Promoting Understanding in Mathematical Problem-Solving through Writing: A Piagetian Analysis

by

Tracy Samantha Craig

A thesis presented for the degree of
Doctor of Philosophy
in the Department of Mathematics and Applied Mathematics
University of Cape Town
April 2007

Promoting Understanding in Mathematical Problem-Solving through Writing: A Piagetian Analysis

Tracy Samantha Craig

April 2007

Abstract

This thesis describes research which suggests that writing about mathematical problem solving processes increases understanding. Piaget's theory of learning, employed, apparently, for the first time in a mathematical problem-solving context, is used to model the process by which such improved understanding is achieved.

Mathematics graduates are expected to be competent problem solvers, yet mathematics undergraduate students are primarily taught algorithms, leaving problem solving to be learnt tacitly. Teaching problem solving explicitly can require significant restructuring of a mathematics course, in content or in resources. The aim of the research project was to investigate the possibilities of using writing as a tool for reflection in mathematical problem solving, without requiring changes to course content or to the physical context of the classes.

The research was carried out in the context of a large first year university mathematics course, at the University of Cape Town. The experiment was carried out in three tutorial groups, two experimental and one control, with the author as tutor. The two experimental groups, in addition to standard tutorial requirements, were required to write explanatory paragraphs on the problem solving processes. One experimental group was required to write about problem solving before carrying out calculations, and the second experimental group wrote about problem solving processes after carrying out calculations.

Data was collected in the form of interviews, submissions of written material and standard assessment tasks. The interviews drew attention to the issue of understanding, particularly to the forcing of understanding which might not have occurred in the absence of the writing activity. The students' written submissions supported the inference of deeper mathematical engagement.

Investigation was undertaken to ascertain whether the writing activities differentially advantaged different language groups, or students with different levels of mathematical preparedness. Results suggest that the writing exercises were similarly advantageous for all student groupings, although second language English speakers found the task more challenging than English main language speakers.

The equitable nature of writing and the suitability of writing assignments for inclusion in an existing course, without substantial changes to the course, strongly recommend the use of writing as an activity to promote mathematical understanding.

Acknowledgements
I owe a debt of gratitude to my supervisors. Professor Tim Dunne, in the Department of Statistical Sciences, UCT, provided strong guidance and was a source of many fine ideas. Professor John Webb, in the Department of Mathematics and Applied Mathematics, provided me with access to the class within which the study took place, and was always ready to listen when I needed to talk.

I have to thank two Tanzanian students in my 2006 MAM1000W class, Max Mwasilu and Cecilia Augustine, for providing me with insight into the current situation of Swahili as a language of learning and teaching in Tanzania.

Acknowledgements are due to the Multilingualism Education Project at UCT for providing statistics on languages spoken at UCT.

I thank Dr Jessica Tiffin the English Department at UCT for our weekly discussions on my data analysis process during the period of the interview analysis and assessment data analysis. It was a fruitful period for my research.

Lastly, I thank my husband, Patrick Vine, for his continual support. We met just before I registered for this degree, before I had any idea of what the final product would look like. Now we are married, with the world's most wonderful little girl, and we all need to get used to Life Without The Thesis, whatever that may be.

Contents

List of Tables and Figures

1 Introduction

The writing of explanatory strategies in the context of mathematical problem-solving increases students' understanding of mathematical topics and deepens their engagement with the problems. The act of writing about mathematics, incorporated into a mathematical activity, makes cognitive and metacognitive demands of the students, requiring them to engage more deeply with the mathematical content than they might otherwise be encouraged to do by the problem requirements, ultimately resulting in the probability of a more insightful problem-solving process and greater understanding of mathematics.

The research project described in this thesis began life as an attempt to improve students' problem-solving skills by encouraging the students to reflect, by writing, on their problem-solving processes. The first step in a successful problem-solving endeavour is to understand the question and to understand the mathematical underpinning of the problem. Results of the research carried out in the writing study project suggest that the writing had at least a self-perceived beneficial effect on students' mathematical understanding and caused them to engage more deeply with the problems than students who were not taking part.

Evidence of the writing exercises undertaken by the students having a prominent effect on the students' problem-solving processes, beyond increased understanding, was not observed, perhaps because the writing intervention was insufficiently incorporated into the introductory course which served as the context of study, and was not summatively assessed. The research project that began with the optimistic intention of analysing what effects writing had on problem-solving skills ended with an analysis of the process by which writing encourages deeper engagement with and understanding of a mathematical problem. The theory by which this process of engagement was modelled is Piaget's stage independent theory of learning, most especially his differentiation between alpha behaviour (the creation of unstable knowledge structures) and beta behaviour (the creation of robust knowledge structures).

1.1 Research question

- What effect does the writing of explanatory strategies have on mathematical problem-solving?

Subquestions include

- Are any observed effects of writing in problem-solving different for students with differing main languages?
- Are any observed effects of writing in problem-solving different for students with differing degrees of mathematical preparedness?

While many studies have been carried out on the usefulness of writing within mathematics, the use of writing explicitly as a tool within mathematical problem-solving has not been well explored. It was considered that the metacognitive activity required in the reflective act of writing would have a beneficial effect on problem-solving processes and abilities. In addition, the use of Piaget's stage independent theory of learning in modelling the observed effects of the writing exercise has not been observed in the literature of mathematical problem-solving, or indeed widely in the field of mathematics education at all.

1.2 Rationale

The author is a lecturer of first and second year mathematics at university level. The rationale for the research question explored in this thesis arose jointly from educational practice and from educational theory of both problem-solving and writing. It was observed in the classroom and in assessment tasks that many students either did not possess much skill in problem-solving or possessed low levels of self confidence in their ability to solve problems. It was simultaneously observed that the first year course, the vehicle for the writing project, did not explicitly teach problem-solving, instead it taught many algorithms and mathematical recipes. In conflict with the content of the course, the lecturers occasionally set problems that demanded a high level of problem-solving ability from the students. It was the aim of this study to address the occasional imbalance between what was taught and what was assessed by creating a course activity which would improve problem-solving skills.

A mathematical problem can be defined as "a task that is difficult for the individual who is trying to solve it. Moreover, that difficulty should be an intellectual impasse rather than a computational one" (Schoenfeld, 1985, p. 74) and mathematical problem-solving simply as the solving of such problems. Problem-solving is distinct from using well-worn algorithms for solving exercises, the former being much more difficult to teach than the latter. Teaching mathematical problem-solving is a matter of some concern to mathematics educators, and, while such teaching (and associated concern) will have been occurring throughout the history of mathematics (see, for example, Stanic and Kilpatrick, 1989), serious scrutiny of the processes of problem-solving first influenced the greater educational community with the publication of George Pólya's (1945) *How To Solve It*. In *How To Solve It*, Pólya breaks the problem-solving process down into the four steps of

- understand the problem,
- devise a plan,
- carry out the plan and
- look back;

four steps that have been quoted repeatedly in the problem-solving literature. In addition to the four step model of problem-solving, a model which has proven influential, Pólya provided a list of heuristic strategies, rough guides to how to respond to particular situations. Pólya's heuristic strategies, or simply heuristics, have proven possibly even more influential than the four step problem-solving process.

In the 1980s and early 1990s problem-solving became a regular topic amongst mathematics educators with major works such as Alan Schoenfeld's *Mathematical Problem Solving* (1985) laying firm groundwork in the domain of teaching problem-solving, likewise edited volumes such as Silver (1983) *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives* and Charles and Silver (1989) *The Teaching and Assessing of Mathematical Problem Solving*. The US National Council of Teachers of Mathematics has also played a role in the increasing attention to problem-solving during the 1980s and early 1990s (see Schoenfeld, 2008, in press, for a good overview).

Specific areas within the field of problem-solving which have received focussed interest include metacognition and expert-novice distinctions, both issues of interest in this thesis. Metacognition can be defined both as knowledge about cognitive phenomena, and monitoring and regulation of cognitive phenomena (Brown et al, 1996; Schoenfeld, 1985; Garofalo and Lester, 1985) with rather less frequent attention being given to a further definition of metacognition as beliefs and affects and their effects on performance (Schoenfeld, 1992). The writing exercises used to develop problem-solving behaviour required the students to reflect on their problem-solving processes, and in so doing to invoke metacognitive processes in their manifestation as declarative knowledge of cognitive processes. The initial intent of the required reflection on cognitive problem-solving processes was to improve metacognitive monitoring and control. The observed positive effect of the activity was that reflection compelled deeper engagement with the mathematical material than might have been the case without reflection. Metacognitive control was not enhanced so much as active cognitive engagement with mathematics deeper than simply carrying out algorithms.

Expert-novice studies which have been particularly influential on this thesis are located in the subject areas of both physics and mathematics. Physics expert-novice studies have found that experts are less likely to be swayed by surface features (although they can, in fact, be swayed), tending more to recognising underlying physics principles. In addition, physics experts are less likely than novices to begin calculation before deciding on a general solution schema (Leonard et al, 1996; Chi et al, 1981; Hardiman et al, 1989). Mathematics expert-novice distinctions tend to highlight the use of heuristic strategies by experts, the greater extent of metacognitive monitoring and regulation than novices and the greater tendency of experts to reflect on a solution and thereby learn from it (Lester, 1994; Schoenfeld, 1992). Expert-novice studies formed part of the grounding for the writing study project, drawing on both mathematics and physics studies. The physics studies influenced the study project by inspiring the activity of writing explanatory strategies, a device used to great effect by Leonard, Dufresne and Mestre (1996) and Leonard, Gerace and Dufresne (1999), while the mathematics studies suggested that the expert-like strategies to be encouraged in the students were use of heuristic strategies, metacognitive control and reflection. In effect it was Pólya's steps of *devise a plan* and *look back* which were the focus of the writing exercises.

The study project described and discussed in this thesis involved requiring students to write explanatory paragraphs on their problem-solving processes, preferably in English sentences, using as little mathematical symbolic notation as possible. Immediately there are two language issues which must be examined. The first is the concern of having designed a learning initiative which might benefit speakers of English as a main language over speakers of other languages, and the other is the problematic embedding of the symbolic and precise technical language of mathematics within the verbal language of English.

South Africa has eleven official languages, all of which, and more besides, are spoken at the University of Cape Town (Adler, 2001; Moodley, 2000). It was one of the intents of the study project to determine whether any effects of the writing exercises involved in the project had a differential effect (either positive or negative) on speakers of different main languages. The final conclusion was that students' main languages did not affect their learning from the writing exercises, which was most encouraging. One aspect of language which did have an effect on some of the students, however, was the technical nature of use of the mathematics register. In many ways, mathematics is a language in its own right, with its own vocabulary, grammar and logical rules (Ellerton and Clements, 1991; Ellerton and Clarkson, 1996; Pimm, 1995). Students of mathematics need to begin to learn the mathematics register and become proficient in its use, a task some students find easier than others, and one that is confused by the necessity of teaching the precise language of mathematics within the less precise language of learning and teaching, in this case English. The writing exercises necessarily required the students to talk about mathematical processes using mathematical terminology, a task some students found daunting, to the point of being unable to begin the process, and others enjoyed, for allowing them to improve their fluency in the register.

A thorough overview of the intertwined concerns of mathematics and language can be obtained from Ellerton and Clements (1991) *Mathematics in Language: A Review of Language Factors in Mathematics Learning* as well as the edited volume of Connolly and Vilardi (1989) *Writing to Learn Mathematics and Science*. Despite more than a decade since either of those books was written, the issues they discuss and the questions they raise are still valid and pertinent. There is widespread support for the usefulness of writing in mathematics in general and in mathematical problem-solving in particular, but the reasons given for the perceived usefulness are not unanimous. There is the school of thought that writing and problem-solving involve exactly the same steps, essentially Pólya' problem-solving steps, and therefore combining the two activities simultaneously brings advantage to both (Kenyon, 1989; Mendez and Taube, 1997). There is the constructivist viewpoint that writing about mathematics requires the writer to form associations and construct cognitive knowledge structures in order to communicate thought processes in an understandable form (Ellerton and Clements, 1992; Sierpinska, 1998). Then there are the supporters of the metacognitive advantages of writing through processes of reflection, monitoring and reaction (Pugalee, 2001;

Kenyon, 1989). All three viewpoints have support and it is entirely possible that all are correct, although Ellerton and Clements repeatedly call for more thorough research to support all the facets of the writing process and the phenomena it is alleged to support.

All three views played a role in the project described in this thesis. The writing under scrutiny in the study project aimed to enhance students' problem-solving abilities by simultaneously addressing cognitive issues such as sense making and metacognitive issues in the form of declarative knowledge of cognitive processes. The writing required the students to describe their solutions strategy in written form and to reflect on their problem-solving processes, directly tying the activities involved in the writing process to Pólya's problem-solving steps of *devise a plan* and *look back*. The results of the project indicated that the enforced deeper engagement with the mathematics involved in the questions about which the students had to write, brought about improved understanding of the mathematics, specifically understanding that classmates not taking part in the writing study project did not always achieve. The observed effect of deeper engagement and improved understanding has been fruitfully analysed using a Piagetian constructivist learning theory.

Over the last few decades, it has become increasingly apparent that university classes are becoming more diverse in language, culture, expectations and academic preparedness (Zevenbergen, 2001; Wood, 2001). Reasons for the increase in diversity abound, the most obvious being the simple increase in numbers of school leavers committing to higher education. One measure of diversity, that of preparedness, has not received much research attention, the primary reaction in universities being to create bridging courses for the less well-prepared students. Even allowing for bridging courses, it is apparent that large mainstream classes still exhibit a wide range of mathematical preparedness. Turning to the literature for ideas about how to equitably teach such a class one can find few suggestions, and those that are found vary widely in form and in applicability. Suggestions for teaching large, academically diverse classes include fully integrated use of multimedia (Cavender and Rutter, 1997), overarching changes to classroom philosophy and epistemology (Northedge, 2003) and development of specific teaching and learning activities, of which explanatory writing is an example (De La Paz, 2005). The study project aimed to determine whether the writing of explanatory strategies in mathematical problem-solving was a suitable activity for a class diverse in mathematical preparedness or whether it advantaged one group of students over another.

Jean Piaget (1896 – 1980), best known for his theories on childhood developmental stages, proposed a constructivist theory of learning through which the epistemic subject learns by the growing and changing of knowledge structures. Suggested readings, among the wealth of Piaget's work, are Piaget (1972) *The Principles of Genetic Epistemology*, translated from *L'Epistémologie Génétique* (1970) and Beth and Piaget (1966) *Mathematical Epistemology and Psychology*, translated from *Épistémologie mathématique et psychologie* (1965). Later in Piaget's career, in *L'Equilibration des Structures Cognitives* (1975), with 1978 (*The Development of Thought – Equilibration of Cognitive Structures*) and 1985 (*The Equilibration of Cognitive Structures*) translations, he embedded his cyclic model of successful learning, involving such concepts as accommodation, assimilation and equilibration, in a three-pronged interpretation of potential learning modes. As the analysis of observations of the study project will reveal, Piaget's model of learning was found to be satisfactorily descriptive of the processes exhibited by student problem-solving practices.

Lev Vygotsky (1896 – 1934) is the major influence in sociohistoric and sociocultural epistemologies. Vygotsky's work was concerned at all times not on the individual learner, but with the social aspects of learning, and how knowledge is socially mediated and constructed. Communication, therefore, plays a central role in Vygotskian studies, and writing is a form of communication. Vygotsky's (1962) *Thought and Language* (edited and translated from the Russian original *Myshlenie i Rech*, 1934) presents Vygotsky's support of writing in the learning process, for its characteristics of abstraction, demand for constructive thought and as a mode of communication. Vygotsky's claim for the constructive use of writing in learning combines successfully with Piaget's constructivist theory of learning to support the place of writing in learning mathematical problem solving.

The course within which the writing experiment took place was a year long course, divided into two semesters. The writing initiative was carried out in the second semester. The class was large, approximately 500 students divided into two smaller classes of approximately 250 students. The class further divided into afternoon tutorial groups of approximately 30 students, which even so are larger than some groups used in problem-solving teaching experiments (Schoenfeld, 1985). Decreasing class size was not an available option. Tutorial classes were limited by the number of available venues as well as the number of available tutors. The course was a preparatory course for second year mathematics, applied mathematics, first and second year physics, chemistry and economics. It was a compulsory course for all Bachelor of Science students as well as Actuarial Science students. Decreasing course content was also not an available option, as every topic covered in the course is important, and lecturers of other courses even sometimes express the wish that more could be taught in the course. It was the aim of the study to design an intervention which could be added to the existing course without a need for content changes, and which would in some way enhance students' problem-solving abilities.

It was determined that such an intervention could not easily be carried out in the lectures of 200 to 250 students. Instead, the writing exercises were introduced in the tutorial classes, of which three (out of approximately twelve) were run with the author as tutor. Two of the three tutorial classes were experimental groups, running two different versions of the writing experiment, and the third was the control, identically run to the remaining non-experimental tutorial groups. One of the experimental groups (the A, After, group) were required to write about the problem-solving processes after having carried out a problem calculation, and the other group (the B, Before, group) were required to write about the planned problem-solving process before carrying out problem calculations. The After group had one extra element, which was to make a brief statement of expectation (a few symbols, a short phrase) on what form the problem solution was anticipated to take. Data was collected on the students in the form of interviews, the written exercises submitted over the course of a semester, and quantitative analyses of the students' assessment tasks throughout the academic year.

As shall be elucidated, the data revealed that the facet of the problem-solving process which the writing initiative particularly supported was Pólya's first step of *understand the problem*; more specifically understand the mathematics underpinning the problem. The process by which the writing encouraged understanding has been fruitfully modelled using Piaget's stage-independent learning theory.

1.3 Chapter summaries

Chapter 2 covers the topic of problem-solving, defining both problems and problem-solving in mathematical terms and comparing those definitions to the corresponding terms in other fields, notably physics. The definitions of both terms are often omitted in the literature, to the detriment of the reader's understanding, since the terms are not always used unambiguously. Mathematical sense making and its associations with classroom epistemology are reviewed, as is the literature on metacognition. The expert-novice studies which have played such a significant role in the design of the writing study project are discussed, in the fields of mathematics and physics; the physics expert-novice studies having been influential far beyond the borders of physics and physics education. Some mention is made of heuristic strategies, and the issue of writing in problem-solving is touched upon, to be covered in more detail in Chapter 4.

In Chapter 3, the issue of language is explored. The contextual setting of the writing project was a course in which students embody a range of main languages, which has to be taken into consideration in a project where the students' use of non-symbolic language is being demanded. The history of language in education in South Africa is one permeated with tension. This history is outlined briefly and the current situation is described against a backdrop of the country's history. Language in education is one issue, but language in mathematics is another. The mathematics register and the simultaneous verbal and symbolic needs of mathematics make mathematics distinctive in any study of subject specific language demands. While the study found that main language played little if no role in students' experience of the writing exercises, it found that the technical requirements of mathematical language did play a role, in both positive and negative ways.

Chapter 4 describes the forms and purposes of writing in and about mathematics. The links between writing and problem-solving are discussed, such as the encouragement of constructive learning through writing, the metacognitive activities encouraged by writing, and the similarities of writing to problem-solving and hence the efficacy of combining the two. Writing in mathematics or writing about mathematics can take many forms and can serve many purposes. In Chapter 4, the various forms of writing (as opposed to purposes of writing) are discussed in brief, these forms being: writing about problem-solving processes, journals, essays, reports on journal papers, writing about mathematical concepts (in contrast to processes), problem posing, investigative projects and projective techniques such as letter writing. The focus of this thesis is writing about problem-solving processes, although other aspects do exert influence, such as journal entry analysis being used usefully in the writing exercise analysis in this project.

The diversity of mathematical preparedness of first year university students is discussed in Chapter 5, a situation found all over the world and, arguably, increasingly in the last couple of decades. A variety of approaches to take when teaching large classes of academically diverse students is summarised, including using multimedia, fundamentally altering the classroom epistemology and philosophy of learning, and using tools such as writing to promote equitable learning.

Piaget's theory of learning is described in detail in Chapter 6, listing and explaining the potentially confusing terms associated with it, such as aliment, assimilation, accommodation, equilibration, perturbation and schemes. Piaget's three-pronged description of learning responses, within which the learning theory can be embedded, is further elucidated. With a brief review of the comparisons between Piaget and Vygotsky, Vygotsky's views on the utility of writing as a tool for learning are given. As an acknowledgement of the presence of radical constructivism in the modern literature on Piagetian theories, a short outline of the theory is given, although it is not directly a topic of this thesis.

The research design of the writing project is described in Chapter 7. The population and sample population are described, as well as the process by which the different experimental groups were defined. The chapter describes what was required of the experimental groups as well as what data was collected. The ethical considerations, and subsequent consent requirements, of the experiment are discussed.

The data analysis of the three forms of data (interviews, writing exercises and problem-solving assessment tasks) is described in Chapter 8. The primary form of informative data was the interviews, with observations drawn from the interviews backed up by observations in the other data types, primarily the writing exercises themselves. A journal was kept throughout the duration of the project, and journal entries made at the time, or shortly after, the writing exercises took place are drawn on to deepen observational findings.

The appropriateness of Piaget's stage independent theory of learning to model the observations described in Chapter 7 is argued for in Chapter 9. Piaget's model is three-pronged, of which one prong (beta behaviour) is often analysed in isolation from the other two in the literature. It is vital to the application of Piaget's theory in this thesis that all three prongs be present. It is an argument of this thesis that writing reflectively on problem-solving processes causes, perhaps forces, beta behaviour when alpha behaviour is more likely to be the outcome in the absence of forced deeper engagement. In the conclusions drawn in Chapter 9, the findings of the study are summarised, constraints of the study are clarified and potential for further study is suggested.

The Appendices include

- an anonymous list of the students involved with all data pertaining to them,
- all the tutorial questions about which students were asked to write explanatory strategies,
- the ethical code and consent form,
- the test and examination questions used as indicators of problem-solving skills,
- the interview questions and the individual analyses and selection of students' original written exercises used as illustrations of observations discussed in the thesis,
- examples of actual student written submissions, and
- the quantitative assessment data and data analysis.

2 Problem-solving

Interest in solving mathematical problems, given in a variety of forms, has a long history. Stanic and Kilpatrick (1989) give examples from ancient Chinese and Egyptian texts of what amount to both algebraic and word problems. Schoenfeld (1989) notes that the view that "mathematics helps you think" (p. 84) is one that is widely held, and it certainly goes back a long way, at least as far as Plato: "those who are by nature good at calculation are, as one might say, naturally sharp in every other study" (Grube, 1974, cited in Stanic and Kilpatrick, 1989). In the 19th Century it was part of "mental discipline theory" (Stanic and Kilpatrick, 1989) that mathematics and the classic languages were the best vehicles for developing the mental faculties of memory, understanding and reasoning, among others. Schoenfeld (1992) considers that the Platonic view that training in mathematics helps you to be a good thinker in general was only challenged in the early 20th Century. While one can find strong signs of interest in theories of thinking and learning in the work of Descartes, Leibnitz, and Bolzano (Schoenfeld, 1992; Stanic and Kilpatrick, 1989) such theories did not develop into an empirical discipline until the late 19th Century and the development of a true field in mathematics education in the 20th Century. Stanic and Kilpatrick (1989) list a variety of texts from the late 19th and early 20th Centuries which present problem-solving in the form of supplying the learner with lists of problems to work through, with the solutions available elsewhere, an interpretation of problem-solving still around in 2007, to the exasperation of researchers of mathematical problem-solving. In 1945 there was a surge of interest in studies of thought and learning, especially in the context of mathematics, and several influential books were published in that year, notably Pólya's *How To Solve It*, but also works by Hadamard, Duncker and Wertheimer. The constructivist epistemology promulgated by Piaget was catching the interest of theorists in the field and interest in problem-solving and theories of learning started to take off.

A major part of Pólya's importance in the problem-solving field is the fact that he was the first problem-solving theorist whose work influenced school curricula (Stanic and Kilpatrick, 1989). Pólya's list of "heuristics" or "heuristic strategies" for the problem-solving process is widely quoted in the problem-solving literature. The term *heuristic* is usually used as an adjective (Oxford English Dictionary), such as in *heuristic process*, *heuristic technique*, and so on, to the point where the adjective form is used to define the noun form. *Heuristic (adjective): serving to find out or discover. Heuristic (noun): A* *heuristic process or method for attempting the solution of a problem; a rule or item of information used in such a process* (online Oxford English Dictionary, consulted 12 December 2006). Pólya's (1945) broad brush heuristics for the problem-solving process are

- Understand the problem
- Devise a plan
- Carry out the plan
- Look back

Pólya's work on problem-solving forms the firm foundation for much of the problem-solving research carried out in the last few decades, notably the body of research carried out by Schoenfeld (1985 among others). Recent collaboration between mathematics educators, educational psychologists and cognitive psychologists to develop programs of study dealing with cognitive learning skills, has broadened the field far beyond a focus on mathematical problems in isolation (Kenyon, 1989). The 1980s in particular were fruitful years for the study of problem-solving (the "problem-solving bandwagon": Ellerton and Clements, 1991, p. 57), with much research being done in the United States, an interest that has waned somewhat over the past decade due to multiple factors. The history of problem-solving research in the US is an interesting one which has been covered fully elsewhere (Lester, 1994; Schoenfeld, 2008 in press).

In the last half century, particularly since the 1980s, much progress has been made on the theory of mathematical problem-solving. Schoenfeld (2008, in press) concludes that problem-solving research over the last 3 decades has achieved the following: recognising the importance of problem-solving, recognising the usefulness of heuristics and associated substrategies, determining that the use of heuristic strategies can be taught, great advances in research on metacognition as well as belief systems and on experience as a shaper of beliefs. Schoenfeld (2008, in press) and Lester (1994) feel there is still much work to be done, such as how much and what kind of practice is needed for students to learn a wide range of problem-solving strategies, as well as a focus on issues such as assessment and transfer of learning. Lester (1994) expressed concern over the waning of interest in problem-solving which he blames on one or more of (a) other issues taking attention away, (b) we think we know all about problem-solving, (c) constructivism has replaced problem-solving as the mathematical ideology, sometimes erroneously being used as a synonym for problem-solving, and (d) problem-solving is even more complex than previously thought. Schoenfeld (2008, in press) agrees with (a), suggesting that the surge of interest in socio-cultural influence on mathematics and mathematics learning has drawn much attention away from problem-solving.

Is mathematical problem-solving important? Stanic and Kilpatrick (1989) report contrasting views in the literature as to why problem-solving is considered to be important, although it is widely regarded as being so. They stress the importance of Dewey's view (1920, 1933, 1963) that reflective thinking (problem-solving) is crucial to everyone's education, not only those destined to become mathematicians. Despite the development of the field of mathematics in the 20th Century there remains conflict among educators and the population in general as what uses mathematics serves, what uses problem-solving serves and what kinds of mathematics should be in the school syllabus and for which pupils (Stanic and Kilpatrick, 1989). Resnick and Glaser (1976) argue that "a major aspect of intelligence is the ability to solve problems, and that careful analysis of problem-solving behavior constitutes a means of specifying many of the psychological processes that intelligence comprises" (p. 205), where they define intelligence as "the ability to learn" (p. 205). Carlson (1999) reports that being a successful mathematics graduate is considered as synonymous with being a good problem solver, a perception which is unfortunately not backed up by observations.

In the author's experience, problem-solving is not a skill customarily taught in the mathematics classroom. However, if problem-solving is as important as many people believe, and if a mathematics graduate is expected to be a good problem solver, then why isn't problem-solving taught explicitly? The answer is as simple as the subject is complicated: teaching mathematical problem-solving is very difficult, in fact, it is not truly understood how to teach it successfully at all. By its nature, problem-solving involves learning a variety of vaguely defined rules of thumb, for application to unspecified and unspecifiable problems. Teaching algorithms (You see Problem X, you use Algorithm Y to solve it) is so much easier, gets through so much more syllabus in a given period, and (this is important) fits so much more easily into most teachers' and students' views and beliefs of classroom culture that algorithms form the bulk, if not the entirety, of any traditional mathematics course. To complicate matters, the terms *problem* and *problem-solving* are encountered with multiple definitions, often given only implicitly. Before embarking on any study related to problem-solving it is important that the terms be defined clearly (Schoenfeld, 1992).

2.1 Problems, defined

Schoenfeld (1985) defines the term *problem*:

> Problem: A task that is difficult for the individual who is trying to solve it. Moreover, that difficulty should be an intellectual impasse rather than a computational one....To state things more formally, if one has ready access to a solution schema for a mathematical task, that task is an exercise and not a problem.
> (Schoenfeld, 1985, p. 74)

To similarly define a problem, Kenyon (1989) cites Bell (1978) as describing a problem by four criteria:

- a person must be aware of the situation
- the person must recognize that the situation requires action
- the person must want or need to act, and must actually take action
- the resolution must not be immediately obvious to the person.

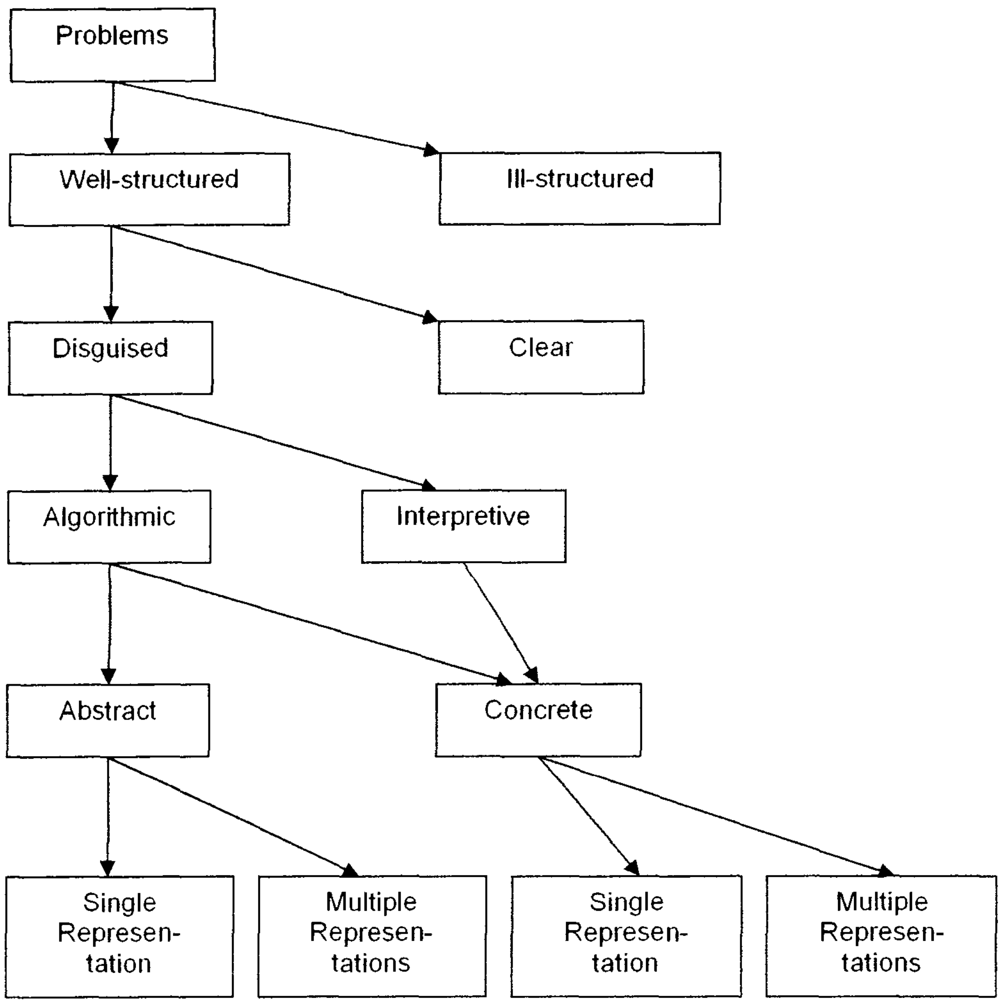
(Kenyon, 1989, p. 75)

To illustrate with an example, suppose Newton's Method of locating the roots of a function is familiar to the student, in that it has been covered in class. A question in a test, requiring the student to find a root of a non-factorisable polynomial is not a problem for the student, merely an exercise, even if the student is not proficient with Newton's Method, and makes errors when attempting it. In the case where a student has never heard of Newton's Method, however, but has the tools necessary to find the root (with a hint, perhaps, depending on the grade of the student) the question becomes a problem, not an exercise.

Schoenfeld continues (1991) to define "good" problems as being relatively accessible, which does not necessarily mean easily solvable. Good problems have a number of different solution routes. The problems and their solutions should relate to important mathematical ideas and should lead into mathematical exploration. Carlson (1999) finds that repeated exposure to challenging and complex problems is one aspect, among several, encouraging students to enjoy mathematics and continue tertiary and postgraduate studies in mathematics. Pólya observed that "although routine problems can be used to fulfil certain pedagogical functions of teaching students to follow a specific procedure or use a definition correctly, only through the judicious use of nonroutine problems can students develop their problem solving ability" (Stanic and Kilpatrick, 1989, p. 16).

The use of the term *problem* often brings to mind disguised well-structured problems, as illustrated by the schematic diagram found in Craig (2002). The diagram is reproduced below and suggests a framework within which different types of problems can be located. Any problem set in a typical school textbook or examination is a well-structured problem in that it is known to have a single solution, accessible by the solution techniques known to the person attempting the problem (or exercise). Disguised problems are ones which are not, at first glance, set in an algebraic format, typically being represented by a paragraph of non-algebraic text. As such, "word" problems are disguised well-structured problems, and, it may be suggested, tend to be the problems people think of when considering problem-solving. In particular, concrete problems are often the only problems considered, that is, ones that are set in a realistic physical context. In fact, problem-solving, as a mathematical skill, can be applied to problems on any branch of this framework. In particular this thesis is concerned with clear problems, not disguised ones; problems that are presented in an algebraic format, albeit in a way that is possibly strange to the student. For a detailed discussion of this framework see Craig (2001).

Figure 2.1 Problem classification framework



A brief illustration of a problem/exercise that can be located within this structure follows: