ELECTROMAGNETIC AND STRONG INTERACTIONS OF HADRONS IN THE QUARK AND DROPLET MODELS

SUMMARY:-

The SU(3) quark model and the droplet model of hadrons are compared in terms of their predictions for hadron electromagnetic form factors and for the elastic scattering of hadrons. For this purpose, a high energy potential scattering theory is developed in Chapter 1 and a simple dynamical model of mesons as quark - antiquark bound states is developed in Chapter 3. Some results derived in Chapter 1 are used to discuss the small and large momentum transfer variation of hadron - hadron elastic differential cross-sections in Chapter 2 from the point of view of both models. Results obtained in Chapter 3 are used in Chapter 4 in the discussion of the nucleon electromagnetic form factor predicted by both models.

INDEX:-

INTRODUCTION.

Many current explanations (1-5) of high energy hadron-hadron strong interactions are based upon the Glauber potential scattering theory.[(6)] Since the latter is restricted to small scattering angles but is often applied indiscriminately to experimental data covering a much wider angular region than for which the theory is strictly valid, it is important to investigate the kinematic and dynamic assumptions and approximations which lead to the Glauber result. For this purpose, a new derivation of the two particle high energy scattering amplitude is given, based upon a distorted wave Born approximation to the solution of the non-relativistic Schroedinger equation (Section A). This is shown (Section B) to satisfy the optical theorem and to reduce to the familiar Glauber small angle of scattering expression obtained from the W.K.B. approximation. In section C, expressions for the scattering amplitude, valid in the intermediate angular region and the large angle limit, are deduced. In section D, the Glauber approximation is investigated. Finally in section E, an analogue between Kirchoff's diffraction theory in optics and the Glauber scattering theory in high energy physics is given.

SECTION A:-

The solution of the Schroedinger equation -

$$(\nabla^2 + k^2)\psi(\underline{r}) = U(\underline{r})\psi(\underline{r}) \quad \text{where } U(\underline{r}) = \frac{2m}{\hbar^2}V(\underline{r})$$

for the motion of a particle of rest mass m in a potential V(r) is

$\psi(\underline{r}) = \int G(\underline{r},\underline{r}^1)\, U(\underline{r}^1)\, \psi(\underline{r}^1)\, d^3\underline{r}^1$ where the free particle Green's function $G(\underline{r}_1\underline{r}^1)$ satisfies $(\nabla^2 + k^2)\, G_o(\underline{r},\underline{r}^1) = \delta(\underline{r}-\underline{r}^1)$ and is given by $G_o(\underline{r},\underline{r}^1) = -\frac{1}{4\pi}\frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|}$.

Consider the distorted wave Green's function $G(\underline{r},\underline{r}^1) = \frac{-1}{4\pi}\frac{e^{i\phi(\underline{r},\underline{r}')}}{|\underline{r}-\underline{r}'|}$ (1)

where $\phi(\underline{r},\underline{r}^1)$ is the phase accumulated at $\underline{r}$ by the particle scattered at $\underline{r}^1$.

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(2) D.H. Harrington and A. Pagnamenta, Phys. Rev. 173, 1599 (1968).
(3) E. Shrauner, L. Benofy and D.W. Cho, Phys. Rev. 177, 2590 (1969).
(4) A.H. Cromer, Phys. Rev. 185, 1731 (1969).
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(6) R. Glauber, in Lectures in Theoretical Physics, 1958, vol. 1. (New York, 1959).

$$\nabla G(\underline{r},\underline{r}') \equiv \hat{n}\frac{\partial G}{\partial r'} = -\frac{1}{4\pi}\left[e^{i\phi}\nabla(|\underline{r}-\underline{r}'|^{-1}) + i|\underline{r}-\underline{r}'|^{-1}e^{i\phi}\nabla\phi\right]$$

where $\hat{n}$ is a unit vector in the direction of $\underline{r}'$

$$\nabla^2 G(\underline{r},\underline{r}') = -\frac{1}{4\pi}\Big[e^{i\phi}\nabla^2(|\underline{r}-\underline{r}'|^{-1}) + ie^{i\phi}\nabla\phi.\nabla(|\underline{r}-\underline{r}'|^{-1}) + i|\underline{r}-\underline{r}'|^{-1}e^{i\phi}\nabla^2\phi + i\nabla\phi.[e^{i\phi}\nabla(|\underline{r}-\underline{r}'|^{-1}) + ie^{i\phi}|\underline{r}-\underline{r}'|^{-1}\nabla\phi]\Big]$$

Noting $\nabla^2(|\underline{r}-\underline{r}'|^{-1}) = 0$

$$\nabla^2 G(\underline{r},\underline{r}') = -\frac{1}{4\pi}\left[2ie^{i\phi}\nabla\phi.\nabla(|\underline{r}-\underline{r}'|^{-1}) + i|\underline{r}-\underline{r}'|^{-1}e^{i\phi}\nabla^2\phi - e^{i\phi}|\underline{r}-\underline{r}'|^{-1}|\nabla\phi|^2\right]$$

$$= -|\nabla\phi|^2 G + iG[\nabla^2\phi + |\underline{r}-\underline{r}'|^2\nabla\phi.\nabla(|\underline{r}-\underline{r}'|^{-2})]$$

$$= -|\nabla\phi|^2 G + i|\underline{r}-\underline{r}'|^2 G[|\underline{r}-\underline{r}'|^{-2}\nabla^2\phi + \nabla\phi.\nabla(|\underline{r}-\underline{r}'|^{-2})]$$

$$= -|\nabla\phi|^2 G + iG|\underline{r}-\underline{r}'|^2\nabla.[|\underline{r}-\underline{r}'|^{-2}\nabla\phi]$$

$$\therefore\ (\nabla^2 + k^2 - U)G = \delta(\underline{r}-\underline{r}') + (-|\nabla\phi|^2 + k^2 - U + i|\underline{r}-\underline{r}'|^2\nabla.[|\underline{r}-\underline{r}'|^{-2}\nabla\phi])G$$

Letting $\underline{\rho} = \underline{r}-\underline{r}'$,

$$\psi(\nabla^2 + k^2 - U)G = \delta(\underline{r}-\underline{r}')\psi(\underline{r}') + \psi(-|\nabla\phi|^2 + k^2 - U + i\rho^2\nabla.[\rho^{-2}\nabla\phi])G$$

Since ψ is a solution of the Schroedinger equation,

$$G(\nabla^2 + k^2 - U)\psi = 0$$

Assuming U is a scalar potential so that $UG = GU$, then

$$\psi\nabla^2 G - G\nabla^2\psi = \delta(\underline{r}-\underline{r}')\psi(\underline{r}') + (-|\nabla\phi|^2 + k^2 - U + i\rho^2\nabla.[\rho^{-2}\nabla\phi])G\psi$$

$$\therefore\ \psi(\underline{r}) = \int_{V'}[\psi\nabla^2 G - G\nabla^2\psi]\,d^3\underline{r}' - \int_{V'}(-|\nabla\phi|^2 + k^2 - U + i\rho^2\nabla.[\rho^{-2}\nabla\phi])G\psi\,d^3\underline{r}'$$

By Green's Theorem, $\int_{V'}[\psi\nabla^2 G - G\nabla^2\psi]\,d^3\underline{r}' = \int_{A'}(\psi\frac{\partial G}{\partial r'} - G\frac{\partial\psi}{\partial r'})\,dA'$
where $d\underline{A}'$ is an infinitesimal element of area normal to the surface enclosing total volume of space V', and of area A'.

$$\therefore\ \psi(\underline{r}) = \int_{A'}(\psi\frac{\partial G}{\partial r'} - G\frac{\partial\psi}{\partial r'})\,dA' - \int_{V'}(-|\nabla\phi|^2 + k^2 - U + i\rho^2\nabla.[\rho^{-2}\nabla\phi])G\psi\,d^3\underline{r}', \quad (2)$$

CLASSICAL APROXIMATION FOR PHASE SHIFT $\phi(\underline{r},\underline{r}')$

It is assumed that at high energies, the motion of the particle through the scattering potential can be approximated by a straight line (backward scattering is ignored). The momentum $\hbar\underline{k_0}$ of the incident particle is changed in the interaction region to $\hbar\underline{k}'$, where, in accordance with the Law of Conservation of Energy,

$$\frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 k'^2}{2m} + V(r'), \qquad k = |\underline{k_0}|$$

(Recoil of the target particle is ignored, i.e. it is assumed that the target is sufficiently massive for its recoil energy to be negligible compared with the kinetic energy of the incident particle.

$$\therefore\ k' = k(1 - \frac{U}{k^2})^{1/2}$$

Assuming $|\frac{U}{k^2}| << 1$, $\quad k' \simeq k - \frac{U}{2k}$

The phase of the scattered wave at $\underline{r}$ due to the particle interacting at

$\underline{r}'$ and moving along the assumed linear trajectory $\rho = \underline{r}-\underline{r}'$ is

$$\phi(\underline{r},\underline{r}') = \int \underline{k}'\cdot d\underline{s} = \int_0^\rho k' ds \simeq \int_0^\rho \left(k - \tfrac{U}{2k}\right) ds$$
$$= k\rho - \tfrac{1}{2k}\int_0^\rho U(\underline{r}'+\hat{\rho}s)\,ds \quad , \quad \hat{\rho} = \tfrac{1}{|\rho|}\underline{\rho}$$
$$\phi(\underline{r}',\underline{r}) = k\rho - \tfrac{1}{2k}\int_0^\rho U(\underline{r}-\hat{\rho}s)\,ds$$

Since $U(\underline{r}'+\hat{\rho}s) = U(\underline{r}-\hat{\rho}s)$,

$$\phi(\underline{r}',\underline{r}) = \phi(\underline{r},\underline{r}') = k\rho + \delta(\underline{r},\underline{r}') \quad (3)$$

where $\delta(\underline{r},\underline{r}') = -\frac{1}{2k}\int_0^\rho U(\underline{r}-\hat{\rho}s)ds = -\frac{1}{2k}\int_0^\rho U(\underline{r}'+\hat{\rho}s)ds$ (4)

is the phase shift due to scattering by the potential $U(\underline{r})$.

From equ.(1), $\frac{\partial G}{\partial r'} = -\frac{1}{4\pi}\left[i\rho^{-1}\frac{\partial\phi}{\partial r'} - \rho^{-2}\frac{\partial\rho}{\partial r'}\right] e^{i\phi}$

$$\rho = |\underline{r}-\underline{r}'| = (r^2 + r'^2 - 2\underline{r}\cdot\underline{r}')^{1/2}$$
$$\frac{\partial\rho}{\partial r'} = \rho^{-1}(r' - \underline{r}\cdot\hat{n})$$

From equ. (3), $\frac{\partial\phi}{\partial r'} = k\frac{\partial\rho}{\partial r'} + \frac{\partial\delta}{\partial r'} = k\rho^{-1}(r'-\underline{r}\cdot\hat{n}) - \frac{1}{2k}U(\underline{r}')\frac{\partial\rho}{\partial r'}$

$$= \left[k - \frac{U(r')}{2k}\right]\rho^{-1}(r'-\underline{r}\cdot\hat{n})$$

$$\therefore \frac{\partial G}{\partial r'} = \left[ik - \frac{iU}{2k} - \rho^{-1}\right] G\,\rho^{-1}(r'-\underline{r}\cdot\hat{n})$$

In equ. (2), $\psi(\underline{r}')$ is replaced by its Born approximation $e^{i\underline{k}_0\cdot\underline{r}'}$

$$\therefore \frac{\partial\psi}{\partial r'} = ik\psi$$

$$\therefore \psi\frac{\partial G}{\partial r'} - G\frac{\partial\psi}{\partial r'} = -G\psi\left[\rho^{-1} + \frac{iU}{2k}\right]\rho^{-1}(r'-\underline{r}\cdot\hat{n})$$

$$\rho \underset{r'\to\infty}{\sim} r' \quad , \quad U(\underline{r}') \underset{r'\to\infty}{\sim} 0$$

$$\therefore \psi\frac{\partial G}{\partial r'} - G\frac{\partial\psi}{\partial r'} \underset{r'\to\infty}{\sim} -G\psi r'^{-1}$$ provided $r'U(r')\to 0$ as $r'\to\infty$

Considering now the surface integral in equ.(2), since A' is arbitrary other than it should enclose no singularities (for Green's Theorem to be valid) it is chosen as an infinite sphere.

$$\therefore \int_{A'}\left(\psi\frac{\partial G}{\partial r'} - G\frac{\partial\psi}{\partial r'}\right)dA' = -\lim_{r'\to\infty} 4\pi r'^2 G\psi r'^{-1}$$

$$= \lim_{r'\to\infty} 4\pi r'\,\frac{e^{i(\phi(\underline{r},\underline{r}')+\underline{k}_0\cdot\underline{r}')}}{4\pi r'} = e^{i\lim_{r'\to\infty}(\phi(\underline{r},\underline{r}')+\underline{k}_0\cdot\underline{r}')}$$

Now $\rho = |\underline{r}-\underline{r}'| \underset{r'\to\infty}{\sim} r' - \underline{r}\cdot\hat{n}$

$$\therefore \underline{r}' = \underline{r}-\underline{\rho} = \underline{r} - \rho\hat{\rho} = \underline{r} - \hat{\rho}(r'-\underline{r}\cdot\hat{n})$$
$$\underset{r'\to\infty}{\sim} -\hat{\rho}r' \underset{r'\to\infty}{\sim} -\hat{k}_0 r'$$

And $\underline{k}_0\cdot\underline{r}' \underset{r'\to\infty}{\sim} -r'\underline{k}_0\cdot\hat{\rho} \underset{r'\to\infty}{\sim} -kr'$;

Since $\hat{\rho} \parallel \underline{k}_0$ approximately, for the assumed undeviated linear path of the scattered particle.

$$\therefore \int_{A'}\left(\psi\frac{\partial G}{\partial r'} - G\frac{\partial\psi}{\partial r'}\right)dA' = e^{i\lim_{r'\to\infty}(\phi(\underline{r},-\hat{k}_0 r') - kr')}$$

Substituting in equ. (2)

$$\psi(\underline{r}) = e^{i\lim_{r'\to\infty}(\phi(\underline{r},-\hat{k}_0 r')-kr')} - \int_{V'}\left(-|\nabla\phi|^2 + k^2 - U + i\rho^2\nabla\cdot[\rho^{-2}\nabla\phi]\right)G\psi\,d^3\underline{r}'$$

$$= e^{i[\underline{k}_o\cdot\underline{r} + \delta_o(\underline{r})]} - \int_{v'} [-|\nabla\phi|^2 + k^2 - U + i\rho^2\nabla\cdot(\rho^{-2}\nabla\phi)]\, G\psi\, d^3\underline{r}'$$

where $\delta_o(\underline{r}) = -\underline{k}_o\cdot\underline{r} + \lim_{r'\to\infty}[\phi(\underline{r}, -\hat{k}_o r') - kr']$

$$= -\underline{k}_o\cdot\underline{r} + \lim_{r'\to\infty}[k|\underline{r}+\hat{k}_o r'| - kr' + \delta(\underline{r}, -\hat{k}_o r')]$$, using equ. (3)

$$|\underline{r}+\hat{k}_o r'| = (r^2 + r'^2 + 2r'\hat{k}_o\cdot\underline{r})^{1/2} \underset{r'\to\infty}{\sim} r' + \hat{k}_o\cdot\underline{r}$$

$$\therefore \delta_o(\underline{r}) = -\underline{k}_o\cdot\underline{r} + kr' + \underline{k}_o\cdot\underline{r} - kr' + \lim_{r'\to\infty}\delta(\underline{r}, -\hat{k}_o r')$$

$$= \lim_{r'\to\infty}\delta(\underline{r}, -\hat{k}_o r') = -\frac{1}{2k}\int_0^\infty U(\underline{r}-\hat{k}_o s)\,ds \qquad \ldots\ (6)$$

From equ. (3), $\nabla\phi = k\nabla\rho + \nabla\delta$

$$\nabla\rho = \hat{n}\frac{\partial\rho}{\partial r'} = \rho^{-1}(r' - \underline{r}\cdot\hat{n})\hat{n}$$

$$\therefore |\nabla\phi|^2 = k^2|\nabla\rho|^2 + |\nabla\delta|^2 + 2k\nabla\rho\cdot\nabla\delta$$

$$= k^2 + |\nabla\delta|^2 + 2k\nabla\rho\cdot\nabla\delta$$

$$\nabla\delta = \hat{n}\frac{\partial\delta}{\partial r'} = -\frac{1}{2k\rho}U(\underline{r}')(r' - \underline{r}\cdot\hat{n})\hat{n}$$

$$\therefore \nabla\delta\cdot\nabla\rho = \frac{-1}{2k\rho^2}U(\underline{r}')(r' - \underline{r}\cdot\hat{n})^2 = -\frac{U(\underline{r}')}{2k}$$

$$\therefore |\nabla\phi|^2 = k^2 + |\nabla\delta|^2 - U$$

$$\nabla\cdot(\rho^{-2}\nabla\phi) = \rho^{-2}\nabla^2\phi + \nabla\phi\cdot\nabla(\rho^{-2})$$

$$= \rho^{-2}\nabla^2(k\rho + \delta) - 2\rho^{-3}\nabla\rho\cdot\nabla(k\rho + \delta)$$

$$\therefore \rho^2\nabla\cdot(\rho^{-2}\nabla\phi) = k\nabla^2\rho + \nabla^2\delta - 2\rho^{-1}(k\nabla\rho + \nabla\delta)\cdot\nabla\rho$$

$$\nabla^2\rho = \nabla\cdot(\rho^{-1}(r' - \underline{r}\cdot\hat{n})\hat{n}) = \rho^{-1}(r' - \underline{r}\cdot\hat{n})\nabla\cdot\hat{n} + \hat{n}\cdot\nabla(\rho^{-1}(r' - \underline{r}\cdot\hat{n}))$$

Noting that $\nabla\cdot\hat{n} = 2r'^{-1}$

$$\nabla^2\rho = 2(\rho r')^{-1}(r' - \underline{r}\cdot\hat{n}) + \hat{n}\cdot\nabla(\rho^{-1}(r' - \underline{r}\cdot\hat{n}))$$

$$\nabla(\rho^{-1}(r' - \underline{r}\cdot\hat{n})) = \rho^{-1}\hat{n} - \rho^{-3}(r' - \underline{r}\cdot\hat{n})^2\hat{n}$$

$$\therefore \nabla^2\rho = 2(\rho r')^{-1}(r' - \underline{r}\cdot\hat{n}) + \rho^{-1} - \rho^{-3}(r' - \hat{n}\cdot\underline{r})^2$$

$$|\nabla\rho|^2 = \rho^{-2}(r' - \underline{r}\cdot\hat{n})^2$$

Substituting in expression above for $\rho^2\nabla\cdot(\rho^{-2}\nabla\phi)$,

$$\rho^2\nabla\cdot(\rho^{-2}\nabla\phi) = 2k(\rho r')^{-1}(r' - \underline{r}\cdot\hat{n}) + k\rho^{-1} - k\rho^{-3}(r' - \underline{r}\cdot\hat{n})^2 + \nabla^2\delta$$
$$- 2k\rho^{-3}(r' - \underline{r}\cdot\hat{n})^2 + \frac{U}{k}\rho^{-3}(r' - \underline{r}\cdot\hat{n})^2$$

$$= \nabla^2\delta + (k\rho)^{-1}U$$

Substituting in expression above for $\psi(\underline{r})$,

$$\psi(\underline{r}) = e^{i[\underline{k}_o\cdot\underline{r} + \delta_o(\underline{r})]} - \int_{v'}[-|\nabla\delta|^2 + i(\nabla^2\delta + (k\rho)^{-1}U)]\,G\psi\,d^3\underline{r}'$$

Now $G = -(4\pi\rho)^{-1}e^{i\phi} = -(4\pi\rho)^{-1}e^{i[k\rho + \delta]}$

$$= G_o e^{i\delta}$$

$$\nabla^2 e^{i\delta} = i\nabla^2\delta\, e^{i\delta} - |\nabla\delta|^2 e^{i\delta}$$

Substituting for $|\nabla\delta|^2$ in expression above for $\psi(\underline{r})$

$$\psi(\underline{r}) = e^{i(\underline{k}_0\cdot\underline{r}+\delta_0(\underline{r}))} - \int_{V'}[\nabla^2 e^{i\delta} + i(k\rho)^{-1} U e^{i\delta}]\, G_0 \psi\, d^3\underline{r}'$$

$$\rho \underset{r\to\infty}{\sim} r - \underline{r}'\cdot\hat{n} + O(r^{-1})$$

$$\underset{r\to\infty}{\sim} r - \underline{r}'\cdot\hat{k}' + O(r^{-1})$$

$$\therefore \psi(\underline{r}) \underset{r\to\infty}{\sim} e^{i[\underline{k}_0\cdot\underline{r}+\delta_0(\underline{r})]} + \int_{V'}[\nabla^2 e^{i\delta} + i[k(r-\underline{r}'\cdot\hat{k}')]^{-1} U e^{i\delta}] \times$$

$$[4\pi(r-\underline{r}'\cdot\hat{k}')]^{-1}\psi\, e^{ik(r-\underline{r}'\cdot\hat{k}')}\, d^3\underline{r}' + O(r^{-2})$$

$$\underset{r\to\infty}{\sim} e^{i[\underline{k}_0\cdot\underline{r}+\delta_0(\underline{r})]} + \frac{e^{ikr}}{4\pi}\int_{V'}(r-\underline{r}'\cdot\hat{k}')^{-1} e^{-i\underline{k}'\cdot\underline{r}'}\psi(\underline{r}')\nabla^2[e^{i\delta_a(\underline{r}')}]\, d^3\underline{r}' + O(r^{-2})$$

where
$$\delta_a(\underline{r}') = \lim_{r\to\infty}\delta(\underline{r},\underline{r}')$$

$$= \lim_{r\to\infty}\left[-\frac{1}{2k}\int_0^{\rho} U(\underline{r}'+\hat{\rho}s)\,ds\right]$$

$$= -\frac{1}{2k}\int_0^{\infty} U(\underline{r}'+\hat{k}'s)\,ds \qquad (7)$$

since $\hat{\rho}\to\hat{k}'$ as $r\to\infty$

$$\therefore \psi(\underline{r}) \underset{r\to\infty}{\sim} e^{i[\underline{k}_0\cdot\underline{r}+\delta_0(\underline{r})]} + r^{-1} f(\underline{k}', \underline{k}_0)\, e^{ikr}$$

where
$$f(\underline{k}', \underline{k}_0) = \frac{1}{4\pi}\lim_{r\to\infty} r\int_{V'}(r-\underline{r}'\cdot\hat{k}')^{-1} e^{-i\underline{k}'\cdot\underline{r}'}\psi(\underline{r}')\nabla^2[e^{i\delta_a(\underline{r}')}]\, d^3\underline{r}', \qquad (8)$$

SECTION B

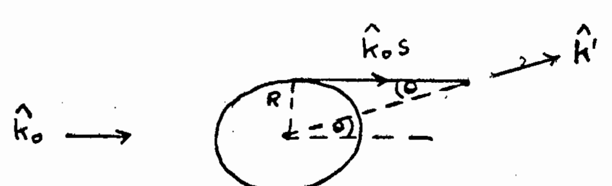
A simple proof is given that (8) above defines the scattering amplitude i.e. that $f(\underline{k}', \underline{k}_0)$ obeys the optical theorem.

PROOF

The scattering potential $U(\underline{r})$ is assumed spherically symmetric and of range R. From equ. (6)

$$\delta_0(\underline{r}) = -\frac{1}{2k}\int_0^{\infty} U(\underline{r}-\hat{k}_0 s)\,ds$$

When point $\underline{r}-\hat{k}_0 s$, at which scattering takes place, lies outside range of potential - $U(\underline{r}-\hat{k}_0 s) \simeq 0$. This occurs for scattering angles $\theta > \theta_0$



where $\sin\theta_0 = \frac{R}{r}$, $\sim\theta_0$, since $R << r$

$$\therefore \delta_0(\underline{r}) \simeq 0 \quad \text{and} \quad \frac{\partial\delta_0}{\partial r} \simeq 0, \quad \text{Except for } \theta \lesssim \frac{R}{r}$$

The scattered particle flux density $J(\underline{r}) = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial\psi}{\partial r} - \psi \frac{\partial \psi^*}{\partial r}\right)$
$= j + j^*$, where $j = \frac{\hbar}{2mi}\psi^* \frac{\partial \psi}{\partial r}$

using (8), $j \underset{r\to\infty}{\sim} \frac{\hbar}{2mi}\left[e^{-i[kr\mu + \delta_0]} + r^{-1} f^* e^{-ikr}\right]\left[ik\mu\, e^{i[kr\mu+\delta_0]} + ikr^{-1} f e^{ikr}\right]$

where $\mu = |\underline{k_0}\cdot\underline{r}|^{-1}\, \underline{k_0}\cdot\underline{r}$

The scattered intensity in cone of semi solid angle $d\Omega$ is

$$dI_{scat} = I_{inc}\, d\sigma = I_{inc}\, |f|^2 d\Omega = I_{inc}\, |f|^2 \frac{dA}{r^2}$$

$$J(r) = v\,|\psi_{scat}(r)|^2 = v\,|f|^2/r^2$$

$$\therefore \frac{1}{I_{inc}} \frac{dI_{scat}}{d\Omega} = \frac{r^2 J}{v} = \frac{r^2 J}{\frac{\hbar k}{m}} = \frac{m r^2 J}{\hbar k}$$

$$\underset{r\to\infty}{\sim} \frac{r^2}{2}\left[r^{-1} f e^{i[kr(1-\mu)-\delta_0]} + \mu r^{-1} e^{-i[kr(1-\mu)-\delta_0]} f^* + r^{-2}|f|^2 + \mu\right] + \text{complex conjugate}$$

$$= r^2\mu + |f|^2 + \left[\frac{r}{2}(1+\mu) f e^{i[kr(1-\mu)-\delta_0]} + c.c.\right]$$

Since total no. of particles is conserved,

$$\int_\Omega \frac{1}{I_{inc}} \frac{dI_{scat}}{d\Omega}\, d\Omega = \frac{m}{\hbar k}\int_A J\, dA = 0$$

Noting $d\Omega = -2\pi d\mu$

then $\int_\Omega |f|^2 d\Omega = -\frac{1}{2}\int_\Omega \left[r(1+\mu) f e^{i[kr(1-\mu)-\delta_0]} + c.c.\right] d\Omega$

$$= \int \frac{d\sigma}{d\Omega}\, d\Omega = \sigma_T = \pi \int_1^{-1} r\left[(1+\mu) f e^{i[kr(1-\mu)-\delta_0]} + c.c.\right] d\mu$$

$$= \pi r\left\{\int_1^{-1} f(1+\mu) e^{ikr(1-\mu)} d\mu + \int_1^{-1} f(1+\mu) e^{ikr(1-\mu)}(e^{-i\delta_0} - 1)\, d\mu + c.c.\right\}$$

the 1st integral becomes

$$-(ikr)^{-1}\left| f(1+\mu) e^{ikr(1-\mu)}\right|_1^{-1} - (ikr)^{-2}\left|\frac{\partial}{\partial\mu}[(1+\mu)f]\, e^{ikr(1-\mu)}\right|_{-1}^{1} + O(r^{-3})$$

$$= 2(ikr)^{-1} f(\underline{k_0}, \underline{k_0}) + O(r^{-2}) + c.c.$$

$\delta_0(\underline{r}) \simeq 0$ for $\theta \gtrsim R/r$, and for $\theta < R/r$

$\delta_0(\underline{r}) = \frac{-1}{2k}\int_0^\infty U(\underline{r} - \hat{k}_0 s)\, ds \lesssim -\frac{U}{2k} R(1-\cos\theta)$ where

$\cos\theta \simeq 1 - \frac{\theta^2}{2} \lesssim 1 - \frac{1}{2}\left(\frac{R}{r}\right)^2$ so that

$\delta_0(r) \lesssim \frac{UR}{4k}\left(\frac{R}{r}\right)^2$ and $|e^{i\delta_0} - 1| \simeq \delta_0$. Therefore the 2nd Integral is 1st integral $\times\ O\left(\frac{R}{r}\right)^2$ and is negligible in limit $r\to\infty$.

$$\therefore\ \sigma_T = \lim_{r\to\infty} \pi r\left[2(ikr)^{-1} f(\underline{k_0}, \underline{k_0}) + c.c.\right]$$

$$= \frac{2\pi}{k}\left[-i f(\underline{k_0}, \underline{k_0}) + i f^*(\underline{k_0}, \underline{k_0})\right] = \frac{4\pi}{k}\, \Im f(\underline{k_0}, \underline{k_0})$$

Q.E.D.

SECTION C

From equ. (8), $f(\underline{k}', \underline{k_0}) = \frac{1}{4\pi}\lim_{r\to\infty}\left[\int \left(1 - \frac{\underline{r}'\cdot\hat{\underline{k}}'}{r}\right)^{-1} e^{-i\underline{k}'\cdot\underline{r}'}\, \psi(\underline{r}')\, \nabla^2[e^{i\delta_0}]\, d^3\underline{r}'\right]$

Making variable change $\underline{r} \to \underline{r_0}$, $\underline{r}' \to \underline{r}$, and choosing z axis

Along direction $-\hat{k}'$, $\quad f(\underline{k}',\underline{k}_0) = \frac{1}{4\pi}\lim_{r_0\to\infty}\left[\int \left(1+\frac{z}{r_0}\right)^{-1} e^{ikz}\,\psi(\underline{r})\nabla^2[e^{i\delta_a}]\,d^3\underline{r}\right]$

$$= \frac{1}{4\pi}\lim_{r_0\to\infty}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}dz\left(1+\frac{z}{r_0}\right)^{-1}e^{ikz}\,\psi\,\nabla^2[e^{i\delta_a}].$$

$$= \frac{1}{4\pi}\lim_{r_0\to\infty}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\Big[\,\Big| -\nabla^2[e^{i\delta_a}]\int_z^{\infty}dz'\left(1+\frac{z'}{r_0}\right)^{-1}e^{ikz'}\psi\Big|_{-\infty}^{\infty}$$

$$+\int_{-\infty}^{\infty}\left\{\frac{d}{dz}[\nabla^2(e^{i\delta_a})]\,k_z\int_z^{\infty}\left(1+\frac{z'}{r_0}\right)^{-1}e^{ikz'}\psi\,dz'\right\}\Big]$$

$$= \frac{1}{4\pi}\lim_{r_0\to\infty}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\Big[\,\Big| -\nabla^2[e^{i\delta_a}]\int_z^{\infty}dz'\left(1+\frac{z'}{r_0}\right)^{-1}e^{ikz'}\psi\Big|_{-\infty}^{\infty}$$

$$-\int_{-\infty}^{\infty}\left\{\frac{i}{2k}\nabla^2(Ue^{i\delta_a})\int_z^{\infty}\left(1+\frac{z'}{r_0}\right)^{-1}e^{ikz'}\psi\,dz'\right\}\Big]$$, using equ. (7)

$\delta_a(\underline{r}) = 0 \quad$ at $\quad z=-\infty$

$\therefore\ \nabla^2(e^{i\delta_a})\big|_{z=-\infty} = 0$

$$\therefore\ f(\underline{k}',\underline{k}_0) = \frac{-i}{4\pi}\lim_{r_0\to\infty}(2k)^{-1}\int\nabla^2(Ue^{i\delta_a})\int_z^{\infty}\left(1+\frac{z'}{r_0}\right)^{-1}e^{ikz'}\psi\,dz'\,d^3\underline{r}$$

$$= -\frac{1}{4\pi}\left(\frac{i}{2k}\right)\int U(\underline{r})e^{i\delta_a}\lim_{r_0\to\infty}\left[\nabla^2\int_z^{\infty}\left(1+\frac{z'}{r_0}\right)^{-1}e^{ikz'}\psi\,dz'\right]d^3\underline{r}$$

Now $\psi(\underline{r}) \underset{r\to\infty}{\sim} e^{i\underline{k}_0\cdot\underline{r}} + \psi_{scat}(\underline{r}) = e^{i[\underline{k}_0\cdot\underline{r}+\delta_0]} + r^{-1}f(\underline{k}',\underline{k}_0)e^{ikr}$

$$\therefore\ \psi_{scat}(\underline{r}) = e^{i\underline{k}_0\cdot\underline{r}}(e^{i\delta_0}-1) + r^{-1}f(\underline{k}',\underline{k}_0)e^{ikr} \qquad \ldots\ldots(9)$$

$\therefore$ Substituting for ψ and noting $\underline{k}'\cdot\underline{r} = -kz$

$$f(\underline{k}',\underline{k}_0) = \frac{-i}{8\pi k}\int U(r)e^{i\delta_a}\lim_{r_0\to\infty}\left[\nabla^2\int_z^{\infty}\left(1+\frac{z'}{r_0}\right)^{-1}\left(e^{i(\underline{k}_0-\underline{k}')\cdot\underline{r}'} + \psi_{scat}e^{ikz'}\right)dz'\right]d^3\underline{r}$$

$\ldots\ldots (10)$

$\underline{k}_0-\underline{k}' = \underline{q}$, the momentum transfer

$$\int_z^{\infty}\left(1+\frac{z'}{r_0}\right)^{-1}e^{i\underline{q}\cdot\underline{r}'}dz' = \frac{i}{q_z}\left(1+\frac{z}{r_0}\right)^{-1}e^{i\underline{q}\cdot\underline{r}} - \frac{i}{q_z r_0}\int_z^{\infty}\left(1+\frac{z'}{r_0}\right)^{-2}e^{i\underline{q}\cdot\underline{r}'}dz'$$

$$\therefore\ \lim_{r_0\to\infty}\nabla^2\int_z^{\infty}\left(1+\frac{z'}{r_0}\right)^{-1}e^{i\underline{q}\cdot\underline{r}'}dz' = \lim_{r_0\to\infty}\left[\frac{i}{q_z}\nabla^2\left(1+\frac{z}{r_0}\right)^{-1}e^{i\underline{q}\cdot\underline{r}}\right] = -\frac{iq^2}{q_z}e^{i\underline{q}\cdot\underline{r}}$$

$$= -2ik\,e^{i\underline{q}\cdot\underline{r}} \quad ?$$

Substituting in (10), $f(\underline{k}',\underline{k}_0) = -\frac{1}{4\pi}\int U(\underline{r})e^{i(\underline{q}\cdot\underline{r}+\delta_a)}d^3\underline{r} - \frac{i}{8\pi k}\int Ue^{i\delta_a}\nabla^2\int_z^{\infty}\psi_{scat}e^{ikz'}dz'd^3\underline{r}$

letting $\underline{k}_0\to-\underline{k}'$, $\underline{k}'\to-\underline{k}_0$, then from (6), (7), $\delta_0\to\delta_a$ and $\delta_a\to\delta_0$.

$$\therefore\ f(-\underline{k}_0,-\underline{k}') = -\frac{1}{4\pi}\int Ue^{i(\underline{q}\cdot\underline{r}+\delta_0)}d^3\underline{r} - \frac{i}{8\pi k}\int Ue^{i\delta_0}\nabla^2\int_z^{\infty}\psi^{(-)}_{scat}e^{ikz'}dz'\,d^3\underline{r}$$

where, from equ. (9), $\psi^{(-)}_{scat} = e^{-i\underline{k}'\cdot\underline{r}'}(e^{i\delta_a}-1) + r'^{-1}f(-\underline{k}_0,-\underline{k}')e^{ikr'}$

$\simeq e^{-i\underline{k}'\cdot\underline{r}'}(e^{i\delta_a}-1)$ (Distorted plane wave Born approximation). Reversibility symmetry of the scattering amplitude gives

$$f(\underline{k}',\underline{k}_0) = f(-\underline{k}_0,-\underline{k}')$$

$$\therefore\ f(\underline{k}',\underline{k}_0) \simeq -\frac{1}{4\pi}\int Ue^{i(\underline{q}\cdot\underline{r}+\delta_0)}d^3\underline{r} - \frac{i}{8\pi k}\int Ue^{i\delta_0}\nabla^2\int_z^{\infty}e^{i\underline{q}\cdot\underline{r}'}(e^{i\delta_a}-1)\,dz'\,d^3\underline{r} \quad \ldots\ldots(11)$$

$$\int_z^\infty e^{i\underline{q}\cdot\underline{r}'}(e^{i\delta_a}-1)\,dz' = (iq_z)^{-1}\left| e^{i\underline{q}\cdot\underline{r}'}(e^{i\delta_a}-1)\right|_z^\infty - \int_z^\infty q_z^{-1}\frac{\partial\delta_a}{\partial z'}e^{i(\underline{q}\cdot\underline{r}'+\delta_a)}\,dz'$$

$$= i q_z^{-1} e^{i\underline{q}\cdot\underline{r}}(e^{i\delta_a}-1) - (iq_z^2)^{-1}\left|\frac{\partial\delta_a}{\partial z'}e^{i(\underline{q}\cdot\underline{r}'+\delta_a)}\right|_z^\infty + \ldots\ldots$$

$$= iq_z^{-1}e^{i\underline{q}\cdot\underline{r}}\left[e^{i\delta_a}-1-q_z^{-1}\frac{\partial\delta_a}{\partial z}e^{i\delta_a}+\ldots\ldots\right] \qquad \ldots.(11)$$

In APPENDIX A, it is shown that $|\delta_a| \lesssim \frac{|U_0|R}{k}$, where $|U_0|$ is maximum value of $U(r)$, and that $\left|\frac{\partial\delta_a}{\partial z}\right| \lesssim \frac{|U_0|}{k}$

$$e^{i\delta_a}-1 = i\delta_a - \frac{\delta_a^2}{2}+\ldots\ldots \lesssim \frac{iU_0R}{k}\left(1-\frac{iU_0R}{2k}+\ldots.\right)$$

$$q_z^{-1}\frac{\partial\delta_a}{\partial z}e^{i\delta_a} \simeq \frac{-U_0}{2k^2\sin^2\theta/2}\left(1-\frac{iU_0R}{k}+\ldots.\right)$$

$\therefore$ Provided $\frac{|U_0|R}{k} << 1$, 2nd term is negligible if,

$$kR\sin^2\tfrac{\theta}{2} >> 1 \qquad \text{i.e.}\quad \sin\tfrac{\theta}{2} >> \frac{1}{\sqrt{kR}}$$

$$\therefore f(\underline{k}',\underline{k}_0) \simeq -\frac{1}{4\pi}\int U e^{i(\underline{q}\cdot\underline{r}+\delta_0)}d^3\underline{r} - \frac{i}{8\pi k}\int U e^{i\delta_0}\nabla^2[iq_z^{-1}e^{i\underline{q}\cdot\underline{r}}(e^{i\delta_a}-1)]\,d^3\underline{r}$$

$$\nabla^2[e^{i\underline{q}\cdot\underline{r}}(e^{i\delta_a}-1)] = e^{i\underline{q}\cdot\underline{r}}[-q^2(e^{i\delta_a}-1)-2e^{i\delta_a}\underline{q}\cdot\nabla\delta_a + ie^{i\delta_a}\nabla^2\delta_a - e^{i\delta_a}|\nabla\delta_a|^2]$$

$$\underset{r\to\infty}{\sim} -q^2e^{i\underline{q}\cdot\underline{r}}(e^{i\delta_a}-1) \qquad \text{since } \delta_a\to 0 \text{ as } r\to\infty$$

$$\therefore f(\underline{k}',\underline{k}_0) = -\frac{1}{4\pi}\int U e^{i(\underline{q}\cdot\underline{r}+\delta_0)}d^3\underline{r} - \frac{1}{4\pi}\int U e^{i(\underline{q}\cdot\underline{r}+\delta_0)}(e^{i\delta_a}-1)\,d^3\underline{r}$$

$$= -\frac{1}{4\pi}\int U(r)\,e^{i(\underline{q}\cdot\underline{r}+\delta_0+\delta_a)}d^3\underline{r} \qquad \ldots \quad (12)$$

Provided $\frac{|U_0|R}{k} << 1$, $\sin\frac{\theta}{2} >> \frac{1}{\sqrt{kR}}$

This is a D.W.B.A. for the large angle scattering amplitude with the assumption of linear trajectories as zeroth approximation. Returning to equ. (11), the second term, which is the dominant first term of a series, is greater than the first term if $kR\sin^2\frac{\theta}{2} << 1$. Thus for small scattering angles,

$$f(\underline{k}',\underline{k}_0) = -\frac{1}{4\pi}\int U e^{i(\underline{q}\cdot\underline{r}+\delta_0)}d^3\underline{r} - (8\pi k q_z^2)^{-1}\int U e^{i\delta_0}\nabla^2\left[\frac{\partial\delta_a}{\partial z}e^{i(\underline{q}\cdot\underline{r}+\delta_a)}\right]d^3\underline{r}$$

$$\nabla^2\left[\frac{\partial\delta_a}{\partial z}e^{i(\underline{q}\cdot\underline{r}+\delta_a)}\right] = e^{i(\underline{q}\cdot\underline{r}+\delta_a)}\Big[\nabla^2\frac{\partial\delta_a}{\partial z} + 2i\underline{q}\cdot\nabla\left(\frac{\partial\delta_a}{\partial z}\right) + 2i\nabla\left(\frac{\partial\delta_a}{\partial z}\right)\cdot\nabla\delta_a + i\nabla^2(\delta_a)\frac{\partial\delta_a}{\partial z} - q^2\frac{\partial\delta_a}{\partial z} - 2\frac{\partial\delta_a}{\partial z}\underline{q}\cdot\nabla\delta_a - \frac{\partial\delta_a}{\partial z}|\nabla\delta_a|^2\Big]$$

$$\nabla^2\frac{\partial\delta_a}{\partial z} \sim \frac{-U_0}{kR^2}$$

$$\underline{q}\cdot\nabla\left(\frac{\partial\delta_a}{\partial z}\right) \sim \frac{U_0 q}{kR} \sim \frac{U_0\sin\frac{\theta}{2}}{R}$$

$$\nabla\left(\frac{\partial\delta_a}{\partial z}\right)\cdot\nabla\delta_a \sim \frac{U_0^2}{k^2R}$$

$$\frac{\partial\delta_a}{\partial z}\nabla^2\delta_a \sim \frac{U_0^2}{k^2R}$$

$$q^2\frac{\partial\delta_a}{\partial z} \sim -U_0 k\sin^2\tfrac{\theta}{2}$$

$$\frac{\partial\delta_a}{\partial z}\underline{q}\cdot\nabla\delta_a \sim \frac{U_0^2\sin\theta/2}{kR}$$

$$\frac{\partial\delta_a}{\partial z}|\nabla\delta_a|^2 \sim -\left(\frac{U_0}{k}\right)^3$$

The dominant term is $\nabla^2\frac{\partial\delta_a}{\partial z}$ provided $\frac{|U_0|R}{k} << 1$, $\sin\frac{\theta}{2} << \frac{1}{kR}$

$$\therefore f(\underline{k}',\underline{k}_0) = -\frac{1}{4\pi}\int U e^{i(\underline{q}\cdot\underline{r}+\delta_0)}d^3\underline{r} - (8\pi k q_z^2)^{-1}\int U e^{i(\underline{q}\cdot\underline{r}+\delta_0+\delta_a)}\nabla^2\left(\frac{\partial\delta_a}{\partial z}\right)d^3\underline{r}$$

$$= \frac{k}{2\pi}\int \mathcal{K}(\underline{r})\,e^{i(\underline{q}\cdot\underline{r}+\delta_0)}\left(1+(2kq_z^2)^{-1}e^{i\delta_a}\nabla^2\left(\frac{\partial\delta_a}{\partial z}\right)\right)d^3\underline{r}$$

where $\mathcal{K}(r) = -\frac{U(r)}{2k}$

The second term in the brackets is $\sim \frac{U_0 R}{k} \cdot \frac{1}{k^3 R^3 \sin^4 \theta/2} \ll 1$
provided $\sin \theta/2 \gg \frac{(U_0 R^2)^{1/4}}{kR}$ i.e. $1 \gg U_0 R^2$

$$\therefore \quad f(\underline{k}', \underline{k}_0) \simeq \frac{k}{2\pi} \int \mathcal{K}(\underline{r})\, e^{i(\underline{q}\cdot\underline{r} + \delta_0)}\, d^3\underline{r} \tag{13}$$

$\underline{r} = \underline{b} + \hat{k}_0 z$, where $\underline{b}$ is the impact parameter vector lying in the plane normal to incident momentum $\hat{k}_0$.

$$\underline{q}\cdot\underline{r} = \underline{q}\cdot\underline{b} + z\hat{k}_0\cdot\underline{q} = \underline{q}\cdot\underline{b} + z\hat{k}_0\cdot(\underline{k}_0 - \underline{k}')$$
$$= \underline{q}\cdot\underline{b} + zk(1-\cos\theta) = \underline{q}\cdot\underline{b} + 2zk\sin^2\theta/2$$

$\therefore \underline{q}\cdot\underline{r} \simeq \underline{q}\cdot\underline{b}$ provided $zk\sin^2\theta/2 \ll \underline{q}\cdot\underline{b} \leq bk\sin\theta/2$

i.e. $\sin\frac{\theta}{2} \ll 1$

Using equ. (6), $f(\underline{k}', \underline{k}_0) = \frac{k}{2\pi}\int \mathcal{K}(r)\, e^{i(\underline{q}\cdot\underline{b} + \int_0^\infty \mathcal{K}(\underline{r} - \hat{k}_0 s)\,ds)}\, d^3\underline{r}$

$\underline{r} - \hat{k}_0 s = \underline{b} + \hat{k}_0(z-s) = \underline{b} + \hat{k}_0 z'$, where $z' = z - s$

$$\therefore f(\underline{k}', \underline{k}_0) = \frac{k}{2\pi}\int \mathcal{K}(b,z)\, e^{i(\underline{q}\cdot\underline{b} - \int_z^{-\infty} \mathcal{K}(b,z')\,dz')}\, d^2\underline{b}\, dz$$

$$= \frac{k}{2\pi}\int e^{i\underline{q}\cdot\underline{b}}\, d^2\underline{b} \int_{-\infty}^{\infty} \mathcal{K}(b,z)\, e^{i\int_{-\infty}^{z}\mathcal{K}(b,z')\,dz'}\, dz$$

$$= \frac{k}{2\pi i}\int e^{i\underline{q}\cdot\underline{b}}\, d^2\underline{b} \int_{-\infty}^{\infty} \frac{d}{dz}\left[e^{i\int_{-\infty}^{z}\mathcal{K}(b,z')\,dz'}\right] dz$$

$$= \frac{ik}{2\pi}\int e^{i\underline{q}\cdot\underline{b}}\left(1 - e^{i\int_{-\infty}^{\infty}\mathcal{K}(b,z')\,dz'}\right) d^2\underline{b}$$

$$= \frac{ik}{2\pi}\int e^{i\underline{q}\cdot\underline{b}}\left(1 - e^{i\chi(\underline{b})}\right) d^2\underline{b}$$

where $\chi(\underline{b}) = \int_{-\infty}^{\infty} \mathcal{K}(b,z)\,dz = -\frac{1}{2k}\int_{-\infty}^{\infty} U(b,z)\,dz = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} V(b,z)\,dz$

This is the familiar Glauber small angle approximation[6] which is valid under the conditions
1. $\frac{|U_0|R}{k} \ll 1 \ll kR$
2. $(|U_0|R^2)^{1/4} \ll kR\sin\frac{\theta}{2} \ll 1$

From (12) and (13), the scattering amplitude can be written

$$f(\underline{k}', \underline{k}_0) = \frac{k}{2\pi}\int \mathcal{K}(\underline{r})\, e^{i(\underline{q}\cdot\underline{r} + \delta_0(\underline{r}) + \alpha(\theta)\delta_a(\underline{r}))}\, d^3\underline{r} \tag{14}$$

where $\alpha(\theta) = 1$ for $\sin\frac{\theta}{2} \gg \frac{1}{\sqrt{kR}}$
$= 0$ for $\sin\frac{\theta}{2} \ll \frac{1}{kR}$

Repeating the above analysis for the general case,

$$f(\underline{k}', \underline{k}_0, \alpha) = \frac{ik}{2\pi}(1-\alpha)^{-1}\int e^{i\underline{q}\cdot\underline{b}}\left(e^{i\alpha\chi(b)} - e^{i\chi(b)}\right) d^2\underline{b} \tag{15}$$

In the impact parameter representation, the large angle scattering amplitude is

$$\lim_{\alpha\to 1} f(\underline{k}', \underline{k}_0, \alpha) = \frac{k}{2\pi}\int e^{i\underline{q}\cdot\underline{b}}\, \chi(b)\, e^{i\chi(b)}\, d^2\underline{b} \tag{16}$$

SECTION D

In this section, the error associated with the Glauber small angle approximation – $\underline{q}\cdot\underline{r} \simeq \underline{q}\cdot\underline{b}$ – is roughly estimated and its energy and dynamical dependance investigated.

From equ. (14), $f(\underline{k}', \underline{k}_0) \equiv f(k, \theta) = \frac{k}{2\pi}\int \mathcal{K}(b,z) e^{i[\underline{q}\cdot\underline{b} + 2kz\sin^2\frac{\theta}{2} + \delta_0 + \alpha\delta_a]} d^2\underline{b}\, dz$

$$= \frac{k}{2\pi}\int \mathcal{K}(b,z)\, e^{i[\underline{q}\cdot\underline{b} + 2kz\sin^2\frac{\theta}{2} + \int_{-\infty}^{z}\mathcal{K}(b,z')dz' + \alpha\int_z^{\infty}\mathcal{K}(b,z')dz']} d^2\underline{b}\, dz .$$

$$\frac{d}{dz}\left[\int_{-\infty}^{z}\mathcal{K}(b,z')dz' + \alpha\int_z^{\infty}\mathcal{K}(b,z')dz'\right] = (1-\alpha)\,\mathcal{K}(b,z)$$

$$\therefore\; f(k,\theta) = \frac{k}{2\pi i}(1-\alpha)^{-1}\int e^{i\underline{q}\cdot\underline{b}} d^2\underline{b}\int_{-\infty}^{\infty} e^{2kzi\sin^2\frac{\theta}{2}}\frac{d}{dz}\left[e^{i[\int_{-\infty}^{z}\mathcal{K}dz' + \alpha\int_z^{\infty}\mathcal{K}dz']}\right]dz$$

$$= \frac{k}{2\pi i}(1-\alpha)^{-1}\int e^{i\underline{q}\cdot\underline{b}} d^2\underline{b}\left[\left. e^{i2kz\sin^2\frac{\theta}{2} + i(\int_{-\infty}^{z}\mathcal{K}dz' + \alpha\int_z^{\infty}\mathcal{K}dz')}\right|_{-\infty}^{\infty}\right.$$

$$\left. - 2ki\sin^2\frac{\theta}{2}\int_{-\infty}^{\infty} e^{i[2kz\sin^2\frac{\theta}{2} + \int_{-\infty}^{z}\mathcal{K}dz' + \alpha\int_z^{\infty}\mathcal{K}dz']} dz\right]$$

$$= \frac{k}{2\pi i}(1-\alpha)^{-1}\int e^{i\underline{q}\cdot\underline{b}} d^2\underline{b}\left[e^{i\chi}\lim_{N\to\infty} e^{iNx} - e^{i\alpha\chi}\lim_{N\to\infty} e^{-iNx}\right.$$

$$\left. - 2ki\sin^2\frac{\theta}{2}\int_{-\infty}^{\infty} e^{i[2kz\sin^2\frac{\theta}{2} + \int_{-\infty}^{z}\mathcal{K}dz' + \alpha\int_z^{\infty}\mathcal{K}dz']} dz\right]$$

where $x = 2k\sin^2\frac{\theta}{2}$

$$e^{i\chi}\lim_{N\to\infty} e^{iNx} - e^{i\alpha\chi}\lim_{N\to\infty} e^{-iNx} = x\left[(e^{i\chi} - e^{i\alpha\chi})\lim_{N\to\infty}\frac{\cos Nx}{x}\right.$$

$$\left. + i(e^{i\chi} + e^{i\alpha\chi})\lim_{N\to\infty}\frac{\sin Nx}{x}\right]$$

In APPENDIX B, it is shown that $\lim_{N\to\infty}\frac{\cos Nx}{x} = -\lim_{N\to\infty}\frac{\sin Nx}{x} = \pi\,\delta(x)$ $x \neq 0$, where $\delta(x)$ is Dirac Delta Function: $\delta(x) = 0$, $x \neq 0$

$$\therefore\; f(k,\theta) = -\frac{(k\sin\frac{\theta}{2})^2}{(1-\alpha)\pi}\int e^{i\underline{q}\cdot\underline{b}} d^2\underline{b}\int_{-\infty}^{\infty} e^{i[xz + \int_{-\infty}^{z}\mathcal{K}dz' + \alpha\int_z^{\infty}\mathcal{K}dz']} dz \qquad (17)$$

$$\int_{-\infty}^{z}\mathcal{K}dz' = \int_{-\infty}^{0}\mathcal{K}dz' + \int_0^z\mathcal{K}dz'$$

Assuming scattering potential is spherically symmetric, then

$$\int_{-\infty}^{0}\mathcal{K}dz' = \int_0^{\infty}\mathcal{K}dz'$$

$$\therefore\; \int_{-\infty}^{z}\mathcal{K}dz' = \int_0^z\mathcal{K}dz' + \int_0^{\infty}\mathcal{K}dz'$$

$$\int_z^{\infty}\mathcal{K}dz' = -\int_0^z\mathcal{K}dz' + \int_0^{\infty}\mathcal{K}dz'$$

Let $\int_0^z\mathcal{K}(b,z')dz' = \xi(b,z)$ so that $\xi(b,0) = 0$

Assuming scattering potential varies with distance sufficiently rapidly that we make allow, as a good approximation, $\delta(b,\infty) = \delta(b,R)$ where R is the range of the potential, then

$$\int_{-\infty}^{z} \kappa\, dz' + \alpha\int_{z}^{\infty} \kappa\, dz' \simeq (\alpha+1)\,\delta(b,R) + (1-\alpha)\,\delta(b,z)$$

$$\therefore \int_{-\infty}^{\infty} e^{i[xz + \int_{-\infty}^{z}\kappa dz' + \alpha\int_{z}^{\infty}\kappa dz']}\,dz \simeq e^{i(\alpha+1)\delta(b,R)}\int_{-\infty}^{\infty} e^{i[xz + (1-\alpha)\delta(b,z)]}\,dz$$

$$= e^{i(\alpha+1)\delta(b,R)}\left[\int_{-\infty}^{0} e^{i[xz+(1-\alpha)\delta(b,z)]}\,dz + \int_{0}^{\infty} e^{i[xz+(1-\alpha)\delta(b,z)]}\,dz\right]$$

Noting $\delta(b,-z) = -\delta(b,z)$

$$\int_{-\infty}^{0} e^{i[xz+(1-\alpha)\delta(b,z)]}\,dz = \int_{0}^{\infty} e^{-i[xz+(1-\alpha)\delta(b,z)]}\,dz$$

$$\therefore \int_{-\infty}^{\infty} e^{i[xz+(1-\alpha)\delta(b,z)]}\,dz = \int_{0}^{\infty}\left[e^{i(A+B)} + e^{-i(A+B)}\right]dz$$

$$= 2\int_{0}^{\infty}\cos(A+B)\,dz \quad \text{where } A = xz, \qquad (18)$$

$B = (1-\alpha)\,\delta(b,z)$

$$\int_{0}^{\infty}\cos(A+B)\,dz = \int_{0}^{R}\cos(A+B)\,dz + \int_{R}^{\infty}\cos(A+B)\,dz$$

$$\int_{0}^{R}\cos(A+B)\,dz = \int_{0}^{R}(\cos A\cos B - \sin A\sin B)\,dz$$

Letting $xR = C$, $(1-\alpha)\,\delta(b,R) = D$ and noting that

$$\delta(b,0) = 0, \quad \frac{d\delta}{dz} = \kappa(b,z), \quad \frac{dB}{dz} = (1-\alpha)\kappa$$

$$\int_{0}^{R}\cos A\cos B\,dz = \left|x^{-1}\sin A\cos B\right|_{0}^{R} + x^{-1}(1-\alpha)\int_{0}^{R}\sin A\sin B\,\kappa(b,z)\,dz$$

$$= x^{-1}\left[\sin C\cos D + (1-\alpha)\int_{0}^{R}\sin A\sin B\,\kappa(b,z)\,dz\right]$$

$$\int_{0}^{R}\sin A\sin B\,dz = x^{-1}\left[-\cos C\sin D + (1-\alpha)\int_{0}^{R}\cos A\cos B\,\kappa(b,z)\,dz\right]$$

$$\therefore \int_{0}^{R}\cos(A+B)\,dz = x^{-1}\left[\sin(C+D) - (1-\alpha)\int_{0}^{R}\kappa(b,z)\cos(A+B)\,dz\right]$$

Making same approximation as before

$$\int_{0}^{R}\kappa(b,z)\cos(A+B)\,dz \simeq \kappa(b,0)\int_{0}^{R}\cos(A+B)\,dz$$

(the scattering potential is assumed rather shallow so that $\frac{\partial\kappa}{\partial z} \ll \frac{\kappa(b,0)}{z}$ This is valid for a wide range of potentials with suitable values of range and well depth).

$$\therefore \int_{0}^{R}\cos(A+B)\,dz \simeq \frac{\sin(C+D)}{(1-\alpha)\kappa(b,0) + x}$$

$$\int_{R}^{\infty}\cos(A+B)\,dz = \int_{R}^{\infty}(\cos A\cos B - \sin A\sin B)\,dz$$

$$\int_{R}^{\infty}\cos A\cos B\,dz = x^{-1}\left[\left|\sin A\cos B\right|_{R}^{\infty} + (1-\alpha)\int_{R}^{\infty}\sin A\sin B\,\kappa(b,z)\,dz\right]$$

$$\simeq -x^{-1}\sin C\cos D\,?, \quad \text{since } \kappa(b,z) \simeq 0 \text{ for } z > R$$

$$\int_{R}^{\infty}\sin A\sin B\,dz = x^{-1}\left[\left|-\cos A\sin B\right|_{R}^{\infty} + (1-\alpha)\int_{R}^{\infty}\cos A\cos B\,\kappa(b,z)\,dz\right]$$

$$\simeq x^{-1}\cos C\sin D$$

$$\therefore \int_{R}^{\infty}\cos(A+B)\,dz \simeq -x^{-1}\sin(C+D)$$

Substituting in (18),

$$\int_{-\infty}^{\infty} e^{i[xz+(1-\alpha)\delta(b,z)]}\,dz \simeq 2\sin(C+D)\left[\left[(1-\alpha)\kappa(b,0)+x\right]^{-1} - x^{-1}\right]$$

$$= -2x^{-1}\sin(c+D)\left[1 + x(1-\alpha)^{-1}\mathcal{K}^{-1}(b,0)\right]^{-1}$$

Substituting in equ. (16),

$$f(k,\theta) \simeq \frac{k}{\pi}(1-\alpha)^{-1}\int e^{i[\underline{q}\cdot\underline{b} + (\alpha+1)S(b,R)]}\sin(c+D)\left[1+x(1-\alpha)^{-1}\mathcal{K}^{-1}(b,0)\right]^{-1}d^2\underline{b}$$

$$= \frac{ik}{2\pi}(1-\alpha)^{-1}\int e^{i[\underline{q}\cdot\underline{b} + (\alpha+1)S(b,R)]}\left(e^{-i(c+D)} - e^{i(c+D)}\right)\left[1+x(1-\alpha)^{-1}\mathcal{K}^{-1}(b,0)\right]^{-1}d^2\underline{b}$$

Now $\chi(b) = \int_{-\infty}^{\infty}\mathcal{K}(b,z')\,dz' = 2\int_0^{\infty}\mathcal{K}(b,z')\,dz' \simeq 2S(b,R)$

$\therefore\ D = (1-\alpha)S(b,R) \simeq \frac{1-\alpha}{2}\chi$

and $\chi(b) \simeq 2\mathcal{K}(b,0)R$

$$\therefore\ f(k,\theta) \simeq \frac{ik}{2\pi}(1-\alpha)^{-1}\int e^{i[\underline{q}\cdot\underline{b} + \frac{\alpha+1}{2}\chi]}\left[e^{-i(c+\frac{1-\alpha}{2}\chi)} - e^{i(c+\frac{1-\alpha}{2}\chi)}\right]\cdot$$

$$\cdot\left[1 + 2Rx(1-\alpha)^{-1}\chi^{-1}\right]^{-1}d^2b$$

$$= \frac{ik}{2\pi}(1-\alpha)^{-1}\int e^{i\underline{q}\cdot\underline{b}}\left[e^{-i(c-\alpha\chi)} - e^{i(c+\chi)}\right]\left[1+2xR(1-\alpha)^{-1}\chi^{-1}\right]^{-1}d^2\underline{b} \quad \ldots\ldots (19)$$

$|\chi| \lesssim \frac{|V_0|R}{k} \ll 1$ (See APPENDIX C) for large k

$2xR(1-\alpha)^{-1}\chi^{-1} = 4kR\sin^2\frac{\theta}{2}(1-\alpha)^{-1}\chi^{-1} \ll 1$

provided $\sin^2\frac{\theta}{2} \ll \frac{(1-\alpha)|\chi|}{kR} \lesssim \frac{|V_0|}{k^2}$

i.e. $\sin^2\frac{\theta}{2} \ll \frac{m|V_0|}{\hbar^2k^2} \sim \frac{|V_0|}{E}$ where V_0 is depth of the scattering potential, E the kinetic energy of the incident particle.

At high energies, $\frac{|V_0|}{E} \ll 1$ so that $\sin^2\frac{\theta}{2} \ll 1$.

Using Binomial Theorem, $\left[1+2xR(1-\alpha)^{-1}\chi^{-1}\right]^{-1} \simeq 1 - 2xR(1-\alpha)^{-1}\chi^{-1}$

$c = xR = 2kR\sin^2\frac{\theta}{2} \ll 1/kR \ll 1$ since $kR \gg 1$

$\therefore$ Expanding the exponentials in equ. (19),

$$e^{i(\alpha\chi-c)} - e^{i(c+\chi)} = 1 + i(\alpha\chi - c) - \tfrac{1}{2}(\alpha\chi - c)^2 + \ldots\ldots$$

$$- \left(1 + i(c+\chi) - \tfrac{1}{2}(c+\chi)^2 + \ldots\ldots\ldots\right)$$

$$= (1-\alpha)\chi\left[-i - \frac{2ic}{(1-\alpha)\chi} + \tfrac{1}{2}(1+\alpha)\chi\left(1 + \frac{2c}{(1-\alpha)\chi}\right)\right] + O(\chi^3)$$

$$\frac{c}{|\chi|} = \frac{2kR\sin^2\theta/2}{|\chi|} \sim \frac{k^2\sin^2\theta/2}{|V_0|} \sim \frac{\hbar^2k^2\sin^2\theta/2}{m|V_0|} \sim \frac{E\sin^2\theta/2}{|V_0|} \ll 1,$$

And so $c \ll |\chi|$

$\therefore$, the expansion becomes approximately $(1-\alpha)\chi\left[-i + \tfrac{1}{2}(1+\alpha)\chi + O(\chi^2)\right]$

Substituting in equ. (19)

$$f(k,\theta) \simeq \frac{ik}{2\pi}(1-\alpha)^{-1}\int e^{i\underline{q}\cdot\underline{b}}\,d^2\underline{b}\,(1-\alpha)\chi\left(-i + \tfrac{1}{2}(1+\alpha)\chi\right)\left(1 - 2xR(1-\alpha)^{-1}\chi^{-1}\right)$$

By making the Glauber approximation $-$ $\underline{q}\cdot\underline{r} \simeq \underline{q}\cdot\underline{b}$, equ. (5) resulted

i.e. $f(k,\theta) = \frac{ik}{2\pi}(1-\alpha)^{-1}\int e^{i\underline{q}\cdot\underline{b}}\,d^2\underline{b}\,\left(e^{i\alpha\chi} - e^{i\chi}\right)$

$$\simeq \frac{ik}{2\pi}(1-\alpha)^{-1}\int e^{i\underline{q}\cdot\underline{b}}\,d^2\underline{b}\,(1-\alpha)\chi\left(-i + \tfrac{1}{2}(1+\alpha)\chi\right)$$

Remembering $2xR(1-\alpha)^{-1}\chi^{-1} \ll 1$, the error associated with this approximation is $\delta f(k,\theta) \simeq -\frac{ik}{2\pi}(1-\alpha)^{-1}\int e^{i\underline{q}\cdot\underline{b}}\,d^2\underline{b}\,(1-\alpha)\chi\left(2xR(1-\alpha)^{-1}\chi^{-1}\right)$

Remembering $|\chi| \ll 1$, $\left|\frac{\delta f(k,\theta)}{f(k,\theta)}\right| \simeq 2xR(1-\alpha)^{-1}\chi^{-1} \sim \frac{E}{|V_0|}\sin^2\frac{\theta}{2} \ll 1$

for fixed scattering angle, the Glauber approximation worsens - the fractional error increasing - as the kinetic energy of the incident particle increases, tending to overestimate the scattering amplitude if scattering potential is attractive and to underestimate, if repulsive. The angular range over which the Glauber theory is valid should become smaller, larger the energy. However, if $\sin^2\frac{\theta}{2} \ll \frac{|V_0|}{E}$, the fractional error remains very small. As expected, the approximation becomes exact in the small angle limit with only two body interactions.

SECTION E:-

The approach to high energy potential scattering developped in the previous sections was prompted by the following analogue with physical optics: for high enough energies (frequencies) the De Broglie (photon) wavelength is small compared with the range of the scattering potential (size of diffraction obstacle). Just as with scattering of light by dielectric media, the rays of geometrical optics (linear, according to the first law of refraction) are first traced and a phase and amplitude assigned to each point on each ray and these added if more than one ray can reach a point in space, so particles propagate along classical linear paths at high energies. The equivalence between Glauber scattering theory and the Kirchoff diffraction theory is mathematical as well as physical. According to Kirchoff's theory, the amplitude of the wave at a point, diffracted by a sheet-like aperture of arbitrary shape, when plane waves of momentum $\underline{k}$ are incident is

$$\psi(\underline{R}) = \frac{ik}{4\pi}\int_A \psi_0(x,y)\,\frac{e^{i\underline{k}\cdot\underline{R}}}{R}\,(1+\cos\theta)\,dx\,dy \quad + \quad O(R^{-2})$$

where θ is angle of diffraction, $\underline{R}$ the position vector of the point from (x,y) on the wavefront of the incident wave (normal to $\underline{k}$). The assumptions and approximations made are:_

1. $kR \gg 1$ - the aperture is much more than k^{-1} away from the point and from the source of waves (the latter at infinity in this case).
2. $|\nabla\psi_0(x,y)| \ll k$ - the variation of amplitude over one wavelength is very small.
3. $\psi_0(x,y)$ and $\nabla\psi_0 = 0$ over obstructed parts of the wavefront and have values over unobstructed parts which they would have in the absence of the obstacle (St. Vernant's Principle). Thus ψ_0 and $\nabla\psi_0$ are assumed discontinuous at the obstacle's edges (obstacle completely black).

Assuming diffraction angle is small, $\cos\theta \simeq 1$

$$\psi(\underline{R}) \simeq \frac{ik}{2\pi}\int_A \frac{\psi_0}{R}(x,y)\, e^{i\underline{k}\cdot\underline{R}}\, dx\, dy$$

By Babinet's Principle, this is the amplitude (apart from sign) of the disturbance at $\underline{R}$ due to an opaque obstacle of the same shape.

For circular symmetry, $dx\,dy = d^2\underline{b}$ where $b^2 = x^2 + y^2$

Also $\underline{r} = \underline{b} + \underline{R}$ where $\underline{r}$ is position vector of point from centre of obstacle.

$\underline{k}\cdot\underline{R} = \underline{k}\cdot\underline{r} - \underline{k}\cdot\underline{b} = \underline{k}\cdot\underline{r}$ since by assumption wave is incident normal to obstacle. $R^{-1} = |\underline{r}-\underline{b}|^{-1} \simeq r^{-1}$ when $r \gg b$.

$$\therefore\ \psi(\underline{r}) \simeq \frac{ik}{2\pi}\int \psi_0(\underline{b})\, r^{-1} e^{i\underline{k}\cdot\underline{r}}\, d^2\underline{b} \equiv f(k,\theta)\, r^{-1} e^{ikr}$$

where the 'scattering amplitude' (amplitude of diffracted wave)

$$f(k,\theta) = \frac{ik}{2\pi}\int \psi_0(\underline{b})\, e^{i(\underline{k}-\underline{k}')\cdot\underline{r}}\, d^2\underline{b} = \frac{ik}{2\pi}\int \psi_0(\underline{b})\, e^{i\underline{q}\cdot\underline{r}}\, d^2\underline{b}, \text{ where } \underline{q} = \underline{k}-\underline{k}'$$

and k' is the wave vector of the diffracted wave.

$\underline{q}\cdot\underline{r} = \underline{q}\cdot\underline{b} + \underline{q}\cdot\underline{R} \simeq \underline{q}\cdot\underline{b}$ when θ is small

$$\therefore\ f(k,\theta) \simeq \frac{ik}{2\pi}\int \psi_0(\underline{b})\, e^{i\underline{q}\cdot\underline{b}}\, d^2\underline{b}$$

Consider the disc shaped obstacle as a circular thin slab of absorbing inhomogeneous dielectric with reflection coefficient $R \equiv R(x,y) = R(\underline{b})$ absorption coefficient $A = A(b)$ and transmission coefficient $T = T(\underline{b})$. Then continuity of the wave at the surface implies $R + A + T = 1$. According to the Kirchoff theory, there is no backward scattered wave as the inclination factor $1+\cos\theta = 0$ for $\theta = \pi$

$$\therefore\ A = 1 - T$$

For an incident wave of unit amplitude, the amplitude of the absorbed wave is $1-T$ and thus $\psi_0(\underline{b}) = 1 - T(\underline{b})$

$$\therefore\ f(k,\theta) = \frac{ik}{2\pi}\int (1 - T(\underline{b}))\, e^{i\underline{q}\cdot\underline{b}}\, d^2\underline{b}$$

The Glauber expression for scattering at high energies when many partial waves enter significantly is $f_G(k,\theta) = \frac{ik}{2\pi}\int (1 - e^{i\chi(\underline{b})})\, e^{i\underline{q}\cdot\underline{b}}\, d^2\underline{b}$

where $\chi(\underline{b}) = e^{i2S_\ell(k)} = e^{i2S_\ell(b)} = S(\underline{b})$, noting $bk = \ell + 1/2$

where $S(b)$ is the partial wave S matrix. Thus the correspondence is exact.

For the usual exponential attenuation through grey disc, thickness a

$$\psi_0(\underline{b}) = e^{-\beta(b) z}$$

$$|\nabla\psi_0| = \left|\frac{\partial\psi_0}{\partial z}\right| = \beta e^{-\beta z} \le \beta.$$

assumption 2 above implies $\beta \ll k$

i.e. $ak \gg a\beta$

if disc is very black, $a\beta \gg 1$

$\therefore\ ak \gg 1$, c.f. the high energy approximation $kR \gg 1$ in Glauber's theory where R is the range of the scattering potential.

CHAPTER 2:-

STRONG INTERACTIONS.

The quark and droplet models of hadrons are compared here in terms of their predictions for high energy hadron-hadron strong interactions. The impact parameter formulism for multiple scattering is reviewed in Section A and applied to point particles (no structure, as far as strong interactions are concerned), finitely and infinitely composite particles. In Section B, the Chou-Yang model is described. The multiple quark scattering model and its applications are discussed in Section C. A modified version of the Frautschi-Margolis infinitely composite model of proton-proton scattering is given in Section D. In Section E, the physics underlying the quark model of hadron scattering is analysed in detail and applied to elastic proton-proton scattering in the asymptatic limit of infinitely high centre of mass energies. Comparison is made with the droplet model.

SECTION A:

Elastic scattering data can be interpretated as approaching a regime at high energies where the differential cross section becomes independant of s and only a function of t(S and t are the Mandelstaum variables: $s = (p_1+p_2)^2$, $t = (p_1-p_1')^2$ where p_1, p_2 are the 4-momenta of the two incident particles, p_1' the 4-momentum of the scattered particle. This behaviour is reminiscent of the classical scattering of waves by opaque obstacles. Furthermore, the structure seen in the cross-sections (dips, changes in slope in different $|t|$ regions, etc.), have similarities to the scattering of fast nucleons by nuclei, where multiple scattering effects are known to be important. Thus, it is natural to exploit these similarities by using a formulism analagous to nuclear scattering, that is, the impact parameter formulism. At high energies, where many partial waves contribute to the scattering, the discrete partial wave sum for the scattering amplitude $f(k,\theta)$

$$f(k,\theta) = \sum_{\ell=0}^{\infty} (2\ell+1)\, a_\ell(k)\, P_\ell(\cos\theta) \qquad (1)$$

can be replaced by an integration over impact parameter b defined by

$$bk = \ell + 1/2 \qquad (2)$$

It is convenient to deal with the amplitude

$$F(\underline{k}',\underline{k}) = -\frac{i}{k}\, f(\underline{k}',\underline{k}) \qquad (3)$$

where $f(\underline{k}',\underline{k})$ is the centre of mass scattering amplitude and k the momentum of a particle in the centre of mass frame. In terms of F, the differential cross-section is

$$\frac{d\sigma}{dt} = \pi |F(\underline{k}',\underline{k})|^2 \qquad (4)$$

The scattering amplitude F can by represented by the two-dimensional Fourier transform

$$F(\underline{k}',\underline{k}) = F(\underline{q}) = (2\pi)^{-1}\int d^2\underline{b}\, e^{i\underline{q}\cdot\underline{b}}(1-S(\underline{b})) \quad (5)$$

where $S(\underline{b}) = e^{i2\delta(b)}$ is the partial wave S matrix and $\underline{q} = \underline{k}-\underline{k}'$ is the momentum transfer $(t=-q^2)$. If S is independant of azimuthal angle ϕ then $F(\underline{q}) = F(q) = \int_0^\infty J_0(bq)(1-S(b))\, b\, db$

SCATTERING OF A STRUCTURELESS PARTICLE BY A POTENTIAL.

If a structureless particle of velocity v is scattered by a potential $V(\underline{r})$, the phase-shift $\delta(\underline{b})$ is, at high energies, given by

$$2\delta(\underline{b}) = -(\hbar v)^{-1}\int_{-\infty}^{\infty} V(\underline{b}+\hat{k}z)\, dz \quad (6)$$

where $\hat{k}$ a unit vector parallel to $\underline{k}$ (or $\underline{k}+\underline{k}'$) defines the z axis.

SCATTERING OF A STRUCTURELESS PARTICLE BY AN N PARTICLE COMPOSITE PARTICLE.

For N fixed scattering centres located at $\underline{r}_j = \hat{k}z_j + \underline{s}_j$ where $\underline{s}_j$ is the transverse coordinate vector $(s_j = \frac{b\cdot r_j}{b})$, the wavefunction of the incident particle accumulates phase from each of the scatterers according to (6). Thus

$$S(b) = e^{2i\sum_{j=1}^{N}\delta_j(\underline{b}-\underline{s}_j)} = \prod_{j=1}^{N} e^{i2\delta_j(\underline{b}-\underline{s}_j)} \quad (7)$$

The amplitude for the scattering of the incident particle by the jth sub particle is from (5).

$$F_j(\underline{q}) = (2\pi)^{-1}\int d^2\underline{b}\, e^{i\underline{q}\cdot\underline{b}}(1-e^{i2\delta_j(b)})$$

The two dimensional inverse Fourier transform of $F_j(q)$ (called the profile function) is

$$\Gamma_j(\underline{b}) = (2\pi)^{-1}\int d^2\underline{q}\, e^{-i\underline{q}\cdot\underline{b}} F_j(\underline{q})$$

Then the complete S matrix for the scattering of an N particle composite is

$$S(\underline{b}) = \prod_{j=1}^{N}[1-\Gamma_j(\underline{b}-\underline{s}_j)] \quad (8)$$

The expansion of this product leads to Glouber's multiple scattering expansion. If the scattering centres are not fixed, as in composite bound states such as nuclei, the internal motion of the scatterers must be taken into account. The assumption of a short collision time, allows mere averaging of their motion by taking an expectation value of S (b) for the ground state of the particle. If independant particle motion is a reasonable approximation.

$$\langle i|S(b)|i\rangle = \prod_{j=1}^{N}\int d^3r_j\, \rho_j(\underline{r}_j)(1-\Gamma_j(\underline{b}-\underline{s}_j)) \quad (9)$$

where $\rho_j(\underline{r}_j)$ is the probability density for the jth particle at $\underline{r}_j$. If all the scatterers are the same and the spatial extension of Γ_j (b) is small compared to the distances over which $\rho_j(\underline{r}_j)$ changes appreciably.

$$\langle i|S(b)|i\rangle = \left(1-\frac{2\pi}{N}\bar{F}(0)D(\underline{b})\right)^N \quad (10)$$

where $\bar{F}(0)$ is the average forward $(q=0)$ amplitude for an individual scattering and the two dimensional density

$$D(\underline{b}) = \int dz\, \rho(\underline{b},z) \quad (11)$$

is a measure of the interacting matter encountered by the incident particle passing through the system at impact parameter b. In (10), $\rho = N\rho_j$ is the total density of interacting matter. For a large nucleus (or in the droplet model of hadrons) the approximation of ρ independent of N as $N\to\infty$ is legitimate.

then (10) becomes $$\langle i | S(\underline{b}) | i \rangle = e^{-2\pi \bar{F}(o) \dot{D}(b)} \qquad (12)$$

SCATTERING OF ONE COMPOSITE BY ANOTHER COMPOSITE PARTICLE.

In this case, the S matrix is generalised to

$$S(\underline{b}) = \prod_{j=1}^{N} \prod_{j'=1}^{N'} [1 - \Gamma_{jj'}(b + \underline{s}_j - \underline{s}'_j)] \qquad (13)$$

where $\Gamma_{jj'}(b)$ is the profile function for scattering for constituent j in one composite by constituent j' in the other. If we view hadrons as extended objects made up of finely divided interacting hadronic matter, then $N \to \infty$, $N' \to \infty$ and $\Gamma_{jj'} \propto \frac{1}{NN'}$. Then the first non-trivial term in the expansion of $S(b)$ is the sum of all single scattering terms in (13). We obtain

$$2i\,\delta(\underline{b}) = -k_{AB} \int d^2 \underline{b}' \, D_A(\underline{b} - \underline{b}') D_B(\underline{b}') \qquad (14)$$

where k_{AB} is a complex interaction parameter for the propagation of composite A through composite B and D_A, D_B are two-dimensional densities of interacting matter defined by (11). The multiple scattering series can now be exhibited.

Defining $$f(\underline{q}) = -(2\pi)^{-1} \int d^2 \underline{b}\, e^{i \underline{q} \cdot \underline{b}}\, 2i\,\delta(\underline{b}) \qquad (15)$$

then $$F(\underline{q}) = f(\underline{q}) - \frac{1}{2!} f(\underline{q}) \otimes f(\underline{q}) + \frac{1}{3!} f(\underline{q}) \otimes f(\underline{q}) \otimes f(\underline{q}) - \ldots\ldots \qquad (16)$$

where $$f(\underline{q}) \otimes f(\underline{q}) = (2\pi)^{-1} \int d^2 \underline{q}'\, f(\underline{q}')\, f(\underline{q} - \underline{q}')$$

SECTION B.

The model of Chou and Yang [1] and the earlier models of Wu and Yang [2] and Byers and Yang [3] are based on the idea of hadrons as extended objects whose ability to interact is given by a well defined density $D(b)$

$$D(\underline{b}) = (2\pi)^{-1} \int d^2 \underline{q}\, e^{-i \underline{q} \cdot \underline{b}}\, F^E(q) = \langle F^E(q) \rangle \qquad (17)$$

where F^E is the electromagnetic form factor of the particle. Noting

$$\langle F^E_A F^E_B \rangle = \langle F^E_A \rangle \otimes \langle F^E_B \rangle$$

then (14) becomes $$2i\,\delta(\underline{b}) = -k_{AB} \langle F^E_A F^E_B \rangle \qquad (18)$$

Using the analogue with the optical model in nuclear theory, k_{AB} was taken to be real and independant of energy, corresponding to a pure imaginary phase shift $\delta(b)$ induced by a purely absorptive medium. Their amplitude is thus an asymptotic one, the first term being, using (15),(16) and (18)

$$f(q) = k_{AB} F^E_A F^E_B$$

For particles with spin, there is some ambiguity as to what electromagnetic form factor to use. Chou and Yang choose the electric charge form factor for proton-proton scattering. If the scaling law $F^E_p = \frac{F^M_p}{\mu}$, where F^E_p, F^M_p, μ are the electric, magnetic form factors and anomalous proton magnetic moment respectively, is obeyed, then this choice is arbitrary. However, recent experimental evidence [4] suggests that the law is broken or that the electric and magnetic charge densities may be different, which is difficult to understand in a droplet model where the distribution of magnetised matter follows the charge matter distribution. Clearly, exchange currents must be present as well. Chou and Yang showed that the calculated electric charge form factor was in agreement with the magnetic form factor of the protor for |t| values as large as $20\ (GeV/c)^2$. However, if further experiments confirm the results of ref (4) it would appear that their choice was wrong. Since the conjecture of Wu and Yang (see Section D) involved the electric charge form factor, scale breaking would imply that the magnetic charge form factor is more relevant, although their prescription [17] remains valid. Durand and Lipes [5] calculated the asymptotic differential cross section for p-p scattering, using a dipole form factor for the proton. This shows deep diffractive minima at about $|t| = 1.3$ and $5.8\ (GeV/c)^2$. Using a complex constant k_{pp} removes these dips. Within the framework of this model, the deep minima are expected to become more and more visible as the incident energy increases, although if the ratio of forward imaginary part of amplitude to real part vanishes slowly (e.g. $(\log s)^{-1}$), the approach to this behaviour may be slow.

SECTION C:-

The quark model differs from the droplet model only in that instead of letting $N, N' \rightarrow \infty$ in equ. (13) above, we take $N = N' = 3$, corresponding to the fundamental triplet model of Gell-Mann and Zweig [6]. Taking the diagonal matrix element of S (b) in equ (9) leads to form factors which are either assumed Gaussian in the 4-momentum transfer or of dipole form or are deduced from shell model or harmonic oscillator wavefunctions. The quark-quark elastic amplitudes are either assumed gaussian in the 4- momentum transfer or are deduced from the observed differential cross-sections in the diffractive peak region by making a single scattering approximation to the total amplitude

(interacting particles are represented by circles, bars represent possible interactions)

This ambiguity has not affected the conclusions drawn by various workers but the freedom of choice in ascribing the scattering either to form factors (point-like quarks), quark-quark amplitudes (quarks non point-like) or to both means little can be deduced about quarks themselves which is not model dependant. The presence, sometimes, of a large number of freely adjustable parameters makes a crucial test of the quark model of high energy scattering impossible. For proton-proton scattering, all applications of the quark model are successful in reproducing the sharp break in the slope of the differential cross section at $|t| \simeq 1 (GeV/c)^2$ but only at the expense of giving a strong momentum transfer dependance to the phase of the quark-quark amplitude and therefore, from the additivity rule, to the phase of the proton-proton amplitude itself. Alternatively, the slope of the quark-quark scattering amplitude may be assumed real, instead of complex, which gives the phase momentum transfer dependance, but then the ratio of real to imaginary parts of the forward scattering amplitude must be put equal to one, contradicting Coulomb interference experiments. Otherwise, deep diffractive minima are obtained which are not observed experimentally. Although multiple scattering analysis of proton-deuteron scattering[7] seems to require a similar dependance of the phase of the nucleon-nucleon amplitude on q at least at low momentum transfers, this is probably due to the neglect of spin dependance which might prevent amplitudes from going through zero in the region of destructive interference between singly scattered and doubly scattered particle amplitudes. However, the presence of a break rather than a dip (as observed in proton-antiproton and pion-proton elastic scattering) is not due to the possible spin dependance of the quark-quark interaction as the complete minima persist when spin is included in the calculation.[8] Hence the unique sharp break in the proton-proton elastic differential cross-section remains problematical for the quark model. The dip is not a result of the proton containing more than 3 quarks since the zero in the differential cross section persists when more than three subparticles in the proton are considered. It may be, however due to the symmetry of the proton wavefunction i.e. an antisymmetric wavefunction demanded for Fermion quarks might lead to the zero being filled in, although this would necessarily result in at least one zero in the proton hadronic form factor which might prove awkward. Thirring[13] has suggested the antisymmetric $\psi(r_1,r_2,r_3) = N(r_1^2-r_2^2)(r_2^2-r_3^2)(r_3^2-r_1^2)\,e^{-\beta^2(r_1^2+r_2^2+r_3^2)}$ for ground state of nucleon which has a node at $q^2 = 17.28\ \beta^2$.[14] Despite its correct symmetry, this wavefunction still gives rise to deep minima, instead of a shoulder, unless $\mathcal{R}[f(0)]$ is comparable with $\mathcal{I}[f(0)]$.[15] Alternatively, the effective quark-quark amplitude may not be purely Gaussian.

Mention should be made of the Durand and Lipes calculation in this context. The zero in the differential cross section, which results when the interaction constant K_{pp} (see equ. 18 Section B) for p-p scattering is taken as real, corresponding to a pure imaginary scattering amplitude at all momentum transfers, was filled in by including a real part in the amplitude (and in K_{pp}). They found also that when spin dependance was included, a minimum in the p-p polarisation near $|t| \simeq 1.2\,(\mathrm{GeV}/c)^2$ resulted due to the presence of this diffraction zero. This has been confirmed by polarisation experiments at $5\ \mathrm{GeV}/c$.[9] So while this implies a spin dependance in proton-proton scattering (singly and doubly scattered protons having nearly opposite spin orientations when scattered in equal measure), when this is taken into account, the quark model still predicts a diffraction zero at finite energies.

The conclusions of various workers as to the structure of the proton are as follows. Shrauner et al [12] conclude that particle number of two or four is ruled out by the scattering data whilst a number of three gives a very good fit to the data. This is confirmed by the results of Harrington and Pagnamenta [10] and Wakaizumi[11]. Others obtain opposite conclusions : Kofoed-Hansen[16] and Cromer [17] both find that the infinitely composite model is preferred. The reason for this divergence is as follows. The two unknown parameters (which can be independantly varied) in any composite model of hadron-hadron scattering are the ratio of the real to imaginary parts of the sub particle - sub particle elastic scattering amplitude and the ratio of the slope of this amplitude to the strong interaction form factor slope for the composite. If one is given an arbitrary value, then so can the other. The small angle data are rather insensitive to the choice of parameter values, although the latter, once fixed, determines the large angle behaviour of the differential cross section uniquely. Clearly large angle scattering would be a better test for distinguishing between the droplet and quark models, but the Glauber theory is not expected to be valid at large angles so that comparison of the models' predictions is then not trustworthy. If some theoretical constraint on quark scattering amplitudes and/or hadronic form factors, either for small or large momentum transfers, could be made, the fitting of the quark model to the data could be done unambiguously and a crucial comparison with the droplet model would become possible at finite energies. At infinite centre of mass energies (forward scattering amplitude pure imaginary), both models give similar diffractive patterns, the first and second diffraction zeros occurring at values of momentum transfers which are predicted to be respectively the same almost in both models. For the fhird zero, the droplet model predicts its occurrence at about $|t| = 5.6\ (\mathrm{GeV}/c)^2$. The quark model however leads to no zeros in the triple scattering region provided that Krisch's plot of present proton-proton scattering cross-sections is truly energy independant, so that it remains valid at infinite energies. Otherwise a diffraction zero is predicted at about the same value of the momentum transfer as the droplet model predicts. Although this in principle would enable a test to be made between the two models, very sensitive counters with high angular resolution would be needed at the very large energies where diffraction minima might be observed. It should be noted that these conclusions are valid for Gaussian quark-quark amplitudes and proton form factors only. If this restriction were relaxed, there would be even more freedom within the models, making it more difficult to choose among them. Using the parameter values, obtained in fitting either quark or droplet models to the experimental data, to predict values of total cross sections, breaks, dips and angular variation of the differential cross section in other reactions would test for self-consistency within each model but could not be made to test one model against the other for it it is assumed that all hadrons are made up of the same sort of hadronic matter (this must be assumed for predictions to be model independant) then comparison with the droplet model would be possible only at infinitely high energies, assuming the Pomeranchuk theorem for particle-particle and p rticle-antiparticle total cross sections, which implies that hadronic and antihadronic matter properties are asymptotically identical. Thus the present evidence leads to the conclusion that the number of constituents in the hadron, whether, baryon or meson, cannot be decided on the basis of strong interactions, though indirect evidence (total cross section sum rules, strong decay rates, etc.) favours the quark model.

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SECTION D:-

THE FRAUTSCHI-MARGOLIS MODEL AND THE WU-YANG CONJECTURE.

The Pomeranchuk or vacuum trajectory occupies a special position in the hierachy of Regge poles because (i) it is the highest lying trajectory (ii) its slope seems abnormally small ($\alpha'_P(0) \simeq 0.0-0.3\,(GeV/c)^2$). (iii) it is doubtful whether it is a Regge pole because of the lack of Regge recurrences. The Chou-Yang model of high energy elastic scattering of hadrons makes a clear distinction between diffraction, which involves exchange of the quantum numbers of the vacuum between colliding particles, and the exchange of other quantum numbers - it contains a degenerate Pomeranchuk trajectory of zero slope. Alternatively one may assume the Pomeranchuk to be a normal Regge pole with normal slope but that multiple scattering corrections are important. The observed flat trajectory is then a consequance of approximating the multiple scattering series by a single Regge pole amplitude. This viewpoint is taken in the Frautschi-Margolis (F-M) model[1]. Starting from the Glauber small angle of scattering expression for the scattering amplitude -

$$f(q) = i\int_0^\infty J_0(bq)\left(1-e^{i\chi(b)}\right) b\,db \qquad (1)$$

where the normalised amplitude satisfies $\frac{d\sigma}{dq^2} = \pi|f(q)|^2$, the n th order multiple scattering contribution to the diffractive amplitude is given by the nth term in the series formed by expanding $e^{i\chi(b)}$ in ascending powers of $\chi(b)$:-

$$f(q) = \sum_{n=1}^{\infty} f_n(q) \qquad \text{with } f_n \text{ given by}$$

$$f_n(q) = -i\int_0^\infty \frac{i^n}{n!}\chi^n(b) J_0(bq)\, b\,db$$

and $f_1(q)$ given by the Born approximation $f_1(q) = iCe^{-aq^2}$, where at high energies $C = C^*$ and $C > 0$. Since their model is based on the small angle of scattering approximation (1), their results should, strictly speaking, be compared only with small angle of scattering data, in particular, their predictions for large momentum transfer diffraction scattering should be meaningful only in the asymptotic energy regime. For purposes of testing the multiple Pomeranchuk exchange mechanism i.e. to see whether exchange of the P trajectory alone can account for the observed exponential dependence of elastic differential cross-sections on $|t|^{1/2}$ where $|t| = q^2$, at currently available laboratory energies, it would appear to be more appropriate to start from a general scattering angle expression for the scattering amplitude. Then, the large $|t|$ prediction can be compared with present data. It is then found that while the $e^{-|t|^{1/2}}$ behaviour is retained, within the limits of the approximations made, the detailed dependance of the predicted differential cross-section on $|t|$ gives, assuming the Wu-Yang conjecture [2] for the large $|t|$ proton electromagnetic form factor, a rather poor fit of the form factor to the SLAC data. It should be noted that Wu and Yang's speculation was for the large angle (not large $|t|$) p-p differential cross section to be proportional to $|F_p^E(t)|^4$ and so can be applied to high (but finite) energy p-p scattering. Oscillations are found in the differential cross section calculated for finite (but large) centre of mass energies which are not observed in the large angle data nor in the proton form factor. These occur in the fixed large angle differential cross section as a function of centre of mass energy as well, violating the Orear formula (3) $s\frac{d\sigma}{d\Omega} = Ae^{-\frac{p\sin\theta}{b}}$ where A, b are constants, although the predicted energy dependence may be weak enough to be consistent with the data in the currently available range of energies. These discrepancies are discussed. It should be noticed that infinite order multiple diffractive scattering is characteristic of hadrons regarded as droplets as opposed to finitely composite objects so that the results described above for p-p scattering should be characteristic of the Chou-Yang model if all orders of scattering are considered, the nth term in the expansion of (1) being the sum of all contributions from n particles in the droplet.

From Section C, equ. (15), of HIGH ENERGY POTENTIAL SCATTERING THEORY, the general scattering amplitude for elastic p-p scattering is

$$F(s,q^2) \equiv F(s,t) = \frac{i}{2\pi}(1-\alpha)^{-1}\int e^{i\underline{q}\cdot\underline{b}}\left(e^{i\alpha\chi} - e^{i\chi}\right)d^2\underline{b}$$

$$= i(1-\alpha)^{-1}\int_0^\infty J_0(b|t|^{1/2})\left(e^{i\alpha\chi} - e^{i\chi}\right)d^2\underline{b}$$

$$= i(1-\alpha)^{-1}\int_0^\infty J_0(b|t|^{1/2})\sum_{n=0}^{\infty}\frac{i^n(\alpha^n-1)}{n!}\chi^n\, b\,db$$

and as $\alpha \to 0$, $F(s,t) \simeq -i\int_0^\infty J_0(b|t|^{1/2})\sum_{n=0}^{\infty}\frac{i^n\chi^n}{n!}\, b\,db$

Writing the single scattering (Born) amplitude as

$F \simeq F_B = ice^{at}$ gives $\chi = \frac{ic}{2a}e^{-\frac{b^2}{4a}}$

from which $F(s,t) = ic\sum_{n=1}^{\infty}\frac{1-\alpha^n}{1-\alpha}\cdot\frac{1}{n.n!}(-c/2a)^{n-1}e^{\frac{at}{n}}$

$= ic\sum_{n=1}^{\infty}\exp\left[at/n + i\pi(n-1) + (n-1)\log D - \log n - \log n! + \log\frac{1-\alpha^n}{1-\alpha}\right]$, $D = c/2a$

for large angle scattering (large momentum transfer scattering when centre of mass momentum is large), the scattering amplitude $= \lim_{\alpha\to 1} F(s,t)$. Noting $\lim_{\alpha\to 1}\frac{1-\alpha^n}{1-\alpha} = n$

then $F(s,t) \underset{|t|\to\infty}{\sim} ic\sum_{n=1}^{\infty}\exp\left[\frac{at}{n} + i\pi(n-1) + (n-1)\log D - \log n!\right]$

$= ic\int_1^\infty e^{f(x)}dx$, where $f(x) = at/x + (i\pi + \log D)(x-1) - \log\Gamma(1+x)$

since large angle scattering is dominated by higher order (large n) multiple scattering, using Method of Steepest Descents to evaluate integral, we obtain

$$F(s,t) \underset{|t|\to\infty}{\sim} ic\left|\frac{-\pi^2 at}{2\log^3(-at)}\right|^{1/4}\left[1 + \mathrm{erf}\left[-\tfrac{1}{2}at\log(-at)\right]^{1/4}\right]\exp\left[-\left(1+\frac{\log D}{\log(-at)}\right)\left(-2at\log(-at)\right)\right]^{\frac{1}{2}}$$

$$1 + \frac{\log D}{\log(-at)} \underset{|t|\to\infty}{\sim} 1, \quad \mathrm{erf}\left[-\tfrac{1}{2}at\log(-at)\right]^{1/4} \underset{|t|\to\infty}{\sim} 1$$

$$\therefore F(s,t) \underset{|t|\to\infty}{\sim} 2ic\left[\frac{-\pi at}{2\log^3(-at)}\right]^{1/4}\exp\left[-\left(-2at\log(-at)\right)^{1/2}\right]$$

The Pomeranchuk exchange amplitude for elastic p-p scattering is

$$F_P(t) = -\frac{c}{\sin\pi\alpha}\left(e^{-i\pi\alpha}+1\right)\left(\frac{s}{s_0}\right)^{\alpha-1}$$

$$= -\frac{ce^{-\frac{i\pi\alpha}{2}}}{\sin\frac{\pi\alpha}{2}}\left(\frac{s}{s_0}\right)^{\alpha-1} \quad \text{where } P \text{ trajectory is}$$

$$\alpha(t) = \alpha(0) + \alpha'(0)t$$

with $\alpha(0) = 1$, assuming asymptotic constancy of total scattering cross-section (no exotics in the direct channel), and $\alpha'(0) \neq 0$, in view of latest Serpukhov data(4) on high energy elastic p-p scattering, which indicates considerable shrinkage of p-p diffraction peak, as centre of mass energy increases, at very high energies.

$$\sin\frac{\pi\alpha}{2} = \sin\frac{\pi}{2}(1+\alpha'(0)t) = \cos[\alpha'(0)t] \simeq 1 \quad \text{if} \quad |t| << \frac{2\pi}{\alpha'(0)}$$

The approximation is reasonable since single P exchange is responsible for the diffraction peak whose width is typically $\lesssim 0.5\,(\frac{GeV}{c})^2$. The signature factor $\frac{e^{-i\pi\alpha}}{\sin\frac{\pi\alpha}{2}}$ $= \cot\frac{\pi\alpha}{2} - i$ has zeros at $-\alpha = 2n+1$ for $n = 0,1,2,$ etc.,

i.e. at $-t = \frac{2}{\alpha'(0)}(n+1) \gtrsim 2,4,6,$ etc., for $\alpha'(0) \lesssim 1$

Hence zeros are unimportant for single P exchange where $|t| \lesssim 1\,(\frac{GeV}{c})^2$ typically. Identifying the single scattering amplitude with P exchange amplitude:

$$F_B = ice^{at} = F_P = ie^{-\frac{i\pi}{2}\alpha'(0)t}c\left(\frac{s}{s_0}\right)^{\alpha-1} = ice^{\alpha'(0)t[\log\frac{s}{s_0} - i\pi/2]}$$

$$\therefore \quad a = \alpha'(0)\left[\log\frac{s}{s_0} - \frac{i\pi}{2}\right] = |a|e^{-i\phi(s)},$$

$$\text{where} \quad |a| = \alpha'(0)\left[\log^2\frac{s}{s_0} + \frac{\pi^2}{4}\right]^{1/2}, \qquad \tan\phi(s) = \pi/2\log\frac{s}{s_0}$$

$$\therefore \quad F(s,t) \underset{|t|\to\infty}{\sim} 2ic\left[\frac{-\pi^2|a|t}{2|\log(-at)|^3}\right]^{1/4} \exp\left[-|A(s,t)|^{1/2}\cos\frac{B}{2}(s,t) + i\left(\frac{3\theta}{4} - \frac{\phi}{4} + |A(s,t)|^{1/2}\sin\frac{B}{2}(s,t)\right)\right]$$

$$\text{where} \quad |A(s,t)| = -2|a|t\left[\phi^2(s) + \log^2(-|a|t)\right]^{1/2}$$

$$\tan B(s,t) = \frac{\phi(s) + \tan\phi(s)\log(-|a|t)}{-\phi(s)\tan\phi(s) + \log(-|a|t)}$$

$$\therefore \quad \frac{d\sigma}{dt} = \pi|F(s,t)|^2 \underset{|t|\to\infty}{\sim} 4\pi c^2\left[-\pi^2|a|t/2|\log(-at)|^3\right]^{1/2} e^{-2|A(s,t)|^{1/2}\cos\frac{B}{2}(s,t)}$$

Now $\tan B(s,t) \underset{|t|\to\infty}{\sim} \tan\phi(s)$ so $B(s,t) \underset{|t|\to\infty}{\sim} \phi(s)$

$$\therefore \quad \frac{d\sigma}{dt} \underset{|t|\to\infty}{\sim} 4\pi c^2\left[\frac{-\pi^2|a|t}{2(\phi^2+\log^2(-|a|t))^{3/2}}\right]^{1/2} e^{-2\left[-2|a|t(\phi^2+\log^2(-|a|t))^{1/2}\right]^{1/2}\cos\frac{\phi}{2}}$$

$$\text{or} \quad \frac{d\sigma}{dt} \propto |t|^{1/2} e^{-\text{const.}|t|^{1/2}} \quad \text{approximately}$$

Applying the Wu-Yang conjecture - $\frac{d\sigma}{dt} \propto |F_p^E(t)|^4$ as $|t| \to \infty$

.Then the large angle eikonal representation of the scattering amplitude due to multiple Pomeranchon exchange predicts the proton change form factor to vary with large momentum transfer $|t|^{1/2}$ as

$$|F^E_p(t)| \propto |t|^{1/8} e^{-\text{const.}|t|^{1/2}}$$

Miyake and Takagi(5) have phenomenologically analysed the SLAC data, for $5 \leq |t| \leq 20\,(\frac{GeV}{c})^2$ for the proton magnetic form factor, using the asymptotic form $F_p^E(t) = C_1|t|^a e^{-C_2|t|^b}$ where C_1 and C_2 are constants, a and b are independent parameters. They obtain a best and loose fit region for a and b (the latter defined by including data at $|t| = 12$ and $20\,(\frac{GeV}{c})^2$, which have large error bars). For $b = 1/2$, the best fit region for a is $-2.0 \leq a \lesssim -1.2$ and the loose fit region is $-2.22 \leq a \lesssim 0.3$. Thus the predicted form factor lies near boundary of the loose fit region and so, although not definitely excluded by the data, it does not fit the data very well. There are various explanations.

1) The SLAC data are not yet in the asymptotic region, as was assumed by Miyake

and Takagi. This is very unlikely unless proton has a core structure at distances less than 0.1F, i.e. quarks (partons) responsible for electron-proton scattering are tightly bunched together at proton centre.

2) Multi P-exchange mechanism is wrong. Present data[4] indicate p-p total cross section is constant between 25 and 65 GeV, instead of rising towards constancy as F-M model predicts. Also, beyond scattering angles of about 60°, the data for p-p elastic differential cross sections deviate significantly from the $e^{-\text{const}|t|^{1/2}}$ behaviour[6], falling off less slowly. The exponential variation is not necessarily due to this mechanism since all multiple scattering models involving unlimited order of scattering are characterised by predicting a transition from Gaussian dependance on momentum transfer at small values to exponential dependance at large values [7] [8]. Also, the exponential fall off is already observed at energies where p-p total cross section is still falling or levelling off.

2) Wu-Yang conjecture is, asymptotically incorrect. It does not appear compatible with the Orear formula[5] although the form factor, deduced from p-p elastic scattering, satisfies the lower bound $|F(t)| \geq e^{-|t|^{1/2}}$ derived by Jaffe from Q.F.T. The quark model confirmation [9] of Wu-Yang's conjecture for small momentum transfers (up to the first break in the slope of the differential cross section) makes the idea difficult to accept if it is applied to the opposite end of the $|t|$ range where high order multiple scattering dominates. Also, the idea implies that the Pomeranchukon is a fixed singularity, contradicting the Serpukhov experiments which indicates shrinkage still of the p-p diffraction peak.

The resolution of these questions will have to wait until larger energy and momentum transfer data from the Serpukhov and Batavia accelerators are available. It should be noted that the quark model, in which the maximum order of multiple scattering is 4, 6 or 9 for meson-meson, meson-baryon or baryon-baryon scattering cannot predict a $|t|^{1/2}$ dependance, when $|t|$ is large, of the elastic differential cross section, since the number of independent Gaussian terms in the multiple scattering expansion of the amplitude is finite, in fact $2^{AB}-1$ where A is the number of quarks in one hadron and B the number in the other. For p-p scattering, the number is 511 and the convergence of the series is slow. The large number of terms means that the series can be approximated by an infinite series, except at very large scattering angles, so that the finite series has the asymptotic variation of $e^{-\text{const}|t|^{1/2}}$. However, the large angle data $(\theta \sim 90^{\circ})$ for p-p scattering is known to fall off less slowly than an exponential in $|t|^{1/2}$. This thus may be evidence for finite compositeness of the proton i.e. a scattering series which is finite so that there are no higher order multiple scattering terms which cause interference and reduce the differential cross-section.

It may, alternatively, be due to the backward scattered proton which is indistinguishable from the proton scattered in the forward hemisphere. As yet, backward scattering cannot be incorporated in the quark model so that the two alternatives cannot be distinguished.

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SECTION E:-

In this section, the kinematic and dynamic assumptions, both implicitly and explicitly made in applying the Glauber formulism to high energy hadron - hadron scattering, viewed as multiple quark-quark scattering, are discussed.

In the quark model, the small angle approximation to the elastic scattering amplitude, originally derived by Glauber, is often applied, with success, to explain the oscillatory diffractive pattern of hadron- hadron scattering at angles considerably larger than that for which one would expect the approximation to be valid. Why is this so? A simple argument shows that this is problematical if the quark is massive compared with the hadron. Consider elastic p-p scattering (this example is quite arbitrary as the argument is true for any particle made up of quarks). The two protons in the centre of mass frame collide with equal and opposite momenta $\underline{k}$ and are scattered elastically with final momenta $\underline{k}'$. The momentum transfer from one proton to the other is $\underline{q} = \underline{k} - \underline{k}'$

Now $\underline{k} = \sum_{j=1}^{3} \underline{k}_j$, $\underline{k}' = \sum_{j=1}^{3} \underline{k}_j'$, where $\underline{k}_j$, $\underline{k}_j'$ are the initial and final momenta of the jth quark relative to centre of mass frame.

$$\therefore \underline{q} = \sum_{j=1}^{3} (\underline{k}_j - \underline{k}_j') = \sum_{j=1}^{3} \underline{q}_j , \quad \text{where} \quad \underline{q}_j = \underline{k}_j - \underline{k}_j'$$

is momentum transfered to jth quark.

$|\underline{q}| = 2k \sin\frac{\theta}{2}$ and $|\underline{q}_j| = 2k_j \sin\frac{\theta_j}{2}$ where θ is the total angle through which proton is scattered, θ_j the jth quark scattering angle, again referred to centre of mass frame. Let $\hat{q}$, $\hat{q}_j$ be unit vectors in the direction of momentum transfer of proton and jth quark respectively.

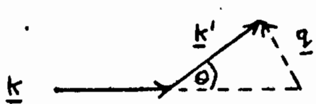
$$\therefore \underline{q} = |\underline{q}|\hat{q} = 2k \sin\frac{\theta}{2}\,\hat{q} \quad \text{and} \quad \underline{q}_j = 2k_j \sin\frac{\theta_j}{2}\,\hat{q}_j$$

$$\therefore 2k \sin\frac{\theta}{2}\,\hat{q} = \sum_{j=1}^{3} 2k_j \sin\frac{\theta_j}{2}\,\hat{q}_j$$

$$\text{or} \quad \hat{q} = \sum_{j=1}^{3} \frac{k_j \sin\frac{\theta_j}{2}}{k \sin\theta/2}\,\hat{q}_j$$

Noting $\underline{V}_j = \underline{V} + \underline{\upsilon}_j$

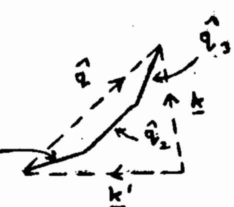
where $\underline{V}$ is velocity of proton in centre of mass frame, $\underline{\upsilon}_j$ is quark velocity in proton centre of mass frame and $\underline{V}_j$ the total jth quark velocity. Then if m_j, M are rest masses of jth quark and proton respectively.



$$\underline{k}_j = m_j \underline{V}_j = m_j(\underline{V} + \underline{\upsilon}_j)$$

$$\underline{k} = M\underline{V}$$

$$\therefore \hat{q} = \sum_{j=1}^{3} \frac{m_j}{M} \left| \frac{\underline{V} + \underline{\upsilon}_j}{V} \right| \frac{\sin\frac{\theta_j}{2}}{\sin\frac{\theta}{2}}\,\hat{q}_j .$$



If the individual scatterings of the quarks are through small angles θ_j, the $\hat{q}_j$ are almost transverse to the incident direction of motion. However, the resultant momentum transfer can be in directions making much larger angles with the incident direction of motion (overall scattering angle can be much longer) if $\frac{m_j}{M} \left| \frac{\underline{V} + \underline{\upsilon}_j}{V} \right| \frac{\sin\frac{\theta_j}{2}}{\sin\frac{\theta}{2}} \gg 1$.

i.e. $\frac{m_i}{M}\left|\frac{\underline{v}+\underline{v}_i}{v}\right| >> \frac{\sin\theta/2}{\sin\frac{\theta_i}{2}}$ and since $\theta > \theta_i$,

$$\frac{m_i}{M}\left|\frac{\underline{v}+\underline{v}_i}{v}\right| >> 1$$

or $$\frac{m_i}{M}\left(1+\frac{v_i^2}{v^2}+\frac{2\underline{v}\cdot\underline{v}_i}{v^2}\right)^{1/2} >> 1$$

$\therefore$ provided $\frac{v_i}{v} << 1$ i.e. motion of quarks in rest frame of proton is non relativistic, then $\frac{m_i}{M} >> 1$. Thus if quark mass is much larger than proton mass, as expected, the overal scattering angle of the proton is much larger than angles through which quarks are scattered. Thus the problem of the Glauber approximation and its angular range of validity is pertinent for its application to multiple quark scattering. Now the small angle approximation is $K R \sin\frac{\theta}{2} << 1$

where R is the range of the potential acting between each proton. At high energies ($k >> M$), each proton is Lorentz contracted to a thin disc so that, roughly, R must be replaced by $R(1-\beta^2)^{1/2}$ where $\beta = \frac{v}{c}$. Thus the effective range is much smaller, which means that the inequality above can be satisfied by larger values of scattering angles than is a priori expected.

Fundamental to the application of Glauber's theory to the quark model is the assumption of additivity of quark phase shifts (see equ.2 in Section E) Effectively this means that the interaction between the incident particle and the composite particle is instantaneous, or, equivalently, that the incident particle approaches the composite with infinite velocity. Since its velocity is bounded by the speed of light, the Fermi motion of the quarks will introduce finite corrections at all energies (including infinite energies). These should be very small for weakly bound systems, like the deuteron, with low internal momenta, a condition consistent with the non-relativistic quark model of hadrons. The assumption of the interaction being instantaneous in the centre of mass frame means that the quark-quark scatterings occur simultaneously as viewed in this frame. Because of time dilation, this will not be true in the rest frame of each composite particle, wherein the interactions will occur successively in time. If v is velocity of one composite relative to the centre of mass frame, then two simultaneous scattering events located at quark positions (x, y, z) and (x', y', z') relative to centre of mass frame will be separated by a time interval in their rest mass frames.

$$\Delta t = \frac{-v\,\Delta x}{c^2\left(1-\frac{v^2}{c^2}\right)^{1/2}}$$

where $\Delta x = x - x' = \Delta x_0\left(1-\frac{v^2}{c^2}\right)^{1/2}$, Δx_0 being the spatial separation in their rest mass frame. Thus $\Delta t = \frac{-v\,\Delta x_0}{c^2}$. As the velocity of the composite increases, the time interval between two scattering events in the rest mass frame increases and Fermi motion of quarks becomes more important (assumption of quarks being 'frozen' during multiple scattering becomes less valid). If V_0 is velocity of quark along x axis with respect to its parent particle rest frame then the distance moved in time Δt is $\frac{vV_0 \Delta x_0}{c^2}$ which is a maximum when $v = c$. The Fermi motion will be small provided $\frac{V_0 \Delta x_0}{c} << \Delta x_0$, i.e. $V_0 << c$, which isassumed in non-relativistic quark models.

Another physical assumption underlyingthe hypothesis of additivity of phase shifts is that the individual interactions are local, two body forces. If the observed saturation of quark content to two and three in mesons andbaryons respectively is, say, due to repulsive three and four body forces respectively, then many body forces, although having much less effect on hadron-hadron elastic scattering than the effective two body quark-quark and quark-antiquark interactions because the former would be of short range and would not influence small angle scattering, might yet be expected to be non negligible, especially at large momentum transfers, when considerable overlap of quarks in the interaction region should occur. Harrington(1) has shown that the corrections to proton-deuteron scattering arising from the presence of a three body force are small (typical correction to total cross-section is about-0.4%), though perhaps larger at larger momentum transfers. Such small effects are to be expected with weakly bound states where the expectation value of the interparticle distance is larger than the De Broglie wavelength so little overlap occurs and the phase shifts accumulated in scattering of a composite particle are additive. In the harmonic oscillator model, however, the De Broglie wavelength of a quark is $\lambda = h/\langle p \rangle$, where $\langle r \rangle = \langle -i\hbar\nabla \rangle$. This is related to the expectation value of the interquark distance (assuming an S state for simplicity) by $\lambda/\langle r \rangle = \pi/4 \sim 1$, so that the considerable overlap (tight binding) predicted by the model appears to make the assumption of only two body forces between actual quarks scattering each other rather dubious, a priori. But if, considering mesons, a repulsive core exists (evidence for which is given in QUARK DYNAMICS, Section E) with an exponential wavefunction in meson core with inverse range a, joined to a Gaussian component, outside the core with inverse range α, then it is simply shown that $\lambda/\langle r \rangle \alpha$ provided $a >> \alpha$. Thus with a short range core, three body effects for mesons should be negligible.

A further approximation to the actual physics of multiple quark scattering is the assumption of the validity of the impulse (or equivalently, the adiabatic) approximation in which the incident hadron passes through the composite target in a time much shorter than that characteristic for a quark to cross the hadron in its bound state motion. This is valid for incident particles with high laboratory momenta, $P_{LAB} >> \frac{\hbar}{R}$, where R is size of composite, and is consistent with non-relativistic motion for quarks inside hadrons.

The particle will then leave the interaction region long before inelastic excitations such as spin flip or transition of quarks to higher orbital angular momentum states are communicated to the composite particle as a whole. The approximation thus ignores off-the-mass shell scattering. One would expect this to be important (if at all) only for weakly bound systems and for large momentum transfer scattering, where, due to dominance of large orders of multiple scattering, such off-shell effects could accumulate and provide a non-negligible correction to multiple elastic scattering processes. For example, Harrington(2) has shown that in π -deuteron elastic scattering, while inelastic contributions are negligible near zero scattering angles, at large angles they are important. Underlying the adiabatic approximation is the neglect of quark recoil. The possible energy transfer between the incident hadron and the target will be small compared with the binding energy if quarks are massive since the former is then very large in order to account for the low lying hadron mass spectrum. It should be important only in weakly bound systems such as the deuteron(3).

ELASTIC PROTON-PROTON SCATTERING TO ALL ORDERS OF MULTIPLE QUARK-QUARK SCATTERING.

In the eikonal representation, the high energy, small angle of scattering elastic scattering amplitude is $f(\underline{q}) = {}^{i}/_{2\pi}\int e^{i\underline{q}\cdot\underline{b}}(1-e^{i\chi(\underline{b})})\,d^2\underline{b}$ (1)
where χ (b) is the phase shift of the $(kb-\frac{1}{2})$th partial wave and $f(q)$ is normalised so that $\frac{d\sigma}{dq^2}=\pi|f(q)|^2$. In the quark model of high energy Proton-Proton elastic scattering, the total phase shift of the scattered proton is the resultant of the phase shifts due to interaction of each of the three quarks in the proton with the other proton, the phase shifts being assumed additive so that

$$\chi(\underline{b}) = \sum_{n=1}^{3}\chi(\underline{b}-\underline{s}_n) = \chi(\underline{b},\underline{s}_1,\underline{s}_2,\underline{s}_3) \qquad (2)$$

where $\underline{s}_n$ is the projection of nth quark spatial coordinate $\underline{r}_n$ (with centre of mass of parent proton as origin of coordinate frame) onto the P-P impact parameter plane containing $\underline{b}$, and $\chi(\underline{b}-\underline{s}_n)$ is phase shift of nth quark due to scattering by proton (isospin invariance is assumed for quark-quark and therefore quark-proton interactions so that phase shifts are independant of quark index n). Noting that

(1) gives $e^{i\chi(b)} = 1 + {}^{i}/_{2\pi}\int f(q)\,e^{-i\underline{q}\cdot\underline{b}}\,d^2\underline{q}$ (3), then

$$e^{i\chi(\underline{b},\underline{s}_1,\underline{s}_2,\underline{s}_3)} = \prod_{n=1}^{3} e^{i\chi(\underline{b}-\underline{s}_n)} = \prod_{n=1}^{3}\left[1 + {}^{i}/_{2\pi}\int f(q_n)\exp[-i\underline{q}_n\cdot(\underline{b}-\underline{s}_n)]\,d^2\underline{q}_n\right]$$

Assuming small angle quark-proton scattering : $\underline{q}_n\cdot\underline{s}_n \simeq \underline{q}_n\cdot\underline{r}_n$,then

$$e^{i\chi(\underline{b},\underline{s}_1,\underline{s}_2,\underline{s}_3)} = 1 + \sum_{n=1}^{3}\left(\tfrac{i}{2\pi}\right)^n\binom{3}{n}\int\exp\Big[-i\sum_{m=1}^{n}\underline{q}_m\cdot(\underline{b}-\underline{r}_m)\Big]\prod_{m=1}^{n}\left[d^2\underline{q}_m\,f(q_m)\right] \qquad (4)$$

(possible spin dependance of quark-quark interactions are ignored so that ordering of quark amplitudes in the binomial expansion (4) is unimportant. Spin effects should be negligible in the asymptotic energy region to which the quark model, in this section, is applied).

The mth quark-proton scattering amplitude, with momentum transfer $\underline{q}_m$ in centre of mass frame of two protons, $f(\underline{q}_m)$ is related through (1) and (3) to the quark-quark scattering amplitude f_Q. One obtains

$$f(\underline{q}_m) \equiv f(\underline{q}_m; \underline{r}_1', \underline{r}_2', \underline{r}_3') = \sum_{\ell=1}^{3} \left(\frac{i}{2\pi}\right)^{\ell-1} \binom{3}{\ell} \int \delta^2\left(\underline{q}_m - \sum_{k=1}^{\ell} \underline{q}_k\right) e^{i \sum_{k=1}^{\ell} \underline{q}_k \cdot \underline{r}_k'} \prod_{k=1}^{\ell} \left[d^2 \underline{q}_k f_Q(\underline{q}_k)\right]$$

$$\therefore e^{i\chi(\underline{b}, \underline{s}_1, \underline{s}_2, \underline{s}_3; \underline{s}_1', \underline{s}_2', \underline{s}_3')} - 1 = \sum_{n=1}^{3} \left(\frac{i}{2\pi}\right)^n \binom{3}{n} \int e^{-i \sum_{m=1}^{n} \underline{q}_m \cdot (\underline{b} - \underline{r}_m)} \prod_{m=1}^{n} \Big[d^2 \underline{q}_m \sum_{\ell=1}^{3} \left(\frac{i}{2\pi}\right)^{\ell-1} \binom{3}{\ell} \cdot$$

$$\int \delta^2\left(\underline{q}_m - \sum_{k=1}^{\ell} \underline{q}_k\right) e^{i \sum_{k=1}^{\ell} \underline{q}_k \cdot \underline{r}_k'} \prod_{k=1}^{\ell} \left[d^2 \underline{q}_k f_Q(\underline{q}_k)\right]\Big] \qquad (5)$$

the P-P elastic scattering amplitude, averaged over spatial coordinates of quarks in both protons is

$$F_{PP}(q) = \langle PP | F(\underline{q}; \underline{r}_1, \underline{r}_2, \underline{r}_3; \underline{r}_1', \underline{r}_2', \underline{r}_3') | PP \rangle \qquad (6)$$

where $F(\underline{q}; \underline{r}_1, \underline{r}_2, \underline{r}_3; \underline{r}_1', \underline{r}_2', \underline{r}_3') = \frac{-i}{2\pi} \int e^{i \underline{q} \cdot \underline{b}} \left(e^{i\chi(\underline{b}; \underline{s}_1, \underline{s}_2, \underline{s}_3; \underline{s}_1', \underline{s}_2', \underline{s}_3')} - 1 \right) d^2 \underline{b}$

The amplitude is determined therefore when diagonal matrix element of overal partial wave amplitude $e^{i\chi(b, s_1, s_2, s_3, s_1', s_2', s_3')} - 1$ is calculated. Denoting n body form factor of either proton as $\Phi(\underline{q}_1, \underline{q}_2, \ldots \underline{q}_n, 0, \ldots 0) = \langle P | e^{i \sum_{m=1}^{n} \underline{q}_m \cdot \underline{r}_m} | P \rangle$

then $\langle PP | e^{i\chi(b, s_1, s_2, s_3; s_1', s_2', s_3')} - 1 | PP \rangle =$

$$= \sum_{n=1}^{3} \left(\frac{i}{2\pi}\right)^n \binom{3}{n} \int e^{-i \sum_{m=1}^{n} \underline{q}_m \cdot \underline{b}}\, \Phi(\underline{q}_1, \ldots \underline{q}_n, 0) \prod_{m=1}^{n} \Big[d^2 \underline{q}_m \sum_{\ell=1}^{3} \left(\frac{i}{2\pi}\right)^{\ell-1} \binom{3}{\ell} \cdot$$

$$\int \Phi(\underline{q}_1, \ldots \underline{q}_\ell, 0)\, \delta^2\left(\underline{q}_m - \sum_{k=1}^{\ell} \underline{q}_k\right) \prod_{k=1}^{\ell} \left[d^2 \underline{q}_k f_Q(\underline{q}_k) \right]\Big] \qquad (7)$$

letting $G_\ell(q_m) = \int \Phi(\underline{q}_1, \ldots \underline{q}_\ell, 0)\, \delta^2\left(\underline{q}_m - \sum_{k=1}^{\ell} \underline{q}_k\right) \prod_{k=1}^{\ell} \left[d^2 \underline{q}_k f_Q(\underline{q}_k) \right]$ (8)

the quark-quark elastic amplitude is approximated by comparing the single scattering contribution to the overal amplitude with the near forward P-P scattering amplitude. The latter is parametrised as

$F_{PP}(q^2) = (i + \alpha_{PP}) \frac{\sigma_T}{4\pi} e^{-\frac{1}{2} S q^2}$ where $\alpha_{PP} = \frac{\mathcal{R}(F_{PP}(0))}{\mathcal{I}[F_{PP}(0)]}$, σ_T is P-P total cross section, S is inverse width of P-P diffraction peak.. The former is

$$F^{(1)}_{PP}(q) = \binom{3}{1}\binom{3}{1} \Phi^2(q, 0, 0)\, f_Q(q)$$

$\therefore$ from the approximation (neglecting higher order contributions to the diffraction peak)

$F^{(1)}_{PP}(q) \simeq F_{PP}(q)$ we obtain

$f_Q(q) = \Phi^{-2}(q, 0, 0)\, f_Q(0)\, e^{-\frac{1}{2} S q^2}$ where

$$f_Q(0) = \frac{\sigma_T}{36\pi}(i + \alpha_{PP})$$

The proton is assumed to be in the $\underline{56}$ irreducible representation of SU(6) with parafermi statistics for quarks. The proton ground state is represented by the $L=0$ harmonic oscillator wavefunction $\Psi(r_1, r_2, r_3) = N e^{-\frac{\alpha^2}{2} \sum_{n<m} (r_n - r_m)^2}$ where N is a normalisation constant.

The n-body proton form factor is then

$\Phi(q_1, q_2, \ldots q_n, 0) = e^{-\frac{\langle r^2 \rangle}{6} \sum_{m=1}^{n} q_m^2}$ where $\langle r^2 \rangle$ is the mean square radius of the proton. Then

(9) $G_\ell(q_m) = [f_Q(0)]^\ell H_\ell(q_m)$ where, from equ (8)

$$H_\ell(q_m) = \int \Phi(q_1, \ldots q_\ell, 0)\, \delta^2\left(\underline{q}_m - \sum_{k=1}^{\ell} \underline{q}_k\right) \prod_{k=1}^{\ell} \left[\Phi^{-2}(q_k, 0, 0)\, e^{-\frac{1}{2} S q_k^2} d^2 \underline{q}_k \right]$$

Noting that

$\Phi(q_1, q_2, \ldots q_\ell, 0) = \prod_{k=1}^{\ell} \Phi(q_k, 0, 0)$ we obtain

$H_\ell(q_m) = \frac{1}{\ell} (\pi/\beta')^{\ell-1} \exp\left[-\frac{\beta'}{\ell} q_m^2\right]$ where $\beta' = \frac{1}{2}\left(S - \frac{\langle r^2 \rangle}{3}\right)$

$\therefore$ equ 9 gives

$$G_\ell(q_m) = \frac{1}{\ell} \left(\frac{\pi}{\beta'}\right)^{\ell-1} [f_Q(0)]^\ell e^{-\frac{\beta'}{\ell} q_m^2}$$

equ (7) becomes $\langle pp| e^{i\chi}-1|pp\rangle = \sum_{n=1}^{3} (\frac{i}{2\pi})^n \binom{3}{n} \int e^{-i\sum_{m=1}^{n} q_m\cdot b}\, \Phi(q_1,\dots q_n,0)\prod_{m=1}^{n} [d^2q_m \sum_{\ell=1}^{3} \frac{1}{\ell}(\frac{i}{2\beta'})^{\ell-1}\binom{3}{\ell}[f_q(0)]^\ell e^{-\frac{\beta'}{\ell}q_m^2}]$

$$= \sum_{n=1}^{3} (\tfrac{i}{2\pi})^n\binom{3}{n}\int\prod_{m=1}^{n}[d^2q_m\sum_{\ell=1}^{3} a_\ell\, e^{-[\alpha_\ell q_m^2+ib\cdot q_m]}]$$

where

$$a_\ell = \frac{1}{\ell}\left(\frac{i}{2\beta'}\right)^{\ell-1}\binom{3}{\ell}[f_q(0)]^\ell \;;\quad \alpha_\ell = \frac{\beta'}{\ell}+\frac{\langle r^2\rangle}{6} \qquad (10)$$

The matrix element becomes

$$\langle pp|e^{i\chi}-1|pp\rangle = \sum_{n=1}^{3}\binom{3}{n}\Big[\sum_{\ell=1}^{3} A_\ell\, e^{-\frac{b^2}{4\alpha_\ell}}\Big]^n$$

where

$$A_\ell = \tfrac{i}{2}\, a_\ell/\alpha_\ell \qquad (11),$$

equ. (6) reduces to $F_{pp}(q) = \frac{-i}{2\pi}\int e^{iq\cdot b}\langle pp|e^{i\chi}-1|pp\rangle d^2b = -i\sum_{n=1}^{3}\binom{3}{n}\int_0^\infty J_0(bq)\,b\Big[\sum_{\ell=1}^{3} A_\ell e^{-\frac{b^2}{4\alpha_\ell}}\Big]^n db$

$= \sum_{n=1}^{3} F_{pp}^{(n)}(q)$ where $F^{(n)}_{pp}(q) = -i\binom{3}{n}\int_0^\infty J_0(bq)\,b\Big[\sum_{\ell=1}^{3} A_\ell\, e^{-b^2/4\alpha_\ell}\Big]^n db$

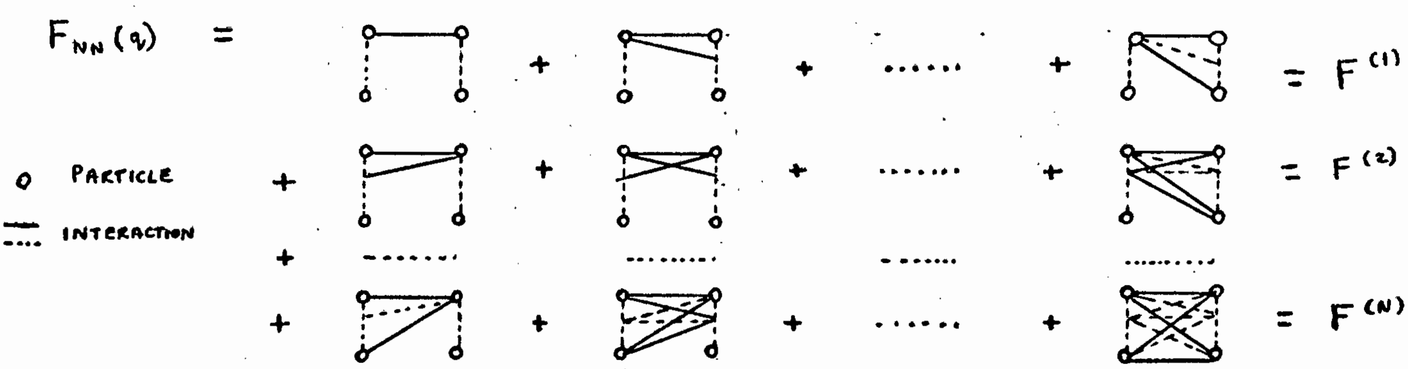
$F^{(n)}_{pp}$ is the contribution to the overall scattering amplitude when n quarks of one proton interact with the quarks of the other proton. Generalisation to identical composite particle scattering is straightforward. For elastic scattering of two particles made up of N identical subparticles, the scattering amplitude is

$$F_{NN}(q) = \sum_{n=1}^{N} F^{(n)}_{NN}(q)$$

where $F^{(n)}_{NN}(q)$ is given by $F^{(n)}_{NN}(q) = -i\binom{N}{n}\int_0^\infty J_0(bq)\,b\Big[\sum_{\ell=1}^{N} A_\ell\, e^{-b^2/4\alpha_\ell}\Big]^n db$

with $A_\ell = \frac{i}{2}\, a_\ell/\alpha_\ell$; $a_\ell = \ell^{-1}(i/2\beta')^{\ell-1}\binom{N}{\ell}[f(0)]^\ell$; $\alpha_\ell = \beta'/\ell + \langle r^2\rangle/6$

where $f(0)$ is forward particle-particle amplitude. The multiple scattering expansion may be represented diagrammatically as shown below.



$$\text{Now}\quad \Big[\sum_{\ell=1}^{3} A_\ell\, e^{-b^2/4\alpha_\ell}\Big]^n = \sum_{r,s,\dots t}^{3} A_r A_{s},\dots A_t\, e^{-b^2/4\alpha_\ell(\alpha_r^{-1}+\alpha_s^{-1}+\dots+\alpha_t^{-1})} \;,\quad \{r,s,t\} = 1,2 \text{ or } 3$$

Writing $A_r, A_s\dots A_t$ as $A_{r,s,\dots t}$ and $\alpha_r^{-1}+\alpha_s^{-1}+\dots+\alpha_t^{-1}$ as $\alpha^{-1}_{r,s,\dots t}$

then $F^{(n)}_{pp}(q) = -2i\binom{3}{n}\sum_{r,s,\dots t} A_{r,s,\dots t}\,\alpha_{r,s,\dots t}\, e^{-\alpha_{r,s,\dots t}q^2}$

Explicitly, $F^{(1)}_{pp}(q) = -2i\binom{3}{1}\sum_{r=1}^{3} A_r(\alpha_r^{-1})^{-1}e^{-(\alpha_r^{-1})^{-1}q^2}$

$$F^{(2)}_{pp}(q) = -2i\binom{3}{2}\sum_{\substack{r=1\\s=1}}^{3} A_rA_s(\alpha_r^{-1}+\alpha_s^{-1})^{-1}e^{-[\alpha_r^{-1}+\alpha_s^{-1}]^{-1}q^2}$$

$$F^{(3)}_{pp}(q) = -2i\binom{3}{3}\sum_{\substack{r=1\\s=1\\t=1}}^{3} A_rA_sA_t(\alpha_r^{-1}+\alpha_s^{-1}+\alpha_t^{-1})^{-1}e^{-[\alpha_r^{-1}+\alpha_s^{-1}+\alpha_t^{-1}]^{-1}q^2}$$

The contributions to the Nth order scattering amplitude $(1\le N\le 9)$ from the various scattering diagrams may be shown by writing

$$F_{pp}(q) = \sum_{N=1}^{9} F^{(N)}_{pp}(q) \qquad (12)$$

where

$$F^{(N)}_{pp}(q) = -2i\sum_{n=1}^{3}\binom{3}{n}A_{r,s,\dots t}\,\alpha_{r,s,\dots t}\,e^{-\alpha_{r,s,\dots t}q^2} \qquad (13)$$

where summation convention is applied to r,s,t and r,s,t have values from 1 to 3 subject to $r+s+t = N$, the restriction arising from the observation that Nth order scattering contributions must contain the quark-quark amplitude raised to the power N and $A_{r,s,\ldots t} \propto a_r a_s \ldots a_t$ from (11)

$\propto [f(0)]^{r+s+t}$ from (10)

The differential cross-section is, for P-P elastic scattering,

$$\frac{d\sigma}{dq^2} = \pi |F_{pp}(q)|^2 = \pi \sum_{N=1}^{q} |F^{(N)}(q)|^2 + 2\pi \mathcal{R} \sum_{\substack{M,N \\ M \neq N}} F^{(M)}(q) F^{(N)*}(q)$$

$$= \sum_{N=1}^{q} \frac{d\sigma^{(N)}}{dq^2}, \quad \text{where } \frac{d\sigma^{(N)}}{dq^2} = \pi \Big[|F^{(N)}|^2 + 2\mathcal{R} \sum_{M, M \neq N} F^{(M)} F^{(N)*} \Big]$$

The first term is the incoherent contribution due to Nth order scattering, the second represents the interference of the Nth order coherent contribution from all other orders, giving rise to diffractive minima. Only the interference term with $M = N+1$ should be important in the interference region where $(N+1)$th order scattering takes over from Nth order.

MULTIPLE SCATTERING CONTRIBUTIONS TO P-P TOTAL CROSS-SECTION.

From (12), (13), using the optical theorem,

$$\sigma_T = 4\pi \mathcal{I} F_{pp}(0) = \sum_{N=1}^{q} \sigma_T^{(N)},$$

where contribution from Nth order quark-quark scattering is $\sigma_T^{(N)} = -8\pi \sum_{n=1}^{3} \binom{3}{n} \alpha_{r,s,\ldots t} \mathcal{R}[A_{r,s,\ldots t}]$, $r+s+t = N$

Denoting $f_Q(0)$ by $f_Q(0) = |f_Q(0)| e^{i\phi}$ with $\pi > \phi = \operatorname{Arg} f_Q(0) \geq \pi/2$, $\alpha_{pp}^{-1} = \tan\phi < 0$

then from (10), (11), $\operatorname{Arg}[A_{r,s,\ldots t}] = (r+s+t)(\phi + \pi/2) = N(\phi + \pi/2)$

$$\therefore \quad \tfrac{3\pi}{2} N > \operatorname{Arg}[A_{r,s,\ldots t}] \geq N\pi$$

Since at high energies, forward elastic P-P amplitude is mostly positive imaginary ($\phi \approx \frac{\pi}{2}$) we see that for odd orders of scattering ($N = 1, 3, \ldots$), $\mathcal{R}(A_{r,s,\ldots t}) < 0$ i.e. odd orders of scattering increase total cross section, whilst for $N = 2, 4, \ldots$, $\mathcal{R}(A_{r,s,\ldots t}) > 0$ i.e. even orders reduce total cross section. The double scattering correction reduces the cross section, as is well known[4]. This result should be true in the asymptotic limit provided the ratio of real part to imaginary part of the forward scattering amplitude approaches zero from the negative side, as is the present trend of data obtained from Coulomb scattering. The alternation in sign of successive terms in the multiple quark scattering series, proved above in the limit of infinitely high centre of mass energies ($\alpha_{pp} = 0$)* when the forward elastic amplitude is pure imaginary, is also characteristic of infinitely composite models of high energy P-P scattering[5] Also, the Regge cut sequence generated by multi Pomeranchon exchange has this sign alternation.[6]

RESULTS:-

The asymptotic P-P differential cross section has been calculated over the (momentum transfer)^2 range of

$$0 \leq |t| \leq 20 \ (\mathrm{GeV}/c)^2$$

Pointlike quarks were assumed so that the charge and matter radii of the protons are the same. The Hofstadter value of 0.81F for the charge.

*FOOTNOTE:-

this is not necessarily true at non-asymptotic energies since, in general, the Nth order contribution to σ_T has sign of $(-1)^{\frac{N-1}{2}} \sin N\phi$ if N is odd and $(-1)^{\frac{N-2}{2}} \cos N\phi$ if N is even so that the signs of the terms may not alternate regularly)

ORDER OF MULTIPLE SCATTERING	MULTIPLE SCATTERING CONTRIBUTION TO P-P TOTAL CROSS SECTION, σ_T (the numbers are statistical weights for each scattering process, σ_{rst} is the contribution to σ_T due to $(r+s+t)$ th order scattering in which 1 quark is scattered r times, the 2nd s times and the 3rd t times)	% CONTRIBUTION TO P-P TOTAL CROSS SECTION
1	$9\ \sigma_{100}$	79
2	$9\ \sigma_{200} + 27\ \sigma_{110}$	19
3	$3\ \sigma_{300} + 54\ \sigma_{210} + 27\ \sigma_{111}$	1
4	$27\ \sigma_{220} + 18\ \sigma_{310} + 81\ \sigma_{211}$	2×10^{-4}
5	$18\ \sigma_{320} + 27\ \sigma_{311} + 81\ \sigma_{212}$	8×10^{-8}
6	$3\ \sigma_{330} + 27\ \sigma_{222} + 54\ \sigma_{321}$	4×10^{-12}
7	$9\ \sigma_{331} + 27\ \sigma_{322}$	4×10^{-16}
8	$9\ \sigma_{332}$	9×10^{-21}
9	$1\ \sigma_{333}$	2×10^{-25}
		100%

radius was assumed. This gives $\langle r^2\rangle = 4\cdot 1\,(\mathrm{GeV}/c)^2$. The following other parameter values were assumed:-

$$\alpha_{pp} = 0$$

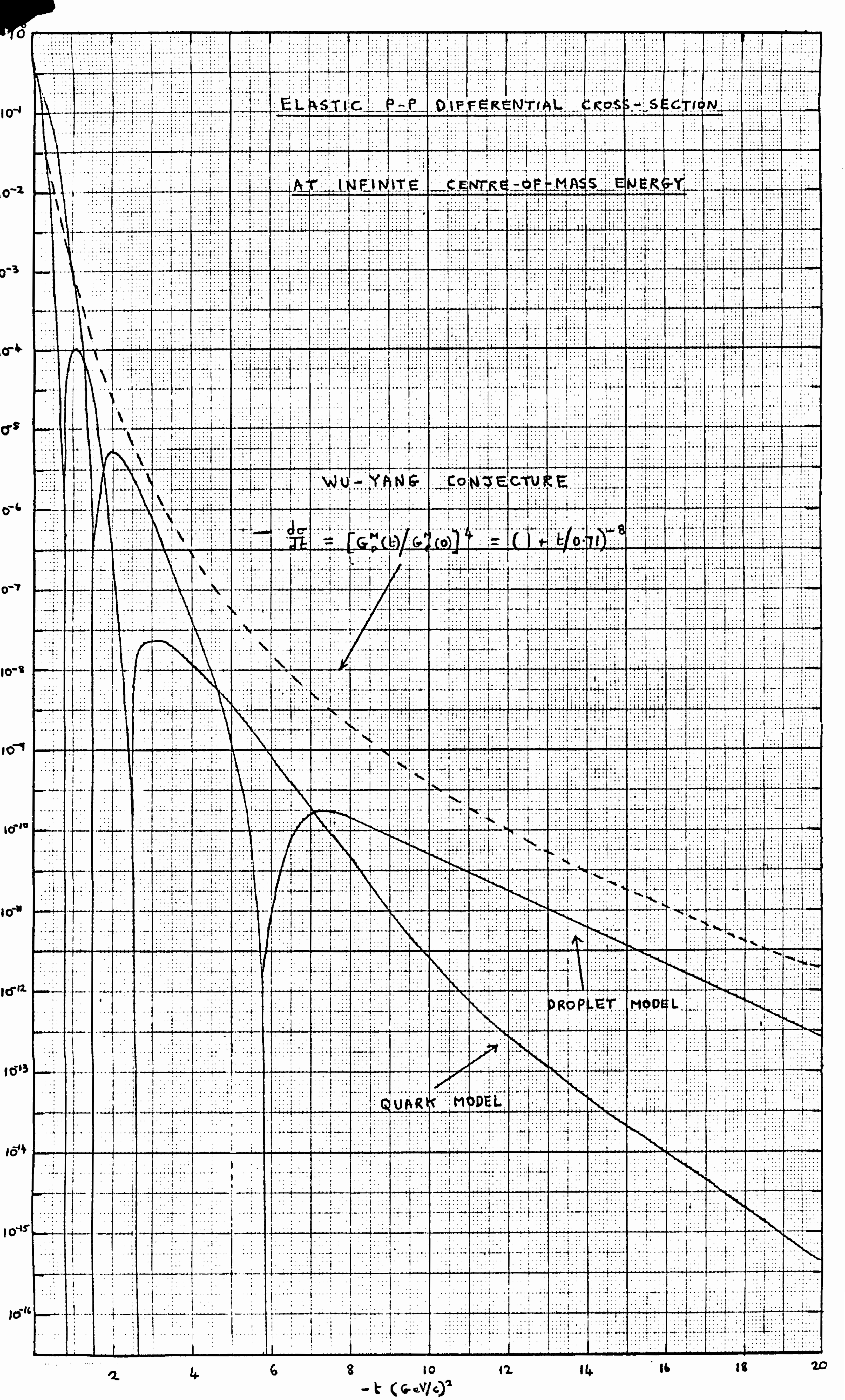
$$\sigma_T = 38\ \mathrm{m.B.}$$

$$\xi = 10\ (\mathrm{GeV}/c)^{-2}$$

The calculated cross section is shown on the graph. For comparison, the droplet model prediction of Durand and Lipes and the Wu-Yang conjecture are included as well. There is little qualitative difference between the quark and droplet model results, both predicting two diffraction zeros, although the onset of the double and triple scattering regions occur closer to the diffraction peak in the quark calculation than do the corresponding regions in the droplet model.
The large momentum transfer differential cross-section predicted by the quark model falls more rapidly, presumably due to the assumption of gaussian matter form factors, instead of dipole form factors which the Durand and Lipes calculation employed.
Contributions from the quark multiple scattering processes to the forward scattering amplitude and therefore, from the optical theorem, to the total cross-section are shown in the table. As expected, these decrease very rapidly as the order of scattering increases. As a check to the statistical weights given to various multiple quark-quark interactions in the table, we note that the number of distinguishable ways in which N quarks out of 3 in one proton and 3 in the other can interact via two body forces to give rise to order scattering is $\binom{9}{N}$. The total number is

$$\sum_{N=1}^{9}\binom{9}{N} = \sum_{N=0}^{9}\binom{9}{N} - 1 = 2^9 - 1 = 511$$

which is the total statistical weight of the terms shown in the table. It



ELASTIC P-P DIFFERENTIAL CROSS-SECTION
AT INFINITE CENTRE-OF-MASS ENERGY
WU-YANG CONJECTURE
$-\frac{d\sigma}{dt} = [G_p^M(t)/G_p^M(0)]^4 = (1 + t/0.71)^{-8}$
DROPLET MODEL
QUARK MODEL
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-11}
10^{-12}
10^{-13}
10^{-14}
10^{-15}
10^{-16}
2
4
6
8
10
12
14
16
18
20
$-t\ (GeV/c)^2$

should be interesting to see whether future experiments show, in P-P elastic scattering, the gradual formation of dips and eventual minima in the differential cross-section, as expected, and in particular whether the fixed, large angle differential cross section falls below the droplet model prediction as this would indicate finite compositeness of the proton (though not necessarily the 3 particle structure provided by the quark model).

N is even so that the signs of the terms may not alternate regularly.

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CHAPTER 3:-

DYNAMICS OF THE QUARK MODEL.

In the quark model of hadrons, baryon (with odd half integer spin) and mesons (with integer spin) are considered as composite bound states of three quarks and quark-antiquark pairs respectively. The spin $\frac{1}{2}$ quarks may be regarded as fermions or parafermions. Internal quantum numbers of the hadrons are given by the corresponding quantum numbers of the constituent quarks whilst their masses depend upon the quantum numbers, masses and nature of interaction of quarks among themselves. Any satisfactory dynamical theory of the quark-quark (Q-Q) and quark-antiquark interaction ($Q\bar{Q}$) must have at least three features.

(1) It must lead to SU (6) symmetry in the non-relativistic limit of quark relative motion and hopefully explain why the static SU (6) model appears to be realised for the hadrons, in particular, why the 56 is lower in mass than the 20 or 70.

(2) It should predict the masses of known baryon and mesons, in particular explaining why the QQQ and $Q\bar{Q}$ configurations lie lowest in the hadron mass spectrum, whilst ruling out the possibility of exotic QQ, QQQQ, $QQ\bar{Q}Q$ etc., states or at least predicting very large masses for them. The zero mass state should be exhibited as a singularity in the dynamical equations of motion.

(3) It should predict linear Regge trajectories for both QQQ and $Q\bar{Q}$ bound states in the low lying end of the mass spectrum of hadrons.

A simple model of mesons is developed below which displays feature 3. In Section A, the Bethe-Salpeter equation for two spinless particles with scalar interactions is shown to lead to a Schroedinger type equation, with $(\text{energy})^2$ instead of energy, dependence. This is solved for a harmonic oscillator binding potential. In Section B, R.M.S. radii for the pion, (and, by extrapolation) for the proton are calculated. These are used to estimate quark change radii (see ELECTROMAGNETIC INTERACTIONS, section C) The strange - non strange quark mass difference is estimated from Y=0 and Y=1 meson Regge trajectories. Estimates of the pion electromagnetic mass splitting are given in Section C. In Section D, phenomenological analysis of high energy proton-antiproton elastic scattering in terms of the multiple scattering quark model is coupled to the non-relativistic harmonic oscillator model to provide a crude estimate of the free quark mass in two limiting cases. Lastly, in Section E, difficulties of the harmonic oscillator model of mesons in relation to possible core in quark-antiquark interactions are discussed.

SECTION A:-

The striking success of SU(6) symmetry[1] as a symmetry for strong interactions means that we expect the superstrong forces responsible for the binding of quarks and antiquarks in mesons to be spin independent and to lead to non relativistic motion for quarks in the bound state so that intrinsic quark spin is separately conserved. Morpurgo[2] and others[3] have shown that such motion is compatible with the huge binding energies of mesons necessary if quark masses are large, as is required by accelerator experiments[4] and cosmic ray data[5]. The question arises - what is the nature of the binding potential? Is it a Yukawa potential, as is the case for the main Coulomb potential in the hydrogen atom, with zero mass photon exchange between the electron and proton? Although this can lead to non-relativistic motion provided the mass of the exchanged particle is much smaller than the free quark mass, the Regge trajectory associated with a light pseudo-scalar boson exchange between a fermion and an anti-fermion in a bound state described by the Bethe-Salpeter equation is much too shallow[6]. For exchange of a single neutral vector meson, quark-antiquark forces are attractive[7] but quark-quark forces are repulsive, so that while the mechanism leads to mesons, it cannot account for baryons. Exchange of all possible vector and pseudo-scalar mesons between the quark and antiquark produces attraction for mesons in <u>35</u> SU(6) multiplet, attraction for quark-quark interactions in baryons belonging to the <u>20</u> and <u>70</u> but repulsion for baryons in the <u>56</u>, implying the latter to be higher in mass than the former, whereas the reverse is the case, experimentally. Discarding exchange forces between quarks and antiquarks leaves direct forces, which may be scalar or vector. Noting that the n- dimensional harmonic oscillator is invariant under SU(n), we choose the 3-dimensional simple harmonic oscillator potential. Since this is an even potential, it is consistent with S wave dominance for the Q-$\bar{Q}$ interaction, argued by Mitra, see A.N. Mitra, Phys. Rev. 142, 1119 (1966). Ignoring the quark spin, the spinor Bethe-Salpeter[6] equation simplifies to

$$(\Box_1 - M_1^2)(\Box_2 - M_2^2)\Psi(x_\mu^{(1)}, x_\mu^{(2)}) = U(x_\mu^{(1)}, x_\mu^{(2)})\Psi(x_\mu^{(1)}, x_\mu^{(2)}) \qquad (1)$$

where M_1, M_2 are rest masses of quark and antiquark, U - the interaction potential and $x^{(1)}, x^{(2)}$ - the 4-coordinates of quark and antiquark. U is assumed to be a two body potential.

$$\therefore \left[-\frac{\Box_1\Box_2}{M_2^2} + \lambda^2\Box_2 + \Box_1 - M_1^2 + \frac{U}{M_2^2}\right]\Psi(x_\mu^{(1)}, x_\mu^{(2)}) = 0 , \qquad \lambda = \frac{M_1}{M_2}$$

we assume quarks are massive but $\lim_{M_1, M_2 \to \infty}\left[\frac{U}{M_2^2} - M_1^2\right] = V \qquad (2)$

and $\lim_{M_1, M_2 \to \infty} M_1/M_2 = \lambda \qquad (3)$

in this limit of very large masses, first term can be ignored.

using metric :-

$$\Box_2 = \sum_{\mu=0}^{3} \partial^{2(2)}_{\mu} = \partial_0^{2(2)} - \partial_\mu^{2(2)}$$ where summation convention is applied to space partial differential operators.

$$\therefore [\lambda^2 \partial_0^{2(2)} + \partial_0^{2(1)} - \lambda^2 \partial_\mu^{2(2)} - \partial_\mu^{2(1)} + \mathcal{V}]\Psi = 0$$

letting $x_\mu = x_\mu^{(1)} - x_\mu^{(2)}$ (relative coordinate)

$$X_\mu = \frac{\lambda x_\mu^{(1)} + x_\mu^{(2)}}{1+\lambda}$$ (C.M. coordinate)

$$\partial_\mu^{(1)} = \partial x_\mu + \frac{\lambda}{1+\lambda}\partial X_\mu \qquad \mu = 1,2,3$$

$$\partial_\mu^{(2)} = -\partial x_\mu + \frac{1}{1+\lambda}\partial X_\mu$$

$$\therefore \partial^{2(1)}_\mu = \partial^2 x_\mu + \frac{\lambda^2}{(1+\lambda)^2}\partial^2 X_\mu + \frac{2\lambda}{1+\lambda}\partial x_\mu \partial X_\mu$$

$$\partial^{2(2)}_\mu = \partial^2 x_\mu + \frac{1}{(1+\lambda)^2}\partial^2 X_\mu - \frac{2}{1+\lambda}\partial x_\mu \partial X_\mu$$

$$\therefore [\lambda^2 \partial_0^{2(2)} + \partial_0^{2(1)} - (1+\lambda^2)\partial^2_{x_\mu} - \frac{2\lambda^2}{(1+\lambda)^2}\partial^2_{X_\mu} - \frac{2\lambda}{1+\lambda}(1-\lambda)\partial x_\mu \partial X_\mu + \mathcal{V}]\Psi = 0$$

We assume quarks have almost same mass (SU(2) almost exact symmetry)

$$\therefore \lambda \doteq 1$$

ignoring last term, $[\lambda^2 \partial_0^{2(2)} + \partial_0^{2(1)} - (1+\lambda^2)\partial^2_{x_\mu} - \frac{2\lambda^2}{(1+\lambda)^2}\partial^2_{X_\mu} + \mathcal{V}]\Psi = 0$

$$\Psi = \Psi(x_m, X_m) = \Phi(x_m)\psi(X_m) \qquad m = 1,2,3,4.$$

$$\Phi(x_m) = \Phi(x_\nu, x_0^{(1)} - x_0^{(2)}), \quad \psi(X_m) = \psi(X_\nu, X_0) \qquad \nu = 1,2,3$$

where $\Phi(x_\nu, x_0^{(1)} - x_0^{(2)}) = \phi(x_\nu)e^{iM(x_0^{(1)}-x_0^{(2)})}$, M-total mass of bound state

$$\psi(X_\nu, X_0) = e^{i[\underset{\sim}{P}\cdot\underset{\sim}{X} - X_0 t]}$$

$$\therefore [-(1+\lambda^2)M^2 - (1+\lambda^2)\nabla_\nu^2 + \frac{2\lambda^2}{(1+\lambda)^2}P^2 + \mathcal{V}]\phi(x_\nu) = 0$$

in C.M. frame, $P = 0$

$$\therefore [M^2 + \nabla_\nu^2]\phi(x_\nu) = \frac{\mathcal{V}}{1+\lambda^2}\phi(x_\nu) \tag{4}$$

$$\therefore [-\frac{\nabla_\nu^2}{2\mu_q} + \frac{\mathcal{V}}{2(1+\lambda^2)\mu_q}]\phi(x_\nu) = \frac{M^2}{2\mu_q}\phi(x_\nu)$$

where $\mu_q = \frac{M_1 M_2}{M_1 + M_2} = \frac{M_1}{1+\lambda}$ is reduced mass of quark.

This is a Schroedinger equation for particle with effective rest mass μ_q kinetic energy $-\frac{\nabla_\nu^2}{2\mu_q} = \frac{P^2}{2\mu_q}$ and total energy $\frac{M^2}{2\mu_q}$, where P is momentum

Noting that when $\nabla_\nu^2 = 0$, $M = M_1 + M_2 = M_1(\frac{1+\lambda}{\lambda})$
$= \frac{1}{\lambda}(1+\lambda)^2\mu_q$

$$\therefore \frac{\mathcal{V}}{(1+\lambda^2)2\mu_q} = V + \frac{(1+\lambda)^4}{\lambda^2}\frac{\mu_q^2}{2\mu_q} = V + \frac{(1+\lambda)^4}{2\lambda^2}\mu_q$$

where V is the potential energy of the particle.

$$\therefore [M^2 + \nabla_\nu^2 - \frac{(1+\lambda)^4}{\lambda^2}\mu_q^2 - 2\mu_q V]\phi(x_\nu) = 0 \tag{5}$$

when masses are equal, $\lambda \doteq 1$, $\mu_q = \frac{M_1}{2} = \frac{M_q}{2}$ where M_q is effective quark mass. Equation of motion of quark-antiquark bound state is

$$[M^2 + \nabla_\nu^2 - 4M_q^2 - M_q V]\phi(x_\nu) = 0 \tag{6}$$

(Note: above, relative frame wavefunction $\Phi(x_m)$ was written $\Phi(x_m) = \phi e^{iM(t_1 - t_2)} = \phi e^{iMt}$ where $t = t_1 - t_2$ is <u>relative</u> time)

This has no singularity at M = O as was required in feature (2). However it exhibits M^2 dependence and hence predicts linear Regge trajectories.

Writing $V = -V_0 + \frac{1}{2}\mu_q\omega^2 r^2$ where $\mu_q = \frac{M_q}{2}$

the solution is $\phi(r,\theta,\phi) = N_{n,L,m}\, e^{im\phi} P_L^m(\cos\theta)(\beta r^2)^{L/2} e^{-\frac{\beta r^2}{2}} L_{n-1}^{L+\frac{1}{2}}(\beta r^2)$

where $$N_{n,L,m} = \left[\frac{\beta^{3/2} (L-m)!\,(n-1)!\,(2L+1)}{2\pi\,(L+m)!\,\Gamma^3(L+n+1/2)} \right]^{1/2} \qquad (7)$$

and $$\beta = M_Q \omega \qquad (8)$$

the $(\text{mass})^2$ eigenvalues are $M^2 = 4M_Q^2 - M_Q V_0 + \frac{3}{2} M_Q \omega + M_Q\omega(2n+L-2)$

with $n = 1, 2, \ldots \infty$

$L = 0, 1, 2, \ldots \infty$

$m = 0, 1, 2, \ldots L$

SECTION B:

For purely orbital excitations of ground state, $M^2 \propto L$

$$\therefore \Delta M^2 \equiv M^2_{L+1} - M^2_L = M_Q \omega \qquad (9)$$

For the pion, $n=1, L=0=m$ (corresponding to ${}^1_0S$)

$$\text{wavefunction of pion} = (M_Q\omega/2\pi)^{3/4}\, e^{-\frac{M_Q \omega r^2}{4}} \qquad (10)$$

giving $\langle r^2 \rangle_\pi = 3(M_Q\omega)^{-1}$

$\sqrt{\langle r^2 \rangle_\pi} = \sqrt{3/\Delta M^2}$ (ignoring mass differences in quark-isospin doublet) (11)

ΔM^2 has been estimated using the ρ Regge trajectory, as the vector meson $SU(3)$ nonet is almost ideal (see Section C), Using data compiled by Rosenfeld et al.[7] and taking as statistical weights for $(\text{mass})^2$ intervals between successive Regge recurrences, the recipricals of standard errors associated with intervals, the mean value of ΔM^2 is found to be

$$\overline{\Delta M^2_\rho} = 1{\cdot}07\ (\text{GeV})^2 \qquad (12)$$

This gives for the R.M.S. radius of the pion

$$\sqrt{\langle r^2 \rangle_\pi} = 0{\cdot}17\,\text{F} \qquad (13)$$

The R.M.S. radius of the proton has been estimated by assuming that it lies in the $\underline{56}$ representation of $SU(6)$ with ground state $(1S)^3$, $L^P = 0^+$.

For this state, the shell model wavefunction for the proton is

$$\Psi_p(r_1, r_2, r_3) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{9/2} e^{-\frac{\alpha^2}{2}(r_1^2 + r_2^2 + r_3^2)} \qquad (14)$$

with $\underline{r}_1 + \underline{r}_2 + \underline{r}_3 = 0$. This gives

$$\langle r^2 \rangle_p = 1/\alpha^2$$

The parameter α is found by transforming the quark coordinates into relative and centre of mass coordinates and comparing resultant harmonic oscillator wave-function with ground state wavefunction of three dimensional harmonic oscillator, whose form was given in Section A, equ 7.

let $\underline{\lambda} = \underline{r}_1 - \underline{r}_2$ be relative coordinate

$\underline{\rho} = \frac{1}{2}(\underline{r}_1 + \underline{r}_2)$ be C.M. coordinate of pair relative to Proton C.M. $\therefore$ from (13) $\underline{r}_3 = -2\underline{\rho}$

$$\therefore \underline{r}_1 + \underline{r}_2 = 2\underline{\rho}$$
$$\therefore 2\underline{r}_1 = 2\underline{\rho} + \underline{\lambda}$$
$$2\underline{r}_2 = 2\underline{\rho} - \underline{\lambda}$$
$$\therefore r_1^2 + r_2^2 + r_3^2 = 6\rho^2 + \frac{\lambda^2}{2}$$

from (13),

$$\Psi_p(r_1, r_2, r_3) = \Psi_p(\lambda, \rho) = \left(\frac{\alpha}{\sqrt{\pi}}\right)^{9/2} e^{-\frac{\alpha^2}{2}\left(\frac{\lambda^2}{2} + 6\rho^2\right)}$$
$$= \phi(\lambda)\,\chi(\rho)$$

$$\therefore \ \Delta = \frac{r-1}{2-r} \cdot 2M \qquad \text{where} \qquad r = \frac{(\Delta M^2)_{Y=1}}{(\Delta M^2)_{Y=0}}$$

For $Y = 0$ Regge trajectory, the ρ is chosen. The vector mesons lying on this are the ρ, A_2^H, R, S, T and U mesons which have $L = 0, 1, 2, 3$, etc. and consist of quark-antiquark pairs with third component of spins parallel so that J=1,2,3, etc., for these. The known Y=1 mesons are $K(495)$ with $J^P=0^-$, $K^*(892)$ with $J^P=1^-$, both S states, and $K^*(1240)$ with $J^P=1^+$, $K^*(1320)$ with $J^P=1^+$ and $K^*(1420)$ with $J^P=2^+$, all P states. A further meson with $J^P=0^+$, $L=1$ should exist. Higher orbital excitations have not been observed yet. Because only two $Y=1$ mesons present themselves for calculation, it is essential that the mass values be corrected for possible spin-orbit and strong spin-spin interactions. The latter can certainly be large since it contributes to the mass splitting of ~400 MeV of the $K(495)$ and $K^*(892)$ mesons, which are in spin singlet and triplet states respectively, and to the splitting of ~600 MeV between the pion and rho mesons. Now to each L excitation correspond four $SU(3)$ nonets ÷ J=L, J=L-1, L, L+1 except for the L=0 states where J=0 or 1. The δ (962), A, (1080), B (1200) and A_2 (1315) are the Y=0, I=1 members of the four nonets with L=1, having $J^P = 0^+, 1^+, 1^+$ and 2^+ respectively. For these, the spin-orbit energy is -2, -1, 0, 1, units. If the mass differences between them is due to the spin orbit interaction, the mass differences between successive mesons should be constant. In fact, the observed mass differences are 118, 120 and 115 MeV. Hence the spin-orbit force between the quark and antiquark seems to dominate, the spin-spin forces being negligible, presumably, as has been suggested,[9] because they are of short range in comparison. For the L=1, Y=1 mesons, the mass differences are 89 and 77 MeV. Again the near equality suggests that spin-orbit forces dominate in $SU(3)$ symmetry breaking. This means, since $K^*(1240)$, $K^*(1320)$ and $K^*(1420)$ correspond to the A, B and A_2 mesons, that the mass of the K^* with L=1 would, in the absence of spin-orbit forces, be that of the $K^*(1320)$ which has no spin-orbit energy since its internal spin is zero. Included still in the observed mass are spin-spin and other contributions but these appear to be negligible. Since the $K^*(892)$ and $K^*(1320)$ have same internal spin state (triplet), the spin-spin coupling may be expected to be the same for each. For these reasons, these two should conform in their bare mass values to two radially excited states with $\Delta L = 1$.

This gives $(\Delta M^2)_{Y=1} = 0.95 \ GeV^2$. Instead of taking an average for the (mass)^2 separations between recurrences on the ρ trajectory (which include spin-orbit and other contributions) symmetry breaking additions to the excitation energies of different L states can probably be minimised by taking only the ρ and B mesons. For the former L=0, for the latter internal spin $S=0$, so that neither has any spin-orbit interaction contributing to its mass. Thus

$$(\Delta M^2)_{Y=0} = 0.94 \text{ GeV}^2$$

This gives $r = 1.01$ and $\Delta = 99$ MeV. This is in good agreement with the value $\Delta = 117$ MeV obtained from the ρ (765) and K^* (892) mass difference, which should be due to the heavier strange quark in the latter, assuming the vector octet is split in mass by the strange quark (This is not true for the pseudoscalar octet in view of the large π-K mass difference, but the former is known to break SU (3) symmetry considerably).

SECTION C:-

In this section, the electromagnetic mass splitting of the pion, viewed as due to the different static Coulomb interaction and magnetostatic coupling of the spins of the quark and antiquark in the $\pi^\pm$ and π^0 is derived semi-classically. Firstly consider the interaction between two magnetic dipoles. From Appendix D, the interaction energy between two dipoles of magnetic moment $\underline{M}_1$ and $\underline{M}_2$ is (proved in Appendix D),

$$V_{ss}(\underline{r}) = -\frac{8\pi}{3}\underline{M}_1\cdot\underline{M}_2\,\delta(\underline{r}) - \frac{1}{r^3}\left(\frac{3(\underline{M}_1\cdot\underline{r})(\underline{M}_2\cdot\underline{r})}{r^2} - \underline{M}_1\cdot\underline{M}_2\right) \qquad (1)$$

for all $\underline{r}$, including the origin .

For a spherically symmetric S state $\quad \left\langle \frac{1}{r^3}\left(\frac{3(\underline{M}_1\cdot\underline{r})(\underline{M}_2\cdot\underline{r})}{r^2} - \underline{M}_1\cdot\underline{M}_2\right)\right\rangle = 0$

$$\therefore \langle V_{ss}\rangle = -\frac{8\pi}{3}\underline{M}_1\cdot\underline{M}_2\,|\Psi(0)|^2, \qquad (2)$$

where $\Psi(r)$ is wavefunction of S state.

For mesons, the question arises whether quark magnetic moment is its Dirac moment, i.e. $\underline{M} = \frac{Q\hbar}{2M_Q c}\cdot\underline{\sigma}$ where Q is charge, M_Q mass of quark, or whether it is anomalous i.e. $\underline{M} = \frac{gQ\hbar}{2M_Q c}\cdot\underline{\sigma}$. In the former case , the z component is

$$\langle M_z\rangle = \frac{e\hbar}{2M_Q c} \qquad \text{for proton of mass } m_p.$$

Since $\langle M_z\rangle = 2.79\, e\hbar/2m_p c$, $\frac{M_p}{M_Q} = 2.79$, implying small quark effective mass.

In the latter case $\langle M_z\rangle = \frac{ge\hbar}{2M_Q c}$ so that $g = \frac{2.79\, M_Q}{M_p} \sim 29$ for quark mass of 10 GeV. Since either case is possible a priori, we will make neither assumption, using only the value of $\underline{M}$ necessary to give the absolute value of the proton magnetic moment i.e. $\underline{M} = 2.79\frac{Q\hbar}{2m_p c}\underline{\sigma} = \mu_p Q\,\underline{\sigma}$ where $\mu_p = 2.79\frac{e\hbar}{2m_p c}$

From equ (2) for an S state meson, $\langle V_{ss}\rangle = -\frac{8\pi}{3}\mu_p^2\,(Q_1 Q_2)\,|\Psi(0)|^2\,\langle\underline{\sigma}_1\cdot\underline{\sigma}_2\rangle$

whilst for a mixed S state, $\langle V_{ss}\rangle = -\frac{8\pi}{3}\mu_p^2\,|\Psi(0)|^2\,\langle Q_1 Q_2\,\underline{\sigma}_1\cdot\underline{\sigma}_2\rangle$

The space-spin $SU(6)$ wavefunctions of the $\pi^\pm$ and π^0 are

$$|\pi^+\rangle = \Psi_\pi(r)\,\chi(\sigma) \quad , \quad |\pi^0\rangle = \Psi_\pi(r)\,\phi(\sigma)$$

where, assuming harmonic oscillator model for pion , $\Psi_\pi(r) = (\beta/\pi)^{3/4} e^{-\frac{\beta r^2}{2}}$

and $\beta = \mu_Q\omega$, μ_Q = reduced quark mass, ω - frequency, and

$$\chi(\sigma) = 2^{-1/2}\left(\underset{p}{\uparrow}\underset{\bar n}{\downarrow} - \underset{p}{\downarrow}\underset{\bar n}{\uparrow}\right), \quad \phi(\sigma) = 2^{-1}\left(\underset{p}{\uparrow}\underset{\bar p}{\downarrow} + \underset{n}{\downarrow}\underset{\bar n}{\uparrow} - \underset{p}{\downarrow}\underset{\bar p}{\uparrow} - \underset{n}{\uparrow}\underset{\bar n}{\downarrow}\right)$$

We get $\quad \langle\pi^+|\,\sigma_1\cdot\sigma_2\,|\pi^+\rangle = -2/3\ , \quad \langle\pi^0|\,Q_1 Q_2\,\underline{\sigma}_1\cdot\underline{\sigma}_2\,|\pi^0\rangle = 5/6$

$$\therefore \langle\pi^+|V_{ss}|\pi^+\rangle = \frac{16\pi}{9}\left(\frac{\beta}{\pi}\right)^{3/2}\mu_p^2$$

and

$$\langle\pi^0|V_{ss}|\pi^0\rangle = -\frac{20\pi}{9}(\beta/\pi)^{3/2}\mu_p^2$$

the Coulomb electrostatic interaction energy is

$\langle V_c \rangle = e^2 \langle \frac{Q_1 Q_2}{r} \rangle$ where Q_1, Q_2 are now regarded as operators i.e. $Q_1 = T_3^{(1)}, Q_2 = T_3^{(2)}$ where T_3 is 3rd component of isospin. Noting isospin states of π^+, π^0 are:- $|\pi^+\rangle = |p\bar{n}\rangle$, $|\pi^0\rangle = 2^{-\frac{1}{2}}(|p\bar{p}\rangle + |n\bar{n}\rangle)$

We get
$$\langle \pi^+ | V_c | \pi^+ \rangle = \frac{2e^2}{9} \langle \pi | \tfrac{1}{r} | \pi \rangle$$
$$\langle \pi^0 | V_c | \pi^0 \rangle = -\frac{5e^2}{18} \langle \pi | \tfrac{1}{r} | \pi \rangle$$

Using the harmonic oscillator wavefunction gives $\langle \pi | \frac{1}{r} | \pi \rangle = 2(\beta/\pi)^{1/2}$
$$\therefore \langle \pi^+ | V_c | \pi^+ \rangle = \frac{4e^2}{9} \left(\frac{\beta}{\pi}\right)^{1/2}$$
and
$$\langle \pi^0 | V_c | \pi^0 \rangle = -\frac{5e^2}{9} (\beta/\pi)^{1/2}$$

The difference in energy of the π^+, π^0 ground state is
$$\Delta M = \langle \pi^+ | V_c + V_{ss} | \pi^+ \rangle - \langle \pi^0 | V_c + V_{ss} | \pi^0 \rangle = e^2 \left(\frac{\beta}{\pi}\right)^{1/2} + \frac{4\pi}{9} (\beta/\pi)^{3/2} \mu_e^2$$

β has been estimated, as outlined in Section B. If a linear mass relation between spin and mass is assumed for mesons, in particular for pion, then mass splitting of π^+, π^0 is found to be

$\Delta M = 3.012$ MeV which is rather too small (4.604 ± 0.016, experimentally[10]) Assuming the quadratic mass relation given by equ. 6. Section A, with eigenvalues given by equ (8)
$$\frac{M^2}{2M_Q} \phi = \left(-\frac{\nabla^2}{2M_Q} + 2M_Q + \tfrac{1}{2} V\right) \phi$$
then to first order in the perturbation $(V_{SS} + V_c)$ of the ground state of the three dimensional harmonic oscillator, the individual mass shifts are

$\frac{M^2}{2M_Q} = \frac{M_0^2}{2M_Q} + \frac{1}{2} \langle 0 | V_{ss} + V_c | 0 \rangle$ where M_0 is mass of unperturbed ground state
$$\therefore M^2(\pi^+) - M^2(\pi^0) = M_Q \Delta M = 3.012 M_Q$$

From data tables[10] $M^2(\pi^+) - M^2(\pi^0) = 1264.0$ MeV^2 There is agreement if quark mass M_Q has an effective value of 420 MeV. This is in very good agreement with value of 426 MeV obtained by averaging baryon SU(3) octet and decuplet masses :- $M_Q = \frac{1}{3}\langle B \rangle$ for $B \in J^P = \frac{1}{2}^-$ and $\frac{3}{2}^+$ octets and exactly same value of 426 MeV obtained by averaging the almost perfect $J^P = 1^-$ vector nonet:-* $M_Q = \frac{1}{2} \langle M \rangle$ for $J^P = 1^-$

Alternatively, regarding effective quark mass as 426 MeV, the mass splitting of the π^0, π^+ is
$$\Delta M = 4.63 \text{ MeV}$$
compared with 4.604 ± 0.016 MeV, found experimentally. The small effective mass implies, for non-relativistic motion, a rather wide potential well. This is confirmed in the next section. The agreement obtained if a small effective

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* FOOTNOTE:-
(whether mass or (mass)^2 operator for vector mesons is used makes little difference as far as almost exact SU(3) symmetry observed for them is concerned. The former gives for the pure unitary singlet and octet states 783 MeV and 765 MeV respectively with mixing angle $\theta_V \simeq 33°$, whilst for the latter the values are 804 MeV and 765 MeV, with mixing angle $\theta \simeq 39°$, comparing well in both cases with $\theta = 35°$ for a perfect SU(3) nonet. As illustrated above for the electromagnetic mass splitting of the pion, the question of mass or (mass)^2 operators operators for pseudoscalar mesons like the pion is important.

mass for quarks (at least in mesons) is assumed means that the gyromagnetic ratio for quarks : $g = \frac{2.79 M_q}{m_p} \simeq 5/4$, i.e. quarks have Dirac magnetic moment which is only slightly anomalous, agreeing with experimental data on electromagnetic decays and consistent with assumption of scalar binding.

SECTION D.

Although the description of high energy scattering of elementary particles in terms of effective scattering potentials is not Lorentz invariant and hence questionable when the energies of the particles far exceed their rest masses, certain insights can be gained by phenomenological analysis of the data. In this section, high energy elastic scattering of hadrons by hadrons is viewed in terms of effective quark-quark scattering potentials which can be derived in two extreme cases:

1. Quarks are point-like, much smaller than hadrons. In this limit, their spatial distribution within the hadron is more important than their scattering properties, which are almost constant over all momentum transfers. The diffraction peaks of elastic hadron-hadron scattering are determined solely by the strong interactions form factors of the hadrons themselves. The similarity of baryon-baryon and baryon-meson diffractive curves results from similarity in baryon and meson form factors which, in the point quark limit, are equal to their corresponding electromagnetic form factors.

2. Quarks have sizes comparable with hadrons. In the diffractive peak region hadron form factors are then approximately constant and equal to one and the shape of the quark-quark diffractive peaks and hadron-hadron diffraction peaks are almost identical. Isospin and charge conjugation invariance for the quark-quark interactions would then imply similar slopes for baryon-baryon and meson-baryon forward diffraction peaks, as observed. This case is treated first.

The additivity assumption for composite particle A - composite particle B elastic scattering at small momentum transfer values of q (q < width of forward diffraction peak) gives for the scattering amplitude :-

$$F_{AB}(q) = f(q)\left(\sum_{i=1}^{A} \mathcal{F}_i^A(q)\right)\left(\sum_{j=1}^{B} \mathcal{F}_j^B(q)\right) \qquad (1)$$

where $f(q)$ is the single particle-particle elastic scattering amplitude (assumed the same, for simplicity, for all pairs of identical particles, $\mathcal{F}_i^A(q)$, $\mathcal{F}_j^B(q)$ are single body form factors of A and B. If the composite particle internal wavefunctions are symmetric with respect to sub-particle spatial coordinates, then

$$\mathcal{Y}_i^A = \mathcal{Y}^A$$

$$\mathcal{Y}_i^B = \mathcal{Y}^B$$

$$\therefore \quad F_{AB}(q) = AB\, f(q)\, \mathcal{Y}^A(q)\, \mathcal{Y}^B(q)$$

Parametrising $F_{AB}(q)$ as

$$F_{AB}(q) = (i + \alpha_{AB}) \cdot \frac{\sigma_{AB}}{4\pi}\, e^{-\frac{1}{2} S q^2} \qquad (2)$$

where $\alpha_{AB} = \dfrac{\mathcal{R}[F_{AB}(0)]}{\mathcal{I}[F_{AB}(0)]}$

σ_{AB} is total scattering cross-section for scattering of A by B, S is inverse width of A-B elastic diffraction peak, then

$$f(q) = [AB\, \mathcal{Y}^A(q)\, \mathcal{Y}^B(q)]^{-1} (i + \alpha_{AB}) \frac{\sigma_{AB}}{4\pi}\, e^{-\frac{1}{2} S q^2} \qquad (3)$$

ignoring the very small multiple scattering contributions to the differential cross-section:-

$$\frac{d\sigma}{dq^2} = \pi\, |F_{AB}(q)|^2$$

at high energies. According to Glauber's scattering theory, the high energy, small angle scattering amplitude is given by

$$f(\underline{q}) = \frac{i}{2\pi} \int e^{i\underline{q}\cdot\underline{b}} \left(1 - e^{i\chi(\underline{b})}\right) d^2\underline{b}$$

where $\chi(\underline{b})$ is related to the scattering potential $U(r)$ by

$$\chi(\underline{b}) = -\frac{1}{2k} \int_{-\infty}^{\infty} U(b,z)\, dz$$

where $b^2 + z^2 = r^2$

Assuming the latter is spherically symmetric and letting

$$b^2 + z^2 = t^{-1}$$

$$b^2 = s^{-1}$$

$$U(r)\, r = \mathcal{U}(r) = \mathcal{U}(t^{-\frac{1}{2}})$$

then $\quad s^{-\frac{1}{2}}\, \chi(s^{-\frac{1}{2}}) = -\dfrac{1}{k} \displaystyle\int_0^s \mathcal{U}(t^{-\frac{1}{2}})\, t^{-1} (s-t)^{-\frac{1}{2}}\, dt$

This is an Abel integral equation whose standard solution is $\quad t^{-\frac{1}{2}}\, \mathcal{U}(t^{-\frac{1}{2}}) = -\dfrac{2k}{\pi} \dfrac{d}{dt} \displaystyle\int_0^t s^{-\frac{1}{2}}\, \chi(s^{-\frac{1}{2}})\, (t-s)^{-\frac{1}{2}}\, ds$

$$\text{or} \quad U(r) = \frac{2k}{\pi} \frac{d}{dr} \left[r \int_r^{\infty} \frac{\chi(b)\, db}{b\sqrt{b^2 - r^2}} \right] \qquad (5)$$

From (4), $\quad e^{i\chi(b)} = 1 + i \displaystyle\int_0^{\infty} J_0(qb)\, q\, f(q)\, dq$

$$\chi(b) = -i \log\left[1 + i \int_0^{\infty} J_0(qb)\, q\, f(q)\, dq\right] \qquad (6)$$

so that, using (3), $U(r)$ can be determined from (5). Notice that because the particle-particle scattering amplitude given by (3) is valid only within the range of scattering angles set by the width of the diffractive cone of B (or A), where single scattering of each particle of A by B dominates, $U(r)$ is the effective particle-particle scattering potential for final particle states within the shadow of the composite only, which may or may not coincide with the diffractive cone of the particles themselves according as the sizes of the particles are the same as or smaller than the size of the composite. $U(r)$ is thus the long range part of the 'actual' scattering potential, being dominant only when the collision involves large impact parameters.

<u>Case 2.</u> here $\mathcal{F}^A(q) \simeq \mathcal{F}^B(q) \simeq 1$ equ (3) becomes

$$f(q) = f(0)\, e^{-\frac{1}{2}\zeta q^2} \qquad \text{where} \qquad f(0) = A^{-1}B^{-1}(i + \alpha_{AB})\frac{\sigma_{AB}}{4\pi}$$

from (6) $\chi(b) = -i \log\left[1 + i f(0)\,\zeta^{-1} e^{-b^2/2\zeta}\right]$

Since this is valid for large b only, $\chi(b) \simeq f(0)\,\zeta^{-1} e^{-b^2/2\zeta}$ (7)

For application to quark-quark scattering ζ must be considered complex, for reasons given in STRONG INTERACTIONS, Section C . Writing $\zeta = |\zeta| e^{-i\phi}$, $f(0) = |f(0)| e^{i\beta}$

where $\tan\beta = \alpha_{AB}^{-1}$, $C = |\zeta|^{-1}|f(0)|\cos(\beta + 2\phi)$, $S = |\zeta|^{-1}|f(0)|\sin(\beta + 2\phi)$

$$B = \tfrac{1}{2}|\zeta|^{-1}\cos 2\phi\,, \quad D = \tfrac{1}{2}|\zeta|^{-1}\sin 2\phi\,,$$

equ (5) is finally

$$U(r) = -\left(\tfrac{2}{\pi}\right)^{1/2} k\,|\zeta|^{-1/2}\left[C\cos(\phi - Dr^2) - S\sin(\phi - Dr^2)\right] e^{-Br^2}$$

$$= \frac{2M}{\hbar^2} V(r)\,, \qquad \text{where } M \text{ is reduced mass of particle,}$$

$V(r)$ the long range part of the physical scattering potential.

$$\therefore V(r) = -(2\pi)^{-1/2}\hbar \cdot \frac{\hbar k}{M}|\zeta|^{-3/2}|f(0)|\cos(3\phi + \beta - Dr^2)\, e^{-Br^2}$$

$\hbar k = Mv \sim Mc$, consistent with non relativistic approximations used at high energies when $v \sim c$. $\therefore V(r) = -(2\pi)^{-1/2} c\hbar|\zeta|^{-3/2}|f(0)|\cos(3\phi + \beta - Dr^2)\, e^{-Br^2}$ (8)

Writing $R = B^{-1/2} = \left(\frac{2|\zeta|}{\cos 2\phi}\right)^{1/2}$ as the range of the potential, $V(r)$ is attractive and its range increases with slope of corresponding diffraction peak i.e. as the peak shrinks, radius of interaction region increases, corresponding to enlargening of diffracting particle and increase of its differential cross-section in the forward direction. $V(r)$ is a pure Gaussian in r when phase of effective quark-quark scattering amplitude is independent of momentum transfer

<u>Case 1.</u> here $f(q) = \left[AB\,\mathcal{F}^A(q)\,\mathcal{F}^B(q)\right]^{-1}(i + \alpha_{AB})\frac{\sigma_{AB}}{4\pi}\, e^{-\frac{1}{2}\zeta q^2}$

$$= \left[\mathcal{F}^A(q)\,\mathcal{F}^B(q)\right]^{-1} f(0)\, e^{-\frac{1}{2}\zeta q^2}$$

$\mathcal{F}^A, \mathcal{F}^B$ are equal to electromagnetic form factors of A, B respectively. Consider now elastic proton-antiproton scattering. The Wilson-Hofstader dipole fit for the proton (and, by charge conjugation invariance, the antiproton) electromagnetic form factor F_p^M is chosen : $\mathcal{F}^A = \mathcal{F}^B = F_p^M = (1 + q^2/q_0^2)^{-2}$

where $q_0^2 = 0{\cdot}71\left(\frac{GeV}{c}\right)^2$ corresponds to point-like quarks and larger values to larger quarks(Deviations from dipole fit occur for $q^2 > 10\left(\frac{GeV}{c}\right)^2$. Since discussion is confined to the diffraction peak for which $0 \le q^2 \le 0{\cdot}5$ typically for $p\bar{p}$ scattering

$\phi(\lambda)$ is the harmonic oscillator wavefunction of one quark with respect to another and has radial dependance $\frac{\alpha^2\lambda^2}{4}$. Comparing with (10)

$$\frac{\alpha^2}{4} = \frac{M_Q\omega}{2}$$

$$\therefore \langle r^2\rangle_p = \tfrac{1}{2}(M_Q\omega)^{-1}$$

The value of $M_Q\omega$ for the proton may be found approximately by regarding the proton as a bound state of a quark and diquark. The mass of a bound state of two particles masses M_1, M_2, is, from (5)

$$M^2 = \frac{(1+\lambda)^4}{\lambda^2}\mu_Q^2 - 2\mu_Q V_0 + 3\mu_Q\omega + 2(2n+L-2)\mu_Q\omega \tag{15}$$

for a quark-diquark bound state $\mu_Q = \frac{M_1 M_2}{M_1+M_2}$ where $M_1 = M_Q$, $M_2 = 2M_Q$ approximately (ignoring SU(2) symmetry breaking quark mass difference).

$$\therefore \mu_Q \simeq 2/3\, M_Q$$

$$\therefore \Delta M^2 \simeq 4/3\, M_Q\omega\,\Delta L$$

The proton lies on the N_α trajectory. The Chew-Frautschi plot of known recurrences gives[8]

$$\alpha_N = -0.39 + 1.01 M^2$$

$$\therefore \Delta\alpha_N = 1.01\,\Delta M^2 = 2$$

Assuming L can be replaced by J, the total spin, i.e. that the $I = 1/2$, $Y = 1$ Regge recurrences are even orbital excitations

then $\Delta L = \Delta J = \Delta\alpha_N = 2$

$$\therefore M_Q\omega \simeq \tfrac{3}{4}\cdot 1.01^{-1} = 0.74\ \mathrm{GeV}^2$$

$$\therefore \sqrt{\langle r^2\rangle_p} \simeq 0.20\ \mathrm{F}$$

STRANGE AND NON-STRANGE QUARK MASS DIFFERENCES:-

The Quark-antiquark harmonic oscillator model of mesons developed above gives for the masses of orbitally excited states of spin $J = L, L\pm 1$

$$M^2 = 2\mu_Q\omega L + C$$

where L is orbital angular momentum of quark-antiquark state, μ_Q is reduced mass of quark and C is a constant

$$\therefore \Delta M^2 = 2\mu_Q\omega\,\Delta L \quad \text{where} \quad \Delta L = 1 \text{ or } 2$$

according as the Regge trajectory is exchange degenerate or not. For $Y = 0$ mesons, made up of two non-strange quarks,

$$\frac{1}{\mu_Q} = \frac{1}{M_1} + \frac{1}{M_2}$$

Assuming SU(2) symmetry for non strange quarks gives $M_1 = M_2 = M$

$$\therefore (\Delta M^2)_{Y=0} = M\omega_0\,\Delta L, \quad \omega_0 \text{ - the oscillator frequency}$$

For $Y = \pm 1$ mesons, made up of 1 strange and 1 non-strange quark,

$$\frac{1}{\mu_Q} = \frac{1}{M} + \frac{1}{M_\lambda}, \quad \text{where } M_\lambda$$

is strange quark mass.

$$\therefore (\Delta M^2)_{Y=1} = 2\frac{M M_\lambda}{M+M_\lambda}\cdot\omega_1\,\Delta L, \quad \omega_1 \text{ - the}$$

frequency of oscillation in $Y = \pm 1$ mesons. Since the superstrong forces between quarks and antiquarks are SU(6) symmetric, we expect $\omega_0 = \omega_1$. Writing

$M_\lambda = M + \Delta$, where Δ is the mass difference between λ and p, n $(Y=0)$ quarks leading to SU (3) symmetry breaking,

$$\frac{(\Delta M^2)_{Y=1}}{(\Delta M^2)_{Y=0}} = \frac{2(M+\Delta)}{2M+\Delta}$$

the formula is accurate enough).

$$\therefore \quad f(q) = \left(1+\frac{q^2}{4a^2}\right)^4 f(0)\, e^{-\frac{1}{2}\xi q^2}$$

This is the effective quark-antiquark elastic scattering amplitude. The equivalent interaction potential is $V(r) = (2\pi)^{-\frac{1}{2}} c\hbar |\xi|^{-3/2} |f(0)|\, G[\phi, B, D, Br^2]\, e^{-Br^2}$ (9)

where $G[\phi, B, D, Br^2]$ is a function whose form is not given, as it is complicated.

From (8),(9) $\qquad V(r) = V(r^2)$

and for small r ($r \ll R$) may be approximated by the zeroth and first order term in its Taylor expansion i.e. $V(r^2) \simeq V(0) + V'(0) r^2$ where $V'(0) = \left(\frac{dV}{dr^2}\right)_{r^2=0}$

and $\qquad V(0) = -(2\pi)^{-1/2} c\hbar |\xi|^{-3/2} |f(0)| \cos(3\phi+\beta)$ (case 2)

$\qquad\qquad = -(2\pi)^{-\frac{1}{2}} c\hbar |\xi|^{-3/2} |f(0)|\, G(\phi, B, D, 0)$ (case 1)

In both cases, $V(0) < 0$ so $V(r^2) \simeq -V_0 + V'(0) r^2$ with $V_0 = -V(0)$

This is a simple harmonic oscillator potential. Now the non-relativistic model for quark-antiquark bound states is reasonable only if the range of the binding potential (whatever its form) is large or more exactly $R^{-1} \ll M_Q^*$ where, for square well potentials for example, $M_Q^* = M_Q - V_0$ is the effective mass of the quark. The condition is thus $R \gg (M_Q - V_0)^{-1}$ It is reasonable to assume that small angle hadron-hadron scattering, within the framework of the quark model, is due to single quark-quark (antiquark) scattering via forces identical to that responsible for the binding i.e. via simple harmonic forces, which is a reasonable approximation to the actual interaction when $\frac{\xi}{R} \ll 1$, as shown above. Thus $V(r^2) = -V_0 + \frac{1}{2}\mu_Q \omega^2 r^2$ where μ_Q is reduced quark mass, ω-angular frequency

$$\therefore \quad V_0 = -V(0) \tag{10. a}$$

$$\frac{1}{2}\mu_Q \omega^2 = V'(0) \tag{10. b}$$

Since $p\bar{p}$ scattering is considered, which involves $\mathcal{P}$ and $\bar{\mathcal{P}}$ quarks only.

$$\mu_Q = \frac{1}{2} M_Q$$

From Section B, equ (9), the inverse slope of Regge trajectories of mesons with equal mass quark and antiquark is $\qquad \Delta M^2 = M_Q \omega$

From (10.b) , $\qquad M_Q = \dfrac{(\Delta M^2)^2}{4V'(0)}$

ΔM^2 has been obtained from the ρ trajectory on which lie orbital encitations of $\mathcal{P}$ and $\bar{n}$ in ground state with triplet spin state. $V'(0)$ has been obtained from following parameters used by Shrauner, Benofy and Cho[11] in their multiple quark scattering analysis of $\bar{p}p$ elastic scattering:-

$$\alpha_{p\bar{p}} = \frac{\mathcal{R}[F_{p\bar{p}}(0)]}{\mathcal{I}[F_{p\bar{p}}(0)]} = -0.109$$

$$\sigma_{p\bar{p}} = 49.9 \text{ mB.}$$

$$\xi = (12.5 - i\,13)\left(\frac{GeV}{c}\right)^{-2}$$

Results are indicated below for cases 1 and 2:-

PARAMETERS	CASE 1		CASE 2.	
	"Point like" quarks		"large" quarks	
M_Q	9.8	GeV	2.0	GeV
V_0	18.5	GeV	14.4	GeV
R	1.8	F	1.8	F

Case 2 is excluded by cosmic ray[12] and accelerator experiments[13] which indicate a lower mass limit of about 5GeV. Case 1 is compatible with both hadron mass spectrum and experiment, and is close to calue of 10GeV obtained by Kaiba[14] from assignment of radially excited SU (3) baryon resonances and lower mass limit of 10GeV suggested by possible quark influence on Regge trajectories[15]. It is not incompatible with low lying meson masses.

SECTION E:-

Sinanoğlu[16] has argued that the three dimensional harmonic oscillator model with a hard core at distances less than about 0.5 F is more compatible with the observed meson spectrum than one without a repulsive core. Information about meson bound state wavefunctions at the origin has been provided by the analysis by Van Royen and Weisskopf [17] of strangeness non conserving leptonic weak decays of the pion and kaon

$$K^+ \rightarrow \mu^+ + \nu$$
$$\pi^+ \rightarrow \mu^+ + \nu$$

They find, for agreement with experimental values of the decay amplitude and decay channel width, the bound state wavefunctions of the pion and kaon at the origin must have the values

$$|\Psi_\pi(0)|^2 = 1.4 \times 10^6 \ (\mathrm{MeV})^3$$
$$|\Psi_K(0)|^2 = 5.1 \times 10^6 \ (\mathrm{MeV})^3$$

and that generally for a meson M,

$$\frac{|\Psi_M(0)|^2}{|\Psi_\pi(0)|^2} = \frac{m_M}{m_\pi}$$

where m_M, m_π are masses of M and pion respectively. This is surprising from the point of view of the simple harmonic oscillator model for mesons since generally $|\Psi(0)|^2 = \left(\frac{\mu\omega}{\pi}\right)^{1/2}$ and, noting that the product $\mu\omega$ is given (from Section B, equ 15) by

$$\Delta M^2 = 2\mu\omega$$

where ΔM^2 is $(\mathrm{Mass})^2$ separation between successive orbitally excited meson states, we have

$$\frac{|\Psi_\pi(0)|^2}{|\Psi_K(0)|^2} = \left[\frac{(\Delta M^2)_\pi}{(\Delta M^2)_K}\right]^{3/2}.$$

Noting $(\Delta M^2)_\pi = [\alpha'_\pi(0)]^{-1}$ and $(\Delta M^2)_K = [\alpha'_K(0)]^{-1}$ where $\alpha'_\pi(0)$, $\alpha'_K(0)$ are slopes of π trajectory (exchange degenerate with B trajectory) and K trajectory, and that $\alpha'_\pi(0) \simeq \alpha'_K(0)$ experimentally, we have

$$\frac{|\Psi_\pi(0)|^2}{|\Psi_K(0)|^2} \simeq 1,$$ instead of 0.27 as indicated experimentally.

This indicates SU(3) symmetry breaking due to unitery spin dependent forces between quark and antiquark. Also, using π trajectory,

$$|\Psi_\pi(0)|^2 \simeq 70 \times 10^6 \ (\mathrm{MeV})^3$$

which is fifty times larger than the experimental value. This suggests that a repulsive core is present in quark-antiquark interaction which strongly reduces the wavefunction value near the origin. Since the asymptotic variation of the pion electromagnetic form factor depends on the behaviour of the pion wavefunction near the origin, the former should not vary smoothly for large momentum transfers, unlike the proton form factor which does. In particular, the π form factor should vary more rapidly than the proton form factor. Present available π form factor data are not extensive enough to test this conclusion.

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CHAPTER 4

ELECTROMAGNETIC INTERACTIONS

INTRODUCTION:-

Electromagnetic form factors and r.m.s. charge radii of various hadrons are considered in relation to the quark and droplet models. In section A, the non-relativistic (rest frame) charge form factor of a composite system of charged particles is derived. Its relativistic modification is indicated and the asymptotic form of the proton magnetic form factor discussed for the evidence it provides for point like entities in the nucleon. However, the experimental upper bound for the former is shown to follow, making reasonable mathematical assumptions about the proton charge density, in a droplet model. SU(3) sum rules for hadron form factors are derived in section B. Although as yet, untestable experimentally, they are identical to results obtained outside the framework of the quark model. In section C, the known charge radii of the p, n and π are expressed in terms of the unknown quark charge radii. The electric form factor of the neutron is shown to be incompatible with a droplet model but is explainable if quarks are almost, but not exactly, point like. Agreement between the SU (6) harmonic oscillator model for quarks and experiment is shown to occur only if quarks are spread out over the proton and the quark charge form factor equals the strong interaction nucleon form factor, both of which assumptions are incompatible with the results of Shrauner et al,[3] and the equality of charge and hadronic matter density distributions of the proton found by Chou and Yang[7] A possible explanation of the discrepancy is provided

SECTION A:-

Consider a spinless cluster N of charged particles, spinless and identical apart from charges $Q_1, Q_2, .. Q_j, ... Q_N$ at positions $\underline{r}_1, \underline{r}_2, ... \underline{r}_j, ... \underline{r}_N$ with respect to the centre of mass O of N which is fixed at the origin by

$$\underline{r}_1 + \underline{r}_2 + ... + \underline{r}_j + ... + \underline{r}_N = 0$$

Let Ψ $(\underline{r}_1, \underline{r}_2, .. \underline{r}_j, ... \underline{r}_N)$ be the normalised internal wavefunction of N. Since the particles are bosons, Ψ is symmetric under all permutations of particle coordinates, with an S state for the ground state of N. Let $\rho_j(\underline{R}_j)$ be the normalised charge density of particle j, where $\underline{R}_j$ is the position vector of j relative to a point P with coordinate $\underline{r}$ relative to the C.M., O.

$$\int e_j(\underline{R}_j)\, d^3\underline{R}_j = 1$$

The charge density at P due to Q_j at $\underline{r}_j$ is $\quad \rho_j(\underline{R}_j) = Q_j e_j(\underline{R}_j)$

The charge density at P due to all charges $Q_1, Q_2, ... Q_j, ... Q_N$ is

$$\rho(\underline{r}) = \rho(\underline{R}_1, \underline{R}_2, ... \underline{R}_j, ... \underline{R}_N) = \sum_{j=1}^{N} \rho_j(\underline{R}_j) = \sum_{j=1}^{N} Q_j e_j(\underline{R}_j)$$

(It is assumed that any interaction of the composite particle with an external potential is instantaneous so that the interactions between the potential and particles Q_j are simultaneous and the individual charge densities have a common time value and are hence additive. This would be true in the static limit (O at rest) but not so in a general Lorentz frame of reference. The rest frame charge form factor of the composite only is considered here) The charge form factor $F_N^E(\underline{q}, \underline{r}_1, \underline{r}_j, \underline{r}_N)$ due to a rigid distribution of particles at $\underline{r}_1, \underline{r}_2, \ldots \underline{r}_j, \ldots \underline{r}_N$ is

$$F_N^E(\underline{q}, \underline{r}_1, \underline{r}_2, \ldots \underline{r}_j, \ldots \underline{r}_N) = \int \rho(r) e^{i\underline{q}\cdot\underline{r}} d^3\underline{r}$$

$$= \sum_{j=1}^{N} Q_j \int \rho_j(R_j) e^{i\underline{q}\cdot\underline{r}} d^3\underline{r} \qquad \text{where}$$

$$\underline{r} = \underline{r}_1 + \underline{R}_1 = \underline{r}_2 + \underline{R}_2 = \ldots\ldots = \underline{r}_j + \underline{R}_j = \ldots\ldots = \underline{r}_N + \underline{R}_N$$

Since $\underline{r}_j$ is a fixed point, origin O may be translated to C.M. of each particle, since the integration over all space is independent of choice of origin i.e. $d^3\underline{r} = d^3\underline{R}_j$

$$\therefore F_N^E(\underline{q}, \underline{r}_1, \underline{r}_2, \ldots \underline{r}_j, \ldots \underline{r}_N) = \sum_{j=1}^{N} Q_j \int \rho_j(\underline{R}_j) e^{i\underline{q}\cdot(\underline{r}_j + \underline{R}_j)} d^3\underline{R}_j$$

$$= \sum_{j=1}^{N} Q_j e^{i\underline{q}\cdot\underline{r}_j} \mathcal{F}_j^E(\underline{q})$$

where $\mathcal{F}_j^E$ is the charge form factor of j in its C.M. frame.

Suppose N interacts with an external potential so that its initial state $|i\rangle$ changes to a final state $|f\rangle$. The charge form factor of N, regarded as an operator, has matrix element $\langle f | F_N^E | i \rangle$ where

$$\langle f | F_N^E | i \rangle \equiv F_{fi}^E(q) = \sum_{j=1}^{N} Q_j \mathcal{F}_j^E(q) \int \Psi_f^*(\underline{r}_1, \underline{r}_2, \ldots \underline{r}_N) e^{i\underline{q}\cdot\underline{r}_j} \Psi_i(\underline{r}_1, \underline{r}_2, \ldots \underline{r}_N) \delta(r_1 + r_2 + \ldots r_N) \times \prod_{k=1}^{N} [d^3 r_k]$$

where delta function is included to impose C.M. constraint

$$F_{fi}^E = \sum_{j=1}^{N} Q_j \mathcal{F}_j^E(q) F_{fi}^j(q) \qquad \text{where} \qquad F_{fi}^j(q) = \langle f | e^{i\underline{q}\cdot\underline{r}_j} | i \rangle$$

$$= F_{fi}^N(q), \qquad \text{independent of } j \text{, by Bose symmetry.}$$

$$\therefore F_{fi}^E(q) = \left(\sum_{j=1}^{N} Q_j \mathcal{F}_j^E(q)\right) F_{fi}^N(q)$$

the elastic charge form factor of the composite ($i \to i$) is

$$F_{ii}^E(q) \equiv F_N^E(q) = \left(\sum_{j=1}^{N} Q_j \mathcal{F}_j^E(q)\right) F_N(q) \qquad (1)$$

where F_N is the elastic one body form factor of composite (for simplicity the charges Q_j are assumed to be unchanged in the transition $i \to f$. In the elastic case this is always true). This result is also valid for a bound composite system of particles with spin, provided their motion is non-relativistic, since in the static limit, the particle spins are separately conserved. However, for relativistic motion of the bound state, the intrinsic spins cannot be decoupled from orbital motions and the fourth component j_0 of the 4-vector current density is Lorentz contracted - $j_0 \to j_0' = \gamma j_0$ where $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = v/c$, where v is the velocity of the charged composite. Assuming the impulse approximation (additivity of quark 4 vector current operators in the Breit frame) and including spin of quarks and the effect of Lorentz contraction of quark spatial coordinates, Licht and Pagnamenta[8] have shown that the rest frame expression (1) is modified (for the nucleon $N=3$) to

$$F_N^E(t) = \alpha^{-1}\left[\sum_{j=1}^{3} Q_j \mathcal{F}_j^E(t)\right] F_N(-t/\alpha) \qquad (2)$$

where $\alpha = 1 - t/4m_N^2$.

$t = -q^2$ is the invariant 4- momentum transfer squared, m_N-nucleon mass, F_N is rest frame strong interaction form factor of nucleon, $\mathfrak{F}_j^E$-electric form factor of quark. If quarks are point like (proper charge densities are Delta functions) (alternatively, by point like is meant quarks with no internal structure or at least with sizes very small compared with that of the proton) then

$$\mathfrak{F}_j^E(t) = 1 \quad , \quad 1 \le j \le N \quad , \qquad \text{equ (3)}$$

For quarks bound in a deep harmonic oscillator potential well with Gaussion 3-quark wavefunctions $\Psi(r_1, r_2, r_3) = N e^{-\frac{\beta^2}{2}(r_1^2+r_2^2+r_3^2)}$ with constraint $r_1 + r_2 + r_3 = 0$, then $F_N(t) = e^{t/6\beta^2}$

from (2), $\quad F_N^E(t) = (1 - t/4m_N^2)^{-1} e^{t[6\beta^2(1-t/4m_N^2)]^{-1}}$

$$\underset{|t|\to\infty}{\sim} |t|^{-1}$$

Since the SLAC data[9] indicate $F_N^E(t) \underset{|t|\to\infty}{\lesssim} |t|^{-2}$, quarks cannot be quite point like. Notice that (2) was derived on the assumption of spin $\frac{1}{2}$ for quarks whereas the SU(6) $\underline{56}$ representation in which the nucleon appears requires the nucleon ground state spatial wavefunction to be antisymmetric in the quark coordinates if quarks are Fermions. The assumption of spatially symmetric nucleon wavefunctions is thus inconsistent with its group symmetry classification, although it is popular,[10-12] unless quarks are para Fermions[13] They are used for convenience and to avoid zeros in baryon form factors which occur with antisymmetric space wave functions.[14]

Though - nodes in the strong, and therefore electromagnetic, form factors can never be ruled out (none up to $|t| = 25 (\frac{GeV}{c})^2$ so far), the above argument should be independent of assumptions about nucleon wavefunctions since

$$F_N(-t/\alpha) \underset{|t|\to\infty}{\sim} F_N(4m_N^2) = \text{a constant}, \quad \text{whatever } F_N(t).$$

It should be noticed that the spurious essential singularity in the time-like region at $t = 4m_N^2$ arises whether spin is considered or not since in the case of composites of n spinless particles Licht and Pagnamenta obtain $F_N^E(t) = (1 - t/m_\rho^2)^{(1-n)/2}_{\alpha} \cdot F_N(\frac{|t|}{\alpha})$ assuming only the ρ meson couples each quark to the photon. Although the expected asymptotic behaviour for $F_N^E(t)$ is then obtained, this, contra Licht and Pagnamenta, does not prove finite compositeness of the proton as an infinitely composite model predicts the $|t|^{-2}$ variation naturally as an upper bound for F_N^E (t) if reasonable, simple assumptions about the proton charge density are made (shown later). The upper bound on the quark electric form factor

$$\mathfrak{F}_Q^E(t) \underset{|t|\to\infty}{\lesssim} |t|^{-1}$$

required for the Licht and Pagnamenta model, with spin included, to agree with the $|t|^{-2}$ asymptotic variation of the nucleon vector current form factors is understandable if quarks have no hadronic structure and their electromagnetic interactions are dominated by vector meson exchange. Then

$$\mathcal{F}_Q^E(t) = (1 - t/m_\rho^2)^{-1}$$

(assuming for simplicity only ρ_0 exchange, though this is not essential). In fact, an excellent fit to the magnetic form factor, better than the ad hoc dipole formula

$$F_p^M(t) = (1 - t/0.71)^{-2}$$

is obtained using this assumption.[8] However the SLAC data indicate, despite large error bars, that the magnetic form factor (and therefore the electric form factor, because of the scaling rule $F_p^M(t)/\mu_p = F_p^E(t)$, where μ_p is the proton anomalous magnetic moment)* falls significantly below the dipole curve at large momentum transfers i.e.

$$F_p^E(t) \underset{|t|\to\infty}{\sim} |t|^{-2-N}$$

with $0 < N < 1$. This implies instead of equ. (3)

$$\mathcal{F}_Q^E(t) \underset{|t|\to\infty}{\sim} |t|^{-1-N}$$

using equ. (2). Such an asymptotic variation should be expected for quarks with non-singular charge densities, invariant under three dimensional rotations, for the quark charge density is then, in terms of the momentum transfer q :-

$$\rho_Q(r) = 4\pi \int_0^\infty \mathcal{F}_Q^E(q^2)\, q^2 \frac{\sin qr}{qr}\, dq$$

$$\text{and} \quad \lim_{r\to 0} \rho_Q(r) = 4\pi \int_0^\infty q^2\, \mathcal{F}_Q^E(q^2)\, dq$$

so that the quark charge density is non-singular at the origin provided $\mathcal{F}_Q^E(q^2) \underset{q^2\to\infty}{\sim} q^{-2(1+N)}$

$$\text{i.e.} \quad \mathcal{F}_Q^E(t) \underset{|t|\to\infty}{\sim} |t|^{-(1+N)}$$

Thus in terms of the Licht and Pagnamenta model, the proton magnetic form factor must fall off faster than the dipole formula indicates in the large momentum transfer region since otherwise the pure dipole variation ($N=0$) would imply a singular charge distribution for quarks. Thus the SLAC data, indicating a proton form factor decreasing as the fourth power or more of the momentum transfer, instead of the second power expected if the virtual photon couples with the proton via vector mesons, is readily understandable in terms of Lorentz contraction of the proton, vector meson coupling of quarks to the virtual photon and almost point-like quarks. The model predicts:-

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* the scaling laws : $F_p^E(t) = F_p^M(t)/\mu_p = F_n^M(t)/\mu_n$ have been verified for $|t| \le 7$ GeV/c (See D. Yount, J. Pine, Phys. Rev. 128, 1842 (1962); K.W. Chen et al., Phys. Rev. 141, 1267 (1966)) The recent experiment of L.E. Price et al. (Phys. Rev. D, Vol. 4, No. 1, 45 (1971) indicates, however, that F_p^E/F_p^M falls below μ_p^{-1} as q^2 increases. Their results have been confirmed by Barger et al. (C.H. Barger et al., Phys. Letters 35B No. 1 (1971) who find that the scaling rule is broken for $10\,F^{-2} \le q^2 \le 50\,F^{-2}$. But since this implies $F_p^E(t)$ decreases more rapidly than $F_p^M(t)$ as $|t|$ increases, the

1. $\dfrac{F_p^E(t)}{F_p^E(\text{DIPOLE})} \underset{|t|\to\infty}{\sim} 1$ if quarks are bosons or para fermions (proton space wavefunction symmetric)

2. $\dfrac{F_p^E(t)}{F_p^E(\text{DIPOLE})} \underset{|t|\to\infty}{\sim} \infty$ if quarks are point fermions, like electrons with vector meson coupling to photons.

3. $\dfrac{F_p^E(t)}{F_p^E(\text{DIPOLE})} \underset{|t|\to\infty}{\sim} 0$ if quarks are non-point-like fermions or para fermions.

with 3, at present favoured experimentally. These conclusions are dependant on assuming the proton contains three sub particles. Apart from the results of Shrauner et al.[3] which exclude the possibility of 2, 4 or more finite numbers of sub-particles present in the proton, the approximate $|t|^{-2}$ asymptotic variation of the proton magnetic form factor is further evidence in favour of the SU (3) quark model as opposed to other finitely composite models[15] of hadrons. For example, for $n = 1$, $F_p^E(t) \sim |t|^{-1}$, for $n = 2$, $F_p^E(t) \sim |t|^{-3/2}$, for $n = 4$, $F_p^E(t) \sim |t|^{-5/2}$. The extension of elastic high energy electron-proton scattering experiments to higher energies and momenta transfers of the scattered electron is of prime importance to the problem of the compositeness of hadrons. If the fall off of the proton observed/dipole form factor ratio persists and tends to zero as the momentum transfer increases but the approximate dipole variation is maintained, this would favour the idea of three almost point-like entities comprising protons, complementing the deep inelastic e-p experiments which imply the existence of point-like structures within the proton and neutron.

THE DROPLET MODEL:-

In this model, elementary particles are conceived as droplets of some basic hadronic matter, conglomerates of infinitesimal particles which scatter each other infinitesimally and when colliding, pass through the other droplet with attenuation and then leave, the resulting attenuated droplet propagating through the vacuum as the scattered particle, the probability amplitude for which depends on the opaqueness of both droplets for transmission of the other through itself. If the proton is conceived as a droplet of finely granulated charged matter, then the following mathematical assumptions about the distribution of charge within it lead naturally to the observed upper bound for the proton charge form factor i.e.

$$F_p^E(q^2) \underset{q^2\to\infty}{\lesssim} q^{-4}.$$

(1) charge density distribution is spherically symmetric - as ground state of proton is an S state:- $\rho(\underline{r}) = \rho(r)$.

(2) charge density is continuous $\rho(r+\epsilon) - \rho(r-\epsilon) = 0$, as $\epsilon \to 0$, for $0 \le |r| \le \infty$

(3) $\rho(0)$ is finite, i.e. proton not a point particle like the electron.

PROOF:-

$$F_p^E(\underline{q}) = \int \rho(\underline{r})\, e^{i\underline{q}\cdot\underline{r}}\, d^3\underline{r}$$

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conclusions arrived at above remain true, qualitatively.

Assuming (1), $F_p^E(\underline{q}) = 4\pi \int_0^\infty \rho(r) r^2 \frac{\sin qr}{qr}\, dr = F_p^E(q)$

Since $\left| \rho(r) r^2 \frac{\sin qr}{qr} \right| \leq r^2 \rho(r)$ for all q in $0 \leq q \leq \infty$ and $\int_0^\infty r^2 \rho(r) dr = 1/4\pi$ is convergent, then by the Weierstrass M-test, $\int_0^\infty \rho(r) r^2 \frac{\sin qr}{qr}\, dr$ is uniformly convergent in the range

$$0 \leq q \leq \infty$$

Assuming (2), then since $r^2 \frac{\sin qr}{qr}$ is continuous in $0 \leq |r| \leq \infty$ $\rho(r) \frac{\sin qr}{qr}$ is continuous in $0 \leq |r| \leq \infty$

$$\therefore \int F_p^E(q) e^{-i\underline{q}\cdot\underline{r}} d^3\underline{q} = \int \left[\int \rho(r') e^{i\underline{q}\cdot(\underline{r}'-\underline{r})} d^3\underline{q} \right] d^3\underline{r}'$$

$$= \int \rho(r') \delta(\underline{r}-\underline{r}') d^3\underline{r}' = \rho(r)$$

From (1), $\rho(-\underline{r}) = \rho(\underline{r}) = \rho(r)$

Letting $\underline{r}' = -\underline{r}$, then $F_p^E(q) = \int \rho(-r) e^{i\underline{q}\cdot\underline{r}} d^3\underline{r} = \int \rho(r') e^{-i\underline{q}\cdot\underline{r}'} d^3\underline{r}'$

$$= F_p^E(-q)$$

$\therefore$ $F_p^E(q)$ is an even function of q i.e. $F_p^E(q) = F_p^E(q^2)$

$$\therefore \quad \rho(r) = 4\pi \int_0^\infty F_p^E(q^2) q^2 \frac{\sin qr}{qr}\, dq$$

$$\rho(0) = 4\pi \int_0^\infty q^2 F_p^E(q^2)\, dq$$

Assuming (3), then integral is convergent i.e. $F_p^E(q^2) \underset{q^2\to\infty}{\lesssim} q^{-2}$

Since $\left| F_p^E(q^2) q^2 \frac{\sin qr}{qr} \right| \leq q^2 F_p^E(q^2)$ for all r in $0 \leq |r| \leq \infty$, then by the M-test, $\int_0^\infty F_p^E(q^2) q^2 \frac{\sin qr}{qr}\, dq$ is uniformly convergent in $0 \leq |r| \leq \infty$

$$\therefore \quad \frac{\partial \rho}{\partial r} = 4\pi/r \int_0^\infty F_p^E(q^2) q^2 \left(\cos qr - \frac{\sin qr}{qr} \right) dq$$

$$\therefore \quad \frac{\partial \rho}{\partial r^2} = \frac{1}{2r}\frac{\partial \rho}{\partial r} = 2\pi/r^2 \int_0^\infty F_p^E(q^2) q^2 \left(\cos qr - \frac{\sin qr}{qr} \right) dq$$

$$= 2\pi \int_0^\infty F_p^E(q^2) q^2 \left[-\tfrac{1}{3} q^2 + O(q^4 r^2) \right] dq$$

$$\therefore \left(\frac{\partial \rho}{\partial r^2} \right)_{r^2=0} = -2\pi/3 \int_0^\infty q^4 F_p^E(q^2)\, dq$$

From (2) and (3), $\left(\frac{\partial \rho}{\partial r^2} \right)_{r^2=0}$ is finite so the integral $\int_0^\infty q^4 F_p^E(q^2) dq$ is bounded. Since $F_p^E(q^2)$ is analytic in the space-like region,*†

$$\therefore \quad F_p^E(q^2) \underset{q^2\to\infty}{\lesssim} q^{-4}$$

The upper bound follows only if assumption (2) is relaxed so as not to include the origin for then the integral diverges.**

No simple model for the charged droplet can reproduce the observed electric form factor of the proton. Gaussian densities give gaussian form factors in the whole momentum transfer domain, which of course vary too rapidly, exponential densities give a pure dipole variation which is ruled out by the large momentum tranfer data, whilst a gaussian charge density with an exponential tail gives a form factor which, at large q values, is dominated by the contribution from the gaussian region :- $F_p^E(q) \underset{q\to\infty}{\sim} C \frac{\sin qa}{qa}$ where C is a constant and a the position of the beginning of the tail. This

---

* Jaffe has established from axiomatic quantum field theory that form factors are analytic in the t plane cut from t_0 to ∞ with $t_0 > 0$, with possible exception of a finite region $|t| < t_0$. See Ref. (16).

† According to dispersion theory, form factors are analytic in the t plane cut

is oscillatory and varies too slowly. The ratio $\frac{F_p^E}{F_{(\text{dipole})}}$ diverges asymptotically. An exponential charge density with a Gaussian tail gives a form factor which asymptotically is dominated by the exponential region contribution

$$F_p^E(q) \underset{q \to \infty}{\sim} \frac{c}{q^2} \cdot \frac{\sin qa}{qa}$$

which has the same faults.

OTHER MODELS

The q^{-4} asymptotic variation of $F_p^E(q)$ has been obtained [17-19] in models where the nucleon is viewed as a composite of two 'elementary' particles bound by exchange of a scalar particle. These cannot account for the apparent faster fall off than q^{-4} though the behaviour is better than the variation $(\log q)^2 q^{-2}$ found for truly elementary particles[18]. The upper bound on F_p^E has been shown to be model dependent for two-particle composites[19] [20] obeying the Bethe-Salpeter equation so nothing can be concluded about the compositeness of the nucleon from these considerations. The variation

$$F_p^E(t) \underset{|t| \to \infty}{\sim} e^{-|t|^{1/2}}$$

obtained as a lower bound on the form factor by Jaffe[16] from local quantum field theory and conjectured by Wu and Yang[1] on the basis of the large angle proton-proton elastic scattering data does not fit the form factor SLAC curve very well[22], the fit being much worse than the dipole fit, which was still the best parametrisation until the work of Licht and Pagnamenta. Bootstrap models[23] which also give this variation, would appear to be ruled out as well.

In conclusion, the approximate smooth fall off as $|t|^{-2}$ of the proton electric form factor is characteristic of any composite model of the nucleon whether finite or infinite in the number of constituents, although the as yet exclusive fit to the data of the quark model favours finite compositeness with three constituents. Future experiments should confirm or invalidate these conclusions.

SECTION B:-

(3) SUM RULES FOR FORM FACTORS:-

If spin is ignored, the electric charge form factor of a composite of N particles is from equ. (1) Section A, $F_N^E(q) = \left(\sum_{j=1}^{N} Q_j \mathfrak{F}_j(q)\right) F_N(q)$

For the nucleon, pion and kaon isospin multiplets, quark model gives

$$F_p^E(q) = \tfrac{1}{3}(4\mathfrak{F}_1 - \mathfrak{F}_2)\,F_p(q)$$
$$F_n^E(q) = \tfrac{2}{3}(\mathfrak{F}_1 - \mathfrak{F}_2)\,F_n(q)$$
$$F_{\pi^+}^E(q) = \tfrac{1}{3}(2\mathfrak{F}_1 + \mathfrak{F}_2)\,F_\pi(q) = F_{\pi^-}^E(q)$$
$$F_{K^+}^E(q) = \tfrac{1}{3}(2\mathfrak{F}_1 + \mathfrak{F}_3)\,F_K(q) = F_{K^-}^E(q)$$
$$F_{K^0}^E(q) = \tfrac{1}{3}(-\mathfrak{F}_2 + \mathfrak{F}_3)\,F_{K^0}(q) = F_{\bar{K}^0}^E(q)$$

where $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3$ are charge form factors of $\mathcal{P}, \mathcal{N}, \lambda$ quarks.

From t_0 to ∞ with possibly some poles between $t=0$ and $t=t_0$. See Ref.(21)

** Noting $\left(\frac{\partial \rho}{\partial r}\right)_{r=0} = 0$, whatever $F_p^E(q)$ is, and $\rho = \rho(r^2)$, the upper bound is obeyed if $0.5 < n < 1$, otherwise $n > 1$ where $\rho(r^2) \underset{r^2 \to 0}{\approx} r^{2n}$.

Exact SU (3) symmetry gives $F_p = F_n = F_\pi = F_K = F_{K^0}$

and $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3$

then $F^E_p(q) = F^E_{\pi^+}(q) = F^E_{K^+}(q)\ ; \quad F^E_n(q) = F^E_{K^0}(q) = 0$

Broken SU (3) symmetry gives $F_p \neq F_\pi \neq F_K$

then $$\frac{F^E_{K^+}}{F_K} - \frac{F^E_{K^0}}{F_{K^0}} = \frac{F^E_{\pi^+}}{F_\pi} = \frac{F^E_p}{F_p} - \frac{F^E_n}{F_n}$$

assuming charge independence for λ-n, λ-p quark interactions gives $F_K = F_{K^0}$. The assumption is made that to first order in the SU (3) symmetry breaking quark antiquark interactions, only meson masses and interactions are modified, not meson bound state wavefunctions. This is suggested for the pion and kaon in particular since their Regge trajectories have the same slope so that if these are associated with $(\text{mass})^2$ eigenvalue solution of the Bethe-Salpeter equation with spin ignored (see page 40) then, since the trajectory slope is

$$\alpha'(0) = (2\mu\omega)^{-1}$$

where μ is the reduced mass of the quark, ω-the oscillator frequency, the product $\mu\omega$ is the same for both mesons. Now the oscillator wavefunction has, as parameter, $\mu\omega^2$ and so is approximately the same for both mesons, ignoring the small mass difference between the strange and non-strange quarks.

$$\therefore\ F_\pi = F_K$$

$$\therefore\ F^E_{\pi^+}(q) = F^E_{K^+}(q) - F^E_{K^0}(q)$$

This result has also been obtained with the assumptions :-

1. same coupling of ρ meson to π, K in the ρ-meson-meson vertex.(24)
2. SU (3) symmetry for coupling constants.(24)
3. One pole approximation in dispersion relations for meson form factors.(25)
4. Vector meson dominance in virtual photon-meson coupling. (26)

SECTION C:-

Mean square radii of electric charge and hadronic matter distributions are defined by

$$\langle r^2\rangle^E = 6\left|\frac{dF^E}{dq^2}\right|_{q^2=0}$$
$$\langle r^2\rangle^H = 6\left|\frac{dF^H}{dq^2}\right|_{q^2=0}$$

where F^E, F^H are the electric and hadronic matter form factors respectively. Denoting the mean square charge radii of the quark by $\langle r^2\rangle^E_Q$, $Q = p$ or n, we have, from Section B,

$$\langle r^2\rangle^E_p = \tfrac{4}{3}\langle r^2\rangle^E_{\mathcal{P}} - \tfrac{1}{3}\langle r^2\rangle^E_{\mathcal{N}} + \langle r^2\rangle_p \qquad (1)$$
$$\langle r^2\rangle^E_\pi = \tfrac{2}{3}\langle r^2\rangle^E_{\mathcal{P}} + \tfrac{1}{3}\langle r^2\rangle^E_{\mathcal{N}} + \langle r^2\rangle_\pi \qquad (2)$$

With assumption of point quarks but SU (3) symmetry breaking for p, π

$$\langle r^2\rangle^E_p - \langle r^2\rangle^E_\pi = \langle r^2\rangle_p - \langle r^2\rangle_\pi \neq 0$$

with assumption of SU (2) symmetry for quarks, point like or not,

$$\langle r^2\rangle^E_{\mathcal{P}} = \langle r^2\rangle^E_{\mathcal{N}}$$
$$\therefore\ \langle r^2\rangle^E_p - \langle r^2\rangle^E_\pi = \langle r^2\rangle_p - \langle r^2\rangle_\pi$$

Since only the L.H.S. is experimentally measurable, model calculation of R.H.S. cannot distinguish between two independant assumptions. R.H.S.

has been calculated from $SU(6)$ shell model wave functions[27] for ρ, π, assuming harmonic oscillator forces, assignments $[56, L^P = 0^+]$ and $[35, L^P = 0^-]$ for the proton and pion respectively and the same spring constants for the simple harmonic motion, calculated from the ρ trajectory slope (see DYNAMICS Section B), the latter being chosen because the $J^{PC} = 1^{-+}$ $SU(3)$ nonet is almost exactly $SU(3)$ symmetric. The results are

$$\langle r^2\rangle_p = 0.040F^2, \quad \langle r^2\rangle_\pi = 0.028F^2$$

The date of Hofstader et al.[28] give $\sqrt{\langle r^2\rangle_p^E} = 0.81F$, the results of Akerlof et al.[29] give $\sqrt{\langle r^2\rangle_\pi^E} = 0.80F$. This gives $L.H.S. = 0.016F^2$, $R.H.S. = 0.012F^2$ with 25% agreement. Since $SU(2)$ is broken for the nonstrange quarks actually $(\langle r^2\rangle_p^E + \langle r^2\rangle_n^E)$ the agreement is quite good.

NEUTRON ELECTRIC FORM FACTOR.

From the general result for the charge form factor of N particle composite

$$F_N^E(q) = \left[\sum_{j=1}^{N} Q_j \mathcal{F}_j^E(q)\right] F_N(q)$$

for a charged droplet, $N = \infty$, $\mathcal{F}_j^E(q) = 1$; $Q_j = \frac{Q}{N}$ where Q is total charge. $\therefore$ $F_N^E(q) = \lim_{N\to\infty} \sum_{j=1}^{N} Q_j F_N = Q F_N$

$\therefore$ $F_N^E(q) = 0$ if $Q = 0$. The electric form factor of the neutron is thus identically zero if it is considered as an infinitely composite particle. In the quark model, $F_n^E(q) = 2/3\,(\mathcal{F}_p^E - \mathcal{F}_n^E)\,F_n(q) \neq 0$ provided quarks are non-point like and their form factors are different. There is the following experimental information on F_n^E. Data about $F_n^E(q)$ has been obtained in three ways. 1. thermal neutron scattering off certain noble gas nuclei[30] (these eliminate Mott scattering as they have no magnetic moments)

2. reflection off and transmission of neutrons through bismuth and liquid oxygen.[32]

3. high energy electron-deuteron scattering.

Ref (30) obtain a weighted average for $\left|\frac{dF_n^E}{dq^2}\right|_{q^2=0}$ of $0.0193 \pm 0.0004F^2$, in reasonable agreement with the value obtained from mirror reflection experiments of $0.02 \pm 0.0019F^2$, but disagreeing rather with the transmission experiment result of $0.0225 \pm 0.0007F^2$. The electron-deuteron results seem to be highly model dependant, varying with choice of hard core radius and percentage of ^3_1D state assumed to be present in deuteron ground state. The results of Benaksas, Drickey and Frèrejaque[31] and Drickey and Hand[32] indicate a mean valve of zero for $3 \leq q^2 \leq 6F^{-2}$. However, although consistent with $F_n^E(q)=0$, their results disagree with thermal neutron experiment slope values of $F_n^E(q)$, giving $\left|\frac{dF_n^E}{dq^2}\right|_{q=0} = 0.0002 \pm 0.0067F^2$. Casper and Goss[33] have, however, shown that the discrepancy is due to using non-relativiatic deuteron wave-functions. When relativistic corrections are made, not only is the calculated slope then completely in agreement with the reliable thermal neutron slope value but also the extracted valves of F_n^E are increased with uncertainties small enough for the deviation from zero to be significant (approximately two or three standard deviations). Thus experiments indicate $F_n^E(q) \neq 0$ when relativistic corrections are made, although the actual values are still somewhat model dependent. Typical values are 0.01 - 0.02 F^2 for $q^2 < 2F^{-2}$. This can be understood in the quark model qualititively as approximate SU (2) symmetry for the quark isospin doublet. We have

$$\langle r^2\rangle_n^E = \tfrac{2}{3}\left[\langle r^2\rangle_p^E - \langle r^2\rangle_n^E\right] \qquad (3)$$

Using the thermal neutron data[31], $\langle r^2\rangle_n^E = 0\cdot1158\ F^2$

$$\therefore \quad \langle r^2\rangle_p^E - \langle r^2\rangle_n^E = 0\cdot1737$$

from equ (2) above;-

$$2\langle r^2\rangle_p^E + \langle r^2\rangle_n^E = 3\left(\langle r^2\rangle_\pi^E - \langle r^2\rangle_\pi\right)$$
$$= 1\cdot836\,F^2$$

$\therefore\ \langle r^2\rangle_p^E = 0\cdot670\,F^2$ and $\langle r^2\rangle_n^E = 0\cdot496\,F^2$,

giving charge radii 0.82 F and 0.70F for p and n respectively. Such charge radii are comparable with the proton change radius[28]. However, their values depend on the rather uncertain pion form factor which at present is compatible with either dipole[34] or ρ dominance forms.[29] The former gives $\langle r^2\rangle_\pi^E = .0\cdot364\,F^2$

and the p and n charge radii are then 0.63F and 0.47F respectively. The latter gives $\langle r^2\rangle_\pi^E = 0\cdot218\,F^2$ and the charge radii are 0.50F and 0.27F respectively. The ambiguity of the pion form factor means firm conclusions as to quark electromagnetic radii cannot be made, other than that at least one is <u>not</u> point-like, in view of the non-vanishing neutron electric form factor.

In view of the surprising result of Chou and Yang[7] that the electric and matter form factor of the proton appear to be the same, it is interesting to deduce the quark R.M.S. radius, assuming such equality. From section B,

$$F_p^E(q) = \tfrac{1}{3}\left[4\mathfrak{F}_p^E - \mathfrak{F}_n^E\right]F_p(q)$$

$\therefore\ F_p^E(q) = F_p(q)$ if 1. $\mathfrak{F}_p^E = \mathfrak{F}_n^E = 1$ i.e. quarks are pointlike.

or 2. $4\mathfrak{F}_p^E - \mathfrak{F}_n^E = 3$ i.e. quarks are non point like with $4\langle r^2\rangle_p^E = \langle r^2\rangle_n^E$.

then $\langle r^2\rangle_n^E = 2\langle r^2\rangle_p^E$

and $\sqrt{\langle r^2\rangle_p^E} = 0\cdot24F$, $\sqrt{\langle r^2\rangle_n^E} = 0\cdot48\,F$. These values are independant of assumptions about nucleon internal wavefunctions since, because of the zero charge of the neutron, the charge radius of the latter is determined only by the quark charge distributions and not by the neutron matter form factor as well (see equ. 3 on previous page).

Since the SU(6) shell model 3-quark wavefunction for the proton is Gaussian in the quark spatial coordinates, the corresponding matter form factor is also Gaussian: $F_p(q) = e^{-\langle r^2\rangle q^2/6}$. If quarks were point like exactly, then the proton charge form factor would also be Gaussian in the momentum-transfer. But, if the proton matter form factor and quark electromagnetic form factor were identical, then $F^E_p = F_p^2 = \mathcal{F}_p^{E\,2} = \mathcal{F}_q^{E\,2}$ and by choosing $\mathcal{F}_p^E(q^2) = (1 + \frac{q^2}{0.71})^{-1}$ the dipole form follows. But then

$$\langle r^2\rangle_p^E = \langle r^2\rangle_q^E = \tfrac{1}{2}\langle r^2\rangle_p^E$$
$$\sqrt{\langle r^2\rangle_p^E} = 0.57F$$

which value is incompatable with the selfconsistent multiple-quark-scattering analysis of Shrauner et al(3), in which too rapid decrease with momentum transfer of the proton-proton elastic differential cross section results if this assumption is made, and also is incompatible with quark sises needed to satisfy the Chou-Yang result. The explanation of the discrepancy between SU (6) shell model from factors and the observed dipole behaviour of F_p^E is that in high energy elastic electron-proton scattering, the former 'sees' a proton contracted in the direction of their relative motion so that the Coulomb scattering is more instantaneous and the proton more point-like, the internal 3-quark structure becoming less important, as the matter distribution collapses into a Delta function-like singularity and so gives a more and more slowly varying hadronic form factor as the momentum transfer increases. It should be noted that Gaussian form factors also violate the lower bound on form factors established by Jaffe from Q.E.T. (16) Although the calculations above could be repeated using the relativistic form factors, this will not be done as they depend, for small momentum transfers, on details of the particle wavefunction behaviour at large distances, which is unknown, other than becoming zero at infinity.

CONCLUSION:-

Present high energy scattering data cannot be used to decide between the quark and infinitely composite models of hadrons, although experimental information prefers the picture of the proton (and antiproton) as made up of three constituents as opposed to any other finite number of particles. Electromagnetic form factor data can be explained successfully in terms of the quark model, although the large momentum transfer variation of the proton magnetic form factor is not inconsistent with a naive droplet model if the electric and magnetic charge distributions in the proton are identical (the scaling law is true at all values of the momentum transfer). Recent experiments may rule this out. The neutron electric charge form factor is incompatible with a droplet model. Thus, experimental information on form factors supports the quark model but not the droplet model.

=== II ===

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APPENDIX A

$$\delta_a(\underline{r}) = -\frac{1}{2k}\int_0^\infty U(\underline{r}+\hat{k}'s)\,ds$$

Choosing the z axis along the incident direction $\hat{k}_0$, the x axis to lie in the plane of $\hat{k}_0$ and $\hat{k}'$, then

$$\begin{aligned} U(\underline{r}+\hat{k}'s) &= U(x+s\sin\theta,\, y,\, z+s\cos\theta) \\ &= \sum_{n=0}^{\infty}\frac{s^n}{n!}\left(\sin\theta\frac{\partial}{\partial x}+\cos\theta\frac{\partial}{\partial z}\right)^n U(x,y,z)\text{, By Taylor's Theorem.} \end{aligned}$$

$$\therefore\ \delta_a(\underline{r}) = \frac{-1}{2k}\sum_{n=0}^{\infty}\int_0^\infty \frac{s^n}{n!}\left(\sin\theta\frac{\partial}{\partial x}+\cos\theta\frac{\partial}{\partial y}\right)^n U(x,y,z)\,ds$$

Assuming

$$\begin{aligned} U(r) &\simeq U_0 , & 0 \le r \le R \\ &\simeq 0 , & r > R \end{aligned}$$

$$\delta_a(\underline{r}) \simeq -\frac{1}{2k}\sum_{n=0}^{\infty}\int_0^R \frac{s^n}{n!}\left(\sin\theta\frac{\partial}{\partial x}+\cos\theta\frac{\partial}{\partial y}\right)^n U(x,y,z)\,ds$$

$$\frac{\partial U}{\partial x} \sim \frac{U_0}{R} \sim \frac{\partial U}{\partial z} \quad\text{and}\quad \frac{\partial^n U}{\partial x^n} \sim \frac{U}{R^n} \sim \frac{\partial^n U}{\partial y^n}$$

$$\begin{aligned} \therefore\ \delta_a(\underline{r}) &\simeq -\frac{1}{2k}\sum_{n=0}^{\infty}\frac{R^{n+1}}{(n+1)!}\left(R^{-1}\sin\theta + R^{-1}\cos\theta\right)^n U_0 \\ &\simeq -\frac{U_0 R}{2k}\sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\cos\theta+\sin\theta)^n \end{aligned}$$

Noting $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

$$\int_0^x e^x\,dx = e^x - 1 = x\sum_{n=0}^{\infty}\frac{x^n}{(n+1)!}$$

$$\therefore\ \sum_{n=0}^{\infty}\frac{x^n}{(n+1)!} = \frac{1}{x}(e^x-1)$$

$$\therefore\ \sum_{n=0}^{\infty}\frac{1}{(n+1)!}(\cos\theta+\sin\theta)^n = \frac{e^{\cos\theta+\sin\theta}-1}{\cos\theta+\sin\theta} \le \frac{e^{\sqrt{2}}-1}{\sqrt{2}} \sim 2.2$$

$$\therefore\ |\delta_a(r)| \lesssim \frac{|U_0|R}{k}$$

$$\text{and}\ \left|\frac{\partial\delta_a}{\partial r}\right| \lesssim \frac{|U_0|}{k}$$

APPENDIX B

The Dirac delta function is $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixy}\,dy$

$$= \frac{1}{2\pi}\lim_{N\to\infty}\int_{-N}^{N} e^{ixy}\,dy = \frac{1}{2\pi}\lim_{N\to\infty}\int_{-N}^{N}(\cos xy + i\sin xy)\,dy$$

$$= \frac{1}{2\pi}\lim_{N\to\infty}\left[\left.\frac{\sin xy}{x}\right|_{-N}^{N} - i\left.\frac{\cos xy}{x}\right|_{-N}^{N}\right]$$

$$= \frac{1}{2\pi}\lim_{N\to\infty}\left[2\frac{\sin Nx}{x} - i\left(\frac{\cos Nx}{x} - \frac{\cos Nx}{x}\right)\right]$$

$$= 1/\pi \lim_{N\to\infty}\frac{\sin Nx}{x}$$

$$\therefore \lim_{N\to\infty}\frac{\sin Nx}{x} = \pi\,\delta(x) = 0 \quad \text{for } x \neq 0$$

Let $x = \frac{\pi}{2N} - x'$

$$\sin Nx = \cos Nx'$$

$$\therefore \lim_{N\to\infty}\frac{\sin Nx}{x} = \lim_{N\to\infty}\frac{\cos Nx'}{\frac{\pi}{2N} - x'} = -\lim_{N\to\infty}\frac{\cos Nx'}{x'}$$

$$\therefore \lim_{N\to\infty}\frac{\cos Nx'}{x'} = -\pi\,\delta(x') = 0 \quad \text{for } x' \neq 0$$

APPENDIX C.

For a wide range of scattering Potentials $V(r)$, the phase shift function $\chi(b)$ for a particle of mass m, defined by

$$\chi(b) = \frac{-1}{2k}\int_{-\infty}^{\infty} U(b,z)\,dz \quad \text{where } U(r) = \frac{2m}{\hbar^2}V(r),\; b^2+z^2=r^2$$

has the upper bound $|\chi(b)| \leq \frac{|U_0|R}{k}$ where R is the range of the potential, $U_0 = \frac{2mV_0}{\hbar^2}$ where V_0 is the depth of the potential well. Examples are given below:-

1. Square potential well

$$U(r) = -U_0 \quad , \quad -R \leq r \leq R$$

$$= 0 \quad , \quad -R > r > R$$

$$\chi(b) = \frac{U_0 R}{k} \quad \text{and} \quad |\chi| = \frac{|U_0|R}{k}$$

2. Exponential potential

$$U(r) = U_0 e^{-r/R} \quad , \quad 0 \leq |r| \leq \infty$$

$$\chi(b) = -\frac{U_0 R}{k}\cdot\frac{b}{R}\,K_1\left(\frac{b}{R}\right)$$

$$|\chi(b)| \lessapprox \frac{|U_0|R}{k} \quad , \quad b \neq 0$$

3. Yukawa potential

$$U(r) = \frac{U_0}{r/R}\,e^{-r/R} \quad , \quad 0 \leq |r| \leq \infty$$

$$\chi(b) = -\frac{U_0 R}{k}\,K_0\left(\frac{b}{R}\right)$$

$$|\chi(b)| \lessapprox \frac{|U_0|R}{k} \quad , \quad b \neq 0$$

4. Gaussian potential

$$U(r) = U_0 e^{-r^2/R^2} \quad , \quad 0 \leq |r| \leq \infty$$

$$\chi(b) = \frac{-U_0 R\sqrt{\pi}}{2k}\,e^{-b^2/R^2}$$

$$|\chi(b)| \lessapprox \frac{|U_0|R}{k}$$

APPENDIX D.

The magnetostatic potential due to a magnetic dipole $\underline{M}$ is

$$\phi(\underline{r}) = \frac{\underline{M}\cdot\underline{r}}{r^3} \tag{1}$$

The static magnetic field due to $\underline{M}$ is

$$\underline{H}(r) = -\nabla\phi = {}^3/_{r^5}(\underline{M}\cdot\underline{r})\underline{r} - \tfrac{1}{r^3}\nabla(\underline{M}\cdot\underline{r}) \tag{2}$$

for the z axis of the coordinate system choosen parallel to $\underline{M}$

$$\underline{M}\cdot\underline{r} = Mz \quad \text{and} \quad \nabla(\underline{M}\cdot\underline{r}) = \underline{M}$$

$$\therefore \underline{H} = {}^3/_{r^5}(\underline{M}\cdot\underline{r})\underline{r} - \tfrac{1}{r^3}\underline{M} \tag{3}$$

The interaction energy of two dipoles of magnetic moment $\underline{M}_1, \underline{M}_2$ is

$$V_{ss}(\underline{r}) = -\underline{M}_2\cdot\underline{H} = r^{-3}\,\underline{M}_1\cdot\underline{M}_2 - 3r^{-5}(\underline{M}_1\cdot\underline{r})(\underline{M}_2\cdot\underline{r}) \tag{4}$$

This has a singularity in r^{-3} at the origin.

<u>Behaviour of $V_{ss}(r)$ at origin</u>

$$\nabla\times\tfrac{1}{r}\underline{M}_1 = r^{-3}\,\underline{M}_1\times\underline{r}$$

$$\therefore \nabla\times\nabla\times\tfrac{1}{r}\underline{M}_1 = r^{-3}\,\nabla\times\underline{M}_1\times\underline{r} - (\underline{M}_1\times\underline{r})\times\nabla(r^{-3})$$

$$\nabla\times\underline{M}_1\times\underline{r} = (\underline{r}\cdot\nabla + \nabla\cdot r)\underline{M}_1 - (\underline{M}_1\cdot\nabla + \nabla\cdot\underline{M}_1)\underline{r} = 3\underline{M}_1 - \underline{M}_1 = 2\underline{M}_1$$

$$\nabla(r^{-3}) = -3r^{-5}\,\underline{r}$$

$$\therefore \nabla\times\nabla\times\tfrac{1}{r}\underline{M}_1 = {}^2/_{r^3}\,\underline{M}_1 + 3[(\underline{M}_1\times\underline{r})\times\underline{r}]\,r^{-5} = \tfrac{3}{r^5}(\underline{M}_1\cdot\underline{r})\underline{r} - \tfrac{1}{r^3}\underline{M}_1$$

$$= \underline{H}, \quad \text{from (3).}$$

$$\therefore V_{ss}(\underline{r}) = -\underline{M}_2\cdot[\nabla\times\nabla\times\tfrac{1}{r}\underline{M}_1] = -\underline{M}_2\cdot[\nabla(\nabla\cdot\tfrac{1}{r}\underline{M}_1) - \nabla^2(\tfrac{1}{r}\underline{M}_1)]$$

$$= \underline{M}_1\cdot\underline{M}_2\,\nabla^2(\tfrac{1}{r}) - (\underline{M}_2\cdot\nabla)(\nabla\cdot\tfrac{1}{r}\underline{M}_1)$$

$$\nabla\cdot\tfrac{1}{r}\underline{M}_1 = \underline{M}_1\cdot\nabla(\tfrac{1}{r})$$

$$\therefore V_{ss}(\underline{r}) = \underline{M}_1\cdot\underline{M}_2\,\nabla^2(\tfrac{1}{r}) - (\underline{M}_2\cdot\nabla)(\underline{M}_1\cdot\nabla)(\tfrac{1}{r}) \tag{5}$$

Consider V_{ss} as an operator acting on some function $f(r)$ regular at origin.

$$\nabla^2(\tfrac{1}{r})f(r) = f(r)\,\nabla\cdot\nabla(\tfrac{1}{r}) = \nabla\cdot[f(r)\nabla(\tfrac{1}{r})] - \nabla(\tfrac{1}{r})\cdot\nabla f(r)$$

integrating both sides on a small sphere of radius R centred at origin,

$$\int_V \nabla^2(\tfrac{1}{r})f(r)\,d^3\underline{r} = \int_V \nabla\cdot[f(r)\nabla(\tfrac{1}{r})]\,d^3\underline{r} - \int_V \nabla(\tfrac{1}{r})\cdot\nabla f(r)\,d^3\underline{r}$$

$$= \int_A f(r)\nabla(\tfrac{1}{r})\cdot d\underline{A} + \int_V \tfrac{1}{r^3}\,\underline{r}\cdot\nabla f(r)\,d^3\underline{r}, \quad \text{Using Green's Theorem.}$$

$$= -\int_A \tfrac{1}{r^2} f(r)\nabla r\cdot d\underline{A} + \int_V \tfrac{1}{r^2}\tfrac{\partial f}{\partial r}\,d^3\underline{r}$$

$$= -\int_A \tfrac{1}{r^2} f(r)\,dA_n + \int_0^R 4\pi\tfrac{\partial f}{\partial r}\,dr,$$ where dA_n is component of $d\underline{A}$ normal to surface of sphere. Noting $dA_n = 4\pi R^2$

the integral is $-4\pi f(R) + 4\pi[f(r)]_0^R = -4\pi f(0)$

$$= -4\pi\int_V f(r)\,\delta(\underline{r})\,d^3\underline{r}$$

$$\therefore \nabla^2(\tfrac{1}{r}) = -4\pi\,\delta(\underline{r}) \tag{6}$$

From (5), $V_{ss}(r) = \tfrac{2}{3}\underline{M}_1\cdot\underline{M}_2\,\nabla^2(\tfrac{1}{r}) - [(\underline{M}_2\cdot\nabla)(\underline{M}_1\cdot\nabla) - \tfrac{1}{3}\underline{M}_1\cdot\underline{M}_2\nabla^2](\tfrac{1}{r})$

the second term vanishes at the origin when integrated with $f(r)$.

From (4),(6), $V(ss(\underline{r})) = -\tfrac{8\pi}{3}\underline{M}_1\cdot\underline{M}_2\,\delta(\underline{r}) - r^{-3}[3r^{-2}(\underline{M}_1\cdot\underline{r})(\underline{M}_2\cdot\underline{r}) - \underline{M}_1\cdot\underline{M}_2]$

for all r, including origin.