Modelling Stochastic Multi-Curve Basis

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A dissertation submitted to the Faculty of Commerce, University of Cape Town, in partial fulfilment of the requirements for the degree of Master of Philosophy.

September 2, 2017

MPhil in Mathematical Finance,
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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy in the University of the Cape Town. It has not been submitted before for any degree or examination in any other University.

Signed by candidate

September 2, 2017

Abstract

As a consequence of the 2007 financial crisis, the market has shifted towards a multi-curve approach in modelling the prevailing interest rate environment. Currently, there is a reliance on the assumption of deterministic- or constant-basis spreads. This assumption is too simplistic to describe the modern multi-curve environment and serves as the motivation for this work. A stochastic-basis framework, presented by Mercurio and Xie (2012), with one- and two-factor OIS short-rate models is reviewed and implemented in order to analyse the effect of the inclusion of stochastic-basis in the pricing of interest rate derivatives. In order to preclude the existence of negative spreads in the model, a constraint on the spread model parameters is necessary. The inclusion of stochastic-basis results in a clear shift in the terminal distributions of FRA and swap rates. In spite of this, stochastic-basis is found to have a negligible effect on cap/floor and swaption prices for the admissible spread model parameters. To overcome challenges surrounding parameter estimation under the framework, a rudimentary calibration procedure is developed, where the spread model parameters are estimated from historical data; and the OIS rate model parameters are calibrated to a market swaption volatility surface.

Acknowledgements

I would like to thank my supervisors, Jörg Kienitz and Thomas McWalter for their guidance, insights and expertise. Additionally, I would like to thank David Taylor and AIFMRM for affording me the opportunity to pursue this rewarding degree. Finally, I would to thank my parents and family for their unconditional support throughout my academic career.

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Chapter 1

Introduction

Prior to the financial crisis of 2007, interbank rates such as LIBOR were considered 'risk-free' since the credit and liquidity risks of large commercial banks were assumed to be non-existent. Consequently, the behaviour of interest rate markets was consistent with explanations provided by textbooks. A typical example is that interbank deposit rates were consistent with those implied from Overnight Index Swap (OIS) rates or that interest rate swap rates were considered independent of the tenor of the underlying floating rate (provided the payment schedule was changed).

In reality there were slight differences. For example non-zero spreads existed on money market basis swaps (interest rate swaps where both legs are floating) as well as between interbank deposit and market OIS rates (Chibane *et al.*, 2009). However, these spreads were minimal (only a few basis points) and thus assumed to be negligible. This can be seen in Figure 1.1 which plots the spreads between interbank deposit rates and those implied from OIS rates — EURIBOR and EONIA in the European context. Prior to August 2007 the spreads were less than 10bps, not very volatile and appeared to be independent of tenor.

The events of 2007 shattered these previously held assumptions as the paradigm that large banks cannot go bankrupt was violated. There was a large divergence in spreads which can be clearly seen in Figure 1.1. This divergence can be explained by noting that OIS contracts have inherently less credit risk than interbank deposits. In the case of OIS, notionals are not exchanged and the overnight credit quality of a counter-party is a lot more certain than for longer periods. Clearly when the credit risk of banks was assumed to be non-existent, interbank deposit and OIS rates had a negligible spread. However, when this assumption disappeared the rates began to differ significantly as credit and liquidity risks associated with this notional amount were now being priced into the interbank rates.

These non-negligible money market basis spreads violate the existence of a unique, well-defined zero-coupon curve and with it, the classic approach to interest rate derivative pricing. Although the widening of the basis was caused by credit and liquidity effects, explicitly modelling these effects at a market level when pricing interest rate derivatives would be extremely complex. Instead, financial institutions have settled on a more empirical multi-curve approach. Distinct forward curves are constructed for each of the common underlying tenors: 3-month, 6-month, 12-month etc. (Mercurio, 2010). The relevant forward curve is then used to forecast future rates while a separate curve is used for discounting.

A large body of literature has focused on the development of this multi-curve environment with Henrard (2007b and 2010b), Chibane *et al.* (2009), Bianchetti (2010) and Kenyon (2010) extending single curve bootstrapping to the multi-curve setting. The use of these multi-curve models has allowed the market to settle on new valuation formulas for vanilla interest rate derivatives. However, significant shortcomings still exist when it comes to the pricing of slightly more complex derivatives where the evolution of multiple curves is required. Currently, there is a reliance on the assumption of deterministic basis spreads, where the evolution of a reference curve is modelled, and all other curves are evolved by adding a deterministic or even constant basis spread to this reference curve (Mercurio and Xie, 2012). Looking at Figure 1.1 it can be seen that the spread is neither deterministic nor constant and assuming these is rudimentary and often inadequate. For example, a LIBOR-OIS swaption derives its value from the uncertainty of the LIBOR-OIS basis spread and thus cannot be priced under these assumptions.

In the post-financial crisis interest rate market, various non-vanilla interest rate derivatives such as money market and cross currency basis swaps and swaptions have gained a lot of relevance due to the aforementioned changes in market behaviour. Consequently, accounting for the stochastic nature of basis spreads is crucial for developing a more rigorous framework for the modelling of these, and other interest rate derivatives, in the modern multi-curve environment. It is these issues that have motivated this work which aims to analyse the effect of the inclusion of stochastic-basis spreads in the pricing of interest rate derivatives through reviewing and implementing a framework proposed by Mercurio and Xie (2012).

This dissertation begins with a brief review of the current approaches to the modelling of stochastic LIBOR-OIS spreads before the Mercurio and Xie (2012) stochastic-basis framework is presented. The pricing of interest rate derivatives in a general multi-curve environment as well as under the Mercurio and Xie (2012) framework is thoroughly investigated before a review of this framework is presented. Subsequently, the effect of stochastic-basis on the pricing of various interest rate derivatives such as FRAs, swaps, caps/floors as well as swaptions is analysed before investigating parameter estimation under the framework. Finally conclusions and recommendations are given.

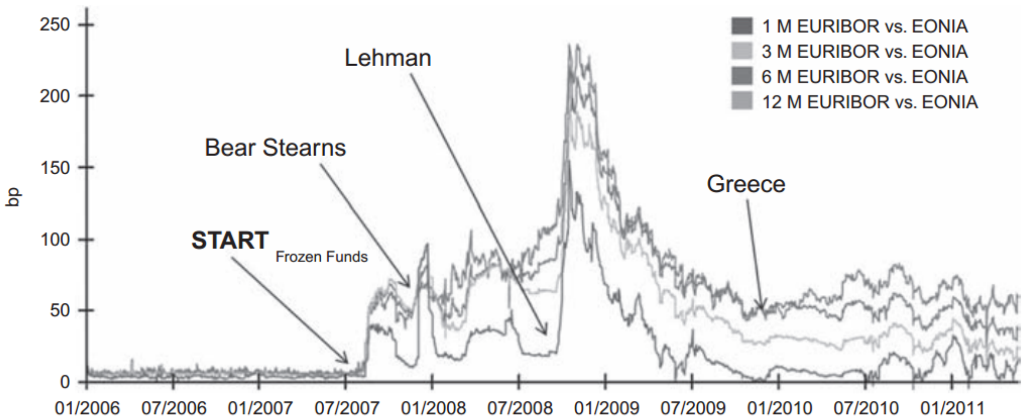


Fig. 1.1: Money market basis spreads (Kienitz, 2014)

Chapter 2

Multi-curve Interest Rate Modelling

2.1 Assumptions and Notation

In order to model the multi-curve interest rate environment, one has to consider the evolution of the discounting curve as well as the various tenored interbank rate forecasting curves.

The existence of distinct forecasting and discounting curves each indexed according to the common market tenors $x = 1m,\ 3m,\ 6m, \ldots$ is assumed. The OIS zero-coupon curve, or market equivalent, is deemed the most suitable proxy for a risk-free curve, and is thus used for discounting (Amin, 2010). Conversely, forward LIBOR curves, or market equivalents, are used for forecasting.

The first important distinction to make is that between OIS forward rates and the LIBOR forward rates. For a particular tenor x and associated time structure $\mathcal{T}^x = \{0 < T_0^x, \ldots, T_{M_x}^x\}$, where $T_k^x - T_{k-1}^x = x$, the OIS discount factor at time t for maturity T_k^x is denoted by $P_D(t, T_k^x)$ and the OIS forward rate can be defined using a single-curve methodology to be

$$F_k^x(t) := F_D(t; T_{k-1}^x, T_k^x) = \frac{1}{\tau_k^x}\left[\frac{P_D(t, T_{k-1}^x)}{P_D(t, T_k^x)} - 1\right], \tag{2.1}$$

for $k = 1, \ldots, M_x$ where τ_k^x is the year fraction between T_{k-1}^x and T_k^x.

Following Mercurio (2010) and Mercurio and Xie (2012), the pricing measures considered are those associated with the OIS curve. We let $Q_D^{T_k^x}$ denote the T_k^x-forward measure whose associated numeraire is the zero-coupon discount bond — $P_D(t, T_k^x)$. While the expectation under this measure is denoted by $\mathbb{E}_D^{T_k^x}$.

The forward LIBOR rate at any time t for the interval $[T_{k-1}^x, T_k^x]$ is defined as the expected future spot LIBOR rate under this measure

$$L_k^x(t) := \mathbb{E}_D^{T_k^x}\left[L^x(T_{k-1}^x, T_k^x)|\mathcal{F}_t\right], \tag{2.2}$$

where $L^x(T^x_{k-1}, T^x_k)$ is the LIBOR rate that is set at T^x_{k-1} with maturity T^x_k.

$L^x_k(t)$ can also be considered the fixed rate in a swap to be exchanged for the floating rate of $L^x(T^x_{k-1}, T^x_k)$ at T^x_k which gives the swap a value of zero at time t. Importantly, it can be easily be shown that $L^x_k(t)$ is a $Q_D^{T^x_k}$ martingale.

In addition, we consider the multi-curve basis spreads. We denote this spread by S^x_k which is defined between the corresponding LIBOR forward rate, L^x_k, and OIS forward rate, F^x_k. It is noted that the actual definition will be further discussed below.

As a result, there are in essence three processes for which the dynamics need to be determined in order to price interest rate derivatives — the OIS forward rates (F^x_k), the LIBOR forward rates (L^x_k) as well as the corresponding LIBOR-OIS spreads (S^x_k). Clearly, the modelling of two of the three processes yields the dynamics of the third, since the definition of the spread will always be a function of L^x_k and F^x_k (the actual definition of S^x_k will be further discussed below). It is this degree of freedom that has governed the classification of literature into implicit and explicit basis spread modelling.

2.2 Implicit Basis Spread Modelling

In the implicit case as seen in Mercurio (2009 and 2010), the joint evolution of L^x_k and F^x_k is modelled in a LIBOR market model framework resulting in S^x_k being implicitly modelled through its definition. Fujii *et al.* (2011) as well as Moreni and Pallavicini (2014) follow a similar approach in an HJM framework. Mercurio (2010) aptly suggests that this allows for the simple extension of single-curve caplet and swaption pricing formulas to those under the multi-curve environment.

That being said, this method does not necessarily guarantee the preservation of the positive nature of the implied basis spreads. This could result in non-realistic behaviour since the credit risk associated with deposit rates implied from OIS rates should always be less than that associated with LIBOR deposits. As a result, new classes of these models have recently gained popularity in the literature to overcome these issues. Nguyen and Seifried (2015) use a multi-currency analogy to model OIS and LIBOR curves with the relevant pricing-kernel processes while Grbac *et al.* (2015) and Cuchiero *et al.* (2016) use a framework of affine LIBOR models to ensure positive and stochastic spreads.

2.3 Explicit Basis Spread Modelling

In the explicit case as seen in Mercurio and Xie (2012), Henrard (2013) as well as Morino and Runggaldier (2014) the evolution of the OIS forward rates as well as the basis spreads are modelled. This allows for the implicit modelling of the LIBOR rates using the spread definition. Mercurio (2010) pertinently indicates that this could be considered more realistic since it is analogous to current market practice where LIBOR forward curves are built at a spread over the OIS curve. This overcomes, to some extent, the problems associated with the earlier implicit basis spread models since S_k^x can be directly modelled by a positive-valued stochastic process ensuring that its sign behaviour is in agreement with historical data and expected future behaviour. However, closed form solutions for interest rate derivatives such as caps, floors and swaptions may not necessarily exist.

In reviewing the modelling framework presented by Mercurio and Xie (2012) this work will focus on explicit basis spread modelling. That being said, this work does not look to invalidate implicit basis spread modelling — particularly the recent approaches.

The literature on explicit basis spread modelling can be divided based on the definition of the LIBOR-OIS spread — S_k^x. In the case of Amin (2010), Mercurio (2010), Fujii *et al.* (2011) and Mercurio and Xie (2012) the spread is defined to be additive such that

$$S_k^x(t) := L_k^x(t) - F_k^x(t),\ k = 1, \ldots, M_x. \tag{2.3}$$

Henrard (2013) uses multiplicative spreads where,

$$1 + \tau_k^x S_k^x(t) := \frac{1 + \tau_k^x L_k^x(t)}{1 + \tau_k^x F_k^x(t)},\ k = 1, \ldots, M_x. \tag{2.4}$$

Alternatively, Anderson & Piterbarg (2010) define instantaneous spreads with

$$P_L(t, T_k^x) := P_D(t, T_k^x) e^{\int_t^{T_k^x} s(u)du}, \tag{2.5}$$

where $P_L(t, T_k^x)$ is the LIBOR discount factor at t for maturity T_k^x. Although $P_L(t, T_k^x)$ is fictitious it can be used to determine the LIBOR forward rates, $L_k^x(t)$.

In each of the three cases, a model for the spread — $S_k^x(t)$ or $s(u)$ — is proposed and together with the spread definition allows for the implicit determination of the LIBOR forward rates. Mercurio and Xie (2012) appropriately suggest that different definitions of spread may be suited to different instruments in terms of the simplicity of pricing formulae.

2.4 Mercurio and Xie (2012) Stochastic-Basis Framework

Mercurio and Xie (2012) appear to provide the first generic framework for modelling stochastic-basis spreads. In this section this framework is presented noting the additive definition of spread (Equation 2.3). Importantly, this is a general framework and can theoretically be combined with any model of OIS rate evolution whether, it be a short-rate, forward-rate or market model.

First the forward basis spread, $S_k^x(t)$, related to each tenor x and corresponding time interval $[T_{k-1}^x, T_k^x]$, is assumed to be a function of its OIS forward rate $F_k^x(t)$ and of an independent martingale χ_k^x such that,

$$S_k^x(t) = \phi_k^x\Big(F_k^x(t), \chi_k^x(t)\Big).$$

For this to give a meaningful model the function ϕ_k^x has to satisfy a number of criteria:

1. The correlation between S_k^x and F_k^x must be modelled;
2. S_k^x must be a martingale under the T_k^x forward measure since S_k^x will be defined as the difference of two T_k^x-martingales (Equation 2.3);
3. The basis volatility must be independent of actual OIS rates (though this is satisfied by introducing the independent basis factors χ_k^x to the model).

While many different functions may satisfy these requirements, Mercurio and Xie (2012) suggest that the most tractable is given by an affine function

$$S_k^x(t) = S_k^x(0) + \alpha_k^x\left[F_k^x(t) - F_k^x(0)\right] + \beta_k^x\left[\chi_k^x(t) - \chi_k^x(0)\right], \quad \chi_k^x(0) = 1, \tag{2.6}$$

where α_k^x and β_k^x are real constant parameters for all k and x. It follows that the α_k^x parameters model the correlation between the OIS forward rates and corresponding spreads while the β_k^x parameters model the basis spread volatility — thus ensuring that ϕ_k^x satisfies the necessary criteria.

It is important to note that the parameters α_k^x and β_k^x must not be chosen to be independent of each other. This dependence is necessary to avoid unrealistic situations where the basis spread is solely a function of OIS rates or where there is zero basis spread volatility but the basis spread is not deterministic. Clearly when β_k^x is zero then α_k^x must also equal zero to ensure that the model reduces to a deterministic-basis model.

Equations 2.3 and 2.6 allow for the implicit modelling of the LIBOR forward rates $L_k^x(t)$ via

$$L_k^x(t) = \xi_k^x + (1 + \alpha_k^x)\, F_k^x(t) + \beta_k^x \chi_k^x(t), \tag{2.7}$$

where the useful quantity ξ_k^x is given by

$$\xi_k^x = S_k^x(0) - \alpha_k^x F_k^x(0) - \beta_k^x \chi_k^x(0). \tag{2.8}$$

Mercurio and Xie (2012) use an equivalent parametrisation of α_k^x and β_k^x which easily allows for this dependence as well as giving the parameters a more intuitive meaning. It is assumed the variances of $F_k^x(t)$ and $\chi_k^x(t)$ under $Q_D^{T_k^x}$ are finite and non-zero, allowing us to characterise the parameters by,

$$\alpha_k^x = \mathrm{Corr}\Big(F_k^x(T_{k-1}^x), S_k^x(T_k^x)\Big)\sqrt{\frac{\mathrm{Var}\big[S_k^x(T_{k-1}^x)\big]}{\mathrm{Var}\big[F_k^x(T_{k-1}^x)\big]}} \tag{2.9}$$

$$\Big(\beta_k^x\Big)^2 = \Big[1 - \mathrm{Corr}\big(F_k^x(T_{k-1}^x), S_k^x(T_k^x)\big)^2\Big]\frac{\mathrm{Var}\big[S_k^x(T_{k-1}^x)\big]}{\mathrm{Var}\big[F_k^x(T_{k-1}^x)\big]}, \tag{2.10}$$

where correlations and variances are taken under $Q_D^{T_k^x}$ — the T_k^x-forward measure. We then set

$$\rho_k^x := \mathrm{Corr}\Big(F_k^x(T_{k-1}^x), S_k^x(T_k^x)\Big) \tag{2.11}$$

$$\nu_k^x := \sqrt{\mathrm{Var}\big[S_k^x(T_{k-1}^x)\big]}. \tag{2.12}$$

This allows us to parametrise the basis spreads in terms of terminal standard deviations ν_k^x and correlations ρ_k^x giving

$$\begin{aligned} S_k^x(t) = S_k^x(0) &+ \frac{\nu_k^x \rho_k^x}{\sqrt{\mathrm{Var}\big[F_k^x(T_{k-1}^x)\big]}}\left[F_k^x(t) - F_k^x(0)\right] \\ &+ \frac{\sqrt{\left(1-\rho_k^x\right)^2}\,\nu_k^x}{\sqrt{\mathrm{Var}\big[\chi_k^x(T_{k-1}^x)\big]}}\left[\chi^x(t) - \chi^x(0)\right]. \end{aligned} \tag{2.13}$$

Under this parametrisation, the model reduces to a constant spread model when the basis spread volatility is zero; while the spread can solely be a function of the corresponding OIS rates only when $\rho_k^x = 1$. Consequently this parametrisation ensures model consistency. At this stage it is also important to point out that the affine nature of the spread model means that it does not preclude negative spreads - a point not mentioned by Mercurio and Xie (2012). As the non-negativity of spreads is seen to be a crucial requirement of a stochastic spread model, this is investigated in detail in Section 4 in particular.

The next issue surrounds the definition of the basis factors χ_k^x. Basis spread volatilities have historically varied with both underlying tenor and the considered term. The definition of the basis factors is general enough that they can be different for different tenors x and indexes k which ensures consistency. That being said, the movements of basis spreads are highly correlated and therefore a simple one or two factor model can be assumed to model the joint evolution of the stochastic processes χ_k^x (Mercurio and Xie, 2012). For example using a one-factor log-normal process and assuming that for each given tenor x, basis factors χ_k^x follow a common Brownian motion, the dynamics of $\chi_k^x(t) = \chi^x(t)$ would be given by

$$d\chi^x(t) = \eta^x(t)\chi^x(t)dZ^x(t), \quad \chi^x(0) = 1, \tag{2.14}$$

where $\eta^x(t)$ is a deterministic process modelling basis volatilities, and Z^x is a Brownian motion independent of OIS rates.

Equations 2.13 and 2.14 together with a chosen model for the OIS forward rates, $F_k^x(t)$ provide the complete specification of a multi-curve model that accounts for stochastic-basis. It is this multi-curve model that is reviewed and implemented.

Chapter 3

Interest Rate Derivative Pricing in the Multi-Curve Environment

In the present chapter the pricing of various interest rate derivatives in the multi-curve environment is reviewed. For each instrument, general and model - independent pricing expressions are derived before the effect of the different definitions of the basis spread, on the complexities of the final pricing formulae, are examined. Pricing formula under the multi-curve model presented by Mercurio and Xie (2012) are then derived for the case of a one-factor OIS rate model. Finally, an extension to the case of a two-factor OIS rate model is presented.

3.1 Forward Rate Agreements

The pay-off of a T^x_{k-1} x T^x_k FRA at the settlement date T^x_{k-1} is given by

$$\frac{\tau^x_k \left(L^x(T^x_{k-1}, T^x_k) - K\right)}{1 + \tau^x_k L^x(T^x_{k-1}, T^x_k)},$$

where K is the fixed rate.

The fair FRA rate at time $t < T^x_{k-1}$, which we denote by $\text{FRA}(t; T^x_{k-1}, T^x_k)$, is the fixed rate that gives the FRA contract zero values at time t. Valuing the FRA under $Q_D^{T^x_{k-1}}$, the T^x_{k-1} forward measure with associated numeraire $P_D(t, T^x_{k-1})$, gives

$$\mathbf{V}_{\mathbf{FRA}}(t) = \mathbb{E}_D^{T^x_{k-1}} \left[\frac{\tau^x_k \left(L^x(T^x_{k-1}, T^x_k) - \mathbf{FRA}(t; T^x_{k-1}, T^x_k)\right)}{1 + \tau^x_k L^x(T^x_{k-1}, T^x_k)} \middle| \mathcal{F}_t \right] = 0$$

$$\therefore\ 0 = \mathbb{E}_D^{T^x_{k-1}} \left[1 - \frac{1 + \tau^x_k \mathbf{FRA}(t; T^x_{k-1}, T^x_k)}{1 + \tau^x_k L^x(T^x_{k-1}, T^x_k)} \middle| \mathcal{F}_t \right].$$

Taking all terms known at time t out of the expectation and rearranging gives

$$\left(1 + \tau^x_k \mathbf{FRA}(t; T^x_{k-1}, T^x_k)\right)\ \mathbb{E}_D^{T^x_{k-1}} \left[\frac{1}{1 + \tau^x_k L^x(T^x_{k-1}, T^x_k)} \middle| \mathcal{F}_t \right] = 1$$

$$\therefore \textbf{FRA}(t; T^x_{k-1}, T^x_k) = \frac{1}{\tau^x_k \mathbb{E}_D^{T^x_{k-1}} \left[\frac{1}{1+\tau^x_k L^x(T^x_{k-1}, T^x_k)} \middle| \mathcal{F}_t \right]} - \frac{1}{\tau^x_k}.$$

Applying the classic change of numeraire technique, the expectation under $Q_D^{T^x_{k-1}}$ can be written as an expectation under $Q_D^{T^x_k}$ whose associated numeraire is $P_D(t, T^x_k)$. It can be shown that under this measure, the fair FRA rate is given by (see Appendix A.1

$$\textbf{FRA}(t; T^x_{k-1}, T^x_k) = \frac{1 + \tau^x_k F^x_k(t)}{\tau^x_k \mathbb{E}_D^{T^x_k} \left[\frac{1+\tau^x_k F^x_k(T^x_{k-1})}{1+\tau^x_k L^x(T^x_{k-1}, T^x_k)} | \mathcal{F}_t \right]} - \frac{1}{\tau^x_k}. \tag{3.1}$$

Equation 3.1 provides a general, model independent, formula to determine FRA rates.

3.1.1 Pricing Using an Additive Spread Definition

In order to evaluate Equation 3.1 when using the additive definition of spread approximations may be required. Using the second-order Taylor series expansion of $\frac{1}{1+x} \approx 1 - x + x^2$, for $|x| < 1$, we can simplify

$$\begin{aligned}
&\mathbb{E}_D^{T^x_k} \left[\frac{1 + \tau^x_k F^x_k(T^x_{k-1})}{1 + \tau^x_k L^x(T^x_{k-1}, T^x_k)} | \mathcal{F}_t \right] \\
&\approx \mathbb{E}_D^{T^x_k} \left[\left(1 + \tau^x_k F^x_k(T^x_{k-1})\right) \left(1 - \tau^x_k L^x(T^x_{k-1}, T^x_k) + (\tau^x_k)^2 L^x(T^x_{k-1}, T^x_k)^2\right) | \mathcal{F}_t \right] \\
&= \mathbb{E}_D^{T^x_k} \Big[1 - \tau^x_k (L^x_k(T^x_{k-1}) - F^x_k(T^x_{k-1})) + (\tau^x_k)^2 \Big(L^x_k(T^x_{k-1})(L^x_k(T^x_{k-1}) - F^x_k(T^x_{k-1}))\ldots \\
&\quad + \tau^x_k F^x_k(T^x_{k-1}) L^x_k(T^x_{k-1})^2 \Big) | \mathcal{F}_t \Big] \\
&= \mathbb{E}_D^{T^x_k} \left[1 - \tau^x_k S^x_k(T^x_{k-1}) + (\tau^x_k)^2 \left(L^x_k(T^x_{k-1}) S^x_k(T^x_{k-1}) + \tau^x_k F^x_k(T^x_{k-1}) L^x_k(T^x_{k-1})^2 \right) | \mathcal{F}_t \right] \\
&\approx 1 - \tau^x_k S^x_k(t) + (\tau^x_k)^2 \left[\text{Corr}\left(S^x_k(T^x_{k-1}), L^x_k(T^x_{k-1}) | \mathcal{F}_t \right) + S^x_k(t) L^x_k(t) \right],
\end{aligned}$$

where we use the facts that $L^x(T^x_{k-1}, T^x_k) = L^x_k(T^x_{k-1})$ from Equation 2.2; $L^x_k(T^x_{k-1}) - F^x_k(T^x_{k-1}) = S^x_k(T^x_{k-1})$ from the definition of additive spreads; and $S^x_k(T^x_{k-1})$, $L^x_k(T^x_{k-1})$ are $Q^{T^x_k}$ martingales.

By using another second order Taylor series expansion Equation 3.1 can be approximated by

$$\textbf{FRA}(t; T^x_{k-1}, T^x_k) \approx L^x_k(t) - \tau^x_k \text{Cov}\left(S^x_k(T^x_{k-1}), L^x_k(T^x_{k-1}) | \mathcal{F}_t \right). \tag{3.2}$$

The term $\tau^x_k \text{Cov}\left(S^x_k(T^x_{k-1}), L^x_k(T^x_{k-1}) | \mathcal{F}_t \right)$ can be considered a FRA convexity correction. Clearly in the single curve case it vanishes since $S^x_k(t) = 0$. Depending on the chosen additive spread multi-curve model it may be possible to derive a closed form approximation of the convexity correction.

In the case of our chosen Mercurio and Xie (2012) stochastic-basis spread model, a closed form approximation of the convexity correction can be derived assuming that the explicit $Q_D^{T_k^x}$ variances for $F_k^x(t)$ exist for the chosen OIS model. Assuming these do exist we can write the covariance as

$$\begin{aligned}
&\text{Corr}\left(S_k^x(T_{k-1}^x), L_k^x(T_{k-1}^x)\right) \\
&= \text{Cov}\left(\alpha_k^x F_k^x(T_{k-1}^x) + \beta_k^x \chi_k^x(T_{k-1}^x), (1+\alpha_k^x)F_k^x(T_{k-1}^x) + \beta_k^x \chi_k^x(T_{k-1}^x)\right) \\
&= \alpha_k^x(1+\alpha_k^x)\text{Var}\left[F_k^x(T_{k-1}^x)\right] + \left(\beta_k^x\right)^2 \text{Var}\left[\chi_k^x(T_{k-1}^x)\right].
\end{aligned}$$

Using the expression for the covariance as well as ξ_k^x (defined in Equation 2.8) we can write

$$\begin{aligned}
\textbf{FRA}(t; T_{k-1}^x, T_k^x) \approx & (1+\alpha_k^x)F_k^x(t) + \beta_k^x \chi_k^x(t) + \xi_k^x - \\
& \tau_k^x \left(\alpha_k^x(1+\alpha_k^x)\text{Var}\left[F_k^x(T_{k-1}^x)\right] + \left(\beta_k^x\right)^2 \text{Var}\left[\chi_k^x(T_{k-1}^x)\right]\right) \\
= & (1+\alpha_k^x)\left(F_k^x(t) - \tau_k^x \alpha_k^x \text{Var}\left[F_k^x(T_{k-1}^x)\right]\right) + \xi_k^x \\
& + \beta_k^x \left(\chi_k^x(t) - \tau_k^x \beta_k^x \text{Var}\left[\chi_k^x(T_{k-1}^x)\right]\right).
\end{aligned} \tag{3.3}$$

Equation 3.3 provides an approximation for the fair FRA rate in a multi-curve framework with stochastic-basis which can be easily implemented.

3.1.2 Pricing Using a Multiplicative Spread Definition

When using an additive spread definition, approximations are required to derive a closed form expression for the fair FRA rate. The expectation in Equation 3.1, which drives the need for approximation, is much easier to handle when using the multiplicative spread definition given by Equation 2.4. Instead of using a Taylor-series approximation we just simplify

$$\begin{aligned}
\mathbb{E}_D^{T_k^x}\left[\frac{1+\tau_k^x F_k^x(T_{k-1}^x)}{1+\tau_k^x L_k^x(T_{k-1}^x)}\middle| \mathcal{F}_t\right] &= \mathbb{E}_D^{T_k^x}\left[\frac{1}{1+\tau_k^x S_k^x(T_{k-1}^x)}\middle| \mathcal{F}_t\right] \\
&= \frac{1}{1+\tau_k^x S_k^x(t)}.
\end{aligned}$$

Since $S_k^x(t)$ is a $Q_D^{T_k^x}$ martingale. The fair FRA rate is therefore given by

$$\begin{aligned}
\textbf{FRA}(t; T_{k-1}^x, T_k^x) = & \frac{1}{\tau_k^x}\left[\left(1+\tau_k^x F_k^x(t)\right)\left(1+\tau_k^x S_k^x(t)\right) - 1\right] \\
= & \frac{1}{\tau_k^x}\left[\left(1+\tau_k^x L_k^x(t)\right) - 1\right] \\
= & L_k^x(t).
\end{aligned}$$

Therefore when assuming a multiplicative definition of spread, fair FRA rates are simply equivalent to forward LIBOR rates and no convexity correction is required. This results in a more straight-forward bootstrapping procedure since the LIBOR forward rates at time-0 are simply the corresponding market FRA rates, assuming the market is in equilibrium.

3.2 Interest Rate Swaps

In this section we value linear interest rate derivatives in the multi-curve environment. We consider an interest rate swap with floating leg payments at times T_k^x based on the LIBOR rate $L^x(T_{k-1}^x, T_k^x)$ set at the previous time T_{k-1}^x for $k = a+1, \dots, b$ and fixed leg payments based on a fixed rate K at T_j^S for $j = c+1, \dots, d$. First we value the floating leg. At time T_k^x, the pay-off of the floating leg is given by

$$\mathbf{FL}(T_k^x; T_{k-1}^x, T_k^x) = \tau_k^x L^x(T_{k-1}^x, T_k^x),$$

where $L^x(T_{k-1}^x, T_k^x)$ is the LIBOR rate that is set at T_{k-1}^x with maturity T_k^x. We determine the value of each T_k^x floating leg cashflow by pricing under the T_k^x-forward measure to give

$$\mathbf{FL}(t; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) \mathbb{E}_D^{T_k^x}\left[L^x(T_{k-1}^x, T_k^x)\right].$$

From the definition given in (5) this reduces to

$$\mathbf{FL}(t; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) L_k^x(t).$$

It is noted that in the multi-curve environment the expected LIBOR rate does not equal the forward rate $F_k^x(t)$ and thus $\mathbf{FL}(t; T_{k-1}^x, T_k^x)$ does not reduce to $P_D(t, T_{k-1}^x)$ $-P_D(t, T_k^x)$ as in the single-curve case. The time-t value of each floating is then summed to give the present value of the swaps floating leg

$$\mathbf{FL}(t; T_a^x, \dots, T_b^x) = \sum_{k=a+1}^{b} \mathbf{FL}(t; T_{k-1}^x, T_k^x) = \sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) L_k^x(t).$$

The present value of the fixed leg is more straight forward since its present value is simply the sum of each discounted fixed payment

$$\mathbf{FIX}(t; T_c^S, \dots, T_d^S) = \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S) K = K \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S).$$

The value of the IRS is simply the difference in the present value of the two legs and is therefore given by (to the fixed rate payer)

$$\mathbf{IRS}(t, K; T_a^x, \dots, T_b^x, T_c^S, \dots, T_d^S) = \sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) L_k^x(t) - K \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S).$$

The fair swap rate is defined as the fixed rate K that gives the IRS a time-t value of 0. We denote the fair swap rate at time t for a swap with floating leg tenor of x, floating leg dates $T_a^x, \ldots, T_b^x$ and fixed leg dates $T_c^S, \ldots, T_d^S$ as $S_{a,b,c,d}^x(t)$ which is given by

$$S_{a,b,c,d}^x(t) = \frac{\sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) L_k^x(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}. \tag{3.4}$$

This is a general formulation of the fair swap rate for $0 < t < (T_a^x \wedge T_c^S)$ and can be used to determine forward starting swap rates.

An important case is the spot-starting swap with floating leg payment dates $T_1^x, \ldots, T_b^x$ and fixed leg payment dates $T_1^S, \ldots, T_d^S$. In this case the fair swap rate is given by

$$S_{0,b,0,d}^x(0) = \frac{\sum_{k=1}^{b} \tau_k^x P_D(0, T_k^x) L_k^x(0)}{\sum_{j=1}^{d} \tau_j^S P_D(0, T_j^S)}.$$

As in the traditional bootstrapping approach, the market expectations of forward LIBOR rates can be implied from market rates of spot-starting swaps by using the relationship given in Equation 8 noting that the discount factors $P_D(0, T_k^x)$ can be obtained from market OIS quotes.

3.2.1 Pricing Using an Additive Spread Definition

The expression for the fair swap rate in a multi-curve environment, given by Equation 3.4, is suited to an additive definition of spread since it is a ratio of two sums. In the case of the chosen Mercurio and Xie (2012) stochastic-basis spread model we simply replace $L_k^x(t)$ using Equation 2.7 to give

$$\begin{aligned} S_{a,b,c,d}^x(t) = {} & \frac{\sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) \xi_k^x}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)} \\ & + \frac{\sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x)\,(1 + \alpha_k^x)\,F_k^x(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)} \\ & + \frac{\sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) \beta_k^x \chi^x(t)}{\sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)}. \end{aligned} \tag{3.5}$$

3.2.2 Pricing Using an Multiplicative Spread Definition

Conversely, it can be clearly seen that a multiplicative definition of spread would result in a complex expression for the fair swap rate. We do not derive the swap rate under this spread definition since the Mercurio and Xie (2012) framework uses an additive definition of spread. Again, as in the case of the fair FRA rates, these differences in the fair swap rate expressions illustrate how something as fundamental as the definition of the spread can have a large effect on the complexity of the pricing formula.

3.3 Interest Rate Caps and Floors

An interest rate cap is a popular vanilla interest rate option often used by corporates to manage interest-rate risk on floating rate debt. A cap pays $\tau_k[L^x(T^x_{k-1}, T^x_k) - K]^+$ at each cap payment date, T^x_k for $k = a+1, ..., b$, where T^x_b is the expiry date of the cap. As a result caps can be considered as a portfolio of adjacent caplets (a simpler interest rate derivative product) and caps are priced as a sum of the component caplet prices. We express the time-0 price of the cap with strike K, start date T^x_a and expiry T^x_b as

$$\textbf{Cap}(0, K, T^x_a, T^x_b) = \sum_{k=a}^{b} \textbf{Cplt}(0, K; T^x_k). \tag{3.6}$$

3.3.1 Interest Rate Caplets

A caplet is simply a call option on forward interest rates. The pay-off of a caplet written on forward LIBOR at time T^x_k is given by

$$\tau^x_k \left[L^x(T^x_{k-1}, T^x_k) - K\right]^+ .$$

The time-0 price can be obtained under the $Q_D^{T^x_k}$ forward measure, which gives

$$\textbf{Cplt}(0, K; T^x_k) = \tau^x_k P_D(0, T^x_k)\, \mathbb{E}_D^{T^x_k} \left\{ \left[L^x(T^x_{k-1}, T^x_k) - K\right]^+ \right\}.$$

In the single-curve case the future spot LIBOR rate $L^x(T^x_{k-1}, T^x_k)$ can be replaced by the LIBOR forward rate using a classic no-arbitrage replication argument. In addition the $Q_D^{T^x_k}$ forward measure is simply the Q^T forward measure, under which forward LIBOR is a martingale. Assuming log-normal dynamics of this forward rate as per the LMM of Brace et al. (1997) leads to the classic Black-like caplet price.

However, in the multi-curve environment the valuation of the caplet is more involved. The problem with pricing in the multi-curve environment is that $L^x(T^x_{k-1}, T^x_k)$ is not necessarily a martingale under the pricing measure ($Q_D^{T^x_k}$) since they relate to different curves. One way to overcome this is to model the LIBOR forward rate ($F_k^{x,L}$) under $Q_x^{T^x_k}$ — the forward measure relating to the LIBOR curve with tenor x. Then one can model the Radon-Nikodym derivative $dQ_x^{T^x_k}/dQ_D^{T^x_k}$ defining the change of measure from $Q_x^{T^x_k}$ to $Q_D^{T^x_k}$. Mercurio (2009) suggest an alternative approach where the LIBOR rate in the caplet pay-off is replaced by an equivalent forward rate which is a martingale under the new pricing measure ($Q_D^{T^x_k}$).

Remembering the definition given by Equation 2.2 we note that

$$L_k^x(t) = \mathbb{E}_D^{T_k^x}\left[L^x(T_{k-1}^x, T_k^x)|\mathcal{F}_t\right]$$
$$\therefore L_k^x(T_{k-1}^x) = L^x(T_{k-1}^x, T_k^x).$$

The caplet can therefore be viewed as call option on $L_k^x(T_{k-1}^x)$ rather than $L^x(T_{k-1}^x, T_k^x)$ since the two rates L^x and L_k^x coincide at the reset date T_{k-1}^x. Thus, the price is rather given by

$$\textbf{Cplt}(0, K; T_k^x) = \tau_k^x P_D(0, T_k^x)\, \mathbb{E}_D^{T_k^x}\left\{\left[L_k^x(T_{k-1}^x) - K\right]^+\right\}. \tag{3.7}$$

Since the price requires the expectation of $L_k^x(T_{k-1}^x)$ one may also consider pricing under the T_{k-1}^x-forward measure which gives

$$\textbf{Cplt}(0, K; T_k^x) = \tau_k^x P_D(0, T_{k-1}^x)\, \mathbb{E}_D^{T_{k-1}^x}\left\{P_D(T_{k-1}^x, T_k^x)\left[L_k^x(T_{k-1}^x) - K\right]^+\right\}. \tag{3.8}$$

Equations 3.7 and 3.8 provide general, model-independent caplet pricing formulae in the multi-curve environment.

Pricing with the Chosen Multi-curve Model

With our chosen multi-curve model with stochastic-basis we replace $L_k^x(T_{k-1}^x)$ using Equations 2.3 and 2.6. Using this replacement together with the definition of $F_k^x(t)$ it can be shown that if OIS rates are driven by a one-factor stochastic process (X), then

$$\tau_k^x P_D(T_{k-1}^x, T_k^x)[L_k^x(T_{k-1}^x) - K]^+ = \left[C\big(X_{T_{k-1}^x}\big)\chi^x(T_{k-1}^x) - D\big(X_{T_{k-1}^x}\big)\right]^+,$$

where

$$C\big(X\big) := \tau_k^x \beta_k^x P_D\big(T_{k-1}^x, T_k^x; X\big) \tag{3.9}$$
$$D\big(X\big) := (1 + \alpha_k^x)\left[P_D\big(T_{k-1}^x, T_k^x; X\big) - 1\right] + \tau_k^x P_D\big(T_{k-1}^x, T_k^x; X\big)(K - \xi_k^x). \tag{3.10}$$

Noting that $P_D\big(T_{k-1}^x, T; X_{T_{k-1}^x}\big)$ denotes the zero-coupon bond price calculated using the chosen OIS rate model which is a function of one stochastic variable X. This allows the caplet price to be expressed as

$$\begin{aligned}
\textbf{Cplt}(0, K) &= P_D(0, T_{k-1}^x)\, \mathbb{E}_D^{T_{k-1}^x}\left\{\left[C\big(X_{T_{k-1}^x}\big)\chi^x(T_{k-1}^x) - D\big(X_{T_{k-1}^x}\big)\right]^+\right\} \\
&= P_D(0, T_{k-1}^x)\, \mathbb{E}_D^{T_{k-1}^x}\left[\mathbb{E}_D^{T_{k-1}^x}\left\{\left[C\big(X_{T_{k-1}^x}\big)\chi^x(T_{k-1}^x) - D\big(X_{T_{k-1}^x}\big)\right]^+ \Big| X = x\right\}\right] \\
&= P_D(0, T_{k-1}^x) \int_{-\infty}^{\infty} \mathbb{E}_D^{T_{k-1}^x}\left\{\left[C\big(x\big)\chi^x(T_{k-1}^x) - D\big(x\big)\right]^+\right\} f_X(x)\, dx,
\end{aligned}$$

where f_X is the probability density function of X under the $Q^{T_{k-1}^x}$ forward measure.

Looking at the expectation within the integral it is noted that since it is conditioned on a specific value of X, $D(x)$ is constant. As a result it can be viewed as the time-0 price of a vanilla European option on some multiple of $\chi^x(t)$ expiring at T^x_{k-1}. By definition $\chi^x(t)$ is log-normal under forward measures which suggests the use of a Black-type formula to analytically evaluate the expectation. To do this requires the consideration of four possible cases relating to the values of $C(x)$ and $D(x)$.

Case 1: $C(x) > 0$ and $D(x) > 0$ — The expectation is simply the time-0 price of a call option on $C(x)\chi^x(t)$ with strike $D(x)$.

Case 2: $C(x) < 0$ and $D(x) < 0$ — In this case we rewrite the expection as:

$$\mathbb{E}_D^{T^x_{k-1}}\left\{[C(x)\chi^x(T^x_k) - D(x)]^+\right\} = \mathbb{E}_D^{T^x_{k-1}}\left\{\left[\Big(-D(x)\Big) - \Big(-C(x)\chi^x(T^x_k)\Big)\right]^+\right\}.$$

This can be seen as the time-0 price of a put option on $-C(x)\chi^x(t)$ with strike $-D(x)$.

Case 3: $C(x) \geq 0$ and $D(x) \leq 0$ — This can be seen as a call option with a negative strike. Since the underlying is log-normal, the option will always expire in the money which allows us to write:

$$\begin{aligned}\mathbb{E}_D^{T^x_{k-1}}\Big\{\big[C(x)\chi^x(T^x_{k-1}) - D(x)\big]^+\Big\} &= \mathbb{E}_D^{T^x_{k-1}}\Big\{C(x)\chi^x(T^x_{k-1}) - D(x)\Big\}\\ &= C(x) - D(x).\end{aligned}$$

Case 4: $C(x) \leq 0$ and $D(x) \geq 0$ — Similarly to case 2, this can be seen as the time-0 price of a put option on $-C(x)\chi^x(t)$ with strike $-D(x)$. However the strike is negative in this case and since $\chi^x(t)$ is log-normal $-C(x)\chi^x(t) \geq 0$. The option is therefore always out the money and therefore worthless.

To deal with these various cases, we define a new function as well as making use of the general log-normal option pricing formula

$$h(A,B,V)\begin{cases}\mathrm{Bl}(A,B,V,1) & \text{if } A,B>0,\\ \mathrm{Bl}(-A,-B,V,-1) & \text{if } A,B<0,\\ A-B & \text{if } A\geq 0,\ B\leq 0,\\ 0 & \text{if } A\leq 0,\ B\geq 0\end{cases} \tag{3.11}$$

where:

$$\mathrm{Bl}(F,K,v,w) := wF\Phi\left(w\frac{ln(F/K)+v^2/2}{v}\right) - wK\Phi\left(w\frac{ln(F/K)-v^2/2}{v}\right).$$

Noting that $\chi^x(0) = 1$, we are able to express the time zero caplet price by

$$\mathbf{Cplt}(0,K) = P_D(0,T^x_{k-1})\int_{-\infty}^{\infty} h\Big(C(x), D(x), V_\chi(T^x_{k-1})\Big) f_X(x)\,dx. \tag{3.12}$$

where $C(x)$ and $D(x)$ are given by Equations 3.9 and 3.10 as before, while $V_\chi(T^x_{k-1})$ is the standard deviation of $\ln\chi^x(T^x_{k-1})$.

While Equation 3.12 provides a simple one-dimensional integral which can be easily evaluated numerically, the OIS rates are only being modelled by a single-factor interest rate model. The limitations of such models has been well documented by many authors such as Longstaff *et al.* (2001). Consequently, the multi-curve model under review needs to be considered with a multi-factor OIS rate model.

In the case of a two-factor OIS rate model we see that all techniques applied in the one-factor case apply. However, one has to condition on two random variables as opposed to one. This results in an extra dimension in the pricing expression which is given by

$$\textbf{Cplt}(0,K) = P_D(0,T^x_{k-1})\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\Big(C(x,y),D(x,y),V_\chi(T^x_{k-1})\Big) f_{XY}(x,y)\,dxdy, \tag{3.13}$$

where f_{XY} is the joint probability density function of X and X under the T^x_{k-1} forward measure. The C and D functions are now also defined to be a function of two-variables since bond prices are driven by two-factors:

$$C\big(X,Y\big) := \tau^x_k\beta^x_k P_D\big(T^x_{k-1},T^x_k;X,Y\big) \tag{3.14}$$

$$D\big(X,Y\big) := (1+\alpha^x_k)\left[P_D\big(T^x_{k-1},T^x_k;X,Y\big)-1\right] + \tau^x_k P_D\big(T^x_{k-1},T^x_k;X,Y\big)(K-\xi^x_k). \tag{3.15}$$

3.4 Interest Rate Swaptions

3.4.1 Physical Delivery Swaptions

A European payer swaption (with physical delivery) gives the holder the right (but not the obligation) to enter into a long IRS position at time $T^x_a = T^S_c$ with floating leg payment times of $T^x_{a+1},\ldots,T^x_b$ and fixed leg payment times of $T^S_{c+1},\ldots,T^S_d$, with $T^x_b = T^S_d$ and fixed rate K. A European receiver swaption, on the other hand, gives the holder the right to enter into a short IRS position at time $T^x_a = T^S_c$. The pay-off of a European swaption at time $T^x_a = T^S_c$ is therefore given by

$$\left[\omega\left(S^x_{a,b,c,d}(T^x_a)-K\right)\right]^+\sum_{j=c+1}^{d}\tau^S_j P_D(T^S_c,T^S_j),$$

where $S^x_{a,b,c,d}(T^x_a)$ is the fair swap rate at time $T^x_a = T^S_c$ (see Equation 3.4) and $\omega = 1$ for a payer swaption and $\omega = -1$ for a receiver swaption.

In the single-curve case, future spot LIBOR rates in the fair swap rate expression can be replaced by forward rates using the classic no-arbitrage replication argument. As a result swaptions are priced under the swap measure Q^A whose numeraire is given by the annuity $A_t^{c,d} = \sum_{j=c+1}^{d} \tau_j^S P_D(t, T_j^S)$; as forward swap rates can be shown to be martingales under this measure. This allows for the derivation of a Black-like pricing formula.

In the multi-curve environment the forward swap rates are no-longer necessarily martingales under the swap measure and thus we rather price under the T_a^x forward measure. The time-t swaption price can then be written as

$$\begin{aligned}&\mathbf{SWPN}(t, K; T_{a+1}^x, \ldots, T_c^x, T_{c+1}^x, \ldots, T_d^x) \\ &\quad = P_D(t, T_a^x)\, \mathbb{E}_D^{T_k^x} \left\{ \left[\omega\left(S_{a,b,c,d}^x(T_a^x) - K\right)\right]^+ \sum_{j=c+1}^{d} \tau_j^S P_D(T_c^S, T_j^S) \middle| \mathcal{F}_t \right\}. \end{aligned} \tag{3.16}$$

Equation 3.16 provides a general expression for the price of a swaption in the multi-curve environment. In this section we do not attempt to use the multiplicative definition of basis spreads due to the untidy expressions obtained when replacing LIBOR forward rates in the fair swap rate expression using this definition.

Pricing with the Chosen Multi-curve Model

Again we derive a pricing formula using the multi-curve model presented by Mercurio and Xie (2012) with a one-factor OIS rate model. Under this model it can be shown that (see Appendix A.2)

$$\left[\omega\left(S_{a,b,c,d}^x(T_a^x) - K\right)\right]^+ \sum_{j=c+1}^{d} \tau_j^S P_D(T_c^S, T_j^S) = \left[\omega C\big(X_{T_a^x}\big)\chi^x(T_a^x) - \omega D\big(X_{T_a^x}\big)\right]^+, \tag{3.17}$$

where the C and D functions are defined differently to those used in the caplet pricing derivation

$$\begin{aligned} C(X) &:= \sum_{k=a+1}^{b} \tau_k^x P_D(T_a^x, T_k^x; X)\beta_k^x \\ D(X) &:= K \sum_{j=c+1}^{d} \tau_j^S P_D(T_c^S, T_j^S; X) - 1 - \alpha_{a+1}^x + P_D(T_a^x, T_b^x; X)\left[1 + \alpha_b^x - \tau_k^x \xi_b^x\right] \\ &\quad - \sum_{k=a+1}^{b-1} P_D(T_a^x, T_k^x; X)\left[\alpha_{k+1}^x - \alpha_k^x + \tau_k^x \xi_k^x\right], \end{aligned}$$

where ξ_k^x is given by Equation 2.8 and noting that $P_D(T_a^x, T; X)$ denotes the zero-coupon bond price determined using the chosen OIS rate model which is a function of one stochastic variable X.

Consequently, the swaption price can be expressed as

$$\mathbf{SWPN}(t,K) = P_D(t,T_a^x)\,\mathbb{E}_D^{T_a^x}\left[\left[\omega C\Big(X(T_a^x)\Big)\chi^x(T_a^x) - \omega D\Big(X(T_a^x)\Big)\right]^+ \Big| \mathcal{F}_t\right].$$

Using the tower property and the fact that $\chi^x(t)$ and OIS rates (and thus X) are independent it can be seen that

$$\begin{aligned}\mathbf{SWPN}(t,K) &= P_D(t,T_a^x)\,\mathbb{E}_D^{T_a^x}\left[\mathbb{E}_D^{T_a^x}\left[\left[\omega C\big(X_{T_a^x}\big)\chi^x(T_a^x) - \omega D\big(X_{T_a^x}\big)\right]^+ \Big| \mathcal{F}_t,\ X = x\right]\right]\\ &= P_D(t,T_a^x)\int_{-\infty}^{\infty}\mathbb{E}_D^{T_a^x}\left[\left[\omega C(x)\chi^x(T_a^x) - \omega D(x)\right]^+\right] f_X(x)\,dx,\end{aligned}$$

where f_X is the probability density function of X under the T_a^x forward measure.

Looking at the expectation within the integral it is noted that, since it is conditioned on a specific value of X, $\omega D(x)$ and $\omega C(x)$ are constant. As a result it can be viewed as the time-t price of a vanilla European option on $\omega C(x)\chi^x(t)$ which, by definition is a log-normal martingale under forward measures. Similarly to deriving the caplet price in Section 3.3, we consider four cases relating to the values of $\omega C(x)$ and $\omega D(x)$. This can be shown to result in the following swaption price semi-analytical formula

$$\mathbf{SWPN}(t,K) = P_D(t,T_a^x)\int_{-\infty}^{\infty} h\Big(\omega C(x), \omega D(x), V_\chi(T_a^x)\Big) f_X(x)\,dx, \tag{3.18}$$

where $V_\chi(T_a^x)$ is the standard deviation of $\ln\chi^x(T_a^x)$ and the h-function is defined above by Equation 3.11.

Again we extend the result to the multi-curve model with a two-factor OIS rate model. As before, we see that all techniques applied in the one-factor case apply. However, one has to condition on two random variables as opposed to one. This results in an extra dimension in the pricing expression which is given by

$$\mathbf{SWPN}(t,K) = P_D(t,T_a^x)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h\Big(\omega C(x,y), \omega D(x,y), V_\chi(T_a^x)\Big) f_X Y(x,y)\,dxdy, \tag{3.19}$$

where f_{XY} is the joint probability density function of X and Y under the T_a^x forward measure. The C and D functions are now also defined to be functions of two-variables since bond prices are driven by two factors

$$\begin{aligned} C(X,Y) &:= \sum_{k=a+1}^{b} \tau_k^x P_D(T_a^x, T_k^x; X, Y)\beta_k^x\\ D(X,Y) &:= K\sum_{j=c+1}^{d} \tau_j^S P_D(T_c^S, T_j^S; X, Y) + P_D(T_a^x, T_b^x; X, Y)\left[1 + \alpha_b^x - \tau_k^x\xi_b^x\right]\\ &\quad - \sum_{k=a+1}^{b-1} P_D(T_a^x, T_k^x; X, Y)\left[\alpha_{k+1}^x - \alpha_k^x + \tau_k^x\xi_k^x\right] - 1 - \alpha_{a+1}^x. \end{aligned}$$

3.4.2 Cash-settled Swaptions

The pay-off of a cash-settled swaption with maturity T_θ^x written on an IRS with start date $T_a^x = T_c^S$, floating payment times of $T_{a+1}^S, \ldots, T_b^S$, fixed payment times of $T_{c+1}^S, \ldots, T_d^S$ and fixed rate K is given by

$$P_D(T_\theta^x, T_a^x)\left[\Psi\left(S_{a,b,c,d}^x(T_\theta^x) - K\right)\right]^+ \sum_{j=c+1}^{d} \frac{\tau_j^S}{\left(1 + \tau_j^S S_{a,b,c,d}^x(T_\theta^x)\right)^j}. \tag{3.20}$$

It is noted that we distinguish between the swaption maturity and underlying swap-start date in this subsection. The summation term is often denoted as the cash-settled annuity, $C(S_{T_a}^x)$, defined by

$$C\left(S_t^x\right) := \sum_{j=c+1}^{d} \frac{\tau_j^S}{\left(1 + \tau_j^S S_{a,b,c,d}^x(t)\right)^j}.$$

It can be seen that the pay-off, while being similar to that of a physical delivery swaption, is discounted using the underlying swap rate which is set at maturity. It is this swap rate that is affected by the inclusion of stochastic-basis and consequently its inclusion may have a larger effect when compared to that of the typical physical delivery swaption.

The standard swaption settlement method in the EUR and GBP interbank markets is cash-settlement (Henrard, 2010a). The cash-annuity ($C(S_t^x)$) relies on one market rate (the fair swap rate) as opposed to a multitude of zero-coupon bond prices in the case of the standard swap-annuity (A_t), making the amount easier to calculate. That being said, cash-settled swaptions are significantly more difficult to price than their physical delivery counterparts since the pay-off is a complex function of the swap rate.

Market Formula

If we consider instead the time-0 price under the general EMM N with associated numeraire N_t, then

$$\mathbf{CSS}(0) = N_0 \mathbb{E}^N \left[\frac{\left[\Psi\left(S_{a,b,c,d}^x(T_\theta^x) - K\right)\right]^+ C(S_{T_\theta}^x)}{N_{T_a^x}} \right].$$

Henrard (2010b) suggests that one can choose $P_D(t, T_\theta^S)C(S_t)$ as the numeraire (to follow a similar process to pricing physical delivery swaptions) which gives

$$\mathbf{CSS}(0) = C(S_0)\mathbb{E}^C\left[\left[\Psi\left(S_{a,b,c,d}^x(T_\theta^x) - K\right)\right]^+\right].$$

The problem here lies in the fact that the swap-rate is not a martingale under this measure. However, the market standard is then to substitute the numeraire C by A where $A_t^{c,d}$ is the standard swap annuity defined by $A_t^{c,d} = \sum_{j=c+1}^{d} \tau_j^S P_D(T_c^S, T_j^S)$. This allows the price to be approximated by

$$\begin{aligned} C(S_0)\mathbb{E}^C\left[\left[\Psi\left(S^x_{a,b,c,d}(T^x_\theta) - K\right)\right]^+\right] \approx & C(S^x_0)\mathbb{E}^A\left[\left[\Psi\left(S^x_{a,b,c,d}(T^x_\theta) - K\right)\right]^+\right] \\ = & \Psi C(S^x_0)\text{Bl}(S^x_0, K, \sigma, \Psi). \end{aligned}$$

However, even this approximation relies on the fact that the swap rate is a martingale under the swap measure with associate numeraire A_t. This no longer holds since in the multi-curve environment, the swap rate is no longer a tradable asset divided by the numeraire A_t since classic no-arbitrage replication of future spot LIBOR rates no longer holds. For more information on other issues with this approximation see Henrard (2010a) and Mercurio (2007).

Pricing with the Chosen Multi-curve Model

Due to the complexity of the payoff given in Equation 3.20 one of the expectations cannot be simplified to a vanilla European option-like payoff as seen in Equation 3.17 for the case of physical delivery swaptions. To overcome this we are forced to make use of an approximation presented by Henrard (2010a) but still price under the T_θ^x forward measure. The pricing formula derivation is done for receiver cash-settled swaptions before we generalise the final results.

Under the T_θ^x forward measure the time-0 price of a receiver cash-settled swaption is given by

$$\textbf{CSS}(0) = P_D(0, T^x_\theta)\mathbb{E}_D^{T^x_\theta}\left[P_D(T^x_\theta, T^x_a)\left[K - S^x_{a,b,c,d}(T^x_\theta)\right]^+ C(S^x_{T_\theta})\right]. \tag{3.21}$$

From Equation 3.4, we know that the fair swap rate $S^x_{a,b,c,d}(t)$ is driven by some stochastic interest rate factor, which we have denoted X, as well as the stochastic-basis factor, denoted χ^x. We again use the tower property however, we condition on a specific value of χ^x as opposed to X, as was the case when deriving the expressions for physical delivery swaption prices. This gives

$$\textbf{CSS}(0) = P_D(0, T^x_\theta)\,\mathbb{E}_D^{T^x_\theta}\left[\mathbb{E}_D^{T^x_\theta}\left[P_D(T^x_\theta, T^x_a)\left[K - S^x_{a,b,c,d}(T^x_\theta)\right]^+ C(S^x_{T_\theta})\middle|\chi^x = x\right]\right]$$

$$= P_D(0, T^x_\theta)\int_{-\infty}^{\infty}\mathbb{E}_D^{T^x_\theta}\left[P_D(T^x_\theta, T^x_a)\left[K - S^x_{a,b,c,d}(T^x_\theta)\right]^+ C(S^x_{T_\theta})\middle|\chi^x = x\right] f_{\chi^x}(x)\,dx,$$

where f_{χ^x} is the probability density function of χ^x under the $Q^{T^x_\theta}$ forward measure.

The conditional expectation inside the integral is a function of one stochastic variable, the stochastic interest rate factor X. This allows us to use the efficient approximation for cash-settled swaption prices presented by Henrard (2010b). This approximation is model dependent and is given for a one-factor Hull-White model for OIS rates — luckily this coincides with our use of this model as our one-factor model of choice for OIS rates in this dissertation.

First we recall some facts about the Hull-White one-factor model. Most importantly, that bond prices under the $Q_D^{T_\theta^x}$ forward measure can be written explicitly in terms of a standard normal random variable Z

$$P_D(T_\theta^x, T_i^x) = \frac{P_D(0, T_i^x)}{P_D(0, T_\theta^x)} \exp\left(-0.5\gamma_i^2 - \gamma_i Z\right), \tag{3.22}$$

where γ_i is defined in Appendix B.2, Equation B.13.

Using this fact, the swap rate (when conditioned on a specific value of χ^x) can be considered as a function of a single standard normal variable Z. The difficult parts in evaluating the conditional expectation are the cash annuity and the swap rate exercise ($S_{T_\theta}^x(Z) < K$ in the receiver swaption case). Henrard (2010a) deals with the swap rate exercise by extending techniques used in the pricing of constant maturity swaps (CMS) presented in Henrard (2007a) where the exercise boundary is defined by a value of κ such that $S_{T_\theta}^x(\kappa) = K$. The exercise condition then becomes $Z < \kappa$, which provides the integration bounds. A third order Taylor series approximation is then used to replace $\left(K - S_{a,b,c,d}^x(T_\theta^x)\right)C(S_{T_\theta}^x)$

$$\left(K - S_{a,b,c,d}^x(T_\theta^x)\right)C(S_{T_\theta}^x) \approx U_0 + U_1(Z - Z_0) + \frac{1}{2}U_2(Z - Z_0)^2 + \frac{1}{3!}U_3(Z - Z_0)^3, \tag{3.23}$$

where the recommended choice for the reference point, Z_0, is κ in order to significantly reduce the approximation error for out-the-money options. It is noted that the Taylor series expansion coefficients (U_0, U_1, U_2 and U_3) will differ to those of Henrard (2010a) due to the presence of stochastic-basis in our definition of the forward swap rate $S_{a,b,c,d}^x(t)$.

Using the Henrard (2010a) approximation, the conditional expectation can be given to the third order in closed form by

$$P_D(0, T_\theta^x)\mathbb{E}_D^{T_\theta^x}\left[P_D(T_\theta^x, T_a^x)\left[K - S_{a,b,c,d}^x(T_\theta^x)\right]^+ C(S_{T_\theta}^x)\middle|\chi^x = x\right]$$

$$= P_D(0, T_a^x)\Bigg[\Big(U_0 - U_1\tilde{\gamma}_a + \frac{1}{2}U_2(1 + \gamma_a^2) - \frac{1}{3!}U_3(\tilde{\gamma}_a{}^3 + 3\gamma_a)\Big)\Phi(\tilde{\kappa}) \tag{3.24}$$

$$-\Big(U_1\tilde{\gamma}_a + \frac{1}{2}U_2(-2\tilde{\gamma}_a + \tilde{\kappa}) - \frac{1}{3!}U_3(-3\tilde{\gamma}_a{}^2 + 3\tilde{\kappa}\tilde{\gamma}_a - \tilde{\kappa}^2 - 2)\Big)\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\tilde{\kappa}^2\right)\Bigg],$$

where $\tilde{\kappa} = \kappa + \gamma_0$ and $\tilde{\gamma_0} = \gamma_0 + Z_0$.

Consequently, the price of a cash-settled swaption, when accounting for stochastic - basis and using a one-factor OIS rate model, can be given by a one-dimensional integral. The closed form approximation for cash-settled swaptions presented by Henrard (2010a) allows one of the integrals to be evaluated analytically.

Chapter 4

Model Review

In this chapter, some of the issues with the Mercurio and Xie (2012) framework are reviewed and methods of overcoming these are suggested.

4.1 General Implementation Notes

First, a brief note on the implementation of the spread model for S_k^x is presented via the following steps:

1. Calculate initial spread values, $S_k^x(0)$:
   - Bootstrap OIS curve from market OIS rates or retrieve bootstrapped OIS curve from Bloomberg/Reuters;
   - Bootstrap relevant x-month Euribor curve using multi-curve bootstrapping techniques [1] or retrieve bootstrapped x-month curve from Reuters or Bloomberg;
   - Determine initial relevant forward rates $F_k^x(0)$ and $L_k^x(0)$ using curves.
2. Calculate α_k^x and β_k^x:
   - Choose ν_k^x and ρ_k^x parameters;
   - Determine $\mathrm{Var}[F_k^x(T_{k-1}^x)]$ (for all k) by implying the $F_k^x(T_{k-1}^x)$ volatility from the OIS caplet price (see Appendix B.2.2 and Appendix B.3.4 for more details in the case of a Hull-White or G2++ OIS model respectively);
   - Determine $\mathrm{Var}[\chi^x(T_{k-1}^x)]$ (for all k) using a log-normal variance expression;

[1] see Chibane *et al.* (2009), Pallavicini and Tarenghi (2010) or a number of other authors for multi-curve bootstrapping algorithms

3. Determine $F_k^x(t)$ and $\chi^x(t)$:
   - Generate standard normal random variables to simulate realisations of $F_k^x(t)$ and $\chi^x(t)$;
   - Use these and the distributions of the OIS forward rates and stochastic-basis factors to simulate realisations of $F_k^x(t)$ and $\chi^x(t)$.
4. Finally calculate $S_k^x(t)$.

This implementation of the spread model to determine $S_k^x(t)$, together with the simulation of $F_k^x(t)$ allows for the implicit simulation of $L_k^x(t)$. It is noted that this explicit implementation to determine $S_k^x(t)$ and consequently $L_k^x(t)$ is only required in this form when analysing the distributions of $S_k^x(t)$ as well as the distribution of fair FRA and Swap rates. However, when pricing instruments such as caps and swaptions the evaluation of the integrals in the pricing formulae given in Equations 3.12, 3.13, 3.18 and 3.19 are required. In this case, realisations of $F_k^x(t)$ and $\chi^x(t)$ are not generated. Rather, $F_k^x(t)$ and $\chi^x(t)$ are determined for the given distributions of the short-rate factor(s)[2] and stochastic-basis factor.

4.2 Possibility of Negative Spreads

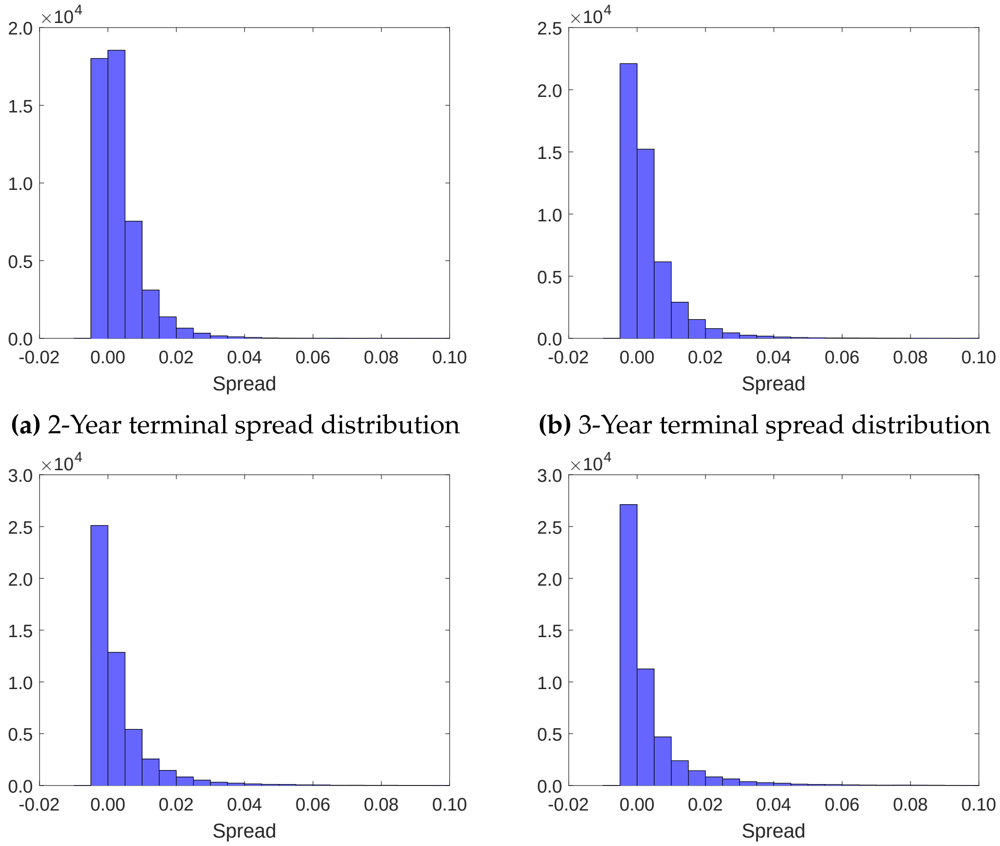
One of the most important considerations when modelling the multi-curve interest environment is ensuring the non-negativity of the spread between interbank deposit rates and those implied from OIS rates. Historically, spreads have almost always been positive and this is likely to be preserved in the future since interbank deposits inherently have more credit risk than their OIS counterparts. This argument only breaks down in the presence of large liquidity shortages OIS market — a highly irregular case which is noted but not accounted for.

Examining the spread model given by Equation 2.6, it can be clearly seen that there is nothing guaranteeing the non-negativity of $S_k^x(t)$. Both $[F_k^x(t) - F_k^x(0)]$ and $[\chi_k^x(t) - \chi_k^x(0)]$ can easily by negative and the predicted spread will be negative as soon as

$$\alpha_k^x \left[F_k^x(t) - F_k^x(0)\right] + \beta_k^x \left[\chi_k^x(t) - \chi_k^x(0)\right] < -S_k^x(0). \tag{4.1}$$

In order to examine the extent of this drawback, a Monte-Carlo experiment is performed to analyse the distribution spreads on a variety of forward rates. The forward rates are those relevant to the pricing of a 5-year swaption on a 5-year swap with the semi-annual floating and fixed leg payments (i.e. 6-month tenor). The alternative formulation of the spread model, given by Equation 2.13, is used together with a Hull-White model for OIS rates. The model parameters follow those used by Mercurio and Xie (2012) for pricing a 5Y5Y swaption given by: $a = 0.001$, $\sigma = 0.008$, $\eta_k^x = \eta = 0.5$, $\nu_k^x = 0.015$, $\rho_k^x = -0.5$. The spreads were simulated independently over-time out to 5 years with a sample size of 50 000. The distributions of the spread after 2-, 3-, 4- and 5-years on the forward rate set at 5-years expiring at 5.5-years are shown in Figure 4.1.

[2] $F_k^x(t)$ is governed by one short-rate factor when using the Hull-White model and two short-rate factors when using the G2++ model for OIS rates



(a) 2-Year terminal spread distribution

(b) 3-Year terminal spread distribution

(c) 4-Year terminal spread distribution

(d) 5-Year terminal spread distribution

Fig. 4.1: Distribution of the spread on the forward rate set at 5-years expiring at 5.5-years for a variety of times

In Figure 4.1 it is noted that the first bar in each histogram represents the number of negative spread realisations. This clearly shows that a significant number of the spread realisations are less than zero. In addition, the proportion of negative spread realisations to total realisations increases as the time-period over which the spreads are simulated. This is as expected, since the longer the simulation period the more uncertainty there is and more variation can be expected. Similar distributions for the spreads on a variety of other forward rates can also be generated. Using these Monte-Carlo experiment results, the number of negative spread realisations can be used to estimate the probability of negative spreads on a variety of forward rates over time. The results are given in Figure 4.2, which clearly show the significant probability of negative spreads.

It can be seen that as the time-period over which the spreads are simulated increases, so too does the probability of the spread being negative. This is expected due to the increased variation in $F_k^x(t)$ and $\chi_k^x(t)$ over time. When pricing a 5Y5Y swaption one would need to simulate these spreads out to 5 years. Worryingly, this means that the a large portion of the realisations used to obtain the price are not consistent with the prevailing environment — resulting in an incorrect price.

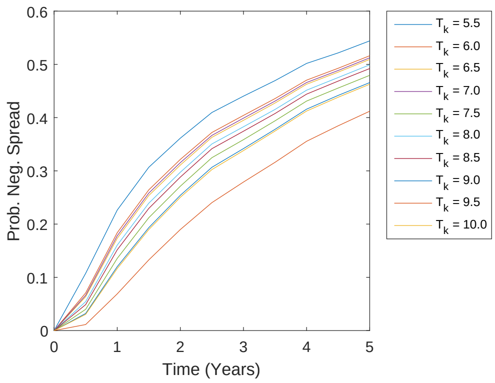


Fig. 4.2: Estimated probability of negative spreads

The different forward rates all exhibit different probabilities of negative spreads and it appears that the further along the yield curve the forward rate lives, the smaller the probability of negative spreads. However, it will be shown that the degree of negative spread probability is rather governed by the shape of the initial yield curves and initial spread levels rather than the position of the forward rate on the yield curve.

The condition for the existence of negative spreads, given by Equation 4.1, is clearly dependent on the spread model parameters α_k^x and β_k^x. This motivates the following subsection in which constraints are placed on the model parameters in order to make the probability of negative spreads negligible.

4.2.1 Constraining the Spread Model Parameters

In deriving the constraint, first the simplifying assumption that the OIS forward rates, $F_k^x(t)$, are strictly positive is made. This assumption is clearly violated in the European interest rate markets. In addition, a number of interest rate models which could be used for $F_k^x(t)$ allow negative rates — for example those belonging to the Gaussian class of short-rate models which are used in this work. That being said, the assumption serves as an analytical starting point.

The assumption allows a lower bound of zero to be placed on the future value of $F_k^x(t)$. This is strictly the case for the log-normal $\chi_k^x(t)$. Using these facts it becomes easy to define the constraint that needs to be met to ensure non-negative spreads. This is given in Equation 4.2

$$\xi_k^x := S_k^x(0) - \alpha_k^x F_k^x(0) - \beta_k^x \geq 0. \tag{4.2}$$

It is noted that the definition of ξ_k^x is conveniently equivalent to the quantity defined in Equation 2.8 which is used in the derivation of most of the interest rate derivative prices. It is interesting to note that Mercurio and Xie (2012) also use this quantity, however no mention of the required constraint is made. The constraint values from the Monte-Carlo experiment performed above for each of the forward rates are given in Table 4.1.

Tab. 4.1: ξ_k^x constraint values

T_k	ξ_k^x
5.5	-0.0049
6.0	-0.0041
6.5	-0.0037
7.0	-0.0034
7.5	-0.0029
8.0	-0.0028
8.5	-0.0025
9.0	-0.0022
9.5	-0.0017
10	-0.0018

It can be seen that for every single one of the forwards rates for which the corresponding spread was simulated, the constraint was violated — explaining the prevalence of negative spreads in Figure 4.2. The constraint parameter values provide further insight into the different probabilities of negative spreads for the various forward rates. It can be seen that the greater the violation of the constraint (that is the more negative ξ_k^x is) the greater the probability of negative spreads. This provides further evidence to the validity of the constraint proposed. To further analyse the ability of the constraint to prevent negative spreads, the Monte-Carlo experiment presented above is repeated with adjusted parameters. The ν_k^x value is varied to adjust the α_k^x and β_k^x parameters keeping ρ_k^x.

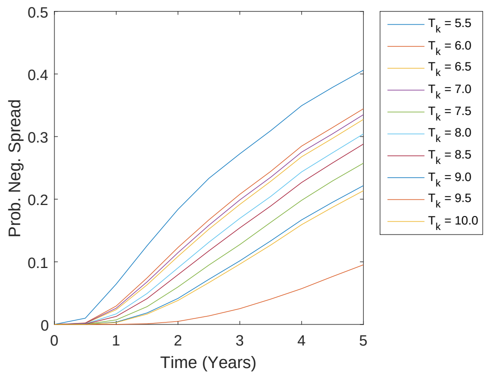


Fig. 4.3: Negative spreads for $\nu_k^x = 0.012$

Tab. 4.2: ξ_k^x constraint values with adjusted spread parameters

T_k	ξ_k^x
5.5	-0.0022
6.0	-0.0016
6.5	-0.0014
7.0	-0.0013
7.5	-0.0009
8.0	-0.0010
8.5	-0.0009
9.0	-0.0006
9.5	-0.0003
10	-0.0005

Clearly from Figure 4.3 and Table 4.2 it can be seen that reducing the standard deviation of the spread reduces the ξ_k^x values and thus results in lower probabilities of negative spreads.

Finally, the effect of assuming non-negative forward rates when deriving a constraint for α_k^x and β_k^x is investigated. For this case ν_k^x and ρ_k^x are not defined but an arbitrary α_k^x is chosen and used to determine β_k^x such that $\xi_k^x = 0$ for all k. The results can be seen in Figure 4.4

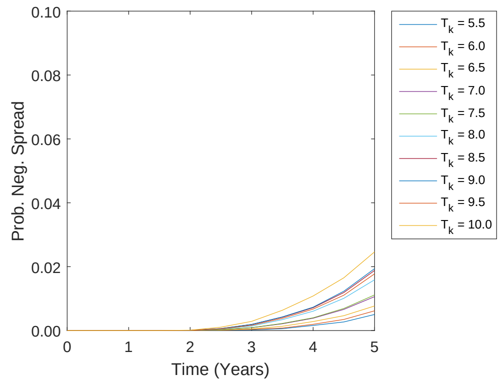


Fig. 4.4: Probability of negative spreads when $\xi_k^x = 0$

Figure 4.4 illustrates that even if the constraint is perfectly met there will still be negative spreads since $F_k^x(t)$ is not bounded below by zero when using the Hull-White short-rate model. However, if the constraint is met it can be seen that the probability of negative spreads is well below 2.5% for each forward rate.

This suggests that although the constraint was derived under the assumption of positive rates, it is still sufficient to ensure negligible probabilities of negative spreads even if negative forward rates are allowed in the model. Re-examining the assumption on the back of these results, validates the approximation even though it is violated in practice. Although negative rates are prevalent in the EUR market, these typically occur on the shorter end of the yield curve. When pricing swaptions one is typically looking towards the longer end of the yield curve and thus these rates are either positive or small negative numbers. As a result, a lower bound of zero on F_k^x is a valid approximation and can be used in defining a constraint on the model parameters. The spread model parameters that satisfy the constraint on ξ_k^x are denoted admissible spread model parameters.

It is noted that although results are presented exclusively when using the Hull-White model for OIS rates; similar trends are observed when using the G2++ model since they are both Gaussian short-rate models.

It is also important to note that this is by no means the only method that can be used to derive constraints on the model parameters. Instead of making the simplifying assumption of non-negative OIS forward rates, the derivation could rather start from any (possibly negative) lower bound on the OIS forward rate. This lower bound could then be treated as an additional parameter in the model. This could possibly reduce the probability of negative spreads however this has to be weighed up with additional calibration costs.

Chapter 5

The Effect of Stochastic-basis on Interest Rate Derivative Pricing

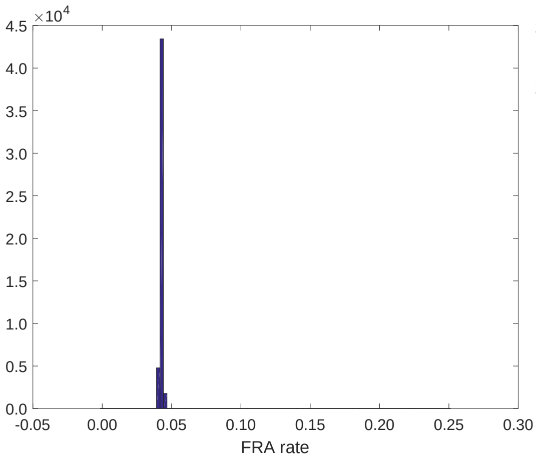
In this chapter the pricing formulae derived in Chapter 3 are implemented in order to illustrate the effect that the inclusion of stochastic-basis has on interest rate derivative pricing. Results are presented for two different interest rate models for the OIS rates where relevent — a one-factor Hull-White model and the two-factor G2++ model. In the case of the one-factor Hull-White model, the parameters used are those given by Mercurio and Xie (2012) in order to produce comparable results so that the consistency of our implementation of the spread model can be confirmed. Furthermore, market data from the same day is used for calibrating the Hull-White model to the initial term structure as well as the initial spread model parameters. This market data is Euro data from August 3, 2012. While the data is relatively old, it is noted that the aim of this work is centred around the implementation of the stochastic spread model and the exact date from which market data is obtained is not an important consideration. Instead, the focus is on producing comparable results to Mercurio and Xie (2012) and examining trends.

5.1 FRA Rates

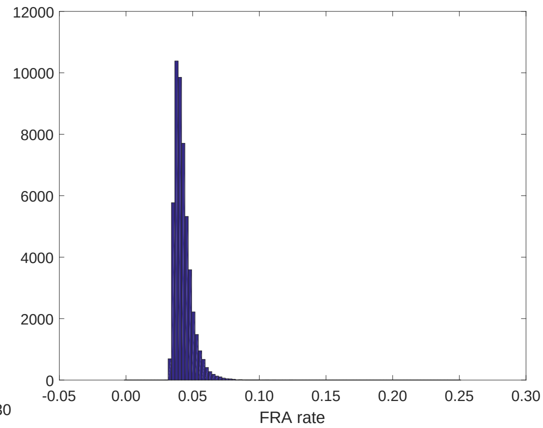
In order to analyse the effect of stochastic-basis on FRA's, the expression given by Equation 3.3 for an approximation of the fair FRA rate is implemented. The fair forward rate for a 6x12 FRA is simulated over a period of 5 years for a sample size of 50 000. The resulting terminal distribution for varying α_k^x and β_k^x parameters is analysed (noting that this is achieved by varying ν_k^x and ρ_k^x since the alternative parametrisation of the spread model given by Equation 2.13 is used).

A Hull-White model for the OIS rates is used with $a = 0.001$ and $\sigma = 0.008$ while the stochastic-basis factor volatility parameter is given by $\eta = 0.25$ as per Mercurio and Xie (2012). The results can be seen in Figure 5.1; noting that the histograms are plotted using the same bin widths for ease of comparison.

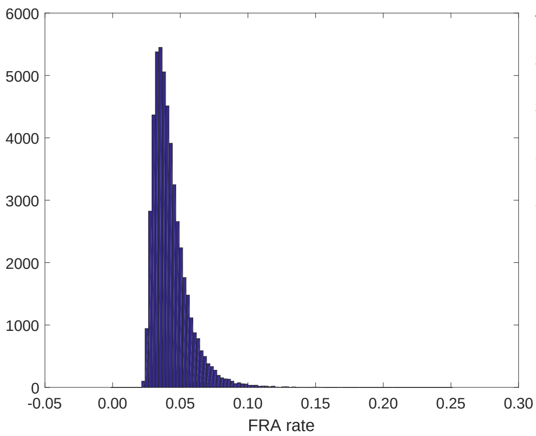
Figure 5.1 clearly shows a shift in the terminal distribution of the fair FRA rates. If the deterministic-basis case (Figure 5.1a) is compared against any of the stochastic-basis cases it can be seen that the inclusion of stochastic-basis clearly increases the variance of the distribution of the fair FRA rates. Additionally the prevalence of the stochastic-basis factors can be seen in the lognormal-like distribu-



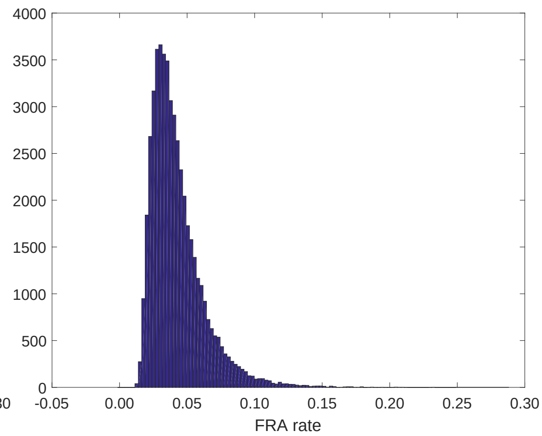
(a) $\nu_k^x = 0.000$



(b) $\nu_k^x = 0.002$



(c) $\nu_k^x = 0.004$



(d) $\nu_k^x = 0.006$

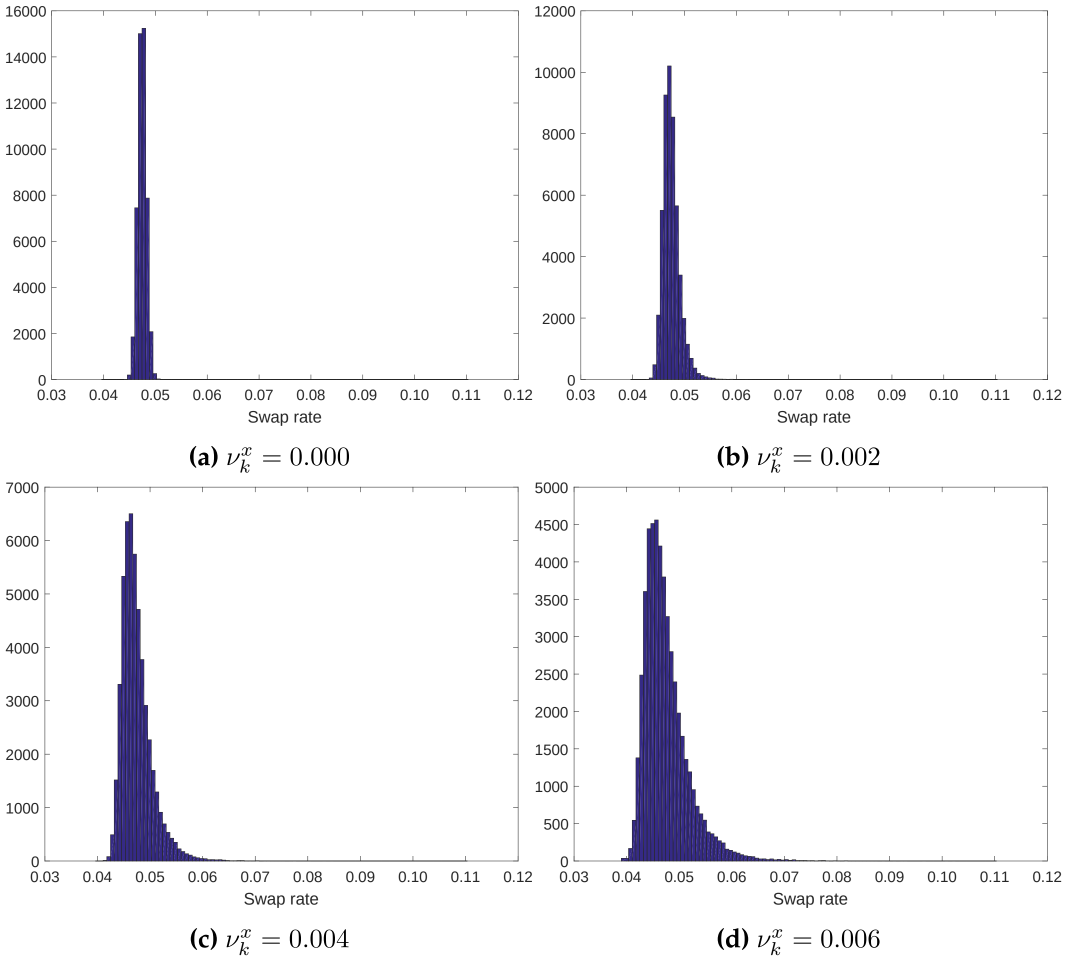
Fig. 5.1: Effect of stochastic-basis on the terminal distribution of FRA rates

tions that result. It is also noted that the larger the standard deviation of the spread (ν_k^x) the greater the variation in the FRA rate; as is expected. Finally, it is noted that when using the G2++ model for the OIS rates similar trends are observed.

5.2 Swap Rates

In order to analyse the effect of stochastic-basis on swaps, the expression given by Equation 3.5 for the fair swap rate is implemented. The fair swap rate for a 5 year swap with semi-annual fixed and floating payments is simulated over a period of 5 years using a sample size of 50 000. The resulting terminal distribution for varying α_k^x and β_k^x parameters is then analysed.

A Hull-White model for the OIS rates is used with $a = 0.001,\ \sigma = 0.008$, while $\eta = 0.25$. The results are given in Figure 5.2 where again the histograms are plotted using the same bin widths.



(a) $\nu_k^x = 0.000$ **(b)** $\nu_k^x = 0.002$

(c) $\nu_k^x = 0.004$ **(d)** $\nu_k^x = 0.006$

Fig. 5.2: Effect of stochastic-basis on the terminal distribution of Swap rates

Similar trends to those observed for the effect on FRA rates can be seen. Again, it can be seen that the inclusion of stochastic-basis results in a more varied distribution of swap rates when compared to the deterministic-basis case (Figure 5.2a). The swap rates also appear log-normal due to the prevalence of the log-normal stochastic-basis factors. Furthermore, the larger the standard deviation of the spread the larger the observed variation in the swap rate distribution as expected. The effect on the fair swap rate shown here is important when considering the pricing interest rate swaptions. It is this swap rate that needs to be evolved to the maturity of the swaption and the variation thereof dictates the price of the option to a large extent.

5.3 Interest Rate Caps

In order to analyse the effect of stochastic-basis on interest rate caps, cap volatility skews for varying α_k^x and β_k^x parameters are generated. To generate these Equation 3.6 is used together with the semi-analytical formulae for caplet prices given by Equation 3.12 for a one-factor OIS rate model and Equation 3.13 for a two-factor OIS rate model. The integrals in Equations 3.12 and 3.13 are evaluated numerically using a one-dimensional and two-dimensional numerical integration scheme respectively with simple quadrature techniques. Since the two OIS rate models used are both Gaussian short-rate models, the integrals over the entire Gaussian densities can be accurately approximated by integrating the short-rate factors over an appropriate number of standard deviations from the mean[1]. Once the relevant cap has been priced, the cap volatility is then implied using the market formula for caps given by Mercurio (2009)

$$\mathbf{Cap}_{\text{mkt}}(0, K, \sigma, T_a^x, T_b^x) = \sum_{k=a}^{b} \tau_k^x P_D(0, T_k^x) \text{Bl}\left(K, L_k^x(0), \sigma\sqrt{T_{k-1}^x}\right). \tag{5.1}$$

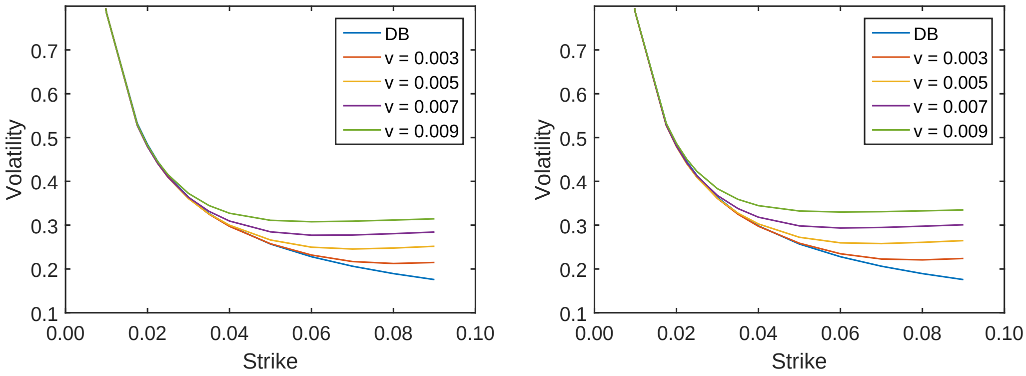
In order to generate a volatility skew, each cap is priced with a variety of strikes. Bloomberg or Reuters provide cap volatilities for at-the-money strikes as well as fixed strikes from 1% up to 10%. Consequently, caps are priced at each of the market quoted strike values.

5.3.1 Caps with Hull-White Model for OIS rates

Using a one-factor Hull-White model for the OIS rates with parameters $a = 0.001$, $\sigma = 0.008$ and stochastic-basis factor volatility parameter $\eta = 0.5$ the cap volatility skews for the above mentioned strikes for varying stochastic-basis model parameters ($\nu_k^x = \nu$, $\rho_k^x = \rho$) are generated. It is noted that our parameter choices are restricted to those values of ν and ρ that ensure $\xi_k^x \geq 0$ for all k.

Following the approach of Mercurio and Xie (2012), the volatility skew for the deterministic-basis model ($v_k^x = 0$) is first determined. Then, for each change in the stochastic-basis model parameters the Hull-White volatility parameter (σ) is recalibrated so that the ATM volatilities match. Using this recalibrated σ the remainder of the volatility skew is then generated. This method follows from typical market practice where some of the interest rate model parameters are calibrated to market data. In this case, the stochastic-basis model is calibrated to ATM data from our deterministic model and thus the effect of stochastic-basis on calibrated volatility skews can be analysed. The results can be seen in Figure 5.3.

[1] ± 6 standard deviations were found to be more than sufficient



(a) $\rho_k^x = -0.5$ **(b)** $\rho_k^x = 0$

Fig. 5.3: Effect of varying stochastic-basis model parameters on cap volatility skews with a Hull-White OIS rate model

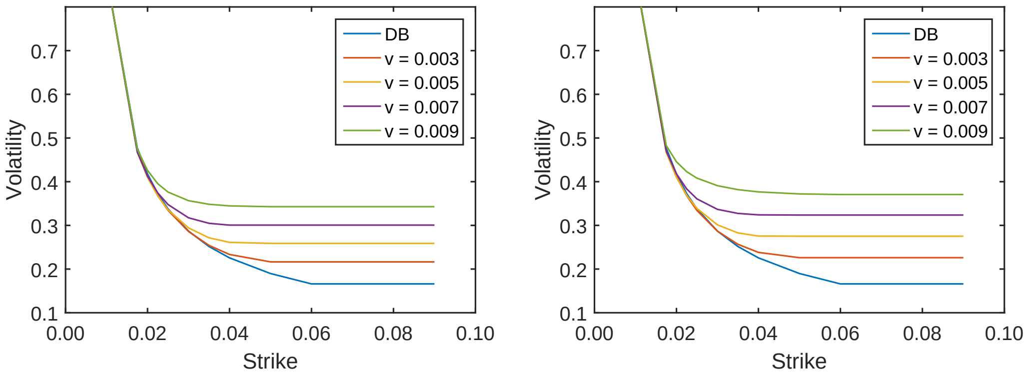
From Figure 5.3 the impact of changing the stochastic-basis model parameters on out-the-money-caps can be seen since the ATM volatilities have been calibrated (noting that the ATM strike ≈ 0.01). It can be seen that the larger the spread standard deviation, the larger the price (and thus volatility) of these caps. This can be explained by considering that the inclusion of stochastic-basis increases the variation of the LIBOR forward rates $L_k^x(t)$. Consequently, this increases the likelihood of these out-the-money caps ending in-the-money — increasing the price of the option.

In terms of the effect of the correlation between the OIS forward rates and the spread (ρ_k^x), it can be seen that a correlation of zero results in a marginal increase in the out-the-money volatilities when compared to a correlation of -0.5. A negative correlation results in a negative α_k^x. In addition, a more negative correlation results in a smaller β_k^x. This results in smaller spreads and consequently lower prices (and thus volatilities). In the case of a positive correlation, the larger the correlation, the smaller the resulting β_k^x. Once again this leads to lower prices (and volatilities).

It is important to note that the effect of stochastic-basis is only evident at strikes of around 4%. This is more than 300bps above the ATM strike which begs the question: what is the relevance of these caps — especially in the prevailing low interest rates in the European market? Typically, the more liquid caps are those traded much closer to the ATM strikes. Interestingly, Figure 5.3 illustrates that the inclusion of stochastic-basis has a very limited effect on the pricing of these caps when admissible spread model parameters are used.

5.3.2 Caps with G2++ Model for OIS rates

The two factor G2++ model for OIS rates is used to generate similar graphs with $a = 0.001$, $b = 0.001$, $\sigma = 0.004$, $\zeta = 0.005$ and $\rho_{G2} = -0.5$ while $\eta = 0.5$. It is noted that by equating a and b, the G2++ model is forced to behave like a single-factor model. This serves as a consistency check of our semi-analytical cap pricing formula with a two-factor OIS rate model. The results are given in Figure 5.4.



(a) $\rho_k^x = -0.5$ **(b)** $\rho_k^x = 0$

Fig. 5.4: Effect of varying stochastic-basis model parameters on cap volatility skews with a G2++ OIS rate model

As expected, similar trends are observed in Figure 5.4 when compared to those in the case of the Hull-White OIS rate model. This confirms the consistency of our extension of the stochastic-basis framework presented by Mercurio and Xie (2012) to include a two-factor OIS rate model in pricing caps. To price caps under this model (which is in fact a three-factor model due to the presence of the stochastic-basis factors) requires the evaluation of a two-dimensional integral for each caplet contained in the cap. Consequently, the use of a two-dimensional numerical integration scheme results in a significant increase in computational time, since the implementation thereof is not conducive to vectorisation techniques. For example, pricing a 5Y cap (containing 10 caplets) across 12 strikes takes 30 seconds when using a two-factor OIS rate model compared to 0.01 seconds when using a one-factor OIS rate model. Consequently, any calibration using this model in its current form is extremely infeasible. It is suggested that closed-form approximations for at least one of the integrals are necessary if any calibration is to be performed. Furthermore, if additional factors are added to either the OIS rate or the stochastic-basis factor models then it is recommended that Monte-Carlo methods be used to evaluate the multi-dimensional integrals required in the pricing formula.

5.4 Interest Rate Swaptions

In order to analyse the effect of stochastic-basis on swaptions, volatility skews for varying α_k^x and β_k^x parameters are generated. To generate these, swaptions are priced using the semi-analytical formulae given by Equation 3.18 for a one-factor OIS rate model and Equation 3.19 for a two-factor OIS rate model. Again, the integrals in Equations 3.12 and 3.13 are evaluated numerically using simple quadrature techniques — integrating the short-rate factor(s) over an appropriate number of standard deviations. Once the relevant swaption is priced, the volatility is implied using the market formula for swaptions given by Mercurio (2009)

$$\mathbf{Swptn}_{\mathrm{mkt}}(0, K, \sigma, \mathcal{T}^x, \mathcal{T}^S) = \sum_{j=c+1}^{d} \tau_j^S P_D(0, T_j^S)\mathrm{Bl}\left(K, S_{a,b,c,d}^x(0), \sigma\sqrt{T_a^x}\right). \quad (5.2)$$

5.4.1 Consistency with Mercurio and Xie (2012)

Swaptions are the only derivative that Mercurio and Xie (2012) present pricing results for. Consequently, this is used as a consistency check for our implementation of the stochastic-basis framework. It is noted that although swaptions are presented as the last instrument in this chapter, the consistency of our implementation was confirmed first and the order of presentation is rather a reflection on the simplicity of the instruments.

To ensure comparable results are produced, Mercurio and Xie (2012) was followed to price 5Y5Y payer swaptions — with semi-annual fixed and floating legs on 3 August 2012 — using the same Hull-White model parameters. In addition, the swaptions are priced at the same strikes (ATM as well as ATM $\pm 100bp$, $\pm 50bp$, $\pm 25bp$).

First, two versions of our deterministic-basis spread model are compared to that of Mercurio and Xie (2012). The first, where the OIS and 6-month curves are bootstrapped from the relevant market quotes using multi-curve bootstrapping techniques. In the second case, bootstrapped OIS and 6-month curves are instead retrieved from data providers such as Reuters or Bloomberg [2]. The results are shown in Figure 5.5.

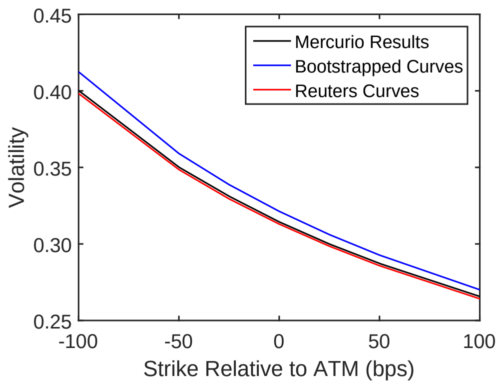


Fig. 5.5: Comparison of deterministic-basis model swaption vol. skews

It can be seen that using different OIS and 6-month curves results in a different deterministic-basis swaption volatility skew. The differences in the input curves result from variations in the multi-curve bootstrapping techniques; as well as the

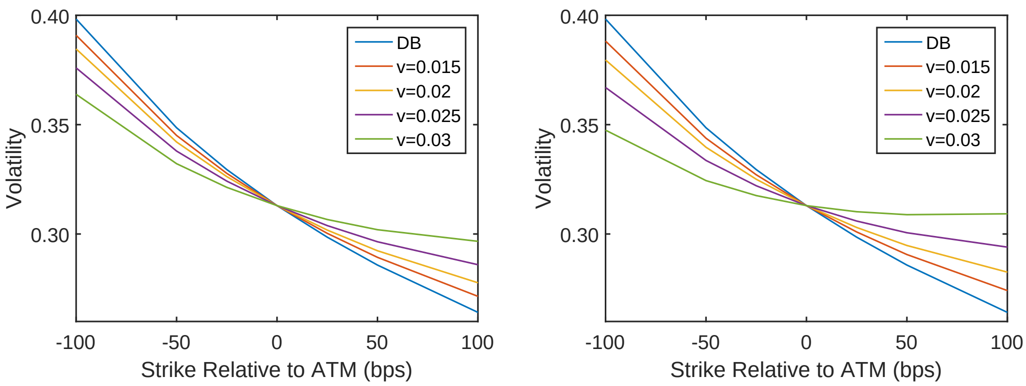
[2] Results are only present when using the Reuters curves since the differences between these and those from Bloomberg are negligible

type of interpolation used. These curves play a role in the calibration of the Hull-White model to the initial OIS term structure; as well as in the initial spread values and thus have a noticeable effect on pricing.

While neither implementation reproduces the Mercurio and Xie (2012) result perfectly, the use of the Reuters curves produce the most consistent results and is thus the method of choice. The mean relative error of less than half a percent is considered negligible — especially due to the uncertainties surrounding the bootstrapping procedures; as well as the interpolation methods actually used by Mercurio and Xie (2012).

5.4.2 Swaptions with Hull-White Model for OIS Rates

Volatility skews are then generated for the above mentioned strikes for varying stochastic-basis model parameters (ν_k^x, ρ_k^x) following those used by Mercurio and Xie (2012). Again, the Hull-White volatility parameter is recalibrated for each variation in the stochastic-basis model parameters in order to reproduce the results presented by Mercurio and Xie (2012). The results can be seen in Figure 5.6:



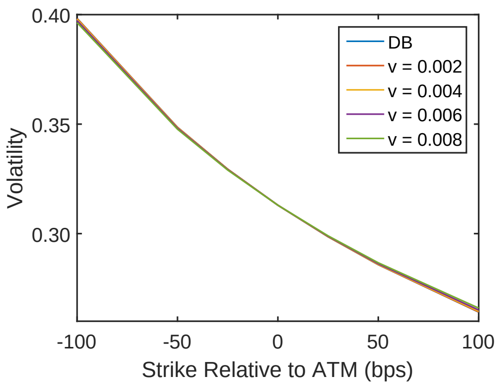
(a) $\rho_k^x = -0.5$ **(b)** $\rho_k^x = 0$

Fig. 5.6: Effect of varying stochastic-basis model parameters on payer swaption volatility skews with a Hull-White OIS rate model

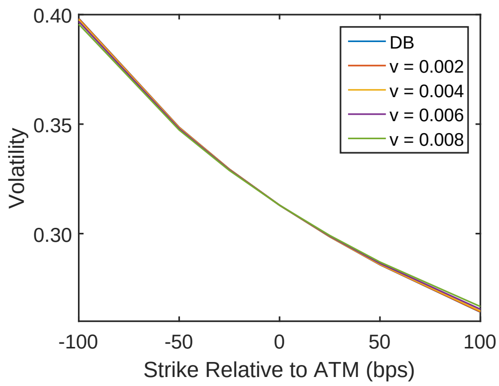
It is first noted that Figure 5.6 accurately reproduces the results given by Mercurio and Xie (2012). It can be seen that increasing the spread standard deviation results in a rotation of the volatility skews. This can be explained by considering that in-the-money payer swaptions have a larger chance of ending out-the-money for larger spread standard deviations; while out-the-money payer swaptions have a larger chance of ending in-the-money. Consequently, the in-the-money swaption prices (and thus volatilities) are lower; while out-the-money swaption prices (and thus volatilities) are higher.

In terms of the effect of the correlation between the OIS forward rates and the spread (ρ_k^x), it can be seen that an increase in the correlation from -0.5 to 0 increases the degree of rotation that is observed. A negative correlation results in a negative α_k^x, while the larger the absolute value of ρ_k^x the smaller β_k^x. This results in smaller spreads and consequently lower prices (and thus volatilities) explaining the increase in the degree of rotation.

It is important to note that although it appears as if the inclusion of stochastic-basis has a large effect on the pricing of swaptions; every ν_k^x parameter choice used by Mercurio and Xie (2012) — and thus presented in Figure 5.6 — violates the constraint placed on the α_k^x and β_k^x parameters. Recalling Figure 4.2 from Section 4.2, it can be seen that even the smallest ν_k^x value of 0.015 results in significant probabilities of negative spreads. Consequently, swaption prices obtained using this parameter cannot be considered consistent and are thus not valid. In Figure 5.7, results when using only ν_k^x and ρ_k^x parameters that ensure $\xi_k^x \geq 0$ for all k are presented.



(a) $\rho_k^x = -0.5$



(b) $\rho_k^x = 0$

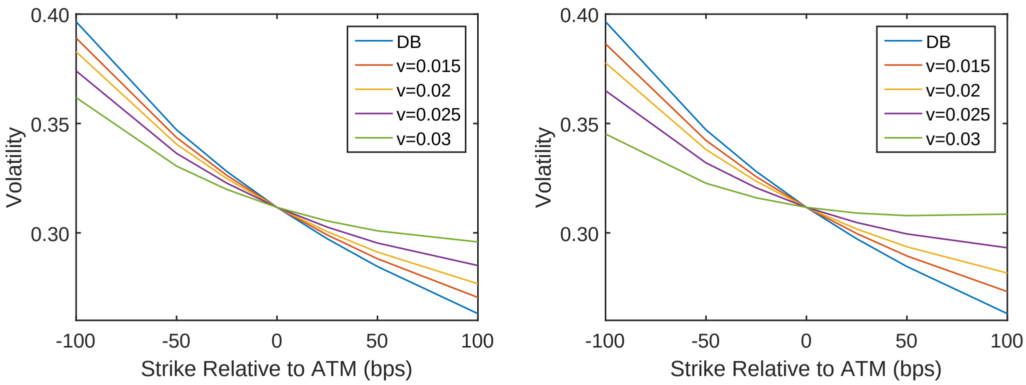
Fig. 5.7: Effect of varying stochastic-basis model parameters on payer swaption volatility skews with a Hull-White OIS rate model

Figure 5.7 illustrates that when the constraint on α_k^x and β_k^x is enforced, the inclusion of stochastic-basis has a minimal impact on swaption pricing in the Mercurio and Xie (2012) framework.

5.4.3 Swaptions with G2++ Model for OIS Rates

The two factor G2++ model for OIS rates is used to generate similar graphs with $a = 0.001$, $b = 0.001$, $\sigma = 0.004$, $\zeta = 0.005$ and $\rho_{G2} = -0.5$ while $\eta = 0.5$. Again, it is noted that by equating a and b, the G2++ model is forced to behave like a single-factor model. This serves as a consistency check of our semi-analytical cap pricing formula with a two-factor OIS rate model. The results are given in Figure 5.8.

Figure 5.8 shows similar trends to those obtained when using a Hull-White model for the OIS rates — confirming the consistency of our extension of the stochastic -basis framework of Mercurio and Xie (2012) to include a two-factor OIS rate model when pricing swaptions. This two-factor OIS rate model, together with the stochastic -basis factor, results in a three-factor multi-curve model. To price swaptions under this model requires the evaluation of a two-dimensional integral however, the implementation thereof is conducive to vectorisation techniques. Conse-



(a) $\rho_k^x = -0.5$ **(b)** $\rho_k^x = 0$

Fig. 5.8: Effect of varying stochastic-basis model parameters on swaption volatility skews with a G2++ OIS rate model

quently, the extra computational requirements moving from 1001 function evaluations in the one-factor case, to evaluating the function on a 1001x1001 grid in the two-factor case, results in a tenfold increase in computational requirements when pricing swaptions across 8 strikes.

While this additional factor can be seen to result in a significant increase in computational time, the current implementation (using numerical integration to evaluate the two-dimensional integral) is still computationally tractable and calibrations using this model are still possible. However, it is suggested that if additional factors are added to either the OIS rate, or stochastic-basis factor model, then Monte-Carlo methods should be used to evaluate the multi-dimensional integrals in the pricing expressions.

Chapter 6

Parameter Estimation

One of the extensions to Mercurio and Xie (2012) that is considered in this dissertation surrounds the parameter estimation in the stochastic-basis framework.

6.1 Issues with Parameter Estimation in this Framework

In the field of interest rate modelling, typically parameter estimation can be divided into two general approaches. The first concerns the use of historical data parameters are found though Maximum Likelihood Estimation (MLE), or Kalman Filtering and MLE, in the case of parameter estimation for hidden processes (i.e., short-rate modelling since the short-rate cannot be observed in the market). The second uses a cross-section of information from the market in the form of cap or swaption volatility surfaces. Subsequently parameters are estimated by finding those that best reproduce the current market prices.

At first glance, the spread model seems best suited to historical parameter estimation. The spreads between OIS and LIBOR rates can be observed in the market, and historical data should provide a wealth of information on the volatilities of the spreads, as well as the spread correlations with OIS rates. That being said, the use of short-rate models for OIS rates (which this work has focused on) means that no closed-form distribution of the spreads $\left(S_k^x(t)\right)$ is available. This is because $F_k^x(t)$ follows a shifted log-normal process (see Appendix B.2.1) while $\chi^x(t)$ follows a log-normal process. Consequently, the use of MLE or Kalman filtering is by no means straight-forward and lies outside the scope of this work due to the time constraints surrounding this dissertation. Furthermore, even if MLE or Kalman filtering was possible, one cannot obtain the stochastic-basis model parameters independently from those of the short-rate model. As a result, the model would be unable to take into account market information contained in swaption or cap volatility surfaces.

This suggests the use of cross-sectional parameter estimation where the short-rate model parameters, as well as the stochastic-basis model parameters, are estimated by finding those that best reproduce the market surfaces. When using a Hull-White model for the OIS rates, this would involve the estimation of five parameters: a, σ, ν^x, ρ^x and η, where it was assumed that $\nu_k^x = \nu^x$ and $\rho_k^x = \rho^x$. Whereas when using a G2++ model for the OIS rates, eight parameters are involved, namely: a, b, σ, ζ, ρ_{G2}, ν^x, ρ^x and η. Unfortunately, due to the number of parameters in both instances, as well as the sensitivity of the pricing formula to the stochastic-basis model parameters one is not even able to calibrate to a model-generated surface and recover the input parameters.

6.2 Rudimentary Calibration of the Multi-curve Model

To overcome some of the issues discussed above with reference to classical parameter estimation techniques, a rudimentary calibration protocol was developed to investigate the feasibility of the calibration of this multi-curve model. A two-step process is proposed:

1. The ν_k^x and ρ_k^x parameters are estimated from a historical time series of yield curves for each k.
2. The short-rate model parameters are then calibrated to a market cap or swaption volatility surface.

Each of these steps is discussed in more detail below.

6.2.1 Historical Estimation of ν_k^x and ρ_k^x

Using a historical time series of OIS and x-month yield curves, one can approximate ν_k^x and ρ_k^x with historical spread standard deviations and correlations respectively. For example, Figure 6.1 shows the historical spread values, $S_k^x(t)$, for a variety of different forward rates with a 6-month underlying tenor for 22 March 2011 to 28 July 2015.

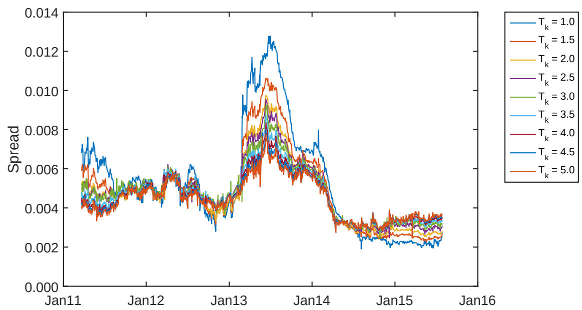


Fig. 6.1: Historical spreads on various forward rates

Figure 6.1 provides an example of the variation of the spread over the past 6 years. Data such as this can be used to determine historical estimations of ν_k^x and ρ_k^x. Table 6.1 shows the estimated parameters relevant to the pricing of a 5Y5Y swaption using the entire historical data sample.

The first thing to note from Table 6.1 is that the estimated parameters satisfy the constraint on α_k^x and β_k^x which ensures non-negative spreads and are thus admissible spread model parameters. If this was not the case, and historical estimations of spread parameters resulted in significant probabilities of negative spreads then

Tab. 6.1: Historical parameter estimation

T_k	ν_k^x	ρ_k^x
5.5	0.0009	0.16
6.0	0.0009	0.17
6.5	0.0008	0.18
7.0	0.0008	0.19
7.5	0.0007	0.20
8.0	0.0007	0.21
8.5	0.0007	0.22
9.0	0.0007	0.23
9.5	0.0006	0.24
10	0.0006	0.26

the stochastic-basis framework would be a poor representation of the prevailing environment. Furthermore, it is noted how small the spread standard deviations are despite the volatile period in 2013. If one was to use more recent data (i.e., after March 2014) these standard deviations would be even smaller. It was shown in Chapter 5 that the inclusion of stochastic-basis on both caps and swaptions is negligible when $\nu_k^x \leq 0.009$. The historical estimations of ν_k^x are all less than a tenth of than this, bringing into question the effect of including stochastic-basis in cap and swaption pricing.

6.2.2 Calibrating to a Swaption Volatility Surface

Using the historical estimates for ν_k^x and ρ_k^x, the calibration of the short-rate model parameters is attempted by finding those that best reproduce a market swaption volatility surface.

First, the calibration to a model-generated surface is attempted. A model - generated swaption volatility surface consisting of 70 ATM swaptions with varying maturities of 1-5 as well as 7 and 10 years, and varying swap lengths of 1 to 10 years is used. Finally, the calibration of the short-rate model parameters to a market swaption volatility surface containing the same ATM swaptions is attempted. For continuity, market data from 3 August 2012 is used and the relative error between the market surface and the calibrated model surface is examined. This relative error is calculated using

$$\text{Relative Error} = \frac{\text{Model Price} - \text{Market Price}}{\text{Market Price}}.$$

Calibrating Hull-White OIS Rate Model

When using a Hull-White OIS rate model, the input parameters can be adequately recovered. This can be seen in Table 6.2.

The parameters obtained when calibrating the Hull-White OIS rate model parameters to market data can be seen in Table 6.3 while the resulting relative error can be seen in the Figure 6.2.

Tab. 6.2: Hull-White calibration to model data

Parameter	Input	Initial Guess	Output
a	0.001	0.02	0.000995
σ	0.008	0.3	0.00800

Tab. 6.3: Hull-White calibration to market data

Parameter	Initial Guess	Output
a	0.02	-0.01216
σ	0.3	0.00742

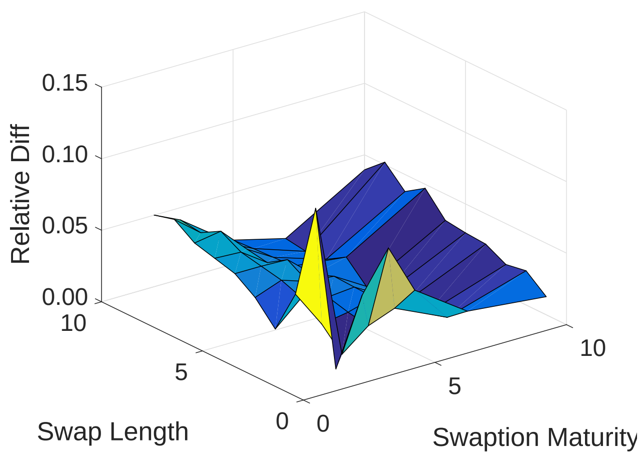


Fig. 6.2: Relative error when calibrating Hull-White OIS rate model parameters to market swaption vol. surface

It is noted that the mean reversion parameter (a) is negative which would actually result in a "mean-diverging" model. However, Brigo and Mercurio (2007) note that this is a common occurrence when calibrating a Hull-White model to either market cap or swaption volatility surfaces.

It can be seen that the calibration is by no means perfect and is especially inept for the swaptions with shorter maturities on shorter swaps. That being said, the relative error is, for the most, part well below 5%. The swaptions with short maturities on short swaps typically perform poorly in calibrations as suggested by Brigo and Mercurio (2007).

It is important to note that typically interest rate models are only calibrated to the most significant swaptions in the market (ignoring those that are illiquid). Alternatively, the calibration will be governed by the product that needs to be priced and the model will only be calibrated to swaptions that are relevant to the product itself (i.e., the product may only be influenced by certain swap rates). The results presented here serve as a feasibility study to investigate the robustness of the implementation, by attempting to calibrate a large swaption surface containing up to 70 instruments. Consequently, it can be concluded that the current implementation of the stochastic-basis framework with a one-factor OIS rate model (namely the Hull-White short-rate model) allows for the calibration of the model parameter to large swaption volatility surfaces.

One concern with this calibration is the dependence of α_k^x on the short-rate model parameters. For each step in the optimisation, a new $\mathrm{Var}[F_k^x(T_{k-1})]$ has to be calculated since the OIS caplet prices (from which the $F_k^x(T_{k-1})$ volatilities are implied) change. Implying these volatilities is fairly computationally intensive — accounting for approximately 40% of the swaption pricing time and consequently significantly affects the calibration time. To speed up the calibration and possibly improve the calibration it is suggested an efficient approximation for $\mathrm{Var}[F_k^x(T_{k-1})]$ under the Hull-White model is found. Alternatively, only interest rate models that have a closed form expressions for the variance of discrete forward rates should be used.

Calibrating the G2++ OIS rate model

When using a G2++ model for the OIS rates the calibration is not as straightforward. The number of parameters to calibrate increases significantly from two to five. In addition, there is an extra dimension in the integrals that have to be determined to calculate the swaption prices; adding to the computational requirements. Consequently, the calibration to model data performs poorly when compared to that of the Hull-White OIS rate model. This can be seen in Table 6.4.

It can be seen that the input model parameters are not adequately recovered. This is concerning, as the calibration to model data is significantly simpler than a calibration to actual market data. If the G2++ correlation parameter, ρ_{G2}, is removed from the calibration then the input parameters can be adequately recovered. This can be seen in Table 6.5.

It can be seen in Table 6.5 how the removal of one of the five parameters from the calibration allows for the adequate recovery of the remaining parameters. The

Tab. 6.4: G2++ calibration to model data

Parameter	Input	Initial Guess	Output
a	0.01	0.015	0.02458
b	0.003	0.005	0.02451
σ	0.005	0.001	0.21969
ζ	0.005	0.008	0.21528
ρ_{G2}	-0.5	-0.2	-0.99958

Tab. 6.5: G2++ calibration to model data excluding ρ_{G2}

Parameter	Input	Initial Guess	Output
a	0.010	0.015	0.00979
b	0.003	0.005	0.00298
σ	0.005	0.001	0.00501
ζ	0.005	0.008	0.00498

parameters obtained when calibrating the G2++ OIS rate model parameters to market data can be seen in Table 6.6, while the resulting relative error can be seen in the Figure 6.3.

Tab. 6.6: G2++ calibration to market data

Parameter	Initial Guess	Output
a	0.004	0.017926
b	0.004	0.026476
σ	0.008	0.008641
ζ	0.008	0.005628

When calibrating the G2++ model parameters to the swaption volatility surface, it can be seen that positive mean reversion parameters are now obtained. As was the case when using the Hull-White OIS rate model, the calibration is by no means perfect and is again inept for the swaptions with shorter maturities on shorter swaps. The relative error is again, for the most, part well below 5%. If the relative errors for the case of the Hull-White OIS rate model are compared to those for the case of the G2++ OIS rate model, a significant difference cannot be observed. Since the G2++ model has extra degrees of freedom and does not assume the perfect correlation of rates when compared to the Hull-White model, it is expected that a significantly better calibration to market data would be achieved. Contrary to this, the sum of the relative errors is in fact less in the case of the Hull-White OIS rate model calibration.

This contradiction speaks to the increase in the computational requirements of pricing swaptions with stochastic-basis with a two-factor OIS rate model. While the feasibility of our implementation has been proven by the reasonable calibration of four of the five parameters to a signifincant swaption surface; it was expected that the use of a two-factor OIS rate model would improve the ability of multi-

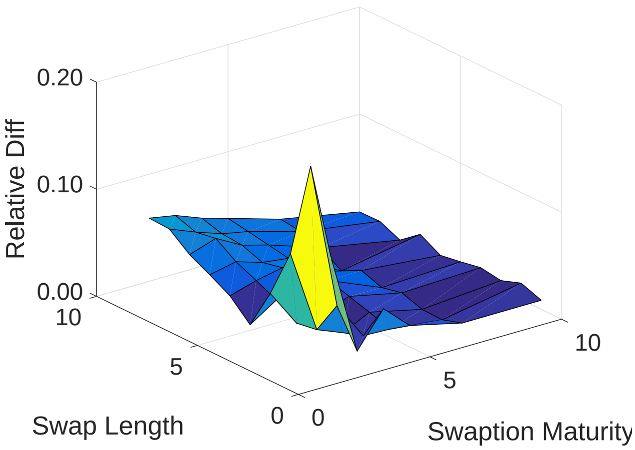


Fig. 6.3: Relative error when calibrating G2++ OIS rate model parameters to market swaption vol. surface

curve model to fit a market volatility surface. This was found not to be the case. Furthermore, each calibration of the G2++ model parameters took at least 20 minutes. While the calibration of a single curve G2++ model to a swaption surface of a similar size takes a few minutes according to Brigo and Mercurio (2007), it can be seen that the inclusion of stochastic-basis greatly increases the computational requirements. If this slowdown, together with the negligible effect that the inclusion of stochastic-basis has been shown to have on swaption prices (especially when historical estimations of the spread model parameters are used) is considered, then the validity of the use of the stochastic-basis framework is brought into question.

Chapter 7

Conclusion

In conclusion, it was found that general pricing formulae in the multi-curve environment for FRA's, swaps, caps/floors as well as physical delivery swaptions could be extended under the Mercurio and Xie (2012) framework to take into account stochastic-basis. Importantly, it was shown how the fundamental choice of the definition of the spread between OIS and LIBOR rates affects the complexity on the resulting pricing formulae. The additive definition of spread used in this framework is better suited for swap and swaption modelled while a multiplicative definition may be better suited to FRA and cap/floor pricing.

Furthermore, it was possible to derive a consistent extension of the Mercurio and Xie (2012) semi-analytical formula for physical delivery swaptions with a one-factor OIS rate model to one with a two-factor OIS rate model. It was found that similar semi-analytical formulae to price caps/floors for a one- and two-factor OIS rate model could also be derived.

In terms of a fundamental review of the model, it has been shown that although the framework uses an explicit spread model it does not preclude the possibility of negative spreads. While this is a significant drawback of the framework, a constraint can be placed on the α_k^x and β_k^x parameters to ensure negligible probabilities of negative spreads. Worryingly, the model parameters used by Mercurio and Xie (2012) violate these constraints which results in prices being obtained where a significant number of the realisations are not consistent with the prevailing environment.

It was also shown that the inclusion of stochastic-basis has a clear effect on the terminal distribution of fair FRA and swap rates. As expected, the inclusion thereof was shown to cause increase in the variation of these distributions. In addition, the prevalence of the stochastic-basis factors was illustrated in the resulting log-normal distributions.

The effect of the inclusion of stochastic-basis on the FRA and swap rate distributions was expected to resonate with the effect on cap/floor and swaption prices. However, when ensuring that the constraint on the α_k^x and β_k^x parameters is met it was shown that the inclusion of stochastic-basis has a small effect on cap/floor and swaption volatility smiles within reasonable distances from the ATM strikes. Importantly, it was found that the parameters used by Mercurio and Xie (2012) to price swaptions violate the necessary constraint and as a result over-estimate the effect that stochastic-basis has on swaption prices.

Finally, issues surrounding the parameter estimation within the stochastic-basis framework were identified. It was shown that it is not well-suited to classic historical parameter estimation techniques such as Maximum Likelihood Estimation as a closed-form distribution of the spread process does not exist when using a short-rate model for the OIS rates. In addition, the number of model parameters makes calibration to a cross-section of market information, in the form of a cap or swaption volatility surface, difficult.

Instead, a rudimentary calibration procedure was found to overcome some of these issues and was used to investigate the feasibility of calibrating this multi-curve model. It was demonstrated that the stochastic-basis model parameters could be estimated from a historical time-series of yield curves. Even over a period of close to six years the variance of the spread was found to be minute, resulting in small ν_k^x parameters; much smaller than those used by Mercurio and Xie (2012). That being said, the estimated parameters were shown to obey the constraints on the spread model parameters; resulting in a consistent model. Moreover, the correlation between the OIS rates and the spread was found to be positive.

Using the historical estimates for the stochastic-basis spread model parameters, it was found that the short-rate model parameters could be calibrated by fitting the model-generated ATM swaption volatility surface to that of the market. When calibrating the Hull-White OIS rate parameters, the mean reversion parameter, a, was found to be negative. Although this would result in a mean diverging model, this was found to be consistent with results from Brigo and Mercurio (2007). The relative calibration errors were found to be all less than 5% — excluding the swaptions with short maturities on short swaps. However, this is in line with Brigo and Mercurio (2007) who suggest that these swaptions present difficulties when using a short-rate model. This calibration confirmed the feasibility of our implementation of the stochastic-basis framework to price swaptions since a reasonable calibration of a relatively large swaption surface of 70 swaptions was achieved. When using this framework to price other product such as CMS spread options one would only need to calibrate to the swaptions relevant to the product; which would improve the calibration error.

In addition to being able to be combined with any OIS rate model, the Mercurio and Xie (2012) stochastic-basis framework was found to allow for the derivation of semi-analytical caps/floors and swaption pricing formulae since the use of the additive definition of spreads allows for the analytical evaluation of one of the pricing integrals. That being said, it was found that the inclusion of stochastic-basis under this framework does not have a significant impact on the pricing of the these interest rate derivatives when admissible spread model parameters are used. The effect of stochastic-basis is only significant when the chosen parameters result in large probabilities of negative spreads which is not consistent with the prevailing environment. In addition, it can be seen that although there has been a large divergence of spreads since the financial crisis of 2007 the spreads remain fairly constant for significant periods of time and are volatile only for short periods. As a result, the possibility of including jump processes in the stochastic-basis factors model is suggested as a log-normal process is not a good representation of these features.

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Appendix A

Supplementary Derivations for Interest Rate Derivatives

A.1 FRA Rate Derivation

$$\mathbb{E}_D^{T_{k-1}^x}\left[\frac{1}{1+\tau_k^x L^x(T_{k-1}^x,T_k^x)}\middle|\mathcal{F}_t\right] = \frac{P_D(t,T_k^x)}{P_D(t,T_{k-1}^x)}\mathbb{E}_D^{T_k^x}\left[\frac{1}{1+\tau_k^x L^x(T_{k-1}^x,T_k^x)}\frac{1}{P_D(T_{k-1}^x,T_k^x)}\middle|\mathcal{F}_t\right].$$

$P_D(T_{k-1}^x,T_k^x)$ can be considered as a forward discount factor and using the definition of OIS forward rates it can be seen that

$$F_k^x(T_{k-1}^x) = \frac{1}{\tau_k^x}\left[\frac{1}{P_D(T_{k-1}^x,T_k^x)} - 1\right]$$

$$\therefore \frac{1}{P_D(T_{k-1}^x,T_k^x)} = 1+\tau_k^x F_k^x(T_{k-1}^x).$$

This gives

$$\mathbb{E}_D^{T_{k-1}^x}\left[\frac{1}{1+\tau_k^x L^x(T_{k-1}^x,T_k^x)}\middle|\mathcal{F}_t\right] = \frac{P_D(t,T_k^x)}{P_D(t,T_{k-1}^x)}\mathbb{E}_D^{T_k^x}\left[\frac{1+\tau_k^x F_k^x(T_{k-1}^x)}{1+\tau_k^x L^x(T_{k-1}^x,T_k^x)}\middle|\mathcal{F}_t\right].$$

Thus we can write

$$\mathbf{FRA}(t;T_{k-1}^x,T_k^x) = \frac{1}{\tau_k^x\frac{P_D(t,T_k^x)}{P_D(t,T_{k-1}^x)}\mathbb{E}_D^{T_k^x}\left[\frac{1+\tau_k^x F_k^x(T_{k-1}^x)}{1+\tau_k^x L^x(T_{k-1}^x,T_k^x)}\middle|\mathcal{F}_t\right]} - \frac{1}{\tau_k^x}.$$

Again using (2) it can be seen that

$$\frac{1}{\frac{P_D(t,T_k^x)}{P_D(t,T_{k-1}^x)}} = 1+\tau_k^x F_k^x(t).$$

This gives

$$\mathbf{FRA}(t;T_{k-1}^x,T_k^x) = \frac{1+\tau_k^x F_k^x(t)}{\tau_k^x\mathbb{E}_D^{T_k^x}\left[\frac{1+\tau_k^x F_k^x(T_{k-1}^x)}{1+\tau_k^x L^x(T_{k-1}^x,T_k^x)}|\mathcal{F}_t\right]} - \frac{1}{\tau_k^x}.$$

Finally, we note that $L^x(T_{k-1}^x,T_k^x) = L_k^x(T_{k-1}^x)$ which gives

$$\mathbf{FRA}(t;T_{k-1}^x,T_k^x) = \frac{1+\tau_k^x F_k^x(t)}{\tau_k^x\mathbb{E}_D^{T_k^x}\left[\frac{1+\tau_k^x F_k^x(T_{k-1}^x)}{1+\tau_k^x L_k^x(T_{k-1}^x)}|\mathcal{F}_t\right]} - \frac{1}{\tau_k^x}.$$

A.2 Swaption Pricing Derivation

Using the definition of $S^x_{a,b,c,d}(T^x_a)$ given by Equation 3.4 and replacing $L^x_k(t)$ with the expression given by Equation 2.7 gives

$$S^x_{a,b,c,d}(T^x_a) \sum_{j=c+1}^{d} \tau^S_j P_D(T^S_c, T^S_j)$$
$$= \sum_{k=a+1}^{b} \tau^x_k P_D(T^x_a, T^x_k) L^x_k(0) + \sum_{k=a+1}^{b} \tau^x_k P_D(T^x_a, T^x_k)\,(1+\alpha^x_k)\,(F^x_k(T^x_a) - F^x_k(0))$$
$$+ \sum_{k=a+1}^{b} \tau^x_k P_D(T^x_a, T^x_k) \beta^x_k\,(\chi^x_k(T^x_a) - \chi^x_k(0))\,.$$

Using the definition of the OIS forward rates (Equation 2.1) the following holds

$$\sum_{k=a+1}^{b} \tau^x_k P_D(T^x_a, T^x_k)\,(1+\alpha^x_k)\,(F^x_k(T^x_a) - F^x_k(0))$$
$$= \sum_{k=a+1}^{b} \tau^x_k \left(P_D(T^x_a, T^x_{k-1}) - P_D(T^x_a, T^x_k)\right)(1+\alpha^x_k) - \sum_{k=a+1}^{b} \tau^x_k P_D(T^x_a, T^x_k) F^x_k(0).$$

Consequently,

$$S^x_{a,b,c,d}(T^x_a) \sum_{j=c+1}^{d} \tau^S_j P_D(T^S_c, T^S_j)$$
$$= \sum_{k=a+1}^{b} \tau^x_k P_D(T^x_a, T^x_k)\,[L^x_k(0) - (1+\alpha^x_k) F^x_k(0) - \beta^x_k \chi^x_k(0)]$$
$$+ \sum_{k=a+1}^{b} \tau^x_k \left(P_D(T^x_a, T^x_{k-1}) - P_D(T^x_a, T^x_k)\right)(1+\alpha^x_k) + \sum_{k=a+1}^{b} \tau^x_k P_D(T^x_a, T^x_k) \beta^x_k \chi^x_k(T^x_a)$$

But

$$\sum_{k=a+1}^{b} \tau^x_k \left(P_D(T^x_a, T^x_{k-1}) - P_D(T^x_a, T^x_k)\right)(1+\alpha^x_k)$$
$$= 1 + \alpha^x_{a+1} + \sum_{k=a+1}^{b-1} P_D(T^x_a, T^x_k)\left(\alpha^x_{k+1} - \alpha^x_k\right) - P_D(T^x_a, T^x_b)\,(1+\alpha^x_b)$$

Defining $\xi^x_k = L^x_k(0) - (1+\alpha^x_k) F^x_k(0) - \beta^x_k \chi^x_k(0)$, therefore gives

$$S^x_{a,b,c,d}(T^x_a) \sum_{j=c+1}^{d} \tau^S_j P_D(T^S_c, T^S_j) = 1 + \alpha^x_{a+1} - P_D(T^x_a, T^x_b)\,[1 + \alpha^x_b - \tau^x_k \xi^x_b]$$
$$+ \sum_{k=a+1}^{b-1} P_D(T^x_a, T^x_k)\left[\alpha^x_{k+1} - \alpha^x_k + \tau^x_k \xi^x_k\right]$$
$$+ \sum_{k=a+1}^{b} \tau^x_k P_D(T^x_a, T^x_k) \beta^x_k \chi^x_k(T^x_a).$$

Finally this gives

$$
\begin{aligned}
&\left[S^x_{a,b,c,d}(T^x_a) - K\right]^+ \sum_{j=c+1}^{d} \tau^S_j P_D(T^S_c, T^S_j) \\
&\quad = \sum_{k=a+1}^{b} \tau^x_k P_D(T^x_a, T^x_k) \beta^x_k \chi^x_k(T^x_a) - K \sum_{j=c+1}^{d} \tau^S_j P_D(T^S_c, T^S_j) + 1 + \alpha^x_{a+1} \\
&\qquad - P_D(T^x_a, T^x_b) \left[1 + \alpha^x_b - \tau^x_k \xi^x_b\right] + \sum_{k=a+1}^{b-1} P_D(T^x_a, T^x_k) \left[\alpha^x_{k+1} - \alpha^x_k + \tau^x_k \xi^x_k\right].
\end{aligned}
$$

Appendix B

Interest Rate Models

B.1 The Hull and White (1993) Short-rate Model

In this section we introduce the Hull and White (1993) short-rate model and derive the results required for its use as the OIS rate model when pricing swaptions. These results include:

- Fitting the model to the initial observed term structure
- The distribution of the short-rate under the $\mathcal{Q}^T$ forward measure
- The volatilities of discrete forward rates

B.1.1 Model Fundamentals

The classic formulation of the Hull and White (1993) extension of the Vasicek (1977) short-rate model is given by:

$$dr(t) = (v(t) - ar(t))dt + \sigma dW_t \tag{B.1}$$

where a and σ are constants and $v(t)$ is a deterministic function which is chosen to fit the initial term structure of interest rates.

The ability of the Hull-White model to fit the initial term structure is clearly attractive. However, determining this $v(t)$ function that allows for this is often a non-trivial task as will be seen below.

The Hull-White model is an Affine Term Structure Model and thus bond prices are of the form:

$$P(t,T) = e^{A(t,T)-B(t,T)r(t)} \tag{B.2}$$

where in the case of the Hull-White model the functions $B(t,T)$ and $A(t,T)$ can be shown to be:

$$B(t,T) = \frac{1}{a}(1 - e^{-a(T-t)}) \tag{B.3}$$

$$A(t,T) = \int_t^T -v(u)B(u,T) + \frac{1}{2}\sigma^2 B^2(u,T)\, du \tag{B.4}$$

B.1.2 Fitting the Model to the Initial Term Structure

Clearly, the function $v(t)$ needs to be chosen to ensure that the model reproduces the term structure of interest rates observed in the market. That is, $P(0,T) = P^M(0,T)$ where $P^M(0,T)$ is the market price of the bond with maturity T at time 0. It can be shown that to ensure this $v(t)$ must be given by:

$$v(t) = \frac{\partial f^M(0,t)}{\partial T} + af^M(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}) \tag{B.5}$$

where $f^M(0,T)$ denotes the market instantaneous forward rate for maturity T defined by:

$$f^M(0,T) = -\frac{\partial \ln P^M(0,T)}{\partial T} \tag{B.6}$$

The issue with this formulation is that it requires observed forward rates as well as their derivatives. The observed term structure of interest rates is typically bootstrapped from a discrete set of market quotes and is thus heavily dependent on interpolation. Consequently, the differentiability of the yield curve and especially the forward curve is very seldom guaranteed.

Certain interpolation methods such as those presented by (Hagan and West, 2006) as well as (Du Preez and Maré, 2013) ensure a continuous forward curve i.e. a differentiable bond curve. However even these continuous forward curves are not differentiable at the knot points (the specific points on the yield curve that the bootstrapped rates apply to). Clearly this makes the numerical implementation of Equation B.5 difficult.

That being said when using the Hull-White model to price bonds, the explicit form of the $v(t)$ function given by Equation B.5 is not needed. This can be shown by solving the SDE given by Equation B.1 which gives:

$$r(t) = e^{-a(t-s)}r(s) + \int_s^t v(u)e^{-a(t-u)}du + \sigma\int_s^t e^{-a(t-u)}dW(u) \tag{B.7}$$

Looking at the second term and substituting Equation B.5 for $v(u)$ gives

$$\begin{aligned}\int_s^t v(u)e^{-a(t-u)}du = &\int_s^t e^{-a(t-u)}\frac{\partial f^M(0,u)}{\partial T}du + a\int_s^t e^{-a(t-u)}f^M(0,u)du \\ &+ \frac{\sigma^2}{2a}\int_s^t e^{-a(t-u)}(1 - e^{-2at})du.\end{aligned}$$

Using integration by parts, the first integral can expressed as

$$\int_s^t e^{-a(t-u)}\frac{\partial f^M(0,u)}{\partial T}du = f^M(0,t) - f^M(0,s)e^{-a(t-s)} - a\int_s^t e^{-a(t-u)}f^M(0,u)du.$$

As a result,

$$\begin{aligned}\int_s^t v(u)e^{-a(t-u)}du &= f^M(0,t) - f^M(0,s)e^{-a(t-s)} + \frac{\sigma^2}{2a}\int_s^t e^{-a(t-u)}(1-e^{-2at})du \\ &= f^M(0,t) - f^M(0,s)e^{-a(t-s)} + \frac{\sigma^2}{2a^2}\left[1 - e^{-a(t-s)} + e^{-2at} - e^{-a(t+s)}\right] \\ &= f^M(0,t) - f^M(0,s)e^{-a(t-s)} + \frac{\sigma^2}{2a^2}\left[(1-e^{-at})^2 - e^{-a(t-s)}(1-e^{-as})^2\right] \\ &= \theta(t) - \theta(s)e^{-a(t-s)},\end{aligned}$$

where $\theta(t)$ is given by

$$\theta(t) = f^M(0,t) + \frac{\sigma^2}{2a^2}(1-e^{-at})^2. \tag{B.8}$$

Consequently, r_t can be written as

$$r(t) = e^{-a(t-s)}r(s) + \theta(t) - \theta(s)e^{-a(t-s)} + \sigma\int_s^t e^{-a(t-u)}dW(u).$$

Defining a new process x (which we denote as the short-rate factor) under the risk-neutral measure $\mathbb{Q}$ by

$$\begin{aligned}dx(t) &= -ax(t)dt + \sigma dW(t), \quad x(0) = 0 \\ \Rightarrow x(t) &= e^{-a(t-s)}x(s) + \sigma\int_s^t e^{-a(t-u)}dW(u).\end{aligned}$$

This allows us to write $r(t)$ as

$$r(t) = \theta(t) + x(t). \tag{B.9}$$

Finally bond prices can be expressed as

$$P(t,T) = e^{A(t,T) - B(t,T)(\theta(t)+x(t))}, \tag{B.10}$$

where $B(t,T)$ is given as before in Equation B.3

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right],$$

while $A(t,T)$ is obtained by integrating the expression given in Equation B.4 to give

$$A(t,T) = ln\left(\frac{P^M(0,T)}{P^M(0,t)}\right) + \left[B(t,T)f^M(0,t) - \frac{\sigma^2}{4a}(1-e^{-2at})B(t,T)^2\right].$$

This formulation can be easily numerically implemented to model bond prices since derivatives of the forward curve are no longer required and $x(t)$ is driven by a simple linear SDE. Consequently calibrating the model to the initial term structure is relatively straightforward so long as the interpolation scheme ensures a continuous forward curve. In addition the distribution of $x(t)$ under the risk-neutral or any forward measure can be easily determined enabling the straightforward modelling of bond prices. The distribution of $x(t)$ (which we denote as the short-rate factor) under forward measures is presented in the following section.

B.1.3 Hull-White Short-rate Factor Distribution

Given the stochastic differential equation of the short-rate factor $x(t)$ under the risk-neutral measure Q:

$$dx(t) = -ax(t)dt + \sigma dW(t), \quad x_0 = 0$$

Since Hull-White is an Affine Term Structure model the volatility of the T-maturity zero-coupon bond ($P(t,T)$) is simply given by $-\sigma B(t,T)$. Consequently, the dynamics under the time-T forward measure Q^T with associated numeraire $P(t,T)$ can be easily shown to be

$$dx(t) = \left[-ax(t) - B(t,T)\sigma^2\right] dt + \sigma dW^T(t), \tag{B.11}$$

where $W^T := W + \int_0^t \sigma B(s,T)ds$ is a Q^T-Brownian motion with

$$B(t,T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right].$$

The SDE given by Equation B.11 can be seen to be linear with additive noise so we define

$$z(t) = e^{-\int_0^t -a\ ds} = e^{at} \qquad Z_t = z(t)x(t).$$

Then by Itô's formula

$$\begin{aligned}
dZ_t &= az(t)x(t)\ dt + z(t)\ dx(t)\\
&= z(t)[ax(t) - B(t,T)\sigma^2 - ax(t)]dt + z(t)\sigma dW^T(t)\\
&= -z(t)B(t,T)\sigma^2 dt + z(t)\sigma dW^T(t)\\
\therefore Z_t &= Z_0 - \sigma^2 \int_0^t z(s)B(s,T)ds + \sigma \int_0^t z(s)\sigma dW^T(s)\\
\therefore z(t)x(t) &= x(0) - \sigma^2 \int_0^t z(s)B(s,T)ds + \sigma \int_0^t z(s)dW^T(s)\\
\therefore x(t) &= e^{-at}x(0) - \sigma^2 \int_0^t B(s,T)e^{-a(t-s)}ds + \sigma \int_0^t e^{-a(t-s)}dW^T(s)\\
\therefore x(t) &= -\sigma^2 \int_0^t \frac{e^{-a(t-s)} - e^{-a(T+t-2s)}}{a}ds + \sigma \int_0^t e^{-a(t-s)}dW^T(s).
\end{aligned}$$

Consequently,

$$x(T) = -\sigma^2 \int_0^T \frac{e^{-a(T-s)} - e^{-2a(T-s)}}{a}ds + \sigma \int_0^T e^{-2a(T-s)}dW^T(s).$$

Recalling Itô's isometry which states that $\int h(s)dW(s) \sim N\left(0, \int h^2(s)ds\right)$ if $h(s)$ is deterministic, it follows that

$$\begin{aligned}\mathbb{E}^T[x(T)] = \mu(T) :=& -\sigma^2 \int_0^T \frac{e^{-a(T-s)} - e^{-2a(T-s)}}{a} ds \\ =& -\frac{\sigma^2}{a^2}\left[1 - e^{-aT}\right] + \frac{\sigma^2}{2a^2}\left[1 - e^{-2aT}\right]\end{aligned}$$

$$\begin{aligned}\text{Var}^T[x(T)] = \nu(T) :=& \sigma^2 \int_0^T e^{-2a(T-s)} ds \\ =& \frac{\sigma^2}{2a}\left[1 - e^{-2aT}\right].\end{aligned}$$

It is also noted that given the above expectation and variance under the Q^T-forward measure the probability density function of $y(T)$ is

$$\frac{1}{\sqrt{2\pi\nu(T)}} \exp\left\{-\frac{[y - \mu(T)]^2}{2\nu(T)}\right\}.$$

B.2 Explicit Form of Discount Factors

As seen in Henrard (2010a), bond prices under the Hull-White model under the $Q_D^{T_k^x}$ can be explicitly represented using a standard normal random variable

$$P_D(T_\theta^x, T_i^x) = \frac{P_D(0, T_i^x)}{P_D(0, T_\theta^x)} \exp\left(-0.5\gamma_i^2 - \gamma_i Z\right). \tag{B.12}$$

where

$$\begin{aligned}\gamma_i =& \int_0^{T_\theta} \left[\sigma B(s, T_i) - \sigma B(s, T_\theta)\right] ds \\ =& \frac{1}{2a}\left[\sigma B(T_\theta, T_i) - \sigma B(T_\theta, T_\theta)\right]^2 - \frac{1}{2a}\left[\sigma B(0, T_i) - \sigma B(0, T_\theta)\right]^2\end{aligned} \tag{B.13}$$

B.2.1 Hull-White Forward Rate Distribution

In this section we attempt to derive the distribution of the discrete forward rates $F_k^x(t)$ in the Hull-White short rate model under the T_k^x forward measure. This is required to determine the volatilities of the discrete forward rates which are one of the input to the model. First we recall the definition of the discrete forward rate at time t:

$$F_k^x(t) := \frac{1}{\tau_k^x}\left[\frac{P_D(t,T_{k-1}^x)}{P_D(t,T_k^x)} - 1\right] \tag{B.14}$$

which is the simple rate applying over the period $[T_{k-1}^x, T_k]$ for $k = 1, \ldots, M_x$ where τ_k^x is the year fraction between T_{k-1}^x and T_k. Clearly:

$$dF_k^x(t) = \frac{1}{\tau_k^x}\, d\left(\frac{P(t,T_{k-1}^x)}{P(t,T_k^x)}\right)$$

Since Hull-White is an affine term structure model bond have been seen to be given by Equation B.2 and thus the risk neutral bond dynamics are given by:

$$\frac{dP(t,T)}{P(t,T)} = rdt - \sigma B(t,T)dW_t^Q$$

First using Ito's lemma and then changing measure to the T_k^x-forward measure with associated numeraire $P(t,T_k^x)$ noting that $\frac{P(t,T_{k-1}^x)}{P(t,T_k^x)}$ is a $Q^{T_k^x}$-martingale it can be shown that:

$$\begin{aligned}\frac{d\left(\frac{P(t,T_{k-1}^x)}{P(t,T_k^x)}\right)}{\left(\frac{P(t,T_{k-1}^x)}{P(t,T_k^x)}\right)} &= [...]dt - \sigma\left[B(t,T_{k-1}^x) - B(t,T_k^x)\right]dW_t^Q \\ &= -\sigma\left[B(t,T_{k-1}^x) - B(t,T_k^x)\right]dW_t^{Q^{T_k}}\end{aligned}$$

where $W_t^{Q^{T_k}}$ is a $Q^{T_k^x}$ Brownian motion.

As a result it can be seen that:

$$\begin{aligned}dF_k^x(t) &= \frac{1}{\tau_k^x}\left(\frac{P(t,T_{k-1}^x)}{P(t,T_k^x)}\right)\sigma\left[B(t,T_k^x) - B(t,T_{k-1}^x)\right]dW_t^{Q^{T_k}} \\ &= \left(F_k^x(t) + \tfrac{1}{\tau_k^x}\right)\sigma\left[B(t,T_k^x - B(t,T_{k-1}^x))\right]dW_t^{Q^{T_k}}\end{aligned}$$

This is the SDE of a shifted log-normal process and thus has no closed form distribution. As a result, we are forced to use an alternate approach to obtain the discrete forward rate volatilities.

B.2.2 Implying Forward Rate Volatilities using Caplet Prices

An alternative approach to obtaining the forward rate volatilities is to imply them from caplet prices. If we consider a caplet written on OIS rates then the time T_k^x pay-off is given by

$$\tau_k^x \left[F_k^x(T_{k-1}^x) - K\right]^+, \tag{B.15}$$

where $F_k^x(T_{k-1}^x)$ is the OIS forward rate for the period $[T_{k-1}^x, T_k^x]$ set at T_{k-1}^x.

Since the OIS forward rates, $F_k^x(t)$, are martingales under the T_k^x forward measure, the pricing follows the classic single-curve approach

$$\textbf{Cplt}(t, K; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x)\, \mathbb{E}_x^{T_k^x} \left\{ \left[F_k^x(T_{k-1}^x) - K\right]^+ |\mathcal{F}_t \right\}.$$

Assuming that the forward rates $F_k^x(t)$ are lognormal which was formally justified by the LMM of Brace *et al.* (1997) allows us to write the $Q^{T_k^x}$ dynamics of $F_k^k(t)$ as:

$$dF_k^x(t) = \sigma_k F_k^x(t) dZ_k(t), \quad t \leq T_{k-1}^x$$

where σ_k is a constant and Z_k is a $Q^{T_k^x}$ Brownian motion. This leads to the Black-like pricing formula

$$\textbf{Cplt}(t, K; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) \text{Bl}\left(K, F_k^x(t), \sigma_k \sqrt{T_{k-1}^x - t}\right), \tag{B.16}$$

where

$$\text{Bl}(K, F, v) = F\Phi\left(\frac{\ln(F/K) + v^2/2}{v}\right) - K\Phi\left(\frac{\ln(F/K) - v^2/2}{v}\right).$$

Under the Hull-White model, closed form solutions for options on zero-coupon bonds and as a result closed form solutions for caplets exist. As a result, one can determine caplet prices and imply the volatility of the forward rates, σ_k, using Equation 3.12.

It can be shown that the price at time t of a European put option with strike X and maturity K written on a zero-coupon bond maturing at time S is given by

$$\textbf{ZBP}(t, T, S, X) = XP(t, T)\Phi(-h + \sigma_p) - P(t, S)\Phi(-h), \tag{B.17}$$

where,

$$\sigma_p = \sigma \sqrt{\frac{1 - e^{-2a(T-t)}}{2a}} B(T, S)$$

$$h = \frac{1}{\sigma_p} \ln\left(\frac{P(t, S)}{P(t, T)X}\right) + \frac{\sigma_p}{2}.$$

Noting that $B(t, T)$ is given by Equation B.3.

To use the expression for zero-coupon bond options prices to price caplets, one notes that the T_k^x pay-off given in Equation B.15 is equivalent to a pay-off at T_{k-1}^x of

$$\frac{\tau_k^x \left[F_k^x(T_{k-1}^x) - K\right]^+}{1 + F_k^x(T_{k-1}^x)\tau_k^x}.$$

However,

$$\frac{\tau_k^x\left[F_k^x(T_{k-1}^x)-K\right]^+}{1+F_k^x(T_{k-1}^x)\tau_k^x}=(1+K\tau_k^x)\left(\frac{1}{1+K\tau_k^x}-\frac{1}{1+F_k^x(T_{k-1}^x)\tau_k^x}\right)^+$$
$$=(1+K\tau_k^x)\left(\frac{1}{1+K\tau_k^x}-P(T_{k-1}^x,T_k^x)\right)^+.$$

This is the same pay-off of $(1+K\tau_k^x)$ many European put options with strike $\frac{1}{1+K\tau_k^x}$ and maturity T_{k-1}^x written on the zero-coupon bond $P_D(t,T_k^x)$. The caplet price can therefore be written as

$$\mathbf{Cplt}(t,K;T_{k-1}^x,T_k^x)=(1+K\tau_k^x)\,\mathbf{ZBP}\left(t,T_{k-1}^x,T_k^x,\frac{1}{1+K\tau_k^x}\right). \tag{B.18}$$

Consequently, we price the relevant OIS caplet using Equation B.18 and then imply the required $F_k^x(T_{k-1}^x)$ volatility using Equation B.16.

B.3 The Two-Additive-Factor Gaussian (G2++) Short-Rate Model

In this section we introduce the G2++ short-rate model and present the results required for its use as the OIS rate model in our multi-curve model with stochastic-basis. Again these results include:

- Fitting the model to the initial observed term structure;
- The distribution of the short-rate under the $\mathcal{Q}^T$ forward measure;
- The volatilities of discrete forward rates.

It is noted we rely heavily on the results presented in Brigo and Mercurio (2007).

B.3.1 Model Fundamentals

The G2++ model is an interest rate model where the addition of two Gaussian factors with correlation ρ as well as a deterministic function models the short-rate process. Clearly the G2++ model is closely related to a two-factor extension of the Hull-White model (Hull and White, 1994). Brigo and Mercurio (2007) in fact prove the natural equivalence of these two approaches however, we use the G2++ formulation due to the ease of implementation and less complicated formulas.

Under this framework, the dynamics of the short-rate under the risk neutral-measure are assumed to be follow

$$r(t) = x(t) + y(t) + \varphi(t), \quad r(0) = r_0, \tag{B.19}$$

where the process $x(t)$ and $y(t)$ are driven by

$$dx(t) = -\,ax(t)dt + \sigma dW_1(t), \quad x(0) = 0 \tag{B.20}$$

$$dy(t) = -\,by(t)dt + \zeta dW_2(t), \quad y(0) = 0, \tag{B.21}$$

with $dW_1(t)dW_2(t) = \rho dt$ where $-1 \leq \rho \leq 1$ and $r_0,\ a,\ b,\ \sigma,\ \zeta$ are positive constants. The $\varphi(t)$ is a deterministic function to ensure the observed term-structure is fitted with $\varphi(0) = r_0$. It is noted that $x(t)$ is the same as the Hull-White short-rate factor defined in the previous section.

Time-t zero-coupon bond prices can be shown to be given by

$$P(t,T) = \exp\left\{ -\int_t^T \varphi(u)du - \frac{1-e^{-a(T-t)}}{a}x(t) - \frac{1-e^{-b(T-t)}}{b}y(t) + \frac{1}{2}V(t,T)\right\}, \tag{B.22}$$

where

$$\begin{aligned}V(t,T) =&\frac{\sigma^2}{a^2}\Big[T-t+\frac{2e^{-a(T-t)}}{a}-\frac{e^{-2a(T-t)}}{2a}-\frac{3}{2a}\\&+\frac{\zeta^2}{b^2}\Big[T-t+\frac{2e^{-b(T-t)}}{b}-\frac{e^{-2b(T-t)}}{2b}-\frac{3}{2b}\\&+2\rho\frac{\sigma\zeta}{ab}\Big[T-t+\frac{e^{-a(T-t)}-1}{a}+\frac{e^{-b(T-t)}-1}{b}-\frac{e^{-(a+b(T-t)}-1}{a+b}. \end{aligned} \tag{B.23}$$

(B.24)

B.3.2 Fitting the Model to the Initial Term Structure

Brigo and Mercurio (2007) show that the G2++ model fits the observed term-structure if and only if for each T:

$$\begin{aligned}\varphi(T) =&f^M(0,T)+\frac{\sigma^2}{2a^2}(1-e^{-aT})^2\\&+\frac{\zeta^2}{2b^2}(1-e^{-bT})^2+\rho\frac{\sigma\zeta}{ab}\left(1-e^{-aT}\right)\left(1-e^{-bT}\right)\end{aligned} \tag{B.25}$$

Similarly to the Hull-White one-factor case, the deterministic function requires the market instantaneous forward rates, $f^M(0,T)$, which were defined in Equation B.6. To obtain these market instantaneous forward rates, one has to differentiate the market bond (discount) curve. However, the market bond curve is built using bootstrapping procedures for a finite set of maturities via interpolation. Interpolation methods do not necessarily guarantee the differentiability of this bond curve which appears to complicate the implementation of the G2++ model. That being said, in order to price bonds, only the integral of the φ function between two times is required. Using Equation B.25 enables us to evaluate this integral and it can be shown that

$$\exp\left\{-\int_t^T\varphi(u)du\right\}=\frac{P^M(0,T)}{P^M(0,t)}\exp\left\{-\frac{1}{2}\left[V(0,T)-V(0,t)\right]\right\}. \tag{B.26}$$

Consequently, the only market curve that is required is the market bond curve which does not need to be differentiated and it is only required for certain maturities - limiting the need for interpolation.

As a result, the time-t zero-coupon bond prices can therefore be expressed as

$$P(t,T)=\frac{P^M(0,T)}{P^M(0,t)}\exp\{\mathcal{A}(t,T)\}, \tag{B.27}$$

where

$$\begin{aligned}\mathcal{A}(t,T):=&\frac{1}{2}\left[V(t,T)-V(0,T)+V(0,t)\right]\\&-\frac{1-e^{-a(T-t)}}{a}x(t)-\frac{1-e^{-b(T-t)}}{b}y(t).\end{aligned} \tag{B.28}$$

This formulation, like the Hull-White one factor model can be easily implemented to model bond prices since $x(t)$ and $y(t)$ are driven by simple linear SDE's. In the next section we present the distributions of the $x(t)$ and $y(t)$ processes under the forward measure.

B.3.3 G2++ Short-rate Factor Distributions

As given in Brigo and Mercurio (2007), the processes $x(t)$ and $y(t)$ evolve under the Q^T forward measure according to

$$dx(t) = \Big[-ax(t) - \frac{\sigma^2}{a}\left(1-e^{-a(T-t)}\right) - \rho\frac{\sigma\zeta}{b}\left(1-e^{-b(T-t)}\right)\Big]dt + \sigma dW_1^T(t) \quad \text{(B.29)}$$

$$dy(t) = \Big[-bx(t) - \frac{\zeta^2}{b}\left(1-e^{-b(T-t)}\right) - \rho\frac{\sigma\zeta}{a}\left(1-e^{-a(T-t)}\right)\Big]dt + \zeta dW_2^T(t), \quad \text{(B.30)}$$

where W_1^T and W_2^T are Q^T Brownian motions with $dW_1^T(t)dW_2^T(t) = \rho dt$

It can also be shown, using a similar derivation to that presented in Section B.1.3, that the explicit solutions for $x(t)$ and $y(t)$ are given by

$$\begin{aligned} x(t) =& x(s)e^{-a(T-t} - M_x^T(s,t) + \sigma\int_s^t e^{-a(T-u)}dW_1^T(u) \\ y(t) =& y(s)e^{-b(T-t} - M_y^T(s,t) + \sigma\int_s^t e^{-b(T-u)}dW_2^T(u), \end{aligned} \quad \text{(B.31)}$$

where

$$\begin{aligned} M_x^T(s,t) =& \left(\frac{\sigma^2}{a^2} + \rho\frac{\sigma\zeta}{ab}\right)\left[1-e^{-a(t-s)}\right] - \frac{\sigma^2}{2a^2}\left[e^{-a(T-t)} - e^{-a(T+t-2s}\right] \\ &- \frac{\rho\sigma\zeta}{b(a+b)}\left[e^{-b(T-t)} - e^{-bT-at+(a+b)s}\right] \\ M_y^T(s,t) =& \left(\frac{\zeta^2}{b^2} + \rho\frac{\sigma\zeta}{ab}\right)\left[1-e^{-b(t-s)}\right] - \frac{\zeta^2}{2b^2}\left[e^{-b(T-t)} - e^{-b(T+t-2s}\right] \\ &- \frac{\rho\sigma\zeta}{b(a+b)}\left[e^{-a(T-t)} - e^{-aT-bt+(a+b)s}\right]. \end{aligned}$$

As a result, the short-rate factors can be seen to bivariate normal with correlation ρ and means and variance given by

$$\mathbb{E}^T[x(T)] = -M_x^T(0,T) \quad \text{(B.32)}$$

$$\text{Var}^T[x(T)] = \int_0^T \sigma^2 e^{-2a(T-u)}du \quad \text{(B.33)}$$

$$= \frac{\sigma^2}{2a}\left[1-e^{-2aT}\right], \quad \text{(B.34)}$$

and

$$\mathbb{E}^T[y(T)] = -M_y^T(0,T) \tag{B.35}$$

$$\text{Var}^T[y(T)] = \int_0^T \zeta^2 e^{-2b(T-u)} du \tag{B.36}$$

$$= \frac{T\zeta^2}{2b}\left[1 - e^{-2bT}\right]. \tag{B.37}$$

B.3.4 Implying Forward Rate Volatilities using Caplet Prices

Since the G2++ model is also a short-rate model, we follow the approach used for the Hull-White one-factor model and imply the discrete forward rate volatilities from caplet prices (see Section B.2.2).

Under the G2++ model, closed form solutions for options on zero-coupon bonds and as a result closed form solutions for caplets exist. As a result, one can determine caplet prices and imply the volatility of the forward rates, σ_k, using Equation 3.12.

It can be shown that the price at time-t of a European put option with strike X and maturity T written on a zero-coupon bond maturing at time S is given by

$$\begin{aligned}\textbf{ZBP}(t,T,S,X) =& XP(t,T)\Phi\left(\frac{\ln\frac{XP(t,T)}{P(t,S)}}{\Sigma(t,T,S)} - \frac{1}{2}\Sigma(t,T,S)\right)\\ &- P(t,S)\Phi\left(\frac{\ln\frac{XP(t,T)}{P(t,S)}}{\Sigma(t,T,S)} + \frac{1}{2}\Sigma(t,T,S)\right),\end{aligned} \tag{B.38}$$

where

$$\begin{aligned}\Sigma(t,T,S)^2 =& \frac{\sigma^2}{2a^3}\left[1 - e^{-a(S-T)}\right]^2\left[1 - e^{-2a(T-t)}\right]\\ &+ \frac{\zeta^2}{2b^3}\left[1 - e^{-b(S-T)}\right]^2\left[1 - e^{-2b(T-t)}\right]\\ &+ 2\rho\frac{\sigma\zeta}{ab(a+b)}\left[1 - e^{-a(S-T)}\right]\left[1 - e^{-b(T-t)}\right]\left[1 - e^{-(a+b)(T-t)}\right].\end{aligned}$$

To use the expression for zero-coupon bond options prices to price caplets, one notes that the pay-off of the caplet with reset date T_{k-1}^x and payment date T_k^x and strike K is equivalent to the pay-off of $(1 + K\tau_k^x)$ many European put options with strike $\frac{1}{1+K\tau_k^x}$ and maturity T_{k-1}^x written on the zero-coupon bond $P_D(t, T_k^x)$. The caplet price can therefore be written as

$$\textbf{Cplt}(t, K; T_{k-1}^x, T_k^x) = (1 + K\tau_k^x)\,\textbf{ZBP}\left(t, T_{k-1}^x, T_k^x, \frac{1}{1 + K\tau_k^x}\right). \tag{B.39}$$