Jordan homomorphisms and derivations on algebras of measurable operators

Martin Weigt

Thesis presented for the Degree of
DOCTOR OF PHILOSOPHY
in the Department of Mathematics and Applied Mathematics
UNIVERSITY OF CAPE TOWN
June 2008

Supervised by: Dr. J.J. Conradie

The financial assistance of the Department of Labour (DoL) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the author and are not necessarily to be attributed to the DoL.

Abstract

Thesis title: Jordan homomorphisms and derivations on algebras of measurable operators.
Name: Martin Weigt.
Date: 11 February 2008.

A few decades ago, Kaplansky raised the question *whether unital linear invertibility preserving maps between unital algebras are Jordan homomorphisms*. This question is still unanswered, and the progress that has been made has mainly been in the context of Banach algebras, including C*-algebras and von Neumann algebras.

Let $\mathcal{M}$ be a von Neumann algebra with a faithful semifinite normal trace τ, and $\widetilde{\mathcal{M}}$ the algebra of τ-measurable operators (measurable for short) affiliated with $\mathcal{M}$. The algebra $\widetilde{\mathcal{M}}$ can be endowed with a topology γ_{cm}, called the *topology of convergence in measure*, such that $\widetilde{\mathcal{M}}$ becomes a complete metrizable topological *-algebra in which $\mathcal{M}$ is dense. One of the aims of this thesis is to find answers to Kaplansky's question in the context of algebras of measurable operators. We prove, amongst other things, that every self-adjoint Jordan homomorphism between algebras of measurable operators is $\gamma_{cm} - \gamma_{cm}$ continuous and can be expressed as a sum of a self-adjoint algebra homomorphism and a self-adjoint algebra anti-homomorphism.

Derivations between algebras of measurable operators are also considered. It is well known that every derivation on a C*-algebra is continuous and that every derivation on a von Neumann algebra is inner. We investigate whether these results carry over to algebras of measurable operators. Motivated by the Singer-Wermer theorem for commutative Banach algebras as well as the non-commutative Singer-Wermer conjecture for Banach algebras, we also ask whether primitive ideals of $\widetilde{\mathcal{M}}$ are invariant under derivations on $\widetilde{\mathcal{M}}$.

The thesis ends with finding answers to Kaplansky's question when the algebras involved are *locally C*-algebras* and *locally W*-algebras*. We also investigate derivations on locally C*-algebras and locally W*-algebras.

Acknowledgements

I would like to thank my advisor Dr. Jurie Conradie of the University of Cape Town for his encouragement, support, patience and time that he has invested in me, including seminars with a fellow student on topics which are of crucial importance in me being able to produce the thesis. Also, a special thanks to him for his constant advice and careful reading of several earlier drafts of my thesis throughout the duration of my research, and for picking out several flaws, without which I would never have managed to complete the thesis. His enthusiasm for measure theory and non-commutative measure theory is highly infectious, and I will always be grateful to him for getting me interested in this exciting field of research.

I would like to thank the Department of Mathematics and Applied Mathematics of the University of Cape Town for providing me with valuable teaching experience in the form of employment as a tutor, and a special thanks to Mrs Diane Loureiro for advising me in how to obtain financial assistance to support my studies.

I would also like to thank the National Research Foundation of South Africa and the Department of Labour for awarding me a Scarce Skills Scholarship during the period 2003-2005 of my studies.

I especially would like to thank the following persons for their help and advice, which has proved to be invaluable to certain aspects of my research:

(i) Professor Sh. A. Ayupov of the University of Uzbekistan for taking an interest in my work on derivations. Also, I would like to thank him for his help and for many useful discussions and comments regarding the subject matter.

(ii) Professor M. Fragoulopoulou of the University of Athens, Greece, for bringing certain references on topological algebras to my attention and for sending me copies of several of her papers unobtainable in South Africa, including [49], a reference which is of utmost importance in Chapter 7 of this thesis.

(iii) Professor J. Hou of the Shanxi Teachers University in Linfen, China, for sending me a copy of his paper [56].

(iv) Professor A. Inoue of Fukuoka University, Japan, for his kindness in sending me copies of several of his papers on topological algebras and derivations of unbounded operator algebras unobtainable in South Africa, including [57].

(v) Professor L. E. Labuschagne of the University of South Africa for bringing to my attention his work on the link between Jordan homomorphisms and composition operators, as outlined in Chapter 3, and for some useful discussions and remarks on these topics. I would also like to express my gratitude to him for his constant help and advice throughout the years with other aspects pertaining to my thesis.

(vi) Dr. F. A. Sukochev of Flinders University in Adelaide, Australia, for his kindness is sending me an earlier draft of [20] while the manuscript was being prepared for publication in an accredited journal. I am also grateful to him for taking an interest in my work on derivations (Chapter 5) and for some interesting discussions which followed, as well as giving me help and advice whenever I needed it.

(vii) All other persons whom I have not mentioned above that helped me during the research process.

Lastly, I would like to thank friends and family for their support during the research process.

Martin Weigt,
February 2008

List of symbols

Contents

Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras. A *Jordan homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$ is a linear map with the property that $\phi(xy + yx) = \phi(x)\phi(y) + \phi(y)\phi(x)$ for every $x, y \in \mathcal{A}$. A *derivation* on $\mathcal{A}$ is a linear map $D : \mathcal{A} \to \mathcal{A}$ such that $D(xy) = xD(y) + D(x)y$ for every $x, y \in \mathcal{A}$. We say that the derivation D is *inner* if there exists $a \in \mathcal{A}$ such that $D(x) = ax - xa$ for every $x \in \mathcal{A}$. In this thesis, we investigate aspects of the theory of Jordan homomorphisms and derivations, concentrating mainly on algebras of unbounded operators.

Jordan homomorphisms between C*-algebras have a special significance in quantum mechanics ([28]) which we will briefly explain. The most suitable interpretation of the observables of a quantum mechanical system is that they are self-adjoint operators in a von Neumann algebra $\mathcal{A}$ ([28]). In 1932-1933, von Neumann and Jordan proposed that the observables of a quantum mechanical system also form a *Jordan algebra* in $\mathcal{A}$, i.e. a non-associative algebra with multiplication defined as $(x, y) \mapsto \frac{1}{2}(xy + yx)$ ([28]). Bijective Jordan homomorphisms are algebra isomorphisms between Jordan algebras. Hence bijective Jordan homomorphisms serve as "quantum mechanical isomorphisms" between quantum mechanical systems ([28]). Motivated by this, one can ask the following question: If one knows all spectral values of the observables of a quantum mechanical system, can one determine the algebraic model of our quantum mechanical system uniquely up to Jordan isomorphism ([75])? A satisfactory answer was given by Aupetit in [11]: If $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras and $\phi : \mathcal{A} \to \mathcal{B}$ is a surjective spectrum preserving linear map, then ϕ is a Jordan isomorphism.

We call a linear map $\phi : \mathcal{A} \to \mathcal{B}$ *invertibility preserving* if $\phi(x)$ is invertible in $\mathcal{B}$ whenever x is invertible in $\mathcal{A}$. Motivated by a result of Marcus and Purves as well as the famous Gleason-Kahane-Żelazko theorem, Kaplansky asked *when unital linear invertibility preserving maps between unital algebras are Jordan homomorphisms* ([29]). This problem is currently still open. Some affirmative answers to this question include the above result of Aupetit as well as the following result of Choi et al. ([32]): If $\mathcal{A}$ and $\mathcal{B}$ are C* algebras, then every unital surjective self-adjoint invertibility preserving linear map $\phi : \mathcal{A} \to \mathcal{B}$ is a Jordan homomorphism.

It is known that every self-adjoint Jordan homomorphism between von Neumann algebras is continuous ([62], [92]) and can be expressed as a sum of a self-adjoint algebra homomorphism and a self-adjoint algebra anti-homomorphism ([9], [92]). By an *algebra anti-homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$, we mean a linear map $\phi : \mathcal{A} \to \mathcal{B}$ satisfying $\phi(xy) = \phi(y)\phi(x)$ for every $x, y \in \mathcal{A}$.

The theory of non-commutative integration was initiated by Segal in [89]. The setting for this theory is the unital *-algebra $S(\mathcal{M})$ consisting of all measurable operators affiliated with a von Neumann algebra $\mathcal{M}$. When $\mathcal{M}$ is commutative, $\mathcal{M} \cong L_\infty(X, \Sigma, \mu)$ for some localizable measure space (X, Σ, μ), and $S(\mathcal{M}) \cong L_0(X, \Sigma, \mu)$. If $\mathcal{M}$ is a semifinite von Neumann algebra, it can be equipped with a faithful semifinite normal trace τ. In this context, we consider the algebra $\widetilde{\mathcal{M}}$ of τ-measurable operators affiliated with $\mathcal{M}$. One can equip $\widetilde{\mathcal{M}}$ with a topology γ_{cm}, called the *topology of convergence in measure*, with respect to which $\widetilde{\mathcal{M}}$ is a complete metrizable topological *-algebra ([78]). When $\mathcal{M} = L_\infty(X, \Sigma, \mu)$ for some localizable measure space (X, Σ, μ) and the trace τ on $\mathcal{M}$ is defined by $\tau(f) = \int_X f \, \mathrm{d}\mu$, $\widetilde{\mathcal{M}}$ is the algebra $\widetilde{L}_\infty(X, \Sigma, \mu)$ of all equivalence classes of complex-valued measurable functions on X that are bounded except possibly on a subset of X having finite measure, and γ_{cm} is the usual measure theoretic topology of convergence in measure.

It is natural to ask to what extent results for Jordan homomorphisms between von Neumann algebras extend to algebras of measurable operators, here "measurable" meaning measurable with respect to a faithful semifinite normal trace on $\mathcal{M}$. In chapter 3, we show, amongst other things, that every self-adjoint Jordan homomorphism between algebras of measurable operators is $\gamma_{cm} - \gamma_{cm}$ continuous and can be expressed as a sum of a self-adjoint algebra homomorphism and a self-adjoint algebra anti-homomorphism. In Section 3.3, we prove results for Jordan homomorphisms between locally convex GB*-algebras. These are generalizations of C*-algebras. We also provide conditions under which a Jordan homomorphism between algebras of measurable operators is an algebra homomorphism. In chapter 4, we give conditions conditions under which unital invertibility preserving linear maps between algebras of measurable operators are Jordan homomorphisms.

In the second part of the thesis, we look at derivations on algebras of τ-measurable operators. Recall that the observables of a quantum mechanical system are regarded as self-adjoint operators in a von Neumann algebra. The symmetries of the given quantum mechanical system are realized as *-automorphisms of $\mathcal{A}$ ([28]). We say that the derivation D is a *generator* of an automorphism group $\{\alpha_t : t \in \mathbb{R}\}$ if $\alpha_t = \exp(tD)$ for every $t \in \mathbb{R}$. If $\mathcal{A}$ is a C*-algebra and $D : \mathcal{A} \to \mathcal{A}$ a linear map, then D is a self-adjoint derivation if and only if D is the generator of a one-parameter group of *-automorphisms of $\mathcal{A}$ ([28]). More generally, if $\mathcal{A}$ is a Banach algebra and D is a derivation on $\mathcal{A}$, then D is the generator of a one-parameter group of automorphisms of $\mathcal{A}$. In the early fifties, it was established that all automorphisms of a semi-simple Banach algebra are continuous ([36]). The automatic continuity of derivations on semi-simple Banach algebras was then conjectured and proved ([36]).

It is well known that every derivation on a C*-algebra is continuous and that every derivation on a von Neumann algebra is inner. An interesting problem is therefore whether these results can be extended to algebras of τ-measurable operators and even measurable operators (in Segal's sense). Recently, in [19] and [20], Ber, Chilin, and Sukochev proved that if $\mathcal{M}$ is commutative, then $S(\mathcal{M})$ admits only inner derivations if and only if the projection lattice of $\mathcal{M}$ is atomic, i.e. every nonzero projection in $\mathcal{M}$ majorizes an atomic projection in $\mathcal{M}$. Motivated by their result, we prove generally in chapter 5 that if the projection lattice of $\mathcal{M}$ is atomic, then $\widetilde{\mathcal{M}}$ has the property that all its derivations are $\gamma_{cm} - \gamma_{cm}$ continuous. After proving this result, the author collaborated briefly with Sh. A. Ayupov, who then strengthened the last mentioned result in [2]. We also investigate the problem of determining whether the projection lattice of $\mathcal{M}$ is atomic if all derivations on $\widetilde{\mathcal{M}}$ are $\gamma_{cm} - \gamma_{cm}$ continuous.

It is still an open problem as to whether every $\gamma_{cm} - \gamma_{cm}$ continuous derivation on $\widetilde{\mathcal{M}}$ is inner ([13]). In the commutative case, a derivation D is $\gamma_{cm} - \gamma_{cm}$ continuous if and only if D is the zero derivation. In chapter 5, we give other results along these lines.

In chapter 6, we discuss issues related to the Singer-Wermer theorem for commutative Banach algebras. Recall that this theorem states that the range of any continuous derivation on a commutative Banach algebra is contained in the (Jacobson) radical of $\mathcal{A}$ ([36]). Since this result appeared in 1955, it has been conjectured that the continuity assumption can be dropped. That this is the case was proved by Thomas in 1988 ([36]). A positive answer to the following conjecture, known as the Singer-Wermer conjecture, would be a non-commutative analogue of Thomas's result ([36]): *If D is a derivation on a Banach algebra $\mathcal{A}$, then $D(\mathcal{I}) \subset \mathcal{I}$ for every primitive ideal $\mathcal{I}$ of $\mathcal{A}$.* In chapter 6, we ask when derivations on $\widetilde{\mathcal{M}}$ have the property that $D(\mathcal{I}) \subset \mathcal{I}$ for all primitive ideals $\mathcal{I}$ of $\widetilde{\mathcal{M}}$. We prove this in the separate cases where $\mathcal{M}$ is commutative and $\tau(1) < \infty$. Some results are also provided when neither of the latter conditions are imposed.

Recall that $\widetilde{\mathcal{M}}$ is a generally non-normed topological *-algebra and therefore, results which are known to hold for Banach algebras do not necessarily remain valid for $\widetilde{\mathcal{M}}$. In proving some of our results about Jordan homomorphisms and derivations, we found it necessary to prove several results about $\widetilde{\mathcal{M}}$ as a topological *-algebra, and these are scattered throughout the thesis. The most important is that $\widetilde{\mathcal{M}}$ is a generalized B*-algebra (*GB*-algebra* for short), a class of topological *-algebras first studied by Allan ([6]) and later by Dixon ([40], [41]).

An important example of a GB*-algebra is a *locally C*-algebra*, i.e. a complete Hausdorff topological *-algebra $\mathcal{A}$ such that the topology of $\mathcal{A}$ is defined by a family of seminorms (p_α) on $\mathcal{A}$ satisfying $p_\alpha(x^*x) = p_\alpha(x)^2$ for every α and every $x \in \mathcal{A}$. Notice that such seminorms automatically satisfy the conditions $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$ for every α and every $x, y \in \mathcal{A}$ ([18], Theorem 38.1). Such algebras are inverse limits of C*-algebras ([88]). A *locally W*-algebra*, a class of algebras first studied in [49], is a locally C*-algebra which is an inverse limit of W*-algebras. It is natural to ask if some of the well known results about Jordan homomorphisms and derivations on C*-algebras and von Neumann algebras carry over to locally C*-algebras and locally W*-algebras. This is the subject of chapter 7. We investigate the problem of Kaplansky mentioned earlier in the context of these algebras. We also give an alternative proof of the known result that every derivation on a locally C*-algebra is continuous ([16], Proposition 2). Next we provide a sufficient condition that a derivation must have for it to be inner. The chapter ends with the result that every derivation on a locally C*-subalgebra $\mathcal{A}$ of $L(\mathcal{H})$, with $\mathcal{H}$ a locally Hilbert space, can be extended to a derivation on the locally von Neumann algebra generated by $\mathcal{A}$.

The thesis starts with two chapters summarizing material that will be needed in the remaining chapters. Chapter 1 is mainly a reminder of standard results from measure theory, von Neumann algebras, unbounded operators and topological algebras, which is dealt with in the first four sections. Section 1.5 is an introduction to the algebra $\widetilde{\mathcal{M}}$ and it also contains some new results about $\widetilde{\mathcal{M}}$ as a topological *-algebra.

Chapter 2 is a summary of some of the well known results on invertibility preserving linear maps, Jordan homomorphisms and derivations on C*-algebras and von Neumann algebras, which will be needed throughout the thesis.

Chapter 1

Preliminaries

In this chapter, we present, without proofs, results from measure theory, and the theory of von Neumann algebras, topological algebras, unbounded operators and the algebra $\widetilde{\mathcal{M}}$. A familiarity with basic functional analysis, including topological vector spaces and Banach algebras, as can be found in [45], [70], [84], [69], [27] and [36], is assumed. All unexplained terms and concepts in measure theory and von Neumann algebras can be found in [23], [33], [52], [63], [86] and [95].

1.1 Measure theory

Lemma 1.1.1 *([99], Lemma 1.7) If (X, Σ, μ) is a semifinite measure space which is not finite, then there exists a sequence of disjoint measurable subsets (F_n) of X such that $1 < \mu(F_n) < \infty$ for every n.*

We say that a measure space (X, Σ, μ) is *localizable* if there exists a collection (A_α) of disjoint measurable subsets of X each having finite measure and such that $X = \cup_\alpha A_\alpha$. If the above collection of sets can be chosen to be countable, then we call (X, Σ, μ) a *σ-finite* measure space. It is clear that every localizable measure space is semifinite. We call a measure space (X, Σ, μ) *atomic* if for every nonempty $A \in \Sigma$, there exists an atom $B \in \Sigma$ such that $B \subset A$ and $\mu(B) > 0$. For a given measure space (X, Σ, μ), we use the notation $L_0(X, \Sigma, \mu)$ to denote the algebra of all equivalence classes of complex-valued measurable functions on X. We denote the algebra of equivalence classes of essentially bounded measurable functions by $L_\infty(X, \Sigma, \mu)$, and the algebra of equivalence classes of measurable functions, which are bounded except possibly on a subset of finite measure, will be denoted by $\widetilde{L}_\infty(X, \Sigma, \mu)$.

Proposition 1.1.2 *([99], Theorem 1.4) Let (X, Σ, μ) be a measure space. For every $\epsilon, \delta > 0$, let*

$$N(\epsilon, \delta) = \{f \in \widetilde{L}_\infty(X, \Sigma, \mu) : \mu(\{x \in X : |f(x)| > \epsilon\}) \leq \delta\}.$$

(i) The class of sets $\{N(\epsilon, \delta) : \epsilon, \delta > 0\}$ forms a basic neighbourhood system at zero defining a vector topology γ_{cu} on $\widetilde{L}_\infty(X, \Sigma, \mu)$, called the topology of convergence in measure. The class of sets $\{N(\epsilon, \epsilon) : \epsilon > 0\}$ is another basic neighbourhood system at zero defining γ_{cu}. We write $N(\epsilon)$ for $N(\epsilon, \epsilon)$.

(ii) Let (f_n) be a sequence in $\widetilde{L}_\infty(X, \Sigma, \mu)$ and let $f \in \widetilde{L}_\infty(X, \Sigma, \mu)$. Then $f_n \to f$ with respect to γ_{cu} if and only if for every $\epsilon > 0$,

$$\mu(\{x \in X : |(f_n - f)(x)| > \epsilon\}) \to 0$$

as $n \to \infty$.

(iii) Equipped with the topology γ_{cu}, $\widetilde{L}_\infty(X, \Sigma, \mu)$ is a complete metrizable topological algebra.

(iv) The completion of $L_\infty(X, \Sigma, \mu)$ under γ_{cu} is $\widetilde{L}_\infty(X, \Sigma, \mu)$ ([99], Theorem 1.15).

Definition 1.1.3 *([17], Definition 1.1, p. 36) Let (X, Σ, μ) be a measure space and let $f \in L_0(X, \Sigma, \mu)$. Define the function*

$$d : \mathbb{R} \to [0, \infty],\ t \mapsto \mu(\{x \in X : |f(x)| > t\}).$$

The function d is called the distribution function of f. The notation $d_t(f)$ is used to indicate the dependence on the function f.

Proposition 1.1.4 *([99], Proposition 2.2) Let (X, Σ, μ) be a measure space and $f \in L_0(X, \Sigma, \mu)$. The following statements are equivalent.*

(i) $f \in \widetilde{L}_\infty(X, \Sigma, \mu)$.

(ii) $d_t(f) \to 0$ *as* $t \to \infty$.

(iii) $d_t(f)$ *is eventually finite.*

Definition 1.1.5 *([17], Definition 1.5, p. 39) Let (X, Σ, μ) be a measure space and $f \in \widetilde{L}_\infty(X, \Sigma, \mu)$. The decreasing rearrangement of f is defined by the formula*

$$\mu : (0, \infty) \to [0, \infty] : \ t \mapsto \inf\{\theta \geq 0 : d_\theta(f) \leq t\}.$$

We use the notation $\mu_t(f)$ to indicate the dependence on the function f.

The rearrangement of $f \in \widetilde{L}_\infty(X, \Sigma, \mu)$ is decreasing and right continuous ([17], Proposition 1.7, p. 41).

1.2 Von Neumann algebras and traces

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators from a Hilbert space $\mathcal{H}$ into itself, and $\mathcal{B}(\mathcal{H})^h$ the set of self-adjoint elements in $\mathcal{B}(\mathcal{H})$.

A self-adjoint element $x \in \mathcal{B}(\mathcal{H})$ is called *positive* if $\langle x\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{H}$; we write $x \geq 0$. A partial order is defined on $\mathcal{B}(\mathcal{H})^h$ by $x \leq y$ if and only if $y - x \geq 0$. We use the notation $x_\alpha \uparrow x$ to mean that (x_α) is an increasing net in $\mathcal{B}(\mathcal{H})^h$ with supremum $x \in \mathcal{B}(\mathcal{H})$. The notation $x_\alpha \downarrow x$ is defined in a similar manner.

The *strong-operator topology* (respectively *weak-operator topology*) on $\mathcal{B}(\mathcal{H})$ is the locally convex topology on $\mathcal{B}(\mathcal{H})$ defined by the family of semi-norms $x \mapsto \|x\xi\|$ (respectively $x \mapsto \langle x\xi, \xi\rangle$), where $\xi \in \mathcal{H}$. The *ultraweak* topology on $\mathcal{B}(\mathcal{H})$ is the $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})_*)$ topology. A *von Neumann algebra* $\mathcal{M}$ is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed under the strong-operator topology and contains the identity operator of $\mathcal{B}(\mathcal{H})$. A *factor* is a von Neumann algebra with its centre consisting only of scalar multiples of the identity operator. A *-subalgebra $\mathcal{M}$ of $\mathcal{B}(\mathcal{H})$ containing the identity operator of $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if $\mathcal{M} = \mathcal{M}''$, the bicommutant of $\mathcal{M}$ in $\mathcal{B}(\mathcal{H})$ ([63], Theorem 5.3.1). We denote by $\mathcal{M}^h$ and $\mathcal{M}^+$ respectively the set of self-adjoint and positive elements in a von Neumann algebra $\mathcal{M}$. With the partial order inherited from $\mathcal{B}(\mathcal{H})$, $\mathcal{M}^h$ is a partially ordered set. We use the symbol $Z(\mathcal{M})$ to denote the centre of the von Neumann algebra $\mathcal{M}$.

Theorem 1.2.1 *([86], Proposition 1.18.1; [39], Theorem 2, p. 132) Let (X, Σ, μ) be a localizable measure space, let $\mathcal{H} = L_2(X, \Sigma, \mu)$ and, for a given $f \in L_\infty(X, \Sigma, \mu)$, let M_f be the bounded linear operator on $\mathcal{H}$ defined by $M_f(g) = fg$ for all $g \in L_\infty(X, \Sigma, \mu)$. Then the mapping $f \mapsto M_f$ is an isometric *-isomorphism of $L_\infty(X, \Sigma, \mu)$ onto the *-subalgebra $\{M_f : f \in L_\infty(X, \Sigma, \mu)\}$ of $\mathcal{B}(\mathcal{H})$. If $\mathcal{M}$ is a commutative von Neumann algebra, there is a localizable measure space (X, Σ, μ) such that $\mathcal{M}$ is *-isomorphic to $\{M_f : f \in L_\infty(X, \Sigma, \mu)\}$, and hence to $L_\infty(X, \Sigma, \mu)$.*

We can think of von Neumann algebra theory as "non-commutative measure theory", as Theorem 1.2.1 confirms. We refer to the operator M_f in Theorem 1.2.1 as the operator of multiplication by f.

Let $\mathcal{M}$ be a von Neumann algebra. Then the set of projections $\mathcal{M}_p$ in $\mathcal{M}$ forms a complete lattice, i.e. the supremum and infimum of each family of projections in $\mathcal{M}$ are projections and lie in $\mathcal{M}$ ([63], Proposition 5.1.8). An *atomic* (or *minimal*) projection of $\mathcal{M}$ is a nonzero projection in $\mathcal{M}$ having no nonzero proper subprojections in $\mathcal{M}$. We say that $\mathcal{M}_p$ is *atomic* if for every nonzero $p \in \mathcal{M}_p$, there exists an atom $q \in \mathcal{M}_p$ such that $q \leq p$.

Two projections p and q are said to be *equivalent*, written $p \sim q$, if and only if there exists a partial isometry $v \in \mathcal{M}$ such that $v^*v = p$ and $vv^* = q$. We will write $p \preceq q$ to mean that there exists $p_1 \in \mathcal{M}_p$ with $p \sim p_1 \leq q$. In $\mathcal{B}(\mathcal{H})$, $p \sim q$ if and only if the dimension of the range p is the same as the dimension of the range of q ([63], p. 402).

We say that a projection $p \in \mathcal{M}$ is *finite* if $p \sim q \leq p$ implies $p = q$. A projection p is called *infinite* if it is not finite. A von Neumann algebra is called *finite* (respectively *semifinite*) if the identity operator in $\mathcal{M}$ is finite (respectively, if for every nonzero $q \in \mathcal{M}_p$, there exists a nonzero $p \in \mathcal{M}_p$ such that p is finite and $p \leq q$). Every finite von Neumann algebra is semifinite ([63], Proposition 6.3.2).

For an explanation of types, type decomposition and tensor products of von Neumann algebras, the reader is referred to [95], [63] and [86]. The tensor product of two von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ will be denoted by $\mathcal{A} \overline{\otimes} \mathcal{B}$.

Theorem 1.2.2 *([86], Theorem 2.3.2 and Theorem 2.3.3) If $\mathcal{M}$ is a type I von Neumann algebra, then there exists a family of orthogonal central projections (p_α) in $\mathcal{M}$ such that $\mathcal{M} = \oplus_\alpha p_\alpha \mathcal{M} p_\alpha$. For each α, there exists a Hilbert space $\mathcal{H}_\alpha$ such that $p_\alpha \mathcal{M} p_\alpha \cong Z(p_\alpha \mathcal{M} p_\alpha) \overline{\otimes} \mathcal{B}(\mathcal{H}_\alpha)$. If $\mathcal{M}$ is finite, then the above family of projections can be chosen to be countable and the Hilbert spaces $\mathcal{H}_\alpha$ finite-dimensional.*

A *trace* on a von Neumann algebra $\mathcal{M}$ is a function $\tau : \mathcal{M}^+ \to [0, \infty]$ such that

(i) $\tau(x + y) = \tau(x) + \tau(y)$ for all $x, y \in \mathcal{M}^+$,

(ii) $\tau(\lambda x) = \lambda \tau(x)$ for every $\lambda \in \mathbb{R}^+$ and for all $x \in \mathcal{M}^+$,

(iii) $\tau(x^* x) = \tau(x x^*)$ for all $x \in \mathcal{M}$.

A trace τ on $\mathcal{M}$ is called

(i) *faithful* if $\tau(x) = 0, x \in \mathcal{M}^+$, implies $x = 0$;

(ii) *normal* if $x_\alpha \uparrow x$, with (x_α) a net in $\mathcal{M}^+$ and $x \in \mathcal{M}^+$, implies $\tau(x_\alpha) \uparrow \tau(x)$;

(iii) *semifinite* if for every $x \in \mathcal{M}^+$, there exists $0 \neq y \in \mathcal{M}^+$ such that $y \leq x$ and $\tau(y) < \infty$;

(iv) *finite* if $\tau(1) < \infty$.

If $p \in \mathcal{M}_p$ has the property that $\tau(p) < \infty$, then p is finite.

Motivated by Example 1.2.3(i) below, the trace may be thought of as a "non-commutative integral" on a von Neumann algebra.

Example 1.2.3 *([99], Theorems 4:3.1 and 4:6.2)*

(i) Let $\mathcal{M} = L_\infty(X, \Sigma, \mu)$. One can define a trace τ on $L_\infty(X, \Sigma, \mu)$ as follows:

$$\tau(f) = \int f \, \mathrm{d}\mu$$

for every f in $L_\infty(X, \Sigma, \mu)$. It can be shown that τ is a faithful semifinite normal trace on $\mathcal{M}$.

(ii) Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and let (ξ_α) be an orthonormal basis for $\mathcal{H}$. The diagonal trace τ on $\mathcal{B}(\mathcal{H})$ is defined by the formula

$$\tau(x) = \Sigma_\alpha \langle x\xi_\alpha, \xi_\alpha \rangle$$

for every $x \in \mathcal{B}(\mathcal{H})$. One can prove that τ is independent of the orthonormal basis (ξ_α) and is a faithful semifinite normal trace on $\mathcal{B}(\mathcal{H})$.

Proposition 1.2.4 *([99], Proposition 3.2) Let τ be a trace on a von Neumann algebra $\mathcal{M}$ and let $x, y \in \mathcal{M}$.*

(i) If $0 \le x \le y$, then $\tau(x) \le \tau(y)$.

(ii) If $0 \le x \le y$ and $\tau(x) < \infty$, then $\tau(y - x) = \tau(y) - \tau(x)$.

(iii) If $p, q \in \mathcal{M}_p$ and $p \wedge q = 0$, then $\tau(p) \le \tau(1 - q)$.

(iv) A trace τ is semifinite if and only if for every $x \in \mathcal{M}^+$, there exists a net (x_α) in $\mathcal{M}^+$ with $x_\alpha \uparrow x$, where $x \in \mathcal{M}^+$, and $\tau(x_\alpha) < \infty$ for every α.

A von Neumann algebra $\mathcal{M}$ is semifinite if and only if $\mathcal{M}$ admits a faithful semifinite normal trace ([95], Theorem V.2.15). A von Neumann algebra $\mathcal{M}$ is said to be *countably decomposable* if every family of mutually orthogonal nonzero projections in $\mathcal{M}$ is at most countable.

Theorem 1.2.5 *([39], Proposition 9, p. 211) A von Neumann algebra $\mathcal{M}$ admits a faithful finite normal trace if and only if $\mathcal{M}$ is finite and countably decomposable.*

Theorem 1.2.6 *([95], Corollary V.2.9) Any finite factor is countably decomposable.*

Theorem 1.2.7 *([63], Theorem 8.5.7) Any factor of type I or II admits a faithful semifinite normal trace. A factor of type III does not admit a faithful semifinite normal trace.*

Proposition 1.2.8 *([95], Corollary V.2.32) Any two traces on a semifinite factor are proportional.*

Let $\mathcal{M}$ be a von Neumann algebra. A *p-ideal* of $\mathcal{M}_p$ ([101], Definition 2.1) is a subset $\mathcal{P}$ of $\mathcal{M}_p$ having the following properties.

(i) $p, q \in \mathcal{P}$ implies $p \vee q \in \mathcal{P}$.

(ii) $p, q \in \mathcal{P}$ implies $p \wedge q \in \mathcal{P}$.

(iii) If $p \in \mathcal{P}$ and $q \in \mathcal{M}$ with $q \sim p$, then $q \in \mathcal{P}$.

Theorem 1.2.9 *([101], Theorem 2.4) The mapping $\mathcal{I} \mapsto \mathcal{I} \cap \mathcal{M}^p$ is a one-one correspondence between the norm closed two-sided ideals in $\mathcal{M}$ and the p-ideals in $\mathcal{M}^p$.*

Theorem 1.2.9 immediately implies that every norm closed two-sided ideal of a von Neumann algebra $\mathcal{M}$ is the norm-closure of the two-sided ideal generated by its projections.

Proposition 1.2.10 *([63], Theorem 6.8.8) If $\mathcal{M}$ is a von Neumann algebra, then every weak-operator closed left (respectively right) ideal $\mathcal{I}$ of $\mathcal{M}$ contains a unique projection p such that $\mathcal{I} = \mathcal{M}p$ (respectively $\mathcal{I} = p\mathcal{M}$). If $\mathcal{I}$ is a two-sided ideal of $\mathcal{M}$, then p is a central projection in $\mathcal{M}$.*

Proposition 1.2.11 *([63], Proposition 6.8.9) Every two-sided ideal of a von Neumann algebra is self-adjoint.*

Theorem 1.2.12 *([63], Theorem 6.8.3) The set $\mathcal{I}$ of operators in a von Neumann algebra having finite range projections, is a two-sided ideal in $\mathcal{M}$. Every nonzero two-sided ideal in a factor contains this ideal.*

Corollary 1.2.13 *([63], Corollary 6.8.4) Every finite factor is simple.*

Theorem 1.2.14 *([63], Theorem 6.8.7) If $\mathcal{A}$ is a countably decomposable factor of type I_∞ or II_∞, then the norm-closure of the two-sided ideal of operators with finite range projection is the only proper norm closed two-sided ideal of $\mathcal{A}$.*

A *W*-algebra* $\mathcal{A}$ is a C*-algebra for which there exists a faithful representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, such that $\pi(\mathcal{A})$ is a von Neumann algebra on $\mathcal{H}$ ([95], Definition III.3.1). A C*-algebra $\mathcal{A}$ is a W*-algebra if and only if $\mathcal{A}$ is the dual of a Banach space $\mathcal{A}_*$, called the *predual* of $\mathcal{A}$, which is uniquely determined by the C*-algebra structure of $\mathcal{A}$ ([95], Theorem III.3.5 and Corollary III.3.9).

Let $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a representation of a W*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. We say that π is *normal* if for every increasing net (x_α) in $\mathcal{A}^+$ with $x_\alpha \uparrow x \in \mathcal{A}^+$, we have $\pi(x_\alpha) \uparrow \pi(x)$.

Theorem 1.2.15 *([95], Proposition III.3.12) If $\mathcal{A}$ is a W*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a normal representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, then $\pi(\mathcal{A})$ is a von Neumann algebra on $\mathcal{H}$.*

A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of a C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is called *irreducible* if and only if $\{0\}$ and $\mathcal{H}$ are the only invariant subspaces of $\mathcal{H}$ under $\pi(\mathcal{A})$.

Proposition 1.2.16 *([95], Proposition I.9.20) Let $\mathcal{A}$ be a C*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ a *-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$. The following statements are equivalent.*

(i) π is irreducible.

(ii) The only operators in $\mathcal{B}(\mathcal{H})$ commuting with all operators in $\pi(\mathcal{A})$ are scalar multiples of the identity operator on $\mathcal{H}$.

Theorem 1.2.17 *([95], Theorem I.9.23) If $\mathcal{A}$ is a C^*-algebra and $x \in \mathcal{A}$ is nonzero, then there exists an irreducible *-representation π of $\mathcal{A}$ on some Hilbert space $\mathcal{H}$ such that $\pi(x) \neq 0$.*

It is an obvious consequence of Theorem 1.2.17 that every C*-algebra $\mathcal{A}$ is semi-simple ([37], Corollary I.9.13).

1.3 Unbounded operators and the spectral theorem

By an *unbounded operator* on $\mathcal{H}$, we mean a linear operator which is not necessarily bounded and everywhere defined. We say that a linear operator x from a Hilbert space $\mathcal{H}$ into itself is *densely defined* if the domain of x, denoted by $\mathcal{D}(x)$, is a dense subspace of $\mathcal{H}$. The operator is said to be *closed* if the graph of x, i.e. the set

$$\mathcal{G}(x) = \{(\xi, \eta) \in \mathcal{H} \times \mathcal{H} : \xi \in \mathcal{D}(x), \eta = x\xi\},$$

is a closed subspace of $\mathcal{H} \times \mathcal{H}$. If x and y are unbounded operators, we write $x \subset y$ (respectively $x = y$) to mean that $\mathcal{G}(x) \subset \mathcal{G}(y)$ (respectively $\mathcal{G}(x) = \mathcal{G}(y)$), and we say that the operator y is an *extension* of the operator x (respectively, x and y are *equal*). The *closure* of x is defined to be the smallest closed extension of x, if it exists, and is denoted by $\overline{x}$. Care must be taken when defining sums and products of unbounded operators, since the domains of unbounded operators generally differ.

Let x, y be unbounded operators from $\mathcal{H}$ into itself. Then the *sum* of x and y is the operator $x + y$, where $\mathcal{D}(x+y) = \mathcal{D}(x) \cap \mathcal{D}(y)$, and $(x+y)(\xi) = x\xi + y\xi$ for all $\xi \in \mathcal{D}(x) \cap \mathcal{D}(y)$. If $\lambda \in \mathbb{C}$, then λx is the operator with

$\mathcal{D}(\lambda x) = \mathcal{D}(x)$ and $(\lambda x)(\xi) = \lambda(x\xi)$ for every $\xi \in \mathcal{D}(x)$. If $\lambda = 0$, we define λx to be the everywhere defined zero operator in $\mathcal{B}(\mathcal{H})$. Lastly, we define the *product* of x and y to be the operator xy, where $\mathcal{D}(xy) = \{\xi \in \mathcal{D}(y) : y\xi \in \mathcal{D}(x)\}$, and $xy(\xi) = x(y\xi)$ for every $\xi \in \mathcal{D}(y)$.

If x is a densely defined operator from $\mathcal{H}$ into itself, we define a linear operator x^*, the *adjoint* of x, as follows:

$$\mathcal{D}(x^*) = \{\eta \in \mathcal{H} : \text{ there exists } \psi \in \mathcal{H} \text{ such that } \langle \xi, \psi \rangle = \langle x\xi, \eta \rangle \text{ for all } \xi \in \mathcal{D}(x)\}.$$

For such η, $x^*\eta = \psi$. We note that x^* is well defined since $\mathcal{D}(x)$ is a dense subspace of $\mathcal{H}$ ([84], p. 330). Therefore we only consider adjoints of densely defined operators.

If $x = x^*$, we say that x is *self-adjoint*. We say that an unbounded operator x is *normal* if $x^*x = xx^*$. An unbounded operator x is said to be *positive* if x is self-adjoint and $\langle x\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{D}(x)$. In some references, such as [93], positive operators are defined without the requirement of self-adjointness. Every self-adjoint operator is closed ([84], Theorem 13.9).

Proposition 1.3.1 *([84], Theorem 13.13) Let x be a closed densely defined operator on a Hilbert space $\mathcal{H}$. Then x^*x is self-adjoint and thus closed. Let $y = 1 + x^*x$. Then there exists $z \in \mathcal{B}(\mathcal{H})$ such that $zy \subset yz = 1$ and such that xz is bounded.*

We say that a closed densely defined operator x is *affiliated* with a von Neumann algebra $\mathcal{M}$ whenever $u^*xu = x$ for all unitary operators u in the commutant of $\mathcal{M}$ ([63], Definition 5.6.2). We write $x\eta\mathcal{M}$ to indicate that x is affiliated with $\mathcal{M}$. We emphasize that the equality $u^*xu = x$ is to be understood in the usual sense that u^*xu and x have the same graphs, in particular the same domains.

Proposition 1.3.2 *([93], Corollary 9:14 and p. 199) Let $\mathcal{M}$ be a von Neumann algebra and let $x\eta\mathcal{M}$ be a positive operator. Then there exists a unique positive operator y such that $x = y^2$. Furthermore, $y\eta\mathcal{M}$.*

We denote the element y in Proposition 1.3.2 by $x^{\frac{1}{2}}$.

Theorem 1.3.3 *([63], Theorem 6.1.11) Suppose that x is a closed densely defined operator on $\mathcal{H}$. Then there exists a partial isometry v with initial space the closure of the range of $(x^*x)^{\frac{1}{2}}$ and final space the closure of the range of x such that*

$$x = v(x^*x)^{\frac{1}{2}} = (xx^*)^{\frac{1}{2}}v.$$

We write $|x| = (x^*x)^{\frac{1}{2}}$. This decomposition is unique and is called the polar decomposition of x. If $x\eta\mathcal{M}$, then $v \in \mathcal{M}$ and $|x|\eta\mathcal{M}$.

Proposition 1.3.4 *([63], Exercise 10.5.11, p. 562) Let $\mathcal{A}$ be a concrete C^*-algebra and let $x \in \mathcal{A}$. If $x = v|x|$ is the polar decomposition of x, then $v|x|^{\frac{1}{2}} \in \mathcal{A}$ (observe that v need not be in $\mathcal{A}$).*

Proposition 1.3.5 *([93], Corollary 9:31 and p. 230) Let $\mathcal{M}$ be a von Neumann algebra and let $x\eta\mathcal{M}$ be self-adjoint. Then there exist positive operators x_1 and x_2 such that $x = x_1 - x_2$. Furthermore, $x_1\eta\mathcal{M}$ and $x_2\eta\mathcal{M}$.*

Let x, y be closed densely defined operators. The operators $\overline{x+y}$ and $\overline{xy}$ are called the *strong sum* and *strong product* of x and y respectively.

If $\mathcal{M}$ is a von Neumann algebra, we use the notation $\mathcal{U}(\mathcal{M})$ to denote the set of closed, densely defined operators affiliated with $\mathcal{M}$. Under strong sum, ordinary scalar multiplication and strong product, $\mathcal{U}(\mathcal{M})$ is a unital *-algebra, provided that $\mathcal{M}$ is finite ([48], Theorem 2). If $x \in \mathcal{U}(\mathcal{M})$ and $\mathcal{M}$ is finite, then the spectrum of x relative to the algebra $\mathcal{U}(\mathcal{M})$ will be denoted by Sp $(x, \mathcal{U}(\mathcal{M}))$.

Theorem 1.3.6 *([63], Theorem 5.6.15(iv)) If $\mathcal{M}$ is a commutative von Neumann algebra and $x, y\ \eta\mathcal{M}$, then $\overline{xy} = \overline{yx}$. Furthermore, $x^*x = xx^* = \overline{x^*x}$.*

The following proposition was proved in [25] for the case where $\mathcal{A}$ is a commutative von Neumann algebra ([25], Lemma 2) and $n = 2$. The same proof as in [25] can be used to show that the result holds in the more general case where $\mathcal{A}$ is a finite von Neumann algebra.

Proposition 1.3.7 *If $\mathcal{A}$ is a finite von Neumann algebra on a Hilbert space $\mathcal{H}$, then, for every n, $M_n(\mathcal{A})$ is a finite von Neumann algebra on the Hilbert space $\oplus_{k=1}^n \mathcal{H}_k$, where $\mathcal{H}_k = \mathcal{H}$ for every $1 \leq k \leq n$, and $\mathcal{U}(M_n(\mathcal{A})) \cong M_n(\mathcal{U}(\mathcal{A}))$, where $M_n(\mathcal{A})$ (respectively $M_n(\mathcal{U}(\mathcal{A}))$) denotes the algebra of $n \times n$ matrices over $\mathcal{A}$ (respectively $\mathcal{U}(\mathcal{A})$).*

The *spectrum* of a closed densely defined operator x on a Hilbert space $\mathcal{H}$, denoted by Sp (x), is defined to be the set of all $\lambda \in \mathbb{C}$ such that $x - \lambda 1$ is not an injective linear mapping from $\mathcal{D}(x)$ onto $\mathcal{H}$ ([63], p. 357). If $\lambda \notin$ Sp (x), then $x - \lambda 1$ is an injective linear mapping from $\mathcal{D}(x)$ onto $\mathcal{H}$ and has a bounded linear inverse y which maps $\mathcal{H}$ onto $\mathcal{D}(x)$ ([63], p. 357). Conversely, if $x - \lambda 1$ has a bounded (everywhere defined) inverse y, then $\lambda \notin$ Sp (x) ([63], p. 357).

If $x \eta \mathcal{M}$, where $\mathcal{M}$ is a commutative von Neumann algebra, and $\lambda \notin$ Sp (x), then it can be shown that the bounded (everywhere defined) inverse y of $x - \lambda 1$ lies in $\mathcal{M}$ ([63], p. 357). Since $x - \lambda 1$ is closed and y is bounded, it follows that $(x - \lambda 1)y$ is closed, and therefore it follows from Theorem 1.3.6 that $1 = (x - \lambda 1)y = \overline{(x - \lambda 1)y} = \overline{y(x - \lambda 1)}$ ([63], p. 357).

Lemma 1.3.8 *([63], p. 357) Suppose that $x \eta \mathcal{M}$, where $\mathcal{M}$ is a commutative von Neumann algebra. Then Sp $(x, \mathcal{U}(\mathcal{M}))$ coincides with the point spectrum of x, i.e. the set of eigenvalues of x.*

Theorem 1.3.9 *([63], Theorem 5.6.18) A linear operator x on a Hilbert space $\mathcal{H}$ is normal if and only if x is affiliated with a commutative von Neumann algebra. Furthermore, if x is normal, there is a smallest von Neumann algebra $\mathcal{A}$ such that $x \eta \mathcal{A}$. The von Neumann algebra $\mathcal{A}$ is commutative.*

Theorem 1.3.10 *([63], Proposition 5.6.35) If $x \eta \mathcal{M}$, where $\mathcal{M}$ is a commutative von Neumann algebra, then $\sum_{k=1}^n \lambda_k x^k$, with $\lambda_k \in \mathbb{C}$ for every k, is a closed linear operator.*

Theorem 1.3.11 (Spectral theorem) *([45], Theorem XII.2.3; [63], Lemma 5.6.7, Theorem 5.2.2) Let x be a self-adjoint (possibly unbounded) oper-*

ator on a Hilbert space $\mathcal{H}$. Then there is a right continuous family of projections $\{e_t(x) : t \in \mathbb{R}\}$, called the spectral resolution of x, which has the following properties.

(i) If $t \leq s$, then $e_t(x) \leq e_s(x)$.

(ii) $e_t(x) \uparrow 1$ as $t \to \infty$.

(iii) $e_t(x) \downarrow 0$ as $t \to -\infty$.

(iv) $xe_t(x) \leq te_t(x)$ for all $t \in \mathbb{R}$.

(v) $t(1 - e_t(x)) \leq x(1 - e_t(x))$ for all $t \in \mathbb{R}$.

The family $\{e_t(x) : t \in \mathbb{R}\}$ is uniquely determined such that

$$\mathcal{D}(x) = \{\xi \in \mathcal{H} : \int_{Sp\ (x)} t^2 \ d\langle e_t(x)\xi, \xi\rangle \text{ is finite}\}$$

and

$$\langle x\xi, \eta\rangle = \int_{-\infty}^{\infty} t \ d\langle e_t(x)\xi, \eta\rangle$$

for all $\xi \in \mathcal{D}(x), \eta \in \mathcal{H}$.

If x is a bounded self-adjoint operator on $\mathcal{H}$, then $e_t(x) = 0$ for all $t \leq -\|x\|$, and $e_t(x) = 1$ for all $t \geq \|x\|$. In this case,

$$x = \int_{-\|x\|}^{\|x\|} t \ de_t(x)$$

in the sense of norm convergence of approximating Riemann sums.

If x is a self-adjoint linear operator in a von Neumann algebra $\mathcal{M}$, then x can be expressed as a limit of a sequence of finite linear combinations of mutually orthogonal projections in $\mathcal{M}$.

Proposition 1.3.12 *([63], Lemma 6.8.1) Let x be a self-adjoint linear operator in a (not necessarily closed) two-sided ideal $\mathcal{I}$ of a von Neumann algebra $\mathcal{M}$. Also, let $\{e_t(x) : t \in \mathbb{R}\}$ be the spectral resolution of x, as defined in Theorem 1.3.11. Then $e_t(x) \in \mathcal{I}$ for all $t < 0$ and $1 - e_t(x) \in \mathcal{I}$ for all $t > 0$.*

1.4 Topological algebras

A *topological algebra* is an algebra $\mathcal{A}$ which is a topological vector space such that multiplication is separately continuous in each variable. If, in addition, $\mathcal{A}$ has continuous involution, then $\mathcal{A}$ is called a *topological $*$-algebra.* Let $\mathcal{A}$ be a topological algebra with identity, denoted by 1. For $x \in \mathcal{A}$, we define the *spectrum* of x in $\mathcal{A}$, written Sp $(x, \mathcal{A})$, by

$$\text{Sp }(x, \mathcal{A}) = \{\lambda \in \mathbb{C} : x - \lambda 1 \text{ is not invertible in } \mathcal{A}\}.$$

Theorem 1.4.1 *([104], Theorem 7.2) If $\mathcal{A}$ is a complete metrizable topological algebra, then the multiplication on $\mathcal{A}$ is jointly continuous.*

We say that a topological algebra $\mathcal{A}$ is a *Q-algebra* if the group of invertible elements of $\mathcal{A}$ is open. A topological algebra $\mathcal{A}$ is said to have *continuous inversion* if the map $x \mapsto x^{-1}$ is continuous on the group of invertible elements of $\mathcal{A}$. For a *-algebra $\mathcal{A}$, the notions of normal, self-adjoint and unitary elements are defined in the same way as for (abstract) C*-algebras. An element x in a *-algebra $\mathcal{A}$ is called *positive* if there exits a $y \in \mathcal{A}$ such that $x = y^*y$. A *positive linear form* on a *-algebra $\mathcal{A}$ is a linear functional f on $\mathcal{A}$ such that $f(x) \geq 0$ for every positive $x \in \mathcal{A}$.

Theorem 1.4.2 *([35], Theorem 11.1) Every positive linear form on a complete metrizable topological *-algebra is continuous.*

1.4.1 Locally C*-algebras

Definition 1.4.3 *([73], Definition III.2.1) Let Λ be a directed set. An inverse system $(E_\alpha, f_{\alpha\beta})$ of topological algebras is a family of topological algebras $\{E_\alpha : \alpha \in \Lambda\}$ along with continuous algebra homomorphisms $f_{\alpha\beta} : E_\beta \to E_\alpha$, with $\alpha \leq \beta$, which satisfy $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ (for $\alpha \leq \beta \leq \gamma$) and $f_{\alpha\alpha} = i_\alpha$, where $i_\alpha : E_\alpha \to E_\alpha$ denotes the identity map of E_α. The inverse limit algebra E of $(E_\alpha, f_{\alpha\beta})$ is defined to be*

$$E = \{x = (x_\alpha) \in \prod_\alpha E_\alpha : f_{\alpha\beta}(x_\beta) = x_\alpha \textit{ for every } \alpha \leq \beta\}$$

(it is easily verified that E is an algebra). The algebra E is further endowed with the initial topology defined on it by the projection maps $f_\alpha : E \to E_\alpha$, i.e. the weakest topology making each f_α continuous. This turns E into a topological algebra. We write $E = \lim_{\leftarrow} E_\alpha$.

In some references, such as [73], an inverse system (respectively inverse limit) of topological algebras is called a *projective system* (respectively *projective limit*) of topological algebras. From here on, the algebra homomorphisms $f_{\alpha\beta}$ in Definition 1.4.3 are called *connecting maps* of the inverse system. It is easy to verify that the projection maps f_α in Definition 1.4.3 are continuous algebra homomorphisms.

Lemma 1.4.4 *([73], Lemma III.3.2) Let $E = \lim_{\leftarrow} E_\alpha$ be the inverse limit of topological algebras E_α, and let B be a subalgebra of E. Then*

$$\overline{B} = \cap f_\alpha^{-1}(\overline{f_\alpha(B)}) = \lim_{\leftarrow} \overline{f_\alpha(B)},$$

where the maps f_α are defined as in Definition 1.4.3. In particular, if B is closed,

$$B = \lim_{\leftarrow} f_\alpha(B) = \lim_{\leftarrow} \overline{f_\alpha(B)}.$$

We say that a topological algebra $\mathcal{A}$ is *locally m-convex* if its topology can be defined by a family of sub-multiplicative seminorms $\{p_\alpha\}$ which separate the points of $\mathcal{A}$. Observe that a locally m-convex topological algebra is always locally convex. If, in addition, $\mathcal{A}$ is a complete *-algebra and $p_\alpha(x^*x) = p_\alpha(x)^2$ for every α and $x \in \mathcal{A}$, we refer to the seminorms p_α as *C*-seminorms* and we call $\mathcal{A}$ a *locally C*-algebra*. Obviously, all C*-algebras are locally C*-algebras. An example of a non-commutative locally C*-algebra can be found in Chapter 7.

Theorem 1.4.5 *([73], Proposition I.1.6) If $\mathcal{A}$ is a locally m-convex topological algebra, then the multiplication on $\mathcal{A}$ is jointly continuous.*

Theorem 1.4.6 *([88], p. 169, Folgerung 5.4) Let $\mathcal{A}$ be a locally C*-algebra with a family (p_α) of C*-seminorms defining the topology of $\mathcal{A}$. Then $\mathcal{A} =$*

$\lim_{\leftarrow}(\mathcal{A}/N_\alpha)$ *within a topological-algebraic* **-isomorphism, where* $N_\alpha = \{x \in \mathcal{A} : p_\alpha(x) = 0\}$ *for each* α*, and the connecting map* $f_{\alpha\beta} : \mathcal{A}/N_\beta \to \mathcal{A}/N_\alpha$ *is defined by* $f_{\alpha\beta}(x + N_\beta) = x + N_\alpha$ *whenever* $\alpha \leq \beta$*. Furthermore, every* $\mathcal{A}/N_\alpha$ *is a C*-algebra with respect to the norm defined by* $\dot{p_\alpha}(x + N_\alpha) = p_\alpha(x)$ *for every* x*.*

The representation $\mathcal{A} = \lim_{\leftarrow}(\mathcal{A}/N_\alpha)$ of Theorem 1.4.6 is sometimes referred to as the *Arens-Michael decomposition* of $\mathcal{A}$ (for example, as in [49] and [73]).

If (p_α) is a family of C*-seminorms defining the topology of a locally C*-algebra $\mathcal{A}$, then we put $\mathcal{A}_s = \{x \in \mathcal{A} : \sup_\alpha p_\alpha(x) < \infty\}$.

Theorem 1.4.7 *([88], Satz 3.1) Let* $\mathcal{A}$ *be a locally C*-algebra and let* (p_α) *be a family of C*-seminorms defining the topology of* $\mathcal{A}$*, and* $\mathcal{A}_s$ *the set defined above. The following statements are equivalent:*

(i) $x \in \mathcal{A}_s$,

(ii) if $x_1 = \frac{1}{2}(x + x^*)$ *and* $x_2 = \frac{1}{2i}(x - x^*)$*, then Sp* $(x_i, \mathcal{A})$ *is a bounded subset of* $\mathbb{R}$ *for* $i = 1, 2$.

Theorem 1.4.7 clearly implies that $\mathcal{A}_s$ is independent of the choice of family of C*-seminorms defining the topology of $\mathcal{A}$.

Theorem 1.4.8 *([8], Theorem 2.3, or [88], Satz 3.1) If* $\mathcal{A}$ *is a locally C*-algebra with* $\mathcal{A}_s$ *as above, then* $\mathcal{A}_s$ *is a C*-algebra with norm* $\|.\|$ *defined as* $\|x\| = \sup_\alpha p_\alpha(x)$ *for every* $x \in \mathcal{A}_s$*. Furthermore,* $\mathcal{A}_s$ *is dense in* $\mathcal{A}$*.*

Proposition 1.4.9 *([81], Proposition 1.11(5)) If* $\mathcal{A}$ *is a locally C*-algebra, then Sp* $(x, \mathcal{A}_s) = \overline{Sp\ (x, \mathcal{A})}$ *for every* $x \in \mathcal{A}_s$*.*

Theorem 1.4.10 *([81], Corollary 1.13) If* $\mathcal{A}$ *and* $\mathcal{B}$ *are* **-isomorphic locally C*-algebras, then* $\mathcal{A}_s$ *and* $\mathcal{B}_s$ *are* **-isomorphic as C*-algebras.*

Recall that a *character* on a topological algebra $\mathcal{A}$ is a nonzero algebra homomorphism of $\mathcal{A}$ into the complex field. Any commutative complete locally m-convex topological algebra $\mathcal{A}$ has at least one continuous character:

Take $a \in \mathcal{A}$ with $a = 1$. Since

$$\text{Sp}\ (x, \mathcal{A}) = \{f(x) : f \text{ a continuous character on } \mathcal{A}\}$$

for every $x \in \mathcal{A}$ ([15], 4:10-8), and Sp $(a, \mathcal{A}) = \{1\}$, it follows that $\mathcal{A}$ has at least one continuous character.

1.4.2 GB*-algebras

The following definition appears in [40] and [41]. For a unital topological *-algebra $\mathcal{A}$, define $\mathcal{C}$ to be the collection of all subsets B of $\mathcal{A}$ such that

(i) B is closed and bounded, and

(ii) $1 \in B$, $B^2 \subset B$, $B^* = B$.

Now let $\mathcal{C}_0$ denote the collection of all those $B \in \mathcal{C}$ which are absolutely convex ([5], [6]). If, for each $B \in \mathcal{C}_0$, $A(B) = \{\lambda x : \lambda \in \mathbb{C}, x \in B\}$ is a Banach algebra with respect to the Minkowski functional of B, and $\mathcal{A}$ is locally convex, we say that $\mathcal{A}$ is *pseudo-complete* ([5], Definition 2.5). An element x of $\mathcal{A}$ is *bounded* if there exists a nonzero $\lambda \in \mathbb{C}$ such that the set $\{(\lambda x)^n : n = 1, 2, \ldots\}$ is a bounded subset of $\mathcal{A}$ ([5], Definition 2.1, and [40], p. 694).

Definition 1.4.11 *([41], Definition 7.1) Let $\mathcal{A}$ be a topological *-algebra with identity* 1. *Suppose that there is a subalgebra $\mathcal{A}_b$ of $\mathcal{A}$ which is a C*-algebra in some norm, and such that $(1 + x^*x)^{-1} \in \mathcal{A}_b$ for every $x \in \mathcal{A}$. If the unit ball B_0 of $\mathcal{A}_b$ is the greatest member of $\mathcal{C}$ with respect to set inclusion, then $\mathcal{A}$ is called a GB*-algebra.*

Observe that the C*-algebra $\mathcal{A}_b$ in Definition 1.4.11 is unique. For if $\mathcal{A}_b'$ were another C*-algebra with the properties in Definition 1.4.11, then the largest member B_0 of $\mathcal{C}$ is the unit ball of $\mathcal{A}_b'$. Hence $\mathcal{A}_b = \mathcal{A}_b'$ since every Banach algebra is the linear span of its unit ball.

The C*-algebra $\mathcal{A}_b$ in Definition 1.4.11 will often be called the *underlying C*-algebra* of the GB*-algebra $\mathcal{A}$.

Another notion of GB*-algebra is [40], Definition 2.5. It can be shown that this definition coincides with that of Definition 1.4.11. A more restricted type of GB*-algebra was considered by Allan in [6].

Since $\mathcal{A}_b$ is the linear span of its unit ball B_0, i.e. $\mathcal{A}_b = A(B_0)$, the Minkowski functional $\|.\|$ on $\mathcal{A}_b$ defines a norm on $\mathcal{A}_b$ in such a way that $(\mathcal{A}_b, \|.\|)$ is a C*-algebra ([40], p. 694-695). Since B_0 is a bounded subset of $\mathcal{A}$, it is easily seen that the GB*-topology restricted to $\mathcal{A}_b$ is weaker than the topology on $\mathcal{A}_b$ defined by $\|.\|$.

Lemma 1.4.12 *([6], p. 95) Every locally C*-algebra $\mathcal{A}$ is a GB*-algebra with $\mathcal{A}_b = \mathcal{A}_s$.*

Examples of GB*-algebras which are not locally C*-algebras is Example 3.3.3. Another example of a GB*-algebra which is not generally a locally C* algebra can be found in Section 1.5 of this chapter (see Theorem 1.5.29).

We denote the one point compactification of $\mathbb{C}$ by $\mathbb{C}^*$. In [40], Definition 4.7, Dixon calls a collection F of continuous $\mathbb{C}^*$-valued functions on a topological space X a **-algebra of functions* if the following conditions are satisfied:

(i) each $f \in F$ takes the value ∞ on at most a nowhere dense subset of X;

(ii) F is a *-algebra of functions under the operations $\lambda f, f+g, fg$ and f^* $(f, g \in F, \lambda \in \mathbb{C})$. These operations are defined pointwise on a dense subset of X where all the values involved are finite, and extending to obtain continuous $\mathbb{C}^*$-valued functions on M.

Theorem 1.4.13 *([40], Theorem 4.6) If $\mathcal{A}$ is a GB*-algebra and M_0 is the set of all characters on $\mathcal{A}_b$, then the Gelfand isomorphism of $\mathcal{A}_b$ onto $C(M_0)$*

extends uniquely to an isomorphism of $\mathcal{A}$ onto a *-algebra of functions on M_0.

Lemma 1.4.14 *([40], Lemma 4.10) Let $\mathcal{A}$ be a GB*-algebra and x a normal element of $\mathcal{A}$. If C is a maximal commutative *-subalgebra of $\mathcal{A}$ containing x, then C, with the induced subspace topology, is a GB*-algebra.*

Lemma 1.4.15 *If $\mathcal{A}$ is GB*-algebra and p is a projection in $\mathcal{A}$, then $p \in \mathcal{A}_b$.*

Proof. Let $p \in \mathcal{A}$ be a projection. Since p is normal, it follows from Lemma 1.4.14 that p generates a commutative GB*-algebra $\mathcal{B}$. Let M_0 denote the set of all characters on $\mathcal{B}_b$. By Theorem 1.4.13, $\mathcal{B}$ is *-isomorphic to a *-algebra of functions on M_0 and $\mathcal{B}_b$ is *-isomorphic to $C(M_0)$. Since $(1+p)^{-1} = (1+p^*p)^{-1} \in \mathcal{B}_b$, it follows from standard arguments that $p \in \mathcal{B}_b \subset \mathcal{A}_b$. ∇

The following proposition is known to hold for commutative locally convex GB*-algebras ([6], Lemma 3.2), and, by Theorem 1.4.13, carries over to all commutative GB*-algebras with exactly the same proof.

Proposition 1.4.16 *If $\mathcal{A}$ is a commutative GB*-algebra, then $x(1+x^*x)^{-1} \in \mathcal{A}_b$ for every $x \in \mathcal{A}$.*

Theorem 1.4.17 *([26], Theorem 2) If $\mathcal{A}$ is a GB*-algebra, then $\mathcal{A}_b$ is sequentially dense in $\mathcal{A}$.*

Theorem 1.4.18 *([26], Proposition 3) If $\mathcal{A}$ is a complete locally m-convex algebra which is a GB*-algebra, then $\mathcal{A}$ is a locally C*-algebra.*

Lemma 1.4.19 *([5], Proposition 2.6) If $\mathcal{A}$ is a sequentially complete locally convex topological algebra, then $\mathcal{A}$ is pseudo-complete.*

Lemma 1.4.20 *([5], Corollary 4.2) If $\mathcal{A}$ is a pseudo-complete locally convex topological algebra with continuous inversion, then $x \in \mathcal{A}$ is bounded if and only if Sp $(x, \mathcal{A})$ is a bounded subset of $\mathbb{C}$.*

Proposition 1.4.21 *([40], Proposition 5.1) If x is a positive element of a GB*-algebra $\mathcal{A}$, then there exists a unique positive element $y \in \mathcal{A}$ such that $x = y^2$.*

Proposition 1.4.22 *([40], Theorem 6.7 and Corollary 6.8) If x is a nonzero positive element of a locally convex GB*-algebra $\mathcal{A}$, then there exists a positive linear form f on $\mathcal{A}$ such that $f(x) > 0$. If $0 \neq x \in \mathcal{A}$ is not positive, then there is a positive linear form f on $\mathcal{A}$ such that $f(x) < 0$.*

1.5 The algebra $\widetilde{\mathcal{M}}$

The study of non-commutative integration was initiated by Segal in [89].

Definition 1.5.1 *([89], Definition 2.1) Let $\mathcal{H}$ be a Hilbert space. A subspace E of $\mathcal{H}$ is strongly dense in $\mathcal{H}$ if $uE = E$ for all unitary operators $u \in \mathcal{M}'$, and there exists a sequence of projections (p_n) in $\mathcal{M}$ such that $p_n(\mathcal{H}) \subset E$ for every n, $1 - p_n$ is finite for every n, and $1 - p_n \downarrow 0$.*

An (unbounded) operator x is called measurable if x is affiliated with $\mathcal{M}$ and has a strongly dense domain. Let $S(\mathcal{M})$ denote the set of measurable operators affiliated with $\mathcal{M}$.

Proposition 1.5.2 *([89], Corollary 5.2) $S(\mathcal{M})$ is a *-algebra under strong sum, strong product, the usual adjunction and scalar multiplication, except that if $\lambda = 0$ and $x \in S(\mathcal{M})$, then λx is defined to be the everywhere defined zero operator on the underlying Hilbert space $\mathcal{H}$.*

Theorem 1.5.3 *([89], Theorem 2) Let $\mathcal{M}$ be a commutative von Neumann algebra. Then, by Theorem 1.2.1, $\mathcal{M} \cong L_\infty(X, \Sigma, \mu)$ for some localizable measure space (X, Σ, μ), and $x \in S(\mathcal{M})$ if and only if x can be identified with a multiplication operator M_f for some $f \in L_0(X, \Sigma, \mu)$.*

We now introduce the notion of measurability which is most important to us in this thesis, namely that of measurability with respect to a trace.

For the remainder of this section, $\mathcal{M}$ will denote a semifinite von Neumann algebra equipped with a faithful semifinite normal trace τ, and $\mathcal{H}$ will denote the underlying Hilbert space of $\mathcal{M}$.

Definition 1.5.4 *([47], Definition 2.1) Let E be a subspace of $\mathcal{H}$. We say that E is τ-dense in $\mathcal{H}$ if for every $\delta > 0$, there exists a projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subset E$ and $\tau(1-p) < \delta$.*

An unbounded operator x is said to be τ-measurable if $x \in \mathcal{U}(\mathcal{M})$ and $\mathcal{D}(x)$ is τ-dense in $\mathcal{H}$.

The set of τ-measurable operators affiliated with $\mathcal{M}$ will be denoted by $\widetilde{\mathcal{M}}$.

Proposition 1.5.5 *([47], p. 271; [89], Corollary 4.1) If $\tau(1) < \infty$, then $\widetilde{\mathcal{M}} = \mathcal{U}(\mathcal{M})$. For any finite von Neumann algebra $\mathcal{M}$, one has $S(\mathcal{M}) = \mathcal{U}(\mathcal{M})$.*

Theorem 1.5.6 *([47], p. 272) Let $\epsilon, \delta > 0$ and let $\widetilde{\mathcal{M}}(\epsilon, \delta)$ denote the set*

$$\{x \in \widetilde{\mathcal{M}} : \text{ there is a } p \in \mathcal{M}_p \text{ such that } p\mathcal{H} \subset \mathcal{D}(x), \|xp\| \leq \epsilon, \tau(1-p) < \delta\}$$

The sets $\{\widetilde{\mathcal{M}}(\epsilon, \delta) : \epsilon, \delta > 0\}$ form a system of basic neighbourhoods of zero for a topology on $\widetilde{\mathcal{M}}$, called the topology of convergence in measure on $\widetilde{\mathcal{M}}$.

Theorem 1.5.7 *([47], p. 272) $\widetilde{\mathcal{M}}$ is a *-algebra under strong sum, strong product, scalar multiplication and ordinary adjunction. Furthermore, when equipped with the topology of convergence in measure, $\widetilde{\mathcal{M}}$ is a complete metrizable topological *-algebra. Lastly, $\mathcal{M}$ is dense in $\widetilde{\mathcal{M}}$ with respect to the topology of convergence in measure.*

It is now immediate from Theorem 1.4.1 that the multiplication on $\widetilde{\mathcal{M}}$ is jointly continuous.

Throughout this thesis, we will denote the topology of convergence in measure on $\widetilde{\mathcal{M}}$ by γ_{cm}. *In the sequel, unless stated otherwise, if $x, y \in \widetilde{\mathcal{M}}$, we will write $x+y$ and xy to mean the strong sum and strong product respectively of x and y.*

Example 1.5.8 Let $\mathcal{M}$ be a commutative von Neumann algebra. Then, by Theorem 1.2.1, $\mathcal{M} \cong L_\infty(X, \Sigma, \mu)$ for some localizable measure space (X, Σ, μ). One can easily verify that the restriction of the topology of convergence in measure to $\mathcal{M}$ is the familiar topology of convergence in measure on $L_\infty(X, \Sigma, \mu)$. By taking completions, we find that $\widetilde{\mathcal{M}}$ is topologically *-isomorphic to $\widetilde{L}_\infty(X, \Sigma, \mu)$.

Example 1.5.9 *([47], p. 271)* Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, and let τ denote the diagonal trace on $\mathcal{B}(\mathcal{H})$ (Example 1.2.3(ii)). Then $\widetilde{\mathcal{M}} = \mathcal{M}$.

Another name for the topology of convergence in measure is the *measure topology*. We will use these two terms interchangeably.

Theorem 1.5.10 *([94], Theorem 2.3(i)) Let $p \in \mathcal{M}_p$ and $\tau_p = \tau|_{(p\mathcal{M}p)^+}$. Then τ_p is a faithful semifinite normal trace on $p\mathcal{M}p$. Furthermore, $\widetilde{p\mathcal{M}p} = p\widetilde{\mathcal{M}}p$, where $\widetilde{p\mathcal{M}p}$ is the algebra of τ_p-measurable operators affiliated with $p\mathcal{M}p$.*

Proposition 1.5.11 *([94], Examples 2.2(3)) The following statements are equivalent.*

(i) $\widetilde{\mathcal{M}} = \mathcal{M}$.

(ii) $\inf\{\tau(p) : 0 \neq p \in \mathcal{M}_p\} > 0$.

(iii) The topology of convergence in measure coincides with the norm topology.

Proposition 1.5.12 *([43], Proposition 1.4) The set of positive elements in $\widetilde{\mathcal{M}}$ is closed with respect to the topology of convergence in measure.*

The following concept is an extension of the decreasing rearrangement of a measurable function (Definition 1.1.5).

Definition 1.5.13 *([47], Definition 2.1) Let $x \in \widetilde{\mathcal{M}}$ and $t > 0$. The generalized singular function of x, denoted by $\mu_t(x)$, is defined by*

$$\begin{aligned} \mu_t(x) &= \inf\{\|xp\| : p \in \mathcal{M}_p, p\mathcal{H} \subset \mathcal{D}(x), \text{ and } \tau(1-p) \leq t\} \\ &= \inf\{\theta \geq 0 : d_\theta(|x|) \leq t\}. \end{aligned}$$

Observe that $\mu_t(x)$ is decreasing as a function of t, so that $\lim_{t\to\infty} \mu_t(x)$ exists, it is denoted by $\mu_\infty(x)$ ([94], p. 74).

Proposition 1.5.14 *([47], Lemma 2.5) Let $x, y, z \in \widetilde{\mathcal{M}}$. The following statements hold for every $t > 0$.*

(i) $\mu_t(x) \leq \|x\|$.

(ii) $\mu_t(x) = 0$ *if and only if* $x = 0$.

(iii) $\mu_t(\alpha x) = |\alpha|\mu_t(x)$ *for all* $\alpha \in \mathbb{C}$.

(iv) $0 \leq x \leq y$ *implies that* $\mu_t(x) \leq \mu_t(y)$.

(v) $\mu_{t_1+t_2}(x + y) \leq \mu_{t_1}(x) + \mu_{t_2}(y)$ *for all* $t_1, t_2 > 0$.

(vi) $\mu_{t_1+t_2}(xy) \leq \mu_{t_1}(x)\mu_{t_2}(y)$ *for all* $t_1, t_2 > 0$.

(vii) $\mu_t(xyz) \leq \|x\|\mu_t(y)\|z\|$, *where the cases* $\|x\| = \infty$ *and* $\|y\| = \infty$ *are allowed.*

(viii) $\mu_t(x^*) = \mu_t(x)$.

The following result can be found in the proof of [94], Theorem 2.3.

Lemma 1.5.15 *([94], p. 77) Let $x \in \widetilde{\mathcal{M}}$ and $p \in \mathcal{M}_p$. If we denote the generalized singular function of pxp in $p\widetilde{\mathcal{M}}p$ (respectively x in $\widetilde{\mathcal{M}}$) by $\mu_t(pxp)$ (respectively $\mu_t(x)$), then $\mu_t(pxp) = \mu_t(x)$ for every $t > 0$.*

Lemma 1.5.16 *([47], Lemma 3.1)*

$$\widetilde{\mathcal{M}}(\epsilon, \epsilon) = \{x \in \widetilde{\mathcal{M}} : \mu_\epsilon(x) \leq \epsilon\}.$$

Consequently, if (x_n) is a sequence in $\widetilde{\mathcal{M}}$ and $x \in \widetilde{\mathcal{M}}$, then $x_n \to x(\gamma_{cm})$ if and only if for every $t > 0$, $\mu_t(x_n - x) \to 0$ as $n \to \infty$.

By Proposition 1.5.14(i) and Lemma 1.5.16, the measure topology of $\widetilde{\mathcal{M}}$ restricted to $\mathcal{M}$ is weaker than the norm topology of $\mathcal{M}$.

Definition 1.5.17 *([94], p. 75) Let $x \in \widetilde{\mathcal{M}}$. We say that x is τ-compact if $\mu_\infty(x) = \lim_{t\to\infty} \mu_t(x) = 0$. The set of τ-compact operators will be denoted by $\widetilde{\mathcal{M}_0}$.*

If $\mathcal{M} = \mathcal{B}(\mathcal{H})$, then $\widetilde{\mathcal{M}_0}$ is the ideal of compact operators of $\mathcal{B}(\mathcal{H})$ ([94], Examples 2.2).

Lemma 1.5.18 *([94], p. 75)*

(i) The set $\widetilde{\mathcal{M}_0}$ is a γ_{cm}-closed two-sided ideal of $\widetilde{\mathcal{M}}$.

(ii) If $p \in \mathcal{M}_p$, then $p \in \widetilde{\mathcal{M}_0}$ if and only if $\tau(p) < \infty$.

Theorem 1.5.19 *([94], Theorem 3.5) Let $\mathcal{M}_0 = \widetilde{\mathcal{M}_0} \cap \mathcal{M}$. Then, if $\mathcal{M}/\mathcal{M}_0$ is equipped with the quotient norm, it is a C^*-algebra. If $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}$ is equipped with the norm $\dot{\mu}_\infty$ defined by $\dot{\mu}_\infty(x + \widetilde{\mathcal{M}_0}) = \mu_\infty(x), x \in \widetilde{\mathcal{M}}$, then $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}$ is isometrically *-isomorphic to $\mathcal{M}/\mathcal{M}_0$, and hence $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}$ is a C^*-algebra.*

Theorem 1.5.20 *([44], Theorem 1.1) The map $x \mapsto x^{\frac{1}{2}}$ is $\gamma_{cm} - \gamma_{cm}$ continuous on the positive cone of $\widetilde{\mathcal{M}}$. Consequently, the map $x \mapsto |x|$ is $\gamma_{cm} - \gamma_{cm}$ continuous on $\widetilde{\mathcal{M}}$.*

Theorem 1.5.21 *([100], Theorem 2.1) Let $\mathcal{I}$ be a γ_{cm}-closed two-sided ideal of $\widetilde{\mathcal{M}}$. Then $\mathcal{I} \cap \mathcal{M}$ is a norm closed two-sided ideal of $\mathcal{M}$ and $\mathcal{I} = \overline{\mathcal{I} \cap \mathcal{M}}^{\gamma_{cm}}$.*

Our discussion will now turn to $\widetilde{\mathcal{M}}$ as a topological *-algebra.

Theorem 1.5.22 *([34], Theorem 1.5.1) If the measure topology on $\widetilde{\mathcal{M}}$ is locally convex, then $\mathcal{M}_p$ is atomic.*

Theorem 1.5.23 *([34], Corollary 1.5.7) Let (X, Σ, μ) be a localizable measure space. The following statements are equivalent.*

(i) *The topology of convergence in measure on* $\widetilde{L}_\infty(X, \Sigma, \mu)$ *is locally convex.*

(ii) (X, Σ, μ) *is an atomic measure space and*

$$\inf\{\mu(A) : A \in \Sigma,\ \mu(A) \neq 0\} > 0, \ \textit{or}$$

(X, Σ, μ) *is an atomic measure space,*

$$\inf\{\mu(A) : A \in \Sigma,\ \mu(A) \neq 0\} = 0,$$

and there exists $K > 0$ *such that*

$$\sum \mu(A) < \infty,$$

where the summation is taken over all atoms $A \in \Sigma$ *with* $\mu(A) < K$.

Let (X, Σ, μ) be a localizable atomic measure space with $\inf\{\mu(A) : A \in \Sigma,\ \mu(A) \neq 0\} = 0$ such that there exists $K > 0$ with $\sum \mu(A) < \infty$, where the summation is taken over all atoms A with $\mu(A) < K$. There are at most countably many atoms A with $\mu(A) < K$. The set of all such atoms can be written as a sequence (A_n) with $\mu(A_n) \downarrow 0$ ([34], p. 29). Let $E = \cup_{n=1}^{\infty} A_n$. Define $\Sigma_E = \{A \in \Sigma : \mu(A) < K\}$ and $\mu_E = \mu|_E$. We define $\Sigma_{X\setminus E}$ and $\mu_{X\setminus E}$ is a similar manner ([34], p. 29).

Theorem 1.5.24 *([34], Theorem 1.7.1) Let* (X, Σ, μ) *be a localizable atomic measure space with*

$$\inf\{\mu(A) : A \in \Sigma,\ \mu(A) \neq 0\} = 0$$

such that there exists $K > 0$ *with* $\sum \mu(A) < \infty$*, where the summation is taken over all atoms* A *with* $\mu(A) < K$. *Then*

$$\left(\widetilde{L}_\infty(X, \Sigma, \mu), \gamma_{cm}\right) = \left(L_0(E, \Sigma_E, \mu_E), \gamma_p\right) \oplus \left(L_\infty(X\setminus E, \Sigma_{X\setminus E}, \mu_{X\setminus E}), \|\cdot\|_\infty\right),$$

where γ_p *denotes the pointwise topology on* $L_0(E, \Sigma_E, \mu_E)$ *and* $\|.\|_\infty$ *the essential supremum norm.*

Theorem 1.5.25 *([34], Theorem 4.3.1) If $\mathcal{M}_p$ is non-atomic, then the dual of $\widetilde{\mathcal{M}_0}$ is trivial.*

Theorem 1.5.26 *If $\widetilde{\mathcal{M}}$ is commutative and locally convex with respect to the measure topology, then $\widetilde{\mathcal{M}}$ is a locally C*-algebra with respect to the measure topology.*

Proof. We can identify $\widetilde{\mathcal{M}}$ with $\widetilde{L}_\infty(X, \Sigma, \mu)$ for some localizable measure space (X, Σ, μ) (Example 1.5.8). If $\widetilde{L}_\infty(X, \Sigma, \mu) = L_\infty(X, \Sigma, \mu)$, then it is obvious that $\widetilde{L}_\infty(X, \Sigma, \mu)$ is a (locally) C*-algebra.

Consider the case $\widetilde{L}_\infty(X, \Sigma, \mu) \neq L_\infty(X, \Sigma, \mu)$. Then, since $\widetilde{L}_\infty(X, \Sigma, \mu)$ is locally convex, it follows from Proposition 1.5.11, Corollary 1.5.23 and Theorem 1.5.24 that

$$\left(\widetilde{L}_\infty(X, \Sigma, \mu), \gamma_{cm}\right) = \left(L_0(E, \Sigma_E, \mu_E), \gamma_p\right) \oplus \left(L_\infty(X \backslash E, \Sigma_{X\backslash E}, \mu_{X\backslash E}), \|.\|_\infty\right).$$

For each n, let $p_n(f) = |f(A_n)| + \|f\chi_{X\backslash E}\|$ for every $f \in \widetilde{L}_\infty(X, \Sigma, \mu)$. It can be shown that the p_n are a family of C*-seminorms defining the measure topology of $\widetilde{L}_\infty(X, \Sigma, \mu)$. ∇

Lemma 1.5.27 *If $x \in \widetilde{\mathcal{M}}$, then $1 + x^*x$ is invertible in $\widetilde{\mathcal{M}}$.*

Proof. For the purposes of this proof only, we distinguish between the ordinary sum $x + y$, the strong sum $\overline{x + y}$, the ordinary product xy and the strong product $\overline{xy}$ of x and y, where $x, y \in \widetilde{\mathcal{M}}$.

Let $y = 1 + x^*x$ with $x \in \widetilde{\mathcal{M}}$. Recall that x^*x is self-adjoint and thus closed (see Proposition 1.3.1). So $x^*x \in \widetilde{\mathcal{M}}$ and y is closed, since the sum of a bounded operator and a closed operator is closed. Therefore $y \in \widetilde{\mathcal{M}}$ since x^*x and 1 are in $\widetilde{\mathcal{M}}$. It is easily verified that y is self-adjoint.

By Proposition 1.3.1, there exists $z \in \mathcal{B}(H)$ such that $zy \subset yz = 1$. Since y is self-adjoint, there is a smallest von Neumann algebra $\mathcal{A}$ with which y is affiliated, and $\mathcal{A}$ is commutative (Theorem 1.3.9). Hence $\mathcal{A} \subset \mathcal{M} \subset \widetilde{\mathcal{M}}$. Therefore, by [63], p.357, $z \in \mathcal{A}$ and thus $z \in \widetilde{\mathcal{M}}$. It follows from Theorem 1.3.6 that $\overline{zy} = \overline{yz}$. So $\overline{zy} = \overline{yz} = \overline{1} = 1$, and so y is invertible in $\widetilde{\mathcal{M}}$. ∇

The next lemma is needed in the proof of Theorem 1.5.29 below, and appears in the proof of [41], Theorem 7.3. We give the proof for completeness.

Lemma 1.5.28 *Let $x \in \widetilde{\mathcal{M}}$ such that x^*x is not in the unit ball of $\mathcal{M}$. Then there exists $k > 1$ and $q \in \mathcal{M}_p$ such that $\|(x^*x)^n\psi\| \geq k^n\|\psi\|$ for all $\psi \in q(\mathcal{H})$ and for all $n \in \mathbb{N}$.*

Proof. By the spectral theorem, we may write $x^*x = \int \lambda de_\lambda$. Let $k > 0$ and $q = 1 - e_k$, where $e_\lambda = e_\lambda(x^*x)$ for every $\lambda \geq 0$. If $\lambda \leq k$, then $e_\lambda q = e_\lambda(1 - e_k) = 0$. Clearly, $\lambda^n > k^n$ if $\lambda > k$. Therefore, if $\psi \in q(\mathcal{H})$, it follows that, for $n \in \mathbb{N}$,

$$\begin{aligned}
\|(x^*x)^n\psi\|^2 &= \int_{\mathbb{R}} \lambda^{2n} d\langle e_\lambda\psi, \psi\rangle \\
&= \int_k^\infty \lambda^{2n} d\langle e_\lambda\psi, \psi\rangle \\
&\geq k^{2n} \int_k^\infty d\langle e_\lambda\psi, \psi\rangle \\
&= k^{2n}(\langle\psi, \psi\rangle - \langle e_k\psi, \psi\rangle) \\
&= k^{2n}\|\psi\|^2.
\end{aligned}$$

Thus $\|(x^*x)^n\psi\| \geq k^n\|\psi\|$ for all $\psi \in q(\mathcal{H})$. Since x^*x is not in the unit ball of $\mathcal{M}$, we can choose $k > 1$. ∇

The proof of Theorem 1.5.29 below is a slight modification of part of the proof of [41], Theorem 7.3, in that Dixon's proof uses the dimension function and we use the trace. We use the same notation as in Section 1.4.

Theorem 1.5.29 *$\widetilde{\mathcal{M}}$ is a GB*-algebra with $(\widetilde{\mathcal{M}})_b = \mathcal{M}$.*

Proof. By Lemma 1.5.27, $(1 + x^*x)^{-1} \in \mathcal{M}$ for every $x \in \widetilde{\mathcal{M}}$. We will show that the unit ball B_0 of $\mathcal{M}$ is the greatest member of the corresponding class of sets $\mathcal{C}$ defined in Section 1.4.

Let $B \in \mathcal{C}$. We show first that if $x \in B$, then $x \in B_0$. Suppose that $x \notin B_0$. Then, by considering the polar decomposition $x = v(x^*x)^{\frac{1}{2}}$ of x, it follows that $x^*x \notin B_0$. By Lemma 1.5.28, there exists $k > 1$ and $q \in \mathcal{M}_p$ such that $\|(x^*x)^n\psi\| \geq k^n\|\psi\|$ for all $\psi \in q(\mathcal{H})$ and for all $n \in \mathbb{N}$. Since τ is faithful and $q \neq 0$, we have $\tau(q) > 0$, say $\tau(q) > r > 0$. Let $\epsilon = \frac{r}{2}$. Since $x \in B \in \mathcal{C}$, it follows that $\{(x^*x)^n : n = 1, 2, \ldots\} \subset B$. By the γ_{cm}-boundedness of B, there exists $\alpha > 0$ such that $\{(x^*x)^n : n = 1, 2, \ldots\} \subset \alpha\widetilde{\mathcal{M}}(\epsilon, \epsilon) = \widetilde{\mathcal{M}}(\alpha\epsilon, \epsilon)$. Hence, for each $n = 1, 2, \ldots$, there exists a projection $p_n \in \mathcal{M}$ such that $\|(x^*x)^n p_n\| < \alpha\epsilon$ and $\tau(1 - p_n) < \epsilon$.

Suppose that $p_n(\mathcal{H}) \cap q(\mathcal{H}) = \{0\}$ for some n. Then, by Proposition 1.2.4(iii), $\tau(q) \leq \tau(1 - p_n) < \epsilon = \frac{r}{2} < r < \tau(q)$. This is a contradiction. Hence $p_n(\mathcal{H}) \cap q(\mathcal{H}) \neq \{0\}$ for every $n = 1, 2, \ldots$.

Therefore for each n, there exists $0 \neq \psi_n \in p_n(\mathcal{H}) \cap q(\mathcal{H})$. Without loss of generality, $\|\psi_n\| = 1$. Since $\psi_n \in p_n(\mathcal{H})$, $\psi_n = p_n(\psi_n)$ and thus $\|(x^*x)^n(\psi_n)\| = \|(x^*x)^n(p_n(\psi_n))\| \leq \|(x^*x)^n p_n\| < \alpha\epsilon$ for all n. Since $\psi_n \in q(\mathcal{H})$, it follows that $\|(x^*x)^n(\psi_n)\| \geq k^n$. A contradiction results for sufficiently large n. Hence $x \in B_0$. Hence $B \subset B_0$ for every $B \in \mathcal{C}$.

It remains to show that $B_0 \in \mathcal{C}$. Certainly, $\overline{B_0}^{\gamma_{cm}}$ is γ_{cm}-closed. Let $\delta > 0$. Then $\widetilde{\mathcal{M}}(\delta, \delta) \cap \mathcal{M}$ is a γ_{cm}-neighbourhood of $0 \in \mathcal{M}$. Since the restriction of the measure topology to $\mathcal{M}$ is weaker than the norm topology on $\mathcal{M}$, $\widetilde{\mathcal{M}}(\delta, \delta) \cap \mathcal{M}$ is a neighbourhood of zero with respect to the norm topology on $\mathcal{M}$. Since B_0 is a bounded subset of $\mathcal{M}$ with respect to the norm topology of $\mathcal{M}$, it follows that there exists $\lambda > 0$ such that $B_0 \subset \lambda(\widetilde{\mathcal{M}}(\delta, \delta) \cap \mathcal{M}) \subset \lambda(\widetilde{\mathcal{M}}(\delta, \delta)$. Hence B_0, and thus $\overline{B_0}^{\gamma_{cm}}$, is γ_{cm}-bounded. Since $1 \in B_0, B_0^2 \subset B_0$ and $B_0^* = B_0$, the same holds for $\overline{B_0}^{\gamma_{cm}}$. It follows that $\overline{B_0}^{\gamma_{cm}} \in \mathcal{C}$. From what we have proved in the previous paragraph, $\overline{B_0}^{\gamma_{cm}} \subset B_0$. Thus $\overline{B_0}^{\gamma_{cm}} = B_0$. Hence $B_0 \in \mathcal{C}$. ∇

Theorem 1.5.30 *$\widetilde{\mathcal{M}}$ is semi-simple, i.e. the (Jacobson) radical Rad($\widetilde{\mathcal{M}}$) of $\widetilde{\mathcal{M}}$ is zero.*

Proof. Since $\mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$ is a two-sided ideal of $\mathcal{M}$, it follows from Proposition 1.2.11 that $\mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$ is self-adjoint. Assume that $0 \neq x \in \mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$. Then $0 \neq x^* \in \mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$. Therefore $y_1 = \frac{1}{2}(x + x^*) \in \mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$ and $y_2 = \frac{1}{2i}(x - x^*) \in \mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$. Since $x \neq 0$, it follows that at least one of y_1 and y_2 are nonzero, implying that $\mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$ has at least one nonzero self-adjoint element.

Let a be a nonzero self-adjoint element of $\mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$. Then, by the spectral theorem, a has a spectral resolution $\{e_\lambda(a) : \lambda \in \mathbb{R}\}$.

Suppose that $e_\lambda(a) = 0$ for every $\lambda < 0$, and $1 - e_\lambda(a) = 0$ for every $\lambda > 0$, i.e. $e_\lambda(a) = 1$ for every $\lambda > 0$. By the spectral theorem, $e_0(a) = \wedge_{\lambda > 0} e_\lambda(a) = 1$. Hence, for every $\lambda \in \mathbb{R}$, $e_\lambda(a) = 0$ or $e_\lambda(a) = 1$. It follows from the spectral theorem that $a = 0$ or $a = \lambda 1$ for some $0 \neq \lambda \in \mathbb{R}$. But $a \neq 0$ by assumption. So $a = \lambda 1$ for some $0 \neq \lambda \in \mathbb{R}$. Since $a \in \mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$ and $\lambda \neq 0$, it follows that $1 \in \mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$, which is a contradiction. Therefore, either there exists $\lambda < 0$ such that $e_\lambda(a) \neq 0$, or there exists $\lambda > 0$ such that $1 - e_\lambda(a) \neq 0$.

By Proposition 1.3.12, it follows that $e_\lambda(a) \in \mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$ for every $\lambda < 0$ and $1 - e_\lambda(a) \in \mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$ for every $\lambda > 0$. Thus $\mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$ contains at least one nonzero projection. But $\mathrm{Rad}(\widetilde{\mathcal{M}})$ has no nonzero projections. This is a contradiction, implying that $\mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M} = \{0\}$.

Now let $x \in \mathrm{Rad}(\widetilde{\mathcal{M}})$. By Proposition 1.3.1, $x(1 + x^*x)^{-1}$ is a bounded everywhere defined operator. Recall that $(1 + x^*x)^{-1} \in \mathcal{M}$. Therefore, since $x(1 + x^*x)^{-1}$ is affiliated with $\mathcal{M}$, $x(1 + x^*x)^{-1} \in \mathrm{Rad}(\widetilde{\mathcal{M}}) \cap \mathcal{M}$. Hence $x(1 + x^*x)^{-1} = 0$ from what we have proved above. Thus $x = 0$, implying that $\mathrm{Rad}(\widetilde{\mathcal{M}}) = \{0\}$. ∇

Chapter 2

Jordan homomorphisms and derivations on operator algebras

In this chapter, we give a brief survey of known results on Jordan homomorphisms and derivations on C*-algebras and von Neumann algebras. No proofs will therefore be given. *All algebras are assumed to have a unit element unless otherwise stated.*

2.1 Jordan homomorphisms and invertibility preserving linear maps of algebras

Definition 2.1.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras, and $\phi : \mathcal{A} \to \mathcal{B}$ a linear map. We say that ϕ is an algebra homomorphism (respectively algebra anti-homomorphism) if $\phi(xy) = \phi(x)\phi(y)$ (respectively $\phi(xy) = \phi(y)\phi(x)$) for every $x, y \in \mathcal{A}$. The map ϕ is called a Jordan homomorphism if $\phi(x^2) = \phi(x)^2$ for every $x \in \mathcal{A}$.*

If ϕ is a bijective algebra homomorphism (respectively bijective algebra anti-homomorphism), then ϕ will be called an algebra isomorphism (respectively algebra anti-isomorphism). We call ϕ a Jordan isomorphism if ϕ is a *bijective Jordan homomorphism.*

Lemma 2.1.2 *([91], p. 13) Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. Then a linear map $\phi : \mathcal{A} \to \mathcal{B}$ is a Jordan homomorphism if and only if $\phi(xy+yx) = \phi(x)\phi(y)+\phi(y)\phi(x)$ for all $x, y \in \mathcal{A}$.*

Proposition 2.1.3 *([59], p. 50) If $\mathcal{R}_1$ and $\mathcal{R}_2$ are rings and $\phi : \mathcal{R}_1 \to \mathcal{R}_2$ is a Jordan homomorphism, then $\phi\Big((xy-yx)^2\Big) = \Big(\phi(x)\phi(y) - \phi(y)\phi(x)\Big)^2$.*

The proofs of the following two propositions are contained in the proof of the corollary on p. 190 in [29].

Proposition 2.1.4 *([29], p. 190) Let $\mathcal{A}$ and $\mathcal{B}$ be algebras and suppose that $\phi : \mathcal{A} \to \mathcal{B}$ is an idempotent preserving linear map. If $x = \Sigma_{i=1}^{n} \lambda_i p_i$, where λ_i $(i = 1, \ldots, n)$ are complex scalars and p_i $(i = 1, \ldots, n)$ are idempotents in $\mathcal{A}$ such that $p_i p_j = p_j p_i = 0$ for all $i \neq j$ $(i, j = 1, \ldots, n)$, then $\phi(x^2) = \phi(x)^2$.*

Proposition 2.1.5 *([29], p. 190) Let $\mathcal{A}$ be a *-algebra, $\mathcal{B}$ an algebra and $\phi : \mathcal{A} \to \mathcal{B}$ a linear map. If $\phi(x^2) = \phi(x)^2$ for all self-adjoint elements of $\mathcal{A}$, then ϕ is a Jordan homomorphism.*

We say that a linear map $\phi : \mathcal{A} \to \mathcal{B}$ between algebras $\mathcal{A}$ and $\mathcal{B}$ *preserves commutativity* if we have that $\phi(x)\phi(y) = \phi(y)\phi(x)$ whenever $xy = yx$ with $x, y \in \mathcal{A}$. For example, every algebra homomorphism and algebra anti-homomorphism between algebras preserves commutativity.

Theorem 2.1.6 *([59], Corollary 1) Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be rings and $\phi : \mathcal{R}_1 \to \mathcal{R}_2$ a Jordan homomorphism. Then ϕ preserves commutativity provided that the ring generated by $\phi(\mathcal{R}_1)$ has no nonzero nilpotent elements in its centre.*

It should be emphasized that Jordan homomorphisms need not be algebra homomorphisms nor algebra anti-homomorphisms. This will be illustrated in the following example.

Example 2.1.7 *([11], p. 918)* Let $\mathcal{A} = \mathcal{B} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$ and $\phi : \mathcal{A} \to \mathcal{B}$ a linear map defined by $\phi(x, y) = \phi(x, y^t)$ for every $x, y \in \mathcal{A}$, where y^t denotes the transpose of y. Then it can be shown that ϕ is a Jordan homomorphism and that ϕ is neither an algebra homomorphism nor an algebra anti-homomorphism.

Theorem 2.1.8 *([54], Theorem 3.1) Let $\mathcal{R}_1$ be a ring and $\mathcal{R}_2$ a prime ring. Then every Jordan homomorphism of $\mathcal{R}_1$ onto $\mathcal{R}_2$ is an algebra homomorphism or an algebra anti-homomorphism.*

We say that a nonzero linear functional f on an algebra $\mathcal{A}$ is a *Jordan functional* if f is a Jordan homomorphism. A linear functional f on an algebra $\mathcal{A}$ will be called a *character* if f is a nonzero multiplicative linear functional on $\mathcal{A}$, i.e. $f(xy) = f(x)f(y)$ for every $x, y \in \mathcal{A}$.

Proposition 2.1.9 *([27], Proposition 16.6) Every Jordan functional on an algebra is a character.*

Definition 2.1.10 *Let $\mathcal{A}$ and $\mathcal{B}$ be algebras with unit elements 1 and $1'$ respectively. A linear map $\phi : \mathcal{A} \to \mathcal{B}$ is called unital if $\phi(1) = 1'$.*

Definition 2.1.11 *If $\mathcal{A}$ and $\mathcal{B}$ are unital algebras, then a linear map $\phi : \mathcal{A} \to \mathcal{B}$ is said to be invertibility preserving if $\phi(x)$ is an invertible element of $\mathcal{B}$ whenever x is an invertible element of $\mathcal{A}$.*

Proposition 2.1.12 *([91], Proposition 1.3) Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. If $\phi : \mathcal{A} \to \mathcal{B}$ is a Jordan homomorphism whose range contains the unit element of $\mathcal{B}$, then ϕ is invertibility preserving.*

In particular, any surjective Jordan homomorphism between algebras is invertibility preserving.

Lemma 2.1.13 *([29], p. 187) Let $\mathcal{A}$ and $\mathcal{B}$ be algebras. A unital linear map $\phi : \mathcal{A} \to \mathcal{B}$ is invertibility preserving if and only if Sp $(\phi(x), \mathcal{B}) \subset$ Sp $(x, \mathcal{A})$ for all $x \in \mathcal{A}$.*

Definition 2.1.14 *A linear map* $\phi : \mathcal{A} \to \mathcal{B}$ *between algebras* $\mathcal{A}$ *and* $\mathcal{B}$ *is spectrum preserving if and only if* $Sp\ (\phi(x), \mathcal{B}) = Sp\ (x, \mathcal{A})$ *for every* $x \in \mathcal{A}$.

Proposition 2.1.15 *([75], p. 266) Let* $\phi : \mathcal{A} \to \mathcal{B}$ *be a surjective linear map between Banach algebras* $\mathcal{A}$ *and* $\mathcal{B}$*. The following statements are equivalent.*

(i) ϕ *is spectrum preserving.*

(ii) ϕ *is unital and* $\phi(x)$ *is an invertible element of* $\mathcal{B}$ *if and only if* x *is an invertible element of* $\mathcal{A}$.

2.2 Jordan homomorphisms of operator algebras

It follows from Proposition 2.1.12 that every unital Jordan homomorphism between algebras is invertibility preserving. This motivated I. Kaplansky to raise the question as to when unital linear invertibility preserving maps between unital algebras are Jordan homomorphisms ([29]). Kaplansky's original question was also motivated by the following results of Gleason, Kahane and Żelazko as well as of Marcus and Purves given below.

Theorem 2.2.1 (Gleason-Kahane-Żelazko) *([10], Theorem 4.1.1, [64]) Let* $\mathcal{A}$ *be a Banach algebra and* f *a nonzero continuous linear functional on* $\mathcal{A}$*. Then* f *is a character on* $\mathcal{A}$ *if and only if* $f(x) \in Sp\ (x, \mathcal{A})$ *for each* $x \in \mathcal{A}$.

It follows from Theorem 2.2.1 and Lemma 2.1.13 that if f is a unital invertibility preserving linear functional on a Banach algebra $\mathcal{A}$, then f is a character on $\mathcal{A}$.

Theorem 2.2.2 (Marcus-Purves) *([74], Theorem 2.1) Suppose that* $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ *is a unital invertibility preserving linear map. Then* ϕ *is either of the form* $\phi(x) = axa^{-1}$ *or* $\phi(x) = ax^ta^{-1}$ *for some invertible matrix* a*, where* x^t *denotes the transpose of the matrix* x.

With the additional help of all known counter examples, the following conjecture seems to be reasonable ([29]):

Conjecture: *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a unital bijective invertibility preserving linear map between semi-simple Banach algebras $\mathcal{A}$ and $\mathcal{B}$. Then ϕ is a Jordan isomorphism.*

One can ask an easier question than that which the conjecture raises: *Let $\mathcal{A}$ and $\mathcal{B}$ be semi-simple Banach algebras, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective unital linear map having the property that $\phi(x)$ is an invertible element of $\mathcal{B}$ exactly when x is an invertible element of $\mathcal{A}$. Is it true that ϕ is a Jordan isomorphism?*

In light of Proposition 2.1.15, we can rephrase this question as follows: Let $\mathcal{A}$ and $\mathcal{B}$ be semi-simple Banach algebras, and let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective spectrum preserving linear map. Is it true that ϕ is a Jordan isomorphism?

Theorem 2.2.3 *([64], Theorem 4) Let $\mathcal{A}$ be a Banach algebra and $\mathcal{B}$ a commutative semi-simple Banach algebra. If $\phi : \mathcal{A} \to \mathcal{B}$ is a unital linear invertibility preserving linear map, then ϕ is an algebra homomorphism.*

The above conjecture can now be rephrased as asking whether the commutativity assumption in Theorem 2.2.3 can be dropped if one makes the additional assumption that $\mathcal{A}$, in Theorem 2.2.3, be semi-simple.

We now give some affirmative answers to the above conjecture in the context of C*-algebras and von Neumann algebras.

Theorem 2.2.4 *([32], Theorem 6) Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras. Every unital surjective self-adjoint invertibility preserving linear map $\phi : \mathcal{A} \to \mathcal{B}$ is a Jordan homomorphism.*

In light of the conjecture, it would be interesting to know if the self-adjointness of ϕ in Theorem 2.2.4 can be removed if one makes the additional assumption that ϕ is injective. In 2000, B. Aupetit provided an answer to the above question regarding spectrum preserving linear mappings, which is the following result.

Theorem 2.2.5 *([11], Theorem 1.3 and Remark 2.7) Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ a semi-simple Banach algebra. If $\phi : \mathcal{A} \to \mathcal{B}$ is a unital bijective invertibility preserving linear map, then ϕ is a Jordan isomorphism.*

Theorem 2.2.6 *([46], Corollary 2) Let ϕ be a unital linear map of a C*-algebra $\mathcal{A}$ into another C*-algebra $\mathcal{B}$ such that ϕ maps unitary elements of $\mathcal{A}$ to unitary elements of $\mathcal{B}$. Then ϕ is a Jordan homomorphism.*

There is an interesting link between Jordan homomorphisms and the Banach-Stone theorem, as pointed out by R. V. Kadison in [62]. First, recall a version of the classical Banach-Stone theorem which says that a linear map $\phi : C(X) \to C(Y)$ is an isometry of $C(X)$ onto $C(Y)$ if and only if ϕ is a self-adjoint algebra isomorphism of $C(X)$ onto $C(Y)$. Here, $C(X)$ and $C(Y)$ denote the algebras of complex-valued continuous functions on X and Y respectively, where X and Y are compact Hausdorff spaces. In 1951, bearing in mind that commutative C*-algebras are up to *-isomorphism simply just the $C(X)$'s, Kadison extended the Banach-Stone result to generally noncommutative C*-algebras, thereby obtaining a "noncommutative Banach-Stone theorem" ([62]). This is the following result.

Theorem 2.2.7 *([62], Theorem 5 and Theorem 7) Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ a unital surjective self-adjoint linear map. Then ϕ is a Jordan isomorphism if and only if ϕ is an isometry.*

Later, it was shown more generally by E. Stϕrmer that every injective self-adjoint Jordan homomorphism of one C*-algebra into another is an isometry ([92], Corollary 3.5).

Theorem 2.2.8 *([92], Lemma 3.2) Let ϕ be a self-adjoint Jordan homomorphism of a von Neumann algebra $\mathcal{A}$ into a C*-algebra $\mathcal{B}$. Then there exist orthogonal projections p and q in $\mathcal{B}$ with $p + q = 1$ such that the map*

$\phi_1 : \mathcal{A} \to \phi(\mathcal{A})p$, defined by $x \mapsto \phi(x)p$, and the map $\phi_2 : \mathcal{A} \to \phi(\mathcal{A})q$, defined by $x \mapsto \phi(x)q$, is a self-adjoint algebra homomorphism and a self-adjoint algebra anti-homomorphism respectively. Furthermore, $\phi = \phi_1 + \phi_2$.

Theorem 2.2.9 *([92], Corollary 3.6) Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and $\phi : \mathcal{A} \to \mathcal{B}$ a self-adjoint Jordan homomorphism. Then ϕ is an algebra homomorphism if and only if there exists $\alpha > 0$ such that $\phi(a^*a) \geq \alpha\phi(a^*)\phi(a)$ for all $a \in \mathcal{A}$.*

2.3 Derivations on operator algebras

We begin by defining the key concept of this section, namely that of a derivation.

Definition 2.3.1 *Let $\mathcal{A}$ be an algebra. A linear map $D : \mathcal{A} \to \mathcal{A}$ is called a derivation on $\mathcal{A}$ if $D(xy) = xD(y) + D(x)y$ for every $x, y \in \mathcal{A}$. If, in addition, D is self-adjoint, we say that D is a *-derivation on $\mathcal{A}$.*

Lemma 2.3.2 *([28], p. 229) Let $D : \mathcal{A} \to \mathcal{A}$ be a derivation on a *-algebra $\mathcal{A}$. Then $D = D_1 + iD_2$, where D_1 (respectively D_2) is defined to be $D_1 = \frac{1}{2}(D(x) + D(x^*)^*)$ for every $x \in \mathcal{A}$ (respectively $D_2(x) = \frac{1}{2i}(D(x) - D(x^*)^*)$ for every $x \in \mathcal{A}$). Furthermore D_1 and D_2 are *-derivations on $\mathcal{A}$.*

Let $\mathcal{A}$ be an algebra and $a \in \mathcal{A}$. If a linear map $D : \mathcal{A} \to \mathcal{A}$ is defined by $D(x) = ax - xa$ for every $x \in \mathcal{A}$, then it can easily be verified that D is a derivation on $\mathcal{A}$ ([36], p. 116). Such derivations are called *inner derivations* ([36], p. 116). Observe that in a commutative algebra, a derivation is inner if and only if it is the zero derivation.

Proposition 2.3.3 *([36]) Let $\mathcal{A}$ be an algebra and let us replace the usual multiplication on $\mathcal{A} \oplus \mathcal{A}$ with a different (non-associative) multiplication defined as follows:*

$$(a, b)(u, v) = (au, av + bu)$$

for all $a, b, u, v \in \mathcal{A}$. Let D be a derivation on $\mathcal{A}$ and define a linear map $\theta : \mathcal{A} \to \mathcal{A} \oplus \mathcal{A}$ by

$$\theta(x) = (x, D(x))$$

for every $x \in \mathcal{A}$. Then θ is an algebra homomorphism with respect to the newly defined multiplication on $\mathcal{A} \oplus \mathcal{A}$.

If, in addition, $\mathcal{A}$ is a topological algebra and $\mathcal{A} \oplus \mathcal{A}$ is equipped with the corresponding product topology, then D is continuous if and only if θ is continuous.

Proposition 2.3.4 *([86], Lemma 4.1.2) The only derivation on a commutative C*-algebra is the zero derivation.*

Theorem 2.3.5 *([83], Theorem 2;[86], Lemma 4.1.3) Every derivation on a C*-algebra $\mathcal{A}$ into a Banach $\mathcal{A}$-module is continuous. In particular, every derivation on a C*-algebra is continuous.*

Theorem 2.3.6 *([86], Theorem 4.1.6) Every derivation D of a von Neumann algebra is inner, i.e. there exists $a \in \mathcal{A}$ such that $D(x) = ax - xa$ for every $x \in \mathcal{A}$.*

Theorem 2.3.7 *([86], Corollary 4.1.7) Let $\mathcal{A}$ be a C*-algebra on a Hilbert space $\mathcal{H}$. If D is a derivation on $\mathcal{A}$, then D is ultraweakly continuous and there exists an element $a \in \overline{\mathcal{A}}$, the ultraweak closure of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$, such that $D(x) = ax - xa$ for every $x \in \mathcal{A}$.*

Example 2.3.8 *([86], Example 4.1.8)* Consider the C*-algebra $K(\mathcal{H})$ of all compact operators on a separable Hilbert space $\mathcal{H}$. For $a \in \mathcal{B}(\mathcal{H})$, let $D(x) = ax - xa$ for every $x \in K(\mathcal{H})$. Then, since $K(\mathcal{H})$ is a two-sided ideal of $\mathcal{B}(\mathcal{H})$, D is a derivation on $K(\mathcal{H})$, but it is not inner if $a \notin K(\mathcal{H}) + \mathbb{C}1_{\mathcal{H}}$.

Chapter 3

Jordan homomorphisms between algebras of measurable operators

Recall that all self-adjoint Jordan homomorphisms between C*-algebras are continuous (Theorem 2.2.7). In particular, self-adjoint algebra homomorphisms between C*-algebras are continuous, in fact, norm reducing. Also, every self-adjoint Jordan homomorphism between von Neumann algebras is expressible as a sum of a self-adjoint algebra homomorphism and a self-adjoint algebra anti-isomorphism (Theorem 2.2.8). Motivated by these results, we explore in this chapter, amongst other things, the extent to which these results carry over to Jordan homomorphisms between algebras of measurable operators.

In Section 3.1,we prove the automatic continuity of characters on complete metrizable GB*-algebras. In Section 3.2, we show, amongst other things, that every Jordan homomorphism between algebras of measurable operators is $\gamma_{cm} - \gamma_{cm}$ continuous. Some results about Jordan homomorphisms between locally convex GB*-algebras are collected in Section 3.3. The main result of Section 3.4 is that every self-adjoint Jordan homomorphism between algebras of measurable operators can be expressed as a sum of a self-adjoint algebra homomorphism and a self-adjoint algebra anti-homomorphism. Section 3.5 gives conditions under which Jordan homomorphisms between algebras of measurable operators are algebra homomorphisms.

Throughout this chapter, the notation $\widetilde{\mathcal{A}}$ (respectively $\widetilde{\mathcal{B}}$) will always stand for the algebra of $\tau_{\mathcal{A}}-$ (respectively $\tau_{\mathcal{B}}-$) measurable operators affiliated with a von Neumann algebra $\mathcal{A}$ (respectively $\mathcal{B}$) equipped with a faithful semifinite normal trace $\tau_{\mathcal{A}}$ (respectively $\tau_{\mathcal{B}}$).

3.1 Continuity of characters on complete metrizable GB*-algebras

In this section, we prove that all characters on $\widetilde{\mathcal{M}}$ are measure continuous (Corollary 3.1.5 below). We first give an example of a character on $\widetilde{\mathcal{M}}$.

Example 3.1.1 Recall from Section 1.4.1 that any commutative complete locally m-convex topological algebra $\mathcal{A}$ has at least one continuous character. In particular, if $\mathcal{A}$ is a commutative $\widetilde{\mathcal{M}}$ which is locally convex with respect to the topology of convergence in measure, then $\mathcal{A}$ is locally m-convex with respect to the topology of convergence in measure (Theorem 1.5.26), and therefore $\mathcal{A}$ has at least one γ_{cm}-continuous character.

Let (X, Σ, μ) be a localizable measure space and $A \in \Sigma$ an atom. The map $\phi_A : \widetilde{L}_\infty(X, \Sigma, \mu) \to \mathbb{C}$, defined as $\phi_A(f) = f(A)$ for every $f \in \widetilde{L}_\infty(X, \Sigma, \mu)$, is a character on $\widetilde{L}_\infty(X, \Sigma, \mu)$. This makes sense since measurable functions are constant on atoms.

Lemma 3.1.2 *Let $\mathcal{A}$ be a unital *-algebra and $x \in \mathcal{A}$. If $1+x^*x$ is invertible in $\mathcal{A}$, then $-x^*x$ is quasi-invertible in $\mathcal{A}$.*

Proof. Let $x \in \mathcal{A}$ be such that $1 + x^*x$ is invertible in $\mathcal{A}$ and let $y = (1 + x^*x)^{-1}(x^*x)$. Then $(1 + x^*x)y = x^*x$, i.e. $y + (x^*x)y = x^*x$. Thus $-x^*x + y - (-x^*x)y = 0$. This implies that $-x^*x$ is right quasi-invertible. Similarly, $-x^*x$ is left quasi-invertible (simply take $z = (x^*x)(1 + x^*x)^{-1}$). Hence $-x^*x$ is quasi-invertible. ∇

Theorem 3.1.3 *([79], Theorem 3) If $\mathcal{A}$ is a complete metrizable topological *-algebra such that $-x^*x$ is quasi-invertible for every $x \in \mathcal{A}$, then every character on $\mathcal{A}$ is continuous.*

Proposition 3.1.4 *Every Jordan functional on a complete metrizable GB*-algebra $\mathcal{A}$ is continuous.*

Proof. Let $x \in \mathcal{A}$. Since $\mathcal{A}$ is a GB*-algebra, $1 + x^*x$ is invertible in $\mathcal{A}$ and therefore, by Lemma 3.1.2, $-x^*x$ is quasi-invertible in $\mathcal{A}$. This holds for each $x \in \mathcal{A}$. Any Jordan functional on a complex algebra is a character (Proposition 2.1.9). The result follows from Theorem 3.1.3. ∇

Since $\widetilde{\mathcal{M}}$ is a GB*-algebra (Theorem 1.5.29), the following corollary is an immediate consequence of Proposition 3.1.4.

Corollary 3.1.5 *Every Jordan functional on $\widetilde{\mathcal{M}}$ is measure continuous.*

3.2 Automatic continuity of Jordan homomorphisms

We begin this section with some examples of algebra homomorphisms between algebras of measurable operators. Let $\mathcal{A}$ be a von Neumann algebra with a faithful finite normal trace $\tau_{\mathcal{A}}$ and denote the algebra of $\tau_{\mathcal{A}}$-measurable operators by $\widetilde{\mathcal{A}}$. Let $\mathcal{M} = M_2(\mathcal{A})$. Then one can define a faithful finite normal trace $\tau_{\mathcal{M}}$ on $\mathcal{M}$ by $\tau_{\mathcal{M}}((x_{ij})) = \tau_{\mathcal{A}}(x_{11}) + \tau_{\mathcal{A}}(x_{22})$, $(x_{ij}) \in \mathcal{M}$ $(i, j = 1, 2)$. Denoting the algebra of $\tau_{\mathcal{M}}$-measurable operators by $\widetilde{\mathcal{M}}$, it follows from Proposition 1.3.7 that

$$\widetilde{\mathcal{M}} = \mathcal{U}(\mathcal{M}) = \mathcal{U}(M_2(\mathcal{A})) \cong M_2(\mathcal{U}(\mathcal{A})) = M_2(\widetilde{\mathcal{A}}).$$

Example 3.2.1 Let $\mathcal{A}$ and $\mathcal{M}$ be as in the previous remark. Given a self-adjoint algebra homomorphism $\theta : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}}$, define a map $\phi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ by $\phi((x_{ij})) = (\theta(x_{ij}))$ for every $(x_{ij}) \in \widetilde{\mathcal{M}}$ $(i, j = 1, 2)$. It can easily be verified that ϕ is a self-adjoint algebra homomorphism. If θ is injective or surjective, then so is ϕ.

Example 3.2.2 *([97])* Let $\mathcal{A}$ and $\mathcal{M}$ be as in Example 3.2.1. Let $p \in \mathcal{A}_p$. The map $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{M}}$ defined as

$$\phi(x) = \begin{pmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{pmatrix}$$

is a self-adjoint algebra homomorphism.

Example 3.2.3 *([97])* Let $\mathcal{A}$ and $\mathcal{M}$ be as in Example 3.2.1. The map $\phi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{A}}$ defined by $\phi((x_{ij})) = x_{11} + x_{12} + x_{21} + x_{22}$ is a self-adjoint algebra homomorphism.

Theorem 3.2.4 *([71], p. 951) Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be finite measure spaces, with X_1 a separable complete metric space and $\mu(X_1) = 1$. A linear map $T : L_0(X_1, \Sigma_1, \mu_1) \to L_0(X_2, \Sigma_2, \mu_2)$ is $\gamma_{cm} - \gamma_{cm}$ continuous if and only if T is of the form*

$$T(f)(t) = \Sigma_{i=1}^{\infty} \phi_i(t) f(\Phi_i(t)), \text{ for almost every } t \in X_2, \text{ where}$$

(i) ϕ_i is a sequence of elements in $L_0(X_2, \Sigma_2, \mu_2)$ such that $\mu_2(\{t : \phi_i(t) \neq 0 \text{ for infinitely many } i\}) = 0$;

(ii) $\Phi_i : X_2 \to X_1$ is a sequence of mappings such that for each i and for each $A \in \Sigma_1$, $\Phi_i^{-1}(A) \in \overline{\Sigma_2}$ (the completion of $\overline{\Sigma_2}$) and $\Phi_i^{-1}(A) \cap \{t : \phi_i(t) \neq 0\}$ is a set of μ_2-measure zero whenever A is a set of μ_1-measure zero.

Example 3.2.5 Let μ and Σ denote the Lebesgue measure and the Lebesgue sigma-algebra of $[0, 1]$ respectively, and let (X, Ω, λ) be a finite measure space. In this case, $\widetilde{L}_\infty(X, \Omega, \lambda) = L_0(X, \Omega, \lambda)$ and $\widetilde{L}_\infty([0, 1]) = L_0([0, 1])$. Let (A_i) be a sequence of disjoint measurable subsets of X. Define a linear map $\phi : \widetilde{L}_\infty([0, 1], \Sigma, \mu) \to \widetilde{L}_\infty(X, \Omega, \lambda)$ as follows:

$$\phi(f)(t) = \Sigma_{i=1}^{\infty} f(\Phi_i(t)) \chi_{A_i}(t)$$

for every $f \in \widetilde{L}_\infty([0,1],\Sigma,\mu)$, where $\Phi_i : X \to [0,1]$ is a sequence of mappings such that, for each i and for each measurable subset A of $[0,1]$, $\Phi_i^{-1}(A) \in \Omega$, and $\Phi_i^{-1}(A) \cap A_i$ has λ-measure zero whenever A has Lebesgue measure zero.

We show that ϕ is a Jordan homomorphism. Let $f \in \widetilde{L}_\infty([0,1],\Sigma,\mu)$ and $t \in X$. Then

$$\phi(f^2)(t) = \Sigma_{i=1}^{\infty} f^2(\Phi_i(t))\chi_{A_i}(t).$$

Now, if $\chi_{A_i}(t) = 1$ for some $i \in \mathbb{N}$, then $\chi_{A_j}(t) = 0$ for all $j \neq i$. Therefore

$$\phi(f)^2(t) = \left(\Sigma_{i=1}^{\infty} f(\Phi_i(t))\chi_{A_i}(t)\right)^2 = \Sigma_{i=1}^{\infty} f^2(\Phi_i(t))\chi_{A_i}(t).$$

Hence ϕ is a Jordan homomorphism.

It is easily verified that the conditions of Theorem 3.2.4 are met, so that ϕ is $\gamma_{cm} - \gamma_{cm}$ continuous.

If $\phi : \mathcal{A} \to \mathcal{B}$ is a linear map between metrizable topological vector spaces $\mathcal{A}$ and $\mathcal{B}$, we define the *separating space* $\mathcal{S}(\phi,\mathcal{B})$ of ϕ to be the set

$$\{b \in \mathcal{B} : \text{ there is a sequence } (x_n) \text{ in } \mathcal{A} \text{ with } x_n \to 0 \text{ and } \phi(x_n) \to b\}.$$

Recall that the closed graph theorem is valid for any complete metrizable topological vector space ([69], p. 101). Therefore, if $\mathcal{A}$ and $\mathcal{B}$ are complete metrizable topological vector spaces, the linear map ϕ is continuous if and only if $\mathcal{S}(\phi,\mathcal{B}) = \{0\}$. It is easily verifiable that $\mathcal{S}(\phi,\mathcal{B})$ is a vector subspace of $\mathcal{B}$.

We now show that every self-adjoint Jordan homomorphism $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is $\gamma_{cm} - \gamma_{cm}$ continuous (Theorem 3.2.8). For this, we require the following two lemmas, in which $\widetilde{\mathcal{M}}^+$ denotes the set of positive operators in $\widetilde{\mathcal{M}}$. The proof of the following lemma is modelled along the lines of [52], Corollary 25C.

Lemma 3.2.6 *Let (x_n) be a sequence in $\widetilde{\mathcal{M}}^+$ with $x_n \to 0$ in measure. Then there is a subsequence (y_n) of (x_n) and a $y \in \widetilde{\mathcal{M}}^+$ such that $2^n y_n \leq y$ for all n.*

Proof. Since $\widetilde{\mathcal{M}}$ is metrizable, there is a subsequence (y_n) of (x_n) such that $z_n = 4^n y_n \to 0$ (γ_{cm}). Indeed, the metrizability of $\widetilde{\mathcal{M}}$ implies that there exists a countable γ_{cm}-neighbourhood base $\{U_k : k \in \mathbb{N}\}$ of zero in $\widetilde{\mathcal{M}}$. Let $k \in \mathbb{N}$. Since $x_n \to 0$ (γ_{cm}), there exists $N_k \in \mathbb{N}$ such that $x_n \in 4^{-k}U_k$ for all $n \geq N_k$. Without loss of generality, the sequence (N_k) is increasing. Hence $4^k x_n \in \cap_{i=1}^k U_i$ for all $n \geq N_k$. Let $y_k = x_{N_k}$ for every k. Then $4^k y_k \in \cap_{i=1}^k U_i$ for every k. Therefore $4^n y_n \to 0$ (γ_{cm}).

Since $\widetilde{\mathcal{M}}$ is complete, it follows that the Cauchy sequence (w_n) of positive elements, defined by $w_n = \Sigma_{k=1}^n 2^{-k} z_k$ for every n, converges to y, say, in the measure topology. By Proposition 1.5.12, $y \geq 0$. Finally, it is evident that $2^n y_n \leq y$ for every n. ∇

The proof of our next lemma is similar to that [52], Proposition 25D.

Lemma 3.2.7 *Let $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ be a positive linear map. If (x_n) is a sequence in $\widetilde{\mathcal{M}}^+$ with $x_n \to 0$ in measure, then there exists a subsequence (y_n) of (x_n) such that $\phi(y_n) \to 0$ in measure.*

Proof. By Lemma 3.2.6, there is a subsequence (y_n) of (x_n) and a $y \in \widetilde{\mathcal{M}}^+$ such that $2^n y_n \leq y$ for every n. Since ϕ is positive, $\phi(y_n) \leq 2^{-n}\phi(y)$ for all n. Therefore, for every n, $0 \leq \mu_t(\phi(y_n)) \leq 2^{-n}\mu_t(\phi(y))$ for all $t > 0$. Since $2^{-n}\phi(y) \to 0$ (γ_{cm}), it follows from Lemma 1.5.16 that $\phi(y_n) \to 0$ (γ_{cm}). ∇.

The following theorem is the main result of this section. For this, observe that every self-adjoint Jordan homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ between GB*-algebras $\mathcal{A}$ and $\mathcal{B}$ is positive: Let $x \in \mathcal{A}$ be positive. By Proposition 1.4.21, there exists a positive element $y \in \mathcal{A}$ such that $x = y^2$. Therefore, since ϕ is a Jordan homomorphism, $\phi(x) = \phi(y^2) = \phi(y)^2$. Now $\phi(y)$ is a self-adjoint element in $\mathcal{B}$, since ϕ is self-adjoint. Hence $\phi(x)$ is a positive element in $\mathcal{B}$. Recall from Theorem 1.5.29 that $\widetilde{\mathcal{M}}$ is a GB*-algebra.

Theorem 3.2.8 *Every self-adjoint Jordan homomorphism $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is $\gamma_{cm} - \gamma_{cm}$ continuous.*

Proof. Let $y \in \mathcal{S}(\phi, \widetilde{\mathcal{B}})$. Then there exists a sequence (x_n) in $\widetilde{\mathcal{A}}$ such that $x_n \to 0$ (γ_{cm}) and $\phi(x_n) \to y$ (γ_{cm}). For every n, let $\mathrm{Re}(x_n) = a_n = \frac{1}{2}(x_n + x_n^*)$. Let $\mathrm{Re}(y) = z = \frac{1}{2}(y + y^*)$. Then $a_n \to 0$ (γ_{cm}) and $\phi(a_n) \to z$ (γ_{cm}) since ϕ is self-adjoint. By Theorem 1.5.20, $b_n = |a_n| \to 0$ (γ_{cm}) and $|\phi(a_n)| \to |z|$ (γ_{cm}). Since ϕ is a self-adjoint Jordan homomorphism, $|\phi(a_n)|^2 = \phi(a_n)^*\phi(a_n) = \phi(a_n)^2 = \phi(a_n^2) = \phi(a_n^* a_n) = \phi(b_n^2)$ for every n. Observe now that $b_n^2 \to 0$ (γ_{cm}) and $\phi(b_n^2) = |\phi(a_n)|^2 \to |z|^2 = z^2$ (γ_{cm}) since z is self-adjoint. By Lemma 3.2.7, there exists a subsequence (c_n) of (b_n^2) such that $\phi(c_n) \to 0$ (γ_{cm}). Since $\phi(b_n^2) \to z^2$ (γ_{cm}), it follows that $\phi(c_n) \to z^2$ (γ_{cm}). By uniqueness of limits, $z^2 = 0$. Since z is self-adjoint, it follows from the polar decomposition of z that $z = 0$.

Similarly, $\mathrm{Im}(y) = \frac{1}{2i}(y - y^*) = 0$. Hence $y = 0$, implying that ϕ is $\gamma_{cm} - \gamma_{cm}$ continuous. ∇.

Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be finite measure spaces with $\mu_1(X_1) = 1$. By Theorem 3.2.8, a self-adjoint Jordan homomorphism $\phi : \widetilde{L}_\infty(X_1, \Sigma_1, \mu_1) \to \widetilde{L}_\infty(X_2, \Sigma_2, \mu_2)$ is a map of the type described in Theorem 3.2.4 with the added condition that $\Sigma_{i=1}^{\infty}\phi_i(t) = \left(\Sigma_{i=1}^{\infty}\phi_i(t)\right)^2$ almost everywhere.

There is an interesting link between Jordan homomorphisms and composition operators which we will now describe ([72]). Let (X_i, Σ_i, μ_i) be σ-finite measure spaces for $i = 1, 2$. A linear operator $C : L_p(X_1, \Sigma_1, \mu_1) \to L_p(X_2, \Sigma_2, \mu_2)$, where $0 \leq p \leq \infty$, is called a (generalized) *composition operator* if there is a $Y \in \Sigma_2$ and a measurable transformation $T : Y \to X_1$ (i.e. $T^{-1}(E) \in \Sigma_2$ whenever $E \in \Sigma_1$) such that

$$C(f)(x) = \begin{cases} (f \circ T)(x) & \text{if } x \in Y \\ 0 & \text{if } x \in X_2 \setminus Y, \end{cases}$$

where $f \in L_p(X_1, \Sigma_1, \mu_1)$, and we write $C = C_T$ ([72]). In this definition, one could even replace the L_p-spaces with $\widetilde{L}_\infty$ and still call C a composition operator. We say that T is *non-singular* if $\mu_2(T^{-1}(E)) = 0$ whenever $\mu_1(E) = 0$, $E \in \Sigma_1$ ([72]).

Proposition 3.2.9 *([72], Proposition 2.1) The operator* $C_T : L_\infty(X_1, \Sigma_1, \mu_1)$

$\to L_\infty(X_2, \Sigma_2, \mu_2)$ is a well defined bounded linear operator if and only if $T : Y \to X_1$ is non-singular.

Proposition 3.2.10 *([72], Proposition 2.2) The operator $C_T : \widetilde{L}_\infty(X_1, \Sigma_1, \mu_1) \to \widetilde{L}_\infty(X_2, \Sigma_2, \mu_2)$ is a well defined $\gamma_{cm} - \gamma_{cm}$ continuous linear operator if and only if T is non-singular and for every $\epsilon > 0$, there exists $\delta > 0$ such that $\mu_1(E) \leq \delta$ ($E \in \Sigma_1$) implies $(\mu_2 \circ T^{-1})(E) \leq \epsilon$.*

Just as in Proposition 3.2.9, the non-singularity condition of T in Proposition 3.2.10 is there to ensure that C_T is well defined. Let $L_\infty(X_i, \Sigma_i, \mu_i)^{\mathbb{R}}$ be the (real) algebra of real-valued functions in $L_\infty(X_i, \Sigma_i, \mu_i)$, $i = 1, 2$. Since a bounded linear operator $\phi : L_\infty(X_1, \Sigma_1, \mu_1)^{\mathbb{R}} \to L_\infty(X_2, \Sigma_2, \mu_2)^{\mathbb{R}}$ is a composition operator if and only if ϕ is a Jordan homomorphism ([72], Proposition 2.6), the automatic continuity of self-adjoint Jordan homomorphisms from one von Neumann algebra into another can be thought of as a non-commutative version of Proposition 3.2.9, i.e. self-adjoint Jordan homomorphisms between von Neumann algebras can be considered to be non-commutative composition operators ([72]). The following result is due to L. E. Labuschagne.

Theorem 3.2.11 *([72], Proposition 4.7) Let ϕ be a self-adjoint Jordan homomorphism from a *-subalgebra $\mathcal{D}(\phi)$ of $\widetilde{\mathcal{A}}$, containing all projections of finite trace, into $\widetilde{\mathcal{B}}$. The following statements are equivalent.*

(i) ϕ is $\gamma_{cm} - \gamma_{cm}$ continuous.

(ii) $\phi(\mathcal{D}(\phi) \cap \mathcal{A}) \subset \mathcal{B}$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that $\tau_{\mathcal{B}}(\phi(p)) \leq \epsilon$ whenever $p \in \mathcal{A}_p$ with $\tau_{\mathcal{A}}(p) \leq \delta$. Furthermore, $\phi|_{(\mathcal{D}(\phi) \cap \mathcal{A})}$ is norm-norm continuous.

By Theorems 3.2.8 and 3.2.11, every self-adjoint Jordan homomorphism $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ satisfies (ii) of Theorem 3.2.11. By Proposition 3.2.10, one can therefore regard every self-adjoint Jordan homomorphism $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ as a non-commutative composition operator.

3.3 Jordan homomorphisms between locally convex GB*-algebras

Lemma 3.3.2 was proved in [72] for the case where $\mathcal{A}$ and $\mathcal{B}$ are algebras of measurable operators. We give a proof that is different to the one given there. For this, we need the following lemma.

Lemma 3.3.1 *If u is a unitary element of a GB*-algebra $\mathcal{A}$, then $u \in \mathcal{A}_b$.*

Proof. Since u is normal, u generates a maximal commutative *-subalgebra $\mathcal{B}$ of $\mathcal{A}$ ([27], Proposition V.7). By Lemma 1.4.14, $\mathcal{B}$ is a GB*-algebra. It follows from the fact that u is unitary and Proposition 1.4.16 that $\frac{1}{2}u = u(1+u^*u)^{-1} \in \mathcal{B}_b \subset \mathcal{A}_b$. Thus $u \in \mathcal{A}_b$. ∇

Lemma 3.3.2 *If $\phi : \mathcal{A} \to \mathcal{B}$ is a unital self-adjoint Jordan homomorphism between GB*-algebras $\mathcal{A}$ and $\mathcal{B}$, then $\phi(\mathcal{A}_b) \subset \mathcal{B}_b$.*

Proof. Let u be a unitary element in $\mathcal{A}$. By Lemma 3.3.1, $u \in \mathcal{A}_b$. We first show that $\phi(u^*)\phi(u) = \phi(u)\phi(u^*)$. Since ϕ is self-adjoint, it follows that $x = \phi(u^*)\phi(u) - \phi(u)\phi(u^*)$ is self-adjoint. By Proposition 2.1.3, it follows that

$$x^2 = \Big(\phi(u^*)\phi(u) - \phi(u)\phi(u^*)\Big)^2 = \phi\Big((u^*u - uu^*)^2\Big) = 0.$$

By Theorem 1.4.13 and Lemma 1.4.14, it follows that $x = 0$.

It now follows from Lemma 2.1.2 that

$$\begin{aligned} 2\phi(u)\phi(u)^* &= 2\phi(u)\phi(u^*) \\ &= \phi(u)\phi(u^*) + \phi(u^*)\phi(u) \\ &= \phi(uu^* + u^*u) \\ &= 2\phi(uu^*) \\ &= 2\phi(1) \\ &= 2.1 \end{aligned}$$

Hence $\phi(u)\phi(u)^* = \phi(u)^*\phi(u) = 1$ for all unitary elements $u \in \mathcal{A}_b$. Therefore ϕ preserves unitary elements. Since $\mathcal{A}_b$ is a C*-algebra, every element of $\mathcal{A}_b$

is a linear combination of four unitary elements of $\mathcal{A}_b$, and so we conclude from Lemma 3.3.1 that $\phi(\mathcal{A}_b) \subset \mathcal{B}_b$. ∇

If $\mathcal{A}$ is a *-algebra, we say that a linear map π is a **-representation* (respectively *Jordan *-representation*) of $\mathcal{A}$ if π is a self-adjoint algebra homomorphism (respectively self-adjoint Jordan homomorphism) of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. In [30], it is proved that every *-representation of a complete metrizable locally m-convex *-algebra is continuous (see [30], Lemma 3.1). This result is also true for every complete metrizable locally convex *-algebra ([50], p. 60). The next proposition is an extension of this result to Jordan *-representations of complete metrizable locally convex GB*-algebras with nearly the same proof, so we only sketch the proof. Proposition 3.3.4 is also needed in the proof of Theorem 3.3.5 below. We first give the following example to show that there are complete metrizable locally convex GB*-algebras which are not locally C*-algebras.

Example 3.3.3 *([6], Example 4)* Let $\mathcal{A}$ be the Arens algebra $L^{\omega}([0,1]) = \cap_{p=1}^{\infty} L_p([0,1])$. When equipped with the topology defined by the collection of L_p-norms, $\mathcal{A}$ is a complete metrizable locally convex GB*-algebra which is not locally m-convex, and therefore not a locally C*-algebra.

Proposition 3.3.4 *Let $\mathcal{A}$ be a complete metrizable locally convex GB*-algebra and $\pi : \mathcal{A} \to B(\mathcal{H})$ a Jordan *-representation. Then π is continuous.*

Proof. Fix $\epsilon > 0$. Let $V = \{a \in \mathcal{A} : \|\pi(a)\| \leq \epsilon\}$ and let $V_{\xi,\eta} = \{a \in \mathcal{A} : |\langle \pi(a)\xi, \eta\rangle| \leq \epsilon\}$, where $\xi, \eta \in \mathcal{H}$. Then $V = \cap\{V_{\xi,\eta} : \|\xi\|, \|\eta\| \leq 1\}$. By linearity of π, it follows that, for every $\xi, \eta \in \mathcal{H}$ with $\|\xi\|$, $\|\eta\| \leq 1$, the set $V_{\xi,\eta}$ is convex and balanced. Every positive linear functional on a complete metrizable topological *-algebra, with continuous involution, is continuous (Theorem 1.4.2).

Hence the linear functional $a \mapsto \langle \pi(a)\psi, \psi\rangle$ is continuous: Let $a \in \mathcal{A}$ with $a \geq 0$. Since $\mathcal{A}$ is a GB*-algebra, it follows from Proposition 1.4.21, that $a = b^2$ for some self-adjoint $b \in \mathcal{A}$. Therefore, since π is a self-adjoint Jordan homomorphism, it follows that

$$\begin{aligned} \langle \pi(a)\psi, \psi \rangle &= \langle \pi(b^2)\psi, \psi \rangle \\ &= \langle \pi(b)^2\psi, \psi \rangle \\ &= \langle \pi(b)^*\pi(b)\psi, \psi \rangle \\ &= \langle \pi(b)\psi, \pi(b)\psi \rangle \\ &\geq 0. \end{aligned}$$

Hence the functional is positive, and therefore continuous for every $\psi \in \mathcal{H}$.

By the polarization formula, the linear functional $a \mapsto \langle \pi(a)\xi, \eta \rangle$ is continuous for every $\xi, \eta \in \mathcal{H}$. Thus, every $V_{\xi,\eta}$ is closed. It follows immediately that V is closed, convex and balanced. Also, by linearity of π, it is easily verified that V is absorbing. Therefore, since $\mathcal{A}$ is metrizable and locally convex, V is a neighbourhood of $0 \in \mathcal{A}$. It follows that π is continuous. ∇

Theorem 3.3.5 *Let $\mathcal{A}$ and $\mathcal{B}$ be complete metrizable locally convex GB^*-algebras. Suppose that $\mathcal{B}$ has a separating family of *-representations, i.e.*

$$\cap\{Ker(\pi) : \pi \text{ a } ^*\text{-representation of } \mathcal{B}\} = \{0\}.$$

Then every self-adjoint Jordan homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is continuous.

Proof. By the closed graph theorem, it suffices to show that if $x_n \to 0$ in $\mathcal{A}$ and if $\phi(x_n) \to y$ in $\mathcal{B}$, then $y = 0$. For every n, let $y_n = \phi(x_n)$ and let π be a *-representation of $\mathcal{B}$. It follows from Proposition 3.3.4 that π is continuous. Since ϕ is a self-adjoint Jordan homomorphism, $\pi \circ \phi$ is a Jordan *-representation on $\mathcal{A}$. By Proposition 3.3.4, we see that $\pi \circ \phi$ is continuous on $\mathcal{A}$. So

$$\begin{aligned} \pi(y) &= \lim_{n\to\infty} (\pi \circ \phi)(x_n) \\ &= (\pi \circ \phi)(\lim_{n\to\infty} x_n) \\ &= 0. \end{aligned}$$

Hence $\pi(y) = 0$ for all *-representations π of $\mathcal{B}$. By hypothesis, it follows that $y = 0$. ∇.

If $\mathcal{A}$ is a metrizable locally C*-algebra, then $\mathcal{A}$ has a separating family of irreducible *-representations ([30], p. 52). In light of Theorem 3.3.5, the following question presents itself: Under what conditions will a GB*-algebra have a separating family of *-representations? The following lemma is known to hold for the case where $\mathcal{A}$ is a C*-algebra (Theorem 1.2.17). It is easily seen that this lemma also holds without the assumption of completeness.

Lemma 3.3.6 *Let* $(\mathcal{A}, \|.\|)$ *be a* **-normed algebra with* $\|x^*x\| = \|x\|^2$ *for each* $x \in \mathcal{A}$*. Then* $\mathcal{A}$ *admits a family of* **-representations which separate the points of* $\mathcal{A}$*.*

Proposition 3.3.7 *For a* **-algebra* $\mathcal{A}$*, the following statements are equivalent.*

(i) $\mathcal{A}$ *admits a separating family of* **-representations.*

(ii) $\mathcal{A}$ *admits a separating family of C***-seminorms* (p_α)*.*

Proof. *(i)* $\Rightarrow$ *(ii):* Let (π_α) be a separating family of *-representations which separate the points of $\mathcal{A}$. For each α, let $p_\alpha(x) = \|\pi_\alpha(x)\|$ for every $x \in \mathcal{A}$. It is easily verified that every p_α is a C*-seminorm and that these seminorms separate the points of $\mathcal{A}$.

(ii) $\Rightarrow$ *(i):* Suppose that (p_α) is a separating family of C*-seminorms which separate the points of $\mathcal{A}$. Let $0 \neq a \in \mathcal{A}$. Then there exists p_α such that $p_\alpha(a) \neq 0$. Let $N_\alpha = \{x \in \mathcal{A} : p_\alpha(x) = 0\}$. Then $\mathcal{A}/N_\alpha$ is a *-normed algebra with the standard quotient norm a C*-norm. By Lemma 3.3.6, there is a *-representation π_α of $\mathcal{A}/N_\alpha$ on some Hilbert space $\mathcal{H}$ such that $\pi_\alpha(a + N_\alpha) \neq 0$ (since $a \notin N_\alpha$, it is clear that $a + N_\alpha \neq N_\alpha$). Let $\pi(x) = \pi_\alpha(x + N_\alpha)$ for every $x \in \mathcal{A}$. Clearly, π is a *-representation of $\mathcal{A}$ on $\mathcal{H}$, and $\pi(a) = \pi_\alpha(a + N_\alpha) \neq 0$. The proof is complete. ∇

3.4 More results on Jordan homomorphisms between algebras of measurable operators

We now give an analogue of Theorem 2.2.6 for algebras of measurable operators.

Proposition 3.4.1 *Suppose that $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is a unital unitary preserving $\gamma_{cm} - \gamma_{cm}$ continuous linear map. Then ϕ is a Jordan homomorphism.*

Proof. All unitary elements in $\widetilde{\mathcal{A}}$ are in $\mathcal{A}$. Similarly, all unitary elements in $\widetilde{\mathcal{B}}$ are in $\mathcal{B}$. Recall that $\mathcal{A}$ and $\mathcal{B}$ are the linear span of unitary elements of $\mathcal{A}$ and $\mathcal{B}$ respectively. Thus, since ϕ is unitary preserving, $\phi(\mathcal{A}) \subset \mathcal{B}$. So, by Theorem 2.2.6, $\phi|_{\mathcal{A}}$ is a Jordan homomorphism. Since ϕ is $\gamma_{cm} - \gamma_{cm}$ continuous, ϕ is a Jordan homomorphism. ∇

The following example is a modification of Example 2.1.7.

Example 3.4.2 Let $\mathcal{M} = \mathcal{B}(\mathbb{C}^2) \oplus \mathcal{B}(\mathbb{C}^2) \cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$, and let $\mathcal{M}_1 = \mathcal{B}(\mathbb{C}^2 \oplus \mathbb{C}^2)$. Now $\mathcal{M} \subset \mathcal{M}_1$. Since $\mathcal{M}_1$ is the algebra of bounded linear operators on $\mathbb{C}^2 \oplus \mathbb{C}^2$, it follows that $\mathcal{M}_1$ admits a faithful finite normal trace τ_1. Thus $\tau = \tau_1|_{\mathcal{M}}$ is a faithful finite normal trace on $\mathcal{M}$.

We conclude that $\mathcal{M} = \widetilde{\mathcal{M}}$. Indeed, since $\mathcal{M}_1 = \widetilde{\mathcal{M}_1}$, it follows from Proposition 1.5.11 that

$$\inf\{\tau(p) : 0 \neq p \in \mathcal{M}_p\} \geq \inf\{\tau_1(p) : 0 \neq p \in (\mathcal{M}_1)_p\} > 0.$$

By Proposition 1.5.11, it follows that $\mathcal{M} = \widetilde{\mathcal{M}}$.

Define a linear map $\phi : \mathcal{M} \to \mathcal{M}$ by $\phi(x, y) = (x, y^t)$, where y^t denotes the transpose of the matrix y. By Example 2.1.7, ϕ is a Jordan homomorphism but not an algebra homomorphism nor an algebra anti-homomorphism. So a Jordan homomorphism between algebras of measurable operators need not be an algebra homomorphism or an algebra anti-homomorphism.

Proposition 3.4.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be GB*-algebras with underlying C*-algebras $\mathcal{A}_b$ and $\mathcal{B}_b$ respectively, and suppose that $\mathcal{A}_b$ is a W*-algebra. Suppose further that the multiplications on $\mathcal{A}$ and $\mathcal{B}$ are jointly continuous. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a continuous self-adjoint Jordan homomorphism such that $\phi(\mathcal{A}_b) \subset \mathcal{B}_b$. Then there exist orthogonal projections p and q in $\mathcal{B}_b$ with $p + q = 1$ such that the maps $\phi_1 : \mathcal{A} \to \phi(\mathcal{A})p$, defined as $x \mapsto \phi(x)p$, and $\phi_2 : \mathcal{A} \to \phi(\mathcal{A})q$, defined as $x \mapsto \phi(x)q$, are self-adjoint homomorphisms and self-adjoint anti-homomorphisms respectively. Furthermore $\phi = \phi_1 + \phi_2$.*

Proof. By hypothesis, $\phi(\mathcal{A}_b) \subset \mathcal{B}_b$. Therefore, Theorem 2.2.8 can be applied to obtain projections p and q in $\mathcal{B}_b$ with $p+q = 1$ such that $\phi_1 : \mathcal{A}_b \to \phi(\mathcal{A}_b)p$ and $\phi_2 : \mathcal{A}_b \to \phi(\mathcal{A}_b)q$ are self-adjoint homomorphisms and self-adjoint anti-homomorphisms respectively, and $\phi|_{\mathcal{A}_b} = \phi_1 + \phi_2$. Since ϕ is continuous and $\mathcal{A}_b$ is dense in $\mathcal{A}$ (Theorem 1.4.17), it follows that the maps ϕ_1 and ϕ_2, first defined on $\mathcal{A}_b$, can be extended by continuity to $\mathcal{A}$. Therefore $\phi_1 : \mathcal{A} \to \phi(\mathcal{A})p$ and $\phi_2 : \mathcal{A} \to \phi(\mathcal{A})q$ is a self-adjoint algebra homomorphism and a self-adjoint algebra anti-homomorphism respectively. Furthermore, $\phi = \phi_1 + \phi_2$. ∇

Corollary 3.4.4 *Suppose that $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is a unital self-adjoint Jordan homomorphism. Then there exist orthogonal projections p and q in $\mathcal{B}$ with $p+q = 1$ such that the maps $\phi_1 : \widetilde{\mathcal{A}} \to \phi(\widetilde{\mathcal{A}})p$, defined as $x \mapsto \phi(x)p$, and $\phi_2 : \widetilde{\mathcal{A}} \to \phi(\widetilde{\mathcal{A}})q$, defined as $x \mapsto \phi(x)q$, are self-adjoint algebra homomorphisms and self-adjoint algebra anti-homomorphisms respectively. Furthermore $\phi = \phi_1 + \phi_2$.*

Proof. First note that Theorem 1.5.29 and Lemma 3.3.2 implies that $\phi(\mathcal{A}) \subset \mathcal{B}$. The claim now follows from Proposition 3.4.3 and Theorem 3.2.8. ∇

An automorphism ϕ on an algebra $\mathcal{A}$ is said to be *inner* if there exists an invertible element $a \in \mathcal{A}$ such that $\phi(x) = axa^{-1}$ for every $x \in \mathcal{A}$.

Theorem 3.4.5 *Let ϕ be a self-adjoint automorphism on $\widetilde{\mathcal{M}}$, where $\mathcal{M}$ is of type I, such that ϕ keeps every element of $Z(\mathcal{M})$ element-wise fixed, i.e. $\phi(x) = x$ for every $x \in Z(\mathcal{M})$. Then ϕ is inner.*

Proof. Since ϕ and ϕ^{-1} are unital, one can apply Lemma 3.3.2 to ϕ and ϕ^{-1} to obtain $\phi(\mathcal{M}) = \mathcal{M}$. Therefore, $\phi|_{\mathcal{M}}$ is an automorphism leaving $Z(\mathcal{M})$ element-wise fixed. Every automorphism on a type I AW*-algebra, leaving its centre element-wise fixed, is inner ([66], Theorem 10. See also [65], Theorem 3). Hence there is an invertible element $a \in \mathcal{M}$ such that $\phi(x) = axa^{-1}$ for every $x \in \mathcal{M}$. Since $\mathcal{M}$ is γ_{cm}- dense in $\widetilde{\mathcal{M}}$ and ϕ is $\gamma_{cm} - \gamma_{cm}$ continuous (Theorem 3.2.8), it follows that $\phi(x) = axa^{-1}$ for every $x \in \widetilde{\mathcal{M}}$. ∇

3.5 Conditions under which a Jordan homomorphism is an algebra homomorphism

Proposition 3.5.1 *Suppose that $\mathcal{A}$ is a commutative algebra and $\mathcal{B}$ is an algebra with no nonzero nilpotent elements in its centre. If $\phi : \mathcal{A} \to \mathcal{B}$ is a surjective Jordan homomorphism, then ϕ is an algebra homomorphism.*

Proof. By Theorem 2.1.6 and the surjectivity of ϕ, it is immediate that ϕ preserves commutativity. Let $x, y \in \mathcal{A}$. Then, since $xy = yx$ and ϕ is a Jordan homomorphism, it follows from Lemma 2.1.2 that

$$\begin{aligned} 2\phi(xy) &= \phi(2xy) \\ &= \phi(xy + yx) \\ &= \phi(x)\phi(y) + \phi(y)\phi(x) \\ &= 2\phi(x)\phi(y) \end{aligned}$$

Hence $\phi(xy) = \phi(x)\phi(y)$. This holds for every $x, y \in \mathcal{A}$. Thus ϕ is an algebra homomorphism. ∇.

Let x be a closed normal (unbounded) linear operator on a Hilbert space. By Theorem 1.3.9, x is affiliated with a commutative von Neumann algebra.

It follows from Theorem 1.3.10 that, for every natural number n, x^n is a closed linear operator. This is important in the proof of our next lemma.

Lemma 3.5.2 *The algebra $\widetilde{\mathcal{M}}$ has no nonzero nilpotent elements in its centre.*

Proof. Suppose that x is a nilpotent element in the centre of $\widetilde{\mathcal{M}}$. Then $xy = yx$ for every $y \in \widetilde{\mathcal{M}}$. In particular, $xa = ax$ for every $a \in \mathcal{M}$. Since x is nilpotent, there exists a natural number n such that $x^n = 0$. As a result, x^n is everywhere defined. Therefore, since $\mathcal{D}(x^n) \subset \mathcal{D}(x)$, it follows that x is everywhere defined. Hence, since x is a closed linear operator, x is a bounded linear operator, implying that $x \in \mathcal{M}$. Therefore $x \in Z(\mathcal{M})$, the centre of $\mathcal{M}$. Since $Z(\mathcal{M})$ is a commutative C*-algebra, $Z(\mathcal{M}) \cong C(X)$, where X is the maximal ideal space of $Z(\mathcal{M})$. Clearly, $C(X)$ has no nonzero nilpotent elements, so $Z(\mathcal{M})$ does not either. Thus $x = 0$. ∇

The next corollary follows immediately from Proposition 3.5.1 and Lemma 3.5.2.

Corollary 3.5.3 *Suppose that $\widetilde{\mathcal{A}}$ is commutative and that $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is a surjective Jordan homomorphism, then ϕ is an algebra homomorphism.*

Corollary 3.5.4 *Suppose that $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is a unital self-adjoint Jordan homomorphism. Then ϕ is an algebra homomorphism if and only if there exists $\alpha > 0$ such that $\phi(a^*a) \geq \alpha\phi(a^*)\phi(a)$ for all $a \in \widetilde{\mathcal{A}}$.*

Proof. Assume that ϕ is an algebra homomorphism. Then, by taking $\alpha = 1$, it is trivial that $\phi(a^*a) \geq \alpha\phi(a^*)\phi(a)$ for all $a \in \widetilde{\mathcal{A}}$.

Now assume that there exists $\alpha > 0$ such that $\phi(x^*x) \geq \alpha\phi(x^*)\phi(x)$ for all $x \in \widetilde{\mathcal{A}}$. This inequality holds in particular for all $x \in \mathcal{A}$. By Theorem 1.5.29 and Lemma 3.3.2, $\phi(\mathcal{A}) \subset \mathcal{B}$. It follows from Theorem 2.2.9 that $\phi|_{\mathcal{A}}$ is an algebra homomorphism. By the $\gamma_{cm} - \gamma_{cm}$ continuity of ϕ (Theorem 3.2.8) and the denseness of $\mathcal{A}$ in $\widetilde{\mathcal{A}}$, it follows that ϕ is an algebra homomorphism. ∇

Recall that the product of two nonzero two-sided ideals $\mathcal{I}_1$ and $\mathcal{I}_2$ of a ring $\mathcal{R}$ is the two-sided ideal of $\mathcal{R}$ consisting of all finite sums of the form x_1x_2, with $x_1 \in \mathcal{I}_1$ and $x_2 \in \mathcal{I}_2$.

Lemma 3.5.5 *([54], p. 47) Let $\mathcal{R}$ be a ring. The following statements are equivalent.*

(i) If $a, b \in \mathcal{R}$ and $a\mathcal{R}b = \{0\}$, then $a = 0$ or $b = 0$.

(ii) The product of two non-zero two-sided ideals of $\mathcal{R}$ is nonzero.

We call a ring satisfying the equivalent conditions of Lemma 3.5.5 a *prime* ring.

Theorem 3.5.6 *Let $\mathcal{M}$ be a von Neumann algebra with a faithful semifinite normal trace. Then $\widetilde{\mathcal{M}}$ is prime if and only if $\mathcal{M}$ is prime.*

Proof. Suppose that $\widetilde{\mathcal{M}}$ is prime and let $a, b \in \mathcal{M}$ with $a\mathcal{M}b = \{0\}$. We will show that $a = 0$ or $b = 0$. Let $x \in \widetilde{\mathcal{M}}$. Then there is a sequence (x_n) in $\mathcal{M}$ with $x_n \to x$ (γ_{cm}). Therefore, since $ax_nb = 0$ for all n, it follows that $axb = a(\lim_{n\to\infty} x_n)b = \lim_{n\to\infty} ax_nb = 0$. This holds for every $x \in \widetilde{\mathcal{M}}$. Thus $a\widetilde{\mathcal{M}}b = \{0\}$. Since $\widetilde{\mathcal{M}}$ is prime, $a = 0$ or $b = 0$, implying that $\mathcal{M}$ is prime.

Suppose that $\mathcal{M}$ is prime. Let $\mathcal{A}$ and $\mathcal{B}$ be two-sided nonzero ideals in $\widetilde{\mathcal{M}}$. We first show that $\overline{\mathcal{A}} \cap \mathcal{M}$ and $\overline{\mathcal{B}} \cap \mathcal{M}$ are nonzero two-sided ideals in $\mathcal{M}$, where the closures are taken with respect to the measure topology.

Since $\mathcal{A}$ and $\mathcal{B}$ are nonzero, $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ are nonzero two-sided ideals in $\widetilde{\mathcal{M}}$. If $\overline{\mathcal{A}} \cap \mathcal{M} = \{0\}$, then, by Theorem 1.5.21, $\overline{\mathcal{A}} = \overline{\overline{\mathcal{A}} \cap \mathcal{M}} = \{0\}$. This is a contradiction. Therefore, we conclude that $\overline{\mathcal{A}} \cap \mathcal{M} \neq \{0\}$. Similarly $\overline{\mathcal{B}} \cap \mathcal{M} \neq \{0\}$.

Since $\mathcal{M}$ is prime, it follows from Lemma 3.5.5 that $(\overline{\mathcal{A}} \cap \mathcal{M})(\overline{\mathcal{B}} \cap \mathcal{M}) \neq \{0\}$. Hence $(\overline{\mathcal{A}})(\overline{\mathcal{B}}) \neq \{0\}$. Therefore there exist elements $x \in \overline{\mathcal{A}}$ and $y \in \overline{\mathcal{B}}$ such that $xy \neq 0$. Since $x \in \overline{\mathcal{A}}$, there is a sequence (x_n) in $\mathcal{A}$ such that $x_n \to x$ (γ_{cm}). Similarly, there is a sequence (y_n) in $\mathcal{B}$ such that $y_n \to y$ (γ_{cm}). Hence $x_ny_n \to xy$ (γ_{cm}). Therefore, since $xy \neq 0$, at least one of the terms of the sequence $x_n y_n$ is nonzero. This implies that $\mathcal{A}\mathcal{B} \neq \{0\}$ since all x_n and y_n are in $\mathcal{A}$ and $\mathcal{B}$ respectively. It follows from Lemma 3.5.5 that $\widetilde{\mathcal{M}}$ is prime. ∇

It is known that a von Neumann algebra is prime if and only if it is a factor ([9], p. 47). Therefore we can deduce the following result from Theorem 3.5.6.

Corollary 3.5.7 *Let $\mathcal{M}$ be a von Neumann algebra with a faithful semifinite normal trace. Then $\widetilde{\mathcal{M}}$ is prime if and only if $\mathcal{M}$ is a factor.*

By an application of Theorem 2.1.8 and Corollary 3.5.7, we obtain the next result.

Corollary 3.5.8 *If $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is a surjective Jordan homomorphism, with $\mathcal{B}$ a factor, then ϕ is an algebra homomorphism or an algebra anti-homomorphism.*

Corollary 3.5.8 is an analogue of the following result of Kadison.

Theorem 3.5.9 *([62], Corollary 11) If $\mathcal{A}$ and $\mathcal{B}$ are factors, then every self-adjoint Jordan isomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is a self-adjoint algebra isomorphism or a self-adjoint algebra anti-isomorphism.*

Chapter 4

A problem of Kaplansky

In Section 2.2, we explored affirmative answers to the following question, raised by I. Kaplansky, in the context of C*-algebras, especially von Neumann algebras:

When is a unital invertibility preserving linear map between unital algebras a Jordan homomorphism?

In Section 4.5 of this chapter, we provide some answers in the case where the unital algebras in question are algebras of measurable operators.

In Section 4.1, we show that every continuous projection preserving linear map between GB*-algebras is a Jordan homomorphism. In Section 4.2, we give some results about positive operators in $\widetilde{\mathcal{M}}$ which we need in the sections that follow. Sections 4.3 and 4.4 deal with automatic continuity of positive linear maps between algebras of measurable operators and projection preserving linear maps respectively. These will culminate in the main results of this chapter.

Throughout this chapter, the notation $\widetilde{\mathcal{A}}$ (respectively $\widetilde{\mathcal{B}}$) will always stand for the algebra of $\tau_{\mathcal{A}}-$ (respectively $\tau_{\mathcal{B}}-$) measurable operators affiliated with a von Neumann algebra $\mathcal{A}$ (respectively $\mathcal{B}$) equipped with a faithful semifinite normal trace $\tau_{\mathcal{A}}$ (respectively $\tau_{\mathcal{B}}$).

4.1 Conditions under which a linear mapping is a Jordan homomorphism

The proof of the following result is similar to the arguments given in [29].

Theorem 4.1.1 *Let $\mathcal{A}$ be a GB^*-algebra with the underlying C^*-algebra $\mathcal{A}_b$ of $\mathcal{A}$ a W^*-algebra, and let $\mathcal{B}$ be a topological *-algebra. Suppose further that the multiplications on $\mathcal{A}$ and $\mathcal{B}$ are jointly continuous. If $\phi : \mathcal{A} \to \mathcal{B}$ is a continuous linear mapping preserving projections, then ϕ is a Jordan homomorphism.*

Proof. Let s be a self-adjoint element in $\mathcal{A}_b$. Then there is a sequence (s_n) of finite linear combinations of mutually orthogonal projections in $\mathcal{A}_b$ such that $s_n \to s$ in norm (Theorem 1.3.11), and hence also with respect to the original topology on $\mathcal{A}$, since the restriction of the original topology of $\mathcal{A}$ to $\mathcal{A}_b$ is weaker than the norm topology of $\mathcal{A}_b$. By Proposition 2.1.4, $\phi(s_n^2) = \phi(s_n)^2$ for every n, and hence, since ϕ is continuous, we have $\phi(s^2) = \phi(\lim_{n\to\infty} s_n^2) = \phi(\lim_{n\to\infty} s_n)^2 = \phi(s)^2$. This holds for any self-adjoint element $s \in \mathcal{A}_b$. By Proposition 2.1.5, $\phi|_{\mathcal{A}_b}$ is a Jordan homomorphism.

Let $x \in \mathcal{A}$. By Theorem 1.4.17, there is a sequence (x_n) in $\mathcal{A}_b$ such that $x_n \to x$. By using a similar argument as above, $\phi(x^2) = \phi(x)^2$. This holds for every $x \in \mathcal{A}$. Thus ϕ is a Jordan homomorphism. ∇

4.2 Positivity in $\widetilde{\mathcal{M}}$

Let $x \in \widetilde{\mathcal{M}}$. Recall that x is positive if and only if x is self-adjoint and $\langle x\xi, \xi\rangle \geq 0$ for all $\xi \in \mathcal{D}(x)$.

Lemma 4.2.1 *Let $x \in \widetilde{\mathcal{M}}$. The following statements are equivalent.*

(i) x is positive.

(ii) There exists a unique self-adjoint $y \in \widetilde{\mathcal{M}}$ such that $x = y^2$.

(iii) There exists $y \in \widetilde{\mathcal{M}}$ such that $x = y^*y$.

Proof. $(i) \Rightarrow (ii)$: Suppose that $x \in \widetilde{\mathcal{M}}$ is positive. By Proposition 1.3.2, there is a unique self-adjoint (and thus closed) densely defined operator y such that $x = y^2$. Also, by Proposition 1.3.2, y is affiliated with $\mathcal{M}$. We now show that $y \in \widetilde{\mathcal{M}}$. Let $\delta > 0$. Since $x \in \widetilde{\mathcal{M}}$, there is a projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subset \mathcal{D}(x)$ and $\tau(1-p) < \delta$. It follows that $p\mathcal{H} \subset \mathcal{D}(y^2) \subset \mathcal{D}(y)$. Thus $y \in \widetilde{\mathcal{M}}$. The implication $(ii) \Rightarrow (iii)$ is obvious.

$(iii) \Rightarrow (i)$: Suppose that $x = y^*y$ for some $y \in \widetilde{\mathcal{M}}$. Then $\mathcal{D}(x) \subset \mathcal{D}(y)$. Therefore $\langle x\xi, \xi\rangle = \langle y^*y\xi, \xi\rangle = \langle y\xi, y\xi\rangle \geq 0$ for all $\xi \in \mathcal{D}(x)$. Also x is self-adjoint. Thus x is positive. ∇

Lemma 4.2.2 *Let $x \in \widetilde{\mathcal{M}}$ be self-adjoint. Then there exist positive operators x_1 and x_2 in $\widetilde{\mathcal{M}}$ such that $x = x_1 - x_2$.*

Proof. By Proposition 1.3.5, there exist positive operators x_1 and x_2 such that $x = x_1 - x_2$. It remains to show that x_1 and x_2 are in $\widetilde{\mathcal{M}}$. By Proposition 1.3.5, it follows that x_1 and x_2 are closed densely defined and affiliated with $\mathcal{M}$. Furthermore, by the proof of [93], Corollary 9:31, $\mathcal{D}(x) \subset \mathcal{D}(x_1)$ and $\mathcal{D}(x) \subset \mathcal{D}(x_2)$. Let $\delta > 0$. Since $x \in \widetilde{\mathcal{M}}$, there is a projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subset \mathcal{D}(x)$ and $\tau(1-p) < \delta$. So we have that for every $\delta > 0$, there is a projection $p \in \mathcal{M}$ such that $p\mathcal{H} \subset \mathcal{D}(x_1)$ and $p\mathcal{H} \subset \mathcal{D}(x_2)$ as well as $\tau(1-p) < \delta$. So x_1 and x_2 are in $\widetilde{\mathcal{M}}$. ∇

If $\mathcal{A}$ is a *-algebra, we say that $b \in \mathcal{A}$ is positive if $b = a^*a$ for some $a \in \mathcal{A}$. Observe that every positive element of $\mathcal{A}$ is self-adjoint. If $\mathcal{A} = \widetilde{\mathcal{M}}$, then it follows from Lemma 4.2.1 that this notion of positivity coincides with the definition of positivity in $\widetilde{\mathcal{M}}$.

Lemma 4.2.3 *Let $\mathcal{A}$ be a *-algebra such that for each self-adjoint $x \in \mathcal{A}$, there exist positive $a, b \in \mathcal{A}$ such that $x = a - b$. Suppose that $\mathcal{B}$ is a *-algebra and $\phi : \mathcal{A} \to \mathcal{B}$ is a positive linear map. Then ϕ is self-adjoint.*

Proof. Let $z \in \mathcal{A}$. Then there exist self-adjoint elements x and y in $\mathcal{A}$ such that $z = x + iy$. By hypothesis, there exist positive elements x_1 and x_2 in

$\mathcal{A}$ such that $x = x_1 - x_2$. Also, there exist positive elements y_1 and y_2 in $\mathcal{A}$ such that $y = y_1 - y_2$. Thus $x = x_1 - x_2 + iy_1 - iy_2$. Hence

$$\phi(x^*) = \phi(x_1) - \phi(x_2) - i\phi(y_1) + i\phi(y_2).$$

On the other hand,

$$\phi(x)^* = \phi(x_1)^* - \phi(x_2)^* - i\phi(y_1)^* + i\phi(y_2)^*.$$

Since ϕ is positive, $\phi(x_1), \phi(x_2), \phi(x_3)$ and $\phi(x_4)$ are positive and thus self-adjoint. Hence

$$\phi(x)^* = \phi(x_1) - \phi(x_2) - i\phi(y_1) + i\phi(y_2).$$

Hence $\phi(x^*) = \phi(x)^*$ for all $x \in \mathcal{A}$, implying that ϕ is self-adjoint. ∇

4.3 Automatic continuity of positive linear mappings

Let $\mathcal{A}$ be a topological algebra. Recall that the *radical* of $\mathcal{A}$, denoted by $Rad(\mathcal{A})$, is the intersection of all maximal left ideals of $\mathcal{A}$. The *strong radical* of $\mathcal{A}$, denoted by $Rad_S(\mathcal{A})$, is defined to be the intersection of all closed maximal left ideals of $\mathcal{A}$. If $Rad(\mathcal{A}) = \{0\}$ (respectively $Rad_S(\mathcal{A}) = \{0\}$), we say that $\mathcal{A}$ is *semi-simple* (respectively *strongly semi-simple*). If $\mathcal{A}$ is a commutative complete locally m-convex topological algebra, then $Rad(\mathcal{A}) = Rad_S(\mathcal{A})$ ([15], 4:11-1).

Theorem 4.3.1 *If $\widetilde{\mathcal{M}}$ is commutative and locally convex, then $\widetilde{\mathcal{M}}$ is strongly semi-simple.*

Proof. By Theorem 1.5.30, $\widetilde{\mathcal{M}}$ is semi-simple. It follows from Theorem 1.5.26 and a preceding remark that $Rad_S(\widetilde{\mathcal{M}}) = Rad(\widetilde{\mathcal{M}}) = \{0\}$. ∇

Let $\mathcal{A}$ and $\mathcal{B}$ be metrizable topological algebras and $\phi : \mathcal{A} \to \mathcal{B}$ a linear map. Recall that the separating space of ϕ, denoted by $\mathcal{S}(\phi, \mathcal{B})$, is defined to be the set

$$\{y \in \mathcal{B} : \text{ there is a sequence } (x_n) \text{ in } \mathcal{A} \text{ with } x_n \to 0 \text{ and } \phi(x_n) \to y\}.$$

Lemma 4.3.2 *If $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is positive, then*

$$\mathcal{S}(\phi, \widetilde{\mathcal{B}}) \subset \cap\{Ker(f) : f \text{ a positive linear form on } \widetilde{\mathcal{B}}\}.$$

Proof. Let $y \in \mathcal{S}(\phi, \widetilde{\mathcal{B}})$. Then there is a sequence (x_n) in $\widetilde{\mathcal{A}}$ such that $x_n \to 0$ (γ_{cm}) and $\phi(x_n) \to y$ (γ_{cm}). Let f be a positive linear form on $\widetilde{\mathcal{B}}$. Then f is measure continuous by Theorem 1.4.2. So $f(\phi(x_n)) \to f(y)$ (γ_{cm}). Since ϕ is positive, $f \circ \phi$ is a positive linear form on $\widetilde{\mathcal{A}}$. So, by Theorem 1.4.2, $f \circ \phi$ is measure continuous on $\widetilde{\mathcal{A}}$. Therefore, since $x_n \to 0$ (γ_{cm}), it follows that $(f \circ \phi)(x_n) \to 0$ (γ_{cm}), i.e. $f(\phi(x_n)) \to 0$ (γ_{cm}). Due to uniqueness of limits, $f(y) = 0$. But f is an arbitrary positive linear form on $\widetilde{\mathcal{B}}$. Hence the result follows. ∇

Theorem 4.3.3 *If $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is positive and $\widetilde{\mathcal{B}}$ is locally convex in measure, then ϕ is $\gamma_{cm} - \gamma_{cm}$ continuous.*

Proof. Let $0 < x \in \widetilde{\mathcal{B}}$. By Theorem 1.5.29 and Proposition 1.4.22, there is a positive linear form f on $\widetilde{\mathcal{B}}$ such that $f(x) > 0$. If $0 \neq x \in \widetilde{\mathcal{B}}$ is self-adjoint, but not positive, there exists a positive linear form g on $\widetilde{\mathcal{B}}$ such that $g(x) < 0$ (Proposition 1.4.22). To summarize, if $0 \neq x \in \widetilde{\mathcal{B}}$ is self-adjoint, there is a positive linear form f on $\widetilde{\mathcal{B}}$ such that $f(x) \neq 0$.

Let $y \in \mathcal{S}(\phi, \widetilde{\mathcal{B}})$. We show that $a = \frac{1}{2}(y + y^*) \in \mathcal{S}(\phi, \widetilde{\mathcal{B}})$. Since $y \in \mathcal{S}(\phi, \widetilde{\mathcal{B}})$, there exists a sequence (x_n) in $\widetilde{\mathcal{A}}$ such that $x_n \to 0$ (γ_{cm}) and $\phi(x_n) \to y$ (γ_{cm}). Observe that $x_n^* \to 0$ (γ_{cm}). Since ϕ is positive and thus self-adjoint (by Lemmas 4.2.2 and 4.2.3), $\phi(x_n^*) = \phi(x_n)^* \to y^*$ (γ_{cm}). Thus $y^* \in \mathcal{S}(\phi, \widetilde{\mathcal{B}})$. Therefore, since $\mathcal{S}(\phi, \widetilde{\mathcal{B}})$ is a vector subspace of $\widetilde{\mathcal{B}}$, it follows that $a \in \mathcal{S}(\phi, \widetilde{\mathcal{B}})$.

Similarly, $b = \frac{1}{2i}(y - y^*) \in \mathcal{S}(\phi, \widetilde{\mathcal{B}})$. Observe that a and b are self-adjoint elements of $\widetilde{\mathcal{B}}$. By Lemma 4.3.2 and the first paragraph, it is immediate that

$a = 0$ and $b = 0$. Hence $y = a + ib = 0$, implying that $\mathcal{S}(\phi, \widetilde{\mathcal{B}}) = \{0\}$. Thus ϕ is $\gamma_{cm} - \gamma_{cm}$ continuous. ∇

4.4 Projection preserving linear mappings

Recall that, for an unbounded linear operator x, we denote the spectrum of x by Sp (x). For $x \in \widetilde{\mathcal{M}}$, we denote the spectrum of x with respect to the algebra $\widetilde{\mathcal{M}}$ by Sp $(x, \widetilde{\mathcal{M}})$, as in Section 1.4.

Lemma 4.4.1 *Let $x \in \widetilde{\mathcal{M}}$ be a normal operator. Then Sp $(x, \widetilde{\mathcal{M}}) \subset$ Sp (x).*

Proof. For the purposes of this proof only, we denote, for elements x and y of $\widetilde{\mathcal{M}}$, the ordinary sum of x and y by $x + y$ and the ordinary product by xy. The strong sum and strong product of x and y will be denoted by $\overline{x + y}$ and $\overline{xy}$ respectively.

Let $\lambda \notin$ Sp (x). Then $x - \lambda 1$ has a bounded inverse y. By Theorem 1.3.9, there is a smallest commutative von Neumann algebra $\mathcal{A}$ with which x is affiliated. Therefore, $\mathcal{A} \subset \mathcal{M} \subset \widetilde{\mathcal{M}}$. Also $y \in \mathcal{A}$ (see [63], p. 357). Thus $y \in \widetilde{\mathcal{M}}$. Note that $1 = (x - \lambda 1)y \supset y(x - \lambda I)$. It follows from Theorem 1.3.6, that

$$1 = \overline{1} = \overline{(x - \lambda 1)y} = \overline{y(x - \lambda 1)}.$$

This says that $x - \lambda 1$ is invertible with respect to $\widetilde{\mathcal{M}}$, i.e. $\lambda \notin$ Sp $(x, \widetilde{\mathcal{M}})$. Hence Sp $(x, \widetilde{\mathcal{M}}) \subset$ Sp (x). ∇

In general the inclusion in Lemma 4.4.1 is strict. We demonstrate this with the following example.

Example 4.4.2 Let $\mathcal{M}$ be the commutative von Neumann algebra $\{M_f : f \in L_\infty([0, 1])\}$. Then $\widetilde{\mathcal{M}} = \mathcal{U}(\mathcal{M}) = \{M_f : f \in L_0([0, 1])\}$. Let $x \in \widetilde{\mathcal{M}}$. By Lemma 1.3.8, it follows that Sp $(x, \widetilde{\mathcal{M}}) =$ Sp $(x, \mathcal{U}(\mathcal{M})) = \sigma_p(x)$, where $\sigma_p(x)$ denotes the point spectrum of x.

We show that $\sigma_p(x)$ may be empty. Let $x = M_f$, where $f(t) = t$ for all $t \in [0, 1]$. It is well known that x is a bounded self-adjoint operator with no eigenvalues, i.e. $\sigma_p(x)$ is empty (see [70], excercise 9, p. 464). Hence Sp $(x, \widetilde{\mathcal{M}}) \neq$ Sp (x) since Sp (x) is non-empty (as x is bounded).

Sometimes, the inclusion of Lemma 4.4.1 is an equality, as the following example confirms.

Example 4.4.3 Let $\mathcal{M}$ be as in Example 4.4.2. If $p \in \widetilde{\mathcal{M}}$ is a projection, then Sp $(p, \widetilde{\mathcal{M}}) =$ Sp (p): Observe that Sp $(p) \subset \{0, 1\}$. Hence

$$\text{Sp } (p) \subset \{0, 1\} = \sigma_p(p) = \text{Sp } (p, \widetilde{\mathcal{M}}).$$

By Lemma 4.4.1, we know that Sp $(p, \widetilde{\mathcal{M}}) \subset$ Sp (p). Thus Sp $(p, \widetilde{\mathcal{M}}) =$ Sp (p).

Theorem 4.4.4 *Let $\mathcal{A}$ be a GB*-algebra and $\mathcal{B}$ a locally C*-algebra. If $\phi : \mathcal{A} \to \mathcal{B}$ is a unital invertibility preserving self-adjoint linear map, then ϕ preserves projections.*

Proof. Let $x \in \mathcal{A}_b$ be self-adjoint. Then, since $\mathcal{A}_b$ is a C*-algebra, Sp $(x, \mathcal{A}_b)$ is bounded. Since ϕ is unital, linear and invertibility preserving, we have from Lemma 2.1.13 that Sp $(\phi(x), \mathcal{B}) \subset$ Sp $(x, \mathcal{A}) \subset$ Sp $(x, \mathcal{A}_b)$, and so Sp $(\phi(x), \mathcal{B})$ is bounded. Since ϕ is self-adjoint, $\phi(x)$ is self-adjoint. Thus, by Theorem 1.4.7, $\phi(x) \in \mathcal{B}_s$.

Let $p \in \mathcal{A}$ be a projection. Since $p \in \mathcal{A}_b$ (Lemma 1.4.15), it follows from the previous paragraph that $\phi(p) \in \mathcal{B}_s$. Since ϕ is unital, linear and invertibility preserving, it follows from Lemma 2.1.13 that Sp $(\phi(p), \mathcal{B}) \subset$ Sp $(p, \mathcal{A}) \subset$ Sp $(p, \mathcal{A}_b) \subset \{0, 1\}$. Therefore, by Proposition 1.4.9, we see that Sp $(\phi(p), \mathcal{B}_s) = \overline{\text{Sp } (\phi(p), \mathcal{B})}$. Hence Sp $(\phi(p), \mathcal{B}_s) \subset \{0, 1\}$. Since p is self-adjoint, $\phi(p)$ is self-adjoint. Since $\mathcal{B}_s$ is a C*-algebra (Theorem 1.4.8), it follows that $\phi(p)$ is a projection in $\mathcal{B}_s$. ∇

4.5 The main results

Lemma 4.5.1 *(i) A unital linear functional f on a unital algebra $\mathcal{A}$ is invertibility preserving if and only if $f(x) \in Sp$ $(x, \mathcal{A})$ for every $x \in \mathcal{A}$.*

(ii) Every unital invertibility preserving linear functional on $\widetilde{\mathcal{M}}$ is a positive linear form.

Proof. *(i).* This is a direct consequence of Lemma 2.1.13.
(ii). Let f be a unital invertibility preserving linear functional on $\widetilde{\mathcal{M}}$, and let $0 \leq x \in \widetilde{\mathcal{M}}$. Then Sp $(x) \subset [0, \infty)$. By Lemma 4.4.1, Sp $(x, \widetilde{\mathcal{M}}) \subset$ Sp $(x) \subset [0, \infty)$. It follows from *(i)* that $f(x) \in$ Sp $(x, \widetilde{\mathcal{M}})$. Hence $f(x) \geq 0$, implying the result. ∇

Since all characters on unital algebras are unital and invertibility preserving, Corollary 3.1.5 also follows immediately from Lemma 4.5.1 and Theorem 1.4.2. Our next result is an analogue of the Gleason-Kahane- Żelazko theorem (Theorem 2.2.1) for $\widetilde{\mathcal{M}}$.

Theorem 4.5.2 *Let f be a linear functional on $\widetilde{\mathcal{M}}$. The following statements are equivalent.*

(i) f is a character on $\widetilde{\mathcal{M}}$.

(ii) f is unital and invertibility preserving.

(iii) $f(x) \in$ Sp $(x, \widetilde{\mathcal{M}})$ for every $x \in \widetilde{\mathcal{M}}$.

Proof. The implication *(i)* $\Rightarrow$ *(ii)* is trivial. The implication *(ii)* $\Rightarrow$ *(iii)* follows from Lemma 4.5.1(i).

(iii) $\Rightarrow$ *(ii)*: If (iii) holds, then f is unital since $f(1) \in$ Sp $(1, \widetilde{\mathcal{M}}) = \{1\}$. Therefore, by Lemma 4.5.1, (ii) follows.

(ii) $\Rightarrow$ *(i)*: Suppose that f is unital and invertibility preserving. In particular, $f|_{\mathcal{M}}$ is unital, linear and invertibility preserving. Therefore, it follows from Theorem 2.2.1 that $f|_{\mathcal{M}}$ is a character. By Lemma 4.5.1(ii), f is positive, and therefore, by Theorem 1.4.2, f is measure continuous. Consequently, f is a character on $\widetilde{\mathcal{M}}$. ∇

We are now ready to give the first main result of this section, namely Theorem 4.5.3. The proof is nearly the same as that of [64], Theorem 4, and we give the proof for completeness.

Theorem 4.5.3 *Suppose that $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is a unital invertibility preserving linear map and that $\widetilde{\mathcal{B}}$ has a separating family of characters, i.e.*

$$\cap\{Ker(f) : f \text{ a character on } \widetilde{\mathcal{B}}\} = \{0\}.$$

Then ϕ is an algebra homomorphism, and thus a Jordan homomorphism.

Proof. Let f be a character on $\widetilde{\mathcal{B}}$ and let $F(x) = f(\phi(x))$ for all $x \in \widetilde{\mathcal{A}}$. Then F is a linear functional on $\widetilde{\mathcal{A}}$. By Theorem 4.5.2 and the fact that ϕ is unital and invertibility preserving, $F(x) = f(\phi(x)) \in \mathrm{Sp}\ (\phi(x), \widetilde{\mathcal{B}}) \subset \mathrm{Sp}\ (x, \widetilde{\mathcal{A}})$ for all $x \in \widetilde{\mathcal{A}}$. By applying Theorem 4.5.2 again, we find that F is a character on $\widetilde{\mathcal{A}}$, i.e. $F(xy) = F(x)F(y)$ for every $x, y \in \widetilde{\mathcal{A}}$. Thus $f(\phi(xy)) = f(\phi(x))f(\phi(y))$ for every $x, y \in \widetilde{\mathcal{A}}$. Therefore, $f(\phi(xy) - \phi(x)\phi(y)) = 0$ for every $x, y \in \widetilde{\mathcal{A}}$. This holds for every character f on $\widetilde{\mathcal{B}}$, i.e. $\phi(xy) - \phi(x)\phi(y) \in \mathrm{Ker}(f)$ for every character f on $\widetilde{\mathcal{B}}$. By hypothesis, ϕ is an algebra homomorphism. ∇

We give an example of an $\widetilde{\mathcal{M}}$ having a separating family of continuous characters.

Example 4.5.4 Suppose that $\widetilde{\mathcal{M}}$ is commutative and locally convex. By Theorems 1.5.26 and 4.3.1, $\widetilde{\mathcal{M}}$ is strongly semi-simple, i.e.

$$\cap\{M : M \text{ a closed maximal two-sided ideal of } \widetilde{\mathcal{M}}\} = \{0\}.$$

Since $\widetilde{\mathcal{M}}$ is commutative and locally convex, a maximal two-sided ideal of $\widetilde{\mathcal{M}}$ is closed if and only if it is the kernel of a continuous character on $\widetilde{\mathcal{M}}$ (see Theorem 1.5.26 and [15], 4.10-4). Thus $\widetilde{\mathcal{M}}$ has a separating family of continuous characters.

In light of Theorem 4.5.3, it would also be interesting to know whether every $\widetilde{\mathcal{M}}$ has a separating family of characters. Proposition 4.5.5 below answers this question in the negative.

Proposition 4.5.5 *If $\mathcal{M}_p$ is non-atomic, then $\widetilde{\mathcal{M}}$ does not have a separating family of (measure continuous) characters on $\widetilde{\mathcal{M}}$.*

Proof. We first show that $f(a) = 0$ for every $a \in \widetilde{\mathcal{M}_0}$ and for every $f \in (\widetilde{\mathcal{M}}, \gamma_{cm})^*$. Suppose that $\mathcal{M}_p$ is non-atomic. Let $f \in (\widetilde{\mathcal{M}}, \gamma_{cm})^*$. Then $f|_{\widetilde{\mathcal{M}_0}} \in (\widetilde{\mathcal{M}_0}, \gamma_{cm})^* = \{0\}$, by Theorem 1.5.25.

By Corollary 3.1.5, all characters on $\widetilde{\mathcal{M}}$ are measure continuous. Therefore, the result follows from the first paragraph. ∇

The following example shows that, in Theorem 4.5.3, the condition that $\widetilde{\mathcal{B}}$ have a separating family of characters cannot be dropped. This is based on the example given in [85].

Example 4.5.6 Let $\mathcal{A}$ be a von Neumann algebra with a faithful finite normal trace $\tau_{\mathcal{A}}$. Let $\mathcal{B} = M_2(\mathcal{A})$, and define a faithful finite normal trace $\tau_{\mathcal{B}}$ on $\mathcal{B}$ by $\tau_{\mathcal{B}}((x_{ij})) = \tau_{\mathcal{A}}(x_{11}) + \tau_{\mathcal{A}}(x_{22})$ for every $(x_{ij}) \in \mathcal{B}$. By Proposition 1.3.7,

$$\widetilde{\mathcal{B}} = \mathcal{U}(\mathcal{B}) = \mathcal{U}(M_2(\mathcal{A})) \cong M_2(\mathcal{U}(\mathcal{A})) = M_2(\widetilde{\mathcal{A}}).$$

Define a map $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ by

$$\phi(x) = \begin{pmatrix} x & x - \psi(x) \\ 0 & x \end{pmatrix},$$

where ψ is an automorphism of $\widetilde{\mathcal{A}}$. It can easily be shown that ϕ is unital, linear and invertibility preserving. However, ϕ is not a Jordan homomorphism unless ψ is the identity automorphism of $\widetilde{\mathcal{A}}$. By Theorem 4.5.3, $\widetilde{\mathcal{B}}$ has no separating family of characters.

We now arrive at the second main result of this section.

Theorem 4.5.7 *Suppose that $\phi : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ is a unital invertibility preserving positive linear map, with $\widetilde{\mathcal{B}}$ a locally C^*-algebra with respect to the measure topology. Then ϕ is a Jordan homomorphism.*

Proof. Recall that $\widetilde{\mathcal{A}}$ is a GB*-algebra (Theorem 1.5.29). By Theorem 4.3.3, ϕ is continuous. Since ϕ is positive, it follows from Lemmas 4.2.2 and 4.2.3 that ϕ is self-adjoint. It is an immediate consequence of Theorem 4.4.4 that ϕ is projection preserving. By Theorem 4.1.1, ϕ is a Jordan homomorphism. ∇

The positivity assumption of ϕ in Theorem 4.5.7 cannot be dropped, as shown by the following example.

Example 4.5.8 *([91], Example 2)* Let $\mathcal{H}$ be a Hilbert space. There exists a basis B of $\mathcal{B}(\mathcal{H})$ such that $1 \in B$. Let $Z = \text{span}\{1, b\}$, where $\lambda 1 \neq b \in B$ for every $\lambda \in \mathbb{C}$. Let $x \in Z$ with $x = \alpha_1 1 + \alpha_2 b$, where $\alpha_1, \alpha_2 \in \mathbb{C}$. Observe that $\{1, b\}$ is a basis for Z. Define a linear functional $f : Z \to \mathbb{C}$ by $f(x) = \alpha_2$. Clearly, f is well defined and $f(1) = 0$ and $f(b) = 1$. By the Hahn-Banach theorem, f can be extended to a a nonzero linear functional g on $\mathcal{B}(\mathcal{H})$. Define a map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ by

$$\phi(x) = \begin{pmatrix} x & g(x)1 \\ 0 & x \end{pmatrix}.$$

It can be shown that ϕ is linear, unital and invertibility preserving but not a Jordan homomorphism. Furthermore, ϕ is not positive. Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$ and $\mathcal{B} = \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$. Then $\mathcal{A} = \widetilde{\mathcal{A}}$ and $\mathcal{B} = \widetilde{\mathcal{B}}$ (Example 1.5.9). Thus, $\widetilde{\mathcal{B}}$ is a locally C*-algebra with respect to the measure topology.

Chapter 5

Derivations on $\widetilde{\mathcal{M}}$

It is well known that all derivations on a C*-algebra are continuous (Theorem 2.3.5) and that every derivation on a von Neumann algebra is inner (Theorem 2.3.6). One can therefore ask if these results also hold for algebras of unbounded operators, such as $\widetilde{\mathcal{M}}$. A recent result of A. F. Ber, V. I. Chilin, and F. A. Sukochev ([19] and [20]) states that the algebra of closed densely-defined operators affiliated with a commutative von Neumann algebra can have non-zero, and hence non-inner, derivations. We will give an outline of their construction in Section 5.4 of this chapter. Motivated by their result, we prove results which give sufficient conditions to ensure that all derivations on a given $\widetilde{\mathcal{M}}$ are $\gamma_{cm} - \gamma_{cm}$ continuous. We also explore some affirmative answers to the question raised by Sh. A. Ayupov in [13], as to whether all $\gamma_{cm} - \gamma_{cm}$ continuous derivations on $\widetilde{\mathcal{M}}$ are inner. All of these issues are explored in Section 5.3. Section 5.1 discusses the basics of non-commutative L_p-spaces which are needed in the proof of Theorem 5.3.20. Section 5.2 gives a characterization of continuous derivations on complete metrizable topological algebras.

5.1 Non-commutative L_p-spaces

For $0 < p < \infty$, we define $L_p(\mathcal{M}, \tau)$ to be the set of all $x \in \widetilde{\mathcal{M}}$ such that

$$\|x\|_p = \Big(\int_0^\infty \mu_t(x)^p \, \mathrm{d}t \Big)^{\frac{1}{p}} < \infty$$

(a special case of [42], Definition 4.1). In addition, we put $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ and denote by $\|.\|_\infty$ the usual operator norm defined on $\mathcal{M}$. It is well known that $L_p(\mathcal{M}, \tau)$ is a Banach space under $\|.\|_p$ whenever $1 \leq p \leq \infty$ ([42], Theorem 4.5). Also, whenever $1 \leq p \leq \infty$, $L_p(\mathcal{M}, \tau)$ is a Banach $\mathcal{M}$-module ([102], Proposition 2.5). If $1 < p < \infty$, then L_p is reflexive (this is a special case of [43], Corollary 5.16).

Lemma 5.1.1 *Let $1 \leq p < \infty$ and let $u \in \mathcal{M}$ be a unitary operator. Then $\|u^* x u\|_p = \|x\|_p$ for every $x \in L_p(\mathcal{M}, \tau)$.*

Proof. We first show that $\mu_t(xu) = \mu_t(x)$ for every $x \in \widetilde{\mathcal{M}}$ and for every $t > 0$. By Proposition 1.5.14, it follows that $\mu_t(xu) \leq \|1\|\mu_t(x)\|u\| = \mu_t(x)$ for every $x \in \widetilde{\mathcal{M}}$ and for every $t > 0$. Now, by similar reasoning, it follows that $\mu_t(x) = \mu_t(xuu^*) \leq \mu_t(xu)$ for every $x \in \widetilde{\mathcal{M}}$ and for every $t > 0$ (since u^* is also a unitary operator in $\mathcal{M}$). Therefore $\mu_t(xu) = \mu_t(x)$ for every $x \in \widetilde{\mathcal{M}}$ and for every $t > 0$.

Hence, by Proposition 1.5.14,

$$\begin{aligned} \mu_t(u^* x u) &= \mu_t(u^* x) \\ &= \mu_t((u^* x)^*) \\ &= \mu_t(x^* u) \\ &= \mu_t(x^*) \\ &= \mu_t(x) \end{aligned}$$

for every $x \in \widetilde{\mathcal{M}}$ and for every $t > 0$. It is now immediate that $\|u^* x u\|_p = \|x\|_p$ for every $x \in L_p(\mathcal{M}, \tau)$. ∇

There are other definitions of non-commutative L_p-spaces which we will not discuss, see for example [47], p. 271, and [102]. These definitions are equivalent to the one given above.

5.2 Derivations on complete metrizable topological algebras

Let X and Y be metrizable topological vector spaces. Recall that the *separating space* of a linear map $\phi : X \to Y$, denoted by $\mathcal{S}(\phi, Y)$, is defined to be the set

$$\{y \in Y : \text{ there is a sequence } (x_n) \text{ in } X \text{ with } x_n \to 0 \text{ and } \phi(x_n) \to y\}.$$

Recall that $\mathcal{S}(\phi, Y)$ is closed ([36], Proposition 5.1.2) and that the closed graph theorem is valid for complete metrizable topological vector spaces (see [69],p. 101). Therefore ϕ is continuous if and only if $\mathcal{S}(\phi, Y) = \{0\}$, provided X and Y are complete and metrizable. From here on, we shall denote the separating space of a derivation D on a metrizable algebra $\mathcal{A}$ by $\mathcal{S}(D)$. It is easy to verify, using the definition of a derivation, that $\mathcal{S}(D)$ is a two-sided ideal of $\mathcal{A}$.

The proof of the following theorem is modelled along the lines of that of [90], Remark 12.3 and Corollary 12.5. Also, the main idea of the proof is well known in the context of Banach algebras. See, for example, the proofs of [36], Theorems 5.3.22 and 5.3.43, and [83], Theorem 2.

Theorem 5.2.1 *Let $\mathcal{A}$ be a unital complete metrizable topological algebra and D a derivation on $\mathcal{A}$. The following are equivalent.*

(i) D is continuous.

(ii) The ideal $\mathcal{I} = \{a \in \mathcal{A} : a\mathcal{S}(D) = \mathcal{S}(D)a = \{0\}\}$ has finite codimension in $\mathcal{A}$, and any sequence (x_n) in $\mathcal{I}$ with $x_n \to 0$ can be written as $x_n = py_n$, with p a nonzero idempotent in $\mathcal{I}$ and (y_n) a sequence in $\mathcal{A}$.

(iii) $\mathcal{I} = \mathcal{A}$, where $\mathcal{I}$ is defined as in (ii).

Proof. *(i) $\Rightarrow$ (iii):* Suppose that D is continuous. Then $\mathcal{S}(D) = \{0\}$, implying that $\mathcal{I} = \mathcal{A}$.

(iii) $\Rightarrow$ (ii): This is trivial since $\mathcal{A}$ is unital.

(ii) $\Rightarrow$ (i): Consider the linear map $\theta : \mathcal{A} \to \mathcal{A} \oplus \mathcal{A}$ defined as $\theta(a) = (a, Da)$. Recall that when $\mathcal{A} \oplus \mathcal{A}$ is equipped with the multiplication defined as $(a, b).(x, y) = (ax, ay + bx)$ for every $a, b, x, y \in \mathcal{A}$, then it is a complete metrizable topological vector space in the product topology. Furthermore, θ is an algebra homomorphism (Proposition 2.3.3). It is easy to verify that $\mathcal{S}(\theta) = \{0\} \oplus \mathcal{S}(D)$, where $\mathcal{S}(\theta) = \mathcal{S}(\theta, \mathcal{A} \oplus \mathcal{A})$. Hence

$$\mathcal{I} = \{a \in \mathcal{A} : \theta(a)\mathcal{S}(\theta) = \mathcal{S}(\theta)\theta(a) = \{(0, 0)\}\}.$$

We show that $\mathcal{I}$ is closed. Let (x_n) be a sequence in $\mathcal{I}$ with $x_n \to x$. Then $x_n\mathcal{S}(D) = \mathcal{S}(D)x_n = \{0\}$ for all n. But $x_n y \to xy$ and $yx_n \to yx$ for every $y \in \mathcal{S}(D)$. Hence $xy = yx = 0$ for all $y \in \mathcal{S}(D)$. Thus $x \in \mathcal{I}$, implying that $\mathcal{I}$ is closed.

Since $\mathcal{S}(D)$ is a two-sided ideal of $\mathcal{A}$, it follows that $\mathcal{I}$ is a two-sided ideal of $\mathcal{A}$.

Let (x_n) be a sequence in $\mathcal{I}$ with $x_n \to 0$. By hypothesis, there exists a nonzero idempotent $p \in \mathcal{I}$ and a sequence (y_n) in $\mathcal{A}$ such that $x_n = py_n$ for all n. Since $p \in \mathcal{I}$, it follows that $\theta(p)\mathcal{S}(\theta) = \{(0, 0)\}$.

We prove that the map $x \mapsto \theta(px)$ is continuous. Suppose that (z_n) is a sequence in $\mathcal{A}$ with $z_n \to 0$ and $\theta(pz_n) \to y$. By the closed graph theorem, it suffices to show that $y = (0, 0)$. Observe that $pz_n \to 0$. Thus $y \in \mathcal{S}(\theta)$. Since $\theta(p)\mathcal{S}(\theta) = \{(0, 0)\}$, it follows that $\theta(p)y = (0, 0)$. Now $\theta(pz_n) = \theta(p)\theta(z_n) = \theta(p^2)\theta(z_n) = \theta(p)\theta(p)\theta(z_n) = \theta(p)\theta(pz_n) \to \theta(p)y = (0, 0)$ since θ is an algebra homomorphism. Therefore $y = (0, 0)$.

Since $py_n = x_n \to 0$, it follows from the previous paragraph that $\theta(x_n) = \theta(py_n) = \theta(ppy_n) \to (0, 0)$. Hence θ is continuous on $\mathcal{I}$. Since $\mathcal{I}$ is closed and of finite codimension in $\mathcal{A}$, it follows that θ is continuous on the whole of $\mathcal{A}$. It follows from Proposition 2.3.3 that D is continuous. ∇

5.3 Derivations on $\widetilde{\mathcal{M}}$

Recall that if D is a derivation on $\widetilde{\mathcal{M}}$, $\mathcal{S}(D)$ is γ_{cm}-closed.

Theorem 5.3.1 *Let D be a derivation on $\widetilde{\mathcal{M}}$ and $\mathcal{I} = \{a \in \widetilde{\mathcal{M}} : a\mathcal{S}(D) = \mathcal{S}(D)a = \{0\}\}$. Then there exists a central projection p in $\mathcal{I}$ such that $\mathcal{I} = p\widetilde{\mathcal{M}}$.*

Proof. Observe that $\mathcal{I}$ is a γ_{cm}-closed two-sided ideal of $\widetilde{\mathcal{M}}$. By Theorem 1.5.21, $\mathcal{I}_0 = \mathcal{I} \cap \mathcal{M}$ is a norm closed ideal of $\mathcal{M}$.

We show that $\mathcal{I}_0$ is strong operator closed in $\mathcal{M}$. Let (a_α) be a net in $\mathcal{I}_0$ with $a_\alpha \to a \in \mathcal{M}$ with respect to the strong operator topology on $\mathcal{M}$. Then $a_\alpha(\eta) \to a(\eta)$ for all $\eta \in \mathcal{H}$, where $\mathcal{H}$ is the underlying Hilbert space of $\mathcal{M}$. Let $x \in \mathcal{S}(D) \cap \mathcal{M}$. Then $a_\alpha\Big(x(\eta)\Big) \to a\Big(x(\eta)\Big)$ for all $\eta \in \mathcal{H}$. So $(a_\alpha x)(\eta) \to (ax)(\eta)$ for all $\eta \in \mathcal{H}$. Since $x \in \mathcal{S}(D)$ and $a_\alpha \in \mathcal{I}$ for all α, it follows that $a_\alpha x = x a_\alpha = 0$ for all α. Hence $(a_\alpha x)(\eta) = 0$ for all $\eta \in \mathcal{H}$. Thus $(ax)(\eta) = 0$ for all $\eta \in \mathcal{H}$. Thus $ax = 0$. This holds for every $x \in \mathcal{S}(D) \cap \mathcal{M}$.

Let $y \in \mathcal{S}(D)$. By Theorem 1.5.21,

$$\overline{\mathcal{S}(D) \cap \mathcal{M}}^{\gamma_{cm}} = \mathcal{S}(D).$$

Hence there is a sequence (y_n) in $\mathcal{S}(D) \cap \mathcal{M}$ with $y_n \to y$ (γ_{cm}). It follows from what we have proved above that $ay_n = 0$ for every n. Thus $ay = 0$.

Also, $ya = 0$: Since $(a_\alpha)(\eta) \to a(\eta)$ for all $\eta \in \mathcal{H}$, $(y_n a_\alpha)(\eta) \to (y_n a)(\eta)$ for all $\eta \in \mathcal{H}$ and for all n. Since $y_n a_\alpha = 0$ for all α and for all n, $y_n a = 0$ for all n. Hence $ya = 0$.

Therefore $a \in \mathcal{I}$, implying that $a \in \mathcal{I}_0$. Hence $\mathcal{I}_0$ is strong operator closed in $\mathcal{M}$, and hence weak-operator closed ([63], Theorem 5.1.2).

Therefore there exists a central projection $p \in \mathcal{I}_0$ such that $\mathcal{I}_0 = p\mathcal{M}$ (Proposition 1.2.10). By Theorem 1.5.21,

$$\mathcal{I} = \overline{\mathcal{I} \cap \mathcal{M}}^{\gamma_{cm}} = \overline{\mathcal{I}_0}^{\gamma_{cm}} = \overline{p\mathcal{M}}^{\gamma_{cm}}.$$

By Theorem 1.5.10, it follows that $\overline{p\mathcal{M}}^{\gamma_{cm}} = p\widetilde{\mathcal{M}}$. Hence $\mathcal{I} = p\widetilde{\mathcal{M}}$. ∇

Corollary 5.3.2 *Let D be a derivation on $\widetilde{\mathcal{M}}$. The following are equivalent.*

(i) D is $\gamma_{cm} - \gamma_{cm}$ continuous.

(ii) The ideal $\mathcal{I} = \{a \in \widetilde{\mathcal{M}} : a\mathcal{S}(D) = \mathcal{S}(D)a = \{0\}\}$ has finite codimension in $\widetilde{\mathcal{M}}$.

(iii) $\mathcal{I} = \widetilde{\mathcal{M}}$.

Proof. By Theorem 5.3.1, there exists a central projection p in $\mathcal{I}$ such that $\mathcal{I} = p\widetilde{\mathcal{M}}$. Therefore, if (x_n) is a sequence in $\mathcal{I}$ with $x_n \to 0$ (γ_{cm}), there is a sequence (y_n) in $\widetilde{\mathcal{M}}$ such that $x_n = py_n$ for all n. The result now follows from Theorem 5.2.1. ∇

Let $\mathcal{M}$ be a type II factor and let D be a derivation on $\widetilde{\mathcal{M}}$. Corresponding to D, consider the ideal $\mathcal{I}$ of $\widetilde{\mathcal{M}}$ as in Corollary 5.3.2. A trivial consequence of Theorem 5.3.1 is then that $\mathcal{I} = \{0\}$ or $\widetilde{\mathcal{M}}$. Therefore, for the case where $\mathcal{M}$ is a type II factor, it follows from Corollary 5.3.2 that it is sufficient to show that $\mathcal{I} \neq \{0\}$ in order to prove that D is $\gamma_{cm} - \gamma_{cm}$ continuous.

Recall that all derivations on a C*-algebra are continuous (Theorem 2.3.5) and that all derivations on a von Neumann algebra are inner (Theorem 2.3.6). The following result is due to A. F. Ber, V. I. Chilin and F. A. Sukochev.

Theorem 5.3.3 *([19], Theorem 3, and [20], Theorem 3.4) Let $\mathcal{M}$ be a commutative von Neumann algebra. Then $S(\mathcal{M})$ admits a nonzero derivation if and only if $\mathcal{M}_p$ is not atomic.*

By recalling that every commutative von Neumann algebra is finite, the following corollary follows from Proposition 1.5.5.

Corollary 5.3.4 *Let $\mathcal{M}$ be a commutative von Neumann algebra with a faithful finite normal trace. Then $\widetilde{\mathcal{M}}$ admits a nonzero derivation if and only if $\mathcal{M}_p$ is not atomic.*

Note that if $\mathcal{M}$ is commutative, then a derivation on $\widetilde{\mathcal{M}}$ is $\gamma_{cm} - \gamma_{cm}$ continuous if and only if it is zero ([2], p.11).

Using Theorem 5.3.3 and the fact that every finite type I von Neumann algebra is a direct sum of finite matrix algebras over commutative von Neumann algebras (Theorem 1.2.2), one can prove the following result.

Theorem 5.3.5 *([21], Theorem 5) If $\mathcal{M}$ is a finite type I von Neumann algebra and every derivation $D : S(\mathcal{M}) \to \mathcal{S}(\mathcal{M})$ is inner, then $\mathcal{M}_p$ is atomic.*

The following corollary is an immediate consequence of Proposition 1.5.5.

Corollary 5.3.6 *Let $\mathcal{M}$ be a finite type I von Neumann algebra with a faithful finite normal trace τ. If every derivation $D : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is inner, then $\mathcal{M}_p$ is atomic.*

Our next result was motivated by Corollary 5.3.4 and the remark immediately thereafter.

Theorem 5.3.7 *If $\mathcal{M}_p$ is atomic, then all derivations on $\widetilde{\mathcal{M}}$ are $\gamma_{cm} - \gamma_{cm}$ continuous.*

Proof. Let D be a derivation on $\widetilde{\mathcal{M}}$ and q an atomic projection in $\mathcal{M}$. Let $y \in \mathcal{S}(D)$. Then there exists a sequence x_n in $\widetilde{\mathcal{M}}$ with $x_n \to 0$ (γ_{cm}) and $D(x_n) \to y$ (γ_{cm}). Therefore

$$\begin{aligned} D(qx_nq) &= qD(x_nq) + D(q)x_nq \\ &= qx_nD(q) + qD(x_n)q + D(q)x_nq \\ &\to qyq \ (\gamma_{cm}). \end{aligned}$$

By Theorem 1.5.10, it follows that $q\widetilde{\mathcal{M}}q = \widetilde{q\mathcal{M}q}$. Since q is an atomic projection, $q\mathcal{M}q = \mathbb{C}q$.

We show that $q\mathcal{M}q = \widetilde{q\mathcal{M}q}$. Since q is an atomic projection, it is the only nonzero projection in $q\mathcal{M}q$. Hence

$$\inf\{\tau(p) : p \text{ a nonzero projection in } q\mathcal{M}q\} = \tau(q) > 0,$$

since $q \neq 0$ and the trace τ is faithful. Therefore, by Proposition 1.5.11, $q\mathcal{M}q = \widetilde{q\mathcal{M}q}$.

Thus $q\widetilde{\mathcal{M}}q = \widetilde{q\mathcal{M}q} = q\mathcal{M}q = \mathbb{C}q$. Since $x_n \to 0$ (γ_{cm}), $qx_nq \to 0$ (γ_{cm}). Since $q\widetilde{\mathcal{M}}q$ is finite-dimensional, $D|_{q\widetilde{\mathcal{M}}q}$ is $\gamma_{cm}-\gamma_{cm}$ continuous, implying that $D(qx_nq) \to 0$ (γ_{cm}). Recall that $D(qx_nq) \to qyq$ (γ_{cm}). Therefore $qyq = 0$. This is true for every atomic projection q in $\mathcal{M}$ and for every $y \in \mathcal{S}(D)$.

We show next that $\mathcal{S}(D)$ contains no atomic projections of $\mathcal{M}$. Suppose that $\mathcal{S}(D)$ contains an atomic projection q_0 of $\mathcal{M}$. Then, from what we have proved above, $q_0q_0q_0 = 0$, i.e. $q_0 = 0$. This is a contradiction since $q_0 \neq 0$. Therefore $\mathcal{S}(D)$ contains no atomic projections of $\mathcal{M}$.

To show that D is $\gamma_{cm} - \gamma_{cm}$ continuous, it suffices to show that $\mathcal{S}(D) = \{0\}$. Suppose that $\mathcal{S}(D) \neq \{0\}$. By Theorem 1.5.21, and the fact that $\mathcal{S}(D)$ is a γ_{cm}-closed two-sided ideal of $\widetilde{\mathcal{M}}$, it follows that $\mathcal{S}(D) \cap \mathcal{M}$ is a norm closed two-sided ideal of $\mathcal{M}$. Furthermore, $\mathcal{S}(D) \cap \mathcal{M} \neq \{0\}$ since, by Theorem 1.5.21,

$$\overline{\mathcal{S}(D) \cap \mathcal{M}}^{\gamma_{cm}} = \mathcal{S}(D) \neq \{0\}.$$

Therefore, by Theorem 1.2.9, $\mathcal{S}(D) \cap \mathcal{M}$ has at least one nonzero projection p. Since $\mathcal{M}_p$ is atomic, there exists an atomic projection q_0 of $\mathcal{M}$ such that $0 < q_0 \leq p$. So $q_0 = q_0p \in \mathcal{S}(D) \cap \mathcal{M}$ because $q_0 \in \mathcal{M}$, p is in $\mathcal{S}(D)$, and $\mathcal{S}(D) \cap \mathcal{M}$ is a two-sided ideal of $\mathcal{M}$. This is a contradiction since $\mathcal{S}(D)$ has no atomic projections of $\mathcal{M}$. ∇

In light of Corollary 5.3.4 and the remark thereafter, it would be interesting to know if the converse of Theorem 5.3.7 holds. We now solve this problem in the affirmative for finite type I von Neumann algebras (Theorem 5.3.14), thereby extending Corollary 5.3.6. For this, we need the following four lemmas. The proof of [21], Theorem 5 (i.e. Theorem 5.3.5), relies strongly on Proposition 1.3.7, which itself depends on the fact that $S(\mathcal{M})$ is a regular algebra whenever $\mathcal{M}$ is finite. In general, $\widetilde{\mathcal{M}}$ is not regular even if $\mathcal{M}$ is commutative (see Corollary 6.5.9). We postpone the discussion on the regularity of $\widetilde{\mathcal{M}}$ until Section 6.5.

Lemma 5.3.8 *Let p be a central projection of $\mathcal{M}$ and D a derivation on $\widetilde{p\mathcal{M}p}$. Then D can be extended to a derivation $\overline{D}$ on $\widetilde{\mathcal{M}}$.*

Proof. Let $\overline{D}(x) = D(pxp)$ for every $x \in \widetilde{\mathcal{M}}$. Since p is a central projection, $D(p) = 0$. Using this, it is easily verified that $\overline{D}$ is a derivation on $\widetilde{\mathcal{M}}$ extending D. ∇

For $p \in \mathcal{M}_p$, we denote by $c(p)$ the least central projection majorizing p, and we call $c(p)$ the *central support* of p. If $p, q \in \mathcal{M}_p$, then $qxp = 0$ for every $x \in \mathcal{M}$ if and only if $c(p)c(q) = 0$ ([95], Corollary V.1.7). This is needed in the proof of the following known result, which we give for completeness.

Lemma 5.3.9 *([56], Theorem 2.1) If q is an atom in a von Neumann algebra $\mathcal{M}$, then the central support $c(q)$ of q is an atom in $Z(\mathcal{M})$.*

Proof. Let p be a central projection of $\mathcal{M}$ such that $0 < p \leq c(q)$. Then $qp \in \mathcal{M}_p$ and $qp \leq q$. Since q is an atom, it follows that $qp = 0$ or $qp = q$.

We show that $qp \neq 0$. Suppose that $qp = 0$. Then $qxp = qpx = 0$ for every $x \in \mathcal{M}$, and so $c(q)c(p) = 0$, i.e. $pc(q) = c(q)p = 0$. Since $p \leq c(q)$, it follows that $p = 0$. This is a contradiction since $0 \neq p$. Thus $qp \neq 0$.

Therefore $qp = q$, implying that $q = qp \leq p$. Hence $c(q) \leq p$, since p is a central projection majorizing q, and thus $c(q) = p$, implying that $c(q)$ is an atom in $Z(\mathcal{M})$. ∇

Lemma 5.3.10 *Let (p_α) be a family of central orthogonal projections in $\mathcal{M}$ such that $\sum_\alpha p_\alpha = 1$. If $(p_\alpha \mathcal{M} p_\alpha)_p$ is atomic for every α, then $\mathcal{M}_p$ is atomic.*

Proof. Suppose $(p_\alpha \mathcal{M} p_\alpha)_p$ is atomic for every α. Let $0 < p \in \mathcal{M}_p$. Since p_α is a central projection for every α, it follows that $pp_\alpha \in \mathcal{M}_p$ for every α. By hypothesis, $\sum_\alpha p_\alpha = 1$, and so $\sum_\alpha pp_\alpha = p$. Since $p \neq 0$, we have that $pp_\alpha \neq 0$ for some α. It is easily verified that $p \geq pp_\alpha$ and that $pp_\alpha \leq p_\alpha$. Therefore, pp_α is a projection in $p_\alpha \mathcal{M} p_\alpha$ and, by hypothesis, there exists an atom q_α in $p_\alpha \mathcal{M} p_\alpha$ such that $p \geq pp_\alpha \geq q_\alpha$. Clearly, q_α is also an atom in $\mathcal{M}$. This completes the proof. ∇

For the next lemma, we recall the notion of a direct sum of von Neumann algebras ([39], p. 20-21). For each α, let $\mathcal{A}_\alpha$ be a von Neumann algebra on the Hilbert space $\mathcal{H}_\alpha$, and let $\|\cdot\|_\alpha$ denote the norm on $\mathcal{A}_\alpha$. If $x_\alpha \in \mathcal{A}_\alpha$ for each α, and $\sup_\alpha \|x_\alpha\|_\alpha < \infty$, define a bounded linear operator x on the direct sum $\mathcal{H}$ of the family of Hilbert spaces $(\mathcal{H}_\alpha)$ by $x(\xi) = (x_\alpha(\xi_\alpha))$, where $\xi = (\xi_\alpha) \in \mathcal{H}$. The *direct sum* $\mathcal{A}$ of the von Neumann algebras $\mathcal{A}_\alpha$ is defined to be the set of all such operators x, and we write $\mathcal{A} = \oplus_\alpha \mathcal{A}_\alpha$. It can be verified that $\mathcal{A}$ is a von Neumann algebra on $\mathcal{H}$ with coordinate-wise operations, and norm $x \mapsto \sup_\alpha \|x_\alpha\|_\alpha$.

Lemma 5.3.11 *Let $\mathcal{M} = \mathcal{A} \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H})$, where $\mathcal{A}$ is a commutative von Neumann algebra and $\mathcal{H}$ a finite-dimensional Hilbert space (so $\mathcal{M}$ is a type I von Neumann algebra). If $\mathcal{A}_p$ is atomic, so is $\mathcal{M}_p$.*

Proof. Since $\mathcal{A}$ is commutative and $\mathcal{A}_p$ is atomic, $\mathcal{A} \cong L_\infty(X, \Sigma, \mu)$ for some localizable atomic measure space (X, Σ, μ). Therefore we can find a disjoint family of atoms $(A_\lambda : \lambda \in \Lambda)$ satisfying $X = \cup A_\lambda$. Recall that measurable functions on atoms are constant. Hence $\mathcal{A} \cong l_\infty(\Lambda)$, where $l_\infty(\Lambda)$ is the space of all bounded nets in $\mathbb{C}$ indexed by Λ. Hence, by [39], p. 29,

$$\begin{aligned} \mathcal{M} &= \mathcal{A} \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H}) \\ &\cong l_\infty(\Lambda) \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H}) \\ &= \oplus_{\alpha\in\Lambda}(\mathbb{C}_\alpha \mathbin{\overline{\otimes}} \mathcal{B}(\mathcal{H})) \\ &\cong \oplus_{\alpha\in\Lambda}\mathcal{B}(\mathcal{H}_\alpha), \end{aligned}$$

where $\mathcal{H}_\alpha = \mathcal{H}$ and $\mathbb{C}_\alpha = \mathbb{C}$ for every $\alpha \in \Lambda$. Every $(\mathcal{B}(\mathcal{H}_\alpha))_p$ is atomic, and one can find a family of central orthogonal projections (p_α) in $\mathcal{M}$ such that $\sum_\alpha p_\alpha = 1$ and $p_\alpha \mathcal{M} p_\alpha \cong \mathcal{B}(\mathcal{H}_\alpha)$, namely $p_\alpha = 1_\alpha$, where 1_α denotes the identity operator on $\mathcal{H}_\alpha$, for every $\alpha \in \Lambda$. By Lemma 5.3.10, $\mathcal{M}_p$ is atomic. ∇

If $\mathcal{M}$ is a finite von Neumann algebra, then $\mathcal{M}$ can be imbedded into the maximal ring of right quotients $\mathcal{Q}_\mathcal{M}$ of $\mathcal{M}$ ([25]). Furthermore, $\mathcal{U}(\mathcal{M}) \cong \mathcal{Q}_\mathcal{M}$ ([25]). Therefore, if $\mathcal{A}$ and $\mathcal{B}$ are *-isomorphic finite von Neumann algebras, then $\mathcal{U}(\mathcal{A}) \cong \mathcal{U}(\mathcal{B})$. In particular, if $\mathcal{A}$ and $\mathcal{B}$ are equipped with faithful finite normal traces, then $\widetilde{\mathcal{A}} = \mathcal{U}(\mathcal{A}) \cong \mathcal{U}(\mathcal{B}) = \widetilde{\mathcal{B}}$. This is needed in the proof of our next proposition.

If $\mathcal{A}$ is a von Neumann algebra with a faithful finite normal trace, then $\widetilde{M_n(\mathcal{A})} = M_n(\widetilde{\mathcal{A}})$ for every $n \in \mathbb{N}$ (Proposition 1.3.7). Let $(a^m_{i,j})$ be a sequence in $M_n(\widetilde{\mathcal{A}})$ and $(a_{i,j}) \in M_n(\widetilde{\mathcal{A}})$. Then $(a^m_{i,j}) \to (a_{i,j})$ in measure as $m \to \infty$ if and only if $a^m_{i,j} \to a_{i,j}$ as $m \to \infty$ for every i, j (this is an immediate consequence of [44], Lemma 2.1).

The proof of the following proposition is a slight modification of the proof [21], Theorem 5, and we give the proof for completeness. It is a special case of Theorem 5.3.14.

Proposition 5.3.12 *Suppose that $\mathcal{M}$ is a type I von Neumann algebra with a faithful finite normal trace. If all derivations on $\widetilde{\mathcal{M}}$ are $\gamma_{cm} - \gamma_{cm}$ continuous, then $\mathcal{M}_p$ is atomic.*

Proof. Since $\mathcal{M}$ is a finite type I von Neumann algebra, there exists a sequence of central projections (p_n) such that $\mathcal{M} = \oplus_{n=1}^{\infty} p_n \mathcal{M} p_n$, and $p_n \mathcal{M} p_n \cong M_{k_n}(\mathcal{A}_n)$, where $\mathcal{A}_n$ is a commutative von Neumann algebra for every n (Theorem 1.2.2), and the k_n are integers. Assume that $\mathcal{M}_p$ is not atomic. Then, by Lemmas 5.3.10 and 5.3.11, it follows that $(\mathcal{A}_r)_p$ is not atomic for some $r \in \mathbb{N}$. By Corollary 5.3.4, there exists a derivation δ on $\widetilde{\mathcal{A}_r}$ which is not $\gamma_{cm} - \gamma_{cm}$ continuous.

We show that there exists a derivation D_r on $\widetilde{M_{k_r}(\mathcal{A}_r)} = M_{k_r}(\widetilde{\mathcal{A}_i})$ which is not $\gamma_{cm} - \gamma_{cm}$ continuous. Define D_r to be the linear map defined by $D_r((a_{i,j})) = (\delta(a_{i,j}))$ for every $(a_{i,j}) \in M_{k_r}(\widetilde{\mathcal{A}_r})$. It is easily verified that D_r is a derivation. Since δ is not $\gamma_{cm} - \gamma_{cm}$ continuous, there exists a sequence (a_m) in $\widetilde{\mathcal{A}_r}$ such that $a_m \to 0$ in measure and $\delta(a_m)$ does not converge to zero in measure. Let $(a^m_{i,j})$ be the sequence in $M_{k_r}(\mathcal{A}_r)$ defined as $a^m_{i,j} = a_m$ for every m and for all i, j. By the preceding remarks, $(a^m_{i,j}) \to (b_{i,j})$ as $m \to \infty$, where $b_{i,j} = 0$ for all i, j. Also, by the preceding remarks, $D_r((a^m_{i,j})) = (\delta(a^m_{i,j}))$ does not converge to $D_r((b_{i,j})) = (\delta(b_{i,j}))$ in measure. Therefore, D_r is not $\gamma_{cm} - \gamma_{cm}$ continuous.

Since $p_r \mathcal{M} p_r \cong M_{k_r}(\mathcal{A}_r)$ and the traces of $p_r \mathcal{M} p_r$ and $M_{k_r}(\mathcal{A}_r)$ are finite, it follows that the algebra $M_{k_r}(\widetilde{\mathcal{A}_r}) = \widetilde{M_{k_r}(\mathcal{A}_r)}$ is isomorphic to $\widetilde{p_r \mathcal{M} p_r}$, and this isomorphism is also $\gamma_{cm} - \gamma_{cm}$ bicontinuous (by Theorem 3.2.8). Therefore, the derivation D_r can be identified with a derivation on $\widetilde{p_r \mathcal{M} p_r}$ which is not $\gamma_{cm} - \gamma_{cm}$ continuous. By Lemma 5.3.8, D_r can be extended to a derivation D on $\widetilde{\mathcal{M}}$. It follows from Lemmas 1.5.15 and 1.5.16 that D is not $\gamma_{cm} - \gamma_{cm}$ continuous. This contradicts the hypothesis, implying that $\mathcal{M}_p$ is atomic. ∇

The following proposition is the first part of the proof of Theorem 5.3.14.

Proposition 5.3.13 *Let $\mathcal{M}$ be a finite type I von Neumann algebra. If all derivations on $\widetilde{\mathcal{M}}$ are $\gamma_{cm} - \gamma_{cm}$ continuous, then $(Z(\mathcal{M}))_p$ is atomic.*

Proof. Suppose that $\mathcal{M}$ is a finite type I von Neumann algebra such that all derivations on $\widetilde{\mathcal{M}}$ are $\gamma_{cm} - \gamma_{cm}$ continuous.

Let q be nonzero central projection such that $\tau(q) < \infty$. Let D be a derivation on $\widetilde{q\mathcal{M}q}$. By Lemma 5.3.8, D can be extended to a derivation $\overline{D}$ on $\widetilde{\mathcal{M}}$. By hypothesis, $\overline{D}$ is $\gamma_{cm} - \gamma_{cm}$ continuous, and thus, D is $\gamma_{cm} - \gamma_{cm}$ continuous (this follows from Lemmas 1.5.15 and 1.5.16). Therefore, all derivations on $\widetilde{q\mathcal{M}q}$ are $\gamma_{cm} - \gamma_{cm}$ continuous. Since $q\mathcal{M}q$ is a type I von Neumann algebra with finite trace, it follows from Proposition 5.3.12 that $(q\mathcal{M}q)_p$ is atomic. Therefore there exists an atom q_0 in $q\mathcal{M}q$ such that $q \geq q_0$. It is clear that q_0 is also an atom in $\mathcal{M}$.

Let $p \in Z(\mathcal{M})_p$. Since $\mathcal{M}$ is finite, it follows that that $\tau|_{Z(\mathcal{M})}$ is semifinite ([39], Proposition 10, p. 12). Therefore, there exists a nonzero $p_1 \in Z(\mathcal{M})_p$ such that $\tau(p_1) < \infty$ and $p \geq p_1$. By the previous paragraph, there is an atom p_2 in $\mathcal{M}$ such that $p_1 \geq p_2$. Hence every central projection majorizes an atom of $\mathcal{M}$. By Lemma 5.3.9, $c(p_2)$ is an atom in $Z(\mathcal{M})$. Since $p \in Z(\mathcal{M})_p$, it follows that $p \geq c(p_2)$. Thus $Z(\mathcal{M})_p$ is atomic. ∇

Theorem 5.3.14 *Let $\mathcal{M}$ be a finite type I von Neumann algebra with a faithful semifinite normal trace τ. If all derivations on $\widetilde{\mathcal{M}}$ are $\gamma_{cm} - \gamma_{cm}$ continuous, then $\mathcal{M}_p$ is atomic.*

Proof. Since $\mathcal{M}$ is a finite type I von Neumann algebra, we can write $\mathcal{M} \cong \oplus_{n=1}^{\infty}\mathcal{M}_n$, where every $\mathcal{M}_n$ is a finite type I von Neumann algebra such that $\mathcal{M}_n \cong Z(\mathcal{M}_n) \,\overline{\otimes}\, \mathcal{B}(\mathcal{H}_n)$, where $\mathcal{H}_n$ is a finite-dimensional Hilbert space for every n. There is a sequence of central projections (p_n) in $\mathcal{M}$ such that, for every n, $\mathcal{M}_n \cong p_n\mathcal{M}p_n$ and $\sum_{n=1}^{\infty} p_n = 1$ (Theorem 1.2.2). By Theorem 1.5.10, $\widetilde{\mathcal{M}_n} = p_n\widetilde{\mathcal{M}}p_n$ for every n.

Consider now a fixed n and let D_n be a derivation on $\widetilde{\mathcal{M}_n}$. It follows from Lemma 5.3.8 that D_n can be extended to a derivation $\overline{D_n}$ on $\widetilde{\mathcal{M}}$. By hypothesis, $\overline{D_n}$ is $\gamma_{cm} - \gamma_{cm}$ continuous, and so the same holds for D_n (this follows from Lemmas 1.5.15 and 1.5.16). Hence all derivations on $\widetilde{\mathcal{M}_n}$ are $\gamma_{cm} - \gamma_{cm}$ continuous. This holds for every n. Since $\mathcal{M}_n$ is a finite von Neumann algebra, $\tau|_{Z(\mathcal{M}_n)}$ is semifinite ([39], Proposition 10, p. 12), and hence, by Proposition 5.3.13, $Z(\mathcal{M}_n)_p$ is atomic for every n. It is an immediate consequence of Lemma 5.3.11, and the fact that $\mathcal{M}_n \cong Z(\mathcal{M}_n) \,\overline{\otimes}\, \mathcal{B}(\mathcal{H}_n)$ for every n, that every $(\mathcal{M}_n)_p$ is atomic. Finally, by Lemma 5.3.10, $\mathcal{M}_p$ is atomic. ∇

The following example demonstrates that the converse of Theorem 5.3.7 does in general not hold.

Example 5.3.15 Let $\mathcal{A} = L_\infty([0,1])$, $\mathcal{H}$ an infinite dimensional Hilbert space, and $\mathcal{M} = \mathcal{A} \,\overline{\otimes}\, \mathcal{B}(\mathcal{H})$. Then $\mathcal{M}$ is a type I_∞ von Neumann algebra.

Now

$$\begin{aligned} Z(\mathcal{M}) &= Z(\mathcal{A} \,\overline{\otimes}\, \mathcal{B}(\mathcal{H})) \\ &= Z(\mathcal{A}) \,\overline{\otimes}\, Z(\mathcal{B}(\mathcal{H})) \\ &= \mathcal{A} \,\overline{\otimes}\, \mathbb{C}1 \\ &\cong \mathcal{A} \\ &= L_\infty([0,1]). \end{aligned}$$

Therefore $Z(\mathcal{M})$ has no atoms. Therefore, by Lemma 5.3.9, $\mathcal{M}$ has no atoms, implying that $\mathcal{M}_p$ is not atomic. However, F. A. Sukochev, A. F. Ber and B.

de Pagter recently proved that all derivations on $\widetilde{\mathcal{M}}$ are $\gamma_{cm} - \gamma_{cm}$ continuous ([22]).

We have already seen that derivations on $\widetilde{\mathcal{M}}$ need not be continuous, nor inner. Therefore, the following problem presents itself.

Problem ([13], Problem 5) Is every $\gamma_{cm} - \gamma_{cm}$ continuous derivation on $\widetilde{\mathcal{M}}$ inner?

Recall that if $\mathcal{M}$ is commutative, then every $\gamma_{cm} - \gamma_{cm}$ continuous derivation on $\widetilde{\mathcal{M}}$ is inner, i.e. the zero derivation is the only $\gamma_{cm} - \gamma_{cm}$ continuous derivation on $\widetilde{\mathcal{M}}$. We now give other results which provide some affirmative answers to this question. One such result is Theorem 5.3.20 below.

Lemma 5.3.16 *Let $\mathcal{A}$ be a GB*-algebra such that the underlying C*-algebra $\mathcal{A}_b$ of $\mathcal{A}$ is a W*-algebra. If D is a continuous derivation on $\mathcal{A}$ such that $D(\mathcal{A}_b) \subset \mathcal{A}_b$, then there exists $a \in \mathcal{A}_b$ such that $D(x) = ax - xa$ for all $x \in \mathcal{A}$, i.e. D is inner.*

Proof. Let D be a derivation as in the hypothesis. Then, since $D(\mathcal{A}_b) \subset \mathcal{A}_b$, there exists $a \in \mathcal{A}_b$ such that $D(x) = ax - xa$ for every $x \in \mathcal{A}_b$ (Theorem 2.3.6). Since D is continuous and $\mathcal{A}_b$ is dense in $\mathcal{A}$ (Theorem 1.4.17), it follows that $D(x) = ax - xa$ for every $x \in \mathcal{A}$, i.e. D is inner. ∇

Corollary 5.3.17 *([13]) If D is a $\gamma_{cm} - \gamma_{cm}$ continuous derivation on $\widetilde{\mathcal{M}}$ such that $D(\mathcal{M}) \subset \mathcal{M}$, then there exists $a \in \mathcal{M}$ such that $D(x) = ax - xa$ for all $x \in \widetilde{\mathcal{M}}$, i.e. D is inner.*

Proof. By Theorem 1.5.29, $\widetilde{\mathcal{M}}$ is a GB*-algebra with $\mathcal{M}$ as underlying C*-algebra. The result follows from Lemma 5.3.16. ∇

Corollary 5.3.24 below demonstrates that not every $\gamma_{cm} - \gamma_{cm}$ continuous derivation on $\widetilde{\mathcal{M}}$ maps $\mathcal{M}$ into itself.

In 2006, the author presented Theorem 5.3.7 at the conference "Great Plains Operator Theory Symposium" in Iowa City, USA ([98]). He then communicated with Sh. A. Ayupov, who, with S. Albeverio and K. K. Kudaybergenov, gave an extension of Theorem 5.3.7 in [2].

Proposition 5.3.18 *Let* $D : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ *be a derivation. If* $D|_{\mathcal{M}}$ *is* $\gamma_{cm} - \gamma_{cm}$ *continuous, then* D *is* $\gamma_{cm} - \gamma_{cm}$ *continuous.*

Proof. By Lemma 2.3.2, we may assume without loss of generality that D is a *-derivation. It suffices to show that $\mathcal{S}(D) = \{0\}$. Let $y \in \mathcal{S}(D)$. Then there is a sequence (x_n) in $\widetilde{\mathcal{M}}$ such that $x_n \to 0$ (γ_{cm}) and $D(x_n) \to y$ (γ_{cm}). Since $x_n(1 + x_n^* x_n)^{-1}$ is affiliated with $\mathcal{M}$ for every n (Lemma 1.5.27), it follows from Proposition 1.3.1 that $x_n(1 + x_n^* x_n)^{-1} \in \mathcal{M}$ for every n. Since inversion is continuous on $\widetilde{\mathcal{M}}$ in the measure topology ([96]) and $1+x_n^* x_n \to 1$ (γ_{cm}), it is immediate that $x_n(1 + x_n^* x_n)^{-1} \to 0$ (γ_{cm}). Observe that

$$D\Big(x_n(1 + x_n^* x_n)^{-1}\Big) = x_n D\Big((1 + x_n^* x_n)^{-1}\Big) + D(x_n)(1 + x_n^* x_n)^{-1},$$

and

$$\begin{aligned} 0 = D(1) &= D\Big((1 + x_n^* x_n)(1 + x_n^* x_n)^{-1}\Big) \\ &= (1 + x_n^* x_n) D\Big((1 + x_n^* x_n)^{-1}\Big) + D(1 + x_n^* x_n)(1 + x_n^* x_n)^{-1}. \end{aligned}$$

Therefore

$$(1 + x_n^* x_n) D\Big((1 + x_n^* x_n)^{-1}\Big) = -D(1 + x_n^* x_n)(1 + x_n^* x_n)^{-1},$$

implying that

$$\begin{aligned} D\Big((1 + x_n^* x_n)^{-1}\Big) &= -(1 + x_n^* x_n)^{-1} D(1 + x_n^* x_n)(1 + x_n^* x_n)^{-1} \\ &= -(1 + x_n^* x_n)^{-1} D(x_n^* x_n)(1 + x_n^* x_n)^{-1} \\ &= -(1 + x_n^* x_n)^{-1}\Big(x_n^* D(x_n) + D(x_n)^* x_n\Big)(1 + x_n^* x_n)^{-1} \\ &\to -1^{-1}(0.y + y^*.0)1^{-1} \\ &= 0 \end{aligned}$$

by continuity of inversion in the measure topology. Hence $D\Big(x_n(1+x_n^* x_n)^{-1}\Big) \to y$ (γ_{cm}). Let $y_n = x_n(1 + x_n^* x_n)^{-1}$ for every n. Then (y_n) is a sequence in

$\mathcal{M}$ such that $y_n \to 0$ (γ_{cm}) and $D(y_n) \to y$ (γ_{cm}). By hypothesis, $y = 0$, implying that $\mathcal{S}(D) = \{0\}$. ∇

In what follows, we will need the following result.

Theorem 5.3.19 (Ryll-Nardzewski fixed point theorem) *([77], p. 444) Suppose that X is a locally convex Hausdorff space and $\emptyset \neq K \subset X$ a weakly compact convex subset of X. Let $\mathcal{F} : K \to K$ be a non-contracting semigroup of weakly continuous affine maps (here, non-contracting means that for every $x, y \in K$ with $x \neq y$, there exists a seminorm p such that $\inf_{\phi\in\mathcal{F}} p(\phi(x) - \phi(y)) > 0$). Then there exists $x \in K$ such that x is a fixed point of $\mathcal{F}$.*

The proof of our next result is similar to that of [58], Lemma 4.1, and [55], Theorem 1. Recall from Section 5.1 the notion of the non-commutative L_p-spaces $L_p(\mathcal{M}, \tau)$, $0 < p < \infty$, and that every $L_p(\mathcal{M}, \tau)$, $p \geq 1$, is a Banach $\mathcal{M}$-module with the norm $\|.\|_p$ as defined in that section.

Theorem 5.3.20 *Let D be a derivation on $\widetilde{\mathcal{M}}$ such that $D(\widetilde{\mathcal{M}}) \subset L_p(\mathcal{M}, \tau)$ for some $1 < p < \infty$. Then there exists an $a \in L_p(\mathcal{M}, \tau)$ such that $D(x) = ax - xa$ for every $x \in \widetilde{\mathcal{M}}$.*

Proof. We denote the unitary group of $\mathcal{M}$ by $\mathcal{M}_u$. Let $\mathcal{K} = \{u^*D(u) : u \in \mathcal{M}_u\}$ and $\mathcal{L}$ be the $\sigma(L_p, (L_p)^*)$-closed convex hull of $\mathcal{K}$ (here $L_p = L_p(\mathcal{M}, \tau)$). Let $D_b = D|_{\mathcal{M}}$. Then D_b is a derivation on $\mathcal{M}$ into L_p. By Theorem 2.3.5, D_b is norm continuous.

Therefore $\mathcal{L}$ is a bounded subset of (the Banach space) L_p: $\|u^*D(u)\|_p \leq \|u^*\|\|D(u)\|_p = \|D(u)\|_p$. Now $\mathcal{M}_u$ is a norm bounded subset of $\mathcal{M}$. Since D_b is (norm) continuous, $\{D(u) : u \in \mathcal{M}_u\}$ is a norm bounded subset of L_p. So $\sup_{u\in\mathcal{M}_u} \|D(u)\|_p < \infty$. Hence $\sup_{u\in\mathcal{M}_u} \|u^*D(u)\|_p < \infty$, meaning that $\mathcal{K}$, and thus $\mathcal{L}$, is a norm bounded subset of L_p.

Since L_p is reflexive, it follows from the Banach-Alaoglu theorem that $\mathcal{L}$ is $\sigma(L_p, (L_p)^*)$-compact. For each $u \in \mathcal{M}_u$, define the affine map $A_u(x) = u^*xu + u^*D(u)$ for all $x \in L_p$. Since L_p is a Banach $\mathcal{M}$-module, $A_u(L_p) \subset L_p$

for every $u \in \mathcal{M}_u$. Since $\|u^*xu\|_p = \|x\|_p$ for every $u \in \mathcal{M}_u$ and every $x \in L_p$ (Lemma 5.1.1), it follows easily that $A_u : L_p \to L_p$ is norm continuous for every $u \in \mathcal{M}_u$. A standard result tells us that A_u is $\sigma(L_p, (L_p)^*)$-continuous for every $u \in \mathcal{M}_u$. Let $u, v \in \mathcal{M}_u$. Then an easy computation shows that $A_v(u^*D(u)) = (uv)^*D(uv)$. Now $uv \in \mathcal{M}_u$. Thus $A_v(\mathcal{K}) \subset \mathcal{K}$. Therefore, since A_v is $\sigma(L_p, (L_p)^*)$-continuous, $A_v(\mathcal{L}) \subset \mathcal{L}$. Since $A_u A_v(x) = A_{vu}(x)$ for every $x \in L_p$, it follows that $\{A_u : u \in \mathcal{M}_u\}$ is a semigroup.

Now let $x, y \in \mathcal{L}$ with $x \neq y$. Then, by Lemma 5.1.1,

$$\begin{aligned} \|A_u(x) - A_u(y)\|_p &= \|u^*xu + u^*D(u) - u^*yu - u^*D(u)\|_p \\ &= \|u^*xu - u^*yu\|_p \\ &= \|u^*(x-y)u\|_p \\ &= \|x - y\|_p \end{aligned}$$

for all $u \in \mathcal{M}_u$ and for all $1 < p < \infty$. Hence

$$\inf_{u \in \mathcal{M}_u} \|A_u(x) - A_u(y)\|_p = \|x - y\|_p > 0$$

for every $1 < p < \infty$. Hence $\{A_u : u \in \mathcal{M}_u\}$ is non-contracting. By the Ryll-Nardzewski fixed point theorem, there exists $a_0 \in \mathcal{L}$ such that $A_u(a_0) = a_0$ for every $u \in \mathcal{M}_u$. Therefore $u^*a_0u + u^*D(u) = a_0$ for every $u \in \mathcal{M}_u$. Let $a = -a_0$. It follows that $D(u) = au - ua$ for every $u \in \mathcal{M}_u$.

Let $x \in \mathcal{M}$. Then it is well known that $x = \sum_{i=1}^{4} \lambda_i u_i$, where $\lambda_i \in \mathbb{C}$ and $u_i \in \mathcal{M}_u$ for all $1 \leq i \leq 4$. Once again, an easy calculation shows that $D(x) = ax - xa$. This holds for every $x \in \mathcal{M}$. Thus $D|_{\mathcal{M}}$ is $\gamma_{cm} - \gamma_{cm}$ continuous. By Proposition 5.3.18, D is $\gamma_{cm} - \gamma_{cm}$ continuous. Therefore, since $\mathcal{M}$ is dense in $\widetilde{\mathcal{M}}$ with respect to the measure topology, it follows that $D(x) = ax - xa$ for every $x \in \widetilde{\mathcal{M}}$, implying that D is inner. ∇

In the proof of [86], Theorem 4.1.6, a similar argument was used for the case where $\mathcal{M}$ is a countably decomposable finite von Neumann algebra: Sakai introduced the maps $T_u(x) = uxu^* + D(u)u^*$ for all $x \in \mathcal{M}$ and for all $u \in \mathcal{M}_u$. He showed, by using Zorn's Lemma, instead of the Ryll-Nardzewski fixed point theorem, that the maps T_u have a fixed point.

It would be interesting to know if all inner derivations on $\widetilde{\mathcal{M}}$ are necessarily defined by bounded operators. We answer this question in the negative (Proposition 5.3.23). First, we give the following lemmas, of which the first is straight-forward, and so we omit the proof.

Lemma 5.3.21 *If $\mathcal{A}$ is a unital algebra and $a \in \mathcal{A}$ is an invertible element in $Z(\mathcal{A})$, the centre of $\mathcal{A}$, then $a^{-1} \in Z(\mathcal{A})$.*

The next result was brought to the author's attention by Sh. A. Ayupov. We give a proof for completeness.

Lemma 5.3.22 *([14]) If $\mathcal{M}$ is a factor, then $Z(\widetilde{\mathcal{M}}) = Z(\mathcal{M}) = \mathbb{C}1$.*

Proof. By the fact that $\mathcal{M}$ is dense $\widetilde{\mathcal{M}}$ with respect to the measure topology, it follows easily that $Z(\mathcal{M}) \subset Z(\widetilde{\mathcal{M}})$. It remains to show that $Z(\widetilde{\mathcal{M}}) \subset Z(\mathcal{M})$. Let $x \in Z(\widetilde{\mathcal{M}})$. Then $x^* \in Z(\widetilde{\mathcal{M}})$ and hence $1 + x^*x \in Z(\widetilde{\mathcal{M}})$. It follows from Lemma 5.3.21 that $(1+x^*x)^{-1} \in Z(\widetilde{\mathcal{M}})$. But $(1+x^*x)^{-1} \in \mathcal{M}$ (Lemma 1.5.27). Hence $(1+x^*x)^{-1} \in Z(\widetilde{\mathcal{M}}) \cap \mathcal{M} \subset Z(\mathcal{M})$. Since $\mathcal{M}$ is a factor, there exists a nonzero $\lambda \in \mathbb{C}$ such that $(1+x^*x)^{-1} = \lambda 1$. Therefore $1+x^*x = \frac{1}{\lambda}1$. Hence $1+x^*x \in \mathcal{M}$. So $x^*x \in \mathcal{M}$. By the polar decomposition of x, it follows that $x \in \mathcal{M}$. Thus $x \in Z(\widetilde{\mathcal{M}}) \cap \mathcal{M} \subset Z(\mathcal{M})$. ∇

The following result was communicated to the author by Sh. A. Ayupov. We give a proof for completeness.

Proposition 5.3.23 *([14]) Suppose that $\mathcal{M}$ is a factor and D is an inner derivation on $\widetilde{\mathcal{M}}$, i.e. there exists $a \in \widetilde{\mathcal{M}}$ such that $D(x) = ax - xa$ for every $x \in \widetilde{\mathcal{M}}$. Suppose $b \in \widetilde{\mathcal{M}}$ is also such that $D(x) = bx - xb$ for every $x \in \widetilde{\mathcal{M}}$. Then $a \in \mathcal{M}$ if and only if $b \in \mathcal{M}$.*

Proof. By hypothesis, $ax - xa = bx - xb$ for all $x \in \widetilde{\mathcal{M}}$. Hence $ax - bx = xa - xb$ for all $x \in \widetilde{\mathcal{M}}$, i.e. $(a-b)x = x(a-b)$ for all $x \in \widetilde{\mathcal{M}}$. Hence $a - b \in Z(\widetilde{\mathcal{M}})$. Since $\mathcal{M}$ is a factor, it follows from Lemma 5.3.22 that $a - b = \lambda 1$ for some $\lambda \in \mathbb{C}$. Therefore $a \in \mathcal{M}$ if and only if $b \in \mathcal{M}$. ∇

Corollary 5.3.24 *If $\mathcal{M}$ is a factor with $\widetilde{\mathcal{M}} \neq \mathcal{M}$, there exists a $\gamma_{cm} - \gamma_{cm}$ continuous derivation D on $\widetilde{\mathcal{M}}$ such that $D(\mathcal{M})$ is not contained in $\mathcal{M}$.*

Proof. Let $a \in \widetilde{\mathcal{M}}$ with $a \notin \mathcal{M}$. Define the inner derivation $D(x) = ax - xa$ for every $x \in \widetilde{\mathcal{M}}$. Then D is a $\gamma_{cm} - \gamma_{cm}$ continuous derivation on $\widetilde{\mathcal{M}}$. By Proposition 5.3.23, there is no $b \in \mathcal{M}$ such that $D(x) = bx - xb$ for very $x \in \widetilde{\mathcal{M}}$.

If $D(\mathcal{M}) \subset \mathcal{M}$, then the restriction of D to $\mathcal{M}$ is a derivation on $\mathcal{M}$, and hence there exits $b \in \mathcal{M}$ such that $D(x) = bx - xb$ for every $x \in \mathcal{M}$ (Theorem 2.3.6). Since D is $\gamma_{cm} - \gamma_{cm}$ continuous, it follows that $D(x) = bx - xb$ for very $x \in \widetilde{\mathcal{M}}$. This is a contradiction. Therefore $D(\mathcal{M})$ is not contained in $\mathcal{M}$. ∇

5.4 Examples of non-inner derivations on $\widetilde{\mathcal{M}}$

In this section, we examine the construction of a non-inner derivation on a commutative $\widetilde{\mathcal{M}}$, where $\mathcal{M}_p$ is not atomic. This construction can be found in the proof of [20], Theorem 5.3.3.

Theorem 5.4.1 *([20], Remark 3.1) Let $\mathcal{A} = L_0(X, \Sigma, \mu)$, where $X = [a, b]$, and let $\mathcal{B} = P([a, b])$, the subalgebra of all polynomials in $\mathcal{A}$. Then every derivation $D : \mathcal{B} \to \mathcal{A}$ can be extended to a derivation on $\mathcal{A}$.*

Example 5.4.2 *([20], Remark 3.1)* Let $\mathcal{A} = L_0(X, \Sigma, \mu)$, where $X = [a, b]$, μ the Lebesque measure on X, and Σ the Lebesque σ-algebra of X. Since μ is a finite measure, we see that $\mathcal{A} = \widetilde{L}_\infty(X, \Sigma, \mu)$. Let $\mathcal{B}$ as in Theorem 5.4.1, and $D(p) = p'$ the standard (differentiation) derivation on $\mathcal{B}$. By Theorem 5.4.1, the derivation D defined on $\mathcal{B}$ extends to a derivation $D_0 : \mathcal{A} \to \mathcal{A}$. Since D is a nonzero derivation, it follows that D_0 is a nonzero derivation on $\mathcal{A}$. Hence D_0 is non-inner, since $\mathcal{A}$ is commutative.

Example 5.4.3 *([2], Example 4.6)* Let $\mathcal{A} = L_\infty([0, 1])$ and $\mathcal{M} = M_2(\mathcal{A})$. Recall that one can define a faithful finite normal trace $\tau_{\mathcal{M}}$ on $\mathcal{M}$ as the sum of the traces of the diagonal elements of (x_{ij}) for every $(x_{ij}) \in \mathcal{M}$. Denoting the algebra of $\tau_{\mathcal{M}}$-measurable operators by $\widetilde{\mathcal{M}}$, it follows from Proposition 1.3.7 that

$$\widetilde{\mathcal{M}} = \mathcal{U}(\mathcal{M}) = \mathcal{U}(M_2(\mathcal{A})) \cong M_2(\mathcal{U}(\mathcal{A})) = M_2(\widetilde{\mathcal{A}}).$$

Let D_A be the nonzero derivation on $\widetilde{\mathcal{A}} = L_0([0,1])$ as in Example 5.4.2. Then the map $D : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}} : (x_{ij}) \mapsto (D_{\mathcal{A}}(x_{ij}))$ $(i, j = 1, 2)$ defines a derivation on $\widetilde{\mathcal{M}}$. It can easily be verified that D is not inner.

Chapter 6

Derivations and primitive ideals of $\widetilde{\mathcal{M}}$

In 1955, Singer and Wermer proved that every continuous derivation D on a commutative Banach algebra $\mathcal{A}$ maps into the (Jacobson) radical $\mathrm{Rad}(\mathcal{A})$ of $\mathcal{A}$ ([36], Corollary 2.7.20). At the same time, they conjectured that the continuity assumption on D can be dropped. This conjecture remained open until 1988, when M. P. Thomas settled this conjecture in the affirmative, i.e. he proved that every derivation on a commutative Banach algebra $\mathcal{A}$ maps into the radical of $\mathcal{A}$ ([36], Theorem 5.2.36).

The *non-commutative Singer-Wermer conjecture* states that if D is a derivation on a (not necessarily commutative) Banach algebra $\mathcal{A}$ with $xD(x)-(Dx)x \in \mathrm{Rad}(\mathcal{A})$ for every $x \in \mathcal{A}$, then D maps into the radical of $\mathcal{A}$ ([36], p. 272). This problem is still open. An equivalent formulation of the non-commutative Singer-Wermer conjecture is the following: If D is a derivation on a Banach algebra $\mathcal{A}$, then $D(\mathcal{I}) \subset \mathcal{I}$ for every primitive ideal $\mathcal{I}$ of $\mathcal{A}$ ([36], p. 272).

This conjecture is already known to hold when D is continuous ([36], Proposition 2.7.22). In this chapter, we explore the Singer-Wermer theorem as well as the non-commutative Singer-Wermer conjecture for the algebra $\widetilde{\mathcal{M}}$. In particular, we show that primitive ideals of $\widetilde{\mathcal{M}}$ are invariant under derivations on $\widetilde{\mathcal{M}}$ in the cases where $\mathcal{M}$ is commutative (Section 6.3) and where the trace on $\mathcal{M}$ is finite (Section 6.5). Section 6.4 looks at the invariance problem for certain ideals of $\widetilde{\mathcal{M}}$. In Section 6.2, we show that primitive ideals of $\widetilde{\mathcal{M}}$ are in general not γ_{cm}-closed and for this, we appeal to some results in Section 6.1, some of which are of independent interest.

6.1 Measure bounded elements of $\widetilde{\mathcal{M}}$ and further properties of $\widetilde{\mathcal{M}}$

Recall from Section 1.4 that an element x in a topological algebra $\mathcal{A}$ is bounded if there there is a nonzero $\lambda \in \mathbb{C}$ such that the set $\{(\lambda x)^n : n = 1, 2, \ldots\}$ is a bounded subset of $\mathcal{A}$.

Proposition 6.1.1 *Let $x \in \widetilde{\mathcal{M}}$. The following statements are equivalent.*

(i) x is γ_{cm}-bounded.

(ii) For every $\delta > 0$, there exist $M > 0$ and $\alpha > 0$, depending on x, such that $\mu_\delta(x^n) \leq M\alpha^n$ for every natural number n.

Proof. *(i) ⇒ (ii)*: Suppose that x is γ_{cm}-bounded. Then there exists a nonzero $\lambda \in \mathbb{C}$ such that the set $\{(\lambda x)^n : n = 1, 2, \ldots\}$ is γ_{cm}-bounded. Let $\epsilon, \delta > 0$. Then there exits $\gamma > 0$ such that

$$\{(\lambda x)^n : n = 1, 2, \ldots\} \subset \gamma\widetilde{\mathcal{M}}(\epsilon, \delta), \text{ i.e.}$$
$$\{\frac{\lambda^n}{\gamma} x^n : n = 1, 2, \ldots\} \subset \widetilde{\mathcal{M}}(\epsilon, \delta).$$

Therefore, by Lemma 1.5.16, $\mu_\delta(\frac{\lambda^n}{\gamma} x^n) \leq \epsilon$ for all n. Thus $\frac{|\lambda|^n}{\gamma}\mu_\delta(x^n) \leq \epsilon$ for all n (Proposition 1.5.14). Hence $\mu_\delta(x^n) \leq \frac{\epsilon\gamma}{|\lambda|^n}$. Let $M = \epsilon\gamma$ and $\alpha = \frac{1}{|\lambda|}$. Then $\mu_\delta(x^n) \leq M\alpha^n$ for all n.

$(ii) \Rightarrow (i)$: Suppose that (ii) holds and take $\epsilon > 0$. Let $\gamma = \frac{M}{\epsilon}$ and let $\lambda \in \mathbb{C}$ be such that $|\lambda| = \frac{1}{\alpha}$. Then $M = \epsilon\gamma$ and $\alpha = \frac{1}{|\lambda|}$. Applying the proof of (i) ⇒ (ii) in reverse, it follows that x is γ_{cm}-bounded. ∇

Recall that $\widetilde{\mathcal{M}}$ is a GB*-algebra (Theorem 1.5.29). The next corollary is known to hold for all GB*-algebras ([40], p. 695). We give a more direct proof for $\widetilde{\mathcal{M}}$ using Proposition 6.1.1.

Corollary 6.1.2 *If $x \in \mathcal{M}$, then x is a γ_{cm}-bounded element of $\widetilde{\mathcal{M}}$.*

Proof. If $x \in \mathcal{M}$, then, by Proposition 1.5.14(i), $\mu_\delta(x^n) \leq \|x^n\| \leq \|x\|^n$. This holds for every n and for every $\delta > 0$. Let $M = 1$ and $\alpha = \|x\|$. Hence $\mu_\delta(x^n) \leq M\alpha^n$ for every n and for every $\delta > 0$. By Proposition 6.1.1, the result follows. ∇.

Lemma 6.1.3 *([40], Lemma 2.6) Let $\mathcal{A}$ be a GB*-algebra with $\mathcal{A}_b$ being the underlying C*-algebra. If $x \in \mathcal{A}$ is a self-adjoint bounded element, then $x \in \mathcal{A}_b$.*

The following result is known for locally convex GB*-algebras ([6], p. 94).

Proposition 6.1.4 *If $x \in \widetilde{\mathcal{M}}$ is γ_{cm}-bounded and normal, then $x \in \mathcal{M}$.*

Proof. If $y \in \widetilde{\mathcal{M}}$ is γ_{cm}-bounded and self-adjoint, then $y \in \mathcal{M}$ (by Theorem 1.5.29 and Lemma 6.1.3).

We first show that x^* is γ_{cm}-bounded. Let $\delta > 0$. Since x is γ_{cm}-bounded, it follows from Proposition 6.1.1 that there exist $M, \alpha > 0$ such that $\mu_\delta(x^n) \leq M\alpha^n$ for all n. Therefore, by Proposition 1.5.14(viii), it follows that $\mu_\delta((x^*)^n) = \mu_\delta((x^n)^*) = \mu_\delta(x^n) \leq M\alpha^n$ for all n. Hence, by Proposition 6.1.1, x^* is γ_{cm}-bounded.

Next, we illustrate that x^*x is γ_{cm}-bounded. Let $\delta > 0$ and $\delta_1 = \delta_2 = \frac{1}{2}\delta$. Then $\delta = \delta_1 + \delta_2$. Since x and x^* are γ_{cm}-bounded, there exist $M_1, M_2, \alpha_1, \alpha_2 > 0$ such that $\mu_{\delta_1}(x^n) \leq M_1\alpha_1^n$ for all n, and $\mu_{\delta_2}((x^*)^n) \leq M_2\alpha_2^n$ for all n. Hence, since x is normal, it follows from Proposition 1.5.14(vi) that

$$
\begin{aligned}
\mu_\delta((x^*x)^n) &= \mu_{\delta_1+\delta_2}((x^*x)^n) \\
&= \mu_{\delta_1+\delta_2}((x^*)^n x^n) \\
&\leq \mu_{\delta_1}((x^*)^n)\mu_{\delta_2}(x^n)
\end{aligned}
$$

$$
\begin{aligned}
&\leq (M_1\alpha_1^n)(M_2\alpha_2^n)\\
&= (M_1M_2)(\alpha_1\alpha_2)^n
\end{aligned}
$$

for all n. Thus x^*x is γ_{cm}-bounded by Proposition 6.1.1.

Now, since x^*x is self-adjoint, it follows that $x^*x \in \mathcal{M}$. By the polar decomposition of x, it follows that $x \in \mathcal{M}$. ∇

The following corollary is already known for locally convex GB*-algebras ([6], p. 94, and [40], p. 695).

Corollary 6.1.5 *If $\mathcal{M}$ is commutative, then $x \in \widetilde{\mathcal{M}}$ is γ_{cm}-bounded if and only if $x \in \mathcal{M}$.*

Proof. By hypothesis, every element of $\widetilde{\mathcal{M}}$ is normal. By Proposition 6.1.4 and Corollary 6.1.2, the result follows. ∇.

Lemma 6.1.6 *([6], Corollary 2.8) Let $\mathcal{A}$ be a barrelled pseudocomplete locally convex GB*-algebra such that every element of $\mathcal{A}$ is bounded. Then $\mathcal{A}$ is a C*-algebra.*

Theorem 6.1.7 *Suppose that $\widetilde{\mathcal{M}}$ is locally convex with respect to the measure topology. The following statements are equivalent.*

(i) $\widetilde{\mathcal{M}} = \mathcal{M}$.

(ii) $\widetilde{\mathcal{M}}$ *is a Q-algebra, i.e. the group of invertible elements of* $\widetilde{\mathcal{M}}$ *is open with respect to the measure topology.*

(iii) $\widetilde{\mathcal{M}}$ *is a C*-algebra in the measure topology.*

(iv) Every element of $\widetilde{\mathcal{M}}$ *is measure bounded.*

Proof. $(i) \Rightarrow (iv)$: By Corollary 6.1.2, every element of $\mathcal{M}$ is a γ_{cm}-bounded element of $\widetilde{\mathcal{M}}$. But $\widetilde{\mathcal{M}} = \mathcal{M}$. Hence every element of $\widetilde{\mathcal{M}}$ is γ_{cm}-bounded.

$(iv) \Rightarrow (iii)$: Recall that $\widetilde{\mathcal{M}}$ is a GB*-algebra (Theorem 1.5.29). By Lemma 1.4.19, $\widetilde{\mathcal{M}}$ is pseudocomplete. Since $\widetilde{\mathcal{M}}$ is complete and metrizable,

$\widetilde{\mathcal{M}}$ is barrelled. It follows from Lemma 6.1.6 that $\widetilde{\mathcal{M}}$ is a C*-algebra with respect to the measure topology.

$(iii) \Rightarrow (ii)$: This is trivial since every Banach algebra is a Q-algebra.

$(ii) \Rightarrow (i)$: Suppose that $\widetilde{\mathcal{M}}$ is a Q-algebra with respect to the measure topology. Assume that $\widetilde{\mathcal{M}} \neq \mathcal{M}$. Then there exists a self-adjoint element $x \in \widetilde{\mathcal{M}}$ with $x \notin \mathcal{M}$. Since $\widetilde{\mathcal{M}}$ is a Q-algebra, Sp $(x, \widetilde{\mathcal{M}})$ is a bounded subset of $\mathbb{C}$ (by [15], 4.8-3, p. 206, the spectrum of every element in a Q-algebra is bounded). By Proposition 6.1.4, every measure bounded self-adjoint element of $\widetilde{\mathcal{M}}$ is in $\mathcal{M}$. Therefore, since $x \notin \mathcal{M}$, x is not a measure bounded element of $\widetilde{\mathcal{M}}$. By Lemma 1.4.19, $\widetilde{\mathcal{M}}$ is pseudocomplete. It follows from Lemma 1.4.20 that $\widetilde{\mathcal{M}}$ does not have continuous inversion. By [104], Theorem 7.4, every complete metrizable Q-algebra has continuous inversion. Since $\widetilde{\mathcal{M}}$ is a complete metrizable Q-algebra, we have a contradiction. Hence $\widetilde{\mathcal{M}} = \mathcal{M}$. ∇

After the author proved Theorem 6.1.7, I. Tembo proved that, in general, $\widetilde{\mathcal{M}}$ is a Q-algebra if and only if $\widetilde{\mathcal{M}} = \mathcal{M}$ ([96]).

6.2 Non-closed primitive ideals of $\widetilde{\mathcal{M}}$

Definition 6.2.1 *If $\mathcal{A}$ is an algebra, then we say that a two-sided ideal $\mathcal{P}$ of $\mathcal{A}$ is a primitive ideal of $\mathcal{A}$ if there exists a maximal left ideal $\mathcal{I}$ of $\mathcal{A}$ such that $\mathcal{P} = \{a \in \mathcal{A} : a\mathcal{A} \subset \mathcal{I}\}$. An algebra is said to be primitive if $\{0\}$ is a primitive ideal of $\mathcal{A}$.*

Every algebra has at least one primitive ideal, since every algebra has at least one maximal left ideal. It is well known that every primitive ideal of a Banach algebra is closed ([36], Proposition 2.28(iii)). In this section, we show that this is not true in general for the algebra $\widetilde{\mathcal{M}}$, not even when $\mathcal{M}$ is commutative. The motivation for this becomes clear later on. The following theorem is due to W. Żelazko.

Theorem 6.2.2 *([105], Corollary 3, [36], p. 594) Let $\mathcal{A}$ be a commutative complete metrizable locally m-convex topological algebra. The following* *statements are equivalent.*

(i) All maximal two-sided ideals of $\mathcal{A}$ are closed.

(ii) $\mathcal{A}$ is a Q-algebra.

Recall that a two-sided ideal of a unital commutative algebra is maximal if and only if it is primitive ([36], Proposition 1.4.36). We now come to the main result of this section which is an immediate consequence of Theorems 1.5.26, 6.1.7 and 6.2.2.

Corollary 6.2.3 *Suppose that $\widetilde{\mathcal{M}}$ is commutative and locally convex with respect to the measure topology. If $\widetilde{\mathcal{M}} \neq \mathcal{M}$, then $\widetilde{\mathcal{M}}$ has at least one primitive ideal which is not γ_{cm}-closed.*

We now give an example of a maximal two-sided (hence primitive) ideal in $\widetilde{\mathcal{M}}$ which is not closed with respect to the measure topology. For this, we first give the following lemma.

Lemma 6.2.4 *Let (X, Σ, μ) be a localizable measure space, $f \in \widetilde{L}_\infty(X, \Sigma, \mu) = \widetilde{L}_\infty$ and $\lambda \in \mathbb{C}$. If $\mu(\{x \in X : f(x) = \lambda\}) > 0$, then $\lambda \in Sp\ (f, \widetilde{L}_\infty)$.*

Proof. Suppose that $\lambda \notin \mathrm{Sp}\ (f, \widetilde{L}_\infty)$. Then $f - \lambda 1$ is invertible in $\widetilde{L}_\infty$, i.e. there exists $g \in \widetilde{L}_\infty$ such that $(f - \lambda 1)g = 1$ almost everywhere. Therefore $\mu(\{x \in X : f(x) = \lambda\}) = 0$. ∇

The following example is modelled along the lines of an argument in the proof of Theorem, p. 293 in [105].

Example 6.2.5 Suppose that $\widetilde{\mathcal{M}} \neq \mathcal{M}$ and that $\widetilde{\mathcal{M}}$ is commutative and locally convex in the measure topology. We know from Example 1.5.8 that $\widetilde{\mathcal{M}}$ can be identified with $\widetilde{L}_\infty(X, \Sigma, \mu)$ for some localizable measure space (X, Σ, μ). By Proposition 1.5.11 and Theorems 1.5.23 and 1.5.24, there is a sequence of atoms (A_n) such that

$$\widetilde{L}_\infty(X, \Sigma, \mu) = L_0(E, \Sigma_E, \mu_E) \oplus L_\infty(X \setminus E, \Sigma_{X \setminus E}, \mu_{X \setminus E}),$$

where $E = \cup_{n=1}^{\infty} A_n$ and $\mu(E) < \infty$. Let $f \in \widetilde{L}_\infty(X, \Sigma, \mu)$ be defined by $f(A_n) = n$ for every $n > 0$, and $f(x) = 0$ for every $x \in X \setminus E$.

We show that $n \in \mathrm{Sp}\ (f, \widetilde{L}_\infty)$ for every $n > 0$, where $\widetilde{L}_\infty = \widetilde{L}_\infty(X, \Sigma, \mu)$. For every n, $\{x \in X : f(x) = n\} = A_n$, and $\mu(A_n) > 0$. By Lemma 6.2.4, $n \in \mathrm{Sp}\ (f, \widetilde{L}_\infty)$ for every $n > 0$.

For every n, let

$$\phi_n(\lambda) = \prod_{k=n}^{\infty} (1 - \frac{\lambda}{k}) \exp\Big(\frac{\lambda}{k} + \frac{1}{2}(\frac{\lambda}{k})^2 + \cdots + \frac{1}{m_k}(\frac{\lambda}{k})^{m_k}\Big),$$

where (m_k) is a sequence of positive integers. It follows from a theorem of Weierstrass ([1], Theorem 7, p. 194) that every ϕ_n is an entire function. By Theorem 1.5.26, $\widetilde{\mathcal{M}}$ is locally m-convex. Using the holomorphic functional calculus for complete metrizable locally m-convex topological algebras ([104], Theorem 11.8), define a sequence (x_n) in $\widetilde{\mathcal{M}}$ by $x_n = \phi_n(f)$ for every n.

Let $\mathcal{I}$ be the smallest ideal of $\mathcal{A}$ containing the sequence (x_n). As in the proof of Theorem, p. 293 in [105], it can be shown that $\mathcal{I}$ is properly contained in $\mathcal{A}$, and therefore there is at least one maximal ideal of $\mathcal{A}$ containing $\mathcal{I}$. One can also prove, as in the proof of Theorem, p. 293 in [105], that any such maximal ideal is not closed, and thus dense in $\mathcal{A}$.

In 2006, W. Żelazko extended Theorem 6.2.2 by proving that a unital complete metrizable topological algebra $\mathcal{A}$ is a Q-algebra if and only if all maximal one-sided ideals of $\mathcal{A}$ are closed ([106]).

We now give a property of primitive ideals of von Neumann algebras, which is of independent interest and does not seem to have appeared in the literature.

Proposition 6.2.6 *If $\mathcal{I}$ is a primitive ideal of a von Neumann algebra $\mathcal{M}$, then $\mathcal{I}$ is strong-operator closed or strong-operator dense in $\mathcal{M}$.*

Proof. We denote by $\overline{\mathcal{I}}^{sot}$ the strong-operator closure of $\mathcal{I}$ in $\mathcal{M}$. Since $\mathcal{I}$ is a primitive ideal, $\mathcal{I}$ is the kernel of an irreducible *-representation π of $\mathcal{M}$. Furthermore, by Proposition 1.2.10, $\mathcal{I} \subset \overline{\mathcal{I}}^{sot} = p\mathcal{M}$ for some central projection p of $\overline{\mathcal{I}}^{sot}$. Since p is central, $\pi(p)\pi(x) = \pi(px) = \pi(xp) = \pi(x)\pi(p)$

for every $x \in \mathcal{M}$. Thus $\pi(p)$ commutes with $\pi(\mathcal{M})$. Hence $\pi(p) = \lambda 1$ for some $\lambda \in \mathbb{C}$ (Proposition 1.2.16). Since p is a projection, so is $\pi(p)$. Thus $\lambda = 0$ or $\lambda = 1$, implying that $\pi(p) = 0$ or $\pi(p) = 1$.

If $\pi(p) = 0$, then $p \in \mathcal{I}$, implying that $\mathcal{I}$ is strong-operator closed. If $\pi(p) = 1$ and $x \in \mathcal{M}$ is arbitrary, then $x = xp + x(1-p)$, implying that $\pi(x(1-p)) = 0$ for every $x \in \mathcal{M}$, i.e. $x(1-p) \in \mathcal{I}$ for every $x \in \mathcal{M}$. In particular, for $x = 1$, $1 - p \in \mathcal{I} \subset \overline{\mathcal{I}}^{sot}$. Therefore $1 = p + (1-p) \in \overline{\mathcal{I}}^{sot}$. Hence $\mathcal{I}$ is strong-operator dense in $\mathcal{M}$. ∇.

6.3 The invariance of primitive ideals under derivations on a commutative $\widetilde{\mathcal{M}}$

In this section, we show that if $\widetilde{\mathcal{M}}$ is commutative, then every primitive ideal of $\widetilde{\mathcal{M}}$ is invariant under derivations on $\widetilde{\mathcal{M}}$ (Corollary 6.3.6). First we review some well-known terminology.

Let $\mathcal{R}$ be a unital ring. If $\mathcal{I}$ is a two-sided ideal of $\mathcal{R}$, then we call $\mathcal{I}$ a *semiprime* (respectively *prime*) ideal if whenever $a \in \mathcal{R}$ and $a\mathcal{R}a \subset \mathcal{I}$ (respectively $a\mathcal{R}b \in \mathcal{I}$ with $b \in \mathcal{R}$), then $a \in \mathcal{I}$ (respectively $a \in \mathcal{I}$ or $b \in \mathcal{I}$). Observe that every prime ideal of $\mathcal{R}$ is semiprime and recall that every primitive ideal of $\mathcal{R}$ is prime ([36], Proposition 1.4.34(iii)). Thus every primitive ideal of $\mathcal{R}$ is semiprime.

Lemma 6.3.1 *Let $\mathcal{R}$ be a unital ring with centre $Z(\mathcal{R})$, and $\mathcal{I}$ a semiprime ideal of $\mathcal{R}$. If $a^2 \in \mathcal{I}$ with $a \in Z(\mathcal{R})$, then $a \in \mathcal{I}$.*

Proof. By hypothesis, it follows that $a\mathcal{R}a = a^2\mathcal{R} \subset \mathcal{I}$. Since $\mathcal{R}$ is semiprime, $a \in \mathcal{I}$. ∇

The following lemma is a special case of [103], p. 63, and we give a proof for completeness.

Lemma 6.3.2 *If $x \in \widetilde{\mathcal{M}}$, then $|x| \in \{x^*x\}''$, the bicommutant of x^*x in $\widetilde{\mathcal{M}}$.*

Proof. Let $y \in \{x^*x\}'$ be self-adjoint. Then $(x^*x)y = y(x^*x)$. Therefore x^*x and y generate a maximal commutative *-subalgebra $\mathcal{A}$ of $\widetilde{\mathcal{M}}$ ([27], Proposition V.35.7). By Theorem 1.5.29 and Lemma 1.4.14, $\mathcal{A}$ is a GB*-algebra. Therefore, by Proposition 1.4.21, $|x| = (x^*x)^{\frac{1}{2}} \in \mathcal{A}$. Since $\mathcal{A}$ is commutative and $|x|, y \in \mathcal{A}$, we have $|x|y = y|x|$. This holds for every self-adjoint $y \in \{x^*x\}'$.

Let $z \in \{x^*x\}'$ (not necessarily self-adjoint). Then $z = a + ib$, where $a = \frac{1}{2}(z + z^*)$ and $b = \frac{1}{2i}(z - z^*)$. Recall that a and b are self-adjoint. It is easily verified that $z^* \in \{x^*x\}'$, and hence $a, b \in \{x^*x\}'$. By the previous paragraph, $|x|a = a|x|$ and $|x|b = b|x|$. It follows that $|x|z = z|x|$. This holds for every $z \in \{x^*x\}'$. Hence $|x| \in \{x^*x\}''$. ∇

Theorem 6.3.3 *Let $\mathcal{I}$ be a semiprime ideal of $\widetilde{\mathcal{M}}$. If $x \in \mathcal{I} \cap Z(\widetilde{\mathcal{M}})$, then $x = ab$ with $a \in \mathcal{I}$ and $b \in \mathcal{I}$.*

Proof. Let $x \in \mathcal{I} \cap Z(\widetilde{\mathcal{M}})$ and $x = v|x|$ the polar decomposition of x. Then $v \in \mathcal{M}$ and, since $\mathcal{I}$ is an ideal, $|x|^2 = x^*x \in \mathcal{I}$. Also, since $x \in Z(\widetilde{\mathcal{M}})$, $x^* \in Z(\widetilde{\mathcal{M}})$. Hence $x^*x \in Z(\widetilde{\mathcal{M}})$, implying that $\{x^*x\}' = \widetilde{\mathcal{M}}$. Therefore, by Lemma 6.3.2, $|x| \in \{x^*x\}'' = (\widetilde{\mathcal{M}})'$. Hence $|x| \in (\widetilde{\mathcal{M}})' \cap \widetilde{\mathcal{M}} = Z(\widetilde{\mathcal{M}})$.

By Lemma 6.3.1, $|x| \in \mathcal{I}$. Observe that $|x| \geq 0$. Once again, $|x|^{\frac{1}{2}} \in \{|x|\}''$ (Lemma 6.3.2). Therefore, since $|x| \in Z(\widetilde{\mathcal{M}})$, it follows that $|x|^{\frac{1}{2}} \in Z(\widetilde{\mathcal{M}})$. Now $(|x|^{\frac{1}{2}})^2 = |x| \in \mathcal{I}$. By Lemma 6.3.1, $|x|^{\frac{1}{2}} \in \mathcal{I}$. Since $v \in \mathcal{M} \subset \widetilde{\mathcal{M}}$, it follows that $v|x|^{\frac{1}{2}} \in \mathcal{I}$. Let $a = v|x|^{\frac{1}{2}} \in \mathcal{I}$ and $b = |x|^{\frac{1}{2}}$. Then $a, b \in \mathcal{I}$ and $x = v|x| = ab$. ∇

Our next result is the main result of this section.

Theorem 6.3.4 *Let $\mathcal{I}$ be a semiprime ideal of $\widetilde{\mathcal{M}}$ and D a derivation on $\widetilde{\mathcal{M}}$. Then $D(\mathcal{I} \cap Z(\widetilde{\mathcal{M}})) \subset \mathcal{I}$.*

Proof. Let $x \in \mathcal{I} \cap Z(\widetilde{\mathcal{M}})$. By Theorem 6.3.3, $x = ab$ with $a \in \mathcal{I}$ and $b \in \mathcal{I}$. Therefore $D(x) = D(ab) = aD(b) + D(a)b \in \mathcal{I}$, implying that $D(\mathcal{I} \cap Z(\widetilde{\mathcal{M}})) \subset \mathcal{I}$. ∇

Proposition 6.3.5 *Let $\mathcal{A}$ an algebra, $\mathcal{I}$ a two-sided ideal of $\mathcal{A}$, and D a derivation on $\mathcal{A}$. If for every $x \in \mathcal{I}$, there exist $a, b \in \mathcal{I}$ such that $x = ab$, then $D(\mathcal{I}) \subset \mathcal{I}$.*

Proof. Let $x \in \mathcal{I}$. By hypothesis, there exist $a, b \in \mathcal{I}$ such that $x = ab$. Hence $D(x) = D(ab) = aD(b) + D(a)b$. Therefore, since $\mathcal{I}$ is a two-sided ideal of $\mathcal{A}$, $D(\mathcal{I}) \subset \mathcal{I}$. ∇

The next corollary follows immediately from Theorem 6.3.4 and Proposition 6.3.5.

Corollary 6.3.6 *Suppose that $\mathcal{M}$ is commutative and let D be a derivation on $\widetilde{\mathcal{M}}$. Then $D(\mathcal{I}) \subset \mathcal{I}$ for every semiprime, and hence primitive, ideal $\mathcal{I}$ of $\widetilde{\mathcal{M}}$.*

6.4 The invariance of ideals under derivations on a general $\widetilde{\mathcal{M}}$

Recall that the zero derivation is the only $\gamma_{cm} - \gamma_{cm}$ continuous derivation on a commutative $\widetilde{\mathcal{M}}$. Therefore, if $\mathcal{M}$ is commutative, then every $\gamma_{cm} - \gamma_{cm}$ continuous derivation on $\widetilde{\mathcal{M}}$ maps into $\mathrm{Rad}(\widetilde{\mathcal{M}})$. Observe that this is an analogue of the Singer-Wermer theorem for commutative Banach algebras ([36], Corollary 2.7.20). The following result follows immediately from Theorem 1.5.30 and Corollary 5.3.4.

Corollary 6.4.1 *If $\mathcal{M}$ is commutative with finite trace, and $\mathcal{M}_p$ is not atomic, then there is a derivation on $\widetilde{\mathcal{M}}$ which does not map into the radical of $\widetilde{\mathcal{M}}$.*

Recall that, in contrast, every derivation on a commutative Banach algebra $\mathcal{A}$ maps into the radical of $\mathcal{A}$ ([36], Theorem 5.2.36).

Theorem 6.4.2 *If D is a derivation on $\widetilde{\mathcal{M}}$, then $D(\mathcal{I}) \subset \mathcal{I}$ for every γ_{cm}-closed two-sided ideal $\mathcal{I}$ of $\widetilde{\mathcal{M}}$.*

Proof. Let $\mathcal{I}$ be a γ_{cm}-closed two-sided ideal of $\widetilde{\mathcal{M}}$. By Theorem 1.5.21, $\mathcal{I} \cap \mathcal{M}$ is a norm closed two-sided ideal of $\mathcal{M}$. Let $x \in \mathcal{I}$. Then, by Theorem 1.5.21, there is a sequence (x_n) in $\mathcal{I} \cap \mathcal{M}$ such that $x_n \to x$ (γ_{cm}). Since $\mathcal{I} \cap \mathcal{M}$ is norm closed, it follows from [95], p. 19, that $|x_n| \in \mathcal{I} \cap \mathcal{M}$ for every n. By Theorem 1.5.20, $|x_n| \to |x|$ (γ_{cm}). Hence $|x| \in \mathcal{I}$ since $\mathcal{I}$ is γ_{cm}-closed. By a similar argument to that given above, $|x|^{\frac{1}{2}} \in \mathcal{I}$. Therefore $v|x|^{\frac{1}{2}} \in \mathcal{I}$, where $x = v|x|$ is the polar decomposition of $x \in \widetilde{\mathcal{M}}$. So $x = v|x|^{\frac{1}{2}}|x|^{\frac{1}{2}} = ab$, where $a = v|x|^{\frac{1}{2}}$ and $b = |x|^{\frac{1}{2}}$. The result now follows from Proposition 6.3.5. ∇

The above result shows that γ_{cm}-closed primitive ideals of $\widetilde{\mathcal{M}}$ are invariant under derivations on $\widetilde{\mathcal{M}}$. By Corollary 6.2.3, primitive ideals of $\widetilde{\mathcal{M}}$ need not be γ_{cm}-closed. Sometimes $\{0\}$ is a primitive ideal of $\widetilde{\mathcal{M}}$. We now characterize the case where $\widetilde{\mathcal{M}}$ has $\{0\}$ as the only primitive ideal. We first prove the following lemma.

Lemma 6.4.3 *An algebra $\mathcal{A}$ is simple if and only if $\{0\}$ is the only primitive ideal of $\mathcal{A}$.*

Proof. Suppose that $\mathcal{A}$ is simple. Then $\{0\}$ is the only proper two-sided ideal of $\mathcal{A}$ and, since primitive ideals are always proper, it is the only primitive ideal of $\mathcal{A}$.

Conversely, suppose that $\{0\}$ is the only primitive ideal of $\mathcal{A}$. Assume that $\mathcal{A}$ is not simple. Then $\mathcal{A}$ has a nonzero proper two-sided ideal $\mathcal{I}$. Therefore there exists a maximal two-sided ideal $\mathcal{I}_0$ of $\mathcal{A}$ with $\mathcal{I} \subset \mathcal{I}_0$. Since every maximal two-sided ideal of an algebra is primitive ([36], Proposition 1.4.34(iv)), we have a contradiction, implying that $\mathcal{A}$ is simple. ∇

Theorem 6.4.4 *The following statements are equivalent.*

(i) $\{0\}$ *is the only primitive ideal of* $\widetilde{\mathcal{M}}$.

(ii) $\widetilde{\mathcal{M}}$ *is simple.*

(iii) $\mathcal{M}$ *is simple.*

(iv) $\mathcal{M}$ *is a finite factor.*

Proof. The equivalence of statements (i) and (ii) follows from Lemma 6.4.3.

$(ii) \Rightarrow (iv)$: Assume that $\widetilde{\mathcal{M}}$ is simple. By Lemma 6.4.3, $\widetilde{\mathcal{M}}$ is primitive and thus prime. By Theorem 3.5.6, $\mathcal{M}$ is prime, and hence a factor ([9], p. 47). Also, since the trace τ on $\mathcal{M}$ is semifinite, $\widetilde{\mathcal{M}_0} \neq \{0\}$. Therefore $\widetilde{\mathcal{M}_0} = \widetilde{\mathcal{M}}$ because $\widetilde{\mathcal{M}}$ is simple and $\widetilde{\mathcal{M}_0}$ is a two-sided ideal of $\widetilde{\mathcal{M}}$. By Lemma 1.5.18, this implies that $1 \in \mathcal{M}$ has finite trace. So the identity element of $\mathcal{M}$ is finite, implying that $\mathcal{M}$ is a finite factor.

$(iv) \Rightarrow (iii)$: This is a restatement of Corollary 1.2.13.

$(iii) \Rightarrow (ii)$: Let $\mathcal{I}$ be a two-sided ideal of $\widetilde{\mathcal{M}}$. Then $\mathcal{I} \cap \mathcal{M}$ is an ideal of $\mathcal{M}$. Since $\mathcal{M}$ is simple, $\mathcal{I} \cap \mathcal{M}$ is $\{0\}$ or $\mathcal{M}$.

Suppose that $\mathcal{I} \cap \mathcal{M} = \{0\}$. Let $x \in \mathcal{I}$. Then, since $x \in \widetilde{\mathcal{M}}$, it follows from Lemma 1.5.27 that $(1+x^*x)^{-1} \in \widetilde{\mathcal{M}}$. By Proposition 1.3.1, $x(1+x^*x)^{-1}$ is a bounded operator. Therefore $x(1+x^*x)^{-1} \in \mathcal{M}$, since $x(1+x^*x)^{-1} \in \widetilde{\mathcal{M}}$, and thus affiliated with $\mathcal{M}$. Hence, since $x \in \mathcal{I}$, we have that $x(1+x^*x)^{-1} \in \mathcal{I} \cap \mathcal{M} = \{0\}$. This implies that $x = 0$. Thus $\mathcal{I} = \{0\}$.

Now suppose that $\mathcal{I} \cap \mathcal{M} = \mathcal{M}$. Then $1 \in \mathcal{I}$, and so $\mathcal{I} = \widetilde{\mathcal{M}}$. Therefore $\mathcal{I}$ is $\{0\}$ or $\widetilde{\mathcal{M}}$. This implies that $\widetilde{\mathcal{M}}$ is simple. ∇

Let $\mathcal{A}$ be an algebra. By an irreducible *-representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, we mean a self-adjoint algebra homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that the only subspaces of $\mathcal{H}$ invariant under $\pi(\mathcal{A})$ are $\{0\}$ and $\mathcal{H}$.

It is well known that if $\mathcal{A}$ is a C*-algebra, then a two-sided ideal $\mathcal{I}$ of $\mathcal{A}$ is primitive if and only if $\mathcal{I}$ is the kernel of an irreducible *-representation of $\mathcal{A}$ on a Hilbert space ([27], Proposition 24.12, and [38], p. 57). This is often taken to be the definition of a primitive ideal of a C*-algebra.

If $\mathcal{A}$ is a metrizable locally C*-algebra, then it is known that $\mathcal{A}$ admits irreducible *-representations on a Hilbert space ([30]). Also, every *-representation of $\mathcal{A}$ is continuous ([30], Lemma 3.1).

Suppose that $\widetilde{\mathcal{M}}$ is commutative and locally convex with respect to the measure topology (thus locally C* with respect to the measure topology by

Theorem 1.5.26), and that $\widetilde{\mathcal{M}} \neq \mathcal{M}$. By Corollary 6.2.3, $\widetilde{\mathcal{M}}$ has at least one primitive ideal which is not γ_{cm}-closed. It follows from the previous paragraph that $\widetilde{\mathcal{M}}$ admits irreducible *-representations which are all continuous. It follows that the kernels of the irreducible *-representations of $\widetilde{\mathcal{M}}$ are all γ_{cm}-closed. Hence primitive ideals of $\widetilde{\mathcal{M}}$ need not be kernels of irreducible *-representations on a Hilbert space.

Example 6.4.5 We give an example of an irreducible *-representation of $\widetilde{\mathcal{M}}$ on a Hilbert space, when the trace on $\mathcal{M}$ is not finite. By Theorem 1.5.19, $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$ is a non-trivial C*-algebra when equipped with the norm $\dot{\mu}_\infty$. Let $\phi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$ be the canonical homomorphism of $\widetilde{\mathcal{M}}$ onto $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$. Since $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$ is a non-trivial C*-algebra, $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$ admits an irreducible *-representation π on a Hilbert space $\mathcal{H}$ (Theorem 1.2.17). Define a *-representation $\pi_0 : \widetilde{\mathcal{M}} \to \mathcal{B}(\mathcal{H})$ by $\pi_0 = \pi \circ \phi$, i.e. $\pi_0(x) = \pi(x + \widetilde{\mathcal{M}}_0)$ for every $x \in \widetilde{\mathcal{M}}$.

We show that π_0 is irreducible. Let $\mathcal{K}$ be subspace of $\mathcal{H}$ such that $\pi_0(x)\mathcal{K} \subset \mathcal{K}$ for every $x \in \widetilde{\mathcal{M}}$. Then $\pi(x + \widetilde{\mathcal{M}}_0)\mathcal{K} \subset \mathcal{K}$ for every $x \in \widetilde{\mathcal{M}}$. Hence $\pi(a)\mathcal{K} \subset \mathcal{K}$ for every $a \in \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$. Since π is irreducible, it follows that $\mathcal{K} = \{0\}$ or $\mathcal{H}$. Hence π_0 is an irreducible *-representation of $\widetilde{\mathcal{M}}$.

Proposition 6.4.6 *Suppose that $\widetilde{\mathcal{M}}$ admits an irreducible *-representation π on a Hilbert space. If D is a derivation on $\widetilde{\mathcal{M}}$, then $D(Ker(\pi)) \subset Ker(\pi)$.*

Proof. By Example 1.5.9, Proposition 1.5.11 and Theorem 3.2.8, π is $\gamma_{cm} - \gamma_{cm}$ continuous when $\mathcal{B}(\mathcal{H})$ admits the diagonal trace in Example 1.2.3(ii), and hence Ker(π) is γ_{cm}-closed. The result follows from Theorem 6.4.2. ∇

We now give an example of a primitive ideal of $\widetilde{\mathcal{M}}$.

Example 6.4.7 Let $\mathcal{M}$ be a countably decomposable factor of type I$_\infty$ or II$_\infty$. By Theorem 1.2.14, $\mathcal{M}_0$ is a maximal ideal of $\mathcal{M}$, and hence $\mathcal{M}/\mathcal{M}_0$ is simple. By Theorem 1.5.19, it follows that $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}_0$ is simple. Therefore $\widetilde{\mathcal{M}}_0$

is a maximal ideal, and hence a primitive ideal of $\widetilde{\mathcal{M}}$. If $\mathcal{M}$ is of type II_∞, then $\widetilde{\mathcal{M}} \neq \mathcal{M}$ ([76], Remark 1(v)).

The next example shows that $\widetilde{\mathcal{M}_0}$ is not always a primitive ideal of $\widetilde{\mathcal{M}}$.

Example 6.4.8 Let $\mathcal{M} = L_\infty(\mathbb{R})$ have the trace $\tau(f) = \int_{\mathbb{R}} f \, \mathrm{d}\mu$ and let $A = (0, \infty) \subset \mathbb{R}$. Let $q = \chi_A$. Let $\mathcal{I}_p$ be the p-ideal generated by $\mathcal{P} = \{p \in \mathcal{M}_p : \tau(p) < \infty\} \cup \{q\}$. By Theorem 1.2.9, $\mathcal{I}_p$ generates a norm-closed two-sided ideal $\mathcal{I}$ of $\widetilde{\mathcal{M}}$. Clearly, $\mathcal{M}_0$ is properly contained in $\mathcal{I}$. Let $F \in \Sigma$ such that $\mu(F) < \infty$ and let $p_0 = \chi_F$. Then $1 \neq p_0 \wedge q$ and $1 \neq p_0 \vee q$. Therefore $1 \notin \mathcal{I}$, implying that $\mathcal{M}_0$ is not a maximal ideal of $\mathcal{M}$. Hence $\mathcal{M}/\mathcal{M}_0$ is not simple, and hence, by Theorem 1.5.19, $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}_0}$ is not simple, implying that $\widetilde{\mathcal{M}_0}$ is not a maximal, and hence not a primitive ideal of $\widetilde{\mathcal{M}}$.

6.5 Derivations on a regular $\widetilde{\mathcal{M}}$

A ring $\mathcal{R}$ is called *regular* if for any $a \in \mathcal{R}$, there exists $x \in \mathcal{R}$ such that $a = axa$.

Lemma 6.5.1 *([53], Corollary 1.2(a)) Every left or right ideal $\mathcal{I}$ of a regular ring has the property that $\mathcal{I}^2 = \mathcal{I}$.*

The next corollary is an immediate consequence of Lemma 6.5.1 and Proposition 6.3.5.

Corollary 6.5.2 *If $\widetilde{\mathcal{M}}$ is regular and D is a derivation on $\widetilde{\mathcal{M}}$, then $D(\mathcal{I}) \subset \mathcal{I}$ for every two-sided ideal $\mathcal{I}$ of $\widetilde{\mathcal{M}}$.*

In light of Corollary 6.5.2, it would be interesting to know when $\widetilde{\mathcal{M}}$ is regular. Recall that $\mathcal{U}(\mathcal{M})$ stands for the set of closed, densely defined operators affiliated with $\mathcal{M}$, and if $\mathcal{M}$ is finite, then $\mathcal{U}(\mathcal{M})$ is a unital *-algebra. The following result is well known.

Theorem 6.5.3 *([48], Theorem 2) If $\mathcal{M}$ is a finite von Neumann algebra, then $\mathcal{U}(\mathcal{M})$ is regular.*

Corollary 6.5.4 *Let $\mathcal{M}$ be a von Neumann algebra with a faithful finite normal trace τ. Then the following statements hold.*

(i) $\widetilde{\mathcal{M}}$ *is regular.*

(ii) If D is a derivation on $\widetilde{\mathcal{M}}$, then $D(\mathcal{I}) \subset \mathcal{I}$ for every two-sided ideal $\mathcal{I}$ of $\widetilde{\mathcal{M}}$.

Proof. *(i)* If $\tau(1) < \infty$, then, by Proposition 1.5.5, $\widetilde{\mathcal{M}} = \mathcal{U}(\mathcal{M})$. By Theorem 6.5.3, $\widetilde{\mathcal{M}}$ is regular.

(ii) This follows from (i) and Corollary 6.5.2. ∇

Let $\mathcal{M}$ be a von Neumann algebra. In [67], an equivalence relation $\leftrightarrow$ on the lattice of projections of $\mathcal{M}$ is defined in the following manner: Let p and q be projections in $\mathcal{M}$. We write $p \leftrightarrow q$ if there exist elements $x \in p\mathcal{M}q$ and $y \in q\mathcal{M}p$ such that $xy = p$ and $yx = q$.

Lemma 6.5.5 *([24], Proposition 1.4) If p and q are projections in $\mathcal{M}$ and $p \sim q$, then $p \leftrightarrow q$.*

We say that a *-algebra $\mathcal{A}$ is **-regular* if $\mathcal{A}$ is a regular algebra and $x^*x = 0$, with $x \in \mathcal{A}$, implies $x = 0$ ([67]). Also, $\mathcal{A}$ is called *complete* **-regular* if $\mathcal{A}$ is *-regular and the projections in $\mathcal{A}$ form a complete lattice ([67]).

Theorem 6.5.6 *([67], Theorem 2) If p and q are projections in a complete *-regular algebra with $p \leftrightarrow q \leq p$, then $p = q$.*

Corollary 6.5.7 *If $\widetilde{\mathcal{M}}$ is regular, then $\mathcal{M}$ is finite.*

Proof. Let $x \in \widetilde{\mathcal{M}}$ with $x^*x = 0$. By the polar decomposition of x, it follows that $x = 0$. Hence $\widetilde{\mathcal{M}}$ is a complete *-regular algebra. Let $p, q \in \mathcal{M}_p$ such that $p \sim q \leq p$. Then, by Lemma 6.5.5, $p \leftrightarrow q \leq p$. It follows from Theorem 6.5.6 that $p = q$, implying that $\mathcal{M}$ is finite. ∇

Theorem 6.5.8 *Let (X, Σ, μ) be a localizable measure space. The following statements are equivalent.*

(i) $\widetilde{L}_\infty(X, \Sigma, \mu)$ *is a regular algebra.*

(ii) (X, Σ, μ) *is a finite measure space.*

(iii) $\widetilde{L}_\infty(X, \Sigma, \mu) = L_0(X, \Sigma, \mu)$.

Proof. $(i) \Rightarrow (ii)$: Suppose that $\widetilde{L}_\infty(X, \Sigma, \mu)$ is a regular algebra and that μ is not finite. Since (X, Σ, μ) is localizable, it is semifinite. By Lemma 1.1.1, there exists a disjoint sequence (F_n) of measurable subsets of X such that $1 < \mu(F_n) < \infty$ for every n. Let

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in F_n \\ 1 & \text{if } x \notin \cup_{n=1}^{\infty} F_n. \end{cases}$$

Then $f \in \widetilde{L}_\infty(X, \Sigma, \mu)$. Since $\widetilde{L}_\infty(X, \Sigma, \mu)$ is regular, there exists $g \in \widetilde{L}_\infty(X, \Sigma, \mu)$ such that $f = f^2 g$. Clearly, $g = \frac{1}{f}$ almost everywhere.

Observe that $\mu(\cup_{n=1}^{n=\infty} F_n) = \infty$. Also, note that

$$g(x) = \begin{cases} n & \text{if } x \in F_n \\ 1 & \text{if } x \notin \cup_{n=1}^{\infty} F_n. \end{cases}$$

We show that $g \notin \widetilde{L}_\infty(X, \Sigma, \mu)$, thereby obtaining a contradiction. Let $t \in \mathbb{R}$. Choose a natural number n_0 such that $t < n_0$ and $n_0 > 1$. Then

$$\begin{aligned} \infty &= \mu(\cup_{n=n_0+1}^{\infty} F_n) \\ &\leq \mu(\{x \in X : g(x) > n_0\}) \\ &\leq \mu(\{x \in X : g(x) > t\}) \\ &= d_t(g). \end{aligned}$$

Hence $d_t(g) = \infty$ for every $t \in \mathbb{R}$. Therefore, by Proposition 1.1.4, it follows that $g \notin \widetilde{L}_\infty(X, \Sigma, \mu)$, a contradiction. Thus (X, Σ, μ) is a finite measure space.

$(ii) \Rightarrow (iii)$: This is immediate.

(iii) $\Rightarrow$ *(i)*: Assume that $\widetilde{L}_\infty(X, \Sigma, \mu) = L_0(X, \Sigma, \mu)$. Since $L_0(X, \Sigma, \mu)$ is regular (by Proposition 1.5.5 and Theorems 6.5.3 and 1.5.3), it follows that $\widetilde{L}_\infty(X, \Sigma, \mu)$ is regular. ∇.

Our next result is now immediate.

Corollary 6.5.9 *Let $\mathcal{M}$ be a commutative von Neumann algebra with a faithful semifinite normal trace τ. The following statements are equivalent.*

(i) $\widetilde{\mathcal{M}}$ *is a regular algebra.*

(ii) *The trace τ is finite.*

(iii) $\widetilde{\mathcal{M}} = \mathcal{U}(\mathcal{M})$.

For the case where $\mathcal{M}$ is not necessarily commutative, we now show that statements (i) and (ii) of Corollary 6.5.9 are equivalent when $\mathcal{M}$ is a factor. Recall that all factors of type I and II admit faithful semifinite normal traces, while type III factors have no faithful semifinite normal traces (Theorem 1.2.7).

Theorem 6.5.10 *Suppose that $\mathcal{M}$ is a factor with a faithful semifinite normal trace τ. The following statements are equivalent.*

(i) $\widetilde{\mathcal{M}}$ *is a regular algebra.*

(ii) *The trace τ is finite.*

Proof. The implication (ii) $\Rightarrow$ (i) follows from Corollary 6.5.4. We now show that the implication (i) $\Rightarrow$ (ii) holds. Suppose that $\widetilde{\mathcal{M}}$ is a regular algebra. By Corollary 6.5.7, $\mathcal{M}$ is a finite factor. Therefore, by Theorem 1.2.6, $\mathcal{M}$ is countably decomposable, and so $\mathcal{M}$ admits a faithful finite normal trace τ_0 (Theorem 1.2.5). It follows from Proposition 1.2.8 that $\tau = k\tau_0$ for some $k > 0$. Hence $\tau(1) = k\tau_0(1) < \infty$. Thus τ is finite. ∇

We now show that in the equivalence of statements (i) and (iii) of Corollary 6.5.9, the commutativity assumption of $\mathcal{M}$ can be replaced with the more general assumption of finiteness of $\mathcal{M}$.

Let $\mathcal{M}$ be a finite von Neumann algebra. Then $\mathcal{M}$ can be enlarged to a regular ring $\mathcal{U}(\mathcal{M})$, the algebra of closed densely defined operators affiliated with $\mathcal{M}$ (Theorem 6.5.3). By [24], p. 211, $\mathcal{U}(\mathcal{M})$ is also the smallest regular ring to contain $\mathcal{M}$, in the sense that the only regular subring of $\mathcal{U}(\mathcal{M})$ to contain $\mathcal{M}$ is $\mathcal{U}(\mathcal{M})$ itself.

Corollary 6.5.11 *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful semifinite normal trace τ. Then $\widetilde{\mathcal{M}}$ is a regular algebra if and only if $\widetilde{\mathcal{M}} = \mathcal{U}(\mathcal{M})$.*

Proof. Suppose that $\mathcal{M}$ is finite and $\widetilde{\mathcal{M}} = \mathcal{U}(\mathcal{M})$. By Theorem 6.5.3, $\mathcal{U}(\mathcal{M})$, and thus $\widetilde{\mathcal{M}}$, is regular.

Suppose that $\widetilde{\mathcal{M}}$ is regular. Since $\mathcal{U}(\mathcal{M})$ is the smallest regular ring to contain $\mathcal{M}$ and $\mathcal{M} \subset \widetilde{\mathcal{M}} \subset \mathcal{U}(\mathcal{M})$, it follows that $\widetilde{\mathcal{M}} = \mathcal{U}(\mathcal{M})$. ∇

In light of Corollary 6.5.9 and Corollary 6.5.11, it would be interesting to know if $\widetilde{\mathcal{M}} = \mathcal{U}(\mathcal{M})$ implies that the corresponding trace on $\mathcal{M}$ is finite. We do not know if there is an $\widetilde{\mathcal{M}}$ having a $\gamma_{cm} - \gamma_{cm}$ continuous derivation D and a primitive ideal $\mathcal{P}$ such that $D(\mathcal{P})$ is not contained in $\mathcal{P}$. The existence of such an example implies the existence of an $\widetilde{\mathcal{M}}$ with a non-inner $\gamma_{cm} - \gamma_{cm}$ continuous derivation.

We end with a characterization of regular C*-algebras, which is of independent interest.

Theorem 6.5.12 *A C*-algebra $\mathcal{A}$ is regular if and only if $\mathcal{A}$ is finite-dimensional.*

Proof. Suppose that $\mathcal{A}$ is regular. Since every regular Banach algebra is finite-dimensional ([68], p. 111), $\mathcal{A}$ is finite-dimensional.

Suppose that $\mathcal{A}$ is finite-dimensional. Then $\mathcal{A} \cong \oplus_{i=1}^{i=n} \mathcal{A}_i$, where $\mathcal{A}_i = M_{n_i}(\mathbb{C})$ for some $n_i \in \mathbb{N}$ for every i ([95], Theorem I.11.9). Since $\mathbb{C}$ is regular, $M_{n_i}(\mathbb{C})$ is regular ([68], Theorem 24, p. 114) for every i. Therefore, since any direct sum of regular rings is regular ([68], p. 110), $\mathcal{A}$ is regular. ∇

Chapter 7

Jordan homomorphisms and derivations on locally W*-algebras

Inverse limits of W*-algebras, called locally W*-algebras, were first studied by M. Fragoulopoulou in [49]. More than a decade later, M. Joita continued this study in the papers [60] and [61]. Derivations on locally C*-algebras were first studied in [49], [16] and [82]. In this chapter, we extend some results on Jordan homomorphisms and derivations for operator algebras to locally C*-algebras and locally W*-algebras. In particular, in Section 7.4, we investigate a problem raised in [49]: Is every derivation on a locally W*-algebra inner? A sufficient condition on a derivation on a locally W*-algebra is given to ensure that it is inner. The automatic continuity of derivations on locally C*-algebras is proved by Becker in [16], and we give an independent proof of his result which differs significantly to that provided by him (Theorem 7.4.1). We give, in Section 7.2, a characterization of those $\widetilde{\mathcal{M}}$ having the property that the measure topology is a locally W*-topology (Corollary 7.2.2). At the same time, we establish another characterization of locally W*-algebras amongst the class of locally C*-algebras (Theorem 7.2.1). Section 7.3 deals with Jordan homomorphisms of locally C*-algebras and locally

W*-algebras. We concern ourselves mainly with the problem as to when invertibility preserving linear maps between locally W*-algebras are Jordan homomorphisms.

7.1 Locally W*-algebras

In this section, we gather some basic concepts and results, mainly from [49], [60] and [61], regarding locally W*-algebras and locally C*-algebras which are not so well known and are needed in this chapter. No proofs will therefore be given since they can all be found in the above mentioned references.

7.1.1 Locally Hilbert spaces and locally C*-algebras

Definition 7.1.1 *([60], Definition 1.1) Let Λ be a directed set and $\{\mathcal{H}_\alpha : \alpha \in \Lambda\}$ a family of Hilbert spaces such that for $\alpha, \beta \in \Lambda$, $\beta \leq \alpha$, $\mathcal{H}_\beta \subset \mathcal{H}_\alpha$ and $\langle \cdot, \cdot \rangle_\beta = \langle \cdot, \cdot \rangle_\alpha |_{\mathcal{H}_\beta}$, where $\langle \cdot, \cdot \rangle_\alpha$ denotes the inner product of $\mathcal{H}_\alpha$. Let $\mathcal{H} = \lim_\rightarrow \mathcal{H}_\alpha = \cup_\alpha \mathcal{H}_\alpha$ be equipped with the inductive limit topology, i.e. the finest topology on $\mathcal{H}$ making the inclusion maps $\mathcal{H}_\beta \to \mathcal{H}_\alpha$ continuous when every $\mathcal{H}_\alpha$ is equipped with the norm topology. Then $\mathcal{H}$ is called a locally Hilbert space.*

Then if $\xi, \eta \in \mathcal{H}$, there is an $\alpha \in \Lambda$ such that $\xi, \eta \in \mathcal{H}_\alpha$. Put $\langle \xi, \eta \rangle = \langle \xi, \eta \rangle_\alpha$. This defines an inner product on $\mathcal{H}$ ([60], Definition 2.1 and Proposition 2.2). The completion of $\mathcal{H}$ under the norm induced by the inner product is denoted by $\widetilde{\mathcal{H}}$. The norm topology on $\mathcal{H}$ is weaker than the inductive limit topology on $\mathcal{H}$ ([60], Corollary 2.5).

As promised in chapter 1, we give an example of a non-commutative locally C*-algebra, namely the algebra described in the next proposition.

Proposition 7.1.2 *([57], Proposition 5.1; [49], Example 3) Let $\mathcal{H} = \lim_\rightarrow \mathcal{H}_\alpha$ be a locally Hilbert space. Denote the algebra of all continuous linear operators on $\mathcal{H}$ by $\mathcal{L}(\mathcal{H})$. Let*

$$L(\mathcal{H}) = \{x \in \mathcal{L}(\mathcal{H}) : \textit{ for every } \alpha \leq \beta,\ x_\beta \circ i_{\beta\alpha} = i_{\beta\alpha} \circ x_\alpha\},$$

where $x_\alpha = x|_{\mathcal{H}_\alpha} \in \mathcal{B}(\mathcal{H}_\alpha)$*, and* $i_{\beta\alpha}$ *the canonical injection of* $\mathcal{H}_\alpha$ *into* $\mathcal{H}_\beta$ *(it is shown in [57] that* $x(\mathcal{H}_\alpha) \subset \mathcal{H}_\alpha$ *for every* α*). The algebra* $L(\mathcal{H})$ *is a locally C*-algebra. The topology of* $L(\mathcal{H})$*, making* $L(\mathcal{H})$ *a locally C*-algebra, is defined by the family of C*-seminorms* $\{p_\alpha : \alpha \in \Lambda\}$*, where, for every* $\alpha \in \Lambda$*,* $p_\alpha(x) = \|x_\alpha\|_\alpha$*, and* $\|.\|_\alpha$ *is the norm on* $\mathcal{B}(\mathcal{H}_\alpha)$*. Moreover,* $L(\mathcal{H}) \cong \lim_{\leftarrow} \mathcal{B}(\mathcal{H}_\alpha)$*, where the connecting maps* $f_{\alpha\beta}$*,* $\alpha \leq \beta$*, are defined by the formulas* $f_{\alpha\beta}(x_\beta) = x_\alpha$*.*

Definition 7.1.3 *If* $\mathcal{H}$ *is a locally Hilbert space, then a locally C*-subalgebra of* $L(\mathcal{H})$ *is a closed *-subalgebra of* $L(\mathcal{H})$*.*

The proof of the following lemma is part of the proof of [49], Scholium 1.2.

Lemma 7.1.4 *Let* $\mathcal{B}$ *be a locally C*-subalgebra of* $L(\mathcal{H})$*, where* $\mathcal{H}$ *is a locally Hilbert space, and let* (p_α) *be the family of C*-seminorms on* $L(\mathcal{H})$ *in Proposition 7.1.2 defining the topology on* $L(\mathcal{H})$*. For every* α*, let* $q_\alpha = p_\alpha|_{\mathcal{B}}$*, and put* $N_\alpha = \{x \in \mathcal{B} : q_\alpha(x) = 0\}$*. For every* α*, denote the projection map of* $L(\mathcal{H})$ *into* $\mathcal{B}(\mathcal{H}_\alpha)$ *by* f_α*. Then, for every* α*, we have that* $\mathcal{B}/N_\alpha$*, with norm* $\dot{q}_\alpha$ *defined as* $\dot{q}_\alpha(x + N_\alpha) = q_\alpha(x)$ *for every* $x \in \mathcal{B}$*, is isometrically *-isomorphic to* $f_\alpha(\mathcal{B})$*, equipped with the norm* $\|.\|_\alpha$ *on* $\mathcal{B}(\mathcal{H}_\alpha)$ *restricted to* $f_\alpha(\mathcal{B})$*. Furthermore,* $f_\alpha(\mathcal{B})$ *is a C*-subalgebra of* $\mathcal{B}(\mathcal{H}_\alpha)$ *for every* α*.*

Proof. For every α, the map $\phi_\alpha : \mathcal{B}/N_\alpha \to f_\alpha(\mathcal{B})$, defined as $\phi_\alpha(x + N_\alpha) = f_\alpha(x)$, is a self-adjoint algebra isomorphism. Furthermore, $\dot{q}_\alpha(x + N_\alpha) = q_\alpha(x) = \|f_\alpha(x)\|_\alpha$ for every $x \in \mathcal{B}$. Since $\mathcal{B}/N_\alpha$ is a C*-algebra for every α (Theorem 1.4.6), it follows that $f_\alpha(\mathcal{B})$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H}_\alpha)$ for every α. ∇

It is a well known fact that every abstract C*-algebra is *-isomorphic to a C*-algebra of operators on some Hilbert space. The next result is an extension of this fact to locally C*-algebras.

Theorem 7.1.5 *([57], Theorem 5.1) If $\mathcal{A}$ is a locally C*-algebra, then there exists a locally Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ is topologically *-isomorphic to a locally C*-subalgebra of $L(\mathcal{H})$.*

We say that a locally C*-subalgebra $\mathcal{A}$ of $L(\mathcal{H})$, where $\mathcal{H}$ is a locally Hilbert space, acts *non-degenerately* on $\mathcal{H}$ if for each $0 \neq \xi \in \mathcal{H}$, there exists $x \in \mathcal{A}$ such that $x\xi \neq 0$ ([60], Definition 3.9).

Proposition 7.1.6 *([60], Proposition 3.10) Let $\mathcal{A}$ be a locally C*-algebra. Then there is a locally Hilbert space $\mathcal{H}$ and an injective homomorphism of locally C*-algebras $\pi : \mathcal{A} \to L(\mathcal{H})$ such that the locally C*-algebra $\pi(\mathcal{A})$ acts non-degenerately on $\mathcal{H}$.*

Lemma 7.1.7 *([61], p. 91) Let $\mathcal{A}$ be a locally C*-algebra which acts non-degenerately on a locally Hilbert space $\mathcal{H}$. Then $\mathcal{A}_s$ can be identified with a C*-subalgebra of $\mathcal{B}(\widetilde{\mathcal{H}})$ which acts non-degenerately on $\widetilde{\mathcal{H}}$.*

Proposition 7.1.8 *([61], Proposition 3.4) If $\mathcal{A}$ is a locally C*-algebra which acts non-degenerately on a locally Hilbert space $\mathcal{H}$, then $(\mathcal{A}'')_s = (\mathcal{A}_s)''$, where $(\mathcal{A}_s)''$ is the bicommutant of $\mathcal{A}_s$ in $\mathcal{B}(\widetilde{\mathcal{H}})$.*

7.1.2 Locally W*-algebras and locally von Neumann algebras

We now come to generalizations of W*-algebras and von Neumann algebras.

Definition 7.1.9 *([49], Definition 1.1) An algebra $\mathcal{A}$ is said to be a locally W*-algebra if it is an inverse limit of W*-algebras, i.e. $\mathcal{A} = \lim_{\leftarrow} \mathcal{A}_\alpha$, where each $\mathcal{A}_\alpha$ is a W*-algebra.*

Before we continue, we give an example of a locally W*-algebra, which is a generalization of the algebra of all bounded linear operators on a Hilbert space.

Example 7.1.10 *([49], Example 3)* Consider the locally C*-algebra $L(\mathcal{H})$ in Proposition 7.1.2, where $\mathcal{H} = \lim_{\rightarrow} \mathcal{H}_\alpha$ is a locally Hilbert space. By Proposition 7.1.2, $L(\mathcal{H}) \cong \lim_{\leftarrow} \mathcal{B}(\mathcal{H}_\alpha)$, proving that $L(\mathcal{H})$ is a locally W*-algebra. Recall that the connecting maps $f_{\alpha\beta}$, $\alpha \leq \beta$, are defined by the formulas $f_{\alpha\beta}(x_\beta) = x_\alpha$. One can also show that these connecting maps are $\sigma_\beta - \sigma_\alpha$ continuous, where, for every α, σ_α is the $\sigma(\mathcal{B}(\mathcal{H}_\alpha), (\mathcal{B}(\mathcal{H}_\alpha))_*)$ topology.

Theorem 7.1.13 below gives an abundance of examples of locally W*-algebras. Another example of a locally W*-algebra is given in the next section.

Theorem 7.1.11 *([49], Scholium 1.2) Every locally W*-algebra $\mathcal{A} = \lim_{\leftarrow} \mathcal{A}_\alpha$, where each $\mathcal{A}_\alpha$ is a W*-algebra, is a locally C*-algebra equipped with the inverse limit topology τ induced on it by the norm topologies of the $\mathcal{A}_\alpha$'s. Furthermore, the Arens-Michael decomposition of $\mathcal{A}$ is given by the *-subalgebras $f_\alpha(\mathcal{A})$ of the $\mathcal{A}_\alpha$'s, and a family (p_α) of C*-seminorms defining the topology of $\mathcal{A}$ is given by $p_\alpha(x) = \|f_\alpha(x)\|_\alpha$ for every $x \in \mathcal{A}$.*

Theorem 7.1.12 *([49], Proposition 1.3) Let $\mathcal{A} = \lim_{\leftarrow} \mathcal{A}_\alpha$ be a locally W*-algebra in such a way that the connecting maps $f_{\alpha\beta}$, $\alpha \leq \beta$, of the inverse system $(\mathcal{A}_\alpha)$ are $\sigma_\beta - \sigma_\alpha$ continuous, where, for each α, σ_α denotes the $\sigma(\mathcal{A}_\alpha, (\mathcal{A}_\alpha)_*)$ topology of $\mathcal{A}_\alpha$. Then $\mathcal{A}$ can be equipped with the inverse limit topology σ defined by equipping each $\mathcal{A}_\alpha$ with its σ_α topology. This topology is coarser than the inverse limit locally C*-topology τ of Theorem 7.1.11.*

Theorem 7.1.13 *([49], Proposition, p. 38) Let $\mathcal{A}$ be a locally W*-algebra admitting the inverse limit topology σ. Then every σ-closed *-subalgebra of $\mathcal{A}$ is a locally W*-algebra.*

Theorem 7.1.14 *([61], Corollary 3.3) If $\mathcal{A}$ is a locally W*-algebra admitting the inverse limit topology σ, then $\mathcal{A}_s$ is a W*-algebra.*

The next definition is a generalization of the notion of strong-operator topology on $\mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on a Hilbert space.

Definition 7.1.15 *([60], Definition 3.1(ii)) Let $L(\mathcal{H})$ be defined as in Proposition 7.1.2. The strong topology of $L(\mathcal{H})$, denoted by st, is the locally convex topology defined by the family of seminorms $(p_\xi)_{\xi\in\mathcal{H}}$, where*

$$p_\xi(x) = \|x\xi\|$$

for every $x \in L(\mathcal{H})$.

We now come to a generalization of von Neumann algebras.

Definition 7.1.16 *([60], Definition 3.7) A locally von Neumann algebra is a strongly closed locally C^*-subalgebra of $L(\mathcal{H})$, where $\mathcal{H}$ is a locally Hilbert space.*

We remark that the strong topology on $L(\mathcal{H})$ is the inverse limit of the strong topologies on the $\mathcal{B}(\mathcal{H}_\alpha)$'s ([60], Remark 1). The following result is now an immediate consequence of Lemma 1.4.4.

Proposition 7.1.17 *Let $\mathcal{H} = \lim_\leftarrow \mathcal{H}_\alpha$ be a locally Hilbert space and $\mathcal{A}$ a locally C^*-subalgebra of $L(\mathcal{H})$ such that $\mathcal{A} = \lim_\leftarrow \mathcal{A}_\alpha$, where $\mathcal{A}_\alpha$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H}_\alpha)$. Then $\overline{\mathcal{A}}^{st} = \lim_\leftarrow \overline{\mathcal{A}_\alpha}^{s_\alpha}$, where s_α denotes the strong-operator topology on $\mathcal{B}(\mathcal{H}_\alpha)$ for every α.*

The next result is a generalization of the well known fact that a W*-algebra can be represented faithfully as a von Neumann algebra on some Hilbert space.

Theorem 7.1.18 *([60], Corollary 3.17) Let $\mathcal{A}$ be a locally W^*-algebra admitting the inverse limit topology σ. Then there exists a locally Hilbert space $\mathcal{H}$ and an injective self-adjoint algebra homomorphism $\pi : \mathcal{A} \to L(\mathcal{H})$ such that $\pi(\mathcal{A})$ is a locally von Neumann algebra on $\mathcal{H}$ containing the identity operator on $\mathcal{H}$.*

It follows from the proof of [60], Corollary 3.17, that the *-homomorphism π in Theorem 7.1.18 is a topological-algebraic isomorphism. By [60], Proposition 3.14 and its proof, every locally von Neumann algebra is a locally W*-algebra which admits the inverse limit topology σ.

The following proposition is an extension of the bicommutant theorem for von Neumann algebras to locally von Neumann algebras.

Proposition 7.1.19 *([60], Corollary 3.12) Let $\mathcal{A}$ be a locally C*-subalgebra of $L(\mathcal{H})$ with strong closure $\mathcal{M}$, where $\mathcal{H}$ is a locally Hilbert space. If $\mathcal{A}$ acts non-degenerately on $\mathcal{H}$, then $\mathcal{M} = \mathcal{A}''$ (the bicommutant of $\mathcal{A}$ in $L(\mathcal{H})$).*

For a locally von Neumann algebra $\mathcal{M}$, we say that $v \in \mathcal{M}$ is a *partial isometry* if v^*v and vv^* are projections in $\mathcal{M}$. Since every locally von Neumann algebra is a locally C*-algebra, and hence a GB*-algebra, it follows from Proposition 1.4.21 that every positive element in a locally von Neumann algebra has a positve square root. This is of importance in the next theorem.

Theorem 7.1.20 (Polar decomposition) *([61], Theorem 4.5) Let $\mathcal{M}$ be a locally von Neumann algebra. For every $x \in \mathcal{M}$, there exists a unique partial isometry $v \in \mathcal{M}$ such that $x = v(x^*x)^{\frac{1}{2}}$.*

7.2 Characterizations of locally W*-algebras

We already know from Theorem 7.1.14 that, if $\mathcal{A}$ is a locally W*-algebra admitting the inverse limit topology σ, then $\mathcal{A}_s$ is a W*-algebra. In Theorem 7.2.1 below, we prove the converse, thereby characterizing locally W*-algebras among the locally C*-algebras.

First we make a remark which will be of importance in the proof of this theorem. If $\mathcal{A}$ and $\mathcal{B}$ are W*-algebras such that $\mathcal{A} \subset \mathcal{B}$, then they can both be represented as von Neumann algebras on the same Hilbert space. Indeed, since $\mathcal{B}$ is a W*-algebra, there exists a Hilbert space $\mathcal{H}$ and a faithful normal representation π of $\mathcal{B}$ on $\mathcal{H}$ such that $\pi(\mathcal{B})$ is a von Neumann algebra on $\mathcal{H}$. Let $\pi_0 = \pi|_{\mathcal{A}}$. Then π_0 is a faithful normal representation of $\mathcal{A}$ on $\mathcal{H}$. By Theorem 1.2.15, $\pi_0(\mathcal{A})$ is a von Neumann algebra on $\mathcal{H}$, implying that $\mathcal{A}$ and $\mathcal{B}$ can both be represented as von Neumann algebras on the same Hilbert space.

Theorem 7.2.1 *Let $\mathcal{A}$ be a locally C*-algebra such that $\mathcal{A}_s$ is a W*-algebra. Then $\mathcal{A}$ is a locally W*-algebra admitting the inverse limit topology σ.*

Proof. Since $\mathcal{A}$ is a locally C*-algebra, it follows from Proposition 7.1.6 that there exists a locally Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ is isomorphic to a locally C*-subalgebra $\mathcal{B}$ of $L(\mathcal{H})$ which acts non-degenerately on $\mathcal{H}$. Therefore there exists a family of Hilbert spaces $\mathcal{H}_\alpha$ such that $\mathcal{H} = \lim_{\rightarrow} \mathcal{H}_\alpha$ and $L(\mathcal{H}) = \lim_{\leftarrow} \mathcal{B}(\mathcal{H}_\alpha)$. For each α, let $\mathcal{B}_\alpha = f_\alpha(\mathcal{B})$. Then, by Lemma 7.1.4, every $\mathcal{B}_\alpha$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H}_\alpha)$ and, by Lemma 1.4.4, $\mathcal{A} \cong \mathcal{B} = \lim_{\leftarrow} \mathcal{B}_\alpha \subset \lim_{\leftarrow} \overline{\mathcal{B}_\alpha}^{s_\alpha} = \mathcal{B}_0$ say, where, for each α, s_α denotes the strong-operator topology of $\mathcal{B}(\mathcal{H}_\alpha)$. By Proposition 7.1.17, $\mathcal{B}_0$ is a locally von Neumann algebra.

We prove that $\mathcal{B} = \mathcal{B}_0$. By Theorem 7.1.14, it follows that $(\mathcal{B}_0)_s$ is a W*-algebra. Since $\mathcal{A}$ is isomorphic to $\mathcal{B}$, we get from Theorem 1.4.10 that $\mathcal{A}_s$ and $\mathcal{B}_s$ are (isometrically) *-isomorphic C*-algebras. Therefore, since $\mathcal{A}_s$ is a W*-algebra, $\mathcal{B}_s$ is also a W*-algebra. Clearly, $\mathcal{B}_s \subset (\mathcal{B}_0)_s$. By Lemma 7.1.7, $\mathcal{B}_s$ and $(\mathcal{B}_0)_s$ can both be represented as von Neumann algebras on the Hilbert space $\widetilde{\mathcal{H}}$, since $\mathcal{B}_0$ acts non-degenerately on $\mathcal{H}$. By Proposition 7.1.19, it follows that $\mathcal{B}_0 = \overline{\mathcal{B}}^{st} = \mathcal{B}''$. Therefore, by Proposition 7.1.8, $(\mathcal{B}_0)_s = (\mathcal{B}'')_s = (\mathcal{B}_s)''$, where $(\mathcal{B}_s)''$ is the bicommutant of $\mathcal{B}_s$ in $\mathcal{B}(\widetilde{\mathcal{H}})$. Since $\mathcal{B}_s$ is a von Neumann algebra with underlying Hilbert space $\widetilde{\mathcal{H}}$, it follows that $(\mathcal{B}_s)'' = \mathcal{B}_s$. Hence $(\mathcal{B}_0)_s = \mathcal{B}_s$. By Theorem 1.4.8, it now follows that $\mathcal{B}_0 = \mathcal{B}$, implying that $\mathcal{B}_0$ is a locally von Neumann algebra. Hence $\mathcal{A}$ is a locally W*-algebra with the inverse limit topology σ. ∇

Corollary 7.2.2 *The following statements are equivalent.*

(i) *$\widetilde{\mathcal{M}}$ is locally m-convex with respect to the measure topology.*

(ii) *$\widetilde{\mathcal{M}}$ is a locally C*-algebra with respect to the measure topology.*

(iii) *$\widetilde{\mathcal{M}}$ is a locally W*-algebra with respect to the measure topology, admitting the inverse limit topology σ.*

Proof. Since $\widetilde{\mathcal{M}}$ is a GB*-algebra (Theorem 1.5.29), the equivalence of statements (i) and (ii) follows from Theorem 1.4.18. The implication (iii) ⇒ (ii) is obvious in light of Theorem 7.1.11.

(ii) ⇒ *(iii)* : Suppose that $\widetilde{\mathcal{M}}$ is a locally C*-algebra with respect to the measure topology. In that case, we know from Theorem 1.5.29 and Lemma 1.4.12 that $\mathcal{M} = (\widetilde{\mathcal{M}})_s$, implying that $(\widetilde{\mathcal{M}})_s$ is a W*-algebra. By Theorem 7.2.1, $\widetilde{\mathcal{M}}$ is a locally W*-algebra. ∇

Definition 7.2.3 *([41], Definition 1.2) Let $\mathcal{A}$ be a set of closed, densely defined operators on a Hilbert space $\mathcal{H}$ which is a *-algebra under strong sum, strong product, adjunction and scalar multiplication (it is understood that $\lambda x = 0$, the zero operator on the whole of $\mathcal{H}$, if $\lambda = 0$). We call $\mathcal{A}$ an EW*-algebra if the following conditions are met:*

(i) $(1 + x^*x)^{-1}$ *exists for every $x \in \mathcal{A}$,*

(ii) the subalgebra $\mathcal{A}_e$ of bounded operators in $\mathcal{A}$ is a W-algebra.*

By [41], Proposition 2.4, the condition (i) of Definition 7.2.3 is equivalent to the condition that $x\eta\mathcal{A}_e$ for every $x \in \mathcal{A}$.

Theorem 7.2.4 *([103], Theorem 3) Every GB*-algebra with underlying C*-algebra a W*-algebra is algebraically *-isomorphic to an EW*-algebra.*

Corollary 7.2.5 *The locally W*-algebras admitting the inverse limit topology σ are, up to self-adjoint algebra isomorphism, precisely the EW*-algebras admitting complete locally m-convex GB* topologies.*

Proof. Suppose that $\mathcal{A}$ is a locally W*-algebra admitting the inverse limit topology σ. Then, by Theorem 7.1.11, $\mathcal{A}$ is a locally C*-algebra with a family of C*-seminorms (p_α) defining the topology of $\mathcal{A}$. By Theorem 7.1.14, $\mathcal{A}_s$ is a W*-algebra. It follows from Lemma 1.4.12 that $\mathcal{A}$ is a GB*-algebra with underlying C*-algebra $\mathcal{A}_b$ a W*-algebra. By Theorem 7.2.4, there is a self-adjoint algebra isomorphism ϕ of $\mathcal{A}$ onto an EW*-algebra $\mathcal{B}$. For every α, let $q_\alpha(\phi(x)) = p_\alpha(x)$ for every $x \in \mathcal{A}$. Since ϕ is a *-algebra isomorphism, every q_α is a C*-seminorm on $\mathcal{B}$, and the topology defined on $\mathcal{B}$ by the family of C*-seminorms (q_α) is a locally C*-algebra, and hence a complete locally m-convex GB*-algebra.

Now suppose that $\mathcal{A}$ is an EW*-algebra admitting a complete locally m-convex GB*-topology. By Theorem 1.4.18, $\mathcal{A}$ is a locally C*-algebra.

We show that $\mathcal{A}_b = \mathcal{A}_e$, where $\mathcal{A}_e$ is the W*-algebra in Definition 7.2.3, and $\mathcal{A}_b$ the underlying C*-algebra of $\mathcal{A}$ as in Definition 1.4.11. Since all unitary elements in $\mathcal{A}$ are bounded operators, it is clear that every unitary element of $\mathcal{A}_b$ is a unitary element in $\mathcal{A}_e$. Conversely, it follows from Lemma 3.3.1 that every unitary element of $\mathcal{A}_e$ is contained in $\mathcal{A}_b$. Since every C*-algebra is the span of its unitary elements, it follows that $\mathcal{A}_b = \mathcal{A}_e$.

By Lemma 1.4.12, $\mathcal{A}_s = \mathcal{A}_b$, implying that $\mathcal{A}_s$ is a W*-algebra. It follows from Theorem 7.2.1 that $\mathcal{A}$ is a locally W*-algebra which admits the inverse limit topology σ. ∇

7.3 Jordan homomorphisms of locally W*-algebras

Proposition 7.3.1 *Let $\mathcal{A}$ be locally W*-algebra admitting the inverse limit topology σ and let $\mathcal{B}$ a locally C*-algebra. If $\phi : \mathcal{A} \to \mathcal{B}$ is a continuous unital self-adjoint invertibility preserving linear map, then ϕ is a Jordan homomorphism.*

Proof. Recall that every locally C*-algebra is a GB*-algebra and therefore, by Theorem 7.1.14 and Lemma 1.4.12, $\mathcal{A}$ is a GB*-algebra with underlying C*-algebra a W*-algebra. The result is now an immediate consequence of Theorem 4.4.4 and Theorem 4.1.1. ∇

The following example demonstrates that not all unital self-adjoint invertibility preserving linear maps between locally C*-algebras are continuous.

Example 7.3.2 *([81], p. 172)* Let F be the family of countable closed subsets of $[0, 1]$ having only finitely many cluster points, and let $\mathcal{A} = C([0, 1])$

be equipped with the topology of uniform convergence on the members of F. Then $\mathcal{A}$ is a locally C*-algebra. Consider the identity map $i : \mathcal{A} \to C([0,1])$, where the codomain of i is equipped with the usual topology. Then it can be shown that i is a discontinuous self-adjoint algebra isomorphism, and hence a discontinuous unital self-adjoint invertibility preserving linear map.

Recall that if $\mathcal{A}$ is a locally C*-algebra, then $\mathcal{A}$ is a GB*-algebra with $\mathcal{A}_b = \mathcal{A}_s$. With this in mind, the following result is an immediate consequence of Theorem 3.3.5, Lemma 3.3.2, Proposition 3.4.3 and Theorem 7.1.14.

Corollary 7.3.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be metrizable locally C*-algebras, with $\mathcal{A}$ a locally W*-algebra admitting the inverse limit topology σ. Then every self-adjoint Jordan homomorphism is continuous and is a sum of a self-adjoint algebra homomorphism and a self-adjoint algebra anti-homomorphism.*

7.4 Derivations on locally C*-algebras and locally W*-algebras

In [49], it is mentioned that it would be of interest to know whether every derivation on a locally W*-algebra is inner. In this section, we give a sufficient condition under which this is the case (Theorem 7.4.2). It is well known that every derivation on a C*-algebra is continuous (Theorem 2.3.5). In [16], Becker proved that every derivation on a locally C*-algebra is continuous. His proof is entirely algebraic. Below, we give another proof of this result using the same strategy as that of Becker, but our proof relies on the fact that every locally C*-algebra can be represented on a locally Hilbert space (Theorem 7.1.5) as well as on some results about locally von Neumann algebras.

Theorem 7.4.1 *([16], Proposition 2) Every derivation on a locally C*-algebra is continuous.*

Proof. Let $\mathcal{A}$ be a locally C*-algebra and D a derivation defined on it. By Theorem 7.1.5, $\mathcal{A}$ can be identified with a locally C*-subalgebra $\mathcal{B}$ of $L(\mathcal{H})$, where $\mathcal{H}$ is a locally Hilbert space, say $\mathcal{H} = \lim_{\rightarrow} \mathcal{H}_\alpha$. Therefore we can identify the derivation D with a derivation D_0 on $\mathcal{B}$. It suffices to show that D_0 is continuous.

Let (p_α) be the family of C*-seminorms defining the topology of $L(\mathcal{H})$, as in Proposition 7.1.2. Now $\mathcal{B}$ is a locally C*-algebra with $(q_\alpha) = (p_\alpha|_{\mathcal{B}})$ being a family of C*-seminorms defining its topology. Recall that $L(\mathcal{H}) \cong \lim_{\leftarrow} \mathcal{B}(\mathcal{H}_\alpha)$. By Lemmas 1.4.4 and 7.1.4, $\mathcal{B} = \lim_{\leftarrow} \mathcal{B}_\alpha$, with every $\mathcal{B}_\alpha$ a C*-subalgebra of $\mathcal{B}(\mathcal{H}_\alpha)$, namely, $\mathcal{B}_\alpha = f_\alpha(\mathcal{B})$, where f_α is the restriction to $\mathcal{B}$ of the projection map of $\prod_\alpha \mathcal{B}_\alpha$ onto $\mathcal{B}_\alpha$ for every α. Let $N_\alpha = \{x \in \mathcal{B} : q_\alpha(x) = 0\}$ for each α.

If $0 \leq x \in N_\alpha$, then there exists a unique $y \in N_\alpha$ such that $x = y^2$. To see this, note that, by Proposition 1.4.21, there already exists $0 \leq y \in \mathcal{B}$ such that $x = y^2$. Now $0 = q_\alpha(x) = q_\alpha(y^2) = q_\alpha(y^*y) = q_\alpha(y)^2$, implying that $y \in N_\alpha$.

Let $x \in N_\alpha$ and consider the polar decomposition $x = v|x|$ of x, where $v \in \overline{\mathcal{B}}^{st}$ (Theorem 7.1.20). Hence $x = (v|x|^{\frac{1}{2}})(|x|^{\frac{1}{2}})$. Since $x \in N_\alpha$, it follows that $x^*x \in N_\alpha$. Hence, by what was proved above, $|x| \in N_\alpha$ because $x^*x \geq 0$. Therefore, once again by what was proved above, $|x|^{\frac{1}{2}} \in N_\alpha$.

We aim to show that $v|x|^{\frac{1}{2}} \in N_\alpha$, so that $x = ab$ with $a, b \in N_\alpha$. Since $|x|^{\frac{1}{2}} \in N_\alpha$, it follows that $p_\alpha(|x|^{\frac{1}{2}}) = 0$. Hence $p_\alpha(v|x|^{\frac{1}{2}}) \leq p_\alpha(v)p_\alpha(|x|^{\frac{1}{2}}) = 0$. So it remains to prove that $v|x|^{\frac{1}{2}} \in \mathcal{B}$, thereby showing that $v|x|^{\frac{1}{2}} \in N_\alpha$. By Lemma 1.4.4, $\mathcal{B} = \lim_{\leftarrow} \mathcal{B}_\alpha \subset \lim_{\leftarrow} \overline{\mathcal{B}_\alpha}^{s_\alpha} = \overline{\mathcal{B}}^{st}$ and $v|x|^{\frac{1}{2}} = (v_\alpha|x_\alpha|^{\frac{1}{2}})_\alpha$, where $v_\alpha|x_\alpha|$ is the polar decomposition of x_α in $\overline{\mathcal{B}_\alpha}^{s_\alpha}$ for every α ([61], Remark 4.6). Therefore, by Proposition 1.3.4, it follows that $v_\alpha|x_\alpha|^{\frac{1}{2}} \in \mathcal{B}_\alpha$ for every α. Hence $v|x|^{\frac{1}{2}} \in \mathcal{B}$.

Since every N_α is a two-sided ideal of $\mathcal{B}$, it follows from Proposition 6.3.5 that $D(N_\alpha) \subset N_\alpha$ for every α, and therefore the map $D_\alpha : \mathcal{B}/N_\alpha \rightarrow \mathcal{B}/N_\alpha : x + N_\alpha \mapsto (D_0(x)) + N_\alpha$ is well defined for every α. One can also easily verify that every D_α is a derivation on $\mathcal{B}/N_\alpha$. By Theorem 1.4.6, every $\mathcal{B}/N_\alpha$ is a C*-algebra, and therefore, by Theorem 2.3.5, every D_α is continuous.

Suppose that (x_λ) is a net in $\mathcal{B}$ with $x_\lambda \to x \in \mathcal{B}$. Then

$$\lim_\lambda \dot{q_\alpha}((x_\lambda + N_\alpha) - (x + N_\alpha)) = \lim_\lambda q_\alpha(x_\lambda - x) = 0$$

for all α. It follows that, for each α, $\lim_\lambda(x_\lambda + N_\alpha) = x + N_\alpha$ in $\mathcal{B}/N_\alpha$. Since every D_α is continuous, it follows that $\lim_\lambda(D_\alpha(x_\lambda + N_\alpha)) = D_\alpha(x + N_\alpha)$ for each α. Hence $\lim_\lambda((D_0(x_\lambda)) + N_\alpha) = (D_0(x)) + N_\alpha$ for each α. Therefore $D_0(x_\lambda) \to D_0(x)$. Hence D_0, and thus D, is continuous. ∇

The zero derivation is the only derivation on a commutative locally C*-algebra ([16], Corollary 3), i.e. every derivation on a commutative locally C*-algebra is inner. In general, one at least has the following result.

Theorem 7.4.2 *Let $\mathcal{A}$ be a locally W*-algebra admitting the inverse limit topology σ. If D is a derivation on $\mathcal{A}$ such that $D(\mathcal{A}_s) \subset \mathcal{A}_s$, then D is inner.*

Proof. Let D be a derivation on $\mathcal{A}$. Since $\mathcal{A}$ is a locally W*-algebra, it follows from Theorem 7.1.11 that $\mathcal{A}$ is a locally C*-algebra. It follows from Theorem 1.4.8 that $\mathcal{A}_s$ is a C*-algebra which is dense in $\mathcal{A}$. By Theorem 7.4.1, D is continuous. By hypothesis, $D(\mathcal{A}_s) \subset \mathcal{A}_s$, i.e. $D|_{\mathcal{A}_s}$ is a derivation on $\mathcal{A}_s$. Now $\mathcal{A}_s$ is a W*-algebra (Theorem 7.1.14), and hence, by Lemma 5.3.16, D is inner. ∇

A derivation D on a locally W*-algebra $\mathcal{A}$ does not always have the property that $D(\mathcal{A}_s) \subset \mathcal{A}_s$, as demonstrated in the following example.

Example 7.4.3 Let $(\mathcal{H}_n)$ be a (countable) family of Hilbert spaces, and let $\mathcal{H} = \lim_\to \mathcal{H}_n$. Consider the locally W*-algebra $\mathcal{A} = L(\mathcal{H})$ in Example 7.1.10.

We show that $Z(\mathcal{A}) = \mathbb{C}1$, where $Z(\mathcal{A})$ denotes the centre of $\mathcal{A}$. Indeed, $\mathcal{A} = \lim_\leftarrow \mathcal{B}(\mathcal{H}_n)$. Now $Z(\mathcal{A}) = \lim_\leftarrow Z(\mathcal{B}(\mathcal{H}_n))$ (this is a special case of [49], Corollary 2.2), implying that $Z(\mathcal{A}) = \lim_\leftarrow \mathbb{C}1_n$, where 1_n is the identity operator on $\mathcal{H}_n$ for every n. Therefore, if $x \in Z(\mathcal{A})$, then $x = (\lambda_n 1_n)$ for scalars λ_n. If $n \leq m$, then $\mathcal{H}_n \subset \mathcal{H}_m$, and so, $\lambda_n 1_n = \lambda_m 1_m|_{\mathcal{H}_n} = \lambda_m 1_n$, implying that $\lambda_n = \lambda_m$. This holds for every $n \leq m$. Thus $\lambda_n = \lambda$, say, for every n. Hence $x = \lambda 1$.

Let $a \in \mathcal{A}$ with $a \notin \mathcal{A}_s$. Consider the inner derivation D on $\mathcal{A}$ defined by $D(x) = ax - xa$ for every $x \in \mathcal{A}$.

We show that $D(\mathcal{A}_s)$ is not contained in $\mathcal{A}_s$. Assume that $D(\mathcal{A}_s) \subset \mathcal{A}_s$. Then, since $\mathcal{A}_s$ is a W*-algebra, it follows from Theorem 2.3.6 that there exists $b \in \mathcal{A}_s$ such that $D(x) = bx - xb$ for every $x \in \mathcal{A}_s$. Since $\mathcal{A}_s$ is dense in $\mathcal{A}$ and D is continuous, it follows that $D(x) = bx - xb$ for every $x \in \mathcal{A}$. Hence $ax - xa = bx - xb$ for every $x \in \mathcal{A}$. Therefore $(a - b)x = x(a - b)$ for every $x \in \mathcal{A}$, i.e. $a - b \in Z(\mathcal{A}) = \mathbb{C}1 \subset \mathcal{A}_s$. Since $b \in \mathcal{A}_s$, we have that $a \in \mathcal{A}_s$, a contradiction.

Our next result was motivated by Theorem 2.3.7. For the proof of this, recall that the von Neumann algebra generated by a C*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, is the closure of $\mathcal{A}$ in the ultraweak and the strong-operator topologies on $\mathcal{B}(\mathcal{H})$ ([39], Corollary 1, p. 45).

Proposition 7.4.4 *Let D be a derivation on a locally C*-subalgebra $\mathcal{B}$ of $L(\mathcal{H})$, where $\mathcal{H}$ is a locally Hilbert space, say $\mathcal{H} = \lim_{\rightarrow} \mathcal{H}_\alpha$. Then D can be extended to a derivation on $\overline{\mathcal{B}}^{st}$.*

Proof. Let (p_α) be the family of C*-seminorms defining the topology of $L(\mathcal{H})$, as in Proposition 7.1.2. Then $\mathcal{B}$ is a locally C*-subalgebra of $L(\mathcal{H})$ with $(q_\alpha) = (p_\alpha|_{\mathcal{B}})$ defining the topology of $\mathcal{B}$. Let $N_\alpha = \{x \in \mathcal{B} : q_\alpha(x) = 0\}$ for each α. Furthermore, by Lemmas 1.4.4 and 7.1.4, $\mathcal{B} = \lim_{\leftarrow} \mathcal{B}_\alpha$, where, for every α, $\mathcal{B}_\alpha$ is a C*-subalgebra of $\mathcal{B}(\mathcal{H}_\alpha)$, namely, $\mathcal{B}_\alpha = f_\alpha(\mathcal{B})$ for each α. For every α, let σ_α denote the topology on $\mathcal{B}(\mathcal{H}_\alpha)$ as defined in Example 7.1.10. In the proof of Theorem 7.4.1, we have shown that $D(N_\alpha) \subset N_\alpha$ for all α, and that, consequently, the derivations $D_\alpha : \mathcal{B}/N_\alpha \rightarrow \mathcal{B}/N_\alpha : x + N_\alpha \mapsto D(x) + N_\alpha$ are well defined. Let the maps $f_{\alpha\beta}$ be defined as in Theorem 1.4.6. It follows from Theorem 1.4.6 that

$$f_{\alpha\beta}(D_\beta(x + N_\beta)) \quad = \quad f_{\alpha\beta}(D(x) + N_\beta)$$

$$\begin{aligned} &= D(x) + N_\alpha \\ &= D_\alpha(x + N_\alpha), \end{aligned}$$

where $\alpha \leq \beta$.

For every α, it follows from Lemma 7.1.4 that the derivation D_α can be identified with a derivation on $\mathcal{B}_\alpha$, which we also denote by D_α. If $x \in \mathcal{B}$, then $x = (x_\alpha) \in \lim_\leftarrow \mathcal{B}(\mathcal{H}_\alpha)$, and $x_\alpha = x|_{\mathcal{H}_\alpha}$ for every α. We may therefore identify the connecting maps associated with $\mathcal{B}$ by the restrictions of the connecting maps of $\mathcal{B}(\mathcal{H}_\alpha)$ to $\mathcal{B}_\alpha$ for every α. We use the same notation $f_{\alpha\beta}$ for these connecting maps, and so $f_{\alpha\beta}(D_\beta(x_\beta)) = D_\alpha(x_\alpha)$ for every $\alpha \leq \beta$, and for every $x = (x_\alpha) \in \mathcal{B}$.

By Theorem 2.3.7, every D_α extends to a derivation $\overline{D_\alpha}$ on $\overline{\mathcal{B}_\alpha}^{s_\alpha} = \overline{\mathcal{B}_\alpha}^{\sigma_\alpha}$, where, for each α, s_α denotes the strong-operator topology on $\mathcal{B}_\alpha$.

Therefore D extends to a derivation $\overline{D}$ on $\overline{\mathcal{B}}^{st} = \lim_\leftarrow \overline{\mathcal{B}_\alpha}^{s_\alpha}$ (Proposition 7.1.17). To see this, let $\overline{D}(x) = (\overline{D_\alpha}(x_\alpha))$, where $x = (x_\alpha) \in \overline{\mathcal{B}}^{st}$ and $x_\alpha \in \overline{\mathcal{B}_\alpha}^{s_\alpha}$ for every α.

We show that $\overline{D}(x) \in \overline{\mathcal{B}}^{st}$ for every $x \in \overline{\mathcal{B}}^{st}$. By Example 7.1.10 and Theorem 2.3.7, the connecting maps $f_{\alpha\beta}$ of $\mathcal{B}$ are $\sigma_\beta - \sigma_\alpha$ continuous and $\overline{D_\alpha}$ is σ_α-continuous. By Theorem 7.1.12 and Lemma 1.4.4, $\lim_\leftarrow \overline{\mathcal{B}_\alpha}^{\sigma_\alpha} = \overline{\mathcal{B}}^{\sigma}$, where σ is the inverse limit topology defined by the σ_α's.

It follows that $f_{\alpha\beta}(\overline{D_\beta}(x_\beta)) = \overline{D_\alpha}(x_\alpha)$ for every $(x_\alpha) \in \lim_\leftarrow \overline{\mathcal{B}_\alpha}^{\sigma_\alpha}$. Indeed, let $x = (x_\alpha) \in \overline{\mathcal{B}}^{\sigma}$. Then one can find a net $((x_{\alpha,\lambda})) \in \mathcal{B} = \lim_\leftarrow \mathcal{B}_\alpha$ such that $\lim_\lambda (x_{\alpha,\lambda}) = (x_\alpha)$ in the σ topology. Therefore, for every α, $\lim_\lambda x_{\alpha,\lambda} = x_\alpha$ in the σ_α topology. Hence

$$\begin{aligned} f_{\alpha\beta}(\overline{D_\beta}(x_\beta)) &= f_{\alpha\beta}(\overline{D_\beta}(\lim_\lambda (x_{\beta,\lambda}))) \\ &= \lim_\lambda f_{\alpha\beta}(D_\beta(x_{\beta,\lambda})) \\ &= \lim_\lambda D_\alpha(x_{\alpha,\lambda}) \\ &= \lim_\lambda \overline{D_\alpha}(x_{\alpha,\lambda}) \\ &= \overline{D_\alpha}(\lim_\lambda x_{\alpha,\lambda}) \\ &= \overline{D_\alpha}(x_\alpha), \end{aligned}$$

where $\alpha \leq \beta$.

Therefore $(\overline{D_\alpha}(x_\alpha)) \in \lim_{\leftarrow} \overline{\mathcal{B}_\alpha}^{\sigma_\alpha} = \lim_{\leftarrow} \overline{\mathcal{B}_\alpha}^{s_\alpha}$, i.e. $\overline{D}(x) \in \overline{\mathcal{B}}^{st}$ for every $x \in \overline{\mathcal{B}}^{st}$. It is easily verified that $\overline{D}$ is a derivation on $\overline{\mathcal{B}}^{st}$. ∇

Bibliography

[1] L. Ahlfors, *Complex analysis*, second edition, McGraw-Hill, New York, 1966.

[2] S. Albeverio, Sh. A. Ayupov, K. K. Kudaybergenov, Derivations on the algebra of measurable operators affiliated with a type I von Neumann algebra, *arXiv:math.OA/0703171*, 6 March 2007.

[3] S. Albeverio, Sh. A. Ayupov, K. K. Kudaybergenov, Derivations on the algebra of τ-compact operators affiliated with a type I von Neumann algebra, *arXiv:math.OA/0703331*, 12 March 2007.

[4] S. Albeverio, Sh. A. Ayupov, K. K. Kudaybergenov, Description of derivations on measurable operator algebras of type I, *arXiv:math.OA/0710.3344*, 17 October 2007.

[5] G. R. Allan, A spectral theory for locally convex algebras, *Proc. London Math. Soc.* **15**, 1965, 399–421.

[6] G. R. Allan, On a class of locally convex algebras, *Proc. London Math. Soc.* **17**, 1967, 91–114.

[7] J. P. Antoine, A. Inoue and C. Trapani, O*-dynamical systems and *-derivations of unbounded operator algebras, *Math. Nachr.* **204**, 1999, 5–28.

[8] C. Apostol, b*-algebras and their representation, *J. London Math. Soc.* **2**, 1971, 30–38.

[9] P. Ara and M. Mathieu, *Local multipliers of C*-algebras*, Springer Monographs in Mathematics, Springer, Berlin-Heidelberg-New York, 2003.

[10] B. Aupetit, *A Primer on Spectral Theory*, Springer, Berlin, 1991.

[11] B. Aupetit, Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras, *J. London Math. Soc.* **62**, 2000, 917–924.

[12] Sh. A. Ayupov, Derivations on algebras of measurable operators, *Dokl. Akad. Nauk RUz.* **3**, 2000, 14–17.

[13] Sh. A. Ayupov, Derivations on unbounded operator algebras, Abstracts of the international conference "Operator algebras and quantum probability", 7–19 September 2005, Tashkent, Uzbekistan.

[14] Sh. A. Ayupov, private communication, 28 July 2006.

[15] E. Beckenstein, L. Narici, C. Suffel, *Topological Algebras*, North-Holland, Amsterdam-New York-Oxford, 1977.

[16] R. Becker, Derivations on LMC*-Algebras, *Math. Nachr.* **155**, 1992, 141–149.

[17] C. Bennet and R. Sharpley, *Interpolation of operators*, Academic Press, Orlando, Florida, 1988.

[18] V. A. Belfi and R. S. Doran, *Characterizations of C*-algebras. The Gelfand-Naimark Theorems*, Dekker, New York, 1986.

[19] A. F. Ber, V. I. Chilin and F. A. Sukochev, Derivations in commutative regular algebras, *Math. Notes* **75**, 2004, 418–419.

[20] A. F. Ber, V. I. Chilin and F. A. Sukochev, Derivations in commutative regular algebras, *Extracta Math.* **21**, 2006, 107–147.

[21] A. Ber, F. Sukochev, Derivations on the algebras of measurable operators, Manuscript presented at the international conference "Operator spaces, non-commutative L^p spaces and applications", 25–29 June 2007, Marseilles, France.

[22] A. F. Ber, B. de Pagter and F. A. Sukochev, private communication, May 2008.

[23] S. K. Berberian, *Measure and integration*, Macmillan, New York, 1965.

[24] S. K. Berberian, *Baer *-rings*, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[25] S. K. Berberian, The maximal ring of quotients of a finite von Neumann algebra, *Rocky Mountain J. Math.* **12**, 1982, 149–164.

[26] S. J. Bhatt, A note on generalized B*-algebras, *J. Indian Math. Soc.* **43**, 1979, 253–257.

[27] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[28] O. Bratteli and D. W. Robinson, *Operator algebras and quantum statistical mechanics I*, Springer-Verlag, New York, 1979.

[29] M. Brešar and P. Šemrl, Spectral Characterization of Idempotents and Invertibility Preserving Linear Maps, *Expo. Math.* **17**, 1999, 185–192.

[30] R. M. Brooks, On representing F*-algebras, *Pacific J. Math.* **39**, 1971, 51–69.

[31] D. M. Burton, *A first course in rings and ideals*, Addison-Wesley, Reading, Massachusetts, 1970.

[32] M. D. Choi, D. Hadwin, E. Nordgren, H. Radjavi and P. Rosenthal, On positive linear maps preserving invertibility, *J. Funct. Anal.* **59**, 1984, 462–469.

[33] D. L. Cohn, *Measure theory*, Birkhauser, Boston, 1980.

[34] C. L. Crowther, *Aspects of duality theory for spaces of measurable operators*, PhD Thesis, University of Cape Town, South Africa, 1997.

[35] H. G. Dales, Automatic continuity : A Survey, *Bull. London. Math. Soc.* **10**, 1978, 129–183.

[36] H. G. Dales, *Banach algebras and automatic continuity*, Clarendon Press, Oxford, 2000.

[37] K. R. Davidson, *C^*-algebras by example*, Fields Institute Monographs, Amer. Math. Soc, Providence, Rhode Island, 1996.

[38] J. Dixmier, *C^*-algebras*, North-Holland Mathematical Library, Amsterdam-New York-Oxford, 1982.

[39] J. Dixmier, *Von Neumann algebras*, North-Holland Mathematical Library, Amsterdam-New York-Oxford, 1981.

[40] P. G. Dixon, Generalized B^*-algebras, *Proc. London Math. Soc.* **21**, 1970, 693–715.

[41] P. G. Dixon, Unbounded operator algebras, *Proc. London Math. Soc.* **23**, 1971, 53–69.

[42] P. G. Dodds, T. K. Dodds and B. de Pagter, Non-commutative Banach function spaces, *Math. Z.* **201**, 1989, 583–597.

[43] P. G. Dodds, T. K. Dodds and B. de Pagter, Non-commutative Köthe duality, *Trans. Amer. Math. Soc.* **339**, 1993, 717–750.

[44] P. G. Dodds, T. K. Dodds, B. de Pagter and F. A. Sukochev, Lipschitz continuity of the absolute value and Riesz projections in symmetric operator spaces, *J. Func. Anal.* **148**, 1997, 28–69.

[45] N. Dunford and J. T. Schwartz, *Linear operators II*, Interscience Publications, New York, 1963.

[46] H. A. Dye, B. Russo, A note on unitary operators in C*-algebras, *Duke Math. J.* **6**, 1966, 413–416.

[47] T. Fack and H. Kosaki, Generalised s-numbers of τ-measurable operators, *Pacific J. Math.* **123**, 1986, 269–300.

[48] J. Feldman, Isomorphisms of finite type II rings of operators, *Ann. of Math.* **63**, 1956, 565–571.

[49] M. Fragoulopoulou, On locally W*-algebras, *Yokohama Math. J.* **34**, 1986, 35–51.

[50] M. Fragoulopoulou, Automatic continuity of *-morphisms between non-normed topological *-algebras, *Pacific J. Math.* **147**, 1991, 57–70.

[51] M. Fragoulopoulou, private communication, 9 March 2006.

[52] D. H. Fremlin, *Topological Riesz spaces and measure theory*, Cambridge University Press, 1974.

[53] K. R. Goodearl, *Von Neumann regular rings*, Pitman, London, 1979.

[54] I. N. Herstein, *Topics in Ring Theory*, University of Chicago Press, Chicago, 1969.

[55] T. B. Hoover, Derivations, homomorphisms and operator ideals, *Proc. Amer. Math. Soc.* **62**, 1977, 293–298.

[56] J. Hou and M. Gao, Elements of rank one and multiplications on von Neumann algebras, *Chinese J. Contemp. Math.* **15**, 1994, 367–374.

[57] A. Inoue, Locally C*-algebra, *Mem. Fac. Sci, Kyushu Univ.* **25**, 1971, 197–235.

[58] A. Inoue, S. Ôta, Derivations on algebras of unbounded operators, *Trans. Amer. Math. Soc.* **261**, 1980, 567–577.

[59] N. Jacobson and C. E. Rickart, Jordan homomorphisms of rings, *Trans. Amer. Math. Soc.* **69**, 1950, 479–502.

[60] M. Joita, Locally von Neumann algebras, *Bull. Math. Soc. Sc. Math. Roumanie Tome* **42**, 1999, 51–64.

[61] M. Joita, Locally von Neumann algebras II, *Rend. Circ. Mat. Palermo* **51**, 2002, 83–94.

[62] R. V. Kadison, Isometries of operator algebras, *Ann. of Math.*, **54**, 1951, 325–338.

[63] R. V. Kadison and J. R. Ringrose, *Fundamentals of the theory of operator algebras*, Academic Press, New York, 1996.

[64] J. P. Kahane and W Żelazko, A characterization of maximal ideals in commutative Banach algebras, *Studia Math.* **29**, 1968, 339–343.

[65] I. Kaplansky, Algebras of type I, *Ann. of Math.* **56**, 1952, 460–472.

[66] I. Kaplansky, Modules over operator algebras, *Amer. J. Math.* **56**, 1953, 839–858.

[67] I. Kaplansky, Any orthocomplemented complete modular lattice is a continuous geometry, *Ann. of Math.* **61**, 1955, 524–541.

[68] I. Kaplansky, *Fields and rings*, second edition, Chicago Lectures in Mathematics, University of Chicago Press, Chicago-London, 1972.

[69] J. L. Kelley and I. Namioka, *Linear topological spaces*, Van Nostrand, Toronto-New York-London, 1963.

[70] E. Kreyszig, *Introductory functional analysis with applications*, New York, 1978.

[71] S. Kwapień, On the form of a linear operator in the space of all measurable functions, *Bull. de L'Acad. Pol. des Sciences* **21**, 1973, 951–954.

[72] L. E. Labuschagne, Composition operators on non-commutative L_p-spaces, *Expo. Math.* 17, 1999, 429–468.

[73] A. Mallios, *Topological Algebras. Selected Topics*, North-Holland, Amsterdam-New York-Oxford-Tokyo, 1986.

[74] M. Marcus and R. Purves, Linear transformations on algebras of matrices: The invariance of the elementary symmetric functions, *Canad. J. Math.* **11**, 1959, 383-396.

[75] M. Mathieu, Spectral Isometries, *Banach Center Publ.* **67**, 2005, 265–269.

[76] M. A. Muratov and V. I. Chilin, *-Algebras of unbounded operators affiliated with a von Neumann algebra, *Zapiski Nauchnyh Seminarov*, 2005, 183–197.

[77] I. Namioka and E. Asplund, A geometric proof of Ryll-Nardzewski's fixed point theorem, *Bull. Amer. Math. Soc.* **73**, 1967, 443–445.

[78] E. Nelson, Notes on non-commutative integration, *J. Funct. Anal.* **15**, 1974, 103–116.

[79] S. B. Ng and S. Warner, Continuity of positive and multiplicative functionals, *Duke Math. J.* **39**, 1972, 281–284.

[80] G. K. Pedersen, *C^*-algebras and their automorphism groups*, Academic Press, London - New York - San Francisco, 1979.

[81] N. C. Phillips, Inverse limits of C*-algebras, *J. of Operator Theory* **19**, 1988, 159–195.

[82] N. C. Phillips, Inner derivations on σ-C*-Algebras, *Math. Nachr.* **176**, 1995, 243–247.

[83] J. R. Ringrose, Automatic continuity of derivations of operator algebras, *J. London Math. Soc.* **5**, 1972, 432-438.

[84] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.

[85] B. Russo, Linear mappings of operator algebras, *Proc. Amer. Math. Soc.* **17**, 1966, 1019–1022.

[86] S. Sakai, *C**-*algebras and W**-*algebras*, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[87] S. Sakai, *Operator algebras in dynamical systems: the theory of unbounded derivations in C**-*algebras*, Cambridge University Press, Cambridge, 1991.

[88] K. Schmüdgen, Über LMC*-Algebren, *Math. Nachr.* **68**, 1975, 167–182.

[89] I. E. Segal, A non-commutative extension of abstract integration, *Ann. of Math.* **57**, 1953, 401-457.

[90] A. M. Sinclair, *Automatic continuity of linear operators*, London Mathematical Society Lecture Note Series, 1976.

[91] A. R. Sourour, Invertibility preserving linear maps on $\mathcal{L}(X)$, *Trans. Amer. Math. Soc.* **348**, 1996, 13–30.

[92] E. Stϕrmer, On the Jordan structure of C*-algebras, *Trans. Amer. Math. Soc.* **120**, 1965, 438–447.

[93] S. Stratila and L. Zsido, *Lectures on von Neumann algebras*, Abacus Press, Tunbridge wells, Kent, England, 1979.

[94] A. Stroh and G. West, τ-compact operators affiliated to a semifinite von Neumann algebra, *Proc. Roy. Irish Acad. Sect. A,* **93**, 1993, 73–86.

[95] M. Takesaki, *Theory of operator algebras I*, Springer-Verlag, Berlin-Heidelberg-New York, 1979.

[96] I. Tembo, Invertibility in the algebra of measurbale operators, Manuscript presented at the international conference “The eighteenth

international workshop on operator theory and applications", 2–6 July 2007, Potchefstroom, South Africa.

[97] N. E. Wegge-Olsen, *K-Theory and C^*-algebras*, Oxford University Press, Oxford-New York-Tokyo, 1993.

[98] M. Weigt, On automatic continuity of derivations on algebras of measurable operators, Abstracts of the international conference "Great Plains Operator Theory Symposium", 23–28 May 2006, Iowa City, USA.

[99] G. P. West, *Non-commutative Banach Function Spaces*, MSc Thesis, University of Cape Town, South Africa, 1990.

[100] G. West, Ideals of τ-measurable operators, *Quaestiones Math.* **18**, 1995, 333–344.

[101] F. B. Wright, A reduction for algebras of finite type, *Ann. of Math.* **60**, 1954, 560–570.

[102] F. J. Yeadon, Non-commutative L^p-spaces, *Math. Proc. Camb. Phil. Soc.* **77**, 1975, 91–102.

[103] B. S. Zakirov and V. I. Chilin, Abstract characterization of EW^*-algebras, *Funct. Anal. Appl.* **25**, 1991, 63–64.

[104] W. Żelazko, *Metric generalizations of Banach algebras*, Rozprawy Matematyczne, Warszawa, 1965.

[105] W. Żelazko, On maximal ideals in commutative m-convex algebras, *Studia. Math.* **58**, 1976, 291–298.

[106] W. Żelazko, When a unital F-algebra has all maixmal left (right) ideals closed?, *Studia. Math.* **175**, 2006, 279–284.

Index