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Amalgamation in Varieties of Algebras

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Contents

Synopsis

One of the most successful approaches to research in universal algebra has been the study of varieties, initiated by Garett Birkhoff in the 1930's. Examples of varieties include many classes of algebras such as groups, semigroups, lattices and Boolean algebras. In 1927, O. Schreier showed that for any set of extensions of a given group, there is another extension of that group that in some sense contains all other extensions in the set. This property of groups, known as the *amalgamation property* was generalized to a universal-algebraic setting by R. Fraïssè in 1954, and the important question as to which varieties satisfied the amalgamation property arose. While some of the answers to this question were positive (such as for the varieties of lattices, distributive lattices and Boolean algebras), many common varieties such as the variety of semigroups and all non-distributive modular lattice varieties were shown to fail to satisfy the amalgamation property. In the light of these negative results, attempts were made to "localize" this property from the variety to its individual members, the most successful being the notions of *amalgamation base* and *amalgamation class*, first introduced by George Grätzer and Henry Lakser in 1971. Investigations into the nature of the amalgamation classes of varieties that fail to satisfy the amalgamation property were carried out in 1970's and 1980's by among others, Clifford Bergman and Henry Rose, the main focus being congruence distributive varieties, of which lattice varieties form the prime example.

The topic of amalgamation has also been studied in fields as diverse as topology, logic and the theory of field extensions. In this dissertation, however, I will focus on the more algebraic results concerning amalgamation. My aim is to present a selection of these results, using as examples varieties of groups, semigroups, lattices and Heyting algebras, in a universal-algebraic framework that is (more or less) self-contained and uniform in its notation.

In Chapter 1, I present some of the basic results in model theory and universal algebra that are required later in the dissertation. Of note are the concepts of *congruence modularity, congruence distributivity* and *residual smallness* for varieties, which play an important role in classifying results on amalgamation. I also formally introduce the concepts of *amalgamation class* and *amalgamation property*, and present some of the elementary results concerning these notions. While the original proofs of these results were largely model-theoretic in nature, I have opted for a more universal-algebraic approach.

In Chapter 2, I consider what can broadly be termed "extension properties" of varieties, of which the amalgamation property is one example. I introduce the concept of the *strong amalgamation property* (first introduced by Bjarni Jónsson in 1956), and present results showing its connection to the amalgamation property. The bulk of the chapter is devoted to a proof that in a residually small congruence modular variety, the amalgamation property implies the congruence extension property. This result flows out of extensive research in the 1980's by Ralph McKenzie, Clifford Bergman and Keith Kearnes, which focused on generalizing the commutator theory of groups to the universal-algebraic setting of congruence modular varieties.

In Chapter 3, I present results which attempt to answer the question as to which varieties of groups, semigroups, lattices and Heyting algebras satisfy the amalgamation property. Work by numerous researchers has led to a complete answer for lattices, but for the other classes of varieties the answer is less satisfactory. For these classes, I present results which completely classify all residually small varieties, the general problem still remaining (to my knowledge) open.

In Chapter 4, I look at some characterizations of the amalgamation class of those varieties that fail to satisfy the amalgamation property. I look particularly at three concepts: *retract extensibility*, *property P* and *property Q*. The latter two concepts were first defined by Clifford Bergman, while the notion of retract extensibility is implicit in work by Peter Jipsen and Henry Rose, but first explicitly defined in this dissertation. I also present applications by Clifford Bergman and Bjarni Jónsson of these characterizations to varieties of lattices and Heyting algebras.

In Chapter 5, I look at the question as to which varieties have amalgamation classes that can be characterized by first-order sentences. I present work by Clifford Bergman which gives a negative answer for some modular lattice varieties, and a positive answer for some discriminator varieties. Lastly, I present work by Peter Bruyns, Colin Naturman and Henry Rose, which gives a positive answer for the variety of lattices generated by the pentagon (but see note after lemma 5.3.13).

Chapter 1

Introduction and General Results

Introduction. Amalgamation is the process of combining two or more structures with something in common to form one large structure that preserves all the features of the smaller ones. In the study of classes of algebras, amalgamation has a very specific meaning: we say a class of algebras has the amalgamation property (defined more rigorously later) if for every algebra in that class, and any two extensions of that algebra, there is a third extension of that algebra that "contains" the other two algebras in some suitable way.

As with many universal-algebraic properties, the amalgamation property had its beginnings in group theory: the work of Schreier [27] and the subsequent investigations of B.H. Neumann [54] and H. Neumann [67] show that the variety of all groups has the amalgamation property. The first discussion of the amalgamation property in a general universal-algebraic setting appears in Fraïssé [54]. Many well-known varieties (such as lattices and Boolean algebras) were also shown to satisfy the amalgamation property, and the question arose whether this property was just a consequence of a class of algebras being a variety. This question was answered in the negative by Kimura [57], who showed that the variety of all semigroups does not possess the amalgamation property.

In the light of this and subsequent negative results, attempts were made to "localize" the concept of amalgamation from a class of algebras to its individual members. The most successful of these was the concept of an amalgamation class, due to Grätzer and Lakser [71], where it was used to discuss amalgamation in varieties of pseudocomplemented distributive lattices. The amalgamation class (denoted $Amal(\mathcal{K})$) of some class of algebras $\mathcal{K}$ can be viewed as a subclass of $\mathcal{K}$ consisting of those members of $\mathcal{K}$ which possess the amalgamation property "locally", and we

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX can define a class $\mathcal{K}$ to have the amalgamation property if all its members are in the amalgamation class (that is, $Amal(\mathcal{K}) = \mathcal{K}$).

In this chapter, I introduce the two concepts of the amalgamation property and the amalgamation class. In section 1.1, I introduce some of the basic model-theoretic notions needed for this dissertation, and in section 1.2, some basic universal-algebraic concepts. Because this dissertation presents a broad overview of the topic, any attempt to make the presentation fully self-contained is impossible. Instead, I will refer the reader to a reference for some of the definitions I consider well-known, or where I consider a rigorous definition not essential to an understanding of the proofs presented here. Also, in the first two subsections, proofs are omitted in the interests of space, the reader being referred to the original proof of the result if known to me. Where I have not been able to trace the original proof of the result, proofs may be found in either Bell and Slomson [71] or Chang and Keisler [73] for model-theoretic results, Burris and Sankappanavar [81] or McKenzie, McNulty and Taylor [87] for universal-algebraic results and Crawley and Dilworth [73] for any lattice-theoretic results.

In section 1.3, I introduce the amalgamation concepts described above, and prove results (largely due to Yasuhara [74]) which describe the contents and properties of the amalgamation class. The original statements and proofs in Yasuhara [74] are largely syntactical and model-theoretic in nature. I have opted, however, for a more universal-algebraic approach, using algebraic formulations of the more model-theoretic properties such as lemma 1.3.5.

1.1 An Introduction to Model Theory

Introduction and Basic Concepts. I assume the reader is familiar with the universal-algebraic concepts of an *algebra*, a *subalgebra*, a *homomorphism*, a *direct product* and the *congruence lattice* (denoted by Con(A)) of an algebra A, and with the notion of a *free algebra* for a class $\mathcal{K}$ (see, for example, McKenzie, McNulty and Taylor [87]). I also assume the reader is familiar with the predicate calculus including the notions of a *term* and an *identity*(also called an *equation*), and the model-theoretic notions of a *language*, a *first-order formula* and *sentence*, a *theory* and a *model* for a theory (see, for example, Chang and Keisler [73]). In this work, I will be dealing exclusively with algebras of finitary type, hence the reader may assume the term *algebra* always means an algebra of finitary type, and that *language* always means a countable language (since algebras of finitary type have countable languages). In this subsection, I define some of the other notions I will be using, as well as presenting some of the basic first-order model theory.

DEFINITION 1.1.1.

(1) Given a theory $\mathcal{T}$ of first-order sentences, denote by *Mod* $\mathcal{T}$ the class of all models of the theory $\mathcal{T}$.

(2) We say a class $\mathcal{K}$ of algebras is an *axiomatizable* (or *elementary*) *class*, if $\mathcal{K} = \text{Mod } \mathcal{T}$ for some theory $\mathcal{T}$ of first-order sentences. If $\mathcal{T}$ can be chosen to be finite, we say $\mathcal{K}$ is *finitely axiomatizable* (or *strictly elementary*).

Given a first-order formula ϕ, an algebra A and a countable sequence $x \in A^{\omega}$ in A (called a *realization* in A), I will denote by $A \models_x \phi$ the notion x *satisfies* ϕ *in* A (for a definition of this notion, see Bell and Slomson [71], page 56). If ϕ is in fact a sentence, note that if $A \models_x \phi$ for some realization x, then $A \models_y \phi$ for all realizations $y \in A^{\omega}$, hence we can just write $A \models \phi$.

DEFINITION 1.1.2. Given a language L and some ordinal β, we can define a new language L^{β} by adding to the symbols of L, β new constants, say $\langle c_{\alpha} : \alpha < \beta \rangle$. Models in L^{β} are now just of the form $\langle A, a \rangle$, where A is a model in the language L and $a \in A^{\beta}$ is a sequence in A.

DEFINITION 1.1.3. Let $A, B \in \mathcal{K}$, with $A \subseteq B$. We say B is an *elementary extension of* A, if, for every formula ϕ and any $x \in A^{\omega}$, $B \models_x \phi$ if and only if $A \models_x \phi$. If $f : A \hookrightarrow B$ is an embedding such that B is elementary extension of $f(A)$, then we call f an *elementary embedding*. We write "$A \prec B$" for "B is an elementary extension of A".

THEOREM 1.1.4. *Let $A \subseteq B$, where A and B are algebras in some language L. The following are equivalent:*

(1) *B is an elementary extension of A.*

(2) *For each ordinal β and $a \in A^{\beta}$ a β-termed sequence, if ϕ is a first-order sentence in the language L^{β}, then $\langle A, a \rangle \models \phi$ if and only if $\langle B, a \rangle \models \phi$.* □

One of the important questions in model theory is what size models (if any) a given theory has. One of the first results was that a finitely axiomatizable theory with a model always possesses a countable model (Löwenheim [15]). Work by T. Skolem and A. Tarski added to these results (see for example, Skolem [34]). Henkin [49] also showed that every first-order theory with arbitrary large finite models has an infinite model. These results are summarised below, the statements following the presentation of Vaught [54].

THEOREM 1.1.5.

(1) *For a first-order theory $\mathcal{T}$, if Mod $\mathcal{T}$ contains members of arbitrarily large finite cardinality, then Mod $\mathcal{T}$ contains an infinite member.*

(2) *Let A be an algebra of cardinality $\alpha \geq \aleph_0$, and let C be a subset of A with $|C| = \gamma$. Then if β is a cardinal satisfying $\max\{\gamma, \aleph_0\} \leq \beta \leq \alpha$, then there is an elementary subalgebra B of A such that $|B| = \beta$ and $C \subseteq B$.*

(3) *(The Löwenheim-Skolem-Tarski Theorem) Let $\mathcal{T}$ be a first-order theory with an infinite model. If $\mathcal{T}$ is finite, then Mod $\mathcal{T}$ contains members of arbitrarily large infinite cardinality. If $\mathcal{T}$ is infinite and of cardinality α, then Mod $\mathcal{T}$ contains members of cardinality β for each cardinal $\beta \geq \alpha$.* ☐

The Reduced Product and Ultraproduct Construction. One of the important connections between model theory and universal algebra is provided by the *ultraproduct* (and the more general *reduced product*) construction, first introduced explicitly in model theory by Łoš [55a]. In this subsection, I introduce these constructions, and give some of the results which exhibit this connection between logic and algebra.

DEFINITION 1.1.6.

(1) For an arbitrary set I, we say $\mathfrak{F}$ is a *filter on I* if $\mathfrak{F}$ is a filter in the power set lattice of I (ordered by set inclusion). We call $\mathfrak{F}$ an *ultrafilter on I* if $\mathfrak{F}$ is a maximal (with respect to set-inclusion) proper filter on I.

(2) Given a direct product $A = \prod\limits_{i \in I} A_i$ of a family of algebras $\{A_i : i \in I\}$, and a filter $\mathfrak{F}$ on I, the congruence $\psi \in \mathrm{Con}(A)$ induced by $\mathfrak{F}$ is given by

$$a \;\psi\; b \quad \text{if and only if} \quad \{i \in I : a_i = b_i\} \in \mathfrak{F}$$

where a_i and b_i are the ith co-ordinates of a and b respectively for $i \in I$.

We call the quotient algebra A/ψ the *reduced product of A modulo $\mathfrak{F}$*, and write it as $\prod_{i \in I} A_i/\mathfrak{F}$ or $A/\mathfrak{F}$. If $\mathfrak{F}$ is an ultrafilter, we call the construction the *ultraproduct of A modulo $\mathfrak{F}$*. In the case where the A_i's are all identical (say $A_i = B$ for all $i \in I$), we call the construction a *reduced power* or an *ultrapower* respectively (and denote it by $B^I/\mathfrak{F}$).

THEOREM 1.1.7. *Let $\{A_i : i \in I\}$ be a set of algebras, let $\mathfrak{F}$ be an ultrafilter on I, and let $A = \prod_{i \in I} A_i/\mathfrak{F}$ be the ultraproduct modulo $\mathfrak{F}$.*

(1) *(Loš* [55a]*) If σ is a first-order sentence in the language of the given algebras, then*

$$A \models \sigma \quad \textit{if and only if} \quad \{i \in I : A_i \models \sigma\} \in \mathfrak{F}$$

(2) *(Keisler* [64]*) A class $\mathcal{K}$ is elementary if and only if $\mathcal{K}$ is closed under ultraproducts, and for some elementary class $\mathcal{M}$ containing $\mathcal{K}$, $\mathcal{M} \setminus \mathcal{K}$ is closed under reduced powers.*

(3) *(Frayne, Morel and Scott* [62]*) If $\mathcal{K}$ is a finite set of finite algebras, then every ultraproduct of members of $\mathcal{K}$ is isomorphic to a member of $\mathcal{K}$.* □

Preservation Theory. I assume the reader is familiar with *universal, existential, universal-existential, positive* and *Horn formulae* (see Chang and Keisler [73] for these definitions). Also, we call a elementary class *inductive* if it is closed under unions of chains.

I define the following operators on classes of algebras:

- $\mathbf{I}\mathcal{K}$ – the class of all isomorphic copies of members of $\mathcal{K}$
- $\mathbf{S}\mathcal{K}$ – the class of all subalgebras of members of $\mathcal{K}$
- $\mathbf{E}\mathcal{K}$ – the class of all extensions of members of $\mathcal{K}$
- $\mathbf{H}\mathcal{K}$ – the class of all homomorphic images of members of $\mathcal{K}$
- $\mathbf{P}\mathcal{K}$ – the class of all direct products of members of $\mathcal{K}$
- $\mathbf{P}_S\mathcal{K}$ – the class of all subdirect products of members of $\mathcal{K}$
- $\mathbf{R}\mathcal{K}$ – the class of all reduced products of members of $\mathcal{K}$
- $\mathbf{P}_U\mathcal{K}$ – the class of all ultraproducts of members of $\mathcal{K}$

(Recall B is a *subdirect product* of a family of algebras $\{A_i : i \in I\}$, if B is a subalgebra of the direct product $A = \prod\limits_{i \in I} A_i$, and for each $i \in I$, the projection map $\pi_i : A \twoheadrightarrow A_i$ restricted to B is an epimorphism.)

Another important connection between universal algebra and model theory is preservation theory, which investigates which first-order sentences are preserved under which algebraic class operators. I present some of these results below.

THEOREM 1.1.8. *Let A and B be algebras of the same type.*

(1) *If A is a subalgebra of B, then every universal sentence satisfied by B is satisfied by A, and every existential sentence satisfied by A is satisfied by B.*

(2) *If B is a homomorphic image of A then every positive sentence satisfied by A is satisfied by B.* □

THEOREM 1.1.9. *Let $\mathcal{K}$ be an elementary class.*

(1) *(Łoš* [55b], *Tarski* [54]*) If $\mathbf{S}\mathcal{K} = \mathcal{K}$, then there is a theory $\mathcal{T}$ whose members all are universal sentences such that $\mathcal{K} = Mod\ \mathcal{T}$.*

(2) *(Łoš* [55b], *Tarski* [54]*) If $\mathbf{E}\mathcal{K} = \mathcal{K}$, then there is a theory $\mathcal{T}$ whose members all are existential sentences such that $\mathcal{K} = Mod\ \mathcal{T}$.*

(3) *(Łoš and Suszko* [57], *Chang* [59]*) If $\mathcal{K}$ is inductive, then there is a theory $\mathcal{T}$ whose members all are universal-existential sentences such that $\mathcal{K} = Mod\ \mathcal{T}$.*

(4) *(Lyndon* [59]*) If $\mathbf{H}\mathcal{K} = \mathcal{K}$, then there is a theory $\mathcal{T}$ whose members all are positive sentences such that $\mathcal{K} = Mod\ \mathcal{T}$.*

(5) *(Keisler* [65], *Galvin* [65]*) If $\mathbf{R}\mathcal{K} = \mathcal{K}$, then there is a theory $\mathcal{T}$ whose members all are Horn sentences such that $\mathcal{K} = Mod\ \mathcal{T}$.* □

1.2 An Introduction to Universal Algebra and Varieties

The Concept of a Variety. One of the most important types of elementary class of algebras is that of a *variety* (also called an *equational class*). Many common classes of algebras such as groups, abelian groups, lattices and semigroups are examples of varieties. In this subsection, I define the concept of a variety formally, and present some of the basic results in varieties.

DEFINITION 1.2.1. A class $\mathcal{K}$ of algebras (all of the same type) is said to be a *variety* if there is some set of identities $\mathcal{T}$ in the language of the algebras such that $\mathcal{K} = \text{Mod}\ \mathcal{T}$.

Since the classes of groups, semigroups, lattices and Boolean algebras are all defined by identities, these classes are all examples of varieties.

Given a variety $\mathcal{V}$, the class of all subvarieties (subclasses of $\mathcal{V}$ that are themselves varieties) is in fact a set closed under arbitrary intersections, so it can be viewed as a complete lattice. Given any subclass $\mathcal{K}$ of a variety $\mathcal{V}$, denote by $\mathcal{K}^{\mathcal{V}}$ be the smallest subvariety of $\mathcal{V}$ containing $\mathcal{K}$. If $\mathcal{K}^{\mathcal{V}} = \mathcal{V}$, we say $\mathcal{K}$ *generates* $\mathcal{V}$.

THEOREM 1.2.2.

(1) *(Birkhoff* [35]*) A class $\mathcal{V}$ is a variety if and only if $\mathcal{V} = \mathbf{H}\mathcal{V} = \mathbf{S}\mathcal{V} = P\mathcal{V}$.*

(2) *(Tarski* [46], *Kogalovskiĭ* [65]*) For any subclass $\mathcal{K}$ of $\mathcal{V}$, a variety,*

$$\mathcal{K}^{\mathcal{V}} = \mathbf{HSP}\mathcal{K} = \mathbf{HP}_S\mathcal{K}. \quad \square$$

Congruences and Subdirect Irreducibility. Let A be an algebra, and let $\Theta \in \mathrm{Con}(A)$. For $a, b \in A$ we denote by:

a/Θ – the *congruence class* of a modulo Θ

A/Θ – the *quotient algebra* of A modulo Θ

$\mathbf{1}_A, \mathbf{0}_A$ – the *top* and *bottom* element of $\mathrm{Con}(A)$ respectively

$\mathrm{con}\,(a, b)$ – the *principal congruence* generated by (a, b) (that is, the smallest congruence that contains (a, b))

One important way of classifying algebras is to classify them according to properties of their congruences. In this subsection, I present one such classification, the notion of *subdirect irreducibility*. Subdirectly irreducible algebras are important in the study of varieties, as a variety is in some sense completely determined by its subdirectly irreducible members.

DEFINITION 1.2.3.

(1) An element $a \in L$, where L is a lattice, is said to be *completely meet irreducible* if for any subset B of L such that the meet of B exists and is equal to a, we have $a \in B$. (If the above is true for only finite subsets B of L, we say a is *meet irreducible*).

(2) An algebra A is said to be *subdirectly irreducible* if any of the following equivalent conditions hold:

  (a) When A is a subdirect product of algebras $\{A_i : i \in A\}$, then A is isomorphic to one of the factors A_i.

  (b) $\mathbf{0}_A$ is completely meet irreducible in Con(A).

  (c) There exist elements $a, b \in A$ such that con (a, b) is the smallest element of Con$(A) \setminus \{\mathbf{0}_A\}$. (We call (a, b) a *critical pair* of A, and con (a, b) the *monolith* of A).

We denote the class of subdirectly irreducible algebras of a variety $\mathcal{V}$ by $\mathcal{V}_{SI}$.

(3) An algebra A is said to be *simple* if Con(A) is isomorphic to the two-element chain. Note that simple algebras are subdirectly irreducible.

THEOREM 1.2.4 (Birkhoff [44]). *Let A be an algebra.*

  (1) *For every $a, b \in A$ such that $a \neq b$, there is some $\Theta \in$ Con(A) such that Θ is completely meet irreducible in Con(A) (or equivalently, A/Θ is subdirectly irreducible) and $(a, b) \notin \Theta$.*

  (2) *A is a subdirect product of its subdirectly irreducible epimorphic images.* □

COROLLARY 1.2.5. *For a variety $\mathcal{V}$,*

$$\mathcal{V} = \mathbf{HSP}(\mathcal{V}_{SI}) = \mathbf{HP}_S(\mathcal{V}_{SI}) = (\mathcal{V}_{SI})^{\mathcal{V}}. \quad \square$$

Distributive and Modular Lattices. In this subsection, I present some of the results in lattice theory required for the dissertation, particularly in discussing congruence lattices of algebras.

DEFINITION 1.2.6.

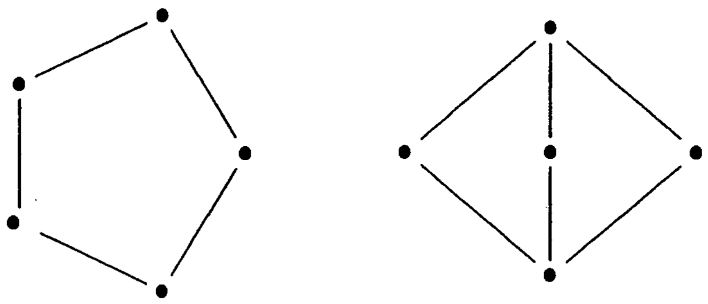
  (1) A lattice is said to be *distributive* if it satisfies the identity

$$a(b + c) = ab + ac.$$

  (2) A lattice is said to be *modular* if it satisfies the identity

$$(ac + b)c = ac + bc.$$

Notice that all distributive lattices are modular. Two important finite lattices are the pentagon N, which is the smallest non-modular lattice, and the diamond M_3, which is the smallest non-distributive modular lattice. Both these lattices have five elements, and their respective orderings are given in figure 1.1.



N M_3

FIGURE 1.1

These lattices can be used to give alternative characterizations of distributivity and modularity.

THEOREM 1.2.7 (Dedekind [00]).

(1) For any lattice L, the following are equivalent:

- (a) *L is modular.*
- (b) *For all $a, b, c \in L$, if $c \leq a$, then $a(b+c) \leq ab + c$.*
- (c) *For any a, b, c, if $a \leq c$, $ab = cb$ and $a + b = c + b$, then $a = c$.*
- (d) *L has no sublattice isomorphic to N.*

(2) For any lattice L, the following are equivalent:

- (a) *L is distributive.*
- (b) *For all $a, b, c \in L$, $a + bc = (a+b)(a+c)$.*
- (c) *For any a, b, c, If $ab = cb$ and $a + b = c + b$, then $a = c$.*
- (d) *L has no sublattice isomorphic to either N or M_3.* □

DEFINITION 1.2.8. Given a lattice L, let $a, b \in L$ with $a \leq b$. Then the set $a/b = \{x \in L : a \leq x \leq b\}$ forms a sublattice of L, called the *interval from a to b*.

The following theorem, also known as *Dedekind's Transposition Principle*, gives a useful result concerning intervals in modular lattices.

THEOREM 1.2.9 (Dedekind [00]). *Let M be a modular lattice, and let $a, b \in M$. Then the maps $\phi : (a+b)/\,b \to a/\,ab$ and $\psi : a/\,ab \to (a+b)/\,b$ are isomorphisms, where $\phi(x) = ax$ for $x \in (a+b)/\,b$ and $\psi(y) = y + b$ for $y \in a/\,ab$.* □

DEFINITION 1.2.10.

(1) Let L be a lattice with top and bottom elements 1 and 0 respectively. Then for $a \in L$, we say $c \in L$ is a *complement of a* if $ac = 0$ and $a + c = 1$.

(2) A lattice with top and bottom element is said to *complemented* if every element has a complement.

(3) A lattice L with top and bottom element is said to *relatively complemented* if every interval in L is complemented.

THEOREM 1.2.11. *Every complemented modular lattice is relatively complemented.* □

DEFINITION 1.2.12.

(1) We define the *length* of a chain C to be the cardinality of the chain with one element removed.

(2) We define the *height* of a lattice L to be the least upper bound (if it exists) of the lengths of all subchains of L. Note that not every lattice has a height.

The following theorem is a consequence of a result in Dilworth [50].

THEOREM 1.2.13. *Every subdirectly irreducible modular lattice of finite height is simple.* □

Lattice Theoretic Properties of the Congruence Lattice. In keeping with classifying algebras by properties of their congruence lattices, I define the notions of *congruence distributivity, congruence modularity* and *congruence permutability*. For a variety, there are connections between identities satisfied in that variety and the congruence lattices of members of the variety (an investigation started by Maltsev [54]). I present some of these connections in theorem 1.2.15. Varieties with congruence distributive algebras are particularly noteworthy, as they have factorable congruences (theorem 1.2.18), their subdirectly irreducible algebras are nicely determined (corollary 1.2.17) and there is a connection between the size of their theories and the sizes of their subdirectly irreducible algebras given by Baker's theorem (theorem 1.2.20).

DEFINITION 1.2.14.

(1) An algebra A is said to be *congruence permutable* if for every $\phi, \psi \in \mathrm{Con}(A)$, we have $\phi \circ \psi = \psi \circ \phi$.

(2) An algebra A is said to be *congruence modular* if $\mathrm{Con}(A)$ is a modular lattice.

(3) An algebra A is said to be *congruence distributive* if $\mathrm{Con}(A)$ is a distributive lattice.

We say a class $\mathcal{K}$ of algebras is *congruence permutable (congruence modular / congruence distributive)* if every member of $\mathcal{K}$ is congruence permutable (congruence modular/congruence distributive).

Any group variety is congruence modular (Dedekind [00]) and any lattice variety is congruence distributive (Funayama and Nakayama [42]).

THEOREM 1.2.15.

(1) *(Maltsev* [54]*) A variety $\mathcal{V}$ is congruence permutable if and only if there is a ternary term p in the language of $\mathcal{V}$ such that the following identities hold in $\mathcal{V}$:*

$$\begin{aligned} p(x,y,y) &= x \\ p(x,x,y) &= y. \end{aligned}$$

(2) *(Day* [69]*) A variety $\mathcal{V}$ is congruence modular if and only if there is a natural number n and a sequence of quaternary terms $m_0, \ldots, m_n$ such that $\mathcal{V}$ satisfies the following identities:*

$$\begin{aligned} &m_0(a,b,c,d) = a \\ &m_n(a,b,c,d) = d \\ &m_i(a,b,b,a) = a \quad \text{for } 0 \leq i \leq n \\ &m_i(a,a,d,d) = m_{i+1}(a,a,d,d) \quad \text{for all even } i < n \\ &m_i(a,b,b,d) = m_{i+1}(a,b,b,d) \quad \text{for all odd } i < n. \end{aligned}$$

(3) *(Jónsson* [67]*) A variety $\mathcal{V}$ is congruence distributive if and only if there is a natural number n and a sequence of ternary terms $d_0, \ldots, d_n$ such that $\mathcal{V}$*

satisfies the following identities:

$$\begin{aligned} d_0(x,y,z) &= x \\ d_i(x,y,x) &= x \quad \text{for } 0 \le i \le n \\ d_i(x,y,y) &= d_{i+1}(x,y,y) \quad \text{for all even } i < n \\ d_i(x,x,y) &= d_{i+1}(x,x,y) \quad \text{for all odd } i < n. \quad \square \end{aligned}$$

The next theorem is commonly known as *Jónsson's lemma*, though its content and usefulness deserve that it be called a theorem.

THEOREM 1.2.16 (Jónsson [67]). *Suppose B is a congruence distributive subalgebra of a direct product $A = \prod_{i \in I} A_i$, and $\Theta \in Con(B)$ is meet irreducible. Then there exists an ultrafilter $\mathfrak{F}$ on I such that $\Phi|_B \subseteq \Theta$, where $\Phi \in Con(A)$ is the congruence induced by $\mathfrak{F}$.* $\square$

COROLLARY 1.2.17 (Jónsson [67]). *Let $\mathcal{K}$ be a subclass of a congruence distributive variety $\mathcal{V}$ such that $\mathcal{K}$ generates $\mathcal{V}$. Then*

(1) $\mathcal{V}_{SI} \subseteq \mathbf{HSP}_U\mathcal{K}$.
(2) $\mathcal{V} = \mathbf{P}_S\mathbf{HSP}_U\mathcal{K}$.
(3) *If $\mathcal{K}$ is a finite set of finitely generated algebras,* $\mathcal{V}_{SI} \subseteq \mathbf{HS}\mathcal{K}$. $\square$

THEOREM 1.2.18. *Let $A = A_1 \times A_2 \in \mathcal{V}$, a congruence distributive variety. Then $Con(A)$ is isomorphic to $Con(A_1) \times Con(A_2)$, with the isomorphism $h : Con(A_1) \times Con(A_2) \to Con(A)$ given by*

$$h(\phi_1, \phi_2) = \phi,$$

where $\phi \in Con(A)$ is defined as follows

$$(a_1, a_2)\ \phi\ (b_1, b_2) \ \text{ if and only if } \ a_1\ \phi_1 b_1 \text{ and } a_2\ \phi_2\ b_2. \quad \square$$

DEFINITION 1.2.19.

(1) We call a variety $\mathcal{V}$ *finitely based* if there is a finite set of identities $\mathcal{T}$ such that $\mathcal{V} = \text{Mod } \mathcal{T}$.
(2) We call a variety $\mathcal{V}$ *finitely generated* if there is a subclass $\mathcal{K}$ of $\mathcal{V}$ such that $\mathcal{K}$ is a finite set of finite algebras, and $\mathcal{K}$ generates $\mathcal{V}$.

THEOREM 1.2.20 (Baker [77]). *Let $\mathcal{V}$ be a congruence distributive variety. Then if $\mathcal{V}$ is finitely generated, $\mathcal{V}$ is finitely based.* □

Residually Small Varieties. One of the most important classes of varieties in amalgamation theory are the residually small varieties, which are roughly speaking varieties with a "size limit" on their subdirectly irreducible members. Their importance to amalgamation is given by an elegant characterization of residually small varieties in terms of *absolute retracts*, which we will later see are special members of the amalgamation class. I define the relevant notions, and state this result below.

DEFINITION 1.2.21. A variety $\mathcal{V}$ is said to be *residually small* if $\mathcal{V}_{SI}$ is (up to isomorphism) a set, or equivalently, if there exists an upper bound on the cardinality of the members of $\mathcal{V}_{SI}$.

DEFINITION 1.2.22.

(1) An extension B of an algebra A is said to be *essential* if every non-trivial congruence on B restricts to a non-trivial congruence on A.

(2) An embedding $f : A \hookrightarrow B$ is called an *essential embedding* if B is an essential extension of $f(A)$. Notice that if A is subdirectly irreducible with (a, b) a critical pair, then if $f : A \hookrightarrow B$ is essential, B is subdirectly irreducible with $(f(a), f(b))$ a critical pair.

(3) An algebra $A \in \mathcal{V}$, a variety, is said to be an *absolute retract in* $\mathcal{V}$, if for every embedding $f : A \hookrightarrow B \in \mathcal{V}$ there is a homomorphism $r : B \twoheadrightarrow A$ such that rf is the identity map on A. (Note that r is necessarily onto). We denote by $\mathcal{V}_{AR}$ the class of all absolute retracts in a variety $\mathcal{V}$.

(4) For a variety $\mathcal{V}$, the class of *weakly maximal irreducibles* $\mathcal{V}_{WMI}$ is defined to be $\mathcal{V}_{WMI} = \{M \in \mathcal{V}_{SI} : M$ has no proper essential extension in $\mathcal{V}_{SI}\}$.

LEMMA 1.2.23.

(1) *If $h : A \hookrightarrow B$ is an embedding, then there is a congruence Θ on B such that h composed with the canonical epimorphism from B to B/Θ is an essential embedding from A into B/Θ.*

(2) *$M \in \mathcal{V}_{WMI}$ if and only if $M \in \mathcal{V}_{SI}$ and M is an absolute retract in $\mathcal{V}$.* □

THEOREM 1.2.24 (Taylor [72]). *The following are equivalent for a variety $\mathcal{V}$:*

(1) *$\mathcal{V}$ is residually small.*

(2) *Every member of* $\mathcal{V}_{SI}$ *has an essential extension in* $\mathcal{V}_{WMI}$.

(3) *For each* $A \in \mathcal{V}$, *there is an extension* B *of* A *such that* $B \in \mathcal{V}_{AR}$. □

1.3 Amalgamation: General Results

Introduction and Basic Concepts. In this section, I introduce the concepts of the amalgamation property and amalgamation class, and prove some elementary facts about the amalgamation class. Two important questions arise for a class $\mathcal{K}$:

1. What is the size of $Amal(\mathcal{K})$? In fact, does it have any members at all?
2. What type of a class is $Amal(\mathcal{K})$? Under what algebraic constructions (such as substructures and products) is it closed?

The answer to the first question was given for inductive classes (which include varieties) by Yasuhara [74], where it was shown that $Amal(\mathcal{K})$ sits cofinally in $\mathcal{K}$, and hence is a proper class if $\mathcal{K}$ is proper. I present a largely universal-algebraic proof of this result. I also present an answer to the second question, showing that the amalgamation class is closed under existential substructures and directed unions, and its complement is closed under direct products (all due to Yasuhara [74]). I end off with a proof that the structure of the amalgamation class is completely determined by the structure of its countable members.

DEFINITION 1.3.1 (Grätzer and Lakser [71]).

1. By a *double extension* in a class $\mathcal{K}$ we mean mean a quintuple (A, f, B, g, C) with $A, B, C \in \mathcal{K}$ and $f : A \hookrightarrow B$, $g : A \hookrightarrow C$ embeddings.
2. By an *amalgam* in $\mathcal{K}$ of a double extension (A, f, B, g, C) in $\mathcal{K}$ we mean a triple (f', g', D) with $D \in \mathcal{K}$, $f' : B \hookrightarrow D$ and $g' : C \hookrightarrow D$ such that $f'f = g'g$. (See figure 1.2)
3. An algebra $A \in \mathcal{K}$ is called an *amalgamation base* in $\mathcal{K}$ if every double extension (A, f, B, g, C) in $\mathcal{K}$ has an amalgam in $\mathcal{K}$. We denote by $Amal(\mathcal{K})$ (called the *amalgamation class of* $\mathcal{K}$), the class of all amalgamation bases in $\mathcal{K}$.
4. If $\mathcal{K} = Amal(\mathcal{K})$, we say $\mathcal{K}$ has the *amalgamation property* (AP).

Note that the term *double extension* is non-standard. Some authors use the rather undescriptive term *diagram*, and others (especially semi-group theorists) use *amalgam*, calling what I term an *amalgam* an *embedding* or *completion* instead.

$$\begin{tikzcd}
& B \arrow[dr, hook, "f'"] & \\
A \arrow[ur, hook, "f"] \arrow[dr, hook, "g"'] & & D \\
& C \arrow[ur, hook, "g'"'] &
\end{tikzcd}$$

FIGURE 1.2

With a few notable exceptions, I will only be concerned with looking at $Amal(\mathcal{K})$ for an elementary class $\mathcal{K}$ (and more particularly, when $\mathcal{K}$ is a variety). Thus for the remainder of this section, the reader may assume that every class $\mathcal{K}$ referred to is in fact elementary.

Suppose $\mathcal{K}$ is an elementary class, and let (A, f, B, g, C) be a double extension in $\mathcal{K}$. We may assume without loss of generality that B and C are distinct. Put $D = (C \setminus g(A)) \cup f(A)$. By identifying $f(A)$ with $g(A)$, we see that D can be viewed as an algebra isomorphic to C. Now if $(f(A), f_1, B, g_1, D)$ has an amalgam, then it can be checked that (A, f, B, g, C) has an amalgam, where f_1 and g_1 are the inclusion maps. Thus we see that the following is a sufficient condition for an elementary class $\mathcal{K}$ to satisfy the amalgamation property: For all $A, B, C \in \mathcal{K}$ such that $A = B \cap C$, (A, f, B, g, C) has an amalgam in $\mathcal{K}$, where f and g are the inclusion maps.

The following two useful lemmas concern the amalgamation class of algebras.

LEMMA 1.3.2 (Grätzer, Jónsson and Lakser [73]). *Let* $h : A \hookrightarrow A' \in Amal(\mathcal{K})$ *be an embedding. If for every embedding* $g : A \hookrightarrow C$*, the double extension* (A, h, A', g, C) *has an amalgam in* $\mathcal{K}$*, then* $A \in Amal(\mathcal{K})$.

Proof. Let (A, f, B, g, C) be a double extension in $\mathcal{K}$, and $h : A \hookrightarrow A'$ as above. By the hypothesis, the double extensions (A, h, A', f, B) and (A, h, A', g, C) have respective amalgams (h_1, f', D_1) and (h_2, g', D_2). Then $A' \in Amal(\mathcal{K})$, so (A', h_1, D_1, h_2, D_2) has an amalgam (h_1', h_2', D), say. Now by the commutativity of the diagram in figure 1.3, $(h_1' f', h_2' g', D)$ is an amalgam of (A, f, B, g, C). □

$$\begin{tikzcd}
& B \arrow[r, hook, "f'"] & D_1 \arrow[dr, hook, "h_1'"] & \\
A \arrow[ur, hook, "f"] \arrow[r, hook, "h"] \arrow[dr, hook, "g"'] & A' \arrow[ur, hook, "h_1"] \arrow[dr, hook, "h_2"] & & D \\
& C \arrow[r, hook, "g'"'] & D_2 \arrow[ur, hook, "h_2'"'] &
\end{tikzcd}$$

FIGURE 1.3

LEMMA 1.3.3 (Grätzer and Lakser [71]). *Let $\mathcal{K}$ be a class closed under direct products.*

(1) *If $A \times A' \in Amal(\mathcal{K})$, and if A' has a one element sub-algebra $\{a\}$, then $A \in Amal(\mathcal{K})$.*

(2) *A double extension (A, f, B, g, C) in $\mathcal{K}$ has an amalgam in $\mathcal{K}$ if and only if for all $u, v \in B$ where $u \neq v$, there exists $D \in \mathcal{K}$ and homomorphisms $f' : B \to D$ and $g' : C \to D$ such that $f'f = g'g$ and $f'(u) \neq f'(v)$, and the same holds for C.*

Proof.

(1) We use lemma 1.3.2. Let $h : A \hookrightarrow A \times A'$ be the embedding given by $h(x) = (x, a)$, and let $g : A \hookrightarrow B \in \mathcal{K}$ be an arbitrary embedding. Now $B \times A' \in \mathcal{K}$. Hence, let $h' : A \times A' \hookrightarrow B \times A'$ be the embedding given by $h(x, y) = (g(x), y)$, and let $g' : B \hookrightarrow B \times A'$ be the embedding $g'(b) = (b, a)$. Now a check shows $(A, h, A \times A', g, B)$ has amalgam $(h', g', B \times A')$.

(2) The condition is necessary. To prove sufficiency we need only observe that the direct product of the above D's as u and v run through distinct pairs of B and C, together with the respective product maps, form an amalgam of the double extension. □

Elementary, Existential and Pure Extensions. In theorem 1.1.8, we saw that if B is an extension of A, then every universal formula is preserved downwards, that is, if a universal formula is satisfied in B, it is satisfied in A. In this subsection, we will consider special types of extensions which preserve other types of formulae as well. The main goal of studying these extensions is to identify some special members of the amalgamation class, which is a first step in trying to determine the 'size' of the amalgamation class.

DEFINITION 1.3.4. Let $A, B \in \mathcal{K}$, with $A \subseteq B$. We say B is an *existential (pure) extension*, if, for every existential formula (finite conjunction of identities) ϕ such that for any $x \in A^\omega$, $B \models_x \phi$ if and only if $A \models_x \phi$. If $f : A \hookrightarrow B$ is an embedding such that B is existential (pure) extension of $f(A)$, then we call f an *existential (pure) embedding.*

We have the following elegant algebraic characterization of elementary (existential/pure) embeddings. The case for elementary emdeddings was first proved in Frayne, Morel and Scott [62], and the case for pure embeddings in Nelson [75]. The case for existential embeddings is an exercise (with reference to the proof in the elementary embedding case) in Bell and Slomson [71].

LEMMA 1.3.5(Nelson [75], Frayne, Morel and Scott [62]). *An embedding* $f : A \hookrightarrow B$ *is elementary (existential/pure) if and only if there exists an elementary embedding (embedding/homomorphism)* $g : B \to A^I/\mathfrak{D}$ *into an ultrapower of* A*, such that* gf *is the diagonal map* $gf : A \hookrightarrow A^I/\mathfrak{D}$.

Proof. $\Longleftarrow$: Let f and g be as above, and let $d : A \hookrightarrow A^I/\mathfrak{D}$ be the diagonal map. If $A \models_x \phi$, then as d is elementary, $A^I/\mathfrak{D} \models_{d(x)} \phi$, so $A^I/\mathfrak{D} \models_{gf(x)} \phi$. By the hypothesis, g is elementary, so $B \models_{f(x)} \phi$. Hence f is elementary.

$\Longrightarrow$: Suppose $f : A \hookrightarrow B$ is an elementary embedding. Let $\underline{b} = \langle b_\xi : \xi < \gamma \rangle$ be a well-ordered indexing of B. Put $I = \{\sigma \in L_B : \langle B, \underline{b} \rangle \models \sigma\}$, where L_B is the set of sentences in the language of the algebra B together with constants $\{c_\xi : \xi < \gamma\}$ corresponding to B. Fix $a_+ \in A$.

$$\text{For all} \quad \xi < \gamma, \text{ put } b'_\xi = \begin{cases} b_\xi & \text{if and only if } b_\xi \in f(A) \\ f(a_+) & \text{otherwise.} \end{cases}$$

Put $\underline{b}' = \left\langle b'_\xi : \xi < \gamma \right\rangle \in (f(A))^\gamma$ and $\underline{a} = \langle a_\xi : \xi < \gamma \rangle$, where $a_\xi = f^{-1}({b'}_\xi), \xi < \gamma$. Let $\phi \in I$, and $c_{\xi_0}, \ldots, c_{\xi_k}$ be constants in ϕ that correspond to elements $b_{\xi_0}, \ldots, b_{\xi_k}$ not in $f(A)$. For $i = 0 \ldots k$, replace each c_{ξ_i} by a v_{j_i} (a free variable) in ϕ to get a new formula ϕ'.

$$\begin{aligned} &\text{Now } \langle B,\underline{b}\rangle \models \phi \\ &\text{implies} \left\langle B,\underline{b}'\right\rangle \models (\exists v_{j_0}\ldots v_{j_k})\phi' \\ &\text{implies} \langle A,\underline{a}\rangle \models (\exists v_{j_0}\ldots v_{j_k})\phi' \quad \text{as } f \text{ is elementary.}\end{aligned}$$

Hence there is some sequence $\underline{a_\phi} = \left\langle a_{\phi_\xi} : \xi < \gamma\right\rangle$ such that $\langle A,\underline{a}_\phi\rangle \models \phi$ where $a_{\phi_\xi} = a_\xi$, except for $\xi_0,\ldots,\xi_k$. Put $J_\phi = \{\psi \in I : \langle A,\underline{a}_\psi\rangle \models \phi\}$ and $\mathfrak{D}^* = \{J_\phi : \phi \in I\}$. Note that $J_\phi \cap J_\psi = J_{\phi\wedge\psi}$, so $\mathfrak{D}^*$ can be extended to an ultrafilter $\mathfrak{D}$ over I. Let $g : B \to A^I/\mathfrak{D}$ be given by $g(b_\xi) = k_\xi/\mathfrak{D}$, where $k_\xi(\sigma) = a_{\sigma_\xi}, \sigma \in I$.

We show: gf is the diagonal map:

Suppose $a \in A$, then $f(a) \in B$, say $f(a) = b_\delta$ where $\delta < \gamma$. Now $gf(a) = k_\delta/\mathfrak{D}$. Since $b_\delta \in f(A)$, $k_\delta(\sigma) = a_{\phi_\delta} = a_\delta = g^{-1}(b_\delta) = a$. Hence gf is the diagonal map.

We now show: $B \models_x \phi$ if and only if $A^I/\mathfrak{D} \models_{g(x)} \phi$ for all formulae ϕ and sequences $\underline{x} = \langle x_i : i < \omega\rangle$. (*)

Let ϕ' be the formula obtained from ϕ as follows: For all $i \in \omega$, if $x_i = b_\xi$, replace each occurence of the free variable v_i in ϕ by c_ξ.

$$\begin{aligned} &\text{Thus } B \models_x \phi \\ &\Longleftrightarrow \langle B,\underline{b}\rangle \models \phi' \\ &\Longleftrightarrow \phi' \in I \\ &\Longleftrightarrow J_{\phi'} \in \mathfrak{D} \\ &\qquad\qquad \text{since for } \psi \in I, \left\langle A,\underline{a}_\psi\right\rangle \models \phi' \text{ implies } \langle B,\underline{b}\rangle \models \phi' \\ &\Longleftrightarrow \{\psi \in I : \left\langle A,\underline{a}_\psi\right\rangle \models \phi'\} \in \mathfrak{D} \\ &\Longleftrightarrow \left\langle A^I/\mathfrak{D}, k/\mathfrak{D}\right\rangle \models \phi' \\ &\qquad\qquad \text{where } k_\psi = a_\psi \text{ for } \psi \in I \text{ and } k = (k_\psi)_{\psi\in I}.\end{aligned}$$

But, by definition, $g(b) = k/\mathfrak{D}$, so we have shown

$$\begin{aligned} &\langle B,\underline{b}\rangle \models \phi' \iff \left\langle A^I/\mathfrak{D}, g(\underline{b})\right\rangle \models \phi' \\ &\text{or } \; B \models_x \phi \iff A^I/\mathfrak{D} \models_{g(x)} \phi\end{aligned}$$

To see that g is a homomorphism, take any n-ary operation λ of A, and apply (*) to the formula $\phi := \lambda(v_0, \dots, v_{n-1}) = v_n$ above. Similarly, g is an embedding, for we can apply (*) to the formula $\phi := \neg(v_0 = v_1)$.

The proof for existential and pure embeddings proceeds similarly, with minor modifications occasioned by the fact that only existential formulae are necessarily preserved by existential embeddings, and that only conjunctions of equations are necessarily preserved by homomorphisms. ☐

THEOREM 1.3.6(Jónsson [84]). *In a class $\mathcal{K}$ which is closed under direct products, if (A, f_0, B, f_1, C) is a double extension in $\mathcal{K}$ such that f_0, f_1 are pure embeddings, then there exists an amalgam (g_0, g_1, D) such that g_0 and g_1 are pure.*

Proof. By lemma 1.3.5, there exist homomorphisms $h_0 : B \to A^I/\mathfrak{F}$ and $h_1 : C \to A^J/\mathfrak{G}$ with $\mathfrak{F}$ and $\mathfrak{G}$ ultrafilters over I and J respectively, such that $h_0 f_0$ and $h_1 f_1$ are diagonal maps. Let $f_0^* : A^J/\mathfrak{G} \to B^J/\mathfrak{G}$ and $f_1^* : A^I/\mathfrak{F} \to C^I/\mathfrak{F}$ be the maps induced in the obvious way by f_0 and f_1, and let $k_0 : B \hookrightarrow B^J/\mathfrak{G}$ and $k_1 : C \hookrightarrow C^I/\mathfrak{F}$ be the respective diagonal maps. Further, let $D = (B^J/\mathfrak{G}) \times (C^I/\mathfrak{F})$, and let $g_0 : B \to D$ and $g_1 : C \to D$ be maps induced by the pairs $\langle k_0, f_1^* h_0\rangle$ and $\langle f_0^* h_1, k_0\rangle$ respectively.

Now if $a \in A$, let $b = f_0(a)$ and $c = f_1(a)$. Let $\underline{b} \in B^J$ and $\underline{c} \in C^I$ be the constant maps given by $\underline{b}(j) = b$, for all $j \in J$ and $\underline{c}(i) = c$, for all $i \in I$. Then $g_0 f_0(a) = (\underline{b}/\mathfrak{G}, \underline{c}/\mathfrak{F}) = g_1 f_1(a)$ and hence $g_0 f_0 = g_1 f_1$, so (A, f_0, B, f_1, C) has an amalgam (g_0, g_1, D). Moreover, since k_0 and k_1 are elementary (and hence pure), it follows that g_0 and g_1 are also pure. ☐

The following result, a corollary of the above result, was first shown for the elementary case by Morley and Vaught [62], and the existential case by Hirschfield and Wheeler [75].

COROLLARY 1.3.7 (Morley and Vaught [62], Hirschfield and Wheeler [75]).
Any double extension (A, f, B, g, C) in $\mathcal{K}$ with f and g elementary (existential) has an amalgam in $\mathcal{K}$. ☐

In fact, the corollary above can be improved, as the next theorem shows.

THEOREM 1.3.8 (Yasuhara [74]). *If (A, f, B, g, C) is a double extension in $\mathcal{K}$ with one of f or g an existential embedding, then the double extension has an amalgam.*

Proof. Let (A, f, B, g, C) be a double extension in $\mathcal{K}$ and suppose, without loss of generality, that f is existential. Then, by lemma 1.3.5, there is a set I, an ultrafilter $\mathfrak{U}$ on I and an embedding $h : B \hookrightarrow A^I/\mathfrak{U}$ such that hf is the diagonal map. Let $g^* : A^I/\mathfrak{U} \hookrightarrow C^I/\mathfrak{U}$ be the embedding induced by g in the obvious way, and let $d : C \hookrightarrow C^I/\mathfrak{U}$ be the diagonal map. A check shows that $(g^*h, d, C^I/\mathfrak{U})$ is an amalgam of the double extension. □

DEFINITION 1.3.9. Let $A \in \mathcal{K}$. We say

(1) A is *equationally closed in* $\mathcal{K}$ if every extension of A in $\mathcal{K}$ is pure.
(2) A is *existentially closed in* $\mathcal{K}$ if every extension of A in $\mathcal{K}$ is existential.

Note that every absolute retract is equationally closed (by lemma 1.3.5), and every existentially closed member is equationally closed.

COROLLARY 1.3.10 (Yasuhara [74]). *Every existentially closed member of $\mathcal{K}$ is a member of Amal($\mathcal{K}$). In addition, if $\mathcal{K}$ is closed under direct products, every equationally closed member (and hence every absolute retract) of $\mathcal{K}$ is a member of Amal($\mathcal{K}$).*

Proof. Evident from theorem 1.3.8 and theorem 1.3.6. □

Amal($\mathcal{K}$) is a Proper Class. One of the obvious questions at this stage is whether a class $\mathcal{K}$ has any existentially closed members, equationally closed members or absolute retracts at all. In this subsection, we will see that for some special classes of algebras (which include varieties), we have that the existentially closed members sit cofinally in the class.

THEOREM 1.3.11 (Yasuhara [74]). *If $\mathcal{K}$ is inductive, then every member M of $\mathcal{K}$ is contained in an existentially closed member of $\mathcal{K}$, say M^*, with $|M^*| \leq |M| + \aleph_0$.*

Proof. Let $M \in \mathcal{K}$, and let $\{\phi_\alpha : \alpha < \tau\}$ be a well-ordered indexing of the set of all existential sentences in the language of $\mathcal{K}$ with constants in M. Note that $\tau = |M| + \aleph_0$.

Define members M_α of $\mathcal{K}$ inductively as follows:

(1) $M_0 = M$.

(2) If $\alpha = \beta + 1$, and there exists $M' \supseteq M_\beta$ such that $M' \in \mathcal{K}$ and $M' \models \phi_\beta$, then put $M_\alpha = M'$, else put $M_\alpha = M_\beta$. Note, by theorem 1.1.5(2), M' (if one exists) can always be chosen such that $|M'| \leq |M| + \aleph_0$.

(3) If α is a limit ordinal, then put $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. Note that $M_\alpha \in \mathcal{K}$.

Put $M^1 = \bigcup_{\alpha<\tau} M_\alpha$, then $M^1 \in \mathcal{K}$ and $|M^1| \leq |M| + \aleph_0$. If ϕ is any existential sentence with constants in M, if $N \models \phi$ for some extension of M^1, by construction we have $M^1 \models \phi$. We now repeat the construction inductively starting from M to obtain a countable chain:

$$M \subseteq M^1 \subseteq M^2 \subseteq M^3 \subseteq \dots$$

Put $M^* = \bigcup_{n<\omega} M^n$. Then $M^* \in \mathcal{K}$. Let ϕ be an existential sentence with constants in M^*. Then ϕ is an existential sentence with constants in some M^n, as the number of terms occurring in ϕ is finite. Now if $N \models \phi$ for some extension of M^*, by construction $M^{n+1} \models \phi$, hence $M^* \models \phi$ (as ϕ is existential). Hence M^* is existentially closed in $\mathcal{K}$ and by construction $|M^*| \leq |M| + \aleph_0$. □

COROLLARY 1.3.12 (Yasuhara [74]). *If $\mathcal{K}$ is inductive, Amal($\mathcal{K}$) is cofinal (with respect to set inclusion) in $\mathcal{K}$. Consequently, Amal($\mathcal{K}$) is a proper class.*

Proof. Follows from corollary 1.3.10 and theorem 1.3.11. □

Closure Properties of the Amalgamation Class. The last two results give some answers to the question asked at the start of this section: Under which algebraic constructions is the amalgamation class closed?

DEFINITION 1.3.13. A *directed set* is a non-empty partially ordered set $\langle P, \leq \rangle$ with the property that for every $x, y \in P$, there is a $z \in P$ such that $x \leq z$ and $y \leq z$. If $\{A_i : i \in I\}$ is a directed set of algebras ordered by inclusion, we call $\bigcup_{i\in I} A_i$ a *directed union.* We say a class $\mathcal{K}$ is *closed under directed unions* if for every directed set $\{A_i : i \in I\}$ with each $A_i \in \mathcal{K}$, we have $\bigcup_{i\in I} A_i \in \mathcal{K}$.

THEOREM 1.3.14. *Let $\mathcal{K}$ be elementary. Then,*

(1) *Amal($\mathcal{K}$) is closed under existential (and hence elementary) substructures.*
(2) *Amal($\mathcal{K}$) is closed under directed unions, and hence unions of chains.*
(3) *The complement of Amal($\mathcal{K}$) is closed under reduced powers, that is, if $A \in \mathcal{K}$ and a reduced power of A is a member of Amal($\mathcal{K}$), then $A \in$ Amal($\mathcal{K}$).*

Proof.

(1) Immediate from lemma 1.3.8.

(2) Let $\{A_\alpha : \alpha \in I\}$ be a directed set in $\mathcal{K}$ such that $A_\alpha \in$ *Amal*($\mathcal{K}$) for all $\alpha \in I$. Put $A = \bigcup_{\alpha \in I} A_\alpha$, and let (A, f, B, g, C) be a double extension. Now, for each $\alpha \in I$, the double extension $(A_\alpha, f|_{A_\alpha}, B, g|_{A_\alpha}, C)$ has an amalgam $(f_\alpha, g_\alpha, D_\alpha)$, say, as $A_\alpha \in$ *Amal*($\mathcal{K}$).

Consider the ultraproduct $D = \prod_{\alpha \in I} D_\alpha / \mathfrak{F}$, where $\mathfrak{F}$ is the ultrafilter generated by $\{F_\alpha : \alpha \in I\}$ where each $F_\alpha = \{\beta \in I : A_\alpha \subseteq A_\beta\}$ (Note that since the set is directed, the set of A_α's has the finite intersection property). Since $\mathcal{K}$ is elementary, $D \in \mathcal{K}$. Now consider the homomorphisms $f' : B \to D$ and $g' : C \to D$ defined by $f'(b) = \underline{b}/\mathfrak{F}$, where $\underline{b}$ is the map $\underline{b} : I \to \prod_{\alpha \in I} D_\alpha$ given by $\underline{b}(\alpha) = f_\alpha(b)$, and g' is defined similarly. Since $\mathfrak{F}$ is a proper filter, f and g are embeddings. Lastly, pick $a \in A$. Then, for some $\beta \in I$, $a \in A_\alpha$ for all $\beta \leq \alpha$, hence $f_\alpha f(a) = g_\alpha g(a)$ for all $\beta \leq \alpha$, and so $f' f(a) = g' g(a)$. Hence, (f', g', D) is an amalgam of (A, f, B, g, C).

(3) Suppose $A^I/\mathfrak{D} \in$ *Amal*($\mathcal{K}$) for some set I and proper filter $\mathfrak{D}$, where $A \in \mathcal{K}$. Let $(A, f, B, d_A, A^I/\mathfrak{D})$ be a double extension in $\mathcal{K}$, where d_A is the diagonal map from A into $A^I/\mathfrak{D}$. Let $d_B : B \hookrightarrow B^I/\mathfrak{D}$ be the diagonal map into $B^I/\mathfrak{D}$, and let $f^* : A^I/\mathfrak{D} \hookrightarrow B^I/\mathfrak{D}$ be the embedding induced from $f : A \hookrightarrow B$ in the obvious way. Note that $(d_B, f^*, B^I/\mathfrak{D})$ is an amalgam of the double extension, and the result follows from lemma 1.3.2. □

The following theorem is stated without proof in Jónsson [90], who refers to Yasuhara [74]. I now give a universal-algebraic proof of this theorem.

Theorem 1.3.15 (Yasuhara [74]). *Let $\mathcal{K}$ be elementary. Then $A \in$ Amal($\mathcal{K}$) if and only if A is a directed union of countable members of Amal($\mathcal{K}$).*

Proof. $\Longleftarrow$: Follows from theorem 1.3.14(2).

$\Longrightarrow$: We may suppose $A \in \mathcal{K}$ is uncountable. Let $\{A_i : i \in I\}$ be the set of all countable subalgebras of A that are members of the amalgamation class. We first show that $\bigcup_{i \in I} A_i = A$: For given any $a \in A$, by theorem 1.1.5(2), we have that there is an elementary subalgebra A_a of A such that $a \in A_a$. By theorem 1.3.14(1), we have that $A_a \in Amal(\mathcal{K})$, so $A_a \in \{A_i : i \in I\}$. Now we show $\{A_i : i \in I\}$ is directed: For if $A_i, A_j \in \{A_i : i \in I\}$ we have again by theorem 1.1.5(2) that there is a countable elementary subalgebra B of A such that B contains $A_i \cup A_j$, since $A_i \cup A_j$ is countable. Again by theorem 1.3.14(1) we have $B \in \{A_i : i \in I\}$. □

The original formulation of this theorem in Yasuhara [74] is somewhat different. There it is shown that a uncountable member (say A) of the amalgamation class is the union of a chain of subalgebras, all members of the amalgamation class, and all having cardinality strictly less than that of A.

Notice that theorem 1.3.15 shows us that the amalgamation class is more or less completely determined by its countable members. In fact, it is a consequence of the above theorem that if all countable members of an elementary class are in the amalgamation class, then that class has the amalgamation property.

Chapter 2

The Amalgamation Property and Other Extension Properties

Introduction. In this chapter I consider what can broadly be termed "extension properties" of varieties of algebras, that is, properties which imply, for a fixed variety $\mathcal{V}$, the existence of certain algebras and homomorphisms in $\mathcal{V}$ which extend some given set of algebras and homomorphisms in $\mathcal{V}$. The amalgamation property itself is a typical extension property: for a given variety $\mathcal{V}$, $\mathcal{V}$ has the amalgamation property if we can extend every double extension (A, f, B, g, C) in $\mathcal{V}$ to an amalgam (D, f', g') in $\mathcal{V}$.

In section 2.1, I present mostly folkloric results relating AP to extension properties such as the congruence extension property(CEP), the transferability property(TP) and the existence of enough injectives(EI). I also introduce here the strong amalgamation property (SAP), a strengthening of AP, and present results due to Isbell [66] relating SAP to the property that epis are surjective (ES).

The rest of this chapter is devoted to a proof that in a congruence modular variety, the amalgamation property and residual smallness imply the congruence extension property, a result that was finally obtained by Kearnes [89] after many partial results by C. Bergman [86] and Bergman and McKenzie [88], among others. The proof of the result depends heavily on the commutator theory for congruence modular varieties, an extension of the the theory of commutator subgroups for groups first introduced by Hagemann and Herrmann [79]. In section 2.2, I present a brief introduction to commutator theory, and in section 2.3, an in-depth look at the non-abelian finitely subdirectly irreducible (NAFSI) algebras, in order to make the proof of the result in section 2.4 more or less self-contained.

As the extension properties involve algebras and their homomorphisms, many of these results can be generalized to a categorical setting. This has been done in an extensive survey by Kiss, Márki, Pröhle and Tholen [83]. I have opted, however, to present these results in the original universal-algebraic setting.

2.1. Extension Properties of Algebras

Enough Injectives, Congruence Extension and Transferability Properties.

DEFINITION 2.1.1. Let $\mathcal{V}$ be a variety and let $A \in \mathcal{V}$.

(1) A is said to be *injective in* $\mathcal{V}$, if for every embedding $f : B \hookrightarrow C$ and homomorphism $g : B \to A$, there is a homomorphism $h : C \to A$ such that $hf = g$.

(2) $\mathcal{V}$ is said to have *enough injectives (EI)* if every algebra in $\mathcal{V}$ can be embedded in an injective in $\mathcal{V}$.

(3) $A \in \mathcal{V}$ is said to be *congruence extensile in* $\mathcal{V}$ if, for every extension B of A in $\mathcal{V}$, every $\theta \in \mathrm{Con}(A)$ can be extended to a congruence $\psi \in \mathrm{Con}(B)$, that is, $\psi|_A = \theta$.

(4) $A \in \mathcal{V}$ is said to have the *congruence extension property (CEP)*, if for every subalgebra B of A, $\mathrm{Con}(B) = \{\theta|_B : \theta \in \mathrm{Con}(A)\}$. A variety $\mathcal{V}$ is said to have *CEP* if all its members have CEP (or equivalently, all its members are congruence extensile).

(5) $\mathcal{V}$ is said to have the *transferability property (TP)* if, for every $A, B, C \in \mathcal{V}$ with $f : A \hookrightarrow B$ an embedding and $g : A \to C$ a homomorphism, there exists $D \in \mathcal{V}$, a homomorphism $f' : B \to D$ and an embedding $g' : C \hookrightarrow D$ such that $f'f = g'g$.

Note, by the first isomorphism theorem, $A \in \mathcal{V}$ is congruence extensile if and only if for every $B, C \in \mathcal{V}$ with $f : A \hookrightarrow B$ an embedding and $g : A \twoheadrightarrow C$ an epimorphism, there exists $D \in \mathcal{V}$, an embedding $g' : C \hookrightarrow D$ and epimorphism $f' : B \twoheadrightarrow D$ such that $ff' = gg'$. In fact by lemma 1.2.23(1), we can choose g' to be essential, so if $C \in \mathcal{V}_{SI}$, then $D \in \mathcal{V}_{SI}$.

The 'enough injectives' property was first introduced for abelian groups by Baer [40], and in the more general universal-algebraic setting by Buchsbaum [55]. The congruence extension property was first introduced by Grätzer and Lakser [71] and the transferability property by Banaschewski [70] (although Bacsich [72] was the first to use the term).

The next theorem relates the properties above to the amalgamation property (AP) and residual smallness (RS). The first result is implicit in Banaschewski [70] and the second appears in Bacsich [72], but see also Day [72].

THEOREM 2.1.2. *Let $\mathcal{V}$ be a variety.*

(1) *$\mathcal{V}$ has EI if and only if $\mathcal{V}$ has TP and RS.*
(2) *$\mathcal{V}$ has TP if and only if $\mathcal{V}$ has AP and CEP.*

Proof.

(1) $\Longrightarrow$: Let $\mathcal{V}$ have EI. Since every every injective is an absolute retract, every algebra can be embedded in an absolute retract, so $\mathcal{V}$ is RS. To show $\mathcal{V}$ has TP, let $f : A \hookrightarrow B$ be an embedding, $g : A \rightarrow C$ a homomorphism, where $A, B, C \in \mathcal{V}$. Letting $g' : C \hookrightarrow D$ be an embedding into an injective $D \in \mathcal{V}$, we see there is a homomorphism $f' : B \rightarrow D$ such that $g'g = f'f$.

$\Longleftarrow$: Let V have RS and TP. We show that absolute retracts are injective: For, if A is an absolute retract in $\mathcal{V}$, $f : B \hookrightarrow C$ is an embedding and $g : B \rightarrow A$ a homomorphism ($B, C \in \mathcal{V}$), then by TP, there is a $D \in \mathcal{V}$, a homomorphism $f' : C \rightarrow D$ and an embedding $g' : A \hookrightarrow D$ such that $g'g = f'f$. Letting $r : D \twoheadrightarrow A$ be a retract of g', and putting $h = rf'$, we see $hf = g$.

(2) $\Longrightarrow$: Let $\mathcal{V}$ have TP. We use lemma 1.3.3(2) to show $\mathcal{V}$ has AP. Let (A, f, B, g, C) be a double extension in $\mathcal{V}$. Let $b_1, b_2 \in B, b_1 \neq b_2$. Then, by TP, there is a $D \in \mathcal{V}$, an embedding $f' : B \hookrightarrow D$ and a homomorphism $g' : C \rightarrow D$ such that $f'f = g'g$. Note $f(b_1) \neq f(b_2)$. Hence $\mathcal{V}$ has AP by lemma 1.3.3(2).

To show $\mathcal{V}$ has CEP, let $A, B, C \in \mathcal{V}$ with $f : A \hookrightarrow B$ an embedding and $g : A \twoheadrightarrow C$ an epimorphism. By TP, there is a $D \in \mathcal{V}$, an embedding $g' : C \hookrightarrow D$ and homomorphism $f' : B \rightarrow D$ with $f'f = g'g$. We may replace D with $f'(B)$, since $g'(C) \subseteq f'(B)$, and regard f' as an epimorphism $f' : B \twoheadrightarrow f'(B)$, to get the requirements for CEP to hold.

$\Longleftarrow$: Let $\mathcal{V}$ have AP and CEP. To prove $\mathcal{V}$ has TP, let $A, B, C \in \mathcal{V}$ with $f : A \hookrightarrow B$ an embedding and $g : A \to C$ a homomorphism. Now there is an epimorphism $\overline{g} : A \twoheadrightarrow g(A)$ obtained from g and an inclusion map $i : g(A) \hookrightarrow C$ such that $i\overline{g} = g$. By CEP, there is a $D \in \mathcal{V}$, an embedding $\overline{g}' : g(A) \hookrightarrow D$ and an epimorphism $s : B \twoheadrightarrow D$ such that $sf = \overline{g}'\overline{g}$. By AP, the double extension $(g(A), \overline{g}', D, i, C)$ has an amalgam, say (t, g', E). Putting $f' = ts$, we see that $g'g = f'f$, thus showing $\mathcal{V}$ has TP. $\square$

The Strong Amalgamation Property. I now define the strong amalgamation property, a strengthening of the amalgamation property due to Jónsson [56]. It has been the subject of very intense study in semigroup theory, where it is also called the special amalgamation property.

DEFINITION 2.1.3. Let $\mathcal{V}$ be a variety, with (A, f, B, g, C) a double extension in $\mathcal{V}$. Then, if the double extension has an amalgam (D, f', g') such that $f'f(A) = f'(B) \cap g'(C)$, we call that amalgam a *strong amalgam*. A variety is said to have the *strong amalgamation property(SAP)* if every double extension in $\mathcal{V}$ has a strong amalgam.

LEMMA 2.1.4. *Let $\mathcal{V}$ be a variety. The following are equivalent:*

(1) *Every double extension in $\mathcal{V}$ that has an amalgam, has a strong amalgam.*

(2) *For every embedding $f : A \hookrightarrow B$, where $A, B \in \mathcal{V}$, the double extension (A, f, B, f, B) has a strong amalgam.*

Proof. (1)$\Longrightarrow$ (2): Notice the double extension (A, f, B, f, B) has an amalgam $(\mathrm{id}_B, \mathrm{id}_B, B)$, where $\mathrm{id}_B : B \to B$ is the identity map, so this direction follows.

(1)$\Longleftarrow$ (2): Let (A, f, B, g, C) be a double extension in $\mathcal{V}$, and suppose (D, f', g') is an amalgam of the double extension. Letting $h = f'f = g'g$, we have by (2) that (A, h, D, h, D) has a strong amalgam, say (s, t, E). A check shows that (sf', tg', E) is a strong amalgam of the original double extension. $\square$

DEFINITION 2.1.5. In the light of lemma 2.1.4, we say a variety satisfies the *intersection property of amalgams(IPA)* if it satisfies any of the equivalent conditions of lemma 2.1.4. Note that as a consequence of lemma 2.1.4, a variety satisfies SAP if and only the variety satisfies IPA and AP.

DEFINITION 2.1.6. Let $\mathcal{V}$ be a variety. We say a homomorphism $f : A \to B, A, B \in \mathcal{V}$ is an *epi in* $\mathcal{V}$ if, for all homomorphisms $g : B \to C$, $h : B \to D$ where $C, D \in \mathcal{V}$ we have $gf = hf$ implying $g = h$. Clearly all epimorphisms are epis, but the converse is not necessarily true. If indeed all epis in $\mathcal{V}$ are epimorphisms, we say $\mathcal{V}$ has the property *epis are surjective (ES).*

The property ES was first introduced by Isbell [57]. To show the important link between ES, AP and SAP, I define the concept of dominion, first introduced in Isbell [66].

DEFINITION 2.1.7.

(1) Let $A, B \in \mathcal{V}$, a variety, with A a subalgebra of B. We define the *dominion of A in B* (denoted $\mathrm{dom}_B(A)$) to be the set of all elements $b \in B$ such that for each pair of homomorphisms $f, g : B \to C \in \mathcal{V}$ such that $f|_A = g|_A$, we have $f(b) = g(b)$. Note that $\mathrm{dom}_B(A)$ is a subalgebra of B containing A.

(2) An algebra $A \in \mathcal{V}$ is said to be *saturated in* $\mathcal{V}$, if for every proper extension B ($B \in \mathcal{V}$) of A, we have $\mathrm{dom}_B(A)$ is a proper subalgebra of B.

(3) An algebra $A \in \mathcal{V}$ is said to be *absolutely closed in* $\mathcal{V}$, if for every extension B ($B \in \mathcal{V}$) of A, we have $\mathrm{dom}_B(A) = A$.

Note that if A is absolutely closed in $\mathcal{V}$, it is saturated in $\mathcal{V}$.

LEMMA 2.1.8 (Isbell [66]).

(1) *$\mathcal{V}$ has IPA if and only if each $A \in \mathcal{V}$ is absolutely closed in $\mathcal{V}$.*

(2) *$\mathcal{V}$ has ES if and only if each $A \in \mathcal{V}$ is saturated in $\mathcal{V}$.*

Proof.

(1) $\Longrightarrow$: Let A be a proper subalgebra of $B \in \mathcal{V}$, and pick $b \in B \backslash A$. Then, as $\mathcal{V}$ has IPA, (A, i, B, i, B) has a strong amalgam, say (f, g, C), where $i : A \hookrightarrow B$ is the inclusion map. But then, if $f(b) = g(b)$, $f'(b) = fi(a)$ for some $a \in A$, and hence $b \in A$, a contradiction. Hence $f(b) \neq g(b)$, so $b \notin \mathrm{dom}_B(A)$, so $\mathrm{dom}_B(A) = A$.

$\Longleftarrow$: Let $f : A \hookrightarrow B$ be an embedding. Since the case where f is an isomorphism is trivial, we may assume f is proper. Now for each $b \in B \setminus A$, since A is absolutely closed, there is a $C_b \in \mathcal{V}$, and homomorphisms g_b, h_b :

$B \to C_b$ such that $g_b(b) \neq h_b(b)$. Put $C = B \times \prod_{b \in B} C_b$ and $g, h : B \to C$ by $g = \mathrm{id}_B \times \prod_{b \in B} g_b$ and $h = \mathrm{id}_B \times \prod_{b \in B} h_b$. A routine check shows that (g, h, C) is a strong amalgam of (A, f, B, f, B).

(2) $\Longrightarrow$: Let $\mathcal{V}$ have ES, and let A a proper subalgebra of some $B \in \mathcal{V}$. Now the inclusion map $i : A \hookrightarrow B$ is not surjective and hence not an epi, so there are homomorphisms $f, g : B \to C$ such that $f \neq g$ but $f|_A = g|_A$. But then there is a $b \in B$ such that $f(b) \neq g(b)$, that is, $b \notin \mathrm{dom}_B(A)$, so A is saturated.

$\Longleftarrow$: Suppose each $A \in \mathcal{V}$ is saturated, and let $f : A \to B$ be an epi in $\mathcal{V}$, where $A, B \in \mathcal{V}$. If f is not surjective, $f(A)$ is a proper subalgebra of B and hence there is a $b \in B \setminus \mathrm{dom}_B(f(A))$. Hence there are maps $g, h : B \to C \in \mathcal{V}$ such that $h(b) \neq g(b)$, but $h|_f(A) = g|_f(A)$. But then $h \neq g$, but $hf = gf$, contradicting the fact that f is an epi. $\square$

Relationships between SAP and ES. I now state and prove the main result of this section, showing the relationships that exist between the properties SAP, AP, ES and IPA.

THEOREM 2.1.9 (Isbell [66]). *Let $\mathcal{V}$ be a variety.*

(1) *If $\mathcal{V}$ has IPA, then $\mathcal{V}$ has ES.*

(2) *If $\mathcal{V}$ has ES and AP, then $\mathcal{V}$ has IPA (and hence SAP).*

Proof.

(1) The result follows immediately from lemma 2.1.8 and the fact that an absolutely closed algebra is saturated.

(2) Let $\mathcal{V}$ have AP and ES. We first claim that if $A \leq B \leq C \in \mathcal{V}$, then $\mathrm{dom}_C(A) \cap B \subseteq \mathrm{dom}_B(A)$:

Proof of claim: Let A, B and C be as above. Pick $b \in B \cap \mathrm{dom}_C(A)$. Suppose $b \notin \mathrm{dom}_B(A)$. Then there are maps $f, g : B \to D \in \mathcal{V}$ such that $f(b) \neq g(b)$, but $f|_A = g|_A$. Let $f^*, g^* : B \to B \times D$ be given by $f^*(b) = (b, f(b))$ and $g^*(b) = (b, g(b))$. Now f^* and g^* are embeddings, hence since $\mathcal{V}$ has AP, $(B, i, C, f^*, B \times D)$ and $(B, i, C, g^*, B \times D)$ have amalgams, say (i_f, f', E_f) and (i_g, g', E_g) respectively. (Note that

$i : B \hookrightarrow C$ is the inclusion map.) Applying AP again, we have that $(B \times D, f', E_f, g', E_g)$ has an amalgam (E, s, t), say. Now $ti_g : C \to E$ and $si_f : C \to E$ are homomorphisms such that $ti_g(b) \neq si_f(b)$, contradicting the fact that $b \in \mathrm{dom}_C(A)$, thus proving our claim.

Now to show $\mathcal{V}$ has IPA, let A be a subalgebra of $B \in \mathcal{V}$. We show A is absolutely closed in $\mathcal{V}$. Now the case $A = B$ is trivial, so we may assume A is a proper subalgebra, and pick $b \in B \setminus A$. Let C_b be the subalgebra generated by $A \cup \{b\}$. Now A is a proper subalgebra of C_b, and since $\mathcal{V}$ has ES, A is saturated, $\mathrm{dom}_{C_b}(A)$ is a proper subalgebra of C_b, that is, $b \notin \mathrm{dom}_{C_b}(A)$. Suppose now that $b \in \mathrm{dom}_B(A)$. Then $b \in \mathrm{dom}_B(A) \cap C_b$, and by the claim above, $b \in \mathrm{dom}_{C_b}(A)$, a contradiction. Thus we have shown that for all $b \in B \setminus A$, $b \notin \mathrm{dom}_B(A)$, thus $\mathrm{dom}_B(A) = A$, proving the theorem. □

2.2. The Commutator Theory

Introduction and Basic Definitions. Commutator theory is an extension of the concept of the theory of commutator subgroups to the universal algebraic setting, first discovered by Hagemann and Herrmann [79]. The commutator operation $[\cdot, \cdot]$, a binary operation on the congruence lattice of an algebra, has particularly nice properties when the congruence lattice is modular (such as for groups). In particular it is meet-dominated (that is, for all congruences α, β, $[\alpha, \beta] \leq \alpha \cdot \beta$) and join preserving (that is, for all congruences α, β, γ, $[\alpha, \beta + \gamma] = [\alpha, \beta] + [\alpha, \gamma]$). It is these two properties which allow one to extend many results from congruence distributive varieties to congruence modular varieties, by observing that the commutator operation in modular varieties behaves rather like meets behave in congruence distributive varieties.

I present in this section a brief introduction to commutator theory, stating most of the basic results without proof. For a more extensive coverage of the commutator theory, the standard reference work is Freese and McKenzie [87].

DEFINITION 2.2.1. Let $\mathcal{V}$ be a variety, $A \in \mathcal{V}$ and let $\alpha, \beta, \delta \in \mathrm{Con}(A)$.

(1) We define $M(\alpha, \beta)$ to be the set of all 2×2 matrices of the form

$$\begin{vmatrix} t(\underline{a}^1, \underline{b}^1) & t(\underline{a}^1, \underline{b}^2) \\ t(\underline{a}^2, \underline{b}^1) & t(\underline{a}^2, \underline{b}^2) \end{vmatrix}$$

where $\underline{a}^1, \underline{a}^2$ are sequences of n elements in A, and $\underline{b}^1, \underline{b}^2$ are sequences of m elements in A such that $a^1_k \ \alpha \ a^2_k$ and $b^1_j \ \beta \ b^2_j$ for $k < n$ and $j < m$, and t is an $n + m$ variable term operation on A.

(2) We say α *centralizes* β *modulo* δ, (written $C(\alpha, \beta; \delta)$, if for every $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \in M(\alpha, \beta)$, $a \ \delta \ b$ implies $c \ \delta \ d$.

(3) $C(\alpha, \beta)$ is defined to be the smallest $\delta \in \mathrm{Con}(A)$ such that $C(\alpha, \beta; \delta)$ holds.

(4) $[\alpha, \beta]$ is defined to be the smallest δ for which both $C(\alpha, \beta; \delta)$ and $C(\beta, \alpha; \delta)$ hold.

Note $C(\alpha, \beta; \alpha \cdot \beta)$ and if $C(\alpha, \beta; \delta_i)$ holds for all $i \in I$, then $C(\alpha, \beta; \bigcap_{i \in I} \delta_i)$ holds, so our definitions are well-defined.

We have the following theorem:

THEOREM 2.2.2. *Let $\mathcal{V}$ be a congruence modular variety, and let $\alpha, \beta, \delta \in Con(A)$, where $A \in \mathcal{V}$.*

(1) *The following are equivalent:*
   (i) $C(\alpha, \beta; \delta)$ holds
   (ii) $C(\beta, \alpha; \delta)$ holds
   (iii) $[\alpha, \beta] \leq \delta$

(2) $C(\alpha, \beta; \delta) = C(\beta, \alpha; \delta) = [\alpha, \beta]$ □

We call the binary operation on $[\cdot, \cdot]$ the *commutator operation.* Note it is a generalization of the commutator operation on normal groups,

$$[N, H] = \left\{ nhn^{-1}h^{-1} \in G : n \in N, h \in H \right\},$$

where N and H are normal subgroups of a group G, in the sense that if θ and ψ are the congruences induced by N and H respectively, $[\theta, \psi]$ is just the congruence that is induced by the normal group $[N, H]$.

DEFINITION 2.2.3. Let $A \in \mathcal{V}$, a congruence modular variety, and let $\theta, \psi \in \mathrm{Con}(A)$.

(1) We define $A(\theta)$ to be θ viewed as a subalgebra of A^2.

(2) We define $A(\theta, \psi) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a \ \psi \ b, c \ \psi \ d, a \ \theta \ c, b \ \theta \ c \right\}$ (Note we can view $A(\theta, \psi)$ as a subalgebra of $A(\psi)^2$ or $A(\theta)^2$.)

(3) We define $\Delta_{\theta,\psi}$ to be the congruence on $A(\theta)$ generated by

$$\left\{\left\langle\begin{pmatrix}u\\u\end{pmatrix},\begin{pmatrix}v\\v\end{pmatrix}\right\rangle : \begin{bmatrix}u & v\\ u & v\end{bmatrix} \in A(\theta,\psi)\right\}$$

(4) We define $\Delta^{\theta,\psi}$ to be the congruence on $A(\psi)$ generated by

$$\left\{\left\langle\begin{pmatrix}x\\y\end{pmatrix},\begin{pmatrix}x\\y\end{pmatrix}\right\rangle : \begin{bmatrix}x & x\\ y & y\end{bmatrix} \in A(\theta,\psi)\right\}$$

THEOREM 2.2.4. *Let $A \in \mathcal{V}$, a congruence modular variety, and let $\theta, \psi \in$ Con(A). Then,*

(1) $\Delta_{\theta,\psi} = \Delta^{\theta,\psi}$.

(2) *For $x, y \in A$, the following are equivalent:*
  (i) $\langle x, y\rangle \in [\theta, \psi]$.
  (ii) $\left\langle \binom{x}{x}, \binom{y}{y} \right\rangle \in \Delta_{\theta,\psi}$.
  (iii) For some $a \in A$, $\left\langle \binom{x}{y}, \binom{a}{a} \right\rangle \in \Delta_{\theta,\psi}$.
  (iv) For some $b \in A$, $\langle \binom{x}{b}, \binom{y}{b} \rangle \in \Delta_{\theta,\psi}$. □

Note: For the rest of the chapter, we will assume $\mathcal{V}$ is congruence modular.

Given $A, B \in \mathcal{V}$, $f : A \to B$ a homomorphism and $\theta \in \mathrm{Con}(B)$, we will take both $\theta|_A$ and $f^{-1}(\theta)$ to be the congruence on A generated by $\{(x,y) \in A^2 : f(x)\ \theta\ f(y)\}$.

In particular, if $\{A_i : i \in I\}$ is a family of algebras, and $\theta \in \mathrm{Con}(A_k)$ for some $k \in I$, we define $\theta_k \in \mathrm{Con}(\prod\limits_{i \in I} A_i)$ to be $\pi_k^{-1}(\theta)$, where $\pi_k : \prod\limits_{i\in I} A_i \twoheadrightarrow A_k$ is the projection map. We will denote $\mathbf{0}_k$ by η_k to avoid confusion.

Properties of the Commutator. I now list some properties of the commutator.

THEOREM 2.2.5. *Let $A, B \in \mathcal{V}$, $\alpha, \beta \in$ Con(A) and $\gamma, \delta \in$ Con(B). The commutator:*

(1) *is additive :* $[\alpha, \sum S] = \sum_{\theta \in S} [\alpha, \theta])$ *for all* $S \subseteq$ *Con*(A)
(2) *is symmetric :* $[\alpha, \beta] = [\beta, \alpha]$
(3) *is meet-dominated :* $[\alpha, \beta] \leq \alpha \cdot \beta$
(4) *is monotone: If $\alpha', \beta' \in$ Con(A) such that $\alpha \leq \alpha', \beta \leq \beta'$, then* $[\alpha, \beta] \leq [\alpha', \beta']$

(5) *respects direct products* : $[\alpha_0 \cdot \gamma_1, \beta_0 \cdot \delta_1] = [\alpha, \beta]_0 \cdot [\gamma, \delta]_1$ *in* $Con(A \times B)$)

(6) *respects homomorphisms* : *If* $f : A \twoheadrightarrow B$ *is an epimorphism, then*

$$f^{-1}([\gamma,\delta]) = [f^{-1}(\gamma), f^{-1}(\delta)] + ker\ f. \quad \square$$

Since the congruence lattice of an algebra has infinite joins, and the commutator preserves joins, it is possible to derive a "pseudo-complementation" operation to the commutator.

DEFINITION 2.2.6. For congruences α, β on an algebra A, define $(\alpha : \beta)$, called the *centraliser of* α *with respect to* β, to be the largest congruence θ such that $[\beta, \theta] \leq \alpha$. The definition is well defined as $[\beta, \alpha] \leq \alpha$ by meet-dominance, and we can just take $(\alpha : \beta)$ to be the join of all θ such that $[\beta, \theta] \leq \alpha$ by join preservation.

Abelian algebras. One of the key generalizations that commutator theory yields from groups to universal algebras is the concept of an abelian algebra. As in group theory, abelian algebras have a particularly nice characterization.

DEFINITION 2.2.7. We call a congruence $\theta \in \mathrm{Con}(A)$ *abelian* if $[\theta, \theta] = \mathbf{0}_A$. An algebra A is said to be *abelian* if $\mathbf{1}_A$ is abelian. Note, for an abelian algebra A, for any $\alpha, \beta \in \mathrm{Con}(A)$ we have $[\alpha, \beta] = \mathbf{0}$ by monotonicity (theorem 2.2.5(4)). We call a variety *abelian* if all its members are abelian.

Note that the abelian algebras in any variety of groups are precisely the abelian groups (that is, groups satisfying $ab = ba$) in that variety.

DEFINITION 2.2.8. Let $A \in \mathcal{V}$, a variety. We call A *affine* if there is an abelian group $\hat{A} = \langle A, +, - \rangle$ having the same universe as A, and a ternary operation of A, $t(x, y, z)$ such that

(1) $t(x, y, z) = x - y + z$

(2) $f(\underline{x} - \underline{y} + \underline{z}) = f(\underline{x}) - f(\underline{y}) + f(\underline{z})$ for every term operation f and $\underline{x}, \underline{y}, \underline{z} \in A^n, n \in \omega$.

When these conditions hold, t is called a *difference term* and the algebra $\langle A, t \rangle$ is called a *ternary group*.

LEMMA 2.2.9. *For every congruence modular variety $\mathcal{V}$ there is a term d in three variables, called the Gumm term,* satisfying

(1) $d(x, y, z) = y$ is an identity in $\mathcal{V}$.

(2) If $(x, y) \in \theta$, where $\theta \in \mathrm{Con}(A)$, $A \in \mathcal{V}$, then $d(x, y, y)\ [\theta, \theta]\ x$

(3) For all $A \in \mathcal{V}$, $\theta \in \mathrm{Con}(A)$ is abelian if and only if for any $a \in A$, the congruence class a/θ is a ternary group with difference term the Gumm term, and whenever $s(u_1, \dots, u_n) = u$ where s is an n-ary term operation then $s : u_1/\theta \times \cdots \times u_n/\theta \to u/\theta$ is a group homomorphism. □

THEOREM 2.2.10. *For a congruence modular variety $\mathcal{V}$, every abelian algebra is affine, and conversely.* □

Abelian algebras have been described in detail by Hagemann and Herrmann [79]. In fact, every abelian algebra is polynomially equivalent to a module over a ring. Using this description, it is possibly to derive the following useful facts about abelian algebras.

THEOREM 2.2.11 (Hagemann and Herrmann [79]).

(1) *Every abelian algebra has the C.E.P.*

(2) *Given a variety $\mathcal{V}$, the abelian algebras form a subvariety of $\mathcal{V}$*

(3) *If $B \in \mathcal{V}$ is abelian, there are isomorphisms $\lambda_0 : B \times B \to B_\nabla \times B$ and $\lambda_1 : B \times B \to B \times B_\nabla$ given by $\lambda_0(x, y) = ((x, y)/\Delta_{1,1}, y)$ and $\lambda_1(x, y) = (x, (x, y)/\Delta_{1,1})$ where $B_\nabla = B/\Delta_{1,1}$, with $\Delta_{1,1}$ as in definition 2.2.3(3).* □

Commutator Identities. Recall that the goal of this chapter is a proof that in a congruence modular variety, AP and RS imply CEP. In Kiss [85], it was discovered that, in a congruence modular variety satisfying CEP, the congruence lattice of every algebra satisfies two identities involving the commutator operation, namely C2 and R (defined below). Also, in Freese and McKenzie [81], it was discovered that every residually small congruence modular variety satisfies another commutator identity, C1, the proof of which I present in this subsection. In the light of this, investigations were carried out as to which commutator identities are satisfied by varieties satisfying RS and AP, and the positive results in this line of investigation yielded the sought-after result which I present fully in section 2.4.

DEFINTION 2.2.12.

(1) We say an algebra A *satisfies C1* if for all $\alpha, \beta \in \mathrm{Con}(A)$, $\alpha \cdot [\beta, \beta] = [\alpha \cdot \beta, \beta]$

(2) We say an algebra A *satisfies C2* if for all $\alpha, \beta \in \mathrm{Con}(A)$, $[\alpha, \beta] = \alpha \cdot \beta \cdot [\mathbf{1}, \mathbf{1}]$.
A variety satisfies C1 (or C2) if every member of the variety satisfies C1 (or C2).

(3) We say a variety $\mathcal{V}$ satisfies R if for every $A \leq B \in \mathcal{V}$, $[\mathbf{1}_B, \mathbf{1}_B]|_A = [\mathbf{1}_A, \mathbf{1}_A]$

Note that C2 implies C1. We have the following useful characterization of C2.

LEMMA 2.2.13. *A variety $\mathcal{V}$ satisfies C2 if and only if every non-abelian subdirectly irreducible algebra has a non-abelian monolith.*

Proof.

$\Longrightarrow$: Let $A \in \mathcal{V}_{SI}$, and suppose A is non-abelian. Let θ be the monolith of A. Now, if $\mathcal{V}$ satisfies C2, then $[\theta, \theta] = \theta \cdot [\mathbf{1}_A, \mathbf{1}_A] = \theta > \mathbf{0}_A$, as $[\mathbf{1}_A, \mathbf{1}_A] > \mathbf{0}_A$ as A is non-abelian.

$\Longleftarrow$: Assume $\mathcal{V}$ does not satisfy C2. Then there is some $A \in \mathcal{V}$ and $\alpha, \beta \in \mathrm{Con}(A)$ such that $[\alpha, \beta] \neq \alpha \cdot \beta \cdot [1, 1]$. In particular, since $[\alpha, \beta] \leq \alpha \cdot \beta \cdot [1, 1]$ by theorem 2.2.5, using monotonicity and meet-dominance, we have

$$(*) \qquad [\alpha, \beta] \ngeq \alpha \cdot \beta \cdot [1, 1].$$

Putting $\lambda = \alpha \cdot \beta \cdot [1, 1]$, we see $[\lambda, \lambda] = [\alpha \cdot \beta \cdot [1, 1], \alpha \cdot \beta \cdot [1, 1]] \leq [\alpha, \beta]$, and hence by (*), $\lambda \neq [\lambda, \lambda]$.

Hence we have $[\lambda, \lambda] < \lambda \leq [1, 1]$. By theorem 1.2.4, there is a $\sigma \in \mathrm{Con}(A)$ such that $[\lambda, \lambda] \leq \sigma$, $\lambda \nleq \sigma$ and $A/\sigma \in \mathcal{V}_{SI}$. Let $\hat{\sigma}$ be the unique cover of σ. Then $\lambda + \sigma \geq \hat{\sigma}$, so

$$\begin{aligned} [\hat{\sigma}, \hat{\sigma}] &\leq [\lambda + \sigma, \lambda + \sigma] \\ &\leq [\lambda, \lambda] + [\sigma, \lambda] + [\sigma, \sigma] \\ &\leq [\lambda, \lambda] + \sigma \\ &\leq \sigma. \end{aligned}$$

Hence the monolith of A/σ is abelian. Since $[1, 1] \nleq \sigma$, we have A/σ is non-abelian and our theorem is proved. ☐

RS implies C1. We now show residual smallness implies C1. To do this, we need the following characterization of C1.

LEMMA 2.2.14. *An algebra A satisfies C1 if and only if for all $\alpha, \beta \in Con(A)$, $\alpha \leq [\beta, \beta]$ implies $\alpha = [\alpha, \beta]$.*

Proof.

$\Longrightarrow$: Let A satisfy C1 and let $\alpha, \beta \in \mathrm{Con}(A)$ such that $\alpha \leq [\beta, \beta]$. Hence

$$\begin{aligned} \alpha = \alpha \cdot [\beta, \beta] &= [\alpha \cdot \beta, \beta] \quad \text{(by C1)} \\ &= [\alpha, \beta] \quad \text{since } \alpha \leq \beta. \end{aligned}$$

$\Longleftarrow$: Let $\alpha, \beta \in \mathrm{Con}(A)$. Now since $\alpha \cdot [\beta, \beta] \leq [\beta, \beta]$, if the implication above is true, we have $\alpha \cdot [\beta, \beta] = [\alpha \cdot [\beta, \beta], \beta] \leq [\alpha \cdot \beta, \beta] \leq \alpha \cdot [\beta, \beta]$. Hence $\alpha \cdot [\beta, \beta] = [\alpha \cdot \beta, \beta]$, so A satisfies C1. □

THEOREM 2.2.15 (Freese and McKenzie [81]). *If $\mathcal{V}$ be residually small, then $\mathcal{V}$ satisfies C1.*

Proof. Suppose $\mathcal{V}$ does not satisfy C1. We will show that there are subdirectly irreducible algebras in $\mathcal{V}$ with arbitrarily large congruence lattices.

Since $\mathcal{V}$ does not satisfy C1, by lemma 2.2.14there is an $A \in \mathcal{V}, \beta, \gamma \in \mathrm{Con}(A)$ with $\beta \leq [\gamma, \gamma]$, but $\beta \neq [\beta, \gamma]$. Since $\beta \leq [\beta, \gamma]$, we have $[\beta, \gamma] < \beta$.

We may suppose A is subdirectly irreducible, with β its monolith and $[\beta, \gamma] = \mathbf{0}$: If not, by theorem 1.2.4, there exists $\theta \in \mathrm{Con}(A)$ such that $[\beta, \gamma] \leq \theta \ngeq \beta$ and A/θ is subdirectly irreducible. If θ^* is the unique cover of θ, put $\gamma^* = \theta + \gamma/\theta$, $\beta^* = \theta^*/\theta$. Note $\theta^* \leq [\theta + \gamma, \theta + \gamma] + \theta$. Also, we have

$$\begin{aligned} [\theta^*, \theta + \gamma] &= [\theta^*, \theta] + [\theta^*, \gamma] \\ &\leq \theta + [\theta + \beta, \gamma] \qquad \text{since } t^* \leq \theta + \beta \\ &\leq \theta + [\beta, \gamma] = \theta. \end{aligned}$$

Hence using the homomorphism property (theorem 2.2.5), note that $\beta^*, \gamma^* \in \mathrm{Con}(A/\theta)$, $\beta^* \leq [\gamma^*, \gamma^*]$ and $[\beta^*, \gamma^*] = \mathbf{0}_{A/\theta} < \beta^*$.

Put $A(\gamma)$ to be the subalgebra of A^2 as defined in definition 2.2.3, and let $\kappa = \Delta_{\gamma,\beta}$. Let η_0, η_1 be restrictions to $A(\gamma)$ of the kernels of the projections from A^2 to A. From the definition of $\Delta_{\gamma,\beta}$ and the fact that $[\beta, \gamma] = \mathbf{0}_A$, we can deduce:

$$(*) \qquad \kappa \cdot \eta_i = \mathbf{0}, \kappa + \eta_i = \beta_i, i = 0, 1$$

Let $\aleph$ be an arbitrary cardinal, and let

$$B = \left\{ (a_\delta) \in A^{\aleph} : a_\delta = a_\epsilon \text{ for all } \delta, \epsilon < \aleph \right\}.$$

For $\psi \in \mathrm{Con}(A)$ we denote by ψ_ϵ the relation $(a_\delta)\ \psi_\epsilon\ (\beta_\delta)$ if and only if $a_\epsilon\ \psi\ b_\epsilon$. Note $\psi_\epsilon \in \mathrm{Con}(B)$, and $\gamma_\delta = \gamma_\epsilon$ for all $\delta, \epsilon < \aleph$. In the light of this observation, we will denote $\gamma = \gamma_\epsilon$, and view γ as a congruence on B.

Let $\kappa_\delta \in \mathrm{Con}(B)$ be the set of pairs

$$\{ \langle (a_\epsilon), (b_\epsilon) \rangle : \langle a_0, a_\delta \rangle \ \Delta_{\gamma,\beta}\ \langle b_0, b_\delta \rangle \text{ and } a_\epsilon = b_\epsilon \text{ for } \epsilon \neq 0, \delta \}.$$

(Thus pairs of elements of B are congruent modulo κ_δ, if they are equal on all co-ordinates other than 0 and δ, and the image of the projection onto A^2 by the 0th and δth co-ordinates is congruent modulo $\Delta_{\gamma,\beta}$). For $\delta < \aleph$, define $\theta_\delta = \{ \langle (a_\epsilon), (b_\epsilon) \rangle : a_\delta\ \beta\ b_\delta \text{ and } a_\epsilon = b_\epsilon, \epsilon \neq \delta \}$. Let $\eta'_\delta = \bigcap_{\epsilon \neq \delta} \eta_\epsilon$, where η_ϵ is the restriction to B of the kernel of the ϵ-th projection on $A^{\aleph}$. Finally, set $\theta = \sum_\delta \theta_\delta$, and let $\kappa = \sum_{\delta > 0} \kappa_\delta$.

Now let $(a_\delta)\ \theta_\delta\ (b_\delta)$. Then $a_0\ \beta\ b_0$ and $a_\epsilon = b_\epsilon$ for all $\epsilon > 0$. Thus for a given $\delta < \aleph$,

$$\begin{aligned}
(a_\epsilon)_{\epsilon<\aleph}\ &\eta'_\delta\ (a_\epsilon)_{\epsilon<\aleph} \\
&\kappa_\delta\ (b_\epsilon)_{\epsilon<\aleph} [\text{ since } \langle a_0, a_0 \rangle\ \Delta\ \langle b_0, b_0 \rangle] \\
&\eta'_\delta\ (b_\epsilon)_{\epsilon<\aleph}
\end{aligned}$$

So $(a_\epsilon)_{\epsilon<\aleph}\ (\eta'_\delta + \kappa_\delta)\ (b_\epsilon)_{\epsilon<\aleph}$. Hence $\theta_0 \leq \eta'_\delta + \kappa_\delta$ for $\delta > 0$. Note $\theta_0 \leq \beta_\delta$ for all $\delta < \aleph$, $\eta'_\delta \cdot \beta_\delta = \theta_\delta$ and $\kappa_\delta \leq \beta_\delta$. Hence $\theta_0 = \theta_0 \cdot \beta_\delta \leq (\eta'_\delta + \kappa_\delta) \cdot \beta_\delta = \theta_\delta + \kappa_\delta$ by modularity, so $\theta_0 \leq \theta_\delta + \kappa_\delta$. Similarly, one can show that $\theta_\delta \leq \theta_0 + \kappa_\delta$.

Hence for $\delta \neq 0 \neq \epsilon$ we have

$$\begin{aligned}
\kappa + \theta_\delta &= \kappa + \kappa_\delta + \kappa_\epsilon + \theta_\delta && \text{since } \kappa_\delta, \kappa_\epsilon \leq \kappa, \\
&= \kappa + \kappa_\epsilon + \theta_0 + \theta_\delta && \text{since } \theta_0 \leq \theta_\delta + \kappa_\delta, \\
&= \kappa + \theta_\epsilon + \theta_0 + \theta_\delta && \text{since } \theta_\epsilon \leq \theta_0 + \kappa_\epsilon, \\
&\geq \theta_\epsilon + \theta_0.
\end{aligned}$$

Hence $\kappa + \theta_\delta \geq \theta$ for all $\delta < \aleph$.

Next we show that for all $\delta < \aleph$, $\theta_\delta \not\leq \kappa$:

We first show this for $\delta = \mathbf{0}$: Since β covers $\mathbf{0}_A$ in Con(A), θ_δ covers $\mathbf{0}_B$ in Con(B), and is thus compact. Hence, if $\theta_0 \leq \kappa$, then $\theta_0 \leq \kappa_{\epsilon_1} + \ldots + \kappa_{\epsilon_n}$ for some $\epsilon_1, \ldots, \epsilon_n > \mathbf{0}$. We proceed by induction on n to show that $\theta_0 \cdot (\kappa_{\epsilon_1} + \ldots + \kappa_{\epsilon_n}) = \mathbf{0}$.

$n = 1$: For any $\epsilon > 0$, $(a_\delta)\ \theta_0 \cdot \kappa\ (b_\delta)$ implies $a_\epsilon = b_\epsilon$ for $\epsilon \neq 0$ and $a_0\ \beta\ b_0$. Also, $\langle a_0, a_\delta \rangle\ \Delta_{\gamma,\beta}\ \langle b_0, a_\delta \rangle$, hence by theorem 2.2.4, $\langle a_0, b_0 \rangle \in [\beta, \gamma] = \mathbf{0}$ so $a_0 = b_0$. Hence $\theta_0 \cdot \kappa = \mathbf{0}$.

Suppose true for $n - 1$, where $n > 1$: Then we have

$$
\begin{aligned}
&\theta_0 \cdot (\kappa_{\epsilon_1} + \ldots + \kappa_{\epsilon_n}) \\
&= \theta_0(\theta_0 + \theta\epsilon_1)((\kappa_{\epsilon_1} + \ldots + \kappa_{\epsilon_n}) \\
&\qquad [\text{since } \theta_0 \leq \theta_0 + \theta_{\epsilon_1}] \\
&= \theta_0 \cdot (\kappa_{\epsilon_1} + (\theta_0 + \theta_{\epsilon_1}) \cdot (\kappa_{\epsilon_2} + \ldots + \kappa_{\epsilon_n})) \\
&\qquad [\text{by modularity using } \kappa_{\epsilon_1} \leq \theta_0 + \theta_{\epsilon_1}, \\
&\qquad\qquad \text{which follows from the definition of } \kappa_{\epsilon_1}] \\
&= \theta_0 \cdot (\kappa_{\epsilon_1} + (\theta_0 + \theta_{\epsilon_1}) \cdot (\theta_0 + \theta_{\epsilon_2} + \ldots + \theta_{\epsilon_n}) \cdot (\kappa_{\epsilon_2} + \ldots + \kappa_{\epsilon_n})) \\
&\qquad [\text{as } \theta_0 + \theta_{\epsilon_2} + \ldots + \theta_{\epsilon_n} \leq \kappa_{\epsilon_2} + \ldots + \kappa_{\epsilon_n}] \\
&= \theta_0 \cdot (\kappa_{\epsilon_1} + \theta_0 \cdot (\kappa_{\epsilon_2} + \ldots + \kappa_{\epsilon_n})) \\
&\qquad [\text{using } (\theta_0 + \theta_{\epsilon_1}) \cdot (\theta_0 + \theta_{\epsilon_2} + \ldots + \theta_{\epsilon_n}) = \theta_0 \text{ by definition of } \theta_{\epsilon_i}\text{'s}] \\
&= \theta_0 \cdot \kappa_{\epsilon_1} = \mathbf{0} \\
&\qquad [\text{by the induction hypothesis}].
\end{aligned}
$$

But then $\theta_0 \leq \kappa_{\epsilon_1} + \ldots + \kappa_{\epsilon_n}$ and $\theta_0 \cdot (\kappa_{\epsilon_1} + \ldots + \kappa_{\epsilon_n}) = \mathbf{0}$ implies $\theta_0 = \mathbf{0}$, a contradiction. Hence $\theta_0 \not\leq \kappa$.

Now for $\delta > \mathbf{0}$, $\kappa + \theta_\delta = \theta = \kappa + \theta_0$. Hence if $\theta_\delta \leq \kappa$, $\kappa + \theta_0 = \kappa + \theta_\delta = \kappa$, so $\kappa \geq \theta_0$, a contradiction. Hence for $\delta < \aleph$, $\theta_\delta \not\leq \kappa$.

From the above discussion, we can see that $\kappa < \theta$. Now by theorem 1.2.4, we can find $\lambda \in$ Con(B) such that B/λ is subdirectly irreducible and $\kappa \leq \lambda \not\geq \theta$. Now since β is a monolith of A, considering the δth projection η_δ of B, we have the interval γ/η_δ will have a unique atom, β_δ. By theorem 1.2.9, $\gamma/\eta_\delta = \eta_\delta + \eta'_\delta/\eta_\delta$ is isomorphic to $\eta_\delta/\eta_\delta \cdot \eta'_\delta = \eta_\delta/\mathbf{0}$, so $\eta_\delta/\mathbf{0}$ has a unique atom. In fact, since θ_δ

is an atom of $\eta_\delta/0$, it must be this atom. Now $\lambda \cdot \eta'_\delta = 0$, for if not, $\lambda \geq \theta_\delta$ hence $\lambda \geq \theta_\delta + \kappa \geq \theta$, a contradiction.

Now for all $\delta < \aleph$, $\lambda + \eta_\delta \not\geq \gamma$: For if $\lambda + \eta_\delta \geq \gamma$, then $[\gamma, \gamma] \leq [\eta_\delta + \eta'_\delta, \eta_\delta + \eta'_\delta] \leq \eta_\delta + \lambda \cdot \eta'_\delta \leq \eta_\delta$. But by the homomorphism property (theorem 2.2.5), using the δth projection, $[\gamma, \gamma] = 0$ in Con(A), contradicting the choice of γ.

Now for $\delta \neq \epsilon$ (where $\delta, \epsilon < \aleph$), we have $\eta_\delta + \lambda \neq \eta_\epsilon + \lambda$: For if $\eta_\delta = \eta_\epsilon$, then $\lambda + \eta_\delta = \lambda + \eta_\delta + \eta_\delta = \lambda + \gamma$ (since $\eta_\delta + \eta_\epsilon = \gamma$). But then $\lambda + \eta_\delta \geq \gamma$, contradicting the statement above.

Hence $|\mathrm{Con}(B/\lambda)| \geq \aleph$, proving the theorem. □

2.3 NAFSI Algebras

Introduction and Basic Definitions. In the general study of varieties, the subdirectly irreducible algebras play an important role, and congruence extension results for a general algebra in that variety can be obtained from congruence extension results involving just the subdirectly irreducibles. In congruence modular varieties, non-abelian finitely subdirectly irreducible(NAFSI) algebras play a similar role, and they were extensively studied by Kearnes [89]. The principal result is that for a congruence modular variety $\mathcal{V}$ satisfying AP and RS, if a NAFSI member $A \in \mathcal{V}$ is an essential extension of B, then B is NAFSI. One can compare this to a similar folkloric result for varieties satisfying CEP: if a subdirectly irreducible member of the variety A is an essential extension of B, then B is subdirectly irreducible.

DEFINITION 2.3.1.

(1) An algebra A is said to be *prime* if and only if, for $\alpha, \beta \in \mathrm{Con}(A)$, $[\alpha, \beta] = 0$ if and only if $\alpha = 0$ or $\beta = 0$.

(2) An algebra A is said to be *non-abelian finitely subdirectly irreducible (NAFSI)* if and only if $[1_A, 1_A] > 0_A$ (non-abelian) and for all $\alpha, \beta \in \mathrm{Con}(A)$, $\alpha \cdot \beta = 0_A$ implies $\alpha = 0_A$ or $\beta = 0_A$ (finitely subdirectly irreducible).

Note that if a variety $\mathcal{V}$ satisfies C2, then it can be seen that $A \in \mathcal{V}$ is NAFSI if and only if A is prime.

DEFINITION 2.3.2. Let L be a lattice. We say $x \in L$ is

(1) *neutral* if and only if for all $y, z \in L$, the lattice generated by $\{x, y, z\}$ is distributive.

(2) *distributive* if and only if for all $y, z \in L$, we have $x \cdot (y + z) = x \cdot y + x \cdot z$.

Now if L is a modular lattice, we have that $a \in L$ is neutral if and only if a is distributive. (See McKenzie, McNulty and Taylor [87], exercise 2.77(5).)

LEMMA 2.3.3. *Let $\mathcal{V}$ be a variety satisfying C2. Let $A_0, A_1 \in \mathcal{V}$, and let $\theta \in$ Con$(A_0), \psi \in$ Con$(A_0 \times A_1)$. Then*

(1) *$\theta \leq [\mathbf{1}_{A_0}, \mathbf{1}_{A_0}]$ implies θ is neutral in Con(A_0).*

(2) *$\psi \leq [\mathbf{1}_{A_0 \times A_1}, \mathbf{1}_{A_0 \times A_1}]$ implies ψ is a product congruence on $A_0 \times A_1$.*

Proof.

(1) Let $\alpha, \beta \in \mathrm{Con}(A)$. Then $\theta \cdot (\alpha + \beta) = \theta \cdot [1, 1] \cdot (\alpha + \beta) = [\theta, \alpha + \beta] = [\theta, \alpha] + [\theta, \beta] = \theta \cdot \alpha \cdot [1, 1] + \theta \cdot \beta \cdot [1, 1] = \theta \cdot \alpha + \theta \cdot \beta$ (using C2 twice and join preservation).

(2) ψ is neutral in $\mathrm{Con}(A_0 \times A_1)$ by (1). Hence

$$\begin{aligned} &(\eta_0 + \psi) \cdot (\eta_1 + \psi) \\ &= \eta_0 \cdot (\eta_1 + \psi) \text{ by modularity} \\ &= \eta_0 \cdot \eta_1 + \psi = \psi \text{ by neutrality.} \end{aligned}$$

Hence $\psi = \alpha_0 \cdot \beta_1$ for some $\alpha \in \mathrm{Con}(A_0), \beta \in \mathrm{Con}(A_1)$. □

LEMMA 2.3.4. *Let $\mathcal{V}$ satisfy C2, and let $A_0, A_1 \in \mathcal{V}$ with A_0 NAFSI be essential extensions of B_0 and B_1 respectively. Then $A = A_0 \times A_1$ is an essential extension of $B = B_0 \times B_1$.*

Proof. Pick $\psi \in \mathrm{Con}(A)$ such that $\psi|_B = \mathbf{0}$. Hence $\psi \cdot [\mathbf{1}_A, \mathbf{1}_A]|_B = \mathbf{0}$. Since $\psi \cdot [\mathbf{1}_A, \mathbf{1}_A]$ is a product congruence, and each factor A_i is an essential extension of B_i, $(i = 0, 1)$, we have $\psi \cdot [\mathbf{1}_A, \mathbf{1}_A] = \mathbf{0}$.

$$\begin{aligned} &\text{Now } (\eta_0 + \psi) \cdot (\eta_0 + [\mathbf{1}_A, \mathbf{1}_A]) \\ &= \eta_0 + \psi \cdot (\eta_0 + [\mathbf{1}_A, \mathbf{1}_A]) \text{ by modularity} \\ &= \eta_0 + \psi \cdot \eta_0 + \psi \cdot [\mathbf{1}_A, \mathbf{1}_A] \\ &\qquad \text{as } \{\psi, \eta_0, [\mathbf{1}_A, \mathbf{1}_A]\} \text{ generate a distributive lattice} \\ &= \eta_0. \end{aligned}$$

Now, as A is finitely subdirectly irreducible, $\eta_0 = \eta_0 + \psi$ or $\eta_0 = \eta_0 + [1_A, 1_A]$, that is, $\eta_0 \geq \psi$ or $\eta_0 \geq [1_A, 1_A]$. But A_0 is non-abelian, so $\eta_0 \ngeq [1_A, 1_A]$. Hence $\eta_0 \geq \psi$, so $\psi = \eta_0 \cdot \alpha_1$ for some $\alpha \in \mathrm{Con}(A_1)$ such that $\alpha|_{B_1} = 0$. But as A_1 is an essential extension of B_1, $\alpha = 0$ so $\psi = \eta_0 \cdot \eta_1 = 0$. □

COROLLARY 2.3.5. *Let V have C2, and let $A, B \in V$ with A an essential extension of B. Then A^2 and $A \times C$ are essential extensions of B^2 and $B \times C$ respectively.* □

Essential Extensions. I now state and prove a number of technical lemmas which explore the connections between NAFSI algebras and essential extensions.

LEMMA 2.3.6. *Let V satisfy C2, and let $A \in V$ be a NAFSI algebra with no proper essential extensions. Let $C \in V$ be an essential extension of A^2. Then there exists a $\psi \in$ Con(C) such that $\psi|_{A^2} = \eta_0^{A^2}$ or $\eta_1^{A^2}$.*

Proof. Let $\delta : A \hookrightarrow A^2$ be the embedding given by $\delta(a) = (a, a)$ for all $a \in A$. By Zorn's lemma, there exists a $\theta \in \mathrm{Con}(C)$ maximal with respect to the property $\theta|_A = 0$.

$$\begin{tikzcd} A \arrow[r, hook, "\delta"] & A^2 \arrow[r, hook] & C \arrow[r, two heads] & C/\theta \end{tikzcd}$$

Now the induced map from A to C/θ is an essential embedding by maximality, hence C/θ is isomorphic to A, as A has no proper essential extensions. Put $\tilde{\theta} = \theta|_{A^2}$. Hence as A^2/θ is isomorphic to A, $\tilde{\theta}$ is meet irreducible and $\tilde{\theta} \ngeq [1, 1]$ since A is NAFSI. But, by modularity and neutrality of $[1, 1]$ and $\eta_1 \cdot [1, 1]$, we have

$$\begin{aligned} & (\tilde{\theta} + \eta_0) \cdot (\tilde{\theta} + \eta_1) \cdot (\tilde{\theta} + [1, 1]) \\ = {} & (\tilde{\theta} + \eta_0) \cdot (\tilde{\theta} + \eta_1 \cdot [1, 1]) \\ = {} & \tilde{\theta}. \qquad \text{(Here } \eta_0, \eta_1 \in \mathrm{Con}(A^2).\text{)} \end{aligned}$$

Hence by the meet-irreducibility and the non-abelian property, $\tilde{\theta} \geq \eta_i$ for some $i = 0, 1$. Now if $\tilde{\theta} > \eta_i$, then $\theta|_A = \tilde{\theta}\big|_A > 0$, a contradiction. Hence $\tilde{\theta} = \theta|_{A^2} = \eta_i^{A^2}$. □

LEMMA 2.3.7. *Let $\mathcal{V}$ satisfy RS, AP and C2. Then if $A \in \mathcal{V}$, with A NAFSI and an essential extension of B, then B is non-abelian.*

Proof. First note that an essential extension C of a NAFSI algebra A is NAFSI: For if A is non-abelian, C contains a non-abelian algebra, and hence is non-abelian. Also, if C is not finitely subdirectly irreducible, then we can find $\alpha, \beta \in \mathrm{Con}(C)$ such that $\alpha \neq \mathbf{0}_C \neq \beta$, but then by essentiality $\alpha|_A \neq \mathbf{0}_A \neq \beta|_A$. Hence $\alpha|_A \cdot \beta|_A = \alpha \cdot \beta|_A = \mathbf{0}_C|_A = \mathbf{0}_A$, and so A is not finitely subdirectly irreducible.

Now since $\mathcal{V}$ is residually small, we may assume A is a maximal essential extension of B.

Now assume B is abelian, and let B_∇, $\lambda_0 : B \times B \to B_\nabla \times B$ and $\lambda_1 : B \times B \to B \times B_\nabla$ be as in theorem 2.2.11. Now since A is an extension of B, we can view λ_0, λ_1 as embeddings $\lambda_0 : B \times B \hookrightarrow B_\nabla \times A$ and $\lambda_1 : B \times B \hookrightarrow A \times B_\nabla$. Hence since $\mathcal{V}$ has AP, there is a $C \in \mathcal{V}$ and embeddings such that the diagram in figure 2.1 commutes.

$$\begin{tikzcd}
 & A^2 \arrow[rd, hook] & \\
B^2 \arrow[ru, hook, "\delta"] \arrow[r, hook, "\lambda_0"] \arrow[rd, hook, "\lambda_1"'] & B_\nabla \times A \arrow[r, hook] & C \\
 & A \times B_\nabla \arrow[ru, hook] &
\end{tikzcd}$$

FIGURE 2.1

(Note $\delta : B^2 \hookrightarrow A^2$ is the embedding given by $\delta(a, b) = (a, b)$ for all $a, b \in B$.)

By Zorn's lemma and commutativity of the diagram, we can choose $\epsilon \in \mathrm{Con}(C)$ maximal with respect to the condition $\epsilon|_{B^2} = \mathbf{0}$. Replacing C with $D = C/\epsilon$, we may assume that all the embeddings in the above diagram are essential.

Now by lemma 2.3.6, there is a $\psi \in \mathrm{Con}(D)$ such that $\psi|_{A^2} = \eta_0^{A^2}$ or $\eta_1^{A^2}$. Without loss of generality, we may assume $\psi|_{A^2} = \eta_0^{A^2}$. Putting $\tilde{\psi} = \psi|_{B_\nabla \times A}$, we see $\tilde{\psi}\big|_{B^2} = \eta_0^{A^2}\big|_{B^2} = \eta_0^{B^2}$, by the commutativity of the diagram. Also, $\eta_0^{B_\nabla \times A}\big|_{B^2} = \Delta_{\mathbf{0},\mathbf{0}}$ and $\eta_1^{B_\nabla \times A}\big|_{B^2} = \eta_1^{b^2}$. Hence for $i = 0, 1$, $\lambda_0^{-1}(\tilde{\psi} \cdot \eta_i^{B_\nabla \times A}) = \mathbf{0}_{B^2}$. Since λ_0 is essential (as B is an essential extension of A), we have $\tilde{\psi} \cdot \eta_i^{B_\nabla \times A} = \mathbf{0}_{B_\nabla \times A}$. Hence $\tilde{\psi} \cdot [\mathbf{1}, \mathbf{1}] \cdot \eta_i^{B_\nabla \times A} = \mathbf{0}$ for $i = 0, 1$ and as $\tilde{\psi} \cdot [\mathbf{1}, \mathbf{1}]$ is a product congruence, $\tilde{\psi} \cdot [\mathbf{1}, \mathbf{1}] = \mathbf{0}$.

Now, as before, using modularity and neutrality of $[1,1]$, $\eta_1 = (\eta_1 + \tilde{\psi}) \cdot (\eta_1 + [\mathbf{1},\mathbf{1}])$. As $B_\nabla \times A/\eta_1$ is isomorphic to A NAFSI, $\eta_1 \not\geq [\mathbf{1},\mathbf{1}]$ and hence $\eta_1 \geq \tilde{\psi}$. But then $\eta_0^{B^2} = \tilde{\psi}\Big|_{B^2} \leq \eta_1^{B_\nabla \times A}\Big|_{B^2} = \eta_1^{B^2}$, so B must be trivial. But then B has only trivial (and hence abelian) essential extensions, contradicting A is NAFSI. □

LEMMA 2.3.8. *Let $\mathcal{V}$ satisfy AP, RS and C2. If $A \in \mathcal{V}$ such that A is NAFSI is an essential extension of $B = B_0 \times B_1$, then either B_0 or B_1 is trivial.*

Proof. Again we may assume A a maximal essential extension of B with $i : B \hookrightarrow A$ the inclusion map. We denote by $\hat{A}$ a copy of A in order to make the notation easier. Let $C = B^2 = B_0 \times B_1 \times B_0 \times B_1$ and let $f_1 : C \hookrightarrow A^2$ and $g_1 : C \hookrightarrow \hat{A}^2$ be given by $f_1(a,b,c,d) = (i(a,b), i(c,d))$ and $g_1(a,b,c,d) = (i(a,d), i(c,d))$. Now the double extension $(C, f_1, A^2, g_1, \hat{A}^2)$ has an amalgam, say (f_2, g_2, F). Now since f_1 and g_1 are essential, we may as in lemma 2.3.7 assume F is chosen so that f_2 and g_2 are essential embeddings. Hence, by lemma 2.3.6, there is a congruence $\psi \in \mathrm{Con}(F)$ such that $g_2^{-1}(\psi) = \eta_0^{\hat{A}^2}$ or $\eta_1^{\hat{A}^2}$. Let $\tilde{\psi} = f_2^{-1}(\psi)$ and set $\beta = \eta_0^{A^2} + \tilde{\psi} \cdot \eta_1^{A^2}$.

We claim: $f^{-1}(\beta) = \eta_0^C$.

Proof of claim:

$\supseteq$:

$$\begin{aligned} f_1^{-1}(\beta) &\supseteq f_1^{-1}(\eta_0^{A^2}) + f_1^{-1}(\tilde{\psi} \cdot \eta_1^{A^2}) \\ &= \eta_{0\times 1}^C + \eta_{0\times 2\times 3}^C \\ &= \eta_0^C. \end{aligned}$$

$\subseteq$: Let $(x,y) \in f_1^{-1}(\beta)$, say $x = (a,b,c,d)$ and $y = (a',b',c',d')$. Then $(f_1(x), f_1(y)) \in \eta_0 + \tilde{\psi} \cdot \eta_1$. Now $\eta_0 + \tilde{\psi} \cdot \eta_1 = \eta_0 \circ (\tilde{\psi} \cdot \eta_1)$ since the projection congruences permute with every congruence of a direct product. Hence there is a $(u,v) \in A^2$ such that $f_1(x) = (i(a,b), i(c,d))\ \eta_0\ (u,v)\ \tilde{\psi} \cdot \eta_1\ (i(a',b'), i(c',d')) = f_1(y)$. Hence $(u,v) = (i(a,b), i(c',d')) \in \mathrm{im}\, f_1$. Hence $(x,y) \in f_1^{-1}(\eta_0) + f_1^{-1}(\psi \cdot \eta_1) = \eta_0^C$.

Similarly, we can let $\gamma = \eta_1^{A^2} + \tilde{\psi} \cdot \eta_0^{A^2}$ and get $f_1^{-1} = \eta_3^C$.

Now A^2/η_0 is isomorphic to A. So let α and α' be congruences on A that correspond to the congruences β/η_0 and γ/η_0 respectively on A^2/η_0. By the claim above, $i^{-1}(\alpha) = \eta_0^{B_0\times B_1}$ and $i^{-1}(\alpha') = \eta_1^{B_0\times B_1}$, so $i^{-1}(\alpha\cdot\alpha') = \mathbf{0}_B$. Since i is essential, $\alpha\cdot\alpha' = \mathbf{0}$. Hence as A is finitely subdirectly irreducible, $\alpha = \mathbf{0}$ or $\alpha' = \mathbf{0}$. Without loss of generality, assume that $\alpha = \mathbf{0}_A$. Then $\eta_0^B = i^{-1}(\alpha) = i^{-1}(\mathbf{0}_A) = \mathbf{0}_B$, so B_1 is trivial. (Similarly if $\alpha' = \mathbf{0}$, then B_0 would be trivial). □

The result below was stated and proved directly in C. Bergman [86], but it follows as a corollary of the previous lemma.

COROLLARY 2.3.9. *Let $\mathcal{V}$ satisfy AP, RS and C2. Let $A \in \mathcal{V}_{SI}$ and assume A is an essential extension of $B_0 \times B_1$. Then either B_0 and B_1 is trivial.*

Proof. Suppose A is abelian, then A satisfies CEP. Hence if $\theta \in \mathrm{Con}(A)$ is the monolith of A, $\theta|_{B_0\times B_1} \neq \mathbf{0}_{B_0\times B_1}$ and for any $\psi \in \mathrm{Con}(B_0 \times B_1), \psi \neq \mathbf{0}$ we have $\psi = \bar{\psi}|_{B_0\times B_1}$ for some $\bar{\psi} \in \mathrm{Con}(A)$. Since $\bar{\psi} \neq \mathbf{0}_A$, $\bar{\psi} \supseteq \theta$ and hence $\psi \supseteq \theta|_{B_0\times B_1}$, so $\theta|_{B_0\times B_1}$ is a monolith of $B_0 \times B_1$, hence $B_0 \times B_1 \in \mathcal{V}_{SI}$, a contradiction. Hence A is NAFSI and the result follows. □

Essential Subalgebras of NAFSI Algebras.

The principal result of this section can now be stated and proved.

THEOREM 2.3.10 (Kearnes [89]). *Let $\mathcal{V}$ satisfy AP, RS and C2. If A is NAFSI, and A is an essential extension of B, then B is NAFSI.*

Proof. By lemma 2.3.7, B is non-abelian, that is $[\mathbf{1},\mathbf{1}] \neq \mathbf{0}$. Now suppose there exist $\alpha', \beta' \in \mathrm{Con}(B), \alpha', \beta' > \mathbf{0}$ such that $\alpha' \cdot \beta' = \mathbf{0}$.

Now if such α', β' exist, we show we can in fact find $\alpha, \bar{\beta} \in \mathrm{Con}(B)$ such that $\alpha\cdot\bar{\beta} = \mathbf{0}$, $\alpha, \bar{\beta} > \mathbf{0}$ and $\alpha \leq [\mathbf{1},\mathbf{1}]$: To see this, put $\alpha = [\mathbf{1},\mathbf{1}], \bar{\beta} = \alpha'$ if $\alpha'\cdot[\mathbf{1},\mathbf{1}] = \mathbf{0}$, else put $\alpha = \alpha' \cdot [\mathbf{1},\mathbf{1}], \bar{\beta} = \beta'$.

Now let $S = \{\gamma \in \mathrm{Con}(B) : \alpha \cdot \gamma = \mathbf{0}\}$. Put $\beta = \sum S > \mathbf{0}$, since $\bar{\beta} > \mathbf{0}$. Also, $\alpha \cdot \beta = \alpha \cdot \beta \cdot [\mathbf{1},\mathbf{1}] = [\alpha,\beta] = \sum_{\gamma\in S}[\alpha,\gamma] \leq \sum_{\gamma\in S}\alpha\cdot\gamma = \mathbf{0}$. Hence we have $\alpha, \beta \in \mathrm{Con}(B), \alpha, \beta > \mathbf{0}, \alpha \leq [\mathbf{1},\mathbf{1}], \alpha \cdot \beta = \mathbf{0}$ and β is the largest congruence such that $\alpha \cdot \beta = \mathbf{0}$.

Let $B_0 = B/\alpha$ and $B_1 = B/\beta$, and let $g : B \hookrightarrow B_0 \times B_1$ be a canonical subdirect embedding. Now the double extension $(B, f, A, g, B_0 \times B_1)$ (where $f : B \hookrightarrow A$ is the inclusion map) has an amalgam, say (f', g', C). We may choose $\epsilon \in \mathrm{Con}(C)$ maximal to the condition $\epsilon|_B = \mathbf{0}_A$ by Zorn's lemma. Let $\tilde{\epsilon} = \epsilon|_{B_0} \times B_1$. Now in $\mathrm{Con}(B_0 \times B_1)$, $\tilde{\epsilon} \cdot [\mathbf{1}, \mathbf{1}]$ is a product congruence, say $\tilde{\epsilon} \cdot [\mathbf{1}, \mathbf{1}] = \gamma_0 \cdot \gamma_1'$, where $\gamma \in \mathrm{Con}(B_0), \gamma' \in \mathrm{Con}(B_1)$. Now labelling $\gamma_0|_B = \tilde{\gamma} \geq \alpha$ and $\gamma_1'|_B = \tilde{\gamma}' \geq \beta$, we see that $\alpha \cdot \tilde{\gamma}' \leq \tilde{\gamma} \cdot \tilde{\gamma}' = \mathbf{0}_B$. Hence by maximality of β, $\tilde{\gamma}' \leq \beta$, so $\tilde{\gamma}' = \beta$. Hence in $\mathrm{Con}(B_0 \times B_1)$, $\epsilon \cdot [\mathbf{1}, \mathbf{1}] \leq \tilde{\gamma}_1' = \eta_1$, hence using neutrality of $[\mathbf{1}, \mathbf{1}]$ and modularity we get $\eta_1 = (\eta_1 + \tilde{\epsilon}) \cdot (\eta_1 + [\mathbf{1}, \mathbf{1}])$. However, $\alpha \cdot (\eta_1|_B) = \alpha \cdot \beta = \mathbf{0}_A$ and $\alpha \cdot (\eta_1 + [\mathbf{1}_{B_0 \times B_1}, \mathbf{1}_{B_0 \times B_1}]|_B \geq \alpha \cdot [\mathbf{1}_B, \mathbf{1}_B] = \alpha$. Hence $\alpha \cdot \eta_1 + \tilde{\epsilon})|_B = \mathbf{0}$. But then, $\beta = \eta_1|_B \leq (\eta_1 + \tilde{\epsilon})|_B \leq \beta$, by maximality of β, hence $\eta_1 + \tilde{\epsilon} = \eta_1$, that is, $\tilde{\epsilon} \leq \eta_1$. Thus $\tilde{\epsilon} = \tau_0 \cdot \eta_1$ for some $\tau \in \mathrm{Con}(B_0)$.

Hence we get induced embeddings $\bar{g} : B \hookrightarrow b_0/\tau \times B_1$, $\bar{g}' : b_0/\tau \times B_1 \hookrightarrow C/\epsilon$ and $\bar{f}' : A \hookrightarrow C/\epsilon$ such that $\bar{g}'g = \bar{f}'f$. Now f is essential by hypothesis, $\bar{f}'$ is essential by our choice of ϵ, and a check shows $\bar{g}'$ is essential. Since A is NAFSI and $\bar{f}'$ essential, we have C/ϵ is NAFSI. Hence by lemma 2.3.8, B_0/τ or B_1 is trivial. But then of the compositions of the projection maps of $b_0/\tau \times B_1$ with $\bar{g}$ is an embedding, forcing either α or β to be $\mathbf{0}$, a contradiction. Thus B is finitely subdirectly irreducible and the result is proved. □

2.4 Congruence Modular Varieties: AP and RS imply CEP

Introduction. I can now present a proof of the main result given by the title of this section. The programme for proving the result is to use the various commutator identities as stepping stones to obtain the desired result. I sketch this programme below:

(1) RS implies C1 (Theorem 2.2.15)
(2) AP, RS and C1 imply C2 (Theorem 2.4.3)
(3) AP, RS and C2 imply R (Theorem 2.4.4)
(4) AP, RS, C2 and R imply CEP (Theorem 2.4.5)

The first result has been proven already. Proofs of the other results follow below.

AP and RS imply C2. To show the second in the series of four results given above, I introduce the following lemmas which examine the nature of subdirectly irreducible algebras in varieties which fail C2.

LEMMA 2.4.1. *Let $\mathcal{V}$ be residually small. If $\mathcal{V}$ fails C2, then there is an $A \in \mathcal{V}_{SI}$ with monolith β, and an endomorphism f such that*

(1) $\mathbf{0} = [\beta, \beta] < \beta = [\beta, \mathbf{1}]$.
(2) $f = f \circ f$.
(3) $x \; \beta \; y$ *if and only if* $f(x) = f(y)$ *and* $f(y) \; \beta \; y$.

Proof. Since $\mathcal{V}$ is residually small, by lemma 2.2.15 we have that $\mathcal{V}$ satisfies C1. If $\mathcal{V}$ fails C2, by lemma 2.2.13, there is a non-abelian subdirectly irreducible $D \in \mathcal{V}$ with abelian monolith γ, that is, $[\gamma, \gamma] = \mathbf{0}$. Let $\kappa = (\mathbf{0} : \gamma)$. Suppose $[\kappa, \kappa] \geq \mathbf{0}$. Then $[\kappa, \kappa] \geq \gamma$, so since $\mathcal{V}$ satisfies C1, by lemma 2.2.14, we have $\gamma \leq [\kappa, \kappa]$ implies $\gamma = [\kappa, \gamma]$, hence $\kappa = \gamma$ contradicting $[\gamma, \gamma] = \mathbf{0}$. Hence $[\kappa, \kappa] = \mathbf{0}$.

Since D is not abelian, $\kappa < \mathbf{1}$, and since $[\gamma, \gamma] = \mathbf{0}$, $k \geq \gamma$.

Let $\Delta = \Delta_{\gamma,\kappa}$ be the congruence on $D(\gamma)$ as in definition 2.2.3. We claim that for $i = 0, 1$ on $D(\gamma)$, $\kappa_i = (\eta_i : \gamma_i)$:

Proof of claim: If $[\delta, \gamma_i] \leq \eta_i$, then $[\delta + \eta_i, \gamma_i] = [\delta, \gamma_i] + [\eta_i, \gamma_i] \leq \eta_i$. Hence $(\eta_1 : \gamma_i) \geq \eta_i$ and by the homomorphism property of theorem 2.2.5, $(\eta_i : \gamma_i) = \kappa_i$.

Furthermore, using the properties of the commutator in theorem 2.2.5, the definition of δ and the fact that $[\gamma, \kappa] = \mathbf{0}$, we have that the following identities hold:

$$\left.\begin{aligned} \eta_i \prec \gamma_i &\leq \kappa_i \\ \gamma_0 = \gamma_1 &= \eta_0 + \eta_1 \\ \Delta + \eta_i &= \kappa_0 = \kappa_1 \\ \Delta \cdot \eta_1 &= \mathbf{0} \end{aligned}\right\} \text{ for } i = 0, 1.$$

Now the interval $\gamma_0 / \eta_0 = \eta_0 + \eta_1 / \eta_0$ is isomorphic to $\eta_1 / \eta_0 \cdot \eta_1 = \eta_1 / \mathbf{0}$ by theorem 1.2.9. A further application shows $\eta_1 / \mathbf{0} = \eta_1 / \Delta \cdot \eta_1$ is isomorphic to $\Delta + \eta_1 / \Delta = \kappa_0 / \Delta$. Similarly, one can show that γ_1 / η_1, $\eta_0 / \mathbf{0}$ and κ_0 / Δ are isomorphic. Hence, sicne $\gamma_0 = \gamma_1$ covers η_0 and η_1, η_0 and η_1 are atoms in $\mathrm{Con}(D(\gamma))$ and κ_0 covers Δ.

We now show that $D(\gamma)/\Delta$ is subdirectly irreducible by showing for all $\lambda > \Delta$, $\lambda \in \mathrm{Con}(D)$, we have $\lambda \geq \kappa_0$: For suppose $\lambda \not\geq \kappa_0$. Now $[\lambda, \gamma_0] \not\leq \eta_0$ as then $\kappa_0 \neq (\eta_0 : \gamma_0)$. Hence $[\gamma, \eta_1] \not\leq \eta_0$, for if $[\lambda, \eta_1] \leq \eta_0$, then $[\lambda, \gamma_0] = [\lambda, \eta_0 + \eta_1] = [\lambda, \eta_0] + [\lambda, \eta_1] \leq \eta_0$. Thus $\gamma \cdot \eta_1 \geq [\gamma, \eta_1] > \mathbf{0}$. But as η_1 is an atom, $\eta_1 \leq \lambda$. But then $\lambda \geq \eta_1 + \Delta = \kappa_0$, a contradiction.

Put $A = D(\gamma)/\Delta$ and note by the above argument that A is subdirectly irreducible with monolith $\beta = \kappa_0/\Delta$. Now $[\kappa_0, \kappa_0] = [\eta_0 + \Delta, \eta_1 + \Delta] \leq \Delta$ as $\eta_0, \eta_1 = \mathbf{0}$, so $[\beta, \beta] = \mathbf{0}$ on A. Now A is not abelian: For if A were abelian, then $[\eta_1, \mathbf{1}] \leq [\mathbf{1}, \mathbf{1}] \leq \Delta$, and hence $[\eta_1, \mathbf{1}] \leq \eta_1 \cdot \Delta = \mathbf{0}$. But then $[\gamma_0, \mathbf{1}] = [\eta_0 + \eta_1, \mathbf{1}] \leq \eta_0$, and hence $\kappa_0 = \mathbf{1}$, a contradiction.

Hence since is A is non-abelian, $\beta < [\mathbf{1}, \mathbf{1}]$ in A, so by C1 we have $\beta = \beta \cdot [\mathbf{1}, \mathbf{1}] = [\beta, \mathbf{1}]$ so (1) is satisfied. Define $f : A \to A$ by $f((x,y)/\Delta) = (x,x)/\Delta$. As $\Delta \leq \kappa_0$ this is well-defined. It is a routine exercise to check that f is an endomorphism satisfying (2) and (3). □

LEMMA 2.4.2. *Let $A \in \mathcal{V}$ be satisfy the conclusions of lemma 2.4.1 above. Then there are automorphisms e_0 and e_1 on $A(\beta)$ given by $e_0\binom{x}{y} = \binom{d(x,f(x),y)}{y}$ and $e_1\binom{x}{y} = \binom{x}{d(x,f(y),y)}$ where d is the difference term for $\mathcal{V}$.*

Proof. We give the proof for e_0, the proof for e_1 being similar.

Suppose $\binom{x}{y}, \binom{u}{v} \in A(\beta)$ with $e_0\binom{x}{y} = e_0\binom{u}{v}$. Then $y = v$, and hence $x \mathrel{\beta} y = v \mathrel{\beta} u$. Also, by lemma 2.4.1, we have $f(x) = f(y) \mathrel{\beta} y$. Thus $f(x), f(y), u, v, x$ and y all lie in the same equivalence class of β, and hence in the same ternary group $M(\beta, x)$ by lemma 2.2.9. Hence $x - f(x) + y = d(x, f(x), y) = d(u, f(x), y) = u - f(x) + y$. Hence $x = u$ and e_0 is injective.

Now given any $\binom{x}{y} \in A(\beta)$,

$$\begin{aligned} e_0\begin{pmatrix} d(x,y,f(x)) \\ y \end{pmatrix} &= e_0\begin{pmatrix} x - y + f(x) \\ y \end{pmatrix} \\ &= \begin{pmatrix} x - y + f(x) - f(x - y + f(x)) + y \\ y \end{pmatrix} \\ &= \begin{pmatrix} x \\ y \end{pmatrix}, \end{aligned}$$

so e_0 is surjective.

That e_0 is a homomorphism follows from the fact that every basic operation (in fact every term) is a ternary group homomorphism on the $M(\beta, x)$-blocks. □

THEOREM 2.4.3 (Bergman and McKenzie [88]). *If $\mathcal{V}$ satisfies AP and RS, then $\mathcal{V}$ satisfies C2.*

Proof. Assume that $\mathcal{V}$ satisfies AP and RS, but not C2. Then there exists an $A \in \mathcal{V}_{SI}$, $\beta \in \mathrm{Con}(A)$ and automorphisms e_0 and e_1 satisfying the conditions of lemma 2.4.2. We will derive a contradiction.

Since $\mathcal{V}$ satisfies RS, A has an extension $E \in \mathcal{V}_{WMI}$ say, with monolith, say μ. We may assume $\mu|_A \geq \beta$. We can view the automorphisms e_0 and e_1 as embeddings from $A(\beta)$ into E^2. A can be diagonally embedded into $A(\beta)$, and we let $e_2 : A(\beta) \hookrightarrow E^2$ be the embedding given by $e_2(a,b) = (a,b)$. Now, applying the amalgamation property twice, we have embeddings $s_i : E^2 \hookrightarrow Q$ for some $Q \in \mathcal{V}, i = 0,1,2$ such that $s_0e_0 = s_1e_1 = s_2e_1$. Let $r : E \hookrightarrow E^2$ be the diagonal embedding. Since $s_2r : E \hookrightarrow Q$ is an embedding and E an absolute retract, we have a retraction $u : Q \twoheadrightarrow E$ such that $us_2r = \mathrm{id}_E$. See figure 2.2. below.

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dd, hook] & & A(\beta) \arrow[dl, hook, "e_2"'] \arrow[d, hook, "e_1"] \arrow[dr, hook, "e_0"] & \\
& E^2 \arrow[dr, hook, "s_2"'] & E^2 \arrow[d, hook, "s_1"] & E^2 \arrow[dl, hook, "s_0"] \\
E \arrow[ur] & & Q \arrow[ll, two heads, "u"] &
\end{tikzcd}$$

FIGURE 2.2

Define $\rho_i = \ker\,(us_i) \in \mathrm{Con}(E^2)$ for $i = 0,1,2$. We claim that for each $i = 0,1,2$, if $\rho_i \neq \mathbf{0}$ then either

$$(+) \qquad \rho_i \geq \eta_0 \cdot \mu_1 \text{ and } \rho_i\eta_1 = \mathbf{0}$$

or

$$(++) \qquad \rho_i \geq \eta_1 \cdot \mu_0 \text{ and } \rho_i\eta_0 = \mathbf{0}.$$

Proof of claim: First note that for $\alpha \in \mathrm{Con}(E)$, $\alpha \neq \mathbf{0}_E$, we have

$$[\mathbf{1}_E, \alpha] \geq [\mathbf{1}_E, \mu] \supseteq [\mathbf{1}_A, \beta] = \beta \neq \mathbf{0}_A.$$

(*) Hence $[\mathbf{1}_E, \alpha] \neq \mathbf{0}$.

By the homomorphism property (theorem 2.2.5), $[\mathbf{1}_{E^2}, \alpha] \neq \mathbf{0}$ for $\alpha \in \mathrm{Con}(E^2)$, $\alpha \neq \mathbf{0}$. Hence, if $\rho_i \neq \mathbf{0}$, then $\mathbf{0}_{E^2} \neq [\mathbf{1}, \rho_i] = [\eta_0 + \eta_1, \rho_i] = [\eta_0, \rho_i] + [\eta_1, \rho_i]$. Hence either $[\eta_0, \rho_i]$ or $[\eta_1, \rho_i]$ is not $\mathbf{0}$.

Suppose $[\eta_0, \rho_i] \neq \mathbf{0}$. Then $\eta_0 \cdot \rho_i \geq [\eta_0, \rho_i] \neq \mathbf{0}$. By theorem 1.2.9, there is an isomorphism between the intervals $\eta_0 / \mathbf{0}$ and $\mathbf{1} / \eta_1$. Now, $\mathbf{1} / \eta_1$ has a unique atom μ_1 as E is subdirectly irreducible, so we have $\eta_0 \cdot \mu_1 \leq \eta_0 \cdot \rho_i \leq \rho_i$, satisfying the first part of (+).

To show the second part, suppose $\rho_i \cdot \eta_1 \neq \mathbf{0}$. By a similar argument to the one above, we have $\rho_i \geq \eta_1 \cdot \mu_0$ so $\rho_i \geq \eta_0 \cdot \mu_1 + \eta_1 \cdot \mu_0 \geq \mu_0 \cdot \mu_1$ by modularity. Choose $(a, b) \in \beta$ such that $a \neq b$. Now $\mu \supseteq \beta$ and $f(a) = f(b)$. Hence, since d is a term, $d(a, f(a), a)\ \mu\ d(b, f(b), b)$, and so $\left\langle e_i\binom{a}{a}, e_i\binom{b}{b} \right\rangle \in \mu_0 \cdot \mu_1 \leq \rho_i$. But then $us_i e_i\binom{a}{a} = us_i e_i\binom{b}{b}, i = 0, 1, 2$, implying by commutativity of the diagram that $a = us_2 e_i\binom{a}{a} = us_2 e_i\binom{b}{b} = b$, a contradiction. Hence $\rho_i \cdot \eta_i = \mathbf{0}$, and (+) is satisfied.

Note, if on the other hand, $[\eta_1, \rho_i] \neq \mathbf{0}$, then a similar argument shows (++) is satisfied, hence the claim is proved.

Now we apply the claim to ρ_2. Firstly $\rho_2 \neq \mathbf{0}$, for if we pick up $x \neq y$ in E and put $z = us_2\binom{x}{y}$ we have $us_2\binom{x}{y} = z = us_2 r(z) = us_2\binom{z}{z}$ so $\binom{x}{y}\ \rho_2\ \binom{z}{z}$. Hence by the claim above, either (+) or (++) holds for i=2. We will show that if (+) holds, we can derive a contradiction. (That a contradiction follows also from (++) is just the dual - in the sense of interchanging subscripts $i = 0$ with $i = 1$ and vice versa - of the proof for (+)).

If (+) holds, $\rho_2 \geq \eta_0 \cdot \mu_1$ and $\rho_2 \cdot \eta_1 = \mathbf{0}$. Hence, $[\mathbf{1}, \rho_2] = [\eta_0 + \eta_1, \rho_2] \leq \eta_0 + \eta_1 \cdot \rho_2 = \eta_0$. Hence $[\mathbf{1}, \rho_2 + \eta_0] \leq \eta_0$. Thus by the homomorphism property (theorem 2.2.5), together with (*), we have $\rho_2 + \eta_0 = \eta_0$, that is, $\rho_2 \leq \eta_0$. In fact, $\rho_2 = \eta_0$: For, letting $x, y \in E$, we have $\binom{x}{y} \equiv rus_2\binom{x}{y} = \binom{z}{z} \bmod \rho_2$ for some $z \in E$, and as $\rho_2 \leq \eta_0$, $x = z$. But then for any $x, y, y' \in E$, $\binom{x}{y}\rho_2\binom{x}{x}\rho_2\binom{x}{y'}$, so $\rho_2 = \eta_0$.

Thus we have $\eta_0^{A(\beta)} = e_2^{-1}(\rho_2) = \ker\,(us_2e_2) = \ker\, us_0e_0 = e_0^{-1}(\rho_0)$. Observe that for all $a \in A$, $e_0\binom{f(a)}{a} = \binom{d(f(a),f(a),a)}{a} = \binom{a}{a}$. Choose $a \neq b$ in A with $a\ \beta\ b$.

$$\begin{aligned}\text{Then } a\ \beta\ b \ &\Longrightarrow f(a) = f(b)\\ &\Longrightarrow \begin{pmatrix} f(a) \\ a \end{pmatrix} \eta_0^{A(\beta)} \begin{pmatrix} f(b) \\ B \end{pmatrix}\\ &\Longrightarrow us_0e_0\begin{pmatrix} f(a) \\ a \end{pmatrix} = us_0e_o\begin{pmatrix} f(b) \\ B \end{pmatrix}\\ &\Longrightarrow \begin{pmatrix} a \\ a \end{pmatrix} \rho_0 \begin{pmatrix} b \\ b \end{pmatrix}.\end{aligned}$$

Now $\rho_0 \neq 0$ as $\beta \neq 0$. Hence, applying the claim to ρ_0, we see that (+) or (++) holds for $i = 0$. If (+) holds, that is $\rho_0 \geq \eta \cdot \mu_1$, then $\binom{a}{b}\rho_0\ \binom{a}{a}\ \rho_0\ \binom{b}{b}$, contradicting $\rho_0 \cdot \eta_1 = 0$.

Hence (++) holds, that is $\rho_0 \geq \eta_1 \cdot \mu_0$. Then $\eta_0 = e_0^{-1}(\rho_0) \geq e_0^{-1}(\eta_1 \cdot \mu_0) \geq \eta_1 \cdot \beta_0$. But then $\binom{a}{b}\ \eta_1 \cdot \beta_0\ \binom{b}{b}$ implies $\binom{a}{b}\ \eta_0\ \binom{b}{b}$, that is $a = b$, a contradiction. Thus since both cases yield a contradiction, our theorem is proved. □

AP, RS and C2 imply R. I now state and prove the third theorem needed for the main result.

THEOREM 2.4.4 (Kearnes [89]). *Let $\mathcal{V}$ satisfy RS, AP and C2. Then $\mathcal{V}$ satisfies R.*

Proof. Suppose $\mathcal{V}$ has RS, AP and C2 but not R. Then there are algebras $B \leq A \in \mathcal{V}$ such that $[1_A, 1_A]|_B > [1_B, 1_B]$. Let $\alpha = [1_A, 1_A]|_B$ and let $\beta = [1_B, 1_B]$. Let $C = B/\beta$. Then, using the properties mentioned in theorem 2.2.5, we have that the lattice in figure 2.3 is a sublattice of $\mathrm{Con}(B \times C)$.

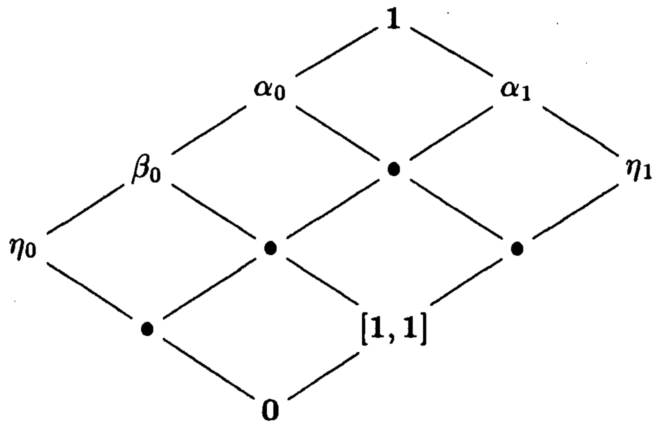


FIGURE 2.3

Considering $\Delta_{\mathbf{1},\alpha} \in \mathrm{Con}(B^2)$, let $\Delta \in \mathrm{Con}(B \times C)$ be the projection of $\Delta_{\mathbf{1},\alpha}$ onto $B \times C$. Now by C2, $[1_B, \alpha] = \alpha \cdot [1_B, 1_B] = \alpha \cdot \beta = \beta$. Hence $\eta_j \cdot \Delta_{\mathbf{1},\alpha} = \eta_j \cdot [1, \alpha]_{1-j} = \eta_j \cdot \beta_{1-j}$ for $j = 0, 1$. Also, using the definition of $\Delta_{\mathbf{1},\alpha}$, we see $\eta_j + \Delta_{\mathbf{1},\alpha} = \alpha_j$. (Here all congruences are in B^2). Hence η_j is a complement of $\Delta_{\mathbf{1},\alpha}$ in $\alpha_j / \eta_1 \cdot \beta_{1-j}$ where $i = 0, 1$. Now using theorem 1.2.11, we see $\Delta_{\mathbf{1},\alpha}$ is a complement of $\alpha_j \cdot \beta_{1-j}$ in $\beta_0 \cdot \beta_1 / \alpha_0 \cdot \alpha_1$ where $i = 0, 1$. Hence in $\mathrm{Con}(B \times C)$, Δ is a complement of $\alpha_0 \cdot \eta_1$ in $\beta_0 \cdot \eta_1 / \alpha_0 \cdot \alpha_1$ where $i = 0, 1$. Hence, we have that the lattice shown in figure 2.4 is a sublattice of $\mathrm{Con}(B \times C)$.

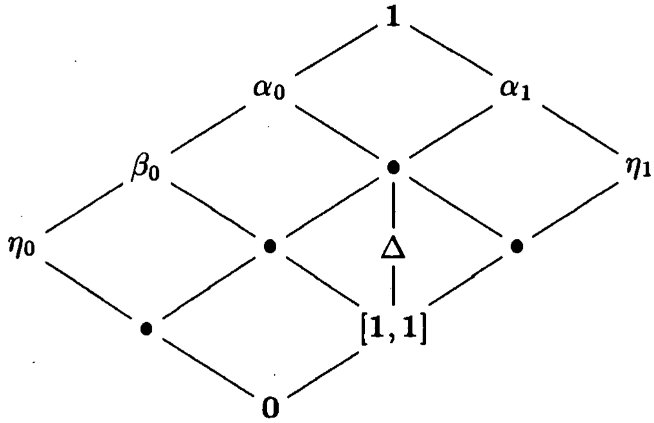


FIGURE 2.4

Let D denote the abelian algebra $(B \times C)/\Delta$, and i the inclusion map from B into A. Let $f_1 : B \times C \hookrightarrow A \times C$ be given by $f_1 = i \times \mathrm{id}_C$ and $g_1 : B \times C \to A \times D$ by $g_1(b, c) = (i(b), (b, c)/\Delta)$. Now g_1 is an embedding since $\ker g_1 = \Delta \cdot \eta_0 = \mathbf{0}$. Since $\mathcal{V}$ has AP, the double extension $(B \times C, f_1, A \times C, g_1, A \times D)$ has an amalgam, say (f_2, g_2, F).

Now $\eta_0 \cdot \alpha_1 \neq 0$, where $\eta_0 \cdot \alpha_1 \in \mathrm{Con}(B \times C)$, so there are $(b_1, c_1), (b_2, c_2) \in B \times C$, $(b_1, c_1) \neq (b_2, c_2)$ such that $(b_1, c_1)\ \eta_0 \cdot \alpha_1\ (b_2, c_2)$. Now $\langle (b_1, c_1), (b_2, c_2) \rangle \notin [\mathbf{1}_{B\times C}, \mathbf{1}_{B\times C}]$ as $\eta_0 \cdot \alpha_1 \cdot [\mathbf{1}_{B\times C}, \mathbf{1}_{B\times C}] = \mathbf{0}_{B\times C}$. Hence, by Zorn's lemma, we can find a maximal congruence $\tau \in \mathrm{Con}(F)$ with respect to the condition $\langle (b_1, c_1), (b_2, c_2) \rangle \notin \tau|_B \times C + [\mathbf{1}_{B\times C}, \mathbf{1}_{B\times C}]$. Now $[\mathbf{1}_F, \mathbf{1}_F]|_B \times C \geq [\mathbf{1}_{A\times C}, \mathbf{1}_{A\times C}]|_B \times C + [\mathbf{1}_F, \mathbf{1}_F]|_B \times C \geq [\mathbf{1}_{A\times D}, \mathbf{1}_{A\times D}]|_B \times C = \alpha_0 \cdot \eta_1 + \alpha_0 \cdot \Delta = \alpha_0 \cdot \alpha_1 \geq \eta_0 \cdot \alpha_1$. Thus since $\langle (b_1, c_1), (b_2, c_2) \rangle \in \eta_0 \cdot \alpha$, we have $\tau \not\geq [\mathbf{1}_F, \mathbf{1}_F]$.

Now τ is meet-irreducible (that is F/τ is finitely subdirectly irreducible): For if $\gamma, \gamma' > \tau$ then $< (b_1, c_1), (b_2, c_2) >$ is an element of both $\gamma|_B \times C + [\mathbf{1}, tc]$ and $\gamma'|_B \times C + [\mathbf{1}, tc]$ by maximality. Now by neutrality of $[\mathbf{1}, \mathbf{1}]$, $< (b_1, c_1), (b_2, c_2) > \in (\gamma|_B \times C + [\mathbf{1}, \mathbf{1}]) \cdot (\gamma'|_B \times C + [\mathbf{1}, \mathbf{1}]) = \gamma \cdot \gamma'|_B \times C + [\mathbf{1}, \mathbf{1}]$, and hence $\gamma \cdot \gamma' > \tau$.

Let $\tilde{\tau} = \tau|_B \times C$. Hence we induced embedding $k : B \times C/\tilde{\tau} \hookrightarrow F/\tau$. In fact, by maximality of τ, k is an essential embedding. Now $\tau \not\geq [\mathbf{1}_F, \mathbf{1}_F]$, that is, F/τ is non-abelian, and hence NAFSI. Thus by theorem 2.3.10, $(B \times C)/\tilde{\tau}$ is NAFSI. Now, in $\mathrm{Con}(B \times C)$, $\tilde{\tau} = (\tilde{\tau} + [\mathbf{1}, \mathbf{1}]) \cdot (\tilde{\tau} + \eta_0 \cdot \alpha_1)$ by neutrality of $[\mathbf{1}, \mathbf{1}]$. Hence by meet irreducibility, $\tilde{\tau} \geq [\mathbf{1}, \mathbf{1}]$ or $\tilde{\tau} \geq \eta_0 \cdot \alpha_1$. The former case is excluded as $B \times C/\tilde{\tau}$ is not abelian, so $\tilde{\tau} \geq \eta_0 \cdot \alpha_1$. But then $\langle (b_1, c_1), (b_2, c_2) \rangle \in \eta_0 \cdot \alpha_0 \leq \tilde{\tau} \subseteq \tau$, a contradiction. □

AP, RS, C2 and R imply CEP. A proof of the last result in the scheme mentioned in the introduction to this section follows.

THEOREM 2.4.5 (Bergman and McKenzie [88]). *Let $\mathcal{V}$ satisfy C2, R, AP and RS. Then $\mathcal{V}$ satisfies CEP.*

Proof. Let $A, B \in \mathcal{V}$, with $A \leq B$ and $\theta \in \mathrm{Con}(A)$ such that A/θ is subdirectly irreducible. To show $\mathcal{V}$ has CEP, we need only show the θ extends to B. By R, $[\mathbf{1}_B, \mathbf{1}_B]|_A = [\mathbf{1}_A, \mathbf{1}_A]$. Thus $A/[\mathbf{1}_A, \mathbf{1}_A]$ can be embedded in $B/[\mathbf{1}_B, \mathbf{1}_B]$.

CASE 1: $\theta \geq [\mathbf{1}_A, \mathbf{1}_A]$ on A. Since $B/[\mathbf{1}_B, \mathbf{1}_B]$ is has CEP, so the congruence $\theta/[\mathbf{1}_A, \mathbf{1}_A]$ in A extends to a congruence $\psi/[\mathbf{1}_B, \mathbf{1}_B]$ on $B/[\mathbf{1}_B, \mathbf{1}_B]$. Then $\psi \in \mathrm{Con}(B)$ and $\psi|_A = \theta$. (see figure 2.5.)

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d] & B \arrow[d] \\
A/[\mathbf{1}_A, \mathbf{1}_A] \arrow[r, hook] \arrow[d] & B/[\mathbf{1}_B, \mathbf{1}_B] \arrow[d] \\
A/\theta \arrow[r, hook] & B/\psi
\end{tikzcd}$$

FIGURE 2.5

CASE 2: $\theta \not\geq [\mathbf{1}_A, \mathbf{1}_A]$ on A. Let $f : A \hookrightarrow B$ be the inclusion map. Define $S = (A/\theta) \times A$ and let $g : A \hookrightarrow S$ be the embedding given by $g(a) = (a/\theta, a)$. Now (A, f, B, g, S) has an amalgam (f', g', D) with $D \in \mathcal{V}$. Let η_0, η_1 be kernels of the two projections on S. If μ is the monolith of A/θ, there is a congruence $\mu_0 \in \mathrm{Con}(S)$ covering η_0. Now by theorem 2.2.5, $[\mathbf{1}_S, \mathbf{1}_S] = [\mathbf{1}, \mathbf{1}]_0 \cdot [\mathbf{1}, \mathbf{1}]_1 \geq \mu_0 \cdot \eta_1$ since A/θ is subdirectly irreducible and A/θ is non-abelian.

CLAIM: For all $\beta \in \mathrm{Con}(S)$, $\beta \not\leq \eta_0$ if and only if $\beta \geq \mu_0 \cdot \eta_1$.

PROOF OF CLAIM: $\Longrightarrow$: If $\beta \not\leq \eta_0$, then $\beta + \eta_0 \geq \mu_0$. Thus

$$\begin{aligned}
\beta &\geq [\beta + \eta_0, \beta + \eta_1] \quad \text{since } \eta_0 \cdot \eta_1 = \mathbf{0} \\
&= (\beta + \eta_0) \cdot (\beta + \eta_1) \cdot [\mathbf{1}_S, \mathbf{1}_S] \quad \text{by C2} \\
&\geq \mu_0 \cdot \eta_1.
\end{aligned}$$

$\Longleftarrow$: $\eta_0 \geq \beta \geq \mu_0 \cdot \eta_1$ implies $\mu_0 \cdot \eta_1 = \mu_0 \cdot \eta_1 \eta_0 = \mathbf{0}$, a contradiction.

Now let γ be a maximal congruence on D such that ${g'}^{-1}(\gamma) \leq \eta_0$. (It exists by Zorn's lemma). Suppose γ were not completely meet irreducible, say $\gamma = \bigcap_{j \in I} \gamma_j$, with $\gamma_j > \gamma$ for all $j \in J$. Then by maximality ${g'}^{-1}(\gamma_j) \not\leq \eta_0$, so by the claim, ${g'}^{-1}(\gamma_j) \geq \mu_0 \cdot \eta_1$ for all $j \in J$. Thus ${g'}^{-1}(\gamma) \geq \mu_0 \cdot \eta_1$ which implies ${g'}^{-1}(\gamma) \not\leq \eta_0$, a contradiction. Hence D/γ is subdirectly irreducible.

Now $\eta_0 \cdot (\eta_1 + {g'}^{-1}(\gamma)) = {g'}^{-1}(\gamma)$ by modularity. Hence there is a $\delta \in \mathrm{Con}(B)$ such that ${g'}^{-1}(\gamma) = \eta_0 \cdot \delta_1$. Hence $S/{g'}^{-1}(\gamma)$ is isomorphic to $A/\theta \times A/\delta$ and we have an induced embedding $g' : A/\theta \times A/\delta \hookrightarrow D/\gamma$.

Moreover, by the maximality of γ, this embedding is essential. Thus by corollary 2.3.9, either A/θ or A/δ is trivial. But $\theta \not\geq [\mathbf{1}, \mathbf{1}]$, so A/δ is trivial. Hence $\delta = 1_A$, so $g'^{-1}(\gamma) = \eta_0$. A check shows $f'^{-1}(\gamma) \in \mathrm{Con}(B)$ extends θ, and the result follows. □

THEOREM 2.4.6 (Kearnes [89]). *Let V be a congruence modular variety. If V satisfies AP and RS, then V satisfies CEP.*

Proof. We have shown for a congruence modular variety,

(1) RS implies C1 (Theorem 2.2.15)
(2) AP, RS and C1 imply C2 (Theorem 2.4.3)
(3) AP, RS and C2 imply R (Theorem 2.4.4)
(4) AP, RS, C2 and R imply CEP (Theorem 2.4.5)

Hence the result follows. □

The results in this chapter are summarised in figure 2.6. Arrows represent implications between the stated properties for a variety (for example, "EI ⟶ TP" means that a variety with enough injectives (EI) satisfies the transferability property (TP)). Where an arrow has two sources, both those properties must be satisfied in order for the property at the target of the arrow to be satisfied. A broken arrow indicates an implication only holding in congruence modular varieties.

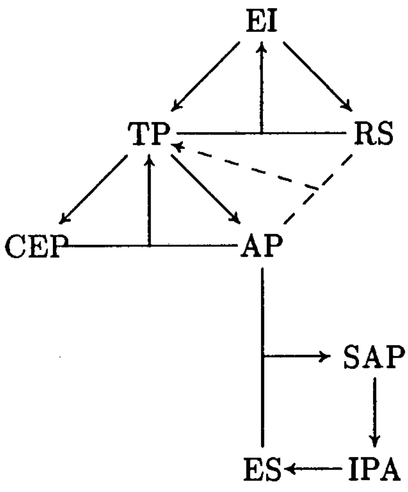


FIGURE 2.6

Chapter 3

Examples of Varieties with the Amalgamation Property

Introduction. In this chapter I present some examples of varieties that satisfy or fail the amalgamation property. As this property has been investigated extensively for many varieties (as well as for other classes of structures as well), it is not possible to present all of these results (for a extensive survey of the amalgamation and related properties, see Kiss, Márki, Pröhle and Tholen [83]). Instead, I have chosen to focus on varieties of groups, semigroups, lattices and Heyting algebras. For varieties of lattices, a complete description of all varieties satisfying the amalgamation property is known (due to Grätzer, Jónsson and Lakser [73] and Day and Ježek [84]). Because of the technical nature of some of the results, I have only sketched some of the proofs. For a more thorough presentation of these results, see Jipsen and Rose [92]. In the cases of group, semigroup and Heyting algebra varieties, the situation is less satisfactory. Here I present results which describe all residually small varieties with the amalgamation property, the more general problem remaining open.

3.1 Varieties of Groups

The Variety of All Groups. One of the earliest results concerning amalgamation is that the variety of all groups satisfies the strong amalgamation property. I present a proof of this result below.

THEOREM 3.1.1 (Schreier [27]). *The variety $\mathcal{G}$ of all groups has the strong amalgamation property.*

Proof. Let (A, f, B, g, C) be a double extension in $\mathcal{G}$. We may suppose, without loss of generality, that B and C are disjoint. Let $\overline{D}$ be the set of finite successions (words) of elements of $B \cup C$ (excluding the void word), that is

$$\overline{D} = \{a_1 a_2 \dots a_t : t \in \mathbb{N} \text{ and } a_i \in B \cup C \text{ for } i = 1 \dots t\}$$

Our goal will be to define an equivalence relation on $\overline{D}$, say ρ, such that $\overline{D}/\rho$ is a group. To do this we we start by defining four elementary equivalences on $\overline{D}$:

Given a word $a_1 a_2 \dots a_t$ in $\overline{D}$,

(E1) If, for some $i = 1 \dots t$, a_i is the indentity on B or C, we have that $a_1 a_2 \dots a_{i-1} a_i a_{i+1} \dots a_t$ is equivalent to $a_1 a_2 \dots a_{i-1} a_{i+1} \dots a_t$;

(E2) If, for some $i = 1 \dots t-1$, a_i and a_{i+1} both belong to B and $a_i a_{i+1} = a^* \in B$, then $a_1 a_2 \dots a_i a_{i+1} \dots a_t$ is equivalent to $a_1 \dots a_{i-1} a^* a_{i+2} \dots a_t$;

(E3) If, for some $i = 1 \dots t-1$, a_i and a_{i+1} both belong to C and $a_i a_{i+1} = a^* \in C$, then $a_1 a_2 \dots a_i a_{i+1} \dots a_t$ is equivalent to $a_1 \dots a_{i-1} a^* a_{i+2} \dots a_t$;

(E4) If, for some $a \in A$, and for some $i = 1 \dots t$, $a_i = f(a)$ and $b_i = g(a)$, then $a_1 \dots a_{i-1} a_i a_{i+2} \dots a_t$ is equivalent to $a_1 \dots a_{i-1} b_i a_{i+2} \dots a_t$.

Given words $v, w \in \overline{D}$ we say v and w are *elementary equivalent* (written $v \sim w$) if v and w are equivalent in any one of the senses (E1) to (E4) above. We now define $v, w \in \overline{D}$ to be *equivalent* (written $v \; \rho \; w$) if there is a finite sequence $v = x_1, x_2, \dots, x_n = w$ of words in $\overline{D}$ such that $x_1 \sim x_2 \sim \dots \sim x_n$. Note ρ is an equivalence relation, so let $D = \overline{D}/\rho$, and denote by $[v]$ the equivalence class of v under ρ, where $v \in \overline{D}$.

Now define a product on D, $[v] \cdot [w] = [vw]$, where $v, w \in \overline{D}$ and vw is the concatenation of the words v and w. This product is well-defined, for if $[v_1] = [v_2]$ and $[w_1] = [w_2]$ we have $v_1 = x_1 \sim x_2 \sim \dots \sim x_n = v_2$ and $w_1 = y_1 \sim y_2 \sim \dots \sim y_m = v_2$ for some words $x_1, \dots, x_n, y_1, \dots, y_m \in \overline{D}$. But then $v_1 w_1 = x_1 y_1 \sim x_2 y_1 \sim \dots \sim x_n y_1 = v_2 w_1$, so $v_1 w_1 \; \rho \; v_2 w_1$. Similarly $v_2 w_1 \; \rho \; v_2 w_2$ so $v_1 w_1 \; \rho \; v_2 w_2$, hence $[v_1] \cdot [w_1] = [v_2] \cdot [w_2]$.

In fact, D is a group under this product: The product is associative since concatenation of words is associative. By E4, if e_B and e_C are the identities of B and C respectively, we have $[e_B] = [e_C]$, and hence using E1, we have $[w][e_B] = [e_B][w] = [w]$ for all $w \in \overline{D}$, so $[e_B] = [e_C]$ is an identity on D. Now if $[a_1 \dots a_t] \in D$, we have $[a_1 \dots a_t][a_t^{-1} \dots a_1^{-1}] = [e_B]$ by E2 and E3, so inverses exist.

Now define $f' : B \to D$ and $g' : C \to D$ by $f'(b) = [b]$, $g'(c) = [c]$ where $b \in B$, $c \in C$. Now from the definition, f' and g' are embeddings, and by E4, $f'f = g'g$. Hence (f', g', D) is an amalgam of (A, f, B, g, C). To see that it is a strong amalgam, let $[w] \in f(B) \cap g(C)$. Then $[w] = [b] = [c]$ for some $b \in B, c \in C$. But then $b \; \rho \; c$, and this is only possible if $b \sim c$ by $E4$, that is, $[w] \in f'f(A)$. Hence (f', g', D) is a strong amalgam. □

Varieties of Abelian Groups. In this subsection, I will present results that show that abelian varieties of groups have the amalgamation property, and that the abelian varieties are precisely the varieties satisfying the congruence extension property. Recall a group G is said to be *abelian* if it satisfies the identity $a+b = b+a$ (By convention I will use $+$ as the operation for an abelian group, and 0 for the identity). We call a variety of groups *abelian* if every member of the variety is abelian.

I will assume the reader is familiar with the concept of *normal subgroup* and the *center* of a group (denoted $Z(G)$). I will also denote the n-element cyclic group by Z_n, and by Z_{p^∞} the group given by union of the chain of subgroups

$$Z_p \hookrightarrow Z_{p^2} \hookrightarrow Z_{p^3} \hookrightarrow \ldots$$

Recall also that for subgroups A and B of a group G, we define the *commutator subgroup* $[A, B]$ to be the smallest normal subgroup containing all elements of the form $[a, b] = aba^{-1}b^{-1}$, where $a \in A$, $b \in B$. As a reference to these concepts and other basic group terminology, see Grove [83].

In Fuchs [70], there is a complete characterization of abelian varieties of groups. It uses the following facts:

(1) Every subdirectly irreducible abelian group G is isomorphic to Z_{p^k} where p is a prime number and k is a natural number or ∞.

(2) If $\mathcal{V}$ is a proper subvariety of the variety of all abelian groups, then $\mathcal{V}$ is determined by an identity of the form $nx = 0$ (and the abelian identity), where n is a natural number.

Using this we have:

(3) If $\mathcal{V}$ is the variety of all abelian groups, then

$$\mathcal{V}_{WMI} = \{Z_{p^\infty} : p \text{ is a prime number}\}.$$

(4) If $\mathcal{V}$ is the proper subvariety of all abelian groups determined by the identity $nx = 0$, where n has prime decomposition $n = p_1^{k_1} \dots p_m^{k_m}$. Then $\mathcal{V}_{WMI} = \{Z_{{p_1}^{k_1}}, \dots, Z_{{p_m}^{k_m}}\}$.

The variety of all abelian groups will be denoted by $\mathcal{A}$, and the subvariety of $\mathcal{A}$ determined by the identity $nx = 0$ will be denoted by $\mathcal{A}_n$.

Given a group G, there is a correspondence between the normal subgroups of G and the congruences on G. Using this correspondence, we are able to give an alternative characterization of the congruence extension property for groups.

LEMMA 3.1.2. *The following are equivalent for a group G:*

(1) *G satisfies the congruence extension property.*

(2) *For every subgroup H of G and K a normal subgroup of H, there is a normal subgroup N of G such that $N \cap H = K$.* □

Using this characterization, I present a proof that the varieties satisfying the congruence extension property are the abelian ones.

THEOREM 3.1.3 (Biró, Kiss and Pálfy [82]). *Let $\mathcal{V}$ be a variety of groups. Then $\mathcal{V}$ has the congruence extension property if and only if $\mathcal{V}$ is abelian.*

Proof. Since all subgroups of an abelian group are normal, by lemma 3.1.2, if $\mathcal{V}$ is abelian, $\mathcal{V}$ satisfies the congruence extension property.

Suppose $\mathcal{V}$ satisfies the congruence extension property. We first prove the following claim:

(*) For $G \in \mathcal{V}, A$ an abelian subgroup of G, we have $[A, G] \cap A = \{e\}$.

Proof of claim: Since $G \times G \in \mathcal{V}$, it satisfies the congruence extension property. Let $\Delta_A = \{(a, a) \in G \times G : a \in A\}$. Now since A is abelian, Δ_A is normal in $A \times A$. Hence by lemma 3.1.2, there is a normal subgroup N of $G \times G$ such that $N \cap (A \times A) = \Delta_A$. Let $a \in A, g \in G$, then $([a, g], e) = (a^{-1}, a^{-1})((g^{-1}, e)(a, a)(g, e)) \in \Delta_A(g, e)^{-1}\Delta_A(g, e) \subseteq N$. Hence $[A, G] \times \{e\} \leq N$, thus $\Delta_A = (A \times A) \cap N \geq (A \times A) \cap ([A, G] \times \{e\}) = (A \cap [A, G]) \times \{e\}$, which implies $A \cap [A, G] = \{e\}$, proving the claim.

Now let $G \in \mathcal{V}_{SI}$ and let M be the unique minimal normal subgroup of G (corresponding to the monolith of G). Then $M \leq Z(G)$: For let $x \in M, x \neq e$, and let A be the subgroup of M generated by x. Now A is abelian, and since $[A, G]$ is normal in G, we have $[A, G] \geq M$ or $[A, G] = \{e\}$. In the former case $[A, G] \cap A \geq M \cap A = A \neq \{e\}$, contradicting the claim (*) above. Hence $[A, G] = \{e\}$, so $xy = yx$ for all $y \in G$, and so $M \leq Z(G)$.

Now pick $g \in G$. Let B be the subgroup generated by $M \cup \{g\}$. Then since $M \leq Z(G)$, B is abelian. Again $[B, G] \geq M$ or $[B, G] = \{e\}$. In the first case, $[B, G] \cap B \subseteq M \neq \{e\}$, contradicting (*). So $[B, G] = \{e\}$, that is $gz = zg$ for all $z \in G$, so G is abelian.

Now since each member of $\mathcal{V}_{SI}$ is abelian, each member of $\mathcal{V}$ is abelian. $\square$

The last part of this subsection is devoted to showing that the abelian varieties have enough injectives, and hence satisfy the amalgamation property. To do this, we need the following definition.

DEFINITION 3.1.4. An abelian group G is said to be *divisible* if for every $a \in G$ and $n \in \mathbb{N}$, there is an $x \in G$ such that $nx = a$.

For example, for every prime p, Z_{p^∞} is divisible.

THEOREM 3.1.5 (Baer [40]). *In the variety of abelian groups $\mathcal{A}$, every divisible group is injective.*

Proof. Let $f : A \hookrightarrow B \in \mathcal{A}$ be an embedding, and let $g : A \to G$ be a homomorphism with G a divisible group. We need to construct a homomorphism $h : B \to G$ extending g.

Let $\mathcal{S} = \{(C, k) : k : C \to G \text{ is a homomorphism}, f(A) \leq C \leq B, kf = g\}$. Now the relation $\geq$ on $\mathcal{S}$ given by

$$(C_1, k_1) \geq (C_2, k_2) \quad \text{if and only if} \quad C_1 \geq C_2 \text{ and } k_1|_{C_2} = k_2$$

is a partial order. Also, since $\mathcal{S}$ is not empty (it contains $(f(A), gf^{-1})$) and $\mathcal{S}$ is closed under unions of chains, by Zorn's lemma, $\mathcal{S}$ contains a maximal element, say (D, h). We show $D = B$.

Suppose not. Then pick $x \in B \setminus D$. Let E be the abelian group generated by $D \cup \{x\}$. If the intersection of the group generated by x (denoted $\langle x \rangle$) with D is just the zero element, then $E = D \oplus \langle x \rangle$, so there is a homomorphism $h' : E \to G$ given by $h'(d + mx) = h(d)$, where $d \in D$ and $m \in \mathbb{Z}$. Note that h' extends h. On the other hand, suppose $D \cap \langle x \rangle \neq \{0\}$, then choose the smallest $k \in \mathbb{N}$ such that $kx \in D$. Now every element in E has unique representation $d + mx$ with $d \in D$ and $0 \leq m < k$. Set $a = h(kx) \in G$. Since G is divisible, there is a $b \in G$ with $a = kb$. Define $h' : E \to G$ by $h'(d + mx) = h(d) + mb, d \in D$, $0 \leq m < k$. Note h' is a homomorphism extending h. Hence, in both cases, we have constructed a pair $(E, h') > (D, h)$, contradicting maximality. Hence $D = B$, so G is injective. □

COROLLARY 3.1.6. *For $\mathcal{A}$, the variety of abelian groups, all members of $\mathcal{V}_{WMI}$ are injective.*

Proof. For every prime p, Z_{p^∞} is divisible. □

In the following lemma, I use the notation (n, m) for the highest common factor of n and m, where $n, m \in \mathbb{N}$.

LEMMA 3.1.7 (Bruyns and Rose [89]). *Let $k, l, m \in \mathbb{N}$, and let $k \leq l \leq m$. If $f : Z_{p^k} \hookrightarrow Z_{p^l}$ and $g : Z_{p^k} \hookrightarrow Z_{p^m}$ are embeddings, then there is a homomorphism $h : Z_{p^l} \to Z_{p^m}$ such that $hf = g$.*

Proof. Let $r = f(1)$ and $y = g(1)$. r has order p^k in Z_{p^l}, so $(r, p^l) = (r, p^m) = p^{l-k}$. y has order p^k in Z_{p_m}, so $(y, p^m) = p^{m-k}$. Hence there is an integer n such that $n(r, p^m) = (y, p^m)$. Now there are integers s and t such that $rs + p^m t = (r, p^m)$, hence $nrs + np^m t = (y, p^m)$. Decomposing y into $y = (p^m, y)n^*$, where n^* is an integer, we have $y \equiv rsnn^* \mod p^m$. Define $h : Z_{p^l} \to Z_{p^m}$ by $h(1) = snn^*$ (since Z_{p^l} is generated by 1, this is sufficient to define h). Since $h(f(1)) = h(r) = rsnn^* \equiv y = g(1) \mod p^m$, we have $hf = g$. □

THEOREM 3.1.8 (Bruyns and Rose [89]). *Let $\mathcal{V} = \mathcal{A}_n$ for some natural number n. Then every member of $\mathcal{V}_{WMI}$ is injective in $\mathcal{V}$.*

Proof. Let $f : A \hookrightarrow B \in \mathcal{V}$ be an embedding and $g : A \to Z_{p^m} \in \mathcal{V}_{WMI}$ be a homomorphism. Then there is a $k \leq m$, and epimorphism $e : A \twoheadrightarrow Z_{p^k}$ and an embedding $m : Z_{p^k} \hookrightarrow Z_{p^l}$ such that $me = g$. Since every abelian variety has the congruence extension property, we have $C \in \mathcal{V}$, an embedding $s : Z_{p^k} \hookrightarrow C$ and an epimorphism $t : B \twoheadrightarrow C$ such that $tf = se$. In fact, we may by lemma 1.2.23 assume s is essential, hence $C \cong Z_{p^l}$ for some $k \leq l \leq m$. Hence by lemma 3.1.7, there is a homomorphism $\overline{h} : C \rightarrow Z_{p^m}$ such that $\overline{h}s = m$. Letting $h = \overline{h}t$, we have $hf = g$, so Z_{p^m} is injective. ☐

LEMMA 3.1.9. *Let $\mathcal{V}$ be a variety. Then injectives in $\mathcal{V}$ are closed under direct products.*

Proof. Let $f : A \hookrightarrow B \in \mathcal{V}$ be an embedding and let $\{C_i : i \in I\}$ be a family of injectives in $\mathcal{V}$. Let $g : A \rightarrow \prod_{i\in I} C_i$ be a homomorphism. If $\pi_j : \prod_{i\in I} C_i \twoheadrightarrow C_j$ is the jth projection, since C_j is injective, there is a map $h_j : B \rightarrow C_j$ such that $h_j f = \pi_j g$. Let $h : B \rightarrow \prod_{i\in I} C_i$ be the map given by the product map $\prod_{i\in I} h_i$. We see $hf = g$. ☐

THEOREM 3.1.10 (B.H.Neumann [60]). *Let $\mathcal{V}$ be an abelian variety. Then $\mathcal{V}$ has enough injectives, hence satisfies the amalgamation property.*

Proof. For every abelian variety $\mathcal{V}$, each member of $\mathcal{V}_{WMI}$ is injective. Since abelian varieties are residually small, every member of $\mathcal{V}$ is embeddable into a product of members of $\mathcal{V}_{WMI}$, which are injective by lemma 3.1.9. Hence $\mathcal{V}$ has enough injectives. ☐

In fact, abelian varieties satisfy the property that epis are surjective. (To see this, let $f : A \rightarrow B \in \mathcal{V}$ be an epi in an abelian variety $\mathcal{V}$. Let $C = B/f(A)$, with $g : B \twoheadrightarrow C$ the canonical map, and let $h : B \rightarrow C$ take every element to the identity in C. Then, since $gf = hf$, $g = h$, and hence $f(A) = B$.) Hence abelian varieties satisfy the strong amalgamation proprty by theorem 2.1.9.

Group Varieties Satisfying AP and RS. We now present the main result of this section.

THEOREM 3.1.11 (Bergman and McKenzie [88]). *The only residually small varieties of groups that satisfy the amalgamation property are the abelian varieties.*

Proof. Since group varieties are congruence modular, a residually small variety satisfying the amalgamation property satisfies the congruence extension property by theorem 2.4.6. By theorem 3.1.3, the only group varieties satisfying the congruence extension property are the abelian ones. ☐

A complete description of which group varieties satisfy the amalgamation property is not known. However, H. Neumann [67] states the following theorem (she refers to a construction of B.H.Neumann [60] as a hint for the proof).

THEOREM 3.1.12 (H. Neumann [67], B.H.Neumann [60]). *Let $\mathcal{V}$ be a proper subvariety of groups satisfying the amalgamation property. Then all its finite members are abelian.* ☐

However, since there are nonabelian varieties whose finite members are abelian (see Ol'shanskiĭ [85]), the following question remains open:

PROBLEM. Are the variety of all groups and the abelian varieties the only group varieties to satisfy the amalgamation property?

3.2 Varieties of Semigroups

The Variety of All Semigroups. I start with an example by Kimura [57], which shows that the variety of all semigroups fails to satisfy the amalgamation property.

THEOREM 3.2.1 (Kimura [57]). *The variety of all semigroups $\mathcal{S}$ does not satisfy the amalgamation property.*

Proof. Let $A = \{u, v, w, 0\}$ be the semigroup with all products equal to 0. Let $B = A \cup \{b\}$ with $bu = ub = v$ and all other products 0 and $C = A \cup \{c\}$ with $cv = vc = w$ and all other products 0. Note that B and C are semigroups. Let (A, f, B, g, C) be the double extension with $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$ the respective inclusion maps. Suppose (f', g', D) is an amalgam of (A, f, B, g, C). Then $g'(w) = g'(v)g'(c) = f'(v)g'(c) = f'(b)f'(u)g'(c) = f'(b)g'(u)g'(c) = f'(b)g'(0) =$

$f'(b)f'(0) = f'(0) = g'(0)$, contradicting $w \neq 0$. Hence $\mathcal{S}$ does not satisfy the amalgamation property. □

In the light of the above theorem, much attention has been devoted to determining which semigroups are in $Amal(\mathcal{S})$. One of the principal results is that $Amal(\mathcal{S})$ contains all inverse semigroups (which includes all groups). For a definition of an *inverse semigroup*, and a proof of this result, see Howie [75].

Semigroup Varieties Satisfying CEP. In this section, I present results due to Biró, Kiss and Pálfy [82] which describe all the semigroup varieties that satisfy the congruence extension property. As the proofs are largely dependent on semigroup theory, the proofs have been ommitted.

DEFINITION 3.2.2.

(1) We denote by N the *two-element null semigroup*, that is, there is a $0 \in N$ such that for all $a, b \in N$, $ab = 0$.
(2) We denote by L the *two-element left-zero semigroup*, that is, for all $a, b \in L, ab = a$.
(3) We denote by R the *two-element right-zero semigroup*, that is, for all $a, b \in R, ab = b$.
(4) We denote by H the *two-element semilattice*, that is $H = \{0, 1\}$ with $ab = 1$ if and only if $a = b = 1$ for all $a, b \in H$.
(5) We denote by Z_n the cyclic group of order n viewed as a semigroup, where n is a natural number.

Note that N, L, R and H are all subdirectly irreducible.

Given any semigroup G, we append an element 0 to G such that 0 is an identity on G. We will denote this new semigroup obtained from G by G^0.

Put

$$\begin{aligned}\mathcal{G}_n = &\{Z_m : m \text{ divides } n \text{ and } m \text{ is a prime power}\}\\ &\cup \left\{Z_m^0 : m \text{ divides } n \text{ and } m \text{ is a prime power}\right\}.\end{aligned}$$

Let $\mathcal{S}_n = \{R, R^0, L, L^0, H, N\} \cup \mathcal{G}_n$.

For $n \in \mathbb{N}$, let $\mathcal{W}_n$ be the subvariety of semigroups determined by the pair of identities:

$$\begin{aligned}(xy)^{n+1} &= xy \\ xyzu &= xzyu.\end{aligned}$$

THEOREM 3.2.3 (Biró, Kiss and Pálfy [82]). *Let $\mathcal{V}$ be a semigroup variety. Then the following are equivalent:*

(1) *$\mathcal{V}$ satisfies the congruence extension property.*
(2) *There is a natural number n such that $\mathcal{V}_{SI} \subseteq \mathcal{S}_n$.*
(3) *There is a natural number n such that $\mathcal{V} \subseteq \mathcal{W}_n$.* □

Residually Small Semigroup Varieties.

McKenzie [81] has described all residually small semigroup varieties. I state without proof some of the results that will be used to describe all residually small varieties satisfying the amalgamation property.

DEFINITION 3.2.4.

(1) Let G be a group, and let $\alpha : G \to S(U)$ be a homomorphism, where $S(U)$ is the symmetric group over some non-void set U. Notice that each element in $S(U)$ can be viewed as a bijection from U to U. In order to avoid confusion, for each $g \in G$, we will denote by α_g the image of g under α, viewed as a bijection $\alpha_g : U \to U$. We define $R(G, U, \alpha)$ to be the semigroup with the universe $G \cup U \cup \{0\}$ (we may assume that these 3 sets are disjoint) with the operation defined as follows for $g, h \in G, u \in U$:

$$\begin{aligned}&g \cdot u = \alpha_g(u),\\ &g \cdot h = \text{ their product in } G, \text{ and}\\ &\text{all other products are } 0.\end{aligned}$$

(2) Let G be a group, and let $\alpha : G \to S(U)$ be a dual homomorphism (that is, for $g, h \in G, \alpha_{gh} = \alpha_h \circ \alpha_g$) where $S(U)$ is the symmetric group over some non-void set U. We define $L(G, U, \alpha)$ to be the semigroup with the universe $G \cup U \cup \{0\}$ (we may assume that these 3 sets are disjoint) with

the operation defined as follows for $g, h \in G, u \in U$:

$$\begin{aligned} u \cdot g &= \alpha_g(u), \\ g \cdot h &= \text{ their product in } G, \text{ and} \\ &\text{all other products are } 0. \end{aligned}$$

For example, let $G = \{e\}$ be the trivial group and let $U = \{1\}$. Since there is only one (dual) homomorphism $\alpha : G \to S(U)$, we can define $L^* = L(\{e\}, \{1\}, \alpha)$ and $R^* = R(\{e\}, \{1\}, \alpha)$. Note $R^* = \{e, 1, 0\}$ with $ex = x$ for all $x \in R^*$ and all other products are 0.

For $n > 1$, define the following semigroup varieties:

(1) $\mathcal{S}_1^n$ is the variety of semigroups satisfying the three identities:

$$\begin{aligned} (xy)^{n+1} &= xy \\ x^n yz &= x^n y x^n z \\ xyz^n &= xz^n yz^n. \end{aligned}$$

(2) $\mathcal{S}_2^n$ is the variety of semigroups satisfying the two identities:

$$\begin{aligned} x^{n+1} y &= xy \\ x^n y^n z &= y^n x^n z. \end{aligned}$$

(3) $\mathcal{S}_3^n$ is the variety of semigroups satisfying the two identities:

$$\begin{aligned} xy^{n+1} &= xy \\ xy^n z^n &= xz^n y^n. \end{aligned}$$

THEOREM 3.2.5 (McKenzie [81]).

(1) *A subdirectly irreducible semigroup in* $\mathcal{S}_1^n$ *is isomorphic to a member of the set* $\{H, N, L, R, L^0, R^0\} \cup \mathcal{G}_n$.

(2) *A subdirectly irreducible semigroup in* $\mathcal{S}_2^n$ *is either isomorphic to member of the set* $\{H, N, R, R^0\} \cup \mathcal{G}_n$ *or to* $R(G, U, \alpha)$ *where* U *is some non-void set,* G *is a group in* $\mathcal{G}_n$ *and* α *is some homomorphism* $\alpha : G \to S(U)$.

(3) *A subdirectly irreducible semigroup in* $\mathcal{S}_3^n$ *is either isomorphic to member of the set* $\{H, N, L, L^0\} \cup \mathcal{G}_n$ *or to* $L(G, U, \alpha)$ *where* U *is some non-void set,* G *is a group in* $\mathcal{G}_n$ *and* α *is some dual homomorphism* $\alpha : G \to S(U)$.

(4) *If* $\mathcal{V}$ *is a residually small semigroup variety, then* $\mathcal{V} \subseteq \mathcal{S}_i^n$ *for some* $n \in \mathbb{N}, i \in \{1, 2, 3\}$. □

COROLLARY 3.2.6 (McKenzie [81]). *Every residually small variety of semigroups satisfies the identity* $x^{2n} = x^n$ *for some* $n \in \omega$.

Proof. A check shows each $\mathcal{S}_i^n$ satisfies $x^{2n} = x^n$ for each $i = 1, 2, 3$. □

Semigroup Varieties Satisfying AP and RS. In this last subsection, I present some lemmas which lead up to a result due to Kearnes [89] that describe fully all residually small semigroup varieties satisfying the amalgamation property. Although semigroups are not congruence modular varieties, we still have residual smallness and the amalgamation property implying the congruence extension property as a consequence.

LEMMA 3.2.7 (Kearnes [89]). *Let* $\mathcal{V}$ *be a variety of semigroups satisfying the amalgamation property. The subclass* $\mathcal{G}$ *of groups in* $\mathcal{V}$ *also satisfies the amalgamation property.*

Proof. Let (A, f, B, g, C) be a double extension in $\mathcal{G}$. It has an amalgam in $\mathcal{V}$, say (f', g', D). Letting D' be the subsemigroup of D generated by $f'(B) \cup g'(C)$, a check on the group axioms shows that D' is in fact a group. Hence (f', g', D') is an amalgam in $\mathcal{G}$. □

LEMMA 3.2.8 (Kearnes [89]). *If* $\mathcal{V}$ *is a residually small variety of semigroups satisfying the amalgamation property, then the subclass of groups in* $\mathcal{V}$ *is a residually small subvariety of* $\mathcal{V}$ *satisfying the amalgamation property.*

Proof. By corollary 3.2.6, $\mathcal{V}$ satisfies $x^n = x^{2n}$ for some n. Let $\mathcal{G}$ be the variety of semigroups satisfying the identities $x^n y = y$ and $y x^n = y$ in addition to those identities satisfied by $\mathcal{V}$. We claim that $\mathcal{G}$ is exactly the class of groups in $\mathcal{V}$. For if $G \in \mathcal{V}$ is a group, then $x^n = x^{2n}$ implies x^n is the identity on G for any $x \in G$, so G satisfies the extra two identities. Conversely, if $G \in \mathcal{G}$, then the identities imply x^n is an identity on G for any $x \in G$, and for every $Y \in G$, y^{n-1} is an inverse of y. □

LEMMA 3.2.9 (Kearnes [89]). *Let $\mathcal{V}$ be residually small, and suppose $\mathcal{V}$ contains either L^* or R^*, the semigroups given as examples after definition 3.2.4. Then $\mathcal{V}$ fails to satisfy the amalgamation property.*

Proof. If $\mathcal{V}$ is residually small and contains R^*, then $\mathcal{V} \subseteq \mathcal{S}_2^n$ for some $n \in \mathbb{N}$. Hence $\mathcal{V}$ satisfies the identity

$$(*) \qquad x^n y^n z = y^n x^n z.$$

Now assume $\mathcal{V}$ satisfies the amalgamation property. Let Z be the semigroup $\{0, u, v\}$ with all products equal to 0. Let $f, g : Z \hookrightarrow R^* \times R^*$ be embeddings given by:

$$\begin{aligned} f(0) &= g(0) = (0,0), \\ f(u) &= g(v) = (1,0), \\ f(v) &= g(u) = (1,1). \end{aligned}$$

where $R^* = \{0, e, 1\}$ as before. Let (f', g', P) be an amalgam of $(Z, f, R^* \times R^*, g, R^* \times R^*)$. Let $w = f'f(v), z = f'f(u), y = f'(e,0)$ and $x = g'(0,e)$. Now $yz = f'((e,0)\cdot(1,1)) = f'(1,0) = f'f(u) = z$. Similarly $yw = z$ and $xz = w = xw$. Hence $x^n y^n z = x^n z = w \neq z = y^n w = y^n x^n z$, contradiction (*). Hence $\mathcal{V}$ does not satisfy the amalgamation property. The case for L^* can be handled dually. ☐

DEFINITION 3.2.10. A variety $\mathcal{V}$ is said to have *hereditary subdirectly irreducible algebras* if for any $A \in \mathcal{V}_{SI}$, every subalgebra of A is subdirectly irreducible or trivial.

THEOREM 3.2.11 (Grätzer and Lakser [71]). *Let $\mathcal{V}$ be a variety satisfying the congruence extension property and having hereditary subdirectly irreducible algebras. Then $\mathcal{V}$ satisfies the amalgamation property if and only if every double extension (A, f, B, g, C) with $A, B, C \in \mathcal{V}_{SI}$ has an amalgam in $\mathcal{V}$.*

Proof. Since the latter statement is weaker, the condition is necessary. To show sufficiency, let (A, f, B, g, C) be a double extension in the variety $\mathcal{V}$ above. Now pick $b, c \in B, b \neq c$. Then by theorem 1.2.4, there is an $S_1 \in \mathcal{V}_{SI}$ and an epimorphism $\phi : B \twoheadrightarrow S_1$ such that $\phi(b) \neq \phi(c)$. Let $s : \phi f(A) \hookrightarrow S_1$ be the inclusion map, and note that $\phi f(A) \in \mathcal{V}_{SI}$ since $\mathcal{V}$ has hereditary subdirectly irreducible algebras. Now since $\mathcal{V}$ satisfies the congruence extension property, there is a $S_2 \in \mathcal{V}$, an embedding $t : \phi f(A) \hookrightarrow S_2$ and epimorphism $\psi : C \twoheadrightarrow S_2$ such that $\psi g = t\phi f$. By lemma 1.2.23, we may assume h is essential and hence $S_2 \in \mathcal{V}_{SI}$. Now by the hypothesis, $(\phi f(A), s, S_1, t, S_2)$ has an amalgam, say (s', t', D). Now put $f' = s'\phi$ and $g' = t'\psi$. Note $f'f = g'g$ and $f'(b) \neq f'(c)$. Hence, by symmetry, the conditions of lemma 1.3.3(2) are satisfied, and the result follows. ☐

THEOREM 3.2.12 (Kearnes [89]). *The following are equivalent for a semigroup variety $\mathcal{V}$:*

1. *$\mathcal{V}$ satisfies the congruence extension property.*
2. *$\mathcal{V}$ is residually small and satisfies the amalgamation property.*
3. *$\mathcal{V}$ is residually small, $R^* \notin \mathcal{V}$, $L^* \notin \mathcal{V}$ and $\mathcal{V}$ contains no non-abelian groups.*

Proof. (1) $\Longrightarrow$ (2): $\mathcal{V}$ is residually small by theorems 3.2.3 and 3.2.5. By doing an exhaustive case study on the subdirectly irreducible members mentioned in theorem 3.2.3(2), it can be seen that $\mathcal{V}$ has hereditary subdirectly irreducible algebras. Hence, to show that $\mathcal{V}$ satisfies the amalgamation property, we need only show that $\mathcal{V}_{SI}$ satisfies the amalgamation property by theorem 3.2.11. This can be done by an exhaustive case study of all possible combinations. For some of the details, see Biró, Kiss and Pálfy [82].

(2) $\Longrightarrow$ (3): By lemma 3.2.9, if $\mathcal{V}$ is residually small and satisfies the amalgamation property, then $R^*, L^* \notin \mathcal{V}$. Now let $\mathcal{G}$ be the subclass of all groups in $\mathcal{V}$. By lemma 3.2.8, it is a residually small variety satisfying the amalgamation property, hence by theorem 3.1.11 it must be abelian.

(3) $\Longrightarrow$ (1): Since $\mathcal{V}$ is residually small, $\mathcal{V} \subseteq \mathcal{S}_i^n$ for $n \in \mathbb{N}$, $i \in \{1,2,3\}$. If $i = 1$, then by theorem 3.2.3, $\mathcal{V}_{SI}$ is contained in $\mathcal{S}_n$, hence by theorem 3.2.3 the conclusion follows. If $i = 2$, suppose $\mathcal{V}_{SI}$ has a member of the form $R(G, U, \alpha)$ as in definition 3.2.4. Then R^* is a subsemigroup of $R(G, U, \alpha)$, a contradiction. Thus $\mathcal{V}_{SI} \subseteq \mathcal{S}_n$, so $\mathcal{V}$ satisfies the congruence extension property. The case $i = 3$ is the dual of the case $i = 2$. ☐

3.3 Varieties of Lattices

The Variety of Distributive Lattices. That the variety of distributive lattices satisfies the amalgamation property was first noted by Pierce [68]. In fact Banaschewski and Bruns [68] showed that this variety has enough injectives. I give a proof of these results below.

THEOREM 3.3.1 (Pierce [68], Banaschewski and Bruns [68]). *The variety of distributive lattices $\mathcal{D}$ has enough injectives, and hence satisfies the amalgamation property.*

Proof. Since $\mathcal{D}_{SI} = \mathcal{D}_{WMI} = \{\underline{2}\}$, where $\underline{2}$ is the two-element chain, we need only show that $\underline{2}$ is injective. For then, since every member of $\mathcal{D}$ is embeddable into a direct product of two-element chains, and injectives are closed under direct products (lemma 3.1.9), we have $\mathcal{D}$ has enough injectives.

Let $f : A \hookrightarrow B \in \mathcal{D}$ be an embedding and let $g : A \to \underline{2}$ be a homomorphism. We need to find a homomorphism $h : B \to \underline{2}$ such that $hf = g$. If $g(A)$ is a singleton, then we may let h be the map that maps all of B onto the singleton $g(A)$. Thus we may assume g is an epimorphism. Then $I = g^{-1}(0)$ and $F = g^{-1}(1)$ are respectively an ideal and filter on A. Now $f(I)$ can be extended to an ideal $I' = \{b \in B : b \leq a \in f(I)\}$ on B, and similarly, $f(F)$ can be extended to a filter $F' = \{b \in B : b \geq a \in f(F)\}$. Now $I' \cap F' = \emptyset$, hence by Zorn's lemma and distributivity of B, we can extend I' to a maximal ideal M on B, which is also disjoint from F'. Define $h : B \to \underline{2}$ by:

$$h(b) = \begin{cases} 0 & \text{if } b \in M \\ 1 & \text{if } b \notin M. \end{cases}$$

A check shows that h is a homomorphism extending g. □

Note that $\mathcal{D}$ does not have the strong amalgamation property: For consider $A = \{0, a, 1\}$, $B = \{0, a, b, 1\}$ and $C = \{0, a, c, 1\}$ where A, B and C are the distributive lattices given in figure 3.1.

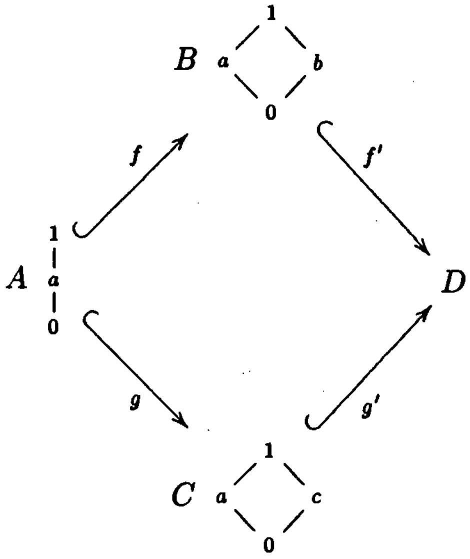


FIGURE 3.1

Let (f', g', D) be an amalgam of (A, f, B, g, C), where $f : A \hookrightarrow B, g : A \hookrightarrow C$ are the inclusion maps. Let D' be the sublattice of D generated by the $f'(B) \cup g'(C)$. Now $f'(b)$ and $g'(c)$ are both complements of $g'(a) = f'(a)$ in D', hence since D' is distributive, $f'(b) = g'(c)$. Hence $f'(B) \cap g'(C) \neq f'f(A)$, so $\mathcal{D}$ does not satisfy the strong amalgamation property.

The Variety of All Lattices. I now present a proof that the variety of all lattices possesses the strong amalgamation property.

THEOREM 3.3.2 (Jónsson [56]). *The variety of all lattices $\mathcal{L}$ satisfies the strong amalgamation property.*

Proof. Let $(\overline{A}, \overline{f}, \overline{B}, \overline{g}, \overline{C})$ be a double extension in $\mathcal{L}$. We may assume $\overline{B}$ and $\overline{C}$ are disjoint. Put $\overline{D} = \overline{B} \cup \overline{C}$, and consider a relation ρ on $\overline{D}$ given by:

$$x \; \rho \; y \quad \text{if and only if} \quad x = y$$
$$\text{or there is an } a \in \overline{A} \text{ such that } \overline{f}(a) = x \text{ and } \overline{g}(a) = y, \text{ or vice versa.}$$

Notice that ρ is an equivalence relation on $\overline{D}$. Let $D = \overline{D}/\rho$, and let $B = \{b/\rho \in D : b \in \overline{B}\}$, $C = \{c/\rho \in D : c \in \overline{C}\}$ and $A = \{\overline{f}(a)/\rho \in D : a \in \overline{A}\} = \{\overline{g}(a)/\rho \in D : a \in \overline{A}\}$. Now $B, C \subseteq D$, and $B \cap C = A$. Let $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$ be the inclusion maps. Observe that $\overline{A}, \overline{B}$ and $\overline{C}$ are respectively isomorphic to A, B and C, and hence we need only show (A, f, B, g, C) has an amalgam to show $(\overline{A}, \overline{f}, \overline{B}, \overline{g}, \overline{C})$ has amalgam. To do this, we start by defining a partial order $\leq_D$ on D as follows (We denote the partial order and lattice operations on B and C by $\leq_B, +_B, \cdot_B$ and $\leq_C, +_C, \cdot_C$ respectively):

$$x \leq_D y \quad \text{if and only if} \quad \begin{cases} x \leq_B y & \text{if } x, y \in B \\ x \leq_C y & \text{if } x, y \in C \\ x \leq_B a, a \leq_C y \text{ for some } a \in A & \text{if } x \in B, y \in C \\ x \leq_C a, a \leq_B y \text{ for some } a \in A & \text{if } x \in C, y \in B. \end{cases}$$

A check shows that $\leq_D$ is indeed a partial order on D. Now define a subset I of D to be a (B, C)-*ideal* of D if I satisfies:

(1) $x \in I$ and $z \leq_D x$ implies $z \in I$,
(2) $x, y \in I \cap B$ implies $x +_B y \in I$, and
(3) $x, y \in I \cap C$ implies $x +_C y \in I$.

Let $\mathcal{I}(B, C)$ be the collection of all (B, C)-ideals of D together with the empty set. $\mathcal{I}(B, C)$ is closed under arbitrary intersections, so it forms a complete lattice with $I \cdot J = I \cap J$ and $I + J$ is the (B, C)-ideal generated by the $I \cup J$, where $I, J \in \mathcal{I}(B, C)$. Now for each $x \in D$, the principal ideal $(x]$ is a (B, C)-ideal, so we can construct maps $f' : B \to \mathcal{I}(B, C)$ and $g' : C \to \mathcal{I}(B, C)$ as follows:

$$\begin{aligned} f'(x) &= (x] \quad \text{for } x \in B \\ f'(y) &= (y] \quad \text{for } y \in C. \end{aligned}$$

We show $(f', g', \mathcal{I}(B, C))$ is a strong amalgam of (A, f, B, g, C) by showing:

(1) f' and g' are embeddings,
(2) $f'f = g'g$, and
(3) $f'(B) \cap g'(C) = f'f(A)$.

Proof of (1): Let $x, y \in B$. Then $f'(x +_B y) = (x +_B y] \supseteq (x] + (y]$ since $x, y \leq_D x +_B y$. Also, $x, y \in ((x] + (y]) \cap B$, hence by (2) of the definition of a (B, C)-ideal, $x +_B y \in (x] + (y]$ so $(x +_B y] \subseteq (x] + (y]$. Hence $f'(x) + f'(y) = f'(x +_B y)$. Similarly $f'(x \cdot_B y) = f'(x) \cdot f'(y)$. Now if $f'(x) = f'(y)$, then $x \leq_D y$ and $y \leq_D x$ so $x = y$, hence f' is an embedding. The case for g' proceeds similarly.

Proof of (2): That $f'f = g'g$ follows from $A = B \cap C$.

Proof of (3): Suppose $I \in f'(B) \cap g'(C)$. Then there is a $b \in B$, $c \in C$ such that $(b] = (c] = I$. But then $b \leq_D c$ and $c \leq_D b$ implies $b = c$. Thus $b \in B \cap C = A$ so $I \in f'f(A)$. □

The Variety of Modular Lattices. When proving that distributive lattices satisfy the amalgamation property, we made use of the fact that every distributive lattice can be embedded in a Boolean algebra, which is just a complemented distributive lattice. The next theorem contains a similar result for modular lattices: If a non-distributive variety of modular lattices satisfies the amalgamation property, then every lattice in the variety can be embedded in a complemented one.

THEOREM 3.3.3 (Grätzer, Jónsson and Lakser [73]). *Let $\mathcal{V}$ be a non-distributive variety of modular lattices satisfying the amalgamation property. Then every member of $\mathcal{V}$ can be embedded in a simple complemented lattice in $\mathcal{V}$.*

Proof. Let $L \in \mathcal{V}$. Since every lattice in a variety can be embedded into a member of the variety having a top and bottom element (for example, the lattice of ideals of a given lattice), we may assume L has a top and bottom element, say 1_L and 0_L respectively. Now $M_3 \in \mathcal{V}$, say $M_3 = \{0, a, b, c, 1\}$ with atoms a, b and c. Let $\underline{3} = \{0, a, 1\}$ be the three-element chain viewed as a sublattice of M_3 with $g : \underline{3} \hookrightarrow M_3$ the inclusion map. Pick any $x \in L \setminus \{0_L, 1_L\}$ and define $f : \underline{3} \hookrightarrow L$ by $f(0) = 0_L, f(a) = x$ and $f(1) = 1_L$. Now since $\mathcal{V}$ has the amalgamation property, $(\underline{3}, f, L, g, M_3)$ has an amalgam, say (f', g', L_1). Again we may assume L_1 is a lattice with top and bottom elements, and that L_1 is in fact an extension of L. Iterating this process now for each $x \in L$ (using transfinite induction, since $\mathcal{V}$ is closed under unions of chains), we arrive at an extension C_0 of L such that for each $x \in L \setminus \{0_L, 1_L\}$, x is contained in a diamond sublattice of C_0. Repeating this process we obtain a countably infinite sequence

$$L = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n \subseteq C_{n+1} \subseteq \ldots$$

of lattices in $\mathcal{V}$, all having the same top and bottom elements as L (which we will denote by 1 and 0 respectively), and having the property that for all $n \in \omega$, if $x \in C_n \setminus \{0, 1\}$, then x is in a diamond sublattice of C_{n+1}, with 1 and 0 as top and bottom element of the diamond respectively.

Letting $C_w = \bigcup_{n\in\omega} C_n$ we see $C_\omega \in \mathcal{V}$ and is complemented. To show C_ω is simple, recall that in a complemented modular lattice every congruence Θ is determined by the ideal of elements congruent to 0 mod Θ. If Θ is not trivial, there is an $x \in C_\omega$, $x \neq 0$ such that $(x, 0) \in \Theta$. If $x = 1$ then Θ is automatically the top congruence, else if $x < 1$, then $x \in C_n$ for some $n \in \omega$. But then there is a diamond sublattice $\{0, x, y, z, 1\}$ of C_{n+1} and hence of C_ω. Since M_3 is simple, it follows that $(1, 0) \in \Theta$, so Θ is the top congruence of C_ω. Hence C_ω is simple. □

In Crawley and Dilworth [73](page 131), a modular lattice is constructed that cannot be embedded into any simple complemented modular lattice. This gives us an immediate consequence to theorem 3.3.3.

COROLLARY 3.3.4 (Grätzer, Jónsson and Lakser [73]). *The variety of modular lattices $\mathcal{M}$ does not satisfy the amalgamation property.* □

Projective Spaces. Projective spaces play an important role in the study of modular lattices. In this subsection, I outline some of the key results that we will be using to show that no non-distributive modular variety satisfies the amalgamation property. For the proofs of most of these results, consult Crawley and Dilworth [73].

DEFINITION 3.3.5.

(1) A *projective space* (also *projective geometry*) is a pair of sets (P, L), where P is a set whose elements are called *points* and L is a collection of subsets of P called *lines*, satisfying the following conditions:

(P1) Each line contains at least two points.

(P2) Any two distinct points p and q are contained in exactly one line (denoted by $\overline{\{p, q\}}$).

(P3) Let p, q and r be three distinct points such that no line contains $\{p, q, r\}$. Then if a line l intersects any two of the lines $\overline{\{p, q\}}, \overline{\{p, r\}}, \overline{\{q, r\}}$ in distinct points, then it intersects the third line.

(2) If a point $p \in P$ is an element of a line $l \in L$ in a projective space (P, L), we say *p lies on l* or *l passes through p*. We call a set of points *colinear* if all the points lie on the same line. A *triangle* (p, q, r) is a triple of non-colinear (and hence distinct) points.

(3) Given a projective space (P, L), we call a subset S of P a *subspace of P* if for every $p, q \in S$ such that $p \neq q$ we have $\overline{\{p,q\}} \subseteq S$. We denote the collection of subspaces of of P by $\mathcal{L}(P)$.

The connection between projective spaces and modular lattices is given by the following theorem, the proof of which is in Birkhoff [48], Frink [46] and Mousinho [50].

THEOREM 3.3.6. *Let P be an arbitrary projective space.*

1. *$\mathcal{L}(P)$ is a complete modular lattice, using set-inclusion as the lattice ordering.*
2. *With every modular lattice M, let $P(M)$ be the pair (P, L), where P is the set of all atoms of M, and $L = \left\{\overline{\{p,q\}} : p, q \in P, p \neq q\right\}$ where $\overline{\{p,q\}} = \{r \in P : r \leq p + q\}$ (that is, a line through distinct atoms p and q is the set of all atoms below $p + q$). Then $P(M)$ is a projective space.*
3. *$P(\mathcal{L}(P))$ is isomorphic to P.* □

DEFINITION 3.3.7.

1. The *dimension* of a projective space P is defined to be the height of the modular lattice $\mathcal{L}(P)$.
2. A projective space is called *non-degenerate* if every line contains at least three points.
3. A projective space P is said to be *Desarguesian* if for any two triples (a_0, a_1, a_2) and (b_0, b_1, b_2) of distinct points in P such that the lines $\overline{\{a_0, b_0\}}$, $\overline{\{a_1, b_1\}}$ and $\overline{\{a_2, b_2\}}$ intersect in a common point d, then the pairs of lines $\overline{\{a_0, a_1\}}$ and $\overline{\{b_0, b_1\}}$, $\overline{\{a_0, a_2\}}$ and $\overline{\{b_0, b_2\}}$ and $\overline{\{a_1, a_2\}}$ and $\overline{\{b_1, b_2\}}$ intersect in three colinear points, say c_2, c_1 and c_0 respectively (see figure 3.2 below).

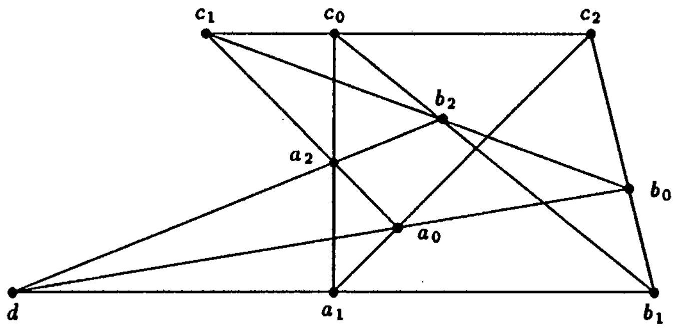


FIGURE 3.2

Important examples of non-degenerate spaces can be constructed from vector spaces over division rings (a *division ring* is a ring with a unit, and where every non-zero element has a multiplicative inverse) as follows:

Let V be an α-dimensional vector space over D, a division ring. The lattice $\mathcal{L}(V, D)$ of all vector subspaces of V is a modular lattice, hence we can construct an α-dimensional projective space $P = P(\mathcal{L}(V, D))$. In fact, we have $\mathcal{L}(V, D)$ is isomorphic to $\mathcal{L}(P)$. (see Crawley and Dilworth [73]).

One of the main results of co-ordinate geometry is the coordinization theorem, due to Frink [46].

THEOREM 3.3.8 (Frink [46]). *Let P be a non-degenerate Desarguesian projective space of dimension $\alpha \geq 3$. Then there exists a division ring D, unique up to isomorphism, such that $\mathcal{L}(P)$ is isomorphic to $\mathcal{L}(D^\alpha, D)$.*

For a full proof of the above result, see Crawley and Dilworth [73]. I make only the follwing observation: To construct the division ring D above, we pick an arbitrary line m say of P, and a point a_∞ on m. Set $D = m \setminus \{a_\infty\}$. Choosing any $a_0 \in D$, it is possible to define ring operations on D with a_0 the additive identity, such that D is a division ring with the properties described above.

Modular Lattice Varieties. All nondistributive subvarieties of modular lattice varieties fail to have the amalgamation property (proof due to Grätzer, Jónsson and Lakser [73]). I present an outline of the proof below. It uses the following lemma.

LEMMA 3.3.9 (Grätzer, Jónsson and Lakser [73]). *Let $\mathcal{V}$ be a variety satisfying the amalgamation property, and let $A, B, C, D \in \mathcal{V}$.*

(1) *If B is an extension of A, and $f : A \hookrightarrow B$ is any embedding, then there is an extension $E \in \mathcal{V}$ of B and embedding $g : B \hookrightarrow E$ such that $g|_A = f$.*

(2) *If D is an extension of C, and α is an automorphism on C, then there exists an extension $\overline{D} \in \mathcal{V}$ of D and an automorphism β on $\overline{D}$ such that $\beta|_C = \alpha$.*

Proof.

(1) Let $i : A \hookrightarrow B$ be the inclusion map, and consider the double extension (A, f, B, i, B), which by assumption has an amalgam (f', i', E), say. The result follows if we identify B with its image $f'(B)$ in E.

(2) Let $D_0 = C$, $D_1 = D$ and let $h_0 : D_0 \hookrightarrow D_1$ be the embedding obtained by composing α with the inclusion map from D_0 into D_1. Applying (1) with $A = D_0$, $B = D_1$ to obtain an extension D_2 of D_1 and an embedding $h_1 : D_1 \hookrightarrow D_2$ such that $h_1|_{h_0(B_0)} = h_0^{-1}$. Continuing this process inductively we obtain a countably infinite sequence of extensions $C = D_0 \subseteq D_1 \subseteq D_2 \subseteq \dots$ and a sequence of embeddings $h_n : D_n \hookrightarrow D_{n+1}$ such that $h_{n+1}|_{h_n(D_n)} = h_n^{-1}$. (See figure 3.3.).

$$\begin{tikzcd}
D_0 \arrow[r, hook] \arrow[d, hook, "\alpha=h_0"'] & D_2 \arrow[r, hook] \arrow[d, hook, "h_2"] & D_4 \arrow[r, hook] \arrow[d, hook, "h_4"] & \cdots & \overline{D} \arrow[d, hook, "\beta"] \\
D_1 \arrow[r, hook] \arrow[ur, hook, "h_1"'] & D_3 \arrow[r, hook] \arrow[ur, hook, "h_3"'] & D_5 \arrow[r, hook] & \cdots & \overline{D}
\end{tikzcd}$$

FIGURE 3.3

We can define

$$\overline{D} = \bigcup_{n \in \omega} D_n \quad \text{and} \quad \beta = \bigcup_{n \in \omega} h_{2n}$$

and we see $\beta : \overline{D} \to \overline{D}$ is an embedding. To see that β is onto, choose any $y \in \overline{D}$. By the definition of $\overline{D}$, there is an $n \in \omega$ such that $y \in D_{2n+1}$. Put $x = h_{2n+1}(y)$, then $\beta(x) = h_{2n+2}(x) = y$, since h_{2n+2} is an extension of h_{2n+1}^{-1}. Hence β is an automorphism on $\overline{D}$ such that $\beta|_C = \alpha$. □

THEOREM 3.3.10 (Grätzer, Jónsson and Lakser [73]). *Any non-distributive subvariety of the variety $\mathcal{M}$ of all modular lattices does not satisfy the amalgamation property.*

Outline of proof. Suppose, for a contradiction, that there is a non-distributive modular lattice variety $\mathcal{V}$ satisfying the amalgamation property. By theorem 3.3.3, every $L \in \mathcal{V}$ can be embedded into a simple complemented lattice $C \in \mathcal{V}$. We may assume C contains an infinite chain. (For if not, embed L into L', the lattice obtained by adding an infinite chain to L. $L' \in \mathcal{V}$, so we may apply theorem 3.3.3 to L'). Now C can be embedded into some $\mathcal{L}(P) \in \mathcal{V}$, where P is a projective geometry. (For the details, see Jónsson [54] and Frink [46]). $\mathcal{L}(P)$ is isomorphic to a product of lattices induced by non-degenerate projective geometries, say $\prod_{i \in I} \mathcal{L}(P_i)$, and in fact, since C is simple, C is embeddable in $\mathcal{L}(P_i)$ for some $i \in I$. (For the details, see Grätzer, Jónsson and Lakser [73]). Hence for any $L \in \mathcal{V}$, L can be embedded into $\mathcal{L}(P) \in \mathcal{V}$ for some infinite dimensional non-degenerate projective space P.

There is a three-dimensional projective space Q such that $\mathcal{L}(Q) \in \mathcal{V}$ and Q has at least six points on each line (for the details see Grätzer, Jónsson and Lakser [73]). Now $\mathcal{L}(Q)$ can be embedded into $\mathcal{L}(P) \in \mathcal{V}$ for some infinite projective geometry P. By theorem 3.3.8, $\mathcal{L}(P)$ is isomorphic to $\mathcal{L}(V, D)$ for some vector space V over a division ring D. But then $\mathcal{L}(Q)$ is isomorphic to a sublattice of subgroups of an abelian group, hence is Desarguesian. Hence by theorem 3.3.8, $\mathcal{L}(Q)$ is isomorphic to some $\mathcal{L}(\overline{V}, \overline{D})$ for some vector space $\overline{V}$ over $\overline{D}$. In fact, we can choose any two points a_0 and a_∞ on some line m, and $\overline{V} = m \setminus \{a_\infty\}$ with a_0 as additive inverse.

Any permutation of points of m induces an automorphism on the interval $m/0$ in $\mathcal{L}(Q)$. Since m has at least six points, there is an $x, y \in \overline{V} \setminus \{a_0\}$ so that $x \oplus y \neq a_0$, where $\oplus$ is the vector addition. Thus we can find an automorphism α on the interval $m/0$ that keeps a_0, a_∞, x and y fixed, but maps $x + y$ onto a point $z \neq x + y$. By lemma 3.3.9, there is an extension $K \in \mathcal{V}$ of $\mathcal{L}(Q)$ such that α extends to an automorphism β on K. Applying the above argument again to K, we K is embeddable into some $\mathcal{L}(V_1, D_1)$ for some vector space V_1 over D_1, letting say $f : K \hookrightarrow \mathcal{L}(V_1, D_1)$ be the embedding. Now the following holds:

$$a \in f(x \oplus y) \quad \text{if and only if for some } b \in f(x), c \in f(y)$$

$$\text{we have } a \ominus b, a \ominus c \in f(a_\infty) \text{ and } a \ominus b \ominus c \in f(a_0)$$

$$(\text{where } a \ominus b = a \oplus (\ominus b) \text{ where } \ominus b \text{ is the additive inverse of } b$$

(For details, see Grätzer, Jónsson and Lakser [73]).

But then, since β fixes x, y, a_∞ and a_0, we have $f(x \oplus y) = f(\beta(x \oplus y)) = f(z)$, contradicting the fact that f is an embedding since $x \oplus y \neq z$. ☐

A-Decomposability. A-Decomposability was first introduced by Slavik [83]. This notion will be used to present a complete characterization of all lattice varieties satisfying the amalgamation property.

DEFINITION 3.3.11 (Slavik [83]). Let L be a finite lattice with L_1 and L_2 proper sublattices of L, and $L = L_1 \cup L_2$. Let $i_1 : L_1 \cap L_2 \hookrightarrow L_1$ and $i_2 : L_1 \cap L_2 \hookrightarrow L_2$ be the respective inclusion maps. L is said to be *A-decomposable by means of L_1 and L_2* (written $L = A(L_1, L_2)$) if, whenever (f_1, f_2, L_3) is an amalgam of $(L_1 \cap L_2, i_1, L_1, i_2, L_2)$, $f = f_1 \cup f_2$ is an embedding of L into L_3.

Thus if $\mathcal{V}$ is a variety satisfying the amalgamation property, and L is a finite lattice with $L_1, L_2 \in \mathcal{V}$ and $L = A(L_1, L_2)$, then $L \in \mathcal{V}$.

The above property is useful for constructing arbitrary large lattices in a lattice variety satisfying the amalgamation property. We will use it to show that if $\mathcal{V}$ is a non-distributive lattice variety satisfying the amalgamation property, it has to contain all lattices.

For any element $z \in L$, a lattice, let $C(z)$ be the set of all covers of z in L, and let $C^d(z)$ be the set of all dual covers of z in L.

THEOREM 3.3.12 (Day and Ježek [84]). *Let $L = L_1 \cup L_2$ be a finite lattice with L_1 and L_2 proper sublattices of L. Then $L = A(L_1, L_2)$ if and only if L_1 and L_2 satisfy*

(1) *For all $\{i, j\} = \{1, 2\}$, if $x \in L_i, y \in L_j$ and $x \leq y$, then there is a $z \in L_1 \cap L_2$ such that $x \leq z \leq y$.*
(2) *If $z \in L_1 \cap L_2$, then $C^d(z) \subseteq L_1$ or $C^d(z) \subseteq L_2$.*
(3) *If $z \in L_1 \cap L_2$, then $C(z) \subseteq L_1$ or $C(z) \subseteq L_2$.*

Proof. $\Longrightarrow$: Suppose $L = A(L_1, L_2)$. Let $(f_1, f_2, \mathcal{I}(L_1, L_2))$ be the amalgam of $(L_1 \cap L_2, i_1, L_1, i_2, L_2)$ given by the construction in theorem 3.3.2. By assumption, the map $f = f_1 \cup f_2 : L \rightarrow \mathcal{I}(L_1, L_2)$ given by $f(x) = (x]$ is a lattice embedding. Let $x \in L_i, y \in L_j$ with $x \leq y$ where $\{i,j\} = \{1,2\}$. Then $f(x) = (x] \subseteq (y] = f(y)$, and as in the proof of theorem 3.3.2, we have $x \leq_D y$, hence there is a $z \in L_1 \cap L_2$ such that $x \leq z \leq y$. So (1) holds. Now suppose that (2) fails: Then there is a $z \in L_1 \cap L_2$ with dual covers x and y such that $x \in L_1 \setminus L_2$ and $y \in L_2 \setminus L_1$. Now $z = x + y$ so $f(z) = (z] = (x] + (y]$. But $(x] \cup (y]$ is already an (L_1, L_2)-ideal, so we have $(x] \cup (y] = (x] + (y]$, a contradiction since $z \notin (x] \cup (y]$. Hence (2) holds. Proving that (3) holds is just the dual of the proof that (2) holds (using the filter completion).

$\Longleftarrow$: Suppose (1), (2) and (3) hold. Let (f_1, f_2, L_3) be an amalgam of $(L_1 \cap L_2, i_1, L_1, i_2, L_2)$. We must show $f = f_1 \cup f_2$ is an embedding from L into L_3.

Now f is strictly order preserving: For if $x < y$ and $x, y \in L_i$ for some $i \in \{1,2\}$, then $f(x) < f(y)$ since f_i is an embedding. On the other hand, if $x < y$ and $x \in L_i \setminus L_j, y \in L_j \setminus L_i$ where $\{i,j\} = \{1,2\}$, then there is a $z \in L_1 \cap L_2$ such that $x < z < y$ (the inequalities are strict as $x, y \notin L_1 \cap L_2$). But then $f_i(x) < f_i(z) = f_j(z) < f_j(y)$, so $f(x) < f(y)$.

Next we show f is a homomorphism: Since f_i is already a homomorphism for $i = 1, 2$, we need only show $f(x+y) = f(x) + f(y)$ and $f(x \cdot y)$ for some $x \in L_1 \setminus L_2$, $y \in L_2 \setminus L_1$.

Define, for $i = 1, 2$, $\mu_i : L \rightarrow L_i$ by $\mu_i(u) = \sum \{v \in L_i : v \leq u\}$ (with the interpretation $\sum \emptyset = 0_{L_i}$). Note that each μ_i is order-preserving. Define two sequences in L inductively as follows:

$$(*) \qquad \begin{aligned} x_0 &= x & y_0 &= y \\ x_{n+1} &= x_n + \mu_1(y_n). & y_{n+1} &= y_n + \mu_2(x_n). \end{aligned}$$

By induction, for all $n \in \omega$, $x_n + y_n = x + y$ and $f(x_n) + f(y_n) = f(x) + f(y)$.

Claim: For some $n \in \omega$, either $x_n = x + y$ or $y_n = x + y$ holds.

Proof of claim: Suppose not, that is, $x_n, y_n < x + y$ for all $n \in \omega$. Now since L is finite, there is a $k \in \omega$ such that $x_{k+1} = x_k$ and $y_{k+1} = y_k$. Hence $\mu_1(y_k) \leq x_k$ and

$\mu_2(x_k) \le y_k$ using (*) above. Also, by the definition of μ_1 and μ_2, $\mu_1(y_k) \le y_k$ and $\mu_2(x_k) \le x_k$. Hence $\mu_1(y_k), \mu_2(x_k) \le x_k y_k$.

Now if $x_k y_k \in L_1$, then $x_k y_k \le \mu_1(y_k)$ so $\mu_1(y_k) = x_k y_k \in L_1$. But $y_k \in L_2$, hence by (1) there is a $z \in L_1 \cap L_2$ such that $\mu_1(y_k) \le z \le y_k$. But then $z \in L_1$ and $z \le y_k$ implies $z \le \mu_1(y_k)$ so $z = \mu_1(y_k) = x_k y_k \in L_1 \cap L_2$. On the other hand, if $x_k y_k \in L_2$, a similar argument shows $\mu_2(x_k) = x_k y_k \in L_1 \cap L_2$.

Now $x_k, y_k \notin L_1 \cap L_2$ (For if $x_k \in L_1 \cap L_2$, say, then $\mu_2(x_k) = x_k$ so $y_{k+1} = y_k + x_k = x + y$, contradicting our assumption, and similarly for y_k). Let u, v be covers of $x_k y_k$ such that $u \le x_k$ and $v \le y_k$. Now $u, v \notin L_1 \cap L_2$, for if say $u \in L_1 \cap L_2$ then $x_k y_k = \mu_2(x_k) < u$, contradicting the definition of μ_2, and similarly for v. But then $u \in L_1 \setminus L_2$ and $v \in L_2 \setminus L_1$, contradicting condition (3), and thus proving our claim.

Now by the claim, for some $n \in \omega$, $x_n = x + y$ or $y_n = x + y$. Without loss of generality, suppose the former holds. Then $f(x) + f(y) = f(x_n) + f(y_n) = f(x + y) + f(y_n) = f(x + y)$ since f is order-preserving.

The proof that $f(x) \cdot f(y) = f(x \cdot y)$ is just the dual of the above argument. Hence, since f is a strictly order preserving homomorphism, it is an embedding. □

COROLLARY 3.3.13 (Day and Ježek [84]).

(1) *If* $L = A(L_1, L_2)$ *and* L_1 *is a sublattice of* L'_i, $i = 1, 2$, *and each* L_i *is a proper sublattice of* L, *then* $L = A({L_1}', {L_2}')$.

(2) *If* $L = [a) \cup (b]$ *for some* $a, b \in L$ *with* $0 < a \le b < 1$, *then* $L = A([a), (b])$. □

Bounded Lattices. Bounded lattices were first introduced by McKenzie [72]. Of particular importance is the fact that the finite bounded lattices generate the variety of all lattices. I state without proof some of the results about bounded lattices below.

DEFINITION 3.3.14.

(1) A lattice L is said to be *semidistributive* if it satisfies the following two implications for all $u, x, y, z \in L$:

$$u = x + y = x + z \text{ implies } u = x + yz \quad \text{and dually,}$$

$$u = xy = xz \text{ implies } u = x(y + z).$$

(2) A lattice homomorphism $f : L \rightarrow L'$ is said to be *upper bounded*, if for every $b \in L'$ the set $f^{-1}(b] = \{x \in L : f(x) \leq b\}$ is either empty or has a greatest element.

(3) A lattice homomorphism $f : L \rightarrow L'$ is said to be *lower bounded*, if for every $b \in L'$ the set $f^{-1}[b) = \{x \in L : f(x) \geq b\}$ is either empty or has a least element.

(4) A lattice homomorphism $f : L \rightarrow L'$ is said to be *bounded* if it is both upper and lower bounded.

(5) A lattice L' is said to be *bounded* if it is the bounded epimorphic image of a free lattice (that is, there is a free L' and a bounded epimorphism $e : L \twoheadrightarrow L'$). We denoted the finite bounded lattices by $\mathcal{B}_F$.

DEFINITION 3.3.15 (Day [70]). Given a lattice L and an interval $I = u/v$ in L, we construct a new lattice $L[I] = (L \setminus I) \cup (I \times \underline{2})$ with the ordering $x \leq y$ in $L[I]$ if and only if at least one of the following holds:

(1) $x, y \in L \setminus I$ and $x \leq y$ in L.
(2) $x \in L \setminus I, y = (b, i)$ and $x \leq b$ in L. $(i = 0, 1)$.
(3) $y \in L \setminus I, x = (a, i)$ and $a \leq y$ in L. $(i = 0, 1)$.
(4) $x = (a, i), y = (b, j)$ and $a \leq b$ in L and $i \leq j$ in $\underline{2}$.

Note there is a natural epimorphism $\gamma : L[I] \twoheadrightarrow L$ given by

$$\gamma(x) = \begin{cases} x & \text{if } x \in L \setminus I \\ a & \text{if } x = (a, i) \text{ for some } a \in I, i \in \underline{2}. \end{cases}$$

In theorem 3.3.16 below, (1) is a consequence of a result in Whitman [41], (2) can be shown using the definition of a bounded lattice, and I give a reference for the proofs for (3) to (5).

THEOREM 3.3.16.

(1) *Every bounded lattice is semi-distributive.*

(2) *$\mathcal{B}_F$ is closed under sublattices, homomorphic images and direct products with finitely many factors.*

(3) *(Day [77]) If $L \in \mathcal{B}_F$ and $I = u/v$ is an interval in L, then $L[I] \in \mathcal{B}_F$.*

(4) *(Day [79]) A finite lattice L is bounded if and only if there is a sequence of lattices $L_0, L_1, \ldots, L_{n+1} = L$, where L_0 is the one-element lattice and a sequence of intervals $u_0/v_0, \ldots, u_n/v_n$ with each $u_i/v_i \subseteq L_i$ such that L_{i+1} is isomorphic to $L_i[u_i/v_i]$, where $i \in \{0, \ldots, n\}$.*

(5) *(Day [77]) The smallest lattice variety containing $\mathcal{B}_F$ is the variety of all lattices $\mathcal{L}$.* □

The Day-Ježek Theorem. If a non-distributive lattice variety satisfies the amalgamation property, using A-decomposibility we can show that all finite bounded lattices are in that variety, and hence it must be the variety of all lattices. I present the details of this argument below, due to Day and Ježek [84]. It enables us to describe all lattice varieties with the amalgamation property.

LEMMA 3.3.17 (Day and Ježek [84]). *Let $I = u/v$ be an interval in a lattice L, $\Theta \in Con(L)$, and let J be the interval $u/\Theta/\, v/\Theta$ on L/Θ. If $I = \bigcup J$, then $L[I]$ is isomorphic to a sublattice of the direct product of L and $L/\Theta[J]$.*

Proof. Recall for an algebra A, if $\Phi, \Psi \in \mathrm{Con}(A)$ such that $\Phi \cap \Psi = \mathbf{0}_A$, then A is a subdirect product of A/Φ and A/Ψ. So, to prove the above statement, we need only find $\Psi, \Phi \in \mathrm{Con}(L[I])$ such that $L[I]/\Psi$ is isomorphic to a sublattice of L, $L[I]/\Phi$ is a sublattice of $L/\Theta[J]$ and $\Psi \cap \Phi = \mathbf{0}_{L[I]}$.

Now let $h : L[I] \twoheadrightarrow L$ be the natural epimorphism, and let $\Psi = \ker h$. Note $L[I]/\Psi$ is isomorphic to L. Define Φ by:

$$x \;\Phi\; y \text{ if and only if } h(x) \;\Theta\; h(y) \text{ and } x, y \in L \setminus I \text{ or } x, y \in I \times \{i\} \text{ where } i = 0, 1.$$

Since h is a homomorphism and $\Theta \in \mathrm{Con}(L)$, $\Phi \in \mathrm{Con}(L[I])$. Also, since $I = \bigcup J$,

$$\begin{aligned} x \in L \setminus I \text{ implies } & \; x/\Phi = x/\Theta \text{ and} \\ (x,i) \in I \times \{i\} \text{ implies } & \; (x,i)/\Phi = (x/\Theta, i) \text{ where } i = 0, 1 \end{aligned}$$

so $L[I]/\Phi$ is a subset of $L/\Theta[J]$. In fact, a case by case analysis of meets and joins shows it is a sublattice.

Now suppose $x, y \in L[I]$ and $x \;(\Psi \cap \Phi)\; y$. Then $h(x) = h(y)$ and $x, y \in L \setminus I$ or $x, y \in I \times \{i\}, (i = 0, 1)$. From the definition of $L[I]$, we see $x = y$ so $\Phi \cap \Psi = \mathbf{0}_{L[I]}$. □

LEMMA 3.3.18 (Day and Ježek [84]). *If L is a finite semidistributive lattice, and $u, v \in L$ with v an atom and u a co-atom in L, then $L[u/v]$ is a sublattice of a product of L and N.*

Proof. Let $\kappa(u) = \sum\{x \in L : xu = 0\}$ and $\lambda(v) = \prod\{y \in L : y + v = 1\}$. Define a map $h : L \to \underline{2} \times \underline{2}$ by

$$h(x) = \begin{cases} (1,1) & \text{if } x \in 1/u + \lambda(v) \\ (0,1) & \text{if } x \in u/v \\ (1,0) & \text{if } x \in \lambda(v)/\kappa(u) \\ (0,0) & \text{if } x \in \kappa(u) \cdot v/0 \end{cases}$$

Using semidistributivity, it can be shown that h is a well-defined homomorphism. Let $\Theta = \ker h$. Now $J = (u/\Theta)/(v/\Theta) = \{u/\Theta\}$, hence $\bigcup J = u/\Theta = u/v$. Hence $(L/\Theta)[J]$ is isomorphic to a sublattice of N, and the result follows by lemma 3.3.17. □

LEMMA 3.3.19 (Day and Ježek [84]). *Let $\mathcal{V}$ be a lattice variety satisfying the amalgamation property and containing N. If $L \in \mathcal{B}_F \cap \mathcal{V}$ and $v \leq u \in L$, then $L_i = (L \times \underline{2})[(u,i)/(v,i)] \in \mathcal{B}_F \cap \mathcal{V}$ for $i = 0, 1$.*

Proof. By theorem 3.3.16(4), $\underline{2} \in \mathcal{B}_F$. Hence $L \times \underline{2} \in \mathcal{B}_F$ by theorem 3.3.16(2), and thus $L_i \in \mathcal{B}_F$ for $i = 0, 1$ by theorem 3.3.16(3).

Fix $i = 1$. Now if $u = 1_L$, then $L_i \leq L \times \underline{3}$ hence $L_1 \in \mathcal{V}$. Hence we may assume $u < 1_L$. We proceed inductively on the size of L. Since L is finite, there is a co-atom w of L such that $u \leq w \prec 1_L$. Let $p = \prod\{y \in L : y + w = 1\}$. Using semidistributivity, it can be checked that $L = (w] \cup [p)$. Hence $L \times \underline{2}$ can be represented as in figure 3.4.

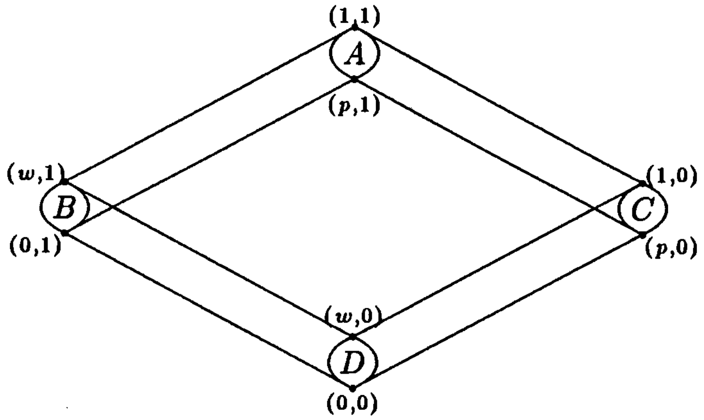


FIGURE 3.4

Let $I = (w,1)/(0,1)$. Then by lemma 3.3.18, $(L \times \underline{2})[I]$ is a sublattice of $(L \times \underline{2}) \times N$. Hence there is an induced homomorphism $h : (L \times \underline{2})[I] \to N$ and thus $(L \times \underline{2})[I]$ can be represented as in figure 3.5.

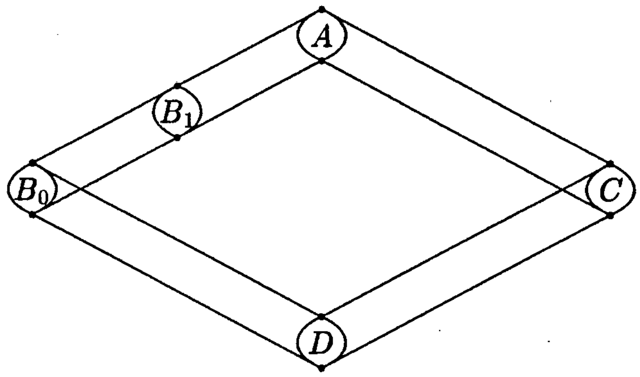


FIGURE 3.5

where A, B_1, B_2, C and D are the congruence classes of $\ker h$. Since B_0 is an interval of $(L \times \underline{2})[I]$, we can use Day's construction to produce $L' = ((L \times \underline{2})[I])[B_0]$ as represented as in figure 3.6.

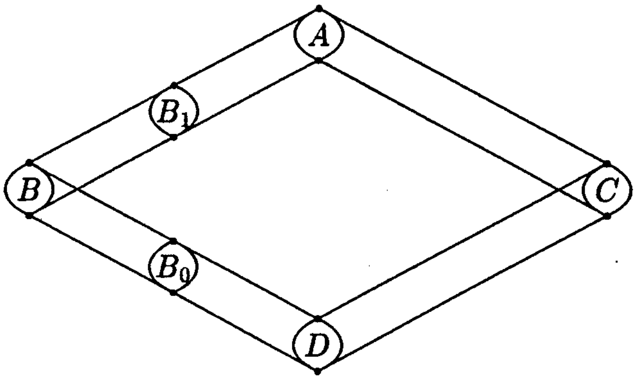


FIGURE 3.6

Now $L' \in \mathcal{V}$ by lemma 3.3.18, since $N \in \mathcal{V}$. Let $J = u/v$, considered as an interval in B. Now consider $L'[J] = A \cup B_0 \cup B_1 \cup C \cup D \cup B[J]$. Define $C_1 = A \cup B_0 \cup B_1 \cup C \cup D$ and $C_2 = B_0 \cup B_1 \cup B[J]$. By checking the conditions of theorem 3.3.12, we have $L'[J] = A(C_1, C_2)$. Now C_1 is isomorphic $(L \times \underline{2})[I] \in \mathcal{V}$, by lemma 3.3.18. By corollary 3.3.13, $C_2 = A(B_0 \cup B[J], B_1 \cup B[J])$ so $C_2 \in \mathcal{V}$ if and only if $B_1 \cup B[J] \in \mathcal{V}$ for $i = 0, 1$. But $B = w/0$, so $|B| < |L|$ and $B_i \cup B[J] = (B \times \underline{2})[(u,i)/(v,i)]$. Hence, by the induction hypothesis, $B_i \cup B[J] \in \mathcal{B}_F \cap \mathcal{V}$ for $i = 0, 1$ so $C_2 \in \mathcal{V}$. Hence $L'[J] \in \mathcal{V}$. But L_1 is a sublattice of $L'[J]$, hence $L_1 \in \mathcal{V}$. The case for $i = 0$ is similar. ☐

THEOREM 3.3.20 (Day and Ježek [84]). *The only lattice varieties satisfying the amalgamation property are $\mathcal{T}$, $\mathcal{D}$ and $\mathcal{L}$.*

Proof. If $\mathcal{V}$ is a non-distributive lattice variety satisfying the amalgamation property, then by theorem 3.3.10, $N \in \mathcal{V}$. By the lemma 3.3.19, if $L \in \mathcal{B}_F \cap \mathcal{V}$ and u/v is an interval in L, we have $L[u/v] \in \mathcal{V}$ since $L[u/v]$ is a sublattice of $(L \times \underline{2})[(u,0)/(v,0)]$. Hence by theorem 3.3.16(4), $\mathcal{B}_F \subseteq \mathcal{V}$, and hence by theorem 3.3.16(5), $\mathcal{V} = \mathcal{L}$. ☐

3.4 Varieties of Heyting Algebras

Introduction. Heyting algebras are generalizations of Boolean algebras, where the complementation operation is weakened to a binary operation called (relative) pseudocomplementation. In this subsection, I introduce the definition of and some elementary results concerning Heyting algebras.

DEFINITION 3.4.1. A *Heyting algebra* $\langle H, +, \cdot, \rightarrow, 0, 1\rangle$ is an algebra with three binary operations $+$, $\cdot$ and $\rightarrow$ (called *(relative) pseudocomplementation)* together with constants 0 and 1, such that $\langle H, +, \cdot, 0, 1\rangle$ is a distributive lattice with 0 and 1 as bottom and top elements respectively, and the following identities are satisfied:

(1) $a \cdot (a \rightarrow b) = a \cdot b$
(2) $(a \rightarrow b) \cdot b = b$
(3) $(a \rightarrow b) \cdot (a \rightarrow b) = a \rightarrow (b \cdot c)$
(4) $a \rightarrow a = 1$

We denote by $\mathcal{H}$ the variety of all Heyting algebras. We will use the following facts about Heyting algebras (see Rasiowa and Sikorski [63] for proofs of these and other basic facts about Heyting algebras).

LEMMA 3.4.2. *Let $x, y, z \in \mathcal{H}$, $H \in \mathcal{H}$. Then*

(1) $x \leq y \rightarrow z$ *if and only if* $xy \leq z$.
(2) *If* $x \leq y$ *then* $x \rightarrow z \geq y \rightarrow z$.
(3) $(x \rightarrow y)(z \rightarrow y) = (x + z) \rightarrow y$.
(4) $x \rightarrow y = 1$ *if and only if* $x \leq y$.

Proof. Found in Rasiowa and Sikorski [63]. □

Heyting algebras are congruence permutable and distributive, for one can check that $m(x, y, z) = ((x \rightarrow y) \rightarrow z)((z \rightarrow y) \rightarrow x)(x + z)$ is Maltsev term. Some examples of Heyting algebras are:

(1) Every bounded chain is a Heyting algebra, if we define

$$a \rightarrow b = \begin{cases} 1 \text{ if and only if } a \leq b \\ b \text{ if and only if } b < a. \end{cases}$$

We will denote the $n + 2$ element chain viewed as a Heyting algebra by C_n.

(2) If $\langle X, \mathcal{T}\rangle$ is a topology, the open subsets of X form a Heyting algebra with lattice order given by set-inclusion, and for any two open subsets A, B of X, $A \rightarrow B = \operatorname{int}((X \setminus A) \cup B)$, where int is the interior operator on the topology. We will denote the Heyting algebra of open subsets of X by X^O. (For a general introduction to topology, see Engelking [89]).

There is a correlation between filters and congruences on Heyting algebras. In fact, for $H \in \mathcal{H}$, Con(H) is lattice isomorphic to Filt(H), where Filt(H) is the lattice of filters on H (with respect to $\subseteq$). The isomorphism is given as follows: For every $\Theta \in \mathrm{Con}(H)$, $F_\Theta = \{a \in H : a\ \Theta\ 1\}$ is the filter corresponding to Θ. Given a filter $F \in \mathrm{Filt}(H)$, $\Theta_F = \{(a,b) \in H^2 : (a \to b)(b \to a) \in F\}$ is the congruence on H corresponding to the filter.

Using the correspondence between filters and congruences we get the following lemmas:

LEMMA 3.4.3.

(1) $a = b$ *if and only if* $(a \to b)(b \to a) = 1$ *for* $a, b \in H$.

(2) *If* $\Theta \in Con(H)$, *then* H/Θ *is finitely subdirectly irreducible if and only if* F_Θ *is a prime filter.*

(3) *The following are equivalent:*

(a) $H \in \mathcal{H}_{SI}$.

(b) $1 \in H$ *is completely join irreducible.*

(c) $H \models (\exists b)(\forall a)(a < 1 \implies a \leq b)$, *that is* H *has a largest non-unit element.* □

In particular, we see that for any subvariety of Heyting algebras $\mathcal{V}$, $\mathcal{V}_{SI}$ is an elementary class. We will call the largest non-unit element of a subdirectly irreducible algebra H the *monolith* of H.

Since Heyting algebras are an example of distributive lattices, we can formulate the following result due to Stone [36] in the setting of Heyting algebras.

LEMMA 3.4.4. *Let* I *be a proper ideal of* H, *and let* $a \notin I$. *Then there is a prime filter* F *of* H *such that* $F \cap I = \emptyset$ *and* $a \notin F$.

For the rest of this chapter, I will assume all prime filters are proper. (Recall a filter F on H is *prime* if for all $a, b \in H$, $a + b \in F$ implies $a \in F$ or $b \in F$). We denote by $\Omega(H)$ the set of all prime filters of a Heyting algebra H.

The Variety of All Heyting Algebras. I now present the result that $\mathcal{H}$ satisfies the strong amalgamation property.

THEOREM 3.4.5 (Day [b]). *The variety of Heyting algebras $\mathcal{H}$ has the strong amalgamation property.*

Proof. Let (A, f, B, g, C) be an amalgam in $\mathcal{H}$. By the remark above lemma 1.3.2, we may assume $A = B \cap C$, and f and g are inclusion maps. Define $X = \{(P, Q) \in \Omega(B) \times \Omega(C) : P \cap A = Q \cap A\}$. Define the map $\sigma : B \times C \to \mathcal{P}(X)$ by $\sigma(b, c) = \{(P, Q) \in X : b \in P \text{ and } c \in Q\}$. Note for a prime filter F, we have:

(1) $ab \in F$ if and only if $a \in F$ and $b \in F$

(2) $a + b \in F$ if and only if $a \in F$ or $b \in F$.

Hence,

$$(*) \qquad \begin{aligned} \sigma(b_1, c_1) \cap \sigma(b_2, c_2) &= \sigma(b_1 \cdot b_2, c_1 \cdot c_2) \\ \text{and } \sigma(b_1, c_1) \cup \sigma(b_2, c_2) &= \sigma(b_1 + b_2, c_1 + c_2). \end{aligned}$$

Also, $\sigma(1, 1) = X$. Hence the set $\mathcal{B} = \{\sigma(b, c) : (b, c) \in B \times C\}$ forms a base for a topology, say $\mathcal{T}$ on X. Put $D = X^O$, the Heyting algebra of open subsets of X with respect to the topology $\mathcal{T}$, and consider the functions $f' : B \to D$ and $g' : C \to D$ given by $f'(b) = \sigma(b, 1)$ and $g(c) = \sigma(1, c)$ respectively for $b \in B, c \in C$. We now proceed to show (f', g', D) is a strong amalgam of (A, f, B, g, C).

Firstly, by (*) above, we see that $f(b_1) \cup f(b_2) = f(b_1 + b_2)$, and $f(b_1) \cap f(b_2) = f(b_1 \cdot b_2)$ for each $b_1, b_2 \in B$. Also $f(1) = \sigma(1, 1) = X$ and $f(0) = \sigma(1, 0) = \emptyset$ as no prime filters contain 0. Hence f, and similarly g, are lattice homomorphisms preserving 0 and 1.

Next we show f and g preserve pseudocomplementation. We need the following claim:

Claim: Let F be a filter on B and Q a prime filter on C such that $F \cap A \subseteq Q \cap A$. Then F can be extended to a prime filter P such that $(P, Q) \in X$.

Proof of claim:

$$\begin{aligned} &\text{Let } G = F + \{x \in B : a \leq x \text{ for some } a \in Q \cap A\} \\ &\text{and } I = \{y \in B : y \leq a \text{ for some } a \in A \setminus (Q \cap A)\}. \end{aligned}$$

Now $Q \cap A$ is a prime filter on A, and I is an ideal on B. Now $G \cap I = \emptyset$: For if not, then there is some $u \in G \cap I$. Hence there is an $v \in F, a_1 \in Q \cap A$ and $a_2 \in A \setminus (Q \cap A)$

with $a_1 \cdot v \leq u \leq a_2$. Hence $v \leq a_1 \to a_2 \in A$. Thus $a_1 \to a_2 \in F \cap A \subseteq Q \cap A$. Hence, since $a_1, a_1 \to a_2 \in Q \cap A$, we have $a_2 \in Q \cap A$, a contradiction.

Hence by lemma 3.4.4, there is a prime filter $P \supseteq G$ such that $P \cap I = \emptyset$. Now, if $x \in P \cap A$, then $x \notin I$, in particular $x \notin A \setminus (Q \cap A)$ so $x \in P \cap A$. Hence $(P, Q) \in X$, proving our claim.

As a special instance of the claim above, if we let $F = \{1\}$, then we have that the following holds: For every $Q \in \Omega(C)$ there is a $P \in \Omega(B)$ such that $(P, Q) \in X$.

We now show f preserves pseudocomplements: Let $b_1, b_2 \in B$. Now

$$\begin{aligned} f'(b_1) \to f'(b_2) &= \operatorname{int}((X \setminus f'(b_1)) \cup f'(b_2)) \\ &= \bigcup \{\sigma(x,y) : (X \setminus \sigma(b_1, 1) \cup \sigma(b_2, 1)) \subseteq \sigma(x,y)\} \\ &= \bigcup \{\sigma(x,y) : \sigma(b_1, 1) \cap \sigma(x,y) \subseteq \sigma(b_2, 1)\} \\ &= \bigcup \{\sigma(x,y) : \sigma(b_1 \cdot x, y) \subseteq \sigma(b_2, 1)\}\,. \end{aligned}$$

Now $\sigma(b_1 \cdot (b_1 \to b_2), 1) = \sigma(b_1 \cdot b_2, 1) \subseteq \sigma(b_2, 1)$ since f' is join preserving. Hence $f(b_1 \to b_2) = \sigma(b_1 \to b_2) \subseteq f(b_1) \to f(b_2)$.

To prove the other direction, suppose $f(b_1 \to b_2) \not\supseteq f(b_1) \cap f(b_2)$ for some $b_1, b_2 \in B$. Then there is an $(x, y) \in B \times C$ such that

(1) $\sigma(x \cdot b_1, y) \subseteq \sigma(b_2, 1)$ and

(2) $\sigma(x, y) \not\subseteq \sigma(b_1 \to b_2, 1)$.

By (2), there is a $(P, Q) \in \sigma(x, y)$ such that $b_1 \to b_2 \notin P$. Now since $b_2 \leq b_1 \to b_2$, $b_2 \notin P$. Hence by (1) $b_1 \notin P$.

Suppose $b_2 \in P + [b_1)$. Then there is a $p \in P$ such that $b_1 \cdot p \leq b_2$, that is $b_1 \to b_2 \geq p$, contradicting $b_1 \to b_2 \notin P$.

So $b_2 \notin P + [b_1)$. Hence we may extend $P + [b_1)$ to a prime filter P' with $b_2 \notin P'$. Thus $Q \cap A = P \cap A \subseteq P' \cap A$ so by the dual of the claim above, we may extend Q to a prime filter Q' on C with $(P', Q') \in X$. Now $y \in Q', x \cdot b_1 \in P'$ and $b_2 \notin P'$, hence $(P', Q') \in \sigma(x \cdot b_1, y) \setminus \sigma(b_2, 1)$, contradicting (1). Hence $f(b_1 \to b_2) = f(b_1) \to f(b_2)$ so f is a homomorphism. Similarly we can show g is a homomorphism.

Now to show f' is one-one, pick $b \in B$ with $b \neq 1$. By theorem 1.2.4, there is a congruence Θ such that $B/\Theta \in \mathcal{H}_{SI}$ and $b/\Theta \neq 1/\Theta$. If P is the filter induced by Θ, we have P is a prime filter not containing b. Hence by the claim above, there is a $Q \in \Omega(C)$ such that $(P,Q) \in X$. Note $(P,Q) \notin \sigma(b,1)$, hence $f'(b) = \sigma(b,1) \neq X = \sigma(1,1) = f'(1)$. Now if $b_1, b_2 \in B$ such that $f'(b_1) = f'(b_2)$, then $f'((b_1 \to b_2)(b_2 \to b_1)) = f'(1)$. Hence $(b_1 \to b_2)(b_2 \to b_1) = 1$, that is, $b_1 = b_2$ by 3.4.3(1), so f' is an embedding. Similarly g' is an embedding.

To show $f'f = g'g$, note for $a \in A$,

$$\begin{aligned} f'(a) &= \sigma(a,1) \\ &= \{(P,Q) \in X : a \in P \text{ and } P \cap A = Q \cap A\} \\ &= \{(P,Q) \in X : a \in P, a \in Q \text{ and } P \cap A = Q \cap A\} \\ &= \{(P,Q) \in X : a \in Q \text{ and } P \cap A = Q \cap A\} \\ &= \sigma(1,a) = g'(a). \end{aligned}$$

We next show that $f'(B) \cap g'(C) = f'f(A)$: Pick $S \in f'(B) \cap g'(C)$. Then $S = \sigma(b,1) = \sigma(1,c)$ for some $b \in B, c \in C$. We claim there is an $a_1 \in A$ such that $b \leq_B a_1$, $a_1 \leq_C c$: Suppose not. Then for every $a \in A \cap [b)$, $a \not\leq_C c$. Now $A \cap [b)$ is non-empty and closed under finite meets, so can be extended to a prime filter on C, say Q, with $c \notin Q$. By the claim above, since $[b) \cap A \subseteq Q \cap A$, we can find a prime filter P on B with $(P,Q) \in X$, $b \in P$. But then $(P,Q) \in \sigma(b,1) \setminus \sigma(1,c)$, a contradiction.

Hence there exists an $a_1 \in A$ such that $b \leq_B a_1, a_1 \leq_C c$. Similarly, one can show there exists an $a_2 \in A$ such that $c \leq_C a_2, a_2 \leq_B b$. But then $a_1 = a_2 = b = c$ so $S \in f'f(A)$. Hence $\mathcal{H}$ has the amalgamation property. □

Varieties Generated by Bounded Chains. As a first step to finding which residually small Heyting algebra varieties satisfy the amalgamation property, I present results which lead to theorem 3.4.8, which gives us an idea of the position varieties satisfying the amalgamation property occupy in the lattice of all Heyting algebra varieties.

DEFINITION 3.4.6. Let C_F be the set of all (non-isomorphic) finite chains (viewed as Heyting algebras), the let C_ω be the variety generated by C_F.

THEOREM 3.4.7 (Day [b]). *The variety $\mathcal{C}_\omega$ has the amalgamation property.*

Proof. $\mathcal{C}_{\omega SI} \subseteq \mathbf{HSP}_U(\mathcal{C}_F)$ by congruence distributivity of Heyting algebras, and since to be a bounded chain is a positive, universal first-order property, all members of $\mathcal{C}_{\omega SI}$ are bounded chains. Note that this also shows $\mathcal{C}_\omega$ has hereditary subdirectly irreducible algebras. Hence if (A, f, B, g, C) is a double extension in $\mathcal{C}_{\omega SI}$, it can be checked that there is an amalgam (f', g', D) where D is a bounded chain. Hence, since every subvariety of Heyting algebra satisfies the congruence extension property, by theorem 3.2.11, $\mathcal{C}_\omega$ satisfies the amalgamation property. $\square$

The finite chains are projective in the variety of Heyting algebras (see Balbes and Horn [70]), hence $\mathcal{H} : C_n$, the class of Heyting algebras which do not have C_n as a subalgebra forms a (splitting) variety. In particular, we denote by $\overline{\mathcal{H}}$ the variety $\mathcal{H} : C_2$, the variety of all Heyting algebras which do not contain the four-element chain as a subalgebra.

THEOREM 3.4.8 (Day [b]). *Let $\mathcal{V}$ be a Heyting algebra variety that satisfies the amalgamation property. Then $\mathcal{V} \supseteq \mathcal{C}_\omega$ or $\mathcal{V} \subseteq \overline{\mathcal{H}}$.*

Proof. Suppose $\mathcal{V}$ satisfies the amalgamation property but $\mathcal{V} \not\subseteq \overline{\mathcal{H}}$. Then $C_2 \in \mathcal{V}$. We will prove that for all $n \in \omega$, $C_n \in \mathcal{V}$, hence $\mathcal{V} \supseteq \mathcal{C}_\omega$.

Firstly, note C_0, C_1 and $C_2 \in \mathcal{V}$. Suppose $C_n \in \mathcal{V}$, $n \geq 2$. Let $C_n = \{0, a_1, \ldots, a_n, 1\}$ with $0 < a_1 < \cdots < a_n < 1$ and $C_{n-1} = \{0, b_1, \ldots, b_{n-1}, 1\}$ with $0 < b_1 < \cdots < b_{n-1} < 1$. Put $f : C_{n-1} \hookrightarrow C_n$ by $f(b_i) = a_i$ for $i = 1 \ldots n-1$, $f(0) = 0$ and $f(1) = 1$. Put $g : C_{n-1} \hookrightarrow C_n$ by $f(b_i) = a_{i+1}$ for $i = 1 \ldots n-1$, $g(0) = 0$ and $g(1) = 1$. Since $\mathcal{V}$ satisfies the amalgamation property, $(C_{n-1}, f, C_n, g, C_n)$ has an amalgam (D, f', g') say. Put $\overline{D} = f(C_n) \cup g(C_n)$. A check shows that $\overline{D}$ is a Heyting algebra isomorphic to C_{n+1}. Hence by induction $\mathcal{V} \supseteq \mathcal{C}_\omega$. $\square$

Varieties Below $\overline{\mathcal{H}}$. While a complete description of the varieties above $\mathcal{C}_\omega$ is not known, a complete description of the varieties below $\overline{\mathcal{H}}$ has been done by Day [a] (see also Lee [69] and Lee [70]). We first need the following definitions:

DEFINITION 3.4.9.

(1) Given a Heyting algebra A with unit 1_A, we define $A \oplus 1$ to be the lattice with universe $A \cup \{1\}$ obtained by adding a new unit $1 > 1_A$. If we further define the operation $\rightarrow$ on $A \oplus 1$ by

$$\text{For all } x, y \in A \oplus 1,\ x \rightarrow y = \begin{cases} x \rightarrow_A y & \text{if } x, y \in A, x \not\leq y \\ 1 & \text{if } x, y \in A, x \leq y \\ y & \text{otherwise,} \end{cases}$$

we see $A \oplus 1$ is in fact a Heyting algebra.

(2) We call a Heyting algebra *Boolean* if it satisfies the identity $a \rightarrow b = (a \rightarrow 0) + b$. Note every Boolean Heyting algebra A is a Boolean algebra in the classical sense if we define the complement of $a \in A$ to be $a \rightarrow 0$. Conversely, every Boolean algebra can be viewed as a Heyting algebra if we define $a \rightarrow b$ to be the join of the complement of a with b. We will denote the finite Boolean algebra with n atoms, viewed as a Heyting algebra, by B_n, and the one element Heyting algebra by B_0.

THEOREM 3.4.10 (Day [a]).

(1) $\overline{\mathcal{H}}_{SI} = \{B \oplus 1 : B$ *is a Boolean Heyting algebra*$\}$.

(2) *Let* $\overline{\mathcal{H}_n}$ *be the variety of Heyting algebras generated by* $B_n \oplus 1$, *where* $n \in \mathbb{N}$. *The lattice of subvarieties of* $\overline{\mathcal{H}}$ *is an* $\omega + 1$ *chain:*

$$\mathcal{T} \subsetneq \overline{\mathcal{H}_0} \subsetneq \overline{\mathcal{H}_1} \subsetneq \cdots \subsetneq \overline{\mathcal{H}}.$$

(Note H_0 *is the variety of all Boolean Heyting algebras, which is polynomially equivalent to the variety of all Boolean algebras).*

(3) *The subdirectly irreducible algebras in* $\overline{H}_n$ *are given (up to isomorphism) by the set* $\{B_m \oplus 1 : m \leq n\}$. □

THEOREM 3.4.11 (Day [b]). *The only subvarieties of* $\overline{\mathcal{H}}$ *satisfying the amalgamation property are* $\overline{\mathcal{H}_0}$, $\overline{\mathcal{H}_1}$, $\overline{\mathcal{H}_2}$ *and* $\overline{\mathcal{H}}$.

Proof. Using theorem 3.2.11, we need only investigate double extensions of the form (S_0, f, S_1, g, S_2) where S_0, S_1 and S_2 are all subdirectly irreducible. (Notice that every variety below $\overline{\mathcal{H}}$ has hereditary subdirectly irreducible algebras).

First consider the case where S_1 and S_2 are both finite, say $S_1 = B_n \oplus 1$, $S_2 = B_m \oplus 1$ with $n \leq m$: Now if $S_0 = B_0 \oplus 1$ or $S_0 = B_1 \oplus 1$, there is only one mapping from S_0 into any $B \oplus 1$, B Boolean. (In the case $S_0 = B_0 \oplus 1$, take 0 to 0 and 1 to 1, and in the case $S_0 = B_1 \oplus 1 = \{0, a, 1\}$ with atom a, take 0 to 0, 1 to 1 and a to the monolith of $B \oplus 1$.) Hence if $S_0 = B_0 \oplus 1$ or $B_1 \oplus 1$, (f', g', S_2) is an amalgam of (S_0, f, S_1, g, S_2), where $f' : B_n \oplus 1 \hookrightarrow B_m \oplus 1$ is any essential embedding, and g' is the identity map on S_2. Hence $\overline{\mathcal{H}_0}$ and $\overline{\mathcal{H}_1}$ satisfy the amalgamation property.

Next we consider $\overline{\mathcal{H}_2}$. If $S_0 = B_0 \oplus 1$ or $B_1 \oplus 1$, by the above observation, the double extension (S_0, f, S_1, g, S_2) has an amalgam. Also, if $S_0 = B_2 \oplus 1$, then as $B_2 \oplus 1$ is maximal subdirectly irreducible algebra, hence an absolute retract, again (S_0, f, S_1, g, S_2) has an amalgam.

To show $\overline{\mathcal{H}}$ satisfies the amalgamation property, let A, B, C be Boolean Heyting algebras and let $f : A \oplus 1 \hookrightarrow B \oplus 1$, $g : A \oplus 1 \hookrightarrow C \oplus 1$ be embeddings. Now there are induced embeddings $\overline{f} : A \hookrightarrow B$ and $\overline{g} : A \hookrightarrow C$ obtained from f and g respectively. Now $A, B, C \in \overline{\mathcal{H}_0}$ and as $\overline{\mathcal{H}_0}$ satisfies the amalgamation property, $(A, \overline{f}, B, \overline{g}, C)$ has an amalgam in $\overline{\mathcal{H}_0}$, say $(\overline{f}', \overline{g}', D)$. But then $(D \oplus 1, f', g')$ is an amalgam of $(A \oplus 1, f, B \oplus 1, g, C \oplus 1)$ where $f' : B \oplus 1 \hookrightarrow D \oplus 1$ and $g' : C \oplus 1 \hookrightarrow D \oplus 1$ are embeddings induced by $\overline{f}'$ and $\overline{g}'$ respectively. Hence $\overline{\mathcal{H}}$ satisfies the amalgamation property.

Now pick $n > 2$. Label the elements $B_2 \oplus 1$ as in figure 3.7.

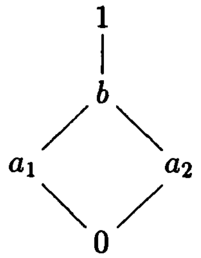


FIGURE 3.7

Let $f : B_2 \oplus 1 \hookrightarrow B_n \oplus 1$ be given as follows: Let $f(0) = 0, f(1) = 1$ and $f(b)$ be the monolith of $B_n \oplus 1$. Let $f(a_1)$ be an arbitrary atom of $B_n \oplus 1$, and let $f(a_2) = f(a_1) \to 0$. Let $g : B_2 \oplus 1 \hookrightarrow B_n \oplus 1$ be given as follows: Let $g(0) = 0, g(1) = 1$ and $g(b)$ be the monolith of $B_n \oplus 1$. Let $g(a_2)$ be an arbitrary atom of $B_n \oplus 1$, and let $f(a_1) = f(a_2) \to 0$. Note $g(a_1)$ is not an atom of $B_n \oplus 1$ as $n > 2$.

Now suppose $\overline{\mathcal{H}_n}$ satisfies the amalgamation property. Then $(B_2 \oplus 1, f, B_n \oplus 1, g, B_n \oplus 1)$ has an amalgam say (f', g', D). Now as $B_n \oplus 1$ is an absolute retract, there is an epimorphism $\rho : D \twoheadrightarrow B_n \oplus 1$ such that $\rho g'$ is the identity on $B_n \oplus 1$. Now, if m is the monolith of $B_n \oplus 1$, $\rho f'(m) = \rho f' f(b) = \rho g' g(b) = g(b) = m$, so $\ker \rho f'$ does not collapse $(m, 1)$. Hence $\rho f' : B_n \oplus 1 \to B_n \oplus 1$ is an embedding, and by finiteness an isomorphism, and hence takes atoms to atoms. But $\rho f' f(a_1) = g(a_1)$ is not an atom, hence we have a contradiction. Hence $\overline{\mathcal{H}_n}$ does not satisfy the amalgamation property for $n > 2$. □

Heyting Algebra Varieties Satisfying AP and RS. I now can describe all the residually small varieties satisfying the amalgamation property.

COROLLARY 3.4.12. *The only residually small varieties of Heyting algebras satisfying the amalgamation property are $\overline{\mathcal{H}_0}$, $\overline{\mathcal{H}_1}$, $\overline{\mathcal{H}_2}$ and $\mathcal{T}$.*

Proof. Let $\mathcal{V}$ be a residually small Heyting algebra variety satisfying the amalgamation property. Suppose $\mathcal{V} \supseteq \mathcal{C}_\omega$. Then $\mathcal{V}_{SI}$ has arbitrary large finite models, hence arbitrary large models by theorem 1.1.5, as $\mathcal{V}_{SI}$ is elementary. Hence by theorem 3.4.8 $\mathcal{V} \subseteq \overline{\mathcal{H}}$. The result now follows from theorem 3.4.11. □

Chapter 4

Some Characterizations of the Amalgamation Class

Introduction. In chapter three, I looked at the question of whether a given variety has the amalgamation property. When a variety fails to satisfy the amalgamation property (as we have seen for a number of varieties), an interesting question arises: Can one find necessary and sufficient conditions for a member of that variety to be in the amalgamation class? In this chapter I will present work that goes some way to answering that question. I present three concepts which will be used to develop characterizations of the amalgamation class: *retract extensibility* (first introduced here, though the idea is implicit in Jipsen and Rose [89]), *property Q* (introduced by C. Bergman [85] and Jónsson [90]) and *property P* (introduced by C. Bergman [85]).

In section 4.1, I introduce retract extensibility and property Q, and prove some basic results concerning these properties. I also give two examples of retract extensile varieties, namely, varieties with the congruence extension property and congruence distributive varieties in which every algebra has a one element subalgebra.

In section 4.2, I consider some finitely generated congruence distributive varieties which have the property that an amalgamation base is a subdirect product of subdirectly irreducible amalgamation bases. The particular varieties that have this property are the semi-simple varieties, the Heyting algebra varieties $\mathcal{H}_n$ (introduced in section 3.4) and the lattice variety generated by the pentagon. This work was done by C. Bergman [85].

In section 4.3, we use the results of section 4.2 to determine a characterization of amalgamation bases for the varieties mentioned above in terms of property P and congruence retract extensibility. In particular, we are able to give an effective test whether a finite algebra is an amalgamation base for these cases.

4.1 Retract Extensibility and Property Q

Maximal Essential Extensions of Amalgamation Bases. For a given variety, maximal essential extensions of algebras in that variety are not necessarily unique. For amalgamation bases, however, this is true (up to isomorphism). This fact will be used later to characterise subdirectly irreducible amalgamation bases.

DEFINITION 4.1.1(C. Bergman [85]).

(1) Let α_0 and α_1 be embeddings of an algebra A into B_0 and B_1 respectively. Define $\alpha_0 \cong \alpha_1$ if and only if there is an isomorphism $\gamma : B_0 \to B_1$ such that $\gamma\alpha_0 = \alpha_1$.

(2) Let $\mathcal{V}$ be a variety, and let $A \in \mathcal{V}$. We define the *envelope of* A *in* $\mathcal{V}$, denoted $\mathcal{E}_{\mathcal{V}}(A)$, to be the class of all maximal essential embeddings of A into some algebra of $\mathcal{V}$. Note $\cong$ is an equivalence relation on $\mathcal{E}_{\mathcal{V}}(A)$, and if $\mathcal{V}$ is residually small, $\mathcal{E}_{\mathcal{V}}(A)$ is not empty by theorem 1.2.24.

(3) We denote by $(A : \mathcal{V})$, the cardinality of $\mathcal{E}_{\mathcal{V}}(A)/\cong$.

THEOREM 4.1.2 (C. Bergman [85]). *Let* $A \in Amal(\mathcal{V})$ *for some variety* $\mathcal{V}$, *and suppose* $|A| > 1$. *Then* $(A : \mathcal{V}) \leq 1$.

Proof. Suppose $\alpha : A \hookrightarrow B$ and $\beta : A \hookrightarrow C$ are both members of $\mathcal{E}_{\mathcal{V}}(A)$. Now (A, α, B, β, C) has an amalgam, say (α', β', D). Since $C \in \mathcal{V}_{AR}$, there is a retraction $\sigma : D \twoheadrightarrow C$ of β. Let $\gamma = \sigma\alpha'$, then we see $\gamma\alpha = \beta$, and $\alpha^{-1}(\ker\gamma) = \ker(\gamma\alpha) = \ker\beta = \mathbf{0}_A$. Hence, as α is essential, $\ker\gamma = \mathbf{0}_B$, so γ is an embedding. Now since C is an essential extension of $\gamma\alpha(A)$, and B is a maximal essential extension of $\alpha(A)$, we must have that γ is an isomorphism, proving the theorem. □

Retract Extensibility and Property Q: Definitions and Basic Results. In this subsection I define these two concepts, and present a result that an algebra having property Q is an amalgamation base for a residually small variety. I present a sufficient condition for the converse to be true, and show every retract extensile variety satisfies this condition. I also show that absolute retracts in retract extensile varieties are closed under direct products, a fact that is not even true in general for finitely generated congruence distributive varieties (see Taylor [73]). Lastly, I show that an algebra possessing property Q is congruence extensile.

The following definition first appears in Jónsson [90], but the idea is implicit in C. Bergman [85].

DEFINITION 4.1.3. An algebra A in a variety $\mathcal{V}$ is said to have *property Q in* $\mathcal{V}$, if for every embedding $f : A \hookrightarrow B \in \mathcal{V}$ and any homomorphism $h : A \to M \in \mathcal{V}_{WMI}$, there is a homomorphism $g : B \to M$ such that $h = gf$.

LEMMA 4.1.4 (C. Bergman [85]). *Let $\mathcal{V}$ be a residually small variety. If $A \in \mathcal{V}$ has property Q in $\mathcal{V}$, then $A \in Amal(\mathcal{V})$.*

Proof. We use lemma 1.3.3(2). Let (A, f, B, g, C) be a double extension in $\mathcal{V}$, and let $u \neq v \in B$. By theorem 1.2.4, there is a $\Theta \in \mathrm{Con}(B)$ such that $B/\Theta \in \mathcal{V}_{SI}$ and $(u, v) \notin \Theta$. Now B/Θ has an extension in $\mathcal{V}_{WMI}$, say M. Now the canonical epimorphism from B onto B/Θ induces a homomorphism $h : B \to M$ such that $h(u) \neq h(v)$. We argue similarly for $u \neq v$ in C, and hence we conclude $A \in Amal(\mathcal{V})$. ☐

DEFINITION 4.1.5. Let $\mathcal{V}$ be a variety. We say:

(1) $\mathcal{V}$ is *retract extensile*, if for every $A \in \mathcal{V}$, $R \in \mathcal{V}_{AR}$ and embedding $f : A \times R \hookrightarrow B$ there is a homomorphism $h : B \to R$ such that $hf = \pi_R$, where $\pi_R : A \times R \twoheadrightarrow R$ is a projection map. Note h is always onto.

(2) $\mathcal{V}$ is a *Q-variety* if $\mathcal{V}$ is residually small, and for all $A \in \mathcal{V}$: $A \in Amal(\mathcal{V})$ if and only if A has property Q in $\mathcal{V}$.

THEOREM 4.1.6. *For every retract extensile variety $\mathcal{V}$, $\mathcal{V}_{AR}$ is closed under direct products.*

Proof. Let $\mathcal{V}$ be a retract extensile variety, and let $R = \prod_{i \in I} R_i$ be a product of absolute retracts, and suppose $f : R \hookrightarrow B \in \mathcal{V}$ is an embedding. Let $\pi_i : R \twoheadrightarrow R_i$ be a projection map. Letting $A = \prod_{j \neq i} R_j$, we see that $R \cong A \times R_i$, hence as $\mathcal{V}$ is retract extensile, there is an $h_i : B \twoheadrightarrow R_i$ such that $h_i f = \pi_i$. Let $h : B \to R$ be the homomorphism defined by $\pi_i h(b) = h_i(b)$, $b \in B$ for each $i \in I$. A check shows that $hf = \mathrm{id}_R$, hence $R \in \mathcal{V}_{AR}$ as required. ☐

THEOREM 4.1.7 (C. Bergman [85]). *Let $\mathcal{V}$ be a residually small variety such that members of $\mathbf{P}(\mathcal{V}_{WMI})$ are congruence extensile. Then $\mathcal{V}$ is a Q-variety.*

Proof. Let $\mathcal{V}$ be as in the statement of the theorem. By lemma 4.1.4, we need only show that $A \in Amal(\mathcal{V})$ implies A has property Q. So let $A \in Amal(\mathcal{V})$, and let $f : A \hookrightarrow B \in \mathcal{V}$ be an embedding and $g : A \to M \in \mathcal{V}_{WMI}$ be a homomorphism. Since $\mathcal{V}$ is residually small, A is embeddable into a product of members of $\mathcal{V}_{WMI}$, say $A \subseteq R = \prod_{i \in I} R_i \in \mathbf{P}(\mathcal{V}_{WMI})$. Since $M \times R$ is a product of members of $\mathcal{V}_{WMI}$, it is congruence extensile.

Let $s : A \hookrightarrow M \times R$ be the embedding given by $s(a) = (g(a), a)$, for $a \in A$. Since $A \in Amal(\mathcal{V})$, $(A, f, B, s, M \times R)$ has an amalgam (f', s', D), say. Since $M \times R$ is congruence extensile, there is an algebra $E \in \mathcal{V}$, an epimorphism $p : D \twoheadrightarrow E$ and an embedding $e : M \hookrightarrow E$ such that $ps' = e\pi_M$, where $\pi_M : M \times R \twoheadrightarrow M$ is the projection map onto M. As $M \in \mathcal{V}_{WMI}$, we can find a retract of e, say $r : E \twoheadrightarrow M$. Putting $h = rpf'$, one can verify that the diagram in figure 4.1 commutes, hence $hf = g$. □

$$\begin{tikzcd}
A \arrow[r, hook, "f"] \arrow[dr, hook, "s"] \arrow[dd, "g"'] & B \arrow[dr, hook, "f'"] & \\
& M \times R \arrow[r, hook, "s'"] \arrow[dl, two heads, "\pi_M"] & D \arrow[dl, two heads, "p"] \\
M \arrow[r, hook, shift left, "e"] & E \arrow[l, two heads, shift left, "r"] &
\end{tikzcd}$$

FIGURE 4.1

COROLLARY 4.1.8. *Every residually small retract extensile variety is a Q-variety.*

Proof. Members of $\mathbf{P}(\mathcal{V}_{WMI})$ are absolute retracts by theorem 4.1.6. □

LEMMA 4.1.9 (C. Bergman [85]). *An algebra B is congruence extensile in $\mathcal{V}$ if and only if for every $C \in \mathbf{P}(\mathcal{V}_{SI})$ extending B, and for every completely meet irreducible congruence Θ on B, there is a congruence Ψ on C such that $\Psi|_B = \Theta$.*

Proof. $\Longrightarrow$: Follows from the definition of congruence extensibility.

$\Longleftarrow$: Let $D \in \mathcal{V}$ be an arbitrary extension of $B \in \mathcal{V}$. By theorem 1.2.4, D is a subdirect product of members of of $\mathcal{V}$, with representation say $D \subseteq C \in \mathbf{P}(\mathcal{V}_{SI})$. Let $\Phi \in \mathrm{Con}(B)$. Then, by theorem 1.2.4, there is a set $\{\Theta_i : i \in I\}$ of completely meet irreducible congruences on B such that $\Phi = \bigcap_{i \in I} \Theta_i$. By our assumption, there are congruences Ψ_i, $i \in I$ on C such that $\Psi_i|_B = \Theta_i$ for each $i \in I$. Let $\Omega = (\bigcap_{i \in I} \Psi_i)|_D$. Then $\Omega|_B = \bigcap_{i \in I} \Psi_i|_B = \Phi$, so Ω extends Φ and the result follows. □

THEOREM 4.1.10 (**Jónsson** [90]). *For every Q-variety $\mathcal{V}$, members of Amal($\mathcal{V}$) are congruence extensile in $\mathcal{V}$.*

Proof. By lemma 4.1.9, we need only show that for $A \in Amal(\mathcal{V})$, and an arbitrary extension $f : A \hookrightarrow B \in \mathcal{V}$, every completely meet irreducible congruence on A extends to B. So let $\Theta \in \mathrm{Con}(A)$ such that Θ is completely meet irreducible in A. Since $\mathcal{V}$ is residually small, there is an $M \in \mathcal{V}_{WMI}$ such that M contains A/Θ. Hence there is a homomorphism $g : A \to M$ obtained from the canonical epimorphism of A onto A/Θ. By property Q, there is a homomorphism $h : B \to M$ extending g, and hence $\ker h|_A = \Theta$, proving the theorem. □

Examples of Retract Extensile Varieties. I now give two examples of retract extensile varieties, the first being varieties with the congruence extension property (such as Heyting algebra varieties), and congruence distributive varieties in which every algebra has a one-element subalgebra (of which varieties of lattices are a prime example).

THEOREM 4.1.11. *Every variety with the congruence extension property is retract extensile.*

Proof. Let $f : A \times R \to B \in \mathcal{V}$, where $\mathcal{V}$ has the congruence extension property, $A \in \mathcal{V}$ and $R \in \mathcal{V}_{AR}$. Let $\pi : A \times R \twoheadrightarrow R$ be the projection map. Since $\mathcal{V}$ has the congruence extension property, there is a $C \in \mathcal{V}$, with $g : R \hookrightarrow C$ an embedding and $k : B \twoheadrightarrow C$ an epimorphism such that $kf = g\pi$. Since $R \in \mathcal{V}_{AR}$, there is a retraction $t : C \twoheadrightarrow R$ of g. Putting $h = tk$, we see $hf = \pi$, hence $\mathcal{V}$ is retract extensile. □

LEMMA 4.1.12 (Jipsen and Rose [89]). *Let A and B be algebras in a congruence distributive variety $\mathcal{V}$, $a \in A$ and suppose $\{a\}$ is a subalgebra of A. Let $h_a : B \hookrightarrow A \times B$ be the embedding given by $h_a(b) = (a, b)$ for all $b \in B$. Then the projection map $\pi_B : A \times B \twoheadrightarrow B$ is the only retraction of h_a onto B.*

Proof. Let $g : A \times B \twoheadrightarrow B$ be a retraction of h_a described above. Since $\mathcal{V}$ is congruence distributive, there are $\Theta_A \in \mathrm{Con}(A)$ and $\Theta_B \in \mathrm{Con}(B)$ such that for $(x, y), (x', y') \in A \times B$:

$$(x, y) \ \ker g \ (x', y') \quad \text{if and only if} \quad x \ \Theta_A \ x' \text{ and } y \ \Theta_B \ y'.$$

Since gh_a is the identity map on B, Θ_B must be trivial. To show $g = \pi_B$, it suffices to show that for any $a' \in A$ we have $a' \ \Theta \ a$. Suppose not. Then there is an $a' \in A$ with $(a, a') \notin \Theta_A$. Pick $b \in B$, then $g(a, b) = b$. Note $g(a', b) \neq b$, for then $(a, a') \in \Theta_A$. Hence let $g(a', b) = b' \neq b$, say. But then $g(a, b') = g(a', b) = b'$. Hence $b' \ \Theta_B \ b$ and $a' \ \Theta_A \ a$, a contradiction. Thus $g = \pi_B$. □

THEOREM 4.1.13 (Jipsen and Rose [89]). *Let $\mathcal{V}$ be congruence distributive, and suppose every member of $\mathcal{V}$ has a one-element subalgebra. Then $\mathcal{V}$ is a retract extensile variety.*

Proof. Let $\mathcal{V}$ be as above, and let $A, B \in \mathcal{V}$, $R \in \mathcal{V}_{AR}$ and suppose $f : A \times R \hookrightarrow B$ is an embedding. Let $\{a\}$ be the one- element subalgebra of A and let $h_a : R \hookrightarrow A \times R$ be given by $h_a(r) = (a, r), r \in R$. Since $R \in \mathcal{V}_{AR}$, fh_a has a retract $h : B \twoheadrightarrow R$. Now $h|_{A \times R}$ is a retract of h_a onto R, hence $h|_{A \times R} = \pi_R$, the projection map $\pi_R : A \times R \twoheadrightarrow R$, by lemma 4.1.12. Hence $hf = \pi_M$, so $\mathcal{V}$ is retract extensile. □

4.2. Subdirect Products of Amalgamation Bases

Semi-simple Varieties. I start this section by taking a brief look at semi-simple varieties. The two main results are theorems 4.2.2 and 4.2.7. The first characterizes simple members of the amalgamation class, and the latter a finite homomorphic image of a product of simple algebras is a product of simple algebras. These will be used to describe subdirect products of amalgamation bases.

DEFINITION 4.2.1. A variety $\mathcal{V}$ is *semi-simple* if every member of $\mathcal{V}_{SI}$ is simple.

THEOREM 4.2.2 (C. Bergman [85]). *Let $\mathcal{V}$ be a residually small semi-simple variety. Then for every $A \in \mathcal{V}_{SI}$, $|A| > 1$: $A \in Amal(\mathcal{V})$ if and only if $(A : \mathcal{V}) = 1$.*

Proof. $\Longrightarrow$: Follows from residual smallness and theorem 4.1.2.

$\Longleftarrow$: We will use lemma 1.3.2. Since $\mathcal{V}$ is residually small, $A \in \mathcal{V}_{SI}$ has a maximal essential extension $B \in \mathcal{V}_{WMI}$, say. Let $\alpha : A \hookrightarrow B$ be that essential embedding. Suppose $(A : \mathcal{V}) = 1$. Let $\beta : A \hookrightarrow C \in \mathcal{V}$ be an arbitrary extension of A. We need only show (A, α, B, β, C) has an amalgam. We use lemma 1.3.3(2).

Pick $a, b \in C$, $a \neq b$. By theorem 1.2.4, there is a $D \in \mathcal{V}_{SI}$, and a epimorphism $\gamma : C \twoheadrightarrow D$ such that $\gamma(a) \neq \gamma(b)$. Now, since A is simple, $\gamma\beta : A \to D$ either collapses A to a single point, or is injective. In the former case, let $\delta : B \to D$ be such that $\delta(B) = \gamma\beta(A)$, and a check shows that $\gamma\beta = \delta\alpha$ satisfying the conditions of the lemma. In the latter case, let E be a maximal essential extension of D, with say $\epsilon : D \hookrightarrow E$. Since A is simple, E is a maximal essential extension of A too, and since $(A : \mathcal{V}) = 1$, there is an isomorphism $\delta : B \to E$ such that $\epsilon\gamma\beta = \delta\alpha$. A check shows that $\epsilon\gamma$, δ and E satisfy the conditions of the lemma.

Now pick $a, b \in B$, $a \neq b$. Since $|A| > 1$, there is at least one $D \in \mathcal{V}_{SI}$ and epimorphism $\gamma : C \twoheadrightarrow D$ such that $\gamma\beta$ is injective. Using the construction above again, since δ is an isomorphism, $\delta(a) \neq \delta(b)$, satisfying the condition of the lemma. Hence $A \in Amal(\mathcal{V})$. □

DEFINITION 4.2.3. We denote $\mathcal{V}_{ASI} = \mathcal{V}_{SI} \cap Amal(\mathcal{V})$.

COROLLARY 4.2.4 (C. Bergman [85]). *If $\mathcal{V}$ is a semi-simple residually small variety, then $\mathcal{V}_{ASI} = Amal(\mathcal{V}_{SI})$.*

Proof. $\mathcal{V}_{ASI} \subseteq Amal(\mathcal{V}_{SI})$: Let $A \in \mathcal{V}_{ASI}$ and let (A, f, B, g, C) be a double extension in $\mathcal{V}_{SI}$. Let B' and C' be maximal essential extensions of B and C respectively. By semi-simplicity, B' and C' are maximal essential extensions of A, hence there is an isomorphism $\gamma : B' \to C'$ such that $\gamma f = g$ by theorem 4.2.2. Hence $(\gamma, \mathrm{id}_{C'}, C')$ is an amalgam in $\mathcal{V}_{SI}$ of (A, f, B, g, C).

$\mathcal{V}_{ASI} \supseteq Amal(\mathcal{V}_{SI})$: Let $A \in Amal(\mathcal{V}_{SI})$. Then $(A : \mathcal{V}) = 1$ since the algebra in the amalgam of $(A, \mathrm{id}_A, B, \mathrm{id}_A, C)$ where B and C are essential extensions of A, must be isomorphic to both B and C. □

LEMMA 4.2.5 (C. Bergman [85]). *Let $\mathcal{V}$ be congruence distributive, semi-simple and finitely generated. Let $\{A_i : i \in I\}$ be a collection of members of $\mathcal{V}_{SI}$, and let Θ be a congruence on $A = \prod_{i \in I} A_i$. The following are equivalent:*

(1) *Θ is a co-atom of Con(A).*

(2) *Θ is completely meet irreducible in Con(A).*

(3) *Θ is induced by an ultrafilter on I.*

Proof. The equivalence of (1) and (2) is evident from the fact that $\mathcal{V}$ is semi- simple.

(2) $\implies$ (3): By theorem 1.2.16, there is an ultrafilter $\mathfrak{D}$ on I such that the congruence induced by $\mathfrak{D}$ is contained in Θ. But then A/Θ is a homomorphic image of $\prod_{i \in I} A_i/\mathfrak{D}$, which is isomorphic to some $A_j \in \mathcal{V}_{SI}$ for some $j \in J$ by corollary 1.2.17. But as A/Θ is non-trivial and A_j is simple we have $A/\Theta \cong \prod_{i \in I} A_i/\mathfrak{D}$ so $\mathfrak{D}$ induces Θ.

(3) $\implies$ (1): For every ultrafilter $\mathfrak{D}$ on I, the ultraproduct $\prod_{i \in I} A_i/\mathfrak{D} \cong A_j \in \mathcal{V}_{SI}$ for some $j \in J$. Hence, since $\prod_{i \in I} A_i/\mathfrak{D}$ is simple, the congruence induced by $\mathfrak{D}$ is a co-atom of Con(A). □

LEMMA 4.2.6 (C. Bergman [85]). *Let $\mathcal{V}$ be finitely generated, congruence distributive and semi-simple. Let $A = \prod_{i \in I} A_i$ be a product of simple algebras in $\mathcal{V}$. Then every congruence of A is induced by a filter in I, and hence A has permuting congruences.*

Proof. Let $\Theta \in$ Con(A). Then, by theorem 1.2.4, $\Theta = \bigcap_{i \in I} \Theta_i$, where each Θ_i is completely meet irreducible in Con(A). Hence by lemma 4.2.5, each Θ_i is induced by an ultrafilter on I, say $\mathfrak{D}_i$. Put $\mathfrak{D} = \bigcap_{i \in I} \mathfrak{D}_i$, we see $\mathfrak{D}$ is a filter inducing Θ. Since congruences induced by filters permute, A has permuting congruences. □

THEOREM 4.2.7 (C. Bergman [85]). *Let $\mathcal{V}$ be finitely generated, congruence distributive and semi-simple. Suppose $A \in P(S)$, where $S \subseteq \mathcal{V}_{SI}$. Then, for every $\Theta \in$ Con(A), if A/Θ is finite, then $A/\Theta \in P(S)$.*

Proof. Put $\Theta = \bigcap_{i \in I} \Psi_i$, where each Ψ_i is completely meet irreducible in Con(A) using theroem 1.2.4. Since A/Θ is finite, Θ has only finitely many members greater than it. Hence we may assume I is finite, and that the Ψ_i's are pairwise distinct. Now by lemma 4.2.6, A has permuting congruences so for each $i \in I$, $\Psi_i \circ (\bigcap_{j \neq i} \Psi_j) = \Psi_i + \bigcap_{j \neq i} \Psi_j = \bigcap_{j \neq i} (\Psi_i + \Psi_j) = 1_A$, using congruence distributivity and the fact that Ψ_i's are the co-atoms of Con(A). Thus $A/\,\Theta \cong \prod\limits_{i \in I} A/\,\psi_i$. Finally, since each Ψ_i is induced by an ultrafilter on I by theorem 1.2.16, $A/\,\Psi_i \in \mathcal{S}$ (up to isomorphism) for each $i \in I$. ☐

Subdirect Products of Amalgamation Bases for Certain Varieties.

I now present results which state that every amalgamation base is a subdirect product of subdirectly irreducible amalgamation bases for some special classes of varieties. These results will be used in section 4.3.

THEOREM 4.2.8 (C. Bergman [85]). *Let $\mathcal{V}$ be congruence distributive, semisimple and finitely generated, and suppose every product of maximally irreducible algebras in $\mathcal{V}$ is congruence extensile. Then $A \in Amal(\mathcal{V})$ implies A is a subdirect product of members of $\mathcal{V}_{ASI}$.*

Proof. Let $A \in Amal(\mathcal{V})$. Now $\mathcal{V}_{SI}$ has only finitely many members, so we may choose a set $\mathcal{S}$ from $\mathcal{V}_{SI}$, minimal under inclusion, such that A is a subdirect product of members of $\mathcal{S}$. To show that for all $T \in \mathcal{S}$, $T \in \mathcal{V}_{ASI}$, we make use of lemma 4.2.2 and show $(T : \mathcal{V}) = 1$. We will break up the proof into a number of steps for clarity, and to aid in the proof of of theorem 4.2.10.

1. Let $A \subseteq \prod\limits_{j \in J} S_j$ be a subdirect product representation, with $S_j \in \mathcal{S}$ for all $j \in J$. For each $j \in J$, the $\pi_j : A \twoheadrightarrow S_j$ be the jth projection restricted to A, and let $\Theta_i = \ker(\pi_j)$. Define $J' = \{j \in J : S_j \neq T\}$ and $\Theta = \bigcap_{j \in J'} \Theta_j$. Since $\mathcal{S}$ is minimal, $\Theta \neq 0$. Thus there is a $k \in J \setminus J'$ such that $\Theta \not\subseteq \Theta_k$. Let k be fixed for the remainder of the proof. Since $A/\,\Theta_k \cong T$ and T is finite, we can choose a finite subset G of A such that the map $\pi_k|_G : G \to T$ is a bijection. Also, since $\Theta \not\subseteq \Theta_k$, there $a, b \in A$ such that $(a, b) \in \Theta$ but $(a, b) \notin \Theta_k$. Let F be the subalgebra of A generated by $G \cup \{a, b\}$. Since $G \subseteq F$, $\pi_k|_F : F \to T$ is an epimorphism. Let $\Phi = \Theta_k|_F$. Since $a, b \in F$, $\Theta_k|_F \not\subseteq \Phi$. Put $R = \{j \in J : \Theta_j|_F = \Phi\}$, and let $Q = J \setminus R$. Now for all $j \in J'$, $\Theta|_F \not\subseteq \Phi$ implies $\Theta_j|_F \not\subseteq \Phi$ since $\Theta \subseteq \Theta_j$, hence $j \notin R$ so $J' \subseteq Q$. For each $j \in R$, there is an isomprphism $\delta_j : S_j \to S_k$ given by

$$(*) \qquad \delta_j = \pi_k|_F \circ (\pi_j|_F)^{-1}$$

which is well-defined as $\ker(\pi_k|_F) = \ker(\pi_j|_F = \Phi$. For $j \in Q$, let $N_j \in \mathcal{V}_{WMI}$ be a maximal essential extension of S_j.

2. Let $\alpha_i : T \hookrightarrow L_i \in \mathcal{V}_{WMI}$ be members of $\mathcal{E}_{\mathcal{V}}(T)$ for $i = 0,1$. We need to show $\alpha_0 \cong \alpha_1$. We define structures B_0 and B_1 as follows: For $i = 0,1$, $B_i = \prod_{j\in R} L_i \times \prod_{j\in Q} N_j$. Also, let $\phi_0 : A \to B_0$ and $\phi_1 : A \to B_1$ be embeddings given co-ordinate-wise by

$$\phi_0 = \Big(\prod_{j\in R}\alpha_0 \times \prod_{j\in Q}\mathrm{id}_{S_j}\Big)\Big|_A$$

$$\phi_1 = \Big(\prod_{j\in R}\alpha_1 \circ \delta_j \times \prod_{j\in Q}\mathrm{id}_{S_j}\Big)\Big|_A .$$

Since $A \in Amal(\mathcal{V})$, the double extension $(A, \phi_0, B_0, \phi_1, B_1)$ has an amalgam, say (μ_0, μ_1, E). Let σ_0 be the kth projection of B_0 onto L_0, and let $\Psi_0 = \ker(\sigma_0)$. Notice that $\sigma_0\phi_0 = \alpha_0\pi_k$ so that the diagram in figure 4.2 commutes.

$$\begin{tikzcd}
T \arrow[d, hook, "\alpha_0"'] & A \arrow[l, two heads, "\pi_k"'] \arrow[r, hook, "\phi_1"] \arrow[d, hook, "\phi_0"'] & B_1 \arrow[d, hook, "\mu_1"'] \\
L_0 & B_0 \arrow[l, two heads, "\sigma_0"'] \arrow[r, hook, "\mu_0"] & E
\end{tikzcd}$$

FIGURE 4.2

3. By our assumption B_0 is congruence extensile, so there is a congruence Ψ on E and an embedding ${\mu_0}' : L_0 \hookrightarrow E/\Psi$ such that for all $b \in B_0$, ${\mu_0}' \circ \sigma_0(b) = \mu_0(b)/\Psi$. In fact, by lemma 1.2.23, we can always choose ${\mu_0}'$ to be an essential embedding. But as $L_0 \in \mathcal{V}_{WMI}$, ${\mu_0}' : L_0 \hookrightarrow E/\Psi$ must be an isomorphism.

4. Put $\Psi_1 = \mu_1^{-1}(\Psi)$. Let ${\mu_1}' : B_1/\Psi_1 \hookrightarrow E/\Psi \cong L_0$ be the induced embedding. Since L_0 is finite, B_1/Ψ_1 is finite. Hence since B_1 is a product of members of $\mathcal{V}_{WMI}$, we have that B_1/Ψ_1 is a product of members of $\mathcal{V}_{WMI}$ by theorem 4.2.7. But then by our assumption, B_1/Ψ_1 is congruence extensile. In particular, every congruence on B_1/Ψ_1 extends to one on $E/\Psi \cong L_0$, but since L_0 is simple we must have B_1/Ψ_1 simple. But then $B_1/\Psi_1 \in \mathcal{V}_{WMI}$, hence ${\mu_1}'$ is an isomorphism. Let $\sigma_1 : B_1 \twoheadrightarrow B_1/\Psi_1$ be the canonical epimorphism. Notice $\phi_1^{-1}(\Psi_1) = \phi_0^{-1}(\Psi_0) =$

Θ_k. Hence we have an induced embedding ${\phi_1}' : T \hookrightarrow B_1/\Psi_1$ such that the diagram in figure 4.3 commutes.

$$\begin{tikzcd}
 & & E/\Psi & & \\
 & & E \arrow[u, two heads] & & \\
L_0 \arrow[uurr, hook, "\mu_0'"] & B_0 \arrow[l, two heads, "\sigma_0"'] \arrow[ur, hook, "\mu_0"] & & B_1 \arrow[ul, hook', "\mu_1"'] \arrow[r, two heads, "\sigma_1"] & B_1/\Psi_1 \arrow[uull, hook', "\mu_1'"'] \\
 & & A \arrow[ul, hook', "\phi_0"'] \arrow[ur, hook, "\phi_1"] \arrow[d, two heads, "\pi_k"] & & \\
 & & T \arrow[uull, hook', "\alpha_0"] \arrow[uurr, hook, "\phi_1'"'] & &
\end{tikzcd}$$

FIGURE 4.3

In particular, ${\mu_0}' \circ \alpha_0 = {\mu_1}' \circ {\phi_1}'$ and since ${\mu_0}'$ and ${\mu_1}'$ are isomorphisms, $\alpha_0 \cong {\phi_1}'$.

5. For each $j \in J$, let Ω_j be the jth projection congruence on B_1. Since Ψ_1 is completely meet irreducible in $\mathrm{Con}(B_1)$, by lemma 4.2.5 we have the Ψ_1 is induced by an ultrafilter $\mathfrak{D}$, say, on J.

CLAIM: $R \in \mathfrak{D}$

Proof of claim: For each $(x,y) \in F \times F$, put $U_{x,y} = \{j \in J : (\phi_1(x), \phi(y)) \in \Omega_j\}$. Notice that $\Psi_1|_F = \Psi_0|_F = \Theta_k|_F = \Phi$. Thus if $(x,y) \in \Phi$, then $(\phi_1(x), \phi(y)) \in \Psi_1$, and hence $\{j : (\phi_1(x), \phi(y)) \in \Omega_j\} = U_{x,y} \in \mathfrak{D}$. Put $U \doteq \bigcap_{(x,y)\in\Phi} U_{x,y}$. Since Φ consists of finitely many pairs, $U \in \mathfrak{D}$. Now $j \in U$ if and only if $(\phi_1(x), \phi(y)) \in \Omega_j$ for all $(x,y) \in \Phi$ if and only if $\Phi \subseteq \Omega_j|_F$. Hence $U = \{j \in J : \Phi \subseteq \Omega_j|_F\}$.

Now since Φ is completely meet irreducible in $\mathrm{Con}(F)$, it has a unique cover in $\mathrm{Con}(F)$, say Φ'. Then $U = \{j \in J : \Phi = \Omega_j|_F\} \cap \{j \in J : \Phi' \subseteq \Omega_j|_F\}$. Now $\Omega_j|_F = \Theta_j|_F$, so $\{j \in J : \Phi = \Omega_j|_F\} = \{j \in J : \Phi = \Theta_j|_F\} = R$. Put $Y = \{j \in J : \Phi' \subseteq \Omega_j|_F\}$. We have $U = R \cup Y$. Now choose $(x,y) \in \Phi' \setminus \Phi$. $(x,y) \notin \Psi_1$, so $Z = \{j \in J : (\phi_1(x), \phi(y)) \notin \Omega_j\} \in \mathfrak{D}$. Notice $Z \cap Y = \emptyset$. Hence $R \supseteq Z \cap R = Z \cap (Y \cup R) = Z \cap U \in \mathfrak{D}$. So $R \in \mathfrak{D}$, proving our claim.

6. In step 4, we have shown $\alpha_0 \cong \phi_1{}'$. We now show $\phi_1{}' \cong \alpha_1$:

For $j \in J$, let ϵ_j be the jth projection map of B_1. We construct a map $\gamma : B_1/\Psi_1 \to L_1$ as follows: Since L_1 is finite, we may assume $L_1 = \{x_1, \ldots, x_n\}$, say. For each $b \in B_1$, we have

$$\bigcup_{i=1}^{n} \{r \in R : \epsilon_r(b) = x_i\} = R \in \mathfrak{D}.$$

The subsets of this union are mutually disjoint, hence as $\mathfrak{D}$ is prime, we have for exactly one $i \in \{1, \ldots, n\}$, $\{r \in R : \epsilon_r(b) = x_i\} \in \mathfrak{D}$. Thus for each $b/\Psi_1 \in B_1/\Psi_1$, put $\gamma(b/\Psi_1) = x$ if and only if $\{r \in R : \epsilon_r(b) = x\} \in \mathfrak{D}$. By the discussion above, γ is well-defined, and it can be checked that it is an isomorphism.

From the diagram in figure 4.3, we see that

$$\text{(A)} \qquad \gamma \circ \phi_1{}' \circ (\pi_k|_F) = \gamma \circ \sigma_1 \circ \phi_1|_F$$

CLAIM: We also have that the following holds:

$$\text{(B)} \qquad \gamma \circ \sigma_1 \circ \phi_1|_F = \alpha_1 \circ (\pi_k|_F)$$

Proof of claim: Let $f \in F$. For each $r \in R$, $\epsilon_r \circ \phi_1(f) = \alpha_1 \circ \delta_r \circ \pi_r(f) = \alpha_1 \circ \pi_k(f)$, the last equality following from (*). Let $x = \alpha_1 \circ \pi_k(f)$. Then $\{r \in R : \epsilon_r \circ \phi_1(f) = x\} = R \in \mathfrak{D}$. So $\gamma \circ \sigma_1 \circ \phi_1(f) = \alpha_1 \circ \pi_k(f)$, proving the claim.

Combining (A) and (B) we have $\gamma \circ \phi_1{}' \circ (\pi_k|_F) = \alpha_1 \circ (\pi_k|_F)$, and by cancellation since $\pi_k|_F$ is an epimorphism, $\gamma \circ \phi_1{}' = \alpha_1$ so $\phi_1{}' \cong \alpha_1$. Hence, $\alpha_0 \cong \alpha_1$, and thus $(T : \mathcal{V}) = 1$, proving the theorem. □

Recall the Heyting algebra varieties $\mathcal{H}_n$ (n a natural number) introduced in section 3.3. We saw that subdirectly irreducible algebras were of the form $B_m \oplus 1$, a Heyting algebra obtained by appending a top element to a Boolean algebra with m atoms. I present a characterization of subdirect products of amalgamation bases in these varieties below.

The following lemma is stated without proof in C. Bergman [85]. The proof given below is adapted from a result in Grätzer and Lakser [71].

LEMMA 4.2.9. *Let n be a natural number, and let $\mathcal{V} = \mathcal{H}_n$. The following are equivalent for $A \in \mathcal{V}_{SI}$:*

(1) $A \in Amal(\mathcal{V})$.

(2) $(A : \mathcal{V}) = 1$.

(3) *A is isomorphic to one of $B_0 \oplus 1$, $B_1 \oplus 1$ or $B_n \oplus 1$.*

Proof. (1) $\Longrightarrow$ (2): Follows from residual smallness of $\mathcal{V}$ and theorem 4.1.2.

(2) $\Longrightarrow$ (3): Since $\mathcal{V}_{WMI} = \{B_n \oplus 1\}$, the only maximal essential extensions of $B_i \oplus 1$, $i = 0, 1$, is $B_n \oplus 1$. In both cases, there is only one such essential extension. Also, any embedding $f : B_n \oplus 1 \hookrightarrow B_n \oplus 1$ is automatically an isomorphism by finiteness. Hence, $(B_i \oplus 1 : \mathcal{V}) = 1$ for $i = 0, 1, n$.

Next we show for $1 < i < n$, $(B_i \oplus 1 : \mathcal{V}) > 1$: we may assume $n \geq 3$. Since $i \geq 2$, as in the proof of theorem 3.4.11, we can construct embeddings $\alpha, \beta : B_i \oplus 1 \hookrightarrow B_n \oplus 1$ such that α, β both map the monolith of $B_i \oplus 1$ onto the monolith of $B_n \oplus 1$, and there is an atom a of $B_i \oplus 1$ such that $\alpha(a)$ is an atom of $B_n \oplus 1$, but $\beta(a)$ is not an atom of $B_n \oplus 1$. Because these embeddings preserve the monolith, they are essential. Suppose there was an automorphism $\gamma : B_n \oplus 1 \to B_n \oplus 1$ such that $\gamma\alpha = \beta$. Then as $\alpha(a)$ is an atom of $B_n \oplus 1$, $\gamma\alpha(a) = \beta(a)$ is an atom of $B_n \oplus 1$, a contradiction. Hence $(B_i \oplus 1 : \mathcal{V}) > 1$.

(3) $\Longrightarrow$ (1): $B_n \oplus 1 \in \mathcal{V}_{WMI}$ so $B_n \oplus 1 \in Amal(\mathcal{V})$.

For i=0, note first that there is only one homomorphism $e : B_0 \oplus 1 \hookrightarrow B_n \oplus 1$, and it is an embedding. So given any embedding $g : B_0 \oplus 1 \hookrightarrow C$, by theorem 1.2.23 we may choose $\Psi \in \mathrm{Con}(C)$ such that hg is essential, where $h : C \twoheadrightarrow C/\Psi$ is the canonical epimorphism. But then $C/\Psi \in \mathcal{V}_{SI}$ so there is an extension $k : C/\Psi \hookrightarrow B_n \oplus 1$. But then by uniqueness, $khg = f$, and thus $B_0 \oplus 1$ has property Q, hence by lemma 4.1.4, $B_0 \oplus 1 \in Amal(\mathcal{V})$.

The proof for $B_1 \oplus 1$ proceeds similarly. □

THEOREM 4.2.10 **(C. Bergman [85]).** *Let $n \geq 1$, and let $\mathcal{V} = \mathcal{H}_n$. Then every $A \in Amal(\mathcal{V})$ is subdirect product of members of $\mathcal{V}_{ASI}$. More specifically, A is a subdirect product of $\{B_0 \oplus 1, B_1 \oplus 1, B_n \oplus 1\}$.*

Proof. The proof is similar to theorem 4.2.8. As in step 1 of theorem 4.2.8, let A be a subdirect product, with factors chosen from some minimal $\mathcal{S} \subseteq \mathcal{V}_{SI}$, and choose T as before. Choose J and J' as before and similarly choose R, Q and δ_j for all $j \in R$.

Let $\alpha_i : T \hookrightarrow L_i \in \mathcal{V}_{WMI}$ be members of $\mathcal{E}_{\mathcal{V}}(T)$ for $i = 0, 1$. Note $L_i \cong B_n \oplus 1$ for $i = 0, 1$ since $\mathcal{V}_{WMI} = \{B_n \oplus 1\}$. Now repeat the process described in step 2 to arrive at algebras B_0, B_1 and E, and maps $\phi_0, \phi_1, \sigma_0, \mu_0$ and μ_1. Notice too that each N_j is isomorphic to $B_n \oplus 1$.

Since B_0 is automatically congruence extensile as $\mathcal{V}$ has the congruence extension property, we can find μ_0' and Ψ as in step 3.

For step 4, we need a new argument. Put $\Psi_1 = \mu_1^{-1}(\Psi)$. Let $\mu_1' : B_1/\Psi_1 \hookrightarrow E/\Psi \cong L_0$ be the induced embedding. Now $E/\Psi \cong B_n \oplus 1$, and one can check that every subalgebra of $B_n \oplus 1$ is subdirectly irreducible, and so $B_1/\Psi_1 \in \mathcal{V}_{SI}$. Thus, by theorem 1.2.16, there is an ultrafilter $\mathfrak{D}$ on J such that B_1/Ψ_1 is a homomorphic image of the ultraproduct $B_1/\mathfrak{D}$. Since factors of B_1 are all isomorphic to $B_n \oplus 1$, by theorem 1.1.7(3), $B_1/\mathfrak{D}$ is isomorphic to $B_n \oplus 1$. Since every proper homomorphic image of $B_n \oplus 1$ is a boolean algebra, we must have B_1/Ψ_1 isomorphic to $B_0 \oplus 1$ or $B_n \oplus 1$. In the first case, since $\phi_1' : T \hookrightarrow B_1/\Psi_1$ is an embedding, $T \cong B_0 \oplus 1$ and this completes the proof.

For the latter case, μ_1' becomes an isomorphism and we conclude as before that $\alpha_0 \cong \phi_1'$. Also, since $B_1/\Psi_1 \cong B_1/\mathfrak{D}$, we have already that Ψ_1 is induced by an ultrafilter $\mathfrak{D}$ on J. The rest of the proof is now exactly the same as steps 5 and 6 for this case. □

Let $\mathcal{N}$ be the lattice variety generated by the pentagon N, the five-element lattice introduced in chapter one. Since subdirectly irreducible algebras are homomorphic images of subalgebras of N, we have that $\mathcal{N}_{SI} = \{\underline{2}, N\}$, where $\underline{2}$ is the two-element chain. I present a characterization of subdirect products of amalgamation bases in this variety below.

THEOREM 4.2.11 (Day [72]). *For any non-distributive finitely generated variety $\mathcal{V}$, we have that the two-element chain $\underline{2} \notin Amal(\mathcal{V})$.*

Proof. Since $\mathcal{V}$ is non-distributive, M_3 or N is a member of $\mathcal{V}$. Let L be M_3 or N, whichever is in $\mathcal{V}$, and let $f : \underline{2} \hookrightarrow L$ be a map into a critical quotient of L such that $f(1)$ is not the top of L. Also, since each $M \in \mathcal{V}_{WMI}$ is finite, we can define a homomorphism (in fact, it is an embedding) $h : \underline{2} \hookrightarrow M$ such that $h(1)$ and $h(0)$ are the top and bottom of M respectively. Now suppose there was a homomorphism $g : L \to M$ such that $gf = h$. Now g cannot be injective as g must take the top of L and $f(1)$ to the top of M. Hence, g must collapse critical quotients of L, hence $gf(0) = gf(1)$, contradicting $h(0) \neq h(1)$. Hence, $\underline{2}$ does not have property Q, hence as $\mathcal{V}$ is a Q-variety, $\underline{2} \notin Amal(\mathcal{V})$. □

COROLLARY 4.2.12. *For the variety $\mathcal{N}$ generated by the pentagon, $\mathcal{V}_{ASI} = \mathcal{V}_{WMI} = \{N\}$.* □

The following result appears in Jónsson [90], but the author credits C. Bergman with the proof.

THEOREM 4.2.13 (Jónsson [90] [by C.Bergman]). *For the variety $\mathcal{N}$, every $A \in Amal(\mathcal{N})$ is a subdirect power of N.*

Proof. Let $A \subseteq \prod_{i \in I} B_i$ be a subdirect product representation of $A \in Amal(\mathcal{N})$. Each B_i is isomorphic to $\underline{2}$ or N, so put $J_1 = \{i \in I : B_i = N\}$ and $J_2 = I \setminus J_1$. For each $i \in I$, let $h_i : A \twoheadrightarrow B_i$ be the ith projection map restricted to A, and let $\Phi_i = \ker h_i$, and let $\Theta = \bigcap_{i \in J_1} \Phi_i$.

We will show $\Theta = \mathbf{0}_A$. Suppose not. We will derive a contradiction to theorem 4.2.11 by showing $\underline{2} \in Amal(\mathcal{N})$:

Choose $x, y \in A$ with $x < y$ and $x \; \Theta \; y$. Define the embedding $f : \underline{2} \hookrightarrow A$ by $f(0) = x, f(1) = y$. Let $g : \underline{2} \hookrightarrow C$ be an arbitrary embedding. By lemma 1.3.2, we need only show $(\underline{2}, f, A, g, C)$ has an amalgam. Let $K_1 = \{i \in I : h_i(x) = h_i(y)\}$ and let $K_2 = I \setminus K_1$. Since at least one h_i separates x and y, $K_2 \neq \emptyset$. Note also that $J_1 \subseteq K_1$.

For $i \in K_1$, define $D_i = B_i, f_i' = h_i', g_i'(u) = h_i(x)$ for all $u \in C$. For $i \in K_2$, define $D_i = B_i \cong \underline{2}, f_i' = gh_i', g_i'(u) = u$ for all $u \in C$. Note, in each case $f_i' : A \to D_i$, $g_i' : C \to D_i$. Now we claim that for each $i \in I$, $f_i'f = g_i'g$: For $i \in K_1$, for $p = 0, 1$,

$f_i'f(p) = h_i(x) = g_i'g(p)$. For $i \in K_2$ as $B_i \cong \underline{2}$, and $h_i(x) = 0, h_i(y) = 1$, and hence $f_i'f(0) = gh_i(x) = g(0) = g_i'g(0)$, and similarly $f_i'f(1) = g_i'g(1)$. Hence letting $D = \prod_{i \in I} D_i$, and $f' : A \to D$ and $g' : C \to D$ be product maps induced by the f_i''s and g_i''s respectively, we have $f'f = g'g$.

It remains to show f' and g' are embeddings to see that (f', g', D) is an amalgam of $(\underline{2}, f, A, g, C)$. For $f' : A \to D$, pick $a, b \in A$ such that $a \neq b$. Then for some $i \in I$, $h_i(a) \neq h_i(b)$ and so $f_i'(a) \neq f_i'(b)$ since g is an embedding. Hence f' is an embedding. For $g' : C \to D$, since $K_2 \neq \emptyset$, there is an $i \in K_2$ such that g_i is the identity on C, so g is an embedding, proving the result. □

4.3 Property P

Introduction. I define property P, and prove a few lemmas which will be used to characterise the amalgamation bases of various classes of varieties in terms of property P and congruence extensibility.

DEFINITION 4.3.1 (C. Bergman [85]). Let $\mathcal{V}$ be a variety and let $A \in \mathcal{V}$.

(1) We denote $\prod \{A/\Theta : \Theta \in \mathrm{Con}(A) \text{ and } A/\Theta \in \mathcal{V}_{ASI}\}$ by $A^{\#}$.

(2) Denote by $\mu_A : A \to A^{\#}$ the product of the canonical maps $e_\Theta : A \to A/\Theta$.

(3) We say A has *property P in* $\mathcal{V}$, if for every $M \in \mathcal{V}_{WMI}$ and homomorphism $\alpha : A \to M$, there is a map $\beta : A^{\#} \to M$ such that $\beta\mu_A = \alpha$.

Note that (3) is generally vacuous if $\mathcal{V}$ is not residually small. Also, if $\mathcal{V}$ is residually small, then (3) implies μ_A is injective.

LEMMA 4.3.2 (C. Bergman [85]). *Let $\mathcal{V}$ be residually small and let $A \in \mathcal{V}$. Let $\alpha : A \hookrightarrow \prod_{i \in I} B_i = B$ be a subdirect embedding in $\mathcal{V}$. The following conditions are sufficient for $A \in Amal(\mathcal{V})$:*

(1) *$B \in Amal(\mathcal{V})$.*

(2) *A is congruence extensile in $\mathcal{V}$.*

(3) *For every $M \in \mathcal{V}_{WMI}$, every homomorphism from A to M extends to a homomorphism from B to M.*

Proof. By lemma 1.3.2, we need only show that for any $C \in \mathcal{V}$ and embedding $\beta : A \hookrightarrow C$, (A, α, B, β, C) has an amalgam. We use lemma 1.3.3(2). For $a, b \in C$,

$a \neq b$, by theorem 1.2.4 there is a completely meet irreducible $\Theta \in \mathrm{Con}(C)$. By residual smallness, C/Θ is embeddable in some $M \in \mathcal{V}_{WMI}$, so we may assume there is a homomorphism $\beta' : C \to M$ such that $\beta'(a) \neq \beta'(b)$. By (3), there is a homomorphism $\alpha' : B \to M$ such that $\alpha'\alpha = \beta'\beta$, thus satisfying the condition of the lemma.

On the other hand, if $a, b \in B$, $a \neq b$ pick $j \in J$ such that $\pi_j(a) \neq \pi_j(b)$, where $\pi_j : B \twoheadrightarrow B_j$ is the jth projection. Since $\pi_j|_A$ is still onto, and A is congruence extensile, there is an embedding $\alpha_j : B_j \hookrightarrow D \in \mathcal{V}$, and an epimorphism $b' : C \twoheadrightarrow D$ such that $\beta'\beta = \alpha_j\pi_j\alpha$. Setting $\alpha' = \alpha_j\pi_j$, we see that $\beta'\beta = \alpha'\alpha$ and $\alpha'(a) \neq \alpha'(b)$, satisfying the conditions of the lemma. ☐

LEMMA 4.3.3. *Let $\mathcal{V}$ be a residually small variety and let $A \in \mathcal{V}$ be congruence extensile, have property P and suppose $A^{\#} \in Amal(\mathcal{V})$. Then $A \in Amal(\mathcal{V})$.*

Proof. Setting $B = A^{\#}$ in lemma 4.3.2, we see (1) and (2) hold, and (3) is just a restatement of property P, hence the conclusion follows. ☐

LEMMA 4.3.4. *Let $\mathcal{V}$ be residually small and let $A \in Amal(\mathcal{V})$, and suppose A is a subdirect product of members of $\mathcal{V}_{ASI}$. Suppose too that all members of $\mathbf{P}(\mathcal{V}_{ASI})$ are congruence extensile. Then A is congruence extensile and has property P.*

Proof. First we show $A \in \mathcal{V}$ congruence extensile. Since A is a subdirect product of members of $\mathcal{V}_{ASI}$, $\mu_A : A \to A^{\#}$ is an embedding. Let $f : A \hookrightarrow C \in \mathcal{V}$ be an arbitrary extension of A. By lemma 4.1.9, we need only show that for every completely meet irreducible $\Theta \in \mathrm{Con}(A)$, there is a $\Psi \in \mathrm{Con}(C)$ extending Θ. Let $M \in \mathcal{V}_{WMI}$ be an essential extension of A/Θ, and let $g : A \hookrightarrow A^{\#} \times M$ be the embedding given by $g(a) = (\mu_A(a), h(a))$, where $h : A \to M$ is the homomorphism induced by the epimorphism for A onto A/Θ. Since $A \in \mathcal{V}$, $(A, f, C, g, A^{\#} \times M)$ has an amalgam, say (α, β, D). Since $A^{\#} \times M \in \mathbf{P}(\mathcal{V}_{ASI})$, $A^{\#} \times M$ is congruence extensile, hence there is a $\Phi \in \mathrm{Con}(D)$ such that $\beta^{-1}(\Phi)$ is the kernel of the projection map from $A^{\#} \times M$ onto M, and hence $(\beta g)^{-1}(\Phi)$ is equal to Θ. Hence the diagram given in figure 4.4 is commutative, and letting $\Psi = \alpha^{-1}(\Phi)$, we have $f^{-1}(\Psi) = \Theta$, proving A is congruence extensile.

$$\begin{tikzcd}
 & A \arrow[r, hook, "c"] \arrow[dl, two heads] \arrow[dr, hook, "g"] \arrow[dd, "h"] & C \arrow[dr, hook, "\alpha"] & \\
A/\Theta \arrow[dr, hook] & & A^{\#} \times M \arrow[r, hook, "\beta"] \arrow[dl, two heads] & D \arrow[dl, two heads] \\
 & M \arrow[r, hook, shift left] & D/\Psi \arrow[l, two heads, shift left, "r"] &
\end{tikzcd}$$

FIGURE 4.4

To show A has property P, let $h : A \to M \in \mathcal{V}_{WMI}$ be a homomorphism. Setting $C = A^{\#}$ and $f = \mu_A$, we can use the same argument above to obtain the same commutative diagram. Now as $M \in \mathcal{V}_{WMI}$, the embedding from M into D/Φ has a retraction $r : D/\Phi \twoheadrightarrow M$. Letting $\epsilon : D \twoheadrightarrow D/\Phi$ be the canonical epimorphism, we see that the composition $r\epsilon\alpha : C \to M$ is a map that extends h. ☐

Semi-simple Varieties and Property P. I present a characterization of amalgamation bases for the following class of semi-simple varieties.

THEOREM 4.3.5 (C. Bergman [85]). *Let $\mathcal{V}$ be congruence distributive, semi-simple and finitely generated. Suppose that*

(α) *all members of* $\mathrm{P}(\mathcal{V}_{ASI})$ *are congruence extensile.*

Then for $A \in \mathcal{V}$: $A \in Amal(\mathcal{V})$ if and only if A is congruence extensile and A has property P.

Proof. $\Longrightarrow$: Follows from lemma 4.3.4, since every $A \in Amal(\mathcal{V})$ is a subdirect product of members of $\mathcal{V}_{ASI}$ by theorem 4.2.8.

$\Longleftarrow$: We use lemma 4.3.3. We need only show $A^{\#} \in Amal(\mathcal{V})$. We do this by showing $A^{\#}$ has property Q:

Let $f : A^{\#} \hookrightarrow C$ be an arbitrary embedding, and let $g : A^{\#} \to M$ be a homomorphism. Let $\Theta = \ker g$. Since M is finite $A^{\#}/\Theta$ is finite, so by theorem 4.2.7, $A^{\#}/\Theta \in \mathrm{P}(\mathcal{V}_{ASI})$. But as M is simple, and by (α), $A^{\#}/\Theta$ is congruence extensile, hence we must have $A^{\#}/\Theta$ is simple, that is $A^{\#}/\Theta \in \mathcal{V}_{ASI}$.

Now (α) applies also to $A^{\#}$, so there is a $\Psi \in \mathrm{Con}(C)$ extending Θ. We may suppose the induced embedding from $A^{\#}/\Theta$ to C/Ψ is essential by lemma 1.2.23, and hence $C/\Psi \in \mathcal{V}_{SI}$. Since $\mathcal{V}$ is residually small, there is an essential embedding $\gamma : C/\Psi \hookrightarrow L \in \mathcal{V}_{WMI}$. Now since $A^{\#}/\Theta \in Amal(\mathcal{V})$, we have by the simplicity of M and L that both the embedding from $A^{\#}/\Theta$ into M and from $A^{\#}/\Theta$ through C/Ψ into L are essential, hence by lemma 4.2.2, there is an isomorphism $\delta : L \to M$ such that the diagram in figure 4.5 commutes.

$$\begin{tikzcd}
A^{\#} \arrow[rr, hook, "f"] \arrow[dr] \arrow[dd, "g"'] & & C \arrow[dr, "\epsilon"] & \\
& A^{\#}/\Theta \arrow[rr, hook] \arrow[dl, hook'] & & C/\Psi \arrow[dl, hook', "\gamma"] \\
M & & L \arrow[ll, "\delta"] &
\end{tikzcd}$$

FIGURE 4.5

Setting $h = \delta\gamma\epsilon$, where $\epsilon : C \twoheadrightarrow C/\Psi$ is the canonical epimorphism, we see that $hf = g$, proving the theorem. □

Filtral Varieties and Property P. In trying to find specific varieties which satisfy the conditions of theorem 4.3.5, the condition (α) is the only unfamiliar one. One way of avoiding it is to assume the variety has the congruence extension property. In fact, these varieties are just the finitely generated filtral varieties (defined below), which were first introduced by Magari [69], with subsequent work done by G. Bergman [71].

DEFINITION 4.3.6 (Magari [69]). A variety $\mathcal{V}$ is *filtral* if for every subdirect product B of a family B_i, $i \in I$ of subdirectly irreducible algebras, and each $\Theta \in \mathrm{Con}(B)$, there is a filter $\mathfrak{D}$ on I such that the congruence induced by the $\mathfrak{D}$ on $\prod_{i \in I} B_i$ restricted to B is Θ.

THEOREM 4.3.7 (G. Bergman [71]). *Let $\mathcal{V}$ be a finitely generated variety. Then $\mathcal{V}$ is filtral if and only if it is congruence distributive, semi-simple and has the congruence extension property.* □

COROLLARY 4.3.8 (C. Bergman [85]). *Let $\mathcal{V}$ be a finitely generated filtral variety. Then for all $A \in \mathcal{V}$: $A \in Amal(\mathcal{V})$ if and only if A has property P.* □

Modular Lattice Varieties and Property P. Another class of varieties satisfying the conditions of theorem 4.3.5 are the finitely generated modular lattice varieties. They are semi-simple and congruence distributive, so we need just check property (α).

LEMMA 4.3.9 (C. Bergman [85]). *Let $\mathcal{V}$ be finitely generated, semi-simple, and congruence distributive. Suppose no non-simple member of $\mathbf{P}(\mathcal{V}_{ASI})$ can be embedded in a simple member of $\mathcal{V}$. Then the members of $\mathbf{P}(\mathcal{V}_{ASI})$ are congruence extensile.*

Proof. We use lemma 4.1.9. Let $A \in \mathbf{P}(\mathcal{V}_{ASI})$ and let $f : A \hookrightarrow B = \prod_{i \in I} B_i$ be an embedding, where $B_i \in \mathcal{V}_{SI}$ for $i \in I$. Let $\Theta \in \mathrm{Con}(A)$ such that $A/\Theta \in \mathcal{V}_{SI}$. By 1.2.16, there is an ultrafilter $\mathfrak{D}$ on I such that if $\Psi \in \mathrm{Con}(B)$ is the congruence induced by $\mathfrak{D}$, we have $\Theta \subseteq \Psi|_A$. Now since $A/\ \Psi|_A$ is embeddable in $B/\Psi \cong B_i$ for some $i \in I$ as $\mathcal{V}$ is finitely generated, we have $A/\ \Psi|_A$ is finite and hence by theorem 4.2.7, we have $A/\ \Psi|_A\ \mathbf{P}(\mathcal{V}_{ASI})$. But then by our assumption, $A/\ \Psi|_A \in \mathcal{V}_{ASI}$. But then as $\Theta \neq 0_A$, $\Theta_A = \Psi|_A$, hence Ψ extends Θ, proving the statement. □

THEOREM 4.3.10 (C. Bergman [85]). *Let $\mathcal{V}$ be a finitely generated variety of modular lattices. Then for $A \in \mathcal{V}$: $A \in Amal(\mathcal{V})$ if and only if A is congruence extensile and has property P.*

Proof. Since a variety of modular lattices is congruence distributive and semi-simple, we need only show that condition (α) of theorem 4.3.5 is satisfied. We use lemma 4.3.9 to do this:

Let $G \in \mathcal{V}_{SI}$ and let $A = \prod_{i \in I} A_i$, where each $A_i \in \mathcal{V}_{ASI}$, and suppose $A \subseteq G$. Since G is finite, we may suppose I is finite and each A_i in non-trivial. Suppose $|I| > 1$. Put $I = \{0, 1, \ldots, n\}$. Pick $a, b \in A_1$ with $a < b$. Fix, for $j = 2, 3, \ldots, n$, elements $c_j \in A_j$. Define embeddings $\alpha, \beta : A_0 \hookrightarrow A$ by

$$\alpha(x) = (x, a, c_2, \ldots, c_n) \text{ and}$$
$$\beta(x) = (x, b, c_2, \ldots, c_n).$$

Now let $M \in \mathcal{V}_{WMI}$ be a maximal essential extension of G. We may suppose α and β are embeddings into M. In fact, since M is simple, α and β are essential, so by lemma 4.2.2, there is an isomorphism $\gamma : M \to M$ such that $\gamma\alpha = \beta$. But for each $x \in A_0$, $\alpha(x) < \beta(x)$. Since M finite, it has no automorphism taking an element to a strictly larger one. Hence we have a contradiction, so $|I| = 1$, proving the theorem. □

Some Heyting Algebra Varieties and Property P. I present another characterization in terms of property P for the Heyting algebra varieties $\mathcal{H}_n$ discussed in section 4.2.

THEOREM 4.3.11 (C. Bergman [85]). *For a natural number n, let $\mathcal{V}$ be the Heyting algebra variety $\mathcal{H}_n$. Then $A \in Amal(\mathcal{V})$ if and only if A satisfies property P.*

Proof. $\Longrightarrow$: Follows from lemma 4.3.4, theorem 4.2.10, and the fact that $\mathcal{V}$ is congruence extensile.

$\Longleftarrow$: To use lemma 4.3.3, since $\mathcal{V}$ has the congruence extension property, we need only show for A having property P, $A^{\#} \in Amal(\mathcal{V})$. In fact, we show $A^{\#}$ has property Q.

Now let $f : A^{\#} \hookrightarrow C \in \mathcal{V}$ be an arbitrary embedding, and let $g : A^{\#} \to B_n \oplus 1$ be a homomorphism. Let $\Theta = \ker g$. By congruence extensibility, there is a $\Psi \in \mathrm{Con}(C)$ extending Ψ. By lemma 1.2.23, we may suppose $C/\Psi \in \mathcal{V}_{SI}$, hence there is an embedding $k : C/\Psi \hookrightarrow B_n \oplus 1$. Now since $A^{\#}/\Theta$ is embeddable in $B_n \oplus 1$, it is subdirectly irreducible. Thus by theorem 1.2.16, there is an ultrafilter $\mathfrak{D}$ on the index set of $A^{\#}$, such that $A^{\#}/\Theta$ is an image of the ultraproduct $A^{\#}/\mathfrak{D}$. But since each factor of $A^{\#}$ is either $B_0 \oplus 1$, $B_1 \oplus 1$ or $B_n \oplus 1$, and their proper homomorphic images are all Boolean, we must have $A^{\#}/\Theta$ isomorphic to $B_0 \oplus 1$, $B_1 \oplus 1$ or $B_n \oplus 1$. Hence $A^{\#}/\Theta \in Amal(\mathcal{V})$. Since $\mathcal{V}$ is Q-variety, $A^{\#}/\Theta$ has property Q, so the embedding from $A^{\#}/\Theta$ to $B_n \oplus 1$ induced by g can be extended to a homomorphism $k : C/\Psi \to B_n \oplus 1$ (see the commutative diagram in figure 4.6).

$$\begin{tikzcd}
A^{\#} \arrow[rr, hook, "f"] \arrow[dr] \arrow[dd, "g"'] & & C \arrow[dr, "\epsilon"] & \\
& A^{\#}/\Theta \arrow[rr, hook] \arrow[dl, hook] & & C/\Psi \arrow[dlll, hook, "k"] \\
B_n \oplus 1 & & &
\end{tikzcd}$$

FIGURE 4.6

Letting $\epsilon : C \twoheadrightarrow C/\Psi$ be the canonical epimorphism, we see $h = k\epsilon$ extends g, that is $hf = g$, proving the theorem. □

The Pentagon Variety and Property P. The last characterization in terms of property P is for the lattice variety generated by the pentagon. Recall that we denote this variety by $\mathcal{N}$.

The following result appears in Jónsson [90], but the author credits C. Bergman.

THEOREM 4.3.12(Jónsson [90] [by C. Bergman]). *$A \in Amal(\mathcal{N})$ if and only if A has property P and is congruence extensile.*

Proof. Note $\mathcal{N}_{ASI} = \mathcal{N}_{WMI} = \{N\}$, and products of members of $\mathcal{N}_{SI}$ are absolute retracts, as $\mathcal{N}$ is retract extendible.

$\Longrightarrow$: Every $A \in Amal(\mathcal{N})$ is a subdirect power of N by theorem 4.2.13, and since each member of $\mathbf{P}(\mathcal{N}_{ASI})$ is an absolute retract (and hence is congruence extensile), the result follows by lemma 4.3.4.

$\Longleftarrow$: Let A be congruence extensile and have property P. $A^{\#}$ is a power of N, so is an absolute retract, and thus is a member of $Amal(\mathcal{N})$. The result now follows from lemma 4.3.3. □

Finite Amalgamation Bases. We have derived characterization theorems for the following finitely generated congruence distributive varieties:

(1) Semi-simple varieties with the properties given in theorem 4.3.5. e.g. modular lattice varieties and filtral varieties.

(2) The Heyting algebra varieties $\mathcal{H}_n$, n a natural number.

(3) The lattice variety generated by the pentagon.

For all these varieties, we can effectively decide which subdirectly irreducible members are in $\mathcal{V}_{ASI}$ (for (1) this can be done using theorem 4.2.2, since in these varieties a finite algebra has only finitely many essential extensions up to isomorphism - see Jónsson [90] and Jipsen and Rose [89]). I now give a result which shows that there is an effective test for deciding whether a finite algebra in any of the above varieties is an amalgamation base. I end this chapter with three more characterizations of finite amalgamation bases, the last two stated with references but without proof, in the interests of space.

THEOREM 4.3.13. *Let $A \in \mathcal{V}$ with A finite and $\mathcal{V}$ a congruence distributive finitely generated variety. Then, if $\mathcal{V}_{ASI}$ is known, it is effectively decidable whether A has property P and A is congruence extensile. Thus for the varieties mentioned in (1), (2) and (3) above, we have an effective test for deciding whether a finite algebra is a member of Amal($\mathcal{V}$).*

Proof. Let $A \in \mathcal{V}$, with A finite. A has only finitely many essential extensions (see Jónsson [90] and Jipsen and Rose [89]), and Con(A) has only finitely many members. Hence it is effectively decidable whether a given congruence on A can be extended to an essential extension of A. Now given any embedding $f : A \hookrightarrow B \in \mathcal{V}$, by lemma 1.2.23, there is a $\Psi \in \mathrm{Con}(B)$ such that that $\epsilon f : A \to B/\Psi$ is an essential embedding, where $\epsilon : B \twoheadrightarrow B/\Psi$ is the canonical epimorphism. If $\Theta \in \mathrm{Con}(A)$, then it is effectively decidable whether there is a $\Phi \in \mathrm{Con}(B/\Psi)$ extending Θ. If so, we can view Φ as a congruence on B, and see that it extends Θ. Thus determining whether A is congruence extensile is effectively decidable.

If $\mathcal{V}_{ASI}$ is known, $A^{\#}$ and μ_A are automatically known. It remains to check whether each homomorphism $g : A \to M \in \mathcal{V}_{WMI}$ can be extended to $A^{\#}$. Since A and $A^{\#}$ are finite, and $\mathcal{V}_{WMI}$ has finitely many finite members, there are only a finite number of of homomorphisms $g : A \to M$ and $h : A^{\#} \to M$ where $M \in \mathcal{V}_{WMI}$. So this process is effectively decidable. □

A similar result for finitely generated Q-varieties was also shown by Jónsson [90], that is, there is an effective test in such a variety for deciding whether a finite algebra is an amalgamation base.

THEOREM 4.3.14 (C. Bergman [85]). *Let $\mathcal{V}$ be finitely generated semi-simple and congruence distributive such that members of* $\mathbf{P}(\mathcal{V}_{ASI})$ *are congruence extensile. Let $A \in \mathcal{V}$ be finite. Then the following are equivalent:*

(1) *$A \in Ps(\mathcal{V}_{ASI})$ and is congruence extensile.*
(2) *A satisfies property P.*
(3) *$A \in Amal(\mathcal{V})$.*

Proof. (3) is equivalent to (1) and (2) combined by theorem 4.3.5, so we need only show that (1) and (2) are equivalent.

(1) $\implies$ (2): Firstly, note $\mu_A : A \hookrightarrow A^{\#}$ is an embedding. Since A is finite we may assume $A^{\#} = \prod_{i=0}^{n} B_i$ with each $B_i \in \mathcal{V}_{ASI}$. Let $\Theta_i \in \mathrm{Con}(A)$ be the kernels of the projection maps onto B_i restricted to A for each $i \in \{0, \ldots, n-1\}$. We may suppose that the Θ_i's are pairwise distinct. Now $\mathrm{Con}(A)$ is finite and distributive, so $\Theta = \Theta_0 \cap \Theta_1 \cap \ldots \cap \Theta_t$ for some $t < n-1$, rearranging the B_i's if necessary.

Let $\delta : A \hookrightarrow M \times \prod_{i=t+1}^{n-1} B_i$ be the embedding given by

$$\delta(a) = (\gamma(a), a/\Theta_{t+1}, \ldots, a/\Theta_{n-1}).$$

Let $\Psi_0, \Psi_{t+1}, \ldots, \Psi_{n-1}$ be the projection kernels of δ. Since A is congruence extensile, there is a congruence Φ on $M \times \prod_{i=t+1}^{n-1} B_i$ extending Θ_t. Using finiteness and congruence distributivity again, $\Phi = \bigcap_{j \in J} \Psi_j$ for some $J \subseteq \{0, t+1, \ldots, n-1\}$. Thus $\Theta_t = \Phi|_A = \bigcap_{j \in J} \Psi_j|_A$. Since Θ_t is meet irreducible, $\Theta_t = \Psi_j|_A$ for some $j \in J$. If $j \neq 0$, $\Theta_t = \Psi_j|_A = \Theta_j$, contradicting pairwise distinctness of the Θ_j's. Hence $\Theta_t = \Psi_0|_A = \Theta$. But then, as $\Theta = \Theta_0 \cap \Theta_1 \cap \ldots \cap \Theta_t$, we must have t = 0. Thus A/Θ is isomorphic to B_0, and hence there is a embedding $f : B_0 \to M$ such that the composition of the projection from $A^{\#}$ onto B_0 and f extends γ.

(2) $\implies$ (1): Suppose a finite $A \in \mathcal{V}$ satisfies property P. Then μ_A is injective, hence $A \in Ps(\mathcal{V}_{ASI})$. In fact, since $\mathrm{Con}(A)$ is finite and distributive, if Θ is a co-atom in $\mathrm{Con}(A)$, $A/\Theta \in \mathcal{V}_{ASI}$. Since A is finite we may assume again that $A^{\#} = \prod_{i=0}^{n} B_i$ with each $B_i \in \mathcal{V}_{ASI}$.

To show A is congruence extensile, it suffices to consider an extension $C \in \mathcal{V}$ of A such that $C = \prod_{j \in J} C_j$ where each $C_j \in \mathcal{V}_{WMI}$. Now for each $j \in J$, let $\gamma_j : A \to C_j$

be the restriction of the projection map to A. By property P, there is an extension of γ_j, say $\delta_j : A^{\#} \to C_j$. Let $\delta : A^{\#} \to C$ be given by the product map $\delta = \prod\limits_{j \in J} \delta_j$. Let $\Psi = \ker \delta$. Now $\Psi|_A = 0_A$ since A is embedded in C. If $\Psi \neq 0_{A^{\#}}$, then using congruence distributivity to split Ψ into its factor congruences of B_i, and the fact that each B_i is simple, we can find $a', b' \in A, a' \neq b'$ such that $\delta\mu_A(a') = \delta\mu_A(b')$, a contradiction. Hence δ is an embedding.

Now let Θ be a completely meet irreducible congruence in Con(A). Θ extends to a congruence on $A^{\#}$ because every completely meet irreducible congruence of A is a projection congruence of $A^{\#}$. By our asssumption, $A^{\#}$ is congruence extensile, so Θ can be extended all the way up to C as $A^{\#}$ is embeddable in C. ☐

COROLLARY 4.3.15 **(C. Bergman [85]).** *For a filtral variety* $\mathcal{V}$ *and a finite* $A \in \mathcal{V}$*:* $A \in$ *Amal*$(\mathcal{V})$ *if and only if* A *is a subdirect product of* $\mathcal{V}_{ASI}$. ☐

THEOREM 4.3.16 **(Grätzer and Lakser [71]).** *Let* A *be a finite Heyting algebra in* $\mathcal{H}_n$ *for some natural number* n*. Then* $A \in$ *Amal*$(\mathcal{H}_n)$ *if and only if* A *has no homomorphism onto* $B_i \oplus 1$ *for all* $1 < i < n$. ☐

THEOREM 4.3.17 **(Jónsson [90]).** *A finite non-trivial lattice* $A \in \mathcal{N}$ *belongs to Amal*$(\mathcal{N})$ *if and only if* A *is a subdirect power of* N *and the three-element chain* $\underline{3}$ *is not an image of* A. ☐

Chapter 5

Elementary Amalgamation Classes

Introduction. In chapter four, I presented some characterizations of the amalgamation class of some varieties. In this chapter, I focus on a particular characterization question: whether or not the amalgamation class of a variety can be described by a set of first-order sentences. The results obtained so far have been surprisingly divergent: For finitely generated varieties of modular lattices (excluding the distributive and trivial variety), the answer is no (due to C. Bergman [89]), but for finitely generated discriminator varieties and the lattice variety generated by the pentagon the answer is yes (but see the remarks made after lemma 5.3.13). Thus the answer to this question seems totally unrelated to the properties P and Q, as all these varieties have similar characterizations of the amalgamation class using these properties. Even the proofs in the affirmative for the discriminator and pentagon varieties are dissimilar: the former is largely constructive (due to C. Bergman [83]), while the latter relies on algebraic methods that are largely non-constructive (due to Bruyns, Naturman and Rose [92]).

5.1 Finitely Generated Discriminator Varieties

Introduction and Basic Concepts. Discriminator varieties were first introduced by Pixley in 1970. Every finitely generated discriminator variety is filtral, and hence by corollary 4.3.8, we have one characterization of the amalgamation class. In this section, I present the even stronger result by C. Bergman [85] that the amalgamation class of a finitely generated discriminator variety can be described by a single first order sentence. It uses extensively a result by McKenzie [75] which relates the quantifier-free formulas to certain terms in the language of the variety.

I begin by introducing the definition of a discriminator variety, and stating (largely without proof) some of the standard facts that are known about such varieties. For a proof of most of these results, consult Burris and Sankappanavar [81] (page 164ff).

DEFINITION 5.1.1 (Pixley [70]). A variety $\mathcal{V}$ is a *discriminator variety* if there is a term $\sigma(x, y, z)$ (called the *discriminator term*) such that $A \in \mathcal{V}$ is subdirectly irreducible or trivial if and only if

$$A \models (\sigma(x,x,z) = z) \text{ and } (x \neq y \implies \sigma(x,y,z) = x).$$

Notice that for a discriminator variety $\mathcal{V}$, $\mathcal{V}_{SI}$ is elementary.

LEMMA 5.1.2. *Let $\mathcal{V}$ be a discriminator variety. Then:*

(1) *$\mathcal{V}$ satisfies the following identities:*

$$\begin{aligned} \sigma(x,x,y) &= y \\ \sigma(x,y,x) &= x \\ \sigma(x,y,y) &= x \\ \sigma(x,\sigma(x,y,z),y) &= y \\ \sigma(x,y,F(v_0,\ldots,v_{n-1})) &= \sigma(x,y,F(\sigma(x,y,v_0),\ldots,\sigma(x,y,v_{n-1}))) \end{aligned}$$

for each n-ary operation F in the type of $\mathcal{V}$.

(2) *$\mathcal{V}$ is congruence permutable and congruence distributive.*

(3) *Every subalgebra (not necessarily proper) of a member of $\mathcal{V}_{SI}$ is simple.*

(4) *Every finite algebra of $\mathcal{V}$ is a product of simple algebras.*

(5) *For every $A \in \mathcal{V}$ and $a, b \in A$, the smallest congruence containing (a, b) is given by:*

$$\operatorname{con}(a,b) = \left\{(x,y) \in A^2 : \sigma(a,b,x) = \sigma(a,b,y)\right\}.$$

(6) *Every finite join of principal congruences is principal.*

(7) *$\mathcal{V}$ satisfies the congruence extension property.* □

LEMMA 5.1.3. *If $\mathcal{V}$ is a discriminator variety, and ϕ is a quantifier-free formula in the language of $\mathcal{V}$, then there are terms α and β in the language of $\mathcal{V}$ satisfying the following conditions:*

(1) *(McKenzie [75]) For all $B \in \mathcal{V}_{SI}$ and $b_0, \ldots, b_{m-1} \in B$, $B \models \phi(\overrightarrow{b})$ if and only if there is a $y \in B$ such that $\alpha(\overrightarrow{b}, y) \neq \beta(\overrightarrow{b}, y)$.*

(2) *(C. Bergman* [83]*) For any $A \in \mathcal{V}$ and $a_0, \ldots, a_{m-1} \in A$, the following are equivalent:*

(i) There exists a co-atom Ψ of Con(A) such that

$$A/\Psi \models \phi(a_0/\Psi, \ldots, a_{m-1}/\Psi).$$

(ii) $A \models ((\exists y)(\alpha(a_0, \ldots, a_{m-1}, y) \neq \beta(a_0, \ldots, a_{m-1}, y))$.

Proof.

(1) See McKenzie [75].

(2) $(i) \implies (ii)$: Let $A \in \mathcal{V}$ and suppose (i) holds. Let $B = A/\Psi$ and $b_i = a_i/\Psi$ for $i < n$. Then $B \in \mathcal{V}_{SI}$, so there exists a $z \in B$ with $\alpha(\overrightarrow{b}, z) \neq \beta(\overrightarrow{b}, z)$. Choose $y \in A$ with $y/\Psi = z$. Then $\alpha(\overrightarrow{a}, y)/\Psi = \alpha(\overrightarrow{b}, z) \neq \beta(\overrightarrow{b}, z) = \beta(\overrightarrow{a}, y)/\Psi$. Hence $\alpha(\overrightarrow{a}, y) \neq \beta(\overrightarrow{a}, y)$, so (ii) is satisfied.

$(ii) \implies (i)$: Suppose there is a $y \in A$ such that $\alpha(\overrightarrow{a}, y) \neq \beta(\overrightarrow{a}, y)$. Then by theorem 1.2.4, there is a co-atom $\Psi \in$ Con(A) (since A/Ψ is simple) such that $\alpha(\overrightarrow{a}, y)/\Psi \neq \beta(\overrightarrow{a}, y)/\Psi$. By labelling B and b_i as before and using (1), we have that (i) holds. □

If B is a finite algebra with cardinality k, then there is a quantifier free formula ϕ with k free variables such that for any algebra A of the same type and $a_0, \ldots, a_{k-1} \in A$ we have $A \models \phi(\overrightarrow{a})$ if and only if $\{a_0, \ldots, a_{k-1}\} \cong B$. Let $Dg_B(\overrightarrow{x})$ denote such a formula.

A Finite Axiomatization of Subdirect Products. As a first step to proving that $Amal(\mathcal{V})$ is strictly elementary for $\mathcal{V}$ a finitely generated discriminator variety, we show that, for a given set of subdirectly irreducibles, we can finitely axiomatize all subdirect products of that set.

THEOREM 5.1.4 (C. Bergman [83]). *Let $\mathcal{V}$ be a finitely generated discriminator variety. Let $S \subseteq \mathcal{V}_{SI}$. Then $Ps(S)$ is a finitely axiomatizable class.*

Proof. Note first that by theorem 1.2.20, $\mathcal{V}$ is finitely axiomatizable. Let Σ be a finite set of identities axiomatizing $\mathcal{V}$. Also S is a finite set of finite algebras, say

$S = \{L_0, \ldots, L_{m-1}\}$. Now fix $j < m$, and put $L = L_j$, and suppose $|L| = r$. Let ϕ be the quantifier-free formula in $r + 2$ variables defined as follows:

$$\phi(x_0, \ldots, x_{r-1}) := (c \neq d \text{ and } Dg_L(\overrightarrow{x})).$$

By lemma 5.1.3, there are terms α and β such that for $A \in \mathcal{V}$, $x_0, \ldots, x_{r-1}$, $c, d \in A$, $A \models (\exists y)(\alpha(\overrightarrow{x}, c, d, y) \neq \beta(\overrightarrow{x}, c, d, y))$ if and only if there is a co-atom $\Psi \in \mathrm{Con}(A)$ such that $c/\Psi \neq d/\Psi$ and $\{x_0/\Psi, \ldots, x_{r-1}/\Psi\} \cong L$. Define the formula $Sep_L(u, v)$ as follows:

$$\begin{aligned} Sep_L(u,v) := (\exists x_0) \ldots (\exists x_{r-1})(\exists y)(\forall z)[(\alpha(\overrightarrow{x}, y, u, v) \neq \beta(\overrightarrow{x}, y, u, v)) \\ \text{and } \bigvee_{i<r} (\sigma(x_i, z, \alpha(\overrightarrow{x}, y, u, v)) \neq \sigma(x_i, z, \beta(\overrightarrow{x}, y, u, v)))] \end{aligned}$$

where σ is the discriminator term of $\mathcal{V}$.

CLAIM 1: For $c, d \in A$, $A \models Sep_L(c, d)$ if and only if there exists $\Psi \in \mathrm{Con}(A)$ such that $c/\Psi \neq d/\Psi$ and $A/\Psi \cong L$.

Proof of claim: $\Longleftarrow$: Suppose there exists $\Psi \in \mathrm{Con}(A)$ with $c/\Psi \neq d/\Psi$ and $A/\Psi \cong L$. Now, since $A/\Psi \cong L$, we can find $a_0, \ldots, a_{r-1} \in A$ such that $A/\Psi \models Dg_L(a_0/\Psi, \ldots, a_r/\Psi)$. Hence by lemma 5.1.3(1), there is a $b \in A$ such that $\alpha(\overrightarrow{a}/\Psi, b/\Psi, c/\Psi, d/\Psi) \neq \beta(\overrightarrow{a}/\Psi, b/\Psi, c/\Psi, d/\Psi)$. Now let $e \in A$. Now $e/\Psi = a_i/\Psi$ for some $i < r$. Hence,

$$\begin{aligned} \sigma(a_i/\Psi,\ e/\Psi,\ \alpha(\overrightarrow{a}, b, c, d)/\Psi) &= \alpha(\overrightarrow{a}/\Psi,\ b/\Psi,\ c/\Psi,\ d/\Psi) \\ \neq \beta(\overrightarrow{a}/\Psi,\ b/\Psi,\ c/\Psi,\ d/\Psi) &= \sigma(a_i/\Psi,\ e/\Psi,\ \beta(\overrightarrow{a}, b, c, d)/\Psi). \end{aligned}$$

So, $\sigma(a_i, e, \alpha(\overrightarrow{a}, b, c, d)) \neq \sigma(a_i, e, \alpha(\overrightarrow{a}, b, c, d))$. Hence as e was arbitrary, $Sep_L(u, v)$ is satisfied.

$\Longleftarrow$: Suppose $A \models Sep_L(c, d)$ for some $c, d \in A$. Let $a_0, \ldots, a_{r-1}, b \in A$ be elements that witness the existential quantifiers of $Sep_L(c, d)$. Denote $\alpha = \alpha(\overrightarrow{a}, b, c, d)$ and $\beta = \beta(\overrightarrow{a}, b, c, d)$. Also, let $A \leq \prod_{t \in T} A_t$ be a subdirect product representation of A, and let $\Theta_t \in \mathrm{Con}(A)$ be the kernel of the tth projection map restricted to A, where $t \in T$. Since each A_t is simple, $\{\Theta_t : t \in T\}$ is a set of co-atoms of $\mathrm{Con}(A)$. Put $U = \{t \in T : \alpha/\Theta_t \neq \beta/\Theta_t\}$ and for each $x \in A$, $V_x = \{t \in T : a_i/\Theta_t = x/\Theta_t \text{ for some } i < r\}$.

CLAIM 2: The set $F = \{U\} \cup \{V_x : x \in A\}$ has the finite intersection property.

Proof of claim 2: Let $x_0, \ldots, x_{k-1} \in A$. Let E be the algebra generated by $\{a_0, \ldots, a_{r-1}, \alpha, \beta, x_0, \ldots, x_{k-1}\}$. Now E is finite, since a finitely generated variety is locally finite. Hence by lemma 5.1.2(4), E is isomorphic to a direct product of of simple algebras, in fact $E \cong \prod\limits_{t \in T_0} E/\Theta_t|_E$ for some finite $T_0 \subseteq T$. Now suppose $U \cap V_{x_0} \cap \ldots \cap V_{x_k}$ is empty. We will derive a contradiction. We have that for all $t \in U$, there is a $t^* < k$ with $t \notin V_{x_{t^*}}$. Now, since E is a direct product of finitely many simple algebras, if $t \in T_0 \cap U$, there is an $e \in E$ such that $e/\Theta_t = x_{t^*}/\Theta_t$. Now $e \in E$ and since $A \models Sep_L(c,d)$, there is an $i < r$ such that $\sigma(a_i, e, \alpha) \neq \sigma(a_i, e, \beta)$. But $\sigma(a_i, e, \alpha), \sigma(a_i, e, \beta) \in E$, hence using again the fact that E is a direct product of simple algebras, we must have $\sigma(a_i, e, \alpha)/\Theta_s|_E \neq \sigma(a_i, e, \beta)/\Theta_s|_E$ for some $s \in T_0$. Hence since $E/\Theta_s|_E$ is simple, $a_i/\Theta_s|_E = e/\Theta_s|_E$ and $\alpha/\Theta_s|_E \neq \beta/\Theta_s|_E$.

Now suppose $s \in U$. Then $s \notin V_{s^*}$, hence $e/\Theta_s = x_{s^*}/\Theta_s \neq a_i/\Theta_s$, so $e/\Theta_{t'}|_E \neq a_i/\Theta_{t'}|_E$, a contradiction. Hence $s \notin U$. But then, $\alpha/\Theta_t = \beta/\Theta_t$, contradicting $\alpha/\Theta_{t'}|_E = \beta/\Theta_{t'}|_E$. Hence in both cases we reach a contradiction, thus claim 2 is proved.

Now since $F \subseteq T$ has the finite intersection property, it can be extended to an ultrafilter $\mathfrak{D}$ over T. Let Θ be the congruence induced by the ultrafilter on $\prod\limits_{t \in T} A_t$, and let $\Psi = \Theta|_A$. Now, since $\mathcal{V}_{SI}$ is elementary $\prod_{t \in T} A_t/\mathfrak{D} \in \mathcal{V}_{SI}$, and hence $A/\Psi \in \mathcal{V}_{SI}$ as A/Ψ is a subalgebra of $\prod_{t \in T} A_t/\mathfrak{D}$. So Ψ is a co-atom of Con(A). Since $U \in \mathfrak{D}$, by lemma 5.1.3 and the definition of α and β, $c/\Psi \not\equiv d/\Psi$ and $\{a_0/\Psi, \ldots, a_{r-1}/\Psi\} \cong L$. But, for $x \in A$, $V_x \in \mathfrak{D}$, hence $x/\Psi = a_i/\Psi$ for some $i < r$. Hence $L \cong \{a_0/\Psi, \ldots, a_{r-1}/\Psi\} \cong A/\Psi$ and claim 1 is proved.

To finish the proof, notice that Σ together with the sentence

$$(\forall u)(\forall v)\left(u \neq v \implies \bigvee_{j<m} Sep_{L_j}(u,v)\right)$$

axiomatizes Ps(S). $\square$

A Finite Axiomatization of the Amalgamation Class. By lemma 5.1.2 and theorem 4.3.7, we have that a finitely generated discriminator variety $\mathcal{V}$ is filtral, hence by theorem 4.1.11 and corollary 4.3.8 we have that for $A \in \mathcal{V}$, $A \in Amal(\mathcal{V})$ if and only if A satisfies property Q if and only if A satisfies property

P. Hence by theorem 4.3.13, the elements of $\mathcal{V}_{ASI}$ can be effectively determined. To prove $Amal(\mathcal{V})$ is finitely axiomatizable, we use the following characterization of $Amal(\mathcal{V})$, which is a reworking of the Property P introduced in chapter four.

LEMMA 5.1.5 (C. Bergman [85]). *Let $\mathcal{V}$ be a finitely generated discriminator variety, and let $A \in \mathcal{V}$. Let $A^{\#}$ and μ_A be as in definition 4.3.1. Then $A \in Amal(\mathcal{V})$ if and only if*

(1) *μ_A is an embedding.*

(2) *For every $M \in \mathcal{V}_{WMI}$, $\Theta \in Con(A)$ and embedding $\eta : A/\Theta \hookrightarrow M$, there exists an $\overline{\Theta} \in Con(A^{\#})$ and an embedding $\overline{\eta} : A^{\#}/\overline{\Theta} \hookrightarrow M$ such that $\overline{\Theta}|_A = \Theta$ and $\overline{\eta} \circ (\mu/\overline{\Theta}) = \eta$, where $\mu/\overline{\Theta} : A/\Theta \hookrightarrow A^{\#}/\overline{\Theta}$ is the induced embedding given by $\mu/\overline{\Theta}(a/\Theta) = \mu(a)/\overline{\Theta}$).*

Proof. $\Longrightarrow$: Let $A \in Amal(\mathcal{V})$. A thus satisfies property P, so μ_A is an embedding. Let $M \in \mathcal{V}_{WMI}$, $\Theta \in \mathrm{Con}(A)$ and $\eta : A/\Theta \hookrightarrow M$ be as above. Then if $e : A \twoheadrightarrow A/\overline{\Theta}$ is the canonical homomorphism, then as A satisfies property P, there is a map $k : A^{\#} \to M$ such that $k \circ \mu_A = \eta \circ e$. Letting $\overline{\Theta} = \ker k$, and composing the isomorphism between $A^{\#}/\overline{\Theta}$ and $k(A^{\#})$ into M to get $\overline{\eta}$, we get $\overline{\Theta}$ and $\overline{\eta}$ satisfying the requirements of (2).

$\Longleftarrow$: Suppose (1) and (2) hold. To show A satisfies property P, let $f : A \to M \in \mathcal{V}_{WMI}$ be a homomorphism. Letting $\Theta = \ker f$, η the composition of the isomorphism between A/Θ and $f(A)$, and the inclusion map $i : f(A) \hookrightarrow M$, and applying (2), we get a map $k = \overline{\eta}e$ (where $e : A^{\#} \hookrightarrow A^{\#}/\Theta$ is the canonical epimorphism) such that $k \circ \mu_A = f$. □

Let $\mathcal{V}$ be a finitely generated discriminator variety, and suppose $K, L \in \mathcal{V}_{SI}$ and $\nu : K \hookrightarrow L$ is an embedding. Let $K = \{k_0, \ldots, k_{r-1}\}$, say. Then, as in the proof of the theorem 5.1.4, we can find formulae Fac_k and $Ext_{L,\nu,K}$ with $|K| + 2$ and $|L| + 2$ free variables such that for $c, d \in A$,

$A \models Fac_K(a_0, \ldots, a_{r-1}, c, d)$ if and only if there is a co-atom Ψ of $\mathrm{Con}(A)$ such that $c/\Psi = d/\Psi$ and $\{a_0/\Psi, \ldots, a_{r-1}/\Psi\} \cong K$, with a_i/Ψ mapping to k_i for each $i < r$.

$A \models Ext_{L,\nu,K}(a_0,\dots,a_{r-1},b_0,\dots,b_{s-1},c,d)$ if and only if there is a co-atom Ψ of Con(A) such that $c/\Psi = d/\Psi$ and $\{a_0/\Psi,\dots, a_{r-1}/\Psi, b_0/\Psi,\dots, b_{s-1}/\Psi\}$ is isomorphic to L with a_i/Ψ corresponding to $\nu(k_j)$ for all $j < r$ and $s = |L \setminus K|$.

THEOREM 5.1.6 (C. Bergman [83]). *Let $\mathcal{V}$ be finitely generated discriminator variety. Then $Amal(\mathcal{V})$ is finitely axiomatizable.*

Proof. Let Σ' be the set of sentences axiomatizing $Ps(\mathcal{V}_{ASI})$, which is obtainable by theorem 5.1.4 and the fact that $\mathcal{V}_{ASI}$ is effectively decidable. Now $A \models \Sigma'$ if and only if μ_A is injective. Let $M \in \mathcal{V}_{WMI}$, $\Theta \in$ Con(A) and $\eta : A/\Theta \hookrightarrow M$ an embedding. We need a sentence that is equivalent to the existence of $\overline{\eta}$ and $\overline{\Theta}$ as in lemma 5.1.5. Now as $\mathcal{V}$ is finitely generated, there are only finitely many pairs $\langle L_i, \nu_i\rangle$ such that $L_i \in \mathcal{V}_{ASI}$, ν_i is an embedding from $K = A/\Theta$ into L_i and there exists $\tau : L_i \to M$ such that $\tau \circ \nu_i = \eta$.

Let $P_{K,\eta}$ be the sentence

$$(\forall \overrightarrow{x}) \bigvee_{i=0}^{m-1} [Fac_K(\overrightarrow{x},u,v) \implies (\exists \overrightarrow{y})Ext_{L_i,\nu_i,K}(\overrightarrow{x}, \overrightarrow{y}, u, v)]$$

We claim that $A \models \Sigma' \cup \{P_{k,\eta}\}$ if and only if $\overline{\Theta}$ and $\overline{\eta}$ exists with the desired properties:

Proof of claim: $\Longrightarrow$: Let $a_0,\dots,a_{r-1}$ be co-set representatives for A by Θ. Since $K = A/\Theta$, for any $(c,d) \in \Theta$ we have $A \models Fac_K(\overrightarrow{a}, c, d)$. Thus by our assumption, there is an $i < m$, such that $A \models (\exists y)Ext_{L_i,\nu_i,K}(\overrightarrow{a}, \overrightarrow{y}, c, d)$.

Let $T = \{\Psi \in \text{Con}(A) : A/\Psi \in \mathcal{V}_{ASI}\}$ and

$U = \{\Psi \in T :$ the map $f : A/\Psi \to L_i$ given by $f(a_j/\Psi) = \nu_i(k_j)$ is an isomorphism$\}$.

Let $V(c,d) = \{\Psi \in T : (c,d) \in \Psi\}$, $(c,d) \in \Theta$.

Consider the family $F = \{U\} \cup \{V(c,d) : (c,d) \in \Theta\}$. We claim F has the finite intersection property: Consider $U \cap V(c_1,d_1) \cap \dots \cap V(c_p,d_p)$, where $(c_q,d_q) \in \Theta$ for $i \le q \le p$. Now $\bigvee_{q=1}^{q=p} \Theta(c_q,d_q)$ is principal by lemma 5.1.2(5), say $\bigvee_{q=1}^{q=p} \Theta(c_q,d_q) = \Theta(c,d)$ for $c,d \in A$. Now since $\Theta(c_q,d_q) \le \Theta$ for $1 \le q \le p$, $\Theta(c,d) \le \Theta$ so $(c,d) \in \Theta$. By our assumption, $A \models (\exists \overrightarrow{y})Ext_{L_i,\nu_i,K}(\overrightarrow{a}, \overrightarrow{y}, c, d)$. Hence there is a $\Psi \in U$ such that $\Psi \geq \Theta(c,d) \geq \Theta(c_q,d_q)$ for $1 \leq q \leq p$, that is, $\Psi \in U \cap V(c_1,d_1) \cap \ldots \cap V(c_p,d_p)$, so F has the finite intersection property.

Thus F can be extended to an ultrafilter $\mathfrak{F}$, say, over T. Let $\overline{\Theta}$ be the congruence induced by the ultrafilter. Since every $V(c,d) \in \mathfrak{F}$, $\overline{\Theta}|_A \subseteq \Theta$. Since $U \in \mathfrak{F}$, we have $\overline{\Theta}|_A \supseteq \Theta$ as well as an isomorphism $f : A^\#/\overline{\Theta} \to L_i$ such that $f \circ \mu/\overline{\Theta} = \nu_i$. Letting $\overline{\eta} = \tau \circ f$, we see $\overline{\eta} \circ (\mu/\overline{\Theta}) = \eta$.

$\Longleftarrow$: Suppose $\overline{\Theta}$ and $\overline{\eta}$ exist with the properties stated in the claim.

$$\begin{tikzcd}
A \arrow[r, hook, "\mu"] \arrow[d] & A^\# \arrow[r, two heads] \arrow[d] & A^\#/\mathfrak{U} \arrow[dl, "e"] \\
A/\Theta = K \arrow[r, hook, "\mu/\overline{\Theta}"] \arrow[d, hook, "\eta"'] & A^\#/\overline{\Theta} \arrow[dl, "\overline{\eta}"] & \\
M & &
\end{tikzcd}$$

FIGURE 5.1

By theorem 1.2.16, there is an ultrafilter $\mathfrak{U}$ such that $e : A^\#/\mathfrak{U} \twoheadrightarrow A^\#/\overline{\Theta}$ is an epimorphism. Now $A^\#/\mathfrak{U} \in \mathcal{V}_{ASI}$ as $\mathcal{V}_{ASI}$ is a finite collection of finite algebras, hence is simple. Hence, e is an isomorphism, so $A^\#/\Theta \cong L_i$ for some $i < m$, hence $P_{K,\eta}$ is satisfied. (see figure 5.1).

Finally, to complete the proof, observe there are only finitely many pairs (K,η) such that η is an embedding into a member of $\mathcal{V}_{WMI}$. Hence if P is the sentence $\bigwedge P_{K,\eta}$, the conjunction of the pairs defined previously, then $\Sigma' \cup \{P\}$ axiomatizes *Amal*$(\mathcal{V})$. $\square$

5.2 Finitely Generated Modular Lattice Varieties

The Bounded Obstruction Property. In Albert and Burris [88], an algebraic property for a class of algebras called the *bounded obstruction property* was introduced, and there it was shown that for certain elementary classes this property is equivalent to the statement that the amalgamation class of the variety is elementary. My aim in this section is to state this result (the proof, being largely model-theoretical, is omitted), and to present in detail a result due to C. Bergman [89] that the amalgamation class of certain modular varieties of lattices are not elementary.

DEFINITION 5.2.1 .

(1) Let $\mathcal{K}$ be an elementary class, and suppose that the double extension (A, f, B, g, C) has no amalgam in $\mathcal{K}$. An *obstruction* is any subalgebra C' of C such that $(A', f|_{A'}, B, g|_{A'}, C')$ has no amalgam in $\mathcal{K}$, where $A' = g^{-1}(C')$.

(2) Let $\mathcal{K}$ be a locally finite elementary class. *Amal*$(\mathcal{K})$ is said to have the *bounded obstruction property (BOP)* with respect to $\mathcal{K}$ if for every $k \in \omega$, there exists an $n \in \omega$ such that the following holds:

If $C \in Amal(\mathcal{K})$, $|B| < k$ and the double extension (A, f, B, g, C) has no amalgam in $\mathcal{K}$, then there is an obstruction $C' \leq C$ such that $|C'| < n$.

THEOREM 5.2.2 (Albert and Burris [88]). *Let $\mathcal{V}$ be a locally finite variety. Then Amal$(\mathcal{K})$ satisfies the bounded obstruction property if and only if Amal$(\mathcal{K})$ is elementary.*

Non-Axiomatizibility of the Amalgamation Class of Certain Modular Varieties.

In chapter four, we saw that the amalgamation class of finitely generated modular lattice varieties also have a characterization in terms of property P. However, in contrast to the result in section 5.1 for discriminator varieties, their amalgamation classes are not elementary, as the next result shows.

THEOREM 5.2.3 (C. Bergman [89]. *Let $\mathcal{V}$ be a finitely generated non-distributive variety of modular lattices. Then Amal$(\mathcal{V})$ does not have the bounded obstruction property, and hence is not elementary.* □

Proof. We may assume $\mathcal{V} = HSP(L)$, where L is a finite non-distributive modular lattice. By theorem 1.2.16, $\mathcal{V}_{SI} \subseteq HS(L)$. Thus every member of $\mathcal{V}_{SI}$ has cardinality at most $|L|$ and is simple. Let M be a lattice of largest cardinality in $\mathcal{V}_{SI}$. By maximality, $M \in \mathcal{V}_{WMI}$, and as L is non-distributive $|M| \geq 5$. Let z and u be the bottom and top elements of M respectively. Let $C = \{x[n] : x \in M \text{ and } n \in \omega\}$ where $x[n] \in M^{\omega}$ is defined by

$$x[n]_j = \begin{cases} u & \text{for } j < n \\ x & \text{for } j = n \\ z & \text{for } j > n \end{cases} \qquad \text{for } j < \omega.$$

Now C is a subset of M^{ω}, in fact, it is a sublattice of M^{ω}: For let $x[n], y[m] \in C$. Then if $n > m$, $x[n] + y[m] = x[n] \in C$, and if $n = m$, $x[n] + y[m] = (x + y)[n] \in C$. Similarly, one can show that C is closed under meets.

Note that $z[n+1] = u[n]$ for all $n \in \omega$, and that $x[n] < y[m]$ if and only if $(n < m)$ or $(n = m$ and $x < y)$. Thus C resembles countably infinitely many copies of M, stacked up so that the top element of one copy is identified with the bottom element of the copy of M immediately above it. (see figure 5.2 for the case $M = M_3$.)

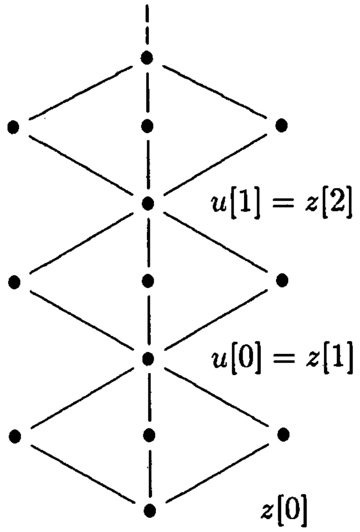


FIGURE 5.2

In fact C is a subdirect power of M, for if $\pi_j : M^{\omega} \twoheadrightarrow M$ is the jth projection, then $\pi_j(x[j]) = x$ for all $x \in M$.

CLAIM 1: Let $\alpha \in \text{Con}(C), \alpha \neq 1_C$, such that $C/\alpha \in \mathcal{V}_{SI}$ (that is, α is completely meet irreducible). Then there is a $j \in \omega$ such that for all $x, y \in C$, $x \equiv y$ mod α if and only if $x_j = y_j$. In particular, $C/\alpha \cong M$. Hence every homomorphic image of C is a subdirect power of M.

Proof of claim 1: $C \leq M^\omega$. Hence as α is completely meet irreducible, by theorem 1.2.16 there is an ultrafilter $\mathfrak{U}$ on ω such that $\Theta|_C \subseteq \alpha$, where Θ is the congruence induced by $\mathfrak{U}$.

Suppose $\mathfrak{U}$ were not principal: Then, for every $x[n] \in C$,

$$\{j \in \omega : x[n]_j = z = z[0]_j\} \supseteq \omega \setminus \{0, 1, \ldots, n\} \in \mathfrak{U}.$$

Hence $x[n] \; \Theta \; z[0] \implies x[n] \; \alpha \; z[0]$. So $\alpha = \mathbf{1}_C$, contradicting our assupmtion.

Hence $\mathfrak{U}$ must be principal. But then, as $C \leq M^\omega$ is a subdirect embedding, $C/\Theta \cong M$ (the jth component). Now since $\Theta|_C \leq \alpha$, C/α is a non-trivial homomorphic image of $C/\Theta \cong M$, and as M is simple, $C/\alpha \cong M$, that is, $\alpha = \Theta|_C$. Hence $x \; \alpha \; y$ if and only if $x_j = y_j$.

Now if $\beta \in \mathrm{Con}(C)$, by theorem 1.2.4, $\beta = \bigcap_{i \in I} \alpha_i$, where each α_i is a completely meet irreducible congruence on C. Then for all $i \in I$, $C/\alpha_i \cong M$, hence we have a subdirect repesentation $C/\beta \leq \prod\limits_{i \in I} C/\alpha_i \cong M^I$.

CLAIM 2: $C \in Amal(\mathcal{V})$

Proof of claim 2: We will show C has property Q. Let $C \leq D$ with inclusion map $i : C \hookrightarrow D$, let $N \in \mathcal{V}_{WMI}$ and let $h : C \to N$ be a homomorphism. Let $\alpha = \ker h$. By claim 1, C/α is a subdirect product of M. Hence $|C/\alpha| \geq M$. But $|N| \geq |C/\alpha|$, hence by maximality $|N| = |C/\alpha| = |M|$, and so, $N \cong C/\alpha \cong M$. Now by theorem 1.2.4, D is a subdirect product of subdirectly irreducible lattices, say $D \subseteq \prod\limits_{i \in I} D_i$. By theorem 1.2.16, there is an ultrafilter $\mathfrak{U}$ on I such that the conguence Θ induced by $\mathfrak{U}$ satisfies $\Theta|_C \subseteq \alpha$.

$$\begin{tikzcd}
C \arrow[r, hook, "i"] \arrow[d] & D \arrow[r, hook] \arrow[d] & \prod\limits_{i \in I} D_i \arrow[d] \\
C/\,\Theta|_C \arrow[r, hook] \arrow[d] & D/\,\Theta|_D \arrow[r, hook] & \prod_{i \in I} D_i/\mathfrak{U} \\
C/\alpha \cong M & &
\end{tikzcd}$$

FIGURE 5.3

Now since $\mathcal{V}_{SI}$ is a finite set of finite lattices, $\prod_{i\in I} D_i/\mathfrak{U} \in \mathcal{V}_{SI}$. Hence, $|M| \leq |C/\,\Theta|_C| \leq |\prod_{i\in I} D_i/\mathfrak{U}|$ implies by maximality of M that $M \cong C/\,\Theta|_C \cong \prod_{i\in I} D_i/\mathfrak{U} \cong C/\alpha$. Thus $\Theta|_C = \alpha$. The desired extension of h to D can now be constructed from the composition of the embedding from D into $\prod\limits_{i\in I} D_i$, the canonical map from $\prod\limits_{i\in I} D_i$ to $\prod_{i\in I} D_i/\mathfrak{U}$ and the isomorphism from $\prod_{i\in I} D_i/\mathfrak{U}$ to C/α (see figure 5.3). Thus claim 2 is proved.

Recall that u and z are the top and bottom elements of M respectively. Let a be an atom of M and let $f : \underline{2} \hookrightarrow M \times \underline{2}$ be the embedding given by $f(0) = (z, 0), f(1) = (a, 1)$. Finally for each $n \in \omega$ let $g_n : \underline{2} \hookrightarrow C$ be the embedding given by $g_n(0) = z[0]$ and $g_n(1) = u[n]$.

CLAIM 3: For every $n \in \omega$, the double extension $(\underline{2}, f, M \times \underline{2}, g_n, C)$ cannot be amalgamated.

Proof of claim 3: Suppose, for a contradiction, that $(\underline{2}, f, M \times \underline{2}, g_n, C)$ has an amalgam (f', g_n', D). By theorem 1.2.4, there is a $D \in \mathcal{V}_{SI}$ and an epimorphism $p : D \twoheadrightarrow D'$ such that $pf'(z, 0) \neq pf'(a, 0)$. Let $\alpha = \ker(pf') \in \mathrm{Con}(M \times \underline{2})$. Now $\mathrm{Con}(M \times \underline{2}) \cong \mathrm{Con}(M) \times \mathrm{Con}(2) \cong \underline{2} \times \underline{2}$. Hence α is one of $\mathbf{1}_{M\times\underline{2}}$, $\mathbf{0}_{M\times\underline{2}}$, η_M or $\eta_{\underline{2}}$, where η_M and $\eta_{\underline{2}}$ are kernels of the projections onto M and $\underline{2}$ respectively. Now $\mathbf{1}_{M\times\underline{2}} \neq \alpha \neq \eta_{\underline{2}}$ since $(z, 0) \not\equiv (a, 0) \mod \alpha$. Also, $\alpha \neq \mathbf{0}_{M\times\underline{2}}$, for then pf is an embedding, hence $|M| < |M \times \underline{2}| \leq |D'|$, contradicting maximality of M. Thus $\alpha = \eta_M$, so $M \times \underline{2}/\alpha \cong M$, and again by maximality of M, $|M| = |D'|$, hence $D' \cong M$. In particular, since a is an atom of M, $pf'(a, 1)$ is an atom of D'. Note that

$$(*) \qquad \begin{aligned} pg_n'g_n(0) &= pf'f(0) \\ &= pf'(z,0) \\ &\neq pf'(a,1) \\ &= pf'f(1) \\ &= pg_n'g_n(1). \end{aligned}$$

Thus $\ker(pg_n')$ is non-trivial. Hence $C/\ker(pg_n')$ is a subdirect power of M by claim 1, and hence by maximality of M, $C/\ker(pg_n') \cong D' \cong M$. In particular, by claim 1, there is a $j \in \omega$ such that $pg_n'(x) = pg_n'(y)$ if and only if $x_j = y_j$. If $j > n$, then $z[0]_j = u[n]_j$, hence $pg_n'g_n(0) = pg_n'(z[0]) = pg_n'(u[n]) = pg_n'g_n(1)$, contradicting (*). Thus $j \leq n$. Now $u[n]_j = u$ is the largest element of M, hence $pg_n{}'g_n(1) = pg_n{}'(u[n])$ will be the largest element of D'. But by (*), $pg_n{}'g_n(1) = pf'(a, 1)$ is an atom of D', contradicting $|M| > 2$. Thus the claim is proved.

Lastly we will show that every obstruction to $(\underline{2}, f, M \times \underline{2}, g_n, C)$ has size greater than n. Hence *Amal*($\mathcal{V}$) fails to satisfy the bounded obstruction property, hence by theorem 5.2.2, *Amal*($\mathcal{V}$) is not elementary.

CLAIM 4: Let A be an obstruction to $(\underline{2}, f, M \times \underline{2}, g_n, C)$, that is, $A \leq C$. Then $|A| > n$.

Proof of claim 4: Let A be an obstruction to $(\underline{2}, f, M \times \underline{2}, g_n, C)$. Then if $\underline{2}' = g_n^{-1}(A)$, we have that $(\underline{2}', f|_{\underline{2}'}, M \times \underline{2}, g_n|_{\underline{2}'}, A)$ cannot be amalgamated. Hence $\underline{2}'$ is non-trivial, that is $\underline{2}' = \underline{2}$. Now $f : \underline{2} \hookrightarrow M \times \underline{2}$ has as a retract the projection map $\pi : M \times \underline{2} \twoheadrightarrow \underline{2}$. Thus $g_n : \underline{2} \hookrightarrow A$ has no retraction $r : A \twoheadrightarrow \underline{2}$, for then $(\underline{2}, f, M \times \underline{2}, g_n, A)$ has an amalgam $(f', g_n{}', A \times (M \times \underline{2}))$ where $f'(x) = (g_n\pi(x), x)$ and $g_n{}'(x) = (x, fr(x))$. (Because for $x \in \underline{2}$, $f'f(x) = (g_n(x), f(x)) = g_n{}'g_n(x)$).

Now $g_n(\underline{2}) = \{z[0], u[n]\} \subseteq A$. We show $|A| > n$ by showing that for all $k \leq n$, $x[k] \in A$ for some $x \in M$, $z \neq x \neq u$: Suppose there was a $k \leq n$ with $x[k] \notin A$ for all $x \in M$, $z \neq x \neq u$. Put $r : A \twoheadrightarrow \underline{2}$ by

$$r(x) = \begin{cases} 0 & \text{if and only if } x \leq z[k] \\ 1 & \text{otherwise.} \end{cases}$$

Note $r(z[0]) = 0$ and $r(u[n]) = 1$. Also r is a homomorphism: One can check that r preserves joins, and if one of $x, y \in A$ is less than or equal to $z[k]$, we have $r(x \cdot y) = 0 = r(x) \cdot r(y)$. Hence we need consider only the case $a, b \in A$ such that $a, b \not\leq z[k]$. Now by our assumption, $a \neq x_1[k], b \neq x_2[k]$ for some $x_1, x_2 \in M \setminus \{z, u\}$. Hence $a, b \geq u[k]$, so $a \cdot b \geq u[k]$ implies $a \cdot b \not\leq z[k]$, so $r(a) \cdot r(b) = 1 = r(a \cdot b)$. So r preserves meets.

But then r is a retraction of g_n, contradicting the observation above. Thus $|A| > n$, proving our claim and hence the theorem. □

5.3 The Lattice Variety Generated by the Pentagon

Introduction. In this section I present a result published in Bruyns, Naturman and Rose [92] that the amalgamation class of the lattice variety $\mathcal{N}$ generated by the pentagon is elementary (and in fact has a theory consisting of Horn sentences). The proof relies on an alternative characterization to the one in chapter four. It uses two key concepts: that of the extension of $\underline{2}$-congruences and the fact that amalgamation bases do not have the three-element chain as a homomorphic image. I present these two concepts in the subsections that follow. I also draw in advance the reader's attention to the remarks following lemma 5.3.13.

$\underline{2}$-Congruence Extendibility. The main result of this subsection is lemma 5.3.5, which says that $\underline{2}$-congruences on amalgamation bases can be extended to any of their extensions.

DEFINITION 5.3.1.

(1) Given a lattice A, we call $\Theta \in \mathrm{Con}(A)$ a *$\underline{2}$-congruence on A* if A/Θ is isomorphic to $\underline{2}$, the two-element chain.

(2) Let u/v be a quotient of a lattice L. We call u/v a *N- quotient of L* if for some $z \in L$ the set $\{u, v, z\}$ generates a sub-lattice of L isomorphic to the pentagon N, with u/v as critical quotient. In this case we write $N(u/v, z)$.

(3) Let $A = \prod_{i \in I} A_i$. A subalgebra B of A is said to be *regular* if, for every two kernels Θ_i and Θ_j of distinct projections on A, we have $\Theta_i|_B \neq \Theta_j|_B$.

The next sequence of lemmas have mostly technical proofs, some of which have been omitted. For the omitted proofs, I refer the reader to Bruyns, Naturman and Rose [92].

LEMMA 5.3.2 (Bruyns, Naturman and Rose [92]). *Let $\mathcal{K}$ be the class of subdirect powers of the pentagon N. Then $\mathcal{K}$ is closed under reduced products.* ☐

LEMMA 5.3.3 (Bruyns, Naturman and Rose [92]). *Let $B \in \mathcal{N}$ be an image of $A \in Amal(\mathcal{N})$. Then if $B \leq N^I$ is a regular subdirect embedding, then every $\underline{2}$-congruence on B can be extended to a $\underline{2}$-congruence on N^I.* ☐

LEMMA 5.3.4 (Bruyns, Naturman and Rose [92]). *Let $f : A \hookrightarrow B \in \mathcal{N}$ be an embedding and assume that every $\underline{2}$-congruence on $f(A)$ can be extended to a $\underline{2}$-congruence on B. Then if C is a finite Boolean algebra and $g : A \twoheadrightarrow C$ is an epimorphism, there is an epimorphism $h : B \twoheadrightarrow C$ such that $g = hf$.*

Proof. Follows from the fact that every finite Boolean algebra is isomorphic to a product of two element chains. □

LEMMA 5.3.5 (Bruyns, Naturman and Rose [92]).
Let $A \in Amal(\mathcal{N})$ and let $A \leq B \in \mathcal{N}$. Then every $\underline{2}$-congruence on A can be extended to a $\underline{2}$-congruence on B.

Proof. Let $e : A \twoheadrightarrow \underline{2}$ be an epimorphism where $A \in Amal(\mathcal{N})$. Let $f : \underline{2} \hookrightarrow N$ be the embedding taking top and bottom elements of $\underline{2}$ to top and bottom elements of N respectively. Notice f has a retract $g : N \twoheadrightarrow \underline{2}$. Now if $A \leq B \in \mathcal{N}$, since A satisfies property Q, we have a homomorphism $h : B \rightarrow N$ such that $h|_A = fe$. But then $\ker gh$ is a $\underline{2}$-congruence on B extending $\ker e$. □

$\underline{3}$ Is Not an Image of an Amalgamation Base. The main result in this subsection is that amalgamation bases in $\mathcal{N}$ do not have an epimorphism onto the three element chain. The next lemma is implicit in Jónsson [90].

LEMMA 5.3.6 (Jónsson [90]). *Let I be a set. Then the three-element chain $\underline{3}$ is not an epimorphic image of N^I.*

Proof. Let $\underline{3} = \{0, a, 1\}$, where $0 \prec a \prec 1$. Suppose $e : N^I \twoheadrightarrow \underline{3}$ is an epimorphism. Let $\Psi_1, \Psi_2 \in \mathrm{Con}(\underline{3})$ be $\underline{2}$-congruences such that $(0, a) \in \Psi_1$ and $(a, 1) \in \Psi_2$. We can view Ψ_1 and Ψ_2 as congruence on N^I. By theorem 1.2.16, there are ultrafilters $\mathfrak{U}_1$ and $\mathfrak{U}_2$ on I such that the congruences Θ_1 and Θ_2 induced by $\mathfrak{U}_1$ and $\mathfrak{U}_2$ respectively have the property $\Theta_i \subseteq \Psi_i$ for $i = 1, 2$. Note, for $i = 1, 2$, $N^I / \Theta_i \cong N$ since N is finite. Now $\mathfrak{U}_1 \neq \mathfrak{U}_2$: For if $\mathfrak{U}_1 = \mathfrak{U}_2$, then $\Theta_1 = \Theta_2 \subseteq \Psi_1 \cap \Psi_2 = \ker e$, hence $\underline{3}$ would be a homomorphic image of N, a contradiction.

Hence, there is an $F \in \mathfrak{U}_1$ such that $F \notin \mathfrak{U}_2$. Pick $x = (x_i)_{i \in I}, y = (y_i)_{i \in I} \in N^I$ such that $e(x) = 1, e(y) = 0$. Put $z = (z_i)_{i \in I} \in N^I$ by

$$z_i = \begin{cases} x_i & \text{if } i \in F \\ y_i & \text{if } i \notin F. \end{cases}$$

Now $x \ \Theta_1 \ z$ and $z \ \Theta_2 \ y$, so viewing Ψ_1 and Ψ_2 as congruences on $\underline{3}$ again, we have $1 \ \Psi_1 \ e(z) \ \Psi_2 0$, a contradiction since $(1,0) \notin \Psi_1 \circ \Psi_2$. Hence $\underline{3}$ is not an image of N^I. □

LEMMA 5.3.7 (Bruyns, Naturman and Rose [92]). *Let N^I be a direct power of the pentagon, and let $f : A \hookrightarrow N^I$ be an embedding. Assume every $\underline{2}$-congruence on $f(A)$ can be extended to a $\underline{2}$-congruence on N^I. If $g : A \rightarrow N$ is a homomorphism such that $g(A) \cong \underline{3}$ and the critical quotient is a sublattice of $g(A)$ then there is no homomorphism $h : N^I \rightarrow N$ such that $g = hf$.*

Proof. Suppose such h exists. We first show h is not onto: For if h is onto, then by the projectivity of N, h induces an N-quotient $u/v \in N^I$. In fact since the critical quotient is a sub-lattice of $g(A)$, we may suppose $u, v \in f(A)$. Let Ψ and Ψ_A be the largest distributive congruence on N^I and A respectively.

$$\begin{aligned}\text{Let } X &= \{\phi \in \mathrm{Con}(N^I) : \phi \text{ is a } \underline{2}\text{-congruence on } N^I\} \\ \text{and } X_A &= \{\phi \in X : \phi|_A \text{ is a } \underline{2}\text{-congruence on } A\}\end{aligned}$$

Note $\bigcap X = \Psi$, and since every $\underline{2}$-congruence on $f(A)$ extends to a $\underline{2}$-congruence on N^I, $\Psi_A = (\bigcap X_A)|_A$. But $\bigcap X_A \supseteq \cap X$, hence $\Psi_A \supseteq \Psi|_A$. Since $\ker g$ is a distributive congruence, we have $\Psi|_A \subseteq \ker g$. But then as $(u,v) \in \Psi$, $(f^{-1}(u), \ f^{-1}(v)) \in \mathit{ker} g$, a contradiction.

Hence, since h is not onto, since $g(A)$ contains the critical quotient of N we must have that $h(N^I)$ is either the 4 or 3 element chain. But this contradicts lemma 5.3.6, hence no such h exists. □

COROLLARY 5.3.8 (Bruyns, Naturman and Rose [92]). *If $A \in Amal(\mathcal{N})$, then A does not have $\underline{3}$ as an homomorphic image.* □

Amalgamating Subdirect Product Representations. Lastly, before the main characterization theorems, I present some technical lemmas about amalgamating double extensions whose embeddings are regular subdirect representations.

LEMMA 5.3.9 (Bruyns, Naturman and Rose [92]). *Let $A \leq B \in \mathcal{N}$. Then every epimorphism $g : A \twoheadrightarrow N$ can be extended to an epimorphism $h : B \twoheadrightarrow N$.*

Proof. Since every $B \in \mathcal{N}$ is embeddable in a direct power of N, we may assume that $B = N^I$ for some set I. Let g be as above. By theorem1.2.16, there is a congruence $\Psi \in \mathrm{Con}(B)$ induced by an ultrafilter on I such that $\Psi|_A \subseteq \ker g$. Now, since N is finite, $B/\Psi \cong N$. But then $\Psi|_A = \ker g$, for if not, N would be the image of a proper subalgebra of N, a contradiction. □

COROLLARY 5.3.10 (Bruyns, Naturman and Rose [92]). *Let $A \in \mathcal{N}$, and let B and C be direct powers of N, say $B = N^I$ and $C = N^J$. If $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$ are subdirect product representations, the the double extension (A, f, B, g, C) has an amalgam in $\mathcal{N}$.*

Proof. We use lemma 1.3.3(2). Let $u, v \in B$ such that $u \neq v$. Now $\pi(u) \neq \pi(v)$ for some projection π on B. Now $\pi f : A \to N$ is an epimorphism, hence by lemma 5.3.9 there is a homomorphism $h : C \to N$ such that $hg = \pi f$. The case for $u, v \in C$ follows by symmetry. □

LEMMA 5.3.11 (Bruyns, Naturman and Rose [92]). *Let A be a subdirect power of N. Then $A \in Amal(\mathcal{N})$ if and only if for any regular subdirect product representation $f : A \hookrightarrow B$ and any homomorphism $k : A \to N$ there is a homomorphism $h : B \to N$ such that $k = hf$.*

Proof. $\Longrightarrow$: Follows since $\mathcal{N}$ is a Q-variety.

$\Longleftarrow$: Let B, C be powers of N, and suppose $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$ are embeddings. Since every member of $\mathcal{N}$ is embeddable in some power of N, we need only show (A, f, B, g, C) has an amalgam to show $A \in Amal(\mathcal{N})$. Now since A is a subdirect power of N, the embeddings f and g induce a regular subdirect representation $f' : A \hookrightarrow B' \leq B$ and $g' : A \hookrightarrow C' \leq C$. Now by corollary 5.3.10, (A, f', B', g', C') has an amalgam (p, g, H), say. By theorem 1.3.11, we may assume $H \in Amal(\mathcal{N})$.

We need only show (A, f, B, f', B') and (A, g, C, g', C') have amalgams, say, (ℓ_1, r, D) and (k_1, s, E) respectively: For then, since B' and C' are powers of N and hence in $Amal(\mathcal{N})$, we have (B', r, D, p, H) and (C', g, H, s, E) have respective amalgams (ℓ_2, t, F) and (u, k_2, G), say. Since $h \in Amal(\mathcal{N})$, (H, t, F, u, G) has an amalgam, say (ℓ_3, k_3, X). Hence we obtain the commutative diagram given in figure 5.4, and a check shows that $(\ell_3\ell_2\ell_1, k_3k_2k_1, X)$ is an amalgam of (A, f, B, g, C).

$$\begin{tikzcd}
B \arrow[rr, hook, "\ell_1"] & & D \arrow[dr, hook, "\ell_2"] & & \\
& B' \arrow[ur, hook, "r"] \arrow[dr, hook, "p"] & & F \arrow[dr, hook, "\ell_3"] & \\
A \arrow[uu, hook, "f"] \arrow[ur, hook, "f'"] \arrow[dr, hook, "g'"] \arrow[dd, hook, "g"'] & & H \arrow[ur, hook, "t"] \arrow[dr, hook, "u"] & & X \\
& C' \arrow[ur, hook, "q"] \arrow[dr, hook, "s"] & & G \arrow[ur, hook, "k_3"] & \\
C \arrow[rr, hook, "k_1"'] & & E \arrow[ur, hook, "k_2"] & &
\end{tikzcd}$$

FIGURE 5.4

To see that (A, f, B, f', B') has an amalgam, let $B = N^I$ and $B' = N^J$ where $J \subseteq I$. We use lemma 1.3.3(2). Let $z \neq w \in B$ such that $\pi(z) \neq \pi(w)$ for some projection π on B. Then $\pi f : A \to N$ is a homomorphism, so by the hypothesis there is a homomorphism $h : B' \to N$ such that $hf' = \pi f$, satisfying the conditions of the lemma. Let $z \neq w$ in B', and let π be a projection on B' such that $\pi(z) \neq \pi(w)$. Since f' is a subdirect representation, $\pi f' : A \to N$ is an epimorphism, hence by lemma 5.3.9, there is an epimorphism $h : B \twoheadrightarrow N$ such that $hf = \pi f'$. Thus the conditions of lemma 1.3.3(2) are satisfied. The case for (A, g, C, g', C') proceeds similarly. ☐

Some Characterizations of the Amalgamation Class. I now state the characterization theorem of Bruyns, Naturman and Rose [92], as well as a technical result which gives a sufficient condition for a member of $\mathcal{N}$ to be an amalgamation base.

THEOREM 5.3.12 (Bruyns, Naturman and Rose [92]). *For $A \in \mathcal{N}$, the following are equivalent:*

(1) $A \in Amal(\mathcal{N})$.

(2) *$\underline{3}$ is not a homomorphic image of A and, if $A \leq B \in \mathcal{N}$, then every $\underline{2}$-congruence on A can be extended to a $\underline{2}$-congruence on B.*

(3) *A is a subdirect power of of N and for any regular subdirect product representation $f : A \hookrightarrow N^I$ and any homomorphism $g : A \rightarrow N$ there is a homomorphism $h : N^I \rightarrow N$ such that $g = hf$.*

(4) *A is a subdirect power of N, $\underline{3}$ is not an image of A and if $f : A \hookrightarrow N^I$ is a regular subdirect representation, then any $\underline{2}$-congruence on $f(A)$ can be extended to a $\underline{2}$-congruence on N^I.*

Proof. (1) $\Longrightarrow$ (2): Follows from lemma 5.3.5 and corollary 5.3.8.

(2) $\Longrightarrow$ (1): If A is as in (2), we show A has property Q: Let $f : A \hookrightarrow B \in \mathcal{N}$ be an embedding and let $g : A \rightarrow N$ be a homomorphism. Then $g(A)$ is either a singleton or isomorphic to one of $\underline{2}$, $\underline{2} \times \underline{2}$ or N. If $g(A)$ is a singleton, set $h : B \rightarrow N$ such that $h(B) = g(A)$, so that $hf = g$. If $g(A) \cong \underline{2}$ or $\underline{2} \times \underline{2}$, then the existence of h such that $hf = g$ follows from lemma 5.3.4. If $g(A) \equiv N$ then the existence of such h follows from lemma 5.3.9. In each case, we have shown that A has property Q, hence is in $Amal(\mathcal{N})$.

(1) $\Longrightarrow$ (3): This is just a restatement of one direction of theorem 4.3.12.

(3) $\Longrightarrow$ (4): Follows from lemma 5.3.7. (That $\underline{2}$-congruences are extendible follows from a similar argument to that in lemma 5.3.5).

(4) $\Longrightarrow$ (1): By lemma 5.3.11 we need only show that if $g : A \rightarrow N$ is a homomorphism then there is a homomorphism $h : N^I \rightarrow N$ such that $hf = g$. By assumption, the non-trivial images of A are isomorphic to N, $\underline{2}$ or $\underline{2} \times \underline{2}$, so the result follows using arguments similar to those in the proof of (2) $\Longrightarrow$ (1). ☐

LEMMA 5.3.13 (Bruyns, Naturman and Rose [92]). *Let Θ_0, Θ_1 be $\underline{2}$-congruences on $B \in \mathcal{N}$. Assume the following: For some $A \in Amal(\mathcal{N})$ and embeddings $f_0, f_1 : A \hookrightarrow B$, the restrictions $\Theta_0|_{f_0(A)}$ and $\Theta_1|_{f_1(A)}$ are distinct $\underline{2}$-congruences on A. Then if $\phi = \Theta_0 \cap \Theta_1$, B/ϕ is not isomorphic to the three-element chain.*

Proof. Suppose $B/\phi \cong \underline{3}$. Then as $\underline{3}$ is projective we may suppose without loss of generality that there exist $u, v, w \in B$ such that $u < v < w$ and $B/\Theta_0 =$

$\{u/\Theta_0, v/\Theta_0\}$, $B/\Theta_1 = \{v/\Theta_1, w/\Theta_1\}$ and $B/\phi = \{u/\phi, v/\phi, w/\phi\}$. We may further suppose that $\{u,v\} \subseteq f_0(A)$ and $\{v,w\} \subseteq f_1(A)$. Let (g_0, g_1, D) be an amalgam of (A, f_0, B, f_1, B). By theorem 5.3.12(2), Θ_0 and Θ_1 can be extended to $\underline{2}$-congruences $\overline{\Theta_0}$ and $\overline{\Theta_1}$ on D. Let $\overline{\phi} = \overline{\Theta_0} \cap \overline{\Theta_1}$. Then $f_0(A)/\,\overline{\phi}|_{f_0(A)}$ is isomorphic to $\underline{3}$, contradicting theorem 5.3.12(2). □

The proof above given for lemma 5.3.13 is given in Bruyns, Naturman and Rose [92]. There are two statements in this proof which I have not been able to justify adequately. The first is that we may suppose that $\{u,v\} \subseteq f_0(A)$ and $\{v,w\} \subseteq f_1(A)$. No indication is given that $f_0(A) \cap f_1(B) \neq \emptyset$. The second statement is that using theorem 5.3.12(2), Θ_0 and Θ_1 can be extended to $\underline{2}$-congruences $\overline{\Theta_0}$ and $\overline{\Theta_1}$ on D. Theorem 5.3.12(2) only applies to extending congruences on members of the amalgamation class, and B is not a member of *Amal*($\mathcal{N}$). A similar assumption seems to be made in the first line of lemma 5.3.14 as well. I have not been able to re-prove these lemmas, or prove theorem 5.3.15 which depends on these lemmas.[1]

LEMMA 5.3.14 (Bruyns, Naturman and Rose [92]). *Let* $B \in \mathcal{N}$ *and assume that, for any distinct* $\underline{2}$*-congruence on* Θ_0 *and* Θ_1 *on* B *there is an* $A \in$ *Amal*($\mathcal{N}$) *and embeddings* $f_0, f_1 : A \hookrightarrow B$ *such that* $\Theta_0|_{f_0(A)}$ *and* $\Theta_1|_{f_1(A)}$ *are distinct* $\underline{2}$*-congruences on* A*. Then* $B \in$ *Amal*($\mathcal{N}$).

Proof. It follows from the given condition that if $B \leq C \in \mathcal{N}$, then any $\underline{2}$-congruence on B can be extended to a $\underline{2}$-congruence on C. Also by lemma 5.3.13, B does not have $\underline{3}$ as a homomorphic image. The result follows from theorem 5.3.12(2). □

Amsl($\mathcal{N}$) is Elementary. I now state and give a proof of the main result of this section. Because this theorem uses lemmas 5.3.13 and 5.3.14, its proof must be judged in the light of the remarks made after lemma 5.3.13.

THEOREM 5.3.15 (Bruyns, Naturman and Rose [92]). *Amal*($\mathcal{N}$) *is an elementary class. It is closed under reduced products and is therefore determined*

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[1] I am indebted to Professor Banaschewski for first raising this problem.

by Horn sentences. Furthermore, if B is an image of $A \in Amal(\mathcal{N})$ and B is a subdirect power of N, then $B \in Amal(\mathcal{N})$.

Proof. We first prove the last statement: Let B be an image of $A \in Amal(\mathcal{N})$. From lemma 5.3.3, it follows that if $B \leq N^I$ is a regular subdirect embedding, then every $\underline{2}$-congruence on B is extendible to a $\underline{2}$-congruence on N^I. Also, since $\underline{3}$ is not an image of A, $\underline{3}$ is not an image of B. Hence by theorem 5.3.12(4), $B \in Amal(\mathcal{N})$.

Next we show $Amal(\mathcal{N})$ is closed under direct products: Let $A = \prod_{\gamma \in \Gamma} A_\gamma$, where A_γ is a non-trivial member of $Amal(\mathcal{N})$ for each $\gamma \in \Gamma$. We use lemma 5.3.14. Let Θ_0, Θ_1 be two distinct $\underline{2}$-congruences on A. We need show that there is a $\gamma \in \Gamma$ and embeddings $f_0, f_1 : A_\gamma \hookrightarrow A$ such that $f_0^{-1}(\Theta_0)$ and $f_1^{-1}(\Theta_1)$ are distinct $\underline{2}$-congruences on A_γ. Now $A/\Theta_0 \cap \Theta_1$ is isomorphic to one of $\underline{2} \times \underline{2}$ or $\underline{3}$. In either case we can use projectivity to find $u, v, z \in A$, with $u > v > z$ such that

$$(u,v) \notin \Theta_0, (v,z) \in \Theta_0 \quad \text{and} \quad (u,v) \in \Theta_1,\ (v,z) \notin \Theta_1.$$

By theorem 1.2.16, there are congruences ϕ_0 and ϕ_1 on A induced by ultrafilters, say $\mathfrak{D}_0$ and $\mathfrak{D}_1$ respectively, on Γ such that $\phi_0 \subseteq \Theta_0$ and $\phi_1 \subseteq \Theta_1$. We have

$$R = \{\alpha \in \Gamma : u_\alpha > v_\beta\} \in \mathfrak{D}_0$$
$$S = \{\beta \in \Gamma : v_\beta > z_\beta\} \in \mathfrak{D}_1.$$

For each $\alpha \in \Gamma$, let a_α be an arbitrary but fixed element of A_α. There are three possible cases:

(1) There is a $\gamma \in \Gamma$ such that $\{\gamma\} \in \mathfrak{D}_0 \cap \mathfrak{D}_1$,
(2) For each $\gamma \in \Gamma$, $\{\gamma\} \notin \mathfrak{D}_0$ and $\{\gamma\} \notin \mathfrak{D}_1$,
(3) There is a $\gamma \in \Gamma$ that belongs to exactly one of $\mathfrak{D}_0$ and $\mathfrak{D}_1$.

If (1) holds, then $\gamma \in R$ and $\gamma \in S$, hence $u_\gamma > v_\gamma > z_\gamma$. Define $f_0 = f_1 = f$ as follows:

$$f(x) = (x_\alpha)_{\alpha \in \Gamma} \text{ where } x_\alpha = \begin{cases} a_\alpha & \text{for } \alpha \neq \gamma \\ x & \text{for } \alpha = \gamma. \end{cases}$$

A check shows that f_0 and f_1 satisfy the required conditions.

If (2) holds, pick any $\gamma \in \Gamma$. We have $(R \setminus \{\gamma\}) \in \mathfrak{D}_0$ and $(S \setminus \{\gamma\}) \in \mathfrak{D}_1$. Now since A_γ is non-trivial, by theorem 4.2.13, it is a subdirect power of N. Thus there are at least two distinct epimorphisms $k, t : A_\gamma \twoheadrightarrow \underline{2}$. Define the embedding $f_0 : A_\gamma \hookrightarrow A$ as follows:

$$f_0(x) = (x_\alpha)_{\alpha \in \Gamma} \quad \text{where } x_\alpha = \begin{cases} x \text{ if } \alpha = \gamma \\ u_\alpha \text{ if } \alpha \in R \setminus \{\gamma\} \text{ and } k(x) = 1 \\ v_\alpha \text{ if } \alpha \in R \setminus \{\gamma\} \text{ and } k(x) = 0 \\ a_\alpha \text{ otherwise.} \end{cases}$$

Define the embedding $f_1 : A_\gamma \hookrightarrow A$ as follows:

$$f_1(y) = (y_\alpha)_{\alpha \in \Gamma} \quad \text{where } y_\alpha = \begin{cases} y \text{ if } \alpha = \gamma \\ v_\alpha \text{ if } \alpha \in S \setminus \{\gamma\} \text{ and } t(x) = 1 \\ z_\alpha \text{ if } \alpha \in S \setminus \{\gamma\} \text{ and } t(x) = 0 \\ a_\alpha \text{ otherwise.} \end{cases}$$

A check shows that f_0 and f_1 satisfy the required conditions.

Case (3) is a combination of (1) and (2). For if $\{\gamma\} \in \mathfrak{D}_0$ and $\{\gamma\} \notin \mathfrak{D}_1$, say, then $S \setminus \{\gamma\} \in \mathfrak{D}_1$. Then we may choose f_0 as for case (1) and f_1 as in case (2).

Next we show *Amal*($\mathcal{N}$) is closed under reduced products. For if B is a reduced product of members of *Amal*($\mathcal{N}$), it is the image of a product of members of *Amal*($\mathcal{N}$), say A. By the above argument $A \in$ *Amal*($\mathcal{N}$). Also, since members of *Amal*($\mathcal{N}$) are subdirect powers of N, we have by lemma 5.3.2 that B is a subdirect power of N. Hence $B \in$ *Amal*($\mathcal{N}$).

Now by theorem 1.3.14, the complement of *Amal*($\mathcal{N}$) is closed under reduced powers. Also, *Amal*($\mathcal{N}$) is closed under ultraproducts as it is closed under reduced products. Hence *Amal*($\mathcal{N}$) is elementary by theorem 1.1.7(2) and theorem 1.3.14(3). In fact by theorem 1.1.9(5), since *Amal*($\mathcal{N}$) is closed under reduced products, it is determined by a set of Horn sentences. ☐

Further Work. In this chapter, I presented results that looked at the question as to whether or not an amalgamation class of some variety is elementary. While the answer to this question is known for finitely generated modular lattice varieties, and (subject to the remarks after lemma 5.3.13) for the variety generated by the pentagon, the question still remains open for all the other finitely generated lattice varieties. Also, very little work has been done in this regard on other congruence distributive varieties that share characterizations in terms of Properties P and Q, such as those finitely generated Heyting algebra varieties mentioned in chapter four.

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