NON-COMMUTATIVE BANACH FUNCTION SPACES

by

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LIST of SYMBOLS

We give a list of the symbols used and a brief indication of their meaning.
Where a concept or symbol is introduced in the text, a reference to the page where it is defined is given. (For more details see also the Prerequisites.)

Symbol	p	Meaning
$\prec\prec$	46	submajorisation
$\prec$	46	majorisation
$\xrightarrow{ae}$		almost everywhere convergence
$\xrightarrow{au}$		almost uniform convergence
$BL(\mathcal{H})$		von Neumann algebra of all bounded linear operators on a Hilbert Space $\mathcal{H}$
$\mathcal{B}(\ )$		Borel σ–algebra
B_ϵ		the ball of radius ϵ in $\mathcal{H}$ with the norm topology
D(S)		Domain of an unbounded operator S
$d_t(f)$	33	distribution function of f
$d_t(S)$	146	distribution function for $S \in \tilde{\mathcal{M}}^{sa}$
$f_a(t)$	45	dilation of f by a
$\mathcal{F}$	214	ideal of elementary operators (elops)
Gr(S)		Graph of an operator S (a subset of $\mathcal{H}_2$, see below)
$\gamma_{c\mu}$	7	the topology of convergence in measure
$\gamma_{lc\mu}$	19	the topology of local convergence in measure
$\mathcal{H}$		Hilbert Space
$\mathcal{H}_2$		$\mathcal{H} \oplus \mathcal{H}$

η	104	affiliation
$\lambda_t(f)$	33	the rearrangement of f
$\lambda_t(S)$	148	spectral scale of $S \in \tilde{\mathcal{M}}^{sa}$ (finite trace)
$L_0(X,\Sigma,\mu)$	6	space of equivalence classes of a.e. $\mathbb{R}$ valued measurable functions on (X,Σ,μ)
$L_\infty(X,\Sigma,\mu)$	6	space of equivalence classes of a.e. $\mathbb{R}$ valued essentially bounded measurable functions on (X,Σ,μ)
$\tilde{L}_\infty(X,\Sigma,\mu)$	18	space of equivalence classes of a.e. $\mathbb{R}$ valued measurable functions essentially bounded except on a set of finite measure.
$L_\infty(\tilde{\mathcal{M}})$	213	$\mathcal{M}$
L_ρ	37	Normed Kőthe Space or Banach Function Space
$L_\rho(\tilde{\mathcal{M}})$	184	(type of) Non–Commutative Banach Function Space
m		Lebesgue measure on (a subset of) the real line
$\mathcal{M}$		a von Neumann algebra
$\mathcal{M}_\tau$	50	definition ideal of a trace τ on $\mathcal{M}$
$\bar{\mathcal{M}}$	104	the set of closed densely defined operators affiliated with $\mathcal{M}$
$\tilde{\mathcal{M}}$		the algebra of τ–measurable operators
$\tilde{\mathcal{M}}(\epsilon, \delta)$	116	basic neighbourhood of 0 in the topology of convergence in measure on $\tilde{\mathcal{M}}$
$\mathcal{M}(\epsilon, \delta)$	116	basic neighbourhood of 0 in the topology of convergence in measure on $\mathcal{M}$
$\mathcal{M}_u$		the group of unitary operators in $\mathcal{M}$
$\mathcal{M}_e$		the von Neumann algebra $\mathcal{M}$ reduced by the projection $e \in \mathcal{M}_p$

$\mathcal{M}'$		the commutant of $\mathcal{M}$
$\mathcal{M}_p$		the lattice of projections in $\mathcal{M}$
$M_n(\mathbb{C})$		algebra of n×n matrices over $\mathbb{C}$
$M_n(\mathcal{M})$	89	the algebra of n×n matrices with entries in $\mathcal{M}$
$\mu_t(f)$	34	rearrangement of $\|f\|$, with $f \in \tilde{L}_\infty$
$\mu_\infty(f)$	35	$\lim_{t\to\infty} \mu_t(f)$
$\mu_t(S)$	155	generalised singular function of $S \in \tilde{\mathcal{M}}$
$\mathcal{N}(\epsilon, \delta)$	6	neighbourhood in $\gamma_{c\mu}$
$\mathcal{N}(\epsilon)$	6	neighbourhood in $\gamma_{c\mu}$
N(S)		the kernel of an operator S
$\mathcal{N}_\tau$	50	"square root" of $\mathcal{M}_\tau$
$\mathcal{P}_\tau$	50	positve part of $\mathcal{M}_\tau$
p_τ	57	identity projection in the so–closure of $\mathcal{M}_\tau$
R(S)		the projection onto the closure of the range of an operator S
ρ	37	function norm
$\rho^\times$	39	first associate norm of ρ
$\rho^{\times\times}$	39	second associate norm of ρ
$\rho(S)$	184	norm of S in a non–commutative Banach Function Space
$\xrightarrow{so}$		strong operator convergence
$s(\tau)$	62	support of normal trace τ
s_e		operator $s \in \mathcal{M}$ reduced by the projection $e \in \mathcal{M}_p$
Σ_f	19	the subset of Σ comprising those sets of finite measure
$S^\sim(t)$	227	Yeadon's rearrangement of an operator
τ	48	trace
τ_e	66	reduced trace oon $\mathcal{M}_e$
τ^e	68	extended trace

$\tilde{\tau}$	87	diagonal trace on tensor product (usually on $M_n(\mathcal{M})$, in particular when n = 2)
$\xrightarrow{us}$		ultrastrong convergence
$\xrightarrow{uw}$		ultraweak convergence
$\xrightarrow{u}$		uniform convergence (of functions)
$\xrightarrow{wo}$		weak operator convergence
(X,Σ,μ)		(arbitrary) measure space
$Z(\mathcal{M})$		the centre of $\mathcal{M}$

We also follow certain conventions in our use of symbols for operators. Although we have tried to avoid introducing a symbol without defining it, the following table may prove useful.

Symbol	Conventional meaning
e, p, q	elements of $\mathcal{M}_p$
u	unitary operator
v	partial isometry
a, r, s	bounded operators, usually elements of $\mathcal{M}$
R, S, T	unbounded operators, usually elements of $\tilde{\mathcal{M}}$
x, y	elements of $\mathcal{H}$

PREREQUISITES

We assume the reader to be familiar with certain topics in analysis and basic von Neumann algebra theory. For the convenience of the reader we give a list of these topics with general references, and state some results in the form that we will use them.

Measure Theory Reference : [C]

We assume a knowledge of elementary measure theory and use notation consistant with [C]

We will make substantial use of measure spaces (A , $\mathcal{B}$(A) , m) for A ⊂ ℝ . Such a measure space will simply be notated as A unless there is danger of confusion; taking it as understood that A is equipped with the induced Borel σ–algebra and the restriction of Lebesgue measure to A . Furthermore, we shall write L_p A , L_ρ A , L_∞ A , L_0 A , etc.

Topological Vector Spaces Reference : [J]

Dual pairs and polars.
Some results on completions of topological vector spaces are used.

Axiom of Choice

We make occasional use of the axiom of choice, usually in the form of Zorn's Lemma

von Neumann algebra theory References : [D] , [KR] , [SZ] , [T]

Elementary von Neumann algebra theory

The lattice of projections [KR] 2.5

Elementary Banach and C^* algebra theory [KR] Chapters 3,4

Von Neumann algebras [KR] 5.1

Topologies on von Neumann algebras [SZ] Chapter 4

we denote the weak operator topology by wo; the strong operator topology by so; the ultrastrong topology by us; the ultraweak topology by uw ($\equiv$ w in the notation of [SZ]); the norm topology by $\|.\|$.

Reduced operators and von Neumann algebras [D] § I.2.1

Dominated monotone convergence theorem for von Neumann algebras [KR] 5.1.4

The double commutant theorem [KR] 5.3.1

Comparison of Projections and classification by types Reference [SZ] Chapter 4

If $p \sim q$ and this equivalence is implemented by the partial isometry v, we shall write $p \overset{v}{\sim} q$.

Tensor Products

Matrix representations [KR] § 2.6

Matrix units [T] IV § 1

Tensor products of Hilbert Spaces, operators, and von Neumann algebras [T] IV § 1

More detailed references will be given in Chapter 4

Unbounded linear operators Reference [KR] § 2.7

We use the terms preclosed and closable interchangeably.

If S is closed then N(S) is closed.

$\mathrm{supp}(S) \equiv 1 - N(S)$

If S is closed, densely defined (in particular, if it is bounded) then

$$R(S) = \mathrm{supp}(S^*) \quad (\text{ and } R(S^*) = \mathrm{supp}(S)\ )$$

$$R(S^*) = R(S^* S)$$

$$N(S^* S) = N(S)$$

If $s \in \mathcal{M}$ then $R(s) \sim R(s^*)$ [KR] 6.1.6

Spectral theory References : [KR] §§ 5.2 , 5.6; [SZ] Chapter 9 ; [DS] Chapter XII

Bounded and unbounded resolutions of the identity

For a self–adjoint operator S there is a uniquely determined spectral family $\{ e_t(S) : t \in \mathbb{R} \}$ satisfying :–

(1) $e_t(S)$ is a projection $\forall\, t \in \mathbb{R}$

(2) $t_1 \leq t_2 \Rightarrow e_{t_1}(S) \leq e_{t_2}(S)$

(3) $e_{t+\epsilon}(S) \downarrow_{so} e_t(S)$ as $\epsilon \downarrow 0$ (the family is right continuous)

(4) $e_t(S) \uparrow_{so} 1$ as $t \uparrow \infty$

(5) $e_t(S) \downarrow_{so} 0$ as $t \downarrow -\infty$

We will exploit the spectral theorem for self–adjoint operators in the 'integral form'.

If S is an (unbounded) self–adjoint operator on $\mathcal{H}$ then there is a uniquely determined resolution of the identity $\{ e_t(S) : t \in \mathbb{R} \}$ such that

$$D(S) = \{ x \in \mathcal{H} : \int_{-\infty}^{\infty} t^2 \, d\|e_t(S)x\|^2 < \infty \}$$

$$S = \int_{-\infty}^{\infty} t \, de_t(S) \quad \text{in the sense of strong operator convergence}$$

S is bounded iff $\exists$ M , m $\in \mathbb{R}$ such that $e_t(S) = 1 \ \forall t \geq M$ and $e_t(S) = 0 \ \forall t < m$.

S is positive iff $e_t(S) = 0 \ \forall t < 0$

If S is positive then $N(S) = e_0(S)$

We make the following identifications :–

$$e_t(S) \equiv e_{(-\infty,t]}(S) \quad (\equiv e_{[0,t]}(S) \text{ if S is positive})$$

$$e_{(t,\infty)}(S) \equiv 1 - e_{(-\infty,t]}(S)$$

$$\sup_{\theta \leq t} e_{(-\infty,\theta]}(S) \equiv e_{(-\infty,t)}(S) \quad (\equiv e_{[0,t)}(S) \text{ if S is positve})$$

This correspondence between unbounded intervals in $\mathbb{R}$ and certain projections generated by the spectral family of S is extended to the Borel σ–algebra.

In particular,

$$e_{(a,b]}(S) = e_{(-\infty,b]}(S) - e_{(-\infty,a]}(S)$$
$$e_{[a,b]}(S) = e_{(-\infty,b]}(S) - e_{(-\infty,a)}(S)$$
$$e_{(a,b)}(S) = e_{(-\infty,b)}(S) - e_{(-\infty,a]}(S)$$
$$e_{[a,b)}(S) = e_{(-\infty,b)}(S) - e_{(-\infty,a)}(S)$$

$e_{(-\infty,t]}(S)\, S \leq t\, e_{(-\infty,t]}(S)$

$e_{(t,\infty)}(S)\, S \geq t\, e_{(t,\infty)}(S)$

If S is positive then $\text{supp}(S) = e_{(0,\infty)}(S)$

Operational Calculus Reference : [DS] Chapter XII

Suppose S is a self–adjoint operator.

If $f : \mathbb{R} \longrightarrow \mathbb{C}$ is a Borel measurable function then f(S) is a closed densely defined operator where

$$D(f(S)) = \{ x \in \mathcal{H} : \int_{-\infty}^{\infty} |f(t)|^2 \, d\|e_t(S)x\|^2 < \infty \}$$

$$f(S) = \int_{-\infty}^{\infty} f(t)\, de_t(S) \quad \text{in the sense of strong operator convergence}$$

If g is another such function then

$$f(S) + g(S) \subset (f + g)(S)$$

$$f(S)\, g(S) \subset (f\, g)(S)$$

If $\| f_i - f \|_\infty \longrightarrow 0$ then $f_i(S) \longrightarrow f(S)$ in the sense of strong operator convergence.

If f is real valued then f(S) is self–adjoint and $e_{(-\infty,t]}(f(S)) = e_{\{ x \in \mathbb{R} : f(x) \leq t \}}(S)$

Polar decomposition Reference : [KR] § 6.1

Suppose S is a closed densely defined operator on $\mathcal{H}$

Then $S = v\,(S^* S)^{1/2} = (S\,S^*)^{1/2}\,v$

where v is a partial isometry with initial space $R((S^* S)^{1/2})$ and final space $R(S)$.

We define $(S^* S)^{1/2} \equiv |S|$ (and $(S\,S^*)^{1/2} \equiv |S^*|$).

$D(S) = D(|S|)$, $N(S) = N(|S|)$ and $R(|S|) = R(S^*) = R(S^* S)$

Thus $v^* v = R(|S|) = \mathrm{supp}(|S|) = e_{(0,\infty)}(S)$

v, $|S|$ are uniquely determined up to $|S|$ being positive and v being a partial isometry with initial space $R(S^*)$.

$S\,S^* = v\,S^*\,S\,v^*$, and so restricted to $R(S^*)$ and $R(S)$ respectively, $S^*\,S$ and $S\,S^*$ are unitarily equivalent, and this equivalence is implemented by v. It follows that $|S^*| = v\,|S|\,v^*$

CONTENTS

INTRODUCTION

It is well known that a commutative von Neumann algebra $\mathcal{M}$ has a representation as a L_∞ space, and that the predual $\mathcal{M}_*$ a representation as the corresponding L_1 space. In particular, the classical integration theory can be applied to commutative von Neumann algebras. The following question then presents itself : how much of the integration theory (and measure theory in general) for commutative von Neumann algebras can be generalised in some way to the non–commutative case. This question essentially defines the field known as *non–commutative integration theory.* Some of the first steps in this theory were taken by Murray and von Neumann ([MvN1] , [MvN2] , [vN3] , [MvN4]) in their definition of *trace*, which generalises the commutative integral. Subsequently the notion of measurable function and the various notions of convergence in commutative spaces of measurable functions such as L_∞ and L_0 spaces were generalised to von Neumann algebras (and the space of unbounded operators affiliated to such algebras) by Segal, Stinespring and Yeadon, amongst others. L_p spaces of such operators, analogous to commutative L_p spaces, were also defined. The topology of convergence in measure was generalised to semifinite von Neumann algebras by Nelson and this work was subsequently improved by Terp. Later some ideas of Yeadon on generalising the theory of rearrangement of functions to von Neumann algebras were developed by Fack and Kosaki. Most recently, Banach Function spaces were generalised to von Neumann algebras in the work of Dodds, Dodds and de Pagter.

Thus we see that there have been a number of different approaches to non–commutative integration theory, in particular the theory of non–commutative Banach Function Spaces. We take the most recent approach which leads to the widest class of non–commutative Banach Function Spaces yet derived. In our opinion some of the essentials of the topic have become obscured, and we have aimed at a self–contained presentation which should be accesible to one possessing only the most basic knowledge of von Neumann algebra theory.

We now give a broad outline of the direction we shall take.

Section I

As already suggested, one of our main goals has been to show how the theory of non–commutative integration (and in particular, the theory of non–commutative Banach Function Spaces) generalises the commutative theory. With this in mind, we introduce all of the commutative results we will need in Section I , before proceeding to the non–commutative case. In Chapter 1 we examine the topology of convergence in measure on $L_0(X,\Sigma,\mu)$ for (X,Σ,μ) an arbitrary measure space in preparation for Chapter 8 and subsequent Chapters. We also examine the topology of local convergence in measure. In Chapter 2 we examine Function Spaces on the semi–axis $(0,\infty)$ – in particular, symmetric Banach Function Spaces for which the norm is lower semicontinuous – in preparation for Chapter 10.

Section II

In Section II we examine traces on von Neumann algebras. In Chapter 3 we define traces and examine such concepts as faithfulness, finiteness and semifiniteness, and normality. This Chapter includes an original proof of the fact that a function satisfying linearity and homogeniety conditions is a trace iff it is unitarily invariant. In Chapter 4 we consider a number of examples of traces, in particular showing how the trace is a generalisation of the commutative integral. In Chapter 5 we show that a von Neumann algebra is finite iff it admits a sufficient (points separating) family of finite normal traces. In Chapter 6 we show that a von Neumann algebra is semifinite iff it admits a faithful semifinite normal trace.

Section III

It quickly becomes apparent that it is neccessary to consider unbounded operators. (For example, $\mathcal{M}$ is not neccessarily complete with respect to the topology defined in Chapter 8, and to identify the completion it is neccessary to 'add unbounded operators to $\mathcal{M}$'.) We will require unbounded operators that in certain respects behave like members of $\mathcal{M}$. Affiliation is the appropiate characterisation and affiliated operators are discussed in Chapter 7. Alternatively, affiliated operators can be viewed as those operators that are generated by the projections belonging to $\mathcal{M}$. This indeed generalises the commutative case; where the space L_0 of unbounded functions (comparable with the affiliated operators) is generated by the characteristic functions (comparible with the projections of a von Neumann algebra).

From Chapter 8 onwards we consider a semifinite von Neumann algebra $\mathcal{M}$ and the faithful semifinite normal trace τ on $\mathcal{M}$ guaranteed by Chapter 6.

In Chapter 8 we introduce the space $\tilde{\mathcal{M}}$ of τ–measurable operators and define the topology of convergence in measure on this space. This approach is essentially that of Nelson [N] and Terp [Tp], the latter being the most recent characterisation of the topologised space of unbounded operators considered appropiate for the purposes of non–commutative integration. Nevertheless it would be extremely unfair not to draw attention to the work done in this context by the pioneers in the field of non–commutative integration. We postpone this to Chapter 11.

In Chapter 9 we define the distribution function $d_t(S)$ and the generalised singular function $\mu_t(S)$ for operators $S \in \tilde{\mathcal{M}}$. The generalised singular function generalises both the rearrangement of a function in $\tilde{L}_\infty$ and the singular value sequence of a positive compact operator in $BL(\mathcal{H})$. The generalised singular function has its origins with Yeadon ([Y1] , he defines the *rearrangement* of an operator) . As Fack and Kosaki point out ([FK] , Introduction) the algebra $\tilde{\mathcal{M}}$ is the natural domain for generalising the rearrangement : " ... the τ–measurability of an operator S exactly corresponds to the property $\mu_t(S) < \infty$, $t > 0$ and the [topology of convergence in measure] can easily and naturally be expressed in terms of μ_t . " We introduce the relation between the generalised singular function and the topology of convergence in measure at a very much earlier stage than in [FK] , and as a consequence results are proved in a simpler and more instructive manner. We also have some new results, especially approximation results, which will be particularly useful in subsequent Chapters.

In Chapter 10 we examine the recent work of Dodds, Dodds and de Pagter. The principle result of [DDd] is establishing that for R , S $\in \tilde{\mathcal{M}}$, $| \mu_t(R) - \mu_t(S) | \prec\prec \mu_t(R - S)$. For $L_\rho(0,\infty)$ a Function Space, one can define a set of operators $L_\rho(\tilde{\mathcal{M}})$ whose members are precisely that subset of $\tilde{\mathcal{M}}$ whose generalised singular function lies in $L_\rho(0,\infty)$. The above result is used to show that if $L_\rho(0,\infty)$ is a symmetric Banach Function Space where ρ is lower semicontinuous, then $L_\rho(\tilde{\mathcal{M}})$ is a Banach Space. Because such spaces generalise the commutative Banach Function Spaces on the interval $(0,\infty)$, such spaces are termed non–commutative Banach Function Spaces. We include some examples of these spaces. It should be pointed out that this result is an improvement of a similar result of Yeadon ([Y4]) ; furthermore the majorisation result given above is of independent value. The approach of Yeadon is substantially different to that of Dodds, Dodds and de Pagter and we shall not explicitly consider it here.

We arrive at a point where we are able to provide some perspective on the history of non–commutative integration theory. In Chapter 11 we discuss the work of Segal, Stinespring, Kunze, Yeadon, and Nelson in defining algebras of unbounded operators which culminated in the work of Terp. We also note that the work of Dodds, Dodds and de Pagter generalises previous work in non–commutative integration theory where much time was devoted to non–commutative L_p spaces, through a variety of approaches. Despite the differences in the approaches of Segal, Dixmier, Stinespring, Kunze, Yeadon, Nelson, Terp and Fack and Kosaki we show that all the variously defined L_p spaces are indeed isomorphic to the $L_p(\tilde{\mathcal{M}})$ spaces of Dodds, Dodds and de Pagter.

In Chapter 12 we consider duality theory. The well known fact that if L_ρ is a Function Space then the associate space $L_{\rho^\times}$ can be identified as the subspace of L_ρ^* that consists of normal linear functionals is mentioned in Chapter 2. In this Chapter we attempt to generalise these facts, and show that in the case $\mathcal{M}$ is non–atomic $L_{\rho^\times}(\tilde{\mathcal{M}})$ may be identified with a subspace of $L_\rho(\tilde{\mathcal{M}})^*$ that we denote $L_\rho(\tilde{\mathcal{M}})^\times$. Dodds, Dodds and de Pagter have announced this result, although to the best of our knowledge have not published it, and so this result appears to be original. We would however like to point out that the approach used generalises some ideas of Yeadon.

1 : TOPOLOGIES of CONVERGENCE in MEASURE

We define the topologies of convergence in measure and local convergence in measure on the spaces $L_0(X,\Sigma,\mu)$ and $L_\infty(X,\Sigma,\mu)$ in order to introduce, and find the canonical commutative example for, the topology of convergence in measure to be defined for semifinite von Neumann algebras in Chapter 8.

1:1 **Definition**

Suppose (X,Σ,μ) is a measure space.

$L_0(X,\Sigma,u)$ is the set of equivalence classes (modulo almost everywhere equality) of real–valued measureable functions on (X,Σ,μ).

We denote this by L_0 unless there is danger of confusion.

We make the requirement that functions are a.e. real valued to ensure that $L_0(X,\Sigma,\mu)$ is a vector space.

$L_\infty(X,\Sigma,u)$ is the set of equivalence classes (modulo locally almost everwhere equality) of real–valued essentially bounded measureable functions on (X,Σ,μ).

We denote this by L_∞ unless there is danger of confusion.

We do not at this stage assume that $L_\infty(X,\Sigma,\mu)$ is neccessarily a von Neumann algebra.

1:2 **Definition**

For ϵ , $\delta > 0$

$$\mathcal{N}(\epsilon\ ,\ \delta) = \{\ f \in L_0 : \mu\{\ x \in X : |f(x)| > \epsilon\} \leq \delta\ \}$$

$$\mathcal{N}(\epsilon) = \mathcal{N}(\epsilon\ ,\ \epsilon) = \{\ f \in L_0 : \mu\{\ x \in X : |f(x)| > \epsilon\} \leq \epsilon\ \}$$

1:3 **Theorem** [W] 12.1.6

If E is a vector space and $\mathcal{F}$ a system of subsets of E satisfying :–

$\mathcal{F}$ is a filter base

every member of $\mathcal{F}$ is symmetric

$\forall\, U \in \mathcal{F} \quad \exists\, V \in \mathcal{F}$ such that $V + V \subset U$.

Then $\mathcal{F}$ induces upon translation a topology with $\mathcal{F}$ a basic neighbourhood system at 0 . □

1:4 **Theorem**

(a) $\forall\, \epsilon_1, \epsilon_2, \delta_1, \delta_2 > 0 \quad \mathcal{N}(\epsilon_1, \delta_1) + \mathcal{N}(\epsilon_2, \delta_2) \subset \mathcal{N}(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$

In particular, $\mathcal{N}(\epsilon_1) + \mathcal{N}(\epsilon_2) \subset \mathcal{N}(\epsilon_1 + \epsilon_2)$

(b) $0 < \epsilon_1 < \epsilon_2,\ 0 < \delta_1 < \delta_2 \Rightarrow \mathcal{N}(\epsilon_1, \delta_1) + \mathcal{N}(\epsilon_2, \delta_2) \subset \mathcal{N}(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$

In particular, $0 < \epsilon_1 < \epsilon_2 \Rightarrow \mathcal{N}(\epsilon_1) \subset \mathcal{N}(\epsilon_2)$

(c) $\{ \mathcal{N}(\epsilon, \delta) : \epsilon, \delta > 0 \}$ forms a basic system at 0 describing upon translation a topology $\gamma_{c\mu}$ on L_0 : *the topology of convergence in measure.*

This topology is also described by the system $\{ \mathcal{N}(\epsilon) : \epsilon > 0 \}$

(d) $\forall\, \epsilon > 0$, $\mathcal{N}(\epsilon)$ is balanced

(e) $f_n \xrightarrow{\gamma_{c\mu}} f$ iff $\forall\, \epsilon > 0 \ \mu\{ x \in X : |(f_n - f)(x)| > \epsilon \} \xrightarrow{n} 0$

(f) [C] 3.1.2

If $f_n \xrightarrow{\gamma_{c\mu}} f$ then there is a subsequence (f_{n_k}) such that $f_{n_k} \xrightarrow{au} f$ and $f_{n_k} \xrightarrow{ae} f$

(g) If $f_n \xrightarrow{au} f$ then $f_n \xrightarrow{\gamma_{c\mu}} f$

(h) [C] 3.1.1

If $\mu(X) < \infty$ and $f_n \xrightarrow{ae} f$ then $f_n \xrightarrow{\gamma_{c\mu}} f$

(i) The subspace topology of $\gamma_{c\mu}$ on L_∞, $\gamma_{c\mu}|_{L_\infty}$, has as a basic system at 0

$\{ \{ f \in L_\infty : \mu\{ x \in X : |f(x)| > \epsilon\} \le \epsilon \} : \epsilon > 0 \}$

Proof

(a)

Suppose $f_i \in \mathcal{N}(\epsilon_i , \delta_i)$ $(i = 1 , 2)$.

Let $E_i = \{ x \in X : |f_i(x)| > \epsilon_i \}$. Then $\mu(E_i) \leq \delta_i$.

If $x \notin E_1 \cup E_2$ then $|(f_1 + f_2)(x)| \leq |f_1(x)| + |f_2(x)| \leq \epsilon_1 + \epsilon_2$

$\Rightarrow$ $\{ x \in X : |(f_1 + f_2)(x)| > \epsilon_1 + \epsilon_2 \} \subset E_1 \cup E_2$

$\Rightarrow$ $\mu\{ x \in X : |(f_1 + f_2)(x)| > \epsilon_1 + \epsilon_2 \} \leq \mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2) \leq \delta_1 + \delta_2$

$\Rightarrow$ $f_1 + f_2 \in \mathcal{N}(\epsilon_1 + \epsilon_2 , \delta_1 + \delta_2)$

(b)

Clearly $\forall\, \epsilon , \delta > 0$ $\quad 0 \in \mathcal{N}(\epsilon , \delta)$.

$$\begin{aligned} \text{Thus } \mathcal{N}(\epsilon_1 , \delta_1) &= 0 + \mathcal{N}(\epsilon_1 , \delta_1) \\ &\subset \mathcal{N}(\epsilon_2 - \epsilon_1 , \delta_2 - \delta_1) + \mathcal{N}(\epsilon_1 , \delta_1) \\ &\subset \mathcal{N}(\epsilon_2 , \delta_2) , \text{ by (a)} \end{aligned}$$

(c)

In accordance with 1:3 , it suffices to show :–

(1) $\{ \mathcal{N}(\epsilon , \delta) : \epsilon , \delta > 0 \}$ is a filter base

(2) $\forall\, \epsilon , \delta > 0$ $\mathcal{N}(\epsilon , \delta)$ is symmetric

(3) $\forall\, \epsilon , \delta > 0$ $\exists\, \epsilon_1 , \delta_1 > 0$ such that $\mathcal{N}(\epsilon_1 , \delta_1) + \mathcal{N}(\epsilon_1 , \delta_1) \subset \mathcal{N}(\epsilon , \delta)$

(1) $\forall\, \epsilon , \delta > 0$ $\quad \frac{\epsilon}{2} . \chi_X \in \mathcal{N}(\epsilon , \delta)$

$\Rightarrow$ $\phi \neq \mathcal{N}(\epsilon , \delta)$ $\forall\, \epsilon , \delta > 0$

It now suffices to recall (b)

(2) Since $|f(x)| = |(-f)(x)|$ $\forall\, x \in X$, this is clear.

(3) From (a) we have that $\mathcal{N}(\frac{\epsilon}{2} , \frac{\delta}{2}) + \mathcal{N}(\frac{\epsilon}{2} , \frac{\delta}{2}) \subset \mathcal{N}(\epsilon , \delta)$

It remains to show that the system $\{ \mathcal{N}(\epsilon) : \epsilon > 0 \}$ forms a basic system at 0 describing upon translation the same topology.

By the arguments already given, it is clear that the system

$\{ \mathcal{N}(\epsilon) : \epsilon > 0 \} = \{ \mathcal{N}(\epsilon , \epsilon) : \epsilon > 0 \}$ indeed forms a basic neighbourhood system at 0 describing upon translation a vector topology.

Suppose $\epsilon , \delta > 0$

Then $\mathcal{N}(\epsilon \wedge \delta) = \mathcal{N}(\epsilon\wedge\delta , \epsilon\wedge\delta) \subset \mathcal{N}(\epsilon , \delta) \subset \mathcal{N}(\epsilon\vee\delta , \epsilon\vee\delta) = \mathcal{N}(\epsilon\vee\delta)$,

and hence the two topologies coincide.

(d)

Suppose $f \in \mathcal{N}(\epsilon)$, $0 < |\lambda| \leq 1$

Then $\mu\{ x \in X : |(\lambda f)(x)| > \epsilon \} = \mu\{ x \in X : |f(x)| > \frac{\epsilon}{|\lambda|} \} \leq \mu\{ x \in X : |f(x)| > \epsilon \} \leq \epsilon$

$\Rightarrow \quad \lambda f \in \mathcal{N}(\epsilon)$

Thus $\mathcal{N}(\epsilon)$ is balanced.

(e)

($\Leftarrow$)

$\forall \epsilon > 0 \;\; \mu\{ x \in X : |(f_n - f)(x)| > \epsilon \} \xrightarrow{n} 0$

$\Rightarrow \quad \forall \epsilon > 0 \; \exists n_\epsilon \in \mathbb{N}$ such that $n \geq n_\epsilon \Rightarrow \mu\{ x \in X : |(f_n - f)(x)| > \epsilon \} \leq \epsilon$

$\Rightarrow \quad f_n \xrightarrow{\gamma_{c\mu}} f$

($\Rightarrow$)

Suppose $\epsilon > 0$

Suppose $0 < \alpha < \epsilon$ is given

$\exists n_\alpha$ such that $n \geq n_\alpha \Rightarrow \mu\{ x \in X : |(f_n - f)(x)| > \alpha \} \leq \alpha$

$\Rightarrow \quad (n \geq n_\alpha \Rightarrow \mu\{ x \in X : |(f_n - f)(x)| > \epsilon \} \leq \alpha)$

$\Rightarrow \quad \mu\{ x \in X : |(f_n - f)(x)| > \epsilon \} \xrightarrow{n} 0$, since α was arbitrary

(f)

Suppose $f_n \xrightarrow{\gamma_{c\mu}} f$

Choose a suitable subsequence (f_{n_k}) so that $f_{n_k} - f \in \mathcal{N}(2^{-k}) \quad \forall\, k \in \mathbb{N}$

Let $A_k = \{ x \in X : |(f_{n_k} - f)(x)| > 2^{-k} \}$

Then $\mu(A_k) \le 2^{-k}$ and $\mu(\bigcup_{i=k}^{\infty} A_i) \le 2^{-(k-1)}$

For $x \notin \bigcup_{i=k}^{\infty} A_i$, $|(f_{n_i} - f)(x)| \le 2^{-i} \le 2^{-k} \quad \forall\, i \ge k$

$\Rightarrow \qquad f_{n_i} \xrightarrow{u} f$ off $\bigcup_{i=k}^{\infty} A_i$

Since $\mu(\bigcup_{i=k}^{\infty} A_i) \xrightarrow{k} 0$, it follows $f_{n_k} \xrightarrow{au} f$

Suppose $x \notin \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k$

$\Rightarrow \qquad \exists\, i \in \mathbb{N}$ such that $x \in A_k \quad \forall\, k \ge i$

$\Rightarrow \qquad |(f_{n_k} - f)(x)| \le 2^{-k} \quad \forall\, k \ge i$

$\Rightarrow \qquad |(f_{n_k} - f)(x)| \longrightarrow 0$ as $k \longrightarrow \infty$

Now $\mu(\bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k) \le \mu(\bigcup_{k=i}^{\infty} A_k) \le 2^{-(i-1)} \quad \forall\, i \in \mathbb{N}$

$\Rightarrow \qquad \mu(\bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} A_k) = 0$, and so $f_{n_k} \xrightarrow{ae} f$.

(g)

Suppose $\epsilon > 0$.

Choose $E_\epsilon \in \Sigma$ such that $\mu(E_\epsilon) < \epsilon$ and $f_n \xrightarrow{u} f$ off E_ϵ

Then for n sufficiently large, $|(f_n - f)(x)| \le \epsilon \ \forall\, x \in X - E_\epsilon$

$\Rightarrow \qquad \{ x \in X : |(f_n - f)(x)| > \epsilon \} \subset E_\epsilon$

$\Rightarrow \qquad f_n - f \in \mathcal{N}(\epsilon)$ for such large n.

(h)

Suppose $\epsilon > 0$

Let $A_n = \{ x \in X : |(f_n - f)(x)| > \epsilon \}$

$$x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

$\Rightarrow$ $\forall n \in \mathbb{N} \ \exists k \geq n$ such that $|(f_k - f)(x)| > \epsilon$

$\Rightarrow$ $f_n(x) \not\to f(x)$

Thus $\mu(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k) = 0$

$\Rightarrow$ $\mu(\bigcup_{k=n}^{\infty} A_k) \xrightarrow{n} 0$ since $\{\bigcup_{k=n}^{\infty} A_k\}$ is a decreasing sequence of sets and μ is finite.

$\Rightarrow$ $\mu(A_n) \xrightarrow{n} 0$

i.e. $f_n \xrightarrow{\gamma_{c\mu}} f$

(i)

Clear by definition of subspace topologies. □

In the remainder of this Chapter we will work exclusively with the system $\{ \mathcal{N}(\epsilon) : \epsilon > 0 \}$. However, the equivalent system $\{ \mathcal{N}(\epsilon, \delta) : \epsilon, \delta > 0 \}$ will prove useful in Chapters 2, 8 and 9.

1:5 **Theorem** [J] 2.2.5 ; 2.8.1

If E is a vector space and $\mathcal{F}$ a system of sets satisfying :–

$\mathcal{F}$ is a filter base

$\forall U \in \mathcal{F} \ \exists V \in \mathcal{F}$ such that $V + V \subset U$

$\forall U \in \mathcal{F}$, U is balanced and absorbing .

Then $\mathcal{F}$ induces upon translation a vector topology with $\mathcal{F}$ a basic neighbourhood system of 0 .
A vector topology is metrisable iff it has a countable base of neighbourhoods of 0. □

1:6 Note

By the results already obtained in 1:4, it follows from 1:5 that the question of whether or not $\gamma_{c\mu}$ or $\gamma_{c\mu|L_\infty}$ are vector topologies depends upon whether or not the basic neighbourhoods $\{ \mathcal{N}(\epsilon) : \epsilon > 0 \}$ are absorbent.

1:7 **Lemma**

If μ is semifinite but not finite then we can choose a disjoint sequence $\{F_n\}_{n\in\mathbb{N}} \subset \Sigma$ such that $1 < \mu(F_n) < \infty \ \forall\, n \in \mathbb{N}$

Proof

It is easy to verify that

$$\mu \text{ is semifinite } \Leftrightarrow \forall\, E \in \Sigma \ \ \mu(E) = \sup \{ \mu(F) : F \in \Sigma , \mu(F) < \infty , F \subset E \}$$

(Some sources e.g. [B] take this as a definition of semifiniteness.)

We construct the sequence $\{F_n\}$ by induction.

Since μ is semifinite but not finite we can choose $F_1 \in \Sigma$ such that $1 < \mu(F_1) < \infty$

Suppose at the n^{th} stage we have sets $F_1, \dots, F_n$ which are mutually disjoint and $1 < \mu(F_i) < \infty \ \ 1 \le i \le n$

Then $X - \bigcup_{i=1}^{n} F_i$ is of infinite measure and hence we can choose $F_{n+1} \in \Sigma$ such that $F_{n+1} \subset X - \bigcup_{i=1}^{n} F_i$ and $1 < \mu(F_{n+1}) < \infty$ □

The following simple argument is used so often that we extract it as a Lemma.

1:8 **Lemma**

Suppose $f \in L_0$

For $n \in \mathbb{N}$ let $X_n = \{ x \in X : |f(x)| > n \}$

If μ is finite, or even if $\mu(X_n)$ is finite for some $n \in \mathbb{N}$, then $\mu(X_n) \xrightarrow{n} 0$.

Proof

$\bigcap_{n=1}^{\infty} X_n = \{ x \in X : |f(x)| = \infty \}$ is of measure 0 .

X_n is a decreasing system of sets.

The result follows by [C] 1.2.3(b) . □

1:9 **Theorem**

(a) If μ is semifinite then $\gamma_{c\mu}$ is a vector topology iff μ is finite.

(b) There exists a non semifinite measure space for which $\gamma_{c\mu}$ is a vector topology.

(c) There exists a non semifinite measure space for which $\gamma_{c\mu}$ is not a vector topology.

(d) $\gamma_{c\mu|L_\infty}$ is a vector topology.

Proof

(a)

($\Leftarrow$)

Suppose μ is finite.

In light of the previous results (see 1:6) , it suffices to show that $\forall\ \epsilon > 0$ $\mathcal{N}(\epsilon)$ is absorbing.

Suppose $f \in L_0$ and $\epsilon > 0$.

By virtue of 1:8 $\exists$ N $\in \mathbb{N}$ such that $\mu\{ x \in X : |f(x)| > N \} \leq \epsilon$

$\Rightarrow$ $\mu\{ x \in X : \frac{\epsilon}{N} |f(x)| > \epsilon \} \leq \epsilon$

$\Rightarrow$ $\frac{\epsilon}{N} f \in \mathcal{N}(\epsilon)$

$\Rightarrow$ $f \in \frac{N}{\epsilon} \mathcal{N}(\epsilon)$

$\Rightarrow$ $\mathcal{N}(\epsilon)$ is absorbent.

($\Rightarrow$)

Suppose μ is semifinite but not finite.

Let $\{F_n\}_{n\in\mathbb{N}}$ be the sequence of sets indicated in 1:7

Let $f = \sum_{n=1}^{\infty} n \chi_{F_n}$.

It is clear that no neighbourhood $\mathcal{N}(\epsilon)$ can absorb this function, and thus $\gamma_{c\mu}$ is not a vector topology.

(b)

Let $X = \{ 0 \}$ and $\mu(\{0\}) = \infty$

Then (X,Σ,μ) is a non semi–finite measure space.

Since $f \in L_0$ must now be finite valued, we can identify L_0 with $\mathbb{R}$ and $\mathcal{N}(\epsilon)$ with $(-\epsilon , \epsilon)$.

Thus $\forall\ \epsilon > 0\ \mathcal{N}(\epsilon)$ is absorbing, so $\gamma_{c\mu}$ is a vector topology (in fact, the canonical topology on $\mathbb{R}$) .

(c)

Consider $(\mathbb{R} , \mathcal{B}(\mathbb{R}))$

Define μ on $(\mathbb{R} , \mathcal{B}(\mathbb{R}))$ by

$\mu(A) = m(A)$ if $0 \notin A$

$\mu(A) = \infty$ if $0 \in A$

It is easily verified $(\mathbb{R} , \mathcal{B}(\mathbb{R}) , \mu)$ is a non–semifinite measure space , and that the neighbourhoods $\mathcal{N}(\epsilon)$ do not absorb the function $f(x) = x$.

Hence $\gamma_{c\mu}$ is not a vector topology .

(d)

Every member of L_∞ (which is, of course, an equivalence class of functions) can be represented by a bounded function from that equivalence class. The neighbourhoods $\mathcal{N}(\epsilon)$ clearly absorb bounded functions, so by 1:6 $\gamma_{c\mu}|_{L_\infty}$ is a vector topology. □

1:10 **Theorem**

(a) $\gamma_{c\mu}$ is a Hausdorff topology.

(b) In the case that $\gamma_{c\mu}$ is a vector topology, it is metrisable.

Proof

(a)

Since the topology is translation invariant, it suffices to separate $0 \neq f \in L_0$ and 0.

So suppose $0 \neq f \in L_0$

$\Rightarrow$ $\exists\, \delta > 0$ such that $\mu\{ x \in X : |f(x)| > \delta \} = c > 0$ (possibly $c = \infty$)

Put $\epsilon = \frac{1}{4} \min \{\delta, c\}$

Then $f \notin \mathcal{N}(\epsilon)$ since $\mu\{ x \in X : |f(x)| > \epsilon\} \geq \mu\{ x \in X : |f(x)| > \delta\} = c > \epsilon$

Thus $-f \notin \mathcal{N}(\epsilon)$ and so $0 = f - f \notin f + \mathcal{N}(\epsilon)$

Assume for a contradiction that $\mathcal{N}(\epsilon) \cap (f + \mathcal{N}(\epsilon)) = \phi$

$\Rightarrow$ $\exists\, g \in L_0$ such that $g \in \mathcal{N}(\epsilon)$, $g \in f + \mathcal{N}(\epsilon)$

$\Rightarrow$ $g \in \mathcal{N}(\epsilon)$ and $g - f \in \mathcal{N}(\epsilon)$

$\Rightarrow$ $f = f - g + g \in \mathcal{N}(\epsilon) + \mathcal{N}(\epsilon) \subset \mathcal{N}(2\epsilon)$

$\Rightarrow$ $\mu\{ x \in X : |f(x)| > \delta \} \leq \mu\{ x \in X : |f(x)| > 2\epsilon \} \leq 2\epsilon < c$, the required contradiction.

(b)

$\gamma_{c\mu}$ is a Hausdorff vector topology with a countable basis $\{ \mathcal{N}(\frac{1}{n}) : n \in \mathbb{N} \}$, so the result follows immediately from 1:5 □

1:11 **Corollary**

$\gamma_{c\mu}|_{L_\infty}$ is metrisable. □

1:12 **Definition** [J] 2.7

Let E be a vector space. An F–seminorm is a map $q : E \to [0,\infty)$ with the properties :–

(1) $q(\lambda f) \leq q(f) \quad \forall f \in E \quad \forall |\lambda| \leq 1$

(2) $\lim_n q(\frac{1}{n} f) = 0 \quad \forall f \in E$

(3) $q(f + g) \leq q(f) + q(g) \quad \forall f, g \in E$

(4) If $q(f) = 0 \Rightarrow f = 0$, then q is said to be an F–norm .

$d(f, g) = q(f - g)$ defines a tranlation invariant (semi)metric in the case that q is a F–(semi)norm .

A collection $\{q_i\}_{i \in I}$ of F–seminorms induces a vector topology on E with a subbasis for neighbourhoods at 0 being $\{ \{ f \in E : q_i(f) < \epsilon \} : \epsilon > 0 , i \in I \}$

1:13 **Theorem**

Suppose $\gamma_{c\mu}$ is a vector topology .

(a) The function $q : L_0 \to [0,\infty) : f \longrightarrow \inf \{\epsilon > 0 : f \in \mathcal{N}(\epsilon) \}$ defines a F–norm on L_0 .

(b) The metric derived from q induces the topology $\gamma_{c\mu}$.

Proof

(a)

We verify the conditions (1) to (4) in 1:12 :–

(1) Follows since every $\mathcal{N}(\epsilon)$ is balanced

(2) Follows since every $\mathcal{N}(\epsilon)$ is absorbent

(3) Follows since $\mathcal{N}(\epsilon_1) + \mathcal{N}(\epsilon_2) \subset \mathcal{N}(\epsilon_1 + \epsilon_2) \quad \forall \epsilon_1 , \epsilon_2 > 0$

(4) Follows since $\bigcap_{\epsilon>0} \mathcal{N}(\epsilon) = \{0\}$ as $\gamma_{c\mu}$ is Hausdorff

(b)

It is clear that $\mathcal{N}(\epsilon) \subset \{ f \in L_0 : d(f,0) < \epsilon\} \subset \mathcal{N}(\alpha) \quad \forall \alpha > \epsilon$

The result follows. □

1:14 **Theorem** (Riesz–Weyl) [B] 21,4

A Cauchy sequence in ($\mathcal{L}_0(X,\Sigma,\mu)$, $\gamma_{c\mu}$) is convergent. Hence if $\gamma_{c\mu}$ is sequentially defined (in particular if the topology is a vector topology, since it is then metrisable), then ($\mathcal{L}_0(X,\Sigma,\mu)$, $\gamma_{c\mu}$) is complete.

Proof

Suppose (f_n) is Cauchy in $(\mathcal{L}_0 , \gamma_{c\mu})$.

By taking a subsequence if neccessary, we may suppose that $\forall\, n \in \mathbb{N}\ \ f_{n+1} - f_n \in \mathcal{N}(2^{-n})$

Let $E_n = \{ x \in X : |(f_{n+1} - f_n)(x)| > 2^{-n} \}$, then $\mu(E_n) \le 2^{-n}$

Let $F_k = \bigcup_{i=n+1}^{\infty} E_i$, then $\mu(F_n) \le 2^{-n}$

If $x \in X - F_n$, then $x \in X - E_k\ \ \forall\, k > n$, so $|(f_{k+1} - f_k)(x)| \le 2^{-k}\ \ \forall\, k > n$

Thus for $x \in X - F_n$, $m > n$, $r \ge 1$

$$|(f_{m+r} - f_m)(x)|$$

$$\le \sum_{k=m}^{m+r-1} |(f_{k+1} - f_k)(x)|$$

$$\le \sum_{k=m}^{m+r-1} 2^{-k}$$

$$\le 2^{-(m-1)}$$

We aim to show that $(f_{n_k})_{k\in\mathbb{N}}$ is almost uniformly Cauchy.

If $\delta > 0\ \exists\, N \in \mathbb{N}$ such that $2^{-N} < \delta$, so it suffices to show $(f_n)_{k\in\mathbb{N}}$ is uniformly Cauchy off F_N .

Given $\epsilon > 0$, choose $s \in \mathbb{N}$ such that $s > N$ and $2^{-(s-1)} < \epsilon$

If $m \geq s$ then $m > n$, so for $x \in X - F_n$, $m \geq s$, $r \geq 1$

$$|(f_{m+r} - f_m)(x)| < 2^{-(m-1)} < 2^{-(s-1)} \leq \epsilon$$

Hence (f_n) is almost uniformly Cauchy.

There exists $f \in \mathcal{L}_0$ such that $f_n \xrightarrow{au} f$ [B] 21,2.

$$\Rightarrow \quad f_n \xrightarrow{\gamma_{c\mu}} f \quad 1:4(g)$$

Thus the subsequence converges, and so the original sequence converges.

The result follows. □

1:15 **Theorem**

The completion of $\mathcal{L}_\infty$ under $\gamma_{c\mu}$ is the space of all functions (essentially) bounded except on a set of finite measure, which we will denote by $\tilde{\mathcal{L}}_\infty$. In particular, if $\mu(X) = 1$, then $\mathcal{L}_0$ is the completion of $\mathcal{L}_\infty$ under $\gamma_{c\mu}$.

Proof

As noted let $\tilde{\mathcal{L}}_\infty$ denote the space of functions (essentially) bounded except on a set of finite measure.

Suppose $f \in \tilde{\mathcal{L}}_\infty$ and (via the canonical identification) that f is bounded except on a set of finite measure.

Let $X_n = \{ x \in X : |f(x)| > n \}$

Let $f_n = f \chi_{X-X_n}$

By 1:8 $\forall \epsilon > 0 \ \exists N \in \mathbb{N}$ such that $n \geq N \Rightarrow \mu(X_n) < \epsilon$

It follows that $n, m \geq N \Rightarrow f_n - f \in \mathcal{N}(\epsilon)$ and $f_m - f_n \in \mathcal{N}(\epsilon)$

Thus the function $f \in \tilde{\mathcal{L}}_\infty$ is the limit of a Cauchy sequence in $\mathcal{L}_\infty$, and so $\tilde{\mathcal{L}}_\infty$ is included in the completion of $\mathcal{L}_\infty$.

Suppose (f_n) is a $\gamma_{c\mu}$ Cauchy sequence in L_∞.

By 1:14, $\exists\, f \in L_0$ such that $f_n \xrightarrow{\gamma_{c\mu}} f$, and by the proof of that theorem, $\exists\ (f_{n_k})$ a subsequence such that $f_{n_k} \xrightarrow{au} f$. Hence there exists $\exists\, E \subset X$, $\mu(E) \leq 1$, such that

$$f_{n_k} \chi_{X-E} \xrightarrow{u} f \chi_{X-E}$$

Since L_∞ is uniformly closed, $f \chi_{X-E} \in L_\infty$

Thus $f \in \tilde{L}_\infty$. □

1:16 **Definition**

For $E \in \Sigma$, $\epsilon > 0$, let $\mathcal{N}(E, \epsilon) = \{ f \in L_0 : \mu\{ x \in E : |f(x)| > \epsilon\} \leq \epsilon \}$

Let $\Sigma_f = \{ E \in \Sigma : \mu(E) < \infty \}$

1:17 **Theorem**

(a) $\forall E \in \Sigma\ \forall \epsilon_1, \epsilon_2 > 0 : \mathcal{N}(E, \epsilon_1) + \mathcal{N}(E, \epsilon_2) \subset \mathcal{N}(E, \epsilon_1 + \epsilon_2)$

(b) $\forall E \in \Sigma\ \ 0 < \epsilon_1 < \epsilon_2 \Rightarrow \mathcal{N}(E, \epsilon_1) \subset \mathcal{N}(E, \epsilon_2)$

(c) $\forall \epsilon > 0\ \ E_1, E_2 \in \Sigma,\ E_1 \subset E_2 \Rightarrow \mathcal{N}(E_2, \epsilon) \subset \mathcal{N}(E_1, \epsilon)$

(d) $\{ \mathcal{N}(E, \epsilon) : E \in \Sigma_f, \epsilon > 0 \}$ forms a basic system at 0 describing upon translation a vector topology $\gamma_{lc\mu}$ on L_0 : *the topology of local convergence in measure.*

(e) $f_n \xrightarrow{\gamma_{lc\mu}} f$

$\Leftrightarrow \quad \forall \epsilon > 0\ \ \forall E \in \Sigma_f\ \ \mu\{ x \in E : |(f_n - f)(x)| > \epsilon \} \xrightarrow{n} 0$

$\Leftrightarrow \quad \forall E \in \Sigma_f\ \ f_n \chi_E \xrightarrow{\gamma_{c\mu}} f \chi_E$

(f) If $f_n \xrightarrow{ae} f$ then $f_n \xrightarrow{\gamma_{lc\mu}} f$

(g) The subspace topology of $\gamma_{lc\mu}$ on L_∞, $\gamma_{lc\mu}|_{L_\infty}$, is a vector topology with a basic system of neighbourhoods at 0 being $\{ \{ f \in L_\infty : \mu\{ x \in E : |f(x)| > \epsilon\} \leq \epsilon \} : \epsilon > 0, E \in \Sigma_f \}$.

Proof

(a)

Similar to 1:4(a).

(b)

Follows from (a).

(c)

Suppose $f \in \mathcal{N}(E_2, \epsilon)$

$\Rightarrow \quad \mu\{ x \in E_2 : |f(x)| > \epsilon \} \leq \epsilon$

$\Rightarrow \quad \mu\{ x \in E_1 : |f(x)| > \epsilon \} \leq \epsilon$

$\Rightarrow \quad f \in \mathcal{N}(E_1, \epsilon)$

(d)

In light of 1:5 it suffices to show :–

(1) $\{ \mathcal{N}(E, \epsilon) : E \in \Sigma_f, \epsilon > 0 \}$ is a filter base

(2) $\forall \epsilon_1 > 0 \ \ \forall E_1 \in \Sigma_f \ \ \exists \epsilon_2 > 0 \ \ \exists E_2 \in \Sigma_f$ such that

$\mathcal{N}(E_2, \epsilon_2) + \mathcal{N}(E_2, \epsilon_2) \subset \mathcal{N}(E_1, \epsilon_1)$

(3) $\mathcal{N}(E, \epsilon)$ is balanced and absorbing.

(1) $\mathcal{N}(\phi, 1) \in \{ \mathcal{N}(E, \epsilon) : E \in \Sigma_f, \epsilon > 0 \}$

$\Rightarrow \quad \phi \neq \{ \mathcal{N}(E, \epsilon) : E \in \Sigma_f, \epsilon > 0 \}$

$\forall E \in \Sigma \ \forall \epsilon > 0 \ \frac{\epsilon}{2} \cdot \mathcal{X}_X \in \mathcal{N}(E, \epsilon)$

$\Rightarrow \quad \phi \neq \mathcal{N}(E, \epsilon) \ \ \forall \epsilon > 0 \ \ \forall E \in \Sigma_f$

Suppose $E_1, E_2 \in \Sigma_f$ and $\epsilon_1, \epsilon_2 > 0$

Let $E = E_1 \cup E_2$ (of course $E \in \Sigma_f$) and $\epsilon = \epsilon_1 \wedge \epsilon_2$

Suppose $f \in \mathcal{N}(E, \epsilon)$

$\Rightarrow \quad f \in \mathcal{N}(E, \epsilon_1)$ by (b)

$\Rightarrow \quad f \in \mathcal{N}(E_1, \epsilon_1)$ by (c)

Likewise $f \in \mathcal{N}(E_2, \epsilon_2)$.

Thus $\mathcal{N}(E, \epsilon) \subset \mathcal{N}(E_1, \epsilon_1) \cap \mathcal{N}(E_2, \epsilon_2)$, and so the system is a filter base.

(2) It follows from (a) that $\mathcal{N}(E, \frac{\epsilon}{2}) + \mathcal{N}(E, \frac{\epsilon}{2}) \subset \mathcal{N}(E, \epsilon)$

(3)

The same reasoning as in 1:4(d) shows that $\mathcal{N}(E, \epsilon)$ is balanced.

Suppose $f \in L_0$, $\epsilon > 0$, $E \in \Sigma_f$

By 1:8 $\exists N \in \mathbb{N}$ such that $\mu\{x \in E : |f(x)| > N\} \leq \epsilon$

$\Rightarrow \quad \mu\{x \in E : \frac{\epsilon}{N} |f(x)| > \epsilon\} \leq \epsilon$

$\Rightarrow \quad \frac{\epsilon}{N} f \in \mathcal{N}(E, \epsilon)$

$\Rightarrow \quad f \in \frac{N}{\epsilon} \mathcal{N}(E, \epsilon)$

$\Rightarrow \quad \mathcal{N}(E, \epsilon)$ is absorbent.

(e)

Follows as before.

(f)

Suppose $f_n \xrightarrow{ae} f$ and $E \in \Sigma_f$

By (e) it suffices to show $\forall \epsilon > 0$ $\mu\{x \in E : |(f_n - f)(x)| > \epsilon\} \xrightarrow{n} 0$

For $k \in \mathbb{N}$, let $E_k = \{x \in E : |(f_k - f)(x)| > \epsilon\}$

Let $F_k = \bigcup_{i=k}^{\infty} E_i$. Note that (F_k) is decreasing.

$$
\begin{aligned}
x \in \bigcap_{k=1}^{\infty} F_k \quad &\Rightarrow \quad x \in F_k \ \ \forall\, k \in \mathbb{N} \\
&\Rightarrow \quad \forall\, k \in \mathbb{N} \ \ \exists\, i \geq k \text{ such that } |(f_i - f)(x)| > \epsilon \\
&\Rightarrow \quad f_n(x) \not\to f(x) \\
&\Rightarrow \quad \mu(\bigcap_{k=1}^{\infty} F_k) = 0 \\
&\Rightarrow \quad \mu(F_k) \xrightarrow{k} 0 \text{ , by [C] 1.2.3(b) , since } \mu(F_1) \leq \mu(E) < \infty \\
&\Rightarrow \quad \mu(E_k) \xrightarrow{k} 0 \quad \text{since } E_k \subset F_k
\end{aligned}
$$

Thus $f_n \xrightarrow{\gamma_{lc\mu}} f$.

(g)

Clear by definition of subspace topologies. □

1:18 **Note**

If $\mu(X) < \infty$, then the topologies of convergence in measure and local convergence in measure coincide.

Proof

$\forall\, \epsilon > 0 \ \ \mathcal{N}(\epsilon) = \mathcal{N}(X , \epsilon)$

$\forall\, E \in \Sigma \ \ \forall\, \epsilon > 0 \ \ \mathcal{N}(\epsilon) = \mathcal{N}(X , \epsilon) \subset \mathcal{N}(E , \epsilon)$ □

Thus we need only discuss the topology of local convergence in measure when $\mu(X) = \infty$.

1:19 **Theorem**

(a) $\gamma_{1c\mu}$ is Hausdorff iff μ is semi–finite.

(b) Suppose μ is σ–finite, with $\{E_n\}_{n\in\mathbb{N}}$ a disjoint sequence in Σ_f such that $X = \bigcup_{n=1}^{\infty} E_n$.

The system $\{ \mathcal{N}(\bigcup_{i=1}^{n} E_i , \frac{1}{m}) : n , m \in \mathbb{N} \}$ is a countable base at 0 for $\gamma_{1c\mu}$.

The system $\{ \mathcal{N}(E_n , \frac{1}{m}) : n , m \in \mathbb{N} \}$ is a countable subbase at 0 for $\gamma_{1c\mu}$.

(c) $\gamma_{1c\mu}$ is metrisable iff μ is σ–finite.

(d) The same results hold for $\gamma_{1c\mu}|_{L_\infty}$.

Proof

(a)

($\Leftarrow$)

Suppose μ is semi–finite. Since $\gamma_{1c\mu}$ is a vector topology, it suffices to separate $0 \neq f \in L_0$ from 0

So suppose $0 \neq f \in L_0$

$\exists\, \delta > 0$ such that $\mu\{ x \in X : |f(x)| > \delta \} = c > 0$ ($c = \infty$ is possible)

Let $E = \{ x \in X : |f(x)| > \delta \}$

By semi–finiteness, choose $E_0 \subset E$ such that $0 < \mu(E_0) < \infty$

$$\begin{aligned} &\mu\{ x \in E_0 : |f(x)| > \tfrac{1}{2} \min \{ \delta , \mu(E_0) \} \} \\ \geq\ & \mu\{ x \in E_0 : |f(x)| > \delta \} \\ =\ & \mu(E_0) \\ >\ & \tfrac{1}{2} \min \{ \delta , \mu(E_0) \} \end{aligned}$$

Thus $f \notin \mathcal{N}(E_0 , \frac{1}{2} \min \{ \delta , \mu(E_0) \})$

($\Rightarrow$)

Suppose $E \in \Sigma$ and $\mu(E) = \infty$.

$\exists\, F \in \Sigma_f$ $\exists\, \epsilon \in (0,1)$ such that $\chi_E \notin \mathcal{N}(F , \epsilon)$

Then $\mu\{ x \in F : |\chi_E(x)| > \epsilon \} = \mu(F \cap E) > \epsilon > 0$

$\Rightarrow$ E has a subset $F \cap E$ of finite non–zero measure.

$\Rightarrow$ μ is semi–finite.

(b)

Consider the countable family $\{ \mathcal{N}(\bigcup_{i=1}^{n} E_i , \frac{1}{m}) : n,m \in \mathbb{N} \}$

Suppose $E \in \Sigma_f$ and $\epsilon > 0$.

Choose $m \in \mathbb{N}$ such that $\frac{1}{m} \leq \epsilon$

By using [C] 1.2.3(a) , choose $n \in \mathbb{N}$ such that $\mu(E - \bigcup_{i=1}^{n} E_i) < \frac{1}{2m}$

Then $f \in \mathcal{N}(\bigcup_{i=1}^{n} E_i , \frac{1}{2m})$

$\Rightarrow$ $\mu\{ x \in \bigcup_{i=1}^{n} E_i : |f(x)| > \frac{1}{2m} \} \leq \frac{1}{2m}$

$\Rightarrow$ $\mu\{ x \in E : |f(x)| > \epsilon \} \leq \frac{1}{2m} + \mu(E - \bigcup_{i=1}^{n} E_i) \leq \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m} \leq \epsilon$

$\Rightarrow$ $f \in \mathcal{N}(E , \epsilon)$

So the system $\{ \mathcal{N}(\bigcup_{i=1}^{n} E_i , \frac{1}{m}) : n,m \in \mathbb{N} \}$ is a countable basis at 0 for $\gamma_{lc\mu}$.

Suppose $\mathcal{N}(\bigcup_{i=1}^{n} E_i , \frac{1}{m})$ is one of the basic sets already considered.

For $f \in \bigcap_{i=1}^{n} \mathcal{N}(E_i , \frac{1}{mn})$,

$\mu\{ x \in \bigcup_{i=1}^{n} E_i : |f(x)| > \frac{1}{m} \}$

$\leq$ $\mu\{ x \in \bigcup_{i=1}^{n} E_i : |f(x)| > \frac{1}{mn} \}$

$=$ $\sum_{i=1}^{n} \mu\{ x \in E_i : |f(x)| > \frac{1}{mn} \}$

$$\leq \sum_{i=1}^{n} \frac{1}{mn}$$

$$= \frac{1}{m}$$

So $f \in \mathcal{N}(\bigcup_{i=1}^{n} E_i , \frac{1}{m})$, and thus $\bigcap_{i=1}^{n} \mathcal{N}(E_i , \frac{1}{mn}) \subset \mathcal{N}(\bigcup_{i=1}^{n} E_i , \frac{1}{m})$

Hence the system $\{ \mathcal{N}(E_n , \frac{1}{m}) : n , m \in \mathbb{N} \}$ is a countable subbase at 0 for $\gamma_{1c\mu}$.

(c)

Recall (1:5) that a vector topology is metrisable iff it has a countable base of neighbourhoods of 0.

Hence if μ is σ–finite then $\gamma_{1c\mu}$ is metrisable by (b).

Conversely, suppose $\gamma_{1c\mu}$ has a countable base of neighbourhoods of 0 .

$\Rightarrow \quad \exists \{E_n\}_{n\in\mathbb{N}} \subset \Sigma_f \; \exists \{\epsilon_n\} \subset (0,\infty)$ such that $\{ \mathcal{N}(E_n , \epsilon_n) : n \in \mathbb{N} \}$ is a base at 0 .

Assume for a contradiction $\mu(X - \bigcup_{n=1}^{\infty} E_n) > 0$.

By (a) μ is semifinite, so $\exists\, Y \in \Sigma$ such that $Y \subset X - \bigcup_{n=1}^{\infty} E_n$ and $0 < \mu(Y) < \infty$

Then $\forall\, n \in \mathbb{N} \quad \mu\{ x \in E_n : |\chi_Y(x)| \geq \epsilon_n \} = \mu(\phi) = 0$

$\Rightarrow \quad 0 \neq \chi_Y \in \mathcal{N}(E_n , \epsilon_n) \;\; \forall\, n \in \mathbb{N}$, the required contradiction.

Thus μ is σ–finite.

(d)

Clear since the functions constructed in (a) and (b) were characteristic functions, hence members of L_∞ . □

We now consider when $\gamma_{lc\mu}$ could be locally convex.

1:20 **Lemma**

Suppose G is a continuous linear functional G on $(\mathcal{L}_0, \gamma_{lc\mu})$

G is of *integrable type* (that is, $0 \le f_n \downarrow 0 \Rightarrow G(f_n) \longrightarrow 0$)

There exists a function $g \in \mathcal{L}_0$ such that $G(f) = \int f\, g\, d\mu \;\; \forall\, f \in \mathcal{L}_0$.

Proof

Suppose $G \in (\mathcal{L}_0, \gamma_{lc\mu})'$.

Suppose $f_n \downarrow 0$ a.e.

$\Rightarrow \quad f_n \xrightarrow{\gamma_{lc\mu}} 0 \quad$ 1:17(f)

$\Rightarrow \quad G(f_n) \longrightarrow 0$ by the continuity of G .

Thus G is of integrable type.

We employ the Radon–Nikodým Theorem to derive the function g.

Define $\nu(E) = G(\chi_E)$

$\nu(\phi) = G(\chi_\phi) = G(0) = 0$

Suppose $\{ E_1, \dots, E_n \} \subset \Sigma$ are disjoint.

It is clear that $\nu(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} \nu(E_i)$ by the linearity of G.

Hence ν is finitely additive.

Suppose $\{E_i\}$ is an increasing sequence of measurable sets.

$$\chi_{E_i} \xrightarrow{ae} \chi_{\bigcup_{i=1}^{\infty} E_i}$$

$$\Rightarrow \quad \chi_{E_i} \xrightarrow{\gamma_{lc\mu}} \chi_{\bigcup_{i=1}^{\infty} E_i} \quad \text{by 1:17(f)}$$

$\Rightarrow \qquad G(\chi_{E_i}) \longrightarrow G(\chi_{\bigcup_{i=1}^{\infty} E_i})$

$\Rightarrow \qquad \nu(E_i) \longrightarrow \nu(\bigcup_{i=1}^{\infty} E_i)$

Thus ν is a measure by [C] 1.2.4(a)

$$\mu(E) = 0 \;\Rightarrow\; G(\chi_E) = 0 \;\Rightarrow\; \nu(E) = 0$$

Thus $\nu \prec\prec \mu$, and the Radon–Nikodým Theorem ([Fr] § 63) can be applied to derive a function $g \in L_0$ such that

$$G(\chi_E) = \nu(E) = \int_E g \, d\mu \qquad \forall\, E \in \Sigma$$

and $G(f) = \int f\, g \, d\mu \qquad \forall\, f \in L_0$ □

1:21 **Theorem**

For (X,Σ,μ) a measure space, exactly one of the following possibilities arises:–

(a) Σ has a non–atomic set.

(b) Σ is purely atomic and all atoms are of infinite measure.

(c) Σ is purely atomic and there is an atom of finite measure.

Then

in (a) $\gamma_{1c\mu}$ is not locally convex;

in (b) $L_0 = \{0\}$;

in (c) $\gamma_{1c\mu}$ is locally convex, and is equal to the topology of pointwise convergence.

Proof

(a)

Suppose there is a non–atomic set E (for which $\mu(E) > 0$).

Assume for a contradiction that $\gamma_{1c\mu}$ is locally convex.

There exists a continuous linear functional G such that $G(\chi_E) \neq 0$, by the Hahn–Banach Theorem.

By 1:20 $\exists\, g \in L_0$ such that $G(f) = \int f\, g\, d\mu \ \ \forall f \in L_0$

$0 \neq G(\chi_E) = \int_E g\, d\mu$, hence $\mu(\text{supp}(g) \cap E) \neq 0$

Since supp(g) ∩ E is non–atomic, we can 'split' this set by an inductive process into a sequence of disjoint sets $\{G_n\}_1^\infty$ such that $\mu(G_n) > 0 \ \forall n \in \mathbb{N}$, $\mu(G_n) \longrightarrow 0$

For $n \in \mathbb{N}$ choose $\lambda_n \in \mathbb{R}$ such that $G(\lambda_n \chi_{G_n}) = 1$

Now $\lambda_n \chi_{G_n} \xrightarrow{\gamma_{1c\mu}} 0$ since $\mu(G_n) \longrightarrow 0$

$\Rightarrow \quad G(\lambda_n \chi_{G_n}) \longrightarrow 0$, by the continuity of G.

This gives the required contradiction, and so $\gamma_{1c\mu}$ is not locally convex.

We make a general observation before proceeding to (b) and (c).

Note that any measurable function is constant on an atom, and thus for the purposes of considering measurable functions we may suppose such atoms comprise single points.

It also follows that $f \in L_0$ vanishes on atoms of infinite measure.

It can be shown that an atomic measure is completely additive. This is an easy consequence of the fact that a summable set of positive numbers contains at most countably many non–zero values. ([Pt] 1.1.5)

(b)

$L_0 = \{0\}$ follows immediately from the above observation.

(c)

It follows from the above observation that we may ignore those atoms of infinite measure. More precisely, all members of L_0 vanish on such sets, and hence (any mode of) convergence of functions is determined on the set of atoms of finite measure.

Suppose $\mathcal{N}(E, \epsilon)$ is a basic neighbourhood of 0 in $\gamma_{1c\mu}$.

Since E is of finite measure, it comprises at most countably many atoms.

Suppose $E = \{ x_n : n \in \mathbb{N} \}$

Choose $\{ x_{i_1}, \dots, x_{i_n}) \subset E$ such that $\mu(E - \bigcup_{j=1}^{n} x_{i_j}) \leq \epsilon$

Then $\bigcap_{j=1}^{n} \{ f \in L_0 : |f(x_{i_j})| \leq \epsilon \} \subset \mathcal{N}(E, \epsilon)$

Thus the topology of pointwise convergence is stronger than $\gamma_{1c\mu}$.

Suppose $P(i, \epsilon) = \{ f \in L_0 : |f(x_i)| \leq \epsilon\}$ is a subbasic neighbourhood in the topology of pointwise convergence.

Then $\mathcal{N}(\{x_i\}, \min \{ \frac{1}{2} \mu(x_i), \epsilon\}) \subset P(i, \epsilon)$

Thus $\gamma_{1c\mu}$ is stronger than the topology of pointwise convergence. □

1:22 **Theorem**

Suppose μ is σ–finite. Then $f_n \xrightarrow{\gamma_{1c\mu}} f$ iff every subsequence of f_n has a subsequence which converges to f μ a.e.

Proof

($\Rightarrow$)

Choose $(E_k) \subset \Sigma_f$ such that $E_k \uparrow X$

Suppose $f_n \xrightarrow{\gamma_{1c\mu}} f$. Then any subsequence is also locally convergent to f in measure, so we can denote such a subsequence again by (f_n).

Now $\forall k \in \mathbb{N}$ $f_n \chi_{E_k} \xrightarrow{\gamma_{c\mu}} f \chi_{E_k}$

Choose a subsequence $(f_{n_{1,i}})$ such that $f_{n_{1,i}} \xrightarrow{i} f$ a.e. on E_1 by 1:4(f)

At the k^{th} stage choose a subsequence $(f_{n_{k,i}})$ of $(f_{n_{k-1,i}})$ such that $(f_{n_{k,i}}) \xrightarrow{i} f$ a.e. on E_k.

Let $F_k = \{ x \in E_k : f_{n_{k,i}}(x) \xrightarrow{i} f(x) \}$. Then $\mu(E_k - F_k) = 0$.

Consider the sequence $(f_{n_{k,k}})_{k \in \mathbb{N}}$, which is indeed a subsequence of (f_n).

For $x \in \bigcup_{p=1}^{\infty} F_p$, $\exists\, p \in \mathbb{N}$ such that $x \in F_p$

$\Rightarrow \quad f_{n_{p,i}}(x) \xrightarrow{i} f(x)$

$\Rightarrow \quad f_{n_{k,k}}(x) \xrightarrow{k} f(x)$ since $(f_{n_{k,k}})$ $(k \geq p)$ is a subsequence of $(f_{n_{p,i}})$

$$\mu(X - \bigcup_{p=1}^{\infty} F_p) = \mu(\, (X - \bigcup_{p=1}^{\infty} E_p) \cup \bigcup_{p=1}^{\infty} (E_p - F_p)\,) \leq \mu(X - \bigcup_{p=1}^{\infty} E_p) + \sum_{p=1}^{\infty} \mu(E_p - F_p) = 0$$

Thus $f_{n_{k,k}} \xrightarrow{ae} f$

($\Leftarrow$)

Suppose every subsequence of (f_n) contains a subsequence converging a.e. to f.

Suppose $\mathcal{N}(E, \epsilon)$ is given.

Let $E_n = \{ x \in E : |(f_n - f)(x)| \geq \epsilon \}$

Assume for a contradiction that $\mu(E_n) \not\to 0$.

$\Rightarrow \quad \exists\, \delta > 0\ \exists\, (n_k)$ such that $\mu(E_{n_k}) \geq \delta\ \forall\, k \in \mathbb{N}$.

By hypothesis $\exists$ subsequence $(f_{n_{k_i}})$ of (f_{n_k}) such that $f_{n_{k_i}} \xrightarrow{ae} f$.

$\Rightarrow \quad f_{n_{k_i}} \chi_E \xrightarrow{ae} f \chi_E$

$\Rightarrow \quad f_{n_{k_i}} \chi_E \xrightarrow{\gamma_{c\mu}} f \chi_E$ by 1:4(h)

$\Rightarrow \quad \mu(E_{n_{k_i}}) \longrightarrow 0$, the required contradiction. □

1:23 **Theorem**

Suppose μ is σ–finite, with $\{E_n\}_{n\in\mathbb{N}} \subset \Sigma_f$ and $E_n \uparrow X$

Define $q_n : L_0 \to [0,\infty) : f \to \inf\{ \epsilon > 0 : f \in \mathcal{N}(E_n , \epsilon) \}$

Then $\{q_n\}_{n\in\mathbb{N}}$ is an increasing family of F–seminorms that are defining for $\gamma_{lc\mu}$.

Proof

The proof that q_n is a F–seminorm for $n \in \mathbb{N}$ is similar to the finite case.

Suppse $q_n(f) = \alpha$ $(n \geq 2)$ and let $\epsilon > \alpha$.

Then $f \in \mathcal{N}(E_n , \epsilon)$ $\Rightarrow$ $f \in \mathcal{N}(E_{n-1} , \epsilon)$ since $E_{n-1} \subset E_n$

$\Rightarrow$ $q_{n-1}(f) \leq \epsilon$

$\Rightarrow$ $q_{n-1}(f) \leq \alpha$ since $\epsilon > \alpha$ was arbitrary.

So $\{q_n\}_{n\in\mathbb{N}}$ is an increasing family of F–seminorms.

It is clear that $\mathcal{N}(E_n , \epsilon) \subset \{ f \in L_0 : q_n(f) \leq \epsilon \} \subset \mathcal{N}(E_n , \alpha)$ $\forall\, \alpha > \epsilon$

So the F–seminorms $\{q_n\}_{n\in\mathbb{N}}$ are defining for $\gamma_{lc\mu}$. □

1:24 **Theorem** Adapted from the Riesz–Weyl theorem.

Suppose μ is σ–finite. Then $(L_0 , \gamma_{lc\mu})$ is complete.

Proof

Let $\{E_r\}_{r\in\mathbb{N}} \subset \Sigma_f$ be a disjoint sequence such that $X = \bigcup_{r=1}^{\infty} E_r$.

Recall from 1:19 that $\{ \mathcal{N}(E_r , \frac{1}{m}) : r , m \in \mathbb{N} \}$ is a subbasic neighbourhood system at 0 for $\gamma_{lc\mu}$ and that $\gamma_{lc\mu}$ is metrisable.

In particular, it suffices to consider Cauchy sequences.

Suppose (f_n) is a Cauchy sequence in $(L_0 , \gamma_{lc\mu})$.

An elementary argument shows that $(f_n \chi_{E_r})$ is Cauchy in $(L_0 , \gamma_{c\mu})$ $\forall\, r \in \mathbb{N}$

For each (fixed) $r \in \mathbb{N}$, $\exists f^r \in L_0$ such that $f_n \chi_{E_r} \xrightarrow{\gamma_{c\mu}} f^r$, by 1:14

It is clear that $f^r(x) = 0$ for $x \notin E_r$

For $x \in X$, $\exists r \in \mathbb{N}$ such that $x \in E_r$

Define $f(x) = f^r(x)$

For $t \in \mathbb{R}$, $\{ x \in X : f(x) > t \} = \bigcup_{r=1}^{\infty} \{ x \in X : f^r(x) > t \} \in \Sigma$, so $f \in L_0$

Note that $f \chi_{E_r} = f^r$

We claim that $f_n \xrightarrow{\gamma_{1c\mu}} f$

Since $\{ \mathcal{N}(E_r, \frac{1}{m}) : r, m \in \mathbb{N} \}$ is a subbasic neighbourhood system at 0 for $\gamma_{1c\mu}$, it suffices by 1:17(e) to show that $f_n \chi_{E_r} \xrightarrow{\gamma_{c\mu}} f \chi_{E_r}$ $\forall r \in \mathbb{N}$

But $f_n \chi_{E_r} \xrightarrow{\gamma_{c\mu}} f^r = f \chi_{E_r}$ $\forall r \in \mathbb{N}$, so the result follows. □

1:25 **Theorem**

Suppose μ is σ–finite. Then $(L_0, \gamma_{1c\mu})$ is the completion of $(L_\infty, \gamma_{1c\mu})$

Proof

Follows as in the case for $\gamma_{c\mu}$ by the above version of the Riesz–Weyl theorem. □

1:26 **Example**

Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), c)$ where c is counting measure.

Since $\epsilon_1 < \epsilon_2 \Rightarrow \mathcal{N}(\epsilon_1) \subset \mathcal{N}(\epsilon_2)$ we need only consider $\{ \mathcal{N}(\epsilon) : \epsilon \in (0,1) \}$

For $\epsilon \in (0,1)$ $\{ (x_n) : c\{n \in \mathbb{N} : |x_n| \geq \epsilon \} \leq \epsilon \} = \{ (x_n) : |x_n| \leq \epsilon \ \forall n \in \mathbb{N} \}$

So $\gamma_{c\mu} = \|\cdot\|_\infty$

By 1:21(c), $\gamma_{1c\mu}$ is the topology of pointwise convergence.

2 : REARRANGEMENTS of FUNCTIONS and BANACH FUNCTION SPACES

The main purpose of this Chapter is to prepare for Section III . Most of the definitions and results here are well known; however, the terminology and notation used has not generally been standardised in the literature. Thus we introduce the concepts with notation analogous to that that will feature in Section III, and give sources that contain proofs of the relevant results. We present proofs for some other results which are less well known.

Supppose (X,Σ,μ) is a measure space.

2:1 **Definition** [L] § 2 and § 4

Suppose $f \in L_0$

Let $d : \mathbb{R} \longrightarrow [0,\infty] : t \longrightarrow \mu\{ x \in X : f(x) > t \}$

We call this function the *distribution function* and use the notation $d_t(f)$, to indicate the dependence on the function f.

Let $\lambda : [0,\infty) \longrightarrow \mathbb{R} \cup \{\infty\} : t \longrightarrow \inf \{ \theta \in \mathbb{R} : d_\theta(f) \leq t \}$

We call this function the *rearrangement of* f and use the notation $\lambda_t(f)$.

2:2 **Proposition**

The following are equivalent

(a) $d_t(|f|) \longrightarrow 0$ as $t \longrightarrow \infty$

(b) $d_t(|f|)$ is eventually finite

(c) $f \in \tilde{L}_\infty$

Proof

(a) ⇒ (b)

Clear

(b) ⇔ (c)

$d_t(|f|)$ is eventually finite

⇔ $\exists\, t > 0$ such that $\mu\{ x \in X : |f(x)| > t \} < \infty$

⇔ $f \in \tilde{L}_\infty$

(b) ⇒ (a)

$\{ x \in X : |f(x)| > n \}$ is a decreasing sequence of sets with intersection $\{ x \in X : |f(x)| = \infty \}$ having zero measure.

One of these sets has finite measure, so the result follows by [C] 1.2.3(b) □

2:3 **Definition** cf. [KPS] II § 2

We now restrict attention to $\tilde{L}_\infty$

Two functions $0 \leq f, g \in \tilde{L}_\infty$ are said to be *equimeasurable* if $d_t(f) = d_t(g)\ \ \forall\, t > 0$

For $f \in \tilde{L}_\infty$ the *rearrangement* of $|f|$ is a decreasing right continuous function equimeasurable with $|f|$. The rearrangement is unique and is determined by the formula

$\mu : (0,\infty) \longrightarrow [0,\infty) : t \longrightarrow \inf\{ \theta \geq 0 : d_\theta(|f|) \leq t \}$

We use the notation $\mu_t(f)$

Clearly $\mu_t(\mu_t(f)) = \mu_t(f)$

For $0 \leq f \in \tilde{L}_\infty$, $\mu_t(f) = \lambda_t(f)$

We list some properties of μ_t [KPS] II § 2

(1) $|f| \le |g| \Rightarrow \mu_t(f) \le \mu_t(g)$

(2) $\|f - g\|_\infty \le \epsilon \Rightarrow \|\mu_t(f) - \mu_t(g)\|_\infty \le \epsilon$

(3) $\mu_t(\alpha f) = \alpha \mu_t(f)$, $\alpha > 0$

(4) $\mu_{t_1+t_2}(f g) \le \mu_{t_1}(f) \mu_{t_2}(g)$

(5) $\mu_{t_1+t_2}(f + g) \le \mu_{t_1}(f) + \mu_{t_2}(g)$

(6) $\|f\|_1 = \|\mu_t(f)\|_1$

2:4 **Proposition** [L] § 2 and § 4

$d_t(f)$ (and hence $d_t(|f|)$) and $\lambda_t(f)$ (and hence $\mu_t(f)$) are decreasing and right continuous.

2:5 **Definition** [KPS] II § 2

It follows that $\lim_{t\to\infty} \mu_t(f)$ exists for $f \in \tilde{L}_\infty$ (since $\mu_t(f)$ is decreasing and positive) . We denote this limit by $\mu_\infty(f)$.

It follows from 2:3 that

(1) $|f| \le |g| \Rightarrow \mu_\infty(f) \le \mu_\infty(g)$

(2) $\|f_n - f\|_\infty \longrightarrow 0 \Rightarrow \mu_\infty(f_n) \longrightarrow \mu_\infty(f)$

(3) $\mu_\infty(\alpha f) = \alpha \mu_\infty(f)$, $\alpha > 0$

(4) $\mu_\infty(f g) \le \mu_\infty(f) \mu_\infty(g)$

(5) $\mu_\infty(f + g) \le \mu_\infty(f) + \mu_\infty(g)$

2:6 **Proposition**

$f \in \mathcal{N}(\epsilon,t) \Leftrightarrow \mu_t(f) \leq \epsilon$

Proof

$$
\begin{aligned}
f \in \mathcal{N}(\epsilon,t) \quad &\Leftrightarrow \quad \mu\{ x \in X : |f(x)| > \epsilon \} \leq t \\
&\Leftrightarrow \quad d_\epsilon(|f|) \leq t \\
&\Leftrightarrow \quad \mu_t(f) \leq \epsilon
\end{aligned}
$$
□

2:7 **Theorem**

$f_n \xrightarrow{\gamma_{c\mu}} f \quad \Leftrightarrow \quad \mu_t(f_n - f) \longrightarrow 0 \quad \forall\, t > 0$

Proof

$$
\begin{aligned}
f_n \xrightarrow{\gamma_{c\mu}} f \quad &\Leftrightarrow \quad \forall\, \epsilon, t > 0\ \exists\, N \in \mathbb{N} \text{ such that } n > N \Rightarrow f_n - f \in \mathcal{N}(\epsilon,t) \\
&\Leftrightarrow \quad \forall\, \epsilon, t > 0\ \exists\, N \in \mathbb{N} \text{ such that } n > N \Rightarrow \mu_t(f_n - f) \leq \epsilon \\
&\Leftrightarrow \quad \mu_t(f_n - f) \xrightarrow{n} 0 \quad \forall\, t > 0
\end{aligned}
$$
□

Our principle sources are [L] , [Z] and [KPS] . We note that [L] , [Z] and [KPS] restrict attention to finite measure, σ–finite measures, and the space $(0,\infty)$ (with Lebesgue measure) respectively. We will be exclusively interested in the space $(0,\infty)$ in Section III, so the approach of [Z] and [KPS] will be sufficient for our purposes.

We thus restrict attention to the measure space $(0,\infty)$. We note however that many of the results of this chapter can be generalised to more general measure spaces – usually the requirement that (X,Σ,μ) is semi–finite and/or *Maharam* is made. For details, see [Fr] §§ 64, 65.

2:8 **Definition** [Z] § 63 , [L] § 11

A *function norm* ρ is a function $\rho : L_0^+ \longrightarrow [0,\infty]$ satisfying :–

(1) $f = 0\ \mu$ a.e. $\Leftrightarrow \rho(f) = 0$

(2) $\rho(\alpha f) = \alpha\,\rho(f)\ \ \forall f \in L_0^+\ \ \forall \alpha > 0$

(3) $\rho(f + g) \le \rho(f) + \rho(g)\ \ \forall f, g \in L_0^+$

(4) $f, g \in L_0^+,\ f \le g \Rightarrow \rho(f) \le \rho(g)$

The domain of definition of ρ is extended to L_0 by defining

$$\rho(f) = \rho(|f|)\ \ \forall f \in L_0$$

and we define $L_\rho = \{ f \in L_0 : \rho(f) < \infty \}$.

L_ρ is a normed space with norm ρ

It follows from (4) that ρ has the special property that

$f \in L_0,\ g \in L_\rho,\ |f| \le |g| \Rightarrow f \in L_\rho$ and $\rho(f) \le \rho(g)$.

Thus L_ρ is *solid* in L_0 and ρ is a lattice norm.

A normed space with this property is usually called a *normed Köthe space.*

If this normed space is complete then L_ρ is termed a *Banach Function Space.*

Some authors (see [KPS] II Introduction) prefer the term *ideal Banach lattice* to emphasise the fact that ρ is a lattice norm.

2:9 **Proposition** cf. [AB] 11.4

Suppose ρ is a function norm.

The following are equivalent :

(a) $0 \le f_1 \le f_2 \le \dots \underset{ae}{\uparrow} f \Rightarrow \rho(f_n) \uparrow \rho(f)$.

(b) $f_n \xrightarrow{ae} f \Rightarrow \rho(f) \le \liminf_n \rho(f_n)$

Proof

(b) ⇒ (a)

Clear.

(a) ⇒ (b)

Suppose $f_n \xrightarrow{ae} f$.

Let $g_n = \sup_{k \geq n} |f_k - f|$.

Then $|f_n - f| \leq g_n \downarrow_{ae} 0$.

$$\begin{aligned} \text{Now } |f_n - f| \leq g_n \quad &\Rightarrow \quad |f| - |f_n| \leq g_n \\ &\Rightarrow \quad |f| - g_n \leq |f_n| \\ &\Rightarrow \quad (|f| - g_n)^+ \leq |f_n| \end{aligned}$$

Furthermore $0 \leq (|f| - g_n)^+ \uparrow_{ae} |f|$

$$\begin{aligned} \text{Hence } \rho(f) &= \rho(|f|) \\ &= \lim_n \rho((|f| - g_n)^+) \text{ , by hypothesis.} \\ &\leq \lim_n \inf \rho(|f_n|) \\ &= \lim_n \inf \rho(f_n) . \end{aligned}$$ □

2:10 **Definition**

A function norm satisfying the equivalent conditions of 2:8 is said to have the *Fatou Property*.

2:11 **Theorem** [Z] § 65 Theorem 1

If ρ has the Fatou property then L_ρ is a Banach Function Space. □

2:12 **Definition**

A function norm ρ is said to be *lower semicontinuous* if $f_n, f \in L_\rho$ and $f_n \xrightarrow{ae} f \Rightarrow \rho(f) \leq \lim_n \inf \rho(f_n)$.

It is clear that the Fatou property implies lower semicontinuity; however, the converse is not true. See 2:17.3

In what follows we suppose that

$\forall\ E \in \Sigma$ such that $\mu(E) > 0\ \exists\ F \subset E$, $F \in \Sigma$ such that $\mu(F) > 0$ and $\chi_F \in L_\rho$

If this is the case then ρ is said to be *saturated* [Z] § 67

We will see in 2:22 that the function norms we will be interested in are always saturated.

2:13 **Definition** [Z] § 68

Suppose $\rho \equiv \rho^{(0)}$ is a saturated function norm

For $n \in \mathbb{N}$, $f \in L_0$ we inductively define

$$\rho^{(n)}(f) = \sup \{ \int |f\, g|\, d\mu : \rho^{(n-1)}(g) \leq 1, g \in L_0^+ \}$$

$\rho^{(n)} = \rho^{(n+2)}$ for $n \geq 1$, and hence we need only consider $\rho^{(0)}$, $\rho^{(1)}$ and $\rho^{(2)}$ which for convenience we notate ρ, $\rho^\times$ and $\rho^{\times\times}$ respectively.

Thus $$\rho^\times(f) = \sup \{ \int |f\, g|\, d\mu : \rho(g) \leq 1, g \in L_0^+ \}$$

$$\rho^{\times\times}(f) = \sup \{ \int |f\, g|\, d\mu : \rho^\times(g) \leq 1, g \in L_0^+ \}$$

$\rho^\times$ is called the *first associate* of ρ and $\rho^{\times\times}$ the *second associate* of ρ □

2:14 The First Associate Space [Z] §§ 68, 69, 71

$\rho^\times$ is a saturated function norm

$\rho^\times$ has the Fatou property (whether or not ρ does)

$\int |f\,g|\, d\mu \le \rho(f)\, \rho^\times(g) \quad \forall f, g \in L_0$ (Holder's inequality).

If $g \in L_{\rho^\times}$ then $\gamma_g(f) = \int f\,g\, d\mu$ defines a bounded linear functional on L_ρ and $\|\gamma_g\| = \rho^\times(g)$.

Thus there is an isometric injection $L_{\rho^\times} \hookrightarrow L_\rho^*$

This result has a converse, as follows

The bounded linear functional G on L_ρ is *of integrable type* if

$$0 \le f_n \downarrow 0, \{f_n\} \subset L_\rho \Rightarrow G(f_n) \longrightarrow 0$$

$G \in L_\rho^*$ is of integrable type $\Leftrightarrow \exists\, g \in L_{\rho^\times}$ such that $G = \gamma_g$

$f \in L_{\rho^\times} \Leftrightarrow \int |f\,g|\, d\mu < \infty \ \forall g \in L_\rho$

2:15 The Second Associate Space [Z] § 66, 68, 71

$\rho^{\times\times} \le \rho$ in the sense that $\rho^{\times\times}(f) \le \rho(f) \ \forall f \in L_0$

For $f \in L_0^+$, $\rho^{\times\times}(f) = \inf\{\lim_n \rho(f_n) : 0 \le f_n \uparrow f\}$ (Lorentz Characterisation)

$\rho^{\times\times}$ is a saturated function norm.

$\rho^{\times\times}$ has the Fatou property (whether or not ρ does).

$\rho = \rho^{\times\times} \Leftrightarrow \rho$ has the Fatou property.

Hence if ρ has the Fatou property then

$L_\rho = L_{\rho^{\times\times}}$

ρ and $\rho^\times$ are associates of each other

For $f \in L_\rho$, $\rho(f) = \rho^{\times\times}(f) = \sup\{\int |f\,g|\, d\mu : \rho^\times(g) \le 1, g \in L_0^+\}$

Suppose ρ is lower semicontinuous.

If $f \in L_\rho$, it follows from the Lorentz Characterisation that $\rho^{\times\times}(f) = \rho(f)$

Thus $L_\rho \hookrightarrow L_{\rho^{\times\times}}$;

and for $f \in L_\rho$, $\rho(f) = \rho^{\times\times}(f) = \sup \{ \int |f\,g|\, d\mu \,:\, \rho^{\times}(g) \leq 1 , g \in L_0^{+} \}$

By restricting attention to the space $(0,\infty)$ we introduce the possibility that $f \in L_\rho \nRightarrow \mu_t(f) \in L_\rho$. This motivates the following definition.

2:16 **Definition** [KPS] II § 4

A Banach Function Space L_ρ is said to be *rearrangement invariant*, or *symmetric* , if whenever $f \in L_\rho$, $g \in L_0$ and $\mu_t(f) = \mu_t(g)$ $\forall\, t > 0$ then $g \in L_\rho$ and $\rho(g) = \rho(f)$.

By putting $g = \mu_t(f)$, it follows from 2:3 that $f \in L_\rho \Rightarrow \mu_t(f) \in L_\rho$ and $\rho(\mu_t(f)) = \rho(f)$.

2:17 **Example**

2.17.1

For $1 \leq p < \infty$, $L_p = \{f \in L_0 : \int |f|^p\, dm < \infty \}$

Then $\|\cdot\|_p : L_p \longrightarrow \mathbb{R} : f \longrightarrow \sqrt[p]{\int |f|^p\, dm}$ is a norm [C] 3.3.4

L_p is complete under $\|.\|_p$ [C] 3.4.1

It follows quite easily that L_p is a Banach Function Space.

$\|.\|_p$ has the Fatou property – this is immediate from the Monotone Convergence Theorem.

L_p is symmetric

2.17.2

L_∞ is complete under $\|.\|_\infty$

It follows quite easily that L_∞ is a Banach Function Space.

$\|.\|_\infty$ has the Fatou property

$L_\infty(X,\Sigma,\mu)$ is symmetric

2:17.3

For $f \in L_0^+$ put $\rho(f) = \begin{cases} \|f\|_\infty & \text{if } \mu_\infty(f) = 0 \\ \infty & \text{if } \mu_\infty(f) \neq 0 \end{cases}$

To show that ρ is a function norm, we verify conditions (1) to (4) in 2:7

(1) $f = 0 \Leftrightarrow \|f\|_\infty = 0 \Leftrightarrow \rho(f) = 0$

(2) By 2:5(3) $\mu_\infty(\alpha f) = 0 \Leftrightarrow \mu_\infty(f) = 0$

Hence the property follows from the corresponding property of the $\|.\|_\infty$ norm

(3) It suffices to consider f , g such that $\mu_\infty(f) = \mu_\infty(f) = 0$

By 2:5(5) $\mu_\infty(f + g) = 0$

Hence the property follows from the corresponding property of the $\|.\|_\infty$ norm

(4) If suffices to consider g for which $\mu_\infty(g) = 0$

Then $0 \leq f \leq g \Rightarrow \mu_\infty(f) = 0$ by 2:5(1)

Hence the property follows from the corresponding property of the $\|.\|_\infty$ norm

If $\|f_n - f\|_\infty \longrightarrow 0$ and $\mu_\infty(f_n) = 0 \ \forall n \in \mathbb{N}$ then $\mu_\infty(f) = 0$ by 2:5(2)

It follows that L_ρ is complete.

If $f \in L_\rho$, $g \in L_\rho$ and $\mu_t(f) = \mu_t(g) \ \forall t > 0$, then $\mu_\infty(g) = \mu_\infty(f) = 0$

It follows that L_ρ is symmetric because L_∞ is symmetric.

If $f_n , f \in L_\rho$ then $f_n , f \in L_\infty$ $\forall\, n \in \mathbb{N}$ and hence $\|f\|_\infty \leq \lim_n \inf \|f_n\|_\infty$

It follows that $\rho(f) \leq \lim_n \inf \rho(f_n)$ and hence ρ is lower semicontinuous.

Let $f_n = \chi_{(0,n]}$. Then $\rho(f_n) = 1$ $\forall\, n \in \mathbb{N}$

$f_n \uparrow \chi_{(0,\infty)}$, and $\rho(\chi_{(0,\infty)}) = \infty$ since $\mu_t(\chi_{(0,\infty)}) = 1$ $\forall\, t > 0$

Hence ρ does not have the Fatou property. □

We now wish to determine the smallest and largest possible symmetric Banach Function Spaces.

2:18 **Definition** [KPS] I § 3

Suppose L_{ρ_1} and L_{ρ_2} are Banach Function Spaces.

The *intersection* $L_{\rho_1} \cap L_{\rho_2}$ is a Banach Function Space with norm

$$\|f\| = \max \{ \rho_1(f) , \rho_2(f) \}$$

The *sum* $L_{\rho_1} + L_{\rho_2} = \{ f_1 + f_2 : f_i \in L_{\rho_i} , i = 1 , 2 \}$ is a Banach Function Space with norm

$$\|f\| = \inf \{ \|f_1\|_{\rho_1} + \|f_2\|_{\rho_2} : f = f_1 + f_2 , f_i \in L_{\rho_i} , i = 1 , 2 \}$$

2:19 **Theorem** [KPS] II § 4 Lemma 4.5

The intersection and sum of symmetric Banach Function Spaces are symmetric Banach Function Spaces. □

2:20 **Theorem**

(a) $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are symmetric Banach Function Spaces

(b) [KPS] II § 3.1

$L_1 + L_\infty$ is the associate space of $L_1 \cap L_\infty$

(c) $L_1 \cap L_\infty$ has the Fatou property

(d) $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are associates of each other.

Proof

(a)

L_1 and L_∞ are symmetric Banach Function spaces (2:17)

Hence $L_1 \cap L_\infty$ and $L_1 + L_\infty$ are symmetric Banach Function spaces by 2:19

(c)

Suppose $0 \leq f_n \uparrow f$

Then $\|f_n\|_1 \uparrow \|f\|_1$ and $\|f_n\|_\infty \uparrow \|f\|_\infty$

Thus $\|f_n\|_{L_1 \cap L_\infty} = \max \{ \|f_n\|_1 ; \|f_n\|_\infty \} \uparrow \max \{ \|f\|_1 ; \|f\|_\infty \} = \|f\|_{L_1 \cap L_\infty}$

(d)

Follows from (b) , (c) and 2:15 □

2:21 **Theorem** [KPS] II § 4 Theorem 4.1

Suppose L_ρ is a symmetric Banach Function Space.

Then $L_1 \cap L_\infty \subset L_\rho \subset L_1 + L_\infty$ (as sets and in the sense of continuous imbeddings)

Furthermore $L_1 + L_\infty \subset \tilde{L}_\infty$. □

It follows immediately that a symmetric Banach Function Space includes the characteristic function of any measurable set of finite measure. In particular, it is saturated.

Furthermore, $\mu_t(f)$ is defined for any $f \in L_\rho$.

2:22 **Theorem** [KPS] II § 4.6

If L_ρ is a symmetric Banach Function Space, then $L_{\rho^\times}$ (and likewise $L_{\rho^{\times\times}}$) are symmetric.

For $f \in L_{\rho^\times}$,

$$\begin{aligned}\rho^\times(f) &= \sup_{\rho(g)\le 1} \int |f\,g|\ dm \\ &= \sup_{\rho(g)\le 1} \int |f|\ \mu_t(g)\ dm \\ &= \sup_{\rho(g)\le 1} \int \mu_t(f)\ g\ dm \\ &= \sup_{\rho(g)\le 1} \int \mu_t(f)\ \mu_t(g)\ dm\end{aligned}$$

Similar characterisations hold for $f \in L_{\rho^{\times\times}}$. □

We will have need of the following results in Chapter 10.

2:23 **Definition** [KPS] II § 4.3

Suppose $f \in L_0$

For $a > 0$, define $f_a(t) = f(at)$

f_a is sometimes referred to as the *dilation of* f *by* a

2:24 **Proposition** [KPS] II § 4 Theorem 4.5 Corollary 1

Suppose L_ρ is a symmetric Banach Function Space.

Suppose $f \in L_\rho$

Then for $a > 0$, $f_a \in L_\rho$ and $\rho(f_a) \le \max\{1, \frac{1}{a}\}\,\rho(f)$. □

2:25 **Definition** [L] 6.1

If f , g $\in L^1$ then f is *submajorised* by g (notated f $\prec\prec$ g) if

$$\int_0^{\theta} \lambda_t(f)\, dt \le \int_0^{\theta} \lambda_t(g)\, dt \quad \forall\, \theta > 0$$

If f , g $\in L^1$ then f is *majorised* by g (notated f $\prec$ g) if

$$f \prec\prec g \quad \text{and} \quad \int_0^{\infty} \lambda_t(f)\, dt = \int_0^{\infty} \lambda_t(g)\, dt .$$

The restriction to functions in L_1 is made to ensure that the above integrals exist.

(f $\in L_1$ iff $\lambda_t(f) \in L_1$)

However, if the functions involved are positive, then the above definitions are extended in the obvious manner.

2:26 **Proposition**

cf. [L] 9.1

If f $\in L^1$ then $\int_0^{\theta} \lambda_t(f)\, dt = \sup_{m(A)=\theta} \int_A f(t)\, dt$

[KPS] II § 2 (2.14)

If $0 \le f \in \tilde{L}_{\infty}$ then $\int_0^{\theta} \mu_t(f)\, dt = \sup_{m(A)=\theta} \int_A f(t)\, dt$ □

2:27 **Theorem** cf. [KPS] II § 4 Theorem 4.9

Suppose L_ρ is a symmetric Banach Function Space for which ρ is lower semicontinuous.

If $0 \le f , g \in L_\rho$ and f $\prec\prec$ g then $\rho(f) \le \rho(g)$.

Proof

$f \prec\prec g$

$$\Rightarrow \quad \int_0^{\theta} \lambda_t(f)\, dt \leq \int_0^{\theta} \lambda_t(g)\, dt \quad \forall\, \theta > 0$$

$$\Rightarrow \quad \int_0^{\theta} \mu_t(f)\, dt \leq \int_0^{\theta} \mu_t(g)\, dt \quad \forall\, \theta > 0$$

$$\Rightarrow \quad \int_0^{\infty} \mu_t(f)\, \mu_t(y)\, dt \leq \int_0^{\infty} \mu_t(g)\, \mu_t(y)\, dt \quad \forall\, y \in L_{\rho^\times} \quad \text{by [KPS] II § 2 (18)}$$

$$\Rightarrow \quad \rho^{\times\times}(f) \leq \rho^{\times\times}(g) \quad \text{by 2:22}$$

$$\Rightarrow \quad \rho(f) \leq \rho(g) \quad \text{by 2:15}$$

□

3 : TRACES

3:1 Definition

A *trace* on a von Neumann algebra $\mathcal{M}$ is a function $\tau : \mathcal{M}^+ \longrightarrow [0,\infty]$ such that

(a) $\tau(r + s) = \tau(r) + \tau(s) \ \forall\, r, s \in \mathcal{M}^+$

(b) $\tau(\lambda\, s) = \lambda\, \tau(s) \ \forall\, \lambda \in \mathbb{R}^+ \ \ \forall\, s \in \mathcal{M}^+$

(c) $\tau(s^* s) = \tau(s\, s^*) \ \forall\, s \in \mathcal{M}$.

In discussing traces we will refer to condition (a) as the *linearity condition*, (b) as the *homogeniety condition*, and (c) as the *commutativity condition.* We make the trivial but pertinent remark that in a commutative von Neumann algebra the commutativity condition is redundant.

3:2 Proposition

Suppose τ is a trace on a von Neumann algebra $\mathcal{M}$.

(a) τ is monotone i.e. it preserves order on $\mathcal{M}^+$.

(b) If $r, s \in \mathcal{M}^+$ and $r \leq s$ and $\tau(r) < \infty$ then $\tau(s-r) = \tau(s) - \tau(r)$.

(c) If $p, q \in \mathcal{M}_p$ and $p \sim q$ then $\tau(p) = \tau(q)$.

(d) If $p, q \in \mathcal{M}_p$ and $p \precsim q$ then $\tau(p) \leq \tau(q)$.

(e) If $p, q \in \mathcal{M}_p$ and $p \wedge q = 0$ then $\tau(p) \leq \tau(1-q)$.

(f) If $p_1, \ldots, p_n \in \mathcal{M}_p$ then $\tau(\bigvee_{i=1}^{n} p_i) \leq \sum_{i=1}^{n} \tau(p_i)$.

Proof

(a)

If $r, s \in \mathcal{M}^+$ and $r \leq s$ then $s-r \in \mathcal{M}^+$ and so $\tau(s) = \tau(r + s-r) = \tau(r) + \tau(s-r) \geq \tau(r)$

(b)

As in (a) we have $\tau(s) = \tau(r) + \tau(s-r)$ and so $\tau(s-r) = \tau(s) - \tau(r)$ since $\tau(r)$ is finite.

(c)

If $p \overset{v}{\sim} q$ then $\tau(p) = \tau(v^* v) = \tau(v v^*) = \tau(q)$ by the commutativity condition.

(d)

Follows directly from (c) and (a).

(e)

If $p \wedge q = 0$ then $p = 1 - p^{\perp} = (p \wedge q)^{\perp} - p^{\perp} = p^{\perp} \vee q^{\perp} - p^{\perp} \sim q^{\perp} - q^{\perp} \wedge p^{\perp} \leq q^{\perp} = 1-q$

by the Kaplansky formula ([KR] 6.1.7)

$\Rightarrow \quad \tau(p) \leq \tau(1-q)$ by (d) .

(f)

We assume that $\tau(p_i) < \infty$ for $i = 1, \dots, n$; the result is clear in other cases.

The proof is by induction.

The case $n = 1$ is trivial, and so we make the inductive hypothesis for $n-1$, namely that

$\tau(\bigvee_{i=1}^{n-1} p_i) \leq \sum_{i=1}^{n-1} \tau(p_i)$.

Then $\bigvee_{i=1}^{n} p_i - p_n \sim \bigvee_{i=1}^{n-1} p_i - (p_n \wedge \bigvee_{i=1}^{n-1} p_i)$ by the Kaplansky formula.

$$\begin{aligned}
\Rightarrow \quad \tau(\bigvee_{i=1}^{n} p_i) - \tau(p_n) &= \tau(\bigvee_{i=1}^{n} p_i - p_n) \quad \text{by (b)} \\
&= \tau(\bigvee_{i=1}^{n-1} p_i - (p_n \wedge \bigvee_{i=1}^{n-1} p_i)) \quad \text{by (c)} \\
&\leq \tau(\bigvee_{i=1}^{n-1} p_i) \quad \text{by (a)}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \quad \tau(\bigvee_{i=1}^{n} p_i) &\leq \tau(\bigvee_{i=1}^{n-1} p_i) + \tau(p_n) \\
&\leq \sum_{i=1}^{n} \tau(p_i) \quad \text{by the inductive hypothesis.}
\end{aligned}$$

□

We now consider ideals of $\mathcal{M}$ determined by the trace τ.

3:3 **Definition** [T] pp 317 , 318.

$$\mathcal{P}_\tau = \{ s \in \mathcal{M}^+ : \tau(s) < \infty \}$$

$$\mathcal{N}_\tau = \{ s \in \mathcal{M} : \tau(s^* s) < \infty \}$$

$$\mathcal{M}_\tau = \{ \sum_{i=1}^{n} r_i s_i : n \in \mathbb{N}, r_i , s_i \in \mathcal{N}_\tau \}$$

3:4 **Proposition** Extracted from [T] V 2.13

$\mathcal{N}_\tau$ is a self–adjoint ideal of $\mathcal{M}$.

Proof

For r , s $\in \mathcal{M}$, x $\in \mathcal{H}$ ($\mathcal{H}$ the underlying Hilbert space)

$$
\begin{aligned}
&\langle (r+s)^*(r+s)x,x\rangle \\
&= \langle (r+s)x,(r+s)x\rangle \\
&= \|(r+s)x\|^2 \\
&\leq [\ \|rx\| + \|sx\|\ ]^2 \\
&= \|rx\|^2 + 2\,\|rx\|\,\|sx\| + \|sx\|^2 \\
&\leq \|rx\|^2 + \|rx\|^2 + \|sx\|^2 + \|sx\|^2 \\
&= 2 \langle rx,rx\rangle + 2 \langle sx,sx\rangle \\
&= \langle 2(r^* r + s^* s)x,x\rangle
\end{aligned}
$$

So $(r + s)^* (r + s) \leq 2 (r^* r + s^* s)$

Hence if r , s $\in \mathcal{N}_\tau$ then by monotonicity of the trace, $r + s \in \mathcal{N}_\tau$.

If $\lambda \in \mathbb{C}$, $s \in \mathcal{N}_\tau$ then $(\lambda s)^*(\lambda s) = |\lambda|^2 s^* s$, so $\lambda s \in \mathcal{N}_\tau$.

For r , s $\in \mathcal{M}$

$r^* r \leq \|r^* r\|\ 1 = \|r\|^2\ 1$, so $(r s)^* (r s) = s^* r^* r s \leq \|r\|^2 s^* s$.

Hence if $r \in \mathcal{M}$, $s \in \mathcal{N}_\tau$ then $r s \in \mathcal{N}_\tau$.

Thus $\mathcal{N}_\tau$ is a left ideal.

$$
\begin{aligned}
s \in \mathcal{N}_\tau \quad &\Rightarrow \quad \tau(s^* s) < \infty \\
&\Rightarrow \quad \tau(s s^*) < \infty \quad \text{by the commutativity condition} \\
&\Rightarrow \quad s^* \in \mathcal{N}_\tau .
\end{aligned}
$$

Thus $\mathcal{N}_\tau$ is self–adjoint.

It is easy to verify that a left (or right) self–adjoint ideal is a two–sided ideal, and thus $\mathcal{N}_\tau$ is a 2–sided ideal. □

3:5 **Proposition**

$\mathcal{M}_\tau$ is a self–adjoint ideal of $\mathcal{M}$.

Proof

$\mathcal{M}_\tau$ is obviously closed under sums.

If $\sum_{i=1}^{n} r_i s_i \in \mathcal{M}_\tau$ $(r_i , s_i \in \mathcal{N}_\tau)$; $\lambda \in \mathbb{C}$; $a \in \mathcal{M}$ then

$$\lambda\left(\sum_{i=1}^{n} r_i s_i\right) = \sum_{i=1}^{n} (\lambda r_i) s_i \in \mathcal{M}_\tau \text{ ; since } \lambda r_i \in \mathcal{N}_\tau$$

and $a\left(\sum_{i=1}^{n} r_i s_i\right) = \sum_{i=1}^{n} (a r_i) s_i \in \mathcal{M}_\tau$; since $a r_i \in \mathcal{N}_\tau$

Thus $\mathcal{M}_\tau$ is a left ideal.

If $\sum_{i=1}^{n} r_i s_i \in \mathcal{M}_\tau$ then $\left(\sum_{i=1}^{n} r_i s_i\right)^* = \sum_{i=1}^{n} s_i^* r_i^* \in \mathcal{M}_\tau$ since $\mathcal{N}_\tau$ is self–adjoint.

Hence $\mathcal{M}_\tau$ is self–adjoint.

Thus $\mathcal{M}_\tau$ is a 2–sided ideal. □

3:6 **Proposition** [T] V 2.16

$\mathcal{P}_\tau = \mathcal{M}_\tau \cap \mathcal{M}^+$

$\mathcal{M}_\tau$ is linearly spanned by $\mathcal{P}_\tau$

The function τ on $\mathcal{P}_\tau$ can be extended to a positve linear functional also denoted τ on $\mathcal{M}_\tau$

such that
$$\tau(s^*) = \overline{\tau(s)} \quad \forall\, s \in \mathcal{M}_\tau$$
$$\tau(a\, s) = \tau(s\, a) \quad \forall\, a \in \mathcal{M},\ s \in \mathcal{M}_\tau$$
$$\tau(r\, s) = \tau(s\, r) \quad \forall\, r, s \in \mathcal{N}_\tau$$

Proof

If $s \in \mathcal{M}_\tau$ then $s = \sum_{i=1}^{n} r_i^* s_i$ for some $n \in \mathbb{N}$, $r_i, s_i \in \mathcal{N}_\tau$

$\forall\ 1 \le i \le n \quad r_i^* s_i = \frac{1}{4} \sum_{k=0}^{3} i^k (s_i + i^k r_i)^*(s_i + i^k r_i)$ (The polarisation identity.)

Thus $s = \frac{1}{4} \sum_{i=1}^{n} \sum_{k=0}^{3} i^k (s_i + i^k r_i)^*(s_i + i^k r_i)$

Now $s_i + i^k r_i \in \mathcal{N}_\tau$ ($\mathcal{N}_\tau$ is an ideal) so $(s_i + i^k r_i)^* \in \mathcal{N}_\tau$ ($\mathcal{N}_\tau$ is self–adjoint)

$\Rightarrow \quad (s_i + i^k r_i)^*(s_i + i^k r_i) \in \mathcal{M}^+ \cap \mathcal{M}_\tau$

$\Rightarrow \quad \mathcal{M}_\tau$ is spanned by $\mathcal{M}^+ \cap \mathcal{M}_\tau$

If further $s \in \mathcal{M}_\tau \cap \mathcal{M}^+$, then

$$
\begin{aligned}
0 &\le s \\
&= \frac{1}{4}\Big[\sum_{i=1}^{n} (s_i + r_i)^*(s_i + r_i) - \sum_{i=1}^{n} (s_i - r_i)^*(s_i - r_i) \Big] \\
&\le \frac{1}{4}\Big[\sum_{i=1}^{n} (s_i + r_i)^*(s_i + r_i) \Big] \\
&\in \mathcal{P}_\tau \text{ since } s_i + r_i \in \mathcal{N}_\tau
\end{aligned}
$$

$\Rightarrow \quad s \in \mathcal{P}_\tau$ by the monotonicity of τ.

Suppose $s \in \mathcal{P}_\tau$

$\Rightarrow \quad s^{1/2} \in \mathcal{N}_\tau$

$\Rightarrow \quad s = (s^{1/2})^2 \in \mathcal{M}_\tau \cap \mathcal{M}^+$.

Thus $\mathcal{M}_\tau \cap \mathcal{M}^+ = \mathcal{P}_\tau$.

We now extend τ to $\mathcal{M}_\tau$.

It is immediate that $\mathcal{M}_\tau \cap \mathcal{M}^h = \mathcal{P}_\tau - \mathcal{P}_\tau$

Define τ on $\mathcal{M}_\tau \cap \mathcal{M}^h$ by $\tau(s-r) = \tau(s) - \tau(r)$ for $r , s \in \mathcal{P}_\tau$

This is well defined for if

$s_1 - r_1 = s_2 - r_2$ for $s_1 , r_1 , s_2 , r_2 \in \mathcal{P}_\tau$

$\Rightarrow \quad s_1 + r_2 = s_2 + r_1$

$\Rightarrow \quad \tau(s_1) + \tau(r_2) = \tau(s_2) + \tau(r_1)$

$\Rightarrow \quad \tau(s_1) - \tau(r_1) = \tau(s_2) - \tau(r_2)$

Define τ on $\mathcal{M}_\tau = (\mathcal{M}_\tau \cap \mathcal{M}^h) + i(\mathcal{M}_\tau \cap \mathcal{M}^h)$ linearly.

If $s \in \mathcal{M}_\tau$ then

$s = s_1 - s_2 + is_3 - is_4 \qquad s_1 , s_2 , s_3 , s_4 \in \mathcal{M}_\tau \cap \mathcal{M}^+$

$s^* = s_1 - s_2 - is_3 + is_4$

So $\tau(s^*) = \overline{\tau(s)}$ follows immediately.

If $r , s \in \mathcal{N}_\tau$ then

$$\tau(r\,s) = \tau\Big(\frac{1}{4}\sum_{k=0}^{3} i^k (r + i^k s^*)^* (r + i^k s^*)\Big) \quad \text{by the polarisation identity.}$$

$$= \frac{1}{4}\sum_{k=0}^{3} i^k\, \tau((r + i^k s^*)^* (r + i^k s^*)) \quad \text{as all the terms are finite.}$$

$$= \quad \frac{1}{4}\sum_{k=0}^{3} i^k\, \tau((r + i^k s^*)(r + i^k s^*)^*) \quad \text{by the commutativity condition.}$$

$$= \quad \tau\Big(\frac{1}{4}\sum_{k=0}^{3} i^k (r + i^k s^*)(r + i^k s^*)^*\Big)$$

$$= \quad \tau\Big(\frac{1}{4}\sum_{k=0}^{3} i^k\, \overline{i^k}\, i^k\, (r^* + i^{-k} s)^*(r^* + i^{-k} s)\Big)$$

$$= \quad \tau\Big(\frac{1}{4}\sum_{k=0}^{3} i^k\, (s + i^k r^*)^*(s + i^k r^*)\Big)$$

$$= \quad \tau(s\, r)$$

If $a \in \mathcal{M}$, $r, t \in \mathcal{N}_\tau$ then

$\tau(a\,(t\,r)) = \tau((a\,t)\,r) = \tau(r\,(a\,t)) = \tau((r\,a)\,t) = \tau(t\,(r\,a)) = \tau((t\,r)\,a)$

$\Rightarrow \qquad \tau(a\,s) = \tau(s\,a) \;\; \forall\, s \in \mathcal{M}_\tau$ by linearity. □

3:7 **Proposition** [T] p 319

$s \in \mathcal{M}_\tau$ iff $|s| \in \mathcal{M}_\tau$

$s \in \mathcal{N}_\tau$ iff $|s| \in \mathcal{N}_\tau$

Proof

$s = v|s|$ and $|s| = v^* s$ for some partial isometry $v \in \mathcal{M}$.

Since $\mathcal{M}_\tau$ and $\mathcal{N}_\tau$ are both ideals, the result follows. □

3:8 **Proposition** [T] p 319

$\mathcal{M}_\tau = \{\, s\,r : s, r \in \mathcal{N}_\tau \,\}$

Proof

The one inclusion is immediate from the definitions.

So suppose $s \in \mathcal{M}_\tau$, and let $s = v|s|$ be its polar decomposition.

Then $s = v\,|s|^{1/2}\,|s|^{1/2}$, so it will suffice to show that $|s|^{1/2} \in \mathcal{N}_\tau$ (for then $v\,|s|^{1/2} \in \mathcal{N}_\tau$).

$$|s|^{1/2^*}|s|^{1/2} = |s| \in \mathcal{M}_\tau \cap \mathcal{M}^+ = \mathcal{P}_\tau$$

$$\Rightarrow \quad \tau(|s|) < \infty$$

$$\Rightarrow \quad |s|^{1/2} \in \mathcal{N}_\tau$$ □

3:9 **Corollary**

$\mathcal{M}_\tau \subset \mathcal{N}_\tau$ □

3:10 **Definition**

$\mathcal{M}_\tau$ is called the *definition ideal* of τ.

$\mathcal{M}_\tau$ is the largest ideal in $\mathcal{M}$ to which τ can be extended as a finite valued linear functional.

In the terminology of [D] , a trace is a function satisfying the linearity and homongeniety conditions and which is unitarily invariant. The following result is equivalent to [D] I . 6 Corollary 1, where it is shown this definition coincides with the usual one. The proof that follows is however entirely original and we feel more instructive.

3:11 **Theorem**

Suppose $\gamma : \mathcal{M}^+ \to [0,\infty]$ satisfies the linearity and homogeniety conditions.

Then γ is a trace iff γ is unitarily invariant.

Proof

($\Rightarrow$)

Suppose γ is a trace.

The claim makes sense, for if $s \in \mathcal{M}^+$ and $u \in \mathcal{M}_u$ then $u^* s u \in \mathcal{M}^+$.

So suppose $u \in \mathcal{M}_u$, $s \in \mathcal{M}^+$.

$$\gamma(u^* s u) = \gamma(u^* s^{1/2} s^{1/2} u)$$

$$= \gamma((s^{1/2} u)^* (s^{1/2} u))$$

$= \quad \gamma((s^{1/2}u)(s^{1/2}u)^*)$ by the commutativity condition

$= \quad \gamma(s)$

($\Leftarrow$)

Suppose γ satisfies the linearity and homogeniety conditions, and is unitarily invariant.

Consider

$$\mathcal{P}_\gamma = \{ s \in \mathcal{M}^+ : \gamma(s) < \infty \}$$
$$\mathcal{N}_\gamma = \{ s \in \mathcal{M} : \gamma(s^* s) < \infty \}$$
$$\mathcal{M}_\gamma = \{ \sum_{i=1}^{n} r_i s_i : n \in \mathbb{N}, r_i, s_i \in \mathcal{N}_\gamma \}$$

If $s \in \mathcal{N}_\gamma$, $u \in \mathcal{M}_u$ then

$(u s)^* u s = s^* s \in \mathcal{P}_\gamma$.

and $(s u)^* s u = u^* s^* s u \in \mathcal{P}_\gamma$ since $s^* s \in \mathcal{P}_\gamma$ and γ is unitarily invariant.

Thus $u s, s u \in \mathcal{N}_\gamma$.

Now if $r, s \in \mathcal{N}_\gamma$ then $(r + s)^*(r + s) \leq 2(r^* r + s^* s)$, and so $r + s \in \mathcal{N}_\gamma$.

Thus $\mathcal{N}_\tau$ is closed under sums.

Hence $a \in \mathcal{M}, s \in \mathcal{N}_\gamma \Rightarrow a s, s a \in \mathcal{N}_\gamma$ since such a can be decomposed into a linear combination of unitary operators.

Thus $\mathcal{N}_\gamma$ is a 2–sided ideal of M.

Thus $\mathcal{N}_\gamma$ is self–adjoint, since 2–sided ideals in von Neumann algebras are always self adjoint. ([KR] 6.8.9)

It follows as in 3:5 that $\mathcal{M}_\gamma$ is a self–adjoint ideal, and as in 3:6 that $\mathcal{P}_\gamma = (\mathcal{M}_\gamma)^+$.

As in 3:6 extend γ linearly from $\mathcal{P}_\gamma$ to $\mathcal{M}_\gamma$.

If $s \in \mathcal{M}_\gamma$, $u \in \mathcal{M}_u$ then $s\,u$, $u\,s \in \mathcal{M}_\gamma$ and

$$\gamma(s\,u) = \gamma(u^* u\,s\,u) = \gamma(u\,s)$$, since γ is unitarily invariant.

$\Rightarrow$ $\gamma(s\,a) = \gamma(a\,s)$ for $a \in \mathcal{M}$ since such a can be decomposed into a linear combination of unitary operators.

Suppose $s^* s \in \mathcal{P}_\gamma$.

$\Rightarrow$ $s \in \mathcal{N}_\gamma$.

$\Rightarrow$ $s^* \in \mathcal{N}_\gamma$, since $\mathcal{N}_\gamma$ is self–adjoint.

$\Rightarrow$ $s\,s^* \in \mathcal{P}_\gamma$.

By replacing s with its adjoint, it follows $s^* s \in \mathcal{P}_\gamma \Leftrightarrow s\,s^* \in \mathcal{P}_\gamma$.

Finally, suppose $s \in \mathcal{M}$

If $s^* s \notin \mathcal{P}_\gamma$, then $s\,s^* \notin \mathcal{P}_\gamma$ and so $\gamma(s^* s) = \infty = \gamma(s\,s^*)$.

If $s^* s \in \mathcal{P}_\gamma$, then $s^* s \in \mathcal{M}_\gamma$ and so $v\,s^* s \in \mathcal{M}_\gamma$; with $s = v\,|s|$ the polar decomposition of s.

Hence $\gamma(s^* s)$

$= \gamma(v^* v\,s^* s)$ since $v^* v\,s^* s = s^* s$

(v has initial space the closure of the range of $s^* s$)

$= \tau(v\,s^* s\,v^*)$ by the commutativity result developed above.

$= \gamma(s\,s^*)$ since $v\,s^* s\,v^* = s\,s^*$

Thus γ is a trace. □

3:12 **Note**

Since $\mathcal{M}_\tau$ is a *–subalgebra, its so–closure is a von Neumann algebra with identity that we will denote by $p_\tau \in \mathcal{M}_p$. Clearly p_τ is the so–supremum of all the projections in $\mathcal{M}_\tau$.

We show that p_τ is central.

Suppose $u \in \mathcal{M}_u$, it suffices to show that $p_\tau u = u\, p_\tau$

Choose $\{s_i\} \subset \mathcal{M}_\tau^{\ +}$ such that $s_i \xrightarrow{so} p_\tau$

Then $u^* s_i u \in \mathcal{M}_\tau^{\ +}$ $\forall\, i \in I$ since τ is unitarily invariant, and $u^* s_i u \xrightarrow{so} u^* p_\tau u$

$\Rightarrow \quad u^* p_\tau u \in (\mathcal{M}_\tau)^{-so}$

Now $u^* p_\tau u$ is a projection, so $u^* p_\tau u \le p_\tau$ since p_τ is the identity here.

Likewise $u\, p_\tau u^* \le p_\tau$ and so $p_\tau \le u^* p_\tau u$

Hence $u^* p_\tau u = p_\tau$ and so $p_\tau u = u\, p_\tau$

The result follows : $p_\tau \in (Z(\mathcal{M}))_p$ □

3:13 **Proposition**

The following are equivalent

(a) $\tau(s) < \infty \;\; \forall\, s \in \mathcal{M}^+$

(b) $\tau(1) < \infty$

(c) $\mathcal{M}_\tau = \mathcal{M}$

Proof

(c) $\Leftrightarrow$ (a) $\Rightarrow$ (b)

Is clear.

(b) $\Rightarrow$ (c)

$\mathcal{M}_\tau$ is an ideal containing 1, and hence $\mathcal{M}_\tau = \mathcal{M}$. □

3:14 **Definition**

A trace τ is said to be *finite* if it satisfies the equivalent conditions in the previous proposition.

3:15 **Proposition** cf. [KR] 8.1.1

If $f : \mathcal{M} \longrightarrow \mathbb{C}$ is a linear mapping (not neccessarily bounded), the following are equivalent

(a) $f(r\,s) = f(s\,r)\ \ \forall\, r, s \in \mathcal{M}$

(b) $f(s\,s^*) = f(s^*\,s)\ \ \forall\, s \in \mathcal{M}$

(c) $f(s) = f(u^*\,s\,u)\ \ \forall\, s \in \mathcal{M}\ \ \forall\, u \in \mathcal{M}_u$

Proof

(a) ⇒ (b)

Clear.

(b) ⇒ (c)

Since $s \in \mathcal{M}$ is a linear combination of positive elements of $\mathcal{M}$, by the linearity of f it suffices to check condition (c) for $s \geq 0$.

The result follows by imitating the method of proof for 3:11(⇒)

(c) ⇒ (a)

Suppose $r, s \in \mathcal{M}$.

For $u \in \mathcal{M}_u$, $f(u\,s) = f(u\,s\,u\,u^*) = f(s\,u)$

⇒ $f(r\,s) = f(s\,r)$ by linearity of f, and the fact that r is the linear combination of four unitary elements. □

3:16 **Definition**

We say that f is a *central form* if it satisfies the conditions of the previous proposition.

3:17 **Proposition**

The finite traces are exactly the positive central forms.

Finite traces are norm continuous.

Proof

Suppose τ is a positive central form.

Then τ is clearly a trace, and is finite.

Suppose τ is a finite trace.

Then τ is a positive linear form, and is thus norm bounded. [KR] 4.3.2

By definition, τ is central. □

3:18 **Definition**

A trace τ is said to be *normal* if $\{s_i\}_{i\in I} \subset \mathcal{M}^+$, $s_i \uparrow_{so} s \Rightarrow \tau(s_i) \uparrow \tau(s)$.

3:19 **Proposition**

The finite normal traces are exactly the positive uw–continuous central forms.

Proof

By the previous proposition, the finite normal traces are the positive central normal forms.

The normal forms on $\mathcal{M}$ are exactly the uw–continuous forms. ([D] 1.4.2 Theorem 1)

The result follows. □

3:20 **Proposition**

Suppose τ is a normal trace and $s_i \downarrow_{so} s \geq 0$

If $\tau(s_{i_0}) < \infty$ for some $i_0 \in I$, then $\tau(s_i) \downarrow \tau(s)$

Proof

$s_i \downarrow_{so} s \Rightarrow s_{i_0} - s_i \uparrow_{so} s_{i_0} - s$

$\Rightarrow \tau(s_{i_0} - s_i) \uparrow \tau(s_{i_0} - s)$ by the normality of τ

$\Rightarrow \tau(s_{i_0}) - \tau(s_i) \uparrow \tau(s_{i_0}) - \tau(s)$ by 3:2(b)

$\Rightarrow \tau(s_i) \downarrow \tau(s)$ □

3:21 **Proposition**

If τ is normal and $\{p_i\}_{i\in I} \subset \mathcal{M}_p$ then $\tau(\bigvee_{i\in I} p_i) \leq \sum_{i\in I} \tau(p_i)$.

Proof

For $J \subset I$, J finite, $\tau(\bigvee_{i\in J} p_i) \leq \sum_{i\in J} \tau(p_i) \leq \sum_{i\in I} \tau(p_i)$ by 3:2(f).

Now $\bigvee_{i\in I} p_i$ is the so limit of the increasing net $(\bigvee_{i\in J} p_i)_{J\subset I,\ J \text{ finite}}$, and so $\tau(\bigvee_{i\in I} p_i) \leq \sum_{i\in I} \tau(p_i)$ by the normality of τ . □

The following three propositions are usually stated for uw–continuous positive forms e.g. [SZ] 5.15 . As almost exclusive use is made of the normality (rather than the continuity as such) there is no difficulty in generalising these results to normal traces as follows :–

3:22 **Proposition**

Suppose τ is a normal trace.

For $r \in \mathcal{M}^+$, $\tau(r) = 0 \Leftrightarrow \tau(s(r)) = 0$.

Proof

($\Rightarrow$)

Suppose $r \in \mathcal{M}^+$ and $\tau(r) = 0$.

Let $\{e_t(r)\}_{t\geq 0}$ be the spectral resolution for r .

Since $r\,(1 - e_t(r)) \geq t\,(1 - e_t(r))\ \ \forall\, t \geq 0$,

$0 = \tau(r) \geq \tau(\,r\,(1 - e_t(r))\,) \geq \tau(\,t\,(1 - e_t(r))\,) = t\,\tau(1 - e_t(r))$.

$\Rightarrow \quad \tau(1 - e_t(r)) = 0\ \ \forall\, t > 0$.

As $s(r) = \sup_{t>0} (1 - e_t(r))$, so $\tau(s(r)) = 0$ by the normality of τ.

($\Leftarrow$)

Suppose $r \in \mathcal{M}^+$ and $\tau(s(r)) = 0$.

Then $r \leq \|r\|\, s(r)$, so $\tau(r) \leq \|r\|\, \tau(s(r)) = 0$. □

3:23 Proposition

Suppose τ is a normal trace.

Then the family $\{ p \in \mathcal{M}_p : \tau(p) = 0 \}$ is upward directed, hence has a supremum whose orthogonal complement will be denoted $s(\tau)$; the *support* of τ .

The said supremum is the greatest projection annihilated by τ .

Proof

It suffices to show that $\{ p \in \mathcal{M}_p : \tau(p) = 0 \}$ is upward directed.

Suppose $p , q \in \mathcal{M}_p$ and $\tau(p) = \tau(q) = 0$.

$\Rightarrow$ $\tau(p + q) = \tau(p) + \tau(q) = 0$.

$\Rightarrow$ $\tau(s(p + q)) = 0$ by 3:22.

$\Rightarrow$ $\tau(p \vee q) = 0$ since $p \vee q = s(p + q)$ □

3:24 Definition

A trace τ is said to be *faithful* if $s \in \mathcal{M}^+$, $\tau(s) = 0 \Rightarrow s = 0$.

A family of traces $\{\tau_i\}_{i \in I}$ is said to be *sufficient* if $0 < s \in \mathcal{M}^+ \Rightarrow \exists\, i \in I$ such that $\tau_i(s) \neq 0$.

Clearly $\{\tau\}$ is sufficient iff τ is faithful.

3:25 Proposition

A family $\{\tau_i\}_{i \in I}$ of normal traces is sufficient iff $\bigvee_{i \in I} s(\tau_i) = 1$.

Proof

Suppose $\bigvee_{i \in I} s(\tau_i) = 1$, $r \in \mathcal{M}^+$, $\tau_i(r) = 0 \ \forall\, i \in I$

$\Rightarrow$ $\tau_i(s(r)) = 0 \quad \forall\, i \in I$ by 3:22

$\Rightarrow$ $s(r) \leq 1 - s(\tau_i) \quad \forall\, i \in I$

$\Rightarrow$ $s(r) \leq \bigwedge_{i \in I} (1 - s(\tau_i)) = 1 - \bigvee_{i \in I} s(\tau_i) = 0$

$\Rightarrow$ $s(r) = 0$

$\Rightarrow$ $r = 0$

Conversely, suppose $\{\tau_i\}_{i\in I}$ is sufficient and let $p = \bigvee_{i\in I} s(\tau_i)$

$$1-p = \bigwedge_{i\in I}(1-s(\tau_i)) \le 1-s(\tau_i) \quad \forall i \in I$$

$$\Rightarrow \quad \tau_i(1-p) = 0 \ \forall i \in I$$

$$\Rightarrow \quad 1-p = 0 \text{ by the sufficiency}$$

□

3:26 **Corollary**

A normal trace τ is faithful iff $s(\tau) = 1$. □

3:27 **Proposition**

If τ is a normal trace, then $s(\tau) \in (Z(\mathcal{M}))_p$

Proof

Suppose $u \in \mathcal{M}_u$, it suffices to show $s(\tau)\,u = u\,s(\tau)$.

Since τ is unitarily invariant, $\tau(1-u^* s(\tau)\,u) = \tau(u^*(1-s(\tau))\,u) = \tau(1-s(\tau)) = 0$

$\Rightarrow \quad 1-u^* s(\tau)\,u \le 1-s(\tau)$ as $1-s(\tau)$ is the largest projection annihilated by τ

$\Rightarrow \quad s(\tau) \le u^* s(\tau)\,u$

The result follows as in 3:12. □

3:28 **Definition**

A trace τ is said to be *semifinite* if $\forall\, 0 < s \in \mathcal{M}^+ \ \exists\, 0 < r \le s$ such that $\tau(r) < \infty$.

The definition of semifiniteness given here (condition (a) in the following proposition) is the most common in references e.g. [T] V 2.1 ; [SZ] 7.13 .

[N] p 105 (implicitly) defines semifiniteness as condition (b) in the following proposition, which we show to be equivalent to (a) .

[D] 1,6,1 defines semifiniteness as condition (e) in the following proposition, which seems only to be equivalent to (a) and (b) under the additional assumption of normality.

3:29 **Proposition**

If τ is a trace on $\mathcal{M}$, the following are equivalent

(a) τ is semifinite i.e. $\forall\ 0 < s \in \mathcal{M}^+\ \exists\ 0 < r \leq s$ such that $\tau(r) < \infty$

(b) $\forall\ s \in \mathcal{M}^+\ \exists\ s_i \uparrow_{so} s$ such that $\tau(s_i) < \infty\ \forall\ i \in I$

(c) $\overline{\mathcal{M}_\tau}^{so} = \mathcal{M}$

(d) $p_\tau = 1$

If in addition, τ is normal, then these conditions are equivalent to :—

(e) $\forall\ s \in \mathcal{M}^+\ \tau(s) = \sup\{\tau(r) : 0 \leq r \leq s,\ \tau(r) < \infty\}$

Proof

(a) ⇒ (b)

Suppose $0 < s \in \mathcal{M}^+$

Let $\{s_i\}_{i\in I}$ be a family of operators maximal with respect to

$$s_i > 0\ \forall i \in I,\ \tau(s_i) < \infty\ \forall i \in I,\ \sum_{i\in J} s_i \leq s\ \forall \text{ finite } J \subset I.$$

Clearly $I \neq \phi$ by hypothesis.

Then $(\sum_{i\in J} s_i)_{\text{finite } J \subset I}$ forms an increasing net bounded above by s, so by the Monotone Convergence Theorem $(\sum_{i\in J} s_i)_{\text{finite } J \subset I} \uparrow_{so} r \leq s$ (say).

Clearly $\tau(\sum_{i\in J} s_i) < \infty\ \forall$ finite $J \subset I$.

Assume for a contradiction that $s - r > 0$.

By (a), choose $0 < t \leq s - r$ such that $\tau(t) < \infty$.

Then t can be included in the family $\{s_i\}_{i\in I}$, contradicting the maximality of I.

Hence $(\sum_{i\in J} s_i)_{\text{finite } J \subset I} \uparrow_{so} s$, and $(\sum_{i\in J} s_i)_{\text{finite } J \subset I}$ is the required net.

(b) ⇒ (c)

By (b) $\mathcal{P}_\tau$ is so–dense in $\mathcal{M}^+$

⇒ $\mathcal{M}_\tau$ is so–dense in $\mathcal{M}$.

(c) ⇒ (d)

Clear.

(d) ⇒ (b)

By definition of p_τ, $\exists \{p_i\}_{i \in I} \subset \mathcal{P}_\tau$ such that $p_i \uparrow_{so} 1$

For $s \in \mathcal{M}^+$, $s^{1/2} p_i s^{1/2} \uparrow_{so} s^{1/2} 1 s^{1/2} = s$

Since $\mathcal{M}_\tau$ is an ideal, $s^{1/2} p_i s^{1/2} \in \mathcal{M}_\tau \ \forall i \in I$

i.e. $\tau(s^{1/2} p_i s^{1/2}) < \infty \ \forall i \in I$

(b) ⇒ (a)

Clear, since if $s_i \uparrow_{so} s > 0$, then $s_i > 0$ for some $i \in I$

Suppose now that τ is normal.

(b) ⇒ (e)

Suppose $s \in \mathcal{M}^+$.

It is clear that $\tau(s) \geq \sup \{ \tau(r) : 0 \leq r \leq s , \tau(r) < \infty \}$

By hypothesis, $\exists$ $s_i \uparrow_{so} s$ such that $\tau(s_i) < \infty \ \forall \ i \in I$, and by normality of τ, $\tau(s_i) \uparrow \tau(s)$.

Hence $\tau(s) \leq \sup \{ \tau(r) : 0 \leq r \leq s , \tau(r) < \infty \}$

(e) ⇒ (a)

Clear. □

The following elementary result will often be useful.

3:30 Proposition

Suppose τ is a faithful semifinite normal trace.

Then $\forall\, 0 < s \in \mathcal{M}^+ \ \exists\, \delta > 0 \ \exists\, p \in \mathcal{M}_p$ such that $0 < \delta p \le s$ and $\tau(p) < \infty$

Proof

Suppose $0 < s \in \mathcal{M}^+$

By the faithfulness of τ, $\tau(s) > 0$

By 3:29(e) $\exists\, 0 \le r \le s$ such that $0 < \tau(r) < \infty$

Then $0 < r$ by the faithfulness of τ

$\exists\, \delta > 0 \ \exists\, p \in \mathcal{M}_p$ such that $0 < \delta p \le r$, by the Spectral Theorem

Then $0 < \tau(\delta p) = \delta\, \tau(p) \le \tau(r) < \infty$

$\Rightarrow \quad 0 < \tau(p) < \infty$ □

3:31 Theorem

Suppose τ is a trace on $\mathcal{M}$.

For $0 \ne e \in \mathcal{M}_p$, define $\tau_e : \mathcal{M}_e^{\,+} \longrightarrow [0,\infty] : s_e \longrightarrow \tau(ese)$

Then τ_e is a trace on M_e

We will call τ_e the *reduction* of τ to $\mathcal{M}_e$

If τ is faithful (semifinite, normal), then so is τ_e

τ_e is finite iff $\tau(e) < \infty$

Proof

Since $(r + s)_e = r_e + s_e$ and $(\lambda\, s)_e = \lambda\, s_e$ for $r, s \in \mathcal{M}$ and $\lambda \in \mathbb{C}$, the linearity and homogeneity conditions are immediate.

For $s \in \mathcal{M}$, $\tau_e(s_e^* s_e) = \tau_e((s^* e s)_e)$

$$= \tau(e s^* e e s e)$$

$$= \tau(e s e e s^* e) \quad \text{by the commutativity condition for } \tau$$

$$= \tau_e((s e s^*)_e)$$

$$= \tau_e(s_e s_e^*) .$$

So τ_e satisfies the commutativity condition.

Suppose τ is normal.

$$s_{i_e} \uparrow_{so} s_e \quad \Rightarrow \quad e s_i e \uparrow_{so} e s e$$

$$\Rightarrow \quad \tau(e s_i e) \uparrow \tau(e s e) \quad \text{by the normality of } \tau$$

$$\Rightarrow \quad \tau_e(s_{i_e}) \uparrow \tau_e(s_e)$$

So τ_e is normal.

Suppose τ is faithful

$$\tau_e(s_e) = 0 \quad \Rightarrow \quad \tau(e s e) = 0$$

$$\Rightarrow \quad e s e = 0 \text{ by the faithfulness of } \tau$$

$$\Rightarrow \quad s_e = 0$$

So τ_e is faithful.

Suppose τ is semifinite.

Suppose $0 < s_e \in \mathcal{M}_e$, $\tau_e(s_e) > 0$.

$\Rightarrow$ $0 < e s e$ and $\tau(e s e) > 0$

$\Rightarrow$ $\exists\, r \in \mathcal{M}^+$ such that $0 < r < e s e$ and $\tau(r) < \infty$, by the semifiniteness of τ

We claim that $r = e r e$

Let $q = 1 - e$

For $x \in \mathcal{H}$, $\langle qrqx,x\rangle = \langle rqx,qx\rangle \leq \langle eseqx,qx\rangle = 0$, so $q r q = 0$

Now $q\,r\,q = (r^{1/2}\,q)^*\,(r^{1/2}\,q)$, so $r^{1/2}\,q = 0$

Hence $r\,q = r^{1/2}\,r^{1/2}\,q = 0$, and $q\,r = (r\,q)^* = 0$

Thus $r = e\,r\,e + e\,r\,q + q\,r = e\,r\,e$

Thus $0 < e\,r\,e = r \leq e\,s\,e$

$\tau(r) < \infty \Rightarrow \tau(e\,r\,e) < \infty$ since $\mathcal{M}_\tau$ is a two sided ideal.

Thus $0 < r_e \leq s_e$ and $\tau_e(r_e) < \infty$

Hence τ_e is semifinite.

τ_e is finite

$\Leftrightarrow \quad \tau_e(1_e) < \infty$

$\Leftrightarrow \quad \tau(e) < \infty$. □

3:32 **Theorem**

Suppose $0 \neq e \in (Z(\mathcal{M}))_p$ and τ is a trace on $\mathcal{M}_e$

Define $\tau^e : \mathcal{M}^+ \longrightarrow [0,\infty] : s \longrightarrow \tau(s_e)$

Then τ^e is a trace on $\mathcal{M}$

We call τ^e the *extention* of τ from $\mathcal{M}_e$

If τ is normal (semifinite), then so is τ^e

If τ is normal then $s(\tau) = s(\tau^e)$

(where $s(\tau)$ is canonically considered to be a projection in $\mathcal{M}$ rather than in $\mathcal{M}_p$)

Proof

By simliar reasoning to 3:31, the linearity and homogeniety conditions are clear.

Suppose $s \in \mathcal{M}$

$$\begin{aligned}
\tau^e(s^*\,s) &= \tau((s^*\,s)_e) \\
&= \tau(s_e^*\,s_e) \quad \text{since } e \text{ is central} \\
&= \tau(s_e\,s_e^*) \quad \text{since } \tau \text{ is a trace}
\end{aligned}$$

$$= \quad \tau((s\,s^*)_e) \quad \text{since e is central}$$
$$= \quad \tau^e(s\,s^*)$$

So τ^e satisfies the commutativity condition.

Suppose τ is normal.

$s_i \uparrow_{so} s$ in $\mathcal{U}$

$$\Rightarrow \quad e\,s_i\,e \uparrow_{so} e\,s\,e$$
$$\Rightarrow \quad s_{i_e} \uparrow_{e\,so} s_e$$
$$\Rightarrow \quad \tau(s_{i_e}) \uparrow \tau(s_e) \quad \text{by the normality of } \tau$$
$$\Rightarrow \quad \tau^e(s_i) \uparrow \tau^e(s)$$

So τ^e is normal.

If τ , and thus τ^e , are normal, then

$$\tau((1-s(\tau^e))_e) = \tau^e((1-s(\tau^e)) = 0$$
$$\Rightarrow \quad (1-s(\tau^e))_e \le 1 - s(\tau)$$
$$\Rightarrow \quad s(\tau^e)_e \ge s(\tau) \qquad \text{(as projections in } \mathcal{U}_e\text{)}$$
$$\Rightarrow \quad e\,s(\tau^e)\,e \ge s(\tau) \qquad \text{(as projections in } \mathcal{U}\text{)}$$
$$\Rightarrow \quad s(\tau^e) = e\,s(\tau^e)\,e \; + \; (1-e)\,s(\tau^e)\,(1-e) \ge s(\tau)$$

and $\tau^e(1-s(\tau)) = \tau(1-s(\tau)) = 0$

$$\Rightarrow \quad 1 - s(\tau) \le 1 - s(\tau^e)$$
$$\Rightarrow \quad s(\tau) \ge s(\tau^e)$$

Hence $s(\tau) = s(\tau^e)$

Suppose τ is semifinite

Suppose $0 < s \in \mathcal{M}$ and $\tau^e(s) > 0$.

$\Rightarrow \quad \tau(s_e) > 0$

$\Rightarrow \quad \exists\ 0 < r_e \leq s_e$ such that $\tau(r_e) < \infty$

Now $\tau^e(e\ r\ e) = \tau(r_e) < \infty$

and $0 < e\ r\ e < e\ s\ e \leq s$, the last inequality following since e is central.

Hence τ^e is semifinite. □

We will have need of the following results in Chapter 10 and Chapter 12.

We suppose τ is a faithful semifinite normal trace on $\mathcal{M}$

3:33 Definition

$\mathcal{M}$ is said to be *non–atomic* if it has no minimal projections.

3:34 Theorem

Suppose $\mathcal{M}$ is non–atomic.

Suppose $p , q \in \mathcal{M}_p$ and $p \leq q$.

If $\theta \in [\tau(p) , \tau(q)]$ then there exists $e_\theta \in \mathcal{M}_p$ such that $p \leq e_\theta \leq q$ and $\tau(e_\theta) = \theta$

Proof

Suppose p , q , θ are given as indicated.

Let $(\mathcal{M}_p)_\theta = \{ e \in \mathcal{M}_p : p \leq e \leq q$ and $\tau(e) \leq \theta \}$ with the usual partial ordering.

$(\mathcal{M}_p)_\theta \neq \phi$ since $p \in (\mathcal{M}_p)_\theta$.

Suppose $\{e_i\}_{i\in I}$ is an increasing chain in $(\mathcal{M}_p)_\theta$. Then $\{e_i\}_{i\in I}$ is bounded above by q, so $e_i \uparrow e$ (say) , $e \in \mathcal{M}$ by the Monotone Convergence Theorem [KR] 5.1.4

Clearly $e \in \mathcal{M}_p$.

By the normality of τ , $\tau(e) \leq \theta$.

Hence $e \in (\mathcal{M}_p)_\theta$.

Thus every chain in $(\mathcal{M}_p)_\theta$ has an upper bound in $(\mathcal{M}_p)_\theta$. By Zorns lemma, there exists a maximal element e_θ.

We contend that $\tau(e_\theta) = \theta$.

Assume for a contradiction $\tau(e_\theta) < \theta$.

Since $\tau(q) \geq \theta$, it follows $\tau(q - e_\theta) > 0$. Then by the faithfulness of τ, $0 < q - e_\theta$.

By the semifiniteness of τ, choose $0 < e_1 \leq q - e_\theta$ such that $\tau(e_1) < \infty$. By 3:30 we may suppose $e_1 \in \mathcal{M}_p$.

We now construct a sequence of projections (e_n) inductively.

At the n^{th} stage $(n \geq 2)$ decompose e_{n-1} as $e_{n-1} = p_{n-1} + q_{n-1}$ where $0 \neq p_{n-1}, q_{n-1} \in \mathcal{M}_p$. This is of course possible by the non–atomicity of $\mathcal{M}$.

Let e_n be that one of p_{n-1}, q_{n-1} with smaller trace.

Then $q - e_\theta \geq e_1 \geq e_2 \geq \dots$

and $\tau(e_n) \downarrow 0$ since $\tau(e_n) \leq 2^{-(n-1)} \tau(e_1)$.

$\Rightarrow \quad \exists\, n \in \mathbb{N}$ such that $\tau(e_n) < \theta - \tau(e_\theta)$.

Then $e_\theta + e_n \in (\mathcal{M}_p)_\theta$, the required contradiction.

Hence $\tau(e_\theta) = \theta$. □

The previous result applies to some fixed $\theta \in [\tau(p), \tau(q)]$. We will need to strengthen this result as follows.

3:35 **Corollary** cf. [MvN2] 3.1.2

Suppose $p , q \in \mathcal{M}_p$ and $p \leq q$

$\forall\, \theta \in [\tau(p) , \tau(q)]$ $\exists\, e_\theta \in \mathcal{M}_p$ such that

$p \leq e_\theta \leq q$

$\tau(e_\theta) = \theta$

$\theta_1 < \theta_2 \Rightarrow e_{\theta_1} < e_{\theta_2}$

$\{e_\theta\}$ is continuous on $[\tau(p) , \tau(q)]$ (in the sense of strong operator convergence)

Proof

It is clear we may suppose $\tau(p) < \infty$

We first deal with thc casc $\tau(q) < \infty$

Put $e_{\tau(p)} = p$ and $e_{\tau(q)} = q$

Choose a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ which is dense in the interval $(\tau(p) , \tau(q))$

We construct a sequence $\{e_{\alpha_n}\}_{n\in\mathbb{N}}$ by induction:

Apply 3:34 to p , q , and α_1 to derive $e_{\alpha_1} \in \mathcal{M}_p$

Suppose we have constructed projections $\{e_{\alpha_1} , \dots , e_{\alpha_n}\}$ satisfying

$p \leq e_{\alpha_i} \leq q \quad 1 \leq i \leq n$

$\tau(e_{\alpha_i}) = \alpha_i \quad 1 \leq i \leq n$

$\alpha_i < \alpha_j \Rightarrow c_{\alpha_i} < c_{\alpha_j} \quad 1 \leq i , j \leq n$

Choose from $\{ p , q , e_{\alpha_1} , \dots , e_{\alpha_n} \}$ the largest possible projection with trace value less than or equal to α_{n+1} ; and the smallest possible projection with trace value greater than or equal to α_{n+1}

Apply 3:34 to the two projections thus chosen and to α_{n+1} to derive $e_{\alpha_{n+1}}$.

It is then clear that

$$p \leq e_{\alpha_i} \leq q \quad 1 \leq i \leq n+1$$

$$\tau(e_{\alpha_i}) = \alpha_i \quad 1 \leq i \leq n+1$$

$$\alpha_i < \alpha_j \Rightarrow e_{\alpha_i} < e_{\alpha_j} \quad 1 \leq i, j \leq n+1$$

It follows by induction that we derive projections $\{e_{\alpha_n}\}_{n \in \mathbb{N}}$ satisfying

$$p \leq e_{\alpha_n} \leq q \quad n \in \mathbb{N}$$

$$\tau(e_{\alpha_n}) = \alpha_n \quad n \in \mathbb{N}$$

$$\alpha_{n_1} < \alpha_{n_2} \Rightarrow e_{\alpha_{n_1}} < e_{\alpha_{n_2}} \quad n \in \mathbb{N}$$

For $\theta \in (\tau(p), \tau(q))$ let $e_\theta = \bigvee_{\alpha_n \leq \theta} e_{\alpha_n}$

This definition is unambiguous in the sense that it agrees with the already defined projections e_{α_n} for $\theta \in \{\alpha_n : n \in \mathbb{N}\}$

$$\begin{aligned}
\tau(e_\theta) &= \tau\Big(\bigvee_{\alpha_n \leq \theta} e_{\alpha_n}\Big) \\
&= \sup_{\alpha_n \leq \theta} \tau(e_{\alpha_n}) \quad \text{by the normality of } \tau \\
&= \sup_{\alpha_n \leq \theta} \alpha_n \\
&= \theta \quad \text{as the sequence } \{\alpha_n\}_{n \in \mathbb{N}} \text{ was dense}
\end{aligned}$$

$\theta_1 < \theta_2$

$\Rightarrow$ $\exists\, n \in \mathbb{N}$ such that $\theta_1 < \alpha_n < \theta_2$, again because the sequence $\{\alpha_n\}_{n \in \mathbb{N}}$ was dense

$\Rightarrow$ $e_{\theta_1} < e_{\theta_2}$

Suppose $\theta \in [\tau(p), \tau(q)]$

If $[\tau(p), \tau(q)] \supset \theta_i \uparrow \theta$ then the net e_{θ_i} is increasing and bounded above by e_θ, hence so–convergent to the upper bound, e (say).

$\tau(e) = \theta$ by the normality of τ.

Thus $\tau(e_\theta - e) = \tau(e_\theta) - \tau(e) = \theta - \theta = 0$, and so $e_\theta = e$ by the faithfulness of τ.

i.e. $e_{\theta_i} \uparrow e_\theta$

A similar argument holds if $[\tau(p), \tau(q)] \supset \theta_i \downarrow \theta$

Now consider the case where $\tau(q) = \infty$

In the above arguments, the continuity of the system $\{e_\theta\}$ at q relied on the finiteness of $\tau(q)$. Nevertheless, a slight modification of the argument shows that continuity at $\tau(q)$ can be arranged.

Since τ is semifinite, we can find $\{q_i\}_{i\in I} \subset \mathcal{M}_p$, $\tau(q_i) < \infty$ $\forall i \in I$ such that $q_i \uparrow q$

Note that $\tau(q_i \vee p) \leq \tau(q_i) + \tau(p) < \infty$, so by replacing each q_i by $q_i \vee p$, we can suppose $q_i \geq p$ $\forall i \in I$

Put $q_0 = p$

We choose a sequence $\{q_n\}_{n\in\mathbb{N}}$ from $\{q_i\}_{i\in I}$ such that

$$q_n \geq q_{n-1}$$
$$\tau(q_n) \geq n + \tau(p)$$
$$q_n \uparrow q$$

Apply the arguments of the first half of this result to each of the pairs q_n, q_{n-1} of projections. The subsystems $\{ e_\theta : \tau(q_n) \leq \theta \leq \tau(q_{n+1}) \}$ are all continuous, and so the system $\{ e_\theta : \tau(p) \leq \theta < \tau(q) \}$ is continuous.

Moreover $e_\theta \uparrow q$ as $\theta \uparrow \infty$ since $q_n \uparrow q$ as $n \uparrow \infty$ □

4 : EXAMPLES of TRACES

We consider various von Neumann algebras and construct what are considered to be the canonical traces on these algebras. In general we aim to construct faithful (semi)finite normal traces for reasons that will become apparent in Chapters 5 and 6.

Commutative von Neumann Algebras

We note that $\mathcal{M}$ is a commutative von Neumann algebra iff $\mathcal{M}$ is *–isomorphic to $L_\infty(X,\mathcal{B}(X),\mu)$ for some localisable measure space $(X,\mathcal{B}(X),\mu)$. We may suppose μ is semifinite.

See [T] III 1.18 ; [S] 1.18 ; [Sg1] .

We recall for the reader's convenience the obvious fact that any commutative von Neumann algebra $\mathcal{M}$ is finite (and hence semifinite) – irrespective of the finiteness or otherwise of any measure space (X,Σ,μ) for which $\mathcal{M} \cong L_\infty(X,\Sigma,\mu)$. We also recall that in a commutative von Neumann algebra the commutativity condition for a trace is redundant.

4:1 $\quad \mathcal{M} = \mathbb{C}\ ,\ \mathcal{H} = \mathbb{C}$

$\tau : \mathbb{C}^+ = [0,\infty) \longrightarrow [0,\infty) : s \longrightarrow s$ is clearly a faithful finite normal trace. □

4:2 $\quad \mathcal{M} = l^\infty,\ \mathcal{H} = l^2$

We define $\tau :\ l^{\infty^+} \longrightarrow [0,\infty] : (x_n) \longrightarrow \sum_{n=1}^{\infty} x_n$.

4:2.1 **Theorem**

τ is a faithful semifinite normal trace on l^∞.

Proof

$l^{\infty^+} = \{ (x_n) \in l^\infty : x_n \geq 0 \ \forall n \in \mathbb{N} \}$, so indeed $\tau : l^{\infty^+} \longrightarrow [0,\infty]$

τ clearly satifies the linearity and homogeniety conditions, and is thus a trace.

If $(x_n) \in l^{\infty^+}$ then

$$\tau((x_n)) = 0$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} x_n = 0$$

$$\Rightarrow \quad x_n = 0 \ \forall n \in \mathbb{N}$$

$$\Rightarrow \quad (x_n) = 0$$

Hence τ is faithful.

If $0 \neq (x_n) \in l^{\infty^+}$ then $\exists\, n \in \mathbb{N}$ such that $x_n \neq 0$

Then the element (y_n) where $y_m = \delta_{nm}\, x_n$ satisfies $0 < (y_n) \leq (x_n)$ and $\tau((y_n)) < \infty$.

Hence τ is semifinite.

Suppose $(x_n)_i \uparrow_{so} (x_n)$, then $(x_n)_i \uparrow (x_n)$ pointwise.

We denote $(x_n)_i$ as (x_{ni})

$$\tau((x_n)) = \sum_{n=1}^{\infty} x_n$$

$$= \sum_{n=1}^{\infty} \sup_i x_{ni}$$

$$= \sup_i \sum_{n=1}^{\infty} x_{ni}$$

(interchanging the summation and supremum since all terms are positive)

$$= \quad \sup_i \tau((x_i))$$

Hence τ is normal. □

4:2 is a special case of :

4:3 $\quad \mathcal{M} = L_\infty(x,\Sigma,\mu) \quad \mathcal{H} = L^2(X,\Sigma,\mu)$, μ is a semifinite measure.

We define $\tau : L_\infty^{\;+} \longrightarrow [0,\infty] : f \longrightarrow \int f \, d\mu$.

4:3.1 **Theorem**

τ is a faithful semifinite normal trace on L_∞ .

Proof

τ is clearly a trace, by the linearity and homogeniety of integration.

Suppose $\tau(f) = 0$ for $f \in L^{\infty^+}$.

$\Rightarrow \quad \mu\{x \in X : f(x) \neq 0 \} = 0$

$\Rightarrow \quad f = 0$ since we are considering equivalence classes of functions.

Hence τ is faithful.

Suppose $0 \leq f_i \underset{so}{\uparrow} f$.

Assume for a contradiction that $\int f_i \, d\mu \uparrow a < \int f \, d\mu$

For $n \in \mathbb{N}$, choose $f_{i_n} \in \{ f_i : i \in I \}$ such that $\int f_{i_n} \, d\mu > a - \frac{1}{n}$ and $f_{i_n} \geq f_{i_{n-1}}$

Then $\{f_{i_n} : n \in \mathbb{N} \}$ is a subnet of $\{f_i : i \in I \}$ and so $f_{i_n} \uparrow f$ a.e.

Thus $\int f_{i_n} \, d\mu \uparrow \int f \, d\mu$ by the monotone convergence theorem.

Thus $\int f \, d\mu \geq a$, the required contradiction.

Hence τ is normal.

Suppose $f \in L^{\infty+}$ and $\int f \, d\mu \neq 0$

$\Rightarrow \quad \exists\, E \in \Sigma$ such that $\mu(E) > 0 \;\; \exists\, \delta > 0$ such that $f \geq \delta \chi_E$

By the semifiniteness of μ, choose $F \subset E$ such that $0 < \mu(F) < \infty$.

Then $\delta \mu(F) = \int \delta \chi_F \, d\mu \leq \int f \, d\mu$ and $\delta \mu(F) < \infty$.

Hence τ is semifinite.

It is clear that τ is finite iff μ is finite. □

<u>Matrix representations and 'diagonal' traces.</u>

4:4 **Matrix Algebras**

Consider any von Neumann subalgebra $\mathcal{M}$ of $M_n(\mathbb{C})$, the algebra of n×n matrices over $\mathbb{C}$.

We write such a matrix A as (A_{ij})

4:4.1 **Lemma**

Suppose $A_\alpha, A \in M_n(\mathbb{C})^+$

(a) $A_{ii} \geq 0 \quad \forall\, 1 \leq i \leq n$

(b) $A = 0 \Leftrightarrow A_{ii} = 0 \;\; \forall\, 1 \leq i \leq n$

(c) $A_\alpha \underset{so}{\uparrow} A \Rightarrow (A_\alpha)_{ii} \uparrow A_{ii} \quad \forall\, 1 \leq i \leq n$

Proof

Let B be the positive square root of A.

Note that $B_{ij} = \overline{B_{ji}}$

(a)

$$A_{ii} = \sum_{j=1}^{n} B_{ij} B_{ji} = \sum_{j=1}^{n} B_{ij} \overline{B_{ij}} = \sum_{j=1}^{n} |B_{ij}|^2 \geq 0$$

(b)

If $A = 0$ then certainly $A_{ii} = 0 \quad \forall\, 1 \leq i \leq n$

Suppose $A_{ii} = 0 \quad \forall\, 1 \leq i \leq n$

$$\Rightarrow \quad \sum_{j=1}^{n} |B_{ij}|^2 = 0 \quad \forall\, 1 \leq i \leq n$$

$$\Rightarrow \quad B_{ij} = 0 \quad \forall\, 1 \leq i,j \leq n$$

$$\Rightarrow \quad B = 0$$

$$\Rightarrow \quad A = 0$$

(c)

Note that $A_{ii} = \langle Ae_i, e_i \rangle$ where $\{ e_1 , \dots , e_n \}$ is the canonical base for $\mathbb{C}^n$.

Hence $A_\alpha \uparrow_{so} A$

$$\Rightarrow \quad \langle A_\alpha e_i, e_i \rangle \uparrow \langle Ae_i, e_i \rangle \quad 1 \leq i \leq n$$

$$\Rightarrow \quad (A_\alpha)_{ii} \uparrow A_{ii} \quad 1 \leq i \leq n$$

□

We define $\tau : \mathcal{U}^+ \longrightarrow [0,\infty) : A \longrightarrow \sum_{i=1}^{n} A_{ii}$

4:4.2 **Theorem**

τ is a faithful finite normal trace on $\mathcal{U}$.

Proof

By 4:4.1(a) , $\tau(A) \geq 0 \quad \forall\, A \in \mathcal{U}^+$

Linearity and homogeniety of τ are clear.

$$
\begin{aligned}
\tau(A^*A) &= \sum_{i=1}^{n} (A^*A)_{ii} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} A^*_{ij} A_{ji} \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} A_{ji} A^*_{ij} \\
&= \sum_{j=1}^{n} (AA^*)_{jj} \\
&= \tau(AA^*)
\end{aligned}
$$

So the commutativity condition follows, and τ is a trace.

$\tau(A) = 0$

$$
\begin{aligned}
&\Rightarrow \quad \sum_{i=1}^{n} A_{ii} = 0 \\
&\Rightarrow \quad A_{ii} = 0 \quad 1 \leq i \leq n \qquad (4{:}4.1(a)) \\
&\Rightarrow \quad A = 0 \quad (4{:}4.1(b))
\end{aligned}
$$

Hence τ is faithful.

Clearly τ is finite valued.

$A_\alpha \underset{so}{\uparrow} A$

$$
\begin{aligned}
&\Rightarrow \quad (A_\alpha)_{ii} \uparrow A_{ii} \quad 1 \leq i \leq n \qquad (4{:}4.1(c)) \\
&\Rightarrow \quad \sum_{i=1}^{n} (A_\alpha)_{ii} \uparrow \sum_{i=1}^{n} A_{ii} \\
&\Rightarrow \quad \tau(A_\alpha) \uparrow \tau(A)
\end{aligned}
$$

Hence τ is normal. □

4:5 Finite dimensional von Neumann Algebras

It is well known that for any finite dimensional von Neumann Algebra $\mathcal{M}$ (= finite dimensional C^*–algebra) there is an isomorphism $\mathcal{M} \longrightarrow \bigoplus_{k=1}^{m} M_{n_k}(\mathbb{C})$ for some positive integers $m, n_1, \ldots, n_m$. m is uniquely determined and the n_i's uniquely determined up to permutation.

([T] I § 11)

With $n = \sum_{k=1}^{m} n_k$, it follows that $\mathcal{M}$ is isomorphic to an algebra of n×n matrices acting on $\mathbb{C}^n$, with the usual matrix sum, scalar product, product and adjunction operations. Thus we have an injection $\mathcal{M} \hookrightarrow M_n(\mathbb{C}) : s \longrightarrow [s]$

We can exploit this isomorphism to construct a trace on $\mathcal{M}$, the *matricial diagonal trace.*

We define $\tau : \mathcal{M}^+ \longrightarrow [0,\infty) : s \longrightarrow \sum_{i=1}^{n} [s]_{ii}$. It follows from the isomorphism and 4:4.2 that τ is a faithful finite normal trace on $\mathcal{M}$.

Both of the concepts already mentioned – the representation of a von Neumann Algebra as a matrix algebra; and the idea of a diagonal trace, can profitably be generalised to the infinite dimensional case.

4:6 The von Neumann Algebra $BL(\mathcal{H})$

Suppose $\mathcal{H}$ is a Hilbert Space, and let $\{e_i\}_{i \in I}$ be an orthonormal basis for $\mathcal{H}$.

Then $s \in BL(\mathcal{H})$ can be represented as a 'matrix' $[s_{ij}]_{i,j \in I}$ with scalar entries where, in fact, $s_{ij} = \langle se_j, e_i \rangle$. It follows from simple calculations that addition, scalar multiplication,

multiplication and adjunction obey rules analogous to the finite dimensional case :–

$$(r + s)_{ij} = r_{ij} + s_{ij}$$

$$(\lambda r)_{ij} = \lambda r_{ij}$$

$$(r s)_{ij} = \sum_{k \in I} r_{ik} s_{kj}$$

$$s^*_{ij} = s^*_{ji}$$

Thus the 'matrices' corresponding to members of $BL(\mathcal{X})$ form an algebra with matrix–type operations, which justifies calling the system $\{ [s_{ij}]_{i,j \in I} : s \in BL(\mathcal{X}) \}$ *matrices.*

([KR] 2.6 : Matrix Representations)

4:6.1 **Lemma**

Suppose s_α, $s \in BL(\mathcal{X})^+$

(a) $s_{ii} \geq 0 \quad \forall i \in I$

(b) $s = 0 \Leftrightarrow s_{ii} = 0 \quad \forall i \in I$

(c) $s_\alpha \uparrow_{so} s \Rightarrow (s_\alpha)_{ii} \uparrow s_{ii} \quad \forall i \in I$

Proof

Similar to 4:4.1 □

We define $\tau : BL(\mathcal{X})^+ \longrightarrow [0,\infty] : s \longrightarrow \sum_{i \in I} s_{ii} = \sum_{i \in I} \langle se_i, e_i \rangle$

4:6.2 **Theorem**

τ is a faithful semifinite normal trace on $BL(\mathcal{X})$, the *diagonal trace* on $BL(\mathcal{X})$.

τ is finite iff $\mathcal{X}$ is of finite dimension.

Proof

τ has range in $[0,\infty]$ by 4:6.1(a) .

τ clearly satisfies the linearity and homogeniety conditions.

For $s \in BL(\mathcal{X})$, $\tau(s^* s) = \sum_{i \in I} <s^* se_i,e_i> = \sum_{i \in I} <e_i,ss^* e_i> = \sum_{i \in I} <ss^* e_i,e_i> = \tau(s s^*)$

So τ satisfies the commutativity condition, and thus is a trace.

Note that the matrix $[s_{ij}]$ representing $s \in BL(\mathcal{X})$ is dependent on the choice of orthonormal base. Nevertheless, we show that τ is defined independently of the choice of orthonormal base. If $\{f_i\}_{i \in I}$ is another such orthonormal base, then there exists a $u \in BL(\mathcal{X})_u$ such that $u e_i = f_i \ \forall i \in I$.

Then for $s \in BL(\mathcal{X})^+$,

$$
\begin{aligned}
\sum_{i \in I} <sf_i,f_i> &= \sum_{i \in I} <sue_i,ue_i> \\
&= \sum_{i \in I} <u^* sue_i,e_i> \\
&= \tau(u^* su) \\
&= \tau(s) \qquad \text{since } \tau \text{ is unitarily invariant.} \\
&= \sum_{i \in I} <se_i,e_i> .
\end{aligned}
$$

Suppose $s \in BL(\mathcal{X})^+$ and $\tau(s) = 0$

$\Rightarrow \quad s_{ii} = 0 \quad \forall i \in I$

$\Rightarrow \quad s = 0 \quad$ by 4:6.1(b)

Thus τ is faithful.

$s_\alpha \uparrow_{so} s$

$\Rightarrow \quad (s_\alpha)_{ii} \uparrow s_{ii} \quad \forall i \in I$

$\Rightarrow \quad \sum_{i \in I} (s_\alpha)_{ii} \uparrow \sum_{i \in I} s_{ii}$

$\Rightarrow \quad \tau(s_\alpha) \uparrow \tau(s)$.

Thus τ is normal.

Note that for any $p \in \mathcal{M}_p$, $\tau(p)$ = the rank of p = the Hilbert dimension of the range of p .
Suppose $0 \neq s \in \mathcal{M}^+$. Then by the spectral theorem $\exists\ \delta > 0\ \exists\ p \in \mathcal{M}_p$ such that $\delta p \leq s$.
Without loss of generality, p is of finite rank, since clearly non–zero projections have non–zero subprojections of finite rank.

Thus $0 < \tau(\delta p) = \delta \tau(p) \leq \tau(s)$, and $\tau(\delta p) < \infty$

Hence τ is semifinite.

τ is finite

iff $\tau(1) < \infty$

iff the rank of 1 is finite

iff H is of finite dimension. □

4:6.3 Note

The above construction fails for an arbitrary von Neumann algebra on $\mathcal{H}$ (i.e. a subalgebra of $BL(\mathcal{H})$) on two counts :–

1 The unitary operators u such that $u e_i = f_i \ \forall i \in I$ may not belong to $\mathcal{M}$.

2 The subprojections of finite rank required in the semifiniteness condition may not belong to $\mathcal{M}$, that is, $\mathcal{M}$ may be non–atomic.

Nevertheless it does follow that any von Neumann algebra admits a faithful normal trace.

4:7 Traces on Tensor products of von Neumann Algebras

A tensor product of von Neumann algebras can be represented as an algebra of matrices, where the entries in the matrices are operators (rather than scalars) .

Suppose $\mathcal{M}_j$ are von Neumann Algebras acting on Hilbert Spaces $\mathcal{H}_j$ (j = 1,2) . Suppose $\{e_i\}_{i \in I}$

is an orthonormal basis for $\mathcal{H}_2$. Then $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ has a representation as an algebra of matrices $[s_{ij}]_{i,j\in I}$ with $s_{ij} \in BL(\mathcal{H}_1)$ $\forall i, j \in I$

The operators s_{ij} are determined as follows :

Let $U_i : \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2 : x \longrightarrow x \otimes e_i$

so $U_i^* : \mathcal{H}_1 \otimes \mathcal{H}_2 \longrightarrow \mathcal{H}_1 : \sum_{j\in I} x_j \otimes e_j \longrightarrow x_i$

and $s_{ij} = U_i^* \, s \, U_j$

The algebraic operations on the matrices satisfy similar properties to the previous case

$$(r + s)_{ij} = r_{ij} + s_{ij}$$

$$(\lambda\, r)_{ij} = \lambda\, r_{ij}$$

$$(r\, s)_{ij} = \sum_{k\in I} r_{ik}\, s_{kj} \quad \text{in the sense of strong operator convergence}$$

$$s^*_{ij} = s_{ji}^*$$

([T] IV § 1)

4:7.1 **Lemma**

Suppose s_α , $s \in (\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)^+$

(a) $s_{ii} \geq 0 \quad \forall i \in I$

(b) $s = 0 \Leftrightarrow s_{ii} = 0 \quad \forall i \in I$

(c) $s_\alpha \uparrow_{so} s \Rightarrow (s_\alpha)_{ii} \uparrow_{so} s_{ii} \quad \forall i \in I$

Proof

Let r be the positive square root of s.

(a)

$$s_{ii} = \sum_{j\in I} r_{ij} r_{ji} = \sum_{j\in I} r_{ij} r_{ij}^* \geq 0 \quad \forall i \in I$$

(b)

Suppose $s_{ii} = 0 \ \forall i \in I$

$$\Rightarrow \quad \sum_{j\in I} r_{ij} r_{ij}^* = 0 \quad \forall i \in I$$

$$\Rightarrow \quad r_{ij} r_{ij}^* = 0 \quad \forall i, j \in I$$

$$\Rightarrow \quad r_{ij} = 0 \quad \forall i, j \in I$$

$$\Rightarrow \quad r = 0$$

$$\Rightarrow \quad s = 0$$

(c)

For $i \in I$, it is clear from 4:7.1(a) that $(s_\alpha)_{ii}$ is increasing and bounded above by s_{ii}.

For $x \in \mathcal{H}_1$

$$\| [(s_\alpha)_{ii} - s_{ii}]x\|$$

$$= \quad \|(s_\alpha - s)_{ii} x\|$$

$$= \quad \|U_i^* (s_\alpha - s) U_i x\|$$

$$\leq \quad \|(s_\alpha - s) U_i x\|$$

$$\longrightarrow \quad 0$$

Thus $(s_\alpha)_{ii} \uparrow_{so} s_{ii} \quad \forall i \in I$ □

We can generalise previous ideas as follows:–

Suppose τ is a normal trace on $\mathcal{M}_1$.

From [T] IV 1.6(iii), we have that $\mathcal{M}_1 \overline{\otimes} BL(\mathcal{H}_2) = \{ s \in BL(\mathcal{H}_1 \otimes \mathcal{H}_2) : s_{ij} \in \mathcal{M}_1 \}$.

Hence $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2 \subset \{ s \in BL(\mathcal{H}_1 \otimes \mathcal{H}_2) : s_{ij} \in \mathcal{M}_1 \}$

In particular, $s_{ii} \in \mathcal{M}_1 \quad \forall s \in \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$

By 4:7.1(a), $s_{ii} \in \mathcal{M}_1^+ \quad \forall i \in I$ if $s \in (\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)^+$

Thus we can define $\tilde{\tau} : (\mathcal{M}_1 \overline{\otimes} BL(\mathcal{H}_2))^+ \longrightarrow [0,\infty] : s \longrightarrow \sum_{i\in I} \tau(s_{ii})$.

4:7.2 **Theorem**

$\tilde{\tau}$ is a normal trace on $\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$ which we also call the *diagonal trace.*

(a) If τ is faithful then $\tilde{\tau}$ is faithful.

(b) [T] V 2.14

If τ is semifinite and $\mathcal{M}_2$ is a type–I factor then $\tilde{\tau}$ is semifinite.

Proof

Additivity and homogeniety are clear.

For $s \in \mathcal{M}_1 \overline{\otimes} \mathcal{M}_2$,

$$\tilde{\tau}(s^* s) = \sum_{i\in I} \tau((s^* s)_{ii})$$

$$= \sum_{i\in I} \tau(\sum_{j\in I} s_{ji}^* s_{ji})$$

where the summation $\sum_{j\in I} s_{ji}^* s_{ji}$ is in the sense of strong operator convergence

$$= \sum_{i\in I} \sum_{j\in I} \tau(s_{ji}^* s_{ji}) \quad \text{by the normality of } \tau$$

$$= \sum_{i\in I} \sum_{j\in I} \tau(s_{ji} s_{ji}^*) \quad \text{by the commutativity condition}$$

$$= \sum_{j\in I} \sum_{i\in I} \tau(s_{ji} s_{ji}^*) \quad \text{since all the terms are positive}$$

$$= \sum_{j\in I} \tau(\sum_{i\in I} s_{ji} s_{ji}^*) \quad \text{by the normality of } \tau$$

$$= \sum_{j\in I} \tau((ss^*)_{jj})$$

$$= \tilde{\tau}(ss^*)$$

Hence $\tilde{\tau}$ satisfies the commutativity condition.

If $0 \leq s_\alpha \uparrow s$ then $(s_\alpha)_{ii} \uparrow s_{ii}$ $\forall\, i \in I$ 4:7.1(c).

Hence $\tilde{\tau}(s) = \sum_{i \in I} \tau(s_{ii})$

$= \sum_{i \in I} \sup_\alpha \tau((s_\alpha)_{ii})$ by the normality of τ.

$= \sup_\alpha \sum_{i \in I} \tau((s_\alpha)_{ii})$ since all the terms are positive.

$= \sup_\alpha \tau(s_\alpha)$

Thus $\tilde{\tau}$ is normal.

(a)

Suppose τ is faithful.

Suppose $s \geq 0$

$\tilde{\tau}(s) = 0$

$\Rightarrow$ $\tau(s_{ii}) = 0$ $\forall i \in I$

$\Rightarrow$ $s_{ii} = 0$ $\forall i \in I$ by the faithfulness of τ

$\Rightarrow$ $s = 0$ by 4:7.1(b)

So $\tilde{\tau}$ is faithful.

(b)

Suppose τ is semifinite and $\mathcal{M}_2$ is a type–I factor.

By [T] V 1.28 we may suppose $\mathcal{M}_2 = BL(\mathcal{H}_2)$.

Thus $\mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{M}_2 = \{ s \in BL(\mathcal{H}_1 \otimes \mathcal{H}_2) : s_{ij} \in \mathcal{H}_1 \}$

We show that $p_{\tilde{\tau}} = 1$ (See 3:29)

Since $Z(\mathcal{M}_1 \mathbin{\overline{\otimes}} \mathcal{M}_2) = Z(\mathcal{M}_1) \mathbin{\overline{\otimes}} Z(\mathcal{M}_2) = Z(\mathcal{M}_1) \mathbin{\overline{\otimes}} \mathbb{C}$ ([T] IV 5.11), we have that $1 - p_{\tilde{\tau}} = p \otimes 1$ for some $p \in (Z(M))_p$.

Assume for a contradiction that $p \neq 0$.

By 3:30, choose $0 \neq q \in \mathcal{M}_p$ such that $q \leq p$ and $\tau(q) < \infty$

Let $r = \begin{bmatrix} q & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & & \ddots \end{bmatrix}$. Note that $r \in (\mathcal{M}_1 \overline{\otimes} \mathcal{M}_2)_p$

Then $0 < r \leq p \otimes 1$, since $p \otimes 1$ has matrix $p_{ij} = \delta_{ij}\, p$ ([T] IV 1.4)
and $\tilde{\tau}(r) = \tau(q) < \infty$.

Thus $r \leq p_{\tilde{\tau}}$, giving the required contradiction.

Hence $1 - p_{\tau} = 0$ and so $\tilde{\tau}$ is semifinite, by 3:33 □

It is easy to see that 4:6.2 is a special case of 4:7.2(b) : put $\mathcal{M}_1 = \mathbb{C}$, $\tau = \mathrm{id}$, $\mathcal{M}_2 = BL(H)$.

The special case follows since $BL(\mathcal{H}) \cong \mathbb{C} \overline{\otimes} BL(\mathcal{H})$.

We have noted that $\mathcal{M}_1 \overline{\otimes} BL(\mathcal{H}_2) = \{ s \in BL(\mathcal{H}_1 \otimes \mathcal{H}_2) : s_{ij} \in \mathcal{M}_1 \}$
This does not however mean that all 'I by I matrices' with entries from $\mathcal{M}_1$ are members of $\mathcal{M}_1 \overline{\otimes} BL(\mathcal{H}_2)$, as certain convergence criteria may fail to hold. If, however, $BL(\mathcal{H}_2)$ is finite dimensional (in particular $M_n(\mathbb{C})$, some $n \in \mathbb{N}$) then this does hold. We will make extensive use of von Neumann algebras $\mathcal{M} \overline{\otimes} M_n(\mathbb{C})$ with diagonal traces; we denote $\mathcal{M} \overline{\otimes} M_n(\mathbb{C})$ by $M_n(\mathcal{M})$.

4.8 Traces on factors

Recall that a factor is a von Neumann algebra $\mathcal{M}$ with $Z(\mathcal{M}) = \mathbb{C}$.
In particular, $(Z(\mathcal{M}))_p = \{0, 1\}$. Thus if τ is a normal trace on a factor, then

$s(\tau) \in \{0, 1\}$ i.e. τ is either 0 or faithful

$p_{\tau} \in \{0, 1\}$ i.e. τ is either everywhere (except at 0) infinite valued or semifinite.

It follows that a normal trace on a factor is either everywhere 0 valued, everywhere (except at 0) infinite valued, or faithful semifinite normal.

A factor is one of the following types : I_n (some $n \in \mathbb{N}$), I_∞ , II_1 , II_∞ , III .

The following are certain uniqueness results about traces on factors. Since we will not make much use of factors, we simply state these results here, with references. In the following theorems, 'unique' means 'unique up to multiplication by a positive constant' .
(If τ is finite, we suppose $\tau(1) = 1$)

4:8.1 **Theorem** [KR] 8.5.3

If $\mathcal{M}$ is a factor of type I_n , then there is a unique faithful semifinite normal trace τ on $\mathcal{M}$, which is in fact finite. $\tau(\mathcal{M}_p) = \{ 0 , \frac{1}{n} , \frac{2}{n} , \dots , 1 \}$ □

4:8.2 **Theorem** [KR] 8.5.5

If $\mathcal{M}$ is a factor of type I_∞ , then there is a unique faithful semifinite normal trace τ on $\mathcal{M}$. $\tau(\mathcal{M}_p) = \{ 0 , 1 , 2 , \dots , \infty \}$ □

4:8.3 **Theorem** [KR] 8.5.3

If $\mathcal{M}$ is a factor of type II_1 , then there is a unique faithful semifinite normal trace τ on $\mathcal{M}$, which is in fact finite. $\tau(\mathcal{M}_p) = [0,1]$. □

4:8.4 **Theorem** [KR] 8.5.5

If $\mathcal{M}$ is a factor of type II_∞ , then there is a unique faithful semifinite normal trace τ on $\mathcal{M}$. $\tau(\mathcal{M}_p) = [0,\infty]$. □

4:8.5 **Theorem** [KR] 8.5.4

If $\mathcal{M}$ is a factor of type III , then there is no faithful semifinite normal trace τ on $\mathcal{M}$. □

5 : TRACES ON FINITE von NEUMANN ALGEBRAS

In this Chapter we show that a von Neumann algebra $\mathcal{M}$ is finite iff it admits a sufficient family of faithful finite normal traces. To achieve this result we state the following two results, with references :—

5:1 **Lemma** (Akemann) [SZ] 5.14

If $F \subset \mathcal{M}_*$ is norm bounded , then the following are equivalent:—

(a) F is $\sigma(\mathcal{M}_*,\mathcal{M})$ relatively compact

(b) $\forall \{e_n\}_1^\infty \subset \mathcal{M}_p$ mutually disjoint, $\varphi(e_n) \xrightarrow{n} 0$ uniformly for $\varphi \in F$ □

5:2 **Theorem** (Ryll–Nardzewski Fixed Point Theorem) [SZ] Appendix

Suppose

X is a locally convex Hausdorff space.

$\phi \neq K \subset X$ is weakly compact convex.

$\mathcal{J} : K \to K$ is a non contracting semi–group of weakly continuous affine maps.

(non contracting means that

$\forall\, x \neq y \in K \;\exists$ defining seminorm p such that $\inf_{j \in \mathcal{J}} p(jx - jy) > 0$)

Then $\exists\, x \in K$ such that x is a fixed point for $\mathcal{J}$. □

5:3 **Proposition** [S] 2.4.2

Suppose $\mathcal{M}$ is finite, $p , p_1 , q , q_1 \in \mathcal{M}_p$

Suppose $p_1 \leq p , q_1 \leq q , p_1 \sim q_1 , p \sim q$

Then $p - p_1 \sim q - q_1$

Proof

By the comparison theorem, $\exists\, e \in (Z(\mathcal{M}))_p$ such that

$(p-p_1)\,e \precsim (q-q_1)\,e$

$(p-p_1)(1-e) \succsim (q-q_1)(1-e)$

Assume for a contradiction that $(p-p_1)\,e \prec (q-q_1)\,e$

$\Rightarrow \quad (p-p_1)\,e \sim e_1 \underset{\neq}{<} (q-q_1)\,e$ for some $e_1 \in \mathcal{M}_p$

$\Rightarrow \quad p\,e = (p-p_1)\,e + p_1\,e \sim e_1 + q_1\,e \underset{\neq}{<} (q-q_1)\,e + q_1\,e = q\,e \sim p\,e\,,$

a contradiction to the finiteness assumption on $\mathcal{M}$

So $(p-p_1)\,e \sim (q-q_1)\,e$

Likewise $(p-p_1)(1-e) \sim (q-q_1)(1-e)$

Hence $p-p_1 \sim q-q_1$ □

5:4 **Lemma** [T] V 2.2

If $\{e_n\}_1^\infty$ is an increasing sequence of projections in a finite von Neumann algebra $\mathcal{M}$, then

$e_n \precsim f \;\; \forall\, n \in \mathbb{N} \Rightarrow \bigvee_{n=1}^{\infty} e_n \precsim f$

Proof

Let $p_0 = e_1$

$p_n = e_{n+1} - e_n \;\; (n \geq 1)$, so $\{p_n\}_1^\infty$ is an orthogonal sequence of projections.

We aim to construct an orthogonal sequence of projections $\{q_n\}_1^\infty$ such that $q_n \sim p_n$,

$q_n \leq f \;\; \forall\, n \in \mathbb{N}$

This will complete the proof, since then $\bigvee_{n=1}^{\infty} e_n = \sum_{n=1}^{\infty} p_n \sim \sum_{n=1}^{\infty} q_n \leq f$

We proceed inductively.

For n=0 : $p_0 = e_1 \precsim f \Rightarrow \exists\, q_0 \leq f$ such that $q_0 \sim e_1 = p_0$

Assume that $q_0, q_1, \ldots, q_{n-1}$ have been constructed.

$\Rightarrow \quad e_n = p_0 + \ldots + p_{n-1} \sim q_0 + \ldots + q_{n-1} \equiv f_n \leq f$

$\Rightarrow \quad f_n \sim e_n \leq e_{n+1} \precsim f$

$\Rightarrow \quad \exists\, g_{n+1}$ such that $f_n \sim e_n \leq e_{n+1} \sim g_{n+1} \leq f$

$\Rightarrow \quad f_n \sim e_n \precsim g_{n+1} \leq f$

$\Rightarrow \quad \exists\, g_n$ such that $e_n \sim f_n \sim g_n \leq g_{n+1} \leq f$

$\Rightarrow \quad p_n = e_{n+1} - e_n \sim g_{n+1} - g_n \leq f - g_n \sim f - f_n$, by applying 5:3

$\Rightarrow \quad \exists\, q_n$ such that $p_n \sim q_n \leq f - f_n$, as required. □

5:5 **Lemma** [T] V 2.3

Suppose $\{e_n\}_1^\infty$ is an orthogonal sequence of projections in a finite von Neumann algebra $\mathcal{M}$.

For any sequence of projections $\{f_n\}_1^\infty$ such that $f_n \sim e_n$, $f_n \xrightarrow{us} 0$

Proof

Suppose $\{f_n\}_1^\infty$ is such a sequence of projections.

Suppose $m \leq n$

$$\bigvee_{i=m}^{n} f_i \quad = \quad f_m + \sum_{i=m+1}^{n} \Big[\bigvee_{j=m}^{i} f_j - \bigvee_{j=m}^{i-1} f_j \Big]$$

$$\sim \quad f_m + \sum_{i=m+1}^{n} \Big[f_i - f_i \wedge \Big(\bigvee_{j=m}^{i-1} f_j \Big) \Big] \quad \text{by the Kaplansky formula}$$

$$\precsim \quad \sum_{i=m}^{n} e_i$$

since the terms in the sum are disjoint and each is majorised by f_i , which is equivalent to e_i

$$\leq \quad \sum_{i=m}^{\infty} e_i$$

$$\Rightarrow \quad p_m \equiv \bigvee_{i=m}^{\infty} f_i \precsim \sum_{i=m}^{\infty} e_i \quad \text{by 5:4}$$

$$\Rightarrow \quad 1 - p_m \succsim 1 - \sum_{i=m}^{\infty} e_i = \sum_{i=0}^{m-1} e_i \quad \text{where } e_0 = 1 - \sum_{i=1}^{\infty} e_i \text{, by 5:3}$$

$$\Rightarrow \quad 1 - \bigwedge_{m=1}^{\infty} p_m \geq 1 - p_m \succsim \sum_{i=0}^{m-1} e_i \quad \forall\, m \in \mathbb{N}$$

$\Rightarrow \quad 1 - \bigwedge_{m=1}^{\infty} p_m \gtrsim \sum_{i=0}^{\infty} e_i = 1$ by 5:4

$\Rightarrow \quad \bigwedge_{m=1}^{\infty} p_m = 0$ by the finiteness of $\mathcal{M}$

$\Rightarrow \quad p_m \xrightarrow{so} 0$ since $\{p_m\}_1^{\infty}$ is decreasing

$\Rightarrow \quad f_m \xrightarrow{so} 0$ since $p_m \geq f_m \geq 0 \ \forall\, m \in \mathbb{N}$

$\Rightarrow \quad f_m \xrightarrow{us} 0$

since the strong and ultrastrong topologies coincide on the unit ball of $\mathcal{M}$. □

5:6 **Theorem** [T] V 2.4

The following are equivalent

(a) $\mathcal{M}$ is finite.

(b) $\mathcal{M}$ admits sufficiently many finite normal traces.

Proof

(b) ⇒ (a)

Suppose $\{\tau_i : i \in I\}$ is a sufficient family of finite normal traces.

Suppose $1 \sim p \in \mathcal{M}_p$

Then $\tau_i(p) = \tau_i(1) \ \forall_{i \in I}$

$\Rightarrow \quad \tau_i(1-p) = 0 \ \forall_{i \in I}$, by the finiteness of τ_i

$\Rightarrow \quad p = 1$ by the sufficiency of the family $\{\tau_i\}$

$\Rightarrow \quad \mathcal{M}$ is finite

(a) ⇒ (b)

Suppose $\mathcal{M}$ is finite.

For $u \in \mathcal{M}_u$, define $T_u : \mathcal{M}_* \longrightarrow \mathcal{M}_* : \gamma \longrightarrow u^* \gamma u$, where $u^* \gamma u(s) = \gamma(u^* s u)$ $(s \in \mathcal{M})$

If $u \in \mathcal{M}_u$, $\gamma \in \mathcal{M}_*$

$$
\begin{array}{ll}
 & s_i \xrightarrow{uw} s \\
\Rightarrow & u^* s_i u \xrightarrow{uw} u^* s u \\
\Rightarrow & \gamma(u^* s_i u) \longrightarrow \gamma(u^* s u) \\
\Rightarrow & u^* \gamma u(s_i) \longrightarrow u^* \gamma u(s)
\end{array}
$$

i.e. $u^* \gamma u \in \mathcal{M}_*$, so the map is well defined.

We claim that $\{T_u : u \in \mathcal{M}_u\}$ is a group of isometries on $\mathcal{M}_*$.

It is clear that

$$T_u T_v = T_{uv} \ ;$$

$$T_1 \text{ is the identity } ;$$

$$(T_u)^{-1} = T_{u^*} \ .$$

$\|T_u(\gamma)\| = \|u^* \gamma u\| = \sup\limits_{\|s\|=1} |\gamma(u^* s u)| \le \|\gamma\|$. In particular, $\|T_u\| \le 1$

$\|\gamma\| = \|T_{u^*} T_u(\gamma)\| \le \|T_{u^*}\| \, \|T_u(\gamma)\| \le \|T_u(\gamma)\|$

Hence $\|T_u \gamma\| = \|\gamma\|$, and $\{T_u : u \in \mathcal{M}_u\}$ is a group of isometries on $\mathcal{M}_*$.

Consider any $\gamma \in \mathcal{M}_*^+$ and fix it (for the time being).

For $u \in \mathcal{M}_u$, $T_u(\gamma) \in \mathcal{M}_*^+$ since $\mathcal{M}^+$ is invariant under u.

Hence $\mathcal{L}_\gamma = \{ T_u(\gamma) : u \in \mathcal{M}_u \} \subset \mathcal{M}_*^+$

Hence $\mathcal{K}_\gamma = \overline{\text{co}}^{\sigma(\mathcal{M}_*,\mathcal{M})}\, \mathcal{L}_\gamma \subset \mathcal{M}_*^{\;+}$ since $\mathcal{M}_*^{\;+}$ is a cone in $\mathcal{M}^*$ (hence convex) , and since $\mathcal{M}_*^{\;+}$ is norm closed in $\mathcal{M}^{*+}$.

We claim that $\mathcal{K}_\gamma$ is $\sigma(\mathcal{M}_*,\mathcal{M})$ compact.
Of course it suffices to show that co $\mathcal{L}_\gamma$ is $\sigma(\mathcal{M}_*,\mathcal{M})$ relatively compact.
For $u \in \mathcal{M}_u$, $\|T_u(\gamma)\| = \|\gamma\|$, hence $\mathcal{L}_\gamma$ and thus co $\mathcal{L}_\gamma$ are norm bounded.
By Akemann's result (5:1) it suffices to prove that $\forall \{e_n\}_1^\infty \subset \mathcal{M}_p$ orthogonal, $\varphi(e_n) \longrightarrow 0$ uniformly for $\varphi \in$ co $\mathcal{L}_\gamma$. But for this it suffices to prove that $\forall \{e_n\}_1^\infty \subset \mathcal{M}_p$ orthogonal, $\varphi(e_n) \longrightarrow 0$ uniformly for $\varphi \in \mathcal{L}_\gamma$, since this will then hold for convex combinations of such members, and hence for co $\mathcal{L}_\gamma$.

Assume for a contradiction that $\exists \{e_n\}_1^\infty \subset \mathcal{M}_p$ orthogonal such that $\varphi(e_n) \not\longrightarrow 0$ uniformly for $\varphi \in \mathcal{L}_\gamma$.
By taking a subsequence if neccessary, this implies

$$\exists\, \delta > 0\ \exists \{u_n\}_1^\infty \subset \mathcal{M}_u \text{ such that } |u_n^*\gamma u_n(e_n)| > \delta\ \forall n \in \mathbb{N}$$

$$\Rightarrow \quad |\gamma(u_n^* e_n u_n)| > \delta\ \forall n \in \mathbb{N}$$

By 5:5, $u_n^* e_n u_n \xrightarrow{us} 0$

$$\Rightarrow \quad u_n^* e_n u_n \xrightarrow{uw} 0$$

$$\Rightarrow \quad \gamma(u_n^* e_n u_n) \longrightarrow 0 \text{ , a contradiction.}$$

Hence $\mathcal{K}_\gamma$ is $\sigma(\mathcal{M}_*,\mathcal{M})$ – compact.

Certainly $\{T_u : u \in \mathcal{M}_u\} \mathcal{L}_\gamma = \{T_u : u \in \mathcal{M}_u\} \gamma = \mathcal{L}_\gamma$

$\Rightarrow$ $\{T_u : u \in \mathcal{M}_u\} \mathcal{K}_\gamma = \mathcal{K}_\gamma$ since the T_u are isometries.

Thus $\{T_u : u \in \mathcal{M}_u\}$ is a group of isometries on $\mathcal{K}_\gamma$.

$\{T_u : u \in \mathcal{M}_u\}$ are isometries, hence trivially non contracting.

$\{T_u : u \in \mathcal{M}_u\}$ are linear, hence trivially affine.

Suppose $\{\varphi_i : i \in I\}$ a net in $\mathcal{K}_\gamma$ and $\varphi_i \xrightarrow{\sigma(\mathcal{M}_*,\mathcal{M})} 0$

$$\Rightarrow \quad \varphi_i(s) \longrightarrow 0 \;\; \forall\, s \in \mathcal{M}$$

$$\Rightarrow \quad u^* \varphi_i u(s) = \varphi_i(u^* s u) \longrightarrow 0 \;\; \forall\, s \in \mathcal{M} \;\; \forall\, u \in \mathcal{M}_u$$

$$\Rightarrow \quad u^* \varphi_i u \xrightarrow{\sigma(\mathcal{M}_*,\mathcal{M})} 0$$

Thus $\{T_u : u \in \mathcal{M}_u\}$ are weakly continuous.

By applying the Ryll – Nardzewski fixed point theorem, (5:2),

$$\exists\, \tau_\gamma \in \mathcal{K}_\gamma \text{ such that } T_u(\tau_\gamma) = \tau_\gamma \;\; \forall\, u \in \mathcal{M}_u$$

$$\Rightarrow \quad \tau_\gamma(u^* s u) = \tau_\gamma(s) \;\; \forall\, s \in \mathcal{M} \;\; \forall\, u \in \mathcal{M}_u$$

Since $\mathcal{K}_\gamma \subset \mathcal{M}_*^{+}$ (thus τ_γ is uw–continuous and positive) it follows τ_γ is a finite normal trace.

We now show $\{\tau_\gamma : \gamma \in \mathcal{M}_*^{+}\}$ is a sufficient family of finite normal traces.

Clearly $\forall\, \gamma \in \mathcal{M}_*^{+}$ $\forall\, u \in \mathcal{M}_u$ $T_u(\gamma)_{|Z(\mathcal{M})} = \gamma_{|Z(\mathcal{M})}$

$$\Rightarrow \quad \forall\, \gamma \in \mathcal{M}_*^{+} \quad \tau_{\gamma_{|Z(\mathcal{M})}} = \gamma_{|Z(\mathcal{M})}$$

since $\tau_\gamma \in \mathcal{K}_\gamma$ is the norm limit of convex combinations of the $T_u(\gamma)$'s

$$\Rightarrow \quad \gamma(1 - s(\tau_{\gamma_{|Z(\mathcal{M})}})) = 0 \text{ and } \tau_\gamma(1 - s(\gamma_{|Z(\mathcal{M})})) = 0$$

$$\Rightarrow \quad s(\tau_\gamma) \leq 1 - (1 - s(\gamma_{|Z(\mathcal{M})})) = s(\gamma_{|Z(\mathcal{M})}) \leq 1 - (1 - s(\tau_\gamma)) = s(\tau_\gamma)$$

Thus $\{\, s(\tau_\gamma) : \gamma \in \mathcal{M}_*^{+}\} = \{\, s(\gamma_{|Z(\mathcal{M})}) : \gamma \in \mathcal{M}_*^{+}\}$

If $r \in \mathcal{M}^+$ then there exists $\gamma \in \mathcal{M}_*$ such that $\gamma(s) \neq 0$ since $<\mathcal{M}_*, \mathcal{M}>$ is a dual pair. Without loss of generality $\gamma \in \mathcal{M}_*^+$ since $\mathcal{M}_*^+$ spans $\mathcal{M}_*$ linearly.

Thus $\quad \{ \gamma : \gamma \in \mathcal{M}_*^+ \}$ is sufficient for $\mathcal{M}$

$\Rightarrow \quad \{ \gamma_{|Z(\mathcal{M})} : \gamma \in \mathcal{M}_*^+ \}$ is sufficient for $Z(\mathcal{M})$

$$\Rightarrow \quad \bigvee_{\gamma \in \mathcal{M}_*^+} s(\gamma_{|Z(\mathcal{M})}) = 1$$

$$\Rightarrow \quad \bigvee_{\gamma \in \mathcal{M}_*^+} s(\tau_\gamma) = 1 \text{, by the above calculation}$$

$\Rightarrow \quad \{\tau_\gamma : \gamma \in \mathcal{M}_*^+ \}$ is sufficient for $\mathcal{M}$ $\square$

6 : TRACES on SEMIFINITE von NEUMANN ALGEBRAS

6:1 **Theorem** (Adapted from [T] V 1.34 V 1.22 V 1.40)

Suppose $\mathcal{M}$ is semifinite and properly infinite.

$\exists\, 0 \neq p \in (Z(\mathcal{M}))_p$ such that $\mathcal{M}_p \cong \mathcal{M}_1 \overline{\otimes} BL(\mathcal{H})$ for some finite $\mathcal{M}_1$ and some Hilbert space $\mathcal{H}$.

Proof

Since $\mathcal{M}$ is semifinite, we can choose finite $0 \neq q \in \mathcal{M}_p$.

Let $\{e_i\}_{i\in I}$ be a maximal family of equivalent orthogonal projections that includes q .

The comparison theorem applied to the pair $(1 - \sum_{i\in I} e_i, q)$ shows that there exists

$p \in (Z(\mathcal{M}))_p$ such that

$$(1 - \sum_{i\in I} e_i)\, p \precsim q\, p \sim e_i\, p \quad \forall\, i \in I$$

$$(1 - \sum_{i\in I} e_i)\,(1-p) \succsim q\,(1-p) \sim e_i\,(1-p) \quad \forall\, i \in I$$

If $e_i\, p \sim (1 - \sum_{i\in I} e_i)\, p$ then $e_i \precsim 1 - \sum_{i\in I} e_i$, contradicting the maximality of $\{e_i\}_{i\in I}$.

Thus $(1 - \sum_{i\in I} e_i)\, p \prec e_i\, p \quad \forall\, i \in I.$

In particular, $p \neq 0$.

We prove that I is infinite; assume for a contradiction that I is finite.

q finite

$\Rightarrow$ e_i finite $\forall\, i \in I$ since $q \sim e_i\ \forall\, i \in I$

$\Rightarrow$ $e_i\, p$ finite $\forall\, i \in I$

$\Rightarrow$ $\sum_{i\in I} e_i\, p$ is finite (since I finite, and the finite projections form a lattice),

and $(1 - \sum_{i\in I} e_i)\, p$ is finite, since $(1 - \sum_{i\in I} e_i)\, p \prec e_i\, p$

$\Rightarrow$ $p = (1 - \sum_{i\in I} e_i)\, p + \sum_{i\in I} e_i\, p$ is finite (since the finite projections form a lattice) a contradiction to the assumption that $\mathcal{M}$ is properly infinite.

Thus I is infinite.

In particular, $\sum_{i\in I} e_i\, p \sim \sum_{\substack{i\in I\\ e_i\neq q}} e_i\, p$ since the cardinality of the sets over which the summation is taken are equal.

Assume for a contradiction that $p \neq \sum_{i\in I} e_i p$

$\Rightarrow \quad p = (1 - \sum_{i\in I} e_i)\, p + \sum_{i\in I} e_i\, p \precsim q\, p + \sum_{\substack{i\in I\\ e_i\neq q}} e_i\, p = \sum_{i\in I} e_i\, p \leq p$, so p is finite, a contradiction.

Thus $p = \sum_{i\in I} e_i\, p$

Thus $\{e_i p\}_{i\in I}$ is a family of mutually orthogonal equivalent projections such that $\sum_{i\in I} e_i p = p$, the identity in $\mathcal{M}_p$.

Let $\{u_i\}_{i\in I}$ be the family of partial isometries such that $qp \overset{u_i}{\sim} e_i p$.

Put $u_{ij} = u_i u_j^*$ $\quad i,j \in I$

Then clearly $\{u_{ij} : i,j\in I\}$ is a matrix unit in $\mathcal{M}_p$.

Hence $\mathcal{M}_p \cong \mathcal{M}_{qp} \,\overline{\otimes}\, BL(\ell^2(I))$ by [T] IV 1.8.

$\mathcal{M}_{qp}$ is finite, since q and hence qp, is finite. □

6:2 **Theorem** Adapted from [T] V 2,15

$\mathcal{M}$ semifinite $\Rightarrow$ $\mathcal{M}$ admits a semifinite normal trace.

Proof

Suppose $\mathcal{M}$ is semifinite.

If $\mathcal{M}$ is finite, then the result follows from 5:6

So suppose $\mathcal{M}$ is not finite, in which case $\mathcal{M}$ has a properly infinite factor $\mathcal{M}_q$ $(0 \neq q \in (Z(\mathcal{M}))_p)$ in its type decomposition.

By 6:1, $\exists\, 0 \neq p \in (Z(\mathcal{M}_q))_p$ such that $(\mathcal{M}_q)_p \cong \mathcal{M}_{qp} \cong \mathcal{M}_1 \mathbin{\overline{\otimes}} BL(\mathcal{K})$ for some finite von Neumann algebra $\mathcal{M}_1$ and some Hilbert space $\mathcal{K}$.

Let τ be the finite normal non–zero trace on $\mathcal{M}_1$ given by 5:6

Let $\tilde{\tau}$ be the trace on $\mathcal{M}_1 \mathbin{\overline{\otimes}} BL(\mathcal{K})$ given by 4:7.2(b), so $\tilde{\tau}$ is a non–zero semifinite normal trace on $\mathcal{M}_{qp}$.

By 3:32, this extends to a non–zero semifinite normal trace on $\mathcal{M}$. □

6:3 **Lemma** [T] V2,12

If $\{\tau_i\}_{i \in I}$ is a family of semifinite normal traces on $\mathcal{M}$ with mutually orthogonal supports, then $\tau = \sum_{i \in I} \tau_i$ is a semifinite normal trace.

Proof

The trace property is clear.

If $s_\alpha \uparrow_{so} s$ then $\tau(s_\alpha) = \sum_{i \in I} \tau_i(s_\alpha) \uparrow \sum_{i \in I} \tau_i(s) = \tau(s)$.

Thus τ is normal.

It remains to show the semifiniteness. Suppose $0 \neq r \in \mathcal{M}^+$ and $\tau(r) > 0$.

$\exists\, i \in I$ such that $s(\tau_i)\, r \neq 0$

$\Rightarrow$ $\exists\, 0 \neq t \leq s(\tau_i)\, r$ such that $\tau_i(t) < \infty$, since τ_i is semifinte.

$\Rightarrow$ $\tau(t) = \tau_i(t) < \infty$, since the supports of the $\{\tau_i\}_{i \in I}$ are orthogonal.

Now $t \leq s(\tau_i)\, r \leq r$, so τ is semifinite. □

6:4 **Theorem** [T] V2,15

The following are equivalent :

(a) $\mathcal{M}$ is semifinite.

(b) $\mathcal{M}$ admits a faithful semifinite normal trace.

Proof

(b) ⇒ (a)

Suppose $\mathcal{M}$ admits a faithful semifinite normal trace τ.

Suppose $0 \neq p \in \mathcal{M}_p$

By 3:30, $\exists\, 0 \neq q \in \mathcal{M}_p$ such that $q \leq p$ and $0 < \tau(q) < \infty$

The reduction τ_q is a faithful finite normal trace (3:31)

⇒ $\mathcal{M}_q$ is finite (5:6)

⇒ q is finite

Thus p majorises a finite q, and so $\mathcal{M}$ is semifinite.

(a) ⇒ (b)

Let $\{\tau_i\}_{i \in I}$ be a maximal family of semifinite normal traces with mutually orthogonal supports.

Let $\tau = \sum_{i \in I} \tau_i$.

It follows from 6:3 that τ is a semifinite normal trace.

To show that τ is faithful, it suffices to show that $s(\tau) = 1$.

Assume for a contradiction that $0 \neq 1 - s(\tau) = p$ (say).

Then $\mathcal{M}_p$ is semifinite, so we can apply 6:2 to construct a semifinite normal trace on $\mathcal{M}_p$.

Certainly $p \in (Z(\mathcal{M}))_p$, so by 3:32 this trace extends to a semifinite normal trace on $\mathcal{M}$ with support included in p.

This contradicts the maximality of $\{\tau_i\}_{i \in I}$.

Hence $s(\tau) = 1$ and τ is faithful. □

7 : AFFILIATED OPERATORS

In this section we wish to consider algebras of unbounded operators. Given a von Neumann algebra $\mathcal{M}$, we want to consider those unbounded operators that are 'generated by the members of $\mathcal{M}'$. *Affiliation* is the appropiate characterisation which will be developed here.

7:1 Proposition

Suppose $\mathcal{M}$ is a von Neumann algebra on a Hilbert Space $\mathcal{H}$, and S an (unbounded) operator on $\mathcal{H}$.

The following are equivalent

(a) $\forall\, r \in \mathcal{M}' \quad r\,S \subset S\,r$

(b) $\forall\, u \in \mathcal{M}'_u \quad u\,S \subset S\,u$

(c) $\forall\, u \in \mathcal{M}'_u \quad u\,S = S\,u$

(d) $\forall\, u \in \mathcal{M}'_u \quad u^*\,S\,u = S$

Proof

(a) $\Rightarrow$ (b)

Clear.

(b) $\Rightarrow$ (c)

Suppose that $\forall\, u \in \mathcal{M}'_u \quad u\,S \subset S\,u$

$\Rightarrow \quad D(S) = D(u\,S) \subset D(S\,u) = u^*\,D(S)$.

$\Rightarrow \quad u\,D(S) \subset D(S)$.

By repeating the argument with u replaced by u^*, it follows that $u^*\,D(S) \subset D(S)$.

Hence $D(u\,S) = D(S\,u)$, and thus $u\,S = S\,u$.

(c) $\Leftrightarrow$ (d)

Clear.

(c) $\Rightarrow$ (a)

Suppose that $\forall\, u \in \mathcal{M}'_u \quad u\,S = S\,u$.

Suppose $r \in \mathcal{M}'$. Then $r = \sum_{i=1}^{4} \alpha_i u_i$, say, $u_i \in \mathcal{M}'_u \quad 1 \le i \le 4$ [KR] 4.1.7.

$$
\begin{aligned}
\text{Then } D(S\,r) &= \{ x \in \mathcal{H} : rx \in D(S) \} \\
&= \{ x \in \mathcal{H} : \sum_{i=1}^{4} \alpha_i u_i x \in D(S) \} \\
&\supset \bigcap_{i=1}^{4} \{ x \in \mathcal{H} : u_i x \in D(S) \} \quad \text{since } D(S) \text{ is a vector space.} \\
&= \bigcap_{i=1}^{4} D(S\,u_i) \\
&= \bigcap_{i=1}^{4} D(u_i\,S) \\
&= D(S) \\
&= D(r\,S)
\end{aligned}
$$

For $x \in D(r\,S)$, $rSx = \sum_{i=1}^{4} \alpha_i u_i Sx = \sum_{i=1}^{4} \alpha_i S u_i x = S\,(\sum_{i=1}^{4} \alpha_i u_i) x = Srx$.

$\Rightarrow \qquad r\,S \subset S\,r$. □

7:2 **Definition**

An operator S on $\mathcal{H}$ satisfying the equivalent conditions of the above proposition (with respect to a von Neumann algebra $\mathcal{M}$) is said to be *affiliated* to $\mathcal{M}$, notated $S\ \eta\ \mathcal{M}$.

We denote by $\overline{\mathcal{M}}$ the set of closed densely defined operators affiliated with $\mathcal{M}$.

7:3 **Note**

s bounded and $s\ \eta\ \mathcal{M} \Leftrightarrow$ s bounded and $s \in \overline{\mathcal{M}} \Leftrightarrow s \in \mathcal{M}$.

Proof

Bounded operators are always closed and densely defined (in fact, everywhere defined) . Hence the result is immediate from the double commutant theorem. □

7:4 **Theorem** [KR] § 5.6 ; [MvN1] 4.1

Suppose $0 \leq S$ is an operator acting on a Hilbert Space $\mathcal{H}$.

(a) S is affiliated with some von Neumann algebra $\mathcal{M}_0$.

There is a resolution of the identity $\{ e_t(S) : t \geq 0 \}$ in $\mathcal{M}_0$ such that $S = \int_0^\infty t \, de_t(S)$.

(b) Suppose $\mathcal{M}$ is a von Neumann algebra acting on $\mathcal{H}$.

Then $S \, \eta \, \mathcal{M} \Leftrightarrow \{ e_t(S) : t \geq 0 \} \subset \mathcal{M}$.

7:5 **Corollary**

Suppose $0 \leq S \in \overline{\mathcal{M}}$, and f is a positive Borel measurable function, so f(S) is a member of the Functional Calculus for S . Then $f(S) \in \overline{\mathcal{M}}$

Proof

f(S) is certainly closed and densely defined.

Now the spectral family for S generates the spectral family for f(S) , so it follows from 7:4(b) that $f(S) \, \eta \, \mathcal{M}$. □

7:6 **Proposition**

(a) If $R, S \, \eta \, \mathcal{M}$ then $R + S \, \eta \, \mathcal{M}$ and $R S \, \eta \, \mathcal{M}$.

(b) If S is preclosed and $S \, \eta \, \mathcal{M}$ then $\overline{S} \, \eta \, \mathcal{M}$.

(c) if S is densely defined and $S \, \eta \, \mathcal{M}$ then $S^* \, \eta \, \mathcal{M}$.

(d) if S is closed and densely defined with polar decomposition $S = v|S|$, then $S \, \eta \, \mathcal{M}$ iff $v \in \mathcal{M}$ and $|S| \, \eta \, \mathcal{M}$. In this case $R(|S|) \sim R(S)$.

(e) [SZ] 9.7

Suppose S is a closed operator. Then Gr(S) is closed, so it is identified with an orthogonal projection in $\mathcal{H}_2$.

$S \, \eta \, \mathcal{M}$ iff $Gr(S) \in M_2(\mathcal{M})$.

Proof

(a)

Suppose $R, S \, \eta \, \mathcal{M}$ and $u \in \mathcal{M}'_u$

$u(R + S) = uR + uS = Ru + Su = (R + S)u.$

So $R + S \, \eta \, \mathcal{M}$.

$u(RS) = (uR)S = (Ru)S = R(uS) = R(Su) = (RS)u.$

So $RS \, \eta \, \mathcal{M}$

(b)

Suppose S is preclosed, $S \, \eta \, \mathcal{M}$ and $u \in \mathcal{M}'_u$

Certainly $u^* S u$ is preclosed and $\overline{u^* S u} = u^* \overline{S} u$.

Suppose $x \in D(\overline{S})$.

Choose $(x_n) \subset D(S)$ such that $(x_n, Sx_n) \to (x, \overline{S}x)$

$\Rightarrow \quad (x_n, u^* Sux_n) = (x_n, Sx_n) \to (x, \overline{S}x)$

$\Rightarrow \quad x \in D(u^* \overline{S} u)$ and $\overline{u^* S u} x = \overline{S} x$.

i.e. $\overline{S} \subset u^* \overline{S} u$

$\Rightarrow \quad \overline{S} \, \eta \, \mathcal{M}$.

(c)

Suppose S is densely defined, thus S^* exists.

Suppose $S \, \eta \, \mathcal{M}$ and $u \in \mathcal{M}'_u$

Then $u S^* \subset (S u^*)^* = (u^* S)^* = S^* u$.

Hence $S^* \, \eta \, \mathcal{M}$.

(d)

Suppose S is closed and densely defined, with polar decomposition $S = v\,|S|$.

$S\ \eta\ \mathcal{M}$ $\Rightarrow$ $\forall\, u \in \mathcal{M}'_u\ \ u^* S u = S$.

$\Rightarrow$ $\forall\, u \in \mathcal{M}'_u\ \ (u^* v\, u)\,(u^* |S|\, u) = u^* v\, |S|\, u = u^* S\, u = S$

$\Rightarrow$ $\forall\, u \in \mathcal{M}'_u\ \ u^* v\, u = v$ and $u^* |S|\, u = |S|$, by the uniqueness of the polar decomposition.

$\Rightarrow$ $v \in \mathcal{M}$, by 7:3, and $|S|\ \eta\ \mathcal{M}$.

The converse follows by 7:3, and (a) .

It is clear that $R(|S|) \overset{v}{\sim} R(S)$ by definition of v.

(e)

Suppose S is closed.

$S\ \eta\ \mathcal{M}$

$\Leftrightarrow$ $\forall\, r \in \mathcal{M}'\ \ r\,S \subset S\,r$

$\Leftrightarrow$ $\forall\, r \in \mathcal{M}'\ \ \forall\, x \in D(r\,S) = D(S)\quad rx \in D(S)$ and $Srx = rSx$.

$\Leftrightarrow$ $\forall\, r \in \mathcal{M}'\ \ \forall\, (x,Sx) \in Gr(S)\quad (rx,rSx) \in Gr(S)$.

$\Leftrightarrow$ $\forall\, r \in \mathcal{M}'\ \ \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} Gr(S) \subset Gr(S)$.

$\Leftrightarrow$ $\forall\, r \in \mathcal{M}'\ \ Gr(S) \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} Gr(S) = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} Gr(S)$

$\Leftrightarrow$ $Gr(S) \in \left\{ \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} : r \in \mathcal{M}' \right\}'$

$\Leftrightarrow$ $Gr(S) \in M_2(\mathcal{M})''$ since $M_2(\mathcal{M})' = \left\{ \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} : r \in \mathcal{M}' \right\}$

$\Leftrightarrow$ $Gr(S) \in M_2(\mathcal{M})$. □

7:7 **Corollary**

Suppose S is a closed densely defined operator and $S = v\,|S|$ is its polar decomposition.

Then $S \in \overline{\mathcal{M}} \Leftrightarrow v \in \mathcal{M}$ and $\{\, e_t(|S|) : t \geq 0 \,\} \subset \mathcal{M}$.

Proof

By 7:4(b) and 7:6(d). □

7:8 **Proposition**

(a) $S \in \overline{\mathcal{M}} \Rightarrow S^* \in \overline{\mathcal{M}}$.

(b) $S \in \overline{\mathcal{M}} \Rightarrow |S| \in \overline{\mathcal{M}}$.

Proof

(a)

Adjoints are always closed.

Since S is closed, S^* is densely defined. [KR] 2.7.8

$S^* \,\eta\, \mathcal{M}$ by 7:6(c) . Hence $S^* \in \overline{\mathcal{M}}$.

(b)

$|S|$ is closed (it is self–adjoint) and densely defined.

$|S| \,\eta\, \mathcal{M}$ by 7:6(d) . Hence $|S| \in \overline{\mathcal{M}}$. □

7:9 **Proposition** [KR] 5.6.4

If (X,Σ,μ) is a σ–finite measure space, $\mathcal{M} = L_\infty(X,\Sigma,\mu)$, $\mathcal{H} = L_2(X,\Sigma,\mu)$ then

$S \in \overline{\mathcal{M}} \Leftrightarrow S$ is the operator of multiplication by g (i.e. $S = M_g$) for some $g \in L_0(X,\Sigma,\mu)$.

In fact, $\overline{\mathcal{M}} \cong L_0(X,\Sigma,\mu)$. □

The case where μ is not σ–finite is far less intuitive. For details, see [KR] 5.6.12.

The main purpose of presenting the material in Section I was to prepare for this section, where most of the concepts of Section I are generalised to (semifinite) von Neumann algebras. For this reason we have used compatible notation in the two sections.

In Section I, the space generally under consideration was $L_0(X,\Sigma,\mu)$, for (X,Σ,μ) an arbitrary measure space.

If $\mathcal{M}$ is a commutative von Neumann algebra, then we have

$\mathcal{M} \cong L_\infty(X,\Sigma,\mu)$ for some measure space (X,Σ,μ)

$\overline{\mathcal{M}} \cong L_0(X,\Sigma,\mu)$, at least in the case that μ is σ–finite, by 7:9

Furthermore, in the next chapter we will define a space $\tilde{\mathcal{M}}$ (for a semifinite von Neumann algebra $\mathcal{M}$) which in the commutative case will coincide with $\tilde{L}_\infty(X,\Sigma,\mu)$.

Most of the results of Section III appply to the space $\tilde{\mathcal{M}}$ (rather than the space $\overline{\mathcal{M}}$) for $\mathcal{M}$ a semifinite von Neumann algebra – for example, the topology of convergence in measure, the distribution function, the spectral scale, the generalised singular function, etc.

The analogous results already developed in Section I apply in general to the space $L_0(X,\Sigma,\mu)$.

Clearly if $\mu(X) = \infty$ then, barring trivialities, $L_0(X,\Sigma,\mu) \underset{\neq}{\supset} \tilde{L}_\infty(X,\Sigma,\mu)$.

Thus when generalising the results of Section I to the non–commutative case in this section, the spaces under consideration will in general be 'smaller' – we will deal mostly with $\tilde{\mathcal{M}}$ (comparable to $\tilde{L}_\infty$) rather than with $\overline{\mathcal{M}}$ (comparable to L_0) .

8 : THE ALGEBRA $\tilde{\mathcal{M}}$

We define the algebra $\tilde{\mathcal{M}}$ and the topology of convergence in measure and show that $\tilde{\mathcal{M}}$ with this topology is a complete metrisable topological *–algebra. The presentation is essentially that of [Tp] . Subsequently we show that this approach yields the same algebra of operators as the original approach of [N] .

We point out that Terp credits much of the contents of [Tp] to U. Haagerup.

We suppose throughout that $\mathcal{M}$ is a semifinite von Neumann algebra on a Hilbert space $\mathcal{H}$ and τ a faithful semifinite normal trace on $\mathcal{M}$ – see Chapter 6.

8:1 **Definition** [Tp] 1.9

A subspace E of $\mathcal{H}$ is called *τ–dense* if $\forall\ \delta > 0\ \exists\ p \in \mathcal{M}_p$ such that $p\mathcal{H} \subset E$ and $\tau(1-p) \leq \delta$

8:2 **Proposition** [Tp] 1.10

E is a τ–dense subspace of $\mathcal{H}$

$\Leftrightarrow$ $\exists$ sequence $\{p_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_p$ such that $p_n \underset{so}{\uparrow} 1$, $\tau(1-p_n) \downarrow 0$, $\bigcup_{n=1}^{\infty} p_n\mathcal{H} \subset E$

Proof

($\Leftarrow$)

Clear.

($\Rightarrow$)

For $n \in \mathbb{N}$ choose $q_n \in \mathcal{M}_p$ such that $q_n\mathcal{H} \subset E$ and $\tau(1-q_n) \leq 2^{-n}$

For $n \in \mathbb{N}$ put $p_n = \bigwedge_{i=n}^{\infty} q_i$

Then $p_n\mathcal{H} = \bigcap_{n=i}^{\infty} q_i\mathcal{H} \subset E$, so $\bigcup_{n=1}^{\infty} p_n\mathcal{H} \subset E$

Since $\{p_n\}$ is increasing, $\bigcup_{n=1}^{\infty} p_n\mathcal{H}$ is indeed a subspace.

$\tau(1-p_n) = \tau(\bigvee_{i=n}^{\infty} (1-q_i)) \leq \sum_{i=n}^{\infty} \tau(1-q_i) \leq \sum_{i=n}^{\infty} 2^{-i} = 2^{-n+1}$

Now p_n is increasing and bounded above, so it converges in the strong operator topology.

Suppose $p_n \underset{so}{\uparrow} p$

$$\begin{aligned} & 1-p \leq 1-p_n \quad \forall\, n \in \mathbb{N} \\ \Rightarrow\quad & \tau(1-p) \leq \tau(1-p_n) \leq 2^{-n+1} \quad \forall\, n \in \mathbb{N} \\ \Rightarrow\quad & \tau(1-p) = 0 \\ \Rightarrow\quad & p = 1 \text{ by the faithfulness of } \tau. \end{aligned}$$

i.e. $p_n \underset{so}{\uparrow} 1$ □

8:3 Examples

8:3.0 $\mathcal{H}$ is τ–dense for any von Neumann algebra $\mathcal{M}$.

8:3.1 If $\mathcal{M} = BL(\mathcal{H})$ then the canonical trace defined in 4:6 takes on integer or infinite values at members of $\mathcal{M}_p$, so the only τ–dense subspace is $\mathcal{H}$ itself.

8:3.2 If $\mathcal{H} = l^2$, $\mathcal{M} = l^\infty$ then the canonical trace defined in 4:2 takes on integer or infinite values at members of $\mathcal{M}_p$, so the only τ–dense subspace is l^2 itself.

8:3.3 Suppose $\mathcal{H} = L_2[0,1]$, $\mathcal{M} = L_\infty[0,1]$ and $\tau = \int dm$

Let $E = \{ f \in L_2[0,1] : \exists\, \delta > 0$ such that $f\chi_{[0,\delta]} = 0 \}$

It is clear $E \neq \mathcal{H}$ and E is a subspace of $\mathcal{H}$.

For $\delta > 0$ put $p = \chi_{(\delta,1]}$, then $p\mathcal{H} \subset E$ and $\tau(1-p) = \delta$

So E is τ–dense since δ was arbitrary.

8:3.4 Suppose $\mathcal{X} = L_2[0,1]$, $\mathcal{M} = L_\infty[0,1]$ and $\tau = \int dm$

Let $E = \{ f \in L_2[0,1] : f(0) = 0$ and f is continuous at 0 $\}$. That is, E comprises of those functions that can be m a.e. identified with such f . This is obviously equivalent to the following condition :–

$\forall \epsilon > 0 \ \exists \delta > 0$ such that $m\{ x \in [0,\delta] : |f(x)| \geq \epsilon \} = 0$

We claim that $E \neq \mathcal{X}$ but that E is τ–dense in $\mathcal{X}$.

To show that $E \neq \mathcal{X}$ it suffices to show that no $f \in E$ is m a.e. equal to $\chi_{[0,1]} \in \mathcal{X}$.

If $f \in E$, then, since f is continuous at 0, $\exists \ \delta > 0$ such that $0 \leq x < \delta \Rightarrow |f(x)| \leq 1/2$.

$\Rightarrow \quad m\{ x \in [0,1] : f(x) \neq 1 \} \geq m([0,\delta]) = \delta$, so $f \neq \chi_{[0,1]}$ m a.e.

E is clearly a subspace of $\mathcal{X}$.

For $\delta > 0$, choose $p = \chi_{(\delta,1]}$

Then $p\mathcal{X} \subset E$ and $\tau(1-p) = \tau(\chi_{[0,\delta]}) = \delta$

So E is τ–dense since δ was arbitrary.

8:3.5 Suppose $\mathcal{X} = L_2[0,1]$, $\mathcal{M} = L_\infty[0,1]$ and $\tau = \int dm$

Using 8:2 it is possible to characterise all τ–dense subspaces of $\mathcal{X}$, of which 8:3.3 and 8:3.4 are examples.

E is a τ–dense subspace of $\mathcal{X}$

$\Leftrightarrow \quad \exists$ sequence $\{p_n\}_{n \in \mathbb{N}} \subset \mathcal{M}_p$ such that $p_n \uparrow 1$, $\tau(1-p_n) \downarrow 0$, $\bigcup_{n=1}^{\infty} p_n\mathcal{X} \subset E$

$\Leftrightarrow \quad \exists$ sequence $\{E_n\}_{n \in \mathbb{N}} \subset \Sigma$ such that $E_n \uparrow X$, $\mu(X-E_n) \downarrow 0$, $\bigcup_{n=1}^{\infty} E_n \, L_2[0,1] \subset E$

$\Leftrightarrow \quad \exists$ sequence $\{E_n\}_{n \in \mathbb{N}} \subset \Sigma$ such that $E_n \uparrow X$, $\bigcup_{n=1}^{\infty} E_n \, L_2[0,1] \subset E$,

by the finiteness of μ . □

8:4 **Corollary** to 8:2 [Tp] 1.11

E τ–dense $\Rightarrow$ E dense.

Proof

In the notation of 8:2, with E τ–dense , $\{p_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_p$, $p_n \uparrow_{so} 1$

$$\Rightarrow \quad \bigvee_{n=1}^{\infty} p_n = 1$$

$$\Rightarrow \quad \mathcal{X} = [\bigcup_{n=1}^{\infty} p_n \mathcal{X}] \subset [E] \qquad \square$$

Of course the converse to 8:4 is not true. While the intersection of dense subspaces may not be dense, τ–dense subspaces are 'so dense' that, for example, the intersection of τ–dense subspaces will be τ–dense. Thus, for example, if two operators have τ–dense domains then their sum will also have a τ–dense domain. We will see that a similar result will hold for products and adjoints of such operators. Thus we may be able to define an algebra of operators with τ–dense domains. This motivates what follows.

8:5 **Definition** [Tp] 1.16

S η $\mathcal{M}$ is called *τ–premeasurable* if

$\forall\, \delta > 0 \;\; \exists\, p \in \mathcal{M}_p$ such that $p\mathcal{X} \subset D(S)$, $\|S\,p\| < \infty$, $\tau(1-p) \leq \delta$

Clearly then D(S) is τ–dense, and so S is densely defined.

8:6 **Examples**

8:6.0 All members of $\mathcal{M}$ are τ–premeasurable.

8:6.1 If $\mathcal{M} = BL(\mathcal{X})$ then the canonical trace takes on integer or infinite values at members of $\mathcal{M}_p$, so the τ–premeasurable operators have domain $\mathcal{X}$ and are bounded. Thus the set of τ–premeasurable operators is exactly $BL(\mathcal{X})$.

8:6.2 If $\mathcal{X} = l^2$, $\mathcal{M} = l^\infty$ then the same arguments as in 8:6.1 show that the set of τ–premeasurable operators is exactly l^∞.

8:6.3 Suppose $\mathcal{X} = L_2[0,1]$, $\mathcal{M} = L_\infty[0,1]$ and $\tau = \int dm$.

Suppose $g \in L_0[0,1]$, then certainly $g \in \tilde{L}_\infty[0,1]$.

Suppose $\delta > 0$

$\exists$ E $\in \Sigma$ such that $g \chi_E \in L_\infty[0,1]$ and $\mu(\chi_{[0,1]} - E) \leq \delta$

i.e. $\exists$ p $\in \mathcal{M}_p$ such that $\|M_g p\| < \infty$ and $\tau(1-p) \leq \delta$

$\Rightarrow$ M_g is τ–premeasurable.

Since $\overline{\mathcal{M}} \cong L_0[0,1]$, by 7:8, it follows that all members of $\overline{\mathcal{M}}$ are τ–premeasurable. □

8:7 **Proposition**

The following conditions are equivalent :–

(a) S is closed and τ–premeasurable.

(b) S $\in \overline{\mathcal{M}}$ and D(S) is τ–dense.

(c) S $\in \overline{\mathcal{M}}$ and $\forall\, \delta > 0$ $\exists$ p $\in \mathcal{M}_p$ such that $p\mathcal{X} \subset D(S)$ and $\tau(1-p) \leq \delta$

Proof

(a) $\Rightarrow$ (b)

S $\eta\, \mathcal{M}$, S is densely defined, S closed, so S $\in \overline{\mathcal{M}}$.

We have already noted D(S) is τ–dense.

(b) $\Rightarrow$ (c)

By the definition of τ–denseness.

(c) $\Rightarrow$ (a)

If $p \in \mathcal{M}_p$ and $p\mathcal{H} \subset D(S)$ then $S\,p$ is everywhere defined.

Suppose $x \in \mathcal{H}$, $x_n \longrightarrow x$, $Spx_n \longrightarrow y$

Then $x_n \longrightarrow x$

$\Rightarrow \quad px_n \longrightarrow px$

$\Rightarrow \quad Spx_n \longrightarrow Spx$ since S is closed.

Hence $y = Spx$, and $S\,p$ is closed.

Hence $\|S\,p\| < \infty$ by the Closed Graph Theorem. □

8:8 **Definition** [Tp] 1.14

If S satisfies the above equivalent conditions then S is said to be *τ–measurable.*

We define $\tilde{\mathcal{M}}$ to be the collection of τ–measurable operators.

8:9 **Examples**

8:9.0 $\mathcal{M} \subset \tilde{\mathcal{M}}$

8:9.1 If $\mathcal{M} = BL(\mathcal{H})$ then the τ–premeasurable operators are exactly $\mathcal{M}$. All these operators are closed, so $\tilde{\mathcal{M}} = \mathcal{M}$.

8:9.2 If $\mathcal{H} = l^2$, $\mathcal{M} = l^\infty$ then the same arguments as in 8:9.1 show that $\tilde{\mathcal{M}} = \mathcal{M}$.

8:9.3 Suppose $\mathcal{H} = L_2[0,1]$, $\mathcal{M} = L_\infty[0,1]$ and $\tau = \int dm$

Then $\tilde{\mathcal{M}} = \bar{\mathcal{M}}$. This follows by 8:6.3. □

To make a previous comment precise, we want to show $\tilde{\mathcal{M}}$ is a *–algebra. We will not be able to use the usual sums and products of unbounded operators in this algebra, for the sums and products of (closed) operators need not be closed. Thus we aim to show that $\tilde{\mathcal{M}}$ is a *–algebra with respect to strong sum (the closure of the sum), strong product (the closure of the product), and (ordinary) adjoint (since adjoints are always closed). This could be achieved directly from the definitions. However, we also want to topologise $\tilde{\mathcal{M}}$ (it will turn out that $\tilde{\mathcal{M}}$ is a complete Hausdorff topological *–algebra) and so to avoid essentially repeating arguments we proceed immediately to define the neighbourhoods of the topology with which we will be able to show that $\tilde{\mathcal{M}}$ is not only closed under the operations strong sum, strong product and adjoint, but also that these operations are continuous.

8:10 **Definition** cf. [Tp] 1.4 and 1.25

For $\epsilon, \delta > 0$

$$A(\epsilon, \delta) = \{ S \,\eta\, \mathcal{M} : \exists\, p \in \mathcal{M}_p \text{ such that } p\mathcal{H} \subset D(S),\ \|S p\| \le \epsilon,\ \tau(1-p) \le \delta \}$$

$$\bar{\mathcal{M}}(\epsilon, \delta) = A(\epsilon, \delta) \cap \bar{\mathcal{M}} = \{ S \in \bar{\mathcal{M}} : \exists\, p \in \mathcal{M}_p \text{ such that } p\mathcal{H} \subset D(S),\ \|S p\| \le \epsilon,\ \tau(1-p) \le \delta \}$$

$$\tilde{\mathcal{M}}(\epsilon, \delta) = A(\epsilon, \delta) \cap \tilde{\mathcal{M}} = \{ S \in \tilde{\mathcal{M}} : \exists\, p \in \mathcal{M}_p \text{ such that } p\mathcal{H} \subset D(S),\ \|S p\| \le \epsilon,\ \tau(1-p) \le \delta \}$$

$$\mathcal{M}(\epsilon, \delta) = A(\epsilon, \delta) \cap \mathcal{M} = \{ s \in \mathcal{M} : \exists\, p \in \mathcal{M}_p \text{ such that } \|s p\| \le \epsilon,\ \tau(1-p) \le \delta \}$$

Note that $A(\epsilon, \delta) \supset \bar{\mathcal{M}}(\epsilon, \delta) \supset \tilde{\mathcal{M}}(\epsilon, \delta) \supset \mathcal{M}(\epsilon, \delta)$.

8:11 **Note** [Tp] 1.17

S is τ–premeasurable iff $\forall\, \delta > 0\ \exists\, \epsilon > 0$ such that $S \in A(\epsilon, \delta)$.

8:12 **Proposition** [Tp] 1.6 , 1.19

Suppose $S \, \eta \, \mathcal{M}$.

(a) If S is preclosed then $S \in A(\epsilon, \delta) \Rightarrow \overline{S} \in A(\epsilon, \delta)$

(b) If S is preclosed and τ–premeasurable, then $\overline{S}$ is τ–premeasurable.

(c) If S is preclosed and τ–premeasurable, then $\overline{S} \in \tilde{\mathcal{M}}$

(d) If $S \in \tilde{\mathcal{M}}$ then $|S| \in \tilde{\mathcal{M}}$

Proof

Suppose S is preclosed and $p\mathcal{H} \subset D(S)$. Then $p\mathcal{H} \subset D(\overline{S})$ and $\overline{S}\, p = S\, p$

In particular, $\|S\, p\| = \|\overline{S}\, p\|$

(a) and (b) follow immediately from this.

(c) follows from (b) by the characterisation 8:7(a) of τ–measurability.

(d)

$$S \in \tilde{\mathcal{M}} \quad \Rightarrow \quad S \in \overline{\mathcal{M}}$$

$$\Rightarrow \quad |S| \in \overline{\mathcal{M}} \text{ by } 7{:}8(b)$$

Furthermore $D(|S|) = D(S)$

Thus $|S| \in \tilde{\mathcal{M}}$. □

8:13 **Proposition** [Tp] 1.5

For $\epsilon_1, \epsilon_2, \delta_1, \delta_2 > 0$

(a) $A(\epsilon_1, \delta_1) + A(\epsilon_2, \delta_2) \subset A(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$

(b) $A(\epsilon_1, \delta_1)\, A(\epsilon_2, \delta_2) \subset A(\epsilon_1 \epsilon_2, \delta_1 + \delta_2)$

Proof

(a)

Suppose $S_i \in A(\epsilon_i, \delta_i)$ (i = 1,2)

$\Rightarrow$ $\exists\ p_i \in \mathcal{M}_p$ such that $p_i\mathcal{X} \subset D(S_i)$, $\|S_i p_i\| \le \epsilon_i$, $\tau(1-p_i) \le \delta_i$. (i = 1,2)

Put $p = p_1 \wedge p_2$

Then $p\mathcal{X} = p_1\mathcal{X} \cap p_2\mathcal{X} \subset D(S_1) \cap D(S_2) = D(S_1 + S_2)$

and $\|(S_1 + S_2)p\| \le \|S_1 p\| + \|S_2 p\| \le \|S_1 p_1\| + \|S_2 p_2\| \le \epsilon_1 + \epsilon_2$

and $\tau(1-p) = \tau((1-p_1) \vee (1-p_2)) \le \tau(1-p_1) + \tau(1-p_2) \le \delta_1 + \delta_2$

Thus p satisfies the required conditions to show that $S_1 + S_2 \in A(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$

(b)

Suppose S_1, S_2, p_1, p_2 are as in (a).

Let q be the projection onto $N((1-p_1) S_2 p_2)$. (This null space is closed, since $S_2 p_2$, and hence $(1-p_1) S_2 p_2$, is bounded; so the definition of q makes sense.)

For $x \in q\mathcal{X}$, $(1-p_1)S_2p_2x = 0$

$\Rightarrow$ $S_2p_2x = p_1S_2p_2x$

$\Rightarrow$ $S_2p_2x \in p_1\mathcal{X}$

$\Rightarrow$ $S_2p_2x \in D(S_1)$

$\Rightarrow$ $x \in D(S_1S_2p_2)$

Put $p = p_2 \wedge q$

For $x \in p\mathcal{X}$, $x \in p_2\mathcal{X}$ and thus $x \in D(S_2)$

$S_2x = S_2p_2x$ since $p_2x = x$

$\in D(S_1)$ since $x \in q\mathcal{X}$, by the above calculation.

$\Rightarrow$ $x \in D(S_1S_2)$, and so it follows that $p\mathcal{X} \subset D(S_1S_2)$

$S_2\,p_2\,q = p_1\,S_2\,p_2\,q$ by definition of q

$\Rightarrow \quad S_2\,p_2\,p = p_1\,S_2\,p_2\,p$

$\Rightarrow \quad S_2\,p = p_1\,S_2\,p_2\,p$

$\Rightarrow \quad S_1\,S_2\,p = S_1\,p_1\,S_2\,p_2\,p$

$\Rightarrow \quad \|S_1\,S_2\,p\| = \|S_1\,p_1\,S_2\,p_2\,p\| \le \|S_1\,p_1\|\,\|S_2\,p_2\|\,\|p\| \le \epsilon_1\,\epsilon_2$

$1-q = 1 - N((1-p_1)\,S_2\,p_2) = R(((1-p_1)\,S_2\,p_2)^*) \sim R((1-p_1)\,S_2\,p_2) \le 1-p_1$

$\Rightarrow \quad \tau(1-q) \le \tau(1-p_1)$

$\Rightarrow \quad \tau(1-p) = \tau((1-p_2) \vee (1-q)) \le \tau(1-p_2) + \tau(1-q) \le \tau(1-p_2) + \tau(1-p_1) \le \delta_1 + \delta_2$

Thus p satisfies the required conditions to show that $S_1\,S_2 \in A(\epsilon_1\,\epsilon_2\,,\,\delta_1 + \delta_2)$ □

8:14 **Lemma** [Tp] 1.6

Suppose S is a closed densely defined operator and $S = v\,|S|$ is its polar decomposition.

Then $\forall\,\epsilon\,,\,\delta > 0\,,\ S \in A(\epsilon\,,\,\delta) \Leftrightarrow v \in \mathcal{M}$ and $|S| \in A(\epsilon\,,\,\delta)$

Proof

Suppose $\epsilon\,,\,\delta > 0$

Suppose $S \in A(\epsilon\,,\,\delta)$

In particular, $S\,\eta\,\mathcal{M}$, so $v \in \mathcal{M}$ and $|S|\,\eta\,\mathcal{M}$ (7:6(d))

$\exists\,p \in \mathcal{M}_p$ such that $p\mathcal{H} \subset D(S)$, $\|Sp\| \le \epsilon$, $\tau(1-p) \le \delta$

Now $D(S) = D(|S|)$ and $\||S|\,p\| = \|v^*\,S\,p\| \le \|S\,p\| \le \epsilon$

So $|S| \in A(\epsilon\,,\,\delta)$

A similar argument holds if $v \in \mathcal{M}$ and $|S| \in A(\epsilon\,,\,\delta)$, and so the result follows. □

We now present a result which will be fundamental in all that follows.

8:15 **Theorem** [Tp] 1.7

Suppose $S \in \overline{\mathcal{M}}$ (respectively $S \in \widetilde{\mathcal{M}}$) and $S = v\,|S|$ is its polar decomposition.

Then for $\epsilon, \delta > 0$, $S \in \overline{\mathcal{M}}(\epsilon, \delta)$ (respectively $S \in \widetilde{\mathcal{M}}(\epsilon, \delta)$) $\Leftrightarrow$ $\tau(e_{(\epsilon,\infty)}(|S|)) \leq \delta$.

Proof

We first consider the case where $S \in \overline{\mathcal{M}}$.

($\Leftarrow$)

Suppose $\epsilon, \delta > 0$ and $\tau(e_{(\epsilon,\infty)}(|S|)) \leq \delta$

Put $p = e_\epsilon(|S|)$

Then $p\mathcal{H} \subset D(|S|)$, $\||S|\,p\| \leq \epsilon$, $\tau(1-p) = \tau(e_{(\epsilon,\infty)}(|S|)) \leq \delta$

$\Rightarrow$ $|S| \in \overline{\mathcal{M}}(\epsilon, \delta)$

Certainly $v \in \mathcal{M}$, so by 8:14, $S \in \overline{\mathcal{M}}(\epsilon, \delta)$

($\Rightarrow$)

Suppose $S \in \overline{\mathcal{M}}(\epsilon, \delta)$

$\Rightarrow$ $|S| \in \overline{\mathcal{M}}(\epsilon, \delta)$ from 8:14, since $|S| \in \overline{\mathcal{M}}$ by 7:8(b)

$\Rightarrow$ $\exists\, p \in \mathcal{M}_p$ such that $p\mathcal{H} \subset D(|S|)$, $\||S|\,p\| \leq \epsilon$, $\tau(1-p) \leq \delta$

Now $\forall\, x \in p\mathcal{H}$ $\||S|x\| = \||S|px\| \leq \epsilon\|x\|$

and $\forall\, 0 \neq x \in (e_{(\epsilon,\infty)}(|S|))\mathcal{H}$, $\||S|x\| > \epsilon\|x\|$

$\Rightarrow$ $p \wedge e_{(\epsilon,\infty)}(|S|) = 0$

$\Rightarrow$ $\tau(e_{(\epsilon,\infty)}(|S|)) \leq \tau(1-p) \leq \delta$ by 3:2(e)

$\widetilde{\mathcal{M}}(\epsilon, \delta) = \widetilde{\mathcal{M}} \cap \overline{\mathcal{M}}(\epsilon, \delta)$, so the case where $S \in \widetilde{\mathcal{M}}$ follows. □

8:16 **Corollary** [Tp] 1.8

(a)

Suppose $S \in \bar{\mathcal{M}}$.

$\forall \epsilon, \delta > 0, \ S \in \bar{\mathcal{M}}(\epsilon, \delta) \Leftrightarrow S^* \in \bar{\mathcal{M}}(\epsilon, \delta) \Leftrightarrow |S| \in \bar{\mathcal{M}}(\epsilon, \delta)$

(b)

Suppose $S \in \tilde{\mathcal{M}}$.

$\forall \epsilon, \delta > 0, \ S \in \tilde{\mathcal{M}}(\epsilon, \delta) \Leftrightarrow |S| \in \tilde{\mathcal{M}}(\epsilon, \delta)$

Proof

(a)

Suppose $S \in \bar{\mathcal{M}}$

Note that by 7:8 , S^* , $|S| \in \bar{\mathcal{M}}$

Thus $S \in \bar{\mathcal{M}}(\epsilon, \delta) \Leftrightarrow |S| \in \bar{\mathcal{M}}(\epsilon, \delta)$ is immediate from 8:15.

Let $S = v\ |S|$ be the polar decomposition of S.

Recall that $S\ S^* = v\ S^*\ S\ v^*$, and so $S\ S^*$ restricted to R(S) and $S^*\ S$ restricted to R(S^*) are unitarily equivalent, and this equivalence is implemented by v .

Thus $|S^*| = v\ |S|\ v^*$.

Furthermore, by the uniqueness of the spectral decomposition, it follows that

$e_{(t,\infty)}(|S^*|) = v\ e_{(t,\infty)}(|S|)\ v^* \quad \forall\, t > 0$

Thus for $t > 0$,

$$\tau(\ e_{(t,\infty)}(|S^*|)\ )$$
$$= \quad \tau(\ v\ e_{(t,\infty)}(|S|)\ v^*\ )$$

$$= \tau(\ (v\, e_{(t,\infty)}(|S|))\ (v\, e_{(t,\infty)}(|S|))^*\ )$$
$$= \tau(\ (v\, e_{(t,\infty)}(|S|))^*\ (v\, e_{(t,\infty)}(|S|))\ )$$ by the commutativity condition.
$$= \tau(\ e_{(t,\infty)}(|S|)\ v^*\ v\ e_{(t,\infty)}(|S|)\ )$$
$$= \tau(\ e_{(t,\infty)}(|S|)\ e_{(0,\infty)}(|S|)\ e_{(t,\infty)}(|S|)\ )$$
$$= \tau(\ e_{(t,\infty)}(|S|)\ )$$

Thus $S \in \bar{\mathcal{M}}(\epsilon\ ,\ \delta)$

$\Leftrightarrow \quad \tau(e_{(\epsilon,\infty)}(|S|)) \leq \delta$

$\Leftrightarrow \quad \tau(e_{(\epsilon,\infty)}(|S^*|)) \leq \delta$

$\Leftrightarrow \quad S^* \in \bar{\mathcal{M}}(\epsilon\ ,\ \delta)$, by 8:15

(b)

If $S \in \tilde{\mathcal{M}}$ then $|S| \in \tilde{\mathcal{M}}$ by 8:12(d) , so this follows from (a) □

8:17 **Proposition** [Tp] 1.21

Suppose $S \in \bar{\mathcal{M}}$ and $S = v\ |S|$ is the polar decomposition.

The following are equivalent :

(a) $S \in \tilde{\mathcal{M}}$

(b) $|S| \in \tilde{\mathcal{M}}$

(c) $\forall\ \delta > 0\ \ \exists\ t > 0$ such that $S \in \bar{\mathcal{M}}(t\ ,\ \delta)$

(d) $\forall\ \delta > 0\ \ \exists\ t > 0$ such that $\tau(e_{(t,\infty)}(|S|)) \leq \delta$

(e) $\tau(e_{(t,\infty)}(|S|)) \longrightarrow 0$ as $t \longrightarrow \infty$

(f) $\exists\ t > 0$ such that $\tau(e_{(t,\infty)}(|S|)) < \infty$

Proof

(a) $\Leftrightarrow$ (b)

S is τ–measurable

$\Leftrightarrow$ $S \in \overline{\mathcal{M}}$ and D(S) is τ–dense

$\Leftrightarrow$ $|S| \in \overline{\mathcal{M}}$ and $D(|S|)$ is τ–dense (by 7:8 and since $D(S) = D(|S|)$)

$\Leftrightarrow$ $|S|$ is τ–measurable

(a) $\Leftrightarrow$ (c)

Clear by 8:11

(c) $\Leftrightarrow$ (d)

8:15

(d) $\Rightarrow$ (f)

Clear

(f) $\Rightarrow$ (e)

By 3:20

(e) $\Rightarrow$ (d)

Clear □

8:18 **Corollary** [Tp] 1. Example 1

If τ is finite, then $\overline{\mathcal{M}} = \tilde{\mathcal{M}}$

Proof

By condition (f) in 8:17 □

8:19 **Lemma** [Tp] 1.20

If R , S are premeasurable then R + S and R S are premeasurable.

Proof

Suppose R , S are premeasurable.

If $\delta > 0$, $\exists\ \epsilon_R , \epsilon_S > 0$ such that $R \in A(\epsilon_R , \frac{\delta}{2})$ and $S \in A(\epsilon_S , \frac{\delta}{2})$ (8:11)

$\Rightarrow$ $R + S \in A(\epsilon_R + \epsilon_S , \delta)$ (8:13(a))

$\Rightarrow$ R + S is τ–premeasurable since δ was arbitrary. (8:11)

and $R\,S \in A(\epsilon_R\,\epsilon_S , \delta)$ (8:13(b))

$\Rightarrow$ R S is τ–premeasurable since δ was arbitrary. (8:11) □

We now want to show that $\tilde{\mathcal{M}}$ is a *–algebra with respect to strong sum and strong product. Of course strong sums and strong products are extentions of the original sums and products, and so in certain circumstances it will suffice to prove results about the original sum and product and then apply these results to the strong sum and products, provided that we have a uniqueness result about extentions of operators. This motivates the following results on uniqueness of extentions.

8:20 **Lemma** [Tp] 1.12

(a) Suppose $q \in \mathcal{M}_p$. If $\forall\ \delta > 0\ \exists\ p \in \mathcal{M}_p$ such that $\tau(1-p) \leq \delta$ and $q \wedge p = 0$, then $q = 0$.

(b) Suppose $p_1 , p_2 \in \mathcal{M}_p$. If $\forall\ \delta > 0\ \exists\ p \in \mathcal{M}_p$ such that $\tau(1-p) \leq \delta$ and $p_1 \wedge p = p_2 \wedge p$, then $p_1 = p_2$.

Proof

(a) If $q \wedge p = 0$ then $q \precsim 1-p$ by 3:2(e)

Hence $\forall\ \delta > 0$, $\tau(q) \leq \delta$

Thus $\tau(q) = 0$ and so $q = 0$ by the faithfulness of τ .

(b) Put $q = p_1 - (p_1 \wedge p_2)$

Suppose $\delta > 0$

$\exists\, p \in \mathcal{M}_p$ such that $\tau(1-p) \leq \delta$ and $p_1 \wedge p = p_2 \wedge p$

Consider $q \wedge p$

$q \wedge p \leq q = p_1 - (p_1 \wedge p_2)$ and $q \wedge p \leq p$

$\Rightarrow$ $q \wedge p \leq p_1$ and $q \wedge p \leq 1 - (p_1 \wedge p_2)$ and $q \wedge p \leq p$

$\Rightarrow$ $q \wedge p \leq p_1 \wedge p$ and $q \wedge p \leq 1 - (p_1 \wedge p_2)$

$\Rightarrow$ $q \wedge p \leq p_1 \wedge p$ and $q \wedge p \leq p_2 \wedge p$ and $q \wedge p \leq 1 - (p_1 \wedge p_2)$ since $p_1 \wedge p = p_2 \wedge p$

$\Rightarrow$ $q \wedge p \leq p_1 \wedge p_2 \wedge p$ and $q \wedge p \leq 1 - (p_1 \wedge p_2)$

$\Rightarrow$ $q \wedge p \leq p_1 \wedge p_2$ and $q \wedge p \leq 1 - (p_1 \wedge p_2)$

$\Rightarrow$ $q \wedge p = 0$

$\Rightarrow$ $q = 0$ by (a)

$\Rightarrow$ $p_1 = p_1 \wedge p_2$

$\Rightarrow$ $p_1 \leq p_2$

By symmetry $p_2 \leq p_1$, and so $p_1 = p_2$. □

8:21 **Proposition** [Tp] 1.12

Suppose $R, S \in \overline{\mathcal{M}}$.

(a) Suppose $E \subset D(R) \cap D(S)$ is τ–dense.

If $R_{|E} = S_{|E}$ then $R = S$

(b) Suppose $\forall\, \delta > 0\ \exists\, p \in \mathcal{M}_p$ such that $\tau(1-p) \leq \delta$, $p\mathcal{H} \subset D(R) \cap D(S)$, $R\,p = S\,p$

Then $R = S$.

Proof

(a)

Consider $M_2(\mathcal{M})$ with the canonical diagonal trace $\tilde{\tau}$.

Since R , S are closed and affiliated it follows from 7:6(e) that $Gr(R)$, $Gr(S) \in M_2(\mathcal{M})$

Let $\delta > 0$.

$\exists\, p \in \mathcal{M}_p$ such that $p\mathcal{H} \subset E$ and $\tau(1-p) \leq \frac{\delta}{2}$

Consider $\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \in M_2(\mathcal{M})$

$\tilde{\tau}(1-\begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}) \leq \delta$

Now $Gr(R) \wedge \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} = Gr(S) \wedge \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ since

$$(x,y) \in (Gr(R) \wedge \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix})\mathcal{H}_2$$

$\Leftrightarrow \quad x \in D(R)$, $y = Rx$, $x \in p\mathcal{H}$, $y \in p\mathcal{H}$

$\Leftrightarrow \quad x \in D(R)$, $y = Rx$, $x \in E$, $x \in p\mathcal{H}$, $y \in p\mathcal{H}$, since $p\mathcal{H} \subset E \subset D(R)$

$\Leftrightarrow \quad x \in D(S)$, $y = Sx$, $x \in p\mathcal{H}$, $y \in p\mathcal{H}$

$\Leftrightarrow \quad (x,y) \in (Gr(S) \wedge \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix})\mathcal{H}_2$

Thus $Gr(R) = Gr(S)$ by 8:20

$\Rightarrow \quad R = S$

(b)

For $n \in \mathbb{N}$ choose $q_n \in \mathcal{M}_p$ such that $\tau(1-q_n) \leq 2^{-n}$, $q_n\mathcal{H} \subset D(R) \cap D(S)$, $R\,q_n = S\,q_n$.

Let $p_n = \bigwedge_{k=n}^{\infty} p_k$. As in the proof of 8:2 , $\tau(1-p_n) \downarrow 0$, $p_n \uparrow 1$, $E = \bigcup_{n=1}^{\infty} p_n\mathcal{H}$ is a τ-dense subspace of $\mathcal{H}$.

Furthermore $E \subset D(R) \cap D(S)$ and $R_{|E} = S_{|E}$.

Hence by (a) , $R = S$. □

8:22 **Corollary** [Tp] 1.15

A τ–premeasurable operator admits at most one extension in $\tilde{\mathcal{M}}$.

Proof

If S_1 , S_2 are extensions in $\tilde{\mathcal{M}}$ of S then $D(S) \subset D(S_1) \cap D(S_2)$ is τ–dense, since S is τ–premeasurable.

Certainly $S_1{}_{|D(S)} = S = S_2{}_{|D(S)}$

Thus $S_1 = S_2$ by 8:21. □

8:23 **Theorem** [Tp] 1.24

$\tilde{\mathcal{M}}$ is a *–algebra with respect to strong sum and strong product.

Proof

We first show that if $R , S \in \tilde{\mathcal{M}}$ then

(1) $R^* \in \tilde{\mathcal{M}}$

(2) $\overline{R+S} \in \tilde{\mathcal{M}}$

(3) $\overline{RS} \in \tilde{\mathcal{M}}$

So suppose $R , S \in \tilde{\mathcal{M}}$

(1)

$R \in \tilde{\mathcal{M}} \Rightarrow \forall \delta > 0 \ \exists \epsilon > 0$ such that $R \in \overline{\mathcal{M}}(\epsilon , \delta)$ by 8:17.

$\Rightarrow \forall \delta > 0 \ \exists \epsilon > 0$ such that $R^* \in \overline{\mathcal{M}}(\epsilon , \delta)$ by 8:16.

$\Rightarrow R^* \in \tilde{\mathcal{M}}$ by 8:17

(2)

By 8:12(c) it suffices to show that R + S is preclosed and τ–premeasurable.

Now R , S , R^* and S^* are τ–premeasurable.

$\Rightarrow$ R + S is τ–premeasurable . (8:19)

Likewise $R^* + S^*$ is τ–premeasurable, in particular, it is densely defined, and so $(R^* + S^*)^*$ exists.

Recall $R = R^{**}$ and $S = S^{**}$ since R and S are both closed.

Thus $R + S = R^{**} + S^{**} \subset (R^* + S^*)^*$, and R + S is preclosed.

(3)

By 8:12(c) it suffices to show that R S is preclosed and τ–premeasurable.

Now R , S , R^* and S^* are τ–premeasurable.

$\Rightarrow$ R S is τ–premeasurable (8:19)

Likewise $S^* R^*$ is τ–premeasurable, in particular, it is densely defined, and so $(S^* R^*)^*$ exists.

$\Rightarrow$ $R S = R^{**} S^{**} \subset (S^* R^*)^*$, and thus R S is preclosed.

Now suppose $R , S , T \in \tilde{\mathcal{M}}$.

By immitating the preceding arguments, it follows that the operators

$R + S + T$; $R S T$; $T R + T S$; $R^* + S^*$; $S^* R^*$ are all τ–premeasurable.

Hence by 8:22, each one admits exactly one extention in $\tilde{\mathcal{M}}$.

It follows that

$$\overline{\overline{R+S} + T} = \overline{R + \overline{S+T}}$$

since both are extentions of $R + S + T$;

$$\overline{\overline{RS}\ T} = \overline{R\ \overline{ST}}$$

since both are extentions of $R\,S\,T$;

$$\overline{\overline{R+S}\ T} = \overline{\overline{RT} + \overline{ST}}$$

since both are extentions of $R\,T + S\,T$

(note that $R\,T + S\,T = (R + S)\,T$) ;

$$\overline{T\ \overline{R+S}} = \overline{\overline{TR} + \overline{TS}}$$

since both are extentions of $T\,R + T\,S$

(note that $T\,R + T\,S \subset T\,(R + S)$) ;

$$\overline{R+S}^{\,*} = \overline{R^{*} + S^{*}}$$

since both are extensions of $R^{*} + S^{*}$

(note that $R^{*} + S^{*} \subset (R + S)^{*}$,

and $(R + S)^{*} = \overline{R+S}^{\,*}$ since $R + S$ is preclosed [KR] 2.7.8) ;

$$\overline{RS}^{\,*} = \overline{S^{*}R^{*}}$$

since both are extensions of $S^{*}\,R^{*}$

(note that $S^{*}\,R^{*} \subset (R\,S)^{*}$,

and $(R\,S)^{*} = \overline{RS}^{\,*}$ since $R\,S$ is preclosed) .

The result follows. □

8:24 **Lemma** [Tp] 1.26

Suppose $\epsilon , \epsilon_1 , \epsilon_2 , \delta , \delta_1 , \delta_2 > 0$ and $0 \neq \lambda \in \mathbb{C}$.

(a) $\tilde{\mathcal{M}}(\epsilon , \delta)^* = \tilde{\mathcal{M}}(\epsilon , \delta)$.

(b) $\{ |R| : R \in \tilde{\mathcal{M}}(\epsilon , \delta) \} = \tilde{\mathcal{M}}(\epsilon , \delta)^+ \equiv \{ R \in \tilde{\mathcal{M}}(\epsilon , \delta) : R \geq 0 \}$.

(c) $\tilde{\mathcal{M}}(|\lambda| \epsilon , \delta) = \lambda \tilde{\mathcal{M}}(\epsilon , \delta)$.

(d) $\epsilon_1 < \epsilon_2 , \delta_1 < \delta_2 \Rightarrow \tilde{\mathcal{M}}(\epsilon_1 , \delta_1) \subset \tilde{\mathcal{M}}(\epsilon_2 , \delta_2)$.

(e) $\tilde{\mathcal{M}}(\epsilon_1 \wedge \epsilon_2 , \delta_1 \wedge \delta_2) \subset \tilde{\mathcal{M}}(\epsilon_1 , \delta_1) \cap \tilde{\mathcal{M}}(\epsilon_2 , \delta_2)$.

(f) $\tilde{\mathcal{M}}(\epsilon_1 , \delta_1) + \tilde{\mathcal{M}}(\epsilon_2 , \delta_2) \subset \tilde{\mathcal{M}}(\epsilon_1 + \epsilon_2 , \delta_1 + \delta_2)$ (strong sum) .

(g) $\tilde{\mathcal{M}}(\epsilon_1 , \delta_1) \tilde{\mathcal{M}}(\epsilon_2 , \delta_2) \subset \tilde{\mathcal{M}}(\epsilon_1 \epsilon_2 , \delta_1 + \delta_2)$ (strong product) .

Proof

(a)

$R \in \tilde{\mathcal{M}} \Leftrightarrow R^* \in \tilde{\mathcal{M}}$ (8:23)

$R \in \bar{\mathcal{M}}(\epsilon , \delta) \Leftrightarrow R^* \in \bar{\mathcal{M}}(\epsilon , \delta)$ (8:16)

So $R \in \tilde{\mathcal{M}}(\epsilon , \delta) \Leftrightarrow R^* \in \tilde{\mathcal{M}}(\epsilon , \delta)$

(b)

The inclusion $\{ |R| : R \in \tilde{\mathcal{M}}(\epsilon , \delta) \} \supset \tilde{\mathcal{M}}(\epsilon , \delta)^+$ is clear.

Suppose $R \in \tilde{\mathcal{M}}(\epsilon , \delta)$

$\Rightarrow$ $R \in \tilde{\mathcal{M}}$ and $R \in \bar{\mathcal{M}}(\epsilon , \delta)$

$\Rightarrow$ $|R| \in \tilde{\mathcal{M}}$ (8:12(d)) and $|R| \in \bar{\mathcal{M}}(\epsilon , \delta)$ (8:16)

$\Rightarrow$ $|R| \in \tilde{\mathcal{M}}(\epsilon , \delta)$

(c)

$R \in \tilde{\mathcal{M}}(|\lambda|\epsilon, \delta)$

$\Leftrightarrow \quad R \in \tilde{\mathcal{M}}$ and $\exists\, p \in \mathcal{M}_p$ such that $p\mathcal{H} \subset D(R)$, $\|R\,p\| \le |\lambda|\epsilon$, $\tau(1-p) \le \delta$

$\Leftrightarrow \quad \frac{1}{\lambda} R \in \tilde{\mathcal{M}}$ and $\exists\, p \in \mathcal{M}_p$ such that $p\mathcal{H} \subset D(\frac{1}{\lambda} R) = D(R)$, $\|\frac{1}{\lambda} R\,p\| = \frac{1}{|\lambda|}\|R\,p\| \le \epsilon$,

$\tau(1-p) \le \delta$

$\Leftrightarrow \quad \frac{1}{\lambda} R \in \tilde{\mathcal{M}}(\epsilon, \delta)$

$\Leftrightarrow \quad R = \lambda \frac{1}{\lambda} R \in \lambda\, \tilde{\mathcal{M}}(\epsilon, \delta)$.

(d)

$R \in \tilde{\mathcal{M}}(\epsilon_1, \delta_1)$

$\Rightarrow \quad R \in \tilde{\mathcal{M}}$ and $\exists\, p \in \mathcal{M}_p$ such that $p\mathcal{H} \subset D(R)$, $\|R\,p\| \le \epsilon_1$, $\tau(1-p) \le \delta_1$

$\Rightarrow \quad R \in \tilde{\mathcal{M}}$ and $\exists\, p \in \mathcal{M}_p$ such that $p\mathcal{H} \subset D(R)$, $\|R\,p\| \le \epsilon_2$, $\tau(1-p) \le \delta_2$

$\Rightarrow \quad R \in \tilde{\mathcal{M}}(\epsilon_2, \delta_2)$

(e)

Follows from (d).

(f)

Suppose $R \in \tilde{\mathcal{M}}(\epsilon_1, \delta_1)$, $S \in \tilde{\mathcal{M}}(\epsilon_2, \delta_2)$

$\Rightarrow \quad R, S \in \tilde{\mathcal{M}}$ and $R \in A(\epsilon_1, \delta_1)$, $S \in A(\epsilon_2, \delta_2)$

$\Rightarrow \quad \overline{R+S} \in \tilde{\mathcal{M}}$ and $R + S \in A(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$ (8:13(a), 8:23)

$\Rightarrow \quad \overline{R+S} \in \tilde{\mathcal{M}}$ and $\overline{R+S} \in A(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$ (8:12(a))

$\Rightarrow \quad \overline{R+S} \in \tilde{\mathcal{M}}(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$

(g)

Suppose $R \in \tilde{\mathcal{M}}(\epsilon_1, \delta_1)$, $S \in \tilde{\mathcal{M}}(\epsilon_2, \delta_2)$

$\Rightarrow$ $R, S \in \tilde{\mathcal{M}}$ and $R \in A(\epsilon_1, \delta_1)$, $S \in A(\epsilon_2, \delta_2)$

$\Rightarrow$ $\overline{RS} \in \tilde{\mathcal{M}}$ and $R\,S \in A(\epsilon_1\, \epsilon_2, \delta_1 + \delta_2)$ (8:13(b), 8:23)

$\Rightarrow$ $\overline{RS} \in \tilde{\mathcal{M}}$ and $\overline{RS} \in A(\epsilon_1\, \epsilon_2, \delta_1 + \delta_2)$ (8:12(a))

$\Rightarrow$ $\overline{RS} \in \tilde{\mathcal{M}}(\epsilon_1\, \epsilon_2, \delta_1 + \delta_2)$ □

8:25 **Theorem** [Tp] 1.27

$\{ \tilde{\mathcal{M}}(\epsilon, \delta) : \epsilon>0, \delta > 0 \}$ is a neighbourhood basis at 0 for a metrisable vector topology on $\tilde{\mathcal{M}}$, the *topology of convergence in measure.*

Proof

$\{ \tilde{\mathcal{M}}(\epsilon, \delta) : \epsilon>0, \delta > 0 \}$ is a filter base :–

$\phi \notin \{ \tilde{\mathcal{M}}(\epsilon, \delta) : \epsilon>0, \delta > 0 \}$ since $0 \in \tilde{\mathcal{M}}(\epsilon, \delta)$ $\forall\, \epsilon>0$ $\forall\, \delta > 0$

The filter base property follows by 8:24(e)

Suppose $S \in \tilde{\mathcal{M}}$, $\epsilon > 0$, $\delta > 0$, $0 < |\lambda| \leq 1$

$\tilde{\mathcal{M}}(\frac{\epsilon}{2}, \frac{\delta}{2}) + \tilde{\mathcal{M}}(\frac{\epsilon}{2}, \frac{\delta}{2}) \subset \tilde{\mathcal{M}}(\epsilon, \delta)$

If $0 < |\lambda| \leq 1$ then $\lambda\, \tilde{\mathcal{M}}(\epsilon, \delta) = \tilde{\mathcal{M}}(|\lambda|\, \epsilon, \delta) \subset \tilde{\mathcal{M}}(\epsilon, \delta)$ (8:24(c) and (d))

So $\tilde{\mathcal{M}}(\epsilon, \delta)$ is balanced.

$\exists\ \epsilon_S > 0$ such that $S \in \tilde{\mathcal{M}}(\epsilon_S, \delta) = \frac{\epsilon_S}{\epsilon}\ \tilde{\mathcal{M}}(\epsilon, \delta)$ (8:19 , 8:24(c))

So $\tilde{\mathcal{M}}(\epsilon, \delta)$ is absorbing.

Thus the system is a neighbourhood basis for a vector topology, by 1:5.

$$S \in \bigcap_{\epsilon, \delta > 0} \tilde{\mathcal{M}}(\epsilon, \delta)$$

$$\begin{aligned}
&\Rightarrow \quad \forall\, \delta > 0\ \forall\, \epsilon > 0\ \ \tau(e_{(\epsilon,\infty)}(|S|)) \leq \delta \qquad (8{:}15)\\
&\Rightarrow \quad \forall\, \epsilon > 0\ \ \tau(e_{(\epsilon,\infty)}(|S|)) = 0\\
&\Rightarrow \quad \tau(e_{(0,\infty)}(|S|)) = 0 \ \text{ by the normality of } \tau\\
&\Rightarrow \quad e_{(0,\infty)}(|S|) = 0 \ \text{ by the faithfulness of } \tau\\
&\Rightarrow \quad |S| = 0\\
&\Rightarrow \quad S = 0
\end{aligned}$$

Thus the topology of convergence in measure is Hausdorff, since it is a vector topology.

By 8:24(d) it follows $\{\ \tilde{\mathcal{M}}(\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{N}\ \}$ is a (countable) neighbourhood base at 0 . Thus the topology of convergence in measure is metrisable. □

8:26 **Proposition**

The topology of convergence in measure is *solid* in the sense that

$$R, S \in \tilde{\mathcal{M}},\ 0 \leq |R| \leq |S|\ ,\ S \in \tilde{\mathcal{M}}(\epsilon, \delta) \Rightarrow R \in \tilde{\mathcal{M}}(\epsilon, \delta)$$

Proof

By 8:24(b) we may suppose $0 \leq R < S$

Suppose $\epsilon > 0$

For $0 \neq x \in e_{(\epsilon,\infty)}(R)\,\mathcal{H}$, $\|Rx\| > \epsilon$, and so $\|Sx\| > \epsilon$

and for $0 \neq x \in e_{[0,\epsilon]}(S)\,\mathcal{H}$, $\|Sx\| \leq \epsilon$

Hence $e_{[0,\epsilon]}(S) \wedge e_{(\epsilon,\infty)}(R) = 0$

$$\Rightarrow \qquad \tau(e_{(\epsilon,\infty)}(R)) \leq \tau(e_{(\epsilon,\infty)}(S)) \text{ by 3:2(e)}$$

Thus $S \in \tilde{\mathcal{M}}(\epsilon, \delta)$

$$\Rightarrow \qquad \tau(e_{(\epsilon,\infty)}(S)) \leq \delta$$

$$\Rightarrow \qquad \tau(e_{(\epsilon,\infty)}(R)) \leq \delta$$

$$\Rightarrow \qquad R \in \tilde{\mathcal{M}}(\epsilon, \delta)$$ □

8:27 **Theorem** [Tp] 1.28

$\tilde{\mathcal{M}}$ with the topology of convergence of measure is a complete metrisable topological *–algebra in which $\mathcal{M}$ is dense.

Proof

To show that $\tilde{\mathcal{M}}$ is a topological *–algebra, it remains to show that adjunction and multiplication are continuous.

Adjunction is continuous by 8:24(a) .

Suppose $R_0, S_0 \in \tilde{\mathcal{M}}$, and that $\tilde{\mathcal{M}}(\epsilon, \delta)$ is a basic neighbourhood of 0 .

Find $\epsilon_{R_0}, \epsilon_{S_0} > 0$ such that $R_0 \in \tilde{\mathcal{M}}(\epsilon_{R_0}, \frac{\delta}{6})$ and $S_0 \in \tilde{\mathcal{M}}(\epsilon_{S_0}, \frac{\delta}{6})$ (8:17)

Put $\epsilon_0 = \epsilon_{R_0} \vee \epsilon_{S_0}$. Then $R_0, S_0 \in \tilde{\mathcal{M}}(\epsilon_0, \frac{\delta}{6})$ (8:24(d)).

Put $\alpha = \sqrt{\epsilon_0^2 + \epsilon} - \epsilon_0$.

Note that this is the positive solution to the equation $\epsilon = \alpha^2 + 2\epsilon_0\alpha$.

Then for $R \in R_0 + \tilde{\mathcal{M}}(\alpha, \frac{\delta}{6})$, $S \in S_0 + \tilde{\mathcal{M}}(\alpha, \frac{\delta}{6})$:

$$\begin{aligned} R\,S - R_0\,S_0 &= (R - R_0)(S - S_0) + R_0(S - S_0) + (R - R_0)\,S_0 \\ &\in \tilde{\mathcal{M}}(\alpha, \tfrac{\delta}{6})\,\tilde{\mathcal{M}}(\alpha, \tfrac{\delta}{6}) + \tilde{\mathcal{M}}(\epsilon_0, \tfrac{\delta}{6})\,\tilde{\mathcal{M}}(\alpha, \tfrac{\delta}{6}) + \tilde{\mathcal{M}}(\alpha, \tfrac{\delta}{6})\,\tilde{\mathcal{M}}(\epsilon_0, \tfrac{\delta}{6}) \\ &\subset \tilde{\mathcal{M}}(\alpha^2, \tfrac{\delta}{3}) + \tilde{\mathcal{M}}(\epsilon_0\alpha, \tfrac{\delta}{3}) + \tilde{\mathcal{M}}(\alpha\epsilon_0, \tfrac{\delta}{3}) \\ &\subset \tilde{\mathcal{M}}(\alpha^2 + 2\epsilon_0\alpha, \delta) \\ &= \tilde{\mathcal{M}}(\epsilon, \delta) \end{aligned}$$

So $(R,S) \longrightarrow R\,S$ is continuous.

We now show that $\mathcal{M}$ is dense in $\tilde{\mathcal{M}}$. We will make use of the following argument in Chapter 9.

Suppose $R = v\,|R| \in \tilde{\mathcal{M}}$

Let $e_n = e_{[0,n]}(|R|) \in \mathcal{M}_p$

By 8:17, $\tau(1-e_n) = \tau(e_{(n,\infty)}(|R|)) \longrightarrow 0$ as $n \longrightarrow \infty$

Suppose ϵ, $\delta > 0$

Choose $N \in \mathbb{N}$ such that $n \geq N \Rightarrow \tau(1-e_n) \leq \delta$

Then the projection e_n demonstrates that $1 - e_n \in \tilde{\mathcal{M}}(\epsilon, \delta)$ for $n \geq N$

Since ϵ, δ were arbitrary, $1 - e_n \longrightarrow 0$ in $\tilde{\mathcal{M}}$ as $n \longrightarrow \infty$.

$\Rightarrow \quad e_n \uparrow 1$ in $\tilde{\mathcal{M}}$ as $n \longrightarrow \infty$

By the continuity of multiplication, $v\int_0^n \lambda\, d(e_\lambda(|R|) = v\,e_n\,|R| \longrightarrow v\,1\,|R| = R$ in $\tilde{\mathcal{M}}$.

Now $\|v\int_0^n \lambda\, d(e_\lambda(|R|)\| \leq \|\int_0^n \lambda\, d(e_\lambda(|R|)\| \leq n$, so $v\int_0^n \lambda\, d(e_\lambda(|R|) \in \mathcal{M}\ \forall n \in \mathbb{N}$

Thus $\mathcal{M}$ is dense in $\tilde{\mathcal{M}}$.

Note that since $v\,e_n\,|R| = v\,|R|\,e_n = R\,e_n$, this argument in particular shows that $\forall\, \delta > 0\ \ \exists\, p \in \mathcal{M}_p$ such that $\tau(1-p) \leq \delta$, $R\,p \in \mathcal{M}$

Finally we show that $\tilde{\mathcal{M}}$ is complete. We note that the proof here is similar to its commutative analogue, the Riesz–Weyl theorem, 1:14. To show that $\tilde{\mathcal{M}}$ is complete, it suffices to show that the completion of $\mathcal{M}$ is included in $\tilde{\mathcal{M}}$, for then the completion of $\tilde{\mathcal{M}}$ is included in $\tilde{\mathcal{M}}$ (since $\mathcal{M}$ is dense in $\tilde{\mathcal{M}}$ and thus $\mathcal{M}$ and $\tilde{\mathcal{M}}$ have the same completion). It would then follow $\tilde{\mathcal{M}}$ is its own completion, and thus complete. Since $\tilde{\mathcal{M}}$ is metrisable, it suffices to consider Cauchy sequences; so suppose $(r_n)_1^\infty \subset \mathcal{M}$ is Cauchy in $\tilde{\mathcal{M}}$.

By taking a subsequence if neccessary, we may suppose that

$$\forall\, n \in \mathbb{N} \quad r_{n+1} - r_n \in \tilde{\mathcal{M}}(2^{-(n+1)}, 2^{-n}).$$

Hence $\exists\ \{p_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_p$ such that $\|(r_{n+1}-r_n)\, p_n\| \leq 2^{-(n+1)}$ and $\tau(1-p_n) \leq 2^{-n}$ $\forall\, n \in \mathbb{N}$.

Put $q_n = \bigwedge_{i=n+1}^{\infty} p_i$. Then $\tau(1-q_n) \leq 2^{-n}$.

Suppose $m > n$ and $t \geq 1$

Then $\|(r_{m+t}-r_m)\, q_n\|$

$$\leq \sum_{k=m}^{m+t-1} \|(r_{k+1} - r_k)\, q_n\| \quad \text{by the triangle inequality}$$

$$\leq \sum_{k=m}^{m+t-1} \|(r_{k+1} - r_k)\, p_k\|$$

since for $k > n$, $q_n \leq p_k$, and so $q_n = p_k q_n$

Thus $\|(r_{k+1} - r_k)\, q_n\| = \|(r_{k+1} - r_k)\, p_k\, q_n\| \leq \|(r_{k+1} - r_k)\, p_k\|$

$$\leq \sum_{k=m}^{m+t-1} 2^{-(k+1)}$$

$$< \sum_{k=m}^{\infty} 2^{-(k+1)}$$

$$= 2^{-m}$$

Now $\bigcup_{n\in\mathbb{N}} q_n\mathcal{X}$ is a linear space as (q_n) is an increasing sequence of projections.

If $x \in \bigcup_{n\in\mathbb{N}} q_n\mathcal{X}$, then $x \in q_n\mathcal{X}$ for some $n \in \mathbb{N}$, and so $(r_m x)_{m\in\mathbb{N}}$ is Cauchy, by the above calculation. By the completeness of $\mathcal{X}$ this sequence converges. Thus we can define an operator R (clearly linear) such that $D(R) = \bigcup_{n\in\mathbb{N}} q_n\mathcal{X}$ and $Rx = \lim_m r_m x \quad \forall\, x \in D(R)$.

Suppose $n \in \mathbb{N}$.

Then for $m > n$

$$
\begin{aligned}
\|(R - r_m)q_n\| &= \sup_{\substack{x\in\mathcal{X}\\ \|x\|=1}} \|(R - r_m)q_n x\| \\
&= \sup_{\substack{x\in\mathcal{X}\\ \|x\|=1}} \|\lim_t (r_{m+t} - r_m)q_n x\| \\
&= \sup_{\substack{x\in\mathcal{X}\\ \|x\|=1}} \lim_t \|(r_{m+t} - r_m)q_n x\| \quad \text{by continuity of the norm.} \\
&\le \sup_{\substack{x\in\mathcal{X}\\ \|x\|=1}} 2^{-m}\|x\| \quad \text{since } \|(r_{m+t} - r_m)q_n\| \le 2^{-m} \\
&\le 2^{-m} \\
&\le 2^{-n}
\end{aligned}
$$

We claim R is τ–premeasurable.

For $n \in \mathbb{N}$

$q_n\mathcal{X} \subset D(R)$;

$\tau(1-q_n) \le 2^{-n}$;

$\|R\, q_n\| \le \|(R - r_{n+1})\, q_n\| + \|r_{n+1}\, q_n\| \le 2^{-n} + \|r_{n+1}\, q_n\| < \infty$.

So R is τ–premeasurable.

Now $r_{n+1}^* - r_n^* = (r_{n+1} - r_n)^* \in \tilde{\mathcal{U}}(2^{-(n+1)}, 2^{-n})^* = \tilde{\mathcal{U}}(2^{-(n+1)}, 2^{-n})$, so the above arguments can be applied to the sequence $\{r_n^*\}_{n \in \mathbb{N}}$, enabling us to define a τ–premeasurable operator S such that $Sx = \lim_m r_n^* x \ \forall x \in D(S)$.

Then $\forall x \in D(R) \ \forall y \in D(S) \ \langle Rx,y\rangle = \lim_m \langle r_m x,y\rangle = \lim_m \langle x, r_m^* y\rangle = \langle x,Sy\rangle$ by the continuity of the inner product.

$\Rightarrow \quad R \subset S^*$

$\Rightarrow \quad R$ is preclosed.

Thus R is τ–premeasurable and preclosed, so by 8:12(c) $\overline{R} \in \tilde{\mathcal{U}}$.

We now show that $r_n \longrightarrow \overline{R}$ in $\tilde{\mathcal{U}}$.

Suppose $n \in \mathbb{N}$.

Then for $m > n$

$\|(\overline{R} - r_m)\, q_n\|$

$= \quad \|(R - r_m)\, q_n\|$ since $\overline{R}\, q_n = R\, q_n$, because $q_n \mathcal{H} \subset D(R)$

$\leq \quad 2^{-n}$, as above.

and $\tau(1-q_n) \leq 2^{-n}$,

so for $m > n$ the projection q_n demonstrates that $\overline{R} - r_m \in \tilde{\mathcal{U}}(2^{-n}, 2^{-n})$,

i.e. $r_m \longrightarrow \overline{R}$.

This completes the proof. □

8:28 **Note**

Since $\mathcal{M}$ is dense in $\tilde{\mathcal{M}}$, $\tilde{\mathcal{M}}$ is the completion of $\mathcal{M}$ with the subspace topology. (Of course the subspace topology is that induced by the system $\{ \mathcal{M}(\epsilon , \delta) : \epsilon>0 , \delta > 0 \}$.) Since $\mathcal{M}$ is a *–subalgebra of $\tilde{\mathcal{M}}$, it follows from 8:27 that $\mathcal{M}$ is a metrisable topological *–algebra when equipped with this subspace topology.

8:29 **Example**

We now confirm that the choice of name for the topology defined is justified – that is, in the commutative case, this topology is the topology of convergence in measure.

Suppose $\mathcal{M} = L_\infty(X,\Sigma,\mu)$ is a commutative von Neumann algebra, $\tau = \int d\mu$.

$$\begin{aligned}
\mathcal{M}(\epsilon , \delta) &= \{f \in \mathcal{M} : \exists\, p \in \mathcal{M}_p \text{ such that } \|f\, p\| \le \epsilon , \tau(1-p) \le \delta \} \\
&= \{f \in \mathcal{M} : \exists\, E \in \Sigma \text{ such that } \|f\, \chi_E\| \le \epsilon , \mu(X-E) \le \delta \} \\
&= \{f \in \mathcal{M} : \mu\{x \in X : |f(x)| > \epsilon \} \le \delta \} \\
&= \mathcal{N}(\epsilon , \delta) \cap \mathcal{M}
\end{aligned}$$

Thus the restriction of the topology of convergence in measure to $\mathcal{M}$ is the commutative topology of convergence in measure restricted to $\mathcal{M}$.

It follows from 1:15 and 8:28 that $\tilde{\mathcal{M}} = \tilde{L}_\infty(X,\Sigma,\mu)$ i.e. the space of functions in $L_0(X,\Sigma,\mu)$ that are bounded except possibly on a set of finite measure. □

We now show that the results derived in this Chapter yield the same algebra of operators as the original approach of [N] .

The topology of convergence in measure was first defined in [N] , as a topology on $\mathcal{M}$.

8:30 **Definition** [N] p 106

For $\epsilon , \delta > 0$ let $\mathcal{M}(\epsilon , \delta) = \{ s \in \mathcal{M} : \exists\, p \in \mathcal{M}_p \text{ such that } \|s\, p\| \le \epsilon \text{ and } \tau(1-p) \le \delta \}$

We have already noted that this is a basic neighbourhood system at 0 for the subspace topology on $\mathcal{M}$ of the topology of convergence in measure.

We denote by $\{ \tilde{\mathcal{M}}_N , \tau_N \}$ the abstract completion of $\mathcal{M}$ with respect to this topology.

Thus $\tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}}_N$ are both completions of the topological *–algebra $\mathcal{M}$. By uniqueness of completions, there exists a *–algebra isomorphism $\phi : \tilde{\mathcal{M}}_N \longrightarrow \tilde{\mathcal{M}}$ such that $\phi_{|\mathcal{M}} = \text{id}$. Clearly under such conditions ϕ is uniquely determined.

Thus the approach of [Tp] gives a 'concrete' description of the abstract completion of [N] .

Recall that the members of $\tilde{\mathcal{M}}$ are densely defined operators, and that their domains are, in general, different subspaces of $\mathcal{H}$. The approach of [N] , however, views the members of $\tilde{\mathcal{M}}_N$ as operators $\tilde{\mathcal{H}} \longrightarrow \tilde{\mathcal{H}}$, for a space $\tilde{\mathcal{H}} \supset \mathcal{H}$ which we now define.

8:31 **Definition** [N] p 107

For $\epsilon , \delta > 0$ let $\mathcal{H}(\epsilon , \delta) = \{x \in \mathcal{H} : \exists \ p \in \mathcal{M}_p$ such that $\|px\| \leq \epsilon , \tau(1-p) \leq \delta \}$

Arguments of a familiar type show that $\{ \mathcal{H}(\epsilon , \delta) : \epsilon , \delta > 0 \}$ is a basic neighbourhood system at 0 for a vector topology $\tau_{\mathcal{H}}$ on H.

Note that since $\mathcal{H}(\epsilon , \delta) \supset B_\epsilon \ \forall \ \epsilon > 0$ (take p = 1 in the definition of $\mathcal{H}(\epsilon , \delta)$), it follows that $\|\cdot\| \geq \tau_{\mathcal{H}}$.

We define $\tilde{\mathcal{H}}$ to be the abstract completion of $\mathcal{H}$ with respect to this topology.

8:32 **Theorem** [N] 1

The map $\mathcal{M} \times \mathcal{X} \longrightarrow \mathcal{X} : (s,x) \longrightarrow sx$ is continuous with respect to τ_N and $\tau_{\mathcal{X}}$.

Hence it has a unique continuous extension as a mapping $\tilde{\mathcal{M}}_N \times \tilde{\mathcal{X}} \longrightarrow \tilde{\mathcal{X}}$; thus the algebra $\tilde{\mathcal{M}}_N$ has a continuous representation on $\tilde{\mathcal{X}}$.

Proof

Suppose $\epsilon_1, \epsilon_2, \delta_1, \delta_2 > 0$.

Suppose $s \in \mathcal{M}(\epsilon_1,\delta_1)$ and $x \in \mathcal{X}(\epsilon_2, \delta_2)$

$\Rightarrow \quad s^* \in \mathcal{M}(\epsilon_1, \delta_1)$ and $x \in \mathcal{X}(\epsilon_2, \delta_2)$

$\Rightarrow \quad \exists\, p_1 \in \mathcal{M}_p$ such that $\|p_1 s\| = \|s^* p_1\| \le \epsilon_1,\ \tau(1-p_1) \le \delta_1$

and $\exists\, p_2 \in \mathcal{M}_p$ such that $\|p_2 x\| \le \epsilon_2,\ \tau(1-p_2) \le \delta_2$

Let $q = N((1-p_2)s^*)$

$$\begin{aligned} \text{Then } 1-q &= 1 - N((1-p_2)s^*) \\ &= R(((1-p_2)s^*)^*) \\ &\sim R((1-p_2)s^*) \\ &\le 1-p_2 \end{aligned}$$

Let $p = p_1 \wedge q$. A familiar argument, see for example 8:13, shows that

$$\|psx\| = \|psp_2x\| = \|pp_1sp_2x\| \le \|p\|\, \|p_1 s\|\, \|p_2x\| \le \epsilon_1\epsilon_2$$

and $\tau(1-p) \le \tau(1-p_1) + \tau(1-q) \le \tau(1-p_1) + \tau(1-p_2) \le \delta_1 + \delta_2$

Hence the projection p demonstrates that $sx \in \mathcal{X}(\epsilon_1 \epsilon_2, \delta_1 + \delta_2)$.

Thus $\mathcal{M}(\epsilon_1, \delta_1)\, \mathcal{X}(\epsilon_2, \delta_2) \subset \mathcal{X}(\epsilon_1 \epsilon_2, \delta_1 + \delta_2)$.

Hence the map $\mathcal{M} \times \mathcal{X} \longrightarrow \mathcal{X} : (s,x) \longrightarrow sx$ is continuous, and so has a unique continuous extension as a mapping $\tilde{\mathcal{M}}_N \times \tilde{\mathcal{X}} \longrightarrow \tilde{\mathcal{X}}$.

Thus the algebra $\tilde{\mathcal{M}}_N$ has a continuous representation on $\tilde{\mathcal{X}}$. □

We now identify the isomorphism $\phi : \tilde{\mathcal{M}}_N \longrightarrow \tilde{\mathcal{M}}$ (such that $\phi_{|\mathcal{M}} = \text{id}$).

8:33 **Definition** [N] p 110

Suppose $S \in \tilde{\mathcal{M}}_N$.

M_S , the operator of multiplication by S , is an operator in $\mathcal{X}$, defined as follows :–

$D(M_S) = \{x \in \mathcal{X} : Sx \in \mathcal{X}\}$

For $x \in D(M_S)$, $M_S x = Sx$.

8:34 **Lemma** [N] 2(i)

$\tau_{\mathcal{X}}$ is Hausdorff.

Proof

Suppose $x \in \mathcal{X}(\epsilon , \delta)$ $\forall \epsilon , \delta > 0$.

$\Rightarrow$ $\forall n \in \mathbb{N}$ $\exists p_n \in \mathcal{M}_p$ such that $\|p_n x\| \leq 2^{-n}$, $\tau(1-p_n) \leq 2^{-n}$.

Let $q_n = \bigwedge_{k=n}^{\infty} p_k$. Then $q_n \underset{so}{\uparrow} 1$.

For $n \in \mathbb{N}$, $\|q_n x\| \leq 2^{-k}$ $\forall k \geq n$

$\Rightarrow$ $\|q_n x\| = 0$

$\Rightarrow$ $q_n x = 0$

Hence $x = \lim_n q_n x = 0$

Thus $\tau_{\mathcal{X}}$ is Hausdorff. □

8:35 **Lemma**

Suppose $S \in \tilde{\mathcal{M}}_N$.

(a) $\forall \delta > 0$ $\exists p \in \mathcal{M}_p$ such that $\tau(1-p) \leq \delta$, $S p \in \mathcal{M}$ cf. [N] 2(ii)

(b) $s \in \mathcal{M} \Rightarrow M_s = s$

(c) $p \in \mathcal{M}_p$, $S p \in \mathcal{M} \Rightarrow M_S p = S p$

Proof

(a)

Recall that the analogous result for $\tilde{\mathcal{M}}$ has already been proved (noted in 8:27) , hence

$$\forall\ \delta > 0\ \ \exists\ p \in \mathcal{M}_p \text{ such that } \tau(1-p) \leq \delta\ ,\ \phi(S)\ p \in \mathcal{M}$$

Now $\phi(S)\ p = \phi(S)\ \phi(p) = \phi(S\ p) = S\ p$, since $\phi(S\ p) \in \mathcal{M}$

(b)

Clear.

(c)

$$\begin{aligned} D(M_S\ p) &= \{\, x \in \mathcal{X} : px \in D(M_S)\,\} \\ &= \{\, x \in \mathcal{X} : Spx \in \mathcal{X}\,\} \\ &= D(S\ p) \end{aligned}$$

For $x \in D(M_S\ p)$, $M_S px = M_S(px) = Spx$. □

8:36 **Proposition** cf. [N] 4

For $S \in \tilde{\mathcal{M}}_N$, $M_S \in \tilde{\mathcal{M}}$.

Proof

Suppose $x \in \mathcal{X}$, $D(M_S) \supset x_n \longrightarrow x$, $M_S x_n \longrightarrow y \in \mathcal{X}$.

$\|\cdot\| \geq \tau_{\mathcal{X}}$

$\Rightarrow$ $x_n \longrightarrow x$, $M_S x_n \longrightarrow y$ in $\tau_{\mathcal{X}}$

and $M_S x_n = Sx_n \longrightarrow Sx = M_S x$ in $\tau_{\mathcal{X}}$

$\Rightarrow$ $y = M_S x$ since $\tau_{\mathcal{X}}$ is Hausdorff (8:34)

Thus M_S is closed.

We now show M_S is affiliated.

Suppose $u \in \mathcal{M}'_u$.

We need to show that $u\, M_S \subset M_S\, u$

$D(u\, M_S) = D(M_S) = \{x \in \mathcal{X} : Sx \in \mathcal{X}\}$

$D(M_S\, u) = \{x \in \mathcal{X} : ux \in D(M_S)\} = \{x \in \mathcal{X} : Sux \in \mathcal{X}\}$

Choose $\{s_n\} \subset \mathcal{M}$ such that $s_n \longrightarrow S$ in $\tilde{\mathcal{M}}_N$

For $x \in D(u\, M_S)$,

$$\begin{aligned} uSx &= \lim_n us_n x \\ &= \lim_n s_n ux \quad \text{since } u \in \mathcal{M}'_u \\ &= Sux \end{aligned}$$

In particular, $Sux \in \mathcal{X}$. Thus $x \in D(M_S\, u)$ and $uM_Sx = Sux = M_Sux$.

Hence $u\, M_S \subset M_S\, u$

Thus M_S is affiliated.

Suppose $\delta > 0$

By 8:35(a), $\exists\, p \in \mathcal{M}_p$ such that $\tau(1-p) \leq \delta$, $S\, p \in \mathcal{M}$

$\Rightarrow \quad D(M_S) = \{x \in \mathcal{X} : Sx \in \mathcal{X}\} \supset p\mathcal{X}$

Hence $D(M_S)$ is τ–dense, since δ was arbitrary.

Thus M_S is closed, affiliated and has τ–dense domain,

and the result follows by 8:7(b). □

8:37 **Theorem**

The unique topological *–algebra isomorphism $\phi : \tilde{\mathcal{M}}_N \longrightarrow \tilde{\mathcal{M}}$ that extends $id : \mathcal{M} \longrightarrow \mathcal{M}$ is

$\phi : \tilde{\mathcal{M}}_N \longrightarrow \tilde{\mathcal{M}} : S \longrightarrow M_S$.

Proof

Since $\tilde{\mathcal{M}}_N$ and $\tilde{\mathcal{M}}$ are both completions of $\mathcal{M}$, it suffices to show

$\{s_n\} \subset \mathcal{M}$, $s_n \longrightarrow S$ in $\tilde{\mathcal{M}}_N$, $s_n \longrightarrow R$ in $\tilde{\mathcal{M}}$ $\Rightarrow$ $M_S = R$.

Suppose $\delta > 0$

Choose $p \in \mathcal{M}_p$ such that $\tau(1-p) \leq \delta$, $p\mathcal{H} \subset D(R)$, $R\,p \in \mathcal{M}$

$s_n \longrightarrow R$ in $\tilde{\mathcal{M}}$

$\Rightarrow$ $s_n\, p \longrightarrow R\, p$ in $\tilde{\mathcal{M}}$

$\Rightarrow$ $s_n\, p \longrightarrow R\, p$ in $\mathcal{M}$ with the subspace topology of $\tilde{\mathcal{M}}$

$\Rightarrow$ $s_n\, p \longrightarrow R\, p$ in $\mathcal{M}$ with the subspace topology of $\tilde{\mathcal{M}}_N$

$\Rightarrow$ $s_n\, p \longrightarrow R\, p$ in $\tilde{\mathcal{M}}_N$

But $s_n \longrightarrow S$ in $\tilde{\mathcal{M}}_N$ $\Rightarrow$ $s_n\, p \longrightarrow S\, p$ in $\tilde{\mathcal{M}}_N$

Thus $R\, p = S\, p$

$\Rightarrow$ $M_S\, p = S\, p = R\, p$

$\Rightarrow$ $M_S = R$, by 8:21(b) , since δ was arbitrary. □

9 : The GENERALISED SINGULAR FUNCTION

In this Chapter we introduce the distribution function, the spectral scale and the generalised singular function. We will see that these are generalisations of their classical commutative analogues, discussed in Chapter 2. We will see that the generalised singular function is also a generalisation of the singular value sequence of a compact operator in $BL(\mathcal{H})$.

We suppose throughout that $\mathcal{M}$ is a semifinite von Neumann algebra and τ a faithful semifinite normal trace on $\mathcal{M}$.

9:1 **Note**

Recall that for a self–adjoint operator $S \ \eta \ \mathcal{M}$, the uniquely determined spectral family $\{ e_t(S) : t \in \mathbb{R} \}$ satisfies :–

(1) $e_t(S) \in \mathcal{M}_p \ \forall \, t \in \mathbb{R}$

(2) $t_1 \leq t_2 \Rightarrow e_{t_1}(S) \leq e_{t_2}(S)$

(3) $e_{t+\epsilon}(S) \downarrow_{so} e_t(S)$ as $\epsilon \downarrow 0$ (the family is right continuous)

(4) $e_t(S) \uparrow_{so} 1$ as $t \uparrow \infty$

(5) $e_t(S) \downarrow_{so} 0$ as $t \downarrow -\infty$

Furthermore, if $0 \leq S$ then

(6) $e_{(0,\infty)}(S) = \text{supp}(S)$

(7) $e_t(S) = 0$ for $t < 0$

9:2 **Definition** cf. [P] p 74 , [FK] 1.3

Suppose $S \in \tilde{\mathcal{M}}^{sa}$

Suppose $\{ e_t(S) : t \in \mathbb{R} \}$ is the spectral family for S .

Let $d : \mathbb{R} \longrightarrow [0,\infty] : t \longrightarrow \tau(e_{(t,\infty)}(S))$

We call this function the *distribution function* and use the notation $d_t(S)$, to indicate the dependence on the operator S . The function is well defined since the spectral family for S is uniquely determined.

Note that if $S \in \tilde{\mathcal{M}}$ then $|S| \in \tilde{\mathcal{M}}^+ \subset \tilde{\mathcal{M}}^{sa}$ and so we may consider the distribution function of $|S|$ for any $S \in \tilde{\mathcal{M}}$. In fact, some sources define the distribution for $S \in \tilde{\mathcal{M}}$ in this manner, e.g. [FK] Definition 1.3. Unfortunately this approach can lead to ambiguities in the case that S is self–adjoint but not positive.

9:3 **Proposition**

Suppose $R , S \in \tilde{\mathcal{M}}^{sa}$.

(a) $d_t(S)$ is a decreasing function.

(b) $d_t(S)$ is right continuous.

(c) $R \leq S \Rightarrow d_t(R) \leq d_t(S)$

Suppose further $\tau(1) < \infty$

(d) $d_t(S) \longrightarrow 0$ as $t \uparrow \infty$

(e) $d_t(S) \longrightarrow \tau(1)$ as $t \downarrow -\infty$

Proof

(a)

$$t_1 \leq t_2 \Rightarrow e_{t_1}(S) \leq e_{t_2}(S) \quad \text{by 9:1(2)}$$

$$\Rightarrow e_{(t_1,\infty)}(S) \geq e_{(t_2,\infty)}(S)$$

$$\Rightarrow d_{t_1}(S) = \tau(e_{(t_1,\infty)}(S)) \geq \tau(e_{(t_2,\infty)}(S)) = d_{t_2}(S) \quad \text{by the monotonicity of } \tau$$

Hence $d_t(S)$ is decreasing.

(b)

$t_i \downarrow t \quad \Rightarrow \quad e_{t_i}(S) \downarrow_{so} e_t(S)$ by 9:1(3)

$\Rightarrow \quad e_{(t_i,\infty)}(S) \uparrow_{so} e_{(t,\infty)}(S)$

$\Rightarrow \quad d_{t_i}(S) \uparrow d_t(S)$ by the normality of τ

(c)

$R \le S \quad \Rightarrow \quad e_{(-\infty,t]}(S) \wedge e_{(t,\infty)}(R) = 0 \;\; \forall\, t \in \mathbb{R}$ (analogous to the proof of 8:26)

$\Rightarrow \quad \tau(e_{(t,\infty)}(R)) \le \tau(e_{(t,\infty)}(S)) \;\; \forall\, t \in \mathbb{R}$ by 3:2(e)

i.e. $d_t(R) \le d_t(S)$

(d)

$1 - e_t(S) \downarrow_{so} 0$ as $t \uparrow \infty$, by 9:1(4)

$\Rightarrow \quad \tau(e_{(t,\infty)}(S)) \downarrow 0$ as $t \uparrow \infty$, by 3:20

(e)

Similar to (d), using 9:1(5) □

9:4 **Proposition** [P] Theorem 1

Suppose $\tau(1) < \infty$ and $S \in \tilde{\mathcal{M}}^{sa}$

For $t \in [0,\tau(1))$, the quantities

$$\inf_{\substack{p \in \mathcal{M}_p \\ p\mathcal{H} \subset D(S) \\ \tau(1-p) \le t}} \; \sup_{\substack{\|x\|=1 \\ x \in p\mathcal{H}}} \langle Sx,x \rangle = A_t \quad \text{(say)};$$

$$\inf\{\, \theta \in \mathbb{R} : d_\theta(S) \le t \,\} = B_t \quad \text{(say)};$$

are equal, and denoted by $\lambda_t(S)$. The function $\lambda_t(S) : [0,\tau(1)) \longrightarrow \mathbb{R} \cup \{\infty\}$ is called the *spectral scale* of S .

Proof

Suppose $t \in [0,\tau(1))$

Suppose $\theta \in \mathbb{R}$ and $d_\theta(S) \leq t$

$\Rightarrow \quad \tau(e_{(\theta,\infty)}(S)) \leq t$

$\Rightarrow \quad A_t \leq \sup_{\substack{\|x\|=1 \\ x \in e_\theta(S)\mathcal{X}}} \langle Sx,x \rangle$ since certainly $e_\theta(S) \subset D(S)$

$\leq \theta$ by the Spectral Theorem.

$\Rightarrow \quad A_t \leq B_t$ by taking the infimum over all admissible θ.

Let $\epsilon > 0$ be given.

Choose $q \in \mathcal{M}_p$ such that $\tau(1-q) \leq t$, $q\mathcal{X} \subset D(S)$ and $\sup_{\substack{\|x\|=1 \\ x \in q\mathcal{X}}} \langle Sx,x \rangle < A_t + \epsilon$.

Then $q \wedge e_{(A_t+\epsilon,\infty)}(S) = 0$ by the Spectral Theorem.

$\Rightarrow \quad \tau(e_{(A_t+\epsilon,\infty)}(S)) \leq \tau(1-q) \leq t$ by 3:2(e)

$\Rightarrow \quad d_{A_t+\epsilon}(S) \leq t$

$\Rightarrow \quad B_t \leq A_t + \epsilon$

$\Rightarrow \quad B_t \leq A_t$ since ϵ was arbitrary. □

9:5 **Note**

The expressions in 9:4 are valid for any von Neumann algebra containing the family $\{ e_t(S) : t \in \mathbb{R} \}$ (since the expression B_t is obviously invariant under such algebras). In particular, they hold for the commutative algebra generated by the family $\{ e_t(S) : t \in \mathbb{R} \}$.

9:6 **Note**

9:3(d) does not neccessarily hold for $d_t(S)$ $(S \in \tilde{\mathcal{M}}^{sa})$ if τ is not finite. However, if $\tau(1) < \infty$ then indeed $d_t(S) \downarrow 0$ as $t \uparrow \infty$, and hence for $t \in (0,\tau(1))$ the set $\{ \theta \in \mathbb{R} : d_\theta(S) \leq t \}$ is always non–empty. Furthermore, $d_t(S) \uparrow \tau(1)$ as $t \downarrow -\infty$ and hence for $t \in (0,\tau(1))$ the set $\{ \theta \in \mathbb{R} : d_\theta(S) \leq t \}$ is bounded below. Hence the assumption that $\tau(1) < \infty$ ensures that the spectral scale is finite valued on $(0,\tau(1))$.

Now consider the case $t = 0$.

If $s \in \mathcal{M}^{sa}$, then the only projection satisfying the requirements of the expression A_t is 1 . Hence $\lambda_0(s) = \sup_{\|x\|=1} \langle sx,x \rangle = \|s^+\|$.

If $S \in \tilde{\mathcal{M}}^{sa} - \mathcal{M}^{sa}$ then there are no projections satisfying the expression A_t – since by faithfulness of τ, p would still have to be 1, but this is impossible as $D(S) \underset{\neq}{\subset} \mathcal{H}$ – and so $\lambda_0(S) = \inf \phi = \infty$.

9:7 **Example**

Suppose $\mathcal{M} = L^\infty(X,\Sigma,\mu)$ and $\tau = \int d\mu$

Then for $f \in \tilde{\mathcal{M}}^{sa}$

$$e_t(f) = \{ x \in X : f(x) \leq t \}$$

$$\Rightarrow \quad d_t(f) = \mu\{ x \in X : f(x) > t \}$$

$$\Rightarrow \quad \lambda_t(f) = \inf \{ \theta \in \mathbb{R} : \mu\{ x \in X : f(x) > \theta \} \leq t \}$$

which is exactly the rearrangement of f seen in Chapter 2. □

9:8 **Proposition**

Suppose $\tau(1) < \infty$ and $R , S \in \tilde{\mathcal{M}}^{sa}$

(a) The infimum in the expression B_t of 9:4 is attained, thus $d_{\lambda_t(S)}(S) \leq t \quad \forall\, t \in (0,\tau(1))$

(b) $\lambda_{d_t(S)}(S) \leq t \quad \forall\, t \in [0,\tau(1))$

(c) $\lambda_t(S)$ is decreasing.

(d) $\lambda_t(S)$ is right continuous.

(e) $R \leq S \Rightarrow \lambda_t(R) \leq \lambda_t(S)$

Proof

(a)

Choose a sequence $(\theta_n) \subset \mathbb{R}$ such that $d_{\theta_n}(S) \leq t \ \ \forall\, n \in \mathbb{N}$, and $\theta_n \downarrow \lambda_t(S)$.

Then $d_{\lambda_t(S)}(S) \leq t$ by the right continuity of $d_t(S)$, and the infimum is attained.

(b)

$\lambda_{d_t(S)}(S) = \inf\{\ \theta \in \mathbb{R} : d_\theta(S) \leq d_t(S)\ \} \leq t$

(c)

$$t_1 \leq t_2 \quad \Rightarrow \quad \{\ \theta \in \mathbb{R} : d_\theta(S) \leq t_1\} \subset \{\ \theta \in \mathbb{R} : d_\theta(S) \leq t_2\}$$

$$\Rightarrow \quad \lambda_{t_2}(S) \leq \lambda_{t_1}(S)$$

(d)

Assume for a contradiction that $\lambda_t(S)$ is not right continuous at some $t \in \mathbb{R}$.

$\Rightarrow \quad \exists\, c > 0$ such that $\lambda_t(S) > c \geq \lambda_{t+\epsilon}(S) \ \ \forall\, \epsilon > 0$, since $\lambda_t(S)$ is decreasing.

$\Rightarrow \quad d_c(S) \leq d_{\lambda_{t+\epsilon}(S)}(S)$ as $d(S)$ is decreasing

$\leq t+\epsilon$ by (a).

Then $d_c(S) \leq t$ (as ϵ was arbitrary), and so $\lambda_t(S) \leq c$, the required contradiction.

(e)

$$R \leq S \quad \Rightarrow \quad d_t(R) \leq d_t(S)$$

$$\Rightarrow \quad \lambda_t(R) = \inf\{\ \theta \in \mathbb{R} : d_\theta(R) \leq t\ \} \leq \inf\{\ \theta \in \mathbb{R} : d_\theta(S) \leq t\ \} = \lambda_t(S)$$

□

We will need the following estimates of the Spectral Scale of a reduced algebra in Chapter 10. We point out that for $s \in \mathcal{M}^{sa}$, $p \in \mathcal{M}_p$,

$$\lambda_t(s_p) = \inf_{\substack{e \in (\mathcal{M}_p)_p \\ \tau_p(p-e) \le t}} \sup_{\substack{\|x\|=1 \\ x \in e\mathcal{X}}} \langle s_p x, x \rangle \qquad t \in [0, \tau(p))$$

which is in general quite different to $\lambda_t(p\, s\, p)$.

9:9 **Proposition** [DDd] 2.2

Suppose $\tau(1) < \infty$.

Suppose $p \in \mathcal{M}_p$ and $s \in \mathcal{M}^{sa}$.

Then $\lambda_{t + \tau(1-p)}(s) \le \lambda_t(s_p) \le \lambda_t(s) \quad \forall\, t \in [0, \tau(p))$

Proof

If $q \in \mathcal{M}_p$ and $\tau(1-q) \le t$, then $p \wedge q \in (\mathcal{M}_p)_p$ and
$\tau_p(p - p\wedge q) = \tau(p - p\wedge q) = \tau(p \vee q - q) \le \tau(1-q) \le t$.

So $$\lambda_t(s_p) = \inf_{\substack{e \in (\mathcal{M}_p)_p \\ \tau_p(p-e) \le t}} \sup_{\substack{\|x\|=1 \\ x \in e\mathcal{X}}} \langle s_p x, x \rangle \qquad t \in [0, \tau(p))$$

$$\le \inf_{\substack{q \in \mathcal{M}_p \\ \tau(1-q) \le t}} \sup_{\substack{\|x\|=1 \\ x \in (p\wedge q)\mathcal{X}}} \langle s_p x, x \rangle$$ by the above observation.

$$= \inf_{\substack{q \in \mathcal{M}_p \\ \tau(1-q) \le t}} \sup_{\substack{\|x\|=1 \\ x \in (p\wedge q)\mathcal{X}}} \langle s x, x \rangle$$

since for $x \in (p\wedge q)\mathcal{X}$, $\langle s_p x, x\rangle = \langle psp(p\wedge q)x, x\rangle = \langle s(p\wedge q)x, px\rangle = \langle sx, spx\rangle = \langle sx, x\rangle$

$$\le \inf_{\substack{q \in \mathcal{M}_p \\ \tau(1-q) \le t}} \sup_{\substack{\|x\|=1 \\ x \in q\mathcal{X}}} \langle s x, x \rangle$$

$$= \lambda_t(s).$$

$$\lambda_{t+\tau(1-p)}(s) = \inf_{\substack{q\in\mathcal{M}_p \\ \tau(1-q)\le t+\tau(1-p)}} \sup_{\substack{\|x\|=1 \\ x\in q\mathcal{X}}} <sx,x>$$

$$\le \inf_{\substack{q\in\mathcal{M}_p \\ q\le p \\ \tau(1-q)\le t+\tau(1-p)}} \sup_{\substack{\|x\|=1 \\ x\in q\mathcal{X}}} <s_p x,x>$$

since for $x \in q\mathcal{X}$, $q \le p$, $<s_p x,x> = <pspqx,qx> = <psqx,qx> = <sqx,qx> = <sx,x>$

$$= \inf_{\substack{q\in(\mathcal{M}_p)_p \\ \tau_p(p-q)\le t}} \sup_{\substack{\|x\|=1 \\ x\in q\mathcal{X}}} <s_p x,x>$$

since $\tau(1-q) = \tau(p-q) + \tau(1-p) = \tau_p(p-q) + \tau(1-p)$

$$= \lambda_t(s_p) .$$ □

We now suppose the trace is not neccessarily finite. As will soon be apparent, it is appropiate to consider positive members of $\tilde{\mathcal{M}}$. Thus we consider $|S|$ for $S \in \tilde{\mathcal{M}}$ in what follows.

9:10 **Proposition** cf. [FK] p272

Suppose R , $S \in \tilde{\mathcal{M}}$.

(a) $d_t(|S|)$ is decreasing.

(b) $d_t(|S|)$ is right continuous.

(c) $|R| \le |S| \Rightarrow d_t(|R|) \le d_t(|S|)$

(d) $d_0(|S|) = \tau(\mathrm{supp}(S))$

(e) $d_t(|S|) = \tau(1)$ for $t < 0$

(f) $S \in \tilde{\mathcal{M}}(\epsilon,t) \Leftrightarrow d_\epsilon(|S|) \le t$

(g) $d_t(|S|) = d_t(|S^*|)$

(h) $d_t(|S|)$ is eventually finite and $d_t(|S|) \longrightarrow 0$ as $t \longrightarrow \infty$

Proof

(a) , (b) and (c)

These follow from 9:3 (a) , (b) and (c)

(d) and (e)

By 9:1 (6) and (7)

(f)

This is a restatement of 8:15

(g)

Follows by the argument included in 8:16

(h)

This is a restatement of 8:17 (e) and (f) □

We thus have that for $S \in \tilde{\mathcal{M}}$, $d_t(|S|)$ is eventually finite and $\lim_{t\to\infty} d_t(|S|) = 0$, regardless of whether the trace is finite valued or not. Hence for $t > 0$, the set $\{ \theta \in \mathbb{R} : d_\theta(|S|) \le t \} = \{ \theta \ge 0 : d_\theta(|S|) \le t \}$ is always non–empty. It follows that the function defined in the next proposition indeed takes values in $[0,\infty)$.

9:11 **Proposition** cf. [FK] 2.2

Suppose $S \in \tilde{\mathcal{M}}$, and $S = v\,|S|$ the polar decomposition of S..

For $t > 0$, the quantities

$$\inf_{\substack{p \in \mathcal{M}_p \\ p\mathcal{H} \subset D(S) \\ \tau(1-p) \leq t}} \; \sup_{\substack{\|x\|=1 \\ x \in p\mathcal{H}}} \|Sx\| = C_t \quad \text{(say)};$$

$$\inf_{\substack{p \in \mathcal{M}_p \\ p\mathcal{H} \subset D(S) \\ \tau(1-p) \leq t}} \|S\,p\| = D_t \quad \text{(say)};$$

$$\inf\{\, \theta \geq 0 : d_\theta(|S|) \leq t \,\} = E_t \quad \text{(say)};$$

$$\inf\{\, \theta \geq 0 : S \in \tilde{\mathcal{M}}(\theta, t) \,\} = F_t \quad \text{(say)};$$

are equal, and notated $\mu_t(S)$. The function $\mu_t(S) : (0,\infty) \longrightarrow [0,\infty)$ is called the *generalised singular function* of S .

Proof

It is clear that $C_t = D_t$

That $E_t = F_t$ follows from 9:10(f)

Let $\theta > 0$, and suppose $S \in \tilde{\mathcal{M}}(\theta, t)$

$\Rightarrow$ $\exists\, p \in \mathcal{M}_p$ such that $p\mathcal{H} \subset D(S)$, $\|S\,p\| \leq \theta$, $\tau(1-p) \leq t$

$\Rightarrow$ $D_t \leq \theta$

$\Rightarrow$ $D_t \leq F_t$ by taking the infimum over all admissible θ.

Let $\epsilon > 0$

Choose $q \in \mathcal{M}_p$ such that $\tau(1-q) \leq t$ and $q\mathcal{H} \subset D(S)$ and $\|S\,q\| < D_t + \epsilon$

Then $\||S|\,q| = \|v^*\,S\,q\| \leq \|v^*\|\,\|S\,q\| \leq \|S\,q\| < D_t + e$

$\Rightarrow \quad q \wedge e_{(D_t+\epsilon,\infty)}(S) = 0$ by the Spectral Theorem.

$\Rightarrow \quad \tau(e_{(D_t+\epsilon,\infty)}(S)) \leq \tau(1-q) \leq t$ by 3:2(e)

$\Rightarrow \quad d_{D_t+\epsilon}(|S|) \leq t$

$\Rightarrow \quad E_t \leq D_t + \epsilon$

$\Rightarrow \quad E_t \leq D_t$ since ϵ was arbitrary. □

9:12 **Note**

Once again we note that the expressions in 9:11 are valid for any von Neumann algebra containing the family $\{ e_t(S) : t \in \mathbb{R} \}$. In particular, they hold for the commutative algebra generated by the family $\{ e_t(S) : t \in \mathbb{R} \}$.

9:13 **Note**

Suppose $S \in \tilde{\mathcal{M}}$ and $\tau(1) < \infty$

It is clear from the expression D_t in 9:11 that $\mu_t(S) = 0$ for $t \geq \tau(1)$

For $t \in (0,\tau(1))$

$$\begin{aligned}
\mu_t(S) &= \inf\{ \theta \geq 0 : d_\theta(|S|) \leq t \} \\
&= \inf\{ \theta \geq 0 : d_\theta(|S|) \leq t \} \\
&= \inf\{ \theta \in \mathbb{R} : d_\theta(|S|) \leq t \} \\
&\qquad \text{since } t \neq \tau(1)\text{, thus } d_\theta(|S|) = \tau(1) > t \text{ for } \theta < 0 \\
&= \lambda_t(|S|)
\end{aligned}$$

Thus we may say that $\mu_t(S) = \lambda_t(|S|)$

In particular, $\mu_t(S) = \lambda_t(S)$ for $S \geq 0$ □

We now show how the generalised singular function is indeed a generalisation of the rearrangement of a function and the singular value sequence of a compact operator.

9:14 Examples

(a)

Suppose $\mathcal{M} = L^{\infty}(X,\Sigma,\mu)$ and $\tau = \int d\mu$

For $f \in \tilde{\mathcal{M}} = \tilde{L}_{\infty}$, $\quad e_t(|f|) = \{ x \in X : |f(x)| \leq t \}$

$\Rightarrow \quad d_t(|f|) = \mu\{ x \in X : |f(x)| > t \}$

$\Rightarrow \quad \mu_t(f) = \inf \{ \theta \geq 0 : \mu\{ x \in X : |f(x)| > \theta \} \leq t \}$

which is exactly the decreasing rearrangement of $|f|$ seen in Chapter 2.

(b)

Suppose $\mathcal{M} = BL(\mathcal{H})$ and τ is the canonical diagonal trace.

Recall that $\tilde{\mathcal{M}} = \mathcal{M}$, and that for $p \in \mathcal{M}_p$, $\tau(p)$ is the Hilbert dimension of p.

For $n \geq 0$, let $P_n = \{ p \in \mathcal{M}_p :$ the Hilbert dimension of $p \leq n \}$

Thus for $t > 0$, $\quad \{ p \in \mathcal{M}_p : \tau(1-p) \leq t \}$

$= \quad \{ p \in \mathcal{M}_p : \tau(1-p) \leq [[t]] \}$ where $[[.]]$ is the *greatest integer function*

i.e. $[[t]] = \max \{ n \in \mathbb{Z} : n \leq t \}$

$= \quad \{ p \in \mathcal{M}_p : p^{\perp} \in P_{[[t]]} \}$

It follows that for $s \in BL(\mathcal{H})$, $\mu_t(s) = \inf_{\substack{p \in \mathcal{M}_p \\ p^{\perp} \in P_{[[t]]}}} \|sp\|$

Furthermore $\mu_t(s)$ is constant on $(0,1)$ (and equal to $\|s\|$)

and is constant on $[n,n+1)$ for $n \in \mathbb{N}$

In particular, the generalised singular function may be identified with a sequence.

Now suppose $s \in BL(\mathcal{H})$ is a compact operator. The *singular sequence* of s is the sequence $\gamma_0(s), \gamma_1(s), \ldots$ of eigenvalues of $|s|$ arranged in decreasing order and counted according to multiplicity. (We index from 0 rather than from 1 to facilitate comparison with the generalised singular function.)

It is well known that $\gamma_n(s) = \inf_{\substack{p \in \mathcal{M}_p \\ p^\perp \in P_n}} \|sp\|$ (See, for example, [GK] II Theorem 2.2)

Hence the singular sequence of a compact operator coincides with the generalised singular function (the latter being identified with a sequence as already indicated). □

The following result will be needed in Chapter 10.

9:15 **Proposition**

(a) Suppose $s \in \mathcal{M}$

Then $\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix} \in M_2(\mathcal{M})$, and $\mu_t(\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}) = \mu_t(s) \quad \forall\, t > 0$

(Of course the generalised singular functions are with respect to different algebras and different traces)

(b) (Adapted [HN] 2.1)

Suppose $s \in \mathcal{M}$, and consider $\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix} \in M_2(\mathcal{M})$.

Then $\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix} \in M_2(\mathcal{M})^{sa}$ and $\mu_t(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+) = \mu_t(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^-) = \mu_t(s) \quad \forall\, t > 0$

Proof

(a)

Note that $\left|\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}\right| = \begin{bmatrix} |s| & 0 \\ 0 & 0 \end{bmatrix}$ and that $e_t(\begin{bmatrix} |s| & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} e_t(|s|) & 0 \\ 0 & 1 \end{bmatrix} \quad \forall\, t \geq 0$

Hence $\tilde{\tau}(\, e_{(t,\infty)}(\begin{bmatrix} |s| & 0 \\ 0 & 0 \end{bmatrix}) = \tilde{\tau}(\begin{bmatrix} e_{(t,\infty)}(|s|) & 0 \\ 0 & 0 \end{bmatrix}) = \tau(e_{(t,\infty)}(|s|))$

$$\Rightarrow \quad d_t(\left|\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}\right|) = d_t(|s|)$$

$$\Rightarrow \quad \mu_t(\begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}) = \mu_t(s)$$

(b)

$$\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^* \begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix} = \begin{bmatrix} ss^* & 0 \\ 0 & s^*s \end{bmatrix}$$

$$\Rightarrow \quad \left|\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}\right| = \begin{bmatrix} (ss^*)^{1/2} & 0 \\ 0 & (s^*s)^{1/2} \end{bmatrix} = \begin{bmatrix} |s^*| & 0 \\ 0 & |s| \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+ = \frac{1}{2}\left(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix} + \left|\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}\right|\right) = \frac{1}{2}\begin{bmatrix} |s^*| & s \\ s^* & |s| \end{bmatrix}$$

$$\text{and} \quad \begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^- = \frac{1}{2}\left(\left|\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}\right| - \begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} |s^*| & -s \\ -s^* & |s| \end{bmatrix}$$

$$\text{Thus} \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^-$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} e_{(t,\infty)}(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = e_{(t,\infty)}(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^-) \quad \forall\, t > 0$$

since $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ is a unitary operator.

$$\Rightarrow \quad \tilde{\tau}(e_{(t,\infty)}(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+)) = \tilde{\tau}(e_{(t,\infty)}(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^-)) \quad \forall\, t > 0 \quad \text{since } \tilde{\tau} \text{ is unitarily invariant.}$$

$$\Rightarrow \quad d_t(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+) = d_t(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^-) \quad \forall\, t > 0$$

$$\Rightarrow \quad \mu_t(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+) = \mu_t(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^-) \quad \forall\, t > 0$$

Furthermore

$$\begin{aligned}
\tau(e_{(t,\infty)}(|s|) &= \tfrac{1}{2}\, [\, \tau(e_{(t,\infty)}(|s|) + \tau(e_{(t,\infty)}(|s^*|))\,] \\
&= \tfrac{1}{2}\, \tilde{\tau}(e_{(t,\infty)}(\begin{bmatrix} |s^*| & 0 \\ 0 & |s| \end{bmatrix}) \\
&= \tfrac{1}{2}\, \tilde{\tau}(e_{(t,\infty)}(\left|\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}\right|) \\
&= \tfrac{1}{2} \left(\tilde{\tau}(e_{(t,\infty)}(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+) + \tilde{\tau}(e_{(t,\infty)}(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^-) \right) \\
&= \tilde{\tau}\,(e_{(t,\infty)}(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+))
\end{aligned}$$

$$\Rightarrow \quad d_t(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+) = d_t(|s|)$$

$$\Rightarrow \quad \mu_t(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+) = \mu_t(s) \qquad \square$$

9:16 **Proposition**

Suppose $S \in \tilde{\mathcal{M}}$.

(a) The infimum in the expression E_t of 9:11 is attained, thus $d_{\mu_t(S)}(|S|) \le t \;\; \forall t > 0$

(b) The infimum in the expression F_t of 9:11 is attained, thus $S \in \tilde{\mathcal{M}}(\mu_t(S), t) \;\; \forall t > 0$.

(c) $\mu_{d_t(|S|)}(S) \le t \;\; \forall t > 0$ for which $d_t(|S|)$ is finite.

(d) $\mu_t(S)$ is decreasing.

(e) $\mu_t(S)$ is right continuous.

(f) $\mu_t(S)$ is discontinuous only countably often, (hence m a.e. continuous).

(g) $\mu_t(S) = \mu_t(S^*)$

Proof

(a), (c), (d), (e) are similar to 9:8 (a), (b), (c), (d) respectively, so their proofs are omitted.

(b) follows by (a) and 9:10(f)

(f)

Note that $\mu_t(S) \ge 0 \;\; \forall t > 0$, $\mu_t(S)$ is decreasing, and hence $\mu_t(S)$ is bounded on $[\frac{1}{n}, \infty) \;\; \forall n \in \mathbb{N}$

Thus there are only countably many discontinuities on $[\frac{1}{n}, \infty)$.

Hence there are only countably many discontinuities on $(0,\infty) = \bigcup_{n=1}^{\infty} [\frac{1}{n}, \infty)$.

(g) follows from 9:10(g) □

9:17 **Proposition** [FK] 3.1

(a) $S \in \tilde{\mathcal{M}}(\epsilon, t) \iff \mu_t(S) \le \epsilon$

(b) $S_i \longrightarrow S$ in $\tilde{\mathcal{M}} \iff \mu_t(S_i - S) \longrightarrow 0 \;\; \forall t > 0$

Proof

(a)

$S \in \tilde{\mathcal{U}}(\epsilon,t) \quad \Leftrightarrow \quad d_\epsilon(|S|) \le t \qquad$ 9:10(f)

$\qquad\qquad\qquad \Leftrightarrow \quad \mu_t(S) \le \epsilon$

(b)

$S_i \longrightarrow S$ in $\tilde{\mathcal{U}} \quad \Leftrightarrow \quad \forall\, \epsilon, t > 0$ there is a tail of (S_i) such that $S_i - S \in \tilde{\mathcal{U}}(\epsilon, t)$

$\qquad\qquad \Leftrightarrow \quad \forall\, \epsilon, t > 0$ there is a tail of (S_i) such that $\mu_t(S_i - S) \le \epsilon$, by (a)

$\qquad\qquad \Leftrightarrow \quad \forall\, t > 0 \;\; \mu_t(S_i - S) \longrightarrow 0$. □

9:18 **Proposition** cf. [FK] 2.5

Suppose $S, S_1, S_2, S_3 \in \tilde{\mathcal{U}}$, $\alpha \in \mathbb{C}$.

(a) $\lim_{t \downarrow 0} \mu_t(S) = \|S\|$

(b) $\mu_t(S) = 0 \Leftrightarrow S = 0$

(c) $\mu_t(\alpha S) = |\alpha|\, \mu_t(S) \quad \forall\, t > 0$

(d) $|S_1| \le |S_2| \Rightarrow \mu_t(S_1) \le \mu_t(S_2) \quad \forall\, t > 0$

(e) $\mu_{t_1+t_2}(S_1 + S_2) \le \mu_{t_1}(S_1) + \mu_{t_2}(S_2) \quad \forall\, t_1, t_2 > 0$

(f) $\mu_{t_1+t_2}(S_1 S_2) \le \mu_{t_1}(S_1)\, \mu_{t_2}(S_2) \quad \forall\, t_1, t_2 > 0$

(g) $\mu_t(S_1 S_2 S_3) \le \|S_1\|\, \mu_t(S_2)\, \|S_3\| \quad \forall\, t > 0$

(h) $|\, \mu_t(S_1) - \mu_t(S_2)\, | \le \|S_1 - S_2\| \quad \forall\, t > 0$

Proof

(a)

Certainly the limit exists (possibly as ∞) since $\mu_t(S)$ is monotone.

By definition of $\mu_t(S)$, $\mu_t(S) \le \|S\| \;\; \forall\, t > 0$

Hence $\lim_{t \downarrow 0} u_t(S) \le \|S\|$

Assume for a contradiction that $\exists\, c > 0$ such that $\|S\| > c \geq \mu_\epsilon(S) \;\; \forall\, \epsilon > 0$

Then $d_c(|S|) \leq d_{\mu_\epsilon(S)}(|S|) \leq \epsilon \;\; \forall\, \epsilon > 0$

$\Rightarrow \quad d_c(|S|) = 0$

$\Rightarrow \quad e_{(c,\infty)}(|S|) = 0$ by the faithfulness of τ .

$\Rightarrow \quad \||S|\| = \|S\| \leq c$, the required contradiction.

(b)

$S = 0 \quad \Leftrightarrow \quad \|S\| = 0$

$\Leftrightarrow \quad \lim_{t \downarrow 0} \mu_t(S) = 0$ by (a)

$\Leftrightarrow \quad \mu_t(S) = 0$ by the monotonicity of $\mu_t(S)$.

(c)

$$\begin{aligned}
\mu_t(\alpha S) &= \inf\{\theta \geq 0 : \alpha S \in \tilde{\mathcal{M}}(\theta, t)\} \\
&= \inf\{\theta \geq 0 : S \in \tilde{\mathcal{M}}(\tfrac{\theta}{|\alpha|}, t)\} \qquad \text{by 8:24(c)} \\
&= \inf\{|\alpha|\,\theta \geq 0 : S \in \tilde{\mathcal{M}}(\theta, t)\} \\
&= |\alpha| \inf\{\theta > 0 : S \in \tilde{\mathcal{M}}(\theta, t)\} \\
&= |\alpha|\, \mu_t(S)
\end{aligned}$$

(d)

Clear since $d_t(|S_1|) \leq d_t(|S_2|)$ (9:10(c))

(e)

$S_i \in \tilde{\mathcal{M}}(\mu_{t_i}(S_i), t_i) \quad i = 1, 2$ by 9:16(b)

$\Rightarrow \quad S_1 + S_2 \in \tilde{\mathcal{U}}(\mu_{t_1}(S_1) + \mu_{t_2}(S_2), t_1 + t_2)$ by 8:24(f)

$\Rightarrow \quad \mu_{t_1+t_2}(S_1 + S_2) \le \mu_{t_1}(S_1) + \mu_{t_2}(S_2)$

(f)

$S_i \in \tilde{\mathcal{U}}(\mu_{t_i}(S_i), t_i) \quad i = 1, 2$

$\Rightarrow \quad S_1 S_2 \in \tilde{\mathcal{U}}(\mu_{t_1}(S_1)\, \mu_{t_2}(S_2), t_1 + t_2)$ by 8:24(g)

$\Rightarrow \quad \mu_{t_1+t_2}(S_1 S_2) \le \mu_{t_1}(S_1)\, \mu_{t_2}(S_2)$

(g)

Suppose $\epsilon > 0$

$\mu_{t+2\epsilon}(S_1 S_2 S_3) \le \mu_\epsilon(S_1)\, \mu_t(S_2)\, \mu_\epsilon(S_3)$ by applying (f) twice

$\le \|S_1\|\, \mu_t(S_2)\, \|S_3\|$ by (a)

$\Rightarrow \quad \mu_t(S_1 S_2 S_3) \le \|S_1\|\, \mu_t(S_2)\, \|S_3\|$ by the right continuity of μ .

(h)

Suppose $\epsilon > 0$

$\mu_{\epsilon+t}(S_1) = \mu_{\epsilon+t}(S_1 - S_2 + S_2)$

$\le \mu_\epsilon(S_1 - S_2) + \mu_t(S_2)$ by (e)

$\le \|S_1 - S_2\| + \mu_t(S_2)$ by (a)

$\Rightarrow \quad \mu_t(S_1) \le \|S_1 - S_2\| + \mu_t(S_2)$ by the right continuity of μ .

$\Rightarrow \quad \mu_t(S_1) - \mu_t(S_2) \le \|S_1 - S_2\|$

By symmetry $| \mu_t(S_1) - \mu_t(S_2) | \le \|S_1 - S_2\|$ □

We will need the following estimates of the generalised singular function of a reduced algebra in Chapter 10.

We point out that for $s \in \mathcal{M}^{sa}$, $p \in \mathcal{M}_p$,

$$\mu_t(s_p) = \inf_{\substack{e \in (\mathcal{M}_p)_p \\ \tau_p(p-e) \leq t}} \sup_{\substack{\|x\|=1 \\ x \in e\mathcal{H}}} \|s_p x\| \qquad t \in (0,\tau(p))$$

which is in general quite different to $\mu_t(psp)$.

9:19 Proposition

Suppose $s \in \mathcal{M}$ and $p \in \mathcal{M}_p$.

Then $\mu_t(s_p) \leq \mu_t(psp)$

Proof

If $q \in \mathcal{M}_p$ and $\tau(1-q) \leq t$, then $p \wedge q \in (\mathcal{M}_p)_p$ and

$\tau_p(p - p\wedge q) = \tau(p - p\wedge q) = \tau(p\vee q - q) \leq \tau(1-q) \leq t$.

Thus

$$\begin{aligned}
\mu_t(s_p) &= \inf_{\substack{e \in (\mathcal{M}_p)_p \\ \tau_p(p-e) \leq t}} \sup_{\substack{\|x\|=1 \\ x \in e\mathcal{H}}} \|s_p x\| \\
&\leq \inf_{\substack{q \in \mathcal{M}_p \\ \tau(1-q) \leq t}} \sup_{\substack{\|x\|=1 \\ x \in (p\wedge q)\mathcal{H}}} \|pspx\| \quad \text{by the above observation.} \\
&\leq \inf_{\substack{q \in \mathcal{M}_p \\ \tau(1-q) \leq t}} \sup_{\substack{\|x\|=1 \\ x \in q\mathcal{H}}} \|pspx\| \\
&= \mu_t(psp) .
\end{aligned}$$

□

We already know from 9:17 that if $S_i \longrightarrow S$ in $\tilde{\mathcal{U}}$, then $\mu_t(S_i - S) \longrightarrow 0$.
We now examine under what conditions we can derive statements of the form
" $\mu_t(S_i) \longrightarrow \mu_t(S)$ ".

9:20 **Theorem** cf. [FK] 3.4

Suppose $S_n \longrightarrow S$ in $\tilde{\mathcal{U}}$.

(a) $u_t(S) \le \liminf_n \mu_t(S_n)$

(b) If $\mu_t(S)$ is continuous at t then $\mu_t(S_n) \longrightarrow \mu_t(S)$

(c) $\mu_t(S_n) \longrightarrow \mu_t(S)$ for m a.e. t .

(d) If $\mu_t(S_n) \le \mu_t(S)$ then $\mu_t(S_n) \longrightarrow \mu_t(S)$

(e) If $\mu_t(S_n) \le \mu_t(S)$ and $\mu_t(S_n)$ are increasing then $\mu_t(S_n) \uparrow \mu_t(S)$

(f) If $0 \le S_n \uparrow S$ then $\mu_t(S_n) \uparrow \mu_t(S)$

Proof

(a)

Suppose $\epsilon > 0$.

$$\begin{aligned} \mu_{t+\epsilon}(S) &= \mu_{t+\epsilon}(S - S_n + S_n) \\ &\le \mu_\epsilon(S - S_n) + \mu_t(S_n) \end{aligned}$$

$\Rightarrow$ $\mu_{t+\epsilon}(S) \le \liminf_n \mu_t(S_n)$ since $\mu_\epsilon(S - S_n) \longrightarrow 0$ by 9:17

$\Rightarrow$ $\mu_t(S) \le \liminf_n \mu_t(S_n)$ by the right continuity of $\mu_t(S)$

(b)

$$\begin{aligned} \mu_t(S_n) &= \mu_t(S_n - S + S) \\ &\le \mu_\epsilon(S_n - S) + \mu_{t-\epsilon}(S) \end{aligned}$$

$\Rightarrow \quad \limsup_n \mu_t(S_n) \;\le\; \mu_{t-\epsilon}(S)$ since $\mu_\epsilon(S_n - S) \longrightarrow 0$ by 9:17

$\Rightarrow \quad \limsup_n \mu_t(S_n) \;\le\; \mu_t(S)$ since $\mu_t(S)$ is (left) continuous at t.

Hence $\mu_t(S) \;\le\; \liminf_n \mu_t(S_n) \;\le\; \limsup_n \mu_t(S_n) \;\le\; \mu_t(S)$, and the result follows.

(c)

This follows from (b) and 9:16(f) .

(d)

$\mu_t(S_n) \le \mu(S)$

$\Rightarrow \quad \limsup_n \mu(S_n) \le \mu_t(S)$

Hence the result follows as in (b).

(e)

Immediate from (d) .

(f)

By 9:18(d) this follows from (e) . □

9:21 **Lemma**

Suppose $S \in \tilde{\mathcal{M}}$, and v a partial isometry such that $v^* v \supset R(S)$

Then $\mu_t(v\,S) = \mu_t(S)$

Proof

$$\begin{aligned} \mu_t(S) &= \mu_t(v^* v\, S) \quad \text{since } v^* v\, S = S \\ &\le \mu_t(v\, S) \\ &\le \mu_t(S) \quad \text{by 9:18(g)} \end{aligned}$$

□

9:22 **Theorem**

(a) Suppose $S \in \tilde{\mathcal{M}}$

Then there exists a sequence $\{e_n\} \subset \mathcal{M}_p$ such that

$$e_n \uparrow 1 \text{ in } \tilde{\mathcal{M}}$$

$$S\, e_n \in \mathcal{M} \quad \forall\, n \in \mathbb{N}$$

$$\mu_t(Se_n) \uparrow \mu_t(S)$$

(b) Suppose $S_1, \dots, S_m \in \tilde{\mathcal{M}}$

Then there exists a sequence $\{e_n\} \subset \mathcal{M}_p$ such that

$$e_n \uparrow 1 \quad \text{in } \tilde{\mathcal{M}}$$

$$S_i\, e_n \in \mathcal{M} \quad \forall\, n \in \mathbb{N} \quad 1 \le i \le m$$

$$\mu_t(S_i e_n) \uparrow u_t(S_i) \quad 1 \le i \le m$$

Proof

(a)

Suppose $S = v\, |S|$ is the polar decomposition of S.

Let $e_n = e_{[0,n]}(|S|) \in \mathcal{M}_p$

Recall from the proof of 8:27 that

$$S\, e_n \in \mathcal{M} \;\forall\, n \in \mathbb{N}$$

$$e_n \uparrow 1 \text{ in } \tilde{\mathcal{M}} \text{ and hence } S\, e_n = v\, |S|\, e_n \longrightarrow v\, |S| = S \text{ in } \tilde{\mathcal{M}}$$

$\mu_t(S\, e_n) \le \mu_t(S)\, \|e_n\| \le \mu_t(S) \;\; \forall\, n \in \mathbb{N}$, so by 9:20(d) $\mu_t(S\, e_n) \longrightarrow \mu_t(S)$

Note that $v^* v = \overline{R(|S|)} \supset R(|S|) \supset R(|S|\, e_n) \;\; \forall\, n \in \mathbb{N}$

Hence 9:21 can be applied :–

If $n_1 < n_2$ then $\mu_t(S\, e_{n_1})$

$$= \mu_t(v\,|S|\,e_{n_1})$$

$$= \mu_t(|S|\,e_{n_1})$$

$$\leq \mu_t(|S|\,e_{n_2}) \quad \text{by 9:18(d)}$$

$$= \mu_t(v\,|S|\,e_{n_2})$$

$$= \mu_t(S\, e_{n_2})$$

Hence $\mu_t(Se_n) \uparrow \mu_t(S)$ by 9:20(e)

(b)

Suppose $S_i = v_i\,|S_i|$ for $1 \leq i \leq m$

Let $e_{n,i} = e_{[0,n]}(|S_i|)$ for $1 \leq i \leq m$, $n \in \mathbb{N}$

Then $e_{n,i} \uparrow 1$ in $\tilde{\mathcal{M}}$ $1 \leq i \leq m$

Let $e_n = \bigwedge_{i=1}^{m} e_{n,i}$

Then $S_i\, e_n = S_i\, e_{n,i}\, e_n \in \mathcal{M}$ $\forall\, n \in \mathbb{N}$ for $1 \leq i \leq m$

We now verify that $e_n \uparrow 1$ in $\tilde{\mathcal{M}}$

$$\tau(1-e_n) = \tau(1 - \bigwedge_{i=1}^{m} e_{n,i})$$

$$= \tau(\bigvee_{i=1}^{m} (1-e_{n,i}))$$

$$\leq \sum_{i=1}^{m} \tau(1-e_{n,i})$$

$$\longrightarrow 0 \quad \text{as } n \longrightarrow \infty .$$

Hence, as argued in 8:27, $e_n \longrightarrow 1$ in $\tilde{\mathcal{M}}$.

Similar arguments to (a) give the final result. □

9:23 **Proposition** cf. [FK] 2.6

(a) Suppose $s \in \mathcal{M}$, $q \in \mathcal{M}_p$.

Then $\mu_t(s\,q) = 0$ for $t \geq \tau(q)$

(b) Suppose $S \in \tilde{\mathcal{M}}$, $q \in \mathcal{M}_p$

Then $\mu_t(S\,q) = 0$ for $t \geq \tau(q)$

Proof

(a)

If $t \geq \tau(q)$ then $\tau(1-[1-q]) = \tau(q) \leq t$

Hence $\mu_t(s\,q)$

$$= \inf_{\substack{p \in \mathcal{M}_p \\ \tau(1-p) \leq t}} \|s\,q\,p\|$$

$$\leq \|s\,q\,(1-q)\|$$

$$= 0$$

(b)

Choose $\mathcal{M} \supset \{s_n\} \longrightarrow S$ in $\tilde{\mathcal{M}}$

Then $s_n\,q \longrightarrow S\,q$ in $\tilde{\mathcal{M}}$

For $t \geq \tau(q)$,

$$0 \leq \mu_t(S\,q)$$

$$\leq \liminf_n \mu_t(s_n\,q) \quad \text{by 9:20(a)}$$

$$= 0 \text{ by (a)} \qquad \square$$

We wish to point out that 9:24 and 9:25 are phrased for operators in $\bar{\mathcal{M}}$ rather than in $\tilde{\mathcal{M}}$. Not much is gained by this, and we make this distinction only for the purposes of some technical points which will be examined in Chapter 11.

9:24 **Definition**

Suppose $0 \leq S \in \bar{\mathcal{M}}$.

Let $S = \int_0^\infty t \, de_t(S)$ be the spectral resolution of S.

Of course $\int_0^n t \, de_t(S) \in \mathcal{M}^+ \ \forall n \in \mathbb{N}$.

We formally define $\tau(S) = \sup_n \tau(\int_0^n t \, de_t(S))$.

Note that $\tau(S) = \lim_n \tau(\int_0^n t \, de_t(S))$ by the monotonicity of τ.

9:25 **Proposition**

Suppose $0 \leq S \in \bar{\mathcal{M}}$.

(a) The function $\nu_S : \mathcal{B}([0,\infty)) \longrightarrow [0,\infty] : B \longrightarrow \tau(e_B(S))$ is a measure.

(b) $\tau(S) = \int_0^\infty t \, d\nu_S(t) = \int_0^\infty t \, d\tau(e_t(S))$

Proof

(a)

$\nu_S(\phi) = \tau(e_\phi(S)) = \tau(0) = 0$

Suppose B_i is a countable disjoint family in $\mathcal{B}([0,\infty))$

$$\begin{aligned}\nu_S(\bigcup_{i=1}^\infty B_i) &= \tau(e_{\bigcup_{i=1}^\infty B_i}(S)) \\ &= \tau(\sum_{i=1}^\infty e_{B_i}(S)) \\ &= \sum_{i=1}^\infty \tau(e_{B_i}(S)) \quad \text{by the normality of } \tau\end{aligned}$$

$$= \sum_{i=1}^{\infty} \nu_S(B_i)$$

Hence ν_S is a measure.

(b)

If suffices to show $\tau\left(\int_0^n t \, de_t(S) \right) = \int_0^n t \, d\tau(e_t(S)) \quad \forall\, n \in \mathbb{N}$

since $\quad \tau\left(\int_0^n t \, de_t(S) \right) \uparrow \tau(S) \quad$ by definition

and $\quad \int_0^n t \, d\tau(e_t(S)) \uparrow \int_0^\infty t \, d\tau(e_t(S))$ by the Monotone Convergence Theorem.

We show this by imitating a standard measure–theoretic argument.

We first show $\tau\left(\int_0^n \chi_B(t) \, de_t(S) \right) = \int_0^n \chi_B(t) \, d\tau(e_t(S)) \quad$ for $B \in \mathcal{B}([0,n])$

Note that $\int_0^n \chi_B(t) \, de_t(S) = e_B(S) \in \mathcal{M}^+$.

$$\begin{aligned}
\text{So } \tau\left(\int_0^n \chi_B(t) \, de_t(S) \right) &= \tau(e_B(S)) \\
&= \nu_S(B) \\
&= \int_0^n \chi_B(t) \, d\nu_S(t) \\
&= \int_0^n \chi_B(t) \, d\tau(e_t(S))
\end{aligned}$$

It follows by the linearity and homogeniety of the trace that

$\tau(\int_0^n f(t)\, de_t(S)) = \int_0^n f(t)\, d\tau(e_t(S))$ for $0 \leq f$ a finite linear combination of characteristic functions.

Choose a sequence $\{f_i\}$ of such functions such that $f_i(t) \uparrow t$ uniformly on $[0,n]$

Then $\int_0^n f_i(t)\, de_t(S) \uparrow \int_0^n t\, de_t(S)$. so

$$
\begin{aligned}
\tau(\int_0^n t\, de_t(S)) &= \lim_i \tau(\int_0^n f_i(t)\, de_t(S)) && \text{by the normality of the trace} \\
&= \lim_i \int_0^n f_i(t)\, d\tau(e_t(S)) \\
&= \int_0^n t\, d\tau(e_t(S)) && \text{by the Monotone Convergence Theorem.}
\end{aligned}
$$

The result follows. □

The use we can make of the formal definition of τ is quite limited. For example, in the case that f is a positive Borel measurable function – so that f(S) in in the Functional Calculus for S – we are unable to determine $\tau(f(S))$ from $\tau(S)$ as $\int_0^\infty f(S)\, de_t(S)$ is *not* the canonical spectral resolution of f(S) .

We thus develop a characterisation of τ which is more easy to work with by showing that for $0 \leq S \in \tilde{\mathcal{M}}$, $\tau(S) = \int_0^\infty \mu_t(S)\, dt$.

In the case that $\tau(1) < \infty$, the following is an interesting alternative proof of an even stronger result. We will need 9:26(b) in Chapter 10.

9:26 **Theorem** cf. [P] Proposition 1

Suppose $\tau(1) < \infty$

(a)

If $0 \leq S \in \tilde{\mathcal{M}}$, then $\tau(S) = \int_0^{\tau(1)} \mu_t(S)\, dt$

(b)

If $s \in \mathcal{M}^{sa}$, then $\tau(s) = \int_0^{\tau(1)} \lambda_t(s)\, dt$

Proof

(a)

Consider Lebesgue measure m on $[0,\tau(1))$.

Consider the Borel measure ν_S on $[0,\infty)$ given by 9:25(a) .

We claim that the function $\mu_t(S) : t \longrightarrow \mu_t(S) : [0,\tau(1)) \longrightarrow [0,\infty)$ is measure preserving with respect to the indicated measures.

To show this, it suffices to show that it is measure preserving for $B = (a,b] \in \mathcal{B}([0,\infty))$, since these sets generate $\mathcal{B}([0,\infty))$.

$$\begin{aligned}
\nu_S(B) &= \tau(e_{(a,b]}(S)) \\
&= \tau(e_{(a,\infty)}(S)) - e_{(b,\infty)}(S)) \\
&= \tau(e_{(a,\infty)}(S)) - \tau(e_{(b,\infty)}(S)) && \text{since } \tau \text{ is finite} \\
&= d_a(S) - d_b(S) . \\
&= m\,(\,[d_b(S)\,,\,d_a(S)\,)\,)
\end{aligned}$$

$$= \quad m\{ t \in (0,\tau(1)) : a < \mu_t(S) \le b \}$$

$$= \quad m([\mu_t(S)]^{-1}(B))$$

Thus $\mu_t(S)$ is measure preserving.

$$\text{Hence} \int_0^{\tau(1)} \mu_t(S)\, dt \quad = \quad \int_0^{\infty} \text{id}\ d(m\,[\mu(S)]^{-1}) \qquad \text{[C] 2.6.5}$$

$$= \quad \int_0^{\infty} \text{id}\ d\nu_S \quad \text{by the above}$$

$$= \quad \tau(S)\ , \ \text{by 9:25(b)}$$

(b)

If $s \in \mathcal{M}^{sa}$, then arguments similar to 9:25 and 9:26(a) apply, namely :

The function ν_s on $\mathcal{B}(\mathbb{R})$ given by $\nu_s(B) = \tau(e_B(s))$ is a measure.

The identity $\int_{-\|s^-\|}^{\|s^+\|} t\ d\nu_s(t) = \int_{\mathbb{R}} t\ d\nu_s(t) = \tau(s)$ is established in a similar manner to 9:25(b) (As in that proof the appropiate identities for characteristic functions and finite linear combinations of such functions are established. Then the identity function on $[-\|s^-\|\ ,\ \|s^+\|\ ]$ is approximated in the uniform norm by such functions; and the result is deduced from the uniform continuity of τ.)

Furthermore $\lambda_t(s) : t \longrightarrow \lambda_t(s) : [0,\tau(1)) \longrightarrow \mathbb{R}$ is measure preserving with respect to the indicated measures.

$$\text{Thus} \int_0^{\tau(1)} \lambda_t(s)\, dt \quad = \quad \int_{\mathbb{R}} \text{id}\ d\nu_s \quad = \quad \tau(s) \qquad \square$$

9:27 **Lemma**

Suppose $p \in \mathcal{H}_p$

Then $\mu_t(p) = \chi_{(0,\tau(p))}(t)$

Proof

$$e_t(p) = \begin{cases} 1 & t \geq 1 \\ 1-p & 0 \leq t < 1 \\ 0 & t < 0 \end{cases}$$

$$\Rightarrow \quad d_t(p) = \begin{cases} 0 & t \geq 1 \\ \tau(p) & 0 \leq t < 1 \\ \tau(1) & t < 0 \end{cases}$$

$$\Rightarrow \quad \mu_t(p) = \begin{cases} 1 & 0 < t < \tau(p) \\ 0 & \tau(p) \leq t < \tau(1) \end{cases}$$

$$= \chi_{(0,\tau(p))}(t)$$

□

9:28 **Proposition**

Suppose $s \in \mathcal{H}$, $S \in \tilde{\mathcal{H}}$, $p \in \mathcal{H}_p$

(a) $\mu_t(p\ s\ p) \leq \mu_t(s\ p) \leq \mu_t(s)\ \mu_t(p) \leq \mu_t(s)$

(b) $\mu_t(p\ S\ p) \leq \mu_t(S\ p) \leq \mu_t(S)\ \mu_t(p) \leq \mu_t(S)$

Proof

(a)

$\mu_t(p\ s\ p) \leq \|p\|\ \mu_t(s\ p) \leq \mu_t(s\ p)$

Now $\mu_t(s\ p) \leq \mu_t(s)\ \|p\| \leq \mu_t(s)$

and $\mu_t(s\ p) = 0$ for $t \geq \tau(p)$ by 9:23(a)

Hence $\mu_t(s\ p) \leq \mu_t(s)\ \chi_{(0,\tau(p))}(t) = \mu_t(s)\ \mu_t(p) \leq \mu_t(s)$

(b)

Use 9:22 to choose $\mathcal{U} \ni s_n \longrightarrow S$ in $\tilde{\mathcal{U}}$ such that $\mu(s_n) \uparrow \mu(S)$.

Then $s_n\, p \longrightarrow S\, p$ in $\tilde{\mathcal{U}}$.

Hence

$$
\begin{aligned}
\mu_t(p\,S\,p) &\le \|p\|\, \mu_t(S\,p) \\
&\le \mu_t(S\,p) \\
&\le \liminf_n \mu_t(s_n\, p) \quad \text{by 9:20(a)} \\
&\le \liminf_n \mu_t(s_n)\, \mu_t(p) \quad \text{by (a)} \\
&= \mu_t(S)\, \mu_t(p) \quad \text{since } \mu(s_n) \uparrow \mu(S) \\
&\le \mu_t(S)
\end{aligned}
$$

□

9:29 **Lemma** cf. [DDd] 2.3

Suppose $s = \sum_{i=1}^{n} \alpha_i\, p_i \in \mathcal{U}^+$ where $\alpha_i \ge 0$ and $1 = p_0 \ge p_1 \ge p_2 \ge \dots \ge p_n \ge p_{n+1} = 0 \in \mathcal{U}_p$.

Then $\mu_t(s) = \sum_{i=1}^{n} \alpha_i\, \mu_t(p_i)$

Proof

Put $\alpha_0 = 0$.

$$
e_t(s) = \begin{cases} 0 & t < 0 \\ 1 - p_k & \sum_{i=0}^{k-1} \alpha_i \le t < \sum_{i=0}^{k} \alpha_i \quad (k = 1, \dots, n) \\ 1 & \sum_{i=0}^{n} \alpha_i \le t \end{cases}
$$

$$\Rightarrow \quad d_t(s) = \begin{cases} \tau(1) & t < 0 \\ \tau(p_k) & \sum_{i=0}^{k-1} \alpha_i \leq t < \sum_{i=0}^{k} \alpha_i \quad (k = 1, \dots, n) \\ 0 & \sum_{i=0}^{n} \alpha_i \leq t \end{cases}$$

$$\begin{aligned} \Rightarrow \quad \mu_t(s) &= \sum_{i=0}^{k} \alpha_i \qquad \tau(p_{k+1}) \leq t < \tau(p_k) \quad (k = 0, \dots, n) \\ &= \sum_{i=1}^{n} \alpha_i \, \chi_{(0,\tau(p_i))}(t) \\ &= \sum_{i=1}^{n} \alpha_i \, \mu_t(p_i) \qquad \text{by 9:27} \end{aligned}$$

□

9:30 **Theorem** [FK] 2.7

Suppose $0 \leq S \in \tilde{\mathcal{M}}$

Then $\tau(S) = \int_0^\infty \mu_t(S)\, dt$

Proof

Suppose s is of the form $\sum_{i=1}^{n} \alpha_i p_i \in \mathcal{M}^+$ where $\alpha_i \geq 0$ and $p_1 \geq p_2 \geq \dots \geq p_n \in \mathcal{M}_p$.

$$\begin{aligned} \text{Then } \int_0^\infty \mu_t(s)\, dt &= \int_0^\infty \sum_{i=1}^{n} \alpha_i \, \chi_{(0,\tau(p_i))}\, dt \qquad \text{by 9:29} \\ &= \sum_{i=1}^{n} \alpha_i \, \tau(p_i) \\ &= \tau\Big(\sum_{i=1}^{n} \alpha_i \, p_i\Big) \\ &= \tau(s) \end{aligned}$$

If $s \in \mathcal{M}^+$, then it follows via the spectral theorem that s can be uniformly approximated from below by a sequence of operators s_n of the above form.

Since $| u_t(s_n) - \mu_t(s) | \leq \| s_n - s \|$ (9:18(h)), $\mu_t(s_n) \longrightarrow \mu_t(s)$.

Thus $\mu(s_n) \uparrow \mu(s)$, by 9:18(d), since $s_n \uparrow s$.

Thus
$$
\begin{aligned}
\tau(s) &= \lim_n \tau(s_n) && \text{by the normality of } \tau \\
&= \lim_n \int_0^\infty \mu_t(s_n)\,dt && \text{by the result already derived} \\
&= \int_0^\infty \lim_n \mu_t(s_n)\,dt && \text{by the Monotone Convergence Theorem} \\
&= \int_0^\infty \mu_t(s)\,dt
\end{aligned}
$$

Finally suppose $0 \leq S \in \tilde{\mathcal{M}}$.

Put $s_n = S\, e_{[0,n]}(S)$

Then $s_n \in \mathcal{M}$ $\forall n \in \mathbb{N}$ and $\mu_t(S_n) \uparrow \mu(S)$, as shown in 9:22

$$
\begin{aligned}
\tau(S) &= \lim_n \tau(s_n) && \text{by definition} \\
&= \lim_n \int_0^\infty \mu_t(s_n)\,dt && \text{by the results already obtained above} \\
&= \int_0^\infty \lim_n \mu_t(s_n)\,dt && \text{by the Monotone Convergence Theorem} \\
&= \int_0^\infty \mu_t(S)\,dt .
\end{aligned}
$$
□

9:31 **Theorem** [FK] 3.5

Suppose $0 \le S_n, S \in \tilde{\mathcal{M}}$.

(a) If $S_n \longrightarrow S$ in $\tilde{\mathcal{M}}$ then $\tau(S) \le \liminf_n \tau(S_n)$

(b) If $S_n \longrightarrow S$ in $\tilde{\mathcal{M}}$ and $\mu_t(S_n) \le \mu_t(S)$ then $\tau(S) = \lim_n \tau(S_n)$

(c) If $S_n \uparrow S$ in $\tilde{\mathcal{M}}$ then $\tau(S) = \lim_n \tau(S_n)$

Proof

(a)

$$\begin{aligned}
\tau(S) &= \int_0^\infty \mu_t(S)\,dt \\
&\le \int_0^\infty \liminf_n \mu_t(S_n)\,dt && \text{9:20(a)} \\
&\le \liminf_n \int_0^\infty \mu_t(S_n)\,dt && \text{by Fatou's lemma} \\
&= \liminf_n \tau(S_n)
\end{aligned}$$

(b)

$\mu_t(S_n) \le \mu_t(S)$

$$\Rightarrow \quad \int_0^\infty \mu_t(S_n)\,dt \le \int_0^\infty \mu_t(S)\,dt$$

$$\Rightarrow \quad \tau(S_n) \le \tau(S)$$

$$\Rightarrow \quad \limsup_n \tau(S_n) \le \tau(S)$$

Hence $\tau(S) = \lim_n \tau(S_n)$ by (a).

(c)

This is a special case of (b), but we can actually implement the classical Monotone Covergence Theorem :—

$$\tau(S) = \int_0^\infty \mu_t(S)\, dt$$

$$= \int_0^\infty \lim_n \mu_t(S_n)\, dt \qquad 9{:}20(f)$$

$$= \lim_n \int_0^\infty \mu_t(S_n)\, dt \qquad \text{by the Monotone Convergence Theorem.}$$

$$= \lim_n \tau(S_n)\,. \qquad \square$$

We now show that this formulation of the trace has some more uses than the formal definition. The following result will be needed in Chapter 10.

9:32 **Proposition** [FK] 2.5(iv)

Suppose $S \in \tilde{\mathcal{M}}$.

Suppose $f : [0,\infty) \longrightarrow [0,\infty)$ is a continuous increasing function.

(a) $\mu_t(f(|S|)) = f(\mu_t(S))$

(b) $\tau(f(|S|)) = \int_0^\infty f(\mu_t(S))\, dt$

Proof

(a)

By 9:14 , we may suppose $\mathcal{M}$ is the commutative von Neumann algebra generated by the spectral resolution of $|S|$. This von Neumann algebra includes the spectral family of $f(|S|)$.

Now $D(f(|S|)) = D(|S|)$, and if $p \in \mathcal{M}_p$ and $p\mathcal{H} \subset D(|S|)$ then $\|f(|S|)p\| = f(\||S|p\|)$

Hence $\mu_t(f(|S|))$

$$= \inf_{\substack{p \in \mathcal{M}_p \\ p\mathcal{X} \subset D(f(|S|)) \\ \tau(1-p) \leq t}} \|f(|S|)p\|$$

$$= \inf_{\substack{p \in \mathcal{M}_p \\ p\mathcal{X} \subset D(f(|S|)) \\ \tau(1-p) \leq t}} f(\||S|p\|)$$

$$= f\left(\inf_{\substack{p \in \mathcal{M}_p \\ p\mathcal{X} \subset D(|S|) \\ \tau(1-p) \leq t}} \||S|p\| \right)$$ by the continuity and monotonicity of f

$$= f(\mu_t(|S|))$$

$$= f(\mu_t(S))$$

(b)

$$\tau(f(|S|)) = \int_0^\infty \mu_t(f(|S|))\, dt = \int_0^\infty f(\mu_t(S))\, dt \qquad \square$$

10 : NON–COMMUTATIVE BANACH FUNCTION SPACES

We suppose throughout that $\mathcal{M}$ is a semifinite von Neumann algebra and τ a faithful semifinite normal trace on $\mathcal{M}$.

In this Chapter we develop the spaces $L_\rho(\tilde{\mathcal{M}})$, due to the work of Dodds, Dodds, and de Pagter [DDd].

In Chapter 9 we have defined the generalised singular function $\mu_t(S)$ for operators $S \in \tilde{\mathcal{M}}$ and developed a number of results concerning this function.
The generalised singular functions are of course functions with domain $(0,\infty)$, so it is a natural question to ask which members of $\tilde{\mathcal{M}}$ have generalised singular functions which satisfy certain conditions – in particular, membership of the classical Normed Köthe Spaces over $(0,\infty)$ discussed in Chapter 2. This generalises the well known fact that the L_p spaces of [Sg], [St], [Ku], [Y1], [Y2], [Y3], [N], [Tp] and [FK] are precisely those members of $\tilde{\mathcal{M}}$ whose generalised singular function is a member of $L_p(0,\infty)$.

Furthermore, we can hope to generalise certain *representation theorems* from the commutative theory to the present theory. For example, it is known that if (X,Σ,μ) is a finite measure space for which μ is *adequate* then all symmetric Fatou norms ρ on $L_0(X,\Sigma,\mu)$ are derived from some symmetric norm ρ' on $L_0(0,\infty)$ via the formula $\rho(f) = \rho'(\mu_t(f))$. See [L] § 12.

These observations motivate the following definitions :

10:1 **Definition** [DDd] 4.1

Suppose $L_\rho(0,\infty)$ is a Normed Köthe Space.

$L_\rho(\tilde{\mathcal{M}}) = \{ S \in \tilde{\mathcal{M}} : \mu_t(S) \in L_\rho(0,\infty) \}$

10:2 **Definition** [DDd] 4.1

For $S \in L_\rho(\tilde{\mathcal{M}})$, define $\rho(S) = \rho(\mu_t(S))$

The question now arises : under what conditions on ρ is $L_\rho(\tilde{\mathcal{M}})$ a normed space with norm ρ ? Furthermore, when is $L_\rho(\tilde{\mathcal{M}})$ complete?

The approach of Dodds, Dodds, and de Pagter which we shall discuss here shows that for $L_\rho(\tilde{\mathcal{M}})$ to be a Banach Space (with the norm ρ) it *suffices* to have $L_\rho(0,\infty)$ a rearrangement invariant Banach Function Space with ρ lower semicontinuous.

The questions of *neccessity* are at this stage open. More specifically, the following questions present themselves :–

(1) What are the neccessary and sufficient conditions on $L_\rho(0,\infty)$ to ensure $L_\rho(\tilde{\mathcal{M}})$ is a vector space?

(2) What are the neccessary and sufficient conditions on $L_\rho(0,\infty)$ to ensure $L_\rho(\tilde{\mathcal{M}})$ is a normed space with norm ρ ?

(3) What are the neccessary and sufficient conditions on $L_\rho(0,\infty)$ to ensure that, if $L_\rho(\tilde{\mathcal{M}})$ is a normed space with norm ρ , that $L_\rho(\tilde{\mathcal{M}})$ is complete, i.e. a non–commutative Banach Function Space?

To answer these questions, it seems likely that an approach substantially different to that of Dodds, Dodds and de Pagter will be required. Under the given approach, the requirement that $L_\rho(0,\infty)$ be a symmetric Banach Function Space is needed as early as proving that $L_\rho(\tilde{\mathcal{M}})$ is a vector space.

Furthermore, the triangle inequality for ρ only follows from the submajorisation

$$|\mu_t(R) - \mu_t(S)| \prec\prec \mu_t(R-S) \quad R\,,\, S \in \tilde{\mathcal{M}} \qquad \text{(established in 10:18)}$$

by assuming in addition that ρ is lower semicontinuous.

We now expand on the above comments.

10:3 **Proposition** [DDd] 4.2

Suppose $L_\rho(0,\infty)$ is a Normed Köthe Space.

(a) If $S \in L_\rho(\tilde{\mathcal{M}})$ and $\alpha \in \mathbb{C}$ then $\alpha S \in L_\rho(\tilde{\mathcal{M}})$ and $\rho(\alpha S) = |\alpha|\ \rho(S)$

(b) $0 \in L_\rho(\tilde{\mathcal{M}})$; $\rho(S) = 0 \Leftrightarrow S = 0$

Proof

(a)

Suppose $S \in L_\rho(\tilde{\mathcal{M}})$ and $\alpha \in \mathbb{C}$.

$\mu_t(\alpha S) = |\alpha|\ \mu_t(S)$ by 9:18(c)

$\Rightarrow \quad \rho(\mu_t(\alpha S)) = |\alpha|\ \rho(\mu_t(S))$

$\Rightarrow \quad \alpha S \in L_\rho(\tilde{\mathcal{M}})$ and $\rho(\alpha S) = |\alpha|\ \rho(S)$

(b)

$0 \in L_\rho(0,\infty)$, and hence $0 \in L_\rho(\tilde{\mathcal{M}})$.

$S = 0 \Leftrightarrow \mu_t(S) = 0 \Leftrightarrow \rho(\mu_t(S)) = 0 \Leftrightarrow \rho(S) = 0$ by 9:18(b) □

10:4 **Proposition** [DDd] 4.2

Suppose $L_\rho(0,\infty)$ is a symmetric Banach Function Space.

Then $L_\rho(\tilde{\mathcal{M}})$ is a vector space.

Proof

Suppose R , S $\in L_\rho(\tilde{\mathcal{M}})$.

$\Rightarrow \quad \mu_t(R)\, ,\, \mu_t(S) \in L_\rho(0,\infty)$

$\Rightarrow \quad \mu_{t/2}(R)\, ,\, \mu_{t/2}(S) \in L_\rho(0,\infty)$

as $L_\rho(0,\infty)$ is symmetric, hence closed under dilations, by 2:24.

$\Rightarrow \quad \mu_{t/2}(R) + \mu_{t/2}(S) \in L_\rho(0,\infty)$

$\Rightarrow \quad \mu_t(R + S) \in L_\rho(0,\infty)$ since $\mu_t(R + S) \le \mu_{t/2}(R) + \mu_{t/2}(S)$ by 9:18(e)

$\Rightarrow \quad R + S \in L_\rho(\tilde{\mathcal{M}})$. □

10:5 **Note**

We now make the additional assumption that ρ is lower semicontinuous – thus $L_\rho(0,\infty)$ is a symmetric Banach Function Space and ρ is lower semicontinuous – from which it follows that the hypotheses of 2:27 are satisfied.

If we are able to determine that for R , S $\in \tilde{\mathcal{M}}$,

$$|\, \mu_t(R) - \mu_t(S)\, | \prec\prec \mu_t(R - S)$$

then the triangle inequality for ρ would follow from 2:27 by the calculation :

$$|\mu_t(R + S) - \mu_t(S)| \prec\prec \mu_t(R)$$

$$\Rightarrow \quad \rho(\,|\mu_t(R + S) - \mu_t(S)|\,) = \rho(\, \mu_t(R + S) - \mu_t(S)\,) \le \rho(\mu_t(R))$$

$$\begin{aligned} \Rightarrow \quad \rho(\, \mu_t(R + S)\,) &= \rho(\, \mu_t(R + S) - \mu_t(S) + \mu_t(S)\,) \\ &\le \rho(\, \mu_t(R + S) - \mu_t(S)\,) + \rho(\mu_t(S)) \\ &\le \rho(\mu_t(R)) + \rho(\mu_t(S)) . \end{aligned}$$

i.e. $\rho(R + S) \le \rho(R) + \rho(S)$

Much of the remainder of this Chapter is devoted to establishing as required above that

$$| \mu_t(R) - \mu_t(S) | \prec\prec \mu_t(R-S) \quad \text{for } R , S \in \tilde{\mathcal{M}} \qquad \text{(Theorem 10:18)}$$

This is achieved via a majorisation result which is of independent interest : in the case that the trace is finite,

$$\lambda_t(r) - \lambda_t(s) \prec \lambda_t(r-s) \quad \text{for } r , s \in \mathcal{M}^{sa} \qquad \text{(Theorem 10:15)}$$

Before proceeding to the details, we give a brief indication of our strategy.

Recall from 2:25 , 2:26 that this majorisation would follow from showing that

$$\int_0^{\tau(1)} \lambda_t(r) - \lambda_t(s) \, dt = \int_0^{\tau(1)} \lambda_t(r-s) \, dt$$

and $\sup_{m(A)=\theta} \int_A \lambda_t(r) - \lambda_t(s) \, dt \leq \sup_{m(A)=\theta} \int_A \lambda_t(r-s) \, dt$ for $0 < \theta < \tau(1)$

The first equality will be an immediate consequence of the finiteness of the trace and the characterisation $\tau(s) = \int_0^{\tau(1)} \lambda_t(s) \, dt$ (9:26(b))

Furthermore, it is clear (since $\lambda_t(r-s)$ is decreasing) that

$$\sup_{m(A)=\theta} \int_A \lambda_t(r-s) \, dt = \int_0^{\theta} \lambda_t(r-s) \, dt$$

Hence it will suffice to show that for $A \in \mathcal{B}([0,\tau(1)))$

$$\int_A \lambda_t(r) - \lambda_t(s) \, dt \leq \int_0^{m(A)} \lambda_t(r-s) \, dt \qquad \text{(Corollary 10:14)}$$

This result is deduced using two simplifying techniques.

The first technique is that it suffices to show certain results in the case that $\mathcal{M}$ is known to be non–atomic. The assumption that $\mathcal{M}$ is non–atomic is not severe; for a (possibly atomic) von Neumann algebra can always be injected into the non–atomic von Neumann algebra

$\mathcal{M} \overline{\otimes} L_\infty[0,1]$ with preservation of trace, distribution function, spectral scale and generalised singular function. The usefulness of the assumption of non–atomicity is demonstrated in the following result, which uses the results of Chapter 9 to generalise 3:34 . We will make extensive use of this result.

10:6 **Proposition** [DDd] 2.5

Suppose $\mathcal{M}$ is non–atomic.

(a) Suppose $\tau(1) < \infty$.

If $\theta \in [0,\tau(1))$ and $s \in \mathcal{M}^{sa}$ then there exists $e_\theta(s) \in \mathcal{M}_p$ such that

$$\tau(e_\theta(s)) = \theta$$

$$e_{(\lambda_\theta(s),\infty)}(s) \leq e_\theta(s) \leq e_{[\lambda_\theta(s),\infty)}(s)$$

(b)

If $\theta > 0$ and $s \in \mathcal{M}$ then there exists $e_\theta(s) \in \mathcal{M}_p$ such that

$$\tau(e_\theta(s)) = \theta$$

$$e_{(\mu_\theta(s),\infty)}(|s|) \leq e_\theta(s) \leq e_{[\mu_\theta(s),\infty)}(|s|)$$

Proof

(a)

Certainly $e_{(\lambda_\theta(s),\infty)}(s) \leq e_{[\lambda_\theta(s),\infty)}(s)$

Now $\tau(e_{(\lambda_\theta(s),\infty)}(s)) = d_{\lambda_\theta(s)}(s) \leq \theta$

and

$\forall\ \alpha < \lambda_\theta(s),\ \tau(e_{(\alpha,\infty)}(s)) > \theta$ and $e_{(\alpha,\infty)}(s) \downarrow_{so} e_{[\lambda_\theta(s),\infty)}(s)$ as $\alpha \uparrow \lambda_\theta(s)$

$\Rightarrow \quad \tau(e_{[\lambda_\theta(s),\infty)}(s)) \geq \theta \qquad$ by 3:20

Hence $\tau(e_{(\lambda_\theta(s),\infty)}) \leq \theta \leq \tau(e_{[\lambda_\theta(s),\infty)})$

The result follows from 3:24.

(b)

Similar arguments apply. □

We introduce the second simplifying technique with a definition.

10:7 **Definition**

We call a set $[a,b) \subset \mathbb{R}$ where $0 \leq a < b < \infty$ a *cell.*

If the trace is finite then it is immediate from the spectral theorem that if $s \in \mathcal{M}^{sa}$ then

$$d_t(s) = 0 \quad \text{for } t \geq \|s^+\|$$
$$d_t(s) = \tau(1) \quad \text{for } t < -\|s^-\|$$

Hence the range of $\lambda_t(s)$ is included in $[-\|s^-\|, \|s^+\|]$, and $\lambda_t(s)$ is bounded.

Since the functions $\lambda_t(r)$, $\lambda_t(s)$, $\lambda_t(r-s)$ are bounded and the interval $[0,\tau(1))$ is bounded, it follows from the Dominated Convergence Theorem that to show that for $A \in \mathcal{B}([0,\tau(1)))$

$$\int_A \lambda_t(r) - \lambda_t(s)\, dt \leq \int_0^{m(A)} \lambda_t(r-s)\, dt$$

it will suffice to consider A to be a finite disjoint union of cells.

The basis for this result is 10:11, which is an inductive argument on the number of cells (in the complement of A). Cells are made to correspond in a certain manner to chosen projections which have trace value the same as the measure of the cells, and satisfy the inequality

$\int_A \lambda_t(r) - \lambda_t(s)\, dt \le \int_0^{\tau(e_A)} \lambda_t((r-s)_{e_A})\, dt$. The assumption of non–atomicity is exploited to construct such projections. By 9:26(b) , the right hand side is $\tau_{e_A}((r-s)_{e_A})$. But this is the same as $\tau((r-s)e_A)$, by the finiteness of τ; finally an inequality for the trace of a product developed in 10:13 enables us to conclude the argument.

We start with a lemma.

10:8 Lemma [DDd] 2.4

Suppose $\tau(1) < \infty$.

Suppose $s \in \mathcal{M}^{sa}$, $e \in \mathcal{M}_p$, $c \in \mathbb{R}$ and $e_{[c,\infty)}(s) \ge e \ge e_{(c,\infty)}(s)$.

Then

(a) $\lambda_t(s_p) = \lambda_t(s)$ for $t \in [0,\tau(e))$, for any $e \le p \in \mathcal{M}_p$

(b) $\lambda_t(s) = \lambda_{t-\tau(1-p)}(s_p)$ for $t \in [\tau(e),\tau(1))$ for any $1-e \le p \in \mathcal{M}_p$

Proof

Note that e (and hence $1 - e$) commute with the spectral family for s.

(a)

s_e has spectral family $\{ e_t(s)_e : t \in \mathbb{R} \}$

$$\text{Now } (1-e_t(s))_e = \begin{cases} 1 - e_t(s) & \text{if } t \ge c \\ e & \text{if } t < c \end{cases}$$

$$\Rightarrow \quad d_t(s_e) = \begin{cases} d_t(s) & \text{if } t \ge c \\ \tau(e) & \text{if } t < c \end{cases}$$

$$\Rightarrow \quad \lambda_t(s_e) = \lambda_t(s) \text{ if } t \in [0,\tau(e)) .$$

Thus for $t \in [0,\tau(e))$

$\lambda_t(s) = \lambda_t(s_e)$

$\le \lambda_t(s_p)$ by 9:9 (Take $\mathcal{M}_p$ to be the algebra and e the projection in 9:9)

$\le \lambda_t(s)$ by 9:9.

(b)

s_{1-e} has spectral family $\{ e_t(s)_{1-e} : t \in \mathbb{R} \}$

Thus $$(1-e_t(s))_{1-e} = \begin{cases} 0 & \text{if } t \geq c \\ 1-e-e_t(s) & \text{if } t<c \end{cases}$$

$$\Rightarrow \quad d_t(s_{1-e}) = \begin{cases} 0 & \text{if } t \geq c \\ d_t(s) - \tau(e) & \text{if } t<c \end{cases}$$

$$\Rightarrow \quad \lambda_{t-\tau(e)}(s_{1-e}) = \lambda_t(s) \qquad t \in [\tau(e),\tau(1))$$

$$\begin{aligned} \lambda_{t+\tau(e)-\tau(1-p)}(s_p) &= \lambda_{t+\tau(p)-\tau(1-e)}(s_p) \\ &= \lambda_{t + \tau_p(p-(1-e))}(s_p) \\ &\leq \lambda_t(s_{1-e}) \qquad t \in [0,\tau(1-e)) \qquad \text{by 9:9} \end{aligned}$$

(Take $\mathcal{M}_p$ to be the algebra and $1 - e$ the projection in 9:9)

So $$\begin{aligned} \lambda_{t-\tau(1-p)}(s_p) &\leq \lambda_{t-\tau(e)}(s_{1-e}) \qquad t \in [\tau(e),\tau(1)) \qquad \text{by a change of variable} \\ &= \lambda_t(s) \qquad \text{by the above calculation} \end{aligned}$$

Furthermore

$$\begin{aligned} \lambda_{t+\tau(1-p)}(s) &\leq \lambda_t(s_p) \qquad t \in [0,\tau(p)) \qquad \text{by 9:9} \\ \Rightarrow \quad \lambda_t(s) &\leq \lambda_{t-\tau(1-p)}(s_p) \qquad t \in [\tau(1-p),\tau(1)) \qquad \text{by a change of variable} \\ \Rightarrow \quad \lambda_t(s) &\leq \lambda_{t-\tau(1-p)}(s_p) \qquad t \in [\tau(e),\tau(1)) \end{aligned}$$

since $1-e \leq p \Rightarrow \tau(1-p) \leq \tau(e)$.

This completes the proof. □

The next result generalises the previous to enable us to deal with the disjoint union of two intervals.

10:9 **Theorem** [DDd] 2.6

Suppose $\tau(1) < \infty$.

Suppose $\mathcal{M}$ is non–atomic.

If $0 < \theta_1 < \theta_2 < \tau(1)$, $r, s \in \mathcal{M}^{sa}$, then

(a) $\exists\, p \in \mathcal{M}_p$ such that

$$\tau(p) = \tau(1) - (\theta_2 - \theta_1)$$

$$\lambda_t(r_p) = \lambda_t(r) \qquad t \in [0,\theta_1)$$

$$\lambda_{t-(\theta_2-\theta_1)}(s_p) = \lambda_t(s) \qquad t \in [\theta_2,\tau(1))$$

(b) In this case

$$\lambda_t(r) - \lambda_t(s) \;\le\; \lambda_t(r_p) - \lambda_t(s_p) \qquad t \in [0,\theta_1)$$

$$\lambda_t(r) - \lambda_t(s) \;\le\; \lambda_{t-(\theta_2-\theta_1)}(r_p) - \lambda_{t-(\theta_2-\theta_1)}(s_p) \qquad t \in [\theta_2,\tau(1))$$

Proof

(a)

Let $e_{\theta_1}(r)$, $e_{\theta_2}(s)$ be as in 10:6(a).

Consider $e_{\theta_1}(r) \vee (1 - e_{\theta_2}(s))$.

$\tau(e_{\theta_1}(r) \vee (1 - e_{\theta_2}(s))) \le \tau(e_{\theta_1}(r)) + \tau((1 - e_{\theta_2}(s)) = \theta_1 + \tau(1) - \theta_2 = \tau(1) - (\theta_2 - \theta_1)$.

$\Rightarrow$ $\exists\, p \in \mathcal{M}_p$ such that $p \ge e_{\theta_1}(r) \vee (1 - e_{\theta_2}(s))$ and $\tau(p) = \tau(1) - (\theta_2 - \theta_1)$, by 3:24

In particular, $\tau(1-p) = \theta_2 - \theta_1$

$p \ge e_{\theta_1}(r)$

$\Rightarrow$ $\lambda_t(r_p) = \lambda_t(r) \qquad t \in [0,\theta_1)$ by 10:8(a)

$p \ge 1 - e_{\theta_2}(s)$

$\Rightarrow$ $\lambda_t(s) = \lambda_{t-\tau(1-p)}(s_p) = \lambda_{t-(\theta_2-\theta_1)}(s_p) \qquad t \in [\theta_2,\tau(1))$ by 10:8(b).

(b)

For $t \in [0,\theta_1)$

$$\lambda_t(r) - \lambda_t(s) \quad = \quad \lambda_t(r_p) - \lambda_t(s) \qquad \text{by (a)}$$

$$\leq \quad \lambda_t(r_p) - \lambda_t(s_p)$$

since $\lambda_t(s_p) \leq \lambda_t(s)$ for $t \in [0,\tau(p))$ (by 9:9), hence for $[0,\theta_1)$, since $\theta_1 < \tau(p)$.

Furthermore

$$\lambda_{t + \tau(1-p)}(r) \quad \leq \quad \lambda_t(r_p) \qquad \text{for } t \in [0,\tau(p)) \text{ by 9:9}$$

$$\Rightarrow \quad \lambda_{t + (\theta_2-\theta_1)}(r) \quad \leq \quad \lambda_t(r_p) \qquad \text{for } t \in [0,\tau(1) - (\theta_2-\theta_1))$$

$$\Rightarrow \quad \lambda_t(r) \quad \leq \quad \lambda_{t - (\theta_2-\theta_1)}(r_p) \qquad \text{for } t \in [\theta_2-\theta_1,\tau(1))$$

by applying a change of variable.

$$\Rightarrow \quad \lambda_t(r) \quad \leq \quad \lambda_{t - (\theta_2-\theta_1)}(r_p) \qquad \text{for } t \in [\theta_2,\tau(1))$$

Thus for $t \in [\theta_2,\tau(1))$

$$\lambda_t(r) - \lambda_t(s) \quad = \quad \lambda_t(r) - \lambda_{t - (\theta_2-\theta_1)}(s_p) \qquad \text{by (a)}$$

$$\leq \quad \lambda_{t - (\theta_2-\theta_1)}(r_p) - \lambda_{t - (\theta_2-\theta_1)}(s_p) \qquad \square$$

10:10 **Definition** [DDd] p 589

For $A \in \mathcal{B}([0,\infty))$ define $m_A : A \longrightarrow [0,m(A)] : t \longrightarrow m(A \cap [0,t))$

It is clear that m_A preserves measure when A is a finite disjoint union of cells.

The following result can be viewed as the crucial step in the argument. The inductive argument is inspired by [M] Theorem 5.1 ; where a majorisation result similar to 10:15 is proved for singular value sequences of compact operators in $BL(\mathcal{X})$.

10:11 **Lemma** [DDd] 2.7

Suppose $\tau(1) < \infty$, $\mathcal{M}$ is non–atomic, and $A \subset [0,\tau(1))$ is a finite disjoint union of cells.

Suppose $r, s \in \mathcal{M}^{sa}$

Then $\exists\, e_A \in \mathcal{M}_p$ such that

$$\tau(e_A) = m(A)$$

$$\lambda_t(r) - \lambda_t(s) \leq \lambda_{m_A(t)}(r_{e_A}) - \lambda_{m_A(t)}(s_{e_A}) \quad \text{for } t \in A$$

Proof

The proof is by induction on the number n of disjoint cells in $[0,\tau(1)) - A$.

Case n=0 :

It must be the case that $A = [0,\tau(1))$, so we choose $e_A = 1$, and the result is immediate.

We make the inductive hypothesis for the n^{th} case.

Suppose we are given a set A_{n+1} which is a finite disjoint union of cells and $[0,\tau(1)) - A_{n+1}$ can be expressed as the disjoint union of n+1 cells.

Let J be any one of these cells and put $A_n = A_{n+1} \cup J$. Then $[0,\tau(1)) - A_n$ is expressed as the disjoint union of n cells, so by the inductive hypothesis

$\exists\, e_n \in \mathcal{M}_p$ such that

$$\tau(e_n) = m(A_n)$$

$$\lambda_t(r) - \lambda_t(s) \leq \lambda_{m_{A_n}(t)}(r_{e_n}) - \lambda_{m_{A_n}(t)}(s_{e_n}) \quad \text{for } t \in A_n .$$

Let $A_1 = [0,m(A_n)) - m_{A_n}(J)$.

If $J = [a,b)$ (say) ; then

$$m_{A_n}(J) = \{\, m(A_n \cap [0,t)) : a \leq t < b \,\}$$

$$= [m(A_n \cap [0,a)) , m(A_n \cap [0,b)))$$

$$\text{Hence } m(m_{A_n}(J)) = m(A_n \cap [0,b)) - m(A_n \cap [0,a))$$
$$= m(A_n \cap [a,b)) \; .$$
$$= m(A_n \cap J)$$
$$= m(J) \; .$$

Hence $m(A_1) = m(A_n) - m(J) = m(A_n - J) = m(A_{n+1})$.

By 10:9 applied to $\mathcal{M}_{e_n}$, with $\theta_1 = m(A_n \cap [0,a))$, $\theta_2 = m(A_n \cap [0,b))$; it follows that

$\exists\, e_{n+1} \in \mathcal{M}_p$, $e_{n+1} \leq e_n$, such that

$$\tau(e_{n+1}) = \tau(e_n) - (\theta_2 - \theta_1)$$
$$= \tau(e_n) - (\, m(A_n \cap [0,b) - m(A_n \cap [0,a))\,)$$
$$= m(A_n) - m(J)$$
$$= m(A_{n+1})$$
$$\lambda_t(r_{e_n}) - \lambda_t(s_{e_n}) \leq \lambda_t(r_{e_{n+1}}) - \lambda_t(s_{e_{n+1}}) \quad t \in [0,\theta_1) \, .$$
$$\lambda_t(r_{e_n}) - \lambda_t(s_{e_n}) \leq \lambda_{t-(\theta_2-\theta_1)}(r_{e_{n+1}}) - \lambda_{t-(\theta_2-\theta_1)}(s_{e_{n+1}}) \quad t \in [\theta_2,\tau(e_n)) \, .$$

Note that $A_1 = [0,\theta_1) \cup [\theta_2,\tau(e_n))$ and thus

for $t \in [0,\theta_1)$, $m_{A_1}(t) = t$

for $t \in [\theta_2,\tau(e_n))$, $m_{A_1}(t) = t - (\theta_2 - \theta_1)$.

Hence $\lambda_t(r_{e_n}) - \lambda_t(s_{e_n}) \leq \lambda_{m_{A_1}(t)}(r_{e_{n+1}}) - \lambda_{m_{A_1}(t)}(s_{e_{n+1}}) \quad t \in A_1$

Therefore for $m_{A_n}(t) \in A_1$

$$\lambda_{m_{A_n}(t)}(r_{e_n}) - \lambda_{m_{A_n}(t)}(s_{e_n}) \leq \lambda_{m_{A_1}(m_{A_n}(t))}(r_{e_{n+1}}) - \lambda_{m_{A_1}(m_{A_n}(t))}(s_{e_{n+1}})$$

Now $m_{A_n}(t) \in A_1$

$\Leftrightarrow \quad m(A_n \cap [0,t)) \in A_1$

$\Leftrightarrow \quad t \in A_{n+1}$;

and for $t \in A_{n+1}$

$$m_{A_1}(m_{A_n}(t)) = m_{A_1}(m(A_n \cap [0,t)) = m_{A_{n+1}}(t) .$$

Therefore

$$\lambda_{m_{A_n}(t)}(r_{e_n}) - \lambda_{m_{A_n}(t)}(s_{e_n}) \leq \lambda_{m_{A_{n+1}}(t)}(r_{e_{n+1}}) - \lambda_{m_{A_{n+1}}(t)}(s_{e_{n+1}}) \qquad t \in A_{n+1}$$

Hence

$\lambda_t(r) - \lambda_t(s)$

$\leq \quad \lambda_{m_{A_n}(t)}(r_{e_n}) - \lambda_{m_{A_n}(t)}(s_{e_n}) \qquad t \in A_n$, by the inductive hypothesis.

$= \quad \lambda_{m_{A_n}(t)}(r_{e_n}) - \lambda_{m_{A_n}(t)}(s_{e_n}) \qquad t \in A_{n+1}$ since $A_{n+1} \subset A_n$.

$\leq \quad \lambda_{m_{A_{n+1}}(t)}(r_{e_{n+1}}) - \lambda_{m_{A_{n+1}}(t)}(s_{e_{n+1}}) \qquad t \in A_{n+1}$.

With this the induction step is completed. □

Before concluding the majorisation argument we will need one further result.

10:12 **Lemma** [DDd] 2.1

Suppose $\tau(1) < \infty$ and $S \in \tilde{\mathcal{M}}^{sa}$

Then

(a) $\lambda_t(-S) = \lim_{\epsilon \downarrow 0} -\lambda_{\tau(1)-t-\epsilon}(S)$ i.e. the right continuous modification of $-\lambda_{\tau(1)-t}(S)$

(b) $\lambda_t(S^+) = \lambda_t^{\ +}(S)$

(c) $\lambda_t(S^-) = (\lim_{\epsilon \downarrow 0} \lambda_{\tau(1)-t-\epsilon}(S))^-$

Proof

(a)

By a similar calulation as in 9:25(a) , the function $\nu : \mathcal{B}(\mathbb{R}) \longrightarrow [0\,,\,\tau(1)] : B \longrightarrow \tau(e_B(S))$ is a finite Borel measure.

Consider the identity function f on $(\mathbb{R}\,,\,\mathcal{B}(\mathbb{R})\,,\,\nu)$

$$\begin{aligned} d_t(f) &= \nu\{\, x \in \mathbb{R} : f(x) > t \,\} \\ &= \nu(t,\infty) \\ &= \tau(e_{(t,\infty)}(S)) \\ &= d_t(S) \end{aligned}$$

It follows that $\lambda_t(S) = \lambda_t(f)$ (Of course $\lambda_t(f)$ is the classical decreasing rearrangement of f; $\lambda_t(S)$ is the spectral scale of S.)

Likewise $d_t(-S) = \tau(e_{(t,\infty)}(-S)) = \tau(e_{(-\infty,-t)}(S)) = d_t(-f)$

It follows that $\lambda_t(-S) = \lambda_t(-f)$

Hence $\lambda_t(-S) = \lambda_t(-f)$

= the right continuous modification of $-\lambda_{\tau(1)-t}(f)$ by [L] 4.9(iii)

= the right continuous modification of $-\lambda_{\tau(1)-t}(S)$

(b)

This can be shown in a similar manner to (a) , using [L] 4.9(ii) ; or directly as follows:–

$$e_t(S^+) = \begin{cases} e_t(S) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\Rightarrow \quad d_t(S^+) = \begin{cases} d_t(S) & t \geq 0 \\ \tau(1) & t < 0 \end{cases}$$

$$\Rightarrow \quad \lambda_t(S^+) = \begin{cases} \lambda_t(S) & t \in [0, d_0(S)) \\ 0 & t \in [d_0(S), \tau(1)) \end{cases} = \lambda_t{}^+(S)$$

(c)

By using the fact that $S^- = (-S)^+$, this is an immediate consequence of (a) and (b) . □

Suppose $s \in \mathcal{M}^{sa}$

We have already noted that $\lambda_t(s)$ is bounded, hence it follows from 9:8 (c) and (d) that $\lambda_t(s)$ is m a.e. continuous. (Similar to 9:16(f))

It then follows from 10:12 that

10:12.1

(a) $\lambda_t(-s) = -\lambda_{\tau(1)-t}(s)$ m a.e.

(b) $\lambda_t(s^+) = \lambda_t{}^+(s)$

(c) $\lambda_t(s^-) = \lambda_{\tau(1)-t}{}^-(s)$ m a.e.

We exploit this in the following theorem, which generalises [L] 8.2. Of course the fact that r and s in the following theorem may not commute means that the following result cannot be deduced from the commutative theory (as was the previous result, for example).

10:13 **Theorem** [DDd] 2.3

Suppose $\tau(1) = a < \infty$.

If $r , s \in \mathcal{M}^{sa}$ then $\int_0^a \lambda_t(r)\,\lambda_{a-t}(s)\,dt \le \tau(rs) \le \int_0^a \lambda_t(r)\,\lambda_t(s)\,dt$

Proof

Suppose first $0 \le r \in \mathcal{M}^{sa}$ and $p \in \mathcal{M}_p$.

Then $\tau(rp) = \tau(prp)$ since the trace is finite

$= \int_0^a \mu_t(prp)\,dt$ by 9:30

$$\leq \int_0^a \mu_t(r)\,\mu_t(p)\,dt \qquad \text{by } 9{:}28$$

$$= \int_0^a \lambda_t(r)\,\lambda_t(p)\,dt \qquad \text{by } 9{:}13$$

And

$$\tau(r) - \tau(rp) = \tau(r(1-p))$$

$$\leq \int_0^a \lambda_t(r)\,\lambda_t(1-p)\,dt \qquad \text{as above}$$

$$= \int_0^{\tau(1-p)} \lambda_t(r)\,dt \quad \text{by } 9{:}27$$

$$= \int_0^a \lambda_t(r)\,dt - \int_{\tau(1-p)}^a \lambda_t(r)\,dt$$

$$= \int_0^a \lambda_t(r)\,dt - \int_0^a \lambda_t(r)\,\chi_{[\tau(1-p),1)}\,dt$$

$$= \int_0^a \lambda_t(r)\,dt - \int_0^a \lambda_t(r)\,\lambda_{a-t}(p)\,dt$$

$$= \tau(r) - \int_0^a \lambda_t(r)\,\lambda_{a-t}(p)\,dt\,.$$

$$\Rightarrow \quad \tau(rp) \geq \int_0^a \lambda_t(r)\,\lambda_{a-t}(p)\,dt\,.$$

Thus the required result follows for $0 \leq r \in \mathcal{M}$ and $p \in \mathcal{M}_p$.

If $s = \sum_{i=1}^{n} a_i p_i$ $(a_i \geq 0 , p_1 \geq p_2 \geq ... \geq p_n \in \mathcal{M}_p)$ then $\lambda_t(s) = \sum_{i=1}^{n} a_i \lambda_t(p_i)$ (9:29) .

Hence the required result follows for such r , s by linearity of the trace and linearity of integration. Since $0 \leq s \in \mathcal{M}$ can be approximated from above and below arbitrarily closely in norm by such combinations of projections, and since τ is norm continuous, since it is finite, the result follows for $r , s \in \mathcal{M}^+$.

If $r , s \in \mathcal{M}^{sa}$ then $r^+ , r^- , s^+ , s^- \in \mathcal{M}^+$.

Thus $\int_0^a \lambda_t(r^+) \lambda_{a-t}(s^+)\, dt \leq \tau(r^+ s^+) \leq \int_0^a \lambda_t(r^+) \lambda_t(s^+)\, dt$

$\Rightarrow$ (1) $\int_0^a \lambda_t^{+}(r) \lambda_{a-t}^{+}(s)\, dt \leq \tau(r^+ s^+) \leq \int_0^a \lambda_t^{+}(r) \lambda_t^{+}(s)\, dt$

by 10:12.1(b)

And $\int_0^a \lambda_t(r^+) \lambda_{a-t}(s^-)\, dt \leq \tau(r^+ s^-) \leq \int_0^a \lambda_t(r^+) \lambda_t(s^-)\, dt$

$\Rightarrow$ (2) $\int_0^a \lambda_t^{+}(r) \lambda_t^{-}(s)\, dt \leq \tau(r^+ s^-) \leq \int_0^a \lambda_t^{+}(r) \lambda_{a-t}^{-}(s)\, dt$

by 10:12.1(b) and (c)

And $\int_0^a \lambda_t(r^-) \lambda_{a-t}(s^+)\, dt \leq \tau(r^- s^+) \leq \int_0^a \lambda_t(r^-) \lambda_t(s^+)\, dt$

$\Rightarrow$ $\int_0^a \lambda_{a-t}^{-}(r) \lambda_{a-t}^{+}(s)\, dt \leq \tau(r^- s^+) \leq \int_0^a \lambda_{a-t}^{-}(r) \lambda_t^{+}(s)\, dt$

by 10:12.1(b) and (c)

$\Rightarrow$ (3) $\int_0^a \lambda_t^{-}(r) \lambda_t^{+}(s)\, dt \leq \tau(r^- s^+) \leq \int_0^a \lambda_t^{-}(r) \lambda_{a-t}^{+}(s)\, dt$

by a change of variable

And $\int_0^a \lambda_t(r^-)\, \lambda_{a-t}(s^-)\, dt \quad \leq \tau(r^- s^-) \leq \quad \int_0^a \lambda_t(r^-)\, \lambda_t(s^-)\, dt$

$\Rightarrow \quad \int_0^a \lambda_{a-t}^-(r)\, \lambda_t^-(s)\, dt \quad \leq \tau(r^- s^-) \leq \quad \int_0^a \lambda_{a-t}^-(r)\, \lambda_{a-t}^-(s)\, dt$

by 10:12.1(c)

$\Rightarrow$ (4) $\int_0^a \lambda_t^-(r)\, \lambda_{a-t}^-(s)\, dt \quad \leq \tau(r^- s^-) \leq \quad \int_0^a \lambda_t^-(r)\, \lambda_t^-(s)\, dt$

by a change of variable.

Hence by calculating (1) – (2) – (3) + (4) we have :–

$$\int_0^a \lambda_t(r)\, \lambda_{a-t}(s)\, dt \quad \leq \tau(rs) \leq \quad \int_0^a \lambda_t(r)\, \lambda_t(s)\, dt$$ □

10:14 **Corollary** to 10:11 [DDd] 2.8

Suppose $\tau(1) < \infty$.

Suppose $A \in \mathcal{B}([0,\tau(1)))$ and $r , s \in \mathcal{M}^{sa}$.

Then $\int_A \lambda_t(r) - \lambda_t(s)\, dt \leq \int_0^{m(A)} \lambda_t(r-s)\, dt$.

Proof

Suppose first that $\mathcal{M}$ is non–atomic, and that A is a finite disjoint union of cells in $[0,\tau(1))$.

Let $e_A \in \mathcal{M}_p$ be the projection derived in 10:11.

Then $\int_A \lambda_t(r) - \lambda_t(s)\, dt$

$\leq \quad \int_A \lambda_{m_A(t)}(r_{e_A}) - \lambda_{m_A(t)}(s_{e_A})\, dt$ by 10:11

$$= \int_0^{m(A)} \lambda_t(r_{e_A}) - \lambda_t(s_{e_A})\, dt \quad \text{since } m_A \text{ is measure preserving.}$$

$$= \int_0^{\tau(e_A)} \lambda_t(r_{e_A})\, dt - \int_0^{\tau(e_A)} \lambda_t(s_{e_A})\, dt$$

$$= \tau_{e_A}(r_{e_A}) - \tau_{e_A}(s_{e_A}) \qquad 9{:}26(b)$$

$$= \tau_{e_A}((r-s)_{e_A})$$

$$= \tau(e_A(r-s)e_A)$$

$$= \tau((r-s)e_A) \quad \text{since } \tau \text{ is finite.}$$

$$\leq \int_0^{\tau(1)} \lambda_t(r-s)\, \lambda_t(e_A)\, dt \qquad 10{:}13$$

$$= \int_0^{\tau(e_A)} \lambda_t(r-s)\, dt \qquad 9{:}27$$

$$= \int_0^{m(A)} \lambda_t(r-s)\, dt\, .$$

As already discussed, the inequality will hold for any $A \in \mathcal{B}([0,1))$ by dominated convergence; and the restriction that $\mathcal{M}$ be non–atomic is removed by embedding $\mathcal{M}$ into the tensor product $\mathcal{M} \,\overline{\otimes}\, L^\infty([0,1])$, which is non–atomic. □

10:15 **Theorem** [DDd] 2.9

Suppose $\tau(1) < \infty$

If $r, s \in \mathcal{M}^{sa}$ then $\lambda_t(r) - \lambda_t(s) \prec \lambda_t(r-s)$

Proof

$$\int_0^{\tau(1)} \lambda_t(r) - \lambda_t(s)\, dt = \int_0^{\tau(1)} \lambda_t(r)\, dt - \int_0^{\tau(1)} \lambda_t(s)\, dt$$

$$= \tau(r) - \tau(s) \quad \text{by 9:26(b)}$$

$$= \tau(r-s)$$

$$= \int_0^{\tau(1)} \lambda_t(r-s)\, dt \quad \text{by 9:26(b)}$$

By 10:14 $\displaystyle\int_A \lambda_t(r) - \lambda_t(s)\, dt \le \int_0^{m(A)} \lambda_t(r-s)\, dt$ for $A \in \mathcal{B}([0,\tau(1)))$

The result follows. □

Recall that we are attempting to establish that for $R, S \in \tilde{\mathcal{M}}$,

$$|\,\mu_t(R) - \mu_t(S)\,| \prec\prec \mu_t(R-S)$$

which in accordance with 2:26 , can be achieved by showing that

$$\int_A |\,\mu_t(R) - \mu_t(S)\,|\, dt \le \int_A \mu_t(R-S) \quad \forall\, A \in \mathcal{B}((0,\infty)) \text{ of finite measure.}$$

To derive this result using 10:15, we can immediately identify two restrictions that need to be overcome.

Firstly, in 10:15, the trace was required to be finite, and the spectral scale was considered for self–adjoint operators, rather than the generalised singular function for arbitrary operators in $\mathcal{M}$.

Secondly, in 10:15, the operators were required to be bounded.

To overcome these restrictions, we consider reduced algebras $\mathcal{M}_e$ for certain $e \in \mathcal{M}_p$ of finite trace. Such projections are considered in 10:16, where the generalised singular function of s^+ for a self adjoint operator s is related to the spectral scale of the reduction s_e .
By using these results and the identity $s^- = (-s)^+$, the restriction to self adjoint operators is then overcome by exploiting 9:15 where it was shown that an arbitrary $s \in \mathcal{M}$ has the same generalised singular function as the positive and negative parts of $\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}$, which is of course self–adjoint.

Thus we will be able to relate the generalised singular function of an arbitrary $s \in \mathcal{M}$ to the spectral scale of certain reductions, and thereby exploit 10:15.
This process is examined in 10:17 and 10:18.

The generalisation from the bounded to the unbounded case is achieved in 10:18 via an approximation argument based on 9:22(b) .

10:16 **Lemma** [DDd] 3.2

Suppose $e \in \mathcal{M}_p$ and $\tau(e) < \infty$.

Suppose $s \in \mathcal{M}^{sa}$.

(a) $\lambda_t(s_e) \leq \mu_t(s^+) \quad \forall\, t \in (0,\tau(e))$.

(b) If in addition $\exists\, c > 0$ such that $e_{(c,\infty)}(s^+) \leq e \leq e_{[c,\infty)}(s^+)$,
then $\lambda_t(s_e) = \mu_t(s^+) \quad \forall\, t \in (0,\tau(e))$.

Proof

(a)

$$\begin{aligned}
\lambda_t(s_e) &\leq \lambda_t((s^+)_e) && (9{:}8(e)) \\
&= \lambda_t((s_e)^+) \\
&= \mu_t((s_e)^+) && (9{:}13) \\
&= \mu_t((s^+)_e) \\
&\leq \mu_t(es^+e) && (9{:}19) \\
&\leq \mu_t(s^+) && (9{:}28)
\end{aligned}$$

(b)

s_e has spectral resolution $\{ e_t(s)_e : t \in \mathbb{R} \}$

Now $(1-e_t(s))_e \quad = \begin{cases} 1-e_t(s) & t \geq c \\ e & t < c \end{cases}$

$\Rightarrow \quad d_t(s_e) \quad = \begin{cases} d_t(s) & t \geq c \\ \tau(e) & t < c \end{cases}$

s^+ has spectral resolution $\begin{cases} e_t(s) & t \geq 0 \\ 0 & t < 0 \end{cases}$

$\Rightarrow \quad d_t(s^+) \quad = \begin{cases} d_t(s) & t \geq 0 \\ \tau(1) & t < 0 \end{cases}$

Hence for $t \geq c \;\; d_t(s) = d_t(s_e) = d_t(s^+)$

$\Rightarrow \quad$ for $t \in (0,\tau(e))$, $\lambda_t(s_e) = \mu_t(s^+)$ □

10:17 **Proposition** [DDd] 3.3

Suppose $\mathcal{M}$ is non–atomic.

Suppose $r, s \in \mathcal{M}^{sa}$.

Suppose $A, B \subset [0,\infty)$ are of finite measure.

Then $\displaystyle\int_A \mu_t(r^+) - \mu_t(s^+)\,dt + \int_B \mu_t(s^-) - \mu_t(r^-)\,dt \le \int_0^{m(A)+m(B)} \mu_t((r-s)^+)\,dt$

Proof

First suppose that $A \subset [0,a)$ and $B \subset [0,b)$, for some $a, b < \infty$. We may suppose $a, b \le \tau(1)$.

The map $\mathcal{M}^+ \longrightarrow \mathcal{M}_2^{\;+} : s \longrightarrow \begin{bmatrix} s & 0 \\ 0 & 0 \end{bmatrix}$ preserves the generalised singular function, 9:15(a).

However, the trace of the identity is doubled. Hence, by making this embedding if neccessary, we may suppose that $c = a + b \le \tau(1)$.

Since $\mathcal{M}$ is non–atomic, $\exists\, e^+, e^- \in \mathcal{M}_p$ such that

$e_{(\mu_a(r^+),\infty)}(r^+) \le e^+ \le e_{[\mu_a(r^+),\infty)}(r^+)$ and $\tau(e^+) = a$

$e_{(\mu_b(s^-),\infty)}(s^-) \le e^- \le e_{[\mu_b(s^-),\infty)}(s^-)$ and $\tau(e^-) = b$ (10:6(b))

Since $\mathcal{M}$ is non–atomic, $\exists\, e \in \mathcal{M}_p$ such that $e \ge e^+ \vee e^-$ and $\tau(e) = a + b = c$ (3:24).

Then for $t \in [0,\tau(e^+)) = [0,a)$

$\mu_t(r^+) - \mu_t(s^+)$	$\le$	$\lambda_t(r_{e^+}) - \mu_t(s^+)$	by 10:16(b)
	$\le$	$\lambda_t(r_{e^+}) - \lambda_t(s_e)$	by 10:16(a)
	$\le$	$\lambda_t(r_e) - \lambda_t(s_e)$	by 9:9

(Take $\mathcal{M}_e$ to be the algebra and e^+ the projection in 9:9)

And for $t \in [0,\tau(e^-)) = [0,b)$

$$
\begin{aligned}
\mu_t(s^-) - \mu_t(r^-) &= \mu_t((-s)^+) - \mu_t((-r)^+) \\
&= \lambda_t((-s)_{e^-}) - \mu_t((-r)^+) && \text{by 10:16(b)} \\
&\le \lambda_t((-s)_e) - \mu_t((-r)^+) && \text{by 9:9}
\end{aligned}
$$

(Take $\mathcal{M}_e$ to be the algebra and e^- the projection in 9:9)

$$
\begin{aligned}
&\le \lambda_t((-s)_e) - \lambda_t((-r)_e) && \text{by 10:16(a)} \\
&= \lambda_t(-s_e) - \lambda_t(-r_e) \\
&= \lambda_{\tau(e)-t}(r_e) - \lambda_{\tau(e)-t}(s_e)
\end{aligned}
$$

for m a.e. $t \in [0,b)$ by 10:12.1(a)

$$
= \lambda_{c-t}(r_e) - \lambda_{c-t}(s_e)
$$

Hence $\int_A \mu_t(r^+) - \mu_t(s^+)\,dt + \int_B \mu_t(s^-) - \mu_t(r^-)\,dt$

$$
\le \int_A \lambda_t(r_e) - \lambda_t(s_e)\,dt + \int_B \lambda_{c-t}(r_e) - \lambda_{c-t}(s_e)\,dt
$$

$$
= \int_A \lambda_t(r_e) - \lambda_t(s_e)\,dt + \int_{c-B} \lambda_t(r_e) - \lambda_t(s_e)\,dt
$$

$$
= \int_{A\cup(c-B)} \lambda_t(r_e) - \lambda_t(s_e)\,dt \qquad \text{since } A \text{ and } c-B \text{ are disjoint}
$$

$$
\le \int_0^{m(A\cup(c-B))} \lambda_t((r-s)_e)\,dt \qquad \text{by 10:14}
$$

$$
= \int_0^{m(A)+m(B)} \lambda_t((r-s)_e)\,dt \qquad \text{since } A \text{ and } c-B \text{ are disjoint}
$$

$$
\le \int_0^{m(A)+m(B)} \mu_t((r-s)^+)\,dt \qquad \text{by 10:16(a), since } m(A)+m(B) \le a+b = \tau(e)
$$

Now suppose A , B $\subset [0,\infty)$ are of finite measure.

Let $A_n = A \cap [0,n)$ and $B_n = B \cap [0,n)$.

Then $\int_A \mu_t(r^+) - \mu_t(s^+)\, dt + \int_B \mu_t(s^-) - \mu_t(r^-)\, dt$

$$\leq \int_A \mu_t(r^+) - \chi_{A_n}(t)\, \mu_t(s^+)\, dt + \int_B \mu_t(s^-) - \chi_{B_n}(t)\, \mu_t(r^-)\, dt$$

$$= \int_{A-A_n} \mu_t(r^+)\, dt + \int_{B-B_n} \mu_t(s^-)\, dt + \int_{A_n} \mu_t(r^+) - \mu_t(s^+)\, dt + \int_{B_n} \mu_t(s^-) - \mu_t(r^-)\, dt$$

$$\leq \int_{A-A_n} \mu_t(r^+)\, dt + \int_{B-B_n} \mu_t(s^-)\, dt + \int_0^{m(A_n)+m(B_n)} \mu_t((r-s)^+)\, dt$$

$$\leq \int_{A-A_n} \mu_t(r^+)\, dt + \int_{B-B_n} \mu_t(s^-)\, dt + \int_0^{m(A)+m(B)} \mu_t((r-s)^+)\, dt \quad \forall\, n \in \mathbb{N}$$

Now $\mu_t(r^+)$ is bounded on $A-A_n$, and $\mu_t(s^-)$ is bounded on $B-B_n$.

Furthermore, $m(A-A_n) \downarrow 0$, $m(B-B_n) \downarrow 0$ as $n \uparrow \infty$.

Hence $\int_{A-A_n} \mu_t(r^+)\, dt \downarrow 0$, $\int_{B-B_n} \mu_t(s^-)\, dt \downarrow 0$ as $n \uparrow \infty$, by the Dominated Covergence Theorem.

The required result thus follows. □

10:18 **Theorem** [DDd] 3.4

If R , S $\in \tilde{\mathcal{M}}$ then $|\mu_t(R) - \mu_t(S)| \prec\prec \mu_t(R-S)$

Proof

By 2:26, it suffices to prove that $\int_A |\mu_t(R) - \mu_t(S)|\, dt \le \int_0^{m(A)} \mu_t(R-S)\, dt$ for all $A \subset [0,\infty)$ of finite measure.

So suppose we are given such A .

Consider first the case where $r , s \in \mathcal{M}$ and $\mathcal{M}$ is non–atomic.

Let $A_1 = \{ t \in A : \mu_t(r) \ge \mu_t(s) \}$

$A_2 = \{ t \in A : \mu_t(r) < \mu_t(s) \}$

Then $\int_A |\mu_t(r) - \mu_t(s)|\, dt$

$$= \int_{A_1} \mu_t(r) - \mu_t(s)\, dt + \int_{A_2} \mu_t(s) - \mu_t(r)\, dt$$

$$= \int_{A_1} \mu_t(\begin{bmatrix} 0 & r \\ r^* & 0 \end{bmatrix}^+) - \mu_t(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^+)\, dt + \int_{A_2} \mu_t(\begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix}^-) - \mu_t(\begin{bmatrix} 0 & r \\ r^* & 0 \end{bmatrix}^-)\, dt \quad \text{by 9:15(b)}$$

$$\le \int_0^{m(A)} \mu_t((\begin{bmatrix} 0 & r \\ r^* & 0 \end{bmatrix} - \begin{bmatrix} 0 & s \\ s^* & 0 \end{bmatrix})^+) \quad \text{by 10:17}$$

$$= \int_0^{m(A)} \mu_t(\begin{bmatrix} 0 & r-s \\ (r-s)^* & 0 \end{bmatrix}^+)$$

$$= \int_0^{m(A)} \mu_t(r-s)\, dt$$

The restriction that $\mathcal{M}$ is non–atomic is removed in the usual manner.

Thus the result follows if $r , s \in \mathcal{M}$.

Suppose now $R, S \in \tilde{\mathcal{M}}$.

$\exists\ (e_n)_1^\infty \subset \mathcal{M}_p$ such that

$$Re_n, Se_n \in \mathcal{M} \quad \forall\, n \in \mathbb{N}$$

$$Re_n \to R,\ Se_n \to S \text{ in } \tilde{\mathcal{M}}$$

$$\mu_t(Re_n) \uparrow \mu_t(R)$$

$$\mu_t(Se_n) \uparrow \mu_t(S) \qquad (9{:}22(b))$$

Hence $\int_A |\mu_t(Re_n) - \mu_t(Se_n)|\, dt$

$$\leq \int_0^{m(A)} \mu_t((R-S)e_n)\, dt \quad \text{by the result already derived above}$$

$$\leq \int_0^{m(A)} \mu_t(R-S)\, dt \qquad (9{:}18(g))$$

Hence $\int_A |\mu_t(R) - \mu_t(S)|\, dt$

$$= \int_A \liminf_n |\mu_t(Re_n) - \mu_t(Se_n)|\, dt$$

$$\leq \liminf_n \int_A |\mu_t(Re_n) - \mu_t(Se_n)|\, dt \quad \text{by Fatou's Lemma}$$

$$\leq \int_0^{m(A)} \mu_t(R-S)\, dt$$

□

We have now established — in accordance with 10:5 — that if $L_\rho(0,\infty)$ is a symmetric Banach Function Space and ρ is lower semicontinuous then $L_\rho(\tilde{\mathcal{M}})$ is a normed space with norm ρ .

Some further results are available.

10:19 **Proposition**

(a) If $S \in L_\rho(\tilde{\mathcal{M}})$, $R \in \tilde{\mathcal{M}}$ and $|R| \le |S|$ then $R \in L_\rho(\tilde{\mathcal{M}})$ and $\rho(R) \le \rho(S)$

(b) $L_\rho(\tilde{\mathcal{M}})$ is a module over $\mathcal{M}$, and ρ is *solid* in the sense that for r_1 , $r_2 \in \mathcal{M}$, $S \in L_\rho(\tilde{\mathcal{M}})$, $\rho(r_1 \, S \, r_2) \le \|r_1\| \, \rho(S) \, \|r_2\|$

(c) $L_\rho(\tilde{\mathcal{M}})$ is self–adjoint and ρ is invariant under adjoints. Hence adjunction is continuous on $L_\rho(\tilde{\mathcal{M}})$.

Proof

(a)

$|R| \le |S|$

$\Rightarrow \quad \mu_t(R) \le \mu_t(S) \qquad$ (9:18(d))

$\Rightarrow \quad \rho(\mu_t(R)) \le \rho(\mu_t(S)$

$\Rightarrow \quad R \in L_\rho(\tilde{\mathcal{M}})$ and $\rho(R) \le \rho(S)$

(b)

Suppose r_1 , $r_2 \in \mathcal{M}$

Then $S \in L_\rho(\tilde{\mathcal{M}}) \quad \Rightarrow \quad \mu_t(S) \in L_\rho(0,\infty)$

$\Rightarrow \quad \|r_1\| \, \mu_t(S) \, \|r_2\| \in L_\rho(0,\infty)$

$\Rightarrow \quad \mu_t(r_1 \, S \, r_2) \in L_\rho(0,\infty)$

since $\mu_t(r_1 \, S \, r_2) \le \|r_1\| \, \mu_t(S) \, \|r_2\|$ by 9:18(g)

$\Rightarrow \quad r_1 \, S \, r_2 \in L_\rho(\tilde{\mathcal{M}})$

and $\rho(r_1 \, S \, r_2)$

$$\begin{aligned} \rho(r_1 S r_2) &= \rho(\mu_t(r_1 S r_2)) \\ &\le \rho(\|r_1\| \, \mu_t(S) \, \|r_2\|) \\ &= \|r_1\| \, \rho(\mu_t(S)) \, \|r_2\| \\ &= \|r_1\| \, \rho(S) \, \|r_2\| \end{aligned}$$

(c)

Immediate from 9:16(g). □

10:20 **Theorem** [DDd] 4.4

The natural injection $L_\rho(\tilde{\mathcal{M}}) \hookrightarrow \tilde{\mathcal{M}}$ is continuous.

Proof

Suppose $\tilde{\mathcal{M}}(\epsilon,\delta)$ is a neighbourhood of 0 in $\tilde{\mathcal{M}}$.

Of course $\rho(\chi_{(0,\delta)}) < \infty$ by 2:21

If $R \in L_\rho(\tilde{\mathcal{M}})$ and $\rho(R) \le \epsilon \, \rho(\chi_{(0,\delta)})$,

then $\mu_t(R) \ge \mu_\delta(R) \, \chi_{(0,\delta)}$ since $\mu_t(R)$ is a decreasing function.

⇒ $\epsilon \, \rho(\chi_{(0,\delta)}) \ge \rho(R) = \rho(\mu_t(R)) \ge \mu_\delta(R) \, \rho(\chi_{(0,\delta)})$

⇒ $\epsilon \ge \mu_\delta(R)$

⇒ $R \in \tilde{\mathcal{M}}(\epsilon,\delta)$ (9:17(a)) □

10:21 **Theorem** [DDd] 4.5

$(L_\rho(\tilde{\mathcal{M}}), \rho)$ is a Banach Space.

Proof

Suppose $(R_n) \subset L_\rho(\tilde{\mathcal{M}})$ is Cauchy in ρ, then (R_n) is Cauchy in $\tilde{\mathcal{M}}$, by 10:20

⇒ $\exists \, R \in \tilde{\mathcal{M}}$ such that $R_n \longrightarrow R$ in $\tilde{\mathcal{M}}$, by the completeness of $\tilde{\mathcal{M}}$.

⇒ $\mu_t(R_n) \longrightarrow \mu_t(R)$ m a.e. by 9:20(c)

Now $\rho(\mu_t(R_n) - \mu_t(R_k)) \leq \rho(R_n - R_k)$ by 10:18, and so $(\mu_t(R_n))$ is Cauchy in $L_\rho(0,\infty)$. Thus $(\mu_t(R_n))$ has a limit, which is clearly $\mu_t(R)$.

Thus $\mu_t(R) \in L_\rho(0,\infty)$ and so $R \in L_\rho(\tilde{\mathcal{M}})$.

It remains to show that $R_n \longrightarrow R$ in $L_\rho(\tilde{\mathcal{M}})$

Now for $n \in \mathbb{N}$, $R_n - R_k \xrightarrow{k} R_n - R$ in $\tilde{\mathcal{M}}$

$$\Rightarrow \quad \mu_t(R_n - R_k) \xrightarrow{k} \mu_t(R_n - R) \text{ m a.e.}$$

$$\begin{aligned}\Rightarrow \quad \rho(R_n - R) &= \rho(\mu_t(R_n - R)) \\ &\leq \liminf_k \rho(\mu_t(R_n - R_k)) \quad \text{by the lower semicontinuity of } \rho \\ &= \liminf_k \rho(R_n - R_k)\end{aligned}$$

$$\Rightarrow \quad \lim_n \rho(R_n - R) \leq \lim_n \liminf_k \rho(R_n - R_k) = 0 .$$ □

Examples of non–commutative Banach Function Spaces

By 2:17, $L_p(0,\infty)$ $(1 \leq p \leq \infty)$ is a symmetric Banach Function Space and $\|.\|_p$ is lower semicontinuous. Hence $L_{\|\cdot\|_p}(\tilde{\mathcal{M}})$ $(1 \leq p \leq \infty)$ are non–commutative Banach Function Spaces.

We notate $L_{\|\cdot\|_p}(\tilde{\mathcal{M}})$ as $L_p(\tilde{\mathcal{M}})$, and for $S \in L_p(\tilde{\mathcal{M}})$, $\|S\|_{\|\cdot\|_p}$ as $\|S\|_p$.

10:22 **Proposition**

$(L_\infty(\tilde{\mathcal{M}}), \|\cdot\|_\infty) = (\mathcal{M}, \|\cdot\|)$

Proof

$$L_\infty(\tilde{\mathcal{M}}) = \{ S \in \tilde{\mathcal{M}} : \mu_t(S) \in L_\infty \}$$

$$= \quad \{ S \in \tilde{\mathcal{M}} : \sup_{t>0} \mu_t(S) < \infty \}$$

$$= \quad \{ S \in \tilde{\mathcal{M}} : \|S\| < \infty \} \quad \text{by 9:18(a)}$$

$$= \quad \mathcal{M}$$

For $s \in L_\infty(\tilde{\mathcal{M}})$, $\|s\|_\infty = \|\mu_t(s)\|_\infty = \sup_{t>0} \mu_t(s) = \|s\|$ □

In accordance with this result we often denote by $\|s\|_\infty$ the operator norm of $s \in \mathcal{M}$.

10:23 **Proposition**

Suppose $1 \leq p < \infty$

If $S \in L_p(\tilde{\mathcal{M}})$ then $\|S\|_p = \tau(|S|^p)^{\frac{1}{p}}$

Proof

$$\|S\|_p^p \quad = \quad \|\mu_t(S)\|_p^p$$

$$= \quad \int_0^\infty \mu_t(S)^p \, dt$$

$$= \quad \tau(|S|^p)$$

by 9:32 , since the function $[0,\infty) \longrightarrow [0,\infty) : t \longrightarrow t^p$ is continuous and increasing.

Hence $\|S\|_p = \tau(|S|^p)^{\frac{1}{p}}$ □

10:24 **Definition** [Sg] 3.1

$s \in \mathcal{M}$ is said to be an *elementary operator* (*elop*) if $\tau(p_{[R(s)]}) < \infty$.

We will denote the set of elementary operators by $\mathcal{F}$

10:25 **Proposition**

(a) $\mathcal{F} \subset \mathcal{M}_\tau \subset L_p(\tilde{\mathcal{M}}) \quad 1 \leq p \leq \infty$

(b) $\mathcal{F}$ is an ideal of $\mathcal{M}$

Proof

(a)

Suppose $s \in \mathcal{F}$

Then $p_{[R(s)]} \in \mathcal{M}_\tau$, so $s = p_{[R(s)]}\, s \in \mathcal{M}_\tau$ since $\mathcal{M}_\tau$ is a two–sided ideal.

The case $p = \infty$ is clear, so suppose $1 \leq p < \infty$

Suppose $s \in \mathcal{M}_\tau$

$\Rightarrow \quad \tau(|s|) < \infty$

$\Rightarrow \quad \tau(|s|^p) = \tau(|s|\ |s|^{p-1}) < \infty$

$\Rightarrow \quad s \in L_p(\tilde{\mathcal{M}})$

(b)

Suppose $s \in \mathcal{F}$ and $a \in \mathcal{M}$

Then $[R(sa)] \leq [R(s)]$, so $\tau(p_{[R(sa)]}) \leq \tau(p_{[R(s)]}) < \infty$

Thus $\mathcal{F}$ is a right ideal.

Since $R(s) \sim R(s^*)$, $\mathcal{F}$ is closed under adjoints.

Hence $\mathcal{F}$ is an ideal. □

10:26 **Theorem**

$\mathcal{F}$ and $\mathcal{M}_\tau$ are dense in $L_p(\tilde{\mathcal{M}})$ $(1 \le p < \infty)$

Thus $L_p(\tilde{\mathcal{M}})$ is the completion of $\mathcal{F}$ (or $\mathcal{M}_\tau$) in the norm $\|s\|_p = \tau(|s|^p)^{\frac{1}{p}}$

Proof

By 10:25(a), it suffices to restrict attention to $\mathcal{F}$.

First suppose $0 \le S \in L_p(\tilde{\mathcal{M}})$.

Let $s_n = S\, e_{[\frac{1}{n},n]}(S)$. Of course $s_n \in \mathcal{M}$ $\forall n \in \mathbb{N}$

We show that $s_n \in \mathcal{F}$

It suffices to show that $e_{[\frac{1}{n},n]}(S) \in \mathcal{F}$ since then $s_n = e_{[\frac{1}{n},n]}(S)\, s_n \in \mathcal{F}$

$$S \ge \frac{1}{n} e_{[\frac{1}{n},n]}(S)$$

$$\Rightarrow \quad \infty > \|S\|_p^{\ p} \ge \|\tfrac{1}{n} e_{[\frac{1}{n},n]}(S)\|_p^{\ p} = \tau(\frac{1}{n^p} \tau(e_{[\frac{1}{n},n]}(S))) = \frac{1}{n^p} \tau(e_{[\frac{1}{n},n]}(S))$$

$$\Rightarrow \quad \infty > \tau(e_{[\frac{1}{n},n]}(S))$$

i.e. $e_{[\frac{1}{n},n]}(S) \in \mathcal{F}$, since it is its own range projection.

$$e_t(S - s_n) = \begin{cases} e_t(S) & n \le t < \infty \\ e_n(S) & \frac{1}{n} \le t < n \\ e_t(S) & 0 \le t < \frac{1}{n} \end{cases}$$

$$\Rightarrow \quad d_t(S - s_n) = \begin{cases} d_t(S) & n \le t < \infty \\ d_n(S) & \frac{1}{n} \le t < n \\ d_t(S) & 0 \le t < \frac{1}{n} \end{cases}$$

$$\Rightarrow \quad \mu_t(S - s_n) \quad = \quad \begin{cases} \mu_t(S) & d_{(\frac{1}{n})^-}(S) < t \\ \frac{1}{n} & d_n(S) \leq t \leq d_{(\frac{1}{n})^-}(S) \\ \mu_t(S) & t < d_n(S) \end{cases}$$

(where $d_{(\frac{1}{n})^-}(S)$ is the left hand limit of $d_t(S)$ at $\frac{1}{n}$)

Thus $\mu_t(S) \geq \mu_t(S - s_n) \downarrow 0$ as $n \longrightarrow \infty$

Now $\mu_t(S)$ is p–integrable, so by the Dominated Convergence Theorem (for p–integrable functions), $\int_0^\infty \mu_t(S - s_n)^p \, dt \downarrow 0$

Thus $\|S - s_n\|_p \downarrow 0$.

In the general case, let $S = v\,|S|$ and let $s_n = e_{[\frac{1}{n},n]}(|S|)\,|S|$

Then $s_n \in \mathcal{F}$, as before, and so $v\,s_n \in \mathcal{F}$

Thus $\|S - vs_n\|_p$

$$\begin{aligned} &= \quad \|v\,|S| - v\,s_n\|_p \\ &\leq \quad \|v\|_\infty\, \||S| - s_n\|_p \qquad \text{by 10:19(b)} \\ &\longrightarrow \quad 0 \quad \text{as } n \longrightarrow \infty \end{aligned}$$

Thus $\mathcal{F}$ is dense in $L_p(\tilde{\mathcal{M}})$.

The final assertion follows by the completeness of $L_p(\tilde{\mathcal{M}})$. □

We are able to generalise 2:21 to the non–commutative case.

10:27 **Proposition**

(a) $(L_1 \cap L_\infty)(\tilde{\mathcal{M}}) \subset L_\rho(\tilde{\mathcal{M}}) \subset (L_1 + L_\infty)(\tilde{\mathcal{M}}) \subset \tilde{\mathcal{M}}$ as sets and in the sense of continuous embeddings.

(b) $\mathcal{M}_\tau = (L_1 \cap L_\infty)(\tilde{\mathcal{M}})$

Proof

(a)

Suppose $S \in (L_1 \cap L_\infty)(\tilde{\mathcal{M}})$

$\Rightarrow \quad \mu_t(S) \in (L_1 \cap L_\infty)(0,\infty)$

$\Rightarrow \quad \mu_t(S) \in L_\rho(0,\infty) \quad$ by 2:21

$\Rightarrow \quad S \in L_\rho(\tilde{\mathcal{M}})$

$S_n \longrightarrow 0$ in $(L_1 \cap L_\infty)(\tilde{\mathcal{M}})$

$\Rightarrow \quad \mu_t(S_n) \longrightarrow 0$ in $(L_1 \cap L_\infty)(0,\infty)$

$\Rightarrow \quad \mu_t(S_n) \longrightarrow 0$ in $L_\rho(0,\infty)$ by 2:21

$\Rightarrow \quad S_n \longrightarrow 0$ in $L_\rho(\tilde{\mathcal{M}})$

Hence $(L_1 \cap L_\infty)(\tilde{\mathcal{M}}) \subset L_\rho(\tilde{\mathcal{M}})$ as sets and in the sense of continuous embeddings.

A similar argument follows for the case $L_\rho(\tilde{\mathcal{M}}) \subset (L_1 + L_\infty)(\tilde{\mathcal{M}})$

The continuity of the embedding $(L_1 + L_\infty)(\tilde{\mathcal{M}}) \subset \tilde{\mathcal{M}}$ is immediate from 10:20

(b)

$$
\begin{aligned}
& r \in \mathcal{M}_\tau \\
\Leftrightarrow\quad & r \in \mathcal{M} \text{ and } \tau(|r|) < \infty \\
\Leftrightarrow\quad & r \in \mathcal{M} \text{ and } r \in L_1(\tilde{\mathcal{M}}) \quad \text{by 10:23} \\
\Leftrightarrow\quad & \mu_t(r) \in L_\infty(0,\infty) \text{ and } \mu_t(r) \in L_1(0,\infty) \\
\Leftrightarrow\quad & \mu_t(r) \in (L_1 \cap L_\infty)(0,\infty) \\
\Leftrightarrow\quad & r \in (L_1 \cap L_\infty)(\tilde{\mathcal{M}})
\end{aligned}
$$

□

Despite 10:27(b) , it is usual to norm $\mathcal{M}_\tau$ with the $L_1(\tilde{\mathcal{M}})$ norm for reasons that will become apparent in Chapter 12.

11: A BRIEF HISTORY of NON–COMMUTATIVE INTEGRATION THEORY

We assume throughout that $\mathcal{M}$ is a semifinite von Neumann algebra and that τ is a faithful semifinite normal trace on $\mathcal{M}$.

We have arrived at a point where we can profitably give a review of the history of non–commutative integration theory, concentrating on the work of Segal, Stinespring, Kunze, Yeadon, Ovchinnikov, Nelson, Terp, Fack and Kosaki, and Dodds, Dodds, and de Pagter.

We have seen much of the work of Nelson and Terp in Chapter 8, Fack and Kosaki in Chapter 9, and Dodds, Dodds and de Pagter in Chapter 10. There can be no denying that the previous work in the field is either improved or even superceded by these papers. Nevertheless, it would be amiss not to draw attention to the work of the pioneers in the field.

As previously noted, the definition of the set of τ–measurable operators $\widetilde{\mathcal{M}}$ and the result that $\widetilde{\mathcal{M}}$ is a complete topological *–algebra (Chapter 8) is the work of Terp; and $\widetilde{\mathcal{M}}$ is presently considered to be the space of unbounded operators most appropiate for the purposes of non–commutative integration theory. The question of finding a suitable such "universal space" – that is, the "optimal" setting for non–commutative function spaces – was one of the key questions in the theory until the publications of Nelson and Terp.

In answering this question, two basic approaches have been taken. Some authors define algebras of unbounded operators large enough to contain all the spaces (such as L_p spaces) of interest, for example Segal's *measurable operators* and *essentially measurable operators*, Yeadon's modifications of these, and Terp's $\tilde{\mathcal{M}}$. Other approaches identify algebras of unbounded operators in an abstract sense, as the completion of subalgebras of $\mathcal{M}$, for example Dixmier's L_p spaces and Nelson's $\tilde{\mathcal{M}}_N$ and L_p spaces.

We will briefly look at the approach each of the main contributors has taken.

We will also devote some attention to the L_p spaces. Most of the main contributors to the field of non–commutative integration prior to Dodds, Dodds and de Pagter have constructed L_p spaces and some considered such questions as duality. Despite the substantial differences in these definitions and constructions we will show that the variously defined L_p spaces are all isomorphic to the spaces $L_p(\tilde{\mathcal{M}})$.

11:1

The general method for achieving the latter in the case $1 \leq p < \infty$ for each of these definitions will be to :–

(a) observe that L_p is complete;

(b) observe that $\mathcal{F}$ (or $\mathcal{M}_\tau$) is dense in L_p;

(c) observe that for the operators in (b) the L_p norm coincides with the $L_p(\tilde{\mathcal{M}})$ norm, $\|s\|_p = \tau(|s|^p)^{\frac{1}{p}}$. (10:23)

It would then follow from 10:26 that L_p and $L_p(\tilde{\mathcal{M}})$ are both completions in $\|.\|_p$ of $\mathcal{F}$ (or $\mathcal{M}_\tau$), and hence that L_p is isometrically isomorphic to $L_p(\tilde{\mathcal{M}})$.

It has been accepted practice since the time of Segal to define L_∞ as $\mathcal{M}$ and to denote by $\|s\|_\infty$ the operator norm of $s \in \mathcal{M}$. Of course 10:22 shows that modern theory is in agreement with this convention.

I. E. Segal ([Sg] , submitted 1951, published 1953)

Segal considered an algebra of *measurable operators* :—

11:2 **Definition** [Sg] Definition 2.1

A subspace E of $\mathcal{H}$ is *strongly dense* in $\mathcal{H}$ if

$uE = E \quad \forall u \in \mathcal{M}'_u$

$\exists \{p_n\} \subset \mathcal{M}_p$ such that

$p_n\mathcal{H} \subset E \quad \forall n \in \mathbb{N}$

$1 - p_n$ is a finite projection $\forall n \in \mathbb{N}$

$1 - p_n \downarrow_{so} 0$

An operator S is *measureable* if

$S \, \eta \, M$

S has a strongly dense domain

S is closed.

An operator S is *essentially measureable* if

$S \, \eta \, M$ and S is preclosed

$\exists \{p_n\} \subset \mathcal{M}_p$ such that

$p_n\mathcal{H} \subset D(S) \quad \forall n \in \mathbb{N}$

$\|S\, p_n\| < \infty \quad \forall n \in \mathbb{N}$

$1 - p_n$ is a finite projection $\forall n \in \mathbb{N}$

$1 - p_n \downarrow_{so} 0$

It is clear from the Closed Graph Theorem that a measurable operator is essentially measurable.

Segal shows that the collection of measurable operators forms an algebra with respect to adjoint, strong sum and strong product. [Sg] Corollary 5.2 .

11:3 **Definition** [Sg] Definition 2.3

A sequence of measurable operators S_n converges *nearly everywhere* to a measurable operator S if $\forall\, \epsilon > 0\ \exists\, \{p_n\} \subset \mathcal{M}_p$ such that

$$p_n \underset{so}{\uparrow} 1$$

$$\|(S_n - S)p_n\| \le \epsilon \quad \forall\, n \in \mathbb{N}$$

$1 - p_n$ is a finite projection $\forall\, n \in \mathbb{N}$

Segal's definition of measurable operators can be compared to that of τ–measurable operators, and nearly everywhere convergence to convergence in measure. Obviously the crucial difference is that the complement of certain projections is required to be algebraically finite, rather than to have finite trace. In fact, the algebra of measurable operators and nearly everywhere convergence are defined independently of the trace. In particular, these constructions are valid for any von Neumann algebra.

Segal uses the trace to define L_1 and L_2 spaces :–

11:4 **Definition** [Sg] Definition 3.2

Suppose $s \in \mathcal{F}$

Then the L_1 norm of s is $\|s\|_1 = \sup \{\ |\tau(as)|\ : a \in \mathcal{M},\ \|a\|_\infty \le 1\ \}$

We will see in 12:1 that this definition of $\|.\|_1$ for members of $\mathcal{F}$ coincides with the modern definition.

11:5 **Definition** [Sg] Definition 3.3

A measurable operator S is called *integrable* if it is the nearly everwhere limit of a sequence $(s_n) \subset \mathcal{F}$ that is Cauchy in L_1. The collection of all integrable operators is denoted as L_1. For such S, $\tau(S)$ is defined as $\lim_n \tau(s_n)$.

This is well defined by [Sg] Remark 3.1 and Theorem 11.

11:6 **Definition** [Sg] Definition 3.4

For $S \in L_1$, $\|S\|_1 = \sup \{ |\tau(r\,S)| : r \in \mathcal{M}, \|r\| \leq 1 \}$

It is shown that L_1 is a normed space under $\|.\|_1$ ([Sg] Corollary 11.3) and subsequently that $\|S\|_1 = \tau(|S|)$ ([Sg] Corollary 11.14)

Segal also defines an L_2 space of measurable operators

11:7 **Definition** [Sg] Definition 3.7 , 3.8

A measurable operator S is *square–integrable* if S can be expressed as $S = S_1 + iS_2$ with S_1, S_2 symmetric measurable operators and $S_1^{\,2}$, $S_2^{\,2}$ integrable.

The collection of all square integrable operators is denoted L_2

$$\|S\|_2 = \tau(|S|^2)^{\frac{1}{2}}$$

It is shown that L_2 is a normed space under $\|.\|_2$ ([Sg] Corollary 12.12)

The approach of Segal is comparable to the classical approach to integration, where the integral is first defined for a set of easily accessible functions and then extended to a larger set by a prescribed limiting or completion process.

In accordance with 11:1, we note that

(a) L_p is complete [Sg] Theorem 13

(b) $\mathcal{F}$ is dense in L_p by definition in the case p = 1, and by construction in the case p = 2 (See [Sg] § 3.4)

(c) For members of $\mathcal{F}$, Segal's $\|.\|_p$ norm coincides with the $L_p(\tilde{\mathcal{M}})$ norms. (p = 1 , 2) .

Hence Segal's L_p spaces are isomorphic to $L_p(\tilde{\mathcal{M}})$ (p = 1 , 2)

W. F. Stinespring ([St] submitted 1956, published 1959)

The work of Segal was accepted without modification by Stinespring.

Stinespring seems to have been the first to introduce *convergence in measure* for sequences of measurable operators.

11:8 **Definition** [St] Definition p 23

A sequence $\{S_n\}$ of measurable operators *converges in measure* to a measurable operator S if $\forall\, \epsilon > 0\ \exists\ \{p_n\} \subset \mathcal{M}_p$ such that $\|(S_n - S)\, p_n\| < \epsilon$ and $\tau(1-p_n) \longrightarrow 0$

Modulo the difference between measurability and τ–measurability, this is the mode of convergence determined by Nelson and Terp's topology of convergence in measure. Thus Stinespring was the first to realise that the finiteness of the trace on the complement of certain projections would be a useful notion, rather than simply (algebraic) finiteness. (Note that a projection with finite trace must be finite, by the faithfulness of τ, whereas the converse is not neccessarily true.)

Furthermore Stinespring defined *gross convergence*, also determined by the trace. This mode of convergence is weaker than both nearly everwhere convergence and convergence in measure. Stinespring's chief motivation for introducing gross convergence seems to be that he was able to derive better continuity results with respect to the algebraic operations under gross convergence than under nearly everywhere convergence or convergence in measure. (See [St] Lemma 4.5 and the note preceeding it.)

R. Kunze ([Ku] submitted 1957, published 1958)

Kunze was the first to give a concrete definition of all the L_p spaces. He uses Segal's measurable operators and extension process for τ, and for a measurable operator S makes the following definition :

11:9 **Definition** [Ku] Definition 3.1

$$\|S\|_p = \tau(|S|^p)^{\frac{1}{p}} \quad (1\leq p<\infty)$$

$$L_p = \{ S \text{ measurable} : \|S\|_p < \infty \}$$

This improves some earlier work of Dixmier ([D1] , 1953) where L_p spaces are identified in an abstract manner, as completions of members of $\mathcal{M}_\tau$ under a p–norm. Kunze also established a Holder inequality for the L_p spaces ([Ku] Lemma 1.4) .

In accordance with 11:1, we note that

(a) L_p is complete [Ku] Theorem 2

(b) $\mathcal{F}$ is dense in L_p [Ku] Corollary 1.2

(c) By definition, Kunze's L_p norm on $\mathcal{F}$ coincides with the $L_p(\tilde{\mathcal{M}})$ norm.

F. J. Yeadon ([Y1] , [Y2] , [Y3] , [Y4] , 1968 to 1980)

Yeadon made some modifications to the definition of measurable and essentially measurable operators ([Y1] 2.1.1) and corresponding modifications to the notions of nearly everwhere convergence ([Y1] 2.2.1) and gross convergence ([Y1] 2.2.4) . Nevertheless, the basic approach is similar to that of Segal and Stinespring and so we shall not discuss these differences here.

In the commutative case, Yeadon's gross convergence reduces to local convergence in measure. In [Y2] a *topology of convergence locally in measure* is defined, under which convergence of sequences coincides with gross convergence. ([Y2] Definition 3.1) This topology is defined via the centre valued dimension function and the representation of the centre as a space $L_\infty(X,\Sigma,\mu)$.

11:10 **Definition** [Y1] 2.2.14

$\mathcal{K} = \{ S \text{ measurable} : \exists\, p \in \mathcal{M}_p \text{ such that } \|S\, p\| < \infty ,\ \tau(1-p) < \infty \}$

$S \in \mathcal{K}$ iff $\exists\, t > 0$ such that $\tau(e_{(t,\infty)}(|S|)) < \infty$

For $S \in \mathcal{K}$, the *rearrangement* of S is

$$S^{\sim}(t) = \inf \{ \theta \geq 0 : \tau(e_{(\theta,\infty)}(|S|)) \leq t \}$$

Clearly $\mathcal{K}$ can be compared to $\tilde{\mathcal{M}}$ and the rearrangement to the generalised singular function.

A metrisable topology is defined on $\mathcal{K}$ which has as a base of neighbourhoods of 0

$$\{ \{ S \in \mathcal{K} : S^{\sim}(t) \leq t \} : t > 0 \}$$ ([Y1] 2.2.17 and 18)

For sequences in $\mathcal{K}$, convergence in this topology agrees with Stinespring's convergence in measure.

L_p spaces are also defined in a way similar to Segal (in the case p = 1) and to Kunze (in the general case) ([Y1] § 3). The L_p spaces are defined as subspaces of the algebra of measurable operators, but it is easy to show that the L_p spaces are included in $\mathcal{K}$. Perhaps therefore it is surprising that Yeadon did not choose as his universal space $\mathcal{K}$ rather than the algebra of measurable operators.

Yeadon also had available the formulation $\|S\|_p^{\,p} = \int S^{\sim}(t)^p \, dt$ ([Y1] 3.4.1)
He uses this to show that for $S, R \in \mathcal{K}$, $\|S R\|_1 \leq \int S^{\sim}(t)\, R^{\sim}(t) \, dt$ ([Y1] 3.4.7)
This result is in turn used to deduce the Holder inequality. ([Y1] 3.4.8)

We will see generalisations of these results in Chapter 12.

Yeadon also established a Radon–Nikodym type theorem for $\mathcal{M}$ non–atomic and uses this to show that L_q is isometrically isomorphic to $L_p^{\,*}$ $(\frac{1}{p} + \frac{1}{q} = 1)$ ([Y1] § 3.5)

Normed modules of measurable operators are considered, generalising commutative Banach Function Spaces. This generalisation is appealing in the sense that many results in the commutative theory have close analogues that also hold in this setting. For example, the classic results on completeness, weak Fatou and Fatou norms, perfect spaces and reflexivity have analogues in this theory. ([Y1] § 4)

11:11 **Definition** [Y1] 4.3.1

For a module $\mathcal{A}$ of measurable operators the *associate space*

$$\mathcal{A}^{\times} = \{ S \text{ measurable} : S R \in L_1 \ \forall R \in \mathcal{A} \}$$

Some duality theory is developed – we will see more on this in Chapter 12.

In [Y1] § 5.1 it is shown that *symmetric modules* of measurable operators (modules closed under rearrangements) are in a 1–1 correspondence with sets of measurable functions on the semiaxis $(0,\infty)$ satisfying properties typical of a symmetric Banach Function Space. This idea has obviously been brought to fruition in the work of Dodds, Dodds and de Pagter.

In subsequent works ([Y2] , [Y3] , [Y4]) Yeadon refined some of the contents of [Y1] . It is unfortunate that the contents of [Y1] were not made more generally available – [Y1] is a dissertation and not a publication as such – as almost all subsequent work in the field is anticipated to a certain extent therein. It is clear that many workers in the field were not aware of the groundwork that had been laid in [Y1] .

Yeadon has ([Y4] , 2.4) a result quite similar to 10:5 . (He does not consider questions of completeness, however.) In this result Yeadon assumes that ρ has the Fatou property, which is a stronger condition than lower semicontinuity.

In accordance with 11:1, we note that

(a) L_p is complete [Y3] Theorem 3.7(iii)

(b) $\mathcal{F}$ is dense in L_p [Y3] Theorem 3.7(ii)

(c) As for Kunze.

V. I. Ovchinnikov ([Ov1] , [Ov2] , 1970)

In [Ov1], Ovchinnikov considers the algebra of measurable operators as defined by Segal. Independently of Yeadon, he defined the distribution function and s–numbers (≡ generalised singular function) for those measurable operators for which $\tau(e_{(\lambda,\infty)}(R)) < \infty \ \forall \ \lambda > 0$.

A number of results anologous to those of Chapter 9 are proved for this class of operators. For such operators $R \geq 0$ a spectral family $p_t(R)$ is derived such that $R = \int_0^\infty \mu_t(R)\, dp_t(R)$. This representation of operators is called a *Schmidt Representation.* Useful generalisations of this and other results will be seen in Chapter 12.

In [Ov2] the notion of symmetric normed spaces of measurable operators is introduced. Some interpolation theory is developed for such symmetric spaces.

E. Nelson ([N] submitted 1972, published 1974)

The paper of Nelson [N] is generally considered to be the starting point of the modern theory of non–commutative integration. As we have seen in Chapter 8, Nelson defined the topology of convergence in measure on $\mathcal{M}$ and defined $\tilde{\mathcal{M}}$ (we have denoted this by $\tilde{\mathcal{M}}_N$ in Chapter 8) to be the abstract completion of $\mathcal{M}$. He also shows ([N] Theorem 4) that each member of $\tilde{\mathcal{M}}_N$ can be represented as a member of $\bar{\mathcal{M}}$. He shows that $S \in \bar{\mathcal{M}}$ is such an operator iff
$\tau(e_{(t,\infty)}(|S|)) \longrightarrow 0$ as $t \longrightarrow \infty$ (compare 8:17)

In [N] § 3, it is claimed that $\mathcal{M}_\tau$ is a normed space in the norm $\|s\|_p = \tau(|s|^p)^{\frac{1}{p}}$ – the proof is difficult, but this is 10:23 – and defines L_p to be the Banach Space completion of this normed space. He shows ([N] Theorem 5) that there is a canonical injection $L_p \hookrightarrow \tilde{\mathcal{M}}_N$.
It follows immediately from 10:26 that Nelson's L_p spaces are isomorphic to $L_p(\tilde{\mathcal{M}})$.

M. Terp (U. Haagerup) ([Tp] 1981)

By defining τ–measurability, Terp used arguments based on those of Nelson to give a concrete description of the algebra $\tilde{\mathcal{M}}$ of τ–measurable operators. We examined this construction in Chapter 8.

Terp defines [Tp] p 23

$$L_p = \{ S \in \overline{\mathcal{M}} : \tau(|S|^p) < \infty \}$$

$$\|S\|_p = \tau(|S|^p)^{\frac{1}{p}}$$

We have the following result:–

11:12 **Proposition**

Suppose $0 \leq S \in \overline{\mathcal{M}}$ and $\tau(|S|^p) < \infty$

Then $S \in \tilde{\mathcal{M}}$ and so $S \in L_p(\tilde{\mathcal{M}})$.

Proof

$$|S| \geq t\, e_{(t,\infty)}(|S|) \quad \forall\, t > 0$$

$$\Rightarrow \quad |S|^p \geq t^p\, e_{(t,\infty)}(|S|) \quad \forall\, t > 0$$

$$\Rightarrow \quad t^p\, \tau(e_{(t,\infty)}(|S|))$$

$$= \quad \tau(t^p\, e_{(t,\infty)}(|S|))$$

$$\leq \quad \tau(|S|^p)$$

$$< \quad \infty$$

$$\Rightarrow \quad \tau(e_{(t,\infty)}(|S|)) < \infty \quad \forall\, t > 0$$

In particular, $S \in \tilde{\mathcal{M}}$. □

It follows that Terp's definition of the L_p spaces coincide with that of $L_p(\tilde{\mathcal{M}})$.

We also point out that Haagerup and Terp have defined L_p spaces for arbitrary von Neumann algebras by using *weights* rather than traces (a weight is similar to a trace, except that it does not satisfy the commutativity condition). For details, see [Tp] § 2. See also [Tp1] for an approach which defines L_p spaces as interpolation spaces between $\mathcal{M}$ and $\mathcal{M}_*$.

T. Fack and H. Kosaki ([FK] 1986)

Fack and Kosaki define generalised s–numbers of the τ–measurable operators. We dealt with this in Chapter 9. This formulation is in part inspired by the rearrangement of Yeadon, and as Fack and Kosaki point out ([FK] , Introduction) the algebra $\tilde{\mathcal{M}}$ is the natural domain for generalising the rearrangement : " .. the τ–measurability of an operator S exactly corresponds to the property $\mu_t(S) < \infty$, $t > 0$ and the [topology of convergence in measure] can easily and naturally be expressed in terms of μ_t. "

We note that Fack and Kosaki's definition of the L_p spaces coincides with that of Terp. [FK] 1.1

P. Dodds, T. K.–Y. Dodds, and B. de Pagter ([DDd] 1989)

The most recent formulation – the definition of the spaces $L_\rho(\tilde{\mathcal{M}})$ – is that of Dodds, Dodds and de Pagter which was presented in Chapter 10.

12 : DUALITY THEORY

We suppose throughout that $\mathcal{M}$ is a semifinite von Neumann algebra and τ a faithful semifinite normal trace on $\mathcal{M}$.

We suppose that $L_\rho(0,\infty)$ is a symmetric Banach Function Space and ρ is lower semicontinuous – thus $L_\rho(\tilde{\mathcal{M}})$ is of the type considered in Chapter 10.

Despite 10:27(b), we choose to norm $\mathcal{M}_\tau$ with the $L_1(\tilde{\mathcal{M}})$ norm for the purposes of the following and subsequent results. Recall that $(\mathcal{M}_\tau , \|\cdot\|_1)$ is dense in $L_1(\tilde{\mathcal{M}})$ (10:26) .

12:1 **Proposition** cf. [T] pp 319, 320

Suppose $s \in \mathcal{M}_\tau$

(a) $|\tau(s)| \leq \tau(|s|) = \|s\|_1$

(b) $\|s\|_1 = \tau(|s|) = \sup \{ |\tau(a\, s)| : a \in \mathcal{M}, \|a\|_\infty \leq 1 \}$

Proof

Let $s = v|s|$ be the polar decomposition of s

(a)

$u|s|^{1/2}, |s|^{1/2} \in \mathcal{N}_\tau$

$$\text{Thus } |\tau(s)|^2 = |\tau(v\,|s|^{1/2}\,|s|^{1/2})|^2 \leq \tau(|s|^{1/2}\, v^* v\, |s|^{1/2})\, \tau(|s|^{1/2}\, |s|^{1/2})$$

by the Cauchy–Schwartz inequality for positive linear functionals. ([T] I 9.5)

$$= \tau(|s|)^2$$

Hence $|\tau(s)| \leq \tau(|s|)$

(b)

Suppose $a \in \mathcal{M}$, $\|a\|_\infty \leq 1$

$$
\begin{aligned}
\text{Then } |\tau(a\,s)| &\leq \tau(|a\,s|) \quad \text{by (a), since } as \in \mathcal{M}_\tau \\
&= \|a\,s\|_1 \\
&\leq \|a\|_\infty \|s\|_1 \quad \text{by 10:19} \\
&\leq \|s\|_1 \\
&= \tau(|s|)
\end{aligned}
$$

Conversely, $\tau(|s|) = \tau(v^* s) = |\tau(v^* s)|$; $v^* \in \mathcal{M}$, $\|v^*\|_\infty \leq 1$. □

12:2 **Definition**

It follows from 12:1(a) that τ (as defined in 3:6) is a continuous positve linear functional on $(\mathcal{M}_\tau , \|.\|_1)$ of norm 1, hence it extends to a continuous linear functional of norm 1 on $L_1(\tilde{\mathcal{M}})$, the completion of $(\mathcal{M}_\tau , \|.\|_1)$ (10:26) , which will also be denoted τ .

Note that adjunction is continuous on $L_1(\tilde{\mathcal{M}})$, and that multiplication by a fixed member of $\mathcal{M}$ is also continuous (10:19) . It thus follows from 3:6 and the continuity of τ that

$$\tau(S^*) = \overline{\tau(S)} \quad \forall\, S \in L_1(\tilde{\mathcal{M}})$$

$$\tau(a\,S) = \tau(S\,a) \quad \forall\, S \in L_1(\tilde{\mathcal{M}}) \quad \forall\, a \in \mathcal{M}$$

12:3 **Note**

We now have four (!) notions of *trace*

the original faithful semifinite normal trace on $\mathcal{M}^+$

a continuous positive linear functional on $\mathcal{M}_\tau$ (3:6)

a continuous linear functional on $L_1(\tilde{\mathcal{M}})$ (12:2)

a $[0,\infty]$ valued function on $\tilde{\mathcal{M}}^+$ (9:24)

However, there is no danger of confusion, as on the intersection of any two of these four given sets, the values of τ agree.

To verify this, it is clear that it suffices to show that the extension process defined in 9:24 coincides with the extension process in 12:2 for operators $S \in L_1(\tilde{\mathcal{M}}) \cap \tilde{\mathcal{M}}^+$.

It follows from the proof of 10:26 that $\exists \{s_n\} \subset \mathcal{M}_\tau$ such that $0 \leq s_n \uparrow S$ in $L_1(\tilde{\mathcal{M}})$. Thus the extension process in 12:2 determines that $\tau(S) = \lim_n \tau(s_n)$. But by 10:20, $s_n \uparrow S$ in $\tilde{\mathcal{M}}$, and so the extension process in 9:24 gives $\tau(S) = \lim_n \tau(s_n)$ by 9:31(c). Hence the extension processes for τ agree.

Before proceeding to the general duality theory, we first highlight the classical result that $L_\infty(\tilde{\mathcal{M}}) \cong L_1(\tilde{\mathcal{M}})^*$

12:4 **Theorem** cf. [T] V 2.18

$(\mathcal{M}_\tau, \|\cdot\|_1)$ can be identified as a subspace of $\mathcal{M}_*$

Furthermore $L_1(\tilde{\mathcal{M}}) \cong \mathcal{M}_*$

$L_\infty(\tilde{\mathcal{M}}) \cong L_1(\tilde{\mathcal{M}})^*$

Proof

Since τ is a linear functional on the ideal $\mathcal{M}_\tau$, we can, for $s \in \mathcal{M}_\tau$, define a linear functional π_s on $\mathcal{M}$ by $\pi_s(a) = \tau(as)$ $(a \in \mathcal{M})$.

Then $\pi_s \in \mathcal{M}^*$ and $\|\pi_s\| = \|s\|_1$ since

$$\begin{aligned} \|\pi_s\| &= \sup\{|\pi_s(a)| : a \in \mathcal{M}, \|a\|_\infty \leq 1\} \\ &= \sup\{|\tau(as)| : a \in \mathcal{M}, \|a\|_\infty \leq 1\} \\ &= \tau(|s|) \quad \text{by 12:1(b)} \\ &= \|s\|_1 \end{aligned}$$

If $s \in (\mathcal{M}_\tau)^+$ and if $0 \leq a_i \underset{so}{\uparrow} a$ in $\mathcal{M}$, then

$$\begin{aligned}
\pi_s(a_i) &= \tau(a_i\, s) \\
&= \tau(a_i\, s^{1/2}\, s^{1/2}) \\
&= \tau(s^{1/2}\, a_i\, s^{1/2}) && \text{by 3:6 since } a_i\, s^{1/2},\ s^{1/2} \in \mathcal{N}_\tau \\
&\uparrow \tau(s^{1/2}\, a\, s^{1/2}) && \text{by the normality of } \tau, \\
& && \text{and since } s^{1/2}\, a_i\, s^{1/2} \uparrow s^{1/2}\, a\, s^{1/2}, \\
&= \tau(a\, s) && \text{since } a\, s^{1/2},\ s^{1/2} \in \mathcal{N}_\tau \\
&= \pi_s(a)
\end{aligned}$$

Thus π_s is normal, ie. uw–continuous, and so $\pi_s \in \mathcal{M}_*$

Since $(\mathcal{M}_\tau)^+$ spans $\mathcal{M}_\tau$ linearly, it follows that $\pi_s \in \mathcal{M}_*$ $\forall\, s \in \mathcal{M}_\tau$

Hence we have an insometry $(\mathcal{M}_\tau \,.\|.\|_1) \hookrightarrow \mathcal{M}_* : s \longrightarrow \pi_s$

We now show $L_1(\tilde{\mathcal{M}}) \cong \mathcal{M}_*$

$L_1(\tilde{\mathcal{M}})$ is the closure of $\mathcal{M}_\tau$ under $\|.\|_1$, so it suffices to show $\mathcal{M}_\tau$ is dense in $\mathcal{M}_*$.

We will see that it suffices to show that $\mathcal{M}_\tau$ separates the points of $\mathcal{M}$.

Suppose $0 \neq a \in \mathcal{M}$

Choose $(\mathcal{M}_\tau)_p \ni p_i \underset{so}{\uparrow} 1$ (possible by 3:12 and 3:29(d))

$\Rightarrow$ $|a^*|\, p_i\, |a^*| \underset{so}{\uparrow} |a^*|\, |a^*| = aa^* \neq 0$

$\Rightarrow$ $\exists\, i \in I$ such that $|a^*|\, p_i\, |a^*| \neq 0$

$\Rightarrow$ $\tau(|a^*|\, p_i\, |a^*|) \neq 0$ by the faithfulness of τ

$$\begin{aligned}
\text{Thus } 0 &\neq \tau(|a^*|\, p_i\, |a^*|) \\
&= \tau(|a^*|^2 p_i) \\
&= \tau(aa^* p_i) \\
&= \pi_{a^* p_i}(a)
\end{aligned}$$

Now certainly $a^* p_i \in \mathcal{M}_\tau$, so $\mathcal{M}_\tau$ separates the points of $\mathcal{M}$

$\mathcal{M}_*{}^* = \mathcal{M}$, so by the theory of bipolars, $\text{cl}\ \mathcal{M}_\tau = \mathcal{M}_\tau{}^{oo}$, (since $\mathcal{M}_\tau$ is a vector space, hence absolutely convex.)

But $\mathcal{M}_\tau{}^o = \{0\}$ since $\mathcal{M}_\tau$ separates the points of $\mathcal{M}$

Thus $\mathcal{M}_\tau{}^{oo} = \{0\}^o = \mathcal{M}_*$.

Thus $(\mathcal{M}_\tau, \|.\|_1)$ is dense in $\mathcal{M}_*$, so it follows $L_1(\tilde{\mathcal{M}}) \cong \mathcal{M}_*$

Now $L_\infty(\tilde{\mathcal{M}}) = \mathcal{M}$ and $\mathcal{M}_*{}^* \cong \mathcal{M}$, hence $L_1(\tilde{\mathcal{M}})^* \cong L_\infty(\tilde{\mathcal{M}})$ □

We now proceed to general duality questions.

12:5 Definition

$L_\rho(\tilde{\mathcal{M}})^\times = \{ R \in \tilde{\mathcal{M}} : R\,S \in L_1(\tilde{\mathcal{M}})\ \ \forall\, S \in L_\rho(\tilde{\mathcal{M}}) \}$

It is clear that $L_\rho(\tilde{\mathcal{M}})^\times$ is a linear subspace of $\tilde{\mathcal{M}}$.

If $a_1, a_2 \in \mathcal{M}$, $R \in L_\rho(\tilde{\mathcal{M}})^\times$ then

$R\,S \in L_1(\tilde{\mathcal{M}})\ \ \forall\, S \in L_\rho(\tilde{\mathcal{M}})$

$\Rightarrow$ $R\,a_2\,S \in L_1(\tilde{\mathcal{M}})\ \ \forall\, S \in L_\rho(\tilde{\mathcal{M}})$ as $L_\rho(\tilde{\mathcal{M}})$ is a module

$\Rightarrow$ $a_1\,R\,a_2\,S \in L_1(\tilde{\mathcal{M}})\ \ \forall\, S \in L_\rho(\tilde{\mathcal{M}})$ as $L_1(\tilde{\mathcal{M}})$ is a module

So $a_1\,R\,a_2 \in L_\rho(\tilde{\mathcal{M}})^\times$ and thus $L_\rho(\tilde{\mathcal{M}})^\times$ is a module.

Suppose $R \in L_\rho(\tilde{\mathcal{M}})^\times$ and $R = v\,|R|$ is the polar decomposition of R.

Then $R\,S = v\,|R|\,S \in L_1(\tilde{\mathcal{M}})\ \ \forall\, S \in L_\rho(\tilde{\mathcal{M}})$

$\Leftrightarrow$ $v^*\,v\,|R|\,S = |R|\,S \in L_1(\tilde{\mathcal{M}})\ \ \forall\, S \in L_\rho(\tilde{\mathcal{M}})$ since $L_1(\tilde{\mathcal{M}})$ is a module

Hence $L_\rho(\tilde{\mathcal{M}})^\times = \{ R \in \tilde{\mathcal{M}} : |R|\,S \in L_1(\tilde{\mathcal{M}})\ \ \forall\, S \in L_\rho(\tilde{\mathcal{M}}) \}$

Furthermore $R\,S \in L_1(\tilde{\mathcal{M}}) \quad \forall\, S \in L_\rho(\tilde{\mathcal{M}})$

$\Rightarrow \quad |R|\,S \in L_1(\tilde{\mathcal{M}}) \quad \forall\, S \in L_\rho(\tilde{\mathcal{M}})$ since $L_\rho(\tilde{\mathcal{M}})$ is a module

$\Rightarrow \quad |R|\,v^*\,S \in L_1(\tilde{\mathcal{M}}) \quad \forall\, S \in L_\rho(\tilde{\mathcal{M}})$ since $L_\rho(\tilde{\mathcal{M}})$ is a module

$\Rightarrow \quad R^*\,S \in L_1(\tilde{\mathcal{M}}) \quad \forall\, S \in L_\rho(\tilde{\mathcal{M}})$ since $R^* = |R|\,v^*$

So $L_\rho(\tilde{\mathcal{M}})^\times$ is self–adjoint, and it follows

$$L_\rho(\tilde{\mathcal{M}})^\times = \{\, R \in \tilde{\mathcal{M}} : R^*\,S \in L_1(\tilde{\mathcal{M}}) \quad \forall\, S \in L_\rho(\tilde{\mathcal{M}}) \,\}$$

Furthermore $R^*\,S \in L_1(\tilde{\mathcal{M}}) \quad \forall\, S \in L_\rho(\tilde{\mathcal{M}})$

$\Leftrightarrow \quad S^*\,R \in L_1(\tilde{\mathcal{M}}) \quad \forall\, S \in L_\rho(\tilde{\mathcal{M}})$ as $L_1(\tilde{\mathcal{M}})$ is self–adjoint

$\Leftrightarrow \quad S\,R \in L_1(\tilde{\mathcal{M}}) \quad \forall\, S \in L_\rho(\tilde{\mathcal{M}})$ as $L_\rho(\tilde{\mathcal{M}})$ is self–adjoint

Hence $L_\rho(\tilde{\mathcal{M}})^\times = \{\, R \in \tilde{\mathcal{M}} : S\,R \in L_1(\tilde{\mathcal{M}}) \quad \forall\, S \in L_\rho(\tilde{\mathcal{M}}) \,\}$

12:6 Definition

It follows from the definition of $L_\rho(\tilde{\mathcal{M}})^\times$ that we may, for any $R \in L_\rho(\tilde{\mathcal{M}})^\times$ define a (obviously linear) mapping $\underline{R} : L_\rho(\tilde{\mathcal{M}}) \longrightarrow L_1(\tilde{\mathcal{M}}) : S \longrightarrow R\,S$

12:7 Proposition

For $R \in L_\rho(\tilde{\mathcal{M}})^\times$, $\underline{R}$ is continuous on $L_\rho(\tilde{\mathcal{M}})$

Proof

$L_\rho(\tilde{\mathcal{M}})$ and $L_1(\tilde{\mathcal{M}})$ are Banach Spaces and $\underline{R}$ is linear. Thus by the Closed Graph Theorem it suffices to show $\underline{R}$ has closed graph.

So suppose $S_n \longrightarrow S$ in $L_\rho(\tilde{\mathcal{M}})$, $R S_n \longrightarrow T$ in $L_1(\tilde{\mathcal{M}})$

We need to show $T = R S$

$S_n \longrightarrow S$ in $L_\rho(\tilde{\mathcal{M}})$

$\Rightarrow$ $S_n \longrightarrow S$ in $\tilde{\mathcal{M}}$ (10:20)

$\Rightarrow$ $R S_n \longrightarrow R S$ in $\tilde{\mathcal{M}}$ by the continuity of multiplication in $\tilde{\mathcal{M}}$

and $R S_n \longrightarrow T$ in $L_1(\tilde{\mathcal{M}})$

$\Rightarrow$ $R S_n \longrightarrow T$ in $\tilde{\mathcal{M}}$ (10:20)

Hence $R S = T$ since $\tilde{\mathcal{M}}$ is Hausdorff. □

12:8 **Definition**

For $R \in L_\rho(\tilde{\mathcal{M}})^\times$ we define $f_R : L_\rho(\tilde{\mathcal{M}}) \longrightarrow \mathbb{C} : S \longrightarrow \tau(RS)$

f_R is continuous as it is the composition $L_\rho(\tilde{\mathcal{M}}) \xrightarrow{R} L_1(\tilde{\mathcal{M}}) \xrightarrow{\tau} \mathbb{C}$, both of which are continuous on the indicated spaces. (12:7 and 12:2)

12:9 **Proposition**

The mapping $L_\rho(\tilde{\mathcal{M}})^\times \longrightarrow L_\rho(\tilde{\mathcal{M}})^*$: $R \longrightarrow f_R$ is injective

Proof

Suppose $R \in L_\rho(\tilde{\mathcal{M}})^\times$.

Suppose $f_R = 0$

$\Rightarrow$ $\tau(R\,S) = 0 \quad \forall\, S \in L_\rho(\tilde{\mathcal{M}})$

$\Rightarrow$ $\tau(R\,s) = 0 \quad \forall\, s \in \mathcal{M}_\tau$

$\Rightarrow \quad \tau(R\, e_{[0,n]}(|R|)\, s) = 0 \quad \forall\, n \in \mathbb{N}\ \forall\, s \in \mathcal{M}_\tau$,

since $\{ e_{[0,n]}(|R|)\, s : n \in \mathbb{N}, s \in \mathcal{M}_\tau \} \subset \mathcal{M}_\tau$ ($\mathcal{M}_\tau$ is an ideal of $\mathcal{M}$)

$\Rightarrow \quad R\, e_{[0,n]}(|R|) = 0 \quad \forall\, n \in \mathbb{N}$ as $\mathcal{M}_\tau$ separates the points of $\mathcal{M}$ (12:4)

$\Rightarrow \quad R = 0$ since $e_{[0,n]}(|R|) \longrightarrow 1$ in $\tilde{\mathcal{M}}$. □

12:10 **Definition**

It follows that we may identify $L_\rho(\tilde{\mathcal{M}})^\times$ with a subspace of $L_\rho(\tilde{\mathcal{M}})^*$, and we norm $L_\rho(\tilde{\mathcal{M}})^\times$ with the norm it so inherits.

i.e. for $R \in L_\rho(\tilde{\mathcal{M}})^\times$ $\quad \|R\| = \|f_R\| = \sup_{\rho(S)\le 1} |f_R(S)| = \sup_{\rho(S)\le 1} |\tau(RS)|$

We now set out to establish that in the case $\mathcal{M}$ is non–atomic, $L_\rho(\tilde{\mathcal{M}})^\times = L_{\rho^\times}(\tilde{\mathcal{M}})$ (as sets and as normed spaces).

We will need a number of preparatory results before we are in a position to prove this result.

The following is similar to a result of Yeadon ([Y1] § 3.4 , [Y3] 3.3) . However, the result follows now in quite a routine manner using the approximation techniques developed in Chapter 9.

12:11 **Theorem**

Suppose S , R $\in \tilde{\mathcal{M}}$

Then $\int_0^\infty \mu_t(S\,R)\, dt \le \int_0^\infty \mu_t(S)\, \mu_t(R)\, dt$, with appropiate interpretations if $\int_0^\infty \mu_t(SR)\, dt = \infty$

Proof

Case 1

Suppose $R = p \in \mathcal{M}_p$

Then $\mu_t(S\,p) \leq \mu_t(S)\,\mu_t(p)$ (9:28)

$$\Rightarrow \int_0^\infty \mu_t(S\,p)\,dt \leq \int_0^\infty \mu_t(S)\,\mu_t(p)\,dt$$

Case 2

Suppose $R = \sum_{i=1}^{n} \alpha_i\, p_i \in \mathcal{M}^+$ where $\alpha_i \geq 0$ and $p_1 \geq p_2 \geq \dots \geq p_n \in \mathcal{M}_p$.

Note that if $\|S\,p_i\|_1 < \infty$ $1 \leq i \leq n$, then $S\,p_i \in L_1(\tilde{\mathcal{M}})$ $1 \leq i \leq n$.

Thus $S\,R = S \sum_{i=1}^{n} \alpha_i\, p_i = \sum_{i=1}^{n} \alpha_i\,(S\,p_i) \in L_1(\tilde{\mathcal{M}})$ as $L_1(\tilde{\mathcal{M}})$ is a vector space.

Thus $\|S\,R\|_1 < \infty$

By taking the contrapositive, it follows that if $\|S\,R\|_1 = \infty$ then $\sum_{i=1}^{n} \alpha_i\,\|S\,p_i\|_1 = \infty$.

In either case a triangle inequality holds, and so

$$\begin{aligned}
\int_0^\infty \mu_t(SR)\,dt &= \|SR\|_1 \\
&\leq \sum_{i=1}^{n} \alpha_i\,\|Sp_i\|_1 \quad \text{by the triangle inequality} \\
&= \sum_{i=1}^{n} \alpha_i \int_0^\infty \mu_t(Sp_i)\,dt \\
&\leq \sum_{i=1}^{n} \alpha_i \int_0^\infty \mu_t(S)\,\mu_t(p_i)\,dt \\
&= \int_0^\infty \mu_t(S) \sum_{i=1}^{n} \alpha_i \mu_t(p_i)\,dt \\
&= \int_0^\infty \mu_t(S)\,\mu_t(R)\,dt \quad \text{by 9:29}
\end{aligned}$$

Case 3

Suppose $R \in \mathcal{M}^+$

As argued in 9:30, R can be approximated by operators (r_n) of the type in Case 2 in such a way that

$$\|r_n - R\|_\infty \longrightarrow 0$$

and $\mu_t(r_n) \uparrow \mu_t(R)$

By 10:20, $r_n \longrightarrow R$ in $\tilde{\mathcal{M}}$ and hence $S\, r_n \longrightarrow S\, R$ in $\tilde{\mathcal{M}}$

Hence $\int_0^\infty \mu_t(S\, R)\, dt$

$\leq \quad \liminf_n \int_0^\infty \mu_t(S\, r_n)\, dt$ by 9:20(a)

$\leq \quad \liminf_n \int_0^\infty \mu_t(S)\, \mu_t(r_n)\, dt$ by case 2

$= \quad \int_0^\infty \mu_t(S)\, \mu_t(R)\, dt$ by the Monotone Convergence Theorem

Case 4

Suppose $R \in \mathcal{M}$

Let $R^* = v\, |R^*|$ be the polar decomposition of R^*

It follows $R = |R^*|\, v^*$

Hence $\int_0^\infty \mu_t(S\, R)\, dt$

$= \quad \int_0^\infty \mu_t(S\, |R^*|\, v^*)\, dt$

$\leq \quad \int_0^\infty \mu_t(S\, |R^*|)\, dt$ by 9:18(g)

$$\leq \int_0^\infty \mu_t(S)\,\mu_t(|R^*|)\,dt \quad \text{by case 3}$$

$$= \int_0^\infty \mu_t(S)\,\mu_t(R)\,dt \quad \text{by 9:16(g)}$$

Case 5

Suppose $R \in \tilde{\mathcal{M}}$

Choose $\{r_n\} \subset \mathcal{M}$ such that $r_n \longrightarrow R$ in $\tilde{\mathcal{M}}$, $\mu_t(r_n) \uparrow \mu_t(R)$ (9:22(a))

Then $S\,r_n \longrightarrow S\,R$ in $\tilde{\mathcal{M}}$.

Hence $\int_0^\infty \mu_t(S\,R)\,dt$

$$\leq \liminf_n \int_0^\infty \mu_t(S\,r_n)\,dt \quad \text{by 9:20(a)}$$

$$\leq \liminf_n \int_0^\infty \mu_t(S)\,\mu_t(r_n)\,dt \quad \text{by Case 4}$$

$$= \int_0^\infty \mu_t(S)\,\mu_t(R)\,dt \quad \text{by the Monotone Convergence Theorem.}$$ □

12:12 **Corollary** (Holder inequality)

$R \in L_{\rho^\times}(\tilde{\mathcal{M}})$, $S \in L_\rho(\tilde{\mathcal{M}}) \Rightarrow R\,S \in L_1(\tilde{\mathcal{M}})$ and $\|R\,S\|_1 \leq \|R\|_{\rho^\times} \|S\|_\rho$

Proof

$$\rho^\times(R)\,\rho(S) = \rho^\times(\mu_t(R))\,\rho(\mu_t(S))$$

$$\geq \int_0^\infty \mu_t(R)\, \mu_t(S)\, dt \quad \text{by the commutative Holder Inequality}$$

$$\geq \int_0^\infty \mu_t(R\, S)\, dt \quad \text{by 12:11}$$

$$= \|R\, S\|_1$$

In particular, $R\, S \in L_1(\tilde{\mathcal{M}})$. □

12:13 **Lemma**

Suppose $R \in \tilde{\mathcal{M}}^+$

Suppose $g \in L_0(0,\infty)$ is m a.e. differentiable, $x \in \mathcal{H}$.

Then for $0 < a < b < \infty$, $\int_a^b g(t)\, d\|e_{[0,t]}(R)x\|^2 = -\int_a^b g(t)\, d\|e_{(t,\infty)}(R)x\|^2$

with appropiate interpretations if either of these expressions is not finite.

(These are, of course, Stieltjies integrals.)

Proof

Note that by the Pythagorean relation, $\|x\|^2 = \|e_{[0,t]}(R)x\|^2 + \|e_{(t,\infty)}(R)x\|^2$ since $e_{[0,t]}(R)$ and $e_{(t,\infty)}(R)$ are orthogonal and $e_{[0,t]}(R) + e_{(t,\infty)}(R) = 1$.

$$\int_a^b g(t)\, d\|e_{[0,t]}(R)x\|^2$$

$$= \Big[\, g(t)\, \|e_{[0,t]}(R)x\|^2 \,\Big]_a^b - \int_a^b \|e_{[0,t]}(R)x\|^2\, dg(t) \qquad \text{(Integration by parts)}$$

$$= \Big[\, g(t)\, \|e_{[0,t]}(R)x\|^2 \,\Big]_a^b - \int_a^b \Big\{ \|x\|^2 - \|e_{(t,\infty)}(R)x\|^2 \Big\}\, dg(t)$$

$$= \left[g(t)\, \|e_{[0,t]}(R)x\|^2 \right]_a^b - \int_a^b \|x\|^2\, dg(t) + \int_a^b \|e_{(t,\infty)}(R)x\|^2\, dg(t)$$

$$= \left[g(t)\, \|e_{[0,t]}(R)x\|^2 \right]_a^b - \left[\|x\|^2\, g(t) \right]_a^b + \int_a^b \|e_{(t,\infty)}(R)x\|^2\, dg(t)$$

$$= \left[g(t) \left\{ \|e_{[0,t]}(R)x\|^2 - \|x\|^2 \right\} \right]_a^b + \int_a^b \|e_{(t,\infty)}(R)x\|^2\, dg(t)$$

$$= -\left[g(t)\, \|e_{(t,\infty)}(R)x\|^2 \right]_a^b + \int_a^b \|e_{(t,\infty)}(R)x\|^2\, dg(t)$$

$$= -\int_a^b g(t)\, d\|e_{(t,\infty)}(R)x\|^2 \qquad \text{(Integration by parts)}$$ □

Suppose $R \in \tilde{\mathcal{M}}$.

Recall that $\mu_t(R)$ is positive and decreasing. Hence $\lim_{t \to \infty} \mu_t(R)$ exists; analogously to 2:5 we will denote this limit by $\mu_\infty(R)$

It is easy to verify that $\mu_\infty(R) = \inf \{ t \geq 0 : d_t(R) < \infty \}$

$$\text{Hence } d_t(R) \begin{cases} < \infty & \text{for } t > \mu_\infty(R) \\ \leq \infty & \text{for } t = \mu_\infty(R) \\ = \infty & \text{for } 0 \leq t < \mu_\infty(R) \end{cases}$$

In particular, $d_{\mu_\infty(R)}(R)$ could be finite or infinite.

We will make use of a special spectral representation of an operator $R \in \tilde{\mathcal{M}}$. This representation is suggested by [MvN2] 3.1.3 and [Y1] 2.2.21 , although we point out that the statement of the latter theorem is false.

There it is claimed that for $\mathcal{M}$ non–atomic, $S \in \mathcal{K}^+$,

$$\int_{S^{\sim}(\infty)}^{\infty} t \, de_t(S) = \int_0^{\tau(1)} S^{\sim}(t) \, dp_t \quad \text{where } S^{\sim}(\infty) = \lim_{t \to \infty} S^{\sim}(t)$$

for a certain spectral family $\{p_t\}_{t \geq 0}$ satisfying $\tau(p_t) = t \;\; \forall \, t \geq 0$.

However, if $\tau(1) = \infty$ then the operator 1 satisfies

$$\mu_t(1) = 1 \;\; \forall \, t > 0 \;\; (\text{equivalently, } 1^{\sim}(t) = 1 \;\; \forall \, t > 0)$$

so $\quad \mu_{\infty}(1) = 1 \;\; (\text{equivalently, } 1^{\sim}(\infty) = 1)$

Thus $\displaystyle\int_{S^{\sim}(\infty)}^{\infty} t \, de_t(S) = \int_1^{\infty} t \, de_t(1) = 0 \neq \int_0^{\infty} 1 \, dp_t = \int_0^{\infty} 1^{\sim}(t) \, dp_t$, and so the claim is false.

This happens precisely because 1 is an operator of case 2 in the following theorem. It seems reasonable to suppose that Yeadon did not realise the need to distinguish between case 1 and case 2.

We denote $\mu_{\infty}(R)$ by μ_{∞} and $d_{\mu_{\infty}(R)}(R)$ by $d_{\mu_{\infty}}$ in the following theorem, (in which R remains fixed throughout) .

Note that we assume τ is not finite in the following theorem. A similar and indeed much simpler argument suffices for the case where τ is finite, which is accordingly omitted.

12:14 **Theorem**

Suppose $\mathcal{M}$ is non–atomic.

Suppose $0 < R \in \tilde{\mathcal{M}}$

Two possible cases arise :–

Case 1 $\quad d_{\mu_\infty} = \infty$

Put $q_R = e_{(\mu_\infty(R),\infty)}$

There exists a spectral family $\{ p_t : t \geq 0 \}$ such that

$$\tau(p_t) = t$$

$$q_R\, R = \int_0^\infty \mu_t(R)\, dp_t$$

Case 2 $\quad d_{\mu_\infty} < \infty$

Put $q_R = e_{[\mu_\infty(R),\infty)}$

There exists a spectral family $\{ p_t : t \geq 0 \}$ such that

$$\tau(p_t) = t$$

$$q_R\, R = \int_0^\infty \mu_t(R)\, dp_t$$

Proof

Case 1

We first construct the spectral family $\{ p_t : t \geq 0 \}$

For $c \in (\mu_\infty , \|R\|_\infty)$ and for $t \in [\tau(e_{(c,\infty)}(R)) , \tau(e_{[c,\infty)}(R))]$ choose $p_{c,t} \in \mathcal{M}_p$ such that

$$e_{(c,\infty)}(R) \leq p_{c,t} \leq e_{[c,\infty)}(R)$$
$$\tau(p_{c,t}) = t$$
$$p_{c,t_1} \leq p_{c,t_2} \quad \text{if } t_1 \leq t_2 \qquad (3{:}35)$$

For $t > 0$ note that $t \in [\ \tau(e_{(\mu_t(R),\infty)}(R)\ ,\ \tau(e_{[\mu_t(R),\infty)}(R)\ ]$ and hence we can define

$$p_t = p_{\mu_t(R),t}$$
$$p_0 = 0$$

$\tau(p_t) = t$, by construction.

$t_1 < t_2$

$\Rightarrow \qquad \mu_{t_1}(R) = \mu_{t_2}(R) \quad \text{or} \quad \mu_{t_1}(R) > \mu_{t_1}(R)$

If $\mu_{t_1}(R) = \mu_{t_2}(R)$ then $p_{t_1} = p_{\mu_{t_1}(R),t_1} = p_{\mu_{t_2}(R),t_1} \leq p_{\mu_{t_2}(R),t_2} = p_{t_2}$

If $\mu_{t_1}(R) > \mu_{t_1}(R)$ then $p_{t_1} = p_{\mu_{t_1}(R),t_1} \leq e_{[\mu_{t_1}(R),\infty)} \leq e_{(\mu_{t_2}(R),\infty)} \leq p_{\mu_{t_2}(R),t_2} = p_{t_2}$

$t_i \downarrow t$

$\Rightarrow \qquad \tau(p_{t_i} - p_t) = \tau(p_{t_i}) - \tau(p_t) = t_i - t \downarrow 0$

$\Rightarrow \qquad p_{t_i} \downarrow_{so} p_t$ by a familiar argument, see for example 3:35

p_t is increasing and bounded above, and thus so–convergent to its supremum.

It follows by construction that

$$
\begin{aligned}
& e_{(\mu_\infty,\infty)}(R) \\
= \quad & \bigvee_{c>\mu_\infty} e_{(c,\infty)}(R) \\
\leq \quad & \bigvee_{t\geq 0} p_t \\
\leq \quad & \bigvee_{c>\mu_\infty} e_{[c,\infty)}(R) \\
= \quad & e_{(\mu_\infty,\infty)}(R)
\end{aligned}
$$

Hence $\bigvee_{t\geq 0} p_t = e_{(\mu_\infty,\infty)}(R)$

Thus the expression $\int_0^\infty \mu_t(R)\, dp_t$ defines an operator on $e_{(\mu_\infty,\infty)}(R)\,\mathcal{X}$; we extend this to an operator on $\mathcal{X}$ by letting it take on 0 value on $e_{[0,\mu_\infty]}(R)\,\mathcal{X}$.

Suppose $t > \mu_\infty$

$\mu_{d_t(R)}(R) \leq t$, so $e_{(t,\infty)}(R) \leq e_{(\mu_{d_t(R)}(R),\infty)}(R)$

But $\tau(e_{(t,\infty)}(R))$

$$
\begin{aligned}
= \quad & d_t(R) \\
= \quad & d_{\mu_{d_t(R)}(R)}(R) \\
= \quad & \tau(e_{(\mu_{d_t(R)}(R),\infty)}(R))
\end{aligned}
$$

Thus $e_{(t,\infty)}(R) = e_{(\mu_{d_t(R)}(R),\infty)}(R)$ by the faithfulness of τ

Furthermore, $p_{d_t(R)}$

$$
= \quad p_{\mu_{d_t(R)}(R),d_t(R)}
$$

$$\geq \quad e_{(\mu_{d_t(R)}(R),\infty)}(R)$$

$$= \quad e_{(t,\infty)}(R)$$

But $\tau(p_{d_t(R)}) = d_t(R) = \tau(e_{(t,\infty)}(R))$

Thus $p_{d_t(R)} = e_{(t,\infty)}(R)$ by the faithfulness of τ

Recall that $q_R = e_{(\mu_\infty(R),\infty)}$

Now $q_R\, R$ has spectral family $f_t = \begin{cases} e_{[0,t]}(R) & \mu_\infty \leq t \\ e_{[0,\mu_\infty]}(R) & 0 \leq t < \mu_\infty \end{cases}$

Hence to show that $q_R\, R = \int_0^\infty \mu_t(R)\, dp_t$ we can show that

$$\forall\, x \in \mathcal{H} \quad \int_0^\infty t^2\, d\|f_t x\|^2 = \int_0^\infty \mu_t(R)^2\, d\|p_t x\|^2$$

from which it follows that the domains of these operators are equal;

$$\forall\, x \in \mathcal{H} \quad \int_0^\infty t\, d\|f_t x\|^2 = \int_0^\infty \mu_t(R)\, d\|p_t x\|^2$$

i.e. that the operators are equal.

$$\int_0^\infty t^2\, d\|f_t x\|^2$$

$$= \int_{\mu_\infty}^\infty t^2\, d\|e_{[0,t]}(R)x\|^2$$

$$= \int_{\mu_\infty+0}^\infty t^2\, d\|e_{[0,t]}(R)x\|^2$$ since $\|e_{[0,t]}(R)x\|^2$ is right continuous

$$= \int_{\infty}^{\mu_\infty+0} t^2 \, d\|e_{(t,\infty)}(R)x\|^2 \quad \text{by 12:13}$$

$$= \int_{\infty}^{\mu_\infty+0} \mu_{d_t(R)}(R)^2 \, d\|e_{(t,\infty)}(R)x\|^2$$

since $e_{(t,\infty)}(R)$ is constant, and hence the integral dissapears, on any intervals where $t^2 \neq \mu_{d_t(R)}(R)^2$

$$= \int_{\infty}^{\mu_\infty+0} \mu_{d_t(R)}(R)^2 \, d\|p_{d_t(R)}x\|^2$$

$$= \int_0^{\infty} \mu_t(R)^2 \, d\|p_t x\|^2 \quad \text{by making the substitution } d_t(R) \to t$$

Note that $d_{\mu_t(R)}(R) \uparrow \infty$ as $t \downarrow \mu_\infty$, as this is case 1.

An entirely similar argument, with t^2 replaced by t and $\mu_t(R)^2$ replaced by $\mu_t(R)$, shows that

$$\int_0^{\infty} t \, d\|f_t x\|^2 = \int_{\mu_\infty}^{\infty} t \, d\|e_{[0,t]}(R)x\|^2 = \int_0^{\infty} \mu_t(R) \, d\|p_t x\|^2$$

With this the proof is completed in case 1.

Case 2

Again we first construct the spectral family $\{ p_t : t \geq 0 \}$

For $c \in (\mu_\infty , \|R\|_\infty)$ and for $t \in [\tau(e_{(c,\infty)}(R)) , \tau(e_{[c,\infty)}(R))]$ choose $p_{c,t} \in \mathcal{M}_p$ in the same manner as before.

For μ_∞ and for $t \in [\, d_{\mu_\infty}, \infty)$ choose $p_{\mu_\infty(R),t}$ such that

$$e_{(\mu_\infty,\infty)}(R) \le p_{\mu_\infty,t} \le e_{[\mu_\infty,\infty)}(R)$$

$$\tau(p_{c,t}) = t$$

$p_{\mu_\infty,t}$ are increasing and $p_{\mu_\infty,t} \uparrow_{so} e_{[\mu_\infty,\infty)}(R)$.

This is possible by 3:35

We define

$$p_t \quad = \quad \begin{cases} 0 & t = 0 \\ p_{\mu_t(R),t} & 0 < t < d_{\mu_\infty} \\ p_{\mu_\infty,\, t} & d\mu_\infty \le t \end{cases}$$

It follows as in case 1 that $\tau(p_t) = t$ and $\{p_t\}$ forms a spectral family.

Note that in this case we have arranged that $p_t \uparrow_{so} e_{[\mu_\infty,\infty)}(R)$

Thus the expression $\int_0^\infty \mu_t(R)\, dp_t$ defines an operator on $e_{[\mu_\infty,\infty)}(R)\,\mathcal{X}$; as before we extend this to an operator on $\mathcal{X}$ by letting it take on 0 value on $e_{[0,\mu_\infty)}(R)\,\mathcal{X}$.

For $t > \mu_\infty(R)$, we once again have that

$$e_{(t,\infty)}(R) = e_{(\mu_{d_t(R)}(R),\infty)}(R)$$

$$p_{d_t(R)} = e_{(t,\infty)}(R)$$

Recall that $q_R = e_{[\mu_\infty(R),\infty)}$

Now $q_R\, R$ has spectral family $f_t = \begin{cases} e_t(R) & t \ge \mu_\infty \\ e_{[0\,,\,\mu_\infty(R))} & 0 \le t < \mu_\infty \end{cases}$

Hence to show that $q_R\, R = \int_0^\infty \mu_t(R)\, dp_t$ we show that

$$\forall\, x \in \mathcal{X} \quad \int_0^\infty t^2\, d\|f_t x\|^2 = \int_0^\infty \mu_t(R)^2\, d\|p_t x\|^2$$

from which it follows that the domains of these operators are equal;

$$\forall\, x \in \mathcal{X} \quad \int_0^\infty t\, d\|f_t x\|^2 = \int_0^\infty \mu_t(R)\, d\|p_t x\|^2$$

i.e. that the operators are equal.

$$\int_0^\infty t^2\, d\|f_t x\|^2$$

$$= \int_0^{\mu_\infty} t^2\, d\|f_t x\|^2 + \int_{\mu_\infty}^\infty t^2\, d\|f_t x\|^2$$

$$= \mu_\infty{}^2\, \|e_{[\mu_\infty]} x\|^2 + \int_{\mu_\infty}^\infty t^2\, d\|f_t x\|^2$$

$$= \mu_\infty(R)^2\, \|e_{[\mu_\infty]} x\|^2 + \int_{\mu_\infty}^\infty t^2\, d\|e_{[0,t]} x\|^2$$

$$= \mu_\infty(R)^2\, \|e_{[\mu_\infty]} x\|^2 + \int_0^{d\mu_\infty} \mu_t(R)^2\, d\|p_t x\|^2$$

by the same calculation as in case 1, except that here

$d_{\mu_t(R)}(R) \uparrow d_{\mu_\infty}$ as $t \downarrow \mu_\infty(R)$, as this is case 2.

$$= \int_{d_{\mu_\infty}}^\infty \mu_t(R)^2\, d\|p_t x\|^2 + \int_0^{d_{\mu_\infty}} \mu_t(R)^2\, d\|p_t x\|^2$$

since $\mu_t(R)$ is constantly μ_∞ on $[d_{\mu_\infty},\infty)$

$p_t \underset{so}{\uparrow} e_{[\mu_\infty,\infty)}(R)$ as $t \uparrow \infty$, by construction

$p_t \underset{so}{\downarrow} e_{(\mu_\infty,\infty)}(R)$ as $t \downarrow d_{\mu_\infty}$, by construction

$$= \int_0^\infty \mu_t(R)^2 \, d\|p_t x\|^2$$

An entirely similar argument, with t^2 replaced by t and $\mu_t(R)^2$ replaced by $\mu_t(R)$, shows that

$$q_R \, R = \int_0^\infty t \, d\|f_t x\|^2 = \int_0^\infty \mu_t(R) \, d\|p_t x\|^2$$

With this the proof is completed in case 2. □

We will subsequently use the projection q_R constructed in 12:14 for $0 < R \in \tilde{\mathcal{M}}$ ($\mathcal{M}$ non–atomic) and denote by $\{ p_t(R) : t \geq 0 \}$ the spectral family derived in 12:14 for such R, without further comment.

12:15 Proposition

Suppose $0 \leq f \in \tilde{L}_\infty(0,\infty)$

Suppose $\mathcal{M}$ is non–atomic.

Choose any spectral family $\{p_t\}_{t\geq 0}$ satisfying $\tau(p_t) = t$. (For example, this family can be the family $\{p_t(R)\}_{t\geq 0}$ for $R \in \tilde{\mathcal{M}}^+$, as derived in 12:14)

Let $S = \int_0^\infty f(t) \, dp_t$

Then $S \in \tilde{\mathcal{M}}$ and $\mu_t(S) = \mu_t(f)$.

If $f \in L_\rho(0,\infty))$ then $\rho(S) = \rho(f)$.

Proof

Let $S = \int_0^\infty f(t)\, dp_t$, and as usual let $\{ e_t(S) : t \geq 0 \}$ be the spectral family for S.

Then $S \in \bar{\mathcal{M}}$ by 7:5

By the Operational Calculus, $e_{(t,\infty)}(S) = p_{\{a>0 \,:\, f(a) > t\}}$

Thus $\tau(e_{(t,\infty)}(S))$

$$= \tau(p_{\{a>0 \,:\, f(a) > t\}}(R))$$

$$= m\{ a > 0 : f(a) > t\} \qquad \text{since } \tau(p_t) = t$$

$$\downarrow \quad 0 \text{ as } t \uparrow \infty \qquad \text{since } f \in \tilde{L}_\infty$$

Thus $S \in \tilde{\mathcal{M}}$

$d_t(S) = \tau(e_{(t,\infty)}(S)) = m\{ a > 0 : f(a) > t\} = d_t(f)$.

Hence $\mu_t(S) = \mu_t(f)$

If $f \in L_\rho(0,\infty)$ then $\rho(S) = \rho(\mu_t(S)) = \rho(\mu_t(f)) = \rho(f)$

In particular, $S \in L_\rho(\tilde{\mathcal{M}})$. □

It follows that in 12:14 , $\mu_t\left(\int_0^\infty \mu_t(R)\, dp_t(R) \right) = \mu_t(R)$

12:16 **Corollary**

Suppose $\mathcal{M}$ is non–atomic.

$\{ \mu_t(S) : S \in L_\rho(\tilde{\mathcal{M}}) \} = \{ \mu_t(f) : f \in L_\rho(0,\infty) \}$

Proof

The one inclusion follows from the definition of $L_\rho(\tilde{\mathcal{M}})$, the other from 12:15. □

12:17 **Note**

Suppose $\mathcal{M}$ is non–atomic.

It follows that $R \in L_{\rho^\times}(\tilde{\mathcal{M}}) \Leftrightarrow \mu_t(R)\,\mu_t(S) \in L_1(0,\infty)\ \ \forall\, S \in L_\rho(\tilde{\mathcal{M}})$ (2:14 and 12:16)

and that for $R \in L_{\rho^\times}(\tilde{\mathcal{M}})$, $\rho^\times(R) = \sup\limits_{\rho(S)\le 1} \int_0^\infty \mu_t(R)\,\mu_t(S)\,dt$ (2:22 and 12:16)

12:18 **Theorem**

Suppose $\mathcal{M}$ is non–atomic.

Consider $L_\rho(\tilde{\mathcal{M}})^\times$ to be normed as in 12:10

Consider the normed space $L_{\rho^\times}(\tilde{\mathcal{M}})$

Then $L_\rho(\tilde{\mathcal{M}})^\times = L_{\rho^\times}(\tilde{\mathcal{M}})$

Proof

$R \in L_{\rho^\times}(\tilde{\mathcal{M}})$ $\Rightarrow$ $\mu_t(R) \in L_{\rho^\times}(0,\infty)$

$\Rightarrow$ $\mu_t(R)\,\mu_t(S) \in L_1(0,\infty)\ \ \forall\, S \in L_\rho(\tilde{\mathcal{M}})$

$\Rightarrow$ $\mu_t(R\,S) \in L_1(0,\infty)\ \ \forall\, S \in L_\rho(\tilde{\mathcal{M}})$ by 12:11

$\Rightarrow$ $R\,S \in L_1(\tilde{\mathcal{M}})\ \ \forall\, S \in L_\rho(\tilde{\mathcal{M}})$

$\Rightarrow$ $R \in L_\rho(\tilde{\mathcal{M}})^\times$

Suppose $R \in L_{\rho^\times}(\tilde{\mathcal{M}})$

Then
$$\|R\|_{L_\rho(\tilde{\mathcal{M}})^\times} = \sup_{\rho(S)\le 1} |\tau(RS)|$$
$$\le \sup_{\rho(S)\le 1} \tau(|RS|) \quad \text{by 12:2}$$
$$= \sup_{\rho(S)\le 1} \int_0^\infty \mu_t(RS)\, dt$$
$$\le \sup_{\rho(S)\le 1} \int_0^\infty \mu_t(R)\, \mu_t(S)\, dt \quad \text{by 12:11}$$
$$= \rho^\times(R) \quad \text{by 12:17}$$

Suppose $R \in L_\rho(\tilde{\mathcal{M}})^\times$ and $R = v|R|$ is its polar decomposition.

Apply 12:14 to $|R|$, so that $q_{|R|}\,|R| = \int_0^\infty \mu_t(R)\, dp_t(|R|)$

Suppose $S \in L_\rho(\tilde{\mathcal{M}})$

Let $S' = \int_0^\infty \mu_t(S)\, dp_t(|R|)$

Then $S' \in L_\rho(\tilde{\mathcal{M}})$ and $\rho(S) = \rho(S')$ by 12:15

Furthermore, $\int_0^\infty \mu_t(R)\, \mu_t(S)\, dp_t(|R|)$
$$= \int_0^\infty \mu_t(R)\, dp_t(|R|) \int_0^\infty \mu_t(S)\, dp_t(|R|)$$
by the Operational Calculus, and since τ–measurable operators cannot be extended (8:22)

$$= \quad q_{|R|}\ |R|\ S'$$

$$= \quad q_{|R|}\ v^*\ R\,S'$$

$$\in \quad L_1(\tilde{\mathcal{M}}) \text{ since } R\,S' \in L_1(\tilde{\mathcal{M}})$$

Thus

$$\begin{aligned}
\infty \quad &> \quad \| q_{|R|}\ |R|\ S' \|_1 \\
&= \quad \| \int_0^\infty \mu_t(R)\,\mu_t(S)\,dp_t(|R|) \|_1 \\
&= \quad \int_0^\infty \mu_t\,(\int_0^\infty \mu_t(R)\,\mu_t(S)\,dp_t(|R|)\ dt \\
&= \quad \int_0^\infty \mu_t(\,\mu_t(R)\,\mu_t(S)\,)\ dt \qquad \text{by 12:15} \\
&= \quad \int_0^\infty \mu_t(R)\,\mu_t(S)\ dt \qquad \text{since clearly } \mu_t(\,\mu_t(R)\,\mu_t(S)\,) = \mu_t(R)\,\mu_t(S)
\end{aligned}$$

Thus $R \in L_\rho(\tilde{\mathcal{M}})^\times \quad \Rightarrow \quad \mu_t(R)\,\mu_t(S) \in L_1(0,\infty) \quad \forall\, S \in L_\rho(\tilde{\mathcal{M}})$

$$\Rightarrow \quad \mu_t(R) \in L_{\rho^\times}(0,\infty) \qquad \text{by 12:17}$$

$$\Rightarrow \quad R \in L_{\rho^\times}(\tilde{\mathcal{M}})$$

and $\rho^\times(R)$

$$= \quad \sup_{\rho(S)\le 1} \int_0^\infty \mu_t(R)\,\mu_t(S)\ dt$$

$$= \quad \sup_{\rho(S)\le 1} \| q_{|R|}\ |R|\ S' \|_1$$

$$= \sup_{\rho(S)\le 1} \int_0^\infty \mu_t(\, q_{|R|}\ |R|\ S'\,)\, dt$$

$$\le \sup_{\rho(S)\le 1} \int_0^\infty \mu_t(|R|\ S')\, dt \quad \text{by 9:18(g)}$$

$$= \sup_{\rho(S)\le 1} \int_0^\infty \mu_t(v|R|\ S')\, dt \quad \text{by 9:21}$$

$$= \sup_{\rho(S)\le 1} \int_0^\infty \mu_t(R\ S')\, dt$$

$$\le \sup_{\rho(S)\le 1} \int_0^\infty \mu_t(R\ S)\, dt$$

since $\{\, S' : S \in L_\rho(\tilde{\mathcal{M}})\ ;\ \rho(S) \le 1 \,\} \subset \{\, S : S \in L_\rho(\tilde{\mathcal{M}})\ ,\ \rho(S) \le 1 \,\}$

$$= \sup_{\rho(S)\le 1} \tau(|RS|)$$

$$= \sup_{\rho(S)\le 1} \tau(w^* RS)$$

where $R\ S = w\ |R\ S|$ is the polar decomposition

$$= \sup_{\rho(S)\le 1} \tau(RSw^*) \quad \text{by 12:2}$$

$$\le \sup_{\rho(S)\le 1} |\tau(RS)| \quad \text{by 10:19(b)}$$

$$= \|R\|_{L_\rho(\tilde{\mathcal{M}})^\times} \qquad \square$$

Recall that in the commutative case, $L_{\rho^{\times}}$ is identified with that subspace of L_{ρ}^{*} that comprises of linear functionals of integrable type, and this identification is achieved via the Radon–Nikodým Theorem. A Radon–Nikodým type theorem has been developed by Yeadon ([Y1] § 3.5 ; [Y3] § 4) in the $L_p(\tilde{\mathcal{M}})$ case, where it is shown in the case that $\mathcal{M}$ is non–atomic, $L_q(\tilde{\mathcal{M}}) \cong L_p(\tilde{\mathcal{M}})^{*}$. $(\frac{1}{p} + \frac{1}{q} = 1)$
In this case the question of generalising *integrable type* to the non–commutative case does not arise, as all continuous linear functionals on L_p are of integrable type. This is obviously not the case in general. Thus it is neccessary to find an appropiate generalisation of integrable type linear functionals and to develop a Radon–Nikodým theorem which will characterise these functionals. More generally, most of the results about duality and reflexivity need to be generalised to the non–commutative theory.

A generalisation of the concept of integrable type for ideals of $\mathcal{M}$ can be found in [DL] .

It is also pertinent to ask if the assumption of non–atomicity of $\mathcal{M}$ can be removed.

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