Aspects of Higher Degree Forms with Symmetries

by

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I hereby certify that this is my own original work and has not been submitted before for any other degree.

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Abstract

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In Chapter One we develop a basis for studying higher degree alternating forms. The concepts and results we present are mostly obvious analogues of Harrison's treatment of higher degree symmetric forms. We explain antisymmetrization; discuss the derivative of an alternating form and its corresponding anticommutative polynomial; define alternating spaces and their direct sum; establish decomposition and cancellation results for alternating spaces; and construct a Witt-Grothendieck group of alternating spaces.

In Chapter Two we discuss hyperbolic alternating space. We compute the centre, algebraic isometry group and its corresponding Lie algebra, and prove a descent result. There are important parallels with Keet's results for hyperbolic symmetric spaces, as well as significant differences, especially in the methods we employ.

In Chapter Three we develop a framework for the study of two aspects of forms of general Young symmetry type: their hyperbolics, and a generalization of the Weil-Siegel duality between symmetric and alternating bilinear forms. We introduce notions like nondegeneracy, derivative of a form, and derivative and integral symmetry types, and are then able to construct a hyperbolic space which is cofinal for spaces equipped with a form of the same symmetry type, and show that symmetry types are Siegel duals in our generalized sense if they have the same derivative symmetry type.

In Chapter Four we present a few results and observations concerning nondegeneracy-type conditions on symmetric forms. These include: an extension of Harrison's proof that nonsingularity implies nonzero Hessian to forms of arbitrary degree; a discussion of s-nondegeneracy and s-regularity; and a relation between a strong nondegeneracy condition on forms of even degree and the catalecticant, a classical invariant.

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Contents

Introduction and Summary

The study of quadratic forms is of great antiquity and has a large literature. Since the work of Weyl, alternating bilinear forms have also attained classic status. The study of higher degree alternating forms, in contrast to higher degree symmetric forms, has been quite limited. Alternating tensors, or "polyvectors", have been studied at least as far back as Grassman; this work was continued in this century by Schouten, Gurevich and others, and more recently, by Cohen and Helminck. Most work seems to have been done on classifying alternating tensors or forms of rank 3, but Cohen and Helminck have also computed their isometry groups.

There has been important recent work on symmetric higher degree forms by Harrison, by Keet, and by Harrison in conjunction with Pareigis. In the first chapter of this thesis, we extend parts of Harrison's treatment of higher degree symmetric forms to the alternating case. The next chapter deals with alternating hyperbolics, and we obtain analogues of several of Keet's results for symmetric forms.

The close parallel between the symmetric and alternating cases, especially in respect of hyperbolics, leads us in the third chapter to investigate generalizing some of these ideas to forms of general Young symmetry type. These have their roots in the work of Frobenius and Young on the representations on the symmetric group S_n, but it appears that Weyl's approach to the representations of $GL(n)$ via "tensors of highest possible symmetry" first emphasized the existence of symmetry types which are given by higher degree representations, as opposed to the two classical ones which are given by one-dimensional representations. In Chapter 3 we present an approach to studying hyperbolics of such general Young symmetry type; we also discuss a generalization of the duality of Weil-Siegel, as developed by Hughes.

The final chapter contains a few results and a number of observations about nondegeneracy-type conditions on symmetric forms.

We give more detail on the content of each chapter:

In **Chapter 1** we provide a basis for the study of higher degree alternating forms, along the lines of Harrison's treatment of symmetric forms (see [H1]). Though we have not seen elsewhere many of the concepts and results we present, they are mainly obvious analogues of Harrison's work, so they are not essentially new. We give details only where the alternating case differs significantly from the symmetric case, usually resulting from simplifications which are available for the symmetric case not being valid in the alternating case. Otherwise we merely cite [H1] or Keet ([K1]).

In §1.1 we introduce basic definitions, notation and terminology, and list some standard results on exterior algebra. We then discuss the isomorphisms between the following spaces:

the grade d component of the exterior algebra $\bigwedge(V^*)$, viz. $\bigwedge^d(V^*)$, which may be identified with the space $F_d\langle x_1, \ldots, x_n\rangle$ of homogeneous anticommutative polynomials of degree d in "variables" $x_1, \ldots, x_n$, actually a dual basis of V ;

the space of alternating tensors in $T^d(V^*)$; and

the space $Alt_d V$ of alternating d-multilinear forms on V.

We then introduce the notions of alternating spaces and their direct sum, and consider their behaviour under extension of the base field.

In §1.2 we discuss derivatives of multilinear alternating forms and anticommutative polynomials, and prove the relation between them. We also discuss nondegeneracy for alternating forms.

In §1.3 we show that Harrison's important decomposition and cancellation results are valid for alternating spaces.

In §1.4 we approach the notion of the centre via the general concept of the adjoint of an endomorphism, and obtain an analogue of Harrison's characterization of indecomposability in terms of the structure of the centre.

The final section, §1.5, uses results from the previous two sections to construct a Witt-Grothendieck group of alternating spaces. We observe that, while this group cannot be given a ring structure in the normal way, it is a module over the Witt-Grothendieck group of symmetric spaces; in fact, the two groups can be combined to form a $\mathbb{Z}_2$-graded ring.

Chapter 2 deals with several aspects of hyperbolic alternating spaces. We obtain results which are identical or similar to corresponding results for hyperbolic symmetric spaces in [K1], but the methods at some key stages often differ markedly.

In §2.1 we define the alternating hyperbolics and prove their basic properties.

In §2.2 we compute the centre of alternating hyperbolic space, and, using Proposition 1.4.1, we show that the alternating hyperbolics are indecomposable. This has important consequences for the structure of the Witt-Grothendieck group we discussed in §1.5.

In §2.3 we discuss general results about algebraic subgroups of $GL(V)$ and their Lie algebras, as preparation for the next two sections. Most of the results included are general and not dependent on symmetry, or extend easily from the symmetric to the alternating case. An exception is Proposition 2.3.5, which requires a different proof.

In §2.4 we compute the algebraic isometry group of alternating hyperbolic space. A key part of the computation, viz. showing that $Alt_d V$ is invariant under isometry, requires a very different proof to the corresponding result for the symmetric case, where nonsingularity makes things very simple. We show that $G(H,f) = GL(V) \bowtie K_\mu(V^*)$, where $K_\mu(V^*)$ is the co-Schur functor of Akin, Buchsbaun and Weyman ([ABW]), and $\mu = (2, 1^{d-1})$. It follows from general theory that $G(H,f)$ is connected and its radical is 2-step solvable.

In §2.5 we show that the Lie algebra is $L(H, f) = End(V) \oplus K_\mu(V^*)$, and its radical is $F \oplus K_\mu(V^*)$, which is 2-step solvable.

In the last section, §2.6, we prove that, if an alternating space extends, under extension of the base field, to an alternating space which is hyperbolic, then the original space is hyperbolic. We use the Lie algebra computed in §2.5, as well as general Lie theory, to prove this.

Chapter 3 presents an approach to studying forms of general Young symmetry type. We do not attempt to develop a completely general rigorous foundation for the subject, but merely a framework within which to discuss

(i) the construction of hyperbolics of general Young symmetry type; and

(ii) a generalized Weil-Siegel duality.

After reviewing in §3.1 the aspects of representation theory which underpin our approach, we elaborate, in §3.2, a scheme of general Young symmetry types using largely the treatment of Akin, Buchsbaum and Weyman ([ABW]), including a discussion of what we regard as "equivalent" general Young symmetry types. We mention the very useful results of Kantor and Trishin ([KT]) on the conditions characterizing forms of general Young symmetry type, and give some examples which we refer to in the sequel.

§3.3 deals with the notion of nondegeneracy and gives a neat characterization.

In §3.4 we introduce the notions of derivative symmetry type and integral symmetry type arising from a given general Young symmetry type. We show that, if f is a form of some general Young symmetry type, then its derivative, appropriately defined, has derivative symmetry type.

We are then able, in §3.5, to construct hyperbolics of general Young symmetry type in such a way as to generalize the alternating and symmetric hyperbolics. We show that, in general, these hyperbolics are cofinal for spaces equipped with a form of the same symmetry type as the hyperbolic form.

The final section, §3.6, begins by reviewing the Weil-Siegel duality between symmetric and alternating bilinear forms. We discuss a particular formulation of Siegel duality due to Hughes ([HUG]), and generalize this to hyperbolics of general Young symmetry type. We obtain conditions for two general Young symmetry types to be Siegel duals, and use this to display two pairs of Siegel dual symmetry types.

Chapter 4 contains a discussion of nondegeneracy-type conditions for symmetric forms.

We present a generalization of a result of Harrison, and discuss a strong nondegeneracy condition on forms of even degree.

Since we have encountered in the literature (for example, [HP], [ORY], [KW], [DK]) references to different concepts which are closely related, we have found it useful to present them in a coherent form, to elaborate on some of them and mention relations between them, as well as to locate our results within the overall context of nondegeneracy-type conditions.

In §4.1 we extend Harrison's proof ([H2]) that nonsingularity implies nonzero Hessian to arbitrary degree; we also give an example of a non-hyperbolic quartic form which is nondegenerate and has zero Hessian.

In §4.2 we use Harrison and Pareigis' notion of s-radical to define what we call s-nondegeneracy; we also elaborate on the condition termed s-regularity by O'Ryan. We note some obvious relations amongst these families of conditions, and give a few examples, including one which "separates" the s-regularity conditions. An important reason for including s-nondegeneracy is its link to the next section.

§4.3: Dolgachev and Kanev define a catalecticant invariant for any form of even degree, but do not show explicitly that their definition extends the classical notion of Sylvester, except via two examples. Although there appears to be no general definition of the catalecticant in the older literature ([ELL], [GY], [SAL]), we show that a commonly used description in terms of partial derivatives does coincide with the [DK] definition. There is also a reference in [DK] to a relation between the catalecticant and what they call nondegeneracy of even degree forms. We had earlier discovered the same relation by other methods, so we present it in rather more detail than in [DK], using their more efficient tools of polarity. We illustrate the relation with a few examples, which also indicate our original approach.

Chapter 1

Alternating Forms : Introduction

The theory of bilinear alternating forms is easy and well known. (See [JAC1] pp332-4 or [SCHA] pp264-5, for example.) Any bilinear alternating form f of dimension n has a basis $\{u_1, v_1, \ldots, u_r, v_r, z_1, \ldots, z_{n-2r}\}$ satisfying $f(u_i, u_j) = f(v_i, v_j) = f(u_i, z_k) = f(v_i, z_k) = f(z_i, z_j) = 0$ for all i, j, k, and $f(u_i, v_i) = -f(v_i, u_i) = 1$ for all i. Relative to this basis, the matrix of f is the block diagonal matrix $diag(S, \ldots, S, 0, \ldots, 0)$, where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Re-ordering the basis, this can be written as $\begin{pmatrix} 0 & I_r & & & \\ -I_r & 0 & & & \\ & & 0 & & \\ & & & \cdot & \\ & & & & 0 \end{pmatrix}$.

f is nondegenerate if and only if its rank is even if and only if f is hyperbolic. In this case, the basis is called *symplectic* (or *hyperbolic*).

In contrast to the symmetric case, not a great deal is known about higher degree (i.e. *degree* > 2) alternating forms. Trilinear alternating forms over algebraically closed fields have been classified up to dimension 8 for characteristic 0 ([GUR] pp390-5), and up to dimension 7 for arbitrary characteristic ([CH] Theorem 2.1 p3). Cohen and Helminck (ibid.) have also computed their isometry groups over certain fields. The classification of alternating forms of degree greater than 3 is known in only the simplest cases ([GUR] p391).

For alternating tensors of arbitrary degree certain general results are known. By the duality $\bigwedge(V^*) \cong \bigwedge(V)^*$ (see §1.1), these also apply to alternating forms. In particular, in a vector space of dimension n, every tensor of degree $n-1$ is decomposable ([MB] Proposition 15 p568); this means that there is no nondegenerate alternating form of degree $n-1$ on a vector space of dimension n. This has an important consequence: in the case of symmetric forms, many arguments are simplified by invoking the existence of nondegenerate forms of arbitrary degree and dimension, but we have to use a different argument in the case of alternating forms. (See Lemma 2.1.1 p31.)

We outline the contents of this chapter.

In §1.1 we describe an action of the symmetric group S_d on the space $Mult_dV$ of d-multilinear forms on a vector space V, and define alternation (or skew-symmetry); we also list a few important facts about the structure of the exterior algebra which we use repeatedly. The antisymmetrization map is discussed in detail. We observe that the grade d component $\bigwedge^d(V^*)$ of the Hopf algebra $\bigwedge(V^*)$ can be identified with the space $F_d\langle x_1, \ldots, x_n\rangle$ of homogeneous anticommutative polynomials of degree d, and describe the space $T^d(V^*)_{alt}$ of alternating tensors in $T^d(V^*)$, showing they are isomorphic to the space Alt_dV of alternating d-multilinear forms on V.

We then set about describing the antisymmetrization map $\bigwedge^d(V^*) \to T^d(V^*)_{alt}$, as well as its inverse, in both Hopf algebra terms and explicitly. The inverse is given in terms of the comultiplication on the Hopf algebra $\bigwedge(V^*)$, while the antisymmetrization map comes from the multiplication.

In the rest of this section we obtain the following analogues from Harrison ([H1]): alternating spaces, isomorphism between them, their direct sum, and their behaviour under base extension.

In §1.2 we define the derivative $\theta^{(v)}$ of an alternating form θ with respect to $v \in V$, and show that, if θ corresponds to the anticommutative polynomial f, then $\theta^{(v)}$

corresponds to the anticommutative polynomial $\frac{1}{d}\sum_{i=1}^{n}\alpha_i\frac{\partial f}{\partial x_i}$, if $v=\sum_{i=1}^{n}\alpha_i e_i$, where $\{e_i\}$ is a basis and $\{x_i\}$ is a dual basis. We also discuss nondegeneracy.

In §1.3 we show that Harrison's theory of decomposition of symmetric spaces extends to alternating spaces. We comment on some of the technical details required to ensure that the proofs remain valid, and show that the two important results, on decomposition and cancellation, carry over to alternating spaces.

In §1.4 we review the notion of the *adjoint* of an endomorphism with respect to a bilinear form, and extend it to higher degree. We show that focusing on the adjointable operators gives a useful alternative approach to introducing the centre of an alternating form, and are able to establish an analogue of the first part of Proposition 4.1 in [H1] in a different way. We end this section by commenting on the important distinction between degree 2 and degree > 2.

In §1.5 we discuss the construction of a Witt-Grothendieck group $\widehat{W}_a(F)$ of alternating spaces by Harrison's method. We observe that, while it is not possible to put a ring structure on $\widehat{W}_a(F)$, there is a $\widehat{W}(F)$-module structure on $\widehat{W}_a(F)$.

1.1 Preliminaries

In this chapter, as well as the next, we shall assume that d is a positive integer ≥ 2, F is a field whose characteristic does not divide $d!$, and V is a vector space of dimension n over F. We denote the dual of V by V^*, its d-fold Cartesian product $V\times\ldots\times V$ by V^d, and the d^{th} tensor power (or grade d component of the tensor algebra $T(V)$) by $T^d(V)$.

There are different possible starting points for the subject of the first two chapters. In this section, we introduce the essential terminology and notation, and demonstrate the equivalence of several concepts.

In cases where the argument for the symmetric case carries over unchanged (except for the objects involved) to the alternating case, we shall omit details and merely cite the reference. It is important to note that, in the symmetric case, the polynomial map ([K1] Ch 1, §2 p12) allows the proofs of several results to be simplified considerably. (See, for example, [K1] Proposition 1.3 pp24-5.) There is no analogue in the alternating case (because we cannot define an evaluation map from the *graded* commutative F-algebra $\bigwedge(V^*)$ to the (commutative) field F), so the proofs often require a different, more cumbersome, approach.

Let θ be a multilinear form of degree d on V, $\theta : V^d \to F$, with $v_1, \ldots, v_n$ a basis for V and $x_1, \ldots, x_n$ the dual basis for V^*.
Denote the space of degree d multilinear forms on V by $Mult_d V$. By the universal property of the tensor product, θ corresponds to a homomorphism $\tilde{\theta} : T^d(V) \to F$, i.e. to an element $\tilde{\theta} \in T^d(V)^*$. Hence $Mult_d V \cong T^d(V)^* \cong T^d(V^*)$. Explicitly, the isomorphism can be described as follows: The elements $x_{\alpha 1} \otimes \ldots \otimes x_{\alpha d}$, where $\alpha : \underline{d} \to \underline{n}$, form a basis for $T^d(V^*)$. ($\underline{n}$ denotes the set $\{1, \ldots, n\}$, etc.) Given $t = \sum_\alpha (t|\alpha 1 \ldots \alpha d) x_{\alpha 1} \otimes \ldots \otimes x_{\alpha d}$, this corresponds to $f \in T^d(V)^*$ given by $f(v_{\alpha 1}, \ldots, v_{\alpha d}) = (t|\alpha 1 \ldots \alpha d)$.
The group $GL(V)$ acts (on the left) on $Mult_d V$ by $\sigma \cdot \theta = \theta \circ \sigma^{-1}$,
i.e. $\sigma \cdot \theta(v_1, \ldots, v_d) = \theta(\sigma^{-1} v_1, \ldots, \sigma^{-1} v_d)$. This defines an action of the symmetric group S_d on $Mult_d V$: $\pi \cdot \theta(v_1, \ldots, v_d) = \theta(v_{\pi 1}, \ldots, v_{\pi d})$. (This action is via a representation of S_d in V^d.)
We say θ is *skew-symmetric* if $\pi \cdot \theta = \epsilon(\pi)\theta$ for all $\pi \in S_d$. ($\epsilon(\pi)$ denotes the sign of π.) Since any $\pi \in S_d$ is a product of transpositions of adjacent integers, it is sufficient to require that $\theta(v_1, \ldots, v_i, v_{i+1}, \ldots, v_d) = -\theta(v_1, \ldots, v_{i+1}, v_i, \ldots, v_d)$ for all $i = 1, \ldots, d-1$.
We say θ is *alternating* if $\theta(v_1, \ldots, v_d) = 0$ whenever $v_i = v_{i+1}$ for some $i = 1, \ldots, d-1$.
Clearly we have: θ alternating implies θ skew-symmetric; and, if $char\, F \neq 2$ (as we assume), the converse also holds.
We shall use the above conditions interchangeably, and usually refer to forms satisfying them as alternating.

Exterior Algebra

Before continuing, we list some well known facts (from [MB] pp558-562) about the functor $\bigwedge(-)$ from the category of modules over a commutative ring to the category of graded commutative Hopf algebras. We state the results for vector spaces, as this is all we need.

Let C be a vector space over a field K. In the tensor algebra $T(C)$ let $D(C^2)$ be the ideal generated by all squares c^2 of elements of C. We define the *exterior algebra* of the vector space C to be the graded quotient algebra $\bigwedge(C) = T(C)/D(C^2)$. The component of $\bigwedge(C)$ in degree p is then $\bigwedge^p(C) = T^p(C)/D^p$, where $D = D(C^2)$. We write the product of two elements $a, b \in \bigwedge(C)$ as $a \wedge b$, called the exterior product.

1.1.1 Proposition: The exterior algebra satisfies the following:

(i) $\bigwedge^0(C) = K$.

(ii) $\bigwedge(C)$ is generated by the set $\bigwedge^1(C) = C$ of its elements of degree 1 (i.e. an element of degree p in $\bigwedge(C)$ is a sum of products of the form $c_1 \wedge \ldots \wedge c_p$, for $c_i \in C = \bigwedge^1(C)$).

(iii) It is graded commutative (i.e. $c \wedge d = (-1)^{pq} d \wedge c$ if $c \in \bigwedge^p(C)$ and $d \in \bigwedge^q(C)$).

(iv) If a has odd degree then $a \wedge a = 0$.

The map $a : C^p \longrightarrow \bigwedge^p(C)$, where $(c_1, \ldots, c_p) \mapsto c_1 \wedge \ldots \wedge c_p$ is alternating because, if $c_i = c_j$ for $i \neq j$, then $c_1 \wedge \ldots \wedge c_p = 0$.

1.1.2 Universal Property: For fixed p, the module $\bigwedge^p(C)$ satisfies the following universal property: Any alternating multilinear function $h : C^p \longrightarrow E$ (E a vector space) factors through $\bigwedge^p(C)$:$(c_1, \ldots, c_p) \mapsto c_1 \wedge \ldots \wedge c_p$, $h(c_1, \ldots, c_p) = t(c_1 \wedge \ldots \wedge c_p)$:

$$\begin{array}{ccc} C^p & \xrightarrow{\quad a \quad} & \bigwedge^p(C) \\ h \downarrow & \swarrow_{t} & \\ E & & \end{array}$$

If C is a vector space with basis $b_1, \ldots, b_n$, its exterior algebra $\bigwedge(C)$ is zero in degrees $> n$, while for degrees $p \leq n$, $\bigwedge^p(C)$ is a vector space with basis the $\binom{p}{n}$ elements $b_{\underline{k}} = b_{k_1} \wedge \ldots \wedge b_{k_p}$, one for each strictly increasing list $\underline{k} : \underline{p} \rightarrow \underline{n}$.

Denote the vector space of alternating forms of degree d on V by $Alt_d V$. If $\theta \in Alt_d V$, then, by the universal property of the exterior algebra, θ corresponds to a homomorphism $\tilde{\theta} : \bigwedge^d(V) \rightarrow F$, i.e. $\tilde{\theta} \in \bigwedge^d(V)^*$. Hence $Alt_d V \cong \bigwedge^d(V)^*$.

Antisymmetrization

We now prepare to discuss, in some detail, the important *antisymmetrization map* (see [ABW] §I.2 p213), which is the analogue of the polarization map in the symmetric case. Suppose V has basis $v_1, \ldots, v_n$, and let $x_1, \ldots, x_n$ be a dual basis for V^*. We have seen that $\bigwedge^d(V^*)$, the grade d component of the Hopf algebra $\bigwedge(V^*)$, can then be given a basis consisting of monomials $x_{i_1} \wedge \ldots \wedge x_{i_d}$, where $1 \leq i_1 < \ldots < i_d \leq n$. It will thus be obvious that $\bigwedge^d(V^*)$ can be identified with $F_d\langle x_1, \ldots, x_n \rangle$, the space of homogeneous polynomials of degree d over F in the anticommuting variables $x_1, \ldots, x_n$. Two elements $f, g \in F_d\langle x_1, \ldots, x_n \rangle$ are equivalent, written $f \cong g$, if f may be obtained from g by an invertible linear change of variables. This corresponds to a change of basis for V^*, which defines an automorphism of $\bigwedge^d(V^*)$. (See [K1] p19 for details.) Thus equivalent anticommutative polynomials correspond to $GL(V)$-equivalent elements of $\bigwedge^d(V^*)$.

We describe the process of differentiating a polynomial f in anticommuting variables x_i. Berezin ([BER] pp74-5) defines the notion of a left (resp. right) derivative of f as follows: If $i = i_s$, then $\frac{\partial}{\partial x_i}(x_{i_1} \ldots x_{i_k})$ is obtained by placing x_{i_s} at the left (resp. right) end in the product $x_{i_1} \ldots x_{i_s} \ldots x_{i_k}$ using anticommutativity, and deleting it; if i does not occur among the $i_1, \ldots, i_k$, then the derivative is zero.

(This definition only works in the symmetric and alternating cases, since there is no straightening for other symmetry types.)

S_d acts on $T^d(V^*)$ by requiring the isomorphism $T^d(V^*) \cong Mult_d V$ to be an S_d-isomorphism: If $t = \sum (t|\alpha 1 \dots \alpha d) x_{\alpha 1} \otimes \dots \otimes x_{\alpha d}$ corresponds to f, where $f(v_{\alpha 1}, \dots, v_{\alpha d}) = (t|\alpha 1 \dots \alpha d)$, then $\pi \cdot t$ corresponds to $\pi \cdot f$, where $\pi \cdot f(v_{\alpha 1}, \dots, v_{\alpha d}) = f(v_{\alpha(\pi 1)}, \dots, v_{\alpha(\pi d)})$.
So $\pi \cdot t = \sum (t|\alpha(\pi 1) \dots \alpha(\pi d)) x_{\alpha 1} \otimes \dots \otimes x_{\alpha d}$.
$t \in T^d(V^*)$ is *alternating* if $\pi \cdot t = \epsilon(\pi) t$ for all $\pi \in S_d$. Let $T^d(V^*)_{alt}$ denote the space of alternating tensors in $T^d(V^*)$.
Since $T^d(V^*) \cong Mult_d V$ is an S_d-isomorphism, we have $T^d(V^*)_{alt} \cong Alt_d V$.
The *antisymmetrization map* is an isomorphism $\bigwedge^d(V^*) \to T^d(V^*)_{alt}$. We shall describe it both in Hopf algebra terms, as well as explicitly.
Now $\bigwedge^d(V^*)$ can be identified with the vector space $F_d\langle x_1, \dots, x_n\rangle$ via $x_{i_1} \wedge \dots \wedge x_{i_d} \mapsto x_{i_1} \dots x_{i_d}$, for every map $i : \{1, \dots, d\} \to \{1, \dots, n\}$ satisfying $1 \le i_1 < \dots < i_d \le n$. (The $x_{i_1} \wedge \dots \wedge x_{i_d}$ form a basis for $\bigwedge^d(V^*)$.) We have already seen that $T^d(V^*)_{alt} \cong Alt_d V$, so we have $F_d\langle x_1, \dots, x_n\rangle \cong Alt_d V$.
The map $T^d(V^*)_{alt} \to \bigwedge^d(V^*)$ is easy to describe. It is the canonical projection $T^d(V^*) \to \bigwedge^d(V^*)$, or, since $\bigwedge^1 V^* = V^*$, it is the component $V^* \otimes \dots \otimes V^* \to \bigwedge^d(V^*)$ of d-fold multiplication in the Hopf algebra $\bigwedge(V^*)$ ([ABW] §I.2 p 213). Explicitly, this map can be described as follows:
d-fold multiplication in $\bigwedge(V^*)$: $\bigwedge(V^*) \otimes \dots \otimes \bigwedge(V^*) \to \bigwedge(V^*)$. Restrict to grade 1: $V^* \otimes \dots \otimes V^* \to \bigwedge(V^*)$, where $x_1 \otimes \dots \otimes x_d \mapsto x_1 \wedge \dots \wedge x_d \in \bigwedge^d(V^*)$. (1)
Now let $t = \sum_\alpha (t|\alpha 1 \dots \alpha d)(x_{\alpha 1} \otimes \dots \otimes x_{\alpha d})$ be an arbitrary tensor in $T^d(V^*)$. If t is alternating, then $\pi \cdot t = \epsilon(\pi) t$ for all $\pi \in S_d$. But $\pi \cdot t = \sum_\alpha (t|\alpha(\pi 1) \dots \alpha(\pi d)) x_{\alpha 1} \otimes \dots \otimes x_{\alpha d}$, so we have $\sum (t|\alpha 1 \dots \alpha d) x_{\alpha 1} \otimes \dots \otimes x_{\alpha d} = \sum \epsilon(\pi)(t|\alpha(\pi 1) \dots \alpha(\pi d)) x_{\alpha 1} \otimes \dots \otimes x_{\alpha d}$ for all $\pi \in S_d$.
Hence $(t|\alpha 1 \dots \alpha d) = \epsilon(\pi)(t|\alpha\pi 1 \dots \alpha\pi d)$ for all $\pi \in S_d$. (2)

If $\alpha_i = \alpha_j$, we can put $\pi = (ij)$ and then deduce that $(t|\alpha 1 \ldots \alpha d) = -(t|\alpha 1 \ldots \alpha d)$, so $(t|\alpha 1 \ldots \alpha d) = 0$ (provided $char\, F \neq 2$).

So in an alternating tensor the only nonzero coefficients (terms) are those for which α is injective. Define an equivalence relation $\sim$ on the set of all injective maps $\alpha : \underline{d} \to \underline{n}$ by $\alpha \sim \alpha'$ iff $Im\, \alpha = Im\, \alpha'$. Since there are $\binom{n}{d} = r$ different images, there are r different equivalence classes $A_1, \ldots, A_r$. Now $\alpha \sim \alpha'$ iff $\alpha' = \alpha\pi$ for some $\pi \in S_d$. In each class A_i, choose the map α_i such that $1 \leq \alpha_i 1 < \ldots < \alpha_i d \leq n$.

Then $A_i = \{\alpha_i \pi : \pi \in S_d\}$. (3)

Hence we have

$$\begin{aligned} t &= \sum_{\text{all } \alpha} (t|\alpha 1 \ldots \alpha d) x_{\alpha 1} \otimes \ldots \otimes x_{\alpha d} \\ &= \sum_{i=1}^{r} \sum_{\alpha \in A_i} (t|\alpha 1 \ldots \alpha d) x_{\alpha 1} \otimes \ldots \otimes x_{\alpha d} \\ &= \sum_{i=1}^{r} \sum_{\pi \in S_d} (t|\alpha_i \pi 1 \ldots \alpha_i \pi d) x_{\alpha_i \pi 1} \otimes \ldots \otimes x_{\alpha_i \pi d} \qquad \text{(by (3))} \\ &= \sum_{i=1}^{r} \sum_{\pi \in S_d} \epsilon(\pi)(t|\alpha_i 1 \ldots \alpha_i d) x_{\alpha_i \pi 1} \otimes \ldots \otimes x_{\alpha_i \pi d} \qquad \text{(by (2))} \end{aligned}$$

Now, for all $\pi \in S_d$,

$$\begin{aligned} x_{\alpha_i \pi 1} \otimes \ldots \otimes x_{\alpha_i \pi d} &\mapsto x_{\alpha_i \pi 1} \wedge \ldots \wedge x_{\alpha_i \pi d} \qquad \text{(by (1))} \\ &= \epsilon(\pi) x_{\alpha_i 1} \wedge \ldots \wedge x_{\alpha_i d} \end{aligned}$$

Hence

$$\begin{aligned} t &\mapsto \sum_{i=1}^{r} \sum_{\pi \in S_d} (t|\alpha_i 1 \ldots \alpha_i d) x_{\alpha_i 1} \wedge \ldots \wedge x_{\alpha_i d} \\ &= \sum_{i=1}^{r} d!(t|\alpha_i 1 \ldots \alpha_i d) x_{\alpha_i 1} \wedge \ldots \wedge x_{\alpha_i d} \\ &= d! \sum_{\alpha} (t|\alpha 1 \ldots \alpha d) x_{\alpha 1} \wedge \ldots \wedge x_{\alpha d}, \end{aligned}$$

where the sum is taken over all α such that $1 \leq \alpha 1 < \ldots < \alpha d \leq n$.

The antisymmetrization map $\bigwedge^d(V^*) \to T^d(V^*)_{alt}$ maps $x_1 \wedge \ldots \wedge x_d$ to $\sum_\sigma \epsilon(\sigma) x_{\sigma 1} \otimes \ldots \otimes x_{\sigma d}$. It may be viewed as the component $\bigwedge^d(V^*) \to V^* \otimes \ldots \otimes V^*$ of d-fold comultiplication in the Hopf algebra $\bigwedge V^*$, and it is a split monomorphism over F ([ABW] §I.2 p213). To understand how the antisymmetrization map works, we examine more closely the comultiplication in the Hopf algebra $\bigwedge(V^*)$.

$\Delta(x_1 \wedge \ldots \wedge x_r) = \Delta x_1 \wedge \ldots \wedge \Delta x_r = (x_1 \otimes 1 + 1 \otimes x_1) \wedge \ldots \wedge (x_r \otimes 1 + 1 \otimes x_r)$.

Consider terms with s factors $x_i \otimes 1$ and $r-s$ factors $1 \otimes x_j$, for $0 \leq s \leq r$. For each s, swop factors in the terms so that each term is of the form

$(x_{\sigma 1} \otimes 1) \ldots (x_{\sigma s} \otimes 1)(1 \otimes x_{\sigma(s+1)}) \ldots (1 \otimes x_{\sigma r})$, with $\sigma 1 < \ldots < \sigma s$ and $\sigma(s+1) < \ldots < \sigma r$, i.e. σ is a *shuffle* of type $(s, r-s)$. The sign of the term changes by $\epsilon(\sigma)$, so

$$\begin{aligned} \Delta(x_1 \wedge \ldots \wedge x_r) &= \sum_{s=0}^{r} \sum_{\sigma} \epsilon(\sigma)(x_{\sigma 1} \otimes 1) \ldots (x_{\sigma s} \otimes 1)(1 \otimes x_{\sigma(s+1)}) \ldots (1 \otimes x_{\sigma r}) \\ &= \sum_{s=0}^{r} \sum_{\sigma} \epsilon(\sigma)(x_{\sigma 1} \wedge \ldots \wedge x_{\sigma s}) \otimes (x_{\sigma(s+1)} \wedge \ldots \wedge x_{\sigma r}) \end{aligned}$$

where the second sum is taken over all shuffles of type $(s, r-s)$.

The antisymmetrization map is the component of d-fold comultiplication:

$$\begin{array}{lccccc} & \bigwedge(V^*) & \rightarrow & \bigwedge(V^*) & \otimes & \bigwedge(V^*) \\ \text{1-fold comultiplication :} & \cup & & \cup & & \cup \\ & \bigwedge^d(V^*) & & \bigwedge^{d-1}(V^*) & & \bigwedge^1(V^*) = V^* \end{array}$$

So the component of $\Delta(x_1 \wedge \ldots \wedge x_d)$ is $\sum_\sigma (x_{\sigma 1} \wedge \ldots \wedge x_{\sigma(d-1)}) \otimes x_{\sigma d}$ (putting $s = d-1$), which equals

$x_1 \wedge \ldots \wedge x_{d-1} \otimes x_d - x_1 \wedge \ldots \wedge x_{d-2} \wedge x_d \otimes x_{d-1} + \ldots + (-1)^{d-1} x_2 \wedge \ldots \wedge x_d \otimes x_1 = \sum_{k=1}^{d} \frac{\partial}{\partial x_k}(x_1 \wedge \ldots \wedge x_d) \otimes x_k$, using the notion of a right derivative of an anticommutative polynomial.

If $p = x_1 \wedge \ldots \wedge x_d \in \bigwedge^d(V^*)$, then the component of Δp in $\bigwedge^{d-1}(V^*) \otimes V^*$ is $\sum_{k=1}^{d} \frac{\partial p}{\partial x_k} \otimes x_k$, and the component of Δp in $\bigwedge^{d-2}(V^*) \otimes V^* \otimes V^*$ is $\sum_{j=1}^{d} \sum_{k=1}^{d} \frac{\partial}{\partial x_j} \frac{\partial p}{\partial x_k} \otimes x_j \otimes x_k$, etc.

Hence the component of Δp in $V^* \otimes \ldots \otimes V^* = T^d(V^*)$ is

$\sum_{k_d=1}^{d} \ldots \sum_{k_1=1}^{d} \frac{\partial}{\partial x_{k_d}} \ldots \frac{\partial p}{\partial x_{k_1}} \otimes x_{k_d} \otimes \ldots \otimes x_{k_1}$.

Now $\frac{\partial}{\partial x_{k_d}} \ldots \frac{\partial p}{\partial x_{k_1}}$ is nonzero only if $k_d, \ldots, k_1$ are all distinct, i.e. $k_d, \ldots, k_1$ must be a permutation σ of $\{1, \ldots, d\}$. In fact, $\frac{\partial}{\partial x_{k_d}} \ldots \frac{\partial p}{\partial x_{k_1}} = \epsilon(\sigma)$. So the component of Δp in $(V^*)^{\otimes d}$ is $\sum_{\sigma \in S_d} \epsilon(\sigma) x_{\sigma 1} \otimes \ldots \otimes x_{\sigma d}$.

If we take an arbitrary element $p \in \bigwedge^d(V^*)$, where $p = \sum_\alpha (p|\alpha 1 \ldots \alpha d) x_{\alpha 1} \wedge \ldots \wedge x_{\alpha d}$, then its antisymmetrization is $\sum_\alpha (p|\alpha 1 \ldots \alpha d) \frac{1}{d!} \sum_{\sigma \in S_d} \epsilon(\sigma) x_{\alpha\sigma 1} \otimes \ldots \otimes x_{\alpha\sigma d}$.

Alternating Spaces

Following Harrison ([H1] p124), we define an *alternating space of degree d* to be a pair (V, θ), where V is a finite dimensional vector space and θ is an alternating multilinear degree d form on V, i.e. $\theta : V^d \to F$. Two alternating spaces (V, θ) and (W, ϕ) are called *isomorphic* (or *isometric*) if there exists a vector space isomorphism $\sigma : V \to W$ such that $\theta(v_1, \ldots, v_d) = \phi(\sigma v_1, \ldots, \sigma v_d) = \sigma^{-1} \cdot \phi(v_1, \ldots, v_d)$ for all $v_1, \ldots, v_d \in V$, i.e. $W = \sigma V$ and $\phi = \sigma \cdot \theta$.

The antisymmetrization map gives an obvious bijective correspondence between alternating spaces (V, θ) of degree d over F and elements of $\bigwedge^d(V^*)$. Since antisymmetrization preserves the actions of $GL(V)$ on $\bigwedge^d(V^*)$ and on $Alt_d V$, isomorphic alternating spaces $(V, \theta), (\sigma V, \sigma \cdot \theta)$ of degree d correspond to $GL(V)$-equivalent elements of $\bigwedge^d(V^*)$ (or equivalent anticommutative polynomials). (The details are similar to the symmetric case — see [K1] Ch 1 §§15,16 pp19-20.)

Direct Sum

Let $(V, \theta), (W, \phi)$ be alternating spaces of degree d over F.

Define $\theta \oplus \phi : (V \oplus W)^d \to F$ by

$$\theta \oplus \phi[(v_1, w_1), \ldots, (v_d, w_d)] = \theta(v_1, \ldots, v_d) + \phi(w_1, \ldots, w_d).$$

It is clear that $\theta \oplus \phi$ is d-multilinear and alternating.

We define $(V, \theta) \oplus (W, \phi)$ to be $(V \oplus W, \theta \oplus \phi)$, called the *direct sum* of the alternating spaces (V, θ) and (W, ϕ). It is easy to see that, if $(V, \theta) \cong (V', \theta')$ and $(W, \phi) \cong (W', \phi')$, then $(V, \theta) \oplus (W, \phi) \cong (V', \theta') \oplus (W', \phi')$ (cf. [K1] Ch 1 §19 p20).

Let $p_1 \in \bigwedge^d(V^*), p_2 \in \bigwedge^d(W^*)$ be anticommutative polynomials in $x_1, \ldots, x_n$; $y_1, \ldots, y_m$, respectively. We define $p_1 \perp p_2$ by $p_1(x_1, \ldots, x_n) + p_2(y_1, \ldots, y_m)$. If p_1, p_2 correspond to $(V, \theta), (W, \phi)$, respectively, then $p_1 \perp p_2$ corresponds to $(V, \theta) \oplus (W, \phi)$

([K1] Ch 1 §18 p20). So, if $p_1 \cong p_1', p_2 \cong p_2'$, then $p_1 \perp p_2 \cong p_1' \perp p_2'$.

Extension of Base Field

Now let K be an extension field of F, and put $V_K = V \otimes_F K$. Suppose the alternating space (V, θ) corresponds to $p \in \bigwedge^d(V^*)$. Then we have a commutative square:

$$\begin{array}{ccc} \bigwedge^d(V^*) & \longrightarrow & \bigwedge^d(V_K^*) \\ \downarrow & & \downarrow \\ Alt_d V & \longrightarrow & Alt_d V_K \end{array}$$

in which the vertical maps are antisymmetrization. We denote the images of p, θ by p_K, θ_K, respectively ([K1] Ch1 §9.1 p16). The form $\theta_K : V_K^d \to K$ is given by $\theta_K(v_1 \otimes 1, \ldots, v_d \otimes 1) = \theta(v_1, \ldots, v_d)$ for all $v_1, \ldots, v_d \in V$. The extensions of isomorphic alternating spaces are isomorphic, because the above diagram is a diagram of $GL(V)$-maps ([K1] Ch 1 §13 pp18-19).

1.2 Derivative and Nondegeneracy

Let (V, θ) be an alternating space of degree d, and suppose $v \in V$.
Define an alternating multilinear form $\theta^{(v)}$ of degree $d-1$ on V by

$$\theta^{(v)}(v_1, \ldots, v_{d-1}) = \theta(v, v_1, \ldots, v_{d-1}) \text{ for all } v_1, \ldots, v_{d-1} \in V.$$

We call $(V, \theta^{(v)})$ (or just $\theta^{(v)}$) the *derivative of* (V, θ) (or just θ) in the direction of v. It is easy to check that, if $(V, \theta) \overset{\sigma}{\cong} (W, \phi)$, then $(V, \theta^{(v)}) \overset{\sigma}{\cong} (W, \phi^{(\sigma \cdot v)})$.
Choose a basis $e_1, \ldots, e_n$ and a dual basis $x_1, \ldots, x_n$, and suppose $v = \alpha_1 e_1 + \ldots + \alpha_n e_n$. Suppose f is the anticommutative polynomial corresponding to (V, θ) (via the given bases). In this section we find it convenient to use the *left* derivative of f (see §1.1 p12). We then have

1.2.1 Proposition: $(V, \theta^{(v)})$ corresponds to $\frac{1}{d} \sum_{i=1}^{n} \alpha_i \frac{\partial f}{\partial x_i}$.

Proof: We denote $\frac{1}{d}\sum_{i=1}^{n}\alpha_i\frac{\partial f}{\partial x_i}$ by $f^{(v)}$ for short. Let $f = d!x_{\alpha 1}\dots x_{\alpha d}$, where $\alpha : \underline{d} \mapsto \underline{n}$ is strictly increasing, i.e. $f \in F_d\langle x_1, \dots, x_n\rangle$.

We discuss the case of a monomial. It will be clear that the argument can be extended to $f = \sum_\alpha (p|\alpha 1 \dots \alpha d)x_{\alpha 1}\dots x_{\alpha d}$ since all the operators involved will be linear.

Then $\theta = \sum_{\sigma\in S_d}\epsilon(\sigma)x_{\alpha\sigma 1}\otimes\dots\otimes x_{\alpha\sigma d}$ is the alternating form corresponding to f (see p15).

Let $v_1 = v_{11}e_1 + \dots + v_{1n}e_n$, where $\{e_i\}$ is a basis for V, with $\{x_i\}$ a dual basis, for $1 \le i \le n$. We show that the alternating form corresponding to the polynomial $f^{(v_1)}$ is the same thing as $\theta^{(v_1)}$. Now

$$\begin{aligned} f^{(v_1)} &= \frac{1}{d}d!\sum_{i=1}^{n} v_{1i}\frac{\partial f}{\partial x_i} \\ &= \frac{d!}{d}\sum_{i=1}^{d} v_{1\alpha i}\frac{\partial f}{\partial x_{\alpha i}} \quad (\text{if } i \notin Im\,\alpha, \text{then } \frac{\partial f}{\partial x_i} = 0) \\ &= \frac{d!}{d}[v_{1\alpha 1}x_{\alpha 2}\dots x_{\alpha d} - v_{1\alpha 2}x_{\alpha 1}x_{\alpha 3}\dots x_{\alpha d} + \dots + (-1)^{d-1}v_{1\alpha d}x_{\alpha 1}\dots x_{\alpha(d-1)}]. \end{aligned}$$

Hence

$$f^{(v_1)} = (d-1)!\sum_{i=1}^{d}(-1)^{i-1}v_{1\alpha i}x_{\alpha 1}\dots\widehat{x_{\alpha i}}\dots x_{\alpha d}.$$

(We use $\hat{a}$ to denote that a is omitted.)

The degree $d-1$ multilinear alternating form induced by $f^{(v_1)}$ is then

$$\tilde{\theta} = \sum_{i=1}^{d}(-1)^{i-1}v_{1\alpha i}\sum_{\sigma\in S_d(i)}\epsilon(\sigma)x_{\alpha\sigma 1}\otimes\dots\otimes\widehat{x_{\alpha\sigma i}}\otimes\dots\otimes x_{\alpha\sigma d},$$

by the usual antisymmetrization process.

($S_d(i)$ denotes the subgroup of S_d which fixes i.)

Now the derivative of θ (with respect to v_1) is

$$\theta^{(v_1)}(v_2\otimes\dots\otimes v_d) = \theta(v_1\otimes v_2\otimes\dots\otimes v_d).$$

For $i = 1, 2, \dots, d$, put $v_i = \sum_{j=1}^{n} v_{ij}e_j$. Then

$$\begin{aligned} \theta^{(v_1)}(v_2\otimes\dots\otimes v_d) &= \theta(v_1\otimes\dots\otimes v_d) \\ &= \sum_{\sigma\in S_d}\epsilon(\sigma)x_{\alpha\sigma 1}\otimes\dots\otimes x_{\alpha\sigma d}(\sum_{j=1}^{n} v_{1j}e_j\otimes\dots\otimes\sum_{j=1}^{n} v_{dj}e_j). \end{aligned}$$

Hence

$$\theta^{(v_1)}(v_2 \otimes \ldots \otimes v_d) = \sum_{\sigma \in S_d} \epsilon(\sigma) v_{1\alpha\sigma 1} v_{2\alpha\sigma 2} \ldots v_{d\alpha\sigma d}.$$

(Each $x_{\alpha\sigma i}$ is one of the $x_1, \ldots, x_n$, and $x_{\alpha\sigma 1} \otimes \ldots \otimes x_{\alpha\sigma d}(\sum_{j=1}^n v_{1j} e_j \otimes \ldots \otimes \sum_{j=1}^n v_{dj} e_j)$ is nonzero provided that in the first argument we have $j = \alpha\sigma 1$, in the second argument $j = \alpha\sigma 2$, etc.)

Note that $\theta^{(v_1)}(v_2 \otimes \ldots \otimes v_d)$ can be written as

$\sum_{\sigma \in S_d, \sigma 1 = 1} \epsilon(\sigma) v_{1\alpha\sigma 1} \ldots v_{d\alpha\sigma d} + \sum_{\sigma \in S_d, \sigma 1 = 2} \epsilon(\sigma) v_{1\alpha\sigma 1} \ldots v_{d\alpha\sigma d} + \ldots$

$+ \sum_{\sigma \in S_d, \sigma 1 = d} \epsilon(\sigma) v_{1\alpha\sigma 1} \ldots v_{d\alpha\sigma d}$

(since S_d can be partitioned by $\sigma 1 = 1, \sigma 1 = 2, \ldots, \sigma 1 = d$)

$= \sum_{i=1}^d v_{1\alpha i} \sum_{\sigma \in S_d, \sigma 1 = i} \epsilon(\sigma) v_{2\alpha\sigma 2} \ldots v_{d\alpha\sigma d}$

$= v_{1\alpha 1} \sum_{\sigma \in S_d, \sigma 1 = 1} \epsilon(\sigma) v_{2\alpha\sigma 2} \ldots v_{d\alpha\sigma d} + v_{1\alpha 2} \sum_{\sigma \in S_d, \sigma 1 = 2} \epsilon(\sigma) v_{2\alpha\sigma 2} \ldots v_{d\alpha\sigma d} + \ldots$

$+ v_{1\alpha d} \sum_{\sigma \in S_d, \sigma 1 = d} \epsilon(\sigma) v_{2\alpha\sigma 2} \ldots v_{d\alpha\sigma d}$.

Now $\tilde{\theta} = v_{1\alpha 1} \sum_{\sigma \in S_d(1)} \epsilon(\sigma) x_{\alpha\sigma 2} \otimes \ldots \otimes x_{\alpha\sigma d} - v_{1\alpha 2} \sum_{\sigma \in S_d(2)} \epsilon(\sigma) x_{\alpha\sigma 1} \otimes x_{\alpha\sigma 3} \otimes \ldots$ $\otimes x_{\alpha\sigma d} + \ldots + (-1)^{d-1} \sum_{\sigma \in S_d(d)} \epsilon(\sigma) x_{\alpha\sigma 1} \otimes \ldots \otimes x_{\alpha\sigma(d-1)}$. Hence

$$\begin{aligned} \tilde{\theta}(v_2 \otimes \ldots \otimes v_d) &= v_{1\alpha 1} \sum_{\sigma \in S_d(1)} \epsilon(\sigma) v_{\alpha\sigma 2} \otimes \ldots \otimes v_{\alpha\sigma d} - \\ & \quad v_{1\alpha 2} \sum_{\sigma \in S_d(2)} \epsilon(\sigma) v_{\alpha\sigma 1} \otimes v_{\alpha\sigma 3} \otimes \ldots \otimes v_{\alpha\sigma d} + \ldots + \\ & \quad (-1)^{d-1} \sum_{\sigma \in S_d(d)} \epsilon(\sigma) v_{\alpha\sigma 1} \otimes \ldots \otimes v_{\alpha\sigma(d-1)} \\ &= \sum_{j=1}^d (-1)^{j-1} v_{1\alpha j} \sum_{\tau \in S_d(j)} \epsilon(\tau) v_{2\alpha\tau i_2} \ldots v_{d\alpha\tau i_d}, \end{aligned}$$

where $i : \{2, \ldots, d\} \to \{1, \ldots, \hat{j}, \ldots, d\}$ is $1-1$ and order-preserving, i.e. $i_2 < i_3 < \ldots < i_d$. (There is clearly only *one* such i for each j.)

We now show that $\theta^{(v_1)} = \tilde{\theta}$ by proving that

$\sum_{\sigma \in S_d, \sigma 1 = j} \epsilon(\sigma) v_{2\alpha\sigma 2} \ldots v_{d\alpha\sigma d} = (-1)^{j-1} \sum_{\tau \in S_d(j)} \epsilon(\tau) v_{2\alpha\tau i_2} \ldots v_{d\alpha\tau i_d}$,

for each $j = 1, \ldots, d$ and i as just described.

If $\sigma \in S_d$ and $\sigma 1 = j$, then σ is a bijection between $\{2, \ldots, d\}$ and $\{1, \ldots, \hat{j}, \ldots, d\}$. Define τ by putting $\sigma 2 = \tau i_2, \ldots, \sigma d = \tau i_d$. Then $\tau \in S_d(j)$. Extend τ to τ' by putting $\tau' k = \tau k$ if $k \neq j$ and $\tau j = j$.

Then clearly $\epsilon(\tau') = \epsilon(\tau)$, and we also have $\epsilon(\tau') = (-1)^{j-1}\epsilon(\sigma)$ because $\sigma 1 = j = \tau' j, \sigma 2 = \tau i_2 = \tau' i_2, \sigma 3 = \tau i_3 = \tau' i_3, \ldots, \sigma d = \tau i_d = \tau' i_d$ and $i_2 < \ldots < i_d$. Hence $\epsilon(\sigma) = (-1)^{j-1}\epsilon(\tau)$, and we obtain

$$\sum_{\sigma \in S_d, \sigma 1 = j} \epsilon(\sigma) v_{2\alpha\sigma 2} \ldots v_{d\alpha\sigma d} = (-1)^{j-1} \sum_{\tau \in S_d(j)} \epsilon(\tau) v_{2\alpha\tau i_2} \ldots v_{d\alpha\tau i_d}.$$

This shows that, if θ corresponds to f (degree d), then $\theta^{(v_1)}$ corresponds to $\tilde{f}$ (degree $d-1$) via the given bases. □

It follows easily that, if $f \cong g$, then $f^{(v)} \cong g^{(v)}$.

Nondegeneracy

We call (V, θ) (or simply θ) *nondegenerate* if $\theta^{(v)} = 0$ implies $v = 0$, i.e. if $\theta(v, v_1, \ldots, v_{d-1}) = 0$ for all $v_1, \ldots, v_{d-1} \in V$ implies $v = 0$. Otherwise, we call (V, θ) degenerate. This condition is clearly preserved by isomorphism ([K1] Ch 2 Proposition 1.2 p24).

There is a corresponding condition for $\bigwedge^d(V^*)$: If $x_1, \ldots, x_n$ is a dual basis and f is the anticommutative polynomial corresponding to (V, θ), then it follows from Proposition 1.2.1 that f is degenerate if and only if there exists $g \cong f$ such that $g(x_1, \ldots, x_n) = g(x_1, \ldots, x_{n-1}, 0)$, i.e. the variable x_n can be "removed".

If f is any anticommutative polynomial of degree d, there exists a nondegenerate anticommutative polynomial h and a zero anticommutative polynomial k such that $f = h \perp k$, and h, k are unique up to equivalence. Hence there is no loss of generality in restricting to nondegenerate forms.

It is easy to see that the direct sum of nondegenerate alternating spaces is nondegenerate.

1.3 Decomposition

A great deal of the theory of decomposition of symmetric spaces of degree $d \geq 3$ developed by Harrison ([H1] §§2,3) extends to alternating spaces.

Since the arguments require little modification to carry over to the alternating case, we confine ourselves to: listing the main results; commenting on the adjustments needed to make the proofs valid for alternating spaces; and supplying a few details not given in [H1].

An alternating space (V,θ) of degree $d \geq 3$ is called *decomposable* if there exist nonzero alternating spaces (U,ϕ) and (W,ψ) such that $(V,\theta) \cong (U,\phi) \oplus (W,\psi)$.
We show that, if (V,θ) is nondegenerate, then so are (U,ϕ) and (W,ψ): Suppose $\phi(u, u_1, \ldots, u_{d-1}) = 0$ for all $u_1, \ldots, u_{d-1} \in U$. Then, for all $v_1, \ldots, v_{d-1} \in V$, we have $\theta(u, v_1, \ldots, v_{d-1}) = \theta(u, u_1 + w_1, \ldots, u_{d-1} + w_{d-1}) = \phi(u, u_1, \ldots, u_{d-1}) + \psi(0, w_1, \ldots, w_{d-1}) = 0$. Since θ is nondegenerate, $u = 0$, as required.
(V,θ) is *indecomposable* if it is nonzero and not decomposable.
We say $u, v \in V$ are *orthogonal*, written $u \perp v$, if $\theta(u, v, v_1, \ldots, v_{d-2}) = 0$ for all $v_1, \ldots, v_{d-2} \in V$.
If A is a subspace of V, we put $A^\perp = \{u \in V \mid u \perp v \text{ for all } v \in A\}$.
Since $u \perp v$ if and only if $v \perp u$, we can deduce quite easily that $A \subset A^{\perp\perp}$. From this it follows that $A \subset B^\perp \Rightarrow B \subset A^\perp$, which is required to prove Lemma 2.2 ([H1] p128).
If A is a subspace of V, let $\theta|_A$ denote the restriction of θ to A. Then $(A, \theta|_A)$ is an alternating space (of degree d).
If $(V,\theta) \cong (U,\phi) \oplus (W,\psi)$, there exist subspaces A, B of V with $V = A \oplus B$ and $A \subset B^\perp$ such that $(A, \theta|_A) \cong (U,\phi)$ and $(B, \theta|_B) \cong (W,\psi)$. Conversely, if $V = A \oplus B$ and $A \subset B^\perp$, then $(V,\theta) \cong (A, \theta|_A) \oplus (B, \theta|_B)$. This follows from

$$\begin{aligned}
&\theta(a_1 + b_1, \ldots, a_d + b_d) \\
&\quad = \theta(a_1, \ldots, a_d) + \theta(a_1, \ldots, a_{d-1}, b_d) + \cdots + \theta(a_1, b_2, \ldots, b_d) + \theta(b_1, \ldots, b_d) \\
&\quad = (\theta|_A \oplus \theta|_B)[(a_1, b_1), \ldots, (a_d, b_d)]
\end{aligned}$$

Notice that alternation is required in the above: since θ is alternating, the condition $A \subset B^{\perp}$ means that all terms in the second expression which have arguments from both A and B, in any position, are zero.

V is called the *orthogonal sum* of the subspaces $A_1, \ldots, A_m$ if $V = A_1 \oplus \cdots \oplus A_m$, and $A_i \subset A_j^{\perp}$ for $i \neq j, i, j = 1, \ldots, m$.

The next two propositions, on decomposition and cancellation of alternating spaces, follow by the same reasoning as in the symmetric case, since we have established the validity of the prerequisite results in the alternating case. We confine ourselves to vector spaces, so we do not need to assume nondegeneracy ([H1] Note 2.5 pp 129-130).

1.3.1 Proposition (cf. [H1] Proposition 2.3 p129): Let (V, θ) be a nonzero alternating space of degree $d \geq 3$. Then there exist finitely many nonzero indecomposable alternating spaces $(U_1, \phi_1), \ldots, (U_m, \phi_m)$, which are unique up to isomorphism and order, such that $(V, \theta) \cong (U_1, \phi_1) \oplus \cdots \oplus (U_m, \phi_m)$.

1.3.2 Remark: We call a subspace A of V a *summand* if $A \oplus A^{\perp} = V$. Then V is the orthogonal sum of its nonzero indecomposable summands.

(We shall sometimes write $\perp$ for the direct sum.)

1.3.3 Proposition (cf. [H1] Proposition 2.4 p129): Let $(V, \theta), (U, \phi), (W, \psi)$ be nonzero alternating spaces of degree $d \geq 3$ with $(V, \theta) \oplus (U, \phi) \cong (V, \theta) \oplus (W, \psi)$. Then $(U, \phi) \cong (W, \psi)$.

1.3.4 Remark: Kanzaki ([KAN] p735) calls a form θ of degree $r \geq 2$ *ζ-skew-symmetric* if $\theta(x_1, x_2, x_3, \ldots, x_r) = \zeta\theta(x_2, x_3, \ldots, x_r, x_1)$ for all x_i, where $\zeta^r = 1$; this clearly includes both symmetry and alternation as special cases.

This notion requires the existence of r^{th} roots of unity in F, and a choice of a particular root ζ which cannot, in general, be fixed canonically.
He proves, in Lemma 2 (pp736-7), a generalization of Harrison's Propositions 2.3 (excluding existence), 2.4 and 4.1 (see §1.5). Since we do not require this kind of generality, we have confined ourselves to the much simpler requirements of the alternating case.

1.4 Adjoint and Centre

The notion of the *centre* of a form is defined by Harrison for symmetric forms of degree ≥ 3 ([H1] p 133) and, more generally, by Kanzaki for ζ-skew-symmetric forms ([KAN] p736).
We discuss the centre within the context of the broader notion of the *adjoint* of an endomorphism (or linear operator) on a vector space equipped with a form; because we adopt a slightly different approach to Harrison's, we discuss the symmetric case in parallel with the alternating case, which is our main concern.

Let (V, θ) be a nondegenerate bilinear symmetric space, and let $T \in End(V)$. It is well known that, because of nondegeneracy, there exists a unique $T^* \in End(V)$ such that $\theta(Tu, v) = \theta(u, T^*v)$ for all $u, v \in V$. (See, for example, [MB] Theorem 10, p396.)
T^* is called the *adjoint* of T (with respect to θ).
By the symmetry and nondegeneracy of θ, we have $T^{**} = T$.
T is called *self-adjoint* (or *symmetric*) if $T^* = T$ ([MB] p397).
All of the above are valid also for bilinear alternating forms.
If T is self-adjoint, the bilinear form θ_T defined by $\theta_T(u, v) = \theta(Tu, v)$ is easily seen to be symmetric (resp. alternating) if θ is symmetric (resp. alternating).
T is called *skew-self-adjoint* (or *skew-symmetric*) if $T^* = -T$. In this case, T converts symmetric forms to alternating forms, and vice versa: if θ is symmetric (resp. alternating), then θ_T is alternating (resp. symmetric).

Now suppose that θ is a nondegenerate symmetric form of degree $d \geq 3$, and let $T \in End(V)$.

We call $S \in End(V)$ an *adjoint* of T if

$$\theta(Tv_1, v_2, v_3, \ldots, v_d) = \theta(v_1, Sv_2, v_3, \ldots, v_d) \text{ for all } v_1, \ldots, v_d.$$

Because of symmetry, the positions of T and S are immaterial.

By nondegeneracy, an adjoint of T, if it exists, is unique; we denote the unique adjoint by T^*. T is called *adjointable* if it has an adjoint.

By symmetry and nondegeneracy we have, as in the bilinear case, *reflexivity*, i.e. $T^{**} = T$.

All of the foregoing holds if θ is alternating. The irrelevance of the positions of T and T^* requires a simple check:

$$\begin{aligned}
\theta(Tv_1, v_2, \ldots, v_d) &= \theta(v_1, T^*v_2, \ldots, v_d) \\
&= -\theta(v_i, T^*v_2, \ldots, v_1, \ldots, v_d) \\
&= -\theta(Tv_i, v_2, \ldots, v_1, \ldots, v_d) \\
&= \theta(v_1, v_2, \ldots, Tv_i, \ldots, v_d)
\end{aligned}$$

(where v_1 is in the i^{th} position in the third and fourth expressions on the RHS); we can likewise shift T^* from v_2 to v_j, say.

The set of adjointable operators is clearly a subspace of $End(V)$, with $(\alpha T)^* = \alpha T^*$ and $(T+S)^* = T^* + S^*$.

We now link our approach to that of Harrison by showing that T adjointable $\Rightarrow$ T self-adjoint. We assume θ is alternating and $d > 2$:

$$\begin{aligned}
\theta(Tv_1, v_2, v_3, \ldots, v_d) &= \theta(v_1, T^*v_2, v_3, \ldots, v_d) \\
&= -\theta(v_3, T^*v_2, v_1, \ldots, v_d) \\
&= -\theta(Tv_3, v_2, v_1, \ldots, v_d) \\
&= \theta(Tv_3, v_1, v_2, \ldots, v_d) \\
&= \theta(v_3, T^*v_1, v_2, \ldots, v_d) \\
&= \theta(T^*v_1, v_2, v_3, \ldots, v_d),
\end{aligned}$$

by repeated application of adjointness and alternation. By nondegeneracy we thus have $T = T^*$.

Note the dependence of this argument, as well as subsequent ones, on the assumption that $d > 2$.

The set of adjointable operators is therefore the *centre* of θ, $Z(\theta)$, as defined by Harrison ([H1] p133).

If θ is symmetric, we get the same result by omitting the minus signs in the above argument.

We check that $Z(\theta)$ is closed under multiplication, i.e. that $Z(\theta)$ is an F-algebra: if S, T are adjointable/self-adjoint and θ is symmetric or alternating then

$$\begin{aligned}\theta(STv_1, v_2, v_3, \ldots, v_d) &= \theta(Tv_1, v_2, Sv_3, \ldots, v_d) \\ &= \theta(v_1, Tv_2, Sv_3, \ldots, v_d) \\ &= \theta(v_1, STv_2, v_3, \ldots, v_d),\end{aligned}$$

hence ST is self-adjoint.

It is now easy to see that $Z(\theta)$ is commutative: $ST = (ST)^* = T^*S^* = TS$.

We have thus established the first part of (cf. [H1] Proposition 4.1 p134):

1.4.1 Proposition: Let (V, θ) be a nondegenerate alternating space of degree $d \geq 3$. Then

(i) $Z(\theta)$ is a commutative F-algebra;

(ii) (V, θ) is indecomposable if and only if $Z(\theta)$ contains no idempotents except 0 and 1.

Proof of (ii): Harrison's proof holds in the alternating case, but we fill in details not provided in [H1].

Suppose V is decomposable, i.e. $V = V_1 \perp V_2$, where V_1, V_2 are proper subspaces of V. Let $p \in End(V)$ denote the projection onto V_1. Then

$$\theta(pv_1, v_2, v_3, \ldots, v_d)$$

$$
\begin{aligned}
&= \theta(p(v_1' + v_1''), v_2' + v_2'', v_3, \ldots, v_d) \quad (\text{where } v_i' \in V_1, v_i'' \in V_2, i = 1, 2) \\
&= \theta(v_1', v_2' + v_2'', v_3, \ldots, v_d) \quad (p \text{ is projection}) \\
&= \theta(v_1', v_2', v_3, \ldots, v_d) \\
&= \theta(v_1' + v_1'', v_2', v_3, \ldots, v_d) \quad (V_1 \perp V_2) \\
&= \theta(v_1, pv_2, v_3, \ldots, v_d) \quad (p \text{ is projection}).
\end{aligned}
$$

Then $p \in Z(\theta)$, and clearly $p \neq 0, 1$ since $V_1 \neq 0, V$.

Now suppose $Z(\theta)$ contains an idempotent $p \neq 0, 1$. Let $K = ker\, p, I = Im\, p$. Then $V = K \oplus I$, since any $v \in V$ can be written $v = (v - p(v)) + p(v)$, with $v - p(v) \in K$ and $p(v) \in I$. (It is clear the $K \cap I = 0$.) Suppose $v_1 \in K, v_2 \in I, v_3, \ldots, v_d \in V$. Then

$$
\begin{aligned}
\theta(v_1, v_2, v_3, \ldots, v_d) &= \theta(v_1, pv_2', v_3, \ldots, v_d) \quad (v_2 \in I = Im\, p) \\
&= \theta(pv_1, v_2', v_3, \ldots, v_d) \quad (p \in Z(\theta)) \\
&= 0 \quad (v_1 \in K, \text{so } pv_1 = 0).
\end{aligned}
$$

So $K \perp I$, hence $V = K \perp I$, with $K, I \neq 0, V$ since $p \neq 0, 1$.

Thus (V, θ) is decomposable. □

We conclude this section by elaborating on the contrast between degree 2 and degree > 2 in relation to adjointness.

In degree 2, adjointable operators do not coincide with self-adjoint operators (as they do in degree > 2). The set of self-adjoint (or symmetric) operators does not constitute a ring; for example

$$
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.
$$

The set of adjointable operators does, however, constitute a ring, since it is the full algebra $End(V)$. This provides some justification for our focusing on adjointability in this section.

The set of self-adjoint operators (i.e. the centre $Z(\theta)$), while not an algebra, is a Jordan algebra ([JAC2] p4): if A, B are self-adjoint, then $\{A,B\}^* = (AB+BA)^* = (AB)^* + (BA)^* = B^*A^* + A^*B^* = BA + AB = \{A,B\}$, so $\{A,B\}$ is self-adjoint. By similar reasoning, we can see that $[A,B]$ is skew-self-adjoint if A, B are, so the set of skew-self-adjoint operators (i.e. the "anti-centre") is a Lie algebra, but not an algebra in general.

1.5 Witt-Grothendieck group

Harrison ([H1] §3 p131) defines a (Witt-)Grothendieck ring $L_r(R)$ of symmetric forms of degree $r \geq 3$ over a field R, consisting of formal differences of isomorphism classes of nondegenerate symmetric spaces of degree r; as an abelian group, $L_r(R)$ is freely generated by the isomorphism classes of nondegenerate *indecomposable* symmetric spaces.

Harrison and Pareigis ([HP]) have adopted an approach to defining a Witt ring of higher degree symmetric forms which uses 1-dimensional subspaces and a notion of diagonalizability, and hence is not directly relevant to the case of alternating forms.

In the case of bilinear alternating forms, the Witt group is defined as in the symmetric case ([SCHA] p239) and is particularly simple. There is one indecomposable space, viz. $\mathcal{H}$, the *hyperbolic/Lagrangian plane*, and every nondegenerate alternating bilinear space is a sum of copies of $\mathcal{H}$. Hence there is precisely one isometry class of nondegenerate alternating spaces in each even dimension, so the Witt-Grothendieck group $\widehat{W}_a(F) \cong \mathbb{Z}$.

(We modify Scharlau's notation of $\widehat{W}(-)$ for the Witt-Grothendieck ring/group and $W(-)$ for the Witt ring/group: the subscript a denotes alternation, and a superscript denotes the degree.)

If we factor out the subgroup generated by the hyperbolics, we obtain the Witt group $W_a(F) = 0$.

In order to define a Witt-Grothendieck group for alternating forms of degree $r \geq 3$, we observe that the results used by Harrison in his construction extend to alternating forms. We have established the results on decomposition and cancellation (§1.3 p22); it is obvious that, as in the symmetric case (see [H1] Proposition 2.1 p127), the direct sum of alternating spaces is associative, commutative and has a zero element.
Hence we can construct a Witt-Grothendieck group for alternating forms of degree r, denoted $\widehat{W}_a^r(F)$.
This group is clearly also freely generated by isomorphism classes of indecomposable alternating spaces of degree r. In contrast to degree 2, where there is *one* indecomposable (of dimension 2), for $r > 2$ the hyperbolics are all indecomposable (see §2.2 pp34-7), and there are countably many non-isomorphic hyperbolics with dimension unbounded.

We can define a tensor product of alternating spaces (V, θ) and (W, ϕ) of degree r as for symmetric spaces, viz. $(V \otimes W, \theta \otimes \phi)$, where

$$\theta \otimes \phi(v_1 \otimes w_1, \ldots, v_r \otimes w_r) = \theta(v_1, \ldots, v_r)\phi(w_1, \ldots, w_r),$$

but now $\theta \otimes \phi$ is *symmetric*, not alternating, so there is no possibility of putting a ring structure on $\widehat{W}_a^r(F)$ with the tensor product.
It is obvious, however, that if θ is symmetric and ϕ is alternating, then $\theta \otimes \phi$ is alternating, so $\widehat{W}_a^r(F)$ is a module over $\widehat{W}^r(F)$. (Again, we observe that the requisite properties of addition, i.e. direct sum, and scalar multiplication, i.e. tensor product, are valid — see [H1] Proposition 2.1 p127.)
In fact, more is true: $\widehat{W}^r(F) \oplus \widehat{W}_a^r(F)$ is a $\mathbb{Z}_2$-graded ring, with tensor product as multiplication, since the tensor product of two alternating forms is symmetric. Nothing seems to be known about the structure of this ring.

Chapter 2

Alternating Spaces: Hyperbolics

In this chapter we introduce hyperbolic alternating spaces and, as Keet has done for symmetric forms ([K1] Ch 4, Ch 5 §1), determine their centre, isometry group, Lie algebra and prove a descent result.

While in many respects there is a great deal of similarity between the symmetric and alternating cases, there are also important divergences which we shall highlight. Even where the arguments are very similar, we sometimes provide details not given in [K1].

We summarize the contents of this chapter.

In §2.1 we define alternating hyperbolic space and prove a simple lemma which we shall often use where, in the symmetric case, merely invoking the existence of nondegenerate forms of arbitrary degree and dimension will suffice.

We prove that these spaces are alternating, multilinear and nondegenerate, and also that they are cofinal for alternating space, i.e. any alternating space can be isometrically embedded in an alternating hyperbolic space.

In §2.2 we compute the centre of the alternating hyperbolic space. Although there is some similarity to the symmetric case, we have to adopt (in (iii)) a different method to show that one of the components of an endomorphism in the centre is scalar. The centre turns out to be F^*Alt_dV, the trivial extension of F by Alt_dV (cf. F^*Sym_dV, where Sym_dV denotes the space of symmetric forms of degree d on V, in the symmetric case — [K1] p41), and it follows from §1.4 that the alternating hyperbolic is indecomposable.

In §2.3 we discuss general results on algebraic groups and their Lie algebras. The proofs of all the results, apart from Proposition 2.3.5, are as in the symmetric case, and we merely cite [K1] as a reference. There is no analogue of Keet's Proposition 1.7 ([K1] p65), so we have to use Proposition 2.3.3 in the next section. We observe that §1.9 in [K1] also carries over to the alternating case, so we have a corresponding result for reducing the calculation of the isometry group to the indecomposable case. In §2.4 we determine the isometry group of the alternating hyperbolic. In the symmetric case, the fact that $Sym_d V$ is the singular locus ([K1] p69) affords a considerable simplification of the computation. Since the concept of nonsingularity makes no sense in the alternating case, we use instead the notion of the i^{th} *domain* of an alternating form, taken from [CH], and the fact that this is invariant under isometry. Proving that $Alt_d V$ is a union of such i^{th} domains constitutes the bulk of the work in this section. We are then able to describe the isometry group as $GL(V) \bowtie \mathcal{F}$, where $\mathcal{F}$ is a space of forms satisfying two symmetry conditions:

(i) alternation in the first d variables; and

(ii) a (signed) Jacobi identity.

We then show that $\mathcal{F}$ is precisely the *co-Schur functor* $K_\mu(V^*)$, where $\mu = (2, 1^{d-1})$ (see [ABW]).

It follows by the same argument as in the symmetric case that the isometry group is connected, and that its radical is 2-step solvable.

In §2.5 we show that the Lie algebra of the alternating hyperbolic is $End(V) \oplus K_\mu(V^*)$. It follows, again by the same reasoning as in the symmetric case, that its radical is 2-step solvable.

In §2.6 we use the results of the previous section, as well as general Lie theory, to prove a descent result for alternating hyperbolics. We follow roughly the approach of Keet ([K1] pp99-105), but have to make several significant adaptations and we give details of these. Most notably, we cannot define the set $T(\theta)$ in the alternating case, so we have to work instead with the (zero set of the) derived algebra of the radical of the Lie algebra.

2.1 Definition and Basic Properties

Let F be a field in which $d! \neq 0$ and let V be a finite-dimensional vector space over F. Let $Alt_{d-1}V$ denote the space of alternating $(d-1)$-multilinear forms on V. Put $W_1 = V$, $W_2 = Alt_{d-1}V, W = W_1 \oplus W_2$. Take $d \geq 2$.

Define the *hyperbolic degree d alternating space* (W, Ψ) by putting

$$\Psi[(x_1, \theta_1), \ldots, (x_d, \theta_d)] = \sum_{i=1}^{d} (-1)^{i-1} \theta_i(x_1, \ldots, \widehat{x_i}, \ldots, x_d).$$

For $d = 2$ this gives the usual hyperbolic bilinear alternating space $(V \oplus V^*, \Psi)$, with $\Psi[(x, f), (y, g)] = f(y) - g(x)$ ([SCHA] p239).

We show that Ψ is (i) alternating (ii) d-multilinear (iii) nondegenerate. First we prove the following:

2.1.1 Lemma: Suppose $f(v_1, \ldots, v_d) = 0$ for all alternating forms f of degree d on a vector space V, and for all $v_2, \ldots, v_d$ in V. Then $v_1 = 0$.

Proof: Let $e_1, \ldots, e_n$ be basis for V. We may assume that $n \geq d$. Put $v_1 = \sum_{i=1}^{n} \alpha_i e_i$. Then we have

$\sum_{i=1}^{n} \alpha_i f(e_i, v_2, \ldots, v_d) = 0$ for all $v_2, \ldots, v_d$ and all f. (1)

Fix $j, 1 \leq j \leq d$. Choose $(v_2, \ldots, v_d) = (e_1, \ldots, \widehat{e_j}, \ldots, e_d)$ and $f = e_j^* \wedge e_1^* \wedge \ldots \wedge \widehat{e_j^*} \wedge \ldots \wedge e_d^*$ in (1). Then $\alpha_j = 0$. This is true for all $j = 1, \ldots, d$, hence $\alpha_1 = \ldots = \alpha_d = 0$. Now fix $j, d+1 \leq j \leq n$. Choose $(v_2, \ldots, v_d) = (e_1, \ldots, e_{d-1})$ and $f = e_j^* \wedge e_1^* \wedge \ldots \wedge e_{d-1}^*$ in (1). Then $\alpha_j = 0$. This is true for all $j = d+1, \ldots, n$, hence $\alpha_{d+1} = \ldots = \alpha_n = 0$. Thus $v_1 = 0$, as required. □

2.1.2 Remark: In the case of symmetric forms, this Lemma follows easily by invoking the existence of nondegenerate symmetric forms of arbitrary degree in arbitrary dimension ([K1] Ch 1 §1.10(iii) p29). We have already observed that this cannot be done in the case of alternating forms (Ch 1 p8).

(i) By §1.1 (p10) it suffices to show that $\Psi = 0$ if two adjacent arguments are equal. Suppose that $(x_i, \theta_i) = (x_{i+1}, \theta_{i+1}) = (x, \theta)$. Then

$$\Psi[(x_1,\theta_1),\dots,(x,\theta),(x,\theta),\dots,(x_d,\theta_d)] =$$
$$\theta_1(x_2,\dots,x,x,\dots,x_d) + \dots + (-1)^{i-1}\theta(x_1,\dots,x_{i-1},x,x_{i+2},\dots,x_d)+$$
$$(-1)^{i}\theta(x_1,\dots,x_{i-1},x,x_{i+2},\dots,x_d) + \dots + (-1)^{d-1}\theta_d(x_1,\dots,x,x,\dots,x_{d-1}) = 0,$$

since all terms are zero except the i^{th} and $(i+1)^{th}$, which cancel one another.

(ii) $\Psi[(x_1,\theta_1),\dots,\alpha(x_j,\theta_j)+\beta(x'_j,\theta'_j),\dots,(x_d,\theta_d)]$
$= \Psi[(x_1,\theta_1),\dots,(\alpha x_j+\beta x'_j, \alpha\theta_j+\beta\theta'_j),\dots,(x_d,\theta_d)]$
$= \sum_{i\neq j}(-1)^{i-1}\theta_i(x_1,\dots,\widehat{x_i},\dots,\alpha x_j+\beta x'_j,\dots,x_d)+$
$(-1)^{j-1}(\alpha\theta_j+\beta\theta'_j)(x_1,\dots,\widehat{x_j},\dots,x_d)$
$= \alpha\sum_{i\neq j}(-1)^{i-1}\theta_i(x_i,\dots,\widehat{x_i},\dots,x_j,\dots,x_d)+$
$\beta\sum_{i\neq j}(-1)^{i-1}\theta_i(x_1,\dots,\widehat{x_i},\dots,x'_j,\dots,x_d) + (-1)^{j-1}\alpha\theta_j(x_1,\dots,\widehat{x_j},\dots,x_d)+$
$(-1)^{j-1}\beta\theta'_j(x_1,\dots,\widehat{x_j},\dots,x_d)$.

Combining the first and third, and second and fourth terms, respectively, this equals

$$\alpha\Psi[(x_1,\theta_1),\dots,(x_j,\theta_j),\dots,(x_d,\theta_d)] + \beta\Psi[(x_1,\theta_1),\dots,(x'_j,\theta'_j),\dots,(x_d,\theta_d)].$$

(iii) Suppose that $\Psi[(v,\theta),(v_2,\theta_2),\dots,(v_d,\theta_d)] = 0$, i.e.
$\theta(v_2,v_3,\dots,v_d) - \theta_2(v,v_3,\dots,v_d) + \dots + (-1)^{d-1}\theta_d(v,v_2,\dots,v_{d-1}) = 0$ for all $v_i, \theta_i, 2 \le i \le d$. (2)
We need to show $v = 0, \theta = 0$. Choose, say, $v_2 = 0$ in (2).Then we obtain $\theta_2(v,v_3,\dots,v_d) = 0$ for all $v_3,\dots,v_d$, and all θ_2. By Lemma 2.1.1, $v = 0$.
If we choose $\theta_2 = \dots = \theta_d = 0$, we obtain $\theta(v_2,\dots,v_d) = 0$ for all $v_2,\dots,v_d$, hence $\theta = 0$. □

We now show that hyperbolic alternating space is *cofinal* for alternating spaces, i.e. every alternating space (V,f) of degree d can be isometrically embedded in some hyperbolic alternating space of degree d:

Define $\phi : (V, f) \to (V \oplus Alt_{d-1}V, \Psi)$ by $\phi(v) = (v, f^{(v)})$, where $f^{(v)}$ denotes the derivative of f with respect to v. It is obvious that ϕ is linear (observe that $f^{(v+w)} = f^{(v)} + f^{(w)}$), and that it is injective. It remains to show that ϕ is an isometry, i.e. that $\Psi[\phi(v_1), \ldots, \phi(v_d)] = f(v_1, \ldots, v_d)$ for all $v_1, \ldots, v_d$. Now

$$\begin{aligned}
\Psi[\phi(v_1), \ldots, \phi(v_d)] &= \Psi[(v_1, f^{(v_1)}), \ldots, (v_d, f^{(v_d)})] \\
&= \sum_{i=1}^{d} (-1)^{i-1} f^{(v_i)}(v_1, \ldots, \widehat{v_i}, \ldots, v_d) \\
&= \sum_{i=1}^{d} (-1)^{i-1} f(v_i, v_1, \ldots, \widehat{v_i}, \ldots, v_d) \\
&= d\, f(v_1, \ldots, v_d).
\end{aligned}$$

Then $\frac{1}{d}\phi$ is the required isometry.

2.1.3 Remarks:

1. All nondegenerate bilinear alternating forms are hyperbolic. For $d > 2$, there exist (simply from dimension considerations) nondegenerate alternating forms which are not hyperbolic.

2. Of the canonical trilinear alternating forms discussed by Cohen and Helminck ([CH] Table 1 p4), f_1 (dimension 3) and f_4 (dimension 6) are hyperbolic. The other dimension 6 form, f_3, as well as all the others, are not hyperbolic.

2.2 Centre of Alternating Hyperbolic Space

Recall (§1.4 p25) that if (V, θ) is any alternating space of degree $d \geq 3$, then the *centre* of (V, θ) is

$$Z(\theta) = \{f \in End(V) : \theta(fv_1, v_2, \ldots, v_d) = \theta(v_1, fv_2, \ldots, v_d) \forall v_1, \ldots, v_d \in V\}.$$

(By alternation, this implies that $\theta(\ldots, fv_i, \ldots, v_j, \ldots) = \theta(\ldots, v_i, \ldots, fv_j, \ldots)$ — see §1.4 p24.)

We determine the centre of the alternating hyperbolic space (W, Ψ) of degree $d \geq 3$ described above. (See [K1] §2.9 pp40-1 for the symmetric case.)

Let $f \in Z(\Psi)$, where $f : W \to W$.

Write f as a matrix $\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$ of linear maps $f_{ij} : W_j \to W_i, i, j = 1, 2$.

(i) First we show that $f_{12} = 0$ (where $f_{12} : W_2 \to W_1$). Take $\theta \in W_2$. Suppose $f(0, \theta) = (x, \theta') \in W$. We show $x = 0$.

Now $f \in Z(\Psi)$ iff $\Psi[f(x_1, \theta_1), (x_2, \theta_2), \ldots, (x_d, \theta_d)] =$ $\Psi[(x_1, \theta_1), f(x_2, \theta_2), \ldots, (x_d, \theta_d)]$ for all $(x_i, \theta_i) \in W$. In particular, $\Psi[f(0, \theta), (x_2, \theta_2), \ldots, (x_d, \theta_d)] = \Psi[(0, \theta), f(x_2, \theta_2), \ldots, (x_d, \theta_d)]$ for all $(x_i, \theta_i), 2 \leq i \leq d$. If we put $f(0, \theta) = (x, \theta'), f(x_2, \theta_2) = (x_2', \theta_2')$, then we have $\Psi[(x, \theta'), (x_2, \theta_2), \ldots, (x_d, \theta_d)] = \Psi[(0, \theta), (x_2', \theta_2'), \ldots, (x_d, \theta_d)]$ for all $(x_2, \theta_2), \ldots, (x_d, \theta_d)$. Expanding this gives

$$\theta'(x_2, \ldots, x_d) - \theta_2(x, x_3, \ldots, x_d) + \theta_3(x, x_2, x_4, \ldots, x_d) - \ldots +$$
$$(-1)^{d-1}\theta_d(x, x_2, \ldots, x_{d-1}) = \theta(x_2', x_3, \ldots, x_d) \text{ for all } (x_i, \theta_i), 2 \leq i \leq d. \quad (1)$$

Putting, say, $x_3 = 0$ in (1), we obtain $\theta_3(x, x_2, x_4, \ldots, x_d) = 0$ for all $x_2, x_4, \ldots, x_d$ and all θ_3. By Lemma 2.1.1, this gives $x = 0$, as required.

(ii) Next we describe f_{21} ($f_{21} : W_1 \to W_2$).

We have $\Psi[f(x_1, 0), (x_2, 0), \ldots, (x_d, 0)] = \Psi[(x_1, 0), f(x_2, 0), \ldots, (x_d, 0)]$ for all $x_1, \ldots, x_d$. Now $f(x_1, 0) = (f_{11}x_1, f_{21}x_1), f(x_2, 0) = (f_{11}x_2, f_{21}x_2)$, so we have $f_{21}x_1(x_2, x_3, \ldots, x_d) = -f_{21}x_2(x_1, x_3, \ldots, x_d)$ for all $x_1, \ldots, x_d$. Define $\alpha : V^d \to F$ by $\alpha(x_1, \ldots, x_d) = f_{21}x_1(x_2, x_3, \ldots, x_d)$. α is linear in $x_2, \ldots, x_d$ because $f_{21}x_1$ is, and is also linear in x_1 because f_{21} is. So α is d-multilinear. To see that α is alternating, we note that $f_{21}x_1$ is alternating, so it suffices to check the first 2 arguments:

$$\begin{aligned} \alpha(x_1, x_2, x_3, \ldots, x_d) &= f_{21}x_1(x_2, x_3, \ldots, x_d) \\ &= -f_{21}x_2(x_1, x_3, \ldots, x_d) \end{aligned}$$

$$= -\alpha(x_2, x_1, x_3, \ldots, x_d).$$

Hence f_{21} is determined by an alternating d-multilinear form.

(iii) We show that f_{11} is multiplication by a scalar ($f_{11} : W_1 \to W_1$). Since $\Psi[f(x_1,0),(x_2,0),\ldots,(x_{d-1},0),(0,\theta)] = \Psi[(x_1,0),f(x_2,0),\ldots,(x_{d-1},0),(0,\theta)]$, using $f(x_1,0) = (f_{11}x_1, f_{21}x_1), f(x_2,0) = (f_{11}x_2, f_{21}x_2)$, we obtain

$$\theta(f_{11}x_1, x_2, \ldots, x_{d-1}) = \theta(x_1, f_{11}x_2, \ldots, x_{d-1}) \text{ for all } x_1, \ldots, x_{d-1} \text{ and all } \theta. \quad (2)$$

Let $\{v_1, \ldots, v_n\}$ be a basis for V. Fix i, j such that $1 \leq i, j \leq n$, and choose $x_3 = v_{i_3}, \ldots, x_{d-1} = v_{i_{d-1}}$, where $1 \leq i_3 < \ldots < i_{d-1} \leq n$ and $i_r \neq i$ or j for $3 \leq r \leq d-1$. Put $\tilde{\theta}(u,v) = \theta(u, v, v_{i_3}, \ldots, v_{i_{d-1}})$ for all $u, v \in V$. Then $\tilde{\theta}$ is an alternating bilinear form, hence it can be represented as an alternating $n \times n$ matrix B. The condition (2) becomes $\tilde{\theta}(f_{11}x_1, x_2) = \tilde{\theta}(x_1, f_{11}x_2)$, i.e.
$(f_{11}x_1)^t B x_2 = x_1^t B(f_{11}x_2)$, i.e. $x_1^t(f_{11}^t B)x_2 = x_1^t(Bf_{11})x_2$ for all $x_1, x_2 \in V$.
Hence $f_{11}^t B = Bf_{11}$. Now choose $\theta = (v_i \wedge v_j \wedge v_{i_3} \wedge \ldots \wedge v_{i_{d-1}})^*$, so that $\tilde{\theta}(v_i, v_j) = -\tilde{\theta}(v_j, v_i) = 1$ and $\tilde{\theta}(v_r, v_s) = 0$ otherwise. The matrix B of $\tilde{\theta}$ is then given by $B = e_{ij} - e_{ji}$, where e_{kl} denotes the usual elementary matrix. Suppose $f_{11} = (a_{rs})$, where $1 \leq r, s \leq n$. A simple calculation shows that, for $1 \leq k, l \leq n$, $(f_{11}^t B)_{ki} = -a_{jk}$, $(f_{11}^t B)_{kj} = a_{ik}, (Bf_{11})_{il} = a_{jl}$ and $(Bf_{11})_{jl} = -a_{il}$. Hence $-a_{ji} = (f_{11}^t B)_{ii} = (Bf_{11})_{ii} = a_{ji}$, so $a_{ji} = 0$, $a_{ij} = (f_{11}^t B)_{jj} = (Bf_{11})_{jj} = -a_{ij}$, so $a_{ij} = 0$ (since *char* $F \neq 2$), and $a_{ii} = (f_{11}^t B)_{ij} = (Bf_{11})_{ij} = a_{jj}$. This is is true for all $i, j = 1, \ldots, n$, and shows that f_{11} is a scalar matrix, as claimed.

(iv) f_{22} is multiplication by a scalar (where $f_{22} : Alt_{d-1}V \to Alt_{d-1}V$). We have $\Psi[f(0,\theta),(x_2,0),\ldots,(x_d,0)] = \Psi[(0,\theta),f(x_2,0),\ldots,(x_d,0)]$, so using $f(0,\theta) = (f_{12}\theta, f_{22}\theta), f(x_2,0) = (f_{11}x_2, f_{21}x_2)$, we obtain
$f_{22}\theta(x_2, \ldots, x_d) = \theta(f_{11}x_2, \ldots, x_d)$ for all $x_2, \ldots, x_d$ and all θ.
Since f_{11} is scalar multiplication, the same applies to f_{22}.

We have thus shown that, if $f \in Z(\Psi)$, then
$f(x,\theta) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}(x,\theta) = (f_{11}x + f_{12}\theta, f_{21}x + f_{22}\theta) = (\lambda x, \alpha^{(x)} + \lambda\theta)$, where $\lambda \in F, \alpha \in Alt_d V$.

Next we show that if f is any endomorphism of this type, then $f \in Z(\Psi)$:

$$\begin{aligned}
&\Psi[f(x_1,\theta_1),(x_2,\theta_2),\dots,(x_d,\theta_d)] \\
&= \Psi[(\lambda x_1, \alpha^{(x_1)} + \lambda\theta),(x_2,\theta_2),\dots,(x_d,\theta_d)] \\
&= \alpha(x_1,x_2,\dots,x_d) + \lambda\theta_1(x_2,\dots,x_d) - \theta_2(\lambda x_1, x_3,\dots,x_d) + \dots + \\
&\quad (-1)^{d-1}\theta_d(\lambda x_1, x_2,\dots,x_{d-1}) \\
&= \alpha(x_1,x_2,\dots,x_d) + \lambda\Psi[(x_1,\theta_1),\dots,(x_d,\theta_d)]
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
&\Psi[(x_1,\theta_1), f(x_2,\theta_2),\dots,(x_d,\theta_d)] \\
&= \Psi[(x_1,\theta_1),(\lambda x_2, \alpha^{(x_2)} + \lambda\theta_2),(x_3,\theta_3),\dots,(x_d,\theta_d)] \\
&= \theta_1(\lambda x_2, x_3,\dots,x_d) - \alpha(x_2,x_1,x_3,\dots,x_d) - \lambda\theta_2(x_1,x_3,\dots,x_d) \\
&\quad + \theta_3(x_1,\lambda x_2,\dots,x_d) + \dots + (-1)^{d-1}\theta_d(x_1,\lambda x_2,\dots,x_{d-1}) \\
&= \alpha(x_1,x_2,\dots,x_d) + \lambda\Psi[(x_1,\theta_1),\dots,(x_d,\theta_d)],
\end{aligned}$$

as required. (The last step uses alternation of α.)

As an F-vector space, $Z(\Psi)$ consists of pairs $(\lambda,\alpha), \lambda \in F, \alpha \in Alt_d V$. So $Z(\Psi) = F \oplus Alt_d V$.

But $Z(\Psi) \subset End(W_1 \oplus W_2)$, so each element in $Z(\Psi)$ is a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Clearly, then, the matrix of (λ,α) is $\begin{pmatrix} \lambda & 0 \\ \alpha^{(-)} & \lambda \end{pmatrix}$, where $\alpha^{(-)}$ denotes the derivative of α with respect to $-$.

Multiplication in $Z(\Psi)$: $\begin{pmatrix} \lambda & 0 \\ \tau & \lambda \end{pmatrix}\begin{pmatrix} \mu & 0 \\ \sigma & \mu \end{pmatrix} = \begin{pmatrix} \lambda\mu & 0 \\ \mu\tau + \lambda\sigma & \lambda\mu \end{pmatrix}$ or $(\lambda,\theta)(\mu,\phi) = (\lambda\mu, \mu\theta + \lambda\phi)$.

Multiplication in Alt_dV is trivial :$(0,\theta)(0,\phi) = (0,0)$. Hence $Z(\Psi)$ is F^*Alt_dV, the trivial extension of F by Alt_dV, which is easily seen to be the unique prime ideal of $Z(\Psi)$. (See [K1] notes following §2.2 p36.) It is also easy to see that the only idempotents of $Z(\Psi)$ are zero $(0,0)$ and the identity $(1,0)$, and hence the alternating hyperbolic space is *indecomposable* (cf. [K1] §2.2(ii) p36): Suppose $(\lambda,\theta)(\lambda,\theta) = (\lambda,\theta)$, i.e. $(\lambda^2, 2\lambda\theta) = (\lambda,\theta)$. Then $\lambda^2 = \lambda$, i.e. $\lambda = 0$ or $\lambda = 1$, and $2\lambda\theta = \theta$ gives $\theta = 0$ if $\lambda = 0$, and gives $2\theta = \theta$, i.e. $\theta = 0$, also if $\lambda = 1$.

2.3 Algebraic Subgroups of $GL(V)$ and their Lie Algebras

We extend the results of [K1] Ch 4 §§1.1-1.9 (pp62-68) to the alternating case. Most of the results do not depend on symmetry at all, and we shall state these without proof. We give the proof of the analogue of Proposition 1.8 in [K1] in full; Proposition 1.7 in [K1] obviously has no analogue.

If V is a vector space of dimension n over F, then $GL(V)$ has the structure of an affine algebraic variety. Given a basis for V, $M \in GL(V)$ is represented by a matrix $(m_{ij}) \in A_F^{n^2}$ with $det(m_{ij}) \neq 0$.
An *algebraic group* is an algebraic variety which is also a group, with multiplication and inversion being morphisms of varieties.
An *algebraic matrix group* is an algebraic group which is a closed subgroup of some $GL_n(F)$ in the Zariski topology.
An *algebraic subgroup of* $GL(V)$ is a subgroup which, given a basis, may be identified with an algebraic matrix group.

An algebraic subgroup of $GL(V)$ has an associated Lie algebra. The Lie algebra of $GL(V)$ itself is denoted $gl(V)$.
As an F-vector space it is $End(V)$, with Lie bracket $[M,N] = MN - NM$.
Let G be an algebraic subgroup of $GL(V)$. We describe its Lie algebra $L(G)$. Let $F[\epsilon]$ denote the F-algebra generated by ϵ with $\epsilon^2 = 0$ (the Study "dual numbers").

Suppose that G satisfies the polynomials $p_\lambda, \lambda \in \Lambda$. Then $L(G)$ consists of all $M \in End V$ such that $I + \epsilon M \in End(V \otimes_F F[\epsilon])$ satisfies all the polynomials p_λ. The set of all such elements of $End(V_{F[\epsilon]})$ constitute an algebraic subgroup of $GL(V_{F[\epsilon]})$, which we denote $G_{F[\epsilon]}$. $L(G)$ is a Lie subalgebra of $gl(V)$.

2.3.1 Proposition ([K1] Proposition 1.2): Let G_1, G_2 be algebraic subgroups of $GL(V_1), GL(V_2)$. Then $G_1 \times G_2$ is isomorphic to an algebraic subgroup of $GL(V_1 \oplus V_2)$ and $L(G_1 \times G_2) \cong L(G_1 \oplus L(G_2)$.

Let f be a multilinear form of degree d on a vector space V, and denote the space of such forms by $Mult_d V$. If σ is an automorphism of V, σ acts on $Mult_d V$ via $\sigma \cdot f(x_1, \ldots, x_d) = f(\sigma^{-1}x_1, \ldots, \sigma^{-1}x_d)$ for all $x_1, \ldots, x_d \in V$. We call σ an *isometry* if $\sigma \cdot f = f$, and the set of all such σ is the *isometry group* of f, denoted $G(f)$.

2.3.2 Proposition ([K1] Proposition 1.4): $G(f)$ is an algebraic subgroup of $GL(V)$.

We denote the Lie algebra associated to $G(f)$ by $L(f)$. In this case, $M \in L(f)$ if and only if $I + \epsilon M \in G(f_\epsilon)$, where f_ϵ denotes the extension of f to $V_{F[\epsilon]} = V \otimes F[\epsilon]$.

2.3.3 Proposition ([K1] Proposition 1.5): The Lie subalgebra $L(f)$ of $gl(V)$ consists of all $M \in End(V)$ such that $\sum_{i=1}^{d} f(v_1, \ldots, v_{i-1}, Mv_i, v_{i+1}, \ldots, v_d) = 0$ for all $v_1, \ldots, v_d$ in V.

We observe next that the algebraic isometry groups and Lie algebras of equivalent forms are conjugate.

2.3.4 Proposition ([K1] Proposition 1.6): Let $\sigma \in GL(V)$. Then (i) $G(\sigma \cdot f) = \sigma.G(f).\sigma^{-1}$ (ii) $L(\sigma \cdot f) = \sigma.L(f).\sigma^{-1}$.

We now show that the Lie algebra of a direct sum of nondegenerate alternating spaces of degree $d \geq 3$ is the direct sum of their Lie algebras (cf. [K1] Proposition 1.8).

2.3.5 Proposition: Let (V, f) be a nondegenerate alternating space of degree $d \geq 3$ over a field F with $d! \neq 0$ in F. If $(V, f) = \bigoplus_{i=1}^{r}(V_i, f_i)$, then $L(f) = \bigoplus_{i=1}^{r} L(f_i)$.

Proof: First assume $r = 2$. Now $M \in L(f)$ iff

$\sum_{i=1}^{d}(f_1 \oplus f_2)(\begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \ldots, M\begin{pmatrix} v_i \\ w_i \end{pmatrix}, \ldots, \begin{pmatrix} v_d \\ w_d \end{pmatrix}) = 0$ for all $v_i \in V_1, w_i \in V_2$ iff

$\sum_{i=1}^{d}[f_1(v_1, \ldots, M_{11}v_i, \ldots, v_d) + f_1(v_1, \ldots, M_{12}w_i, \ldots, v_d) +$

$f_2(w_1, \ldots, M_{21}v_i, \ldots, w_d) + f_2(w_1, \ldots, M_{22}w_i, \ldots, w_d)] = 0$ for all $v_i \in V_1, w_i \in V_2$.

If all $w_i = 0$, then we obtain $\sum f_1(v_1, \ldots, M_{11}v_i, \ldots, v_d) = 0$ for all $v_i \in V_1$, i.e. $M_{11} \in L(f_1)$. Similarly, we obtain $M_{22} \in L(f_2)$.

Hence $\sum f_1(v_1, \ldots, M_{12}w_i, \ldots, v_d) + \sum f_2(w_1, \ldots, M_{21}v_i, \ldots, w_d) = 0$ for all v_i, w_i.

If $v_1 = 0$ and $w_2 = w_3 = 0$, then we obtain $f_1(M_{12}w_1, v_2, \ldots, v_d) = 0$ for all $v_2, \ldots, v_d \in V_1, w_1 \in V_2$. Since f is nondegenerate, so is f_1, hence $M_{12}w_1 = 0$ for all $w_1 \in V_2$.

Thus $M_{12} = 0$. A similar argument shows that $M_{21} = 0$.

This shows that $L(f) \subset L(f_1) \oplus L(f_2)$.

The converse is obvious: If $M_1 \in L(f_1), M_2 \in L(f_2)$, then

$\sum_{i=1}^{d}(f_1 \oplus f_2)(\begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \ldots, (M_1 \oplus M_2)\begin{pmatrix} v_i \\ w_i \end{pmatrix}, \ldots, \begin{pmatrix} v_d \\ w_d \end{pmatrix})$

$= \sum_{i=1}^{d}(f_1 \oplus f_2)(\begin{pmatrix} v_1 \\ w_1 \end{pmatrix}, \ldots, \begin{pmatrix} M_1v_i \\ M_2w_i \end{pmatrix}, \ldots, \begin{pmatrix} v_d \\ w_d \end{pmatrix})$

$= \sum f_1(v_1, \ldots, M_1v_i, \ldots, v_d) + \sum f_2(w_1, \ldots, M_2w_i, \ldots, w_d) = 0$ for all $v_i \in V_1, w_i \in V_2$. Hence $L(f) = L(f_1) \oplus L(f_2)$.

The case $r > 2$ follows by induction. □

Keet reduces the calculation of the isometry group of a symmetric form to the case of an indecomposable ([K1] §1.9 pp66-68). The entire discussion carries over to alternating forms, since it depends on Proposition 2.3 of [H1] (p129), for which we have an analogue in Proposition 1.3.1 (p22), as well as general results about algebraic groups and their Lie algebras, and in algebraic geometry. Hence we have:

2.3.6 Proposition: Let (V, θ) be a nondegenerate alternating space of degree $d \geq 3$, and suppose $V = V_1 \perp \ldots \perp V_s$, where the V_i are the nonzero indecomposable summands of V. Then the connected component of $G(V, \theta)$ containing the identity equals the direct product of the connected components of the $G(V_i, \theta)$ containing the identity.

2.4 Isometry Group of Alternating Hyperbolic Space

The isometry group of a bilinear alternating form, known as its *symplectic group*, may be described in two ways:

If (V, f) is a nondegenerate bilinear alternating space of dimension $2n$ over F, its symplectic group $Sp_{2n}(F)$ is generated by the following matrices ([SCHA] Lemma 7.4 p263): (i)$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$, with $A \in GL_n(F)$; (ii)$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$, $\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$,where $B, C \in M_n(F)$ are skew-symmetric; and (iii)$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Hence all elements of $Sp_{2n}(F)$ have determinant 1.

Geometrically, the symplectic group is generated by the *symplectic transvections*, i.e. transformations $\sigma : V \to V$ given by $\sigma v = v + c.f(a, v)a$, where a is a fixed nonzero vector and c is a constant. (If $c = 0$, $\sigma = 1_V$; if $c \neq 0$, σ leaves v fixed if and only if $f(a, v) = 0$, meaning that v belongs to the hyperplane $H = (F \cdot a)^*$, the orthogonal complement of the line through a; σ is also the identity on the line $F \cdot a$ for any value of c.) The centre of $Sp_{2n}(F)$ is $\pm 1_V$, so $Sp_{2n}(F)$ is not simple. (See

[ART] pp139-140 for details.)

We now determine the isometry group of an alternating *hyperbolic* space of arbitrary degree and dimension. Recall (§1) the degree $d+1$ alternating hyperbolic, which consists of the space $H = V \oplus Alt_d V$ together with the alternating form f defined by $f[(v_1, \theta_1), \ldots, (v_{d+1}, \theta_{d+1})] = \sum_{i=1}^{d}(-1)^{i-1}\theta_i(v_1, \ldots, \hat{v}_i, \ldots, v_{d+1})$. We want to determine the isometry group $G(H, f)$. The case $dim\, V = d$ is easily dispensed with:

2.4.1 Proposition :
If $dim\, V = d$, then $G(H, f) = SL_{d+1}(F)$.

Proof: $dim\, H = n + \binom{n}{d} = d + \binom{d}{d} = d + 1$. If θ is *any* alternating form of degree $d+1$ on a vector space of dimension $d+1$ and σ is an isometry then $\theta(v_1 \wedge \ldots \wedge v_{d+1}) = \theta(\sigma v_1 \wedge \ldots \wedge \sigma v_{d+1}) = (det\, \sigma)\theta(v_1 \wedge \ldots \wedge v_{d+1})$, essentially because the determinant is the only alternating form of degree $d+1$ in $d+1$ variables. Hence $det\, \sigma = 1$, i.e. $\sigma \in SL_{d+1}(F)$. □

In the case $dim\, V > d$ there is a great deal of analogy with the symmetric situation (see [K1] §§2.1-2.4 pp 69-72) but a key part of the proof requires a quite different approach. In the symmetric case, $Sym_d V =$ the singular set of the hyperbolic form, which is easily seen to be invariant under isometry; in the alternating case, we cannot define the singular set, so proving the invariance of $Alt_d V$ is done as follows:

Let $x \in V$. Recall the derivative of f with respect to x is the degree $d-1$ alternating form $f^{(x)}$ given by
$f^{(x)}(x_1, \ldots, x_{d-1}) = f(x, x_1, \ldots, x_{d-1})$ for all $x_1, \ldots, x_{d-1} \in V$.
The *kernel* of f is
$ker\, f = \{x \in V : f(x, x_1, \ldots, x_{d-1}) = 0$ for all $x_1, \ldots, x_{d-1} \in V\}$, i.e. $x \in ker\, f$ if and only if $f^{(x)} = 0$,

and the *rank* of f is $rk\, f = dim\, V - dim\, ker f$.

Following Cohen and Helminck ([CH] p2), we define, for any nonnegative integer i, the i^{th} *domain of* f, $R_i(f) = \{x \in V : rk\, f^{(x)} = i\}$.

2.4.2 Proposition: $\sigma R_i(f) \subset R_i(f)$ for any isometry σ of f.

Proof: First we show that $rk(\sigma \cdot f)^{(x)} = rk\, f^{(\sigma^{-1}x)}$ for any automorphism $\sigma \in GL(V)$.

$$\begin{aligned}
& u \in ker(\sigma \cdot f)^{(x)} \\
\Leftrightarrow\ & (\sigma \cdot f)^{(x)}(u, v_3, \ldots, v_d) = 0 \quad \text{for all } v_3, \ldots, v_d \in V \\
\Leftrightarrow\ & \sigma \cdot f(x, u, v_3, \ldots, v_d) = 0 \quad \text{for all } v_3, \ldots, v_d \in V \\
\Leftrightarrow\ & f(\sigma^{-1}x, \sigma^{-1}u, \sigma^{-1}v_3, \ldots, \sigma^{-1}v_d) = 0 \quad \text{for all } v_3, \ldots, v_d \in V \\
\Leftrightarrow\ & f(\sigma^{-1}x, \sigma^{-1}u, w_3, \ldots, w_d) = 0 \quad \text{for all } w_3, \ldots, w_d \in V \\
\Leftrightarrow\ & f^{(\sigma^{-1}x)}(\sigma^{-1}u, w_3, \ldots, w_d) = 0 \quad \text{for all } w_3, \ldots, w_d \in V \\
\Leftrightarrow\ & \sigma^{-1}u \in ker\, f^{(\sigma^{-1}x)} \\
\Leftrightarrow\ & u \in \sigma ker\, f^{(\sigma^{-1}x)}
\end{aligned}$$

So $dim\, ker(\sigma \cdot f)^{(x)} = dim[\sigma ker\, f^{(\sigma^{-1}x)}] = dim[ker\, f^{(\sigma^{-1}x)}]$, hence $rk(\sigma \cdot f)^{(x)} = rk\, f^{(\sigma^{-1}x)}$.

If σ is moreover an isometry, then $\sigma \cdot f = f$. Let $x \in R_i(f)$, i.e. $rk\, f^{(x)} = i$. Then $rk\, f^{(\sigma x)} = rk(\sigma^{-1} \cdot f)^{(x)} = rk\, f^{(x)} = i$. Hence $\sigma x \in R_i(f)$, so $x \in \sigma^{-1}R_i(f)$. Thus $R_i(f) \subset \sigma^{-1}R_i(f)$, i.e. $\sigma R_i(f) \subset R_i(f)$. □

2.4.3 Remark If we put $R_0 \cup R_1 \cup \ldots \cup R_i = R_{\leq i}$, then $R_{\leq i}$ is still invariant under isometry. This will play an important role in our proof.

In order to determine the isometry group $G(H, f)$, we write an isometry as a matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A : V \to V$, $B : Alt_d V \to V$, $C : V \to Alt_d V$ and $D : Alt_d V \to Alt_d V$ are linear maps.

First we show that $M(Alt_dV) \subset Alt_dV$, which implies that $B = 0$. We do this by showing that $Alt_dV = R_{\leq n}(f)$ and using Remark 2.4.3.

2.4.4 Proposition: If $H = V \oplus Alt_dV$ is the alternating hyperbolic space with form f, then $R_{\leq n}(f) = Alt_dV$.

Proof: We choose a basis for $H = V \oplus Alt_dV$ as follows : Let $e_1, \ldots, e_n$ be a basis for V. Let $A = \{\alpha : \underline{d} \to \underline{n} \,|\, 1 \leq \alpha 1 < \ldots < \alpha d \leq n\}$. Then $\{(e_{\alpha 1} \ldots e_{\alpha d})^* | \alpha \in A\}$ is a basis for Alt_dV. (For brevity we shall often write e_α for $(e_{\alpha 1} \ldots e_{\alpha d})^*$.) We see that $dim\, H = n + \binom{n}{d}$. Suppose $y \in ker f^{(x)}$,where $x = \sum_{i=1}^n a_i e_i + \sum_{\alpha \in A} a_\alpha e_\alpha$ and $y = \sum_{i=1}^n b_i e_i + \sum_{\alpha \in A} b_\alpha e_\alpha$. Then

$y \in ker f^{(x)}$ iff $f(x, y, z_1, \ldots, z_{d-1}) = 0 \;\forall\, z_1, \ldots, z_{d-1} \in H$
iff for all $t_j \in \underline{n} \cup A, j = 1, \ldots, d-1$,

$$f(\sum a_i e_i + \sum a_\alpha e_\alpha, \sum b_i e_i + \sum b_\alpha e_\alpha, e_{t_1}, \ldots, e_{t_{d-1}}) = 0 \quad (1)$$

Nontrivial equations arise only when at most one $t_j \in A$, so we distinguish two types of equations which can occur:

Type I: $t_j \in \underline{n} \;\forall j = 1, \ldots, d-1$. It suffices to consider those t_j satisfying $1 \leq t_1 < \ldots < t_{d-1} \leq n$ (i.e. $t \in A' = \{t : \underline{d-1} \to \underline{n} \,|\, 1 \leq t_1 < \ldots < t_{d-1} \leq n\}$), since other choices of the t_j will give, by alternation, trivial equations or the same equations as for $t \in A'$. Expanding (1) gives, for all $t \in A'$:

$$\sum_i \sum_j a_i b_j f(e_i, e_j, e_{t_1}, \ldots, e_{t_{d-1}}) + \sum_i \sum_\alpha a_i b_\alpha f(e_i, e_\alpha, e_{t_1}, \ldots, e_{t_{d-1}}) +$$
$$\sum_\alpha \sum_j a_\alpha b_j f(e_\alpha, e_j, e_{t_1}, \ldots, e_{t_{d-1}}) + \sum_\alpha \sum_\beta a_\alpha b_\beta f(e_\alpha, e_\beta, e_{t_1}, \ldots, e_{t_{d-1}}) = 0.$$

By definition of f the first and last sums are zero (since they do not have exactly *one* argument $e_t, t \in A$), and using the alternation of f in the middle sums we obtain : $\sum_i \sum_\alpha (a_i b_\alpha - a_\alpha b_i) f(e_i, e_\alpha, e_{t_1}, \ldots, e_{t_{d-1}}) = 0, \forall t \in A'$.

Now fix t (i.e. choose one of these equations). Those terms on the LHS for which $i \in Im\, t$ are zero, so we consider those for which $i \notin Im\, t$. Suppose $1 \leq \ldots < t_{k_i} <$

$i < t_{k_i+1} < \ldots \leq n$, where $0 \leq k_i \leq d-1$ and we define $t_0 = 0, t_d = \infty$ to allow for $i < t_1$ or $i > t_{d-1}$.Then

$$\begin{aligned} f(e_i, e_\alpha, e_{t_1}, \ldots, e_{t_{d-1}}) &= -f(e_\alpha, e_i, e_{t_1}, \ldots, e_{t_{d-1}}) \\ &= -(-1)^{k_i} f(e_\alpha, e_{t_1}, \ldots, e_{t_{k_i}}, e_i, e_{t_{k_i+1}}, \ldots, e_{t_{d-1}}) \quad \text{(by alternation)} \\ &= (-1)^{k_i-1} f(e_\alpha, e_{t_1}, \ldots, e_{t_{k_i}}, e_i, e_{t_{k_i+1}}, \ldots, e_{t_{d-1}}). \end{aligned}$$

So for each $i \notin Im\, t$, we have

$$\begin{aligned} &\sum_\alpha (a_i b_\alpha - a_\alpha b_i) f(e_i, e_\alpha, e_{t_1}, \ldots, e_{t_{d-1}}) \\ &= \sum_\alpha (-1)^{k_i-1} (a_i b_\alpha - a_\alpha b_i) f(e_\alpha, e_{t_1}, \ldots, e_i, \ldots, e_{t_{d-1}}) \\ &= (-1)^{k_i-1} (a_i b_{t_1 \ldots t_{k_i} i t_{k_i+1} \ldots t_{d-1}} - a_{t_1 \ldots t_{k_i} i t_{k_i+1} \ldots t_{d-1}} b_i), \end{aligned}$$

since there is only one nonzero term in the middle expression, viz. the term for which $\alpha 1, \ldots, \alpha d$ corresponds exactly to $t_1, \ldots, t_{k_i}, i, t_{k_i+1}, \ldots, t_{d-1}$.
So the equation is $\sum_i (-1)^{k_i-1} (a_i b_{t_1 \ldots i \ldots t_{d-1}} - a_{t_1 \ldots i \ldots t_{d-1}} b_i) = 0$.
For an *arbitrary* injection $\alpha : \underline{d} \to \underline{n}$ we define $a_\alpha = \epsilon(\sigma) a_{\alpha'}$, where $\alpha' \in A$ and $\sigma\alpha = \alpha'$. Then we have $\sum_i (-1)^{k_i-1} (-1)^{k_i} (a_i b_{it_1 \ldots t_{d-1}} - a_{it_1 \ldots t_{d-1}} b_i) = 0$.
Hence $\sum_{i=1}^n (a_i b_{it_1 \ldots t_{d-1}} - a_{it_1 \ldots t_{d-1}} b_i) = 0$, for each $t \in A'$.

Type II: $t_{j_0} \in A$ for exactly one $j_0, 1 \leq j_0 \leq d-1$, and $t_j \in \underline{n}$ for $j \neq j_0$.
Suppose $t_{j_0} = \alpha_{j_0} 1 \ldots \alpha_{j_0} d = \alpha_{j_0}$ (by abuse of notation). Expanding (1) gives

$$\begin{aligned} &\sum_i \sum_j a_i b_j f(e_i, e_j, e_{t_1}, \ldots, e_{\alpha_{j_0}}, \ldots, e_{t_{d-1}}) + \\ &\quad \sum_i \sum_\alpha a_i b_\alpha f(e_i, e_\alpha, e_{t_1}, \ldots, e_{\alpha_{j_0}}, \ldots, e_{t_{d-1}}) + \\ &\quad \sum_\alpha \sum_i a_\alpha b_i f(e_\alpha, e_i, e_{t_1}, \ldots, e_{\alpha_{j_0}}, \ldots, e_{t_{d-1}}) + \\ &\quad \sum_\alpha \sum_\beta a_\alpha b_\beta f(e_\alpha, e_\beta, e_{t_1}, \ldots, e_{\alpha_{j_0}}, \ldots, e_{t_{d-1}}) = 0 \end{aligned}$$

By definition of f the last three sums are zero (since they do not have exactly one argument $e_t, t \in A$), so we obtain

$\sum_i \sum_k a_i b_k f(e_i, e_k, e_{t_1}, \ldots, e_{\alpha_{j_0}}, \ldots, e_{t_{d-1}}) = 0$ for all $\alpha_{j_0} \in A$, $j_0 = 1, \ldots, d-1$, and all $t_j \in \underline{n}, j \neq j_0$.

As before, because of alternation, it is sufficient to consider t satisfying $1 \leq t_1 < \ldots < t_{j_0-1} < t_{j_0+1} < \ldots < t_{d-1} \leq n$. Moreover, we need consider only those t for which $Im\, t \subset Im\, \alpha_{j_0}$, since other t will give either trivial equations or the same equations as these t give.

By alternation, we can write these equations as

$\sum_i \sum_{k>i}(a_i b_k - a_k b_i) f(e_i, e_k, e_{t_1}, \ldots, e_{\alpha_{j_0}}, \ldots, e_{t_{d-1}}) = 0$, for all $\alpha_{j_0} \in A, j_0 = 1, \ldots, d-1$, and all t such that $1 \leq t_1 < \ldots < t_{j_0-1} < t_{j_0+1} < \ldots < t_{d-1} \leq n$, $Im\, t \subset Im\, \alpha_{j_0}$.

Now $f(e_i, e_k, e_{t_1}, \ldots, e_{\alpha_{j_0}}, \ldots, e_{t_{d-1}}) = \pm f(e_{\alpha_{j_0}}, e_{t_1}, \ldots, e_i, \ldots, e_k, \ldots, e_{t_{d-1}})$, so

$\sum_i \sum_{k>i}(a_i b_k - a_k b_i) f(e_i, e_k, e_{t_1}, \ldots, e_{\alpha_{j_0}}, \ldots, e_{t_{d-1}}) =$

$\pm \sum_i \sum_{k>i}(a_i b_k - a_k b_i) f(e_{\alpha_{j_0}}, e_{t_1}, \ldots, e_i, \ldots, e_k, \ldots, e_{t_{d-1}})$.

Fix $1 \leq j_0 \leq d-1, \alpha_{j_0} \in A$ and t such that $1 \leq t_1 < \ldots < t_{j_0-1} < t_{j_0+1} < \ldots < t_{d-1} \leq n$ and $Im\, t \subset Im\, \alpha_{j_0}$. (2)

Then exactly one pair (i, k) with $i < k$ will give a nonzero term in the double sum, viz. the two elements in $Im\, \alpha_{j_0} \backslash Im\, t$. So we get $a_i b_k - a_k b_i = 0$ (for this pair). If we range through all choices of j_0, α_{j_0} and t satisfying the conditions (2) above, we obtain equations

$$a_i b_k - a_k b_i = 0 \quad \text{for all } i, k \in \underline{n}.$$

We now show $Alt_d V \subset R_{\leq n}(f)$:

If $x \in Alt_d V$, i.e. $a_1 = \ldots = a_n = 0$, then all Type II equations are redundant and Type I equations become :

$$\sum_{k=1}^{n} a_{k t_1 \ldots t_{d-1}} b_k = 0.$$

Hence the b_α are arbitrary, so $dim\, ker\, f^{(x)} \geq \binom{n}{d}$, i.e. $rk\, f^{(x)} \leq n + \binom{n}{d} - \binom{n}{d} = n$, hence $x \in R_{\leq n}(f)$, as required.

Finally we show that $R_{\leq n}(f) \subset Alt_d V$:

Suppose $x \notin Alt_d V$, i.e. some $a_i \neq 0$. Renumbering, if necessary, we suppose $a_1 \neq 0$. From Type II equations we have : $a_r b_s - a_s b_r = 0\, \forall r, s \in \underline{n}$. For all $s \geq 2$ (taking $r = 1$) we have $b_s = \frac{a_s}{a_1} b_1 = c_s b_1$, and for $s = 1$ we put $c_1 = 1$.

It is easy to see that all Type II equations are satisfied by the b_s : $a_r b_s - a_s b_r = a_r \frac{a_s}{a_1} b_1 - a_s \frac{a_r}{a_1} b_1 = 0$. Now substitute these b_s into the Type I equations : $\sum_{k=1}^{n}(a_{kt_1\ldots t_{d-1}} b_1 - a_k b_{kt_1\ldots t_{d-1}}) = 0$ for all $t \in A'$.

If $t \in A'$ and $1 \notin Im\, t$ we can write the equation as

$b_{1t_1\ldots t_{d-1}} = \frac{1}{a_1}[a_{1t_1\ldots t_{d-1}} c_1 b_1 + \sum_{k=2}^{n}(a_{kt_1\ldots t_{d-1}} b_1 - a_k b_{kt_1\ldots t_{d-1}})]$.

We check that *all* Type I equations are satisfied by the b_i and the b_α. It is obviously sufficient to check only the remaining Type I equations, i.e. those for which $1 \in Im\, t$. Suppose that $t_j = 1$ for some j. Type I equations are then $\sum_{i=1}^{n}(a_{1it_1\ldots\hat{t_j}\ldots t_{d-1}} b_i - a_i b_{1it_1\ldots\hat{t_j}\ldots t_{d-1}}) = 0$, for all $t \in A'$. The first term is zero, so we have $\sum_{i=2}^{n}(a_{1it_1\ldots\hat{t_j}\ldots t_{d-1}} b_i - a_i b_{1it_1\ldots\hat{t_j}\ldots t_{d-1}}) = 0$.

Substitute for b_i and $b_{1it_1\ldots\hat{t_j}\ldots t_{d-1}} (i \geq 2)$:

$$
\begin{aligned}
&\sum_{i=2}^{n}[a_{1it_1\ldots\hat{t_j}\ldots t_{d-1}} c_i b_1 - \\
&\qquad \frac{a_i}{a_1}\{a_{1it_1\ldots\hat{t_j}\ldots t_{d-1}} c_1 b_1 + \sum_{k=2}^{n}(a_{kit_1\ldots\hat{t_j}\ldots t_{d-1}} c_k b_1 - a_k b_{kit_1\ldots\hat{t_j}\ldots t_{d-1}})\}] \\
= \;& -\sum_{i=2}^{n}\sum_{k=2}^{n}(c_i c_k a_{kit_1\ldots\hat{t_j}\ldots t_{d-1}} - c_i a_k b_{kit_1\ldots\hat{t_j}\ldots t_{d-1}}) \\
= \;& 0,
\end{aligned}
$$

because of alternation of the a_α and b_β.

So all the b_i and b_β can be written in terms of b_1 and the $b_{t_1\ldots t_{d-1}k}$, where $k \geq 2$ and $1 \notin Im\, t$.

Now $card(\{b_i, i \geq 2\} \cup \{b_{1t_1\ldots t_{d-1}}\}) = n - 1 + \binom{n-1}{d-1}$, hence $rk\, f^{(x)} \geq n - 1 + \binom{n-1}{d-1}$. Since we assume $n > d$, $n-1 > d-1$, so $\binom{n-1}{d-1} > 1$, and hence $rk\, f^{(x)} > n-1+1 = n$. Thus $x \notin R_{\leq n}(f)$, as required. □

We now return to the determination of $G(H,f)$. Since M is invertible and $B=0$, A is also invertible, i.e. $A \in GL(V)$. Now the isometry condition gives, for all $v_1,\ldots,v_{d+1}$ and $\theta_1,\ldots,\theta_{d+1}$,

$$f[\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}\begin{pmatrix} v_1 \\ \theta_1 \end{pmatrix},\ldots,\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}\begin{pmatrix} v_{d+1} \\ \theta_{d+1} \end{pmatrix}] = f[\begin{pmatrix} v_1 \\ \theta_1 \end{pmatrix},\ldots,\begin{pmatrix} v_{d+1} \\ \theta_{d+1} \end{pmatrix}], \text{ i.e.}$$

$$\begin{aligned} &(Cv_1 + D\theta_1)(Av_2,\ldots,Av_{d+1}) - (Cv_2 + D\theta_2)(Av_1, Av_3,\ldots,Av_d) + \ldots \\ &\qquad + (-1)^d(Cv_{d+1} + D\theta_{d+1})(Av_1,\ldots,Av_d) \\ &= \theta_1(v_2,\ldots,v_{d+1}) - \theta_2(v_1,v_3,\ldots,v_{d+1}) + \ldots + (-1)^d\theta_{d+1}(v_1,\ldots,v_d). \qquad (3) \end{aligned}$$

Choose $\theta_2 = \ldots = \theta_{d+1} = 0, v_1 = 0$. Then
$D\theta_1(Av_2,\ldots,Av_{d+1}) = \theta_1(v_2,\ldots,v_{d+1})$, i.e.
$A^{-1}\cdot(D\theta_1)(v_2,\ldots,v_{d+1}) = \theta_1(v_2,\ldots,v_{d+1})$ for all $v_2,\ldots,v_{d+1}$ and all θ_1.
Hence $A^{-1}\cdot(D\theta_1) = \theta_1$, i.e. $D\theta_1 = A\cdot\theta_1$ for all θ_1.
This means that D is determined by A.
In order to describe C, put $\theta_1 = \ldots = \theta_{d+1} = 0$ in (3) :
$\sum_{i=1}^{d+1}(-1)^{i-1}Cv_i(Av_1,\ldots,\hat{Av_i},\ldots,Av_{d+1}) = 0\ \forall v_1,\ldots,v_{d+1}$.
Define $\phi : V^{d+1} \to F$ by

$$\phi(v_1,\ldots,v_{d+1}) = Cv_{d+1}(Av_1,\ldots,Av_d) = A^{-1}\cdot(Cv_{d+1})(v_1,\ldots,v_d).$$

ϕ is linear in $v_1,\ldots,v_{d+1}$ because C is linear in v_{d+1} and Cv_{d+1} is linear in $v_1,\ldots,v_d$.
ϕ is alternating in $v_1,\ldots,v_d$ because Cv_{d+1} is alternating.
ϕ also satisfies :

$\sum_{i=1}^{d+1}(-1)^{i-1}\phi(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1},v_i) =$
$\sum_{i=1}^{d+1}(-1)^{i-1}Cv_i(Av_1,\ldots,\hat{Av_i},\ldots,Av_{d+1}) = 0.$

So ϕ is a multilinear form satisfying

(i) ϕ is alternating in $v_1,\ldots,v_d$;
(ii) $\sum_{i=1}^{d+1}(-1)^{i-1}\phi(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1},v_i) = 0.$

Hence C is determined by a d-multilinear form ϕ satisfying the above conditions.

As a set we may thus identify $G(H,f)$ with $GL(V)\times\mathcal{F}$,where $\mathcal{F}$ denotes the space of $(d+1)$-multilinear forms ϕ satisfying conditions (i) and (ii). We can recover $M = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in G(H,f)$ from $(A,\phi)\in GL(V)\times\mathcal{F}$ using $D\theta = A\cdot\theta$ and $Cv_{d+1}(v_1,\ldots,v_d) = \phi(A^{-1}v_1,\ldots,A^{-1}v_{d+1})$.

We now establish that $G(H,f)\cong GL(V)\bowtie\mathcal{F}$, the semidirect product of $GL(V)$ and $\mathcal{F}$. We obtain the operation on $GL(V)\times\mathcal{F}$ by using the compatibility requirement: Suppose $M_i = \begin{pmatrix} A_i & 0 \\ C_i & D_i \end{pmatrix}$ corresponds to (A_i,ϕ_i) for $i=1,2$. Now suppose $M_1M_2 = M = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$ and $(A_1,\phi_1)(A_2,\phi_2) = (A,\phi)$.

Then $M = \begin{pmatrix} A_1A_2 & 0 \\ C_1A_2 + D_1C_2 & D_1D_2 \end{pmatrix}$, so obviously $A = A_1A_2$, $D = D_1D_2$, $C = C_1A_2 + D_1C_2$, and ϕ must satisfy

$$\begin{aligned}
\phi(v_1,\ldots,v_{d+1}) &= (C_1A_2 + D_1C_2)v_{d+1}(Av_1,\ldots,Av_d) \\
&= C_1(A_2v_{d+1})(Av_1,\ldots,Av_d) + D_1(C_2v_{d+1})(Av_1,\ldots,Av_d) \\
&= \phi_1(A_1^{-1}Av_1,\ldots,A_1^{-1}Av_d, A_2v_{d+1}) + A_1\cdot(C_2v_{d+1})(Av_1,\ldots,Av_d) \\
&= \phi_1(A_2v_1,\ldots,A_2v_{d+1}) + C_2v_{d+1}(A_2v_1,\ldots,A_2v_d) \\
&= A_2^{-1}\cdot\phi_1(v_1,\ldots,v_{d+1}) + \phi_2(v_1,\ldots,v_{d+1})
\end{aligned}$$

This shows that the operation on $GL(V)\times\mathcal{F}$ is

$$(A_1,\phi_1)(A_2,\phi_2) = (A_1A_2, A_2^{-1}\cdot\phi_1 + \phi_2),$$

so the product is semidirect as claimed.

The space $\mathcal{F}$ turns out to be isomorphic to a certain co-Schur functor ([ABW] Definition II.1.3 p220). First we observe that it is easy to show that condition (ii) can be written in other ways:

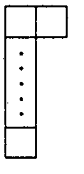
(ii)$'$ $\sum_{i=1}^{d+1} \epsilon(\sigma_i)\sigma_i \cdot \phi = 0$, where $\sigma_i = (i, d+1)$ for $1 \leq i \leq d+1$;

(ii)$''$ $\phi(v_1, \dots, v_{d+1}) + (-1)^d \phi(v_2, \dots, v_{d+1}, v_1) + \phi(v_3, \dots, v_{d+1}, v_1, v_2) + \dots +$
$\phi(v_d, v_{d+1}, v_1, \dots, v_{d-1}) + (-1)^d \phi(v_{d+1}, v_1, \dots, v_d) = 0$;

(ii)$'''$ $\sum_{i=1}^{d+1} \epsilon(\gamma^{i-1})\gamma^{i-1} \cdot \phi = 0$, where $\gamma = (1, 2, \dots, d+1)$.

The last two of these give a Jacobi identity which alternates in sign when $deg(\phi) = d+1$ is even. We shall sometimes refer to forms satisfying conditions (i) and (ii) as *hook-alternating.*

The next result is proved by a direct method, which contrasts with Keet's use of the letter-place algebra in the symmetric case ([K1] Ch 4 §3.9-3.17 pp81-6).

2.4.5 Proposition: The space $\mathcal{F}$ is isomorphic to the co-Schur functor $K_\mu(V^*)$, where μ is the partition $(2, 1^{d-1})$, i.e. the Young diagram  (with d blocks in the first column).

The Proposition is proved as follows:

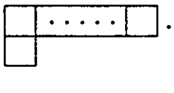
(a) We describe a standard basis for $K_\mu(V^*)$, and use it to show that $\mathcal{F} \subset K_\mu(V^*)$.

(b) We show that every basis element of $K_\mu(V^*)$ satisfies (i) and (ii), which implies that $K_\mu(V^*) \subset \mathcal{F}$.

The co-Schur functor $K_\mu(V^*)$ may be described as follows: Let $\{v_1, \dots, v_n\}$ be a basis for V, with $\{x_1, \dots, x_n\}$ the dual basis for V^*.

The co -Schur functor is the image of $d'_\mu(V^*) : S_\mu(V^*) \to \bigwedge^{\tilde{\mu}}(V^*)$, i.e.

$d'_\mu(V^*) : S_2(V^*) \otimes V^* \otimes \dots \otimes V^* \to \bigwedge^d(V^*) \otimes V^*$,

where $\tilde{\mu} = (d, 1) =$ .

$d'_\mu(V^*)[(x_{p_1} \odot x_{p_{d+1}}) \otimes x_{p_2} \otimes \ldots \otimes x_{p_d}]$ can be visualized as follows ([ABW] top of p222):

$$\begin{array}{c|c|c|} S^2 & p_1 & p_{d+1} \\ \cline{2-3} S^1 & p_2 & \\ \cline{2-2} \vdots & \vdots & \\ \cline{2-2} S^1 & p_d & \\ \cline{2-2} \end{array} \quad \longrightarrow \quad \begin{array}{|c|c|} \multicolumn{1}{c}{\bigwedge^d} & \multicolumn{1}{c}{\bigwedge^1} \\ \hline p_1 & p_{d+1} \\ \hline p_2 & \\ \cline{1-1} \vdots & \\ \cline{1-1} p_d & \\ \cline{1-1} \end{array} \quad + \quad \begin{array}{|c|c|} \multicolumn{1}{c}{\bigwedge^d} & \multicolumn{1}{c}{\bigwedge^1} \\ \hline p_{d+1} & p_1 \\ \hline p_2 & \\ \cline{1-1} \vdots & \\ \cline{1-1} p_d & \\ \cline{1-1} \end{array}$$

(The rows on the LHS are multilinearized.)

Now take the exterior product down the columns on the RHS, and tensor the results to obtain:

$d'_\mu(V^*)[(x_{p_1} \odot x_{p_{d+1}}) \otimes x_{p_2} \otimes \ldots \otimes x_{p_d})] =$
$(x_{p_1} \wedge \ldots \wedge x_{p_d}) \otimes x_{p_{d+1}} + (x_{p_{d+1}} \wedge x_{p_2} \wedge \ldots \wedge x_{p_d}) \otimes x_{p_1}$.

By the Standard Basis Theorem ([ABW] Theorem II.3.16 p235) , a basis for $K_\mu(V^*)$ is given by those elements for which the diagram is co-standard, i.e. for which $p_1 < \ldots < p_d$ and $p_1 \leq p_{d+1}$.

Proof of (a): An arbitrary $(d+1)$-multilinear form ϕ can be (uniquely) written as $\phi = \sum_{i \in I}(a|i_1 \ldots i_{d+1})x_{i_1} \otimes \ldots \otimes x_{i_{d+1}}$, where $I = \{i : \underline{d+1} \to \underline{n}\}$. We now consider what happens when $\phi \in \mathcal{F}$. By (i), if $i_s = i_t$ for some $1 \leq s < t \leq d$, then
$\phi(\ldots e_{i_s} \otimes \ldots \otimes e_{i_t} \ldots) = (a|\ldots i_s \ldots i_t \ldots) = 0$.
So $\phi = \sum_{i \in I_1}(a|i_1 \ldots i_{d+1})x_{i_1} \otimes \ldots \otimes x_{i_{d+1}}$, where $I_1 = \{i \in I \,|\, i_1, \ldots, i_d \text{ are distinct}\}$.
Choose $j \in I_1$. Then for $\pi \in S_d$, $\pi \cdot \phi(e_{j_1} \otimes \ldots \otimes e_{j_d} \otimes e_{j_{d+1}}) = \epsilon(\pi)\phi(e_{j_1} \otimes \ldots \otimes e_{j_d} \otimes e_{j_{d+1}})$ (by (i)), i.e. $\sum_{i \in I_1}(a|i_1 \ldots i_{d+1})(x_{i_1} \otimes \ldots \otimes x_{i_{d+1}})(e_{j_{\pi 1}} \otimes \ldots \otimes e_{j_{\pi d}} \otimes e_{j_{d+1}}) =$
$\epsilon(\pi)\sum_{i \in I_1}(a|i_1 \ldots i_{d+1})(x_{i_1} \otimes \ldots \otimes x_{i_{d+1}})(e_{j_1} \otimes \ldots \otimes e_{j_{d+1}})$, i.e.
$(a|j_{\pi 1} \ldots j_{\pi d} j_{d+1}) = \epsilon(\pi)(a|j_1 \ldots j_{d+1})$.
Let $I_2 = \{i \in I_1 | i_1 < \ldots < i_d\}$. Clearly, the set of all $i_{\pi 1} \ldots i_{\pi d} i_{d+1}$ as i ranges over I_2 and π ranges over S_d is just I_1. Hence

$$\phi = \sum_{i \in I_2} \sum_{\pi \in S_d} (a|i_{\pi 1} \ldots i_{\pi d} i_{d+1}) x_{i_{\pi 1}} \otimes \ldots \otimes x_{i_{\pi d}} \otimes x_{i_{d+1}}$$

$$= \sum_{i \in I_2} \sum_{\pi \in S_d} \epsilon(\pi)(a|i_1 \ldots i_{d+1}) x_{i_{\pi 1}} \otimes \ldots \otimes x_{i_{\pi d}} \otimes x_{i_{d+1}}$$
$$= \sum_{i \in I_2} (a|i_1 \ldots i_{d+1}) \sum_{\pi \in S_d} \epsilon(\pi) x_{i_{\pi 1}} \otimes \ldots \otimes x_{i_{\pi d}} \otimes x_{i_{d+1}}.$$

Using the embedding $\bigwedge^d(V^*) \hookrightarrow T^d(V^*)$, we can identify ϕ with
$d! \sum_{i \in I_2} (a|i_1 \ldots i_{d+1})(x_{i_1} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_{d+1}}$. (4)
Now suppose that for some $j \in I_2$ we have $j_1 > j_{d+1}$. By condition (ii), we have:
$(-1)^d \phi[(v_{j_1} \wedge \ldots \wedge v_{j_d}) \otimes v_{j_{d+1}}] =$
$-\sum_{r=1}^d (-1)^{r-1} \phi[(v_{j_1} \wedge \ldots \wedge \widehat{v_{j_r}} \wedge \ldots \wedge v_{j_{d+1}}) \otimes v_{j_r}]$, i.e.

$$(-1)^d (a|j_1 \ldots j_{d+1}) = -\sum_{r=1}^{d} (-1)^{r-1} (a|j_1 \ldots \widehat{j_r} \ldots j_{d+1} j_r)$$
$$= -\sum_{r=1}^{d} (-1)^{r-1} (-1)^{d-1} (a|j_{d+1} j_1 \ldots \widehat{j_r} \ldots j_d j_r).$$

Hence $(a|j_1 \ldots j_{d+1}) = \sum_{r=1}^d (-1)^{r-1} (a|j_{d+1} j_1 \ldots \widehat{j_r} \ldots j_d j_r)$. (5)
This shows that any coefficient $(a|j)$ for which $j_{d+1} < j_1$ can be written as a sum of coefficients $(a|j^s)$ with all j^s distinct, and $j_1^s < \ldots < j_d^s, j_1^s < j_{d+1}^s$ for all s. Also, for any other coefficient $(a|l)$ for which $l_{d+1} < l_1$, the l^t (which occur on the RHS of (5)) are distinct from the j^s.
It follows that the sum (4) may be taken over $I_3 = \{i \in I_2 | i_1 \leq i_{d+1}\}$; also, since the coefficients on the RHS of (5) are all distinct, any coefficient $(a|i)$ with $i \in I_3$ occurs at most twice in the sum. It remains to show that every term in the sum can be written as $(a|i_1 \ldots i_{d+1})[(x_{i_1} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_{d+1}} + (x_{i_{d+1}} \wedge x_{i_2} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_1}]$, where $i \in I_3$. We consider three cases:
(i) If $i_{d+1} = i_1$,then
$(x_{i_1} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_{d+1}} = \frac{1}{2}[(x_{i_1} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_{d+1}} + (x_{i_{d+1}} \wedge x_{i_2} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_1})]$.
(ii) If $i_{d+1} = i_r$, for some $r = 2, \ldots, d$, then
$(x_{i_1} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_{d+1}} = (x_{i_1} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_{d+1}} + (x_{i_{d+1}} \wedge x_{i_2} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_1}$, since the last term is zero.
(iii) If the $i_s, 1 \leq s \leq d+1$, are all distinct, we suppose $1 \leq i_1 < \ldots < i_{r-1} < i_{d+1} < i_{r+1} < \ldots < i_d \leq n$, i.e. i_{d+1} is in the r^{th} position in the sequence. Then

$(a|i_2 \ldots i_{d+1} \ldots i_d i_1)[(x_{i_2} \wedge \ldots \wedge x_{i_{d+1}} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_1}]$
$= (a|k_1 \ldots k_{r-1} \ldots k_d k_{d+1})[(x_{i_2} \wedge \ldots \wedge x_{i_{d+1}} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_1}]$ (re-labelling)
$= \sum_{l=1}^{d}(-1)^{l-1}(a|k_{d+1}k_1 \ldots \widehat{k_l} \ldots k_d k_l)[(x_{i_2} \wedge \ldots \wedge x_{i_{d+1}} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_1}]$
(since $k_1 < \ldots < k_d$ and $k_{d+1} < k_1$, we can use (5)).

The $(r-1)^{th}$ term in the sum is

$$\begin{aligned}
&(-1)^{r-2}(a|k_{d+1}k_1 \ldots \widehat{k_{r-1}} \ldots k_d k_{r-1})[(x_{i_2} \wedge \ldots \wedge x_{i_{d+1}} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_1}] \\
&\quad = (-1)^{r-2}(a|i_1 i_2 \ldots i_d i_{d+1})[(x_{i_2} \wedge \ldots \wedge x_{i_{d+1}} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_1}] \\
&\quad = (-1)^{r-2}(-1)^{r-2}(a|i_1 \ldots i_{d+1})[(x_{i_{d+1}} \wedge x_{i_2} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_1}] \\
&\quad = (a|i_1 \ldots i_{d+1})[(x_{i_{d+1}} \wedge x_{i_2} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_1}]
\end{aligned}$$

So the coefficient of $(a|i_1 \ldots i_{d+1})$ in this case is also
$(x_{i_1} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_{d+1}} + (x_{i_{d+1}} \wedge x_{i_2} \wedge \ldots \wedge x_{i_d}) \otimes x_{i_1}$, as required.

Proof of (b): We show that a basis element $\Phi = (x_{p_1} \wedge \ldots \wedge x_{p_d}) \otimes x_{p_{d+1}} + (x_{p_{d+1}} \wedge x_{p_2} \wedge \ldots \wedge x_{p_d}) \otimes x_{p_1}$ of $K_\mu(V^*)$ satisfies (i) and (ii). Then by linearity of the permutation action, all elements of $K_\mu(V^*)$ satisfy (i) and (ii). Since $\Phi \in \bigwedge^d(V^*) \otimes V^* \cong (\bigwedge^d V)^* \otimes V^* \cong (\bigwedge^d V \otimes V)^*$, it is obvious that Φ satisfies (i). To show that Φ satisfies (ii), we observe first that two (apparently) different actions of the symmetry group on a form are in fact the same. We consider only a basis element $f = x_{i_1} \otimes \ldots \otimes x_{i_k} \in T^k(V^*)$, as both actions can be extended to an arbitrary form by linearity. The two actions are
$\pi \cdot f(v_{j_1} \otimes \ldots \otimes v_{j_k}) = f(v_{j_{\pi 1}} \otimes \ldots \otimes v_{j_{\pi k}})$ (usual action), and
$\pi * f = \pi * (x_{i_1} \otimes \ldots \otimes x_{i_k}) = x_{i_{\pi^{-1}1}} \otimes \ldots \otimes x_{i_{\pi^{-1}k}}$ (tensor action).
Now

$$\begin{aligned}
\pi \cdot f(v_{j_1} \otimes \ldots \otimes v_{j_k}) &= f(v_{j_{\pi 1}} \otimes \ldots \otimes v_{j_{\pi k}}) \\
&= (x_{i_1} \otimes \ldots \otimes x_{i_k})(v_{j_{\pi 1}} \otimes \ldots \otimes v_{j_{\pi k}}) \\
&= x_{i_1}(v_{j_{\pi 1}}) \ldots x_{i_k}(v_{j_{\pi k}}),
\end{aligned}$$

while

$$\begin{aligned}\pi * f(v_{j_1} \otimes \ldots \otimes v_{j_k}) &= (x_{i_{\pi^{-1}1}} \otimes \ldots \otimes x_{i_{\pi^{-1}k}})(v_{j_1} \otimes \ldots \otimes v_{j_k}) \\ &= x_{i_{\pi^{-1}1}}(v_{j_1}) \ldots x_{i_{\pi^{-1}k}}(v_{j_k}).\end{aligned}$$

It is obvious that the two actions are identical.
Hence to show that $\sum_{i=1}^{d+1} \epsilon(\sigma_i)\sigma_i \cdot \Phi = 0$, we show instead that $\sum_{i=1}^{d+1} \epsilon(\sigma_i)\sigma_i * \Phi = 0$.
By antisymmetrization (see p14), we have

$$\begin{aligned}\Phi &= (x_{p_1} \wedge \ldots \wedge x_{p_d}) \otimes x_{p_{d+1}} + (x_{p_{d+1}} \wedge x_{p_2} \wedge \ldots \wedge x_{p_d}) \otimes x_{p_1} \\ &= (x_{p_1} \wedge \ldots \wedge x_{p_d}) \otimes x_{p_{d+1}} + (x_{p_{\sigma_1 1}} \wedge x_{p_{\sigma_1 2}} \wedge \ldots \wedge x_{p_{\sigma_1 d}}) \otimes x_{p_{\sigma_1(d+1)}} \\ &= \sum_{\alpha \in S_{d+1}(d+1)} \epsilon(\alpha) x_{p_{\alpha 1}} \otimes \ldots \otimes x_{p_{\alpha d}} \otimes x_{p_{\alpha(d+1)}} + \\ &\quad \sum_{\beta \in S_{d+1}(d+1)} \epsilon(\beta) x_{p_{\sigma_1 \beta 1}} \otimes \ldots \otimes x_{p_{\sigma_1 \beta d}} \otimes x_{p_{\sigma_1 \beta(d+1)}},\end{aligned}$$

where $\sigma_1 = (1, d+1)$ (as on p49) and $S_{d+1}(d+1)$ denotes (as on p18) the subgroup of S_{d+1} which fixes $d+1$, and which may be identified with S_d.
Then we have

$$\begin{aligned}\sum_{i=1}^{d+1} \epsilon(\sigma_i)\sigma_i * \Phi &= \sum_{i=1}^{d+1} \sum_{\alpha \in S_{d+1}(d+1)} \epsilon(\alpha)\epsilon(\sigma_i) x_{p_{\alpha\sigma_i 1}} \otimes \ldots \otimes x_{p_{\alpha\sigma_i(d+1)}} + \\ &\quad \sum_{i=1}^{d+1} \sum_{\beta \in S_{d+1}(d+1)} \epsilon(\beta)\epsilon(\sigma_i) x_{p_{\sigma_1\beta\sigma_i 1}} \otimes \ldots \otimes x_{p_{\sigma_1\beta\sigma_i(d+1)}}.\end{aligned}$$

It is easy to see that as i ranges from 1 to $d+1$ and α (resp. β) ranges over $S_{d+1}(d+1)$, $\alpha\sigma_i$ (resp. $\sigma_1\beta\sigma_i$) ranges over S_{d+1}. Put $\alpha\sigma_i = \gamma$ and $\sigma_1\beta\sigma_i = \delta$. Then $\epsilon(\gamma) = \epsilon(\alpha)\epsilon(\sigma_i)$ and $\epsilon(\delta) = \epsilon(\sigma_1)\epsilon(\beta)\epsilon(\sigma_i) = -\epsilon(\beta)\epsilon(\sigma_i)$.
Hence we have

$$\sum_{i=1}^{d+1} \epsilon(\sigma_i)\sigma_i * \Phi = \sum_{\gamma \in S_{d+1}} \epsilon(\gamma) x_{p_{\gamma 1}} \otimes \ldots \otimes x_{p_{\gamma(d+1)}} - \sum_{\delta \in S_{d+1}} \epsilon(\delta) x_{p_{\delta 1}} \otimes \ldots \otimes x_{p_{\delta(d+1)}} = 0,$$

as required. □

2.4.6 Remark: We have thus proved that the dual of the space of tensors $\mathcal{T} \subset T^{d+1}(V)$ satisfying the symmetry conditions

(i) $\sigma \cdot (v_1 \otimes \ldots \otimes v_{d+1}) = \epsilon(\sigma) v_1 \otimes \ldots \otimes v_{d+1}$ for $\sigma \in S_d$; and

(ii) $v_1 \otimes \ldots \otimes v_{d+1} + v_2 \otimes v_3 \otimes \ldots \otimes v_{d+1} \otimes v_1 + \ldots + v_{d+1} \otimes v_1 \otimes \ldots \otimes v_d = 0$

corresponds to the co-Schur functor $K_\mu(V^*)$, where $\mu = (2, 1^{d-1})$.

This is the analogue of the antisymmetrization isomorphism in the alternating case (§1.1 p13), or polarization in the symmetric case ([K1] Ch 1 §§3-6 pp12-15). Since we have no analogue of the exterior (or symmetric) algebra for these forms, there is no possibility of describing the isomorphism in terms of Hopf algebra structure. We can only give the above explicit description, as we do in §1.1 (p14) for alternating tensors.

The structure of $G(f)$, where f is alternating hyperbolic, therefore differs from the symmetric case only in that the co-Schur functor $K_\mu(V^*)$ replaces the Schur functor $L_\lambda(V^*)$. (See [K1] Ch 4 Propn 2.4 p72.) We therefore have the following ([K1] Ch 4 §§2.5,2.6 pp72-3):

2.4.7 Proposition: $G(f)$ is connected. If $charF \neq 0$, then the radical of $G(f)$ is $F^* \bowtie K_\mu(V^*)$, which is 2-step solvable.

2.5 Lie Algebra of Alternating Hyperbolic Space

We determine the Lie algebra of the alternating hyperbolic space $H = (V \oplus Alt_d V, f)$.

Suppose that $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in End(H)$ is in the Lie algebra of (H, f). We have seen (§3 p38) that this means that $I + \epsilon M$ is in $G(f_\epsilon)$, where f_ϵ denotes the extension of f to $H \otimes_F F[\epsilon]$. Now $I + \epsilon M = \begin{pmatrix} I_n + \epsilon A & \epsilon B \\ \epsilon C & I_m + \epsilon D \end{pmatrix}$, so $\epsilon B = 0$ by the comment preceding Proposition 2.4.4 (p43), hence $B = 0$.

By Proposition 2.3.3 (p38) we have

$\sum_{i=1}^{d+1} f[(v_1, \theta_1), \ldots, M(v_i, \theta_i), \ldots, (v_{d+1}, \theta_{d+1})] = 0$ for all $(v_i, \theta_i) \in H$, i.e.

$\sum_{i=1}^{d+1} f[(v_1, \theta_1), \ldots, (Av_i, Cv_i + D\theta_i), \ldots, (v_{d+1}, \theta_{d+1})] = 0$ for all $(v_i, \theta_i) \in H$. (1)

Putting $v_1 = 0$ this becomes:

$f[(0, D\theta_1), (v_2, \theta_2), \ldots, (v_{d+1}, \theta_{d+1})] + f[(0, \theta_1), (Av_2, Cv_2 + D\theta_2), \ldots, (v_{d+1}, \theta_{d+1})]$
$+ \ldots + f[(0, \theta_1), \ldots, (v_d, \theta_d), (Av_{d+1}, Cv_{d+1} + D\theta_{d+1})] = 0$ for all
$v_2, \ldots, v_{d+1} \in V, \theta_1, \ldots, \theta_{d+1} \in Alt_d V$, i.e.
$D\theta_1(v_2, \ldots, v_d) + \sum_{i=2}^{d+1} \theta_1(v_2, \ldots, Av_i, \ldots, v_{d+1}) = 0$ for all $v_2, \ldots, v_{d+1}, \theta_1$.
Put $d_A\theta_1(v_2, \ldots, v_{d+1}) = \sum_{i=2}^{d+1} \theta_1(v_2, \ldots, Av_i, \ldots, v_{d+1})$. ($d_A\theta_1(v_2, \ldots, v_{d+1})$ is the *directional derivative* of θ_1 at $(v_2, \ldots, v_{d+1})$ in the direction $(Av_2, \ldots, Av_{d+1})$ — see [K1] Ch 4 §2.7 p74.)
Then we have $D\theta_1 = -d_A\theta_1$ for all θ_1, so D is completely determined by $A \in End(V)$ (since d_A depends only on A).

In order to describe C, we put all $\theta_i = 0$ in (1):
$\sum_{i=1}^{d+1} f[(v_1, 0), \ldots, (Av_i, Cv_i), \ldots, (v_{d+1}, 0)] = 0$ for all v_i, i.e.
$\sum_{i=1}^{d+1} (-1)^{i-1} Cv_i(v_1, \ldots, \hat{v}_i, \ldots, v_{d+1}) = 0$ for all v_i.
Now put $\phi(v_1, \ldots, v_{d+1}) = Cv_{d+1}(v_1, \ldots, v_d)$. We can use the same reasoning as before (Proposition 2.4.5 p49) to see that C may be identified with an element of the co-Schur functor $K_\mu(V^*)$.

Hence we can identify $L(f)$, as a set, with $\mathcal{L} = End(V) \oplus K_\mu(V^*)$. We can recover $M = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \in L(f)$ from $(A, \phi) \in End(V) \oplus K_\mu(V^*)$ using $D\theta = -d_A\theta$ and $Cv_{d+1}(v_1, \ldots, v_d) = \phi(v_1, \ldots, v_{d+1})$.
The operation on $L(f) \subset End(V \oplus Alt_d V)$ is the Lie bracket $[M_1, M_2] = M_1M_2 - M_2M_1$. We use the compatibility requirement again to describe the operation on $\mathcal{L}$:
Suppose $M_i = \begin{pmatrix} A_i & 0 \\ C_i & D_i \end{pmatrix}$ corresponds to (A_i, ϕ_i) for $i = 1, 2$, and that
$[M_1, M_2] = M$ and $(A_1, \phi_1)(A_2, \phi_2) = (A, \phi)$. Then

$$\begin{aligned} M &= [\begin{pmatrix} A_1 & 0 \\ C_1 & D_1 \end{pmatrix}, \begin{pmatrix} A_2 & 0 \\ C_2 & D_2 \end{pmatrix}] \\ &= \begin{pmatrix} A_1A_2 - A_2A_1 & 0 \\ (C_1A_2 + D_1C_2) - (C_2A_1 + D_2C_1) & D_1D_2 - D_2D_1 \end{pmatrix}. \end{aligned}$$

Hence $A = [A_1, A_2], D = [D_1, D_2]$ and $C = (C_1A_2 + D_1C_2) - (C_2A_1 + D_2C_1)$, so ϕ must satisfy

$$\begin{aligned}
\phi(v_1, \ldots, v_{d+1}) &= [(C_1A_2 + D_1C_2) - (C_2A_1 + D_2C_1)]v_{d+1}(v_1, \ldots, v_d) \\
&= [C_1(A_2v_{d+1}) - D_2(C_1v_{d+1}) - C_2(A_1v_{d+1}) + D_1(C_2v_{d+1})](v_1, \ldots, v_d) \\
&= \phi_1(v_1, \ldots, v_d, A_2v_{d+1}) + d_{A_2}C_1v_{d+1}(v_1, \ldots, v_d) - \phi_2(v_1, \ldots, v_d, A_1v_{d+1}) \\
&\quad - d_{A_1}C_2v_{d+1}(v_1, \ldots, v_d) \\
&= \phi_1(v_1, \ldots, v_d, A_2v_{d+1}) + \sum_{i=1}^{d} C_1v_{d+1}(v_1, \ldots, A_2v_i, \ldots, v_d) \\
&\quad - \phi_2(v_1, \ldots, v_d, A_1v_{d+1}) - \sum_{i=1}^{d} C_2v_{d+1}(v_1, \ldots, A_1v_i, \ldots, v_d) \\
&= (d_{A_2}\phi_1 - d_{A_1}\phi_2)(v_1, \ldots, v_{d+1}).
\end{aligned}$$

Thus $(A_1, \phi_1)(A_2, \phi_2) = ([A_1, A_2], d_{A_2}\phi_1 - d_{A_1}\phi_2)$.

The following result follows by the same argument as in the symmetric case (see [K1] Ch 4 §2.12 p76):

2.5.1 Proposition: Assume *char* $F = 0$. Then the radical of $L(f)$ is $F \oplus K_\mu(V^*)$, which is 2-step solvable.

2.6 Descent of Alternating Hyperbolics

We prove an analogue of a result of Keet's ([K1] Ch 5 §1 pp99-105): If an alternating form over a field of characteristic zero extends (under extension of the base field) to a form which is equivalent to a hyperbolic form, then the original form is (equivalent to) a hyperbolic form. The proof follows the lines of the symmetric case, but it also differs from that case in important respects. Most notably, the set $T(\theta)$ ([K1] Definition 1.3 p100) cannot be employed; instead we work directly with the derived algebra of the radical of the Lie algebra of the form.

2.6.1 Proposition: Let F be a field of characteristic 0 and (W, θ) an alternating space of degree $d+1 \geq 3$ over F, with $dim\, W \geq d+1$. Let $K|F$ be a field extension, and suppose that the alternating space (W_K, θ_K) is equivalent to a hyperbolic alternating space over K. Then (W, θ) is equivalent to a hyperbolic alternating space over F.

Proof: We may assume, without loss of generality, that $W = V \bigoplus Alt_d V$, with $dim V > d$. Let ψ denote the hyperbolic alternating form on W, and suppose that $\theta_K = \sigma \cdot \psi_K$ for some $\sigma \in GL(W_K)$. The proof is split up into a sequence of lemmas:

2.6.2 Lemma (cf. [K1] Lemma 1.2): Suppose ϕ is a nondegenerate alternating form of degree $d+1$ on W such that:

(a) $W = U \oplus S$, with $dim_F S = dim_F Alt_d V$ (V as above);

(b) $\phi(s_1, s_2, w_3, \ldots, w_{d+1}) = 0$ for all $s_1, s_2 \in S$ and $w_3, \ldots, w_{d+1} \in W$;

(c) $\phi(u_1, \ldots, u_{d+1}) = 0$ for all $u_1, \ldots, u_{d+1} \in U$.

Then ϕ is equivalent to the hyperbolic form ψ.

Proof: Let $v_1, \ldots, v_n$ be a basis for V, $x_1, \ldots, x_n$ a dual basis for V^*.
Choose the basis $x_{i_1} \ldots x_{i_d}$, $1 \leq i_1 < \ldots < i_d \leq n$ for $Alt_d V$, and let $x_{i_1 \ldots i_d}$ be the corresponding element of the dual basis of $\bigwedge^d(V)$.
Relative to the basis $x_1, \ldots, x_n; \ldots, x_{i_1 \ldots i_d}, \ldots$ of $(V \bigoplus Alt_d V)^* = V^* \bigoplus \bigwedge^d(V)$, the anticommutative polynomial corresponding to ψ is $\sum_i x_{i_1 \ldots i_d} x_{i_1} \ldots x_{i_d}$, where $i : \underline{d} \to \underline{n}$ satisfies $1 \leq i_1 < \ldots < i_d \leq n$.
Let $\{s_{i_1 \ldots i_d}\}$ (i as before) be a basis for S^*, and $u_1, \ldots, u_n$ be a basis for U^*. Together they form a basis for $(U \bigoplus S)^*$.
By conditions (b) and (c) we see that the anticommutative polynomial corresponding to ϕ is $\sum_i t_{i_1 \ldots i_d} u_{i_1} \ldots u_{i_d}$, where $t_{i_1 \ldots i_d}$ is a linear form in the $s_{i_1 \ldots i_d}$.

Since ϕ is nondegenerate, the $t_{i_1\dots i_d}$ are linearly independent. We can thus define an isometry between ϕ and ψ by $u_j \leftrightarrow x_j$, $t_{i_1\dots i_d} \leftrightarrow x_{i_1\dots i_d}$, and hence obtain the result. □

We now define the space which we use in a similar way to the set $T(\theta)$ of [K1].

2.6.3 Definition: For any alternating form ϕ of degree $d+1$ on W, put $\mathcal{Z}(\phi) = \{w \in W \mid [L(\phi)_r, L(\phi)_r].w = 0\}$, where $L(\phi)$ is the Lie algebra of ϕ, $L(\phi)_r$ is its radical, and $l(\phi) = [L(\phi)_r, L(\phi)_r]$ is the first derived algebra of the radical.
It is clear that $\mathcal{Z}(\phi)$ is a subspace of W.

Part (a) of the next lemma is similar to Lemma 1.4 of [K1], but it requires a different proof.

2.6.4 Lemma: (a) For any $\tau \in GL(W)$, $\mathcal{Z}(\tau \cdot \phi) = \tau\mathcal{Z}(\phi)$.
(b) $\mathcal{Z}(\tau \cdot \phi)$ satisfies condition (b) of Lemma 2.6.2 if and only if $\mathcal{Z}(\phi)$ does.

Proof: (a) (Adapted from [K1] Lemma 1.6 pp101-2.) By Proposition 2.3.4(ii) (p38), $L(\tau \cdot \phi) = \tau L(\phi)\tau^{-1}$. Since conjugation by τ maps the lattice of solvable ideals of $L(\phi)$ isomorphically onto the lattice of solvable ideals of $L(\tau \cdot \phi)$, we have $L(\tau \cdot \phi)_r = \tau L(\phi)_r \tau^{-1}$. Now

$$\begin{aligned}
w \in \mathcal{Z}(\tau \cdot \phi) &\Leftrightarrow l(\tau \cdot \theta)w = 0 \\
&\Leftrightarrow [L(\tau \cdot \phi)_r, L(\tau \cdot \phi)_r].w = 0 \\
&\Leftrightarrow \tau^{-1}[L(\tau \cdot \phi)_r, L(\tau \cdot \phi)_r]\tau\tau^{-1}w = 0 \\
&\Leftrightarrow [\tau^{-1}L(\tau \cdot \phi)_r\tau, \tau^{-1}L(\tau \cdot \phi)_r\tau]\tau^{-1}w = 0 \\
&\Leftrightarrow [L(\phi)_r, L(\phi)_r].\tau^{-1}w = 0 \\
&\Leftrightarrow \tau^{-1}w \in \mathcal{Z}(\phi) \\
&\Leftrightarrow w \in \tau\mathcal{Z}(\phi).
\end{aligned}$$

(b) Suppose $\mathcal{Z}(\phi)$ satisfies 2.6.2(b). By (a), we can choose two arbitrary elements of $\mathcal{Z}(\tau\cdot\phi)$ of the form $\tau s_1, \tau s_2$, where $s_1, s_2 \in \mathcal{Z}(\phi)$. Then

$$\begin{aligned}\tau\cdot\phi(\tau s_1, \tau s_2, w_3, \ldots, w_{d+1}) &= \phi(\tau^{-1}\tau s_1, \tau^{-1}\tau s_2, \tau^{-1}w_3, \ldots, \tau^{-1}w_{d+1}) \\ &= \phi(s_1, s_2, \tau^{-1}w_3, \ldots, \tau^{-1}w_{d+1}) \\ &= 0 \quad \text{for all } w_3, \ldots, w_{d+1} \in W,\end{aligned}$$

since $s_1, s_2 \in \mathcal{Z}(\phi)$ and we assume $\mathcal{Z}(\phi)$ satisfies 2.6.2(b). The converse is obvious. □

2.6.5 Lemma (cf. [K1] Lemma 1.5(a)): $\mathcal{Z}(\psi) = Alt_dV$ (where ψ is the hyperbolic described in Proposition 2.6.1 and $W = V \oplus Alt_dV$, etc.)

Proof: Let $\begin{pmatrix} 0 \\ \theta \end{pmatrix} \in Alt_dV$, and let $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ be an arbitrary element of $l(\theta)$ (by Proposition 2.5.1 p56).
Then $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \theta \end{pmatrix} = 0$, hence $\begin{pmatrix} 0 \\ \theta \end{pmatrix} \in \mathcal{Z}(\psi)$. So $Alt_dV \subset \mathcal{Z}(\psi)$.

Now suppose $\begin{pmatrix} v \\ \theta \end{pmatrix} \notin Alt_dV$, i.e. $v \neq 0$. We construct an element $\pi \in l(\psi)$ such that $\pi\begin{pmatrix} v \\ \theta \end{pmatrix} \neq 0$.
Choose a basis $v_1, \ldots, v_n = v$ of V, with dual basis $x_1, \ldots, x_n$ of V^* ($n \geq d$). Let π be the element of $l(\psi)$ corresponding to the (hook-alternating degree $d+1$) form $\phi = (x_1 \wedge \ldots \wedge x_d) \otimes x_n + (x_n \wedge x_2 \wedge \ldots \wedge x_d) \otimes x_1$. (If $n = d$, the second term is zero.) Then $Cv(v_1, \ldots, v_d) = Cv_n(v_1, \ldots, v_d) = \phi(v_1, \ldots, v_d, v_n) = 1$ (see p55), so $Cv \neq 0$.
Hence $\pi\begin{pmatrix} v \\ \theta \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} v \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ Cv \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, so $\begin{pmatrix} v \\ \theta \end{pmatrix} \notin \mathcal{Z}(\psi)$. Thus $\mathcal{Z}(\psi) \subset Alt_dV$, and the result follows. □

2.6.6 Remark: Lemma 2.6.5 shows that $\mathcal{Z}(\psi)$ satisfies condition (b) of Lemma 2.6.2, and, by the same reasoning, so does $\mathcal{Z}(\psi_K)$, or any other hyperbolic. By Lemma 2.6.4(b), it follows that if ϕ is equivalent to a hyperbolic, then $\mathcal{Z}(\phi)$ satisfies

2.6.2(b). Thus $\mathcal{Z}(\theta_K)$ and $\mathcal{Z}(\rho \cdot \theta_K) = \mathcal{Z}((\rho \cdot \theta)_K)$, for any $\rho \in GL(W)$, also satisfy 2.6.2(b).

Part (a) of the next lemma is similar to the last part of Lemma 1.7 in [K1].

2.6.7 Lemma: For any $\rho \in GL(W)$, we have
(a) $\mathcal{Z}(\rho \cdot \theta)$ has the same dimension as $Alt_d V$;
(b) $\mathcal{Z}(\rho \cdot \theta)$ satisfies 2.6.2(b).

Proof: (a) (Adapted from [K1] Lemma 1.7 pp102-3.)
Define $\alpha : W \to Hom_F[l(\rho \cdot \theta), W]$ by $\alpha(w)(f) = f(w)$. Then $ker\, \alpha = \mathcal{Z}(\rho \cdot \theta)$.
(Henceforth we omit F from $- \otimes_F -$.)
Since $K|F$ is flat, $ker(\alpha \otimes 1) = ker\, \alpha \otimes K$, hence $dim_K[ker(\alpha \otimes 1)] = dim_F(ker\, \alpha)$.
We have

$$\begin{aligned} L(\rho \cdot \theta)_r \otimes K &= (L(\rho \cdot \theta) \otimes K)_r \quad \text{(by [CHE2] Proposition 3 p107)} \\ &= L((\rho \cdot \theta)_K)_r \quad \text{(by [CHE1] Proposition 2 p129).} \end{aligned}$$

So if we apply $- \otimes K$ to α, we obtain:

$$\begin{aligned} \alpha \otimes 1 : \; W_K \to \; & Hom_F([L(\rho \cdot \theta)_r, L(\rho \cdot \theta)_r], W) \otimes K \\ \cong \; & Hom_K([L(\rho \cdot \theta)_r, L(\rho \cdot \theta)_r] \otimes K, W_K) \\ \cong \; & Hom_K([L((\rho \cdot \theta)_K)_r, L((\rho \cdot \theta)_K)_r], W_K) \end{aligned}$$

Thus $ker(\alpha \otimes 1) = \mathcal{Z}((\rho \cdot \theta)_K)$. By Lemma 2.6.4(a), $\mathcal{Z}((\rho \cdot \theta)_K)$, $\mathcal{Z}(\theta_K)$ and $\mathcal{Z}(\psi_K)$ are all isomorphic, hence they all have the same dimension. From Lemma 2.6.5, we know $\mathcal{Z}(\psi_K) = (Alt_d V)_K$, whose dimension over K equals $dim_F Alt_d V$.
(b) We have already observed, in the proof of (a), that $l((\rho \cdot \theta)_K) = l(\rho \cdot \theta) \otimes K$. It follows easily from this that $\mathcal{Z}((\rho \cdot \theta)_K) = \mathcal{Z}(\rho \cdot \theta) \otimes K$.
If $s_1, s_2 \in \mathcal{Z}(\rho \cdot \theta)$, then

$\rho \cdot \theta(s_1, s_2, w_3, \ldots, w_{d+1}) = (\rho \cdot \theta)_K(s_1 \otimes 1, s_2 \otimes 1, w_3 \otimes 1, \ldots, w_{d+1} \otimes 1) = 0$, by Remark 2.6.6. □

2.6.8 Lemma (cf. [K1] Lemma 1.8): There exists a canonical epimorphism of Lie algebras $L(\theta) \to gl(W/\mathcal{Z}(\theta))$.

Proof: First we show that if $f \in L(\theta)$, then $f.\mathcal{Z}(\theta) \subset \mathcal{Z}(\theta)$, so that f induces $\bar{f}$ on $W/\mathcal{Z}(\theta)$.
If $g \in L(\psi_K)$, then $g.\mathcal{Z}(\psi_K) \subset \mathcal{Z}(\psi_K)$, since ψ_K is hyperbolic and $\mathcal{Z}(\psi_K) = Alt_d V_K$ is invariant under g (by §5, second par. p54). We have $L(\theta_K) = L(\sigma \cdot \psi_K) = \sigma L(\psi_K)\sigma^{-1}$ (Proposition 2.3.4(ii) p38) and $L(\theta_K) = L(\theta) \otimes K$ ([CHE1] Proposition 2 p129); also, by Lemma 2.6.4, $\mathcal{Z}(\theta_K) = \sigma\mathcal{Z}(\psi_K)$, so

$$\begin{aligned} f \in L(\theta_K) &\Leftrightarrow f \in \sigma L(\psi_K)\sigma^{-1} \\ &\Leftrightarrow \sigma^{-1} f \sigma \in L(\psi_K) \\ &\Rightarrow \sigma^{-1} f \sigma \mathcal{Z}(\psi_K) \subset \mathcal{Z}(\psi_K) \\ &\Leftrightarrow f \sigma \mathcal{Z}(\psi_K) \subset \sigma \mathcal{Z}(\psi_K) \\ &\Leftrightarrow f \mathcal{Z}(\theta_K) \subset \mathcal{Z}(\theta_K). \end{aligned}$$

We have $\mathcal{Z}(\theta_K) = \mathcal{Z}(\theta) \otimes K$ (taking $\rho = 1$ in the proof of Lemma 2.6.7(b)) and $L(\theta_K) = L(\theta) \otimes K$, hence $f.\mathcal{Z}(\theta) \subset \mathcal{Z}(\theta)$ for $f \in L(\theta)$ as claimed. Thus we have a homomorphism of Lie algebras $L(\theta) \to gl(W/\mathcal{Z}(\theta))$, given by $f \mapsto \bar{f}$.
It remains to show this is an epimorphism. We know the structure of $L(\psi_K)$ (§2.5 p55), and $\mathcal{Z}(\psi_K) = Alt_d V_K$ (Lemma 2.6.5), hence the canonical homomorphism of Lie algebras $L(\psi_K) \to gl(W_K/\mathcal{Z}(\psi_K))$ is an epimorphism. The following square is commutative:

$$\begin{array}{ccc} \sigma L(\theta_K)\sigma^{-1} = L(\psi_K) & \longrightarrow & gl(W_K/\sigma\mathcal{Z}(\theta_K)) = gl(W_K/\mathcal{Z}(\psi_K)) \\ \uparrow & & \downarrow \\ L(\theta_K) & \longrightarrow & gl(W_K/\mathcal{Z}(\theta_K)) \end{array}$$

The horizontal maps are the canonical ones; the upward map is $f \mapsto \sigma f \sigma^{-1}$; and the downward map is $\bar{f} \mapsto \bar{\sigma}\bar{f}\bar{\sigma}^{-1}$. All the homomorphisms, except perhaps the bottom one, are epimorphisms, so the bottom one is too. But $L(\theta_K) \cong L(\theta) \otimes K$ and $gl(W_K/\mathcal{Z}(\theta_K)) \cong gl(W/\mathcal{Z}(\theta)) \otimes K$, so the bottom epimorphism comes from applying $- \otimes K$ to the homomorphism $L(\theta) \to gl(W/\mathcal{Z}(\theta))$. Since $K|F$ is faithfully flat, this gives the result. □

2.6.9 Lemma (cf. [K1] Lemma 1.9): There exists $\rho \in GL(W)$ such that $\mathcal{Z}(\rho \cdot \theta)$ satisfies condition (c) of Lemma 2.6.2.

Proof: Let U be a subspace of W complementary to $\mathcal{Z}(\theta)$: $W = U \oplus \mathcal{Z}(\theta)$. By Lemma 2.6.8 there exists some $f = \begin{pmatrix} 1 & 0 \\ C & D \end{pmatrix} \in L(\theta)$.

This means $\sum_{i=1}^{d+1} \theta[\begin{pmatrix} u_1 \\ t_1 \end{pmatrix}, \dots, f\begin{pmatrix} u_i \\ t_i \end{pmatrix}, \dots, \begin{pmatrix} u_{d+1} \\ t_{d+1} \end{pmatrix}] = 0$ for all $u_1, \dots, u_{d+1} \in U, t_1, \dots, t_{d+1} \in \mathcal{Z}(\theta)$, i.e.

$$(d+1)\theta[\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u_{d+1} \\ 0 \end{pmatrix}] +$$

$$d\{\sum_{i=1}^{d+1} \theta[\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u_{i-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t_i \end{pmatrix}, \begin{pmatrix} u_{i+1} \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u_{d+1} \\ 0 \end{pmatrix}]\} +$$

$$\sum_{i=1}^{d+1} \theta[\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u_{i-1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Cu_i + Dt_i \end{pmatrix}, \begin{pmatrix} u_{i+1} \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u_{d+1} \\ 0 \end{pmatrix}] = 0$$

(by multilinearity, and Lemma 2.6.7(b) with $\rho = 1$).

Putting $u_{d+1} = 0$, this becomes:

$$d\theta[\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u_d \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t_{d+1} \end{pmatrix}] + \theta[\begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} u_d \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ Dt_{d+1} \end{pmatrix}] = 0$$ for all $u_1, \dots, u_d \in U, t_{d+1} \in \mathcal{Z}(\theta)$. (1)

Now $\theta[\begin{pmatrix} u_1 \\ t_1 \end{pmatrix}, \dots, \begin{pmatrix} u_d \\ t_d \end{pmatrix}, \begin{pmatrix} 0 \\ Dt_{d+1} + dt_{d+1} \end{pmatrix}] =$

Chapter 3

Forms of general Young symmetry type

In this chapter we extend some of the concepts and results about symmetric and alternating forms to forms of general Young symmetry type. Examples of such forms have already made an appearance in the isometry group of the symmetric hyperbolic ([K1] Proposition 3.12 p83) and the alternating hyperbolic forms (Proposition 2.4.5 p49).

In general, these are forms which satisfy symmetry conditions corresponding to higher dimensional (or degree) irreducible representations of the symmetric group S_n over $\mathbb{C}$; in this context, symmetry and alternation correspond, of course, to the 1-dimensional trivial and sign representations, respectively.

We note that, in contrast to symmetry and alternation, there is no Hopf algebra structure for forms of (non-standard) general Young symmetry type.

There is no systematic treatment (as far as we are aware) of such forms of general Young symmetry type, except for a recent paper by Kantor and Trishin ([KT]). Other (non-Young) symmetry types have been studied, for example by Kanzaki ([KAN]), and cyclic symmetry has been extensively studied in connexion with Connes' noncommutative geometry.

§3.1 summarizes the ordinary representation theory of S_n, using mainly the excellent overview of Fulton and Harris ([FH]), but also important parts of the paper of Akin et al. ([ABW]), which we use to establish our notation and terminology.

In §3.2 we develop our approach to forms of general Young symmetry type, justify our notation and terminology, and explain what we mean by equivalent general Young symmetry types.

We state the very useful results of Kantor and Trishin ([KT]) on conditions characterizing forms of general Young symmetry type, but re-cast them in our notation and terminology.

We end this section by giving several examples — perhaps more than strictly necessary — simply because we have not encountered them in explicit form elsewhere, and because we shall use them to illustrate the ideas we introduce in the sequel.

In §3.3 we discuss a generalized notion of nondegeneracy, and show that nondegeneracy-d implies nondegeneracy in general.

§3.4: Given some general Young symmetry type, we explain what we mean by its *derivative symmetry type* and *integral symmetry type*, indicate when an integral symmetry type is not unique, and illustrate with a few examples.

We then define the derivative of a form f of some general Young symmetry type, and show it has derivative symmetry type.

In §3.5 we take the alternating and symmetric hyperbolics as models for a generalized hyperbolic by asking: Given $H = V \oplus \mathcal{F}$, where $\mathcal{F}$ is a space of forms of some general Young symmetry type, can we define a generalized hyperbolic form ψ of some general Young symmetry type on H?

We use general properties of Young symmetrizers and a generalized polarity to describe such a hyperbolic form ψ and show (H, ψ) is cofinal for spaces equipped with a form of the same general Young symmetry type as ψ. We end the section by giving a few new examples of such hyperbolics of general Young symmetry type, and also show these are nondegenerate.

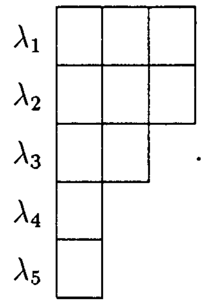
In §3.6 we outline the Weil-Siegel duality between symmetric and alternating bilinear forms, and then give a formulation which lends itself to generalization to hyperbolics of general Young symmetry type.
This requires us to define a notion of Lagrangian subspace of a hyperbolic space of general Young symmetry type, and to investigate some of its properties. We describe conditions for two general Young symmetry types to be Siegel duals (in our sense), and conclude by exhibiting examples of two pairs of Siegel dual symmetry types obtained in this manner.

3.1 Representations of S_n: A brief review

There are several excellent references which elaborate the ordinary representation theory of the symmetric group, first worked out by Frobenius at the end of the nineteenth century. (See, for example, Curtis and Reiner [CR], James and Kerber [JK], Sagan [SAG], Fulton and Harris [FH], Martin [MAR], Boerner [BOE] or, of course, Weyl [WEY].)

We eschew the more efficient approach via Specht modules (see [SAG] or [JK]) in favour of Young's classical approach using the group algebra, which is directly relevant to our purpose. We follow largely the quick overview of Fulton and Harris ([FH] p44 et seq.). There is some variation in terminology, notation and conventions (with regard to filling of diagrams, definition of symmetrizers, etc.) in the literature; we follow mainly Fulton and Harris ([FH]), Akin, Buchsbaum and Weyman ([ABW]) and James and Kerber ([JK]), and, when the use of different conventions is uanvoidable and significant, we shall reconcile these.

To a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ (i.e. $\lambda_1 + \ldots + \lambda_k = n, \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$) is associated a *Young diagram* (or Young frame, or Ferrers diagram), consisting of λ_i boxes (cells) in the i^{th} row. If an integer λ_j is repeated, say, s times, we write this as λ_j^s. For example, the Young diagram associated to $\lambda = (3^2, 2, 1^2)$ is



(We shall use λ to denote the partition as well as the diagram.)

The partition *conjugate* to λ is $\tilde{\lambda} = (\tilde{\lambda_1}, \ldots, \tilde{\lambda_r})$, obtained by interchanging the rows and columns of the Young diagram λ.

A *tableau* T_λ on a Young diagram λ is a filling of the boxes with the numbers from 1 to n. We adopt the convention (unless stated otherwise) of *canonical numbering*, as used by Fulton and Harris ([FH] p45) and Akin et al. ([ABW] p276), i.e. we number the boxes from left to right across the rows, starting at the top row. (A different numbering of T_λ would give an isomorphic representation.)

Given a tableau T_λ, define subgroups

$$R_\lambda = \{\sigma \in S_n \,|\, \sigma \text{ preserves each row}\}$$

$$C_\lambda = \{\sigma \in S_n \,|\, \sigma \text{ preserves each column}\},$$

and the following two elements of the group algebra $\mathbb{C}\, S_n$:

$$a_\lambda = \sum_{\sigma \in R_\lambda} \epsilon(\sigma)\sigma, \quad b_\lambda = \sum_{\sigma \in C_\lambda} \sigma.$$

The *Young symmetrizer* is $c_\lambda = b_\lambda a_\lambda$ ([ABW] p276); c_λ is pseudo-idempotent, i.e. $c_\lambda^2 = n_\lambda c_\lambda$, where n_λ is a scalar which will be described shortly.

It is important, from our viewpoint, to describe the actions of a_λ and b_λ:

If V is a vector space and S_n acts on $T^n(V)$ by permuting factors (i.e. $\sigma(v_1 \otimes \ldots \otimes v_n) = v_{\sigma^{-1}1} \otimes \ldots \otimes v_{\sigma^{-1}n}$), the image of $b_\lambda \in \mathbb{C}\, S_n \to End(T^n(V))$ is just the subspace

$Im(b_\lambda) = S^{\lambda_1}(V) \otimes \ldots \otimes S^{\lambda_k}(V) \subset T^n(V)$, and

$Im(a_\lambda) = \bigwedge^{\mu_1}(V) \otimes \ldots \otimes \bigwedge^{\mu_r}(V) \subset T^n(V)$, where $\mu = \tilde{\lambda}$.
When $\lambda = (n)$, $c_{(n)} = b_{(n)}$, so $c_{(n)} \cdot T^n(V) = S^n(V)$; and when $\lambda = (1^n)$, $c_{(1^n)} = a_{(1^n)}$, so $c_{(1^n)} \cdot T^n(V) = \bigwedge(V)$.
In general, $c_\lambda \cdot T^n(V)$ is the *Schur functor* $L_\lambda(V)$ of Akin et al. ([ABW] p276). (It is also called the *Weyl module*, or *Weyl's construction.*) Note that for $L_\lambda V$ to be nonzero the number of rows of λ should not exceed the dimension of V ([FH] last paragraph p76). In order to generalize the standard dualities $S(V^*) \cong S(V)^*$ (polarization, assuming $char\, F = 0$) and $\bigwedge(V^*) \cong \bigwedge(V)^*$ (antisymmetrization), we define the *co-symmetrizer* $c'_\lambda = a_\lambda b_\lambda$; then $c'_\lambda \cdot T^n(V)$ is the *co-Schur functor* $K_{\tilde{\lambda}}(V)$ ([ABW] p276), and we have: $L_\lambda(V)^* \cong K_{\tilde{\lambda}}(V^*)$ ([ABW] Proposition II.4.1 p236).

The spaces $L_\lambda(V)$ (likewise $K_\lambda(V)$), where $|\lambda| = d$ and $\lambda_1 \leq n$, give a complete set of distinct irreducible polynomial representations of $GL(V)$ of degree d.
We also have the following result ([FH] Theorem 4.3 p46): For any partition λ, the image of c_λ (by right multiplication on $\mathbb{C}\, S_n$) is an irreducible representation V_λ of S_n; and every irreducible representation of S_n can be obtained in this way for some unique partition λ.
For example, $V_{(n)}$ is the trivial representation, and $V_{(1^n)}$ is the sign (or alternating) representation.
The dimension (or degree) f^λ of the irreducible representation V_λ is given by the simple combinatorial Hooklength formula ([FH] 4.12 p50 or [SAG] Theorem 3.1.2 p92), or else by the Frobenius formula ([FH] (4.11) p50). Alternatively, it may be described as the number of *standard* tableaux T_λ, i.e. tableaux in which the entries in the rows and columns are increasing. The scalar n_λ referred to earlier is given by $n_\lambda = \frac{n!}{f^\lambda}$ ([FH] Lemma 4.26 p54). We will sometimes find it more convenient to use *Young idempotents* than (co-)symmetrizers. These are defined by $I_\lambda = \frac{f^\lambda}{n!} c_\lambda$ and $I'_\lambda = \frac{f^\lambda}{n!} c'_\lambda$.

Our focus in the sequel is on using Young symmetrizers (and co-symmetrizers), and Schur (and co-Schur) functors to investigate forms which are of general Young symmetry type.

3.2 Symmetry conditions; Examples

The action of S_d on a multilinear form ϕ of degree d (i.e. $\sigma \cdot \phi(v_1, \ldots, v_d) = \phi(v_{\sigma 1}, \ldots, v_{\sigma d})$) extends in obvious way to an action of the group algebra $\mathbb{C}\, S_d$ on ϕ.

We know, for example, that ϕ is alternating if and only if $\phi \in \bigwedge^d(V)^* \cong \bigwedge^d(V^*) = c_\lambda \cdot T^d(V^*) = L_\lambda(V^*)$, where $\lambda = (1^d)$. It follows easily (using $c_\lambda^2 = \frac{d!}{f^\lambda} c_\lambda$) that ϕ is alternating if and only if $c_\lambda \cdot \phi = \frac{d!}{f^\lambda}\phi = d!\,\phi$.

We now generalize this. If λ is any partition, we say ϕ *has general Young symmetry type* $L(\lambda)$ if $\phi \in L_\lambda(V)^* \cong K_{\tilde{\lambda}}(V^*)$. We shall shortly discuss symmetry conditions which characterize a general Young symmetry type. It will then be clear that a different choice of numbering (i.e. non-canonical), which corresponds to an isomorphic representation, gives a general Young symmetry type which is equivalent to this in an obvious sense — simply re-number the coordinates.

Note that we name the symmetry type according to the space on which ϕ operates, viz. $L_\lambda(V)$, rather than the space in which ϕ lives, viz. $K_{\tilde{\lambda}}(V^*)$. This is because the way S_d acts on ϕ gives ϕ the "same" symmetry conditions as the space $L_\lambda(V)$ (and not the space $K_{\tilde{\lambda}}(V^*)$). For example, if $T_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, then $c_\lambda = b_\lambda a_\lambda = (e + (13))(e - (12)) = e + (13) - (12) - (123)$ (where e denotes the identity permutation). Thus $c_\lambda(v_1 \otimes v_2 \otimes v_3) = v_1 \otimes v_2 \otimes v_3 + v_3 \otimes v_2 \otimes v_1 - v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_1 \otimes v_2$, so the tensors in $L_\lambda(V) = c_\lambda \cdot T^3(V)$ are alternating in v_1, v_2 (and satisfy a Jacobi identity). Now $c'_\lambda = a_\lambda b_\lambda = e - (12) + (13) - (132)$, so $c'_\lambda \cdot \phi(v_1, v_2, v_3) = \phi(v_1, v_2, v_3) - \phi(v_2, v_1, v_3) + \phi(v_3, v_2, v_1) - \phi(v_3, v_1, v_2)$. Hence $c'_\lambda \cdot \phi$ is also alternating in v_1, v_2 (and satisfies a Jacobi identity). On the other hand, $K_{\tilde{\lambda}}(V^*) = c'_\lambda \cdot T^3(V^*)$, and

$c'_\lambda(x_1 \otimes x_2 \otimes x_3) = x_1 \otimes x_2 \otimes x_3 - x_2 \otimes x_1 \otimes x_3 + x_3 \otimes x_2 \otimes x_1 - x_2 \otimes x_3 \otimes x_1$, which is symmetric in x_1, x_3 (and satisfies a Jacobi identity). In the alternating and symmetric cases this distinction is irrelevant because the two spaces in question satisfy the same symmetry conditions.

Thus ϕ has general Young symmetry type $L(\lambda)$ if and only if $c'_\lambda \cdot \phi = \frac{d!}{f^\lambda}\phi$, since $K_{\tilde{\lambda}}(V^*) = c'_\lambda \cdot T^d(V^*)$.

3.2.1 Remark: General Young symmetry types correspond to irreducible representations of S_d. We do not consider here symmetry types corresponding to *reducible* representations of S_d, or, more generally, representations of subgroups of S_d. For example, $c = \frac{1}{2}(c_{(3)} + c_{(1^3)})$ is the (non-primitive) *cyclic symmetrizer* of degree 3: the form $c \cdot \phi$ has cyclic symmetry, i.e. $(123)c \cdot \phi = (132)c \cdot \phi = c \cdot \phi$.

Kantor and Trishin ([KT]) prove symmetry conditions which characterize "forms with Young symmetry". We summarize their results but re-cast them to be consistent with the conventions we have adopted.

Let e_λ, e^*_λ be the symmetrizers defined by Kantor and Trishin ([KT] pp309, 313). Then it is easily checked that $c_{\tilde{\lambda}} = e^*_\lambda$ and $c'_\lambda = e_{\tilde{\lambda}}$, and hence we have: ϕ has symmetry type $L(\lambda)$ if and only if $\phi \in L_\lambda(V)^* \cong K_{\tilde{\mu}}(V^*)$ if and only if $e_{\tilde{\lambda}} \cdot \phi = c'_\lambda \cdot \phi = \frac{d!}{f^\lambda}\phi$ if and only if "ϕ has $e_{\tilde{\lambda}}$-symmetry" ([KT] p309). (We assume that $\tilde{\lambda}$ appearing in $c_{\tilde{\lambda}}$ and $e_{\tilde{\lambda}}$ has the entries of λ transposed together with the cells.)

Likewise, we say ϕ *has symmetry type* $K(\lambda)$ if and only if $\phi \in K_\lambda(V)^* \cong L_{\tilde{\lambda}}(V^*)$. This is equivalent to Kantor and Trishin's "e^*_λ-symmetry". ($e^*_\lambda \cdot \phi = c_{\tilde{\lambda}} \cdot \phi = \frac{d!}{f^\lambda}\phi$, since $L_{\tilde{\lambda}}(V^*) = c_{\tilde{\lambda}} \cdot T^d(V^*)$.)

In terms of Young idempotents, we have: ϕ has general Young symmetry type $L(\lambda)$ if and only if $I'_\lambda \cdot \phi = \phi$, and ϕ has general Young symmetry type $K(\lambda)$ if and only if $I_{\tilde{\lambda}} \cdot \phi = \phi$.

The same comment made with regard to equivalence of general Young symmetry types $L(\lambda)$ applies here. But it must be noted that, even though $L_\lambda(V) \cong K_{\tilde{\lambda}}(V)$ if *char* $F = 0$ ([ABW] p209), the general Young symmetry types $L(\lambda)$ and $K(\tilde{\lambda})$ are not, in general, equivalent. They are equivalent in the "extreme" cases, i.e. $\lambda = (d)$ or $\lambda = (1^d)$: Symmetric forms have general Young symmetry type $L(1^d)$ or $K(d)$, and alternating forms have general Young symmetry type $L(d)$ or $K(1^d)$; but in even the simplest non-standard case, i.e. $\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$, it is easy to see they are not equivalent.

We have previously encountered forms of symmetry type $L(d-1,1)$ ("hook-alternating") in describing the isometry group of the alternating hyperbolic (Proposition 2.4.5 p49); forms of symmetry type $K(d-1,1)$ appear in the isometry group of the symmetric hyperbolic ([K1] Proposition 3.12 p83). (Fulton and Harris call the representations corresponding to these symmetry types *standard* representations ([FH] Ex 4.6* p48).)

We now describe Kantor and Trishin's main result. A form ϕ is said to satisfy the (generalized) *Jacobi identity* in the variables $x_{i_1}, \ldots, x_{i_k}$ if $\sum_{s=0}^{k-1}(-1)^{(k-1)s}t^s \cdot \phi = 0$, where t is the cycle $(i_1, \ldots, i_k)$ ([KT] (5) p309). If j_1 and j_2 are two elements in the same column of λ, and $r(j_2) \leq r(j_1)$, where $r(j)$ denotes the number of elements in the row containing j, let $i_1, \ldots, i_{k-1}$ be all the elements which occur in the row containing j_1, and let $i_k = j_2$. Define $J_{j_1 j_2} = \sum_{s=0}^{k-1}(-1)^{(k-1)s}t^s$, called a *Jacobi element* of the group algebra $\mathbb{C}\, S_d$ ([KT] (6) p310). We then have:

3.2.2 Theorem ([KT] Theorem 2.2 p317): ϕ has symmetry type $L(\lambda)$ if and only if:

(1) ϕ is skew-symmetric in every pair of variables with indices in the same row of λ;

(2) ϕ satisfies each Jacobi identity $J_{j_1 j_2} \cdot \phi = 0$, where the indices j_1, j_2 are in the same column of λ.

We can also define Jacobi elements $J^*_{j_1j_2} = \sum_{s=0}^{k-1} t^s$, where j_1, j_2 occur in the same column of λ. Then:

3.2.3 Theorem ([KT] Theorem 2.2′ p317): ϕ has symmetry type $K(\lambda)$ if and only if:

(1) ϕ is symmetric in every pair of variables with indices in the same row of λ;

(2) ϕ satisfies each Jacobi identity $J^*_{j_1j_2} \cdot \phi = 0$, where the indices j_1, j_2 are in the same column of λ.

3.2.4 Notes:

1. It is sufficient (because S_d is generated by transpositions of adjacent integers) to consider pairs of neighbouring indices i_1, i_2 and j_1, j_2 ([KT] Remark 2 p318).
2. If j_1 is the only element in its row, then $J_{j_1j_2} \cdot \phi = 0$ (resp. $J^*_{j_1j_2} \cdot \phi = 0$) just gives symmetry (resp. skew-symmetry) in x_{j_1}, x_{j_2}.
3. The condition $r(j_2) \leq r(j_1)$ implies $j_1 < j_2$; or $j_1 > j_2$ and $r(j_2) = r(j_1)$. We observe that in the latter case $J_{j_1j_2} \cdot \phi = 0$ and $J_{j_2j_1} \cdot \phi = 0$ are equivalent, and hence it is sufficient to consider $J_{j_1j_2}$ with $j_1 < j_2$ in Theorem 3.2.2. For suppose we have $r(k) = r(l)$: $\begin{array}{|c|c|c|c|c|}\hline k_1 & \cdots & k & \cdots & k_s \\ \hline l_1 & \cdots & l & \cdots & l_s \\ \hline\end{array}$. Then J_{kl} "cycles" $k_1, \ldots, k_s, l$ and J_{lk} "cycles" $l_1, \ldots, l_s, k$. For each $i = 1, \ldots, s$, we can swop the k_i^{th} and l_i^{th} variables by Theorem 3.2.2(1), so it is clear that $J_{kl} \cdot \phi = 0 \Leftrightarrow J_{lk} \cdot \phi = 0$.

The same reasoning applies to the J^*_{kl}.

Next we prove a simple lemma which will be used later. We consider only a form of general Young symmetry type $L(\lambda)$; the case $K(\lambda)$ is similar.

3.2.5 Lemma: Let θ be a form of general Young symmetry type $L(\lambda)$ of degree d. Given $u_1, \ldots, u_d \in V$, fix some $i = 1, \ldots, d$. Then we can write $\theta(u_1, \ldots, u_i, \ldots, u_d)$

as a linear combination of terms $\theta(u_{j_1}, \ldots, u_{j_d})$ in which the first variable is always u_i.

Proof: Suppose the first row of λ is $1, \ldots, k$.
If $i \leq k$, the result follows by Theorem 3.2.2(1).
If i is not in the first row of λ, suppose i_1 is the first entry in the row containing i.
By Theorem 3.2.2(1),

$$\theta(u_1, \ldots, \bar{u_{i_1}}, \ldots, u_i, \ldots, u_d) = -\theta(u_1, \ldots, \bar{u_i}, \ldots, u_{i_1}, \ldots, u_d),$$

where the bar denotes the variable in the i_1^{th} position.
Now i_1 is in the same column as 1, so by Theorem 3.2.2(2) θ satisfies the Jacobi identity $J_{1i_1} \cdot \phi = 0$, i.e. $\theta(u_1, \ldots, u_k, \ldots, \bar{u_i}, \ldots)$ is a linear combination of $\theta(u_2, \ldots, u_k, u_i, \ldots, \bar{u_1}, \ldots), \theta(u_3, \ldots, u_k, u_i, u_1, \ldots, \bar{u_2}, \ldots), \ldots$
$\ldots, \theta(u_i, u_1, \ldots, u_{k-1}, \ldots, \bar{u_k}, \ldots)$, with coefficients 1 or -1 as appropriate.
By Theorem 3.2.2(1) u_i can be shifted to the first position in all terms in this linear combination, and hence we obtain the result. □

We conclude this section with several examples, both to illustrate the above ideas, as well as for use in later sections.

3.2.6 Examples

1. Symmetry: If ϕ has symmetry type $L(1^d)$, where $(1^d) = \begin{array}{|c|}\hline v_1 \\ \hline \vdots \\ \hline \vdots \\ \hline v_d \\ \hline\end{array}$, then ϕ is symmetric in $v_1, \ldots, v_d$, by Theorem 3.2.2 and Note 3.2.4(2). But symmetry type $K(d)$, where $(d) = \begin{array}{|c|c|c|}\hline v_1 & \cdots\cdots & v_d \\ \hline\end{array}$, also gives symmetry in $v_1, \ldots, v_d$ (by Theorem 3.2.3).

2. Alternation: Using similar reasoning as in Example 1, we see that symmetry types $K(1^d)$ and $L(d)$ both give alternation in all coordinates.

3. Hook-symmetry: If ϕ has symmetry type $K(d-1,1)$, where $(d-1,1) = \begin{array}{|c|c|c|}\hline v_1 & \cdots\cdots & v_{d-1} \\ \hline v_d \\ \cline{1-1}\end{array}$, then

(1) ϕ is symmetric in $v_1, \ldots, v_{d-1}$;

(2) $\phi(v_1, \ldots, v_d) + \phi(v_2, v_3, \ldots, v_d, v_1) + \ldots + \phi(v_d, v_1, \ldots, v_{d-1}) = 0$ (Jacobi identity).

(Note that $\phi \in L_{(2,1^{d-2})}(V^*)$ — see [K1] Proposition 3.12 p83.)

4. Hook-alternation: If ϕ has symmetry type $L(d-1,1)$, then

(1) ϕ is alternating in $v_1, \ldots, v_{d-1}$;

(2) $\phi(v_1, \ldots, v_d) + (-1)^{d-1}\phi(v_2, v_3, \ldots, v_d, v_1) + \ldots + (-1)^{d-1}\phi(v_d, v_1, \ldots, v_{d-1}) = 0.$

(Note that $\phi \in K_{(2,1^{d-2})}(V^*)$ — see §2.4 p49.)

We now discuss some "new" symmetry types.

5. Symmetry type $L(2,1^{d-2})$: $\begin{array}{|c|c|}\hline v_1 & v_2 \\ \hline v_3 \\ \cline{1-1} \vdots \\ \cline{1-1} \vdots \\ \cline{1-1} v_d \\ \cline{1-1}\end{array}$

(1) ϕ is alternating in v_1, v_2;

(2) ϕ is symmetric in $v_3, \ldots, v_d$ (see Note 3.2.4(2));

(3) $\phi(v_1, v_2, v_3, v_4, \ldots, v_d) - \phi(v_2, v_3, v_1, v_4, \ldots, v_d) + \phi(v_3, v_1, v_2, v_4, \ldots, v_d) = 0$ (Jacobi identity on the first three coordinates).

This is "dual" to the symmetry type $K(d-1,1)$, in the sense of the duality result referred to earlier (p68).

6. <u>Symmetry type $K(2, 1^{d-2})$</u>:

(1) ϕ is symmetric in v_1, v_2;

(2) ϕ is alternating in $v_3, \ldots, v_d$;

(3) $\phi(v_1, v_2, v_3, v_4, \ldots, v_d) + \phi(v_2, v_3, v_1, v_4, \ldots, v_d) + \phi(v_3, v_1, v_2, v_4, \ldots, v_d) = 0$.

(Compare with Example 5.)

7. <u>Symmetry type $L(d-2, 1^2)$</u>:

v_1	$\cdots\cdots$	v_{d-2}
v_{d-1}		
v_d		

(1) ϕ is alternating in $v_1, \ldots, v_{d-2}$;

(2) ϕ is symmetric in v_{d-1}, v_d;

(3) $\phi(v_1, \ldots, v_{d-1}, v_d) + (-1)^{d-2}\phi(v_2, \ldots, v_{d-1}, v_1, v_d) + \phi(v_3, \ldots, v_{d-1}, v_1, v_2, v_d) + \ldots + (-1)^{d-2}\phi(v_{d-1}, v_1, \ldots, v_{d-2}, v_d) = 0$ ((signed) Jacobi identity on $v_1, \ldots, v_d$).

Compare with Example 6: (1) and (2) can be obtained by re-numbering, but (3) is different.

8. <u>Symmetry type $K(d-2, 1^2)$</u>:

(1) ϕ is symmetric in $v_1, \ldots, v_{d-2}$;

(2) ϕ is alternating in v_{d-1}, v_d;

(3) $\phi(v_1, \ldots, v_{d-1}, v_d) + \phi(v_2, \ldots, v_{d-1}, v_1, v_d) + \phi(v_3, \ldots, v_{d-1}, v_1, v_2, v_d) + \ldots + \phi(v_{d-1}, v_1, \ldots, v_{d-2}, v_d) = 0$.

Compare with Example 5.

9. <u>Symmetry type $L(3, 1^{d-3})$</u>:

v_1	v_2	v_3
v_4		
$\vdots$		
$\vdots$		
v_d		

(1) ϕ is alternating in v_1, v_2, v_3;

(2) ϕ is symmetric in $v_4, \ldots, v_d$;

(3) $\phi(v_1, \ldots, v_4, v_5, \ldots, v_d) - \phi(v_2, \ldots, v_1, v_5, \ldots, v_d) + \phi(v_3, \ldots, v_2, v_5, \ldots, v_d) - \phi(v_4, \ldots, v_3, v_5, \ldots, v_d) = 0$.

We discuss the simplest "new" symmetry type in more detail:

10. Symmetry type $L(2,2)$:

v_1	v_2
v_3	v_4

(1) ϕ is alternating in v_1, v_2;

(2) ϕ is alternating in v_3, v_4;

(3) $\phi(v_1, v_2, v_3, v_4) + \phi(v_2, v_3, v_1, v_4) + \phi(v_3, v_1, v_2, v_4) = 0$.

We also get another Jacobi identity, but this follows easily from (2) and (3). We can deduce from the above, or more obviously using the Standard Basis Theorem for Schur functors ([ABW] Theorem II.2.16 p232), as in §2.4, other conditions:

(4) ϕ is symmetric in v_1, v_3 and v_2, v_4 simultaneously;

(5) analogues of (3) with the first (or second) coordinate fixed (see Note 3.2.4(3)).

These conditions show that the elements of $L_{(2,2)}(V)$ are precisely the Riemann-Christoffel tensors ([TOW2] Ex 2.3 p422).

11. Symmetry type $K(2,2)$: This gives the same conditions as Example 10, except that symmetry should replace alternation.

3.3 Nondegeneracy

Let f be a multilinear form of degree d on a vector space V over a field F. Following Milnor and Husemoller ([MH] Definition (1.1) p1), we define, for each $i = 1, \ldots, d$, $t_i : V \to T^{d-1}(V)^*$ by $t_i(v) = f(\ldots, v, \ldots)$, where v is inserted in the i^{th} position. This generalizes the notion of a derivative of an alternating (or symmetric) form, and we shall denote it $f_i^{(v)}$, the subscript indicating the position of insertion of v.

We call f *nondegenerate-i* if t_i is injective; *nondegenerate* if t_i is injective for *all* i. Differently stated, f is nondegenerate-i if and only if $f_i^{(v)} = 0$ implies $v = 0$.

If f is alternating (or symmetric), we clearly have f nondegenerate if and only if f is nondegenerate-i for *some* i. In fact, this is true more generally for ζ-skew-symmetric forms ([KAN] p735).

If $d = 2$, it is easy to check that nondegeneracy-1 and nondegeneracy-2 are equivalent for any multilinear form ([JAC1] Theorem 6.1 pp328-9).

We prove the next result for a general Young symmetry type $L(\lambda)$; a similar result obviously holds for general Young symmetry type $K(\lambda)$.

3.3.1 Proposition: If f has general Young symmetry type $L(\lambda)$, where $|\lambda| = d$, and f is nondegenerate-d, then f is nondegenerate.

Proof: We need to show that f is nondegenerate-i for all $1 \leq i \leq d-1$. This is clear from Proposition 3.2.2(1) if i is in the same row as d.
If i is in some row above d, we can for the same reason assume it is in the same column as d. By Proposition 3.2.2(2), f satisfies the Jacobi identity $J_{id} \cdot f = 0$. Suppose the indices in the row containing i are $i_1, \ldots, i, \ldots, i_r$. Then for all $v_1, \ldots, v_d \in V$, we have

$$f(\ldots, v_{i_1}, \ldots, v_i, \ldots, v_{i_r}, \ldots, v_d) \pm f(\ldots, v_{i_2}, \ldots, v_i, \ldots, v_{i_r}, v_d, \ldots, v_{i_1}) \pm \ldots$$
$$\pm f(\ldots, v_d, v_{i_1}, \ldots, v_i, \ldots, v_{i_{r-1}}, \ldots, v_{i_r}) = 0. \quad (1)$$

Now suppose $f_i^{(v)} \equiv 0$. By Proposition 3.2.2(2) this means $f_{i_1}^{(v)} \equiv \cdots \equiv f_{i_r}^{(v)} \equiv 0$. Putting $v_i = v$ in (1), we obtain $f(\ldots, v) \equiv 0$, i.e. $f_d^{(v)} \equiv 0$, since all other terms are zero. This gives $v = 0$, as required. □

3.4 Derivative and Integral of a general Young symmetry type

If ϕ is an alternating form of degree d, then its derivative with respect to $v \in V$, $\phi^{(v)}$, is an alternating form of degree $d-1$ (§1.2). This notion was used to establish the cofinality of hyperbolic alternating space (§2.1).

We now introduce the idea of a *derivative symmetry type* and *integral symmetry type* of a given general Young symmetry type. We shall relate these to a notion of *derivative of a form* of general Young symmetry type.

Our approach is an adaptation of the "surgery" on Young diagrams employed in the Branching Rule for restricting and inducing irreducible representations of S_d to S_{d-1} and S_{d+1}, respectively. (See [SAG] Definition 2.8.1 p76, Theorem 2.8.3 p77.)

Since we do not assume ϕ is symmetric or alternating, the forms $\phi_i^{(v)}$, $i = 1, \ldots, d$ (see §3), are, in general, radically distinct. We shall show that, given the canonical filling, one particular derivative, viz. $\phi_d^{(v)}$, is always of some general Young symmetry type. Whether other derivatives are also of some general Young symmetry type depends on the particular symmetry conditions on ϕ. For example, if ϕ is of general Young symmetry type $K(2,1)$ (i.e. $\phi(v_1, v_2, v_3) = \phi(v_2, v_1, v_3)$ and $\phi(v_1, v_2, v_3) + \phi(v_2, v_3, v_1) + \phi(v_3, v_1, v_2) = 0$), then $\phi_3^{(v)}$ is symmetric, but $\phi_1^{(v)} = \phi_2^{(v)}$ is not of general Young symmetry type (which could only be symmetry or alternation).

3.4.1 Definitions: Let T_λ be a Young tableau of weight $|\lambda| = d$ with the canonical numbering. The *derivative tableau* ∂T_λ is obtained from T_λ by deleting the cell containing d. An *integral tableau* $\int T_\lambda$ is obtained by adding a cell containing $d+1$ to T_λ so that the resulting tableau is a canonically numbered tableau.

The general Young symmetry types associated with ∂T_λ and $\int T_\lambda$ are called *derivative symmetry types* and *integral symmetry types*, respectively, of the general Young symmetry type associated with T_λ.

3.4.2 Remark: For a given T_λ, the derivative tableau ∂T_λ is clearly unique. For the integral tableau $\int T_\lambda$, there are two cases:

(a) If the last row has fewer cells than the previous row, then the new cell containing $d+1$ can be placed either

(i) adjacent to the last cell; or

(ii) below the last row as the only entry in a new row. We denote these $\int_s T_\lambda$ and $\int_a T_\lambda$, respectively, following the examples of symmetry and alternation.

(b) If the last two rows have the same number of cells, then the new cell containing $d+1$ can be placed only below the last row as the only entry in a new row.

3.4.3 Examples

1. If $T_\lambda = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4 \\ \cline{1-1}\end{array}$, then $\partial T_\lambda = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}$ and $\int_s T_\lambda = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array}$, $\int_a T_\lambda = \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$.

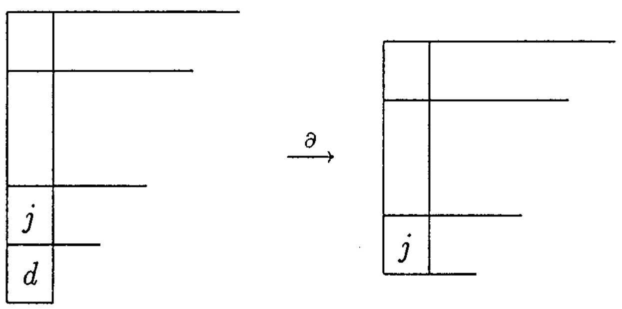
2. If $T_\lambda = \begin{array}{|c|c|}\hline 1&2\\ \hline 3&4 \\ \hline\end{array}$, then $\partial T_\lambda = \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array}$ and $\int T_\lambda = \begin{array}{|c|c|}\hline 1&2\\ \hline 3&4 \\ \hline 5 \\ \cline{1-1}\end{array}$.

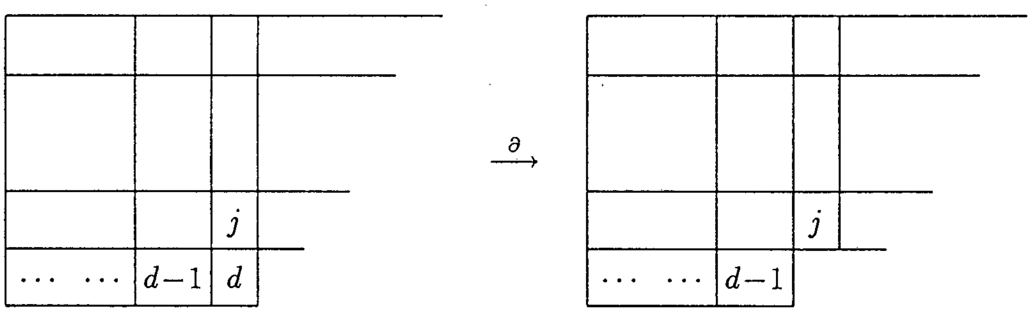
3. If $T_\lambda = \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$, then $\partial T_\lambda = \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1}\end{array}$ and $\int T_\lambda = \begin{array}{|c|c|}\hline 1&2\\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$.

3.4.4 Remark: The existence of only one integral symmetry type in certain cases has implications for Siegel duality (see §6), and is a consequence of defining derivative and integral symmetry types via evaluation.

The notions we use seem to give the simplest and most natural extension of the bilinear case, but ours is not intended to be the most general approach to derivative and integral symmetry types. For example, differentiation could consist of evaluation combined with a (co-)symmetrizer, and integration could be described by using the Littlewood-Richardson rule ([FH] pp78-9) for decomposing the tensor product of a (co-)Schur functor and V.

We now carry out the easy check that, if ϕ is a form of degree d of general Young symmetry type $L(\lambda)$ (or $K(\lambda)$), then $\phi^{(v_d)}$ (of degree $d-1$) has general Young symmetry type $L(\partial T_\lambda)$ (or $K(\partial T_\lambda)$). (We check only general Young symmetry type $L(\lambda)$; the result for $K(\lambda)$ is similar.)
There are two possibilities to consider:

(i) 

(ii) 

Skew-symmetry: In both (i) and (ii), if i_1, i_2 are adjacent entries in the same row of λ, and $i_2 < d$, they remain so for $\partial\lambda$. Then ϕ and $\phi^{(v_d)}$ are both skew-symmetric in v_{i_1}, v_{i_2}. In (i), neither ϕ nor $\phi^{(v_d)}$ is skew-symmetric in v_d. In (ii), ϕ is skew-symmetric in v_{d-1}, v_d, but $\phi^{(v_d)}$ is (obviously) not. Thus $\phi^{(v_d)}$ satisfies the condition (1) of Theorem 3.2.2 for general Young symmetry type $L(\partial T_\lambda)$.
Jacobi identity: In both (i) and (ii): If j_1, j_2 are adjacent entries in the same column of λ, and $j_2 < d$, they remain so for $\partial\lambda$. Then $J_{j_1 j_2} \cdot \phi = 0$ and $J_{j_1 j_2} \cdot \phi^{(v_d)} = 0$. If

j, d are adjacent entries in the same column of λ, then $J_{jd} \cdot \phi = 0$, but there is no corresponding identity for $\phi^{(v_d)}$.

Thus $\phi^{(v_d)}$ satisfies condition (2) of Theorem 3.2.2 for general Young symmetry type $L(\partial T_\lambda)$. Hence, by Theorem 3.2.2 and Note 3.2.4.1, $\phi^{(v_d)}$ is of general Young symmetry type $L(\partial T_\lambda)$, as claimed. We thus have

3.4.5 Proposition: If ϕ is a form of degree d of general Young symmetry type $L(\lambda)$ (resp. $K(\lambda)$), then its derivative $\phi^{(v_d)}$ is of general Young symmetry type $L(\partial T_\lambda)$ (resp. $K(\partial T_\lambda)$).

3.5 Hyperbolics

The hyperbolics discussed by Keet ([K1] Ch 2.1.10 pp28-9) and ourselves (§2.1) have the form $H = V \oplus \mathcal{F}$, where $\mathcal{F}$ is a space of symmetric or alternating forms, and the multilinear form ψ is defined by applying an appropriate symmetrizer ($c_{(d)}$ or $c_{(1^d)}$) to the evaluation of an element of $\mathcal{F}$.

We investigate generalizing this procedure to a situation where $\mathcal{F}$ and c_λ (or c'_λ) have non-standard Young symmetry type. We consider only the case where $\mathcal{F}$ is a Schur functor; the co-Schur functor case is parallel.

Let $H = V \oplus L_\lambda(V^*)$, where V is a vector space of dimension n over a field of characteristic 0, and $L_\lambda(V^*) \cong K_{\tilde{\lambda}}(V)^*$ is the space of multilinear forms of degree d of general Young symmetry type $K(\tilde{\lambda})$ and $|\lambda| = |\tilde{\lambda}| = d$ (see §2).

We wish to determine, given such H, what the general Young symmetry type is of a hyperbolic form ψ defined on H. Our requirement is that such a hyperbolic space should be cofinal for spaces equipped with a form of the same general Young symmetry type as ψ.

Suppose U is any vector space equipped with a form f of degree $d + 1$ of the same general Young symmetry type as ψ.

Define $\rho : (U, f) \to (H, \psi)$ by $\rho(u) = (u, f^{(u)})$, as in the alternating case (§2.1). (We use here the derivative introduced in §4, viz. $f^{(u)} = f^{(u)}_{d+1}$.)

We need to ensure that $f^{(u)} \in L_\lambda(V^*) \cong K_{\tilde{\lambda}}(V)^*$, i.e. the derivative of f should have general Young symmetry type $K(\tilde{\lambda})$. This means (by §4) that f (and hence also ψ) should have general Young symmetry type $\int K(\tilde{\lambda})$. For a given λ, there are one or two possibilities for $\int K(\tilde{\lambda})$, the general Young symmetry type of f and ψ.

Now define, for $f \in K_{\tilde{\lambda}}(V)^*$ and $v_2 \otimes \ldots \otimes v_{d+1} \in K_{\tilde{\lambda}}(V)$,

$\phi : K_{\tilde{\lambda}}(V)^* \otimes K_{\tilde{\lambda}}(V) \to F$ by $\phi(f \otimes v_2 \otimes \ldots \otimes v_{d+1}) = f(v_2, \ldots, v_{d+1})$.

(This generalizes the notion of an apolarity/dual pairing — see §4.3 p107.)

Let c_μ be the symmetrizer corresponding to some integral symmetry type of $K(\tilde{\lambda})$.

Define, for $(v_i, f_i) \in H$, $1 \leq i \leq d+1$,

$$\psi[(v_1, f_1), \ldots, (v_{d+1}, f_{d+1})] = c_\mu \cdot \phi(f_1, v_2, \ldots, v_{d+1}),$$

where the action of $\sigma \in S_{d+1}$ on ϕ is given by

$$\sigma \cdot \phi(f_1, v_2, \ldots, v_{d+1}) = \phi(f_{\sigma 1}, v_{\sigma 2}, \ldots, v_{\sigma(d+1)}).$$

(It is clear that this gives the hyperbolic symmetric and hyperbolic alternating forms when $\mu = (d)$ and $\mu = (1^d)$, respectively.) Then

$$\begin{aligned} c_\mu \cdot \psi[(v_1, f_1), \ldots, (v_{d+1}, f_{d+1})] &= c_\mu^2 \cdot \phi(f_1, v_2, \ldots, v_{d+1}) \\ &= n_\mu c_\mu \cdot \phi(f_1, v_2, \ldots, v_{d+1}) \text{ (where } n_\mu = \frac{(d+1)!}{f^\mu}) \\ &= n_\mu \psi[(v_1, f_1), \ldots, (v_{d+1}, f_{d+1})]. \end{aligned}$$

Thus ψ has general Young symmetry type $K(\tilde{\mu})$.

Next we check the cofinality requirement:

$$\begin{aligned} \psi[\rho(u_1), \ldots, \rho(u_{d+1})] &= \psi[(u_1, f^{(u_1)}), \ldots, (u_{d+1}, f^{(u_{d+1})})] \\ &= c_\mu \cdot \phi(f^{(u_1)}, u_2, \ldots, u_{d+1}) \\ &= c_\mu \cdot f(u_2, \ldots, u_{d+1}, u_1) \\ &= c_\mu \cdot [\tau \cdot f(u_1, \ldots, u_{d+1})] \quad \text{(where } \tau = (1, \ldots, d+1)) \\ &= (c_\mu \cdot \tau \cdot c_\mu) \cdot f(u_1, \ldots, u_{d+1}). \end{aligned}$$

($c_\mu \cdot f = f$ because f has general Young symmetry type $K(\tilde{\mu})$.)

We know that, for any $\tau \in \mathbb{C}\, S_{d+1}$, $c_\mu \tau c_\mu = m_\mu c_\mu$ for some scalar m_μ ([FH] Lemma 4.23(2) p53), hence

$\psi[\rho(u_1), \ldots, \rho(u_{d+1})] = m_\mu c_\mu \cdot f(u_1, \ldots, u_{d+1}) = m_\mu f(u_1, \ldots, u_{d+1})$.

Adjusting by a suitable scalar factor, we see that ρ is an isometric embedding of (U, f) into (H, ψ). We have thus established:

3.5.1 Proposition: Let $H = V \oplus L_\lambda(V^*)$, where the number of rows of λ does not exceed *dim* V. Then we can define a form ψ on H whose symmetry type is an integral of the general Young symmetry type $K(\tilde{\lambda})$, and (H, ψ) is cofinal for spaces equipped with a form of such integral symmetry type.

We now compute a few hyperbolic forms of general Young symmetry type; we shall use these in the next section.

3.5.2 Examples

1. If $\mathcal{F} = L_{(1^d)}(V^*)$ is the space of symmetric forms, usually denoted $Sym_d V$, one of its integral symmetry types is symmetry, whose hyperbolic we have already referred to. But the other integral symmetry type is $K(d, 1)$, or hook-symmetry. The symmetrizer in this case is $c_{\tilde{\lambda}}$, where $\tilde{\lambda} = (2, 1^{d-1}) =$

1	$d+1$
2	
$\vdots$	
$\vdots$	
d	

.

So $c_{\tilde{\lambda}} = b_{\tilde{\lambda}} a_{\tilde{\lambda}} = (\sum_{\sigma \in S_d} \sigma)(e - (1, d+1))$. Hence

$\psi[(v_1, f_1), \ldots, (v_{d+1}, f_{d+1})] = c_{\tilde{\lambda}} \cdot \phi(f_1, v_2, \ldots, v_{d+1})$

$= \sum_{\sigma \in S_d} \sigma \cdot [\phi(f_1, v_2, \ldots, v_{d+1}) - \phi(f_{d+1}, v_2, \ldots, v_1)]$

$= \sum_{\sigma \in S_d} \sigma \cdot \phi(f_1, v_2, \ldots, v_{d+1}) - \sum_{\sigma \in S_d} \sigma \cdot \phi(f_{d+1}, v_2, \ldots, v_1)$

$= (d-1)!\sum_{i=1}^{d} f_i(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1}) - d!\, f_{d+1}(v_1,\ldots,v_d)$ (by symmetry of the f_i)

$= (d-1)![\sum_{i=1}^{d} f_i(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1}) - d\, f_{d+1}(v_1,\ldots,v_d)]$.

By the general arguments above, $(V \oplus Sym_d V, \psi)$ has general Young symmetry type $K(d,1)$ and is cofinal for spaces equipped with a form of this symmetry type. We show also that ψ is nondegenerate-$(d+1)$, hence nondegenerate (by Proposition 3.3.1):

Suppose $\psi[(v_1,f_1),\ldots,(v_{d+1},f_{d+1})] = 0$ for all $v_i, f_i, i = 1,\ldots,d$. Put $f_i = 0$ for $i = 1,\ldots,d$. Then $d\, f_{d+1}(v_1,\ldots,v_d) = 0$ for all $v_1,\ldots,v_d$, hence $f_{d+1} = 0$. Choose $v_1 = 0$ and let f_1 be any nondegenerate symmetric form; then $f_1(v_2,\ldots,v_d) = 0$ for all $v_2,\ldots,v_d$, so $v_{d+1} = 0$. Hence ψ is nondegenerate.

2. If $L_{(d)}(V^*)$ is the space of alternating forms, usually denoted $Alt_d V$, then we obtain the hyperbolic of symmetry type $L(d,1)$ (hook-alternating) in a similar fashion to Example 1:

$H = V \oplus Alt_d V$, with $\psi[(v_1,f_1),\ldots,(v_{d+1},f_{d+1})] =$

$(d-1)![\sum_{i=1}^{d}(-1)^{i-1} f_i(v_1,\ldots,\hat{v}_i,\ldots,v_{d+1}) + (-1)^{d-1} d\, f_{d+1}(v_1,\ldots,v_d)]$.

ψ is easily seen to be nondegenerate by the same reasoning as in Example 1, except we have to use Lemma 2.1.1 to show that $v_{d+1} = 0$.

Finally, we discuss the hyperbolics which can be defined when $\mathcal{F} = L_{(2,1)}(V^*) \cong K_{(2,1)}(V)^*$. There are two integral symmetry types for the general Young symmetry type $K(2,1)$:

3. Hyperbolic of symmetry type $K(2,1^2)$: In this case $\tilde{\lambda} = (3,1) = \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1} \end{array}$, so the symmetrizer is $c_{\tilde{\lambda}} = (e+(12))(e-(13)-(14)-(34)+(134)+(143))$. Hence $\psi[(v_1,f_1),\ldots,(v_4,f_4)] = f_1(v_2,v_3,v_4) - f_1(v_2,v_4,v_3) + f_2(v_1,v_3,v_4) + f_2(v_1,v_4,v_3) - 3f_3(v_1,v_2,v_4) + 3f_4(v_1,v_2,v_3)$, after simplification using the $K(2,1^2)$ symmetry properties of the f_i.

We check nondegeneracy: Suppose $\psi[(v_1, f_1), \ldots, (v_4, f_4)] = 0$ for all v_1, v_2, v_3 and f_1, f_2, f_3. Put $f_1 = f_2 = f_3 = 0$. Then $3f_4(v_1, v_2, v_3) = 0$ for all v_1, v_2, f_3. Choosing f_3 nondegenerate, or using an argument as in Lemma 2.1.1 (p31), gives $v_4 = 0$. Thus ψ is nondegenerate-4, hence nondegenerate by Proposition 3.3.1 (p77).

4. Hyperbolic of symmetry type $K(2,2)$: Here $\tilde{\lambda} = (2,2) = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$, so the symmetrizer is $c_{\tilde{\lambda}} = (e + (12) + (34) + (12)(34))(e - (13) - (24) + (13)(24))$, which gives

$\psi[(v_1, f_1), \ldots, (v_4, f_4)] = -3[f_1(v_3, v_4, v_2) + f_2(v_3, v_4, v_1) + f_3(v_1, v_2, v_4)$
$+f_4(v_1, v_2, v_3)]$ (since the f_i have symmetry type $K(2,1)$).

ψ is easily seen to be nondegenerate using an argument very similar to that in Example 3.

3.6 Siegel Duality

Our problem here is to generalize the interesting duality, which first occurred in the work of Siegel (see [WEI]), between symmetric and alternating bilinear forms, and which has recently been developed by Hughes ([HUG]).

First we outline the 2-variable case:

Let (W, f) be a nondegenerate (i.e. hyperbolic) alternating bilinear space. In symplectic geometry, a subspace L is called *Lagrangian* if $L = L^{\perp}$, where $L^{\perp} = \{x \in W \mid f(x, y) = 0 \ \forall y \in L\}$ ([LV] p8). This means that (i) $f(x, y) = 0$ for all $x, y \in L$, i.e. L is *isotropic*; and (ii) $f(x, L) = 0$ implies $x \in L$. It is easily checked that (ii) is equivalent to L being a *maximal* isotropic subspace.

The *conformal symplectic Lie algebra* of W, $csp(W)$, consists of all $A \in End(W)$ satisfying

$$f(Au, v) + f(u, Av) = \mu_A f(u, v) \text{ for all } u, v \in W,$$

where μ_A is a scalar. (If f is symmetric bilinear, this yields the usual conformal algebra.)

We then have

3.6.1 Proposition ([GS] Proposition 2.2 p117-8): Let X be a fixed Lagrangian subspace of (W, f). Then the following sets are in 1-1 correspondence:

(i) Lagrangian subspaces Y such that $Y \cap X = \{0\}$ (i.e. Y transversal to X);

(ii) $P \in csp(W)$ such that $\mu_P = 1$ and $P|_X \equiv 0$: and

(iii) symmetric bilinear forms q on W such that $q(x, v) = -\frac{1}{2}f(x, v)$ for all $x \in X, v \in W$.

(Here $Y = ker(P-I)$, while P and q are related by $q_P(x, y) = f(Px, y) - \frac{1}{2}\mu_P f(x, y)$.)

The correspondence between Lagrangian subspaces of the *alternating* bilinear space (W, f) and *symmetric* bilinear forms on W constitutes Siegel duality.

We investigate a notion of generalized Siegel duality for the hyperbolics H of general Young symmetry type discussed in §3.5.

3.6.2 Definition: If W is a space equipped with a form f of degree d of some general Young symmetry type, the *conformal Lie algebra* $cl(W)$ consists of all $A \in End(W)$ satisfying, for all $u_i \in W$,

$$f(Au_1, u_2, \ldots, u_d) + f(u_1, Au_2, \ldots, u_d) + \cdots + f(u_1, \ldots, u_{d-1}, Au_d) = \mu_A f(u_1, \ldots, u_d) \qquad (1)$$

for some scalar μ_A.

3.6.3 Remark: It is easy to see that $A \in cl(W)$, with parameter μ_A, if and only if $I - A \in cl(W)$, with parameter $d - \mu_A$.

Let $H = V \oplus \mathcal{F}$ be the hyperbolic space of some general Young symmetry type of degree d, and write, for brevity, its form ψ as a repeated dot product.

3.6.4 Definition: A subspace L of H is *Lagrangian* if
(a) L is a complementary subspace to $\mathcal{F}$, i.e. $H = L + \mathcal{F}$;
(b) L is isotropic, i.e $u_1 \cdot u_2 \cdots u_d = 0$ for all $u_i \in L$.

If L is Lagrangian, let P_L denote the projection onto L. We show that $P_L \in cl(H)$. By 3.6.4(a), it is sufficient to check condition (1) of 3.6.2 for elements of L or $\mathcal{F}$. The cases are:
More than 2 elements in $\mathcal{F}$: Both sides of (1) are 0 by definition of H and ψ;
2 elements in $\mathcal{F}$, say f_i and f_j: The RHS of (1) is 0, while the LHS contains the terms $P_L f_i = 0$ and $P_L f_j = 0$, and all other terms have 2 elements from $\mathcal{F}$, so are also 0;
All elements in L: Both sides are 0 by 3.6.4(b);
1 element in $\mathcal{F}$, say f_i: On the LHS the i^{th} term contains $P_L f_i = 0$, and all other terms are $l_1 \cdots f_i \cdots l_d$. So the LHS$=(d-1)l_1 \cdots f_i \cdots l_d$.
Hence $P_L \in cl(H)$, with $\mu = d - 1$.
On the other hand, if L is complementary to $\mathcal{F}$ and $P_L \in cl(H)$ with $\mu = d - 1$, then, for all $l_i \in L$,

$$P_L l_1 \cdots l_d + l_1 \cdot P_L l_2 \cdots l_d + \cdots + l_1 \cdots l_{d-1} \cdot P_L l_d = (d-1) l_1 \cdots l_d.$$

This gives $d l_1 \cdots l_d = (d-1) l_1 \cdots l_d$, i.e. $l_1 \cdots l_d = 0$, so L is isotropic.

We have thus proved the following analogue of (i)$\Leftrightarrow$ (ii) of Proposition 3.6.1:

3.6.5 Proposition: If L is a Lagrangian subspace of a hyperbolic space H of some general Young symmetry type, then $P_L \in cl(H)$; and if L is a complementary subspace to $\mathcal{F}$ and $P_L \in cl(H)$, then L is Lagrangian.

Remark 3.6.6: By Remark 3.6.3 it follows that $I - P_L = P_{\mathcal{F}} \in cl(H)$ with $\mu = 1$.

We shall formulate our version of Siegel duality by looking at a special type of Lagrangian subspace.

First consider the bilinear *alternating* hyperbolic space $H(V)^2_a = V \oplus V^*$, with form $\psi_a[(v_1, f_1), (v_2, f_2)] = f_1(v_2) - f_2(v_1)$. Now let f be any *symmetric* bilinear form on V, and put $\mathcal{L}_f = \{(v, f^{(v)}) \mid v \in V\}$. Then it is easy to see that $\mathcal{L}_f$ is a Lagrangian subspace of $H(V)^2_a$.

Analogously, for the bilinear *symmetric* hyperbolic space $H(V)^2_s = V \oplus V^*$ with form $\psi_s[(v_1, f_1), (v_2, f_2)] = f_1(v_2) + f_2(v_1)$, if f is any *alternating* bilinear form on V, then $\mathcal{L}_f$ (defined as above) is a Lagrangian subspace of $H(V)^2_s$.

Siegel duality thus appears here in the following form: The Lagrangian subspaces of the type $\mathcal{L}_f$ of the *alternating* (resp. symmetric) hyperbolic are those $\mathcal{L}_f$ where f is *symmetric* (resp. alternating).

It is in this sense that we attempt to formulate our generalized Siegel duality, i.e. the Lagrangian subspaces of the type $\mathcal{L}_f$ of the generalized hyperbolic (H, ψ) should be those where f has *dual* symmetry type to ψ.

Let $H = V \oplus L_\lambda(V^*)$ ($\cong K_{\tilde{\lambda}}(V)^*$) be a hyperbolic space of general Young symmetry type $\int K(\tilde{\lambda})$, say $K(\mu)$. (We consider only a Schur functor; the other case being parallel.)

We want to determine the symmetry type of a form f such that $\mathcal{L}_f = \{(v, f^{(v)}) \mid v \in V\}$ is a Lagrangian subspace of H. Clearly, we must have $f^{(v)} \in L_\lambda(V^*) \cong K_{\tilde{\lambda}}(V)^*$, so the symmetry type of f must be an integral of the general Young symmetry type $K(\tilde{\lambda})$, just as the symmetry type of ψ is.

If $(v, f^{(v)}) \in \mathcal{L}_f \cap \mathcal{F}$, then $v = 0$, so $f^{(v)} = 0$, hence $\mathcal{L}_f \cap \mathcal{F} = \{0\}$. Also $dim\, \mathcal{L}_f = dim\, V$, so 3.6.4(a) is satisfied.

We now check under what condition(s) 3.6.4(b) is satisfied. We want

$\psi[(v_1, f^{(v_1)}), \ldots, (v_{d+1}, f^{(v_{d+1})})] = 0$ for all $v_1, \ldots, v_{d+1}$.

By the same reasoning as in the cofinality argument (§5), we require $c_{\tilde{\mu}}\tau \cdot f = 0$, where $c_{\tilde{\mu}}$ is the symmetrizer for ψ.

Let c_ν be the symmetrizer for f, where $|\tilde{\mu}| = |\nu|$. Then $f = c_\nu \cdot f$, so we require $c_{\tilde{\mu}}\tau c_\nu \cdot f = 0$. But this holds whenever $\nu \neq \tilde{\mu}$ (see [FH] Example 4.24 p53). So we merely require f and ψ to have distinct integral symmetry types of the general Young symmetry type $K(\tilde{\lambda})$. We have already seen in §4 under what conditions this obtains.

We thus have

3.6.7 Proposition: The Lagrangian subspaces of the type $\mathcal{L}_f$ of the general hyperbolic space $(V \oplus \mathcal{F}, \psi)$ are those where f and ψ are distinct integral symmetry types of the symmetry type of $\mathcal{F}$.

In terms of our formulation of generalized Siegel duality, two symmetry types are Siegel dual if they are different and have the same derivative symmetry type.

(This includes the classical case, since the derivatives of symmetric and alternating bilinear forms are both just linear forms/functionals.)

We have already observed (p86) that Proposition 3.6.1 gives Siegel duality in the bilinear case via the following relation:

$$g(x,y) = f(Px,y) - \frac{1}{2}\mu f(x,y) \quad (2)$$

If f is alternating and P is conformal, then g is symmetric. We aim to generalize this part of Proposition 3.6.1 and link it to our generalized Siegel duality.

First we observe that (2) can be written as follows:

$$\begin{aligned} g(x,y) &= f(Px,y) - \frac{1}{2}[f(Px,y) + f(x,Py)] \\ &= f(Px,y) - \frac{1}{2}[f(Px,y) - f(Py,x)] \end{aligned}$$

If we put $\theta(x,y) = f(Px,y)$, then we have

$$g(x,y) = \theta(x,y) - I_a \cdot \theta(x,y), \quad (3)$$

where I_a denotes the skew-symmetrizing idempotent.

The form with Siegel dual symmetry type, viz. g, is thus obtained by subtracting from the form $f(Px,y)$ its skew-symmetrized part. We extend this idea to our general situation.

As before, let (H,f) be a hyperbolic space of general Young symmetry type $L(\lambda)$ of degree d, and let I_λ denote the Young idempotent for this symmetry type. Let L be a Lagrangian subspace, and let P denote the projection onto L. Then P is conformal, i.e.

$f(Pu_1, u_2, \ldots, u_d) + f(u_1, Pu_2, \ldots, u_d) + \cdots + f(u_1, u_2, \ldots, Pu_d) =$

$(d-1)f(u_1, u_2, \ldots, u_d) \quad (4).$

By Lemma 3.2.5 (p72), every term on the LHS of (4) can be replaced by terms with Pu_i in the first position.

Now put $\theta(u_1, \ldots, u_d) = f(Pu_1, u_2, \ldots, u_d)$. Then every term on the LHS of (4) has the form $\theta(u_{j_1}, \ldots, u_{j_d})$. If we now apply the Young idempotent I_λ to (4), the RHS just becomes $(d-1)I_\lambda \cdot f = (d-1)f$, while each term on the LHS becomes $I_\lambda \cdot \theta(u_{j_1}, \ldots, u_{j_d})$, which has symmetry type $L(\lambda)$.

We can thus apply Lemma 3.2.5 again (in reverse), and obtain

$I_\lambda \cdot \theta(u_1, \ldots, u_d) + I_\lambda \cdot \theta(u_1, \ldots, u_d) + \cdots + I_\lambda \cdot \theta(u_1, \ldots, u_d) = (d-1)f(u_1, \ldots, u_d)$,

i.e. $dI_\lambda \cdot \theta(u_1, \ldots, u_d) = (d-1)f(u_1, \ldots, u_d)$, i.e.

$$I_\lambda \cdot \theta(u_1, \ldots, u_d) = \frac{d-1}{d} f(u_1, \ldots, u_d). \quad (5)$$

Now consider the form $\phi(u_1, \ldots, u_d) = f(u_1, \ldots, u_{d-1}, Pu_d)$. By Lemma 3.2.5 we can write $\phi(u_1, \ldots, u_d)$ as a linear combination of terms $f(Pu_d, u_{j_2}, \ldots, u_{j_d}) = \theta(u_d, u_{j_2}, \ldots, u_{j_d})$.

Then $I_\lambda \cdot \phi(u_1, \ldots, u_d)$ is a linear combination (with the same coefficients and same order of variables in corresponding terms) of terms $I_\lambda \cdot \theta(u_d, u_{j_2}, \ldots, u_{j_d}) =$

$\frac{d-1}{d}f(u_d, u_{j_2}, \ldots, u_{j_d})$ (by (5)). We can apply Lemma 3.2.5 in reverse and replace the linear combination of the $f(u_d, u_{j_2}, \ldots, u_{j_d})$ by $f(u_1, \ldots, u_d)$. We thus obtain

$$I_\lambda \cdot \phi(u_1, \ldots, u_d) = \frac{d-1}{d} f(u_1, \ldots, u_d).$$

Now put

$$\begin{aligned} g(u_1, \ldots, u_d) &= \phi(u_1, \ldots, u_d) - I_\lambda \cdot \phi(u_1, \ldots, u_d) \quad \text{(cf. (3))} \\ &= f(u_1, \ldots, u_{d-1}, Pu_d) - \frac{d-1}{d} f(u_1, \ldots, u_d). \end{aligned}$$

In the bilinear case the form g turns out to have dual symmetry type, i.e. g is symmetric when f is alternating. In the general situation it is not easy to describe the symmetry type of g so explicitly, but there is nonetheless a relation to our generalized Siegel duality in the following sense: if we fix u_d, we can see that the resulting derivative of g has the same symmetry type as the corresponding derivative of f. We have thus established

3.6.8 Proposition: If P is the projection onto a Lagrangian subspace of some hyperbolic space of general Young symmetry type, then P defines a form whose derivative has the same symmetry type as the corresponding derivative of the hyperbolic form.

We conclude by illustrating how this Siegel-type duality works for higher degree symmetry, higher degree alternation, and a less familiar general Young symmetry type:

1. Higher degree symmetric hyperbolics: $H(V)_s^{d+1} = (V \oplus Sym_d V, \psi_s)$, with $d \geq 2$. Let f be any form of symmetry type $K(d,1)$ (i.e. hook-symmetric). (Note that both f and ψ_s have integral symmetry type to that of $Sym_d V$.)
We check that $\mathcal{L}_f$ is isotropic:
$\psi_s[(v_1, f^{(v_1)}), , \ldots, (v_{d+1}, f^{(v_{d+1})})]$
$= f^{(v_1)}(v_2, \ldots, v_{d+1}) + f^{(v_2)}(v_1, v_3, \ldots, v_{d+1}) + \cdots + f^{(v_{d+1})}(v_1, \ldots, v_d)$

$= f(v_2, v_{d+1}, \ldots, v_1) + f(v_1, v_3, \ldots, v_{d+1}, v_2) + \cdots + f(v_1, \ldots, v_{d+1})$

$= f(v_2, v_{d+1}, \ldots, v_1) + f(v_3, \ldots, v_{d+1}, v_1, v_2) + \cdots + f(v_1, \ldots, v_{d+1})$

(f is symmetric in the first d variables)

$= 0$ (by the Jacobi identity).

2. Hook-symmetric hyperbolics (Symmetry type $K(d,1)$)

$H(V) = (V \oplus Sym_d V, \psi)$, where $d \geq 2$ and $\psi[(v_1, f_1), \ldots, (v_d, f_d)] =$

$f_1(v_2, \ldots, v_{d+1}) + \cdots + f_d(v_1, \ldots, v_{d-1}, v_{d+1}) - d\, f_{d+1}(v_1, \ldots, v_d)$.

Let f be any symmetric form of degree $d + 1$.

Then $\psi_s[(v_1, f^{(v_1)}), , \ldots, (v_{d+1}, f^{(v_{d+1})})]$

$= f^{(v_1)}(v_2, \ldots, v_{d+1}) + \cdots + f^{(v_d)}(v_1, \ldots, v_{d-1}, v_{d+1}) - d\, f^{(v_{d+1})}(v_1, \ldots, v_d)$

$= f(v_2, v_{d+1}, \ldots, v_1) + \cdots + f(v_1, \ldots, v_{d-1}, v_{d+1}, v_d) - d\, f(v_1, \ldots, v_{d+1})$

$= 0$ (by symmetry of f).

Thus $\mathcal{L}_f$ is isotropic.

Examples 1 and 2 thus illustrate the Siegel-type duality between *symmetry* and *hook-symmetry* for degrees ≥ 3 which we have elaborated in general terms above.

It is easy to show, using very similar reasoning to Examples 1 and 2, that there is a Siegel-type duality between *alternation* and *hook-alternation* for degrees ≥ 3.

Our final example concerns the non-standard symmetry types which have appeared repeatedly.

3. Hyperbolic of symmetry type $K(2, 1^2)$ (See Example 3, §5 p84.)

Let f be any form of symmetry type $K(2,2)$. Then $\psi[(v_1, f^{(v_1)}), \ldots, (v_4, f^{(v_4)})] =$ $f(v_2, v_3, v_4, v_1) - f(v_2, v_4, v_3, v_1) + f(v_1, v_3, v_4, v_2) - f(v_1, v_4, v_3, v_2) - 3\, f(v_1, v_2, v_4, v_3) +$ $3\, f(v_1, v_2, v_3, v_4) = f(v_2, v_3, v_4, v_1) - f(v_2, v_4, v_3, v_1) + f(v_4, v_2, v_1, v_3) - f(v_3, v_2, v_1, v_4)$

$-3\,f(v_1, v_2, v_4, v_3) + 3\,f(v_1, v_2, v_3, v_4) = 0$ (by the symmetry conditions on f — see Example 11, §2). Thus $\mathcal{L}_f$ is isotropic.

4.Hyperbolic of symmetry type $K(2,2)$ (See Example 4, §5 p85.)

Let f be any form of symmetry type $K(2,1^2)$. Then $\psi[(v_1, f^{(v_1)}), \ldots, (v_4, f^{(v_4)})] =$
$-3[f(v_3, v_4, v_2, v_1) + f(v_3, v_4, v_1, v_2) + f(v_1, v_2, v_4, v_3) + f(v_1, v_2, v_3, v_4)]$
$= -3[f(v_3, v_4, v_2, v_1) - f(v_3, v_4, v_2, v_1) + f(v_1, v_2, v_4, v_3) - f(v_1, v_2, v_4, v_3)] = 0$ (by the symmetry conditions on f — see Example 6 or 8, §2).

The last two examples illustrate the Siegel-type duality between the symmetry types $K(2,1^2)$ and $K(2,2)$.

Chapter 4

Symmetric Forms: Nondegeneracy-type Conditions

Several nondegeneracy-type conditions have been used in the study of symmetric forms. Harrison ([H2]) and Keet ([K1] Ch2 §1 pp23-35) discuss the hierarchy of conditions nonsingularity, nonzero Hessian and nondegeneracy for symmetric higher degree forms, which are stratified by covariants such as the Hessian and discriminant. Harrison and Pareigis ([HP] p1288) define a notion of s-radical, which can be used to generalize nondegeneracy, and O'Ryan ([ORY] Definition 1.5 p969) defines a condition of s-regularity (called s-nondegeneracy in [HP]); there is also a special condition on forms of even degree, linked to the first-mentioned.

We summarize the content of this chapter.

We begin §4.1 by reviewing the conditions nonsingularity, nonzero Hessian and nondegeneracy, giving characterizations and mentioning the covariants testing for two of them. We then extend Harrison's result that nonsingularity implies nonzero Hessian for cubic forms to all degrees, using his generic methods. The validity of the converse of this result, as well as the converse of the (easily proved) result that nonzero Hessian implies nondegeneracy, are discussed next.

Each of the conditions nonsingularity and nondegeneracy gives rise, in a natural way, to a family of related conditions. In §4.2 we use the *s-radical* of [HP] to define *s-nondegeneracy*; we also discuss the s-regularity of [ORY], using the approach of [HP]. We note some connexions between them and give examples. The notion of s-nondegeneracy turns out to be related to a special condition on even degree forms, which we discuss in the next section.

Any bilinear symmetric form f on V induces a homomorphism $t : V \to V^*$ via $t(v) = f(v, -)$, and nondegeneracy of f is equivalent to t being an isomorphism, with the discriminant Δ being a covariant (in fact, invariant) testing for nondegeneracy; for symmetric forms f of degree $d \geq 3$, nondegeneracy is equivalent to the induced homomorphism $t : V \to S^{d-1}(V)^*$, given by $t(v) = f(v, -, \ldots, -)$, being a monomorphism. If $d = 2k$ is even, however, we also have an induced homomorphism $\alpha : S^k(V) \to S^k(V)^*$, given by $\alpha(v_1 \odot \cdots \odot v_k) = f(v_1, \ldots, v_k, -, \ldots, -)$, and there is the possibility of this being an isomorphism.

In investigating this question, we discovered, by direct methods, a relation between this condition and a classical invariant called the catalecticant. We subsequently found reference to the same relation in the paper by Dolgachev and Kanev ([DK]), based on the more efficient theory of polarity. We fill in some details, make explicit the connection with the classical notion of the catalecticant, and illustrate our earlier approach with examples.

We begin by reviewing the main properties of the catalecticant, as they appear in three classical texts, by Elliott ([ELL]), Grace and Young ([GY]) and Salmon ([SAL]). We give the symbolic expressions for the catalecticant in the modern notation of Grosshans et al. ([GRS]) in a few cases, and show the catalecticant is not additive. We then review the aspects of the theory of polarity which we require to give the general definition of the catalecticant in [DK], and we show that it specializes to the cases defined classically. We then discuss, using the induced bilinear form, the relation between the condition we call *strong nondegeneracy* and the catalecticant; we also illustrate this relation in explicit terms.

Finally we mention some relations between strong nondegeneracy and the conditions discussed in §4.1 and §4.2. There are several other possible connexions between the conditions discussed in this chapter which remain to be investigated. We conclude by listing a few which we have been unable to resolve.

4.1 Nonsingularity, nonzero Hessian and Nondegeneracy

We review briefly the conditions nonsingularity, nonzero Hessian and nondegeneracy. Let f be a symmetric multilinear form of degree d on a vector space V of dimension n over a field F. Suppose V has basis $\{e_i\}$ and $\{x_i\}$ is a dual basis. We shall generally assume $charF = 0$ or $charF \not| d!$, but sometimes this will clearly not be necessary.

Nonsingularity: f is nonsingular iff $f(v, \ldots, v, w) = 0$ for all w implies $v = 0$. This means its discriminant $disc(f) \neq 0$, where $disc(f)$ is the Eliminant of the first order partial derivatives of f. It is easy to see that $\frac{\partial f}{\partial x_i} = d\, f(e_i, x, \ldots, x)$. (Here we use f to denote both the multilinear form and its associated homogeneous polynomial.)

Nonzero Hessian: The Hessian of f, $H(f) = det(\frac{\partial^2 f}{\partial x_i \partial x_j})$. We say f has nonzero Hessian if $H(f) \neq 0$. It is easy to see that $\frac{\partial^2 f}{\partial x_i \partial x_j} = d(d-1)f(e_i, e_j, x, \ldots, x)$, so f has nonzero Hessian iff there exists $v \in V$ such that $det(f(e_i, e_j, v, \ldots, v)) \neq 0$, i.e. such that $f(-, -, v, \ldots, v)$ is a nondegenerate quadratic form. This characterization can be used to deduce very easily that nonzero Hessian implies nondegeneracy (which we discuss next).

Nondegeneracy: f is nondegenerate iff $f(v_1, \ldots, v_d) = 0$ for all $v_1, \ldots, v_{d-1}$ implies $v_d = 0$, or, equivalently, $f(v, \ldots, v, w) = 0$ for all v implies $w = 0$. (See [H1] p125, [KAN] p735, [R] p969.) f induces a homomorphism $f_1 : V \to S^{d-1}(V)^*$ (where $S^r(V)$ denotes the r^{th} symmetric power of V) with $f_1(v)(v_1, \ldots, v_{d-1}) = f(v, v_1, \ldots, v_{d-1})$, which is injective iff f is nondegenerate. By duality, this means the obvious induced homomorphism from $S^{d-1}(V)$ to V^* is surjective.

Harrison proved that nonsingularity implies nonzero Hessian over an algebraically closed field of characteristic zero ([H2] Proposition 1.1 p519). We now extend Harrison's proof to arbitrary degree. Although Keet ([K1] Proposition 1.8 p34) has proved the same result using algebraic geometry, Harrison's approach has the merit of using simple algebraic methods, and being potentially useful in proving results in the next section.

The key step in the generalization is Lemma 4.1.3.

4.1.1 Proposition: Let (V, θ) be a symmetric space of degree d over an algebraically closed field k of characteristic zero. If θ is nonsingular then $H(\theta) \neq 0$.

Proof: Let $n = dim_k V$ and let K be an algebraic closure of $k(t_1, \ldots, t_n)$, where the t_i are transcendentals. Let $\{v_i\}$, $\{x_i\}$ be dual bases for V, V^*, respectively, and denote the form (polynomial) associated to (V, θ) by $f = f(x_1, \ldots, x_n)$. By nonsingularity, $\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}$ do not have a common nontrivial zero over k. A common nontrivial zero over K would correspond to a k-homomorphism $k[x_1, \ldots, x_n]/\langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle \to K$, whose image would be a finite algebraic extension of k (by Corollary 5.24 of [AM]), and thus k itself (k is algebraically closed). So a nontrivial zero over K is in fact in k, and hence cannot exist. Thus f is still nonsingular, considered as a form over K, the form associated to the symmetric degree d space $(V \otimes_k K, \theta_K)$ where $\theta_K(v_1 \otimes a_1, \ldots, v_d \otimes a_d) = a_1 \ldots a_d \theta(v_1, \ldots, a_d)$. This means that $(V \otimes_k K, \theta_K)$ is nonsingular. For $i = 1, \ldots, n$, there is a unique (because the extension is transcendental) k-linear derivation D_i on $k(t_1, \ldots, t_n)$ with $D_i(t_j) = \delta_{ij}$, and D_i can be uniquely extended to K. Define $B_i : V \otimes K \to V \otimes K$ by

$B_i(v_1 \otimes a_1 + \ldots + v_n \otimes a_n) = v_1 \otimes D_i(a_1) + \ldots + v_n \otimes D_i(a_n)$.

This is well-defined and independent of the choice of basis.

4.1.2 Lemma: For all $w_1, \ldots, w_d \in V \otimes K$,

$D_i \theta_K(w_1, \ldots, w_d) = \theta_K(B_i(w_1), w_2, \ldots, w_d) + \ldots + \theta_K(w_1, \ldots, w_{d-1}, B_i(w_d))$.

Proof: We show the relation holds on basis elements.

$D_i \theta_K(v_{i_1} \otimes a_{i_1}, \ldots, v_{i_d} \otimes a_{i_d})$

$= D_i(a_{i_1} \ldots a_{i_d}) \theta(v_{i_1}, \ldots, v_{i_d})$

$= D_i(a_{i_1})a_{i_2}\dots a_{i_d}\theta(v_{i_1},\dots,v_{i_d}) + a_{i_1}D_i(a_{i_2})a_{i_3}\dots a_{i_d}\theta(v_{i_1},\dots,v_{i_d}) +$

$\dots + a_{i_1}\dots a_{i_{d-1}}D_i(a_{i_d})\theta(v_{i_1},\dots,v_{i_d})$

$= \theta_K(D_i(a_{i_1})\otimes v_{i_1}, a_{i_2}\otimes v_{i_2},\dots,a_{i_d}\otimes v_{i_d}) +$

$\dots + \theta_K(a_{i_1}\otimes v_{i_1},\dots,a_{i_{d-1}}\otimes v_{i_{d-1}}, D_i(a_{i_d})\otimes v_{i_d})$

$= \theta_K(B_i(a_{i_1}\otimes v_{i_1}), a_{i_2}\otimes v_{i_2},\dots,a_{i_d}\otimes v_{i_d}) +$

$\dots + \theta_K(a_{i_1}\otimes v_{i_1},\dots,a_{i_{d-1}}\otimes v_{i_{d-1}}, B_i(a_{i_d}\otimes v_{i_d}))$. □

Now put $z = v_1\otimes t_1 + \dots + v_n\otimes t_n$. Then $B_i(z) = v_i\otimes 1$, since $D_i(t_j) = \delta_{ij}$. For $r = 2,\dots,d$, denote by $\theta_K^{(z^{d-r})}$ the form of degree r obtained by substituting z in the first $d-r$ coordinates in θ_K: $\theta_K^{(z^{d-r})}(w_1,\dots,w_r) = \theta_K(z,\dots,z,w_1,\dots,w_r)$.

4.1.3 Lemma: Suppose that $\theta_K^{(z^{d-r})}$ is nonsingular for some $r \geq 3$. Then $\theta_K^{(z^{d-r+1})}$ is nonsingular (of degree $r-1$).

Proof: Assume $\theta_K^{(z^{d-r+1})}(u,\dots,u,w) = 0$, i.e. $\theta_K(\underbrace{z,\dots,z}_{d-r+1},\underbrace{u,\dots,u}_{r-2},w) = 0$ for all w. (We show $u=0$.) In particular, $\theta_K(z,\dots,z,\underbrace{u,\dots,u}_{r-1}) = 0$. For all $i = 1,\dots,n$,

$0 = D_i(0) = D_i(\theta_K(z,\dots,z,\underbrace{u,\dots,u}_{r-1})) =$

$(d-r+1)\theta_K(B_i(z),z,\dots,z,\underbrace{u,\dots,u}_{r-1}) + (r-1)\theta_K(z,\dots,z,B_i(u),\underbrace{u,\dots,u}_{r-2})$. (By Lemma 4.1.2 and symmetry.) The last term is zero by hypothesis, so

$\theta_K(B_i(z),\underbrace{z,\dots,z}_{d-r},\underbrace{u,\dots,u}_{r-1}) = 0$ for all $i = 1,\dots,n$ ($char\, F = 0$).

This gives $\theta_K(v_i\otimes 1,\underbrace{z,\dots,z}_{d-r},\underbrace{u,\dots,u}_{r-1}) = 0$ for all i, i.e. $\theta_K(y,\underbrace{z,\dots,z}_{d-r},\underbrace{u,\dots,u}_{r-1}) = 0$ for all y, since $\{v_i\otimes 1\}$ is a basis. This means $\theta_K^{(z^{d-r})}(y,\underbrace{u,\dots,u}_{r-1}) = 0$ for all y, so, by assumption, $u = 0$. □

Since $\theta_K = \theta_K^{(z^0)}$ is nonsingular, we can use Lemma 2 successively with $r = d,\dots,3$ and obtain $\theta_K^{(z^{d-2})}$ is nonsingular/nondegenerate (quadratic).

Finally, we show that this implies that $H(\theta) \neq 0$:

$\theta_K^{(z^{d-2})}$ is nondegenerate iff $\theta_K^{(z^{d-2})}(u,w) = 0$ for all $w \in V \otimes K$ implies $u = 0$ iff the homogeneous system $\theta_K^{(z^{d-2})}(u, v_j \otimes 1) = 0, j = 1, \ldots, n$ has unique solution $(u = 0)$ iff the homogeneous system $\theta_K^{(z^{d-2})}(\sum v_i \otimes a_i, v_j \otimes 1) = 0, j = 1, \ldots, n$ has unique solution $a_1 = \ldots = a_n = 0$ (put $u = \sum v_i \otimes a_i$) iff the homogeneous system $\sum_i a_i \theta_K^{(z^{d-2})}(v_i \otimes 1, v_j \otimes 1) = 0, j = 1, \ldots, n$ has unique solution $a_1 = \ldots = a_n = 0$ iff the system of n linear equations in n unknowns $a_1, \ldots, a_n$, $\alpha_{1j}a_1 + \ldots + \alpha_{nj}a_n = 0, j = 1, \ldots, n$ has unique solution $a_1 = \ldots = a_n = 0$. Putting $\alpha_{ij} = \theta_K^{(z^{d-2})}(v_i \otimes 1, v_j \otimes 1)$, this means $det(\alpha_{ij}) \neq 0$. Now

$$\begin{aligned}
&\theta_K^{(z^{d-2})}(v_i \otimes 1, v_j \otimes 1) \\
&= \theta_K(z, \ldots, z, v_i \otimes 1, v_j \otimes 1) \\
&= \theta_K(v_k \otimes t_k, \ldots, v_k \otimes t_k, v_i \otimes 1, v_j \otimes 1) \\
&= \sum_{1 \leq k_1, \ldots, k_{d-2} \leq n} t_{k_1} \ldots t_{k_{d-2}} \theta_K(v_{k_1} \otimes 1, \ldots, v_{k_{d-2}} \otimes 1, v_i \otimes 1, v_j \otimes 1) \\
&= \sum t_{k_1} \ldots t_{k_{d-2}} \theta(v_{k_1}, \ldots, v_{k_{d-2}}, v_i, v_j).
\end{aligned}$$

Thus each entry in $det(\theta_K^{(z^{d-2})}(v_i \otimes 1, v_j \otimes 1))$ is a homogeneous polynomial of degree $d-2$ in the transcendentals $t_1, \ldots, t_n$, so the (nonzero) determinant is a homogeneous polynomial $g(t_1, \ldots, t_n)$ of degree $n(d-2)$. Since
$g(t_1, \ldots, t_n) = det(\sum_{1 \leq k_1, \ldots, k_{d-2} \leq n} t_{k_1} \ldots t_{k_{d-2}} \theta(v_{k_1}, \ldots, v_{k_{d-2}}, v_i, v_j)) \neq 0$, there exist $s_1, \ldots, s_n \in k$ with $g(s_1, \ldots, s_n) \neq 0$, i.e.
$det(\sum_{1 \leq k_1, \ldots, k_{d-2} \leq n} s_{k_1} \ldots s_{k_{d-2}} \theta(v_{k_1}, \ldots, v_{k_{d-2}}, v_i, v_j)) \neq 0$, i.e.
$det(\theta(\sum_{k_1} s_{k_1} v_{k_1}, \ldots, \sum_{k_{d-2}} s_{k_{d-2}} v_{k_{d-2}}, v_i, v_j)) \neq 0$. Put $u = \sum s_k v_k$.
Then $det(\theta(u, \ldots, u, v_i, v_j)) \neq 0$, i.e. $\theta^{(u^{d-2})}$ is nondegenerate (quadratic). We have seen earlier (§1) that this implies that θ has nonzero Hessian. □

We conclude this section by discussing other connexions between nondegeneracy, nonzero Hessian and nonsingularity.
We have seen earlier in this section that nonsingularity⇒ nonzero Hessian⇒ nondegeneracy in general. We now consider the converses.

For arbitrary one-dimensional forms and arbitrary quadratic forms, it is obvious that nonsingularity$\Leftrightarrow$ nonzero Hessian$\Leftrightarrow$ nondegeneracy.

The simplest counterexample to the first equivalence is the binary cubic form $f = xy^2$, which clearly has nonzero Hessian but is singular.

For arbitrary binary forms (see [DIC1] Ex 5 p7), as well as ternary cubics (first shown by Poincaré in a series of 4 papers in J. école polyt. and Comptes Rendus (1880-2); see [DIC2] p260), it is known that nonzero Hessian$\Leftrightarrow$ nondegeneracy. We now discuss counterexamples to this. In response to a comment of Harrison's, Keet shows that the symmetric hyperbolics $H = V \oplus Sym_d V$ are all nondegenerate, but, for $n \geq 2$ and $d \geq 2$, all have zero Hessian ([K1] Ch 2 §1.12 p30). Keet's minimal quartic example is thus of dimension 6. Here is a "smaller" non-hyperbolic example, viz. a quartic of dimension 5: $f = x_1^3x_3 + x_1^2x_2x_4 + x_2^3x_5$. (Direct calculation shows that $H(f) = 0$; it is easy to see that the homomorphism $S^3(V) \to V^*$ given by $v_1 \odot v_2 \odot v_3 \mapsto f(v_1, v_2, v_3, -)$ is onto, i.e. f is nondegenerate.)

4.2 Families of Nondegeneracy-type Conditions

In their treatment of Witt Rings of higher degree forms, Harrison and Pareigis introduce the notion of an *s-radical*, and use it to reduce the degree of symmetric spaces ([HP] §2 pp1288 et seq.). We use this to define a family of conditions we call *s-nondegeneracy* (a term used by Harrison and Pareigis for a different concept), which is related to ordinary nondegeneracy as well as to a special condition we discuss in §4.3.

s-nondegeneracy: If f is a symmetric form of degree d, the induced homomorphism $f_1 : V \to S^{d-1}(V)^*$ (§1 p96) is generalized as follows, assuming $charF \not| d!$ ([HP] p1288): For $1 \leq s < d$, f induces a homomorphism $f_s : S^s(V) \to S^{d-s}(V)^*$, with $f_s(v_1 \odot \ldots \odot v_s)(v_{s+1} \odot \ldots \odot v_d) = f(v_1, \ldots, v_d)$. ($f_s$ is a polarization-type map; notice that $f_s(v_1 \odot \cdots \odot v_s)$ is the polarization of $ap_s(f)(v_1 \odot \cdots \odot v_s)$ — see p107.) Define the s-radical of f, $s\text{-}rad(f) = ker(f_s)$. ($s\text{-}rad(f) \neq 0$ if $2s > d$ because of dimension.) Clearly, $1\text{-}rad(f) = rad(f)$ is the subspace of V for whose elements the derivative $f^{(v)}$ vanishes; in general, *s-rad*(f) is the subspace of S^sV for whose elements the derivatives of order s vanish. (We return to this later.)

We shall call f *s-nondegenerate* if *s-rad*(f) $= 0$, i.e. f_s is injective. (Note that Harrison and Pareigis use this term for another notion — see below.)
Clearly, for f to be s-nondegenerate we must have $s \leq d - s$, i.e. $2s \leq d$ (for dimension reasons).
For $s \geq 2$ we have s-nondegeneracy$\Rightarrow$ $(s-1)$-nondegeneracy. This gives a chain of conditions d-nondegeneracy$\Rightarrow$ $(d-1)$-nondegeneracy$\Rightarrow$ $\ldots$ $\Rightarrow$2-nondegeneracy$\Rightarrow$ 1-nondegeneracy($\equiv$ nondegeneracy).
1-nondegeneracy is just ordinary nondegeneracy, as used by Harrison ([H1]), O'Ryan ([ORY]) and others. If d is even, we get a condition (for $s = \frac{d}{2}$) which Dolgachev and Kanev just call "nondegeneracy" ([DK] Definition (2.8), p226). We shall return to this later.

Harrison and Pareigis ([HP] §4 p1302 et seq.) also introduce a notion they call s-nondegeneracy, but which we prefer to call *s-regularity*, and use it to study maximal symmetric spaces. They obtain a stronger version (see Lemma 4.1 p1303) of Keet's result that, if (V, θ) is nonsingular indecomposable, then $Z(\theta)$ is a field ([K1] §2.7 p 38).

s-regularity: If $1 \leq s \leq d$, an element $v \in V$ is called a $(d-s+1)$-*zero* if $f(v, \ldots, v, v_{s+1}, \ldots, v_d) = 0$ for all $v_{s+1}, \ldots, v_d \in V$. This is equivalent to $v^s = v \otimes \ldots \otimes v \in$ *s-rad*(f). In particular, $v \in$ *rad*(f) iff v is a d-zero ($s = 1$). We call f *s-regular* if it has only trivial $(d-s+1)$-zeroes, i.e. if $v^s \in$ *s-rad*(f) implies $v = 0$, or, equivalently, $f(v, \ldots, v, v_{s+1}, \ldots, v_d) = 0$ for all $v_{s+1}, \ldots, v_d \in V$ implies $v = 0$. We follow here the terminology of O'Ryan ([ORY] Definition 1.5 p969) and Kanzaki and Watanabe ([KW] p224). Harrison and Pareigis ([HP] p1303) use the term s-nondegenerate, but we prefer to reserve the latter term for the notion described earlier. It is clear that s-regularity implies t-regularity for $t < s$ (put $v_{t+1} = \ldots = v_s = v$); that 1-regularity is just nondegeneracy; and that $(d-1)$-regularity is nonsingularity. If we extend the definition of *(an)isotropy* for bilinear forms ([MH] p55), then d-regularity means anisotropy.

So we get a chain of conditions: $(d-1)$-regularity ($\equiv$nonsingularity)$\Rightarrow$ $(d-2)$-regularity$\Rightarrow \ldots \Rightarrow$ 2-regularity$\Rightarrow$ 1-regularity ($\equiv$nondegeneracy).

We can also see that s-nondegeneracy$\Rightarrow$ s-regularity for $1 \leq s \leq d$.

We now show that none of the converses in the chain of s-regularity conditions holds in general.

Using the notion of the k^{th} mixed polar of a polynomial (for details, see §4.3 p106), it is easy to see that a polynomial f is s-regular if and only if all its partial derivatives of order s have only trivial zeroes.

Simple calculation of the derivatives then shows that:

1. The form $f = x^{d-r}y^r$ is r-regular but not $(r+1)$-regular for $1 \leq r \leq \frac{d}{2}$. (The next example does not have this restriction on r.) Hence the hyperbolic form $f = x^{d-1}y$ is 1-regular, i.e. nondegenerate, but not 2-regular. In fact, its Hessian is nonzero (in contrast to hyperbolic forms in higher dimension), so nonzero Hessian does not imply 2-regularity.

2. The form $f = x^d + x^r y^{d-r}$ is $(d-r)$-regular but not $(d-r+1)$-regular, for $2 \leq r \leq d-1$.

4.3 A condition on forms of even degree

In the case of a form f of even degree $d = 2k$, the condition we have called k-nondegeneracy (§4.2 p101) has a particularly nice interpretation. For then $f_k : S^k(V) \to S^k(V)^*$ is injective so, by duality, we have $S^k(V) \overset{f_k}{\cong} S^k(V)^*$.

In this section we discuss how the *catalecticant* invariant tests for this condition in the same way that the discriminant tests for nonsingularity. First we review classical accounts of the catalecticant. We give explicit and symbolic expressions in a few cases; we also give the latter in modern notation, and show that the catalecticant is not additive. There appears to be no general definition of the catalecticant in this literature, only a description in terms of partial derivatives.

Next we list parts of polarity theory required to give the general definition of [DK] for the catalecticant of arbitrary even degree forms, and show how this extends the cases given classically.

We then discuss what we call *strong nondegeneracy* for forms of even degree, and show explicitly how this is related to the induced bilinear form.

We conclude by mentioning certain connexions to the conditions discussed earlier, and listing a few unresolved questions in this regard.

The catalecticant

The catalecticant, usually denoted $J(f)$, was first defined by Sylvester, and we begin by reviewing the fairly scattered material on the subject in the older literature. Our sources are Elliott ([ELL] pp267-8, 293-5), Grace and Young ([GY] pp122, 231-2, 313) and Salmon ([SAL] p265).

The catalecticant is described in the cases where f is a binary form of arbitrary even degree, or a quartic form up to degree 5. It is shown that a binary form f of degree $2k$ is a sum of k $2k$'th powers if and only if $J(f) = 0$, and that a ternary (resp. quaternary, quinary) quartic f is a sum of 5 (resp. $9, 14$) fourth powers iff $J(f) = 0$.

We now give explicit expressions for $J(f)$ in the cases where f is binary of arbitrary degree $d = 2k$, or ternary quartic:

1. If $f = a_0x^d + \binom{d}{1}a_1x^{d-1}y + \ldots + a_dy^d$, then $J(f)$ is the determinant

$$\begin{vmatrix} a_0 & a_1 & \cdot & \cdot & a_k \\ a_1 & a_2 & \cdot & \cdot & a_{k+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_k & \cdot & \cdot & \cdot & a_d \end{vmatrix}.$$

(See [DK] Example (2.6) p225, [ELL] p268.)

When $d = 4$, this gives $J(f) = \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} = a_0a_2a_4 - a_0a_3^2 - a_1^2a_4 + 2a_1a_2a_3 - a_2^3.$

When $d = 6$, then $J(f) = \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & . & . & . \\ . & . & . & . \\ a_3 & a_4 & a_5 & a_6 \end{vmatrix}.$

2. Ternary quartic ([ELL] p294): Let $f = \sum_{\underline{i}} \binom{4}{\underline{i}} a_{\underline{i}} x^{\underline{i}}$, where $\underline{i} = (i_1, i_2, i_3)$ satisfies $i_1 + i_2 + i_3 = 4$ and $x^{\underline{i}} = x_1^{i_1} x_2^{i_2} x_3^{i_3}$.
Writing $f = a_{(400)}x^{(400)} + 4a_{(310)}x^{(310)} + \ldots + a_{(004)}x^{(004)}$, we have

$$J(f) = \begin{vmatrix} a_{400} & a_{310} & a_{301} & a_{220} & a_{211} & a_{202} \\ a_{310} & a_{220} & a_{211} & a_{130} & a_{121} & a_{112} \\ a_{301} & a_{211} & a_{202} & a_{121} & a_{112} & a_{103} \\ a_{220} & a_{130} & a_{121} & a_{040} & a_{031} & a_{022} \\ a_{211} & a_{121} & a_{112} & a_{031} & a_{022} & a_{013} \\ a_{202} & a_{112} & a_{103} & a_{022} & a_{013} & a_{004} \end{vmatrix}.$$

(This example is also discussed by Dolgachev and Kanev ([DK] Example (2.7) pp225-6) using a different notation.)

Classically, the catalecticant (as all other invariants or covariants), is given by symbolic expressions.
For the binary quartic it is $\frac{1}{6}(ab)^2(bc)^2(ca)^2$ ([GY] p122); for the binary sextic it is $(bc)^2(ca)^2(ab)^2(ad)^2(bd)^2(cd)^2$ ([GY] p232); and for the ternary quartic it is $J(f) = \frac{1}{4!}(abc)^2(bcd)^2(def)^2(efa)^2$.
These can be written in the more modern notation of [GRS] (pp27, 48), but become unwieldy for higher degrees and dimensions. For the binary and ternary quartics,

the expressions are, respectively

$$\left\langle U, Tab\begin{pmatrix} a_1 & a_2 & 1 & 2 \\ a_1 & a_2 & 1 & 2 \\ a_2 & a_3 & 1 & 2 \\ a_2 & a_3 & 1 & 2 \\ a_3 & a_1 & 1 & 2 \\ a_3 & a_1 & 1 & 2 \end{pmatrix}\right\rangle \; ; \; \left\langle U, Tab\begin{pmatrix} a_1 & a_2 & a_3 & 1 & 2 & 3 \\ a_1 & a_2 & a_3 & 1 & 2 & 3 \\ a_2 & a_3 & a_4 & 1 & 2 & 3 \\ a_2 & a_3 & a_4 & 1 & 2 & 3 \\ a_4 & a_5 & a_6 & 1 & 2 & 3 \\ a_4 & a_5 & a_6 & 1 & 2 & 3 \\ a_5 & a_6 & a_1 & 1 & 2 & 3 \\ a_5 & a_6 & a_1 & 1 & 2 & 3 \end{pmatrix}\right\rangle.$$

The closest thing to a manageable general definition of the catalecticant in the older literature is its description (for the cases dealt with) as the determinant of the coefficients of the k'th order partial derivatives of the form. This is what we use to show that the [DK] definition does extend the classical notion.

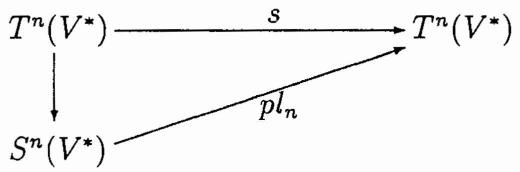
We conclude with the following:

4.3.1 Remark: The catalecticant is not additive, in the sense used by Keet ([K1] Ch 3, 2.1 Definition p53). Take $g = h = x^2y^2$, so that $J(g) = J(h) = -1$ (since $a_2 = 1$ and all other $a_i = 0$ — see p104). Put $f = g \perp h = x^2y^2 + z^2w^2$. Then $J(g)J(h) = 1$, but we can see that $J(f) = 0$: for example, $\frac{\partial^2 f}{\partial x \partial z} = 0$, so the (10×10) determinant of coefficients of the second order partial derivatives of f is zero.

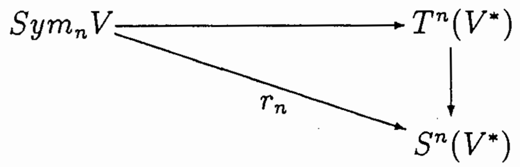
Polarity: A brief review

We summarize parts of the theory of polarity and a modern approach to the catalecticant, as expounded by Dolgachev and Kanev ([DK] §§1,2 pp219-227).

Let V be a vector space of dimension $r+1$, V^* its dual. The symmetrization map $s : T^n(V^*) \to T^n(V^*)$ given by $t \mapsto \sum_{\sigma \in S_n} \sigma(t)$ factors through $S^n(V^*)$ and defines the *polarization map* $pl_n : S^n(V^*) \to T^n(V^*) \cong T^n(V)^*$. Its image equals $Sym_n V$, the space of symmetric n-linear forms on V, which is naturally isomorphic to $S^n(V)^*$.



If we restrict the projection $T^n(V^*) \to S^n(V^*)$ to the subspace $Sym_n V$, we obtain the *restitution map* $r_n : S^n(V)^* \cong Sym_n V \to S^n(V^*)$.



It is easy to see that both $pl_n \circ r_n$ and $r_n \circ pl_n$ are $n!$ times the identity. So, if $char F = 0$, we have $S^n(V^*) \cong S^n(V)^*$. (Re-define pl_n by multiplying by $\frac{1}{n!}$.)

If we choose a basis $u_0, \ldots, u_r$ for V and $x_0, \ldots, x_r$ for V^*, then we can identify $S^n(V)$ (resp. $S^n(V^*)$) with the space of homogeneous polynomials of degree n in $u_0, \ldots, u_r$ (resp. $x_0, \ldots, x_r$). Then a basis of $S^n(V)$ consists of all monomials $u_0^{i_0} \ldots u_r^{i_r}$, where $i_0 + \ldots + i_r = n$, denoted $u^{\underline{i}}$ for short, and likewise for $S^n(V^*)$.

The *polarization* of a polynomial $F \in S^n(V^*)$ is the unique symmetric multilinear function $\tilde{F}$ on V^n such that, for all $x \in V$, $F(x) = \tilde{F}(x, \ldots, x)$.

The k^{th} *mixed polar* of F with respect to $a_1, \ldots, a_k \in V$ is

$P_{a_1 \ldots a_k}(F)(x) = \tilde{F}(a_1, \ldots, a_k, x, \ldots, x)$.

This is clearly symmetric in the a_i; $P_{a^i}(F)$ has the obvious meaning. Clearly $P_{a_1 \ldots a_k}(F) = P_{a_1}(P_{a_2}(\ldots(P_{a_k}(F))\ldots))$. It is easy to see that the *first polar* of F with respect to $a = a_0 u_0 + \ldots + a_r u_r \in V$ is $P_a(F) = \frac{1}{n}\sum_{i=0}^{r} a_i \frac{\partial F}{\partial x_i}$.

Hence $P_{a_1 \ldots a_k}(F) = \frac{(n-k)!}{n!} \sum_{0 \leq i_1, \ldots, i_k \leq r} a_{1 i_1} \ldots a_{k i_k} \frac{\partial^k F}{\partial x_{i_1} \ldots x_{i_k}}$. (1)

In coordinate-free terms, the k^{th} *polarization map* $pl_{k,n} : T^k(V) \otimes S^n(V^*) \to S^{n-k}(V^*)$ is obtained as follows:

$S^n(V^*) \overset{pl_n}{\to} Sym_n V \hookrightarrow T^k(V^*) \otimes Sym_{n-k} V \overset{1\otimes r_{n-k}}{\longrightarrow} T^k(V^*) \otimes S^{n-k}(V^*)$,

and then tensoring on the left by $T^k(V)$.

Then $pl_{k,n}(a_1 \otimes \ldots \otimes a_k \otimes F) = P_{a_1 \ldots a_k}(F)$.

When $k = n$, we get the polarization map, $pl_n : S^n(V^*) \to T^n(V^*)$.

If we compose $pl_k^* \otimes 1 : S^k(V) \otimes S^n(V^*) \to T^k(V) \otimes S^n(V^*)$ with $pl_{k,n}$, we obtain $spl_{k,n} : S^k(V) \otimes S^n(V^*) \to S^{n-k}(V^*)$. ($pl_k^*$ is the obvious map obtained by replacing V^* by V in pl_k.)

The *polar of F with respect to* Φ, $P_\Phi(F) = spl_{k,n}(\Phi, F) \in S^{n-k}(V^*)$, where $\Phi \in S^k(V), F \in S^n(V^*)$. When $\Phi = a_1 \ldots a_k = (\sum a_{1i}u_i) \ldots (\sum a_{ki}u_i)$ is a product of linear polynomials, then $P_\Phi(F) = P_{a_1 \ldots a_k}(F)$. (2)

The *apolarity pairing* is $spl_{n,n} : S^n(V) \otimes S^n(V^*) \to S^0(V) \cong F$.

Put $spl_{n,n}(\Phi, F) = \langle \Phi, F \rangle$. The apolarity pairing can be viewed as the map $S^n(V^*) \to S^n(V)^*$, which is just the polarization map. In particular, it is nondegenerate.

Explicitly, if $\Phi = u^{\underline{i}} = u_0^{i_0} \ldots u_r^{i_r}, F = x^{\underline{j}} = x_0^{j_0} \ldots x_r^{j_r}$ are monomials of degree n, then

$$\langle u^{\underline{i}}, x^{\underline{j}} \rangle = \left\{ \begin{array}{cc} \binom{n}{\underline{i}}^{-1} & \text{if } \underline{i} = \underline{j} \\ 0 & \text{otherwise} \end{array} \right\} \quad (3)$$

(where $\binom{n}{\underline{i}} = \frac{n!}{i_0! \ldots i_r!}$).

4.3.2 Proposition([DK] Proposition (1.7) pp222-3): Let $\Phi \in S^k(V), \Phi' \in S^{n-k}(V), F \in S^n(V^*)$. Then $\langle \Phi', P_\Phi(F) \rangle = \langle \Phi\Phi', F \rangle$.

Let $F \in S^n(V^*)$ and define $ap_k(F) : S^k(V) \to S^{n-k}(V^*)$ by $ap_k(F)(\Phi) = P_\Phi(F)$. The matrix $Cat(F)$ of the linear map $ap_k(F)$ with respect to the bases $u^{\underline{i}}$ of $S^k(V)$ and $x^{\underline{j}}$ of $S^{n-k}(V^*)$ (ordered in some way) is called the k^{th} *catalecticant matrix of F*. If $n = 2k$, $det(Cat(F))$ is called the *catalecticant of F* and denoted $C(F)$.

We give an explicit description of $Cat(F)$: Let $\Phi = u^{\underline{j}} = u_0^{j_0}\dots u_r^{j_r}$ (where $j_0 + \dots + j_r = k$) be a generator of $S^k(V)$, and the set of all $x_{\underline{i}} = x_0^{i_0}\dots x_r^{i_r}$ (where $i_0 + \dots + i_r = n-k$) be a basis for $S^{n-k}(V^*)$.
Suppose $ap_k(F)(u^{\underline{j}}) = P_\Phi(F) = \sum_{\underline{i}} \binom{n-k}{\underline{i}} c_{\underline{i}\underline{j}} x^{\underline{i}}$, so that $Cat(F) = (c_{\underline{i}\underline{j}})$.
By Proposition 4.3.2, putting $\Phi' = u^{\underline{k}}$ and $\Phi = u^{\underline{j}}$, we have

$$\begin{aligned}
\langle u^{\underline{k}+\underline{j}}, F\rangle &= \langle u^{\underline{k}} u^{\underline{j}}, F\rangle \\
&= \langle u^{\underline{k}}, P_\Phi(F)\rangle \\
&= \langle u^{\underline{k}}, \sum_{\underline{i}} \binom{n-k}{\underline{i}} c_{\underline{i}\underline{j}} x^{\underline{i}}\rangle \\
&= \sum_{\underline{i}} \binom{n-k}{\underline{i}} c_{\underline{i}\underline{j}} \langle u^{\underline{k}}, x^{\underline{i}}\rangle \\
&= \left\{ \begin{array}{cc} c_{\underline{i}\underline{j}} & \text{if } \underline{k} = \underline{i} \\ 0 & \text{otherwise} \end{array} \right\} \quad \text{(see (3) p107).}
\end{aligned}$$

So if $F = \sum_{\underline{r}} \binom{n}{\underline{r}} a_{\underline{r}} x^{\underline{r}}$, then $c_{\underline{i}\underline{j}} = \langle u^{\underline{i}+\underline{j}}, F\rangle = \sum_{\underline{r}} \binom{n}{\underline{r}} a_{\underline{r}} \langle u^{\underline{i}+\underline{j}}, x^{\underline{r}}\rangle = a_{\underline{i}+\underline{j}}$ (by (3)), i.e. $Cat(F) = (a_{\underline{i}+\underline{j}})$. ($\underline{i}+\underline{j}$ obviously means $(i_0 + j_0, \dots, i_r + j_r)$.)

We are now in a position to demonstrate that the above definition does indeed specialize to the cases given explicitly in the older literature.
If $\Phi = u_0^{i_0}\dots u_r^{i_r}$ (where $i_0 + \dots + i_r = k$), then

$$\begin{aligned}
ap_k(F)(\Phi) &= P_\Phi(F) \\
&= P_{u_0^{i_0}\dots u_r^{i_r}}(F) \quad \text{(by (2) p107)} \\
&= \frac{k!}{(2k)!} \frac{\partial^k F}{\partial^{i_0} x_0 \dots \partial^{i_r} x_r} \quad \text{(by (1) p106).}
\end{aligned}$$

But $Cat(F)$ is the matrix of $ap_k(F)$ with respect to the bases $u^{\underline{i}}$ and $x^{\underline{j}}$, so it is clear that $Cat(F)$ also gives the coefficient matrix of the k^{th} order partial derivatives of F with respect to the monomials $x^{\underline{j}}$. Hence $C(F) = det(Cat(F))$ generalizes the notion of the catalecticant, $J(F)$, classically defined for binary forms of arbitrary even degree and quartics up to degree 5.

Strong nondegeneracy

Any symmetric form f of degree $2k$ on V induces a symmetric bilinear form on $S^k(V)$ as follows: $b_f(u_1 \odot \ldots \odot u_k, v_1 \odot \ldots \odot v_k) = f(u_1, \ldots, u_k, v_1, \ldots, v_k)$. We call f *strongly nondegenerate* if b_f is nondegenerate. Clearly b_f is nondegenerate if and only if $f_k : S^k(V) \to S^k(V)^*$ is an isomorphism, i.e. f is k-nondegenerate. But also b_f is nondegenerate if and only if $disc(b_f) \neq 0$. We now describe the matrix M_{b_f} of b_f: If $\{u^{\underline{i}}\}$ is a basis of $S^k(V)$ as before, then $b_f(u^{\underline{i}}, u^{\underline{j}}) = f(u^{\underline{i}} u^{\underline{j}}) = f(u^{\underline{i}+\underline{j}}) = c_{\underline{ij}}$ (p108). Hence $M_{b_f} = (c_{\underline{ij}}) = Cat(f)$. Thus $disc(b_f) = J(f)$, and we conclude that f is strongly nondegenerate if and only if $J(f) \neq 0$.

4.3.3 Note: Dolgachev and Kanev call this condition nondegeneracy ([DK] Definition (2.8) p226). We have used the convention of Harrison and others, whose notion of nondegeneracy extends the classical one to arbitrary degree in a very natural way. Although the catalecticant is a general discriminant in the sense of Gelfand et al. ([GKZ] pp14-16), the non-additivity of the catalecticant (see Remark 4.3.1 p105) could be considered as reason for not viewing it as the true generalization of the discriminant of quadratic forms, as Dolgachev and Kanev do.

We conclude this section by giving explicit calculations which illustrate the above results and observations.

Let f be a binary quartic form on a space V with basis e_1, e_2 and dual basis x_1, x_2. Write

$$\begin{aligned} f &= \sum_{1 \leq i,j,k,l \leq 2} a_{ijkl} x_i x_j x_k x_l \\ &= a_{1111} x_1^4 + 4a_{1112} x_1^3 x_2 + 6a_{1122} x_1^2 x_2^2 + 4a_{1222} x_1 x_2^3 + a_{2222} x_2^4, \end{aligned}$$

where $a_{ijkl} = f(e_i \odot e_j \odot e_k \odot e_l)$, for $1 \leq i, j, k, l \leq 2$.

Now f induces a quadratic form $\tilde{f}$ on $S^2(V)$. Choose the basis $g_1 = e_1 \odot e_1, g_2 = e_1 \odot e_2, g_3 = e_2 \odot e_2$ for $S^2(V)$, with dual basis y_1, y_2, y_3.

Then $\tilde{f} = \sum_{1\le i,j\le 3} b_{ij}y_iy_j = b_{11}y_1^2 + 2b_{12}y_1y_2 + 2b_{13}y_1y_3 + b_{22}y_2^2 + 2b_{23}y_2y_3 + b_{33}y_3^2$, where $b_{ij} = \tilde{f}(g_i \odot g_j)$. Hence $b_{11} = \tilde{f}(g_1 \odot g_1) = f(e_1 \odot e_1 \odot e_1 \odot e_1) = a_{1111}, b_{12} = a_{1112}, b_{13} = a_{1122}, b_{22} = a_{1122}, b_{23} = a_{1222}, b_{33} = a_{2222}$.

The matrix of $\tilde{f}$ (relative to the basis $\{g_i\}$) is then

$$M_{\tilde{f}} = \begin{pmatrix} a_{1111} & a_{1112} & a_{1122} \\ a_{1112} & a_{1122} & a_{1222} \\ a_{1122} & a_{1222} & a_{2222} \end{pmatrix}.$$

This illustrates $disc(\tilde{f}) = det M_{\tilde{f}} = J(f)$.

We outline how this extends to an arbitrary quartic: Suppose V has basis $e_1, \ldots, e_n$, and $x_1, \ldots, x_n$ is a dual basis. Write $f = \sum_{1\le i,j,k,l\le n} a_{ijkl}x_ix_jx_kx_l$. Choose the following basis for $S^2(V)$: $g_{rs} = e_r \odot e_s$, where $1 \le r \le s \le n$. For $1 \le p \le q \le n$ and $1 \le r \le s \le n$, put $b_{pq,rs} = \tilde{f}(g_{pq} \odot g_{rs}) = f(e_p \odot e_q \odot e_r \odot e_s) = a_{pqrs}$. Then $\tilde{f} = \sum_{1\le p,q\le n, 1\le r,s\le n} b_{pq,rs}y_{pq}y_{rs}$.

Order the g_{rs} lexicographically: $g_{11}, g_{12}, \ldots, g_{1n}, g_{22}, \ldots, g_{2n}, \ldots, g_{nn}$. Relative to this basis, the matrix of $\tilde{f}$ is $M_{\tilde{f}} = (\tilde{f}(g_{pq} \odot g_{rs})) = (a_{pqrs})$.

The above notation obviously becomes unwieldy in higher degrees, so we discuss the ternary quartic (see [DK] Example (2.7) pp225-6) in a notation which makes the relation between the quartic form, the induced quadratic form, and the catalecticant clear: Let $f = \sum_{\underline{i}} \binom{4}{\underline{i}} a_{\underline{i}} x^{\underline{i}} = a_{(400)}x^{(400)} + 4a_{(310)}x^{(310)} + \ldots$.

A basis for $S^2(V^*)$ consists of $x^{\underline{i}}$, where $i_1 + i_2 + i_3 = 2$. We order it as follows: $x^{(200)}, x^{(020)}, x^{(002)}, x^{(110)}, x^{(101)}, x^{(011)}$.

Then $C(f) = (c_{\underline{i}\underline{j}}) = (a_{\underline{i}+\underline{j}})$, where $c_{(200),(200)} = a_{(200)+(200)} = a_{(400)}, c_{(200),(020)} = a_{(200)+(020)} = a_{(220)}, c_{(200),(002)} = a_{(200)+(002)} = a_{(202)}$, etc.

The induced quadratic form on $S^2(V)$ is $\tilde{f} = \sum_{\underline{i},\underline{j}} a_{\underline{i}+\underline{j}} x^{\underline{i}} x^{\underline{j}}$, where $\underline{i} = (i_1, i_2, i_3)$ satisfies $i_1 + i_2 + i_3 = 2$, and likewise for $\underline{j}$.

Finally, we list some relations between strong nondegeneracy and the earlier conditions:

1. The form $f = x^k y^k$ $(k \geq 2)$ is easily seen to be singular. However, the catalecticant matrix has nonzero entries all along the reverse diagonal, and zeroes elsewhere, so $J(f) \neq 0$. Hence strong nondegeneracy$\not\Rightarrow$ nonsingularity.

2. The form $f = x^d + y^d$ $(d \geq 2)$ is easily seen (by direct calculation) to be nonsingular but not strongly nondegenerate. Thus nonsingularity$\not\Rightarrow$ strong nondegeneracy.

3. The symmetric hyperbolic space of degree $2k$, $H = V \bigoplus Sym_{2k-1}V$ is not 2$-$nondegenerate: if $(0, g_1), (0, g_2)$ are two elements of the canonical basis of H (see [K1] Ch 2 §1.11 pp29-30), then $\psi_2[(0, g_1), (0, g_2)] \equiv 0$. Hence it is not strongly nondegenerate (i.e. not k-nondegenerate).

4. We have noted that $t-$nondegeneracy$\Rightarrow$ $t-$regularity (§2 p102). If degree $d = 2k$, this means strong nondegeneracy$\Rightarrow$ k-regularity.

Open Questions

1. We have seen that nonzero Hessian does not imply 2-regularity in general; can the hypothesis of Proposition 4.1.1 be weakened to show $(d-2)$-regularity implies nonzero Hessian?

2. Does the condition nonzero Hessian fit into the chain of s-nondegeneracy conditions (p101) and, if so, where?

3. The simplest form, as far as we are aware, which is nondegenerate but has zero Hessian is Keet's 5-dimensional hyperbolic cubic. Is there a non-hyperbolic quaternary cubic with this property (as there is a quinary quartic)?

4. Is there a family of forms which is $(s-1)$-nondegenerate but not s-nondegenerate for all s, as there is for s-regularity (p102)?

5. Is there a family of forms which is s-regular but not s-nondegenerate for all s?

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