1 : C_1 \longrightarrow C, \iota_2 : C_2 \longrightarrow C) \sim (\eta_1 : D_1 \longrightarrow C, \eta_2 : D_2 \longrightarrow C)$$

if and only if there exist isomorphisms $\phi_1 : D_1 \longrightarrow C_1$ and $\phi_2 : D_2 \longrightarrow C_2$ such that $\iota_1 \phi_1 = \eta_1$ and $\iota_2 \phi_2 = \eta_2$. The quotient

$$\mathsf{CompSub(C)} = P/\sim$$

is called the set of complemented subobjects of C.

Remark 3.26. For a given $C \in \mathbb{C}$, sending each $f \in \hom_{\mathbb{C}}(C,2)$ to the complemented subobject it determines obviously constitutes a surjective map $\hom_{\mathbb{C}}(C,2) \longrightarrow \mathsf{CompSub(C)}$. Moreover, this map is easily seen to be a bijection by the universality of coproduct injection morphisms. Transportation of structure along this bijection therefore permits us to refer to $\hom_{\mathbb{C}}(C,2) \cong \mathsf{CompSub(C)}$ as the *Boolean algebra of complemented subobjects of* C.

Finally, if $\mathbb{C}$ further satisfies the conditions of Proposition 3.21, then the functor I is left adjoint to H constructed in Section 3.2 (see [4, Theorem 2.3]). That is, we have:

Theorem 3.27. *For a lextensive category $\mathbb{C}$ satisfying the conditions of Proposition 3.21, there is a Galois structure $(\mathbb{C}, \mathsf{Stone}, I, H)$ determined by the adjoint functors $I \dashv H$ of (3.4) and Proposition 3.21, respectively.*

When C is connected, Theorem 1.32 shows $\hom_{\mathbb{C}}(C,2)$ is obviously the two-element Boolean algebra. That is, $I(C)$ is a terminal object in Stone when C is connected. Moreover, C is connected if and only if it is I-connected (see Section 2.3).

Returning to $\mathbb{C} = \mathsf{CSemiRing}^{\mathrm{op}}$, we therefore also have the functor

$$I = \hom_{\mathsf{CSemiRing}}(\mathbf{N}^2, -) : \mathsf{CSemiRing} \longrightarrow \mathsf{Bool}$$

and the Galois structure

$$\Gamma = (\mathsf{CSemiRing}^{\mathrm{op}}, \mathsf{Stone}, I, H)$$

called the *Galois theory of commutative semirings*, where H is as in Theorem 3.24.

3.5 Which objects are admissible?

Focusing on the Galois theory of commutative semirings Γ, recall from Definition 2.3 that for each object $B \in \mathsf{CSemiRing}$ there is an induced adjunction

$$\left(I^B, H^B\right) : (B \downarrow \mathsf{CSemiRing})^{\mathrm{op}} \longrightarrow (\mathsf{Stone} \downarrow I(B))$$

where $I^B(A,\alpha) = (I(A), I(\alpha))$ and $H^B(X,\phi) = (B \times_{HI(B)} H(X), \mathrm{proj}_1)$. Recall also that B is admissible if the right adjoint H^B is fully faithful.

Suppose, from now on, that $B \in \mathsf{CSemiRing}$ is connected, so that $(\mathsf{Stone} \downarrow I(B)) \cong \mathsf{Stone}$. Note that H^B obviously preserves cofiltered limits and so, by Proposition 3.3, it is the essentially unique cofiltered-limit preserving functor such that $H^B(X) = (B^X, \delta_B)$ on finite sets X, where $\delta_B : B \longrightarrow B^X$ is the diagonal map.

Proposition 3.28. *B is admissible with respect to Γ.*

Proof. Consider the restriction $H^B_{\mathrm{fin}} : \mathsf{FinSets} \longrightarrow (B \downarrow \mathsf{CSemiRing})^{\mathrm{op}}$ of H^B to finite sets. Since $(B \downarrow \mathsf{CSemiRing})^{\mathrm{op}}$ is lextensive and B is connected, H^B_{fin} is the essentially unique (and fully faithful) functor preserving terminal objects and finite copowers of it, by Lemma 3.19. Furthermore, from Proposition 3.23, $H^B_{\mathrm{fin}}(X)$ is finitely presentable in $(B \downarrow \mathsf{CSemiRing})$ for each finite set X. Therefore, by Proposition 3.4, $H^B : \mathsf{Stone} \longrightarrow (B \downarrow \mathsf{CSemiRing})^{\mathrm{op}}$ is fully faithful. $\square$

Corollary 3.29. *If E is a (finite) product $\prod_{i=1}^{n} B_i$ of connected commutative semirings, then E is admissible with respect to Γ.*

Proof. Certainly, the finite set determined by the finite Boolean algebra $I\big(\prod_i B_i\big) = \hom_{\mathsf{CSemiRing}}\big(\mathbf{N}^2, \prod_i B_i\big)$ is the set $\underline{n} = \{1,2,\ldots,n\}$ of n elements. Therefore, we have the induced adjunction

$$(I^E, H^E) : (E \downarrow \mathsf{CSemiRing})^{\mathrm{op}} \longrightarrow (\mathsf{Stone} \downarrow \{1,2,\ldots,n\})$$

Since both $\mathsf{CSemiRing}^{\mathrm{op}}$ and Stone are lextensive categories, we obtain

$$(E \downarrow \mathsf{CSemiRing})^{\mathrm{op}} \approx \prod_{i=1}^{n} (B_i \downarrow \mathsf{CSemiRing})^{\mathrm{op}}$$

and

$$\begin{aligned}(\mathsf{Stone} \downarrow \{1,2,\ldots,n\}) &\approx (\mathsf{Stone} \downarrow 1)^n \\ &\approx \mathsf{Stone}^n\end{aligned}$$

Therefore, we have the diagram

$$\begin{array}{ccc}(E \downarrow \mathsf{CSemiRing})^{\mathrm{op}} & \xrightarrow{\quad I^E \quad} & (\mathsf{Stone} \downarrow \{1,2,\ldots,n\}) \\ \approx \big\downarrow & & \big\uparrow \approx \\ \prod_{i=1}^{n} (B_i \downarrow \mathsf{CSemiRing})^{\mathrm{op}} & \xrightarrow[\prod_i I^{B_i}]{} & \mathsf{Stone}^n\end{array}$$

We claim that, up to isomorphism, the above diagram is commutative. That is, up to the vertical equivalences, $I^E \cong \prod_i I^{B_i}$. For, using a simple diagram chase, $(A,\alpha) \in (E \downarrow \mathsf{CSemiRing})^{\mathrm{op}}$ has $(A,\alpha) \longmapsto \big(I(A), I(\alpha)\big)$ along the top horizontal arrow, and

$$\begin{aligned}(A,\alpha) \longmapsto \big(A_i,\alpha_i\big)_{i=1}^{n} \longmapsto \big(I(A_i), I(\alpha_i)\big)_{i=1}^{n} \longmapsto & \left(\coprod_i^n I(A_i), \coprod_i^n I(\alpha_i)\right) \\ \cong & \left(I\Big(\prod_i^n A_i\Big), I\Big(\prod_i^n \alpha_i\Big)\right) \\ \cong & \big(I(A), I(\alpha)\big)\end{aligned}$$

along the bottom path. Therefore $I^E \cong \prod_i I^{B_i}$, as desired.

Next, since H^E is right adjoint to I^E, it is right adjoint to $\prod_i I^{B_i}$, and thus, up to the vertical equivalences, $H^E \cong \prod_i H^{B_i}$. Finally, by Corollary 3.28, H^{B_i} is fully faithful for each i. Therefore, so is $\prod_i H^{B_i}$ and thus so is H^E. □

3.6 Descent in $\mathsf{CSemiRing}^{\mathrm{op}}$

In an exact category regular epimorphisms coincide precisely with those morphisms which are monadic extensions of their codomains (see, for example, [25]). Furthermore, the category of models $\mathsf{Mod}_{\mathscr{T}}$ over an algebraic theory $\mathscr{T}$ is monadic over the category of sets, and hence is exact. Therefore, CSemiRing is exact.

A natural starting point then for our investigation of monadic extensions in $\mathsf{CSemiRing}^{\mathrm{op}}$ would be to hope that $\mathsf{CSemiRing}^{\mathrm{op}}$ were also exact, since then the descent problem (of which morphisms are effective descent morphisms) for the Galois theory of commutative semirings Γ is easy.

Unfortunately, $\mathsf{CSemiRing}^{\mathrm{op}}$ is not even a regular category. However, for our purposes, we will not need to consider *all* monadic extensions, but rather, as we shall soon see in Chapter 4, the morphisms we wish to characterize in $\mathsf{CSemiRing}^{\mathrm{op}}$ are those homomorphisms of semirings which are split as monomorphisms of B-semimodules.

We begin by noting that any morphism of monads $f : S \longrightarrow R$ over an arbitrary category $\mathbb{X}$ has a "forgetful" functor $\mathbb{X}^f : \mathbb{X}^R \longrightarrow \mathbb{X}^S$. If $\mathbb{X}$ additionally has reflexive coequalizers then $\mathbb{X}^f$ has the left adjoint *change-of-base* functor $L^f : \mathbb{X}^S \longrightarrow \mathbb{X}^R$ (see, for example, [15, Section 4]). The following theorem then provides sufficient conditions under which the unit of the adjunction $L^f \dashv \mathbb{X}^f$ is a split monomorphism (of natural transformations), and therefore, by the split monadicity in Theorem 1.52, ensures the functor L^f comonadic. A coequalizer diagram (f,g,h) in which the parallel pair of morphisms (f,g) are split epimorphisms with a common splitting is called a *reflexive* coequalizer.

Theorem 3.30. *[16, Theorem 4.1, with a misprint corrected] Suppose a category $\mathbb{X}$ has equalizers and reflexive coequalizers. The change-of-base functor $L^f : \mathbb{X}^S \longrightarrow \mathbb{X}^R$ induced by a morphism $f : (S,\mu^S,\eta^S) \longrightarrow (R,\mu^R,\eta^R)$ of monads on $\mathbb{X}$ is comonadic whenever*

1. *R preserves reflexive coequalizers;*
2. *there exists a natural transformation $g : R \longrightarrow S$ with $gf = 1$, $\mu^S(Sg) = g(\mu^R(fR))$ and $\mu^S(gS) = g(\mu^R(Rf))$.*

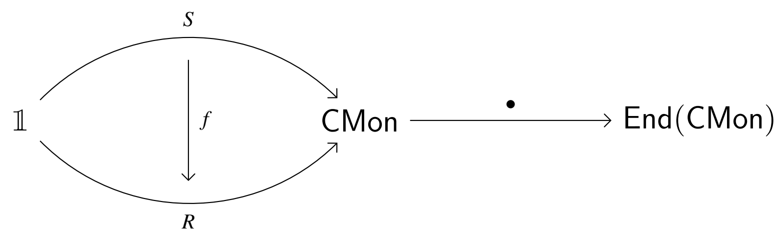
Remark 3.31. We do not actually need to require the existence of *all* coequalizers in the split monadicity theorem, and hence, similarly, we do not require all equalizers in Theorem

3.30 above. For instance, we could only require the existence of reflexive coequalizers since all the coequalizers needed for the Beck monadicity criterion in Theorem 1.51 have that property. So the category $\mathbb{X}$ in the above theorem can simply be required to have reflexive equalizers and coequalizers.

Each semiring (commutative or otherwise) S determines the category $S-\mathsf{SemiMod}$ of (left) S-semimodules, which is monadic over CMon because the forgetful functor from the category of S-semimodules to CMon is *algebraic* (see, for example, [1, Section 3.7]). Furthermore, CMon obviously has all equalizers and reflexive coequalizers. These considerations suggest an application of Theorem 3.30 to the category $\mathbb{X} = \mathsf{CMon}$ of commutative monoids. Let us first note that:

1. the collection of all monoidal categories, *monoidal functors* between them and *monoidal natural transformations* form a 2-category MON (see, for example [24] or [3] for details);
2. for any monoidal category $\mathbb{C}$, the category of monoids $\mathsf{Mon}(\mathbb{C})$ over $\mathbb{C}$ is in fact the category $\hom_{\mathsf{MON}}(\mathbb{1}, \mathbb{C})$ of monoidal functors $\mathbb{1} \longrightarrow \mathbb{C}$;
3. CMon "acts" on itself via the monoidal functor $\bullet : \mathsf{CMon} \longrightarrow \mathsf{End}(\mathsf{CMon})$ given by $\bullet(M)(X) = M \otimes_{\mathbf{N}} X$ for commutative monoids M and X.

Therefore, in the case $\mathbb{X} = \mathsf{CMon}$, the monads (S, μ^S, η^S) and (R, μ^R, η^R) listed in Theorem 3.30 can be taken as the composite of $\bullet : \mathsf{CMon} \longrightarrow \mathsf{End}(\mathsf{CMon})$ with the semirings S and R, considered as monoids $\mathbb{1} \longrightarrow \mathsf{CMon}$ over the monoidal category CMon, as displayed in the following diagram in MON:



Similarly, a homomorphism of semirings $f : S \longrightarrow R$, considered as a monoidal natural transformation $S \longrightarrow R$, induces a morphism of monads $f : (S, \mu^S, \eta^S) \longrightarrow (R, \mu^R, \eta^R)$ by horizontal composition in the above diagram.

Finally, the underlying functor $R = R \otimes_{\mathbf{N}} (-) : \mathsf{CMon} \longrightarrow \mathsf{CMon}$ of the monad (R, μ^R, η^R) induced by the semiring R obviously preserves reflexive coequalizers because it is a left adjoint, and a further series of routine but long calculations show that if the homomorphism of semirings f is a split monomorphism of left and right S-linear maps, then condition (2) in Theorem 3.30 is satisfied. Therefore, having a homomorphism of *commutative* semirings $f : S \longrightarrow R$ which is a split monomorphism of S-linear maps, the change-of-base (also called the *extension-of-scalars*) functor $R \otimes_S (-) : S\text{−}\mathsf{SemiMod} \longrightarrow R\text{−}\mathsf{SemiMod}$ is comonadic.

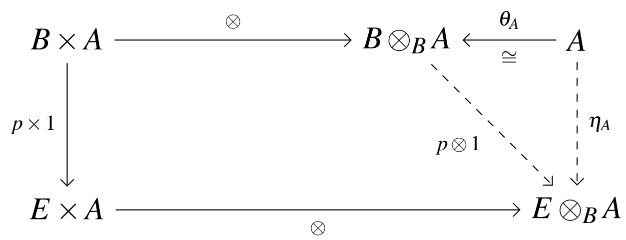
Instead of writing down these long calculations proving the claims we have made for the case $\mathbb{X} = \mathsf{CMon}$, we show now, much more *directly*, that the extension-of-scalars functor $R \otimes_S (-)$ is comonadic whenever we have a homomorphism of commutative semirings $f : S \longrightarrow R$ of the kind just described.

Theorem 3.32. *Given a homomorphism of commutative semirings $p : B \longrightarrow E$, the extension-of-scalars functor*

$$E \otimes_B (-) : B\text{−}\mathsf{SemiMod} \longrightarrow E\text{−}\mathsf{SemiMod}$$

is comonadic if p is a split monomorphism of B-semimodules.

Proof. We denote by $(-)_p : E\text{−}\mathsf{SemiMod} \longrightarrow B\text{−}\mathsf{SemiMod}$ the restriction-of-scalars ("forgetful") functor. Since $p : B \longrightarrow E$ is a homomorphism of semirings, p is a B-linear map and so the unit of the adjunction $E \otimes_B (-) \dashv (-)_p$ has components for each A given by the unique B-linear map $\eta_A = (p \otimes 1_A)\theta_A : A \longrightarrow E \otimes_B A$ as in the diagram



Therefore, if there is a B-linear map $q : E \longrightarrow B$ with $qp = 1_B$, the universal property of the B-bilinear maps $\otimes : B \times A \longrightarrow B \otimes_B A$ and $\otimes : E \times A \longrightarrow E \otimes_B A$ gives $(q \otimes 1_A)(p \otimes 1_A) = qp \otimes 1_A = 1_{B \otimes A}$. Therefore, $\theta_A^{-1}(q \otimes 1_A)\eta_A = 1_A$. Furthermore, the family of B-linear maps $\big(\theta_A^{-1}(q \otimes 1_A) : E \otimes_B A \longrightarrow A\big)_{A \in B\text{−}\mathsf{SemiMod}}$ is obviously a natural transformation

$$\big(E \otimes_B (-)\big)_p \longrightarrow 1_{B\text{−}\mathsf{SemiMod}}$$

Therefore, the unit of the adjunction η is a split monomorphism (of natural transformations). By the split monadicity theorem of Theorem 1.52, the extension-of-scalars functor is comonadic. □

Corollary 3.33. *A homomorphism of commutative semirings $p : B \longrightarrow E$, considered as a morphism $E \longrightarrow B$ in $\mathsf{CSemiRing}^{op}$, is a monadic extension of B if p is a split monomorphism of B-semimodules.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
(B \downarrow \mathsf{CSemiRing}) & \xrightarrow{E \otimes_B (-)} & (E \downarrow \mathsf{CSemiRing}) \\
\Big\downarrow {\scriptstyle U_B} & & \Big\downarrow {\scriptstyle U_E} \\
B-\mathsf{SemiMod} & \xrightarrow[E \otimes_B (-)]{} & E-\mathsf{SemiMod}
\end{array}$$

where the bottom horizontal arrow is the (comonadic) extension-of-scalars functor of Theorem 3.32 and the top horizontal arrow is the (opposite) of the pullback functor $p^* : (B \downarrow \mathsf{CSemiRing})^{\mathrm{op}} \longrightarrow (E \downarrow \mathsf{CSemiRing})^{\mathrm{op}}$.

Note that since the vertical arrows are algebraic functors, they reflect isomorphisms, and reflect and preserve limits. Therefore, Theorem 1.53 shows that the functor $E \otimes_B (-) : (B \downarrow \mathsf{CSemiRing}) \longrightarrow (E \downarrow \mathsf{CSemiRing})$ is comonadic. $\square$

Chapter 4

Quadratic Galois theory

In the previous chapter, we described the Galois structure $\Gamma = (\mathsf{CSemiRing}^{\mathrm{op}}, \mathsf{Stone}, I, H)$. We have also shown that each finite product of connected objects from CSemiRing is admissible. This is a perfectly useful Galois theory in its own right. That is, saying no more than we have already, any monadic extension $p : B \longrightarrow E$ of B, considered as a morphism $E \longrightarrow B$ in $\mathsf{CSemiRing}^{\mathrm{op}}$, where E is a finite product of connected semirings, will now determine, by a straightforward application of the results from Chapter 2, an equivalence

$$\mathsf{Spl}(E, p) \approx \mathsf{Stone}^{\mathsf{Gal}(E,p)}$$

where $\mathsf{Gal}(E, p)$ is the Galois pregroupoid of the monadic extension (E, p). Moreover, as was shown in Chapter 2, if (E, p) is a normal extension of B, $\mathsf{Gal}(E, p)$ would in fact be a groupoid internal to Stone.

We will go slightly further and construct, towards the end of this chapter, a concrete example of a normal extension in this Galois theory of commutative semirings, which, as a monadic extension of commutative semirings, is of the kind detailed in Corollary 3.33 at the end of the previous chapter. Our search for an example of such a normal extension, however, begins elsewhere, in the *Galois theory of commutative rings*.

It is well-known from the classical Galois theory of fields that each finite Galois extension of fields $B \subseteq E$ arises as the splitting field of a separable polynomial over B (see, for example [20, Chapter XIII]). Conversely, every such splitting field produces a finite Galois field extension. Therefore, the simplest non-trivial finite Galois extension of fields $B \subseteq E$ are those arising from a separable monic quadratic polynomial $f(x) = x^2 + qx + r$ irreducible over B, in which the splitting field $E \cong B[x]/f(x)B[x]$ of f is two-dimensional as a B-vector space.

It turns out that we may generalize some of these results to the Galois theory of commutative rings, of which the classical Galois theory of fields is a special case. In this setting, normal extensions of rings $p : B \longrightarrow E$ generalize the Galois extensions of fields: every finite Galois extension of fields $B \subseteq E$ is a normal extension of commutative rings. Then, every separable monic quadratic polynomial $f(x) = x^2 + qx + r$ over a connected commutative ring B with a (multiplicatively) invertible discriminant $q^2 - 4r$ produces a normal extension of commutative rings, in which $E \cong B[x]/f(x)B[x]$ is a B-algebra which has *rank two* as a B-module (see [17, Section 2]). This is the so-called *classical theory of quadratic equations over commutative rings*.

This brief discussion naturally invites us to consider if the classical theory of quadratic equations over commutative rings might be extended, in some precise way, to commutative semirings. If such an extension were possible, it would constitute one of the "simplest kinds" of a normal extension of commutative semirings one could hope to find.

We will describe our very particular proposed generalization to commutative semirings of the classical theory of quadratic equations over commutative rings in the first part of this chapter. We will show that our generalization is *not a strict one*: it does not hold for connected commutative semirings which are not rings.

The constructions of our proposed generalization of the classical theory of quadratic equations over commutative rings do, however, inspire the subsequent work in the last part of the chapter, leading to the first known normal extension of commutative semirings.

4.1 Rank-two B-semimodules

Let B be a commutative semiring and X a nonempty subset of a B-semimodule M. The intersection of all B-subsemimodules of M containing X is a B-subsemimodule of M called the *B-subsemimodule generated by* X and denoted by BX. If there is a nonempty subset X of M such that $BX = M$, then X is called a *generating set* for M. A B-semimodule M having a finite generating set is called *finitely generated*. The *rank* of a B-semimodule M, denoted by $r(M)$, is the least n for which there exists a generating set for M having cardinality n. The rank $r(M)$ of a finitely generated B-semimodule M always exists. See, for example, [26, Section 2] for more details.

For a commutative semiring B, fix elements s and t in B. Let $\mathbb{B}_*$ denote the category whose objects are pairs (A, a), where $A = (A, \alpha) \in (B \downarrow \mathsf{CSemiRing})$ and $a \in A$, and whose morphisms are homomorphisms of B-semialgebras $f : A \longrightarrow B$ such that $f(a) = b$. Let $\mathbb{B}$ be the full subcategory of $\mathbb{B}_*$ of pairs (A, a) with the axiom

$$a^2 = sa + t1 \tag{4.1}$$

being satisfied in A.

For the view point of universal algebra, we may regard each object (A,a) of $\mathbb{B}_*$ as the system $A = (A,0,1,a,+,\cdot,\omega)$, where $(A,0,1,+,\cdot,\omega)$ is a commutative B-semialgebra, and where a is a nullary operation selecting an element $a \in A$. Then, the variety $\mathbb{B}$ is formed as the class of all algebras with operations $(0,1,+,a,\cdot,\omega)$ and the axiom (4.1), in addition to the other usual axioms of commutative B-semialgebras.

Therefore, the inclusion functor $I : \mathbb{B} \longrightarrow \mathbb{B}_*$ has the obvious left adjoint $Q : \mathbb{B}_* \longrightarrow \mathbb{B}$ (quotient by the congruence generated by $(a^2, sa+t1)$). Since the canonical inclusion homomorphism of semirings $i : B \longrightarrow B[x]$ is a free object in $(B \downarrow \mathsf{CSemiRing})$ on the single-element set, $(B[x],x)$ is an initial object in $\mathbb{B}_*$. Denote by R the congruence

$$R = \overline{(x^2, sx+t)} \subseteq B[x] \times B[x] \tag{4.2}$$

on $B[x]$ generated by the pair $(x^2, sx+t)$. Therefore, we conclude (in obvious notation) that

$$Q\big(B[x],x\big) = \big(B[x]/R, \mathrm{cls}_R(x)\big)$$

is an initial object in $\mathbb{B}$.

Next, consider the commutative semiring E, where the underlying commutative monoid structure is the canonical structure of the commutative monoid product $B \times B$, and the multiplication is given by:

$$(w,x)(y,z) = (wy + xzt, wz + xy + xzs) \tag{4.3}$$

With this structure, the map $p : B \longrightarrow E$ defined by:

$$p(b) = (b,0) \tag{4.4}$$

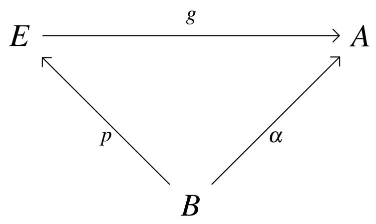
is a homomorphism of semirings and $E = (E,p) \in (B \downarrow \mathsf{CSemiRing})$.

Proposition 4.1. *Consider the pair* $\big(E,e\big) = \big((E,p),e\big)$*, where* e *is the element* $(0,1) \in E$*. Then* $\big(E,e\big)$ *is an initial object in* $\mathbb{B}$*.*

Proof. For a pair $(A,a) = \big((A,\alpha),a\big) \in \mathbb{B}$, where $A = (A,\alpha) \in (B \downarrow \mathsf{CSemiRing})$, suppose $g : E \longrightarrow A$ is a homomorphism of semirings, with $gp = \alpha$ and $g(0,1) = g(e) = a$. Then,

$$\begin{aligned} g(b_1,b_2) &= g(b_1,0) + g(b_2,0)g(0,1) \\ &= gp(b_1) + gp(b_2)a \\ &= \alpha(b_1) + \alpha(b_2)a \end{aligned}$$

Therefore, if such a g exists, it is uniquely determined. Also, it is easily shown that a map defined as $g(b_1,b_2) = \alpha(b_1) + \alpha(b_2)a$ is a homomorphism of semirings, makes the diagram



commute and, moreover, sends the pair $e = (0,1)$ to a. Indeed, we check only that g so defined is a homomorphism of multiplication in the semirings E and A:

$$\begin{aligned}
g\big((b_1,b_2)(b_3,b_4)\big) &= g(b_1b_3 + b_2b_4q, b_1b_4 + b_2b_3 + b_2b_4p) \\
&= \alpha(b_1b_3 + b_2b_4q) + \alpha(b_1b_4 + b_2b_3 + b_2b_4p)a \\
&= \alpha(b_1)\alpha(b_3) + t\alpha(b_2)\alpha(b_4) + \alpha(b_1)\alpha(b_4)a + \alpha(b_2)\alpha(b_3)a + s\alpha(b_2)\alpha(b_4)a \\
&= \alpha(b_1)\alpha(b_3) + \big(\alpha(b_2)\alpha(b_3) + \alpha(b_1)\alpha(b_4)\big)a + \alpha(b_2)\alpha(b_4)(sa+t) \\
&= \alpha(b_1)\alpha(b_3) + \big(\alpha(b_2)\alpha(b_3) + \alpha(b_1)\alpha(b_4)\big)a + \alpha(b_2)\alpha(b_4)a^2 \\
&= \big(\alpha(b_1) + \alpha(b_2)a\big)\big(\alpha(b_3) + \alpha(b_4)a\big) \\
&= g(b_1,b_2)g(b_3,b_4)
\end{aligned}$$

□

Since $\big(B[x]/R, \mathrm{cls}_R(x)\big)$ is an initial object in $\mathbb{B}$, the above proposition shows that

$$\big(B[x]/R, \mathrm{cls}_R(x)\big) \cong \big(E, e\big)$$

in $\mathbb{B}$. Obviously, $E = B \times B$ as B-semimodules.

Selecting elements s' and t' in E, we may now construct $\mathbb{E}$, defined similarly to $\mathbb{B}$, as a category of pairs (D,d) with $D = (D,\delta) \in (E \downarrow \mathsf{CSemiRing})$ and $d \in D$ in which the same axiom

$$d^2 = s'd + t'1$$

is satisfied.

Corollary 4.2. *Let R be the congruence on $B[x]$ generated by the pair $(x^2, sx+t)$, and R' the congruence on $E[x]$ generated by the pair $(x^2, s'x+t')$ where $s' = (s,0)$ and $t' = (t,0)$. Then*

$$\big(E \otimes_B \big(B[x]/R\big), \iota_E\big) \cong \big(E[x]/R', incl_E\big)$$

in $(E \downarrow \mathsf{CSemiRing})$.

Proof. If the category $\mathbb{B}$ is constructed by selecting the elements s and t in B, then $\mathbb{E}$, constructed by selecting the elements $s' = p(s) = (s,0)$ and $t' = p(t) = (t,0)$ in E, obviously has the initial object $\Big(E[x]/R', \mathrm{cls}_{R'}(x)\Big) = \Big(\big(E[x]/R', \mathrm{incl}_E\big), \mathrm{cls}_{R'}(x)\Big)$.

Next, as shown in Definition 1.27, $p : B \longrightarrow E$, considered as a morphism $E \longrightarrow B$ in $\mathsf{CSemiRing}^{\mathrm{op}}$, determines an adjunction

$$(p_!, p^*) : (E \downarrow \mathsf{CSemiRing})^{\mathrm{op}} \longrightarrow (B \downarrow \mathsf{CSemiRing})^{\mathrm{op}}$$

by pulling back along p. Since the pushout in CSemiRing is calculated as the tensor product, this adjunction determines the (opposite) adjunction

$$(B \downarrow \mathsf{CSemiRing}) \underset{U_p}{\overset{E \otimes_B (-)}{\rightleftarrows}} (E \downarrow \mathsf{CSemiRing})$$

with left adjoint $E \otimes_B (-) : (B \downarrow \mathsf{CSemiRing}) \longrightarrow (E \downarrow \mathsf{CSemiRing})$, sending a commutative B-semialgebra (A, α) to the E-semialgebra $(E \otimes_B A, \iota_E)$. Furthermore, this adjunction determines another adjunction

$$\big(E \otimes_B (-), U_p\big) : \mathbb{B} \longrightarrow \mathbb{E}$$

Here, $E \otimes_B (-)$ sends the pair $(A, a) \in \mathbb{B}$ to the pair $(E \otimes_B A, 1 \otimes a) \in \mathbb{E}$. Since $E \otimes_B (-) : \mathbb{B} \longrightarrow \mathbb{E}$ is a left adjoint, it sends initial objects in $\mathbb{B}$ to initial objects in $\mathbb{E}$. By Proposition 4.1, $\big(E \otimes_B \big(B[x]/R\big), 1 \otimes \mathrm{cls}_R(x)\big) \cong \big(E \otimes_B E, 1 \otimes e\big)$ is an initial object in $\mathbb{E}$. But then we also have

$$\Big(E \otimes_B \big(B[x]/R\big), \iota_E\Big) \cong \Big(E[x]/R', \mathrm{incl}_E\Big)$$

in $(E \downarrow \mathsf{CSemiRing})$. $\square$

Let $E = (E, p)$ be the commutative B-semialgebra with the multiplication given by (4.3), and with the homomorphism of semirings $p : B \longrightarrow E$ defined by (4.4). Consider the commutative semiring uE, where u is a non-zero idempotent in E, and the map $\gamma_u : B \longrightarrow uE$

defined by

$$\gamma_u(b) = up(b) = u(b,0) \tag{4.5}$$

A routine calculation shows that γ_u is a homomorphism of semirings, so that $(uE, \gamma_u) \in (B \downarrow \mathsf{CSemiRing})$. Furthermore, the map γ_u satisfies another simple, but useful, property.

Lemma 4.3. *If the commutative B-semialgebra (E, p) has a nonzero idempotent u, the element $u(0,1) \in uE$ is the image of some $b_0 \in B$ under $\gamma_u : B \longrightarrow uE$ if and only if γ_u is a surjection.*

Proof. If γ_u is surjective, then obviously there exists such an element in B.

On the other hand, suppose there is $b_0 \in B$ such that $\gamma_u(b_0) = u(0,1)$. Then in uE,

$$\begin{aligned} u(y,z) &= u(y,0) + u(0,z) \\ &= u(y,0) + u(z,0)u(0,1) \\ &= \gamma_u(y) + \gamma_u(z)\gamma_u(b_0) \\ &= \gamma_u(y + zb_0) \end{aligned}$$

Therefore, γ_u is a surjection. □

4.2 The classical theory of quadratic equations

In this section we describe the classical theory of quadratic equations over commutative rings, showing that very little knowledge of the theory of separable algebras over commutative rings–in, for example, [17] or [23]–is needed for it's development, as long as we restrict ourselves to the case where our "base" connected commutative ring B is a field.

Let B be a field, and suppose $f(x) = x^2 + qx + r$ is a quadratic polynomial over B. Regarding B as a commutative semiring, we construct the category $\mathbb{B}$ of Section 4.1 by selecting $s = -q$ and $t = -r$. Then, as in the equality (4.2), we obtain the congruence R on $B[x]$ generated by the pair $(x^2, sx+t) = (x^2, -qx-r)$. Straightforward calculations now show that R is nothing more than the congruence on $B[x]$ corresponding to the principal ideal $f(x)B[x]$ generated by the polynomial $f(x)$. Recalling $E = (E, p)$ is the commutative B-algebra with the multiplication given by

$$(w,x)(y,z) = (wy + xzt, wz + xy + xzs) = (wy - xzr, wz + xy - xzq) \tag{4.6}$$

as in (4.3), and with the homomorphism of rings $p : B \longrightarrow E$ defined by

$$p(b) = (b, 0)$$

as in (4.4), Proposition 4.1 now shows that $\big(E, e\big) = \big(E, (0,1)\big) \cong \big(B[x]/f(x)B[x], x + f(x)B[x]\big)$ is then a two-dimensional B-vector space, where $x + f(x)B[x]$ is the coset $\{x + f(x)k(x) \mid k(x) \in B[x]\}$.

Next, the Galois theory of commutative rings (see [2, Chapter 4]) is the Galois structure

$$\Lambda = \big(\mathsf{CRing}^{\mathrm{op}}, \mathsf{Stone}, I, H\big) \tag{4.7}$$

where:

- the functor $I : \mathsf{CRing}^{\mathrm{op}} \longrightarrow \mathsf{Stone}$ is defined by (up to Stone Duality) the Boolean algebra $I(A) = \hom_{\mathsf{CRing}}(\mathbf{Z}^2, A)$;
- the functor $H : \mathsf{Stone} \longrightarrow \mathsf{CRing}^{\mathrm{op}}$ is the unique (up to isomorphism) cofiltered-limit preserving functor satisfying $H(X) = \mathbf{Z}^X$ for each *finite set* X.

Recall from Definition 2.3 that for each object $B \in \mathsf{CRing}$ there is an induced adjunction

$$\big(I^B, H^B\big) : (B \downarrow \mathsf{CRing})^{\mathrm{op}} \longrightarrow (\mathsf{Stone} \downarrow I(B))$$

where $I^B(A, \alpha) = (I(A), I(\alpha))$ and $H^B(X, \phi) = (B \times_{HI(B)} H(X), \mathrm{proj}_1)$. Furthermore, since $\mathsf{CRing}^{\mathrm{op}}$ is a geometric category, each finite product of connected objects in CRing is admissible.

As a B-linear map, p is clearly a split monomorphism of B-vector spaces. Therefore, the extension-of-scalars functor

$$E \otimes_B (-) : B\text{--}\mathsf{Vect} \longrightarrow E\text{--}\mathsf{Mod}$$

is comonadic by Theorem 1.52, the proof being almost identical to that of Theorem 3.32. Finally, by Theorem 1.53 (the "induced monadicity theorem"), the next result easily follows:

Proposition 4.4. *The homomorphism of rings $p : B \longrightarrow E$, considered as a morphism $E \longrightarrow B$ in CRing^{op}, is a monadic extension of B.*

Let us recall some simple facts from commutative ring theory.

Proposition 4.5. *For a quadratic polynomial $f(x) = x^2 + qx + r$ in $B[x]$, where B is a commutative ring, denote the discriminant polynomial of f by $d_f(x) = x^2 - \Delta$ where $\Delta = q^2 - 4r$. Let F and D be the set of all roots of f and the set of all roots of d_f in B, respectively. If 2 has a multiplicative inverse in B then the maps*

$$D \underset{k}{\overset{l}{\rightleftarrows}} F$$

defined by $n \longmapsto l(n) = \frac{1}{2}(n - q)$ and $m \longmapsto k(m) = 2m + q$, for each $n \in D$ and $m \in F$, are bijections inverse to one another.

Recall that, if B is an integral domain then by the division algorithm of commutative rings, d has either zero or two (possibly non-distinct) roots in B. Therefore, if 2 is invertible, f also has zero or two roots using the bijection above.

We are now in a position to state our most important result of this section.

Theorem 4.6. *Suppose B is a field and $f(x) = x^2 + qx + r$ is a quadratic polynomial over B such that $q \neq 0 \neq \Delta = q^2 - 4r$. Let $E = (E, p)$ be the B-algebra defined previously. An element $(y, z) \in E$ is a non-trivial idempotent if and only if there is a root n of $d_f(x) = x^2 - \Delta$ with $y = \frac{1}{2}(1 + zb)$ and $z = n^{-1}$.*

Proof. "Only if": Suppose $(y, z) \in E$ is a non-trivial idempotent. Then using the ring multiplication for E defined in (4.6),

$$y^2 - z^2 r = y \tag{4.8}$$

$$2yz - z^2 q = z \tag{4.9}$$

Now $z = 0$ gives $y = y^2$. In such a case, $(y, z) = 1$ or $(y, z) = 0$, contrary to (y, z) non-trivial. Hence $z \neq 0$ and equality (4.9) gives $z = q^{-1}(2y - 1)$. Substituting (the square of) this last equality into (4.8) we get, after simplification

$$\Delta\left(y^2 - y - r\Delta^{-1}\right) = 0$$

Therefore, y is a root of the polynomial $\phi(x) = x^2 - x - r\Delta^{-1}$. Since y is a root of this polynomial, the bijection k of Proposition 4.5 (applied to ϕ) gives $n_1 = k(y) = 2y - 1$ is a root of the discriminant polynomial $d_\phi(x) = x^2 - \left(1 + 4r\Delta^{-1}\right) = x^2 - q^2\Delta^{-1}$, and so $n = qn_1^{-1}$ is a root of $d_f(x) = x^2 - \Delta$. Thus $2y - 1 = n_1 = qn^{-1}$ giving $y = \frac{1}{2}\left(1 + qn^{-1}\right)$, and $z = q^{-1}(2y - 1) = q^{-1}n_1$ giving $z = n^{-1}$.

"If": On the other hand, suppose n is a root of $d_f(x) = x^2 - \Delta$. Therefore, there is a root $n_2 = qn^{-1}$ of $d_\phi(x) = x^2 - q^2\Delta^{-1}$, which is the discriminant polynomial of $\phi(x) = x^2 - x - r\Delta^{-1}$. Then, using the bijection l of Proposition 4.5 (applied to ϕ), there is a root $y = l(n_2) = \frac{1}{2}(n_2+1) = \frac{1}{2} + \frac{1}{2}qn^{-1}$ of ϕ. Furthermore, taking $z = q^{-1}(2y-1) = n^{-1}$ gives a pair (y,z) which satisfies equalities (4.8) and (4.9) above. Therefore, (y,z) is a non-trivial idempotent in E. □

In other words, we have actually shown that there are bijections $j : U \longrightarrow D$ and $h : D \longrightarrow U$, between the set U of all non-trivial idempotents of E and the set D of all roots of the discriminant polynomial d_f of f, defined, respectively, by $u = (y,z) \longmapsto j(u) = q(2y-1)^{-1} = (qz)^{-1}$, and $n \longmapsto h(n) = \left(\frac{1}{2} + \frac{1}{2}qn^{-1}, n^{-1}\right)$, for each $u \in U$ and $n \in D$. Moreover, j and h are inverse to one another. Using the bijections k and l of Proposition 4.5, we now have the diagram

$$U \underset{h}{\overset{j}{\rightleftarrows}} D \underset{k}{\overset{l}{\rightleftarrows}} F \tag{4.10}$$

giving a bijection $F \longrightarrow U$. In fact, we may prove more under the additional assumption that f is separable.

Theorem 4.7. *Let B be a field and $f(x) = x^2 + qx + r$ a separable quadratic polynomial over B in which $q \neq 0 \neq \Delta = q^2 - 4r$. Let $E = (E, p)$ be the commutative B-algebra with the multiplication given by (4.6), and with the homomorphism of rings $p : B \longrightarrow E$ defined by (4.4). Let I be the set of non-trivial complementary idempotents in E. Let F be the set of roots of f in B. Then there is a bijection $g : F \longrightarrow I$. Furthermore, writing $g(m_1) = (y_1, z_1)$ and $g(m_2) = (y_2, z_2)$ for the roots m_1 and m_2 in F, the following conditions hold in E:*

1. $(y_1,z_1)(m_1,0) = (y_1,z_1)(0,1)$ *and* $(y_2,z_2)(m_2,0) = (y_2,z_2)(0,1)$
2. $(y_1,z_1)(b,0) = (y_1,z_1)(b',0)$ *implies* $b = b'$, *and* $(y_2,z_2)(b,0) = (y_2,z_2)(b',0)$ *implies* $b = b'$, *for every b and b' in B.*

Proof. Since f is separable, F is either empty or has two elements. If F is non-empty, call m_1 and m_2 the (distinct) roots of f in B. Therefore, there are two distinct non-trivial idempotents $hk(m_1) = (y_1,z_1)$ and $hk(m_2) = (y_2,z_2)$ in E by Theorem 4.6, where h and k are the bijections displayed in diagram (4.10). Since $1-(y_1,z_1)$ is a non-trivial idempotent in E, $(y_2,z_2) = 1-(y_1,z_1)$. Thus $(y_1,z_1)(y_2,z_2) = 0$ and $(y_1,z_1)+(y_2,z_2) = 1$. Therefore, $I = J$ and we may take $g = hk$.

Next, we show that for the two roots m_1 and m_2 of f, $(y_1,z_1)(m_1,0)=(y_1,z_1)(0,1)$ and $(y_2,z_2)(m_2,0)=(y_2,z_2)(0,1)$. Indeed, with n a root of $d_f(x)=x^2-\Delta$, we have $m_1=l(n)=\frac{1}{2}(n-q)$, $y_1=\pi_1 h(n)=\frac{1}{2}(1+z_1q)$ and $z_1=\pi_2 h(n)=n^{-1}$,

$$
\begin{aligned}
(y_1,z_1)(m_1,0)=(y_1m_1,z_1m_1) &= \Big(\frac{1}{2}(1+n^{-1}q)m_1, n^{-1}m_1\Big)\\
&= \Big(\frac{1}{4}(1+n^{-1}q)(n-q), \frac{1}{2}n^{-1}(n-q)\Big)\\
&= \Big(\frac{1}{4}(n-q+q-n^{-1}q^2), \frac{1}{2}(1-n^{-1}q)\Big)\\
&= \Big(\frac{1}{4}(n^2-q^2)n^{-1}, \frac{1}{2}(1-z_1q)\Big)\\
&= \Big(\frac{1}{4}(-4r)n^{-1}, y_1-z_1q\Big)\\
&= (-n^{-1}r, y_1-z_1q)\\
&= (-z_1r, y_1-z_1q)\\
&= (y_1,z_1)(0,1)
\end{aligned}
$$

A similar calculation will show $(y_2,z_2)(m_2,0)=(y_2,z_2)(0,1)$.

Finally, because $(y_1,z_1)(b,0)=(y_1b,z_1b)$ and $(y_2,z_2)(b,0)=(y_2b,z_2b)$, and $z_1={n_1}^{-1}$ and $z_2={n_2}^{-1}$ for a roots n_1 and n_2 of $d_f(x)=x^2-\Delta$, the second condition above obviously holds. □

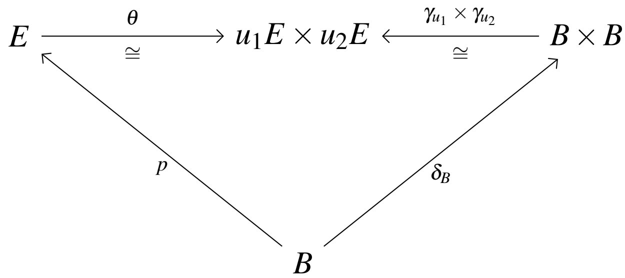
Corollary 4.8. *Under the hypothesis of Theorem 4.7, f has a root in B if and only if $E\cong B\times B$ as B-algebras.*

Proof. If f has a root m_1, then since f is separable, it has another (distinct) root m_2. Therefore, the bijection g of Theorem 4.7, and Proposition 1.7 give the ring isomorphism $\theta: E\longrightarrow u_1E\times u_2E$ defined by $e\longmapsto\big((y_1,z_1)e,(y_2,z_2)e\big)$, where $g(m_1)=u_1=(y_1,z_1)$ and $g(m_2)=u_2=(y_2,z_2)$. Recall, from equality (4.5), that $\gamma_{u_1}: B\longrightarrow u_1E$ is defined by:

$$\gamma_{u_1}(b)=(y_1,z_1)(b,0)=(y_1b,z_1b)$$

and a similar definition for γ_{u_2}. We show γ_{u_1} is an isomorphism of rings. Indeed, γ_{u_1} is an injection by condition (2) of Theorem 4.7. Furthermore, by condition (1) in Theorem 4.7, and Lemma 4.3, γ_{u_1} is a surjection. Similarly $\gamma_{u_2}: B\longrightarrow u_2E$ is an isomorphism of rings.

Finally with $\delta_B: B\longrightarrow B\times B$ the diagonal map, the diagram



is obviously commutative. Therefore, $(E,p) \cong (B\times B, \delta_B)$ in $(B \downarrow \mathsf{CRing})$.

On the other hand, suppose $E \cong B\times B$. Then E has two (distinct) non-trivial idempotents, since B is a field. Therefore, by the bijection in Theorem 4.7, f has two (distinct) roots in B. □

Note that, if f has no root in B, then f is irreducible over B and hence E itself is a field.

Corollary 4.9. *Under the hypothesis of Theorem 4.7, E is a field or $E \cong B\times B$ as B-algebras.*

We may finally formulate these results in the language of the Galois theory of commutative rings. Recall from Proposition 4.4 that $p : B \longrightarrow E$ is a monadic extension of B.

Theorem 4.10. *Let B be a field and $f(x) = x^2 + qx + r$ a separable quadratic polynomial over B in which $q \neq 0 \neq \Delta = q^2 - 4r$. Let $E = (E,p)$ be the commutative B-algebra with the multiplication given by (4.6) (taking $s = -q$ and $t = -r$), and with the homomorphism of rings $p : B \longrightarrow E$ defined by (4.4), considered as a morphism $E \longrightarrow B$ in CRing^{op}. Then $(E,p) \in \mathsf{Spl}(E,p)$.*

Proof. By Corollary 4.9, E is either a field or $E \cong B\times B$ as B-algebras. In the latter case

$$\big(E\otimes_B (B\times B), \iota_E\big) \cong \big((E\otimes_B B)\times(E\otimes_B B), \delta\big) \cong \big(E\times E, \delta_E\big)$$

Otherwise, if E is a field, then $x^2 + bx + c$, where $b = p(q) = (q,0)$ and $c = p(r) = (r,0)$, is a separable quadratic polynomial over E in which $b \neq 0 \neq b^2 - 4c$, and which now has a root $\mathrm{cls}_R(x)$ in E. By Corollary 4.8,

$$\big(E[x]/R', \mathrm{incl}_E\big) \cong \big(E\times E, \delta_E\big)$$

where R' is the congruence on $E[x]$ generated by the pair $(x^2, -bx-c)$. But $\big(E\otimes_B \big(B[x]/R\big), \iota_E\big)$ $\cong \big(E[x]/R', \mathrm{incl}_E\big)$ by Corollary 4.2. So, in both cases, we obtain $(E\otimes_B E, \iota_E) \cong (E\times E, \delta_E)$.

Next, in both cases above E is admissible with respect to the Galois structure Λ in (4.7). Therefore, considering $p : B \longrightarrow E$ as a morphism $E \longrightarrow B$ in $\mathsf{CRing}^{\mathrm{op}}$, Proposition 2.8 gives that $(E,p) \in \mathsf{Spl}(E,p)$ if and only if, for the pullback functor p^* of p, $p^*(E,p) \cong H^E(X,\phi)$ for some $\big(X,\phi\big) \in \big(\mathsf{Stone} \downarrow I(E)\big)$. Calculating $p^*(E,p)$ in its opposite category $(E \downarrow \mathsf{CRing})$, we have in both cases $p^*(E,p) \cong (E \times E, \delta_E)$, and therefore, taking $X = \mathbf{2}$, the two-element set, when E is a field, and $X = \mathbf{4}$, the four-element set, when $E \cong B \times B$ as B-algebras, gives the desired result. □

4.3 Extensions by quadratic equations in CSemiRing

Let B be a commutative semiring and consider an equation in a commutative B-semialgebra A

$$x^2 = sx + t1 \tag{4.11}$$

where s and t are fixed elements of B, and x is a variable over A. Furthermore, if in every commutative B-semialgebra A any two solutions a_1 and a_2 of the equation (4.11) are distinct, then we shall call an equation of this form a *separable quadratic equation over B*.

Selecting the elements s and t above, we may now construct, as in Section 4.1, the category $\mathbb{B}$ which has as an initial object the object $\big(E,e\big) = \big(E,(0,1)\big)$, where E is the commutative B-semialgebra with the semiring multiplication given by

$$(w,x)(y,z) = (wy + xzt, wz + xy + xzs)$$

as in (4.3), and with the homomorphism of semirings $p : B \longrightarrow E$ defined by (4.4). Again, there is the congruence R on $B[x]$ generated by the pair $(x^2, sx+t)$ as in equality (4.2).

Our attention returns now specifically to Theorem 4.7. We would like to generalize such a result to CSemiRing. Since we do not have the bijections of Proposition 4.5, we will add additional assumptions to our proposed hypothesis, which is the following natural conjecture intended to extend Theorem 4.7:

Conjecture 4.11. *Let B be a connected commutative semiring and $x^2 = sx + t$ a separable quadratic equation over B in which $s \neq 0 \neq \Delta = s^2 + 4t$ are multiplicatively invertible in B. Suppose also that this equation has zero or two solutions in B. Let $E = (E,p)$ be the commutative B-semialgebra defined previously. Let I be the set of non-trivial complementary idempotents in E. Let S be the set of solutions of $x^2 = sx + t$ in B. Then there is a bijection $g : S \longrightarrow I$. Furthermore, writing $g(m_1) = (y_1,z_1)$ and $g(m_2) = (y_2,z_2)$ for the solutions m_1 and m_2 in S, the following conditions hold in E:*

1. $(y_1,z_1)(m_1,0) = (y_1,z_1)(0,1)$ *and* $(y_2,z_2)(m_2,0) = (y_2,z_2)(0,1)$*;*

2. $(y_1,z_1)(b,0) = (y_1,z_1)(b',0)$ *implies* $b = b'$, *and* $(y_2,z_2)(b,0) = (y_2,z_2)(b',0)$ *implies* $b = b'$, *for every* b *and* b' *in* B;

3. *if* $x^2 = sx + t$ *has no solution in* B, *then, in addition to* $(0,1) \in E$, *the equation* $x^2 = sx + t1$ *has exactly one more solution in* E.

Remark 4.12. In Theorem 4.7, whenever the separable quadratic polynomial $f(x) = x^2 + qx + r$ has no root in B, the B-algebra E is a field and obviously now contains the two roots of f because B is a field. Therefore, reapplication of Theorem 4.7 to f as a separable quadratic polynomial over E allows one to deduce $E \otimes_B E \cong E \times E$. Condition (3) in Conjecture 4.11 replaces this condition when B is simply a connected commutative semiring.

If we are able to prove Conjecture 4.11 then certainly Corollary 4.8 would have a corresponding generalization.

Conjecture 4.13. *Under the hypothesis of Conjecture 4.11, the equation* $x^2 = sx + t$ *has a solution in* B *if and only if* $E \cong B \times B$ *as* B*-semialgebras.*

Corollary 4.9 would also have a corresponding generalization.

Conjecture 4.14. *Under the hypothesis of Conjecture 4.11,* E *is connected or* $E \cong B \times B$ *as* B*-semialgebras.*

The semiring homomorphism $p : B \longrightarrow E$ is a split monomorphism of B-semimodules, and so by Corollary 3.33, $p : E \longrightarrow B$, considered as a morphism $B \longrightarrow E$ in $\mathsf{CSemiRing}^{\mathrm{op}}$ is a monadic extension of B. Therefore, we would also have a reformulation of these results in terms of the Galois theory of commutative semirings, corresponding to Theorem 4.10.

Conjecture 4.15. *Let* B *be a connected commutative semiring and* $x^2 = sx + t$ *a separable quadratic equation over* B *in which* $s \neq 0 \neq \Delta = s^2 + 4t$ *are multiplicatively invertible in* B. *Suppose also that this equation has zero or two solutions in* B. *Let* $E = (E,p)$ *be the commutative* B*-semialgebra with the multiplication given by (4.3), and with the homomorphism of semirings* $p : B \longrightarrow E$ *defined by (4.4), considered as a morphism* $E \longrightarrow B$ *in* $\mathsf{CSemiRing}^{op}$. *Then* $(E,p) \in \mathsf{Spl}(E,p)$.

The proof would follow an almost identical proof to that in Theorem 4.10, with the Galois structure now $\Gamma = (\mathsf{CSemiRing}^{\mathrm{op}}, \mathsf{Stone}, I, H)$ defined in the previous chapter.

Unfortunately, as we show below, a generalization of Theorem 4.7 necessarily puts us back into CRing: *no such generalization exists for connected commutative semirings which are not rings.*

Theorem 4.16. *Let B be a connected commutative semiring and consider the B-semialgebra E previously defined. If $E \cong B \times B$ as B-semialgebras, then B is a commutative ring.*

Proof. Let $\tau : B \times B \longrightarrow E$ be the B-semialgebra isomorphism in the theorem statement. Since B is connected, E contains the two non-trivial idempotents $\tau(1,0) = u_1 = (c_1, d_1)$ and $\tau(0,1) = u_2 = (c_2, d_2)$ complementary to each other. Then, $B \times B \cong E \cong u_1E \times u_2E$ as B-semialgebras, by Proposition 1.7.

Next, τ induces isomorphisms $\gamma_{u_1} : B \longrightarrow u_1E$ and $\gamma_{u_2} : B \longrightarrow u_2E$, which are defined by $\gamma_{u_1}(b) = u_1(b,0) = (c_1b, d_1b)$ and $\gamma_{u_2}(b) = u_2(b,0) = (c_2b, d_2b)$, respectively. Choosing b and b' in B with $\gamma_{u_1}(b) = u_1(0,1) = (d_1t, c_1 + d_1s)$ and $\gamma_{u_2}(b') = u_2(0,1) = (d_2t, c_2 + d_2s)$, we obtain

$$c_1b = d_1t \tag{4.12}$$
$$d_1b = c_1 + d_1s \tag{4.13}$$

and

$$c_2b' = d_2t \tag{4.14}$$
$$d_2b' = c_2 + d_2s \tag{4.15}$$

Also since $u_1 + u_2 = 1$ and $u_1u_2 = 0$ in E we have

$$c_1 + c_2 = 1 \tag{4.16}$$
$$d_1 + d_2 = 0 \tag{4.17}$$

By equality (4.17), both of d_1 and d_2 are additively invertible. Then from (4.13), $0 = (d_1 + d_2)b = d_1b + d_2b = c_1 + (d_1s + d_2b)$, hence c_1 is additively invertible. A similar calculation shows c_2 is also additively invertible. By (4.16), 1 is additively invertible, and hence B is a ring. □

Remark 4.17. Theorem 4.16 shows that Conjecture 4.11 is impossible for connected commutative semirings B which are not rings. For, if it were, consider a connected commutative semiring B which is not a ring and a separable quadratic equation $x^2 = sx + t$ over B satisfying the hypothesis of Conjecture 4.11. Then, $x^2 = sx + t$ has no solutions in B, because otherwise $E \cong B \times B$ as B-semialgebras, so that B is a commutative ring, by Theorem 4.16. Therefore, E is a connected semiring. Furthermore, considering the congruence R' over $E[x]$ generated by the pair $(x^2, s'x + t')$ where $s' = p(s) = (s,0)$ and $t' = p(t) = (t,0)$, we have $\left(E \otimes_B E, \iota_E\right) \cong \left(E[x]/R', \mathrm{incl}_E\right)$ as E-semialgebras. But since $x^2 = s'x + t'$ is a separable quadratic equation over E satisfying the hypothesis of Conjecture 4.11 and having a solution $(0,1)$ in E, $E \otimes_B E \cong E \times E$ as E-semialgebras. Therefore, E is a ring, again by Theorem 4.16. Therefore, B is a ring, a contradiction.

Let us introduce a few final definitions, simple results and remarks.

Definition 4.18. A semiring B is called congruence-free if each congruence R on B is trivial; that is $R = B \times B$ or $R = \{(b,b) \mid b \in B\}$.

Theorem 4.19. *Let B be a congruence-free commutative semiring. Then B is multiplicatively cancellative; that is, for all $a \neq 0$ in B, $ax = ay$ implies $x = y$ for any x, y in B.*

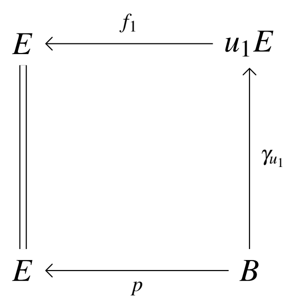
Proof. For each $a \in B$, the set $R_a = \{(x,y) \in B \times B \mid ax = ay\}$ is a congruence on B. Therefore, $R_a = B \times B$, in which case $a = a1 = a0 = 0$, or $R_a = \{(b,b) \mid b \in B\}$, in which case B is multiplicatively cancellative. □

Example 4.20. Every integral domain, regarded as a commutative semiring, is congruence-free.

Theorem 4.21. *[6, Theorem 3.2] Let B be a congruence-free commutative semiring. Then B is a field or the semiring $\{0,1\}$ in which $1+1 = 1$.*

One may consider if Conjecture 4.15 is still possible for connected commutative semirings B which are not rings. Certainly, if E of that conjecture has no non-trivial complementary idempotents, then, by Theorem 4.16 and by considering the rank of the underlying E-semimodule of $E \otimes_B E$, we conclude $(E,p) \notin \mathsf{Spl}(E,p)$. The only possibility is then that E may have non-trivial complementary idempotents u_1 and u_2 such that $E \otimes_B (u_1E) \cong E \cong E \otimes_B (u_2E)$, but that $u_1E \times u_2E \ncong B \times B$.

For example, suppose that the homomorphisms of semirings $\gamma_{u_1} : B \longrightarrow u_1E$ and $\gamma_{u_2} : B \longrightarrow u_2E$, defined by $\gamma_{u_1}(b) = u_1(b,0)$ and $\gamma_{u_2}(b) = u_2(b,0)$, respectively, are epimorphisms. This will happen, for example, if there exist b_1 and b_2 in B such that $\gamma_{u_1}(b_1) = u_1(0,1)$ and $\gamma_{u_1}(b_2) = u_2(0,1)$, by Lemma 4.3. In the field case, such b_1 and b_2 correspond to the roots of the quadratic polynomial defining E. Furthermore, suppose there exist homomorphisms of semirings $f_1 : u_1E \longrightarrow E$ and $f_2 : u_2E \longrightarrow E$ such that $f_1\gamma_{u_1} = p = f_2\gamma_{u_2}$. Then the diagram



is a pushout in CSemiRing, and likewise for a similar diagram involving $f_2 : u_2E \longrightarrow E$. Therefore, $E \otimes_B (u_1E) \cong E \cong E \otimes_B (u_2E)$, and so (E, p), considered as an object in $\big(B \downarrow \mathsf{CSemiRing}\big)^{\mathrm{op}}$, is an object of $\mathsf{Spl}(E, p)$. Note that, if there exist b_1 and b_2 in B such that $\gamma_{u_1}(b_1) = u_1(0,1)$ and $\gamma_{u_1}(b_2) = u_2(0,1)$, then B cannot be even a multiplicatively cancellative semiring, otherwise γ_{u_1} and γ_{u_2} will in fact be isomorphisms, making B a ring.

4.4 Normal extensions of semirings which are rank-two as B-semimodules

The semiring multiplication

$$(w,x)(y,z) = (wy + xzt, wz + xy + xzs)$$

of (4.3) is motivated by the ring multiplication in the B-algebra $B[x]/f(x)B[x]$ when B is a commutative ring and where $f(x)$ is a separable quadratic polynomial over B. But, having in mind Corollary 4.9 we now think to modify the above semiring multiplication so as to arrive at a commutative B-semialgebra E for a connected commutative semiring B such that $E \cong B \times B$ as B-semialgebras. In fact, this modification, displayed in (4.19) below, is a very natural generalization which includes (4.3) as a special case.

Beginning then with a commutative semiring B, consider an arbitrary commutative B-semialgebra E whose underlying B-semimodule structure is the canonical B-semimodule structure of the product $B \times B$. The semiring multiplication in E then determines three elements (a_1, b_1), (a_2, b_2) and (a_3, b_3) in $B \times B$ by the following equalities:

$$(1,0)^2 = (a_1, b_1), \quad (1,0)(0,1) = (a_2, b_2) \quad \text{and} \quad (0,1)^2 = (a_3, b_3) \tag{4.18}$$

By introducing familiar conditions on the commutative semiring B, these equalities allow us to deduce the following simple observation.

Lemma 4.22. *Suppose B is a commutative semiring such that, for each b and b' in B, the following conditions hold:*

1. *$b + b' = 0$ implies $b = 0 = b'$;*
2. *$bb' = 0$ implies $b = 0$ or $b' = 0$.*

Let $E = (B \times B, 0, 1, +, \cdot, \phi)$ be a commutative B-semialgebra where $(B \times B, 0, +, \phi)$ is the canonical B-semimodule product. Denoting the identity element 1 in E by (e_1, e_2), exactly *one of the following four conditions holds in E for the values of* (a_1,b_1), (a_2,b_2) *and* (a_3,b_3) *in (4.18):*

1. e_1 *and* e_2 *are multiplicatively invertible, and* $a_1 = e_1^{-1}$; $a_2 = 0$; $a_3 = 0$; $b_1 = 0$; $b_2 = 0$; *and* $b_3 = e_2^{-1}$.
2. e_1 *and* e_2 *are multiplicatively invertible, and* $a_1 = 0$; $a_2 = e_2^{-1}$; $a_3 = 0$; $b_1 = 0$; $b_2 = e_1^{-1}$; *and* $b_3 = 0$.
3. e_1 *is multiplicatively invertible and* $e_2 = 0$, *and* $a_1 = e_1^{-1}$; $a_2 = 0$; $b_1 = 0$; *and* $b_2 = e_1^{-1}$.
4. e_2 *is multiplicatively invertible and* $e_1 = 0$, *and* $a_2 = e_2^{-1}$; $a_3 = 0$; $b_2 = 0$; *and* $b_3 = e_2^{-1}$.

Proof. The equalities in (4.18) show that the semiring multiplication in E is given by

$$\big(w,x\big)\big(y,z\big) = \big(wya_1 + (wz+xy)a_2 + xza_3, wyb_1 + (wz+xy)b_2 + xzb_3\big) \tag{4.19}$$

for each (w,x) and (y,z) in E. Since $(e_1,e_2)(1,0) = (1,0)$ and $(e_1,e_2)(0,1) = (0,1)$ we obtain:

$$(e_1a_1 + e_2a_2, e_1b_1 + e_2b_2) = (1,0) \quad \text{and} \quad (e_1a_2 + e_2a_3, e_1b_2 + e_2b_3) = (0,1)$$

which in turn give

$$e_1a_1 + e_2a_2 = 1 \tag{4.20}$$
$$e_1b_1 + e_2b_2 = 0 \tag{4.21}$$

and

$$e_1a_2 + e_2a_3 = 0 \tag{4.22}$$
$$e_1b_2 + e_2b_3 = 1 \tag{4.23}$$

From condition (1) on B above, the equalities (4.21) and (4.22) give:

$$e_1b_1 = 0 \qquad e_2b_2 = 0 \qquad e_1a_2 = 0 \qquad e_2a_3 = 0 \tag{4.24}$$

Having in mind (4.20) and (4.23), this last set of equalities then gives:

$$(e_1a_1)(e_2a_2) = (e_1a_2)(e_2a_1) = 0$$
$$(e_1b_2)(e_2b_3) = (e_1b_3)(e_2b_2) = 0$$

Since condition (2) obviously implies that B is connected, this last set of equalities and (4.20) and (4.23) show that

$$\big(e_1a_1 = 1 \quad \text{and} \quad e_2a_2 = 0\big) \quad \text{or} \quad \big(e_1a_1 = 0 \quad \text{and} \quad e_2a_2 = 1\big)$$

and

$$\big(e_1b_2 = 0 \quad \text{and} \quad e_2b_3 = 1\big) \quad \text{or} \quad \big(e_1b_2 = 1 \quad \text{and} \quad e_2b_3 = 0\big)$$

Considering each of these four possibilities and using the equalities (4.24), we obtain the four cases as claimed. □

Proposition 4.23. *Suppose B is as in Lemma 4.22. Let M be the set of all pairs (m,i) with $m : (B\times B)\times(B\times B) \longrightarrow B\times B$ and $i = (e_1,e_2) \in B\times B$ a binary and a nullary operation on $B\times B$, respectively, such that $(B\times B, 0, i, +, m, \phi)$ is a commutative B-semialgebra having non-trivial complementary idempotents where $(B\times B, 0, +, \phi)$ is the canonical B-semimodule product. Let P be the set of pairs (s,t) of multiplicatively invertible elements of B. Consider the diagram:*

$$M \underset{h}{\overset{g}{\rightleftarrows}} P \tag{4.25}$$

in which:

- *the map g is defined by:*

$$g(m,i) = i$$

- *the map h is defined by:*

$$h(s,t)_1 : \big((a,b),(c,d)\big) \longmapsto \big(acs^{-1}, bdt^{-1}\big)$$

  and

$$h(s,t)_2 = (s,t)$$

Then, the maps g and h in diagram (4.25) are inverse to each other, and therefore establish a bijection between the sets M and P.

Proof. If $E = (B \times B, 0, i, +, m, \phi)$ is a commutative B-semialgebra having non-trivial complementary idempotents (u_1, v_1) and (u_2, v_2), then, with $i = (e_1, e_2)$,

$$(u_1 + u_2, v_1 + v_2) = (e_1, e_2)$$

and

$$(u_1, v_1)(u_2, v_2) = (0, 0)$$

This second equality, by (4.19), gives

$$u_1 u_2 a_1 + (u_1 v_2 + v_1 u_2) a_2 + v_1 v_2 a_3 = 0 \tag{4.26}$$

$$u_1 u_2 b_1 + (u_1 v_2 + v_1 u_2) b_2 + v_1 v_2 b_3 = 0 \tag{4.27}$$

while the first equality gives:

$$u_1 + u_2 = e_1 \tag{4.28}$$

$$v_1 + v_2 = e_2 \tag{4.29}$$

Being a B-semialgebra previously described in Lemma 4.22, there are just the four possibilities for the the values of (a_1, b_1), (a_2, b_2) and (a_3, b_3). Starting with case (3) from the list in Lemma 4.22, $v_1 = 0 = v_2$ from (4.29) and condition 1 on B. Because $u_1 u_2 = 0$ by (4.26) and $u_1 + u_2 = e_1 \neq 0$ by (4.28), exactly one of u_1 and u_2 is 0, and the other is e_1. Therefore in this case, the complementary idempotents are simply the trivial pair.

Similarly, we will find case (4) will give all complementary idempotents again being the trivial pair.

The case (2) gives $u_1 v_2 = 0 = v_1 u_2$ from (4.26) and (4.27). So suppose $u_1 \neq 0$. Then $v_2 = 0$, so $v_1 = e_2 \neq 0$ so $u_2 = 0$. Hence $u_1 = e_1$. On the other hand, $u_1 = 0$ gives $u_2 = e_1 \neq 0$ so that $v_1 = 0$ and so $v_2 = e_2$. In both these scenarios we get the complementary idempotents again being the trivial pair.

The case (1) gives $u_1 u_2 = 0$ and $v_1 v_2 = 0$, and so for the same reasons as above, exactly one of u_1 and u_2 is 0 and exactly one of v_1 and v_2 is zero. The only non-trivial pairs here are then $(u_1, v_1) = (e_1, 0)$ and $(u_2, v_2) = (0, e_2)$ or $(u_1, v_1) = (0, e_2)$ and $(u_2, v_2) = (e_1, 0)$.

Therefore, E is the commutative B-semialgebra with the multiplication

$$m\big((w,x),(y,z)\big) = (wye_1^{-1}, xze_2^{-1}) \tag{4.30}$$

which admits non-trivial complementary idempotents $(e_1,0)$ and $(0,e_2)$.

Conversely, it is easily checked that selecting multiplicatively invertible elements e_1 and e_2 in B and defining $m : (B\times B)\times(B\times B) \longrightarrow B\times B$ by (4.30) and $i=(e_1,e_2)$ does in fact make $E=(B\times B, 0, i, +, m, \phi)$ a commutative B-semialgebra having non-trivial complementary idempotents $(e_1,0)$ and $(0,e_2)$. □

If (e_1,e_2) is a pair of multiplicatively invertible elements in B and E the corresponding commutative B-semialgebra as in Proposition 4.23, a routine calculation now also shows that the map $p : B \longrightarrow E$ given by

$$p(b) = (be_1, be_2) \tag{4.31}$$

is a homomorphism of semirings and $(E,p) \in (B \downarrow \mathsf{CSemiRing})$ is this commutative B-semialgebra corresponding to (e_1,e_2).

Theorem 4.24. *For a chosen pair (e_1,e_2) of invertible elements in a commutative semiring B as in Lemma 4.22, let E be the corresponding commutative B-semialgebra determined by the bijection h of Proposition 4.23. Then $E \cong B\times B$ as B-semialgebras.*

Proof. The pair of complementary idempotents $u=(e_1,0)$ and $v=(0,e_2)$ presents $E\cong uE\times vE$ by Proposition 1.7. The canonical commutative B-semialgebra structures on the commutative semirings uE and vE are determined by the homomorphisms of semirings $\gamma_u : B \longrightarrow uE$ and $\gamma_v : B \longrightarrow vE$ defined by

$$\gamma_u(b) = up(b) = (be_1, 0) \quad \text{and} \quad \gamma_v(b) = vp(b) = (0, be_2)$$

respectively, with $p : B \longrightarrow E$ as in (4.31).

We first show that $(uE,\gamma_u) \cong (B, 1_B)$ in $(B \downarrow \mathsf{CSemiRing})$. Indeed, consider the map $h : E \longrightarrow B$ given by

$$h(x,y) = xe_1^{-1} \tag{4.32}$$

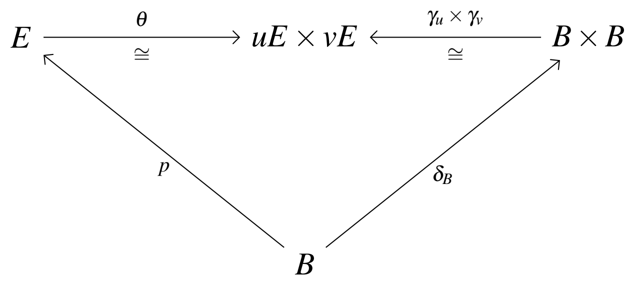
An easy check shows h is a homomorphism of B-semialgebras $(E,p) \longrightarrow (B,1_B)$. Additionally, since $h(u) = h(e_1,0) = e_1e_1^{-1} = 1$ and $\gamma_u(b) = up(b)$ for each $b\in B$, the restriction $h_u : uE \longrightarrow B$ of h to the semiring uE is a homomorphism of B-semialgebras

$(uE,\gamma_u) \longrightarrow (B,1_B)$. Furthermore, since $uE = \{(b,0) \mid b \in B\}$, if $x = (b,0) \in uE$ then

$$\gamma_u\big(h_u(x)\big) = \gamma_u\big(h(b,0)\big) = \gamma_u\big(be_1^{-1}\big) = (be_1e_1^{-1},0) = (b,0) = x$$

That is, $\gamma_u h_u = 1_{uE}$. As h_u already satisfies $h_u\gamma_u = 1_B$ since it is a homomorphism of B-semialgebras $(uE,\gamma_u) \longrightarrow (B,1_B)$, we conclude $\gamma_u : (B,1_B) \longrightarrow (uE,\gamma_u)$ is an isomorphism. A similar argument shows the same is true of $\gamma_v : (B,1_B) \longrightarrow (vE,\gamma_v)$.

Finally, with $\theta : E \longrightarrow uE \times vE$ the semiring isomorphism of Proposition 1.7, the diagram



commutes since

$$\begin{aligned}\theta(p(b)) &= \theta(be_1,be_2)\\ &= \big((be_1,0),(0,be_2)\big)\\ &= (\gamma_u\times\gamma_v)(b,b)\\ &= (\gamma_u\times\gamma_v)(\delta_B(b))\end{aligned}$$

Therefore, $(E,p) \cong (B\times B,\delta_B)$, as desired. □

Corollary 4.25. *The homomorphism of semirings $p : B \longrightarrow E$, considered as a morphism $E \longrightarrow B$ in $\mathsf{CSemiRing}^{op}$, is a normal extension of B.*

Proof. By the definition of the homomorphism of semirings $h : E \longrightarrow B$ in (4.32), p is obviously a split monomorphism of B-semialgebras and therefore a split monomorphism of the underlying B-semimodules. By Corollary 3.33, $p : B \longrightarrow E$ is a monadic extension of B. Finally, by Theorem 4.24

$$\big(E\otimes_B E,\iota_E\big) \cong \big(E\otimes_B (B\times B),\iota_E\big) \cong (E\times E,\delta_E)$$

and so $(E,p) \in \mathsf{Spl}(E,p)$, as desired. □

Example 4.26. Let B be either of the commutative semirings $\mathbf{Q}_{\geq 0}$ or $\mathbf{R}_{\geq 0}$ of non-negative rational numbers and non-negative real numbers, respectively. Then B clearly satisfies the two conditions in Lemma 4.22. Furthermore, each $x \neq 0$ in B is multiplicatively invertible.

Example 4.27. Consider the full subcategory $\text{Étale}(X)$ of $(\mathsf{Top} \downarrow X)$ of *étale bundles* over the topological space X, consisting of pairs (A, α) where A is a topological space and $\alpha : A \longrightarrow X$ is a local homeomorphism. $\text{Étale}(X)$ is closed under colimits and finite limits in $(\mathsf{Top} \downarrow X)$ (see, for example [2, Proposition 6.4.7]). Furthermore, since $(\mathsf{Top} \downarrow X)$ is lextensive, so too is $\text{Étale}(X)$ by Corollary 1.16. Proposition 1.26 now says that each continuous map $(U, \mu) \longrightarrow 0$ in $\text{Étale}(X)$ into an initial object is a homeomorphism. Therefore, if $(U, \mu) + (V, \nu) \cong 0$, then both $(U, \mu) \cong 0$ and $(V, \nu) \cong 0$ in $\text{Étale}(X)$

Binary products in $\text{Étale}(X)$ have a special property when X is *hyperconnected*, that is; a space in which every two non-empty open subsets have non-empty intersection. Specifically, since for étale bundles (U, μ) and (V, ν), their product is given by the pullback $\big(U \times_{\mu,\nu} V, \mu\text{proj}_U\big)$ in Top, and, since μ and ν are local homeomorphisms, if U and V are non-empty topological spaces, the product $(U, \mu) \times (V, \nu)$ in $\text{Étale}(X)$ is not initial whenever X is hyperconnected.

Note that examples of hyperconnected spaces include any partial order X having a least element and endowed with the left order topology, or any infinite set X having the cofinite topology.

Therefore, the collection of isomorphism classes of objects of the category $\text{Étale}(X)$ over a hyperconnected space X determines a (large) commutative semiring B when addition and multiplication are induced by (binary) coproducts and products in $\text{Étale}(X)$, respectively, which further satisfies the two conditions in Lemma 4.22. However, unlike Example 4.26, B has no non-trivial multiplicatively invertible elements since $(U, \mu) \times (V, \nu) \cong (X, 1_X)$ implies both $(U, \mu) \cong (X, 1_X)$ and $(V, \nu) \cong (X, 1_X)$ in $\text{Étale}(X)$.

As shown in Example 4.27, (large) commutative semirings determined by the collection of isomorphism classes of objects of a lextensive category necessarily satisfy the first condition in Lemma 4.22 since they have strict zeroes. We continue to find suitable lextensive categories which satisfy condition (2) in Lemma 4.22 and have a large class of multiplicatively invertible elements.

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Appendix A

Construction of Ind$(\mathbb{A})$

We now restate and prove Theorem 3.8.

Theorem A.1. *In* $\mathsf{Ind}(\mathbb{A})$:

1. *Each object* $F : J \longrightarrow \mathbb{A}$ *is the filtered colimit (over* J*) of the objects* $F(j) \in \mathbb{A}$ *considered as functors* $\mathbb{1} \longrightarrow \mathbb{A}$.

2. *Each* $H : I \longrightarrow \mathsf{Ind}(\mathbb{A})$ *for* I *filtered has a colimit in* $\mathsf{Ind}(\mathbb{A})$.

3. *For each object* $A \in \mathbb{A}$ *considered as a functor* $\mathbb{1} \longrightarrow \mathbb{A}$*, the functor*

$$hom_{\mathsf{Ind}(\mathbb{A})}(A, -) : \mathsf{Ind}(\mathbb{A}) \longrightarrow \mathsf{Sets}$$

   preserves filtered colimits.

4. *The canonical inclusion functor* $\mathbb{A} \longrightarrow \mathsf{Ind}(\mathbb{A})$ *is fully faithful.*

Proof. (Of Theorem A.1(2)) Suppose I is filtered, and consider a functor $H : I \longrightarrow \mathsf{Ind}(\mathbb{A})$. For each $i \in I$, $H(i)$ is an object of $\mathsf{Ind}(\mathbb{A})$, hence a functor $H(i) : I_i \longrightarrow \mathbb{A}$ with I_i filtered. Consider all diagrams $H(i)$ in $\mathbb{A}$ in the shape of I_i for every $i \in I$, together with all elements (x, λ) from each equivalence class $\overline{(f(s), \phi_s)}$ (with respect to the equivalence relation $\sim$ in (3.2)), where $(f, \phi) = H(\xi)$, for every $s \in I_i$ and $\xi : i \longrightarrow i'$. Such a collection of objects and morphisms in $\mathbb{A}$ is a subcategory J of $\mathbb{A}$ because (as shown in Remark 3.7):

1. if $(x, \lambda) \in \overline{(f(s), \phi_s)}$, where $s \in I_i$, $x \in I_{i'}$ and $(f, \phi) = H(\xi)$ for $\xi : i \longrightarrow i'$, then $\big(g(x), \psi_x \lambda\big) \in \overline{\big(g\big(f(s)\big), \psi_{f(s)} \phi_s\big)}$, where $(g, \psi) = H(\xi')$ for $\xi' : i' \longrightarrow i''$;

2. if $(y, \tau) \in \overline{(g(x), \psi_x)}$, then $(y, \tau\lambda) \in \overline{(g(x), \psi_x \lambda)}$.

Furthermore, J is filtered. This is easily seen by simply examining the possible cases.

Firstly, suppose j_1 and j_2 are objects of J. Then $j_1 = H(i)(s)$ and $j_2 = H(i')(t)$ for some i and i' in I, and $s \in I_i$ and $t \in I_{i'}$.

- If $i \neq i'$, then since I is filtered, there is some $k \in I$ and morphisms $\alpha : i \longrightarrow k$ and $\beta : i' \longrightarrow k$, giving morphisms $(f,\phi) = H(\alpha) : H(i) \longrightarrow H(k)$ and $(g,\psi) = H(\beta) : H(i') \longrightarrow H(k)$ in $\mathsf{Ind}(\mathbb{A})$. Therefore, in J, there is some ${j_1}' = H(k)(x)$, where $x \in I_k$, and $\lambda : j_1 \longrightarrow {j_1}'$ such that $(x,\lambda) \in \overline{(f(s),\phi_s)}$, and ${j_2}' = H(k)(y)$, where $y \in I_k$, and $\tau : j_2 \longrightarrow {j_2}'$ such that $(y,\tau) \in \overline{(g(t),\psi_t)}$. Since the image of I_k under $H(k)$ in $\mathbb{A}$ is a filtered subcategory of $\mathbb{A}$, there is $j_3 = H(k)(u)$ for some $u \in I_k$, and morphisms $\theta : x \longrightarrow u$ and $\sigma : y \longrightarrow u$ in I_k giving $H(k)(\theta) : {j_1}' \longrightarrow j_3$ and $H(k)(\sigma) : {j_2}' \longrightarrow j_3$. Therefore, there are morphisms $H(k)(\theta)\lambda : j_1 \longrightarrow j_3$ and $H(k)(\sigma)\tau : j_2 \longrightarrow j_3$ in J.
- The case $i = i'$ is obvious.

Secondly, suppose again $j_1 = H(i)(s)$ and $j_2 = H(i')(t)$ are objects of J, for some i and i' in I, and $s \in I_i$ and $t \in I_{i'}$, and $u : j_1 \longrightarrow j_2$ and $n : j_1 \longrightarrow j_2$ are a pair of parallel morphisms in J.

- If $i \neq i'$, then (t,u) is in some equivalence class $\overline{(f(s),\phi_s)}$ where $(f,\phi) = H(\mu)$ for some $\mu : i \longrightarrow i'$, and (t,n) is in some equivalence class $\overline{(g(s),\psi_s)}$ where $(g,\psi) = H(\eta)$ for some $\eta : i \longrightarrow i'$. Since I is filtered, there is some $k \in I$ and $\kappa : i' \longrightarrow k$ such that $(l,\rho)\cdot(f,\phi) = (l,\rho)\cdot(g,\psi)$, where $(l,\rho) = H(\kappa)$. Furthermore, since $(t,u) \in \overline{(f(s),\phi_s)}$

$$\overline{(l(t),\rho_t u)} = \overline{(l(f(s)),\rho_{f(s)}\phi_s)}$$

and, similarly, since $(t,n) \in \overline{(g(s),\psi_s)}$

$$\overline{(l(t),\rho_t n)} = \overline{(l(t),\rho_{g(s)}\psi_s)}$$

But because $(l,\rho)\cdot(f,\phi) = (l,\rho)\cdot(g,\psi)$, we obtain $\overline{\big(l(f(s)),\rho_{f(s)}\phi_s\big)} = \overline{\big(l(g(s)),\rho_{g(s)}\psi_s\big)}$. Therefore,

$$\overline{\big(l(t),\rho_t u\big)} = \overline{\big(l(t),\rho_t n\big)}$$

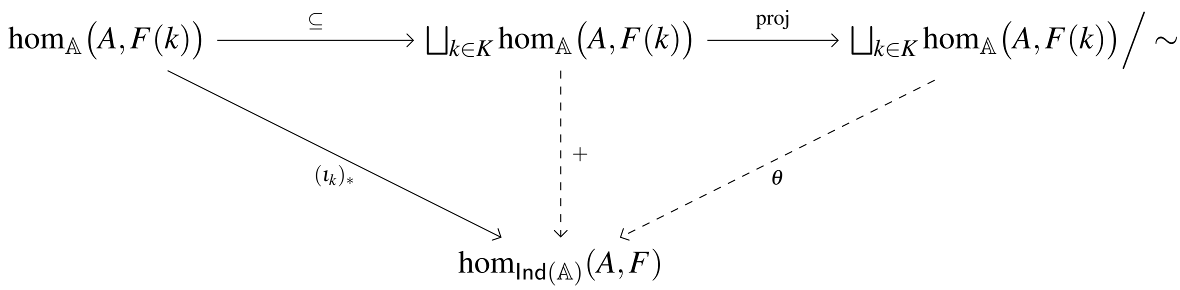
Finally, a diagram chase shows that if (x,λ) and (y,τ) are elements of $\overline{\big(l(t),\rho_t\big)}$, there is some $w \in I_k$, and morphisms $\theta : x \longrightarrow w$ and $\sigma : y \longrightarrow w$ in I_k such that $H(k)(\theta)\lambda = H(k)(\sigma)\tau : j_2 \longrightarrow j_3$ and $\big(H(k)(\theta)\lambda\big)u = \big(H(k)(\sigma)\tau\big)n$, where $H(k)(w) = j_3$.

- The case $i = i'$ is again obvious.

Therefore, J is filtered, and taking the inclusion functor $L : J \longrightarrow \mathbb{A}$ gives an object of $\mathsf{Ind}(\mathbb{A})$. Furthermore, for each $i \in I$, $\mathrm{in}_i = (\mathrm{incl}_i, \iota_i)$ with $\mathrm{incl}_i : I_i \longrightarrow J$ sending s to $H(i)(s)$, and $(\iota_i)_s = 1_{H(i)(s)}$ is indeed is a morphism $H(i) \longrightarrow L$ in $\mathsf{Ind}(\mathbb{A})$ since I_i is filtered. Finally, by the definition of $\sim$, the family $\mathrm{in} = (\mathrm{in}_i : H(i) \longrightarrow L)_{i \in I}$ is a cone $H \longrightarrow L$, which is in fact a limiting cone. $\square$

Note that a similar construction will provide a proof of Theorem A.1(1).

Proof. (Of Theorem A.1(3)) To show each object $A \in \mathbb{A}$ considered as a functor $\mathbb{1} \longrightarrow \mathbb{A}$ is finitely presentable, by Theorems A.1(1) and A.1(2) it suffices to show that for $F : K \longrightarrow \mathbb{A}$ with K filtered, the unique coproduct map $+ : \bigsqcup_{k \in K} \hom_{\mathbb{A}}\big(A, F(k)\big) \longrightarrow \hom_{\mathsf{Ind}(\mathbb{A})}(A, F)$, induced by the canonical inclusion morphisms $\iota_k : F(k) \longrightarrow F$ in $\mathsf{Ind}(\mathbb{A})$, has its kernel equivalence equal to $\sim$ and is surjective.



Both of these claims follow by the definition of $\sim$ and the definition of the disjoint union $\bigsqcup_{k \in K}$ operator. Therefore, $\hom_{\mathsf{Ind}(\mathbb{A})}(A, -) : \mathsf{Ind}(\mathbb{A}) \longrightarrow \mathsf{Sets}$ preserves filtered colimits as desired. $\square$