University of Cape Town

PhD Thesis

The Gravity of Modern Amplitudes

Using on-shell scattering amplitudes to probe gravity

Author:
Daniel Johannes Burger

Supervisor:
J Murugan
A Weltman

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Declaration

I, Daniel Johannes Burger, declare that this thesis titled, *The Gravity of Modern Amplitudes: Using on-shell scattering amplitudes to probe gravity* and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a research degree at the University of Cape Town.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- Chapters 3 through 5 is based in part on work done in collaboration with others.
- Chapter 3 is based on the work in *2019; Rotating Black Holes in Cubic Gravity; D. J. Burger, W. T. Emond, N. Moynihan. [arXiv:1910.11618].*
- Chapter 4 is based on the work in *2017;Amplitudes for Astrophysicists: Known Knowns; D. J. Burger, R. Carballo-Rubio, N. Moynihan, J. Murugan and A. Weltman. General Relativity and Gravitation, 50(12):156 (2017). ISSN 1572-9532. [arXiv:1704.05067]*
- Chapter 5 is based on the work in *2020; On-Shell Perspectives on the Massless Limit of Massive Supergravity; D. J. Burger, N. Moynihan and Jeff Murugan. [arXiv:2005.14077].*

Signed by candidate

Daniel Johannes Burger

Abstract

In this thesis, we explore the use of on-shell scattering amplitudes as a way to understand various gravitational phenomena. We show that amplitudes are a viable way of studying certain aspects of gravity and showcase three such novel results here.

First is the computation of the deflection angle of both light and gravitational waves due to a massive static body. We compute this from a purely on-shell amplitude perspective and find that the result is in complete agreement with the corresponding calculation in General Relativity.

The second is the ability to derive classical results from the amplitudes. In this section we use on-shell scattering amplitudes to derive the perturbative metric of a rotating black hole in a generic form of Einstein gravity that has additional terms cubic in the Riemann tensor. We show that the metric we derive reduces to correct static metric in the zero angular momentum limit. We show that at first order in the coupling, the classical potential can be written to all orders in spin as a differential operator acting on the non-rotating potential. Further we compute the classical impulse and scattering angle of such a black hole.

The third is the resolution of a classical discontinuity in $\mathcal{N} = 1$ super gravity. Here we use on-shell methods for massive particles and use them to compute the supersymmetric version of the van Damme-Veltman-Zakharov (vDVZ) discontinuity. We construct the amplitudes of massive gravitinos (the superpartner of massive gravitons) and show that in the massless limit of the gravitinos there is the same discontinuity as found in massive gravity. This method sheds light on intricacies of the discontinuity that is obscured when handled classically.

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"To love the journey is to accept no such end. I have found, through painful experience, that the most important step a person can take is always the *next* one."

— Brandon Sanderson, Oathbringer

Chapter 1

Introduction

In modern theoretical physics there have been two great advances that have fundamentally altered the way we describe the universe. These are the Theory of General Relativity (GR) and Quantum Mechanics (QM). The former describes gravity in geometrical terms as the curvature in spacetime while the latter codifies the discrete nature of fundamental particles. In the attempt to elevate quantum mechanics to be relativistically consistent Quantum Field Theory (QFT) was discovered in which particles are describes as localised excitations in a fundamental field. In this description one can now consider a number of particles approaching one another from asymptotic infinity, incoming particles. Once close enough the particles interact, the scattering event, and some number of particles move off to infinity, the outgoing particles, possibly with some different properties. To describe this scattering event one can compute the s-matrix which is a unitary matrix codifying the probabilities of the initial particles scattering into the outgoing particles. The scattering amplitude is another way to compute this and captures the scattering process in a very specific way.

From the above descriptions of the scattering event one can compute the differential cross-section, the mod squared of the amplitude, which can be used to compute some observable such as the deflection of an electrically charged particle in an electric field. The incredible thing about this formulation is that one can view nearly any physical process in the universe as a scattering process. This means that if one wanted to learn about a physical system, say an atom, all one has to do is probe it with some appropriate particle and calculate the scattering amplitude. Upon measuring the resultant outgoing particles and knowing how the atom interacts with the probe, one can make inferences about the structure of the atom.

At first QFT and the scattering amplitudes were formulated in terms of integral equations, [1]. These are particularly complex and the technology of the time limited their use to calculate physically relevant processes to few people. So to calculate some scattering amplitude one had to write down the action of the theory, from that the relevant integrals and them compute them. This did not particularly lend itself to garnering an intuitive description of nature but rather a rigorous mathematical one. Being more intuitive in nature Feynman introduced a new tool to help with the ordering of the strenuous integrals required for QFT [2]. The method of using the appropriately named Feynman diagrams to codify the different ways in which a scattering process can happen, and the individual diagrams representing the integral required. This is a lot less labour intensive than just computing the scattering amplitude from the integral formulation alone since the diagrams allowed one to order the processes in a much more efficient way. The wide acceptance of Feynman diagrams as a tool for theoretical physicists is in large part due to the work of Dyson [3].

Even thought the Feynamn diagram was the industry standard for computing scattering amplitudes for six decades there were still problems that were untenable even with the computational power brought to the table by computers. These include but are not limited to gravitational interactions and the huge amount of scattering events that have to be computed in high energy scattering experiments. In the case of gravity this problem was due to the sheer number of diagrams required to compute even the simplest of processes. Consider for example that the three-graviton vertex consists of sixteen terms and the graviton propagator[1] of three terms this is due to the complicated tensor structure of gravity in the Einstein-Hilbert action. The four graviton amplitude at classical level has only three Feynman diagrams but given the complicated vertex structure there are hundreds of terms to consider. This number ramps up extremely quickly with the addition of more external particles. The field was ripe for a new advance.

In the early 2000's Witten reformulated in scattering amplitudes in terms of Penrose's twistor variables [4]. Twistor variables [5] turned out to be the natural language in which to frame amplitudes due to the manifest conformal symmetry and soon it had attracted much attention from other theorists and the contemporary amplitudes program was born. Over the course of the next few years the program was developed by among others Cachazo, Feng, Britto, Arkani-Hamed, Hodges, Trnka and many more. The most notable shift in understanding of amplitudes that came out of this was that the twistor variables exploit the hidden structure of the *integrands* in the original formulation of QFT. The great advantage of this is that it allows for a lot of simplification of the computation even before having to perform the integral. In specific cases it even localised the integral over the propagator completely requiring that one only performs integrals over Dirac delta functions. Effectively this recast the process of computing amplitudes from a calculus problem to an algebraic one of multiplying integrand constituents. One of the prime things working in these variables buys one is the level of constraint it places on the structure of the amplitudes. Given the helicities of the particles interacting and the dimension of the coupling one can uniquely construct the three particle scattering amplitude. Next, by forcing the propagator to be on-shell one can factorise higher number of particle interaction into a product of lower particle number interactions. The shift from calculus to algebra. The development of amplitudes and the acceptance in mathematical and formal particle physics made for the quick and expansive developments in the understanding of amplitudes and is now making its way into other areas of physics.

In the field of mathematical physics, amplitudes offers a wealth of structure to study. Take

[1]The mathematical representation of the particle that mediates the interaction between the two vertices.

for example the amplituhedron [6] which is an example of a geometric structure called the positive Grassmannian. This interesting mathematical structure was found in the context amplitudes and offered significant advancement of the study of Grassmannian geometry in pure mathematics. In high energy phenomenology, the natural arena of the scattering amplitude, these new variables allowed for the more efficient computation of scattering processes in experiments, e.g. the detection of the Higgs boson [7]. This allowed for a more detailed analysis of data garnered from experiment allowing physicists to gain an even deeper understanding quantum chromodynamics.

Recently the amplitudes program has made the natural leap to gravity. Since the discovery of gravitational waves the there has been a big drive to develop amplitudes able to calculate precision data for black hole mergers. In essence this is what motivated the work in this thesis since the introduction of this new technology into the realm of gravity may offer the same kind of advances in our understanding of gravity as it did for other theories. As for the application to gravitational waves, computing precision data would allow astronomers a an additional observable with which to probe large scale structure.

Since the focus of my research is on amplitudes as applied to gravity we elaborate a bit about the state of the art. As stated before we can reformulate many physical phenomena as scattering processes. Consider for example the two massive orbiting bodies. This is nothing but a very massive particle, e.g. the sun, interacting with some other massive particle, a planet, via gravitons, the force carriers of the gravitational field. Formulated in the language of scattering, a scalar exchanging a graviton with another scalar. This is no coincidence. It turns out that gravity amplitudes are intimately related to gauge theory ones by a remarkable equivalence called the KLT relations [8]. Even though these were first derived in the context of string theory, they are particularly clearly expressed in the on-shell language of amplitudes. Phrased in another way, the KLT relations allow for the computation of gravity amplitudes by simply squaring (relatively simpler) gauge theory ones. Until recently the use of amplitudes in gravity theories has been rather limited due to the fact that the amplitudes program formulated in terms of the spinor-helicity variable[2] was geared toward massless particles. In the last few years several articles such as [9] have appeared that have adapted the formalism to massive particles making gravity computations even easier. Another notion that has had a profound impact on the study of scattering amplitudes in gravity is that of the *leading singularity* [10]. This offers a method to completely localise the integrals in certain loop level amplitudes allowing us to efficiently compute *classical contributions* from loops. Due to the non-linear structure of gravity it becomes necessary to study these contributions in contexts of strong gravity such as black holes.

In this treatise we focus on studying three aspects of the amplitudes program applied to gravity. After a self contained review of the basics spinor helicity variables and scattering amplitudes in chapter 2 in chapter 3 we use amplitudes to derive a rotating black hole solution in *Einsteinian Cubic Gravity* (ECG). This is made possible since the scattering amplitudes have to satisfy the

[2]More about this shortly.

equations of motion derived from the action describing the theory, Einstein's equations. In the case of a black hole the geometry of the spacetime is just a solution to Einstein's equations. Therefore if we know the amplitudes of a test particle probing the spacetime generated by a massive particle they can be related to the geometry of the black hole. In particular this is useful in certain theories of gravity where the sheer number of terms in Einstein's equations make it impractical to solve the equations of motion analytically.

In chapter 4 we collect the results produced in an article in which we introduced the astrophysics community to the amplitudes program. We focus on the well-understood example of lightbending by a massive body and then move on to compute the deflection of a gravitational wave due to a massive body. Again this would be impractical using other methods due to the sheer amount of terms in the interaction vertex.

In chapter 5 we tackle a problem for which amplitudes is imminently suited. Discontinuities in massive supergravity. When considering interactions in massive gravity the graviton mediating the force acquires additional modes that couples to matter in a different manner. For example the massive graviton gains a scalar mode commonly called the dilaton that now couples to a scalar particle but not to a vector due to the vector having a traceless energy momentum tensor. These contributions do not necessarily vanish when the massless limit of the graviton is taken resulting in a discontinuity that violates the equivalence principle. A similar discontinuity exists in $\mathcal{N} = 1$ supergravity where the exchange particle is now the superpartner of the graviton, the gravitino. This spin-3/2 particle can be treated analogously to the graviton exchange process. The amplitude formulation of this makes it very clear where the discontinuity arises.

Chapter 2

Technical Review

Due to the inherent complexity of computing scattering amplitudes from the integral formulation [1] it came as a great relief when Feynman redefined the field in terms of diagrams [2] as a mnemonic for doing these brutal computations. Recently there has been a renaissance in the understanding and computation of scattering amplitudes in the form of spinor helicity variables [4]. This is the representation of interacting particle theories represented as spinors which in turn stems from their representation in twistor space [5]. Originally the main focus of these techniques were focussed in the areas of physics one would expect; firstly high energy particle physics since this new approach allowed the efficient computation of the background scattering found in collider experiments. Secondly computations in supersymmetry due to its enhanced structure allowing a concrete framework in which to explore the mathematical structure of scattering amplitudes. Over the course of the last decade the amplitudes community has evolved from only using it as a tool for phenomenology to studying the field for its own merits. This has lead to great advancement in the understanding of the structure of amplitudes which in turn has lead to a greater understanding of all areas of physics where it is applied.

This section serves as an introduction to the powerful machinery that we use throughout this thesis and will build draw heavily on several key works in the field [11, 9, 12, 13, 14, 15, 16, 17, 18]. We start with a general formulation of the amplitudes program, building in complexity as we demand more of the formalism.

2.1 Massless Spinor Helicity Formalism

First let us define what we mean by a spinor in this formalism. Working in 4-dimensional Minkowski space consider a four-momentum p_μ, with $p_\mu p^\mu = -m^2$. One should first observe that the Lorentz group of rotations and boosts can be mapped to the group of 2×2 matrices with complex entries and unit determinant,

$$SO(1,3) \simeq SL(2,C). \tag{2.1.1}$$

This observation allows us to decompose any Lorentz four vector, in our case the afore mentioned four-momentum, into a so-called bi-spinor, a 2×2 matrix with two indices from $SL(2, C)$

$$p_\mu \gamma^\mu = \begin{pmatrix} 0 & p_{a\dot{b}} \\ p^{\dot{a}b} & 0 \end{pmatrix}, \tag{2.1.2}$$

where we have defined the γ-matrix as the 4×4 matrix constructed from the Pauli spin matrices

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & 0 \end{pmatrix}, \tag{2.1.3}$$

with $(\sigma^\mu)_{a\dot{b}} = (1, \sigma^i)_{a\dot{b}}$ and $(\bar{\sigma}^\mu)^{\dot{a}b} = (1, -\sigma^i)^{\dot{a}b}$, and the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.1.4}$$

In effect, we have traded one Lorentz index for two independent spinor indices, where the dotted and undotted indices correspond to the row and column label of the 2×2 matrix respectively. Note that instead of (2.1.2) one can define the bi-spinors as

$$\begin{aligned} p_{a\dot{b}} &\equiv p_\mu (\sigma^\mu)_{a\dot{b}} \\ p^{\dot{a}b} &\equiv p_\mu (\bar{\sigma}^\mu)^{\dot{a}b}. \end{aligned} \tag{2.1.5}$$

These momentum bi-spinors define commuting complex 2-component column and row vectors, in this case commonly known as Weyl spinors. These are the fundamental building blocks of the spinor helicity formalism and the variables in which we will encode all kinematics of scattering processes. If in addition the 4-momentum p_μ is null i.e. $\partial_\mu p^\mu = 0$, we can write these Weyl spinors as

$$\begin{aligned} \lambda_a = \langle p|_a &= \begin{pmatrix} \sqrt{p_0 + p_3} & \frac{p_1 - ip_2}{\sqrt{p_0+p_3}} \end{pmatrix} \\ \lambda^a = |p\rangle^a &= \begin{pmatrix} -\frac{p_1 - ip_2}{\sqrt{p_0+p_3}} \\ \sqrt{p_0 + p_3} \end{pmatrix} \\ \tilde{\lambda}_{\dot{a}} = |p]_{\dot{a}} &= \begin{pmatrix} \sqrt{p_0 + p_3} \\ \frac{p_1 + ip_2}{\sqrt{p_0+p_3}} \end{pmatrix} \\ \tilde{\lambda}^{\dot{a}} = [p|^{\dot{a}} &= \begin{pmatrix} \frac{p_1 - ip_2}{\sqrt{p_0+p_3}} & -\sqrt{p_0 + p_3} \end{pmatrix}. \end{aligned} \tag{2.1.6}$$

For the rest of this section we will work only with null-vectors which, in terms of momentum, correspond to massless particles. We will only reinstitute massive particles in a later section of this chapter where we expand the formalism to incorporate massive spinors. There are now several relations to note, first is that for real momenta λ_a is the Hermitian conjugate of $\tilde{\lambda}_{\dot{a}}$ and vice versa. Next is the fact that the spinor indices are raised and lowered with the antisymmetric Levi-Civita symbol, i.e. $\lambda_b = \epsilon_{ba}\lambda^a$, taking note that we always raise or lower using the second index of the Levi-Civita symbol and our convention is

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.1.7}$$

Now going back to (2.1.5), for any null-vector we can write the bi-spinor as an outer product of the corresponding Weyl spinors

$$\det(p_{a\dot{b}}) = 0 \iff p_{a\dot{b}} = -\lambda_a\tilde{\lambda}_{\dot{b}}. \tag{2.1.8}$$

The spinors as defined in (2.1.6) are antisymmetric. Therefore contracting same spinors results in zero and since the bi-spinors are written in terms of the spinors we can contract them with the spinors. This leads to the massless Weyl equations. which are the two-component spinor equivalent of the massless Dirac equation. These are

$$p_{a\dot{b}}|p]^{\dot{b}} = 0, \qquad [p|_{\dot{a}}p^{\dot{a}b} = 0, \qquad |p\rangle^a p_{a\dot{b}} = 0, \qquad p^{\dot{a}b}\langle p|_b = 0. \tag{2.1.9}$$

Considering the rich structure of the spinors there are numerous identities and relations such as enumerated in [11]. We will restrict ourselves to only the most useful. For a start we want to suppress the indices on contracted spinors to avoid the tedium of trying to keep track of them. To this end we can write, while taking into account the antisymmetry of the spinors,

$$\langle pq\rangle = \langle p|_a|q\rangle^a = -\langle qp\rangle, \quad [pq] = [p|^{\dot{a}}|q]_{\dot{a}} = -[qp]. \tag{2.1.10}$$

Next we see that two null-vectors satisfy

$$(p+q)^2 = 2p\cdot q = \langle pq\rangle[qp], \tag{2.1.11}$$

where it is useful to note that the mapping of Lorentz vectors into bi-spinors is implicit. The next important identity to consider is the re-formulation of momentum conservation in term of spinor variables. We are only considering massless particles labelled $i = 1, ..., n$ in an all outgoing interaction. Meaning momentum conservation can be written as

$$0 = \sum_{i=1}^{n} p_i^{\mu} = \sum_{i=1}^{n} p_{i\dot{a}b} = \sum_{i=1}^{n} -|i]_{\dot{a}} \langle i|_b \,. \tag{2.1.12}$$

Here we have used the notation $|p_i\rangle \longrightarrow |i\rangle$ to simplify the complex expressions we will get for the amplitudes later. We can now dot in on the left and right with $[j|^{\dot{a}}$ and $|k\rangle^b$ where $\{j, k\} \in i$ to write momentum conservation in the very convenient spinor form

$$\sum_{i=1}^{n} \langle ki\rangle [iq] = 0. \tag{2.1.13}$$

Lastly, since the spinors live in a two dimensional space we can write a basis for the space using any two linearly independent spinors. This allows us to write an analogue of the Jacobi identity for spinors,

$$|i\rangle \langle jk\rangle + |j\rangle \langle ki\rangle + |k\rangle \langle ij\rangle = 0. \tag{2.1.14}$$

This is known as the Schouten identity and will be one of the most useful tools to simplify amplitudes.

2.2 Little Group Scaling

Now that we have the basic building blocks we can start to construct amplitudes. In the standard approach to deriving scattering amplitudes we would start with an action describing the theory from which we would then derive Feynman rules. For many theories this method gives rise to unwieldy and extremely complicated expressions for these rules. This is more often than not due to the necessary introduction of virtual particles to enforce locality. These virtual particles are in the end removed from the final answer. In the on-shell formulation this issue is circumvented by using the symmetries of the problem to constrain the possible answers that one can have as well as never introducing the notion of particles[1].

The next level of our construction is the 3-point amplitude, or in terms of particle physics lingo the three particle interaction vertex. This is constructed by using that great tool of physics, dimensional analysis. Applying this in the amplitude formulation to derive the 3-point

[1] Although we will still refer to the constituents of the scattering process as such.

amplitudes goes by the name of little group scaling. The little group is a subgroup of some group that leaves a particular state invariant. Specifically, if G is a group that acts on a space M and $m \in M$ is some fixed element. Then if $H \subset G$ is a subgroup that acts on m leaving it invariant then H is a little group of G. For example if we consider a four-momentum aligned along the z-direction then clearly rotations in the xy-plane leaves it invariant, hence rotations in the xy-plane in this case from a little group of the Lorentz group. More precisely the group we are interest in is the Poincare group in four dimensions, which acts on the space of 4-vectors p^μ. If p^μ is timelike then the little group is the $SO(3) \simeq SU(2)$ subgroup of the Poincare group. If the 4-vector is spacelike the little group is $SO(1,2)$ and if lightlike the little group is $SO(2) \simeq U(1)$. Since we can represent a null-vector as bi-spinor $p_{\dot{a}b} = -|p]_{\dot{a}} \langle p|_b$ we can clearly see that rescaling the individual spinors by some complex phase t as

$$|p\rangle \longrightarrow t\,|p\rangle\,, \qquad |p] \longrightarrow t^{-1}|p] \tag{2.2.15}$$

leaves the bi-spinor invariant. For the next piece in constructing the 3-points we introduce the notion of three particle special kinematics. To this end consider the all-outgoing interaction of three massless particles with momenta p_1, p_2 and p_3. Conservation of momentum demands that $p_1 + p_2 + p_3 = 0$ and therefore $(p_1 + p_2)^2 = p_3^2 = 0$. In spinor notation this means $\langle 12\rangle[21] = 0$, and since $\langle 12\rangle$ and $[21]$ are independent for complex momenta at least one of the factors must be zero. Suppose we choose $[21] = 0$ then by conservation of momentum so must $[13] = [23] = 0$. We see that if even one square bracket is zero then all of them must be and the 3-point can only depend on angle brackets. Similarly for the converse. But how to determine whether it should be squares or angles? To do this we need to first consider the helicity of our particles. Hence the 'helicity' part of the spinor-helicity formalism. For this we need to notice that the spinors we previously identified in (2.1.6) correspond to specific helicities, angle spinors to negative helicity and square spinors to positive helicity.

Consider a particle i with helicity h_i. The helcity is the projection of the spin of the particle along the direction of momentum. Let us for a moment allow both angle and square spinors in the 3-point amplitude containing particle i. This amplitude should then contain some powers of angle and square i's which under the little group will scale as $|i\rangle^x\,|i]^y \longrightarrow t^{x-y}\,|i\rangle^x\,|i]^y$. But by virtue of the fact that each individual spinor is a spin-1/2 object we know that this construction should also scale as t^{-2h_i}. Of course it is clear that $x - y = -2h_i$. This is commonly called the *little group weight* and the method for using this to determine an amplitude's dependence on square and angle brackets of a specific particle is called little group counting.

The last ingredients required are much simpler. The first is the mass dimension of the coupling constant of the theory in question, second noticing that angle and square brackets have mass dimension one and lastly that an n-point amplitude in four dimensions has mass dimension $4-n$. All the ingredients have now been assembled to construct a 3-point amplitude. Consider the 3-point with all outgoing particles and helicities h_1, h_2, h_3. The amplitude must now consist of some combination of either angle or square brackets, hence we have two options

$$A(1^{h_1}, 2^{h_2}, 3^{h_3}) = c\langle 12\rangle^{x_{12}} \langle 13\rangle^{x_{13}} \langle 23\rangle^{x_{23}}$$
$$A(1^{h_1}, 2^{h_2}, 3^{h_3}) = c[12]^{y_{12}} [13]^{y_{13}} [23]^{y_{23}}. \tag{2.2.16}$$

In the above amplitudes each particle scales in a unique way under the little group, constraining the exponents in terms of the helicities as

$$-2h_1 = x_{12} + x_{13}, \quad -2h_2 = x_{12} + x_{23}, \quad -2h_3 = x_{23} + x_{13}$$
$$2h_1 = y_{12} + y_{13}, \quad 2h_2 = y_{12} + y_{23}, \quad 2h_3 = y_{23} + y_{13}. \tag{2.2.17}$$

To make this example concrete let us consider the interaction of an electron emitting a photon, in the context of QED and the interaction has dimensionless coupling e. One possible helicity structure for this is the amplitude $A(1^{-1/2}, 2^{+1/2}, 3^{-1})$, giving

$$x_{12} = -1 = -y_{12}, \quad x_{13} = 2 = -y_{13}, \quad x_{23} = 0 = -y_{23}. \tag{2.2.18}$$

This means that the only viable option for the amplitude is

$$A(1^{-1/2}, 2^{+1/2}, 3^{-1}) = e\frac{\langle 13\rangle^2}{\langle 12\rangle}. \tag{2.2.19}$$

Note that any amplitude that cannot be constructed in this manner will vanish. An example of this is $A(1^{-1/2}, 2^{-1/2}, 3^{-1})$ with dimensionless coupling, which has only the mass dimensions $[A^{angle}] = 2$ and $[A^{square}] = -2$. Neither of these have the correct mass dimension and as such this is not an allowed helicity choice making the fact that the electron has to switch its helicity during photon emission manifest.

The last object of interest here is to define a polarisation vector. This is not strictly necessary when considering a purely on-shell little group construction, but in certain cases it is easier to construct amplitudes from the stripped vertex functions obtained from the action and then dot in polarisation vectors for the external particles. The polarisation vector for positive and negative helicity for particle with momentum p can be written in spinor helicity variables as

$$\varepsilon_+^\mu(p,q) = -\frac{\langle q|\,\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle}, \quad \varepsilon_-^\mu(p,q) = -\frac{\langle p|\,\gamma^\mu|q]}{\sqrt{2}[qp]}, \tag{2.2.20}$$

where q is some arbitrary reference spinor. In this formulation of the polarisation vector the arbitrariness of the reference vector captures the gauge invariance of the amplitude. Clearly the final amplitude should be independent of the choice of q as long as $q_i \neq p_i$.

2.3 BCFW Recursion Relations

Now that we have established an effective way to construct the most basic scattering amplitude we can move on to construct higher point amplitudes. These are of course the more interesting interactions where we have n-point amplitudes consisting of n particles interacting, and commonly denoted here as A_n. In the usual sense of Feynman diagrams one would have to derive the Feynman rules for the various interaction vertices from the perturbative expansion of the action. One would then connect these vertices with a virtual particle mathematically represented by the propagator also derived from the action. The virtual particle is virtual in the sense that it cannot be observed by experiment and hence has no physical interpretation. They are also off-shell meaning that $p^2 \neq 0$. The external amplitude then has to be dressed with polarisation vectors. This process is extremely computationally intensive for all but the simplest theories, but luckily in the on-shell method a simpler approach exists. This is based on the factorization properties of n-point amplitudes into products of lower-point amplitudes around simple poles. So that with a bit of complex analysis we can construct arbitrarily high tree level amplitudes with little effort. The most common of these recursion relations is the Britto, Cachazo, Feng and Witten (BCFW) recursion relations [15], on which we will focus in this section. Before we can jump into constructing these amplitudes in the spinor helicity formalism we first have to establish with the help of some complex analysis that this is indeed a consistent way to proceed.

Let us consider the basics that make up the BCFW recursion relations: complex shifts and poles. Amplitudes in general are written in terms of the external physical momenta but contain some dependence in the denominator from the propagators of the form (remember that we are still considering only massless particles)

$$\frac{1}{P^2_{abc...}} = \frac{1}{(p_a + p_b + p_c^2 + ...)^2}. \tag{2.3.21}$$

This is the transfer momentum from one vertex to another, i.e. the previously mentioned virtual particle. One can consider this as the photon that allows for the interaction between two electrons. It is necessarily off-shell since if $P^2_{abc...} = 0$ the amplitude would be divergent and the way to deal with this is to make some minor complex shift to this momentum before integrating over all transfer momentum. But we want to stick to the on-shell method of doing things, so we need to find some way of forcing the propagator momentum on-shell. We draw on the lessons of quantum field theory for this and institute a complex shift in this momentum by making complex shifts on the external momenta. And this is the quantity that will be forced on-shell allowing for a consistent way to recursively build amplitudes.

Since this propagator momentum is written in terms of the external momentum and we consider only null momenta we need to choose some complex shift that preserves this as well as conservation of momentum fo all the n external momenta p_i^μ. To this end we introduce n complex-valued vectors r_i^μ with the following properties

$$\sum_{i=1}^{n} r_i^\mu \qquad = 0 \tag{2.3.22}$$

$$r_i \cdot r_j \quad = 0, \quad \text{for all } i, j \tag{2.3.23}$$

$$p_i \cdot r_i \quad = 0, \quad \text{no sum.} \tag{2.3.24}$$

We use these vectors to define the complex shifted momenta

$$\hat{p}_i^\mu \equiv p_i^\mu + z r_i^\mu \quad \text{with} \quad z \in \mathbb{C} \tag{2.3.25}$$

Notice now that by property (2.3.22) that $\sum_i^n \hat{p}_i = 0$ meaning that the shifted momenta are conserved. By property (2.3.23) and (2.3.24) the shifted momenta are also null, which means that we can study the original amplitude in terms of the shifted momenta as a function of z. We are interested in the poles leading from this shift in momentum and how they relate to the shifted and unshifted transfer momentum. To this end consider a non-trivial subset of generic momenta $\{p_i\}_{i\in I}$ and define

$$P_I^\mu = \sum_{i\in I} p_i^\mu \quad R_I^\mu = \sum_{i\in I} r_i^\mu, \tag{2.3.26}$$

then

$$\hat{P}_I^2 = P_I^2 + z 2 P_I \cdot R_I, \tag{2.3.27}$$

showing that the sifted propagator is linear in z and we can additionally write

$$\hat{P}_I^2 = -P_I^2 \frac{z - z_I}{z_I} \quad \text{and} \quad z_I = -\frac{P_I^2}{2P_I \cdot R_I}. \tag{2.3.28}$$

Notice that from the above the poles in the amplitude arise from $1/\hat{P}_I^2$. What is more this pole is at z_I and is simple since any amplitude can only have one propagator of the subset P_I. For 4-point amplitudes this would simply be the s,t and u channel Feynman diagrams all having simple poles at different positions in the z-plane. Next we also have that for generic momenta $z_I \neq 0$, meaning all the poles are away from the origin. Using this we can now start the study of the n-point amplitude in terms of the shifted momenta $\hat{A}_n(z)$ and clearly to recover the original amplitude all one has to is evaluate the shifted amplitude at $z = 0$, meaning $\hat{A}_n(z = 0) = A_n$. First let us consider finding the residue at a specific pole z_I. Due to the structure of $\hat{P}_I^2$ the shifted amplitude has a pole of the form $(z - z_I)^{-1}$ meaning that the residue at $z = z_I$ of the shifted amplitude can be found using the standard definition of the residue of a simple pole

$$\text{Res}_{z=z_I}\hat{A}_n(z) = \lim_{z\longrightarrow z_I}(z-z_I)\hat{A}_n(z). \tag{2.3.29}$$

Given this we want to evaluate $\hat{A}_n(z=0)$ to recover the original unshifted amplitude. To make this manifest we consider the complex function $f(z) = \hat{A}_n(z)/z$ which has a simple pole at $z=0$. Finding the corresponding residue to be

$$\text{Res}_{z=0}f(z) = \lim_{z\longrightarrow =} zf(z) = \hat{A}_n(0) = A_n. \tag{2.3.30}$$

The two above equations are not particularly useful by themselves but putting them together using Cauchy's residue theorem provides the basis of the recursion relations. Cauchy's residue theorem is one of the basics of complex analysis which applies to meromorphic functions. The theorem states that the integral of a complex function along a closed curve that does not meet any poles is equal to all the residues of the poles it encloses, up to a $2\pi i$ factor. Due to how the theorem is structured the curve we choose can be arbitrarily large, for example we can choose the curve γ as the circle with radius $R \longrightarrow \infty$ ensuring we enclose all the possible poles. Then Cauchy's residue theorem implies

$$B_n = \frac{1}{2\pi i}\oint_\gamma dz\, f(z) = \text{Res}_{z=0}\frac{\hat{A}_n(z)}{z} + \sum_{z_I}\text{Res}_{z=z_I}\frac{\hat{A}_n(z)}{z}, \tag{2.3.31}$$

where we have defined the boundary term B_n as the integral over the curve.

From our previous discussion, the residue at $z=0$ yields back A_n. This permits to write the physical n-point amplitude as

$$A_n = -\sum_{z_I}\text{Res}_{z=z_I}\frac{\hat{A}_n(z)}{z} + B_n. \tag{2.3.32}$$

This may not seem particularly useful since we still don't know what $\hat{A}_n(z)$ looks like and now we also have boundary term to get rid of. Luckily the boundary term vanishes for a large number of possible complex shifts, which we will get to in a minute, but let us for now just set $B_n = 0$.

Then the n-point amplitude A_n is determined entirely by the residues of $\hat{A}_n(z)/z$ at the $z = z_I$ poles. Which is an extremely powerful statement. To understand why recall that the poles in the shifted n-point amplitude arises from forcing a complex-valued propagator linking two lower point amplitudes on-shell. Meaning that any n-point amplitude can be factorised into a product of lower point amplitudes and on-shell propagators. So for a specific factorisation channel we can write

$$\text{Res}_{z=z_I}\frac{\hat{A}_n(z)}{z} = -i\hat{A}_L(z_I)\frac{1}{P_I^2}\hat{A}_R(z_I). \tag{2.3.33}$$

Unpacking this: firstly the propagator is the on-shell propagator since it is required for the factorisation to hold. Secondly the amplitudes $\hat{A}_L(z_I)$ and $\hat{A}_R(z_I)$ are the complex shifted on-shell amplitudes where for example $\hat{A}_L(z_I)$ contains all the particles in the subset I and the transfer particle and $\hat{A}_R(z_I)$ contains the transfer particle and all the rest of the external particles $J \notin I$. We call these subamplitudes and they necessarily have fewer than n-points meaning we can build an n-point amplitude using this by recursively factorising down to 3-point amplitudes which are fixed by little group scaling.

Now that we have shown the validity of this method it is time to make this concrete in terms of spinors. It makes more sense to make the complex shifts on specific spinors, since we are writing everything in terms of spinors, rather than attempting to shift entire momenta. Even more simply than this we need only shift two specific momenta, this is the BCFW shift, rather than all of the external momenta as was done above. So let us pick two particles i and j in our n-point amplitude and shift the spinors in what is called an $[i|\,|j\rangle$-shift

$$|\hat{i}] = |i] + z|j], \quad |\hat{j}] = |j], \quad |\hat{j}\rangle = |j\rangle - z\,|i\rangle\,, \quad |\hat{i}\rangle = |i\rangle\,. \tag{2.3.34}$$

No other spinors are shifted and this satisfies the original restrictions on the shift momenta. Notice as well that $\langle\hat{j}k\rangle$ and $[\hat{i}k]$ are linear in z with all other spinor contractions remaining unshifted. The last bit we need is the ability to show that the shift we have chosen is valid. What we mean by this is that the boundary term in Cauchy's residue theorem actually vanish. Form the definition of the boundary term

$$B_n = \frac{1}{2\pi i}\oint_\gamma dz\, f(z) \tag{2.3.35}$$

and considering that the contour goes to infinity if we require $B_n = 0$ this is equivalent to $\lim_{|z|\longrightarrow\infty} zf(z) = 0$ or in terms of the shifted amplitude

$$\lim_{|z|\longrightarrow\infty}\hat{A}_n(z) = 0. \tag{2.3.36}$$

So all we have to do is choose a shift and check that the z dependence in the shifted amplitude obeys $\hat{A}_n(z) \propto z^{-a}$ where a is some real positive number. The simplest way to do this is to evaluate the subamplitudes and propagators individually. One choice that makes a considerable difference is choosing that the shifted particles are in separate subamplitudes, i.e. on opposite sides of the propagator, this provides an additional power of $1/z$. One of the best results of this recursion relation is the fact that once a valid shift has been chosen only the diagrams in which that shift is valid need to be evaluated in order to compute the entire n-point amplitude. For example if we wanted to compute a 4-point amplitude and the shift with valid boundary behaviour is chosen in such a way that only the s-channel factorisation is valid. Then only the s-channel needs to be constructed since it captures all the relevant information and the other channels can be ignored. We will see this in action later when computing the gravitational wave bending.

Finally we have the ability to construct arbitrarily high n-point three level amplitudes by starting with nothing more than some group theory arguments, some complex analysis and physical constraints. This allows us to construct these amplitudes by only knowing the particle momenta, helicities and coupling, but this has two serious deficiencies. Firstly this still only deals with massless particles, ruling out the study of any phenomena containing massive particles at all but the highest energies. There are several ways to apply BCFW to massive particles, see for example [19] but they are tedious and come with many restrictions. In this work we will focus instead on a recent development [9] that introduced a consistent way to handle massive spinors. This we will introduce in detail in the next section. Secondly we have only been able to construct tree level amplitudes, meaning that we are in essence restricted to studying classical effects since we are as of yet unable to compute amplitudes that contain loops. In this work however the focus is on gravity which is currently non-renormalisable and as such lacks a consistent quantum description. But there are some interesting effects that we studied that are attributed to classical loops, hence we have to deal with them. Luckily there are several recent [13] and many other older [16] ways of dealing with loop amplitudes especially in the on-shell regime. We will introduce these methods in a subsequent section in this chapter.

2.4 Massive Spinors

In recent years there have been several works [9, 14] that have adapted the spinor helicity formalism to massive particles. This has come as a great advance allowing the on-shell regime to gain even greater footing in the physics community. Even before this there were two ways to handle massive particles. The one is only viable in 4-dimensions, we accomplish the extension to massive particles by working in 6-dimensions. By letting the six momentum be null we are able to use the power of spinor helicity variables but when integrating out the additional two dimensions the 4-momentum is not necessarily zero, but equal to the mass of the particle. This is a great way to approach the problem but we then face the problem of compactifying the additional dimensions. The other is what served as the basis for the recent advances. We represent massive momenta by projecting it out onto the light-cone and treating each of the projected momenta as particles. Again this is a useful treatment but in most cases it is immensely difficult to write the amplitude in terms of its original momenta and we have the additional problem of having two times the number of particles to deal with. Next we go through the new enlightened method.

Consider the massive momenta $\mathbf{p}^{\mu}$ where $\mathbf{p}^2 = -m^2$. As is standard in this formalism we use bold notation to indicate massive spinors, bi-spinors and momenta. This notation requires that we establish certain conventions for the contraction of massive spinors to account for relative minus signs, we will do this as it becomes necessary. We can then do the same as the second method stated above and decompose the momentum into a sum of two null vectors

$$\mathbf{p}^{\mu} = p^{\mu} + \eta_p^{\mu}, \tag{2.4.37}$$

where $p^{\mu} = \lim_{m\longrightarrow 0} \mathbf{p}^{\mu}$ with $\eta_p^2 = 0 = p^2$. Keeping in tune with the spinors structure we can promote these to bi-spinors and then spinors, meaning we can write

$$\mathbf{p}^{a\dot{b}} = p^{a\dot{b}} + \eta_p^{a\dot{b}}. \tag{2.4.38}$$

Now thinking back to when we wrote the massless bi-spinors as an outer product of two spinors. We could do this since $det(p^{a\dot{b}}) = 0$ meaning it is a rank one matrix and as such is the outer product of two vectors of the same dimension. The massive bi-spinor can be treated in a similar way, since $det(\mathbf{p}^{a\dot{b}}) = -m^2$, it is a rank two matrix and can then be written as an outer product of two rank one matrices. To encompass this additional degree of freedom we introduce another $SU(2)$ index on the spinor and denote it using uppercase Latin letters and we call them massive indices to distinguish from the spinor indices. These are raised and lowered using the antisymmetric Levi-Civita symbol with the same convention as the spinor indices. During computation these are suppressed similarly to the spinor indices. This means we can rewrite (2.4.38) as

$$\mathbf{p}^{a\dot{b}} = |\mathbf{p}\rangle^{aI}\, {}_I[\mathbf{p}|^{\dot{b}} = -\,|p\rangle^{a}\,[p|^{\dot{b}} - |\eta_p\rangle^{a}\,[\eta_p|^{\dot{b}}. \tag{2.4.39}$$

This is the first point where care should be taken to with established convention of the massive spinors. Since we raise and lower massive indices with the Levi-Civita symbol if we were to swap the up and down indices on the angle and square spinors in (2.4.39) we would pick up a relative minus sign

$$\mathbf{p}^{a\dot{b}} = |\mathbf{p}\rangle^{aI}\, {}_I[\mathbf{p}|^{\dot{b}} = -\,|\mathbf{p}\rangle^{a}_{I}\, {}^{I}[\mathbf{p}|^{\dot{b}} = -\mathbf{p}^{a\dot{b}}. \tag{2.4.40}$$

One should obviously just keep track of this to ensure the correct sign in computations. From (2.4.39) we can immediately see that since $\det(\mathbf{p}^{a\dot{b}}) = -m^2$ then $\det(|\mathbf{p}\rangle^{aI}) = M$ and $\det({}_I[\mathbf{p}|^{\dot{b}}) = -\tilde{M}$ where $M\tilde{M} = m^2$. It is also convenient to expand the massive spinors in two dimensional basis vectors ζ^{-I} and ζ^{+I}

$$\begin{aligned} |\mathbf{p}\rangle^{aI} &= \zeta^{-I}\,|p\rangle^{a} + \zeta^{+I}\,|\eta_p\rangle^{a} \\ [\mathbf{p}|^{\dot{b}}_{I} &= \zeta^{-}_{I}\,[p|^{\dot{b}} - \zeta^{+}_{I}\,[\eta_p|^{\dot{b}}, \end{aligned} \tag{2.4.41}$$

where

$$\zeta^{-I} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{I} \quad \zeta^{+I} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{I}. \tag{2.4.42}$$

For our purposes it makes sense to define $M = m = \tilde{M}$ and from the structure of (2.4.39),(2.4.41) and their properties we have that

$$\langle p\eta_p \rangle = [p\eta_p] = m = -\frac{1}{2}{}^{I}\langle \mathbf{pp}\rangle_I = -\frac{1}{2}{}^{I}[\mathbf{pp}]_I. \tag{2.4.43}$$

Before we proceed to constructing amplitudes with these newly fashioned spinors it is necessary to take an in depth look at some of the properties that this construction offers. First is that these spinors hold in the massive Weyl equation

$$\mathbf{p}^{a\dot{b}}|\mathbf{p}]_{\dot{b}I} = -m\,|\mathbf{p}\rangle^{a}_{I} \quad , \quad \mathbf{p}^{a\dot{b}}\,\langle\mathbf{p}|^{I}_{a} = m[\mathbf{p}|^{\dot{b}I}, \tag{2.4.44}$$

the greatest benefit of this is that this equation allows us to switch between square and angle bracket and more specifically $|p\rangle \leftrightarrow |\eta_p]$, $|p] \leftrightarrow |\eta_p\rangle$ for the correct choices of the massive index. This also sheds light on one of the stranger aspects of this formalism. In the massless formalism we very clearly had that angle spinors represented negative helicity states and square spinors positive helicity states. In the massive formalism this is not so clear since the spinor represents a massive spin-1/2 state it is a superposition of both the negative and positive states, as is consistent with particle theory. In the up index position once the choice for the massive spinor is made it isolates the spin state, $I = 1$ for negative spin-1/2 state and $I = 2$ for the positive spin-1/2 sate, and oppositely in the down position. Concretely

$$\begin{aligned} |\mathbf{p}\rangle^{1} = |p\rangle \quad &, \quad |\mathbf{p}]^{1} = -|\eta_p] \\ |\mathbf{p}\rangle^{2} = |\eta_p\rangle \quad &, \quad |\mathbf{p}]^{2} = |p]. \end{aligned} \tag{2.4.45}$$

Given the above equation it is natural to ask what about the η spinors, are they not also representative of some helicty state? Technically yes. But this only enters when taking the massless limits and as a result having to pick specific helicity states. This allows us a very simple way in which to take the massless limits of amplitudes. To illustrate this consider that the basis in which we expanded the massive spinors forms natural eigenstates of spin-1/2 states in the direction of the spatial momentum. This means that we can effectively write the spinors in terms of the energy and momentum of the particle in the following way

$$|p\rangle \longrightarrow \sqrt{E+p}\,\zeta^{-} \quad , \quad |\eta_p\rangle \longrightarrow \sqrt{E-p}\,\zeta^{+}. \tag{2.4.46}$$

We restrict ourselves to angle spinors to illustrate these properties but similar arguments hold for the square spinors as well. Taking the high energy limit in (2.4.46) we get $\sqrt{E+p} \longrightarrow \sqrt{2E}$

and $\sqrt{E-p} \longrightarrow \frac{m}{\sqrt{2E}}$. Hence we see that the η projection spinors are proportional to the mass of the particle. Therefore when taking the massless limit of an amplitude any term that contains an η spinor will go to zero leaving only the terms that contain the massless projection of the massive particles. This also means that when constructing 3-point amplitudes it will generally contain both angle and square spinors as opposed to the massless formalism.

Lastly before we construct the amplitudes we collect here some useful identities. Note also that the Schouten identity holds for the massive spinors as long as the massive indices are retained by the initial particle.

$$
\begin{gathered}
\mathbf{p}^{a\dot{b}} = |\mathbf{p}\rangle^{aI}{}_I[\mathbf{p}|^{\dot{b}} = -\,|p\rangle^a\,[p|^{\dot{b}} - |\eta_p\rangle^a\,[\eta_p|^{\dot{b}}\,, \quad \mathbf{p}^2 = \det(\mathbf{p}^{a\dot{b}}) = -m_p^2\,, \quad \langle p\eta_p\rangle = [p\eta_p] = m_p \\
\langle\mathbf{p}|^I_a\,\mathbf{p}^{a\dot{b}} = m[\mathbf{p}|^{\dot{b}I}\,, \quad [\mathbf{p}|^{\dot{a}I}\mathbf{p}_{\dot{a}b} = -m\,\langle\mathbf{p}|^I_b\,, \quad \langle i|\mathbf{p}\mathbf{p}|j\rangle = -m^2\langle ij\rangle\,, \quad {}^I\langle\mathbf{p}\mathbf{p}\rangle^J = m\epsilon^{IJ} \\
{}_I[\mathbf{p}\mathbf{p}]_J = -m\epsilon_{IJ}\,, \quad \langle i\mathbf{p}\rangle^I{}_I\langle\mathbf{p}j\rangle = m\langle ij\rangle\,, \quad [i\mathbf{p}]^I{}_I[\mathbf{p}j] = m[ij]\,, \quad |\mathbf{p}\rangle^I{}_J[\mathbf{p}| = -\,|\mathbf{p}\rangle_I{}^J[\mathbf{p}|.
\end{gathered}
\tag{2.4.47}
$$

There are four distinct types of 3-points that we would like to construct, specified by the classes of mass of the particles

1. One massive, two massless.
2. Two massive, different mass.
3. Two massive, same mass.
4. Three massive.

2.4.1 One massive

Let us start with the first and simplest configuration. Suppose that we have one particle of mass m_1 and spin s_1 and two massless particles with respective helicities h_2 and h_3, hence the amplitude $M_3^{I_1 I_2 \cdots I_{2s_1}}(\mathbf{1}^{s_1}, 2^{h_2}, 3^{h_3})$. Since we have two massless particles which can easily be decomposed into spinors they can be used to form a basis for the amplitude in question, $(v^a, u^a) = (|2\rangle^a, |3\rangle^a)$. Due to little group scaling we can uniquely fix the order of each of the spinors. Specifically there needs to be $2h_i$ for each of the massless spinors and $2s_1$ for the massive spinor and therefore $2s_1$ free massive indices. These indices are kept free since different choices represent different configurations of the spin super-position. One configuration for the 3-point is then according to [9]

$$
\begin{aligned}
&M_3^{\{I_1 I_2 \ldots I_{2s_1}\}}(\mathbf{1}^{s_1}, 2^{h_2}, 3^{h_3}) \\
&= \tilde{g}(|2\rangle^{s_1-h_2+h_3}\,|3\rangle^{s_1-h_3+h_2})^{\{a_1 a_2 \ldots a_{2s_1}\}}[23]^{s_1+h_2+h_3 I_1}\,\langle\mathbf{1}|_{a_1}{}^{I_2}\,\langle\mathbf{1}|_{a_2}\ldots{}^{I_{2s_1}}\,\langle\mathbf{1}|_{a_{2s_1}}\,.
\end{aligned}
\tag{2.4.48}
$$

This expression requires some in unpacking. Let us start with the term in parentheses

$(|2\rangle^{s_1-h_2+h_3}\,|3\rangle^{s_1-h_3+h_2})^{\{a_1,a_2,\ldots,a_{2s_1}\}}$. The superscripts on the spinors are exponents and the spinor indices are distributed in all possible distinct ways onto the spinors. Note also that the braces on the spinor indices indicate that we symmetrise over all of them. Once these get contracted with the massive spinors the symmetrisation gets transferred to the massive spinor indices hence the braces on the massive indices on the left hand side of the expression. Next note that the powers of the various spinors ensures that the all have the correct little group weight. The last bit that comes in is the dimensionfull scaling $\tilde{g}$. Since one of the particles is massive we can have the mass showing up in g along with the coupling of the given theory g. From dimensional analysis and the fact that $[M_3] = 1$ we can establish the form of $\tilde{g} = gm^{-(3s+h_2+h_3+[g]-1)}$. Note also that due to conservation of momentum we can exchange $[23] = m^2/\langle 23\rangle$. As an example let us compute the amplitude of a massive graviton decaying into two photons. In the construction below we also suppress the massive indices, we can do this because of the symmetrisation of the indices allows us to always reinstate them in a unique manner. We should also note the required number of free indices, if there are more massive spinors than the number of required indices any remaining should obviously be contracted with some other massive spinor in the term. The amplitude is

$$\begin{aligned} M_3^{\{I_1I_2I_3I_4\}}(\mathbf{1}^2, 2^{-1}, 3^{+1}) &= \frac{\kappa}{m^4}{}^{\{I_1}\langle \mathbf{12}\rangle^{I_2}\langle \mathbf{12}\rangle^{I_3}\langle \mathbf{12}\rangle^{I_4\}}\langle \mathbf{12}\rangle[23]^2 \\ &= \kappa\frac{\langle \mathbf{12}\rangle^4}{\langle 23\rangle^2}, \end{aligned} \tag{2.4.49}$$

which is a wonderfully simple and compact form.

2.4.2 Two massive, distinct masses

Next we consider the case of having two different massive particles and following the same conventions as above, results in the amplitude $M_3^{\{I_1I_2\ldots I_{2s_1}\}\{J_1J_2\ldots J_{2s_2}\}}(\mathbf{1}^{s_1}, \mathbf{2}^{s_2}, 3^{h_3})$. Since we only have one massless spinor we need to construct a slightly more complicated basis, $(v^a, u^a) = (|3\rangle^a, \mathbf{p}_1^{a\dot{b}}|3]_{\dot{b}}/m_1)$. We can use either of the massive momenta to construct the basis, we should just normalise using the appropriate mass. The 3-point amplitude is

$$\begin{aligned} &M_3^{\{I_1I_2\ldots I_{2s_1}\}\{J_1J_2\ldots J_{2s_2}\}}(\mathbf{1}^{s_1}, \mathbf{2}^{s_2}, 3^{h_3}) \\ &= \sum_{i=1} \tilde{g}_i (v^{s_1+s_2-h_3} u^{s_1+s_2+h_3})^{\{a_1a_2\ldots a_{2s_1}\}\{b_1b_2\ldots b_{s_22}\}\{I_1} \langle \mathbf{1}|_{a_1}{}^{I_2}\langle \mathbf{1}|_{a_2}\ldots{}^{I_{2s_1}\}}\langle \mathbf{1}|_{a_{2s_1}}{}^{\{J_1}\langle \mathbf{2}|_{b_1}{}^{J_2}\langle \mathbf{2}|_{b_2}\ldots{}^{J_{2s_2}\}}\langle \mathbf{2}|_{b_{2s_2}}. \end{aligned} \tag{2.4.50}$$

Where the sum is over all distinct ways in which the spinor indices can be distributed onto the basis spinors. The example we explore below to make this construction manifest will be relevant in later sections. Consider a vector multiplet interacting with a massive gravitino with κ coupling, i.e. a massless spin-1 particle (photon), a massive spin-1/2 particle (fermion of mass m_1) and a massive spin-3/2 particle (gravitino of mass m_2), or $M_3^{I\{J_1J_2J_3\}}(\mathbf{1}^{1/2}, \mathbf{2}^{3/2}, 3^{-1})$. There are two ways the four spinor indices, one for the fermion three for the gravitino, can be distributed onto the basis $v^a v^{b1} v^{b2} u^{b3}$ and $v^{b1} v^a v^{b2} u^{b3}$. There is no point in symmetrising the gravitino indices since we can just transfer that to the massive indices. The amplitude is

$$
\begin{aligned}
&M_3^{I\{J_1J_2J_3\}}(\mathbf{1}^{1/2}, \mathbf{2}^{3/2}, 3^{-1}) \\
&= \tilde{g}_1{}^I \langle \mathbf{13}\rangle^{\{J_1} \langle \mathbf{21}\rangle [\mathbf{1}3]^{J_2} \langle \mathbf{21}\rangle [\mathbf{1}3]^{J_3\}} \langle \mathbf{21}\rangle [\mathbf{1}3]/m_1^3 + \tilde{g}_2{}^I \langle \mathbf{11}\rangle [\mathbf{1}3]^{\{J_1} \langle \mathbf{2}3\rangle^{J_2} \langle \mathbf{21}\rangle [\mathbf{1}3]^{J_3\}} \langle \mathbf{21}\rangle [\mathbf{1}3]/m_1^3 \\
&= -\frac{\tilde{g}_1 m_2}{m_1^3}{}^I \langle \mathbf{1}3\rangle^{\{J_1} [\mathbf{2}3]^{J_2} \langle \mathbf{21}\rangle [\mathbf{1}3]^{J_3\}} \langle \mathbf{21}\rangle [\mathbf{1}3] + \frac{\tilde{g}_2}{m_1^2}{}^I [\mathbf{1}3]^{\{J_1} \langle \mathbf{2}3\rangle^{J_2} \langle \mathbf{21}\rangle [\mathbf{1}3]^{J_3\}} \langle \mathbf{21}\rangle [\mathbf{1}3] \\
&= \tilde{g}_1 m_g \langle \mathbf{1}3\rangle [\mathbf{2}3] \langle \mathbf{2}3\rangle^2 + \tilde{g}_2 m_f [\mathbf{1}3] \langle \mathbf{2}3\rangle^3 .
\end{aligned}
\tag{2.4.51}
$$

Again we see that the 3-point amplitude consists of both angle and square spinors while also ensuring the correct little group weight of each spinor. The complicated thing now is deciding how to determine the mass dependence structure of the couplings, since there are now two masses which can be used. There is some ambiguity in this and we will expand on this in a following section.

2.4.3 Two massive, same mass

In the case of the two massive particles having the same mass the two previously chosen basis vectors are not linearly independent i.e. $\frac{\langle 3|\mathbf{p}_1|3]}{m} = 0$ and as such cannot be used for a basis. There as a constant of proportionality between them that can be used to gain more information

$$
x\,|3\rangle^a = \frac{\mathbf{p}_1^{a\dot{b}}|3]_{\dot{b}}}{m} \quad , \quad \frac{[3|^{\dot{b}}}{x} = \frac{\langle 3|_a\, \mathbf{p}_1^{a\dot{b}}}{m}. \tag{2.4.52}
$$

We can then dot in with an arbitrary reference spinor ξ in such a way that x is independent of ξ, dimensionless and carries little group weight $+1$ of the massless spinor, yielding

$$
x_{13} = \frac{\langle \xi|\,\mathbf{p}_1|3]}{m\langle \xi 3\rangle} \quad \text{or} \quad x_{13} = \frac{m[\xi 3]}{\langle 3|\,\mathbf{p}_1|\xi]}. \tag{2.4.53}
$$

If this factor appears in the 3-point it then has an additional pole in ξ but when constructing the 4-points one can choose ξ to be a spinor of one of the external legs on the other 3-point which produces a pole in a different factorisation channel. This gives a sufficient constraint for fixing the amplitude and using the anti-symmetric tensor along with $|3\rangle$ it is

$$M_3^{\{I_1 I_2 \ldots I_{2s_1}\}\{J_1 J_2 \ldots J_{2s_2}\}}(\mathbf{1}^{s_1}, \mathbf{2}^{s_2}, 3^{h_3})$$
$$= \sum_{i=|s_1-s_2|}^{s_1+s_2} \tilde{g}_i x_{13}^{i+h} (|3\rangle^{2i} \, \epsilon^{s_1+s_2-i})^{\{a_1 a_2 \ldots a_{2s_1}\}\{b_1 b_2 \ldots b_{s2_2}\} I_1} \langle \mathbf{1}|_{a_1} {}^{I_2} \langle \mathbf{1}|_{a_2} \ldots {}^{I_{2s_1}} \langle \mathbf{1}|_{a_{2s_1}} {}^{J_1} \langle \mathbf{2}|_{b_1} {}^{J_2} \langle \mathbf{2}|_{b_2} \ldots {}^{J_{2s_2}} \langle \mathbf{2}|_{b_{2s_2}} . \tag{2.4.54}$$

For an example consider the same configuration as in the previous section but now with the fermion and gravitino having the same mass. The amplitude for this is

$$M_3^{I\{J_1 J_2 J_3\}}(\mathbf{1}^{1/2}, \mathbf{2}^{3/2}, 3^{-1})$$
$$= \tilde{g}_1({}^I\langle \mathbf{13}\rangle^{\{J_1}\langle \mathbf{23}\rangle^{J_2}\langle \mathbf{22}\rangle^{J_3\}} + {}^I\langle \mathbf{12}\rangle^{\{J_1 J_2}\langle \mathbf{23}\rangle^{J_3\}}\langle \mathbf{23}\rangle) + \tilde{g}_2 x_{13} {}^I\langle \mathbf{13}\rangle^{\{J_1}\langle \mathbf{23}\rangle^{J_2}\langle \mathbf{23}\rangle^{J_3\}}\langle \mathbf{23}\rangle$$
$$= \tilde{g}_1(\langle \mathbf{13}\rangle\langle \mathbf{23}\rangle\langle \mathbf{22}\rangle + \langle \mathbf{12}\rangle\langle \mathbf{23}\rangle^2) + \tilde{g}_2 x_{13}\langle \mathbf{13}\rangle\langle \mathbf{23}\rangle^3. \tag{2.4.55}$$

Fixing the mass dependence in $\tilde{g}_i$ in this case is quite simple since the masses are the same. At this point it is necessary to mention that the minimal coupling piece of this amplitude (as well as the three massive amplitude in the next section) is contained in the terms that has the highest order of epsilons in the construction. When considering certain processes this is the only term that need be considered.

2.4.4 Three massive

In the three massive particle case we no longer have any massless spinors to construct a basis. Instead we have to use tensors for this we use ($\mathcal{O}^{ab} = \mathbf{p}_1^{\dot{b}\{a} \mathbf{p}_{2\dot{b}}^{b\}}, \epsilon^{ab}$). The 3-point amplitude is then given by

$$M_3^{\{I_1 I_2 \ldots I_{2s_1}\}\{J_1 J_2 \ldots J_{2s_2}\}\{K_1 K_2 \ldots K_{2s_3}\}}(\mathbf{1}^{s_1}, \mathbf{2}^{s_2}, \mathbf{3}^{s_3})$$
$$= \sum_{i=0}^{s_1+s_2+s_3} \tilde{g}_i((\mathcal{O}^i \epsilon^{s_1+s_2+s_3}))^{\{a_1 a_2 \ldots a_{2s_1}\}\{b_1 b_2 \ldots b_{s2_2}\}\{c_1 c_2 \ldots c_{s2_3}\}} \times \tag{2.4.56}$$
$$\times {}^{I_1}\langle \mathbf{1}|_{a_1} {}^{I_2}\langle \mathbf{1}|_{a_2} \ldots {}^{I_{2s_1}}\langle \mathbf{1}|_{a_{2s_1}} {}^{J_1}\langle \mathbf{2}|_{b_1} {}^{J_2}\langle \mathbf{2}|_{b_2} \ldots {}^{J_{2s_2}}\langle \mathbf{2}|_{b_{2s_2}} {}^{K_1}\langle \mathbf{3}|_{c_1} {}^{K_2}\langle \mathbf{3}|_{c_2} \ldots {}^{K_{2s_3}}\langle \mathbf{3}|_{c_{2s_3}} .$$

As an example we construct another 3-point that will be useful later, a massive scalar multiplet consisting of a massive scalar and a massive fermion interacting with a massive gravitino

$$
\begin{aligned}
&M^{I\{J_1J_2J_3\}}(\mathbf{1}^{1/2},\mathbf{2}^{3/2},\mathbf{3}^0)\\
&=\tilde{g}_0{}^I\langle\mathbf{12}\rangle^{\{J_1J_2}\langle\mathbf{22}\rangle^{J_3\}}+\tilde{g}_1({}^I\langle\mathbf{12}\rangle^{\{J_1J_2}\langle\mathbf{21}\rangle[\mathbf{12}]\langle\mathbf{22}\rangle^{J_3\}}+{}^{\{J_1}\langle\mathbf{22}\rangle^{J_2J_3\}}\langle\mathbf{21}\rangle[\mathbf{12}]\langle\mathbf{21}\rangle^I\\
&+{}^{\{J_1}\langle\mathbf{22}\rangle^{J_2J_3\}}\langle\mathbf{22}\rangle[\mathbf{21}]\langle\mathbf{11}\rangle^I)+\tilde{g}_2({}^I\langle\mathbf{11}\rangle[\mathbf{12}]\langle\mathbf{22}\rangle^{\{J_1J_2\}}\langle\mathbf{21}\rangle[\mathbf{12}]\langle\mathbf{22}\rangle^{J_3\}}\\
&+{}^I\langle\mathbf{12}\rangle[\mathbf{21}]\langle\mathbf{12}\rangle^{\{J_1J_2\}}\langle\mathbf{21}\rangle[\mathbf{12}]\langle\mathbf{22}\rangle^{J_3\}}).
\end{aligned}
\tag{2.4.57}
$$

2.4.5 Coupling mass dependence and Massless Limits

As mentioned in previous cases there are certain ambiguities when attempting to constrain the couplings found in these massive amplitudes. Take for example a generic dimensionful coupling g_0 with mass dimension -2. So let us assume a coupling of κ which has dimension -1 we are still left with a mass in the denominator. The form of this factor is at this stage undetermined, it could take on any polynomial of the masses in the problem over any other polynomial of one order in mass higher to work, for example

$$
\frac{a_1m_1^n+a_2m_2^n+a_3m_1m_2^{n-1}+a_4m_2m_1^{n-1}+...}{b_1m_1^{n+1}+b_2m_2^{n+1}+b_3m_1m_2^n+b_4m_2m_1^n+...}
$$

where n need not even be an integer. The only consistent way to constrain the form of these couplings is to take the massless limit of these massive amplitudes and compare them to lower mass order amplitudes constructed independently. There is also the option of using additional off-shell information to constrain the couplings or perhaps experimental results. But this begs away from the notion that the amplitude formalism is completely deterministic in and of itself. We will explore the effects of these various ways to constrain the couplings in a later chapter.

2.4.6 Massive n-Point Amplitudes

The next logical step, having established concrete rules for determining the 3-point amplitudes of massive particles, is to be able to compute higher point amplitudes. The first way is to refer back to the section on BCFW and see that we can do complex shifts on certain spinors and as such use complex analysis to determine the higher point amplitudes. The drawback of this method is that BCFW only works with massless spinors. This means that we would first have to decompose the massive spinors into massless projections and then perform the shifts. This not only doubles the number of spinors in an all massive amplitude but we lose all the convenience of the new massive notation established in the previous sections.

The current method is to simply draw on quantum field theory and on-shell methods and discounting the simplification that BCFW offers for the process. To illustrate the method let us consider an all massless 4-point interaction with arbitrary particle content $M_4(1^{h_1},2^{h_2},3^{h_3},4^{h_4})$. If we want to construct the 4-point amplitude sans BCFW we need to consider all the possible factorisation channels. This is simply the s,t and u channels. So we can write the 4-point as

$$
\begin{aligned}
M_4(1^{h_1},2^{h_2},3^{h_3},4^{h_4}) = &\frac{M_3(1^{h_1},2^{h_2},P_{12}^{h_p})\tilde{M}_3(3^{h_3},4^{h_4},-P_{12}^{h_p})}{P_{12}^2} \\
&+\frac{M_3(1^{h_1},3^{h_3},P_{13}^{h_p})\tilde{M}_3(2^{h_2},4^{h_4},-P_{13}^{h_p})}{P_{13}^2} \\
&+\frac{M_3(1^{h_1},4^{h_4},P_{14}^{h_p})\tilde{M}_3(3^{h_3},2^{h_2},-P_{14}^{h_p})}{P_{14}^2}.
\end{aligned}
\tag{2.4.58}
$$

Where we define $P_{ij} \equiv p_i + p_j$ and $s = P_{12}^2$, $t = P_{13}^2$ and $u = P_{14}^2$. So at the level of the 4-point there is at most three distinct 4-point amplitudes to compute. The problem comes in at the higher points, consider the 5-point for which there are thirty distinct amplitudes. This experiences quite rapid growth as we increase the number of points in the interaction. Fortunately there is a lot of interesting processes at the 4-point level. So let us move on to massive particles. Let us consider a scattering process in which we have a massless particle p_1 of helicity h_1 and a massive particle p_2 with spin s_2 interacting via a another massive particle of some particular spin-s and mass m_p. The 4-point is then simply

$$
M_4^{\{I_1I_2\cdots I_{2s_2}\}\{J_1J_2\cdots J_{2s_4}\}}(1^{h_1},\mathbf{2}^{s_2},3^{h_3},\mathbf{4}^{s_4}).
\tag{2.4.59}
$$

Note that by convention we keep the external particles' massive indices in the up position. Let us consider a factorisation channel in which the 3-point amplitudes are

$$
\begin{aligned}
&M_3^{\{I_1I_2\cdots I_{2s_2}\}\{K_1K_2\cdots K_{2s}\}}(1^{h_1},\mathbf{2}^{s_2},\mathbf{P}^s) \\
&\tilde{M}_3^{\{I_1I_2\cdots I_{2s_2}\}\{L_1L_2\cdots L_{2s}\}}(3^{h_3},\mathbf{4}^{s_4},-\mathbf{P}^s).
\end{aligned}
\tag{2.4.60}
$$

Now to replicate the procedure we just did in the massless example we need to contract the transfer momentum in the two 3-points. and divide by the appropriate propagator, which in this channel is $\mathbf{P}_{12}^2 + m_p^2$. The 4-point from this channel is

$$
\begin{aligned}
M^{\{I_1I_2\cdots I_{2s_2}\}\{K_1K_2\cdots K_{2s}\}}&(1^{h_1},\mathbf{2}^{s_2},\mathbf{P}^s)\frac{\epsilon_{K_1L_1}\epsilon_{K_2L_2}\cdots\epsilon_{K_{2s}L_{2s}}}{\mathbf{P}_{12}^2+m_p^2}\tilde{M}^{\{J_1J_2\cdots J_{2s_4}\}\{L_1L_2\cdots L_{2s}\}}(3^{h_3},\mathbf{4}^{s_4},-\mathbf{P}^s) \\
&= M^{\{I_1I_2\cdots I_{2s_2}\}\{K_1K_2\cdots K_{2s}\}}(1^{h_1},\mathbf{2}^{s_2},\mathbf{P}^s)\frac{1}{\mathbf{P}_{12}^2+m_p^2}\tilde{M}^{\{J_1J_2\cdots J_{2s_4}\}}_{\{K_1K_2\cdots K_{2s}\}}(3^{h_3},\mathbf{4}^{s_4},-\mathbf{P}^s).
\end{aligned}
\tag{2.4.61}
$$

Holding to the definitions in the massless example this is clearly the s-channel. Hence we still need to add the possible t and u channels, but we can see that if this is the only possible interaction there is no t-channel since there is no 3-point that contains the two massless particles as well as the propagator particle. Hence we only need the s and u-channels. The thing that makes this a particularly frustrating piece of computation is the symmeterisation over the massive indices of the propagator in each of the 3-points. This means that there are $(2s)!$ terms in each of the three points $(2s)^2$ terms in the s-channel 4-point. Since the indices are summed over there are actually only $2s$ terms in the s-channel 4-point and possible symmetries in the problem can reduce this further. But this is only at the level of the 4-point where we have only three factorisation channels. Once we go to 5-points there are thirty channels which starts to make the construction of these higher point amplitudes untenable without the use automation.

2.5 Loop Amplitudes

The next aspect of interest is the ability to construct loop amplitudes. At the tree level to which we have restricted ourselves till now we can only extract classical information from the amplitudes. To get any information on the quantum properties of a scattering system we need to include loop processes see for example [20, 21]. In gravity this is a bit strange since no concrete theory of quantum gravity exists, any quantum information we extract about gravity is fundamentally unreliable. So where does this leave us?

What we need to understand about the perturbative expansion in order of loops is that it corresponds to a perturbative expansion in $\hbar$ in the action, hence

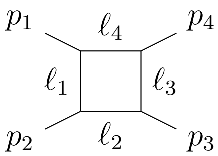
$$M^{tree} + M^{1-loop} + M^{2-loop} + \cdots \propto \hbar^0 + \hbar^1 + \hbar^2 + \cdots \,. \tag{2.5.62}$$

And since we generally measure "quantumness" by the order of $\hbar$ we would naively exclude loops in classical calculations. But as has been established in various other works [22, 23] there are certain terms in, for example, the 1-loop contribution that is proportional to $\frac{1}{\hbar}$ thus actually making a classical contribution. We therefore need to apply the loop machinery to the gravity problem, fortunately it is a general construction and is directly applicable. There are also other scenarios where the lowest order classical contribution comes from the 1-loop contribution [13, 24, 25].

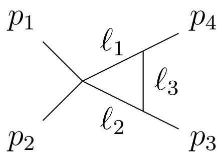
2.5.1 Generalised Unitarity

Modern amplitude methods are particularly well suited to problems involving multiple interactions and loops. To this end, we need to introduce the first method of calculating loops, the idea of *generalised unitarity*. For a more detailed introduction of the method see [11] and [12].

The basis of generalised unitarity comes down to the well established fact that in integral form a generic one-loop amplitude can be written as a linear combination of scalar integrals involving what we call box, triangle, bubble and tadpole diagrams. This technique is often referred to as the Passarino-Veltman reduction [26]. When discussing loops we generally denote the amplitude of an n-point l-loop amplitude as M_n^l. To illustrate the techniques here we will use a generic 4-point, 1-loop amplitude. The reduction in terms of diagrams can be expressed as



$+$



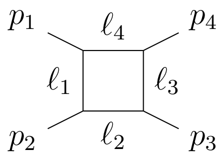
$+$ bubble and tadpole diagrams, (2.5.63)

or in terms of integrals

$$M_4^1 = \sum_i \left[c_4^i I_4^i + c_3^i I_3^i + c_2^i I_2^i + c_1^i I_1^i\right] + \mathcal{R}, \tag{2.5.64}$$

where the sum is over all the possible configurations of the external momenta. The c_j's are coefficients consisting of kinematic invariants and the I_j's are the scalar integrals, and $\mathcal{R}$ is a rational part remnant from dimensional reduction. I_4 is the scalar box integral given by

$$I_4(\{p_i^2\}; s_{12}, s_{23}; \{m_i^2\}) =$$



$$= \int \frac{d^{4-2\varepsilon}\ell_1}{(2\pi)^{4-2\varepsilon}} \frac{1}{(\ell_1^2 + m_1^2 - i\varepsilon)(\ell_2^2 + m_2^2 - i\varepsilon)(\ell_3^2 + m_3^2 - i\varepsilon)(\ell_4^2 + m_4^2 - i\varepsilon)}. \tag{2.5.65}$$

Here, the m_i's are the masses of the internal lines.

This technique of integral reduction gets rid of the problem of computing integrals with complicated tensor structures instead translating it to determining the coefficients of the various scalar integrals and the rational part. To compute the coefficients, we turn to the very efficient method of generalised unitarity, which in essence is a generalisation of the optical theorem, which relates different orders in perturbation theory, i.e. higher loop order to lower loop order all the way down to tree amplitudes. We can then compute loop amplitudes by analysing the discontinuities of the various kinematic channels similar to what is done in the BCFW recursion relations, and utilising what we know about the factorisation property of amplitudes.

The optical theorem in QFT uses the unitary nature of the S-matrix, $SS^\dagger = 1$. We expand the S-matrix into its trivial and non trivial part, $S = 1 + iT$. Taken together, and written in terms of matrix elements, we find

$$i(T - T^\dagger) = TT^\dagger \implies i\langle f|T|i\rangle - i\langle f|T^\dagger|i\rangle = \int d\mu \,\langle f|T|\mu\rangle \langle \mu|T^\dagger|i\rangle\,. \tag{2.5.66}$$

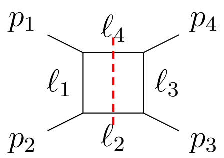
This is known as the generalised optical theorem, and we can see exactly how it relates the contributions of T at different orders once one considers the perturbative expansion of T in terms of the coupling constant g,

$$T = g^2 T^{(\text{Tree})} + g^4 T^{(1-\text{loop})} + O(g^6). \tag{2.5.67}$$

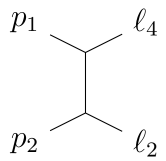
Plugging this in equation (2.5.66), we immediately see that

$$i(T^{(1-\text{loop})} - T^{(1-\text{loop})\dagger}) = T^{(\text{Tree})}T^{(\text{Tree})\dagger}. \tag{2.5.68}$$

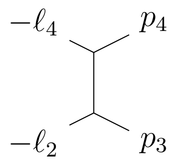
This is known as Cutkosky's rule [27], which allows us to express loop amplitudes in terms of tree-level amplitudes. We can represent this diagrammatically as



$$= \sum_{\text{Internal states}} \int d^4 l_2 \delta(l_2^2)\delta[(l_2 - p_1 - p_2)^2]$$



$\times$



(2.5.69)

Keeping to the on-shell formulation, what the cut does is effectively force the cut internal momenta on-shell. This in turn gives the discontinuity around which the amplitude can be factorised. We can then again use the all the power of complex analysis to evaluate the integrals. The above diagram only represents the s-channel (vertical) cut of the box diagram, to get the complete box contribution we also need to do a u-channel (horizontal) cut. But this is also not yet the entire amplitude contribution, only the box diagram, we then have to also consider possible triangle, bubble and tadpole diagrams as well. All of which can be computed in a similar way. Generalised unitarity does have limits and one often finds that the integrals are divergent. Therefore there is another way to compute the classical contributions of loops that we will introduce in the next section.

2.5.2 Leading Singularity (LS)

In the approach of generalised unitarity the two-particle cuts and the solutions to the conditions they impose ensure that the loop momenta remain real and the integrals can be evaluated on the real solutions. However as we have seen in the previous sections in the modern on-shell approach using analytic functions of complex variables is actually the more natural way in which to approach scattering problems. This generally offers great simplification to the problem as well as lend us the use of all the established machinery of complex analysis. To this end we introduce here the method of the Leading Singularity (LS) [13, 16], the highest codimension singularity found by fully localising all the loop integrals. At one loop is accomplished by cutting all the loop momenta and taking them on-shell. At higher loop order this is no longer clear since there are not enough cut loop propagators whose on-shell conditions can localise all the integrals. Another note is that the cut propagator at tree level is also a LS, as was explained in the sections dealing with constructing n-point amplitudes 2.3,2.4.6. In doing this we generally find that the solutions to these cut conditions are complex implying that the computation of computing loops, at least at the 1-loop level, reduces to finding the residue of a product of tree amplitudes. It has also recently been conjectured that the leading singularity contains all the information required to compute classical contributions of the loop amplitudes [13, 28, 29, 30]. Therefore this is the ideal method for use in the application of amplitudes to the realm of gravity.

To illustrate the techniques we use we will focus on the computation of a 4-point 1-loop amplitude. Specifically the triangle 1-loop amplitude since this will be of most use later. The triangle amplitude with all the cuts can be given as

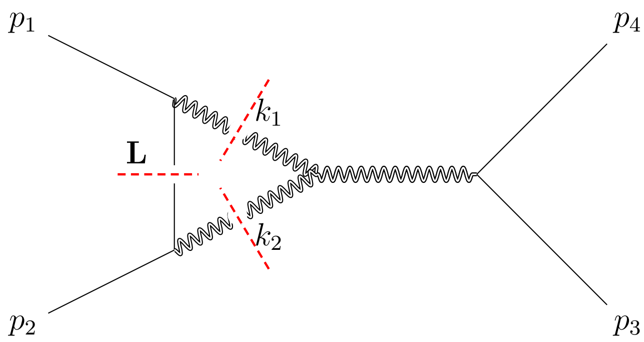


Figure 2.1: LS Triangle Diagram

Where solid lines represent scalars of mass m and wiggly lines massless particles with appropriate helicity. This process can be adapted to other particle content without much trouble. It is necessary to have at least one massive particle in the loop to extract classical information from the amplitude. We can then write this amplitude as the integral

$$\mathcal{I} = \sum_{h_1,h_2} \oint_\Gamma \frac{1}{(2\pi)^4} \frac{d^4\mathbf{L} M_3(\mathbf{p}_1, -\mathbf{L}, k_1^{-h_1}) M_3(\mathbf{L}, \mathbf{p}_2, k_2^{-h_2}) M_4(-k_1^{-h_1}, -k_2^{-h_2}, \mathbf{p}_3, \mathbf{p}_4)}{(\mathbf{L}^2 + m^2) k_1^2 k_2^2}, \tag{2.5.70}$$

where $k_1 = \mathbf{L} + \mathbf{p}_1$ and $k_2 = \mathbf{L} - \mathbf{p}_2$, and each of the tree amplitudes given in the amplitude are fully on-shell. To evaluate the integral we are required to parametrise $\mathbf{L}$ in the following way

$$L = z\ell + \omega q, \quad l_{a\dot{b}} = -\left|\ell\right\rangle_a [\ell|_{\dot{b}}, \tag{2.5.71}$$

where the variables to be integrated over are $z, \omega \in \mathbb{C}$ and the massless spinors $\left|\ell\right\rangle_a [\ell|_{\dot{b}}$. q is just some fixed reference spinor. Now when we cut the $L^2 + m^2$ propagator we end dealing with a Lorentz invariant phase space integral of a massive vector. The integration measure can be rewritten as

$$\frac{1}{(2\pi)^4} \frac{d^4\mathbf{L}}{(\mathbf{L}^2 + m^2)} = \frac{1}{(2\pi)^4} z dz \langle \ell d\ell \rangle [\ell d\ell] \frac{d\omega}{4(\omega - \frac{m^2}{2z\ell \cdot q})}. \tag{2.5.72}$$

The residue can then be easily extracted by performing the contour integral in the ω plane, which fixes ω and dropping the integrand for expediency we can rewrite the leading singularity

(2.5.70) as

$$\mathcal{I} = \sum_{h_1,h_2} \frac{1}{4(2\pi i)^3} \oint_\Gamma \frac{zdz\langle \ell d\ell\rangle[\ell d\ell]}{k_1^2 k_2^2}. \tag{2.5.73}$$

To compute the residues of the poles corresponding to the two massless propagators we make the introduction of two massless vectors $p_1 = |1\rangle [1|$ and $p_2 = |2\rangle [2|$, such that

$$\mathbf{p}_1 = p_1 + xp_2, \quad \mathbf{p}_2 = p_2 + xp_1, \quad x = \frac{m^2}{2p_1 \cdot p_2}, \tag{2.5.74}$$

where we have used the on-shell condition to fix x. We can use x to easily parametrise the t channel as

$$\frac{(1+x)^2}{x} = \frac{t}{m^2}, \quad \frac{(1-x)^2}{x} = \frac{t-4m^2}{m^2}. \tag{2.5.75}$$

At this point it is useful to introduce another two mixed auxiliary reference vectors $q_1 = |1\rangle [2|$ and $q_2 = |2\rangle [1|$. Since these are linearly independent we can expand the massless projection vector l as

$$\ell = Ap_1 + Bp_2 + Cq_1 + Dq_2. \tag{2.5.76}$$

Any overall scale of ℓ can be absorbed into z and by the on-shell condition we find that $C = AB$. We can now regard $A, B \in \mathbb{C}$ as the integration variables corresponding to $\langle \ell d\ell\rangle[\ell d\ell]$. Performing this change of variables we find the leading singularity as

$$\mathcal{I} = \sum_{h_1,h_2} \frac{1}{(2\pi i)^3} \frac{2p_1 \cdot p_2}{16} \oint_\Gamma \frac{zdzdAdB}{(m^2 + zp_1 \cdot p_2(B + xA))(-m^2 + zp_1 \cdot p_2(A + xB))}. \tag{2.5.77}$$

Resulting in the poles in the propagators being $A = -B = \frac{2x}{z(1-x)}$. Finding the residues gives us

$$\mathcal{I} = \frac{x}{4m^2(1-x^2)} \sum_{h_1,h_2} \frac{1}{2\pi i} \oint_\Gamma \frac{dz}{z} M_3(\mathbf{p}_1, -\mathbf{L}, k_1^{-h_1}) M_3(\mathbf{L}, \mathbf{p}_2, k_2^{-h_2}) M_4(-k_1^{-h_1}, -k_2^{-h_2}, \mathbf{p}_3, \mathbf{p}_4). \tag{2.5.78}$$

Now that we have localised all the integrals we want to perform the integral by finding the residue. The important thing to note here is the possible dependence on z in the tree amplitudes in the integrand. The trick here is to parametrise the internal massless spinors k_1 and k_2 in the same way we have done for ℓ. This has to be done on a case by case basis, the next section deals with a parametrisation that we use throughout our computation here.

2.5.3 Holomorphic Classical Limit (HCL)

One very useful parametrisation to use when doing the leading singularity is the HLC parametrisation. We make heavy use of this in chapter 3 when computing the classical contributions of higher derivative gravity to spinning black holes. We review the HCL parametrisation here for that purpose. Consider in this case that we are using a light probe particle of mass m_A and momentum p_1 to probe the spacetime generated by a massive spinning particle of mass m_B and momentum p_3. After the interaction and exchange of momentum K the probe has momentum p_2 and the generating particle has momentum p_4. We can therefore define the transfer momentum as

$$K \equiv p_1 - p_2 = |\lambda]\,\langle\lambda| = (0, \mathbf{q}), \qquad K^2 = t = -|\mathbf{q}|^2. \tag{2.5.79}$$

Like in section 2.5.2 we make use of the 1-loop triangle diagram to showcase the discussion. Considering that we parametrised the loop momenta, and some of the external momenta in 2.5.2 to make the computation of the leading singularity easier we do a similar parametrisation here which allows us to more easily extract the classical contributions from the amplitudes, e.g.

$$\begin{aligned} p_3 &= |\eta]\langle\lambda| + |\lambda]\langle\eta|\,, \\ p_4 &= \beta|\eta]\langle\lambda| + \frac{1}{\beta}|\lambda]\langle\eta| + |\lambda]\langle\lambda|\,, \\ \frac{t}{m_B^2} &= \frac{(\beta-1)^2}{\beta}\,, \\ \langle\lambda\eta\rangle &= [\lambda\eta] = m_B\,. \end{aligned} \tag{2.5.80}$$

Parametrising the loop momentum L is simply

$$L = z\ell + \omega K, \qquad |\ell] = |\eta] + B|\lambda], \qquad \langle\ell| = \langle\eta| + A\,\langle\lambda|\,. \tag{2.5.81}$$

The on-shell cut conditions $k_{3,4}^2 = L^2 - m_B^2$ (and imposing $\ell^2 = 0$) fixes $\omega = -\frac{1}{z}$ with $A = -B = -\frac{1}{z}\frac{2\beta}{1+\beta}$. Given this parametrisation, the LS is given by

$$\mathcal{I} = \frac{1}{16\sqrt{-t}m_B}\oint_\Gamma \frac{\mathrm{d}y}{y} M_3(\mathbf{p}_1, -\mathbf{L}, k_1^{-h_1}) M_3(\mathbf{L}, \mathbf{p}_2, k_2^{-h_2}) M_4(-k_1^{-h_1}, -k_2^{-h_2}, \mathbf{p}_3, \mathbf{p}_4), \tag{2.5.82}$$

where we have taken the $\beta \longrightarrow 1$ limit.

Given the above choices we have a handy parametrisation for the massless loop momenta k_3 and k_4 as well

$$
\begin{aligned}
|k_3] &= \frac{1}{\beta+1}\left(|\eta](\beta^2-1)y + |\lambda](1+\beta y)\right), \\
\langle k_3| &= \frac{1}{\beta+1}\left(\langle\eta|(\beta^2-1) - \frac{1}{y}\langle\lambda|(1+\beta y)\right), \\
|k_4] &= \frac{1}{\beta+1}\left(-\beta|\eta](\beta^2-1)y + |\lambda](1-\beta^2 y)\right), \\
\langle k_4| &= \frac{1}{\beta+1}\left(\frac{1}{\beta}\langle\eta|(\beta^2-1) + \frac{1}{y}\langle\lambda|(1-y)\right).
\end{aligned}
\tag{2.5.83}
$$

Some other useful parameters are

$$
u \equiv [\lambda|p_1|\eta\rangle\ , \quad v \equiv [\eta|p_1|\lambda\rangle\ , \tag{2.5.84}
$$

with $u+v=2p_1\cdot p_3$ in the HCL. These are related to the mandelstam variables in the HCL via

$$
\begin{aligned}
u &= m_A m_B(\rho+\sqrt{\rho^2-1}), \\
v &= m_A m_B(\rho-\sqrt{\rho^2-1}),
\end{aligned}
\tag{2.5.85}
$$

where

$$
m_A m_B\sqrt{\rho^2-1} = \sqrt{(s-(m_A+m_B)^2)(s-(m_A-m_B)^2)}
$$

and the non-dispersive limit is given by $\rho\to 1$.

Given these definitions and kinematics, we find the following relations

$$
\begin{aligned}
\langle\lambda|\,p_1|\lambda] &= -\frac{(\beta-1)^2}{\beta}m_B^2 + (1-\beta)(v-\frac{u}{\beta}), \\
\langle\eta|\,p_1|\eta] &= \frac{uv-m_A^2m_B^2}{(u-v)(\beta-1)} + \mathcal{O}(\beta-1)^0\ .
\end{aligned}
\tag{2.5.86}
$$

2.6 Spin Operators and their Representation in the On-Shell Formalism

Spin operators from an integral part of our ability to compute classical spin effects using the on-shell amplitude formalism. We therefore need to formulate these in the language of spinors and on-shell quantities. One formulation of the Gordon decomposition is

$$
\bar{u}(p_1)\gamma^\mu u(p_2) = \bar{u}(p_1)\left[\frac{p_1^\mu+p_2^\mu}{2m} + \frac{i\sigma^{\mu\nu}(p_{1\nu}-p_{2\nu})}{2m}\right]u(p_2). \tag{2.6.87}
$$

This for example expresses, in terms of form factors, the interaction of a massless photon with two massive fermions where the form factors correspond to a spin-independent and a spin-dependent piece. In [9] it was shown that in on-shell language one can express this identity by choosing a purely chiral spinor basis and in so doing expose the spin-dependence. Let us continue with the above example and recast the decomposition in an on-shell form. From 2.4.3 the amplitude in the undotted basis is given by

$$M_3^{IJ}(\mathbf{1}^{1/2}, \mathbf{2}^{1/2}, 3^{-1}) = g x_{13} \epsilon_{ab} \left|\mathbf{1}\right\rangle^{aI} \left|\mathbf{2}\right\rangle^{bJ}, \tag{2.6.88}$$

for the purposes of this section we can strip off the massive spinors and write the partial amplitude as

$$M_{ab} = x\epsilon_{ab}. \tag{2.6.89}$$

From the massive Weyl equation we know that we can switch between the undotted basis (angle spinors) and the dotted basis (square spinors) by using the operator $\mathbf{p}^{a\dot{b}}/m$. Identities that are useful here are

$$\mathcal{O}_{ab} \equiv \frac{\mathbf{p}_{1a}^{\dot{b}}\mathbf{p}_{2\dot{b}b}}{m^2} = \epsilon_{ab} - x\frac{\lambda_{3a}\lambda_{3b}}{m}, \quad \mathcal{O}_{\dot{a}\dot{b}} = \epsilon_{\dot{a}\dot{b}} - \frac{1}{x}\frac{\tilde{\lambda}_{3\dot{a}}\tilde{\lambda}_{3\dot{b}}}{m}. \tag{2.6.90}$$

Now we can write

$$M_{ab} \longrightarrow M_{\dot{a}\dot{b}} = x\epsilon_{\dot{a}\dot{b}} - \frac{\tilde{\lambda}_{3\dot{a}}\tilde{\lambda}_{3\dot{b}}}{m}. \tag{2.6.91}$$

Relating this to the spin requires us to consider the Pauli-Lubanski pseudo-vector

$$S^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}p_\nu\sigma_{\rho\sigma}, \tag{2.6.92}$$

where

$$(\sigma_{\mu\nu})_a^{\ b} = \frac{i}{2}(\sigma_{[\mu}\bar{\sigma}_{\nu]})_a^{\ b}, \quad (\bar{\sigma}_{\mu\nu})^{\dot{a}}_{\ \dot{b}} = -\frac{i}{2}(\bar{\sigma}_{[\mu}\sigma_{\nu]})^{\dot{a}}_{\ \dot{b}}. \tag{2.6.93}$$

The general spin-s generator $\bar{\sigma}_{\mu\nu}$ can be written in a simpler form for the chiral representations of massive states as

$$(\bar{\sigma}_{\mu\nu})^{\dot{a}_1,\dot{a}_2,\dots\dot{a}_{s2}}_{\dot{b}_1,\dot{b}_2,\dots\dot{b}_{s2}} = \sum_i (\bar{\sigma}_{\mu\nu})^{\dot{a}_i}_{\dot{b}_i}\bar{\mathbb{I}}_i, \quad \bar{\mathbb{I}}_i = \delta^{\dot{a}_1}_{\dot{b}_1}\cdots\delta^{\dot{a}_i}_{\dot{b}_i}\cdots\delta^{\dot{a}_{2s}}_{\dot{b}_{2s}}. \tag{2.6.94}$$

Hence we can write the Pauli-Lubanski pseudo vector as

$$
\begin{aligned}
(S_\mu)^{\dot{a}}_{\dot{b}} &= \frac{i}{m} p_\nu (\bar{\sigma}^{\mu\nu})^{\dot{a}}_{\dot{b}} \\
&= \frac{1}{4m}[(p\cdot\sigma)\bar{\sigma}_\mu - \sigma_\mu(p\cdot\bar{\sigma})]^{\dot{a}}_{\dot{a}}.
\end{aligned} \tag{2.6.95}
$$

We can generalise this to any spin-s in spinor notation by realising that $\sum_i (\bar{\sigma}_{\mu\nu})^{\dot{a}_i}_{\dot{b}_i} \bar{\mathbb{I}}_i = 2s(\bar{\sigma}_{\mu\nu})^{\dot{a}_1}_{\dot{b}_1} \bar{\mathbb{I}}_1$ and writing

$$
(S_\mu)^{\dot{a}_1\cdots\dot{a}_{2s}}_{\dot{b}_1\cdots\dot{b}_{2s}} = \frac{s}{2m}(\langle\mathbf{p}|\,\sigma_\mu|\mathbf{p}] + [\mathbf{p}|\bar{\sigma}_\mu\,|\mathbf{p}\rangle)\bar{\mathbb{I}}_1. \tag{2.6.96}
$$

This can now be contracted with the massless momentum p_3 giving

$$
(p_3\cdot S)^{\dot{a}}_{\dot{b}} = -\frac{|3][3|}{2x} \tag{2.6.97}
$$

Now we can write the general 3-point amplitude of a massless particle of helicity h and two massive particles of the same mass m and spin s in a from where the spin dependence is explicit

$$
\begin{aligned}
M^{\{I_1\cdots I_{2s}\}\{J_1\cdots J_{2s}\}}(\mathbf{1}^s,\mathbf{2}^s,3^h) &= g(mx)^h\frac{\langle\mathbf{12}\rangle^{2s}}{m^{2s}} \\
&= -g(mx)^h\left[[\mathbf{1}|\left(1-\frac{|3][3|}{mx}\right)|\mathbf{2}]\right]^{2s}.
\end{aligned} \tag{2.6.98}
$$

This method of writing the formula is particularly useful when studying classical spin effects since we would normally consider such spin to have large s and would normally be untenable in the spinor helicity formalism. The down side of this chiral representation is that we lose any information that might be captured in the anti-chiral part. Luckily due to the symmetries of the formalism these can be restored at a later stage by correctly normalising the resultant amplitude.

Chapter 3

Rotating Black Holes in Cubic Gravity

With the recent developments in the detection of gravitational waves and the precision with which we can start to observe this phenomena there has been a fundamental shift in gravitational wave astronomy [31, 32]. Due to the current physical limitations of the experiments used to observe gravitational waves the natural subjects for observation are those that can produce the clearest and most powerful signals, i.e. black holes. Hence there is a need within the field to develop precision theoretical data for the mergers of black holes. In essence, and from a very simplified point of view, this is nothing but a scattering amplitude in which two massive particles (the initial black holes) interact to produce another massive particle (the newly formed black hole) and a massless graviton (representative of the gravitational wave). This is of course an over simplification due to the fact that the interaction of black holes are an effect smeared out over spacetime as opposed to the understanding that a scattering amplitude is by definition an interaction at a specific point in spacetime [33]. As such what one actually has to consider the scattering amplitude in an effective field theory [34, 35]. This is beyond the scope of the work done in this treatise. In this chapter we will instead focus on the constituents that are necessary for the above calculations, the various aspects of black holes that make up the interactions and what can be gleaned about them from amplitudes.

The initial analysis of the data from the gravitational wave detection from the merger of black holes suggest strongly that General Relativity is the correct low-energy description of gravity. General Relativity does still have its problems especially from the view point of cosmology. This includes the decades old problem of the dark sector of the universe and the more recent H_0-problem. These discrepancies in the description of gravity by GR leaves the door open to study gravitational theories that extend beyond GR. Such considerations are necessary even if the only result is the ability to rule out some of these theories of gravity. In this chapter we specifically study the effect that adding a term cubic in the Riemann tensor to the Einstein-Hilbert action has on rotating solutions to the field equations. I.e. we are studying what effect additional cubic curvature contributions to GR has on the rotating black hole solutions.

Given that angular momentum is conserved we would expect that nearly all astrophysical black holes would be spinning, this is irrespective of the theory of gravity being considered. Generally finding solutions to the field equations in gravity is difficult in part due to the complexity of rotating solutions and in part due to the non-linearity of gravity theories. Modern on-shell amplitude techniques are useful for these kinds of problems due to their inherent gauge invariance as well as the technology developed during the last decade that greatly simplify computations [36, 37, 38, 34, 13, 39, 40, 29, 41, 42, 43, 44, 45, 18, 46, 47, 48, 49]. One extremely effective method for computing the perturbative classical from amplitudes is the Leading Singularity [13] as set out in chapter 2 especially for loops.

As stated previously we are considering a theory of gravity in which we add terms cubic in the Riemann tensor to the Einstein-Hilbert action, this six derivative theory is given by the action

$$S = \int \mathrm{d}^4x\sqrt{-g}\left(\frac{2}{\kappa^2}R + \lambda\mathcal{P}\right), \tag{3.0.1}$$

where the coupling has mass dimension $[\lambda] = -2$ and

$$\begin{aligned}\mathcal{P} = \beta_1 R^{\mu}{}_{\alpha\nu\beta}R^{\alpha\lambda\beta\sigma}R_{\lambda\mu\sigma}{}^{\nu} + \beta_2 R_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta}{}^{\lambda\sigma}R_{\lambda\sigma}{}^{\mu\nu}\\ + \beta_3 R_{\mu\nu\alpha\beta}R^{\mu\alpha}R^{\nu\beta} + \beta_4 R_{\mu}{}^{\nu}R_{\nu}{}^{\alpha}R_{\alpha}{}^{\mu},\end{aligned} \tag{3.0.2}$$

where β_i are generic coefficients for the time being. Gravity theories of this form have been studied extensively in for example [50, 51, 52, 53, 54, 55, 56]. An interesting theory that results from the specific choice $\beta_1 = 12$, $\beta_2 = 1$, $\beta_3 = -12$, $\beta_4 = 8$ is that of Einsteinian Cubic Gravity (ECG) [52, 51] which propagates the same massless degrees of freedom as GR, only two on-shell. For now we will keep the coefficients general, to better understand how each of these terms contribute to the solution. The field equations for this action can be found for example in [53]. The sheer number of terms found in these equations put even GR to shame in terms of complexity. But this is where amplitude techniques start to shine since the tensorial complexity of a theory is inconsequential all that matters in the particle content.

To understand the solutions to the field equations we will consider a light scalar particle of mass m_A and momentum p_1 probing the spacetime generated by a heavy spinning particle of mass m_B, momentum p_3 and spin s.

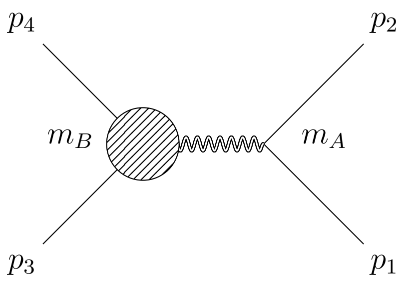


Figure 3.1: Gravitational probe of spinning particles in cubic gravity

In gravity the spin-effects can be found in the post-Minkowskian multipole expansion [33, 57, 58, 59, 34, 60, 17, 61, 62]. To get the classical contributions we only need to consider up to the first post-Minkowskian order (1PM) and classical effective matching as in [17] gives the all order in spin expansion in the two body problem. A similar analysis has been found directly from amplitudes by matching the results from amplitudes to an effective action [29, 18, 41, 45]. We will follow this approach throughout this section and in order to compute the classical contributions from the interaction described above we compute the Leading singularities in the Holomorphic classical limit (HCL) [28], established in section 2.5.3 and will be expanded upon as it becomes necessary. We do at this point let the external particles have the on-shell condition

$$p_1^2 = p_2^2 = m_A^2 \,, \quad p_3^2 = p_4^2 = m_B^2 \,, \tag{3.0.3}$$

and the Mandelstam variables are given by

$$s \equiv (p_1 + p_3)^2 \,, \quad t \equiv (p_1 - p_2)^2 \,, \quad u \equiv (p_1 - p_4)^2 \,. \tag{3.0.4}$$

The exchanged momentum is defined as

$$K \equiv p_1 - p_2 = |\lambda] \langle\lambda| = (0, \mathbf{q}), \qquad K^2 = t = -|\mathbf{q}|^2. \tag{3.0.5}$$

Before we get into the LS calculation let us first have a look at the pure gravity 3-point amplitudes and how they differ in GR and in cubic theories of gravity.

3.1 Pure Gravity Three Point Amplitudes

For now we are considering massless gravity theories therefore we only consider massless spin-2 gravitons. We can use little group scaling as set out in 2.2 to fix the 3-point amplitudes. Let us start by constructing all the possible helicity configurations

$$\begin{aligned} M(1^{-2}, 2^{-2}, 3^{+2}) &= g_1 \frac{\langle 12\rangle^6}{\langle 23\rangle^2 \langle 31\rangle^2}, \quad M(1^{+2}, 2^{+2}, 3^{-2}) = g_2 \frac{[12]^6}{[23]^2[31]^2}, \\ M(1^{-2}, 2^{-2}, 3^{-2}) &= g_3 \langle 12\rangle^2 \langle 23\rangle^2 \langle 31\rangle^2, \quad M(1^{+2}, 2^{+2}, 3^{+2}) = g_4 [12]^2[23]^2[31]^2. \end{aligned} \tag{3.1.6}$$

Now recall that the coupling in GR is $\kappa = \sqrt{8\pi G}$ which has mass dimension $[\kappa] = -1$. From this and the fact that n-point amplitudes in 4 dimensions have to have mass dimension 1 we can see we get the correct mass dimension for the first two amplitudes if $g_1 = g_2 = \kappa$. This is not the case for the second two, hence the only allowed 3-point in pure gravity in GR are $M(1^{-2}, 2^{-2}, 3^{+2})$ and $M(1^{+2}, 2^{+2}, 3^{-2})$. The other two amplitudes require the couplings to have dimension -5 which when taking the λ coefficient of the cubic term into consideration can be set to $g_3 = g_4 = \kappa^3\lambda$. Upon closer inspection we can think of the linearised metric normally expanded in κ as $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$, where η is the flat background and h is some small perturbation in spacetime which we generally call the graviton. Due to this we can make the assertion that terms cubic in the Riemann tensor will couple with $\kappa^3\lambda$. Hence the last two amplitudes in (3.1.6) only arise in cubic theories of gravity. The more applicable observation is that contributions to gravity due to terms in the action that are cubic in the Riemann tensor will, at the level of the amplitude, only enter if a pure gravity 3-point appears.

3.2 Tree Level Leading Singularity

Since we have established in the previous section that contributions of $\mathcal{O}(\lambda)$ only enter at a graviton three point we know that with the current set up only GR will contribute at tree level. The only diagram we need consider is

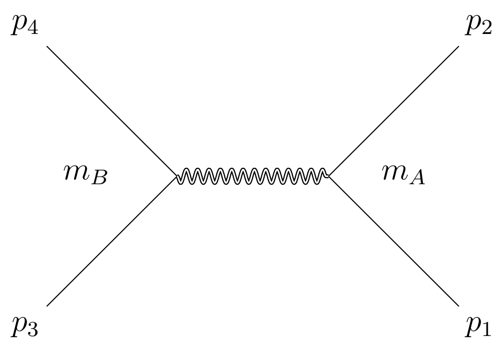


Figure 3.2: Tree Level LS

To study this diagram we are required to construct the two massive same mass 3-points on either side of the propagator according to 2.4.3. And since all we require is the minimally coupled piece we need only the term in the construction that has only epsilons. The minimally coupled three-particle amplitude with one graviton and two spin-s particles is given by [9]

$$\begin{aligned} M_3(\bar{1}^s, \bar{2}^s, K^{+2}) &= \frac{\kappa}{2}(mx_{12})^2 \frac{\langle \mathbf{12} \rangle^{2s}}{m^{2s}} \\ M_3(\bar{1}^s, \mathbf{2}^s, K^{-2}) &= \frac{\kappa}{2}\left(\frac{m}{x_{12}}\right)^2 \frac{[\mathbf{12}]^{2s}}{m^{2s}}, \end{aligned} \tag{3.2.7}$$

where x_{ij} is defined in the same way as in 2.4.3

$$x_{ij}\lambda_i^\alpha = \frac{\tilde{\lambda}_{i\dot{\alpha}} P_j^{\dot{\alpha}\alpha}}{m}, \qquad \frac{\tilde{\lambda}_i^{\dot{\alpha}}}{x_{ij}} = \frac{p_j^{\dot{\alpha}\alpha}\lambda_{i\alpha}}{m}. \tag{3.2.8}$$

The tree-level LS is simply the residue in the t channel, in this case given by the product of the above amplitudes as per the construction of higher point amplitudes in 2.4.6 and summed over the helicity of the internal particle.

$$\begin{aligned} M_4^s(\mathbf{1}^0, \mathbf{2}^0, \mathbf{3}^s, \mathbf{4}^s) &= \sum_h M_3(\bar{1}^0, \bar{2}^0, K^h) M_3(\bar{3}^s, \bar{4}^s, -K^{-h}) \\ &= \frac{\kappa^2}{4t} m_A^2 m_B^2 \left(\frac{x_{12}^2}{x_{34}^2} \frac{[\mathbf{34}]^{2s}}{m_B^{2s}} + \frac{x_{34}^2}{x_{12}^2} \frac{\langle \mathbf{34} \rangle^{2s}}{m_B^{2s}} \right). \end{aligned} \tag{3.2.9}$$

This is a bit unwieldy to extract the classical spin information so we write it in the chiral representation as defined in section 2.6. To make this even more convenient we now implement the HCL parametrisation as set up in section 2.5.3, i.e.

$$
\begin{aligned}
K \equiv p_1 - p_2 &= |\lambda]\,\langle\lambda| = (0,\mathbf{q}), \qquad K^2 = t = -|\mathbf{q}|^2, \\
p_3 &= |\eta]\langle\lambda| + |\lambda]\langle\eta|\,, \\
p_4 &= \beta|\eta]\langle\lambda| + \frac{1}{\beta}|\lambda]\langle\eta| + |\lambda]\langle\lambda|\,, \\
\frac{t}{m_B^2} &= \frac{(\beta-1)^2}{\beta}\,, \\
\langle\lambda\eta\rangle &= [\lambda\eta] = m_B\,.
\end{aligned}
\tag{3.2.10}
$$

We also define the mass rescaled form of the Pauli-Lubanski spin-vector as

$$
\tilde{a}^\mu = -\frac{1}{m^2}(P_i^\nu \bar{\sigma}_{\mu\nu}), \tag{3.2.11}
$$

and subsequently define the spin-vector $a = 2s\tilde{a}$. The spin dependence of the amplitude is characterised by identifying [36]

$$
\epsilon_{\mu\nu\rho\sigma}\mathbf{P}_1^\mu\mathbf{P}_3^\nu K^\rho a^\sigma = (E_A + E_B)(\mathbf{a}\cdot\mathbf{p}\times\mathbf{q}), \tag{3.2.12}
$$

where $\mathbf{p}$ is the relative momentum between the two initial particles and $\mathbf{q}$ is the transfer momentum. By noting that $u = m_A m_B \frac{x_{34}}{x_{12}}$ and $v = m_A m_B \frac{x_{12}}{x_{34}}$ with use of the HCL parametrisation we can rewrite the amplitude in the chiral basis

$$
\begin{aligned}
M_4^s(\mathbf{1}^0,\mathbf{2}^0,\mathbf{3}^s,\mathbf{4}^s) &= -\left(\frac{\kappa}{2}\right)^2 \frac{m_A^2 m_B^2}{t}\left(\frac{x_{34}^2}{x_{12}^2}\left(\mathbb{I} + \frac{K\cdot a}{s}\right)^{2s} + \frac{x_{12}^2}{x_{34}^2}\right), \\
&= -\left(\frac{\kappa}{2}\right)^2 \frac{1}{t}\left(u^2\left(\mathbb{I} + \frac{K\cdot a}{s}\right)^{2s} + v^2\right).
\end{aligned}
\tag{3.2.13}
$$

3.3 One Loop Leading Singularity

To compute the contributions at $\mathcal{O}(\lambda)$ we need to consider at least 1-loop in our expansion and more to the point the triangle diagram to ensure that we have the required graviton 3-point interaction. This means we want to consider the diagram

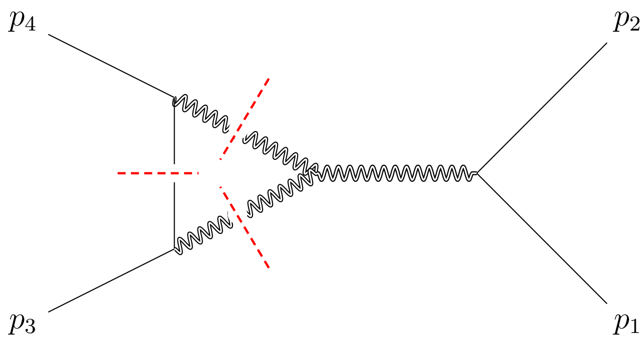


Figure 3.3: LS Triangle Diagram

The amplitude from this diagram corresponds to computing the integral

$$\mathcal{I}=\sum_{h_1,h_2}\oint_{\Gamma}\frac{1}{(2\pi)^4}\frac{d^4\mathbf{L}M_3(\mathbf{p}_1,-\mathbf{L},k_1^{-h_1})M_3(\mathbf{L},\mathbf{p}_2,k_2^{-h_2})M_4(-k_1^{-h_1},-k_2^{-h_2},\mathbf{p}_3,\mathbf{p}_4)}{(\mathbf{L}^2+m^2)k_1^2k_2^2}, \tag{3.3.14}$$

from section 2.5.2. Instead of going through the whole process of re-deriving the LS in its simplest form we recall that we have already set up most of the definitions required to work in the HCL parametrisation in section 2.5.3. We give these here for easy reference

$$\begin{aligned}
K \equiv p_1-p_2 &= |\lambda]\,\langle\lambda| = (0,\mathbf{q}), \qquad K^2=t=-|\mathbf{q}|^2,\\
L &= z\ell+\omega K, \qquad |\ell]=|\eta]+B|\lambda], \qquad \langle\ell|=\langle\eta|+A\,\langle\lambda|\,,\\
p_3 &= |\eta]\langle\lambda|+|\lambda]\langle\eta|\,,\\
p_4 &= \beta|\eta]\langle\lambda|+\frac{1}{\beta}|\lambda]\langle\eta|+|\lambda]\langle\lambda|\,,\\
\frac{t}{m_B^2} &= \frac{(\beta-1)^2}{\beta}\,,\\
\langle\lambda\eta\rangle &= [\lambda\eta]=m_B\,,\\
|k_3] &= \frac{1}{\beta+1}\left(|\eta](\beta^2-1)y+|\lambda](1+\beta y)\right),\\
\langle k_3| &= \frac{1}{\beta+1}\left(\langle\eta|(\beta^2-1)-\frac{1}{y}\langle\lambda|(1+\beta y)\right),\\
|k_4] &= \frac{1}{\beta+1}\left(-\beta|\eta](\beta^2-1)y+|\lambda](1-\beta^2 y)\right),\\
\langle k_4| &= \frac{1}{\beta+1}\left(\frac{1}{\beta}\langle\eta|(\beta^2-1)+\frac{1}{y}\langle\lambda|(1-y)\right).
\end{aligned} \tag{3.3.15}$$

The on-shell cut conditions $k_{3,4}^2=L^2-m_B^2$ (and imposing $\ell^2=0$) fixes $\omega=-\frac{1}{z}$ with $A=-B=-\frac{1}{z}\frac{2\beta}{1+\beta}$. Given this parametrisation, the LS is given by

$$\mathcal{I}=\frac{1}{16\sqrt{-t}m_B}\oint_\Gamma\frac{\mathrm{d}y}{y}\sum_{h_3,h_4=\pm2}M_3(\mathbf{3}^s,-\mathbf{L}^s,k_3^{h_3})M_3(\mathbf{4}^s,\mathbf{L}^s,k_4^{h_4})M_4(\mathbf{1}^0,\mathbf{2}^0,-k_3^{-h_3},-k_4^{-h_4}). \tag{3.3.16}$$

3.3.1 Tree-Level Components

The tree level components required to compute this LS can all be found from the 3-point amplitudes of a particle of generic spin emitting a graviton, which are basically (3.2.7). But using everything we have established in the previous section we can write them in the positive helicity chiral basis as

$$M_3(1^s,2^s,K^{+2})=-\frac{\kappa}{2}m^2x_{12}^2\left(\mathbb{I}+\frac{K\cdot a}{s}\right)^{2s}, \tag{3.3.17}$$

$$M_3(1^s,2^s,K^{-2})=-\frac{\kappa}{2}\left(\frac{m^2}{x_{12}^2}\right), \tag{3.3.18}$$

We can therefore work out the product of three-particle amplitudes that go into the LS

$$M_3(\mathbf{3}^s,-\mathbf{L}^s,k_3^{h_3})M_3(\mathbf{4}^s,\mathbf{L}^s,k_4^{h_4})=\left(\frac{\kappa}{2}\right)^2m_B^4y^4\left(1+\frac{(1+y)^2}{4y}\frac{K\cdot a}{s}\right)^{2s}\left(1-\frac{(1-y)^2}{4y}\frac{K\cdot a}{s}\right)^{2s}, \tag{3.3.19}$$

where we have used the fact that $x_{3L}=x_{4L}=-y$ in this parametrisation.

For the 4-point note that we can express

$$k_3\cdot\mathbf{p}_1=k_3\cdot\mathbf{p}_2+\mathcal{O}(\beta-1)^2=(\beta-1)\frac{(1-y^2)(v-u)}{8y}, \tag{3.3.20}$$

we find

$$\begin{aligned}
&M_4(\mathbf{1}^0,\mathbf{2}^0,-k_3^{-h_3},-k_4^{-h_4})\\
&=\frac{3}{16}\frac{\kappa^4\lambda\langle k_3k_4\rangle^4}{(p_1-p_2)^2}\left[\beta_1\left((k_3\cdot p_1+k_3\cdot p_2)^2-m_A^2k_3\cdot k_4\right)+8\beta_2(k_3\cdot p_1)(k_3\cdot p_2)\right]\\
&=-\frac{3}{64}\frac{\kappa^4\lambda}{t}\frac{m_B^4(\beta-1)^6}{y^4}\left[2\beta_1m_A^2m_B^2-(\beta_1+2\beta_2)\frac{(1-y^2)^2(v-u)^2}{4y^2}\right]\\
&=-\frac{3}{64}\kappa^4\lambda\frac{t^2}{m_B^2y^4}\left[2\beta_1m_A^2m_B^2-(\beta_1+2\beta_2)\frac{(1-y^2)^2(v-u)^2}{4y^2}\right].
\end{aligned}$$

Putting this all together, we find the LS to be evaluated is

$$\mathcal{I}^s = -\frac{3}{4096}\kappa^6\lambda m_B(-t)^{3/2}\oint\frac{\mathrm{d}y}{y}\left(1+\frac{(1+y)^2}{4y}\frac{K\cdot a}{s}\right)^{2s}\left(1-\frac{(1-y)^2}{4y}\frac{K\cdot a}{s}\right)^{2s}$$
$$\times\left[2\beta_1 m_A^2 m_B^2 - (\beta_1+2\beta_2)(v-u)^2(1+y)^2\right]. \qquad (3.3.21)$$

Taking appropriate limits, specifically $s \longrightarrow 0$, this result matches those found in [24] in which the static solution was found.

3.4 All Order In Spin Classical Potential

In momentum space, the classical potential is a related to the amplitude M by

$$V(\mathbf{q},\mathbf{p}) = \frac{\langle M\rangle}{4E_A E_B}, \qquad (3.4.22)$$

where $E_i = \sqrt{m_i^2+\mathbf{p}^2+\frac{\mathbf{q}^2}{4}}$. Recall that since we constructed the leading singularity amplitudes in the chiral basis we are required to normalise the amplitudes with the general expectation value (GEV) to re-acquire any information that we may have lost in the following way $\langle M\rangle = e^{-K\cdot a}M$.

The spin s at this stage still relates to the spin of a single point particle. But we expect that spinning astrophysical black holes to have much larger spin than a single point particle. To get a consistent classical limit we have to take the limit $s \longrightarrow \infty$ and $\hbar \longrightarrow 0$ keeping $s\hbar$ fixed, since the intrinsic angular momentum of a spin s particle is proportional to $s\hbar$ [63]. We now make a further identification for the variables u and v as being

$$u = m_A m_B e^{w}, \qquad v = m_A m_B e^{-w}, \qquad (3.4.23)$$

where w is the rapidity. Plugging this into the tree-level four point and taking the infinite spin limit gives

$$M_4^\infty = -\frac{\kappa^2}{4}\frac{m_A^2 m_B^2}{t}(e^{2w}e^{2K\cdot a}+e^{-2w}). \qquad (3.4.24)$$

After normalisation with the GEV, we then find

$$\langle M_4^\infty\rangle = -\frac{\kappa^2}{4}\frac{m_A^2 m_B^2}{t}\left(e^{2w}e^{K\cdot a}+e^{-2w}e^{-K\cdot a}\right). \qquad (3.4.25)$$

Doing the same for the loop amplitude yields

$$\mathcal{I}^\infty = -\frac{3\kappa^6\lambda}{2048}(-t)^{3/2}m_A^2 m_B^3\beta(w)(1+e^{2K\cdot a}), \qquad (3.4.26)$$

where we have defined $\beta(w) \equiv \beta_1 + 2(\beta_1+2\beta_2)\sinh^2 w$. Again normalising this using the GEV gives

$$\langle\mathcal{I}^\infty\rangle = -\frac{3\kappa^6\lambda}{1024}m_A^2 m_B^3\beta(w)\mathbf{q}^3\cosh\mathbf{q}\cdot\mathbf{a}. \qquad (3.4.27)$$

The momentum space potential is therefore given by

$$V = \frac{\kappa^2}{16}\frac{m_A m_B}{\mathbf{q}^2}\left(e^{2w}e^{\mathbf{q}\cdot\mathbf{a}} + e^{-2w}e^{-\mathbf{q}\cdot\mathbf{a}}\right) - \frac{3\kappa^6\lambda m_A m_B^2}{4096}\beta(w)\mathbf{q}^3\cosh\mathbf{q}\cdot\mathbf{a}. \tag{3.4.28}$$

Restricting ourselves to the non-dispersive terms (i.e. $w = 0$) and taking the Fourier transform, we can find the all-order in spin potential. Some useful Fourier transforms are listed here

$$\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{r}}\frac{1}{\mathbf{q}^2} = \frac{1}{4\pi r} \tag{3.4.29}$$

$$\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{r}}\frac{q_i}{\mathbf{q}^2} = \frac{ix_i}{4\pi r^3} \tag{3.4.30}$$

$$\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{r}}|\mathbf{q}|^n = \frac{(n+1)!}{2\pi^2 r^{3+n}}\sin\left(\frac{3\pi n}{2}\right), \tag{3.4.31}$$

$$\int\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{r}}q_i|\mathbf{q}|^n = i\frac{(n+1)!(3+n)x_i}{2\pi^2 r^{5+n}}\sin\left(\frac{3\pi n}{2}\right), \tag{3.4.32}$$

$$\int\frac{\mathrm{d}^4 q}{(2\pi)^3}\delta(q^0)\delta(\gamma q^1 - \beta\gamma q^3)e^{i\mathbf{q}\cdot\mathbf{b}}q^\mu|\mathbf{q}|^n = -\frac{i}{2\pi|\beta\gamma|}\mathcal{H}_1[r^{n+1}]\hat{b}^\mu, \tag{3.4.33}$$

$$\mathcal{H}_\nu[r^n] = \int_0^\infty r^{n+1}J_\nu(kr) = \frac{2^{n+1}}{k^{n+2}}\frac{\Gamma(\frac{1}{2}(2+\nu+n))}{\Gamma(\frac{1}{2}(\nu-n))}. \tag{3.4.34}$$

Applying these the all-order in spin potential is given by

$$\begin{aligned} V(r) &= -\frac{\kappa^2}{32\pi}m_A m_B\cos\left(\mathbf{a}\cdot\nabla\right)\left(\frac{1}{r} - \frac{9\kappa^4\lambda\beta_1}{32\pi}\frac{m_A+m_B}{r^6}\right), \\ &= -Gm_A m_B\cos\left(\mathbf{a}\cdot\nabla\right)\left(\frac{1}{r} - 288\pi G^2\lambda\beta_1\frac{m_A+m_B}{r^6}\right). \end{aligned} \tag{3.4.35}$$

We note that this form of the potential allows us to interpret the attachment of spin-factors to the amplitudes generating the spacetime as the on-shell avatar of the Newman-Janis algorithm for higher derivative gravity, in precisely the same way as it does in the Kerr and Kerr-Newman cases [47, 48]. This is because attachment of such factors gives rise to a differential operator which performs a complex deformation of the coordinates $r \to r + ia$ (a simple extension of the translation operator), i.e.

$$\cos(a\cdot\nabla)f(r) = 2\Re\left(f(r+ia)\right), \tag{3.4.36}$$

$$\sin(a\cdot\nabla)f(r) = 2\Im\left(f(r+ia)\right), \tag{3.4.37}$$

for any holomorphic function $f(r)$.

3.5 Metric Construction

In momentum space, the classical potential for a gravitomagnetic system is of the form

$$V(\mathbf{q}) = m\Phi(\mathbf{q}) + \mathbf{p}\cdot\mathbf{w}, \tag{3.5.38}$$

where Φ is the gravito-electric field, $\mathbf{w}$ is gravito-magnetic field and $\mathbf{p}$ is kinetic momentum. The perturbative metric can be decomposed in terms of these fields as

$$h_{00} = 2\Phi, \qquad h_{0i} = -w_i, \qquad h_{ij} = 2\Phi\delta_{ij}. \tag{3.5.39}$$

We can then identify the relevant components of the metric

$$\Phi = \lim_{m_A\longrightarrow 0}\frac{1}{m_A}V, \qquad w_i = \lim_{m_A\longrightarrow 0} 2\frac{\partial V}{\partial p^i}, \tag{3.5.40}$$

where we note that we will often write $p_i = m_A u_i$ in the limit we are interested in.

In order to identify the spin components, we are required to identify the dispersive terms that multiply $K\cdot a$ via

$$m_A m_B \sinh w \; K\cdot a = i\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_3^\nu K^\rho a^\sigma. \tag{3.5.41}$$

In the centre of mass frame, we can use this on-shell condition to write

$$K\cdot a = \frac{i\epsilon_{\mu\nu\rho\sigma}p_1^\mu p_3^\nu K^\rho a^\sigma}{m_A m_B \sinh w} = -i\mathbf{u}\cdot(\mathbf{a}\times\mathbf{q}), \tag{3.5.42}$$

where $\mathbf{u}$ is the relative four velocity.

With this in hand, we can therefore rewrite the potential, keeping only the necessary dispersive terms, as

$$V = \cos(\mathbf{u}\cdot(\mathbf{a}\times\mathbf{q}))\left[\frac{\kappa^2}{8}\frac{m_A m_B}{\mathbf{q}^2} - \frac{3\kappa^6\lambda m_A m_B^2}{4096}\beta(w)\mathbf{q}^3\right]. \tag{3.5.43}$$

We find then, in momentum space,

$$\Phi = \cosh(\mathbf{q}\cdot\mathbf{a})\left(\frac{\kappa^2}{8}\frac{m_B}{\mathbf{q}^2} - \frac{3}{4096}\beta(w)\kappa^6\lambda m_B^2\mathbf{q}^3\right), \tag{3.5.44}$$

$$w_i = -\sinh(\mathbf{q}\cdot\mathbf{a})\times \left[\frac{\kappa^2}{4}\frac{m_B}{\mathbf{q}^2} - \frac{3\kappa^6\lambda m_B^2}{2048}\beta(w)\mathbf{q}^3\right](i\mathbf{a}\times\mathbf{q})_i\,, \tag{3.5.45}$$

which in position space is

$$\Phi = \cos(\mathbf{a}\cdot\nabla)\left(\frac{Gm_B}{r} - \frac{288G^3\pi\lambda\beta(w)m_B^2}{r^6}\right),$$
$$= \cos(\mathbf{a}\cdot\nabla)\left(\Phi_{Kerr} + \Phi_{R^3}\right), \tag{3.5.46}$$

$$
\begin{aligned}
w_i &= -\sin(\mathbf{a}\cdot\nabla)\left(\frac{2Gm_B}{r} - 3456\frac{G^3\pi\lambda\beta(w)m_B^2}{r^6}\right)\frac{(\mathbf{a}\times\mathbf{r})_i}{r^2} \\
&= -2\sin(\mathbf{a}\cdot\nabla)\left(\Phi_{Kerr}+6\Phi_{R^3}\right)\frac{(\mathbf{a}\times\mathbf{r})_i}{r^2},
\end{aligned} \tag{3.5.47}
$$

from which the components of the metric can be determined. Specialising to the case of ECG [52, 51], we choose $\beta_1 = 12,\ \beta_2 = 1$ and $\lambda = -\frac{G\tilde{\lambda}}{16\pi}$, we find the metric

$$
\begin{aligned}
g_{00}^{ECG} &= 1 - \frac{2GM}{r} - 432\frac{G^4M^2\tilde{\lambda}}{r^6} + \cdots \\
g_{0i}^{ECG} &= \left(1 + \frac{2GM}{r} + 2592\frac{G^4M^2\tilde{\lambda}}{r^6}\right)\frac{(\mathbf{a}\times\mathbf{r})_i}{r^2} + \cdots \\
g_{ij}^{ECG} &= \left(1 + \frac{2GM}{r} + 432\frac{G^4M^2\tilde{\lambda}}{r^6}\right)\delta_{ij} + \cdots
\end{aligned} \tag{3.5.48}
$$

3.6 Scattering Angle

We can now derive the scattering angle, given in terms of the LS by [64, 29, 23]

$$
\theta = 2\sin\left(\frac{\theta}{2}\right) = \frac{-E}{(2m_Am_B\sinh w)^2}\frac{\partial}{\partial b}\langle M^\infty(b)\rangle, \tag{3.6.49}
$$

where $\langle M^\infty(b)\rangle$ is the LS in impact parameter space.The tree-level LS is given by

$$
\langle M_4^\infty(b)\rangle = \left(\frac{\kappa^2}{8\pi}\right)m_A^2m_B^2\sum_\pm e^{\pm 2w}\ln|b\pm a|, \tag{3.6.50}
$$

and the loop LS (3.4.27) by

$$
\langle \mathcal{I}^\infty(b)\rangle = -\frac{27\kappa^6\lambda}{4096\pi}m_A^2m_B^3\sum_\pm\frac{\beta(w)}{(b\pm a)^5}. \tag{3.6.51}
$$

Plugging these back into (3.6.49) we find the angle

$$
\theta = \frac{-GE}{\sinh^2 w}\sum_\pm\left(\frac{e^{\pm 2w}}{b\pm a} + 270\pi^2G^2\lambda m_B\frac{\beta(w)}{(b\pm a)^6}\right). \tag{3.6.52}
$$

In this chapter, we have derived a new black hole solution in cubic gravity using modern amplitude methods. In particular, we present the all-order (in spin) classical potential, to leading-order in the cubic coupling λ. Further, we have shown how the form of this potential allows for an interpretation of the on-shell avatar of the Newman-Janis algorithm, extending it to higher-derivative gravity. This is certainly good motivation to try and establish the precise algorithm that deforms coordinates in some particular coordinate system and allows one to derive the rotating solution directly from the static one.

It should be emphasised that deriving such a black hole solution using traditional geometric methods is a difficult endeavour, a fact which highlights the benefits of using modern amplitude techniques to understand gravitational phenomena. Indeed, *this is a rare case in which it has been possible to derive a novel result via modern amplitude methods before it has been done so through the geometric approach*, in which the presence of cubic-order curvature terms makes the task almost intractable.

In addition to finding the black hole solution, we also present results for the scattering angle. Given some reliable observational data, these quantities could be used to place a bound on the coupling λ, whose parameter space has been only partially constrained [55] – in the model presented, λ can assume arbitrarily large or small values. Due to the equivalence principle the scattering angle should be the same for all forms of matter. In the next chapter we focus on deriving the scattering angle for light as well as gravitational waves due to a massive body as was done in [19].

Chapter 4

Lightbending

This chapter is based in part on the work done in [19], in which we attempted to introduce the astrophysics community to the amplitudes program. We did this by showcasing the various ways in which to compute the deflection of light by a massive stellar body in General Relativity. We start with the classic geodesic approach in GR and then move on to do the computation using standard off-shell Feynman techniques. Neither of which are particularly difficult or time consuming. The purpose of the paper was to draw attention the ease with which one can use amplitude techniques to perform more computationally intensive processes like gravitational lensing of gravitational waves, computed for the first time in this paper using BCFW. Due to the development of better computational techniques in the on-shell amplitudes formalism since the release of this paper we will instead recast the results in this more modern and even simpler approach.

4.1 The old way of doing things

This section serves only as a comparison for the amplitudes method and illustrate why it is necessary. As such we will give the minimal detail here and when at all possible avoid calculating anything.

4.1.1 Light Bending in General Relativity

We will start by briefly reviewing the classical deflection of light due to a massive body in the context of General Relativity. To determine the path of a massless particle, the photon in this case, in a gravitational field, we need to know the null geodesics in the corresponding spacetime. Assuming the simplest stationary spacetime with spherical symmetry which naturally leads us to the Schwarzschild geometry. The metric of the Schwarzschild solution for a body of mass M is

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{4.1.1}$$

First it is necessary to recall that if we define the tangent to a geodesic as $u^\alpha = dx^\alpha/d\lambda$ where λ is an affine parameter along the geodesic, the inner product of u^α with the Killing field of the geometry, $u^\mu \xi_\mu$ is a constant. In fact, from this we can read off the constants of motion,

$$u \cdot \xi = u^t \xi_t + u^\varphi \xi_\varphi = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\lambda} + (r^2 \sin^2 \theta) \frac{d\varphi}{d\lambda} = E + L, \tag{4.1.2}$$

since, if the affine parameter, λ, is normalized such that u^α coincides with the momentum of a null vector then, E and L are to be understood as the energy and the angular momentum of the photon respectively. The rotational symmetry of the Schwarzschild metric implies that if a null geodesic starts in, say, the equatorial plane then the entire geodesic remains in the plane, meaning we can set $\theta = \pi/2$ without loss of generality. We require that the tangent vector be null which, by the geodesic equation, tells us that

$$0 = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = -\left(1 - \frac{2M}{r}\right) \left(\frac{dt}{d\lambda}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{d\lambda}\right)^2 + r^2 \left(\frac{d\varphi}{d\lambda}\right)^2, \tag{4.1.3}$$

which after plugging in Eq. (4.1.2) and with a little algebraic manipulation gives,

$$\frac{1}{2} E^2 = \frac{1}{2} \left(\frac{dr}{d\lambda}\right)^2 + \frac{L^2}{2r^2} \left(1 - \frac{2M}{r}\right). \tag{4.1.4}$$

The deflection angle of a light ray is usually framed in terms of the impact parameter which, in flat space is defined by $b = L/E$. Since we consider only paths in the weak field regime, $r \gg M$, b will serve as our apparent impact parameter. From eq. (4.1.4), we find that the effective potential of massless orbits is $V(r) = L^2(r - 2M)/2r^3$. We next define the point of closest approach of the particle to the center of the geometry as $r = R_0$, the point at which the photon will have a turning point as it passes near the massive body.

From equation (4.1.4) and using the definition of the angular momentum that, for $\theta = \pi/2$, takes the form $L = r^2 d\varphi/d\lambda$ we find

$$\frac{d\varphi}{dr} = \frac{L}{r^2} \left(E - \frac{L^2}{r^3}(r - 2M)\right)^{-1/2}. \tag{4.1.5}$$

This is now sufficient for us to calculate the change in the photon's trajectory. Assuming that the particle approaches from and proceeds to infinity, this change is captured by the angle between these trajectories as a result of the deflection of a photon due to gravity, $\Delta\varphi = \varphi_{+\infty} - \varphi_{-\infty}$. However, as a consequence of the symmetries of the geometry, the contributions to the integrals before and after the turning point are equal. To determine the opening angle $\Delta\varphi$, we have to integrate eq. (4.1.5). this is easiest to do by introducing the new variable $u = 1/r$ and using the effective potential to eliminate b to give

$$\Delta\varphi = 2 \int_0^{1/R_0} du \left(R_0^{-2} - 2M R_0^{-3} - u^2 + 2Mu^3\right)^{-1/2}. \tag{4.1.6}$$

As a check, we set $M = 0$ to give

$$\Delta\varphi = 2 \arcsin 1 = \pi, \tag{4.1.7}$$

which is of course the expected result in flat spacetime. Evaluating the integral for $M \neq 0$ to first order in M, we need to treat M and R_0 as independent variables, and then vary the integrand with respect to M. This allows us to calculate the deflection angle to first order in M as a function of mass but at a fixed radius R_0. It is important to note that the physical parameter we want in the result is the impact parameter, but if $M = 0$ we have $b = R_0$. Differentiating with respect to M and evaluating the result at $M = 0$ gives

$$\left.\frac{\partial\varphi}{\partial M}\right|_{M=0} = 2\int_0^{1/b} du \left(b^{-3} - u^3\right)\left(b^{-2} - u^2\right)^{-3/2} = \frac{4}{b}. \tag{4.1.8}$$

So, to first order in M,

$$\Delta\varphi = \pi + M \left.\frac{\partial\varphi}{\partial M}\right|_{M=0} = \pi + \frac{4M}{b}. \tag{4.1.9}$$

Of course, in order to compute the deflection angle, we are interested in the deviation from the flat spacetime trajectory induced by the Schwarzschild geometry, *i.e.*

$$\varphi_{\mathrm{D}} = \Delta\varphi - \pi = \frac{4GM}{R_0}. \tag{4.1.10}$$

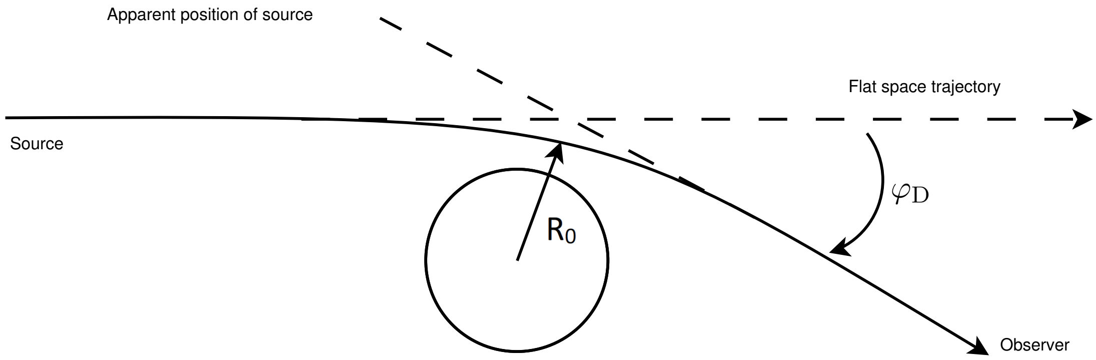


Figure 4.1: Graphical representation of the deflection angle φ_{D}.

At this point it is worth noting that apart from assuming that the lensed beam of photons obeys the weak field Einstein equations and moves on a null geodesic in the geometric optics limit, we did not have to specify anything else about the probe. Essentially then, the derivation of the deflection angle is applicable to any massless particle moving in a stationary, spherically symmetric space time exterior to a massive object with mass M.

4.1.2 Light bending in Quantum Field Theory

As we have previously stated the deflection of light can be considered to be nothing more than a scattering event between a photon and some massive particle representative of the massive body generating the spacetime in which the photon travels mediated by a graviton exchange. To compute this in the QFT sense we start with the Einstein-Maxwell-scalar action,

$$S[A,h,\phi] = \int d^4x\sqrt{-g}\left(\frac{2}{\kappa^2}R - \frac{1}{4}g^{\mu\nu}g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{m^2}{2}\phi^2\right). \tag{4.1.11}$$

Now the process we want to calculate corresponds to the following Feynman diagram, where we have chosen some specific states for the external particles and again assume the most basic massive particle, a scalar, to be the particle avatar of the Schwarzschild geometry.

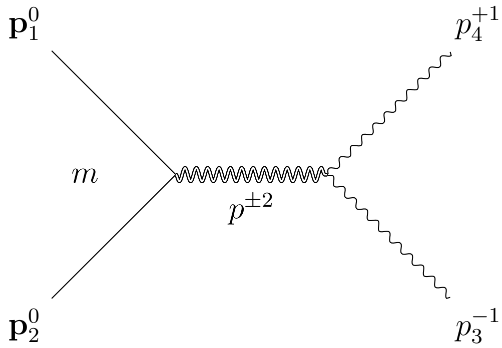


Figure 4.2: Gravitational Light Bending

From the action we can derive the necessary vertex functions and propagator to compute this process. We do not go through the full calculation here since it is basic QFT and not in the on-shell amplitude focus of this work, the reader is welcome to see [19] for the full derivation. We find that the scalar-scalar-graviton vertex is

$$V^{\mu\nu}(1^0 2^0) = -\frac{1}{4}i\kappa\left[p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \eta^{\mu\nu}\left(p_1\cdot p_2 - m^2\right)\right], \tag{4.1.12}$$

and the photon-photon-graviton vertex is given by

$$\begin{aligned} V^{\rho\sigma\gamma\delta}(1^{+1}2^{-1}) = V^{\rho\sigma\gamma\delta}(1^{-1}2^{+1}) = \frac{1}{4}i\kappa\,[\, & p_1\cdot p_2(\eta^{\rho\gamma}\eta^{\sigma\delta} + \eta^{\rho\delta}\eta^{\sigma\gamma} - \eta^{\rho\sigma}\eta^{\gamma\delta}) \\ & + \eta^{\rho\sigma}p_1^\delta p_2^\gamma + \eta^{\gamma\delta}(p_1^\rho p_2^\sigma + p_1^\sigma p_2^\rho) \\ & - (\eta^{\rho\delta}p_1^\sigma p_2^\gamma + \eta^{\sigma\delta}p_1^\rho p_2^\gamma + \eta^{\sigma\gamma}p_1^\delta p_2^\rho + \eta^{\rho\gamma}p_1^\delta p_2^\sigma)\,]. \end{aligned} \tag{4.1.13}$$

The graviton propagator with transfer momentum p is

$$P_{\mu\nu\alpha\beta}(p) = \frac{1}{2p^2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right). \tag{4.1.14}$$

Now we can construct the scattering process by appropriately contracting (4.1.12), (4.1.13) and (4.1.14) as well as dotting in the appropriate polarisation vectors for the external particle to get

$$A_4(\mathbf{1}^0, \mathbf{2}^0, 3^{-1}, 4^{+1}) = V^{\mu\nu}(1^0 1^0) P_{\mu\nu\rho\sigma}(p) V^{\rho\sigma}{}_{\gamma\delta}(3^{-1}4^{+1}) \epsilon_-^{\gamma}(p_3) \epsilon_+^{\delta}(p_4). \tag{4.1.15}$$

Note that since all the particles are outgoing we require the helicity to be different for the incoming and outgoing photons. In order to extract a measurable quantity out of this, we will work in the center of mass frame and make the following approximations and substitutions:

- First, taking the static limit for the scalar, as one would expect for a massive star say, requires that we take $(p_1)_\mu = (p_2)_\mu = m\eta_{\mu 0}$.
- Next, we assume that the photon deflection angle, θ_D, is small. This is equivalent to the approximation of small momentum transfer, or $P_{34}^2 = (p_3 + p_4)^2 \approx 0$.
- Finally, keeping the static limit and the small angle approximation in mind, we use momentum conservation at the photon vertex, $P_{34}^2 = (p_3+p_4)^2 = 2p_3 \cdot p_4$, and the fact that the change in energy between the incoming and outgoing photons is small, $E_3 - E_4 \approx 0$ to write $P_{34}^2 \approx -4E_3^2 \sin^2(\theta/2)$.

Next we need to evaluate the differential cross-section which we can relate to the deflection angle observable. In general for this kind of process the cross-section is given by

$$\frac{d\sigma^{(-1,+1)}}{d\Omega} = \frac{1}{64\pi^2 s} |A_4(\mathbf{1}^0, \mathbf{2}^0, 3^{-1}, 4^{+1})|^2, \tag{4.1.16}$$

where we define the Mandelstam invariants as $s = (\mathbf{p}_1 + p_4)^2, t = (\mathbf{p}_1 + \mathbf{p}_2)^2, u = (\mathbf{p}_1 + p_3)^2$. Using the low energy limit, in which the energy of the photon is much less than the mass of the scalar and the small angle approximation we can write the scattering cross-section as

$$\frac{d\sigma}{d\Omega} = \frac{16G^2m^2}{\theta^4}. \tag{4.1.17}$$

To compare this with the more familiar result from GR, we need to relate the cross-section to the impact parameter b, the perpendicular offset of the incoming photons. Some elementary geometry shows that $\sigma = \pi b^2$, or, infinitesimally,

$$b\ db = -\frac{d\sigma}{d\Omega} \sin\theta d\theta. \tag{4.1.18}$$

The scattering angle can be found by integrating this equation, using Eq. (4.1.17) in the small angle approximation

$$\int b\ db = \frac{b^2}{2} = -\int \frac{d\sigma}{d\Omega} \theta d\theta = \frac{8G^2m^2}{\theta^2}, \tag{4.1.19}$$

where the integration constant can be set to zero by comparing to the flat space ($m = 0$) case. Physically, we expect the maximum deflection angle θ_D when the photon just grazes the surface of the lens where $b = R_0$ and

$$\theta_{\rm D} = \frac{4Gm}{R_0}. \tag{4.1.20}$$

This is nothing but the classical result for the gravitational light-bending angle that we obtained in Eq. (4.1.10), if we make the natural identification between the mass of the scalar m and the Schwarzschild mass M.

As computations go both of these are relatively simple and there is little need to involve the modern amplitudes program. But what if we want to study more complicated systems. Consider the GR calculation where we had the simplest metric for a massive body in four dimensions if we now want to say make the black hole a spinning one we need to consider the Kerr metric which is much more complicated or we could consider cubic in curvature gravity, as is done in chapter 3, the calculation becomes considerably more difficult and requires the use of computers to make any headway. Taking the QFT route we need not even leave the realm of the Schwarzschild solution in GR to make the calculation more effort than it is worth, all we need to do is replace the photon with a graviton. The three graviton vertex as derived from the action in all its glory is

$$\begin{aligned}
V^{\mu\nu}_{\alpha\beta\gamma\delta}(1^{\pm2}2^{\pm2}) = -\frac{i\kappa}{2}\Bigg(&\mathcal{P}_{\alpha\beta\gamma\delta}\left[{p_1}^{\mu}{p_1}^{\nu} + (p_1-p_2)^{\mu}(p_1-p_2)^{\nu} + {p_2}^{\mu}{p_2}^{\nu} - \frac{3}{2}\eta^{\mu\nu}(p_2)^2\right] \\
&+ 2p_{2\lambda}p_{2\sigma}\left[{I_{\alpha\beta}}^{\sigma\lambda}{I_{\gamma\delta}}^{\mu\nu} + {I_{\gamma\delta}}^{\sigma\lambda}{I_{\alpha\beta}}^{\mu\nu} - {I_{\alpha\beta}}^{\mu\sigma}{I_{\gamma\delta}}^{\nu\lambda} - {I_{\gamma\delta}}^{\mu\sigma}{I_{\alpha\beta}}^{\nu\lambda}\right] \\
&+ \Bigg[p_{2\lambda}{p_2}^{\mu}\left(\eta_{\alpha\beta}{I_{\gamma\delta}}^{\nu\lambda} + \eta_{\gamma\delta}{I_{\alpha\beta}}^{\nu\lambda}\right) + p_{2\lambda}{p_2}^{\nu}\left(\eta_{\alpha\beta}{I_{\gamma\delta}}^{\mu\lambda} + \eta_{\gamma\delta}{I_{\alpha\beta}}^{\mu\lambda}\right) \\
&- (p_2)^2\left(\eta_{\alpha\beta}{I_{\gamma\delta}}^{\mu\nu} - \eta_{\gamma\delta}{I_{\alpha\beta}}^{\mu\nu}\right) - \eta^{\mu\nu}p_{2\sigma}p_{2\lambda}\left(\eta_{\alpha\beta}{I_{\gamma\delta}}^{\sigma\lambda} + \eta_{\gamma\delta}{I_{\alpha\beta}}^{\sigma\lambda}\right)\Bigg] \\
&+ \Bigg[2p_{2\lambda}\big({I_{\alpha\beta}}^{\lambda\sigma}{I_{\gamma\delta\sigma}}^{\nu}(p_1-p_2)^{\mu} + {I_{\alpha\beta}}^{\lambda\sigma}{I_{\gamma\delta\sigma}}^{\mu}(p_1-p_2)^{\nu} - {I_{\gamma\delta}}^{\lambda\sigma}{I_{\alpha\beta\sigma}}^{\nu}{p_1}^{\mu} - {I_{\gamma\delta}}^{\lambda\sigma}{I_{\alpha\beta\sigma}}^{\mu}{p_1}^{\nu}\big) \\
&+ (p_2)^2\left({I_{\alpha\beta\sigma}}^{\mu}{I_{\gamma\delta}}^{\nu\sigma} + {I_{\alpha\beta}}^{\nu\sigma}{I_{\gamma\delta\sigma}}^{\mu}\right) + \eta^{\mu\nu}p_{2\sigma}p_{2\lambda}\left({I_{\alpha\beta}}^{\lambda\rho}{I_{\gamma\delta\rho}}^{\sigma} + {I_{\gamma\delta}}^{\lambda\rho}{I_{\alpha\beta\rho}}^{\sigma}\right)\Bigg] \\
&+ \Bigg\{[(p_1)^2 + (p_1-p_2)^2]\left[{I_{\alpha\beta}}^{\mu\sigma}{I_{\gamma\delta\sigma}}^{\nu} + {I_{\gamma\delta}}^{\mu\sigma}{I_{\alpha\beta\sigma}}^{\nu} - \frac{1}{2}\eta^{\mu\nu}\mathcal{P}_{\alpha\beta\gamma\delta}\right] \\
&- \left({I_{\gamma\delta}}^{\mu\nu}\eta_{\alpha\beta}(p_1)^2 + {I_{\alpha\beta}}^{\mu\nu}\eta_{\gamma\delta}(p_1-p_2)^2\right)\Bigg\}\Bigg),
\end{aligned} \tag{4.1.21}$$

with $\mathcal{P}_{\mu\nu\alpha\beta} = (\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\beta} - \eta_{\mu\nu}\eta_{\alpha\beta})/2$ and $I_{\mu\nu\alpha\beta} = (\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\beta})/2$.

This is a nightmare to compute in the standard QFT sense. So let us replicate these computations in the on-shell formalism.

4.2 Modern Light Bending

To start with we need to construct the required on-shell 3-point amplitudes, these are diagrammatically

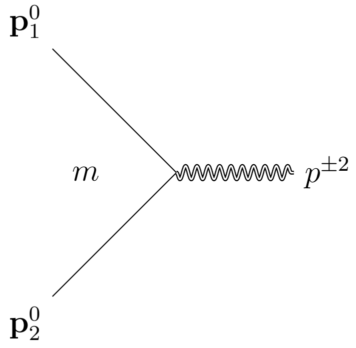


Figure 4.3: Scalar 3-Point

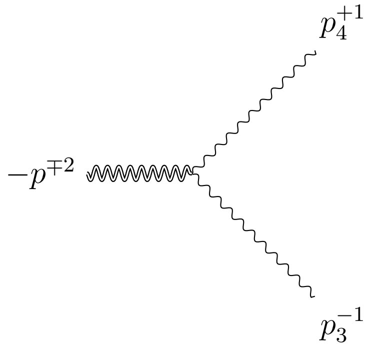


Figure 4.4: Photon 3-Point

Firstly we compute the scalar 3-points. Since we are working in the weak field limit and the transfer momentum is small we can assume that the scalars have the same mass m, hence we have to use (2.4.54) to construct the amplitudes. Since the exchange graviton is massless we can define

$$x_{1p} = \frac{\langle\xi| \, \mathbf{p}_1 |p]}{m\langle\xi p\rangle}, \tag{4.2.22}$$

giving the 3-points as

$$\begin{aligned} M_3(\mathbf{1}^0, \mathbf{2}^0, p^{-2}) &= \kappa m^2 x^{-2} = \kappa \frac{m^4 \langle\xi p\rangle^2}{\langle\xi| \, \mathbf{p}_1 |p]^2} \\ M_3(\mathbf{1}^0, \mathbf{2}^0, p^{+2}) &= \kappa m^2 x^2 = \kappa \frac{\langle\xi| \, \mathbf{p}_1 |p]^2}{\langle\xi p\rangle^2}. \end{aligned} \tag{4.2.23}$$

Next up is the photon 3-point and since all three particles are massless we can just use little group scaling as in section 2.2. Recall as well that the photons have to have opposite helicity, the non-zero amplitudes are

$$\begin{aligned} M_3(3^{-1}, 4^{+1}, -p^{-2}) &= \kappa \frac{\langle 3p\rangle^4}{\langle 34\rangle^2} \\ M_3(3^{-1}, 4^{+1}, -p^{+2}) &= \kappa \frac{[4p]^4}{[34]^2}. \end{aligned} \tag{4.2.24}$$

Since these amplitudes are determined by only dimensional analysis there is the option of having some numerical factor currently absorbed in the coupling. As the amplitude stands above to get the correct observable result the coupling is $\kappa^2 = 8\pi G$. Then very simply the 4-point amplitude is

$$
\begin{aligned}
M_4(\mathbf{1}^0, \mathbf{2}^0, 3^{-1}, 4^{+1}) &= \frac{1}{t}(M_3(\mathbf{1}^0, \mathbf{2}^0, p^{-2})M_3(3^{-1}, 4^{+1}, -p^{+2}) + M_3(\mathbf{1}^0, \mathbf{2}^0, p^{+2})M_3(3^{-1}, 4^{+1}, -p^{-2})) \\
&= \frac{\kappa^2}{t}\left(\frac{m^4\langle\xi p\rangle^2}{\langle\xi|\,\mathbf{p}_1|p]^2}\frac{[2p]^4}{[12]^2} + \frac{\langle\xi|\,\mathbf{p}_1|p]^2}{\langle\xi p\rangle^2}\frac{\langle 1p\rangle^4}{\langle 12\rangle^2}\right).
\end{aligned}
\tag{4.2.25}
$$

To simplify this expression first recognise that we can choose $\xi = 3$, this then allows us to immediately determine that the first term in parentheses is zero by conservation of momentum, $\langle 3p\rangle[p4] = -\langle 33\rangle[34] - \langle 34\rangle[44] = 0$. After some more simplification the amplitude may be written as

$$
M_4(\mathbf{1}^0, \mathbf{2}^0, 3^{-1}, 4^{+1}) = \frac{\kappa^2}{t}\langle 3|\,\mathbf{p}_1|4]^2. \tag{4.2.26}
$$

Now to get the scattering angle from this amplitude we simply use the definition of the scattering cross-section as defined in section 4.1.2 and recalling that $|M_4(\mathbf{1}^0, \mathbf{2}^0, 3^{-1}, 4^{+1})| = |M_4(\mathbf{1}^0, \mathbf{2}^0, 3^{+1}, 4^{-1})|$ hence we can write

$$
\begin{aligned}
\frac{d\sigma}{d\Omega} &= \frac{1}{64\pi^2 s}|M_4(\mathbf{1}^0, \mathbf{2}^0, 3^{-1}, 4^{+1})|^2 \\
&= \frac{2\kappa^4}{64\pi^2 s t^2}(\langle 3|\,\mathbf{p}_1|4]\,\langle 4|\,\mathbf{p}_1|3])^2.
\end{aligned}
\tag{4.2.27}
$$

We can now write the factor in the parentheses as a trace of slashed momenta. We do this by writing all the constituent spinor indices explicitly and rewrite them as bispinors to get

$$
\begin{aligned}
\langle 3|\,\mathbf{p}_1|4]\,\langle 4|\,\mathbf{p}_1|3] &= \mathbf{p}_1^{a\dot{b}}p_{4\dot{b}b}\mathbf{p}_1^{b\dot{a}}p_{3\dot{a}a} \\
&= \mathrm{Tr}(\mathbf{p}_1 p_4 \mathbf{p}_1 p_3) \\
&= 2[(\mathbf{p}_1\cdot p_4)(\mathbf{p}_1\cdot p_3) - (\mathbf{p}_1\cdot\mathbf{p}_1)(p_4\cdot p_3) + (\mathbf{p}_1\cdot p_4)(\mathbf{p}_1\cdot p_3)]
\end{aligned}
\tag{4.2.28}
$$

Using the definitions of the Mandelstam invariants and the fact that $s + t + u = -2m^2$ we get the cross-section as

$$
\frac{d\sigma}{d\Omega} = \frac{\kappa^4}{64\pi^2 s t^2}(su - m^4)^2, \tag{4.2.29}
$$

and given that we use the same approximations made in the QFT calculation we can make the following approximations in the Mandalstam invariants

$$\begin{aligned} t &\simeq \vec{P}^2 = 4E^2\sin^2(\theta/2), \\ u &\simeq m^2 - 2mE - 4E^2\sin^2(\theta/2), \\ s &\simeq (m+E)^2 \simeq m^2 + 2mE. \end{aligned} \tag{4.2.30}$$

Plugging this in and using the fact that $m + 2E \simeq m$ and $\sin^2(\theta/2) \simeq (\frac{\theta}{2})^2$. We can now recall that $\kappa^2 = 8\pi G$ to find that

$$\frac{d\sigma}{d\Omega} = \frac{16G^2m^2}{\theta^4}, \tag{4.2.31}$$

which is exactly the cross-section in the QFT case and will therefore give the same scattering angle when related to the impact parameter.

4.3 Gravitational Wave Bending

Now that we have established the deflection angle for photons we can do the same for massless gravitons in the same background. Quite fortunately we already have all the ingredients necessary to compute the 4-point amplitude. The scalar 3-points are the same as in (4.1.12) and the three graviton amplitude we derived in section 3.1and given here by

$$M(3^{-2}, 4^{+2}, -p^{-2}) = \kappa\frac{\langle 3p\rangle^6}{\langle 4p\rangle^2\langle 43\rangle^2}, \quad M(3^{-2}, 4^{+2}, -p^{+2}) = \kappa\frac{[4p]^6}{[34]^2[3p]^2}. \tag{4.3.32}$$

Therefore, choosing $\xi = 3$ and simplifying we can compute the 4-point amplitude as

$$\begin{aligned} M_4(\mathbf{1}^0, \mathbf{2}^0, 3^{-2}, 4^{+2}) &= \frac{1}{t}(M_3(\mathbf{1}^0, \mathbf{2}^0, p^{-2})M_3(3^{-2}, 4^{+2}, -p^{+2}) + M_3(\mathbf{1}^0, \mathbf{2}^0, p^{+2})M_3(3^{-2}, 4^{+2}, -p^{-2})) \\ &= \frac{\kappa^2}{t}\left(\frac{m^4\langle\xi p\rangle^2}{\langle\xi|\,\mathbf{p}_1|p]^2}\frac{[4p]^6}{[34]^2[3p]^2} + \frac{\langle\xi|\,\mathbf{p}_1|p]^2}{\langle\xi p\rangle^2}\frac{\langle 3p\rangle^6}{\langle 34\rangle^2\langle p4\rangle^2}\right) \\ &= \frac{\kappa^2\,\langle 3|\,\mathbf{p}_1|4]^4}{t(s+m^2)^2}. \end{aligned} \tag{4.3.33}$$

Using the same tricks as in section 4.2 we can write the differential cross-section as

$$\frac{d\sigma}{d\Omega} = \frac{\kappa^4(su - m^4)^4}{64\pi^2 st^2(s+m^2)^2}. \tag{4.3.34}$$

We can now substitute the appropriate Mandelstam approximations and simplify to get

$$\frac{d\sigma}{d\Omega} = \frac{16G^2m^2}{\theta^4}, \tag{4.3.35}$$

which again matches the cross-section of the photon bending interaction. Hence we can see that both photons and gravitons experience the same deflection angle due to a gravitational interaction with a massive body. This is exactly what is expected from GR since both of these particles should follow the null geodesics.

4.4 Comparison to results derived in Chapter 3

Next we can take a quick look at the scattering angle derived in the case of a rotating black hole as in chapter 3

$$\theta = \frac{-GE}{\sinh^2 w} \sum_{\pm} \left(\frac{e^{\pm 2w}}{b \pm a} + 270\pi^2 G^2 \lambda m_B \frac{\beta(w)}{(b \pm a)^6} \right). \tag{4.4.36}$$

Recall that this is a massive scalar probing a rotating and cubic in curvature spacetime and since we want to compare with the results in this chapter we need to take some appropriate limits. First we can get rid of the contributions due to cubic in curvature terms by letting $\lambda \longrightarrow 0$. Next we want the static limit of the rotating black hole so we can let $a \longrightarrow 0$. We are working in natural units, i.e. $c = 1$, so we can rewrite the parts dependent on the rapidity in terms of the relative velocity of the probing particle v giving

$$\theta = \frac{-2GE(1+v^2)}{v^2 b}, \tag{4.4.37}$$

where E is the centre of mass energy of the system. Lastly we want to compare lightlike particles so we let the probe particle have near to zero mass and speed close to 1 meaning that $E \approx M$ the mass of the black hole and we find

$$\theta \approx \frac{-4GM}{b}. \tag{4.4.38}$$

The minus is just due to different conventions in chapter 3, where we considered some of the particles being outgoing. This is a nice representation of the equivalence principle. It does not matter which particle species gravity couples to, the same observable quantity is found. This is exactly what we expect form classical GR since all massless particles should follow null-geodesics. This result is of course limited to the realm of massless gravity. As soon as the graviton is given some mass we find some deviation from this fundamental principle and we explore this in the next chapter.

Chapter 5

Discontinuities Massive Super Gravity

In the previous chapter we made the observation that when computing some observable, in this case the scattering angle, of some particle by a massive source mediated by a massless graviton the particle species does not effect the observable. I.e. gravity couples in the same way to all particles, which is of course just the equivalence principle that lead to GR. But there are several reasons why one might want to endow the graviton with some small mass, one such reason is a credible explanation to the observed late-time acceleration of the Universe without the need to invoke exotic forms of matter and energy. This of course is not without its problems, one of which is a ghost mode, resolved recently by de Rham *et. al.*[65] and another is the discontinuity in the 2-point function of the theory in the massless limit. This second problem will be the focus of this chapter.

This discontinuity found in the massless limit of massive gravity comes down to a noncommutativity of limits. For this, there are two possibilities:

- turning off interactions does not necessarily commute with the massless limit. In other words, in order to resolve the vDVZ discontinuity, it is necessary to go beyond the linearised Fierz-Pauli action, leading to the famed Vainshtein screening mechanism of [66], or
- the massless limit does not commute with the limit of vanishing cosmological constant.

Either case will break Birkhoff's theorem resulting in a van Dam, Veltman and Zakharov (vDVZ) like discontinuity. One can very clearly see this at the level of the action when computing the propagator in the linearised Einstein-Hilbert action, see for example [21] in which there is a nice pedagogical derivation. Another pertinent description can be found in [67] in which if one considers linearised gravity coupled to a conserved stress energy tensor $T_{\mu\nu}$ it leads to the interaction term for massless gravity

$$I_{m=0} \sim \int d^4x d^4x' [T_{\mu\nu} D(x-x') T'^{\mu\nu} - \frac{1}{2} T^{\mu}_{\mu} D(x-x') T'^{\nu}_{\nu}], \tag{5.0.1}$$

and in the case of starting with massive gravity and taking the massless limit

$$I_{m\longrightarrow 0} \sim \int d^4xd^4x'[T_{\mu\nu}D(x-x')T'^{\mu\nu} - \frac{1}{3}T^{\mu}_{\mu}D(x-x')T'^{\nu}_{\nu}]. \tag{5.0.2}$$

Clearly there is a discrepancy in the second term of these interactions given by the different numerical factors. This can be attributed to the fact that the massless graviton famously only has two degrees of freedom, the two tensor modes. But when one endows the graviton with a mass it has five on-shell degrees of freedom, two tensor modes, two vector modes and one scalar mode. These additional modes will of course couple differently to different forms of matter. In [68] some of my colleagues studied this discontinuity in terms of the on-shell scattering amplitudes. They were able to resolve where the discontinuity arises by showing that when a massive graviton couples to a massive scalar, the contribution to the amplitude, of two scalars interacting via a graviton, due to the scalar mode of the graviton coupling to the massive scalar does not vanish in the massless limit of the graviton. As opposed to the additional vector modes completely decoupling and the scalar mode being unable to couple to the photon in a similar amplitude construction, since its energy momentum tensor is of course traceless.

This massive to massless discontinuity of the spin-2 graviton is also shared by a spin-3/2 Rarita-Schwinger field coupled to a conserved vector-spinor current j^{μ}. This is not an unexpected result since, when the spin-3/2 field is coupled to the current, the supermatter interactions resulting from single-fermion interactions have precisely the form required by supersymmetry to complement single-graviton exchange between stress tensor sources [69]. Giving the gravitino mass results in two additional fermionic modes, corresponding to four on-shell degrees of freedom that are grouped by spin as $\{\pm\frac{3}{2}, \pm\frac{1}{2}\}$. These additional fermionic degrees of freedom coupling to the gamma-trace of the current in a different way depending on the current particle content. This is demonstrated in [67] as well by

$$I_{m=0} \sim i\int d^4xd^4x'[\bar{j}_{\mu}D(x-x')j'^{\mu} - \frac{1}{2}\bar{\not{j}}D(x-x')\not{j}'], \tag{5.0.3}$$

and in the case of starting with a massive Rarita-Schwinger field and taking the massless limit

$$I_{m\longrightarrow 0} \sim i\int d^4xd^4x'[\bar{j}_{\mu}D(x-x')j'^{\mu} - \frac{1}{3}\bar{\not{j}}D(x-x')\not{j}']. \tag{5.0.4}$$

Furthermore, we are motivated by the fact that, while experimental evidence so far suggests a massless graviton, it suggests quite the opposite for the Rarita-Schwinger field. Firstly, we would have expected to observe the gravitino should it have been massless, and secondly various supersymmetry breaking mechanisms (such as the super-Higgs effect, or gravitationally induced SUSY breaking) have been shown to endow the Rarita-Schwinger particle with a non-zero mass [70, 71, 72].

In this chapter we approach the spin-3/2 discontinuity from a purely on-shell point of view by constructing the massive amplitudes, taking their massless limits and comparing to the independently constructed massless amplitudes. To be concrete, we consider scattering amplitudes in $\mathcal{N} = 1$ 4D supergravity, whose gauge multiplet consists entirely of a (spin-2) graviton and one Majorana (spin-3/2) spinor gravitino. This can be coupled to matter multiplets that preserve the supersymmetry, specifically an $\mathcal{N} = 1$ vector multiplet consisting of a gauge-boson and gaugino (1,1/2), and an $\mathcal{N} = 1$ chiral multiplet consisting of a spin-1/2 fermion and a complex scalar (1/2,0). As will be shown later this construction requires that the particles in the vector multiplet both be massless and the particles in the scalar multiplet both have the same mass. We also illustrate what happens when one breaks supersymmetry by relaxing this above constraint, i.e. endowing the gluino with some mass but keeping the gauge boson massless. The work in this chapter is based on the article [73].

5.1 Field Theory Analysis

To begin, we should first clarify what is the analog of the vDVZ discontinuity in the supersymmetric context. Toward this end, we will utilise the Stükelberg formalism, appropriately adapted. Consider then, the free massive Rarita-Schwinger action

$$S = -\int d^4x \; e \left(\frac{1}{2} \overline{\Psi}^{\alpha}_{\mu} \gamma^{\mu\rho\nu} \partial_\rho \Psi_{\nu\alpha} - \frac{m}{2} \overline{\Psi}^{\alpha}_{\mu} \gamma^{\mu\nu} \Psi_{\nu\alpha} + \overline{\Psi}^{\alpha}_{\mu} j^{\mu}_{\alpha} \right) \equiv -\int d^4x \; e \; \mathcal{L}, \tag{5.1.5}$$

where α is a spinor index, μ, ν, ρ are Lorentz indices and e is, as usual, the determinant of the frame field $e^a_\mu(x)$. Our γ-conventions read

$$\gamma^{\mu\rho\nu} \equiv i\epsilon^{\mu\sigma\rho\nu} \gamma_5 \gamma_\sigma, \qquad \gamma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{5.1.6}$$

Without the mass term, this action is invariant under

$$\Psi^{\alpha}_{\mu} \longrightarrow \Psi^{\alpha}_{\mu} + \partial_\mu \chi^\alpha, \qquad \overline{\Psi}^{\alpha}_{\mu} \longrightarrow \overline{\Psi}^{\alpha}_{\mu} + \overline{\chi}^{\alpha} \overleftarrow{\partial}_\mu, \tag{5.1.7}$$

for some spinor χ^α. This symmetry is broken by the mass term but can be restored if we introduce the supercovariant derivative

$$D_\mu = \partial_\mu + \frac{1}{2} m \gamma_\mu, \tag{5.1.8}$$

in terms of which the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2} \overline{\Psi}^{\alpha}_{\mu} \gamma^{\mu\rho\nu} D_\rho \Psi_{\nu\alpha} + \overline{\Psi}^{\alpha}_{\mu} j^{\mu}_{\alpha}, \tag{5.1.9}$$

after judicious use of the identity $\epsilon^{\mu\nu\rho\sigma}\gamma_5\gamma_\nu = 2\gamma^\mu\gamma^{\rho\sigma}$. We can now introduce χ as a (spinorial) Stückelberg field via the transformation[1]

$$\Psi^{\alpha}_{\mu} \longrightarrow \Psi^{\alpha}_{\mu} + \frac{1}{\sqrt{3}m} D_\mu \chi^\alpha. \tag{5.1.10}$$

[1]The factor of $\frac{1}{\sqrt{3}m}$ is to ensure a canonical fermionic kinetic term.

Under this transformation, the Lagrangian density varies as

$$\delta\mathcal{L} = -\frac{\sqrt{3}m}{2}\left(\overline{\Psi}^{\alpha}\chi_{\alpha} + \overline{\chi}^{\alpha}\Psi_{\alpha}\right) - \overline{\chi}^{\alpha}(\partial\!\!\!/ + m)\chi_{\alpha} + \frac{1}{\sqrt{6}}\overline{\chi}^{\alpha} j\!\!\!/_{\alpha}\,. \tag{5.1.11}$$

Subsequently, taking the massless limit does not lead to the original massless Lagrangian since,

$$\mathcal{L}^{massive}\Big|_{m\longrightarrow 0} = \mathcal{L}^{massless} + \frac{1}{\sqrt{6}}\overline{\chi}^{\alpha} j\!\!\!/_{\alpha}. \tag{5.1.12}$$

It is this that we identify as the SUSY equivalent of the vDVZ discontinuity, since any matter with a γ-traceless current will couple differently than matter with a non-vanishing γ-trace.

As alluded to in the introduction, we wish to couple to a chiral $(1/2, 0)$ multiplet and a vector multiplet $(1, 1/2)$. The corresponding vector-spinor currents are, by Noethers theorem,

$$j^{\mu}_{\alpha}[\Phi, \psi] = \left[i\gamma^{\nu}\partial_{\nu}(\phi_1 - i\gamma_5\phi_2) - m(\phi_1 + i\gamma_5\phi_2)\right]\gamma^{\mu}\psi_{\alpha} \tag{5.1.13}$$

$$j^{\mu}_{\alpha}[A, \lambda] = \gamma^{\rho}\gamma^{\nu}\gamma^{\mu}\lambda_{\alpha}F_{\rho\nu}, \tag{5.1.14}$$

where $\Phi = \phi_1 + i\phi_2$ is a complex scalar, ψ_{α} a Majorana fermion (both with mass m_{Φ}), λ_{α} a massless photino and $F_{\rho\nu}$ the Maxwell tensor for photon A_{μ}. These are both conserved, i.e. that $\partial_{\mu}j^{\mu}_{\alpha}[\Phi, \psi] = \partial_{\mu}j^{\mu}_{\alpha}[A, \lambda] = 0$, however only one has a non-zero Dirac trace, e.g.

$$j\!\!\!/_{\alpha}[\Phi, \psi] \neq 0, \qquad j\!\!\!/_{\alpha}[A, \lambda] = 0. \tag{5.1.15}$$

While this formulation is off-shell, we will use it to guide our on-shell investigation in the proceeding sections.

5.2 $\mathcal{N} = 1$ Supersymmetric Discontinuity

First we consider the case of the supersymmetric multiplets. This means that we need to ensure that all of the particles contained in the multiplet have the same mass. To this end we will draw on the vector- and scalar-multiplet currents as they are stated in the previous section. Note that many of the techniques and conventions used throughout this chapter will be set up during this first section to ensure that we need not have as much detail in following sections.

5.2.1 Vector Mulitplet

Let us start by computing the amplitudes necessary for the vector multiplet $(1, \frac{1}{2})$. This produces a 3-point vertex consisting of a gauge boson, a fermion and a gravitino interacting with a coupling κ. We require that the gravitino, as the propagator, be massive to study the effects of taking its mass to zero. This means that we want to compute following 4-point diagram

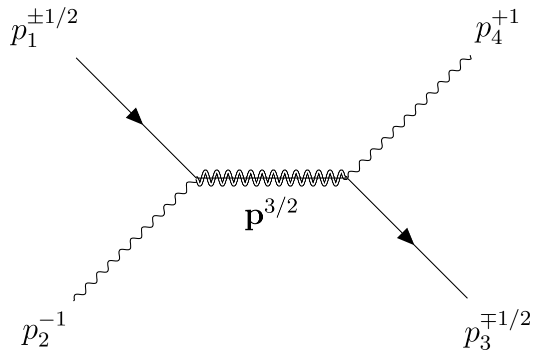


Figure 5.1: Vector multiplet 4-point

We need only construct the left hand 3-point amplitude for all the possible helicity configurations. We can find the right hand 3-point amplitude by complex conjugating since we only consider opposite helicity photons. The amplitude has two massive and one massless particle hence we can use the one massive formula from [9] given in chapter 2 equation (2.4.48). In this case we can write it as

$$M_3^{\{JKL\}}(1^{\pm 1/2}, 2^{-1}, \mathbf{p}^{3/2}) = \tilde{g}\, \langle 1|^{3/2\mp 1/2-1} \langle 2|^{3/2\pm 1/2+1} |\mathbf{p}\rangle^3 \, [12]^{3/2\pm 1/2-1}, \tag{5.2.16}$$

where the spinors are contracted appropriately and the coupling $\tilde{g}$ can have some mass structure and contains a κ since the gravitino needs to couple in the same way as the graviton. This gives us the possible 3-points with all outgoing particles as

$$M_3^{\{JKL\}}(1^{+1/2}, 2^{-1}, \mathbf{p}^{3/2}) = \frac{\kappa}{m^2}\langle 2\mathbf{p}\rangle^3[12] = \kappa\frac{\langle 2\mathbf{p}\rangle^3}{\langle 12\rangle}, \tag{5.2.17}$$

$$M_3^{\{JKL\}}(1^{-1/2}, 2^{-1}, \mathbf{p}^{3/2}) = \frac{\kappa}{m}\langle 1\mathbf{p}\rangle\langle 2\mathbf{p}\rangle^2. \tag{5.2.18}$$

Note that we have used conservation of momentum $\mathbf{p}^2 = -m^2 = \langle 12\rangle[21]$ to simplify (5.2.17). Before we proceed we need to make a few comments on the massless limits of the above 3-points. We first construct the relevant massless amplitudes and since the massive gravitino technically contains both spin-3/2 and spin-1/2 modes the relevant 3-points constructed via little group methods in section 2.2 are

$$\begin{gathered} M_3(1^{+1/2}, 2^{-1}, p^{-3/2}) = \kappa\frac{\langle 2p\rangle^3}{\langle 12\rangle}, \quad M_3(1^{-1/2}, 2^{-1}, p^{-1/2}) = \kappa\langle 12\rangle\langle 2p\rangle \\ M_3(1^{\pm 1/2}, 2^{-1}, p^{+3/2}) = 0, \quad M_3(1^{\pm 1/2}, 2^{-1}, p^{\mp 1/2}) = 0. \end{gathered} \tag{5.2.19}$$

Before we start taking massless limits we have to be clear on an intricacy of the formalism. Since the massive fermion contains both the positive and negative helicity modes of the massless projection of the fermion when we naively unbar the massive spinors we retain both terms which is of course incorrect, only terms with appropriate helicity should survive. A way to designate which term survives is to make a choice for the massive spinor index, for which we have the following results,

$$\begin{aligned} |\mathbf{1}\rangle^1 &= |1\rangle \quad & |\mathbf{1}\rangle^2 &= |\eta_1\rangle \\ |\mathbf{1}]^1 &= -|\eta_1] \quad & |\mathbf{1}]^2 &= -|1]. \end{aligned} \tag{5.2.20}$$

In either case some of the terms, those containing η_1, will die in the massless limit, this can be very clearly seen when considering the high energy limit as is done in [9, 14]. Hence we have the correct result for the chosen helicities in the massless limit. The other option is that when one unbars the massive spinors one just needs to keep the appropriate term based on the helicity needed. Since we know that $|p\rangle$ corresponds to negative helicity and that $|p]$ to positive helicity this greatly simplifies figuring out which one survives.

According to section 2.4 when we take the massless limits we want to be able to just unbold the massive spinors, which is what we will do in most cases, rather than having to choose the modes of the massive particle before doing so. But in general we need to be a bit more careful when taking the massless limit in the 3-point. First recall that if one chooses the massive indices in such a way that $|\eta\rangle$ becomes explicit we can, in the massless limit, rewrite it as $m\,|\tilde{\eta}\rangle$ where $\tilde{\eta}$ is just an arbitrary spinor. This is where the above argument originates from. Now consider equation (5.2.17), if we make any choice resulting in η's the amplitude is zero in the massless limit. Hence the only choice we can make is $IJK = 111$ which is just equivalent to unbolding and getting the $-3/2$ helicity mode of the gravitino. Hence we have that

$$M_3^{\{JKL\}}(1^{+1/2}, 2^{-1}, \mathbf{p}^{3/2})|_{m\longrightarrow 0} = M_3(1^{+1/2}, 2^{-1}, p^{-3/2}), \tag{5.2.21}$$

and all other massless modes of the gravitino in this amplitude zero. Next we consider (5.2.18). Choosing the $-3/2$ helicity mode we have

$$M_3^{111}(1^{-1/2}, 2^{-1}, \mathbf{p}^{3/2}) = \frac{\kappa}{m}\langle 1p\rangle\langle 2p\rangle^2. \tag{5.2.22}$$

Now obviously we cannot naively take the massless limit since we have the mass in denominator so we first have to find a vay to make the mass dependence in the numerator explicit. Therefore we multiply with $[p2]^2/[p2]^2$ and write $\langle 2p\rangle[p2] = \langle 21\rangle 12 + \langle 2\eta\rangle[\eta 2] = -m^2(1 - \langle 2\tilde{\eta}\rangle[\tilde{\eta}2])$. Which now allows us to take the limit $m \longrightarrow 0$ and show that just unbolding the massive spinors is zero. For the $+3/2$ helicity mode $IJK = 222$ we can just immediately show the amplitude is zero in the massless limit by pulling masses from the η's. Similarly for the $+1/2$ helicity mode $IJK = 122$. Lastly we need to consider the $-1/2$ helicity mode $IJK = 112$ which due to the symmetrisation of the indices is

$$\begin{aligned} M_3^{112}(1^{-1/2}, 2^{-1}, \mathbf{p}^{3/2}) &= \frac{\kappa}{3m}(\langle 1\eta\rangle\langle 2p\rangle^2 + 2\langle 1p\rangle\langle 2\eta\rangle\langle 2p\rangle) \\ &= \kappa(\frac{1}{m}\langle 1\eta\rangle\langle 2p\rangle^2 + \frac{2}{3}\langle 12\rangle\langle 2p\rangle), \end{aligned} \tag{5.2.23}$$

where we have used the Schouten identity to get the second line. The second term is now in a form in which we can take the massless limit but for the first term we can rewrite the mass as $m = \langle p\eta\rangle$ and multiplying with $[21]/[21]$ we can write

$$\frac{1}{3m}\langle 1\eta\rangle\langle 2p\rangle^2 = \frac{[21]\langle 1\eta\rangle\langle 2p\rangle^2}{3[21]\langle p\eta\rangle} = \frac{\langle 2p\rangle\langle 2p\rangle[p2]}{3[21]} = -\langle 12\rangle\langle 2p\rangle - m^2\frac{\langle 2p\rangle\langle 2\tilde{\eta}\rangle[\tilde{\eta}2]}{[21]}. \tag{5.2.24}$$

Plugging this back in and taking the massless limit we find that

$$M_3^{112}(1^{-1/2}, 2^{-1}, \mathbf{p}^{3/2})|_{m\longrightarrow 0} = -\frac{\kappa}{3}\langle 1p\rangle\langle 2p\rangle = -\frac{1}{3}M_3(1^{-1/2}, 2^{-1}, p^{-1/2}). \tag{5.2.25}$$

Alternately we can write it in a form in which we can just unbold the massive spinors. We do this by using the massive Weyl equation (2.4.44) and $-m^2 = \langle 12\rangle[21] = -\langle 2\mathbf{p}\rangle[\mathbf{p}2]$ to get

$$\begin{aligned} M_3^{\{JKL\}}(1^{-1/2}, 2^{-1}, \mathbf{p}^{3/2}) &= \frac{\kappa}{m^2}\langle 1|\, p_2|\mathbf{p}]\langle 2\mathbf{p}\rangle^2 \\ &= -\kappa\frac{\langle 12\rangle\langle 2\mathbf{p}\rangle\langle 2\mathbf{p}\rangle[\mathbf{p}2]}{\langle 2\mathbf{p}^I\rangle[\mathbf{p}_I 2]}. \end{aligned} \tag{5.2.26}$$

Now taking the massless limit we can simply unbold to get

$$M_3^{JKL}(1^{-1/2}, 2^{-1}, \mathbf{p}^{3/2})|_{m\longrightarrow 0} = -\alpha\kappa\langle 1p\rangle\langle 2p\rangle = -\alpha M_3(1^{-1/2}, 2^{-1}, p^{-1/2}). \tag{5.2.27}$$

Comparing this to the previous result we know that $\alpha = 1/3$ this is due to the fact that we still need to symmetrise over the massive indices in the numerator but the numerator is symmetric in two indices, hence the factor of $1/3$. A similar process can be followed for other configurations.

This means that (5.2.18) only propagates the spin-$1/2$ mode of the gravitino in the massless limit which holds with the helicity structures of the massless amplitudes (5.2.19). Before we move on to compute the 4-points with a massive propagator let us first do all the relevant 4-points with a massless gravitino. These are

$$\begin{aligned} M_4(1^{+1/2}, 2^{-1}, 3^{-1/2}, 4^{+1})|_{m=0} &= \sum_{\pm} M(1^{+1/2}, 2^{-1}, p^{\pm 3/2})\frac{1}{t}M(\mathbf{3}^{-1/2}, 4^{+1}, -p^{\mp 3/2}) \\ &= \frac{\kappa^2}{t}\frac{\langle 2|\, p|4]^3}{\langle 12\rangle[34]}, \end{aligned} \tag{5.2.28}$$

$$M_4(1^{-1/2},2^{-1},3^{+1/2},4^{+1})|_{m=0} = \sum_{\pm} M(1^{+1/2},2^{-1},p^{\pm 3/2})\frac{1}{t}M(\mathbf{3}^{-1/2},4^{+1},-p^{\mp 3/2}) = 0. \quad (5.2.29)$$

Now that we have something to compare the massless limits to let us compute the 4-points with massive internal gravitino. The first of which is

$$\begin{aligned} M_4(1^{+1/2},2^{-1},3^{-1/2},4^{+1}) &= M^{\{IJK\}}(1^{+1/2},2^{-1},\mathbf{p}^{3/2})\frac{1}{\mathbf{p}^2+m^2}\tilde{M}_{\{IJK\}}(3^{-1/2},4^{+1},-\mathbf{p}^{3/2}) \\ &= \frac{\kappa^2}{\mathbf{t}}\frac{\langle 2|\,\mathbf{p}|4]^3}{\langle 12\rangle[34]}. \end{aligned} \quad (5.2.30)$$

Where we have defined $\mathbf{t} = \mathbf{p}^2 + m^2$. This amplitude has no explicit mass dependence and we can easily take the massless limit in which to see that

$$M_4(1^{+1/2},2^{-1},3^{-1/2},4^{+1})|_{m\longrightarrow 0} = M_4(1^{+1/2},2^{-1},3^{-1/2},4^{+1})|_{m=0}. \quad (5.2.31)$$

For the next amplitude notice that the 3-point amplitudes are symmetric in two of the indices and due to the summation of the internal massive spinors we can write

$$M_3^{\{JKL\}}M_{3\{JKL\}} = \frac{1}{6}M_3^{JKL}M_{3\{JKL\}} = \frac{1}{3}M_3^{JKL}(M_{3JKL}+M_{3KJL}+M_{3LKJ}). \quad (5.2.32)$$

This allows us to write the 4-point amplitude as

$$\begin{aligned} M_4(1^{-1/2},2^{-1},3^{+1/2},4^{+1}) &= M_3^{\{IJK\}}(1^{-1/2},2^{-1},\mathbf{p}^{3/2})\frac{1}{\mathbf{t}}\tilde{M}_{3\{IJK\}}(3^{+1/2},4^{+1},-\mathbf{p}^{3/2}) \\ &= \frac{\kappa^2}{3m^2\mathbf{t}}(\langle 1|\,\mathbf{p}|3]\,\langle 2|\,\mathbf{p}|4]^2 + 2\,\langle 1|\,\mathbf{p}|4]\,\langle 2|\,\mathbf{p}|3]\,\langle 2|\,\mathbf{p}|4]) \\ &= \frac{\kappa^2}{m^2\mathbf{t}}(-\frac{1}{3}m^2\langle 12\rangle[34]\,\langle 2|\,\mathbf{p}|4] + \frac{2}{3}\,\langle 1|\,\mathbf{p}|4]\,\langle 2|\,\mathbf{p}|3]\,\langle 2|\,\mathbf{p}|4]), \end{aligned} \quad (5.2.33)$$

where we have used the Schouten identity to get the last line. In this amplitude the second term needs some work. First we multiply by $[12]^2/[12]^2$ and use conservation of momentum to get

$$-\frac{1}{m^2[12]^2}[21]\,\langle 1|\,\mathbf{p}|4][12]\,\langle 2|\,\mathbf{p}|3] = -\frac{1}{m^2[12]^2}[2|\mathbf{p}\mathbf{p}|4][1|\mathbf{p}\mathbf{p}|3] = -\frac{m^2[24][13]}{[12]^2} \quad (5.2.34)$$

With this done we can take the limit to get

$$M_4(1^{-1/2}, 2^{-1}, 3^{+1/2}, 4^{+1})|_{m\longrightarrow 0} = -\frac{\kappa^2}{3t}\langle 12\rangle[34]\,\langle 2|\,p|4]. \tag{5.2.35}$$

Comparing this to (5.2.29) we have a discontinuity. This seems strange immediately since there appears to be a discontinuity for one specific helicity structure of external particles but not the other. A likely conclusion one can draw from this is that by some mechanism one helicity is chosen in the interaction of the theory, which in simpler terms would equate to a considering a chiral theory. At this point we cannot draw any concrete conclusions without using information outside of the amplitudes formalism. But this is not exactly correct. From the field theory we can see that the vector multiplet is actually chiral and as such picks out a specific helicity for the fermion. In this case the fermion with opposite helicity to the photon which, in the all outgoing set up, is chiral. As such the last 4-point amplitude does not actually contribute to the process and we find that there is no discontinuity in the vector multiplet.

5.2.2 Scalar Mulitplet

In the scalar multiplet we now need to consider the interaction of a massive fermion and a massive scalar, both with the same mass. The corresponding 4-point diagram is

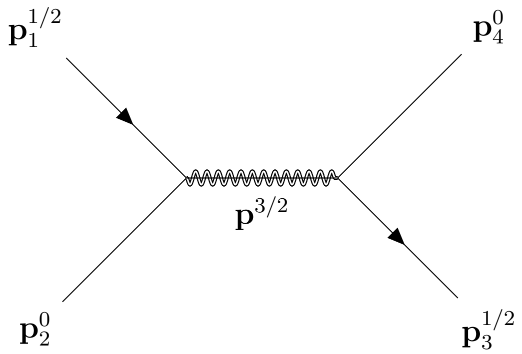


Figure 5.2: Scalar multiplet 4-point

For the scalar multiplet 3-points we need to use the formula from [9] for three massive particles, given in section 2.4.4 by equation (2.4.56), to construct the amplitudes with a massive gravitino. We write them here in the form that will be most convenient and stick to the negative helicities since the others can just be found by complex conjugating. For the left hand 3-point this is given by

$$\begin{aligned} M_3^{I\{JKL\}}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{p}^{3/2}) = g_1\langle\mathbf{1p}\rangle\langle\mathbf{pp}\rangle \\ + g_2(\langle\mathbf{1p}\rangle\,\langle\mathbf{p}|\,\mathbf{p}_1\mathbf{p}\,|\mathbf{p}\rangle + \langle\mathbf{pp}\rangle(\langle\mathbf{1}|\,\mathbf{p}_1\mathbf{p}\,|\mathbf{p}\rangle + \langle\mathbf{p}|\,\mathbf{p}_1\mathbf{p}\,|\mathbf{1}\rangle)) \\ + g_3(\langle\mathbf{1}|\,\mathbf{p}_1\mathbf{p}\,|\mathbf{p}\rangle + \langle\mathbf{p}|\,\mathbf{p}_1\mathbf{p}\,|\mathbf{1}\rangle)\,\langle\mathbf{p}|\,\mathbf{p}_1\mathbf{p}\,|\mathbf{p}\rangle), \end{aligned} \tag{5.2.36}$$

where we have chosen the basis ($\mathcal{O}^{ab} = \mathbf{p}_1^{\{a\dot{b}}\mathbf{p}_{\dot{b}}^{b\}}, \epsilon^{a,b}$) and we the couplings g_i with general mass structure that will be fixed shortly. To simplify this we first note that since we symmetrise over the massive indices of the gravitino any term that has a factor $\langle \mathbf{p}^I \mathbf{p}^J \rangle = m\epsilon^{IJ}$ where IJ are free indices will be zero once symmetrised and can be ignored. Next we have only a few simple structures that this amplitude can be reduced to. For now keeping the masses distinct, the fermion with mass m_f, the scalar with mass m_s and the gravitino with mass m, we get

$$\begin{aligned} \langle \mathbf{p}^I \mathbf{p}^J \rangle &= m\epsilon^{IJ}, \\ \langle \mathbf{1} | \, \mathbf{p}_1 \mathbf{p} \, | \mathbf{p} \rangle &= -m_f m [\mathbf{1p}], \\ \langle \mathbf{p} | \, \mathbf{p}_1 \mathbf{p} \, | \mathbf{1} \rangle &= m m_f [\mathbf{1p}] - (m^2 + m_f^2 - m_s^2) \langle \mathbf{1p} \rangle, \\ \langle \mathbf{p} | \, \mathbf{p}_1 \mathbf{p} \, | \mathbf{p} \rangle &= -m \, \langle \mathbf{p} | \, \mathbf{p}_1 | \mathbf{p}]. \end{aligned} \tag{5.2.37}$$

Putting all this together we have

$$M_3^{I\{JKL\}}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{p}^{3/2}) = m(-g_2 + g_3(m^2 + m_f^2 - m_s^2)) \langle \mathbf{1p} \rangle \, \langle \mathbf{p} | \, \mathbf{p}_1 | \mathbf{p}] + 2m^2 m_f g_3 [\mathbf{1p}] \, \langle \mathbf{p} | \, \mathbf{p}_1 | \mathbf{p}]. \tag{5.2.38}$$

This specific amplitude will be used again later but in the case of supersymmetry we still need to impose that $m_f = m_s = m_1$ resulting the following amplitude

$$M_3^{I\{JKL\}}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{p}^{3/2}) = m(-g_2 + g_3 m^2) \langle \mathbf{1p} \rangle \, \langle \mathbf{p} | \, \mathbf{p}_1 | \mathbf{p}] + 2m^2 m_1 g_3 [\mathbf{1p}] \, \langle \mathbf{p} | \, \mathbf{p}_1 | \mathbf{p}]. \tag{5.2.39}$$

Now the last thing in this amplitude we need to constrain is the structure of the masses in the coupling since there are still two possible masses that can be used to get the correct dimension for the coupling. To do this we will take the various massless limits and compare them to the appropriate independently constructed 3-point amplitudes. To this end let us start by listing all the possible all massless 3-points

$$\begin{aligned} M_3(1^{-1/2}, 2^0, p^{-3/2}) = \kappa \frac{\langle 1p \rangle^2 \langle 2p \rangle}{\langle 12 \rangle}, &\quad M_3(1^{\pm 1/2}, 2^0, p^{\mp 3/2}) = 0, \\ M_3(1^{\pm 1/2}, 2^0, p^{\pm 1/2}) = 0, &\quad M_3(1^{\pm 1/2}, 2^0, p^{\mp 1/2}) = 0. \end{aligned} \tag{5.2.40}$$

Next we need the one massive amplitudes with the gravitino massive and the rest massless, for this we use the formula from [9] given in section 2.4.1 by equation (2.4.48), to get

$$M_3^{\{IJK\}}(1^{-1/2}, 2^0, \mathbf{p}^{3/2}) = \frac{\kappa}{m}\langle 1\mathbf{p}\rangle \langle \mathbf{p}|\, p_1|\mathbf{p}],$$
$$M_3^{\{IJK\}}(1^{+1/2}, 2^0, \mathbf{p}^{3/2}) = \frac{\kappa}{m}[1\mathbf{p}] \langle \mathbf{p}|\, p_1|\mathbf{p}]. \tag{5.2.41}$$

Lastly we need to construct the two massive same mass amplitudes with a massless gravitino which given by equation (2.4.54) in section 2.4.3. Constructing the 3-point amplitudes gets us

$$M_3^I(\mathbf{1}^{1/2}, \mathbf{2}^0, p^{-1/2}) = \kappa m_1 \langle \mathbf{1} p\rangle$$
$$M_3^I(\mathbf{1}^{1/2}, \mathbf{2}^0, p^{-3/2}) = \kappa \frac{\langle \mathbf{1} p\rangle \langle p|\, \mathbf{p}_1|\xi]}{[\xi p]}, \tag{5.2.42}$$

where ξ is just some arbitrary reference vector. Now quite clearly all the massless limits of (5.2.41) and (5.2.42) reduce to the correct all massless amplitudes. What we now have to ensure is the correct mass dependence in (5.2.39) by checking its massless limits. We start by taking $m_1 \longrightarrow 0$ to and assuming the at the limits exist we should get

$$M_3^{I\{JKL\}}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{p}^{3/2})|_{m_1 \longrightarrow 0} \simeq \begin{cases} \frac{\kappa}{m}\langle 1\mathbf{p}\rangle \langle \mathbf{p}|\, p_1|\mathbf{p}], & h_1 = -1/2 \\ \frac{\kappa}{m}[1\mathbf{p}] \langle \mathbf{p}|\, p_1|\mathbf{p}], & h_1 = +1/2. \end{cases} \tag{5.2.43}$$

This implies that $\lim_{m_1 \longrightarrow 0} m(-g_2 + g_3 m^2) \simeq \frac{\kappa}{m}$ and $\lim_{m_1 \longrightarrow 0} 2m^2 m_1 g_3 \simeq \frac{\kappa}{m}$ since the three massive amplitude just exactly reduces to the correct amplitudes when picking the correct helicity structure after unbolding. Next we take the limit $m \longrightarrow 0$ to get

$$M_3^{I\{JKL\}}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{p}^{3/2})|_{m_1 \longrightarrow 0} \simeq \begin{cases} \kappa m_1 \langle \mathbf{1} p\rangle, & h_p = -1/2, \\ \kappa \frac{\langle \mathbf{1} p\rangle \langle p|\mathbf{p}_1|\xi]}{[p\xi]}, & h_p = -3/2, \\ \kappa m_1 [\mathbf{1} p], & h_p = +1/2, \\ \kappa \frac{[\mathbf{1} p]\langle \xi|\mathbf{p}_1|p]}{\langle \xi p\rangle}, & h_p = +3/2 \end{cases} \tag{5.2.44}$$

For this we need to recall that in the two massive case $\langle p|\, \mathbf{p}_1|p] = 0$ by momentum conservation and in the three massive case we have that $\langle \mathbf{p}^I|\, \mathbf{p}_1|\mathbf{p}_I] = m^2$. Considering first the $-3/2$ helicity limit of the gravitino note that if we have a gravitio mass in the denominator of the three massive amplitude and we choose $JKL = 111$ we can write the following

$$
\begin{aligned}
M_3^{I\{111\}}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{p}^{3/2}) &= \kappa\frac{\langle\mathbf{1}p\rangle\langle p|\,\mathbf{p}_1|\eta]}{m}+\kappa\frac{[\mathbf{1}\eta]\langle p|\,\mathbf{p}_1|\eta]}{m}\\
&= \kappa\frac{\langle\mathbf{1}p\rangle\langle p|\,\mathbf{p}_1|\eta]}{[p\eta]}+\kappa\frac{[\mathbf{1}\eta]\langle p|\,\mathbf{p}_1|\eta]}{m}\\
&= \kappa\frac{\langle\mathbf{1}p\rangle\langle p|\,\mathbf{p}_1|\xi]}{[p\xi]}+\kappa m[\mathbf{1}\xi]\langle p|\,\mathbf{p}_1|\xi],
\end{aligned}
\tag{5.2.45}
$$

where in the last line we let $\eta \longrightarrow m\xi$. Now we can take the massless limit to recover the appropriate amplitude. The same process works for the $+3/2$ helicity mode. This implies that $m(-g_2+g_3m^2)\propto\frac{\kappa}{m}$ and $2m^2m_1g_3\propto\frac{\kappa}{m}$. We need not attempt to get any information from the $\pm1/2$ helicity mode of the graviton. From the $m_1\longrightarrow 0$ limit the most general coupling we can write down is $\kappa/(m_1+m)$ but from the latest argument we know there has to be an overall multiplicative $1/m$ meaning the only real choice is κ/m resulting in the amplitude

$$
M_3^{I\{JKL\}}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{p}^{3/2}) = \frac{\kappa}{m}(\langle\mathbf{1p}\rangle+[\mathbf{1p}])\langle\mathbf{p}|\,\mathbf{p}_1|\mathbf{p}]. \tag{5.2.46}
$$

Now that we have the 3-point we can construct the 4-point amplitude with relative ease. Note that the amplitude is not symmetric in any of its indices which means we have six distinct tensor structures to consider in the 4-point. So the 4-point, after some trivial simplification, in all its glory is

$$
\begin{aligned}
M_4^{I_1I_3}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{3}^{1/2},\mathbf{4}^0) &= \frac{M_3^{I_1\{JKL\}}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{p}^{3/2})M_3^{I_3}{}_{\{JKL\}}(\mathbf{3}^{1/2},\mathbf{4}^0,-\mathbf{p}^{3/2})}{\mathbf{p}^2+m^2}\\
&= \frac{\kappa^2}{6\mathbf{t}m^2}\Big[(\langle\mathbf{1}|\,\mathbf{p}|\mathbf{3}]+m\langle\mathbf{13}\rangle)\times\\
&\times(\langle\mathbf{p}^I|\,\mathbf{p}_1|\mathbf{p}^J][\mathbf{p}_J|\mathbf{p}_3\,|\mathbf{p}_I\rangle-\langle\mathbf{p}^I|\,\mathbf{p}_1\mathbf{p}\mathbf{p}_3|\mathbf{p}_I])\\
&-\langle\mathbf{1}|\,\mathbf{p}\mathbf{p}_3\mathbf{p}\mathbf{p}_1\mathbf{p}|\mathbf{3}]\\
&+m(\langle\mathbf{1}|\,\mathbf{p}\mathbf{p}_3\mathbf{p}\mathbf{p}_1\,|\mathbf{3}\rangle-\langle\mathbf{1}|\,\mathbf{p}\mathbf{p}_3\mathbf{p}_1\mathbf{p}\,|\mathbf{3}\rangle-\langle\mathbf{1}|\,\mathbf{p}_3\mathbf{p}\mathbf{p}_1\mathbf{p}\,|\mathbf{3}\rangle)\\
&+m^2(\langle\mathbf{1}|\,\mathbf{p}_3\mathbf{p}_1\mathbf{p}|\mathbf{3}]-\langle\mathbf{1}|\,\mathbf{p}\mathbf{p}_3\mathbf{p}_1|\mathbf{3}]-\langle\mathbf{1}|\,\mathbf{p}_3\mathbf{p}\mathbf{p}_1|\mathbf{3}])\\
&-m^3\langle\mathbf{1}|\,\mathbf{p}_3\mathbf{p}_1\,|\mathbf{3}\rangle+\text{C.C.}\Big].
\end{aligned}
\tag{5.2.47}
$$

The additive complex conjugate term is for all the other terms. Now given much application of the Schouten identity and using the Mandalstam invariants we can write

$$
\begin{aligned}
\langle \mathbf{p}^I|\, \mathbf{p}_1|\mathbf{p}^J][\mathbf{p}_J|\mathbf{p}_3|\mathbf{p}_I] &= -m^2(2\mathbf{p}_1\cdot\mathbf{p}_3)\\
\langle \mathbf{p}^I|\, \mathbf{p}_1\mathbf{p}\mathbf{p}_3|\mathbf{p}_I] &= m^2(2\mathbf{p}_1\cdot\mathbf{p}_3)+(2\mathbf{p}\cdot\mathbf{p}_1)(2\mathbf{p}\cdot\mathbf{p}_3)\\
\langle \mathbf{1}|\,\mathbf{p}\mathbf{p}_3\mathbf{p}\mathbf{p}_1\mathbf{p}|\mathbf{3}] &= m^2\,\langle \mathbf{1}|\,\mathbf{p}\mathbf{p}_3\mathbf{p}_1|\mathbf{3}] + m^2 m_f\langle \mathbf{13}\rangle(2\mathbf{p}\cdot\mathbf{p}_1)+\langle \mathbf{1}|\,\mathbf{p}|\mathbf{3}](2\mathbf{p}\cdot\mathbf{p}_1)(2\mathbf{p}\cdot\mathbf{p}_3)\\
\langle \mathbf{1}|\,\mathbf{p}\mathbf{p}_3\mathbf{p}\mathbf{p}_1\,|\mathbf{3}\rangle &= m^2\,\langle \mathbf{1}|\,\mathbf{p}_3\mathbf{p}_1\,|\mathbf{3}\rangle - m_f\,\langle \mathbf{3}|\,\mathbf{p}|\mathbf{1}](2\mathbf{p}\cdot\mathbf{p}_3)+\langle \mathbf{13}\rangle(2\mathbf{p}\cdot\mathbf{p}_3)^2\\
\langle \mathbf{1}|\,\mathbf{p}\mathbf{p}_3\mathbf{p}_1\mathbf{p}\,|\mathbf{3}\rangle &= m^2\,\langle \mathbf{1}|\,\mathbf{p}_3\mathbf{p}_1\,|\mathbf{3}\rangle - m_f\,\langle \mathbf{1}|\,\mathbf{p}|\mathbf{3}](2\mathbf{p}\cdot\mathbf{p}_1)+\langle \mathbf{13}\rangle(2\mathbf{p}\cdot\mathbf{p}_1)^2 - m_f\,\langle \mathbf{3}|\,\mathbf{p}|\mathbf{1}](2\mathbf{p}.\mathbf{p}_1)\\
\langle \mathbf{1}|\,\mathbf{p}_3\mathbf{p}\mathbf{p}_1\mathbf{p}\,|\mathbf{3}\rangle &= -m^2\,\langle \mathbf{1}|\,\mathbf{p}_3\mathbf{p}_1\,|\mathbf{3}\rangle + m_f\,\langle \mathbf{1}|\,\mathbf{p}|\mathbf{3}](2\mathbf{p}\cdot\mathbf{p}_1)-\langle \mathbf{13}\rangle(2\mathbf{p}\cdot\mathbf{p}_1)^2\\
\langle \mathbf{1}|\,\mathbf{p}_3\mathbf{p}_1\mathbf{p}|\mathbf{3}] &= -m_f^2\,\langle \mathbf{3}|\,\mathbf{p}|\mathbf{1}] - m_f[\mathbf{13}](2\mathbf{p}\cdot\mathbf{p}_3)+\langle \mathbf{1}|\,\mathbf{p}|\mathbf{3}](2\mathbf{p}_1\cdot\mathbf{p}_3)\\
\langle \mathbf{1}|\,\mathbf{p}_3\mathbf{p}\mathbf{p}_1|\mathbf{3}] &= m_f^2\,\langle \mathbf{3}|\,\mathbf{p}|\mathbf{1}] - m_f\langle \mathbf{13}\rangle(2\mathbf{p}\cdot\mathbf{p}_1)-\langle \mathbf{1}|\,\mathbf{p}|\mathbf{3}](2\mathbf{p}_1\cdot\mathbf{p}_3)+m_f[\mathbf{13}](2\mathbf{p}\cdot\mathbf{p}_3)\\
\langle \mathbf{1}|\,\mathbf{p}\mathbf{p}_3\mathbf{p}_1|\mathbf{3}] &= -m_f^2\,\langle \mathbf{3}|\,\mathbf{p}|\mathbf{1}] + m_f\langle \mathbf{13}\rangle(2\mathbf{p}\cdot\mathbf{p}_1)+\langle \mathbf{1}|\,\mathbf{p}|\mathbf{3}](2\mathbf{p}_1\cdot\mathbf{p}_3)
\end{aligned}
\tag{5.2.48}
$$

Putting this all back we can write the amplitude as

$$
\begin{aligned}
M_4^{I_1I_3}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{3}^{1/2},\mathbf{4}^0) = \frac{\kappa^2}{6\mathbf{t}m^2}&(\langle \mathbf{1}|\,\mathbf{p}|\mathbf{3}]-\langle \mathbf{3}|\,\mathbf{p}|\mathbf{1}])(-6m^2(2\mathbf{p}_1\cdot\mathbf{p}_3)+2(2\mathbf{p}\cdot\mathbf{p}_1)^2\\
&+4m^2m_1^2)-4mm_1^2(\langle \mathbf{1}|\,\mathbf{p}|\mathbf{3}]+\langle \mathbf{3}|\,\mathbf{p}|\mathbf{1}])(2\mathbf{p}\cdot\mathbf{p}_1)\\
&+4m(\langle \mathbf{13}\rangle+[\mathbf{13}])(2\mathbf{p}\cdot\mathbf{p}_1)^2)+\mathcal{O}(m).
\end{aligned}
\tag{5.2.49}
$$

Now, using the fact that $2p\cdot p_1 = -2p\cdot p_3 = -m^2$, we take the massless limit to get,

$$
M_4^{I_1I_3}\left[\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{3}^{1/2},\mathbf{4}^0\right]\Big|_{m\longrightarrow 0} = -\frac{\kappa^2}{t}(\langle \mathbf{1}|\,p|\mathbf{3}]-\langle \mathbf{3}|\,p|\mathbf{1}])(2\mathbf{p}_1\cdot\mathbf{p}_3-\frac{2}{3}(m_1^2))\,. \tag{5.2.50}
$$

We now compute the 4-point amplitude with a massless gravitino for which we will use the amplitude (5.2.42) to get

$$
\begin{aligned}
M_4^{3/2I_1I_3}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{3}^{1/2},\mathbf{4}^0)|_{m=0} &= \sum_{\pm 3/2} M_3^{I_1\{JKL\}}(\mathbf{1}^{1/2},\mathbf{2}^0,p^{\pm 3/2})\frac{1}{t}M_3^{I_3}{}_{\{JKL\}}(\mathbf{3}^{1/2},\mathbf{4}^0,-p^{\mp 3/2})\\
&= \frac{\kappa^2}{t}(\langle \mathbf{1}|\,p|\mathbf{3}]\frac{\langle \zeta|\,\mathbf{p}_3 p\mathbf{p}_1|\xi]}{\langle \zeta|\,p|\xi]}-C.C.),
\end{aligned}
\tag{5.2.51}
$$

where ζ and ξ are just reference spinors. To get this in a form that is comparable to (5.2.50) we can choose $\zeta=\xi$ and employing conservation of momentum and the Schouten identity we can write the amplitude as

$$M_4^{3/2 I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m=0} = -\frac{\kappa^2}{t}(\langle \mathbf{1}| p |\mathbf{3}] - \langle \mathbf{3}| p|\mathbf{1}])(2\mathbf{p}_1 \cdot \mathbf{p}_3). \tag{5.2.52}$$

To make a concrete comparison of (5.2.50) with its massless propagator counterparts we also need to compute the four particle amplitude with a massless fermion exchange. This is straightforwardly computed as

$$\begin{aligned} M_4^{1/2 I_1 I_3}[\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0]\Big|_{m=0} &= \sum_{\pm 1/2} \mathcal{M}_3^{I_1\{JKL\}}(\mathbf{1}^{1/2}, \mathbf{2}^0, p^{\pm 1/2})\frac{1}{t}\mathcal{M}_3^{I_3}{}_{\{JKL\}}(\mathbf{3}^{1/2}, \mathbf{4}^0, -p^{\mp 1/2}) \\ &= \frac{\kappa^2}{t}(\langle \mathbf{1}| p|\mathbf{3}] - \langle \mathbf{3}| p|\mathbf{1}])(m_1^2). \end{aligned} \tag{5.2.53}$$

Very clearly we see that

$$M_4^{I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m\longrightarrow 0} = M_4^{3/2 I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m=0} + \frac{2}{3}M_4^{1/2 I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m=0}$$

Hence we have a discontinuity in the scalar sector as expected form the field theory analysis.

In this section we have now shown exactly where the discontinuity arises when the mass of the gravitino in $\mathcal{N} = 1$ supergravity is taken to zero. Crucially the analysis in this section only takes into account when the supergravity couples to supersymmetric matter. It does however require that we take some guidance from the field theory such as realising that the vector multiplet must be chiral. Perhaps further development of the formalism used will allow us to do away with this external information requirement. Another interesting question that arises from the analysis above is whether this discontinuity persists below the supersymmetry breaking scale. Indeed, from the field theory perspective, none of the arguments for the existence of the discontinuity depend in any substantial way on whether the supersymmetry is broken or not, just that a massive spin-3/2 propagator couples to a current with non-vanishing trace. We can test this hypothesis quite easily in the on-shell formalism by considering multiplets with distinct masses, e.g. a massless photon and for the massive scalar and fermion $m_s \neq m_f$. The following sections deal with this.

5.3 Non-SUSY Vector Multiplet Amplitude

Firstly consider the gravitino coupling to the vector multiplet $(1, \frac{1}{2})$. This produces a 3-point vertex consisting of a gauge boson, a fermion and a gravitino interacting with a coupling κ as in the previous section. We require that the gravitino, as the propagator, be massive to study the effects of taking its mass to zero. Now since we are breaking the symmetry of the vector multiplet we endow the fermion with some mass m_f and keep the photon massless. By necessity we choose the helicities of the massless particles and with all external particles outgoing the process is given by

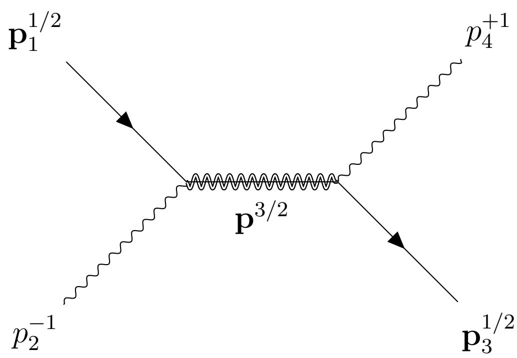


Figure 5.3: Vector multiplet 4-point

5.3.1 3-point Construction

Now that the process with the appropriate particle content it setup we need to construct the relevant 3-point amplitudes. We use the same methadology as in section 5.2.1 but now we are required to use the two massive distinct masses formula found in this work in section 2.4.2 equation (2.4.50). Choosing an appropriate basis $(v, u) = (|2\rangle, \mathbf{p}_1|2])$ we get the 3-point amplitude

$$M^{I\{JKL\}}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{p}^{3/2}) = g_1 m_f [\mathbf{1}2]\langle \mathbf{p}2\rangle^3 - g_2 m \langle \mathbf{1}2\rangle [\mathbf{p}2]\langle \mathbf{p}2\rangle^2. \tag{5.3.54}$$

we can also rewrite the amplitude using momentum conservation which will be useful when we are considering certain massless limits

$$M^{I\{JKL\}}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{p}^{3/2}) = (g_1 + g_2) m m_f [\mathbf{1}\mathbf{p}]\langle \mathbf{p}2\rangle^2 - (g_1 m_f^2 + g_2 m^2)\langle \mathbf{1}\mathbf{p}\rangle\langle \mathbf{p}2\rangle^2. \tag{5.3.55}$$

Now that we have the general 3-point amplitude the next step is to fix the coupling. Currently the couplings g_i are dimensionful and have some form off mass dependence. Since we know that any 3-point amplitude in four dimensions has to have mass dimension 1 we can use this to fix the general mass dependence of the coupling. But currently there is some ambiguity in deciding what information can be used to constrain the functional form of the coupling. See for example [14] in which great care is taken to constrain the form of the coupling for the standard model. There are three general ways in which to constrain the couplings, these are

1. The simplest general polynomial structure of the masses that can be fixed using dimensional analysis.

2. Fixing the mass dependence by requiring that the massless limits of the massive amplitude recover the correct lower mass order or massless amplitudes.

3. Requiring option 2 above but also requiring that the longitudinal modes of massive particles vanish when taking their massless limit at the level of the 3-point.

In this chapter we will focus on the last two methods.

Since the gravitino is the supersymmetric partner of the graviton we know that it has to have coupling κ which has mass dimension -1. From dimensional analysis we see that g_i should have a mass dependence of the order $[f(m, m_f)] = -3$. Let us begin by constructing the necessary amplitudes to which we want to compare the massless limits of the two massive 3-point. Some of these have already been done in section 5.2 in equations (5.2.19),(5.2.17) and (5.2.18). In this section we also require the one massive amplitudes where the fermion is massive, these are

$$\begin{gathered} M_3^I(\mathbf{1}^{1/2}, 2^{-1}, p^{-1/2}) = \kappa\langle\mathbf{1}2\rangle\langle p2\rangle, \quad M_3^I(\mathbf{1}^{1/2}, 2^{-1}, p^{-3/2}) = \frac{\kappa}{m_f}\langle\mathbf{1}p\rangle\langle p2\rangle^2, \\ M_3^I(\mathbf{1}^{1/2}, 2^{-1}, p^{+3/2}) = 0, \quad M_3^I(\mathbf{1}^{1/2}, 2^{-1}, p^{+1/2}) = 0. \end{gathered} \tag{5.3.56}$$

We stick to the left hand 3-point in 5.3 since the right hand 3-point amplitude can just be recovered by complex conjugation.

Let us attempt to recover (5.2.19) from (5.3.56). Starting with the first equation in (5.3.56) it is simple to let $m_f \longrightarrow 0$ since there is no mass dependence in the denominator and we can just unbar the massive spinors, hence

$$M_3^I(\mathbf{1}^{1/2}, 2^{-1}, p^{-1/2})|_{m_f \longrightarrow 0} = \kappa\langle 12\rangle\langle p2\rangle = M_3(1^{-1/2}, 2^{-1}, p^{-1/2}). \tag{5.3.57}$$

Super simple stuff. The next option is more intricate due to the mass in the denominator. So we need to be a bit more careful, first multiply top and bottom with $-m_f^2 = \langle 2p\rangle[p2]$.Then use conservation of momentum to get rid of the m_f in the denominator. So in the numerator we use $\langle\mathbf{1}p\rangle[p2] = \langle\mathbf{1}\mathbf{1}\rangle[\mathbf{1}2] = m_f[\mathbf{1}2]$ and in the denominator $\langle 2p\rangle[p2] = \langle 2\mathbf{1}\rangle[\mathbf{1}2]$. Now this puts the amplitude an a manageable form but there is still one last intricacy that needs to be resolved and will be used though out this paper when taking the massless limit. It stems from the earlier argument of choosing the correct helicity structure when taking the massless limits. First note that the corresponding massless amplitude this should reduce to is dependent only on angle brackets and from three particle special kinematics we can see that the in the massless case $[12] = [1p] = [2p] = 0$. Meaning that in the structure we now have a factor that looks like $\frac{[\mathbf{1}2]}{\langle 2\mathbf{1}\rangle[\mathbf{1}2]}$ in which we cannot cancel the square brackets. But as we take the massless limits $\{[\mathbf{1}2], [\mathbf{1}2]\} \longrightarrow [12]$ and can be cancelled as the limit is taken. Note that since we are only looking for the general mass structure we do not need to consider the symmetrisation of the massive indices as we did in section 5.2.1, these numerical factors will again be taken care of when we construct the 4-point amplitudes. Hence we get

$$\begin{aligned} M_3^I(\mathbf{1}^{1/2},2^{-1},p^{-3/2})|_{m_f\longrightarrow 0} &= -\kappa\frac{\langle 2p\rangle^3[\mathbf{12}]}{\langle\mathbf{21}\rangle[\mathbf{12}]}|_{m_f\longrightarrow 0} \\ &= \kappa\frac{\langle 2p\rangle^3}{\langle 12\rangle} = M_3(1^{+1/2},2^{-1},p^{-3/2}). \end{aligned} \tag{5.3.58}$$

There is no need to check for the other one massive amplitudes since they were covered in section 5.2.1. Now we can start to focus on the only actually relevant amplitude in this section that has ambiguity in the coupling the two massive amplitude (5.3.54) and (5.3.55). To start with let us take $m_f \longrightarrow 0$ to get

$$M^{I\{JKL\}}(\mathbf{1}^{1/2},2^{-1},\mathbf{p}^{3/2})|_{m_f\longrightarrow 0} = \begin{cases} \frac{\kappa}{m}\langle\mathbf{p}1\rangle\langle\mathbf{p}2\rangle^2 & ,h_1=-1/2 \\ \frac{\kappa}{m}[\mathbf{p}1]\langle\mathbf{p}2\rangle^2 & ,h_1=+1/2. \end{cases} \tag{5.3.59}$$

From this we can very clearly see that we want $g_1 \sim \frac{\kappa}{m^2}$ and we want $-(g_1 m_f^2 + g_2 m^2) \sim \frac{\kappa}{m}$. Now let us take $m \longrightarrow 0$ in (5.3.54) that should give

$$M^{I\{JKL\}}(\mathbf{1}^{1/2},2^{-1},\mathbf{p}^{3/2})|_{m\longrightarrow 0} = \begin{cases} \frac{\kappa}{m_f}\langle\mathbf{1}p\rangle\langle p2\rangle^2 & ,h_p=-3/2 \\ \frac{\kappa}{m_f}[\mathbf{1}p]\langle p2\rangle^2 & ,h_p=-1/2. \end{cases} \tag{5.3.60}$$

Therefore $-(g_1 m_f^2 + g_2 m^2) \sim \frac{\kappa}{m_f}$ and we need $(g_1+g_2)mm_f \sim \frac{\kappa}{m_f}$ which is a nice mirror result from the other limit. Given these results the simplest choice of the couplings is

$$g_1 = \frac{\kappa}{m_f(m^2-m_f^2)}, \quad g_2 = -\frac{\kappa}{m(m^2-m_f^2)}, \tag{5.3.61}$$

there fore constraining the couplings in this method gives the 3-point as

$$M^{I\{JKL\}}(\mathbf{1}^{1/2},2^{-1},\mathbf{p}^{3/2}) = \frac{\kappa}{m+m_f}([\mathbf{1p}]\langle\mathbf{p}2\rangle^2 + \langle\mathbf{1p}\rangle\langle\mathbf{p}2\rangle^2). \tag{5.3.62}$$

Where we also stick to the convention that when unbarring the massive spinors in the massless limit to retain the term that produces the correct little group weight for the particle mode in question. This coupling structure reproduces all the possible modes of the gravitino in the massless limit which corresponds to the second option of couplling constrains we mentioned earlier. To ensure that only the transverse modes of the gravitino survive we need to make the following rather painful alteration

$$g_1 = \frac{\kappa(m^2+m_f^2)}{m_f(m^2-m_f^2)(m_f+m)^2}, \quad g_2 = \frac{2\kappa m_f}{(m_f^2-m^2)(m+m_f)^2}, \tag{5.3.63}$$

which in its simplest form produces the amplitude

$$M_3^{I\{JKL\}}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{p}^{3/2}) = \frac{\kappa}{(m+m_f)^2}(m[\mathbf{1p}]\langle\mathbf{p}2\rangle^2 + m_f\langle\mathbf{1p}\rangle\langle\mathbf{p}2\rangle^2). \tag{5.3.64}$$

We next note that the braces on the massive spinor indices indicate that we should consider all permutations of how the indices can be distributed on the spinors. But with the structure of the amplitudes above we can clearly see that it is symmetric in the last two indices, therefore,

$$M^{I\{JKL\}} = \frac{1}{3}\left(M^{IJKL} + M^{IKLJ} + M^{ILJK}\right). \tag{5.3.65}$$

The other tree point in this instance can be constructed in a way similar to above. The result is the complex conjugate of the 3-point above with $1 \to 3, 2 \to 4, p \to -p$ and lowered massive-spinor indices. These are also symmetric in two of the massive indices. Next we move on to construct the 4-point amplitudes for each of the different constraints.

5.3.2 Constructing the 4-points

Now that we have the building blocks we can find the 4-point amplitude by contracting the massive spinor indices across the factorisation pole of the two 3-points using

$$M_4^{I_1I_3}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{3}^{1/2}, 4^{+1}) = M_3^{I_1\{JKL\}}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{p}^{3/2})\frac{1}{\mathbf{p}^2+m^2}\tilde{M}_3^{I_3}{}_{\{JKL\}}(\mathbf{3}^{1/2}, 4^{+1}, -\mathbf{p}^{3/2}). \tag{5.3.66}$$

Substituting the 3-points as above and expanding we see that the expression can be simplified significantly by relabelling some dummy indices and applying some of the simplifications found in 2.4. Before we rush headlong down this road let us first construct the 4-point amplitude with a massless propagator since this is what we want to compare to the massless limits of the other 4-points.

From (5.3.56) we can construct 4-point amplitudes with the same external particle content but starting with the massless propagator. For the spin-3/2 mode of the propagator we get the 4-point as,

$$M_4^{3/2 I_1 I_3}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{3}^{1/2}, 4^{+1})|_{m=0} = \sum_{\pm} M_3^{I_1}(\mathbf{1}^{1/2}, 2^{-1}, p^{\pm 3/2})\frac{1}{t}M_3^{I_3}(\mathbf{3}^{1/2}, 4^{+1}, -p^{\mp 3/2})$$
$$= M^{I_1}(\mathbf{1}^{1/2}, 2^{-1}, p^{-3/2})\frac{1}{t}M^{I_3}(\mathbf{3}^{1/2}, 4^{+1}, -p^{+3/2}) \tag{5.3.67}$$
$$= \frac{\kappa^2}{tm_f^2}\langle\mathbf{1}|\,p|\mathbf{3}]\,\langle 2|\,p|4]^2.$$

Similarly constructing the 4-point with a spin-1/2 mode in the propagator gives,

$$M_4^{1/2 I_1 I_3}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{3}^{1/2}, 4^{+1})|_{m=0} = \sum_{\pm} M_3^{I_1}(\mathbf{1}^{1/2}, 2^{-1}, p^{\pm 1/2})\frac{1}{t}M_3^{I_3}(\mathbf{3}^{1/2}, 4^{+1}, -p^{\mp 1/2})$$
$$= M_3^{I_1}(\mathbf{1}^{1/2}, 2^{-1}, p^{-1/2})\frac{1}{t}M_3^{I_3}(\mathbf{3}^{1/2}, 4^{+1}, -p^{+1/2}) \tag{5.3.68}$$
$$= \frac{\kappa^2}{t}\langle\mathbf{1}2\rangle[4\mathbf{3}]\,\langle 2|\,p|4].$$

As in the supersymmetric case what we are interested in seeing is whether the massless limit of the massive propagator amplitudes we will construct below will propagate only the spin-3/2 mode of the gravitino, i.e. it will match (5.3.67) or will it also propagate a spin-1/2 mode equivalent to (5.3.68). This is the source of the discontinuity. Specifically if it arises in either the vector multiplet or the scalar multiplet but not in both since then all that is required to solve it is a rescaling of the coupling κ.

Since we have the two differently constrained amplitudes we have two possible four points to consider. So lets jump right in and consider the second constraint regime for which we construct the 4-point from (5.3.62) and its complex conjugate. Simplifying using the numerous tricks in the spinor helicity formalism we get

$$M_4{}^{I_1 I_3}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{3}^{1/2}, 4^{+1})$$
$$= \frac{\kappa^2}{3\mathbf{t}(m+m_f)^2}[(\langle\mathbf{1}|\,\mathbf{p}|\mathbf{3}] - \langle\mathbf{3}|\,\mathbf{p}|\mathbf{1}] + m(\langle\mathbf{13}\rangle + [\mathbf{13}]))\,\langle 2|\,\mathbf{p}|4]^2 \tag{5.3.69}$$
$$+ 2(m^2[\mathbf{1}4]\langle 2\mathbf{3}\rangle + m(\langle\mathbf{1}|\,\mathbf{p}|4]\langle 2\mathbf{3}\rangle + [\mathbf{1}4]\,\langle 2|\,\mathbf{p}|\mathbf{3}]) + \langle\mathbf{1}|\,\mathbf{p}|4]\,\langle 2|\,\mathbf{p}|\mathbf{3}])\,\langle 2|\,\mathbf{p}|4]]$$

Now taking the massless limit of the gravitino we get

$$
\begin{aligned}
&M_4^{I_1 I_3}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{3}^{1/2}, 4^{+1})|_{m\longrightarrow 0} \\
&= \frac{\kappa^2}{3tm_f^2}[(\langle\mathbf{1}|\,p|\mathbf{3}] - \langle\mathbf{3}|\,p|\mathbf{1}])\,\langle 2|\,p|4]^2 + 2\,\langle\mathbf{1}|\,p|4]\,\langle 2|\,p|\mathbf{3}]\,\langle 2|\,p|4]] \\
&= \frac{\kappa^2}{tm_f^2}(\langle\mathbf{1}|\,p|\mathbf{3}] - \frac{1}{3}\,\langle\mathbf{3}|\,p|\mathbf{1}])\,\langle 2|\,p|4]^2.
\end{aligned}
\tag{5.3.70}
$$

In this case the discontinuity is manifested by the second term which is due to the spin-1/2 mode of the propagator and matches the corresponding massless gravitino amplitude (5.3.68). This is easily seen by noting that $\langle\mathbf{3}p\rangle[p4] = m_1[\mathbf{3}4]$ and $\langle 2p\rangle[p\mathbf{1}] = -m_1\langle 2\mathbf{1}\rangle$. The factor of a third also hold with literature [67]. We now have only the last constraint regime left, using (5.3.64) we get the following 4-point amplitude

$$
\begin{aligned}
&M_4{}^{I_1 I_3}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{3}^{1/2}, 4^{+1}) \\
&= \frac{\kappa^2}{3\mathbf{t}(m+m_f)^4}[(m_f^2\,\langle\mathbf{1}|\,\mathbf{p}|\mathbf{3}] - m^2\,\langle\mathbf{3}|\,\mathbf{p}|\mathbf{1}] + m^2 m_f(\langle\mathbf{13}\rangle + [\mathbf{13}]))\,\langle 2|\,\mathbf{p}|4]^2 \\
&+ 2(m^4[\mathbf{1}4]\langle 2\mathbf{3}\rangle + m^2 m_f(\langle\mathbf{1}|\,\mathbf{p}|4]\langle 2\mathbf{3}\rangle + [\mathbf{1}4]\,\langle 2|\,\mathbf{p}|\mathbf{3}]) + m_f^2\,\langle\mathbf{1}|\,\mathbf{p}|4]\,\langle 2|\,\mathbf{p}|\mathbf{3}])\,\langle 2|\,\mathbf{p}|4]]
\end{aligned}
\tag{5.3.71}
$$

Again taking the massless limit of the gravitino we get

$$
M_4^{I_1 I_3}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{3}^{1/2}, 4^{+1})|_{m\longrightarrow 0} = \frac{\kappa^2}{tm_f^2}\,\langle\mathbf{1}|\,p|\mathbf{3}]\,\langle 2|\,p|4]^2. \tag{5.3.72}
$$

So this regime does not have a discontinuity in the vector multiplet. Next we follow the same school of thought when computing the scalar multiplet amplitudes.

5.4 Non-SUSY Scalar Multiplet Amplitude

5.4.1 3-points

In this section we follow exactly the same procedure as in the photon-fermion amplitude but now with the photon replaced by a scalar, i.e. the other multiplet in the interaction. We therefore consider the interaction of a fermion p_1, a scalar p_2 and a gravitino p, represented by the diagram

This section follows largely in the same way as the supersymmetric scalar multiplet but there are some distinct differences which at times makes this more tedious but also simpler. To start off with there are some problems when attempting to isolate the different modes of the gravitino. To illustrate this we will first construct the amplitudes in which the gravitino is massless. For this we can use the two massive general formula to construct the amplitude

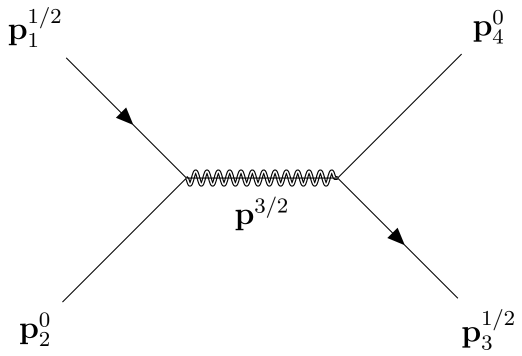


Figure 5.4: Scalar multiplet 4-point

$M_3^I(\mathbf{1}^{1/2}, \mathbf{2}^0, p^{\pm 3/2})$ with the basis $(|p\rangle, |\mathbf{1}\rangle [\mathbf{1}p])$. Unlike the SUSY case this is in fact impossible since the exponents on the basis spinors are $s_1 + s_2 \pm h_p$ is either 2 or -1 which has the correct overall dimension since we only have one external spinor to dot in and as such we will have an uncontracted spinor in the denominator which is not allowed. Therefore this amplitude should be zero

$$M_3^I(\mathbf{1}^{1/2}, \mathbf{2}^0, p^{\pm 3/2}) = 0. \tag{5.4.73}$$

If we consider the spin-1/2 mode of the gravitino $M_3^I(\mathbf{1}^{1/2}, \mathbf{2}^0, p^{\pm 1/2})$ we find that $s_1 + s_2 \pm h_p$ is either 0 or 1 which is allowed and we can construct the amplitude as

$$\begin{aligned} M_3^I(\mathbf{1}^{1/2}, \mathbf{2}^0, p^{-1/2}) &= g\langle \mathbf{1}p\rangle \\ M_3^I(\mathbf{1}^{1/2}, \mathbf{2}^0, p^{+1/2}) &= g\langle \mathbf{12}\rangle[\mathbf{2}p] = gm_f[\mathbf{1}p]. \end{aligned} \tag{5.4.74}$$

Now this seems strange at first glance since we only have the spin-1/2 mode of the gravitino but this will still allow us to study the discontinuity. The other option is to promote the external fermion to a gravitino. This lets the internal gravitino propagate a spin-3/2 mode as well as the spin-1/2 mode. We can then at a later stage project out the spin-1/2 mode of the external gravitino so that it can be compared to an external fermion amplitude. This does make the computation more tedious and does not offer a significant advantage. In the classical analysis of the discontinuity as is done in [67] and showcased at the start of this chapter we can clearly the second term is the bit that produces the discontinuity i.e. the spin-1/2 mode of the gravitino but the first terms are exactly the same meaning that if the spin-3/2 mode vanishes it will do so in both cases. Hence it is fine if they are zero. So let us start by constructing our ladder of amplitudes from massless all the way to all massive using the same methodology as in the vector multiplet. The relevant massless amplitudes are already given in section 5.2.2 in equation (5.2.40), and some of the relevant one massive amplitudes as well. But recall that in the massless amplitude the only non-zero amplitude had the spin-3/2 mode of the gravitino. Again this seems disconcerting at first glance. In these massless amplitudes with κ coupling the scalar multiplet only couples to the spin-3/2 mode of the gravitino and not the spin-1/2 mode which is the direct opposite of the two massive case we just established. This stems from the order in which we take certain particle masses to zero and the fact that angles and squares of massive particles carry both the positive and negative helicity modes of the massless projection. In the spin-2 discontinuity where we want to compare massive scalar amplitudes with massless photon amplitudes coupled to massive and massless spin-2 particles. For spin-2 if the scalar is taken to be massless then no discontinuity appears. We use this as inspiration and will in fact attempt to steer clear of having to take the scalar to be massless. Hence the lowest order of mass amplitude we want to use to constrain anything is the one massive amplitude with the massive scalar. For completeness we will write down the whole array of relevant one massive amplitudes, leaving out those that can be found by complex conjugating or swapping particles.

$$
\begin{aligned}
M_3(1^{\pm 1/2}, \mathbf{2}^0, p^{\mp 1/2}) &= 0, \\
M_3(1^{-1/2}, \mathbf{2}^0, p^{-1/2}) &= \kappa m_s \langle 1p \rangle = \frac{\kappa}{m_s} \langle 1p \rangle \langle p | p_1 | p], \\
M_3^I(\mathbf{1}^{1/2}, 2^0, p^{-1/2}) &= \kappa m_f \langle \mathbf{1}p \rangle = \frac{\kappa}{m_f} \langle \mathbf{1}p \rangle \langle p | \mathbf{p}_1 | p], \\
M_3^I(\mathbf{1}^{1/2}, 2^0, p^{\pm 3/2}) &= 0, \\
M_3^I(1^{-1/2}, 2^0, \mathbf{p}^{1/2}) &= \kappa m \langle 1\mathbf{p} \rangle, \\
M_3^{\{IJK\}}(1^{-1/2}, 2^0, \mathbf{p}^{3/2}) &= \frac{\kappa}{m} \langle 1\mathbf{p} \rangle \langle \mathbf{p} | \mathbf{p}_1 | \mathbf{p}].
\end{aligned}
\tag{5.4.75}
$$

Taking the massless limits we see that there is only one problematic amplitude $M(\mathbf{1}^{1/2}, 2^0, p^{\pm 3/2}) = 0$ that should have one zero result and one non-zero result. The only conclusion we can draw from this is that once the fermion mass is taken to zero it specifically picks out the helicity mode such that it reduces only to the zero result. Next we have to consider the two massive amplitudes again explicitly giving all the possible amplitudes and simplifying as much as possible we get

$$
\begin{aligned}
M_3^I(\mathbf{1}^{1/2}, \mathbf{2}^0, p^{\pm 3/2}) &= 0, \\
M_3^I(\mathbf{1}^{1/2}, \mathbf{2}^0, p^{-1/2}) &= g_1 \langle \mathbf{1}p \rangle \\
M_3^{\{JKL\}}(1^{-1/2}, \mathbf{2}^0, \mathbf{p}^{3/2}) &= g_2 \langle 1\mathbf{p} \rangle \langle \mathbf{p} | p_1 | \mathbf{p}] \\
M_3^{I\{JKL\}}(\mathbf{1}^{1/2}, 2^0, \mathbf{p}^{3/2}) &= g_3 \langle \mathbf{1}\mathbf{p} \rangle \langle \mathbf{p} | \mathbf{p}_1 | \mathbf{p}] + g_4 [\mathbf{1}\mathbf{p}] \langle \mathbf{p} | \mathbf{p}_1 | \mathbf{p}]
\end{aligned}
\tag{5.4.76}
$$

We have used the symmetrisation over the massive indices to kill off any terms with an ϵ. Next we need the three massive amplitude but we have already constructed it in section 5.2.2 equation (5.2.38), which was in its simplest form

$$
M_3^{I\{JKL\}}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{p}^{3/2}) = m(-h_2 + h_3(m^2 + m_f^2 - m_s^2)) \langle \mathbf{1}\mathbf{p} \rangle \langle \mathbf{p} | \mathbf{p}_1 | \mathbf{p}] + 2m^2 m_f h_3 [\mathbf{1}\mathbf{p}] \langle \mathbf{p} | \mathbf{p}_1 | \mathbf{p}].
\tag{5.4.77}
$$

For option two we were required to structure the couplings in such a way that when taking a massless limit we would get the appropriate lower mass order amplitude. To this end let us start with all the nonzero amplitudes in (5.4.76) and take massless limits. In the first instance we have

$$
\begin{aligned}
M_3^I(\mathbf{1}^{1/2},\mathbf{2}^0,p^{-1/2})|_{m_f\longrightarrow 0} &= g_1\langle\mathbf{1}p\rangle|_{m_f\longrightarrow 0} = \kappa m_s\langle 1p\rangle \implies g_1\propto \kappa m_s \\
M_3^I(\mathbf{1}^{1/2},\mathbf{2}^0,p^{-1/2})|_{m_s\longrightarrow 0} &= g_1\langle\mathbf{1}p\rangle|_{m_s\longrightarrow 0} = \kappa m_f\langle\mathbf{1}p\rangle \implies g_1\propto \kappa m_f \\
&\therefore g_1 = \kappa(m_s+m_f)
\end{aligned}
\tag{5.4.78}
$$

Some of the next amplitudes require us to follow the convention we established previously of only retaining certain terms that have the correct little group scaling for the corresponding amplitude once unbolding. We are light on the details here but the reader is free to check. This gives us the following

$$
M_3^{\{JKL\}}(1^{-1/2},\mathbf{2}^0,\mathbf{p}^{3/2})|_{m_s\longrightarrow 0} = \begin{cases}\kappa m\langle 1\mathbf{p}\rangle \\ \frac{\kappa}{m}\langle 1\mathbf{p}\rangle\,\langle\mathbf{p}|\,p_1|\mathbf{p}]\end{cases} \implies g_2\propto -\frac{\kappa}{m}
\tag{5.4.79}
$$

$$
M_3^{\{JKL\}}(1^{-1/2},\mathbf{2}^0,\mathbf{p}^{3/2})|_{m\longrightarrow 0} = \begin{cases}0 \\ \kappa m_s\langle 1\mathbf{p}\rangle\end{cases} \implies g_2\propto -\frac{\kappa}{m_s}
\tag{5.4.80}
$$

Giving us $g_2 = \frac{\kappa}{m-m_s}$. Lastly at this level we have that

$$
M_3^{I\{JKL\}}(\mathbf{1}^{1/2},2^0,\mathbf{p}^{3/2})|_{m\longrightarrow 0} = \begin{cases}0 \\ \kappa m_f\langle\mathbf{1}p\rangle\end{cases}
\tag{5.4.81}
$$

$$
M_3^{I\{JKL\}}(\mathbf{1}^{1/2},2^0,\mathbf{p}^{3/2})|_{m_f\longrightarrow 0} = \begin{cases}\kappa m\langle 1\mathbf{p}\rangle \\ \kappa m[1\mathbf{p}] \\ \frac{\kappa}{m}\langle 1\mathbf{p}\rangle\,\langle\mathbf{p}|\,p_1|\mathbf{p}] \\ \frac{\kappa}{m}[1\mathbf{p}]\,\langle\mathbf{p}|\,p_1|\mathbf{p}].\end{cases}
\tag{5.4.82}
$$

This implies that $g_3 = g_4 = \frac{\kappa}{m+m_f}$. Given these constraints we can now establish what the constraints have to be on (5.4.77). First looking at the positive helicity modes of the gravitino by taking the various limits

$$M_3^{I\{JKL\}}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{p}^{3/2})\begin{cases}\xrightarrow[m_f\longrightarrow 0]{} \frac{\kappa}{m-m_s}[1\mathbf{p}]\langle\mathbf{p}|\,p_1|\mathbf{p}] \\ \xrightarrow[m_s\longrightarrow 0]{} \frac{\kappa}{m+m_f}[\mathbf{1p}]\langle\mathbf{p}|\,\mathbf{p}_1|\mathbf{p}] \\ \xrightarrow[m\longrightarrow 0]{} \frac{\kappa}{m_f-m_s}[\mathbf{1}p]\langle p|\,\mathbf{p}_1|p] \end{cases} \tag{5.4.83}$$

So for h_3 this is a rather trivial with $h_3 = \frac{\kappa}{2m^2 m_f(m+m_f-m_s)}$. We can follow a similar procedure for h_2 to get the amplitude with couplings

$$M_3^{I\{JKL\}}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{p}^{3/2}) = \frac{\kappa}{m+m_f-m_s}(\langle\mathbf{1p}\rangle\langle\mathbf{p}|\,\mathbf{p}_1|\mathbf{p}] + [\mathbf{1p}]\langle\mathbf{p}|\,\mathbf{p}_1|\mathbf{p}]). \tag{5.4.84}$$

Lastly we had the regime in which we required the longitudinal mode of the gravitino to vanish when its mass is taken to zero. Looking at the previous amplitude we see that in fact only the longitudinal mode of the gravitino ever survives when we take its mass to zero. And since we are only interested in taking the gravitino mass to zero in the 4-point amplitude we can make the simplest choice

$$M_3^{I\{JKL\}}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{p}^{3/2}) = 0 \tag{5.4.85}$$

5.4.2 Constructing the 4-points

Now that we have the relevant 3-point amplitudes we can compute the 4-points,

$$M_4^{I_1I_3}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{3}^{3/2},\mathbf{4}^0) = M_3^{I_1\{JKL\}}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{p}^{3/2})\frac{1}{\mathbf{p}^2+m^2}\tilde{M}_3^{I_3}{}_{\{JKL\}}(\mathbf{3}^{1/2},\mathbf{4}^0,-\mathbf{p}^{3/2}) \tag{5.4.86}$$

We first compute the 4-point with a massless gravitino, and for the sake of completeness and comparison we also compute a spin-1/2 mode of the massless propagator.

$$\begin{aligned} M^{3/2I_1I_3}(\mathbf{1}^{1/2},\mathbf{2}^0,\mathbf{3}^{1/2},\mathbf{4}^0)|_{m=0} &= \sum_{\pm} M^{I_1}(\mathbf{1}^{1/2},\mathbf{2}^0,p^{\pm 3/2})\frac{1}{t}M^{I_3}(\mathbf{3}^{1/2},\mathbf{4}^0,-p^{\mp 3/2}) \\ &= 0, \end{aligned} \tag{5.4.87}$$

$$M^{1/2 I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m=0} = \sum_{\pm} M^{I_1}(\mathbf{1}^{1/2}, \mathbf{2}^0, p^{\pm 1/2}) \frac{1}{t} M^{I_3}(\mathbf{3}^{1/2}, \mathbf{4}^0, -p^{\mp 1/2})$$
$$= \frac{\kappa^2 (m_f + m_s)^2}{t} (\langle \mathbf{1}| p |\mathbf{3}] - \langle \mathbf{3}| p |\mathbf{1}]). \quad (5.4.88)$$

From this we see that if there is no discontinuity in this sector the massless limit of the 3-point must be zero. If we in turn see some term that looks like (5.4.88) a discontinuity exists. The only thing in this section that is different to the supersymmetric case when constructing the all massive 4-point amplitude is the coupling. And due to the structure of the coupling we can just trivially take the massless limit in the 4-point. So we can look again at the 4-point amplitude (5.2.47) with the new appropriate coupling from (5.4.84). There are some additional changes that are required. First note that previously me made extensive use of the fact that $\langle \mathbf{1p} \rangle [\mathbf{p1}] = m^2$ if the fermion and the scalar have the same mass. If they have distinct mass we find that $\langle \mathbf{1p} \rangle [\mathbf{p1}] = m^2 + m_f^2 - m_s^2$. This leads to

$$\begin{aligned} \langle \mathbf{p}^I | \, \mathbf{p}_1 | \mathbf{p}^J] [\mathbf{p}_J | \mathbf{p}_3 | \mathbf{p}_I] &= m^2 (s + 2m_1^2) \\ \langle \mathbf{p}^I | \, \mathbf{p}_1 \mathbf{p} \mathbf{p}_3 | \mathbf{p}_I] &= m^2 (s + 2m_1^2) - (m^2 + m_f^2 - m_s^2)^2, \end{aligned} \quad (5.4.89)$$

which allows us to write the amplitude as

$$M_4^{I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0) = \frac{\kappa^2}{3\mathbf{t}(m + m_f - m_s)^2} (((\langle \mathbf{1}| \, \mathbf{p} |\mathbf{3}] - \langle \mathbf{3}| \, \mathbf{p} |\mathbf{1}])(m_f^2 - m_s^2)^2 + \mathcal{O}(m)). \quad (5.4.90)$$

Now taking the massless limit and noting that in this case $- \langle p | \, \mathbf{p}_1 | p] = \langle p | \, \mathbf{p}_3 | p] = m_f^2 - m_s^2$ we find that

$$M_4^{I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m \longrightarrow 0} = \frac{\kappa^2 (m_f + m_s)^2}{3t} (\langle \mathbf{1}| \, p |\mathbf{3}] - \langle \mathbf{3}| \, p |\mathbf{1}]). \quad (5.4.91)$$

From this we can see the discontinuity in the scalar sector as, i.e

$$M_4^{I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m \longrightarrow 0} = M_4^{3/2 I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m=0} + \frac{1}{3} M_4^{1/2 I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m=0}$$

which differs from the supersymmetric case by a factor of two. For the coupling constraints that the longitudinal modes of the gravitino vanishes the 3-point (5.4.85) is just zero and so without further ado

$$M_4{}^{I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{3/2}, \mathbf{4}^0)|_{m \longrightarrow 0} = 0, \tag{5.4.92}$$

and hopefully you can see there is no discontinuity.

5.5 Realization of the Discontinuity

We have found the discontinuity in a similar way as in the spin-2 propagator scenario, but with several qualitative differences. One such difference is that in the vector multiplet the amplitudes of the spin-3/2 and spin-1/2 modes of the propagator have different structures. In the cross-section this is not the case but it is not apparent at the level of the amplitude. Another is the discontinuity arises only in the case where the gravitino couples to chiral supersymmetric matter.

First let us revisit the supersymmetric matter analysis in section (5.2). In the scalar multiplet the discontinuity arises neatly in the form

$$M_4^{I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m \longrightarrow 0} = M_4^{3/2 I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m=0} + \frac{2}{3} M_4^{1/2 I_1 I_3}(\mathbf{1}^{1/2}, \mathbf{2}^0, \mathbf{3}^{1/2}, \mathbf{4}^0)|_{m=0},$$

which matches exactly what we would expect from the field theory. The vector multiplet on the other hand is split into two parts: the chiral and non-chiral part. Since we know from the field theory that the multiplet has to be chiral we need only consider the chiral amplitude which in the massless limit is

$$M_4(1^{+1/2}, 2^{-1}, 3^{-1/2}, 4^{+1})|_{m \longrightarrow 0} = M_4^{3/2}(1^{+1/2}, 2^{-1}, 3^{-1/2}, 4^{+1})|_{m=0}.$$

Hence we see that the vDVZ discontinuity arises exactly in the same form as the field theory even mirroring the Stückelberg decomposition. We also computed the non-chiral vector multiplet which in the massless limit yields

$$M_4(1^{-1/2}, 2^{-1}, 3^{+1/2}, 4^{+1})|_{m \longrightarrow 0} = \frac{1}{3} M_4^{1/2}(1^{-1/2}, 2^{-1}, 3^{+1/2}, 4^{+1})|_{m=0}.$$

This does give rise to a discontinuity in the vector multiplet but only the non-chiral case and as such is non-physical. We can draw from this that this construction of the amplitudes acts like the on-shell avatar of the Stückelberg expansion by isolating the different modes of the gravitino to different terms or even amplitudes. This effect is even more pronounced once we break supersymmetry.

As we launch into analysing the results of the amplitudes below the supersymmetry breaking scale in sections (5.3) and (5.4) we first address some inconsistencies in taking the limits of the various three particle amplitudes. As we established when fixing the mass dependence of the couplings in the above sections the current formalism has a discrepancy. Specifically we find for example the one-massive amplitude $M_3^I(\mathbf{1}^{1/2}, 2^0, p^{\pm 3/2}) = 0$. We also have the following all massless amplitudes $M_3(1^{-1/2}, 2^0, p^{-3/2}) = \kappa\frac{\langle 1p\rangle^2\langle 2p\rangle}{\langle 12\rangle}$ and $M_3(1^{\pm 1/2}, 2^0, p^{\mp 3/2}) = 0$, both of which should be recoverable from the one-massive amplitude by taking the massless limit. Clearly the first of the massless amplitudes is non-zero and as such cannot be recovered from the one-massive amplitude. We argue that when taking the massless limit there is some mechanism that fixes the helicity of the massive fermion such that only the zero massless amplitude is recovered. There is of course another option. One is to require that since the one-massive amplitude above has no consistent massless limit it cannot form part of the tower of massive amplitudes and can not be used to constrain the mass dependence of two- or three-massive amplitudes. If this school of thought is followed throughout the four-particle amplitudes, in which supersymmetry has been broken, all have order $\mathcal{O}(m^{-1})$ terms and hence no consistent massless limit for the gravitino. This discrepancy requires further study to resolve whether it is some particular mechanism that restricts the massless limits, a problem with the formalism or that these amplitudes do not exist one supersymmetry is broken.

Following the argument we made in the previous sections we found in the vector case

$$M_4^{I_1I_3}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{3}^{1/2}, 4^{+1})|_{m\longrightarrow 0} = M_4^{3/2I_1I_3}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{3}^{1/2}, 4^{+1})|_{m=0}+\frac{1}{3}M_4^{1/2I_1I_3}(\mathbf{1}^{1/2}, 2^{-1}, \mathbf{3}^{1/2}, 4^{+1})|_{m=0}$$

and in the scalar case

$$M_4^{I_1I_3}(\mathbf{1}^{1/2}, \mathbf{2}^{0}, \mathbf{3}^{1/2}, \mathbf{4}^{0})|_{m\longrightarrow 0} = M_4^{3/2I_1I_3}(\mathbf{1}^{1/2}, \mathbf{2}^{0}, \mathbf{3}^{1/2}, \mathbf{4}^{0})|_{m=0}+\frac{1}{3}M_4^{1/2I_1I_3}(\mathbf{1}^{1/2}, \mathbf{2}^{0}, \mathbf{3}^{1/2}, \mathbf{4}^{0})|_{m=0}.$$

From this we see even though the spin-1/2 mode of the gravitino couples to the external particles and that this contribution to the amplitude does not vanish in the massless limit it enters in the same way in both the scalar and vector cases. Thus there is a discontinuity between the massless and massless limit propagators. Since this additional contribution enters in the same way for all matter it can be fixed by rescaling the coupling, unlike the vDVZ discontinuity. Lastly we computed the amplitudes in the regime where we constrain the mass dependence of the couplings by only allowing the transverse modes to survive in the massless limit at the level of the three particle amplitudes. Naturally as one might expect this leads to no contributions due to the spin-1/2 mode of the gravitino to the four particle amplitudes in the massless limit.

Chapter 6

Conclusion

In the course of this work we have utilised some of the state of the art amplitudes technology to study various aspects of gravitation. As an example of this technological advancement, in chapter 4 we showed how the deflection angle of gravitational waves passing by a massive body may be computed with minimal effort, once the corresponding computation is known for light. Since the presentation of this result and other relevant articles efforts in the area of gravitational wave calculation has shifted from merely tracking their path through space to calculating the signals produced in black hole mergers. The state of the art approach is to relate amplitudes to the *effective one-body theory* [34]. This is the natural continuation of this work.

In chapter 3 we used the leading singularity method to compute the classical corrections of a rotating black hole in Einsteinian Cubic Gravity. As stated in that chapter the equations of motion for this theory of gravity are too complicated to obtain any form of analytic solution. Indeed, prior to our construction, the only known solutions were numerical [53]. From our solution we're also able to compute the contribution to the deflection angle of light due to the additional terms which could then be used to constrain the parameters of the theory.

Finally in chapter 5 we were able to get a clear picture on where the discontinuity in the massive Rarita-Schwinger field comes from. We also investigate the breaking of supersymmetry by giving some of the particles in the multiplet that the gravitino couples to a mass and exploring how this effects the discontinuity. We discover that in certain cases the amplitude mirrors the Stückelberg decomposition of the field theory. If it is indeed the case that the amplitude mirrors the Stückelberg decomposition it may offer valuable insight into the structure of amplitudes in general. We find that the vDVZ discontinuity manifests quite clearly when the spin-3/2 gravitino is coupled to supersymmetric matter. Once supersymmetry is broken we find that the gravitino couples to all matter in the same way i.e. even though there is a discontinuity it is not the vDVZ discontinuity and is resolvable by scaling the coupling. We also find irregularities when constraining the mass dependence of massive three particle amplitudes. This provides a compelling direction in which to further the on-shell amplitude formalism.

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