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Convexity in quasi-metric spaces

by

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May 2012

A thesis presented for the degree of Doctor of Philosophy prepared under the supervision of Professor Hans-Peter Albert Künzi.

Abstract

Over the last fifty years much progress has been made in the investigation of the hyperconvex hull of a metric space. In particular, Dress, Espinola, Isbell, Jawhari, Khamsi, Kirk, Misane, Pouzet published several articles concerning hyperconvex metric spaces. The principal aim of this thesis is to investigate the existence of an injective hull in the categories of T_0-quasi-metric spaces and of T_0-ultra-quasi-metric spaces with nonexpansive maps. Here several results obtained by others for the hyperconvex hull of a metric space have been generalized by us in the case of quasi-metric spaces. In particular we have obtained some original results for the q-hyperconvex hull of a T_0-quasi-metric space; for instance the q-hyperconvex hull of any totally bounded T_0-quasi-metric space is joincompact. Also a construction of the ultra-quasi-metrically injective (= u-injective) hull of a T_0-ultra-quasi-metric space is provided. Furthermore, we show that the u-injective hull of a totally bounded T_0-ultra-quasi-metric space is joincompact. We prove that a commuting family of nonexpansive maps on a bounded q-hyperconvex T_0-quasi-metric space into itself has a common fixed point and the common fixed point set is q-hyperconvex. Some interesting examples related to the q-hyperconvex hull and u-injective hull are also discussed.

Acknowledgements

I would like to thank my supervisor, Professor H.-P. Künzi for his infinite support, patience, proper guidance, stimulating discussions on the subject and encouragement given to me throughout my studies.

I would also like to thank the Department of Mathematics and Applied Mathematics of the University of Cape Town in general, and in particular the Topology and Category Theory Research Group for all kinds of support and a marvelous research environment.

I would like to express my sincerest gratitude to the National Research Foundation (NRF) for providing a fund under Grant FA200622300009, for PhD Studies. Further financial assistance was provided by the University of Cape Town under an International Scholarship.

I wish to thank the Laboratory of Mathematical Logic and Theoretical Computer Science of the University of Siegen, which accepted me as a visitor in May 2011.

To my friends Patrick Abedi, Joshua Bosange, Alphonse Kalala, Eric Madudu, Charly Makitu, Zachariah Mushaandja, and Jean-Marie Rusimbuka thank you for your support and encouragement.

Dedication

I dedicate this thesis to my parents Pauline Konge Esangowale and Médard Amundala Odimo, to my wife Nathalie Ntandunzadi, my boy Liouville Olela Otafudu and my girl Christella Olela Otafudu, for their patience and understanding.

To my family and friends, who are willing to carry out any scientific work.

Declaration

I, OLIVIER OLELA OTAFUDU

hereby declare that this thesis is my own unaided work and is being submitted for the degree of Doctor of Philosophy at the University of Cape Town. It has not been submitted for any degree or examination to any other university.

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Contents

Introduction

The concept of "hyperconvexity" for metric spaces was first introduced by Aronszajn and Panitchpakdi [4](1956) who proved that a hyperconvex metric space is a nonexpansive absolute retract, i.e. it is a nonexpansive retract of any metric space in which it is isometrically embedded. The hyperconvex metric spaces called *injective* metric spaces were fully characterized by these authors. In particular they pointed out the connection to the classical Hahn-Banach Theorem in normed spaces. As the term "hyperconvex" suggests, this characterization involves a type of convexity which turns out to be stronger than metric convexity. Although this fact, and mainly the properties that are deduced from this concept, might tell us that since hyperconvex spaces enjoy nice properties there might be a few such spaces, Isbell [29] has shown that every metric space has an envelope (injective hull) which is hyperconvex. In particular, a metric space and its completion have exactly the same hyperconvex hull. The corresponding theory for normed linear spaces was developed by Gleason, Goodner, Kelley and Nachbin (see for instance [44]). In the nonlinear theory interesting fixed point theorems were proved for nonexpansive mappings in bounded hyperconvex metric spaces [32].

Isbell [29], Dress [20], and Chrobak and Larmore [12] independently established that every metric space (X, d) has an injective hull, which is compact if X is compact (in a more general framework a similar result was presented in [32]).

Such a space is called the *injective envelope* (denoted by $\epsilon(X)$) by Isbell, the *convex hull* by Chrobak and Larmore, and the *maximal tight extension* or *tight span* (denoted by T_X) by Dress.

Since one of the most interesting concepts in the theory of hyperconvex metric spaces is the notion of the injective hull of a metric space [29], it is very natural to wonder about the existence of an injective hull in the field of generalized metric spaces.

In this thesis we continued some investigations originally started by Salbany and Kemajou (a former student of my supervisor) on a reasonable concept of hyperconvexity in T_0-quasi-metric spaces (compare [33]). It is proved by Salbany in [53] that a T_0-quasi-metric space is q-hyperconvex if and only if it is metrically convex and q-hypercomplete.

The main purpose of this thesis is to investigate the hyperconvex hull of a T_0-quasi-metric space, which we call the **q-hyperconvex hull** of the T_0-quasi-metric space. The main result is that the q-hyperconvex hull of a T_0-quasi-metric space is unique up to isometries and we show that the q-hyperconvex hull of any totally bounded T_0-quasi-metric space is joincompact. Secondly we investigate the ultra-quasi-metrically injective hull of a T_0-ultra-quasi-metric space in parallel to the q-hyperconvex hull of a T_0-quasi-metric space. The main result is that a T_0-ultra-quasi-metric space is ultra-quasi-metrically injective [which we call **u-injective**] if and only if is it **q-spherically complete** (see Definition 5.3.1). We present an explicit construction of the u-injective hull of a T_0-ultra-quasi-metric space and we also show that the u-injective hull of a totally bounded T_0-ultra-quasi-metric space is joincompact. Note that the q-hyperconvex hull and the u-injective hull of an ultra-quasi-metric space need not coincide. And of course, we achieve all this by tackling several interesting questions, and as expected this leads us to some new open problems. The thesis is organized as described below.

A Brief Outline of The Thesis

Chapter 0. In the first chapter we give a brief overview of certain well-known basic concepts from the theory of quasi-pseudometric and ultra-quasi-pseudometric spaces. Some interesting examples of T_0-quasi-metric spaces and T_0-ultra-quasi-metric spaces (see Example 0.2.1) to be used throughout the thesis are presented.

Chapter 1. In this chapter we collect the fundamental results about hyperconvex metric spaces. We begin by defining metrical convexity, hyperconvexity and metrical injectivity. We recall that a metric space is hyperconvex if and only if it is injective (see Theorem 1.1.1) and any hyperconvex space is complete (see Proposition 1.1.1). In the second section we present Isbell's ideas about the construction of the injective hull of a metric space. In the last section we summarize the well-known tight span construction. We remember in Remark 1.3.1 that the tight span and injective hull or hyperconvex hull of a metric space are equivalent.

Chapter 2. The notion of hyperconvexity is well-known and developed for metric spaces (see previous chapter). In [8] Bayod and Martínez-Maurica presented a related notion (namely, spherical completeness) suitable for the category of ultra-metric spaces. In the first section, we begin by recalling, the extension property and the definition of spherical completeness. Then we recall that an ultra-metric space is spherically complete if and only if it has the extension property (see Theorem 2.1.1). The ultra-metrically tight extension is presented in the second section and in the last section we summarize the construction of the ultra-metrically injective envelope of an ultra-metric space.

Chapter 3. In this chapter we start our own investigations into the q-hyperconvex hull of a T_0-quasi-metric space. In the first section we introduce concepts of convexity in T_0-quasi-metric spaces, namely q-hyperconvexity in T_0-quasi-metric spaces and q-hypercompleteness in T_0-quasi-metric spaces. We provide an interesting example of a q-hyperconvex T_0-quasi-metric space (see Example 3.1.1) and we show that any q-hyperconvex T_0-quasi-metric space is bicomplete (see Corollary 3.1.3.) In the second section we study the space of nonnegative function pairs of a T_0-quasi-metric space. We define an extended T_0-quasi-metric D on the set of these function pairs (see Definition 3.2.1) and we show that the set of all function pairs equipped with D is q-hyperconvex. Therefore $D^s = \max\{D, D^{-1}\}$ is complete (see Remark 3.2.1). The third section deals with detailed investigations of the concept of a q-injective hull or q-hyperconvex hull of a T_0-quasi-metric space. We show that a T_0-quasi-metric space is q-hyperconvex if and only if it is q-injective (see Theorem 3.3.1). In Proposition 3.3.4 we show that for a metric space the hyperconvex hull is embedded in its q-hyperconvex hull and we successfully prove that the q-hyperconvex hull of a T_0-quasi-metric space is unique up to isometries. Let us mention that total boundedness is preserved by the q-hyperconvex hull of a T_0-quasi-metric space. At the end of this chapter we compute two examples (see Example 3.3.1 and Example 3.3.2) of q-hyperconvex hulls.

Chapter 4. This chapter deals with further properties of q-hyperconvex spaces. In Theorem 4.1.1, we show that the fixed point set of a nonexpansive map into itself of a bounded q-hyperconvex T_0-quasi-metric space is nonempty and q-hyperconvex. In the last two sections we introduce respectively the theory of approximation of fixed points in q-hyperconvex spaces and a new concept called external q-hyperconvexity.

Chapter 5. In this chapter we investigate the concept of the ultra-quasi-metrically injective hull of a T_0-ultra-quasi-metric space. In Section 1 we introduce the notion of strongly tight function pairs. We define an extended T_0-ultra-quasi-metric space N on the set of all such function pairs (see Definition 5.1.1). The second section deals with the construction of the injective hull of a T_0-ultra-quasi-metric space. In the third section we introduce the notion of q-spherical completeness. In Example 5.3.1 we provide an interesting example of a q-spherically complete space. We show that any q-spherically complete T_0-ultra-quasi-metric space is bicomplete (Proposition 5.3.2) and Theorem 5.3.1 states that a T_0-ultra-quasi-metric space is q-spherically complete if and only if it is u-injective. In Proposition 5.3.4 we show that for an ultra-metric space the injective hull is embedded in its u-injective hull and Theorem 5.3.6 establishes that the ultra-quasi-metrically injective hull of a T_0-ultra-quasi-metric space is unique up to isometries. In the last section we treat total boundedness in T_0-ultra-quasi-metric spaces.

Chapter 6. In this last chapter we conclude this work by reflecting on the main results of the thesis and highlight some connections of this current work with old work in the literature. Furthermore we mention some open problems which can constitute the topics for further research. The study of q-hyperconvex T_0-quasi-metric spaces and ultra-quasi-metrically injective T_0-ultra-quasi-metric spaces leads to many open problems. For instance one could investigate whether the theory of q-hyperconvex and u-injective spaces can be applied to asymmetrically normed linear spaces. The construction of the q-hyperconvex hull of an asymmetric normed linear space, with all of its related areas may be an interesting application of our theory to the theory of asymmetric normed linear spaces (see [15]).

The notion of T-theory was introduced by A. Dress, V. Moulton and W. Terhalle [22]. T-theory is the name that they adopted for the theory of trees, injective envelopes of metric spaces, and all areas that are connected with those topics, which have been developed over the last 27–32 years. Similarly one can guess that one could also investigate a T-theory of oriented trees.

Chapter 0

Preliminaries

In this chapter, firstly we recall the definition, notation and some properties of quasi-metric spaces. We refer the reader to [27] or [37]. Secondly we also summarize facts about the ultra-quasi-metric spaces which are often called *non-achimedean quasi-pseudometric spaces* (see e.g. [26]) and we establish some interesting examples (Examples 0.2.1, Examples 0.2.3) and results (Lemma 0.2.1, Corollary 0.2.2) that help us to construct the ultra-quasi-metrically injective hull of a T_0-ultra-quasi-metric space.

0.1 Concepts of quasi-pseudometrics

We start this section by recalling some basic concepts from the theory of quasi-pseudometric spaces.

0.1.1 Definition. *Let X be a set and let $d : X \times X \longrightarrow [0, \infty]$ be a function mapping into the set $[0, \infty]$ of the non-negative reals plus infinity. Then d is called an* extended quasi-pseudometric *on X if* *(a)* $d(x,x) = 0$ *whenever* $x \in X$,

(b) $d(x,z) \leq d(x,y) + d(y,z)$ *whenever* $x, y, z \in X$

(where for d *attaining the value* ∞ *the triangle inequality is interpreted in the obvious way).*

We shall say that d *is an (extended)* T_0-quasi-metric *provided that* d *also satisfies the following condition: For each* $x, y \in X$,

$d(x,y) = 0 = d(y,x)$ *implies that* $x = y$.

Furthermore we shall say that d *is a* quasi-pseudometric *provided that* d *maps* $X \times X$ *into* $[0, \infty)$.

We are mainly interested in quasi-pseudometrics in the following.

0.1.1 Remark. *Let* d *be a(n extended) quasi-pseudometric on* X, *then* $d^{-1} : X \times X \longrightarrow [0, \infty]$ *defined by* $d^{-1}(x,y) = d(y,x)$ *whenever* $x, y \in X$ *is also a(n extended) quasi-pseudometric, called the* conjugate quasi-pseudometric *of* d. *As usual, a(n extended) quasi-pseudometric* d *on* X *such that* $d = d^{-1}$ *is called a(n extended)* pseudometric. *Note that for any quasi-pseudometric* d, $d^s = \max\{d, d^{-1}\} = d \vee d^{-1}$ *is a pseudometric.*

Let (X, d) be a(n extended) quasi-pseudometric space. For each $x \in X$ and $\epsilon > 0$, $B_d(x, \epsilon) = \{y \in X : d(x,y) < \epsilon\}$ denotes the open ϵ ball at x. The collection of all "open" balls yields a base for a topology $\tau(d)$. It is called the *topology induced by* d *on* X.

Similarly we set for each $x \in X$ and $\epsilon > 0$, $C_d(x, \epsilon) = \{y \in X : d(x,y) \leq \epsilon\}$. Note that this set is $\tau(d^{-1})$-closed, but not $\tau(d)$-closed in general.

If $a, b \in [0, \infty)$ we shall put $a \dot{-} b = \max\{a - b, 0\}$. The set $[0, \infty)$ will be equipped with the T_0-quasi-metric $r(x,y) = x \dot{-} y$ whenever $x, y \in [0, \infty)$.

A map $f : (X, d) \longrightarrow (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called an *isometric map* or *isometry* provided that $e(f(x), f(y)) = d(x, y)$ whenever $x, y \in X$. Observe that if $f : X \longrightarrow Y$ is an isometric map between two quasi-pseudometric spaces X and Y and X is a T_0-quasi-metric space, then f is injective (see [36, Lemma 4]).

Two quasi-pseudometric spaces (X, d) and (Y, e) will be called *isometric* provided that there exists a bijective isometry $f : (X, d) \longrightarrow (Y, e)$.

A map $f : (X, d) \longrightarrow (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called *nonexpansive* provided that $e(f(x), f(y)) \leq d(x, y)$ whenever $x, y \in X$.

The following result is well-known and immediate (see [52]).

0.1.1 Proposition. *A T_0-quasi-metric space (X, d) is bicomplete if and only if the metric space (X, d^s) is complete.*

0.2 Concept of ultra-quasi-pseudometrics

Let us mention that the ultra-quasi-pseudometric spaces should not be confused with *quasi-ultra-metric spaces* as they are discussed in the theory of dissimilarities (see e.g. [19]).

In this thesis we shall consider $\sup A$ for many subsets $A \subseteq [0, \infty)$. In particular recall that $\sup A = 0$, if $A = \emptyset$.

0.2.1 Definition. *Let X be a set and let $u : X \times X \longrightarrow [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the nonnegative reals. Then u is an ultra-quasi-pseudometric on X if*

(a) $u(x, x) = 0$ whenever $x \in X$,

(b) $u(x,z) \leq \max\{u(x,y), u(y,z)\}$ *whenever* $x, y, z \in X$.

Note that the so-called conjugate u^{-1} *of* u, *where* $u^{-1}(x,y) = u(x,y)$ *whenever* $x, y \in X$, *is an ultra-quasi-metric, too.*

If u *also satisfies the condition*

(c) for any $x, y \in X$, $u(x,y) = 0 = u(y,x)$ *implies that* $x = y$, *then* u *is called a* T_0*-ultra-quasi-metric.*

Observe that then $u^s = u \vee u^{-1}$ *is an ultra-metric on* X.

0.2.1 Remark. *In [51, Definition 1],* Seda *mentions that a* T_0*-quasi-metric space* (X,d) *is called a* T_0-ultra-quasi-metric *if it satisfies the strong triangle inequality:* $d(x,y) \leq \max\{d(x,z), d(z,y)\}$ *for all* $x, y, z \in X$.

0.2.1 Example. *Let* $X = [0,\infty)$ *be equipped with* $n(x,y) = x$ *if* $x > y$ *and* $n(x,y) = 0$ *if* $x \leq y$. *It is easy to check that* (X,n) *is a* T_0*-ultra-quasi-metric space: We verify the strong triangle inequality* $n(x,z) \leq \max\{n(x,y), n(y,z)\}$ *whenever* $x, y, z \in X$. *The case that* $n(x,y) = x$ *is trivial, since then* $n(x,z) \leq n(x,y)$. *Similarly the case* $n(x,y) = 0$ *and* $n(y,z) = y$ *are obvious, since then* $x \leq y$ *and* $n(x,z) \leq n(y,z)$. *In the remaining case that* $n(x,y) = 0 = n(y,z)$, *we see by transitivity of* $\leq$ *that we have that* $x \leq z$, *and thus* $n(x,z) = 0$. *It is also obvious that* n *satisfies that* T_0*-condition* (c).

Note also that for $x, y \in [0,\infty)$ *we have* $n^s(x,y) = \max\{x,y\}$ *if* $x \neq y$ *and* $n(x,y) = 0$ *if* $x = y$. *Observe that the ultra-metric* n^s *is complete on* $[0,\infty)$. *Note that* 0 *is the only non-isolated point of* $\tau(n^s)$. *Indeed* $A = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ *is a compact subspace of* $([0,\infty), n^s)$.

In some cases we need to replace $[0,\infty)$ by $[0,\infty]$ (where for an ultra-quasi-pseudometric u attaining the value ∞ the strong triangle inequality (b) is interpreted in the obvious way). In such a case we shall speak of an *extended ultra-*

quasi-pseudometric. In the following we sometimes apply concepts from the theory of (ultra-)quasi-pseudometrics to extended (ultra-)quasi-pseudometrics (without changing the usual definitions of these concepts).

0.2.1 Lemma. *[9, Proposition 2.1] Let $a, b, c \in [0, \infty)$. Then the following conditions are equivalent:*

(a) $n(a, b) \leq c$*;*

(b) $a \leq \max\{b, c\}$.

Proof. $(a) \Rightarrow (b)$: In order to reach a contradiction, suppose that $a > \max\{b, c\}$. Since $a > b$, then $n(a, b) = a \leq c$ by (a) and the definition of n. Thus $a \leq \max\{b, c\} < a$—a contradiction.

$(b) \Rightarrow (a)$: In order to reach a contradiction suppose that $n(a, b) > c$. Then $n(b, c) = a$ and $a > b$ by definition of n. Thus $a > c$, which implies that $a > \max\{b, c\}$. But $a \leq \max\{b, c\}$ by (b)—a contradiction. □

0.2.1 Corollary. *Let (X, u) be an ultra-quasi-pseudometric space. Consider $f : X \to [0, \infty)$ and let $x, y \in X$. Then the following are equivalent:*

(a) $n(f(x), f(y)) \leq u(x, y)$*;*

(b) $f(x) \leq \max\{f(y), u(x, y)\}$. □

0.2.2 Corollary. *Let (X, u) be an ultra-quasi-metric space.*

(a) Then $f : (X, u) \to ([0, \infty), n)$ is a contracting map if and only if $f(x) \leq \max\{f(y), u(x, y)\}$ whenever $x, y \in X$.

(b) Then $f : (X, u) \to ([0, \infty), n^{-1})$ is a contracting map if and only if $f(y) \leq \max\{f(x), u(x, y)\}$ whenever $x, y \in X$. □

0.2.2 Example. *Let $X = \{0, 1\}$ be equipped with the discrete metric $u(x, x) =$*

0 whenever $x \in X$ and $u(x,y) = 1$ whenever $x \neq y$. Observe that u is an ultra-metric on X.

0.2.3 Example. *Let $X = \{0,1\}$ endowed with $u(0,1) = 0$, $u(1,0) = 1$, $u(0,0) = 0$ and $u(1,1) = 0$. Note that u is a T_0-ultra-quasi-metric.*

Chapter 1

On hyperconvex metric spaces

It is hard to believe that hyperconvex metric spaces have already been investigated for more than fifty years. Let us recall that the concept of hyperconvex metric spaces was introduced in [4] and was investigated later by many authors (see [20],[25],[29], [30], [31], [34]).

In this chapter we mainly give an overview of the terminology and some elementary results about hyperconvex metric spaces, most of which already appear in the literature, to be generalized throughout the thesis to quasi-metric spaces. For more on these basics, the reader is referred to, among others, [25] and [34].

1.1 Hyperconvexity

In this section we summarize the concept of hyperconvexity.

The following recalls the definition of a metrically convex space.

1.1.1 **Definition.** *Let (X, d) be a metric space. We say that (X, d) is* metrically convex *or* Menger convex *if for any points* $x, y \in X$ *and nonnegative real numbers* α *and* β *such that* $d(x, y) \leq \alpha + \beta$, *there exists* $z \in X$ *such that* $d(x, z) \leq \alpha$ *and* $d(z, y) \leq \beta$, *or equivalently* $z \in C_d(x, \alpha) \cap C_d(y, \beta)$.

1.1.1 Remark. *([25]) Let* (X, d) *be a metric space. Using the triangle inequality, we have* $C_d(x, \alpha) \cap C_d(y, \beta) \neq \emptyset$ *implies* $d(x, y) \leq \alpha + \beta$ *for any* $x, y \in X$ *and nonnegative real numbers* α, β. *The converse is true on the real line and corresponds to the concept of "metrically convex" in a metric space.*

We next recall the definition of an injective metric space.

1.1.2 Definition. *([34]) A metric space* (X, d) *is said to be* injective *if it has the following extension property: Whenever* Z *is a subspace of a metric space* Y *and* $f : Z \longrightarrow X$ *is nonexpansive, then* f *has a nonexpansive extension* $\tilde{f} : Y \longrightarrow X$.

We next define the concept of a hyperconvex metric space which was first introduced by Aronszajn and Panitchpakdi (see [4]). It will be generalized to quasi-metric spaces later (see Definition 3.1.1).

1.1.3 Definition. *([34]) A metric space* (X, d) *is called hyperconvex if for any indexed family of closed balls* $C_d(x_i, r_i), i \in I$, *of* X *which satisfy* $d(x_i, x_j) \leq r_i + r_j$ *where* $i, j \in I$, *it is necessarily the case that* $\bigcap_{i \in I} C_d(x_i, r_i) \neq \emptyset$.

Any hyperconvex metric space is complete and similarly later it will be shown that any q-hyperconvex T_0-quasi-metric space will be bicomplete (see Corollary 3.1.3).

1.1.1 Proposition. *([34, Proposition 4.4]) Let* (X, d) *be a hyperconvex metric space. Then* (X, d) *is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in a hyperconvex space (X, d). For each integer n, set $\rho_n = \sup\{d(x_n, x_m) : m > n\}$. Then for $m > n$, $d(x_n, x_m) \leq$

$\rho_n \leq \rho_n + \rho_m$. Thus by hyperconvexity there exists a point z in the intersection $\bigcap_{n\geq 1} C_d(x_n, \rho_n)$. Since $\lim_{n\longrightarrow\infty}\rho_n = 0$, then $\lim_{n\longrightarrow\infty}x_n = z$. □

1.1.1 Theorem. *([25, Theorem 4.2]) Let (X, d) be a metric space. The following statements are equivalent:*

(a) (X, d) is hyperconvex.

(b) (X, d) is injective.

Proof. [25, Theorem 4.2] See also Theorem 3.3.1, where this proof has been generalized in the case of quasi-metric spaces. □

1.2 Isbell's hyperconvex hull of a metric space

The notion of the hyperconvex hull was introduced by Isbell in [29]. In this section, we will discuss Isbell's ideas for the construction of the hyperconvex hull of a metric space. (In the light of Theorem 1.1.1 we may use terms "injective" and "hyperconvex" interchangeably, see [25], [29] and [34]).

1.2.1 Definition. *Let (X, d) be a metric space. For any $x \in X$, we define the positive real-valued function $h_x : X \longrightarrow [0, \infty)$ by $h_x(y) = d(x, y)$ whenever $y \in X$.*

1.2.1 Remark. *Let (X, d) be a metric space. By using the triangle inequality and for any $a \in X$, we get $d(x, y) \leq h_a(x) + h_a(y)$ and $h_a(x) \leq d(x, y) + h_a(y)$ for all $x, y \in X$.*

1.2.2 Definition. *([20]) Let (X, d) be a metric space. A function $f : X \to [0, \infty)$ is called* tight *if $d(x, y) \leq f(x) + f(y)$ whenever $x, y \in X$.*

1.2.1 Lemma. *Let (X, d) be a metric space and a function $f : X \longrightarrow [0, \infty)$ be such that $d(x, y) \leq f(x) + f(y)$, for all $x, y \in X$. Let $a \in X$. If $f(x) \leq h_a(x)$ for all $x \in X$ then $f = h_a$.*

Proof. For a fixed $a \in X$, assume that $f(x) \leq h_a(x)$, for all $x, y \in X$. Indeed, first we have $f(a) \leq h_a(a) = 0$, which implies $f(a) = 0$. Then $h_a(x) = d(x, a) \leq f(x) + f(a) = f(x)$, for all $x \in X$, combined with $f(x) \leq h_a(x)$, we get $f(x) = h_a(x)$, whenever $x \in X$. □

1.2.3 Definition. *Let (X, d) be a metric space and $A \subseteq X$. Let a function $f : A \longrightarrow [0, \infty)$ be such that $d(x, y) \leq f(x) + f(y)$ for all $x, y \in A$. We say that f is* extremal*, if $g : A \longrightarrow [0, \infty)$ such that $d(x, y) \leq g(x) + g(y)$ and $g(x) \leq f(x)$ for all $x \in A$, then we must have $f = g$.*

1.2.4 Definition. *Let (X, d) be a metric space and $A \subseteq X$. We denote by $\epsilon(A)$, the set of all extremal functions defined on A,which will be called the* injective envelope *or* hyperconvex hull *of A.*

In particular from Lemma 1.2.1 $h_a \in \epsilon(A)$ for any $a \in A$.

1.2.2 Remark. *Let (X, d) be a metric space and $A \subseteq X$. Consider the map $e : A \longrightarrow \epsilon(A)$, defined by $e(a) = h_a$ for all $a \in A$. Then e is an isometry. In other words, we have*

$$d_\infty(e(a), e(b)) = \sup_{x \in A} \mid h_a(x) - h_b(x) \mid = \sup_{x \in A} \mid d(a, x) - d(b, x) \mid = d(a, b).$$

So A and $e(A)$ are isometric spaces and we may identify A and $e(A)$ via $a \longleftrightarrow e(a)$.

To show that $\epsilon(A)$ is hyperconvex, we will need the following lemma.

1.2.2 Lemma. *([34, Lemma 4.4]) Let (X, d) be a metric space and $A \subseteq X$.*

Let $r : A \longrightarrow [0, \infty)$ be such that $d(x, y) \leq r(x) + r(y)$ for any $x, y \in A$.

Then there exists $R : X \longrightarrow [0, \infty)$ which extends r such that

$d(x, y) \leq R(x) + R(y)$ for any $x, y \in X$.

Moreover, there exists an extremal function f defined on X such that $f(x) \leq R(x)$ for any $x \in X$.

Proof. The proof will be presented for the more general case of quasi-metric spaces in Proposition 3.3.7. □

The next proposition shows that $\epsilon(A)$ is compact if A is compact. A similar result is given for quasi-metric spaces (see Corollary 3.3.2), but the proof is very different.

1.2.1 Proposition. *([34, Proposition 4.6]) Let (X, d) be a metric space and $A \subseteq X$. The following statements are true.*

(1) If $f \in \epsilon(A)$, then $f(x) \leq d(x, y) + f(y)$ for all $x, y \in A$.

Moreover, we have $f(x) = \sup_{y \in A} \mid f(y) - h_x(y) \mid = d_\infty(f, e(x))$.

(2) For any $f \in \epsilon(A)$, $\delta > 0$, and $x \in A$, there exists $y \in A$ such that $f(x) + f(y) < d(x, y) + \delta$.

(3) If A is compact, then $\epsilon(A)$ is compact.

(4) If s is an extremal function on the metric space $\epsilon(A)$, then $s \circ e$ is extremal on A.

Proof. See [34, Proposition 4.6] for the proof of $(1), (2)$ and (4). They will be presented later in the quasi-metric setting.

(3). From the property 1, we get $\mid f(x) - f(y) \mid \leq d(x, y)$ for any $x, y \in A$. This implies $\epsilon(A) \subseteq Lip_1(A)$, where $Lip_1(A)$ is the space of all Lipschitzian real-valued functions with Lipschitz constant equal 1. Hence, the set of extremal functions is equicontinuous. Also, it is quite easy to show that the pointwise-limit of extremal functions is an extremal function. Since A is compact, the Arzela-Ascoli theorem implies that $\epsilon(A)$ is compact. □

1.2.3 Lemma. *([20]) Let (X, d) be a metric space. For any $f, g \in \epsilon(X)$ we have:*

$$d_\infty(f, g) = \sup\{f(x) - g(x) : x \in X\}.$$

Proof. See [20, Theorem 3]. □

The following proposition makes a link between Isbell's ideas and hyperconvexity.

1.2.2 Proposition. *([34, Proposition 4.7]) Let (X, d) be a metric space and $A \subseteq X$. The following statements are true.*

(1) $\epsilon(A)$ is hyperconvex.

(2) $\epsilon(A)$ is an injective envelope of A, that is, no proper subset of $\epsilon(A)$ which contains A (metrically) is hyperconvex. Moreover, any hyperconvex metric space H which contains A metrically and is minimal (i.e., any proper subset of H which contains A is not hyperconvex), is isometric to $\epsilon(A)$.

Proof. See [34, Proposition 4.7]. The proof for the quasi-metric case is given in Proposition 3.3.6. □

In the following we recall the definition of the hyperconvex hull.

1.2.5 Definition. *Let (X, d) be a metric space and $A \subseteq X$. The hyperconvex space $\epsilon(A)$ is called the* hypercovex hull *of (X, d).*

We next mention the definition of a fixed point set.

1.2.6 Definition. *([25]) Let (X, d) be a metric space. If $T : X \longrightarrow X$ is a map, then $x \in X$ is a fixed point of T if $T(x) = x$.*

Moreover we denote the fixed point set *of T by $Fix(T)$, where $Fix(T) = \{x \in X : T(x) = x\}$.*

The next theorem describes an important connection between hyperconvexity and fixed point theory.

1.2.1 Theorem. *([25, Theorem 6.1]) Let H be a bounded hyperconvex metric space. Any nonexpansive map $T : H \longrightarrow H$ has a fixed point. Moreover, the fixed point set of T is hyperconvex.*

Proof. [25, Theorem 6.1]. See also Theorem 4.1.1, where this proof is presented in the case of quasi-metric spaces. □

We have the following uniqueness property of hyperconvex hulls (the second statement of Proposition 1.2.2) which we prove next using fixed point theory.

1.2.3 Proposition. *([25, Proposition 5.6]) Let (X, d) be a metric space. Assume that H_1, H_2 are two hyperconvex hulls of (X, d). Then H_1 and H_2 are isometric.*

Proof. Since H_1 and H_2 are hyperconvex and from Theorem 1.1.1, we see that they are injective. So there exists a nonexpansive map $T_1 : H_1 \longrightarrow H_2$ such that the restriction of T_1 to X is the identity map. Keep in mind that H_1 as well as H_2 contains X isometrically. Similarly, there exists another nonexpansive map $T_2 : H_2 \longrightarrow H_1$ such that the restriction of T_2 to X is the identity map.

The map $T_1 \circ T_2$ is defined on H_2 into H_2. Its restriction to X is the identity map of X. So we have $X \subseteq \text{Fix}(T_1 \circ T_2)$, by Theorem 1.2.1, $T_1 \circ T_2$ is nonexpansive and $\text{Fix}(T_1 \circ T_2)$ is nonexpansive and contains X. By minimality of H_2, we have $\text{Fix}(T_1 \circ T_2) = H_2$, hence $T_1 \circ T_2$ is the identity map of H_2. By a similar argument we have $T_2 \circ T_1$ is the identity map of H_1. So T_1 and T_2 are inverse of each other and are nonexpansive. Therefore both are isometric maps. □

1.2.3 Remark. *Though hyperconvex hulls are not unique, the previous proposition shows that up to an isometry that they are indeed unique. So if (X, d)*

is a metric space and $\epsilon(X)$ its hyperconvex hull such that X is subset of a hyperconvex space H, then $X \subseteq \epsilon(X) \subseteq H$.

1.3 The T-construction

In this section we are going to discuss the notion of the T-construction. The notion of the T-construction was introduced by Dress (see [20], [21] and [22]) independently of Isbell's ideas (see Section 1.2 and [29]).

1.3.1 Definition. *([20]) Let (X, d) be a metric space. We denote by P_X and T_X the sets*

$$P_X := \{f : X \to \mathbb{R} \mid d(x, y) \leq f(x) + f(y) \quad \textit{for all} \quad x, y \in X\}$$

and

$$T_X := \{f : X \to \mathbb{R} \mid f(x) = \sup_{y \in X}(d(x, y) - f(y)) \quad \textit{for all} \quad x, y \in X\}.$$

For any $f, g \in T_X$,

$$d_\infty(f, g) := \sup_{x \in X}(\mid f(x) - g(x) \mid).$$

We next recall the tight extension definition.

1.3.2 Definition. *([20]) Let (X, d) be a metric space. An extension $(Y, d^{'})$ of (X, d) is defined to be a* tight extension *if for any pseudo-metric $d^{"} : Y \times Y \to \mathbb{R}$ satisfying the conditions*

$$d^{"}(x_1, x_2) = d(x_1, x_2), \quad \textit{whenever} \quad x_1, x_2 \in X$$

and

$$d^{"}(y_1, y_2) \leq d^{'}(y_1, y_2), \quad \textit{whenever} \quad y_1, y_2 \in Y,$$

one necessarily has $d^{"}(y_1, y_2) = d^{'}(y_1, y_2)$, whenever $y_1, y_2 \in Y$.

The next theorem characterizes the tightness of a metric extension. For any metric space (X, d) there exists an essentially unique maximal tight extension, which will be called the *tight span* or *metric envelope* of (X, d).

1.3.1 Theorem. *Let* (X, d) *be a metric space. Let* $(Y, d^{'})$ *be an extension of* (X, d)*. Then* $(Y, d^{'})$ *is tight if and only if*

$$d^{'}(y_1, y_2) = \sup_{x_1, x_2 \in X}(d(x_1, x_2) - d(x_1, y_1) - d(y_2, x_2))$$

holds whenever $y_1, y_2 \in Y$*.*

Proof. [20, Theorem 1] See also Remark 3.3.2, where this proof has been generalized in the case of quasi-metric spaces. □

In the following we make a connection between the injective hull or hyperconvex hull and the tight span or metric envelope.

1.3.1 Remark. *Theorem 1.1.1, Proposition 1.2.2 and Remark 1.2.2 tell us that for any metric space* (X, d) *the map* $h_X : X \to T_X$ *is an injective envelope of* X*, that is,*

(i) *For any metric space* X *there exists an injective envelope, and*

(ii) T_X *together with the map* h_X *can be characterized–up to canonical isomorphism–by the property of providing that injective envelope.*

Chapter 2

On ultra-metrically injective spaces

In this chapter we discuss the theory of ultra-metrically injective spaces. Ultra-metric spaces used to raise interest only among pure mathematicians; this situation has changed recently, when the concept of ultra-metricity appeared in a natural way in several physical modellings of natural phenomena (see [42]). Let us mention that ultra-metric spaces and hyperconvex metric spaces share many common properties, yet they are quite different in very distinctive ways. The most striking similarity has to do with the injective extension property; the most striking difference is likely the fact that while hyperconvex metric spaces are always metrically convex, nontrivial ultra-metric injective spaces do not exist [34, p. 116].

2.1 Spherical completeness

In this section we are going to define ultra-metric spaces, the extension property and spherical completeness.

The following definition can be found in ([8], [34] and [51]).

2.1.1 Definition. *A metric space* (X, d) *is an ultra-metric space if, in addition to the usual metric axioms, the following property called* strong triangle inequality *holds for each* $x, y, z \in X$*:*

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$

This condition strengthens the triangle inequality.

2.1.2 Definition. *([34]) An ultra-metric space* (X, d) *is said to have the* extension property *if given any ultra-metric space* (Y, e) *and any subspace* Z *of* Y*, every nonexpansive mapping* $f : Z \longrightarrow X$ *has a nonexpansive extension* $F : Y \longrightarrow X$*.*

We next recall the definition of an ultra-metric spherically complete space.

2.1.3 Definition. *([8]) An ultra-metric space* (X, d) *is said to be* spherically complete *if every collection of closed balls in* X *with the binary intersection property has a nonempty intersection.*

The next theorem characterizes spherical completeness and is similar to Theorem 1.1.1.

2.1.1 Theorem. *([34, Theorem 5.10]) An ultra-metric space is spherically complete if and only if it has the extension property.*

Proof. [34, Theorem 5.10]. The proof has been generalized to the ultra-quasi-metric case and is given in Theorem 5.3.1. □

2.2 Ultra-metrically tight extensions

The notion of an ultra-metrically tight extension was introduced by Bayod and Martínez-Maurica in [8].

2.2.1 Definition. *Let* (Y, d) *be an ultra-metric space and* $X \subseteq Y$*. We say that* Y *is an* ultra-metrically tight extension *of* X *if for every map* $d^{'} : Y \times Y \longrightarrow [0, \infty)$*, the following properties imply* $d^{'} = d$ *over* $Y \times Y$*:*

(a) $d^{'}(x, y) = d(x, y)$ *whenever* $x, y \in Y$*;*

(b) $d^{'}(x, y) \leq \max\{d^{'}(x, z), d^{'}(z, y)\}$ *whenever* $x, y, z \in Y$*;*

(c) $d^{'}(y, y^{'}) \leq d(y, y^{'})$ *whenever* $y, y^{'} \in Y$*;*

(d) $d^{'}(x, x^{'}) = d(x, x^{'})$ *whenever* $x, x^{'} \in X$*.*

The following recalls the definition of a basis of a metric space (see [8]).

2.2.2 Definition. *Let* (Y, d) *be a metric space and* $X \subseteq Y$*. Then* X *is said to be a* basis *of* Y *if* $d(x, z) = d(x, y)$ *whenever* $x \in X$ *implies* $z = y$*.*

The following recalls the definition of an immediate extension of a metric space (see [8]).

2.2.3 Definition. *Let* (Y, d) *be a metric space and* $X \subseteq Y$ *such that* X *is a closed subset of* (Y, d)*. Then* Y *is said to be an* immediate extension *of* X *if* $d(y, X) < d(y, x)$ *whenever* $x \in X$ *and* $y \in Y \setminus X$*.*

The proof of the following result can be found in [8, Theorem 1].

2.2.1 Theorem. *Let* (Y, d) *be an ultra-metric space and let* X *be a closed subspace of* Y*. Then, the following properties are equivalent:*

(1) Y *is an ultra-metrically tight extension of* X.

(2) X *is a basis of* Y *and* Y *is an immediate extension of* X.

(3) $C_d(y, d(y, X)) = \{y\}$ *whenever* $y \in Y \setminus X$.

Proof. See [8, Theorem 1]. Compare Problem 6.1.1, where we hope that it should be generalized to the case of ultra-quasi-metric spaces in future research. □

The following is another characterization of spherical completeness (see [8, Theorem 2]).

2.2.2 Theorem. *Let* (X, d) *be a complete ultra-metric space. Then the following properties are equivalent:*

(1) (X, d) *is spherically complete.*

(2) *Every* X*-valued uniformly continuous mapping from a subset of any ultra-metric space* Z *can be extended, with the same modulus of continuity (see below), to* Z.

(3) (X, d) *is ultra-metrically injective.*

(4) *Every isometric embedding* $T : Y \longrightarrow X$ *can be extended to an isometric embedding* $\overline{T} : Z \longrightarrow X$ *where* Z *is any ultra-metrical extension.*

(5) (X, d) *has no proper ultra-metrically tight extension of* Y.

(6) (X, d) *has no proper immediate extension.*

Proof. [8, Theorem 2] (1) $\Longrightarrow$ (2). Let $T : Y \longrightarrow X$ be a uniformly continuous mapping with $Y \subseteq Z$ and consider the modulus of continuity of T, i.e., the map $\delta_T : (0, \infty) \to (0, \infty)$ defined by

$$\delta_T(\epsilon) = \sup\{d(T(y), T(y^{'})) : y, y^{'} \in Y, d(y, y^{'}) \leq \epsilon\}.$$

Let Θ be the set of all possible X-valued uniformly continuous maps S defined on a subspace of Z which contains Y and satisfy: $S|_Y = T$ and $\delta_S = \delta_T$.

For the usual ordering, Θ satisfies the hypothesis of Zorn's lemma. Let $\overline{T} : \overline{Y} \to X$ be a maximal element of Θ. We only need to prove that $\overline{Y} = Z$. Otherwise, take $z \in Z \setminus \overline{Y}$ and consider the collection of balls in X,

$$C_d[\overline{T}(y), \delta_T(d(z, y))], \quad y \in \overline{Y}.$$

For any $y, y^{'} \in \overline{Y}$ we have

$$d(\overline{T}(y), \overline{T}(y^{'})) \leq \delta_T(d(y, y^{'})) \leq \delta_T(\max\{d(y, z), d(y^{'}, z)\})$$

$$= \max\{\delta_T(d(y, z), \delta_T(d(y^{'}, z))\}.$$

So, by the spherical completeness of X, we can take an $x \in \bigcap C_d[\overline{T}(y), \delta_T(d(z, y))]$. Now we extend $\overline{T}$ to $T^{'} : \overline{Y} \cup \{z\} \to X$ by defining $T^{'}(z) = x$. Then $T^{'} \in \Theta$, and this contradicts the maximality of $\overline{T}$.

(2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (5) are obvious.

(5) $\Longrightarrow$ (6). If Y is a proper immediate extension of X, take $y \in Y \setminus X$. By Theorem 2.2.1, $X \cup \{y\}$ is a proper ultra-metrically tight extension of X.

(6) $\Longrightarrow$ (1). Assume that X is not spherically complete, and let $C_d(x_\alpha, r_\alpha)$ $(\alpha \in A)$ be a system of closed balls in X with $C_d(x_\alpha, r_\alpha) \cap C_d(x_\beta, r_\beta) \neq \emptyset$ and thus $C_d(x_\alpha, r_\alpha) \subseteq C_d(x_\beta, r_\beta)$ or $C_d(x_\beta, r_\beta) \subseteq C_d(x_\alpha, r_\alpha)$ whenever $\alpha, \beta \in A$, but $\bigcap C_d(x_\alpha, r_\alpha) = \emptyset$. Note that for $x \in X \setminus (C_d(x_\alpha, r_\alpha) \cup C_d(x_\beta, r_\beta))$ one has $d(x, x_\alpha) = d(x, x_\beta)$. Hence we may define an immediate extension $Y = X \cup \{y\}$ of X by choosing for each $x \in X$ some $\alpha \in A$ with $x \notin C_d(x_\alpha, r_\alpha)$ which exists in view of $\bigcap C_d(x_\alpha, r_\alpha) = \emptyset$, and then putting $d(y, x) = d(x_\alpha, x)$. □

2.3 Ultra-metrically injective envelopes

In this section we recall the construction of the ultra-metrically injective envelope for any given ultra-metric space. This direct construction follows the spirit of similar ideas used by Dress in the context of arbitrary metric spaces and it is similar for the case of ultra-metric spaces to Section 1.3.

2.3.1 Definition. *Let (X, d) be an ultra-metric space. Then we define UP_X and UT_X by:*

$$UP_X = \{f : X \longrightarrow [0, \infty) \quad \textit{for all} \quad x, y \in X \quad \textit{such that} \quad d(x, y) \leq \max\{f(x), f(y)\}\}$$

and

$$UT_X = \{f \in UP_X \quad \textit{for all} \quad x \in X \quad \textit{such that} \quad f(x) = \sup\{d(x, y) : f(y) < d(x, y)\}\}$$

(where the last sup *is understood to be zero in case $f(y) \geq d(x, y)$ for every $y \in X$).*

The following lemmas are useful in the proof of Theorem 2.3.1.

2.3.1 Lemma. *([8, Lemma 3]) Let (X, d) be an ultra-metric space. Then the following are true:*

(a) For any $z \in X$, we define the positive real-valued function $h_z : X \longrightarrow [0, \infty)$ by $h_z(x) = d(x, z)$. Then $h_z \in UT_X$.

(b) Let $f \in UT_X$. Then f is exactly a minimal element of UP_X with respect to the pointwise ordering.

Proof. [8, Lemma 3]. □

The following lemma is needed.

2.3.2 Lemma. *([8, Lemma 4]) Let* (X, d) *be an ultra-metric space. Then the following are true:*

(a) If $f \in UP_X$ *and* $x, y \in X$*, the map* $h : X \longrightarrow [0, \infty)$ *defined by* $h(z) = f(z) \quad \text{if} \quad z \neq x$, $h(x) = \max\{d(x, y), f(y)\}$*, is also in* UP_X.

(b) For $f \in UT_X$ *and* $x, y \in X$ *one has* $f(x) \leq \max\{d(x, y), f(y)\}$ *and therefore one has either* $f(x) = f(y) \leq d(x, y)$ *or* $f(x) = d(x, y) > f(y)$ *or* $f(y) = d(x, y) > f(x)$.

Proof. [8, Lemma 3]. □

The following theorem is similar to Theorem 5.2.1 for ultra-quasi-metric spaces.

2.3.1 Theorem. *([8, Theorem 5]) Let* (X, d) *be an ultra-metric space. For any* $f, g \in UT_X$*, we define* E *by*

$$E(f, f) = 0$$

and

$$E(f, g) = \inf_{x \in X}\{\max\{f(x), g(x)\}\} \quad \text{if} \quad f \neq g.$$

Then we have:

(a) For any $f \in UT_X$ *and* $z \in X$, $E(f, h_z) = f(z)$.

(b) E *is an ultra-metric on* UT_X.

(c) The map $i : X \longrightarrow UT_X$, $i(z) = h_z$*, is an isometric embedding of* X *into* UT_X *which maps* X *onto* $\{f \in UT_X : f^{-1}(0) \neq \emptyset\}$.

Proof. [8, Theorem 5] See also Theorem 5.2.1, where this proof has been generalized in the case of ultra-quasi-metric spaces. □

2.3.2 Definition. *([8]) Let* (X, d) *and* (Y, u) *be two ultra-metric spaces and let a map* $e : X \longrightarrow Y$ *be given. We call* Y *an* ultra-metrically injective envelope *of* X *or* spherical completion *of* X *if* Y *is spherically complete,* e *is an isometric embedding and no spherically complete proper subspace of* Y *contains* $e(X)$*.*

2.3.2 Theorem. *([8, Theorem 6]) Given any ultra-metric space* (X, d)*, then* (UT_X, E) *is an ultra-metrically injective envelope of* X*.*

Proof. [8, Theorem 6]. □

The next theorem will be generalized to the ultra-quasi-metric space setting (see Proposition 5.3.6).

2.3.3 Theorem. *([8, Theorem 7]) Let* (X, d) *be any ultra-metric space.*

(a) *For any two ultra-metrically injective envelopes (or spherical completions)* Y *and* Z *of* X *there exists always a unique isometry of* Y *onto* Z *over* X*.*

(b) *There exists an ultra-metrically injective envelope of* X *within every spherically complete ultra-metric space which contains* X*.*

Proof. This proof is given in the case of ultra-quasi-metrics in Proposition 5.3.6. □

Chapter 3

The q-hyperconvex hull of a T_0-quasi-metric space

In this chapter we start our own investigation. We study the concept of q-hyperconvexity. An explicit construction of the corresponding hull (called q-hyperconvex hull of a T_0-quasi-metric space) will be given. Our investigation develops quite naturally in parallel with the well-known metric theory of hyperconvexity presented in Chapter 1. While many classical ideas about hyperconvexity can be generalized properly from the metric to the quasi-pseudometric setting, these generalizations are not always trivial and sometimes the asymmetric setting requires interesting new variations of old arguments; compare with [33].

We start this chapter by introducing the concept of quasi-pseudometric convexity.

3.1 Convexity

In this section we shall define q-hyperconvexity and q-hypercompleteness in a quasi-pseudometric space.

3.1.1 Definition. *A (n extended) quasi-pseudometric space (X, d) will be called* q-hyperconvex *provided that for each family $(x_i)_{i\in I}$ of points in X and families of non-negative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ the following condition holds:*

If $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$, then $\bigcap_{i\in I} C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i) \neq \emptyset$.

3.1.1 Remark. *In the following we are interested in working in T_0-quasi-metric spaces. So we do not require that r_i or s_i (where $i \in I$) attain only positive values in Definition 3.1.1. We also note that we can assume without loss of generality that the points x_i $(i \in I)$ are pairwise distinct in Definition 3.1.1: Indeed if this is not the case, then for each $x \in X$, set $T(x) = \{i \in I : x_i = x\}$ and consider only those points x of X that satisfy $T(x) \neq \emptyset$. Furthermore set $r(x) = \inf\{r_i : i \in T(x)\}$ and $s(x) = \inf\{s_i : i \in T(x)\}$. Then we have $d(x, y) \leq r_i + s_j$ whenever $i \in T(x)$ and $j \in T(y)$. Thus $d(x, y) \leq r(x) + s_j$ whenever $j \in T(y)$, and consequently $d(x, y) \leq r(x)+s(y)$. Applying the definition of q-hyperconvexity to the family $(x)_{T(x)\neq\emptyset}$ of pairwise distinct points of X and the families $(r(x))_{T(x)\neq\emptyset}$ and $(s(x))_{T(x)\neq\emptyset}$ of nonnegative reals we find that $\emptyset \neq \bigcap_{T(x)\neq\emptyset}(C_d(x, r(x)) \cap C_{d^{-1}}(x, s(x))) \subseteq \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$. Hence the apparently weaker condition is indeed equivalent to our definition.*

The following examples are not surprising, but they are so important that we have to mention them.

3.1.1 Example. *Let the set $\mathbb{R}$ of the reals be equipped with the T_0-quasi-metric $r(x, y) = x \dot{-} y$ whenever $x, y \in \mathbb{R}$. Then $(\mathbb{R}, r)$ is q-hyperconvex.*

Proof. Note that $C_r(x,\epsilon) = [x-\epsilon,\infty)$ and $C_{r^{-1}}(x,\epsilon) = (-\infty, x+\epsilon]$ whenever $x \in \mathbb{R}$ and $\epsilon \geq 0$.

Let $(x_i)_{i\in I}$ be a family of points in $\mathbb{R}$ and $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be families of non-negative real numbers such that $r(x_i,x_j) \leq r_i + s_j$ whenever $i,j \in I$. Suppose that $\bigcap_{i\in F}(C_r(x_i,r_i) \cap C_{r^{-1}}(x_i,s_i)) = \emptyset$ for some finite subset F of I. It follows that $\min\{x_i - r_i : i \in F\} > \max\{x_i + s_i : i \in F\}$. Consequently there are $i_0, j_0 \in F$ such that $x_{i_0} - r_{i_0} > x_{j_0} + s_{j_0}$. In particular $x_{i_0} > x_{j_0}$. Thus $r(x_{i_0}, x_{j_0}) = x_{i_0} - x_{j_0} > r_{i_0} - s_{j_0}$ —a contradiction. We conclude that $\bigcap_{i\in F}(C_r(x_i,r_i) \cap C_{r^{-1}}(x_i,s_i)) \neq \emptyset$ whenever F is a finite subset of I. Since for any $i \in I$, $C_r(x_i,r_i) \cap C_{r^{-1}}(x_i,s_i)$ is compact with respect to the topology $\tau(r^s)$ on $\mathbb{R}$, we conclude that $\bigcap_{i\in I}(C_r(x_i,r_i) \cap C_{r^{-1}}(x_i,s_i)) \neq \emptyset$. Hence $(\mathbb{R}, r)$ is q-hyperconvex. □

3.1.1 Corollary. *The subspace $[0,\infty)$ of $(\mathbb{R}, r)$ is q-hyperconvex.*

Proof. In the proof above we can work with the balls $C_r(x,\epsilon) \cap [0,\infty)$ and $C_{r^{-1}}(x,\epsilon) \cap [0,\infty)$, where $x \in [0,\infty)$ and $\epsilon \geq 0$. □

3.1.2 Example. *Let $\mathbb{R}$ be equipped with its standard metric $r^s(x,y) = \mid x-y \mid$ whenever $x,y \in \mathbb{R}$. Then $(\mathbb{R}, r^s)$ is not q-hyperconvex.*

Proof. For any $i \in [0,1]$ set $r_i = \frac{1}{4}$ and $s_i = \frac{3}{4}$. Then for any $i,j \in [0,1]$ we have that $r^s(i,j) \leq 1 = r_i + s_j$.

But $\bigcap_{i\in[0,1]}(C_{r^s}(i,r_i) \cap C_{r^s}(j,s_j)) \subseteq C_{r^s}(0,\frac{1}{4}) \cap C_{r^s}(1,\frac{1}{4}) = [-\frac{1}{4},\frac{1}{4}] \cap [\frac{3}{4},\frac{5}{4}] = \emptyset$. □

3.1.3 Example. *Consider the product of $(\mathbb{R}, r)$ and $(\mathbb{R}, r^{-1})$, hence the plan $\mathbb{R}^2$ is equipped with $D((\alpha,\beta),(\alpha',\beta')) = (\alpha-\alpha') \vee (\beta'-\beta) \vee 0$. Then the diagonal $\{(\alpha,\alpha) : \alpha \in \mathbb{R}\}$ in this T_0-quasi-metric space is isometric to $(\mathbb{R}, r^s)$.*

We next generalize the concept of metric convexity (see Definition 1.1.1) to quasi-pseudometric spaces as follows:

3.1.2 Definition. *A quasi-pseudometric space* (X, d) *will be called* metrically convex (or radially convex) *if for any point* $x, y \in X$ *and nonnegative numbers* r *and* s *such that* $d(x, y) \leq r + s$*, there exists* $z \in X$ *such that* $d(x, z) \leq r$ *and* $d(z, y) \leq s$.

We next give an example of a space which is not metrically convex.

3.1.4 Example. *We consider the so-called Sorgenfrey quasi-pseudometric on* $\mathbb{R}$ *which is defined for each* $x, y \in \mathbb{R}$ *as follows:*

$d(x, y) = x - y$ *if* $x \geq y$ *and* $d(x, y) = 1$ *otherwise.*

Then d *is not metrically convex. Indeed we have* $d(\frac{1}{2}, 1) = 1 \leq \frac{1}{2} + \frac{1}{2}$*. But there is no* $z \leq \frac{1}{2}$ *and* $z \geq 1$.

Furthermore $d(\frac{1}{2}, a) = \frac{1}{2}$ *has a unique solution* $a = 0$*, but* $d(a, 1) \neq \frac{1}{2}$.

3.1.3 Definition. *Let* (X, d) *be a(n extended) quasi-pseudometric space. A family of balls* $(C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i))_{i \in I}$ *with* $r_i, s_i \geq 0$ *and* $x_i \in X$ *is said to have the* mixed binary intersection property *if for all indices* $i, j \in I$,
$C_d(x_i, r_i) \cap C_{d^{-1}}(x_j, s_j) \neq \emptyset$.

3.1.4 Definition. *A (n extended) quasi-pseudometric space is called* q-hypercomplete *if every family* $(C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i))_{i \in I}$ *of balls with the mixed binary intersection property has* $\bigcap_{i \in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset$.

3.1.1 Lemma. *A quasi-pseudometric space* (X, d) *is q-hyperconvex if and only if it is metrically convex and q-hypercomplete.*

Proof. Suppose that (X, d) is q-hyperconvex. Let $x_1, x_2 \in X$, $r_1, s_2 \geq 0$ such that $d(x_1, x_2) \leq r_1 + s_2$. By q-hyperconvexity of (X, d) we have $C_d(x_1, r_1) \cap C_{d^{-1}}(x_2, s_2) \neq \emptyset$. Then this implies that there is $z \in X$ such that $d(x_1, z) \leq r_1$ and $d(z, x_2) \leq s_2$. So (X, d) is metrically convex. Let $(C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i))_{i \in I}$

have the mixed binary intersection property. Thus $d(x_i, x_j) \leq r_i + s_j$, $i, j \in I$. Then $\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset$ by q-hyperconvexity of (X, d). So (X, d) is q-hypercomplete.

For the converse, assume that (X, d) is metrically convex and q-hypercomplete. Suppose that $(x_i)_{i\in I}$ is a family of points in X, and $(r_i)_{i\in I}$, and $(s_i)_{i\in I}$ are families of nonnegative reals such that $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$. Then $(C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i))_{i\in I}$ has the mixed binary intersection property by metric convexity of (X, d). Therefore $\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset$. Hence (X, d) is q-hyperconvex. □

3.1.1 Proposition. *(a) If (X, d) is a q-hyperconvex (resp. q-hypercomplete, radially convex) quasi-pseudometric space, then (X, d^{-1}) is q-hyperconvex (resp. q-hypercomplete, radially convex).*

(b) If (X, d) is a q-hyperconvex (resp. q-hypercomplete) quasi-pseudometric space, then (X, d^s) is hyperconvex (resp. hypercomplete). The corresponding statement for "metrically convex" or "radially" convex does not hold.

Proof. (a) Assume (X, d) is q-hyperconvex. Let $(x_i)_{i\in I}$ be a family of points in X and $(r_i)_{i\in I}$, $(s_i)_{i\in I}$ two families of nonnegative real numbers, such that $d^{-1}(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$. Find $x \in C_{d^{-1}}(x_i, r_i) \cap C_d(x_i, s_i)$ for any $i \in I$. By the q-hyperconvexity of (X, d), $d^{-1}(x_i, x_j) = d(x_j, x_i) \leq r_i + s_j$ whenever $i, j \in I$ implies that there exists $x_0 \in C_d(x_i, s_i) \cap C_{d^{-1}}(x_i, r_i)$. Take $x = x_0$ and then (X, d^{-1}) is q-hyperconvex.

Assume (X, d) is q-hypercomplete. Let a family of balls $(C_{d^{-1}}(x_i, r_i), C_d(x_i, s_i))_{i\in I}$ be given with the mixed binary intersection property. Since (X, d) is q-hypercomplete, $(C_{d^{-1}}(x_i, r_i), C_d(x_i, s_i))_{i\in I}$ with the mixed binary intersection property, has $\bigcap_{i\in I}(C_{d^{-1}}(x_i, r_i)$ $\cap\, C_d(x_i, s_i)) \neq \emptyset$ then (X, d^{-1}) is q-hypercomplete.

Assume that (X, d) is metrically convex. We shall show that (X, d^{-1}) is metrically convex. Let $x, y \in X$, $r_1, s_2 \in [0, \infty)$ be such that $d^{-1}(x, y) \leq r_1 + s_2$. By the radial convexity of (X, d) and $d(y, x) = d^{-1}(x, y) \leq r_1 + s_2$ we have $C_d(y, s_2) \cap C_{d^{-1}}(x, r_1) \neq \emptyset$. Then (X, d^{-1}) is metrically convex.

(b) Assume (X, d) is q-hyperconvex. Let $(x_i)_{i \in I}$ be a family of points in X and $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ be two families of nonnegative real numbers such that $d^s(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$. By q-hyperconvexity of (X, d) we have $\emptyset \neq \bigcap_{i \in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, r_i)) = \bigcap_{i \in I} C_{d^s}(x_i, r_i)$. It follows that (X, d^s) is hyperconvex.

Suppose that (X, d) is q-hypercomplete. Let the family of balls $(C_{d^s}(x_i, r_i))_{i \in I}$ have the binary intersection property. Then $(C_d(x_i, r_i), C_{d^{-1}}(x_i, r_i))_{i \in I}$ has the mixed binary intersection property. Consequently $\emptyset \neq \bigcap_{i \in I} C_{d^s}(x_i, r_i)$. Thus (X, d^s) is hypercomplete. The final statement follows from Example 3.1.5 below.□

3.1.2 Corollary. *Each metric space (X, m) that is q-hyperconvex (q-hypercomplete) is hyperconvex (hypercomplete).*

Proof. The assertion is obvious by Proposition 3.1.1(b). □

3.1.3 Corollary. *Each q-hyperconvex T_0-quasi-metric space (X, d) is bicomplete.*

Proof. By Proposition 3.1.1 d^s is hyperconvex. Since hyperconvex metric spaces are complete, we conclude that the T_0-quasi-metric space (X, d) is bicomplete.□

3.1.5 Example. *Consider the points of the unit circle in the Euclidean plane C. The distance $d(p_1, p_2)$ from p_1 to $p_2 \in C$ is set equal to the arc length measured counter clockwise from p_1 to p_2. Note that d is a T_0-quasi-metric on C, which is clearly metrically convex: Suppose that $d(x, y) \leq r + s$ where $r, s \in [0, \infty)$. We assume without loss of generality that $r + s \neq 0$. Then we can find z such* *that* $d(x,z) = \frac{r}{r+s}d(x,y) \leq r$ *and* $d(x,z) = \frac{s}{r+s}d(x,y) \leq s$*, which proves the assertion. But* d^s *is not metrically convex, since evidently* d^s *does not admit any values between* 0 *and* π*.* $\square$

3.2 A space of nonnegative function pairs of the T_0-quasi-metric space (X,d)

3.2.1 Definition. *Let* (X,d) *be a quasi-pseudometric space and let* $\mathcal{FP}(X,d)$ *be the set of all pairs of functions* $f = (f_1, f_2)$ *where* $f_i : X \longrightarrow [0,\infty)$ $(i = 1,2)$*. We define an extended* T_0*-quasi-metric* D *on* $\mathcal{FP}(X,d)$ *as follows:*

$$D(f,g) = \sup_{x\in X}(f_1(x)\dot{-}g_1(x)) \vee \sup_{x\in X}(g_2(x)\dot{-}f_2(x))$$

whenever $f,g \in \mathcal{FP}(X,d)$*. (For clarity we may also write* D_d *instead of* D*.)*

3.2.1 Remark. *We note that* $(D^s)(f,g) = \sup_{x\in X}|(f_1(x)-g_1(x))| \vee \sup_{x\in X}|(f_2(x)-g_2(x))|$ *whenever* $f,g \in \mathcal{FP}(X,d)$ *is a complete extended metric. Furthermore* $(\mathcal{FP}(X,d), D)$ *is q-hyperconvex. Hence* D^s *is complete by Corollary 3.1.3.*

Proof. *In Definition 3.2.1 it is straightforward to check that* D *is an extended* T_0-*quasi-metric on* $\mathcal{PF}(X,d)$*. Therefore q-hyperconvexity of* $(\mathcal{FP}(X,d), D)$ *follows from q-hyperconvexity of the factors* $([0,\infty), r)$ *and* $([0,\infty), r^{-1})$ *in this product (see Example 3.1.1 and Proposition 3.1.1): Indeed suppose that* $((f_i)_1, (f_i)_2)_{i\in I}$ *is a family of points in* $\mathcal{FP}(X,d)$ *and* $(r_i)_{i\in I}$ *and* $(s_i)_{i\in I}$ *are two families of nonnegative reals such that*

$$\sup_{x\in X}((f_i)_1(x)\dot{-}(f_j)_1(x)) \vee \sup_{x\in X}((f_j)_2(x)\dot{-}(f_i)_2(x)) \leq r_i + s_j$$

whenever $i,j \in I$*.*

Then for each $x \in X$ *there is* $g_2(x) \in \cap_{i\in I}(C_r((f_i)_2(x), r_i) \cap C_{r^{-1}}((f_i)_2(x), s_i))$ *and* $g_1(x) \in \cap_{i\in I}(C_r((f_i)_1(x), r_i) \cap C_{r^{-1}}((f_i)_1(x), s_i))$*.*

Hence

$$(g_1, g_2) \in \cap_{i\in I}(C_D((f_i)_1, (f_i)_2, r_i) \cap C_{D^{-1}}((f_i)_1, (f_i)_2, s_i)).$$

Consequently $(\mathcal{PF}(X,d), D)$ *is q-hyperconvex.* □

The following defines the canonical function pairs, compare with Definition 1.2.1.

3.2.2 Definition. *Let* (X,d) *be a quasi-pseudometric space. We define*
$e_X : (X,d) \longrightarrow (\mathcal{FP}(X,d), D)$, $a \mapsto e_X(a)$
where
$e_X(a) : (X,d) \longrightarrow [0,\infty)^2$, $x \mapsto e_X(a)(x) = (d(a,x), d(x,a))$.

The following result is similar to Remark 1.2.2.

3.2.1 Lemma. *For any* $a \in X$, *set* $f_a(x) := f_{\{a\}}(x) = (d(a,x), d(x,a))$ *whenever* $x \in X$. *For any* $a, b \in X$ *we have* $d(a,b) = D(f_a, f_b)$. *Therefore* e_X *is an isometric embedding. In case that* (X,d) *is a* T_0*-quasi-metric space, e is injective.*

Proof. Obviously $\sup_{x\in X}(d(a,x) - d(b,x)) = d(a,b)$, as we see by setting $x = b$ and using the triangle inequality. Similarly $\sup_{x\in X}(d(x,b) - d(x,a)) = d(a,b)$ whenever $a, b \in X$. Hence e_X is an isometric map. If for $a, b \in X$ we have that $e_X(a) = e_X(b)$, then $0 = d(a,a) = d(a,b)$ and $d(b,a) = d(b,b) = 0$. Consequently $a = b$ by the T_0-property. □

A generalization of this kind of function leads to the Hausdorff quasi-pseudometric, as we show next.

3.2.1 Example. *Let* (X,d) *be a quasi-pseudometric space and let* $\mathcal{P}_0(X)$ *be the set of nonempty subsets of* X. *For each* $A \in \mathcal{P}_0(X)$ *set* $(f_A)_1(x) = d(A,x)$

and $(f_A)_2(x) = d(x, A)$ *whenever* $x \in X$ *where, as usual, for instance* $d(A, x) = \inf\{d(a, x) : a \in A\}$. *We shall call* $d(A, x)$ *also* $dist(A, x)$ *later.*

Then for $f_A = ((f_A)_1, (f_A)_2)$ *we see that* $(f_A)_1$ *is nonexpansive on* (X, d^{-1}) *and* $(f_A)_2$ *is nonexpansive on* (X, d).

Proof. *It is well-known that for any* $x, y \in X$ *and* $A \in \mathcal{P}_0(X)$, $d(A, x) - d(A, y) \leq d(y, x) = d^{-1}(x, y)$. *Similarly* $d(x, A) - d(y, A) \leq d(x, y)$ *whenever* $x, y \in X$. □

The following uses the Hausdorff quasi-pseudometric of a quasi-pseudometric space introduced by G. Berthiaume [7].

3.2.2 Remark. *(compare [46, Lemma 3.1]) We note that* $d_H(A, B) = D(f_A, f_B)$ *whenever* $A, B \in \mathcal{P}_0(X)$ *where* $d_H(A, B) = \sup_{b \in B} d(A, b) \vee \sup_{a \in A} d(a, B)$ *is the extended Hausdorff quasi-pseudometric on* $\mathcal{P}_0(X)$.

Proof. *It suffices to prove that for given* $A, B \in \mathcal{P}_0(X)$, $\rho_1(A, B) := \sup_{b \in B} d(A, b)$ *is equal to* $\rho_2(A, B) := \sup_{x \in X}(d(A, x) \dot{-} d(B, x))$.

Let $b \in B$. *Then* $d(A, b) - 0 \leq d(A, b) - d(B, b) \leq \rho_1(A, B)$ *and thus* $\rho_2(A, B) \leq \rho_1(A, B)$. *For the converse choose a sequence* $(x_n)_{n \in \mathbb{N}} \in X$ *such that* $\rho_1(A, B) = \lim_{n \to \infty} d(A, x_n) \dot{-} d(B, x_n)$. *For each* $n \in \mathbb{N}$ *find* $b_n \in B$ *such that* $d(b_n, x_n) \leq d(B, x_n) + \frac{1}{n+1}$. *Hence for each* $n \in \mathbb{N}$, $d(A, x_n) - d(B, x_n) \leq d(A, b_n) + d(b_n, x_n) - d(B, x_n) \leq \rho_2(A, B) + \frac{1}{n+1}$. *Thus* $\rho_1(A, B) \leq \rho_2(A, B)$.

For the converse choose a sequence $(x_n)_{n \in \mathbb{N}} \in X$ *such that*

$$\rho_1(A, B) = \lim_{n \longrightarrow \infty} d(A, x_n) \dot{-} d(B, x_n)$$

For each $n \in \mathbb{N}$ *find* $b_n \in B$ *such that* $d(b_n, x_n) \leq d(B, x_n) + \frac{1}{n+1}$. *Hence for each* $n \in \mathbb{N}$,

$$d(A, x_n) - d(B, x_n) \leq d(A, b_n) + d(b_n, x_n) - d(B, x_n) \leq \rho_1(A, B) + \frac{1}{n+1}.$$

Thus $\rho_2(A, B) \leq \rho_1(A, B)$. □

3.3 The q-hyperconvex hull $\epsilon_q(X, d)$ of a T_0-quasi-metric space (X, d)

3.3.1 Definition. *We say that a pair* $f \in \mathcal{PF}(X, d)$ *is* q-tight *if for all* $x, y \in X$*, we have* $d(x, y) \leq f_2(x) + f_1(y)$.

The set of all q-tight function pairs on a T_0-quasi-metric space (X, d) will be denoted by $\mathcal{T}_q(X, d)$.

3.3.1 Lemma. *For each* $a \in X$*,* f_a *is q-tight function pair.*

Proof. Indeed $d(x, y) \leq (f_a)_2(x) + (f_a)_1(y) = d(x, a) + d(a, y)$. □

We say that a pair f is *minimal* or *extremal* (among the q-tight pairs) if it is q-tight and if g is q-tight such that for each $x \in X$, $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$ then $f = g$.

3.3.2 Lemma. *Any minimal q-tight function pair* f *satisfies* $f_1(x) - f_1(y) \leq d^{-1}(x, y)$ *whenever* $x, y \in X$ *and* $f_2(x) - f_2(y) \leq d(x, y)$ *whenever* $x, y \in X$*. (Hence* f_1 *is nonexpansive on* (X, d^{-1}) *and* f_2 *is nonexpansive on* (X, d)*, when considered as maps into* $([0, \infty), r)$*, where* r *denotes the restriction of the* T_0*-quasi-metric considered in Example 3.1.1. (In particular for each minimal q-tight function pair* f*, the function* f_1 *is* $\tau(d)$*-upper semi-continuous and* $\tau(d^{-1})$*-lower semi-continuous, and* f_2 *is* $\tau(d)$*-lower semi-continuous and* $\tau(d^{-1})$*-upper semi-continuous. For these concepts see [24].))*

Proof. Let us consider f_2. Suppose that $x_0, y_0 \in X$ such that $f_2(x_0) > d(x_0, y_0) + f_2(y_0)$. Set $g_2(x) = f_2(x)$ if $x \neq x_0$ and $g_2(x) = d(x_0, y_0) + f_2(y_0)$ if $x = x_0$. Clearly

$(f_1, g_2) < (f_1, f_2)$. Let $x, y \in X$. Then $d(x, y) \leq f_2(x) + f_1(y) = g_2(x) + g_1(y)$ if $x \neq x_0$.

So assume $x = x_0$ and $y \in X$. Then $d(x, y) = d(x_0, y) \leq d(x_0, y_0) + d(y_0, y) \leq d(x_0, y_0) + f_2(y_0) + f_1(y) \leq g_2(x) + f_1(y)$. It is q-tight and we have reached a contradiction.

Similarly one shows that $f_1(x) - f_1(y) \leq d^{-1}(x, y)$ whenever $x, y \in X$. □

Let (X, d) be a T_0-quasi-metric space. In the following by $\epsilon_q(X, d)$ (or more briefly by $\epsilon_q(X)$) we shall denote the set of all minimal q-tight function pairs in $\mathcal{T}_q(X, d)$. We shall show that $(\epsilon_q(X, d), D)$ is isometric to the q-injective hull of (X, d), where, for convenience, D also denotes the restriction of D to $\epsilon_q(X, d)$.

3.3.1 Proposition. *Let $f = (f_1, f_2)$ be a q-tight function pair on a T_0-quasi-metric space (X, d) such that f_1 is nonexpansive on (X, d^{-1}) and f_2 is nonexpansive on (X, d) (compare Lemma 3.3.2).*

Furthermore suppose that there is a sequence $(a_n)_{n \in \mathbb{N}}$ in X with $lim_{n \longrightarrow \infty} f_1(a_n) = 0$ and $lim_{n \longrightarrow \infty} f_2(a_n) = 0$. Then f is a minimal q-tight pair.

Proof. Suppose otherwise. Then there is a q-tight pair g such that $g < f$, say there is $x_0 \in X$ such that $g_2(x_0) < f_2(x_0)$ and $g_1 = f_1$. (The second case is dealt with analogously.) Then by nonexpansiveness and q-tightness of g, we have that $f_2(x_0) - f_2(a_n) \leq d(x_0, a_n) \leq g_2(x_0) + g_1(a_n) \leq f_2(x_0) + f_1(a_n)$ whenever $n \in \mathbb{N}$. Thus $\lim_{n \longrightarrow \infty} d(x_0, a_n) = g_2(x_0)$.

So g cannot be a q-tight function pair, since otherwise $f_2(x_0) = \lim_{n \longrightarrow \infty} d(x_0, a_n) \leq g_2(x_0) + \lim_{n \longrightarrow \infty} f_1(a_n)$—a contradiction. Hence f is a minimal q-tight function pair. □

Our next result describes a connection between the two functions of a minimal q-tight function pair.

3.3.2 Proposition. *Let (X, d) be a T_0-quasi-metric space. If $(f_1, f_2) \in \epsilon_q(X, d)$ then*

$$f_1(x) = \sup_{y \in X}(d^{-1}(x, y) \dot{-} f_2(y)) = \sup_{y \in X}((f_x)_2(y) \dot{-} f_2(y))$$

and

$$f_2(x) = \sup_{y \in X}(d(x, y) \dot{-} f_1(y)) = \sup_{y \in X}((f_x)_1(y) \dot{-} f_1(y))$$

whenever $x \in X$.

Proof. Since $(f_1, f_2) \in \epsilon_q(X, d)$, we have $d(y, x) \leq f_2(y) + f_1(x)$ whenever $x, y \in X$, which implies that $\sup_{x \in X}(d^{-1}(x, y) - f_2(y)) \leq f_1(x)$ whenever $x \in X$.

Suppose that there is x_0 such that $\sup_{x \in X}(d^{-1}(x_0, y) \dot{-} f_2(y)) < f_1(x_0)$. Set $k_1(x) = f_1(x)$ if $x \in X$ and $x \neq x_0$, and $k_1(x_0) = \sup_{y \in X}(d^{-1}(x_0, y) \dot{-} f_2(y))$. Then (k_1, f_2) is q-tight: Indeed for any $y \in X$ we have $d(y, x_0) - f_2(y) \leq \sup_{a \in X}(d(a, x_0) - f_2(a))$. Thus

$$d^{-1}(x_0, y) \leq f_2(y) + \sup_{a \in X}(d^{-1}(x_0, a) \dot{-} f_2(a)) = f_2(y) + k_1(x_0)$$

whenever $y \in X$. It follows that (h_1, f_2) is q-tight and $(k_1, f_2) < (f_1, f_2)$, but (f_1, f_2) is minimal q-tight. We have reached a contradiction and conclude that $f_1(x) = \sup_{y \in X}(d(x, y) \dot{-} f_2(y))$ whenever $x \in X$. Given $x \in X$, the definition of f_x (see Lemma 3.2.1) yields the second equality $\sup_{y \in X}(d^{-1}(x, y) \dot{-} f_2(y)) = \sup_{y \in X}((f_x)_2(y) \dot{-} f_2(y))$.

Similarly one shows that $f_2(x) = \sup_{y \in X}(d(x, y) \dot{-} f_1(y))$ whenever $x \in X$. □

The next result gives a formula for the distance between two minimal q-tight function pairs.

3.3.3 Lemma. *Let $(f_1, f_2), (g_1, g_2)$ be minimal q-tight pairs on a T_0-quasi-metric space (X, d). Then*

$$\sup_{x \in X}(f_1(x) \dot{-} g_1(x)) = \sup_{x \in X}(g_2(x) \dot{-} f_2(x))$$

and hence

$$D((f_1, f_2), (g_1, g_2) = \sup_{x\in X}(f_1(x)\dot{-}g_1(x)) = \sup_{x\in X}(g_2(x)\dot{-}f_2(x)).$$

Proof. Let $x, a \in X$. Then $f_1(x) - g_1(x) \leq d(a,x) - g_1(x) + f_1(x) - d(a,x) \leq g_2(a) + f_1(x) - d(a,x)$, because (g_1, g_2) is q-tight. Given $x \in X$ and $\epsilon > 0$ we can find $a \in X$ such that $f_1(x) - \epsilon \leq d(a,x) - f_2(a)$, because $f_1(x) = \sup_{y\in X}(d(x,y) - f_2(y))$ by Lemma 3.3.2. Hence given $x \in X$, there is $a \in X$ such that $f_1(x) - g_1(x) \leq g_2(a) + f_1(x) - d(a,x) \leq g_2(a) - f_2(a) - \epsilon \leq \sup_{a\in X}(g_2(a) - f_2(a)) + \epsilon$. Consequently $\sup_{x\in X}(f_1(x) - g_1(x)) \leq \sup_{a\in X}(g_2(a) - f_2(a)) + \epsilon$. Since ϵ was arbitrary, we have that $\sup_{x\in X}(f_1(x) - g_1(x)) \leq \sup_{x\in X}(g_2(x) - f_2(x))$. Similarly, $0 < \sup_{x\in X}(g_2(x) - f_2(x))$ which implies that $\sup_{x\in X}(g_2(x) - f_2(x)) \leq \sup_{x\in X}(f_1(x) - g_1(x))$.

Therefore $\sup_{x\in X}(f_1(x)\dot{-}g_1(x)) = \sup_{x\in X}(g_2(x)\dot{-}f_2(x))$. □

The following gives the distance between the canonical q-tight function pair with any minimal q-tight function pair and is similar to Proposition 1.2.1(1).

3.3.4 Lemma. *Let (X, d) be a T_0-quasi-metric space. Consider a minimal q-tight pair f and $a \in X$. Then $D(f, f_a) = f_1(a)$ and $D(f_a, f) = f_2(a)$.*

Proof. We have $f_1(a) \leq \sup_{x\in X}(f_1(x) - d(a,x))$, since $d(a,a) = 0$. Furthermore for any $x \in X$, $f_1(x) - f_1(a) \leq d^{-1}(x,a)$, since f_1 is nonexpansive on (X, d^{-1}). Thus $f_1(x) - d(a,x) \leq f_1(a)$ whenever $x \in X$. Hence $\sup_{x\in X}(f_1(x) - d(a,x)) = f_1(a)$. Furthermore $d(x,a) - f_2(x) \leq f_1(a)$ whenever $x \in X$, since f is q-tight. So $\sup_{x\in X}((f_a)_2(x)\dot{-}f_2(x)) \leq f_1(a)$. (In fact it is stated in Lemma 3.3.2 that equality holds in the last inequality.) According to the definition of D, certainly $D(f, f_a) = f_1(a)$.

Similarly one verifies that $f_2(a) = \sup_{x\in X}(f_2(x) - d(x,a))$ and $\sup_{x\in X}(d(a,x)\dot{-}f_1(x)) \leq f_2(a)$, where indeed by Lemma 3.3.2 the last equality is an inequality. In particular by the definition of D, certainly $D(f, f_a) = f_2(a)$. □

3.3.1 **Remark.** *We note that $D(f,g) < \infty$ whenever $f, g \in \epsilon_q(X,d)$. Indeed we have for any $a \in X$, $D(f,g) \leq D(f, f_a) + D(f_a, g) \leq f_1(a) + g_2(a)$ by Lemma 3.3.4. Hence D is a T_0-quasi-metric space on $\epsilon_q(X,d)$.*

The following lemma should be compared with Proposition 1.2.1.

3.3.5 **Lemma.** *For each $a \in X$, the pair f_a belongs to $\epsilon_q(X,d)$.*

Proof. This follows from Proposition 3.3.1 in the light of Example 3.2.1, since $(f_a)_1(a) = 0 = (f_a)_2(a)$. □

3.3.6 **Lemma.** *Suppose that (X,d) is a T_0-quasi-metric space and $(f_1, f_2) \in \epsilon_q(X,d)$ such that $f_1(a) = 0 = f_2(a)$ for some $a \in X$. Then $(f_1, f_2) = e_X(a)$.*

Proof. By q-tightness of (f_1, f_2) we have $d(x,a) \leq f_2(x) + 0$ and $d(a,x) \leq 0 + f_1(x)$ whenever $x \in X$. Therefore $e_X(a) \leq (f_1, f_2)$ and thus $e_X(a) = (f_1, f_2)$, since (f_1, f_2) is minimal. □

In the next remark, we state the basic ideas of a tight extension in the asymmetric setting similarly to Theorem 1.3.1.

3.3.2 **Remark.** *For all $f, g \in \epsilon_q(X,d)$ we have*

$$D(f,g) = \sup_{x_1, x_2 \in X}[d(x_1, x_2) - f_2(x_1) - g_1(x_2)] \vee 0 :$$

Assume first that for some $f, g \in \epsilon_q(X)$ we have $D(f,g) > 0$. Then for each $\epsilon > 0$ there is $x_1 \in X$ such that $g_2(x_1) - f_2(x_1) > 0$ and $D(f,g) - \epsilon \leq g_2(x_1) - f_2(x_1)$ by Lemma 3.3.3. Since $g_1(x_1) > 0$, by Lemma 3.3.2 there is $x_2 \in X$ such that $g_1(x_1) - \epsilon \leq d(x_1, x_2) - g_1(x_2)$ and hence $g_2(x_1) - d(x_1, x_2) \leq -g_1(x_1) + \epsilon$.

Thus $g_2(x_1) - f_2(x_1) \leq d(x_1, x_2) - f_2(x_1) + g_2(x_1) - d(x_1, x_2) \leq d(x_1, x_2) - f_2(x_1) - g_2(x_2) + \epsilon \leq d(x_1, x_2) - D(f_{x_1}, f) - D(g, f_{x_2}) + \epsilon$.

Consequently $D(f,g) \leq \sup_{x_1,x_2 \in X}[(D(f_{x_1}, f_{x_2}) - D(f_{x_1}, f) - D(g, f_{x_2})] \vee 0$, *which also holds in the remaining case where for* $f, g \in \epsilon_q(X)$ *we have* $D(f,g) = 0$.

Furthermore $D(f_{x_1}, f_{x_2}) - D(f_{x_1}, f) - D(g, f_{x_2}) \leq D(f,g)$ *whenever* $f, g \in \epsilon_q(X)$ *and* $x_1, x_2 \in X$ *by the triangular inequality. Thus* $[D(f_{x_1}, f_{x_2}) - D(f_{x_1}, f) - D(g, f_{x_2})] \vee 0 \leq D(f,g)$ *whenever* $f, g \in \epsilon_q(X)$ *and* $x_1, x_2 \in X$, *which establishes the claimed equality* $D(f,g) = \sup_{x_1,x_2 \in X}[d(x_1, x_2) - f_2(x_1) - g_1(x_2)] \vee 0$ *whenever* $f, g \in \epsilon_q(X)$.

It follows from this formula that the isometric map $e_X : (X, d) \to (\epsilon_q(X, d), D)$ *has the following "tightness" property: If* q *is any quasi-pseudometric on* $\epsilon_q(X, d)$ *such that* $q \leq D$ *and* $q(e_X(x), e_X(y)) = D(e_X(x), e_X(y))$ *whenever* $x, y \in X$, *then* $D(f,g) = q(f,g)$ *whenever* $f, g \in \epsilon_q(X, d)$: *In fact let* $f, g \in \epsilon_q(X, d)$. *Then*

$$D(f,g) = \sup_{x_1,x_2 \in X}[D(f_{x_1}, f_{x_2}) - D(f_{x_1}, f) - D(g, f_{x_2})] \vee 0 \leq$$

$$\sup_{x_1,x_2 \in X}[q(f_{x_1}, f_{x_2}) - q(f_{x_1}, f) - q(g, f_{x_2})] \vee 0 \leq q(f,g).$$

Therefore $q(f,g) = D(f,g)$ *whenever* $f, g \in \epsilon_q(X, d)$.

The following defines the concept of a q-injective T_0-quasi-metric space similarly to Definition 1.1.2.

3.3.2 Definition. *Let* (Y, d_Y) *be a* T_0*-quasi-metric space. Then it is called* q-injective *provided that for any* T_0*-quasi-metric space* (X, d_X), *any subspace* A *of* (X, d) *and any nonexpansive map* $f : A \longrightarrow (Y, d_Y)$, f *can be extended to a nonexpansive map* $g : (X, d_X) \longrightarrow (Y, d_Y)$.

The following result is analogous to Theorem 1.1.1.

3.3.1 Theorem. *A* T_0*-quasi-metric space is* q*-hyperconvex if and only if it is* q*-injective.*

Proof. First assume that X is a q-hyperconvex T_0-quasi-metric space. Let A be a T_0-quasi-metric space and $T : A \longrightarrow X$ be a nonexpansive map. Let M be a T_0-quasi-metric space containing A quasi-metrically. Consider the following set $C = \{(T_F, F) : T_F : F \longrightarrow X, A \subseteq X \subseteq M$, as a quasi-metric subspace$\}$ where T_F is a nonexpansive extension of T. We have $(T, A) \in C$. Therefore, C is nonempty. On the other hand, one can partially order C by $(T_F, F) \leq (T_G, G)$ if and only if $F \subseteq G$ and the restriction of T_G to F is T_F. It is easy to see that C satisfies the hypothesis of Zorn's Lemma. Therefore, C has a maximal element. Let (T_1, F_1) be a maximal element of C.

Let us show that $F_1 = M$. Assume not. Let $z \in M$ but $z \notin F_1$ and set $F = F_1 \cup \{z_1\}$. Let us extend T_1 to F. The problem is to find a point z_1, which will play the role of the value of the extension at z. Since we need the extension to be nonexpansive, we must have $d(T_1(x), z_1) \leq d(x, z)$ for all $x \in F_1$ and $d(z_1, T_1(x)) \leq d(z, x)$ for all $x \in F_1$. Consider the family of balls $(C_d(T_1(x), d(x,z)), C_{d^{-1}}(T_1(y), d^{-1}(y,z)))_{x \in F_1}$. Since $d(T_1(x), T_1(y)) \leq d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y \in F_1$, the q-hyperconvexity of X implies that

$\bigcap_{x \in F_1} (C_d(T_1(x), d(x,z)) \cap C_{d^{-1}}(T_1(x), d^{-1}(x,z))) \neq \emptyset.$

Let z_1 be any fixed point in this intersection. Set $T^* : F \longrightarrow X$.

For arbitrary $x \in F$,

set $T^*(x) = T_1(x)$ if $x \neq z$, $T^*(x) = z_1$ if $x = z$. We have for all $x, y \in F$,

$d(T^*(x), T^*(y)) = d(T_1(x), T_1(y))$ if $x \neq z$,

$d(T^*(x), T^*(y)) = d(z_1, T_1(y))$ if $x = z$,

and

$d(T^*(y), T^*(x)) = d(T_1(y), T_1(x))$ if $x \neq z$, $d(T^*(x), T^*(y)) = d(T_1(y), z_1)$ if $x = z$, Then, we can see that for arbitrary $x, y \in F$, $d(T^*(x), T^*(y)) \leq d(x, y)$ and $d(T^*(y), T^*(x)) \leq d(y, x)$. Therefore, T^* is a nonexpansive extension of T, thus (T^*, F) belongs to C, hence $(T_1, F_1) \leq (T^*, F)$ and $(T_1, F_1) \neq (T^*, F)$. This contradicts the maximality of (T_1, F_1). Therefore $F_1 = M$ and hence T has a nonexpansive extension to M. Consequently, X is q-injective.

For the converse, assume that X is q-injective. We want to prove that X is q-hyperconvex. Suppose that there is given $(x_i)_{i \in I}$, a family of pairwise distinct points in X, and two families of nonnegative real numbers $(r_i)_{i \in I}$ and $(s_j)_{j \in I}$ such that $d(x_i, x_j) \leq r_i + s_j$ for any $i, j \in I$.

Similarly as before, consider the set $\mathcal{PF}(A, d)$ of all nonnegative real-valued pair functions $f = (f_1, f_2)$ defined on the set $A = \{x_i : i \in I\}$ such that $d(x_i, x_j) \leq f_2(x_i) + f_1(x_j)$ for all $i, j \in I$. The distance between the elements of A is the one inherited from X. The pair of functions (r, s) such that:

$r : A \longrightarrow [0, \infty), x_i \mapsto r(x_i)$

$s : A \longrightarrow [0, \infty), s_i \mapsto s(x_i)$

belongs to $\mathcal{PF}(A, d)$.

The set $\mathcal{PF}(A, d)$ is partially ordered by the afore-mentioned pointwise order on the pair functions. Obviously, any descending chain of elements of $\mathcal{PF}(A, d)$ has a lower bound. Hence, Zorn's Lemma implies the existence of a minimal element

$(f_1, f_2) \in \mathcal{PF}(A, d)$ smaller than (r, s), i.e. $f_1(x_i) \leq s(x_i) = s_i$, $f_2(x_i) \leq r(x_i) = r_i$ whenever $i \in I$.

Using the minimality of (f_1, f_2), it follows that, for any $i, j \in I$, $f_1(x_i) \leq d(x_j, x_i) + f_1(x_j)$ and $f_2(x_i) \leq d(x_i, x_j) + f_2(x_j)$, see Lemma 3.3.2.

We now consider two cases: Case 1: There is $a \in A$ such that $(e_X(a))(x) =$

$(f_1, f_2) = (d(a, x), d(x, a))$ whenever $x \in X$. Then

$$a \in \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset.$$

Case 2: $(f_1, f_2) \neq e_X(a)$ whenever $a \in A$.

Let w be a point from X not in the set A. Consider $A^* = A \cup \{w\}$. For the new point w, set $d(w, x_i) = f_1(x_i)$ and $d(x_i, w) = f_2(x_i)$, for any $i \in I$, as well as $d(w, w) = 0$. Since for any $a \in A$, $f_1(a)$ or $f_2(a)$ is positive by Lemma 3.3.1, A^* is a T_0-quasi-metric space which contains A as a quasi-metric subspace.

Then according to our assumption, there exists a nonexpansive extension R of the inclusion map (defined from A to X). It is clear that

$d(R(x_i), R(w)) = d(x_i, R(w)) \leq d(x_i, w) = f_2(x_i) \leq r_i$, whenever $i \in I$ and $d(R(w), R(x_i)) = d(R(w), x_i) \leq d(w, x_i) = f_1(x_i) \leq r_i$, whenever $i \in I$.

Thus $R(w) \in \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset$. We have shown that X is q-hyperconvex. □

Our next result shows that the q-hyperconvex hull of a T_0-quasi-metric space is isometric to the conjugate space of the q-hyperconvex hull of its conjugate.

3.3.3 Proposition. *Let (X, d) be a T_0-quasi-metric space. Then $(f_1, f_2) \in \epsilon_q(X, d)$ implies that $(f_2, f_1) \in \epsilon_q(X, d^{-1})$. It follows that*

$$s : (\epsilon_q(X, d), D_d) \longrightarrow (\epsilon_q(X, d^{-1}), D_{d^{-1}})$$

where $D_{d^{-1}} = {D_d}^{-1}$, defined by $s((f, g)) = (g, f)$ whenever $(f, g) \in \epsilon_q(X, d)$ is a bijective isometric map.

Proof. Suppose that $(f_1, f_2) \in \epsilon_q(X, d)$. Then $d^{-1}(x, y) = d(y, x) \leq f_1(x) + f_2(y) = f_2(y) + f_1(x)$ whenever $x, y \in X$. Obviously (f_2, f_1) is minimal q-tight on (X, d^{-1}), since (f_1, f_2) is minimal q-tight on (X, d). By the preceding observation, it is evident that s is a bijection. It is an isometry, since $D^{-1}(s(f_1, f_2), s(g_1, g_2)) = D((g_2, g_1), (f_2, f_1)) = D((f_1, f_2), (g_1, g_2))$ whenever $(f_1, f_2), (g_1, g_2) \in \epsilon_q(X, d)$. $\square$

The next result makes a connection between hyperconvex and q-hyperconvex hulls.

3.3.4 Proposition. *Let (X, m) be a metric space. Then $h(f) = (f, f)$ defines an isometric embedding of $(\epsilon_m(X, m), E)$ into $(\epsilon_q(X, m), D)$.*

Proof. Given $f \in \epsilon_m(X, m)$, it is evident that the pair (f, f) is q-tight. Suppose that (k_1, k_2) is a q-tight function pair such that $k_1 \leq f$ and $k_2 \leq f$. Then obviously $(\frac{k_1+k_2}{2}, \frac{k_1+k_2}{2})$ is a q-tight function pair, too. Thus $\frac{k_1+k_2}{2}$ is a tight function on (X, m). By minimality of f, $\frac{k_1+k_2}{2} = f$. But then $k_1 = f$ and $k_2 = f$, since $k_1 \leq f$ and $k_2 \leq f$. Thus $(f, f) \in \epsilon_q(X, m)$. Let $f, g \in \epsilon_m(X, m)$.

Then

$$d_\infty(f, g) = \sup_{x \in X} \mid f(x) - g(x) \mid = \sup_{x \in X}(f(x) \dot{-} g(x)) \vee \sup_{x \in X}(g(x) \dot{-} f(x))$$

$$= D((f, f), (g, g)).$$

Hence $h : (\epsilon_m(X, m), E) \longrightarrow (\epsilon_q(X, m), D)$ is an isometric embedding, where $h(f) = (f, f)$ whenever $f \in \epsilon_m(X, m)$. $\square$.

The following auxiliary result will be useful in the following.

3.3.7 Lemma. *Let A be a nonempty subset of a T_0-quasi-metric space (X, d), let $(r_1, r_2) : A \longrightarrow [0, \infty)$ be such that $\forall x, y \in A$, $d(x, y) \leq r_2(x) + r_1(y)$. Then there exists $(R_1, R_2) : X \longrightarrow [0, \infty)$ which extends the pair (r_1, r_2) such that for all $x, y \in X$, $d(x, y) \leq R_2(x) + R_1(y)$. Moreover, there exists a minimal pair of functions (f_1, f_2) defined on X such that for all $x \in X$, $f_1(x) \leq R_1(x)$ and $f_2(x) \leq R_2(x)$.*

Proof. Choose $x_0 \in A$ fixed. Define $(R_1, R_2) : X \longrightarrow [0, \infty)$ by setting

$R_1(x) = r_1(x)$ if $x \in A$ and $R_1(x) = d(x_0, x) + r_1(x_0)$ if $x \notin A$

and

$R_2(x) = r_2(x)$ if $x \in A$ and $R_2(x) = d(x, x_0) + r_2(x_0)$ if $x \notin A$. We next check our claim that (R_1, R_2) is tight. Let $x, y \in X$: We consider 4 cases.

Case 1: $x, y \in A \Rightarrow d(x, y) \leq R_2(x) + R_2(y)$ by assumption.

Case 2: $x \notin A, y \in A \Rightarrow d(x, y) \leq d(x, x_0) + d(x_0, y)$ by the triangle inequality

$\Rightarrow d(x, y) \leq d(x, x_0) + r_2(x_0) + r_1(y)$ by assumption

$\Rightarrow d(x, y) \leq R_2(x) + R_1(y)$ by definitions.

Case 3: $x \in A, y \notin A \Rightarrow d(x, y) \leq d(x, x_0) + d(x_0, y)$

$\Rightarrow d(x, y) \leq r_2(x) + r_1(x_0) + d(x_0, y)$ as in Case 2.

$\Rightarrow d(x, y) \leq R_2(x) + R_1(y)$

Case 4: $x \notin A, y \notin A \Rightarrow d(x, y) \leq d(x, x_0) + d(x_0, y)$ by the triangle inequality

$\Rightarrow d(x, y) \leq d(x, x_0) + r_2(x_0) + d(x_0, y) + r_1(x_0)$ by assumption

$\Rightarrow d(x, y) \leq R_2(x) + R_1(y)$ by definitions.

Then $(R_1, R_2) : X \longrightarrow [0, \infty)$ extends (r_1, r_2) such that for all

$$x, y \in X, d(x, y) \leq R_2(x) + R_1(y).$$

So (R_1, R_2) is a q-tight pair on X. Since $(\mathcal{FP}(X, d), D)$ is partially ordered by the pointwise order on function pairs, Zorn's Lemma implies the existence of a minimal pair element $f = (f_1, f_2)$ defined on X such that $f_1(x) \leq R_1(x)$ and $f_2(x) \leq R_2(x)$ for any $x \in X$. $\square$

The following result is useful to show that the q-hyperconvex hull of a T_0-quasi-metric space is unique up to isometries and is similar to Theorem 1.2.1(4).

3.3.5 Proposition. *Let (X, d) be a T_0-quasi-metric space. If $s = (s_1, s_2)$ is a minimal q-tight pair of functions on the T_0-quasi-metric space $\epsilon_q(X)$, then $s \circ e_X$ is a minimal q-tight pair of functions on X.*

Proof. Let $s = (s_1, s_2)$ be a minimal q-tight pair of functions on the T_0-quasi-metric space $\epsilon_q(X)$. Note that for any $x, y \in X$, we have $d(x, y) = D(f_x, f_y) = D(e_X(x), e_X(y)) \leq s_2(e_X(x)) + s_1(e_X(y))$, because (s_1, s_2) is q-tight on $\epsilon_q(X)$. Assume that $s \circ e_X$ is not a minimal q-tight pair of functions on X. Then there exists a pair $(h_1, h_2) \in \epsilon_q(X)$ such that $h_1(x) \leq s_1(e_X(x))$ and $h_2(x) \leq s_2(e_X(x))$ whenever $x \in X$, and one of the inequalities is strict at some point $x_0 \in X$.

In the following we shall consider the case that $h_1(x_0) \leq s_1(e_X(x_0))$. (The case $h_2(x_0) \leq s_2(e_X(x_0))$ can be dealt with similarly.) Define the function pair (t_1, t_2) on $\epsilon_q(X)$ where $t_2(f) = s_2(f)$ whenever $f \in \epsilon_q(X)$, and for $f \in \epsilon_q(X)$, $f_1(f) = s_1(f)$ if $f \neq e_X(x_0)$ and $f_1(f) = s_1(f)$ if $f = e_X(x_0)$.

Let us show that (t_1, t_2) satisfies the inequality $D(f, g) \leq t_2(f) + t_1(g)$ whenever $f, g \in \epsilon_q(A)$, which will contradict the fact that (s_1, s_2) is a minimal q-tight pair of functions on $\epsilon_q(X)$ so that our assumption that (h_1, h_2) exists is false. Since each (t_1, t_2) and (s_1, s_2) is a q-tight pair of functions, we only need to prove the above inequality for $g = e_X(x_0)$ and $f \neq e_X(x_0)$, i.e. $D(f, e_X(x_0)) \leq t_2(f) + t_1(e_X(x_0))$.

We distinguish two cases: Case 1: $f_1(x_0) = 0$. Then $D(f, e_X(x_0)) = f_1(x_0) = 0$ by Lemma 3.3.2 and our claim is obviously satisfied.

Case 2: $f_1(x_0) > 0$. By Proposition 3.3.2 $f_1(x) = \sup_{y \in X}(d(y, x_0) - f_2(y))$. Therefore for any $\delta > 0$ there exists $y \in X$ such that $f_1(x_0) - \delta \leq d(y, x_0) - f_2(y)$ and, thus, $f_2 + f_1(x_0) \leq d(y, x_0) + \delta$. If $y = x_0$, then we must have $f_1(x_0) < \delta$. Hence $D(f, e_X(x_0)) = f_1(x_0) \leq \delta + t_2(f) + t_1(e_X(x_0))$.

On the other hand, if $y \neq x_0$ and $f \neq e_X(x_0)$, then $f_2(y) + D(f, e_X(x_0)) - \delta = f_2(y)+f_1(x_0)-\delta < d(y, x_0)$ and $d(y, x_0) \leq h_2(y)+h_1(x_0) \leq t_2(e_X(y))+t_1(e_X(x_0))$.

Since (s_1, s_2) is a minimal q-tight pair of functions on $\epsilon_q(X)$, then by Lemma 3.3.2 $t_2(e_X(y)) = s_2(e_X(y)) \leq s_2(f) + D(e_X(y), f) = t_2(f) + f_2(y)$ whenever $f \in \epsilon_q(X)$, where we have used the fact that $f = (f_1, f_2)$ is a minimal q-tight pair of functions on X. So we have the two inequalities $f_2(y) + D(f, e_X(x_0)) - \delta < t_2(e_X(y)) + t_1(e_X(x_0))$ and $t_2(e_X(y)) \leq t_2(f) + f_2(y)$.

Adding the two inequalities, we get $f_2(y) + D(f, e_X(x_0)) - \delta + t_2(e_X(y)) < t_2(e_X(y)) + t_1(e_X(x_0)) + t_2(f) + f_2(y)$ which leads to $D(f, e_X(x_0)) - \delta \leq t_2(f) + t_1(e_X(x_0))$. Since δ is arbitrary, we get the desired inequality $D(f, e_X(x_0)) \leq t_2(f) + t_1(e_X(x_0))$. We conclude that $s \circ e_X \in \epsilon_q(X)$. □

Our next result shows that the q-hyperconvex hull is unique up to isometries.

3.3.6 Proposition. *The following statements are true for any T_0-quasi-metric space (X, d).*

(a) $\epsilon_q(X)$ is q-hyperconvex;

(b) $\epsilon_q(X)$ is a q-injective hull of X, i.e. no proper subset of $\epsilon_q(X)$ which contains X as a quasi-metric subspace is q-hyperconvex. The q-hyperconvex hull of the T_0-quasi-metric space (X, d) is unique up to isometries.

Proof. (a) In order to prove that $\epsilon_q(X)$ is q-hyperconvex, let $(f)_{i\in I}$ be a family of pairwise distinct points $f_i = ((f_i)_1, (f_i)_2) \in \epsilon_q(X)$ and $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be two families of nonnegative real numbers such that $D(f_i, f_j) \leq r_i + s_j$ whenever $i, j \in I$. Set $Y = \{f_i : i \in I\}$. Define a map $s : Y \longrightarrow [0, \infty)$ by $s(f_i) = s_i$ and $r : Y \longrightarrow [0, \infty)$ by $r(f_i) = r_i$ whenever $i \in I$. By Lemma 3.3.7, we extend r and s to (R, S) on the entire set $\epsilon_q(X)$ such that $D(f, g) \leq R(f) + S(g)$ whenever $f, g \in \epsilon_q(X)$.

Using Lemma 3.3.7, there exists a minimal q-tight pair $h = (h_1, h_2)$ of functions on $\epsilon_q(X)$ such that $h_2 \leq R$ and $h_1 \leq S$ where R and S are the extensions of r and s, respectively.

Using the property established in Proposition 3.3.5 for a minimal q-tight pair of functions on $\epsilon_q(X)$, we know that $h \circ e_X \in \epsilon_q(X)$. It is then easy to see that

$$h \circ e_X \in \bigcap\nolimits_{f \in \epsilon_q(X)} (C_D(f, R(f)) \cap C_{D^{-1}}(f, S(f)))$$

$$\subseteq \bigcap\nolimits_{i \in I} (C_D(f_i, r_i) \cap C_{D^{-1}}(f_i, s_i)) :$$

Indeed, the distance D between $h \circ e_X$ and $f = (f_1, f_2) \in \epsilon_q(X)$ is defined by

$$D(h \circ e_X, f) = \sup\nolimits_{x \in X} \{h_1(e_X(x)) \dot{-} f_1(x)\} \vee \sup\nolimits_{x \in X} \{f_2(x) \dot{-} h_2(e_X(x))\}.$$

Using Lemma 3.3.4, we can write $f_1(x) = D(f, e_X(x))$ and $f_2(x) = D(e_X(x), f)$. Furthermore, since $h = (h_1, h_2)$ is a minimal q-tight pair of functions on $\epsilon_q(X)$ and using Lemma 3.3.2, we have:

$$h_1(e_X(x)) - D(f, e_X(x)) \leq h_1(f).$$

By q-tightness of (h_1, h_2), we see that

$$D(e_X(x), f) - h_2(e_X(x)) \leq h_1(f).$$

By the choice of $h = (h_1, h_2)$, we have

$$h_1(f) \leq S(f).$$

Therefore we get that

$$D(h \circ e_X, f) \leq h_1(f) \leq S(f)$$

whenever $f = (f_1, f_2) \in \epsilon_q(X)$. Similarly we see that

$$D(f, h \circ e_X) \leq h_2(f) \leq R(f)$$

whenever $f = (f_1, f_2) \in \epsilon_q(X)$. The proof is therefore complete.

(b) Let H be a subset of $\epsilon_q(X)$ such that $X \subseteq H$. Assume that H is q-hyperconvex, hence q-injective by Theorem 3.3.1. There exists a nonexpansive map (R_1, R_2) extending the inclusion map $i : X \longrightarrow H$ such that

$$R = (R_1, R_2) : \epsilon_q(X) \longrightarrow H$$

$$f = (f_1, f_2) \longmapsto (R_1(f), R_2(f)).$$

Using Lemma 3.3.4 and nonexpansivity of $R = (R_1, R_2)$, we have

$$(R_1(f))(x) = D(R(f), f_x) = D(R(f), R(f_x)) \leq D(f, f_x) = f_1(x)$$

whenever $x \in X$. Similarly $(R_2(f))(x) \leq f_2(x)$ whenever $x \in X$.

Since $f = (f_1, f_2)$ is a minimal q-tight pair of functions on X, we must have $R_1(f) = f_1$ and $R_2(f) = f_2$. This implies that R is the identity map and $H = \epsilon_q(X)$. Consequently, no proper subset of $\epsilon_q(X)$ which contains X is q-hyperconvex.

Let H be any q-hyperconvex T_0-quasi-metric space which contains X as a subspace such that no proper subset of H which contains X as a subset is q-hyperconvex. Consider a nonexpansive map $\phi : \epsilon_q(X) \longrightarrow H$ extending the inclusion map $i : e_X(X) \longrightarrow H$ defined by $i(e_X(x)) = x$ whenever $x \in X$. Furthermore consider a nonexpansive map $\varphi : H \longrightarrow \epsilon_q(X)$ extending the map $i^{-1} : X \longrightarrow \epsilon_q(X)$. Then the contraction $\varphi \circ \phi : \epsilon_q(X) \longrightarrow \epsilon_q(X)$ extends the identity map on $e_X(X)$. The argument described in the preceding step of the proof implies that $\varphi \circ \phi$ is the identity map on $\epsilon_q(X)$. Hence φ is surjective. It also follows that ϕ and φ are isometries, because ϕ and φ are nonexpansive. Hence φ is also injective, since H is a T_0-space by [25, Lemma 4]. Thus φ is bijective and $\phi = \varphi^{-1}$. We have shown that $\epsilon_q(X)$ and H are isometric T_0-quasi-metric spaces. $\square$

3.3.1 **Corollary.** *The following statements are equivalent for a T_0-quasi-metric space (X, d):*

(a) (X, d) is q-hyperconvex.

(b) For each $f \in \epsilon_q(X)$ there is $x \in X$ such that $f_1 = (f_x)_1$ and $f_2 = (f_x)_2$.

(c) For each $f \in \epsilon_q(X)$ there is $x \in X$ such that $f_1(x) = f_2(x) = 0$.

Proof. This a consequence of Proposition 3.3.6 and Lemma 3.3.6. □

We next show that total boundedness is preserved by the q-hyperconvex hull.

3.3.7 **Proposition.** *If (X, d) is a totally bounded T_0-quasi-metric space, then the T_0-quasi-metric space $(\epsilon_q(X, d), D)$ is totally bounded, too.*

Proof. By total boundedness of (X, d) there are $k \in \mathbb{N}$ and $y_1, ..., y_k \in X$ such that for each $x \in X$, there is $i \in \{1, ..., k\}$ such that $d^s(x, y_i) \leq 1$. It follows from the triangle inequality that for any $x, y \in X$ we have $d(x, y) \leq b := \max\{d(y_i, y_j) + 2 : i, j \in \{1, ..., k\}\}$, that is, d is bounded. Obviously boundedness of d by b implies that $f_1(X), f_2(X) \subseteq [0, b]$, whenever (f_1, f_2) is a minimal tight pair of functions.

Let $\epsilon > 0$. By total boundedness of (X, d), there exist $x_1, ..., x_n \in X$ such that for any $x \in X$, there exists $i \in \{1, ..., n\}$ such that $d^s(x, x_i) < \epsilon$.

There are $c_1, ..., c_m \in [0, b]$ such that for any $c \in [0, 1]$ there is $i \in \{1, ..., m\}$ such that $\mid c - c_i \mid \leq \epsilon$.

Consider any pair $(\psi_1, \psi_2) \in \{1, ..., m\}^{\{1,...,n\}} \times \{1, ..., m\}^{\{1,...,n\}}$. Define $\lambda_{(\psi_1,\psi_2)} = \{(f_1, f_2) \in \epsilon_q(X, d) : \sup_{1\leq i\leq n}(\mid f_1(x_i) - c_{\psi_1(i)} \mid \vee \mid f_2(x_i) - c_{\psi_2(i)} \mid)\}$. Note that $\epsilon_q(X, d)$ is the union of the finitely many sets $\lambda_{(\psi_1,\psi_2)}$. We show that

$\bigcup_{(\psi_1,\psi_2)\in\{1,...,m\}^{\{1,...,n\}}\times\{1,...,m\}^{\{1,...,n\}}} \lambda_{(\psi_1,\psi_2)}{}^2 \subseteq U_{D,4\epsilon}$.

For $(\psi_1, \psi_2) \in \{1, ..., m\}^{\{1,...,n\}} \times \{1, ..., m\}^{\{1,...,n\}}$, let $(f_1, f_2), (g_1, g_2) \in \lambda_{(\psi_1,\psi_2)}$.

For given $x \in X$, there exists $i \in \{1, ..., n\}$ such that $d^s(x, x_i) \leq \epsilon$. Then we have $f_1(x)\dot{-}g_1(x) \leq (f_1(x)\dot{-}f_1(x_i)) + (f_1(x_i)\dot{-}c_{\psi_1(i)}) + (c_{\psi_1(i)}\dot{-}g_1(x_i)) + (g_1(x_i)\dot{-}g_1(x))$ which implies that $f_1(x)\dot{-}g_1(x) \leq 4\epsilon$. Hence $\sup_{x\in X}(f_1(x)\dot{-}g_1(x)) \leq 4\epsilon$.

Analogously one shows that $\sup_{x\in X}(g_2(x)\dot{-}f_2(x)) \leq 4\epsilon$.

We conclude that $D((f_1, f_2), (g_1, g_2)) \leq 4\epsilon$. Hence we are done. □

3.3.2 Corollary. *If (X, d) is a T_0-quasi-metric space such that $\tau(d^s)$ is compact, then D^s induces a compact topology on $\epsilon_q(X, d)$.*

Proof. If $\tau(d^s)$ is compact, then the T_0-quasi-metric d is totally bounded. By Proposition 3.3.7 $(\epsilon_q(X, d), D)$ is totally bounded. The result follows, since the metric D^s is always complete on $\epsilon_q(X, d)$, because $(\epsilon_q(X, d), D)$ is q-hyperconvex. □

In the following, we are going to compute for two simple T_0-quasi-metric spaces their q-hyperconvex hulls.

3.3.1 Example. *Let $X = \{0, 1\}$ be equipped with the discrete metric $d(x, x) = 0$ whenever $x \in X$ and $d(x, y) = 1$ whenever $x \neq y$. As expected, the q-hyperconvex hull $\epsilon_q(X, d)$ of (X, d) is strictly larger than the (metric) hyperconvex hull $\epsilon_m(X, d)$ of (X, d).*

For each $(\alpha, \beta) \in [0, 1]^2$ define $(\alpha, \beta) = ((\alpha, \beta)_1, (\alpha, \beta)_2)$ as follows: $(\alpha, \beta)_1(0) = \alpha$, $(\alpha, \beta)_1(1) = \beta$, $(\alpha, \beta)_2(0) = 1 - \beta$ and $(\alpha, \beta)_2(1) = 1 - \alpha$. It is readily checked that these are exactly the minimal tight pairs belonging to $\epsilon_q(X, d)$.

Note that $D((\alpha, \beta), (\alpha', \beta')) = (\alpha\dot{-}\alpha') \vee (\beta\dot{-}\beta')$ whenever (α, β), $(\alpha', \beta') \in [0, 1]^2$.

Hence the quasi-metric q-hyperconvex hull $\epsilon_q(X, d)$ of (X, d) can be identified with $([0, 1] \times [0, 1], D)$. Obviously the (metric) hyperconvex hull $\epsilon_m(X, d)$ is isometric to the subspace of pairs (f_1, f_2) satisfying $f_1 = f_2$, that is the subspace $\{(\alpha, 1-\alpha)$:

$\alpha \in [0,1]\}$.

3.3.2 Example. *Let* $X = \{0,1\}$ *and* $d(0,1) = 0$, $d(1,0) = 1$, $d(0,0) = 0$ *and* $d(1,1) = 0$. *For each* $\alpha \in [0,1]$ *set* $(f_\alpha)_1(0) = \alpha$, $(f_\alpha)_1(1) = 0$, $(f_\alpha)_2(0) = 0$ *and* $(f_\alpha)_2(1) = 1 - \alpha$.

It is readily checked that in this way we get $\epsilon_q(X,d) = \{f_\alpha : \alpha \in [0,1]\}$. *Note that* $D(f_\alpha, f_{\alpha'}) = \alpha \dot{-} \alpha'$ *whenever* $\alpha, \alpha' \in [0,1]$. *Thus* $(\epsilon_q(X,d), D)$ *is isometric to the real unit interval with its standard* T_0*-quasi-metric* r *restricted to that interval (see Example 3.1.1).*

Chapter 4

Properties of q-hyperconvex spaces

We first summarize some of the properties of the hyperconvex metric spaces, which were investigated by many authors see ([4], [20], [25], [29], [32], and [34]). Secondly we start an investigation of the properties of q-hyperconvex quasi-metric spaces; for instance, the intersection of any descending family of q-hyperconvex quasi-metric spaces is a q-hyperconvex quasi-metric space. We have tried to indicate where possible the correspondence (or lack thereof) between the results obtained for q-hyperconvexity and those obtained for hyperconvexity. The approximation of the concept of a fixed point of nonexpansive map and the class of externally q-hyperconvex spaces will be described separately.

4.1 Intersection of q-hyperconvex spaces and fixed point theorems

In this section we begin by generalizing some notations (refer to [34, p. 79]), which are useful in proving some properties of q-hyperconvex spaces. But note that the definition of the bicover of a set A is new.

Let (X, d) be a T_0-quasi-metric space. For a nonempty bounded subspace A of X, we set:

$r_x(A)_d := \sup\{d(x, y) : y \in A\}, \text{where} \quad x \in X$ and

$r_x(A)_{d^{-1}} := \sup\{d^{-1}(x, y) : y \in A\}, \text{where} \quad x \in X.$

Moreover let $r_x(A) := r_x(A)_d \vee r_x(A)_{d^{-1}}$ where $x \in X$.

$r_X(A) := \inf\{r_x(A) : x \in X\}$ and $r(A) := \inf\{r_x(A) : x \in A\}$.

Also set $\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}$.

Furthermore $C(A) := \{x \in A : r_x(A) = r(A)\}$ and $C_X(A) := \{x \in X : r_x(A) = r(A)\}$.

Finally $\operatorname{cov}(A)_d := \cap\{C_d(x, r) : A \subseteq C_d(x, r), x \in X, r \geq 0\}$ and

$\operatorname{cov}(A)_{d^{-1}} := \cap\{C_{d^{-1}}(x, s) : A \subseteq C_{d^{-1}}(x, s), x \in X, s \geq 0\}$ and

$\operatorname{bicov}(A) := \operatorname{cov}(A)_d \cap \operatorname{cov}(A)_{d^{-1}}$.

Note that the values of $\operatorname{diam}(A)$, $r_x(A), r(A), C_X(A), C(A)$ do not change when defined for the space (X, d^s) instead of (X, d).

$r_X(A)$ is called the *radius* of A (relative to X), $\operatorname{diam}(A)$ is called the *diameter* of A, $C(A)$ is called the *center* of A (relative to X), $C_X(A)$ is called the *Chebyshev* *center* of A, $r(A)$ is called the *Chebyshev radius* of A and $\text{bicov}(A)$ is called the *bicover* of A.

4.1.1 Remark. *Let (X,d) be a T_0-quasi-metric space. Let A be a nonempty bounded subset in X. Then*

$$cov(A)_{d^s} = \cap\{C_{d^s}(x,r) : A \subseteq C_{d^s}(x,r), x \in X, r \geq 0\}.$$

Obviously we have $bicov(A) \subseteq cov(A)_{d^s}$.

4.1.1 Example. *Let $X = [0,1] \times [\frac{1}{4}, \frac{3}{4}]$ be equipped with the T_0-quasi-metric defined by $D((\alpha,\beta),(\alpha',\beta')) = (\alpha \dot{-} \alpha') \vee (\beta \dot{-} \beta')$ whenever $(\alpha,\beta),(\alpha',\beta') \in X$.*

Consider $A = \{(0,\frac{1}{2}),(1,\frac{1}{2})\} \subseteq X$. Then $bicov(A)$ is equal to the line segment from $x = (0,\frac{1}{2})$ to $y = (1,\frac{1}{2})$. This follows from the fact that for each $\epsilon \in [0,\frac{1}{4}]$, $y \in C_D(x,\epsilon) = [0,1] \times [\frac{1}{2} - \epsilon, \frac{3}{4}]$ and $x \in C_{D^{-1}}(y,\epsilon) = [0,1] \times [\frac{1}{4}, \frac{1}{2} + \epsilon]$ and that segment is a subset of any set of the form $C_D(a,r) \cap C_{D^{-1}}(b,s)$ for which $\{x,y\} \subseteq C_D(a,r) \cap C_{D^{-1}}(b,s)$. Indeed assume that z belongs to this segment. Then $D(z,y) = 0 = D(x,z)$ and therefore $z \in C_D(a,r) \cap C_{D^{-1}}(b,s)$ by the triangle inequality.

On the other hand $cov(A)_{d^s} = X$, since $\{x,y\} \subseteq C_{D^s}(z,\epsilon)$ with $z \in X$ implies that $\epsilon \leq \frac{1}{2}$. Indeed assume that $z = (a,b)$. Then $a \leq D^s((a,b),(0,\frac{1}{2})) \leq \epsilon$ and $1 - a \leq D^s((a,b),(a,\frac{1}{2})) \leq \epsilon$. Thus $\epsilon \geq \max\{a, 1-a\} \geq \frac{1}{2}$. It follows that $X \subseteq C_{D^s}(z,\epsilon)$, because the interval $[\frac{1}{4},\frac{3}{4}]$ has length $\frac{1}{2}$. Therefore $cov_{D^s}(A) = X$.

The following lemma should be compared with [34, Lemma 4.1].

4.1.1 Lemma. *Let A be a nonempty bounded subspace of a q-hyperconvex T_0-quasi-metric space (X,d). Then:*

(1) *$bicov(A) = \bigcap_{x \in X}(C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}}))$.*

(2) *$r_x(bicov(A)) = r_x(A)$, for any $x \in X$.*

(3) $r(bicov(A)) = r(A)$.

(4) $r(A) = \frac{1}{2} diam(A)$.

(5) $diam(bicov(A)) = diam(A)$.

Proof. (1) Let $x \in X$. For $y \in A$, we have $d(x,y) \leq \sup\{d(x,y) : y \in X\}$. Then $d(x,y) \leq r_x(A)_d$ which implies $y \in C_d(x, r_x(A)_d)$. Hence $A \subseteq C_d(x, r_x(A)_d)$ whenever $x \in X$. It must therefore be the case that $\text{cov}(A)_d \subseteq C_d(x, r_x(A)_d)$ whenever $x \in X$.

Similarly one can show that $A \subseteq C_{d^{-1}}(x, r_x(A)_{d^{-1}})$ whenever $x \in X$. We then have $\text{cov}(A)_{d^{-1}} \subseteq C_{d^{-1}}(x, r_x(A)_{d^{-1}})$ whenever $x \in X$.

Then

$$\text{bicov}(A) \subseteq \bigcap\nolimits_{x \in X} (C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}})). \tag{4.1}$$

On the other hand, suppose that $A \subseteq C_d(x,r)$ and $A \subseteq C_{d^{-1}}(x,s)$ for some $x \in X$ and $r, s \geq 0$. For any $y \in A$, we have $d(x,y) \leq r$ and $d^{-1}(x,y) \leq s$ which implies $r_y(A)_d \leq r$ and $r_y(A)_{d^{-1}} \leq s$. Thus $C_d(x, r_x(A)_d) \subseteq C_d(x,r)$. Hence $C_d(x, r_x(A)_d) \subseteq \text{cov}(A)_d$ whenever $x \in X$.

Similarly one can show that $C_d(x, r_x(A)_{d^{-1}}) \subseteq \text{cov}(A)_{d^{-1}}$ whenever $x \in X$.

Hence $C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}}) \subseteq \text{bicov}(A)$ whenever $x \in X$.

Furthermore

$$\bigcap\nolimits_{x \in X} (C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}})) \subseteq \text{bicov}(A). \tag{4.2}$$

Combination of (4.1) and (4.2) yields

$$\text{bicov}(A) = \bigcap\nolimits_{x \in X} (C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}})).$$

(2) By (1) we have that $r_x(\text{bicov}(A)) = \sup\{d(x,y) : y \in \bigcap_{x\in X}(C_d(x, r_x(A)_d) \cap C_{d^{-1}}(x, r_x(A)_{d^{-1}}))$. In particular, $y \in \text{bicov}(A)$ implies that $y \in C_d(x, r_x(A)_d)$ and $y \in C_{d^{-1}}(x, r_x(A)_{d^{-1}})$ whenever $x \in X$.

Hence $d(x,y) \leq r_x(A)_d$ and $d^{-1}(x,y) \leq r_x(A)_{d^{-1}}$, which implies

$$r_x(\text{bicov}(A))_d \leq r_x(A)_d \leq r_x(A)$$

and

$$r_x(\text{bicov}(A))_{d^{-1}} \leq r_x(A)_{d^{-1}} \leq r_x(A).$$

Altogether we have $r_x(\text{bicov}(A)) = r_x(\text{bicov}(A))_d \vee r_x(\text{bicov}(A))_{d^{-1}} \leq r_x(A)$. The reverse inequality is obvious since $A \subseteq \text{bicov}(A)$.

(3) This is immediate from the definition of r and property (2).

The proof of (4) and (5) can be completed similarly as in the proof of [34, Lemma 4.1]. Moreover $r(A)$ and $\text{diam}(A)$ are concepts from symmetric topology. □

We next define a q-admissible subset of a T_0-quasi-metric space similarly to [34, Definition 4.2].

4.1.1 Definition. *Let (X,d) be a T_0-quasi-metric space. A nonempty bounded subset D of X is* q-admissible *if $D = bicov(D)$.*

The collection of all q-admissible subsets of a T_0-quasi-metric space (X,d) will be denoted by $\mathcal{A}_q(X)$.

4.1.2 Remark. *Let (X,d) be a T_0-quasi-metric space.*

(a) Note that a subset of X is q-admissible if and only if it can be written as the intersection of a family of sets of the form $C_d(x,r) \cap C_{d^{-1}}(x,s)$ with $r,s \geq 0$ and $x \in X$. For this reason, the family $\mathcal{A}_q(X)$ is closed under nonempty intersection of nonempty families.

(b) For any $A \in \mathcal{A}_q(X)$ let $\delta = diam(A)$. We have that

$$C(A) = \bigcap_{a\in A}(C_d(a, \frac{\delta}{2}) \cap C_{d^{-1}}(a, \frac{\delta}{2})) \cap A \in \mathcal{A}_q(X).$$

Moreover, $diam(C(A)) \leq diam(A)/2$. So we have $A = C(A)$ if and only if $A \in \mathcal{A}_q(X)$ and $diam(A) = 0$, i.e. A is reduced to one point.

Indeed, let $y \in \bigcap_{a\in A}(C_d(a, \frac{\delta}{2}) \cap C_{d^{-1}}(a, \frac{\delta}{2})) \cap A$. Then $d(a, y) \leq \frac{\delta}{2}$ and $d^{-1}(a, y) \leq \frac{\delta}{2}$. Therefore $r_y(A)_d \leq \frac{\delta}{2}$ and $r_y(A)_{d^{-1}} \leq \frac{\delta}{2}$, hence $r_y(A) \leq \frac{\delta}{2}$ whenever $y \in A$.

From Lemma 4.1.1(4), we have that $\frac{\delta}{2} = r_X(A) \leq r(A) = r_y(A) \leq \frac{\delta}{2}$ whenever $y \in A$. Then for any $y \in A$, $r_y(A) = r(A) = \frac{\delta}{2}$. Therefore $y \in C(A)$.

On the other hand, consider $t \in C(A)$. Then $t \in A$ and $r_t(A) = r(A)$. From Lemma 4.1.1(4), $\frac{\delta}{2} = r_X(A) \leq r(A) = r_t(A)$ for any $t \in A$. But by $r_t(A) = r(A) = \inf\{r_y(A) : y \in A\} = \frac{\delta}{2}$ we have that $r_t(A)_d \leq \frac{\delta}{2}$ and $r_t(A)_{d^{-1}} \leq \frac{\delta}{2}$. Hence $t \in C_d(a, \frac{\delta}{2})$ and $t \in C_{d^{-1}}(a, \frac{\delta}{2})$ whenever $a \in A$. Therefore $t \in \bigcap_{a\in A}(C_d(a, \frac{\delta}{2}) \cap C_{d^{-1}}(a, \frac{\delta}{2})) \cap A$. □

The following result shows that any q-admissible subset of a T_0-quasi-metric space is q-hyperconvex. It should be compared with [34, Proposition 4.5].

4.1.1 Proposition. *Suppose (X, d) is a q-hyperconvex T_0-quasi-metric space. Then $D \in \mathcal{A}_q(X)$ is itself q-hyperconvex.*

Proof. Since $D \in \mathcal{A}_q(X)$ implies that D is q-admissible, so we can write $D = \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset$, $x_i \in X$, r_i and s_i are nonnegative real numbers whenever $i \in I$.

Let $(C_d(x_\alpha, r_\alpha)$,$C_{d^{-1}}(x_\alpha, s_\alpha))_{\alpha\in A}$ be a family of balls, where $x_\alpha \in D$ whenever $\alpha \in A$ and $d(x_\alpha, x_\beta) \leq r_\alpha + s_\beta$ whenever $\alpha, \beta \in A$. Then by q-hyperconvexity of X, $\bigcap_{\alpha\in A}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \neq \emptyset$.

Now consider the family of balls

$((C_d(x_\alpha, r_\alpha))_{\alpha\in A}, (C_{d^{-1}}(x_\alpha, s_\alpha))_{\alpha\in A}, ((C_d(x_i, r_i))_{i\in I}, (C_{d^{-1}}(x_i, s_i))_{i\in I})$.

We have for each $\alpha \in A$ and $i \in I$,

$$d(x_\alpha, x_i) \le s_i \le r_\alpha + s_i$$

and

$$d(x_i, x_\alpha) \le r_i \le r_i + s_\alpha.$$

Furthermore for all $i, j \in I$ and any $\alpha \in A$ we have that

$$d(x_i, x_j) \le d(x_i, x_\alpha) + d(x_\alpha, x_j) \le r_i + s_j,$$

so it again follows from the q-hyperconvexity of X that

$(\bigcap_{\alpha\in A}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha))) \cap (\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)))$

$= (\bigcap_{\alpha\in A}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha))) \cap D \neq \emptyset.$

Hence the subspace D of X is q-hyperconvex. □

4.1.1 Corollary. *Let (X, d) be a q-hyperconvex T_0-quasi-metric space. Let A be a nonempty bounded subset in X. Then $(bicov(A), d)$ is q-hyperconvex, while $(cov(A)_{d^s}, d^s)$ is hyperconvex, but not q-hyperconvex.*

Proof. This a consequence of Lemma 4.1.1 and Proposition 3.1.1. □

The proposition below is the quasi-metric space analogue of [34, Theorem 4.8] for metric spaces.

4.1.1 Theorem. *If (X, d) is a bounded q-hyperconvex T_0-quasi-metric space and if $T : X \longrightarrow X$ is nonexpansive, then the fixed point set $Fix(T)$ is nonempty and q-hyperconvex.*

Proof. Consider a nonexpansive map $T : (X, d) \longrightarrow (X, d)$. We first show that $Fix(T) \neq \emptyset$.

By nonexpansivity of T, we have

$$d(T(x), T(y)) \leq d(x, y) \leq d^s(x, y)$$

and

$$d^{-1}(T(x), T(y)) \leq d^{-1}(x, y) \leq d^s(x, y) \quad \text{whenever} \quad x, y \in X.$$

Then $d^s(T(x), T(y)) \leq d^s(x, y)$ whenever $x, y \in X$. Hence $T : (X, d^s) \longrightarrow (X, d^s)$ is a nonexpansive map and (X, d^s) is bounded, since (X, d) bounded. By Proposition 3.1.1 (b) (X, d^s) is a hyperconvex metric space. Since (X, d^s) is a bounded hyperconvex space and $T : (X, d^s) \longrightarrow (X, d^s)$ is a nonexpansive map by [34, Theorem 4.8], we have that $Fix(T) \neq \emptyset$.

We need now to show that $Fix(T)$ is q-hyperconvex. Let $(C_d(x_i, r_i), C_{d^{-1}}(x_i, s_i))_{i \in I}$ be a family of balls, where $x_i \in Fix(T)$ such that $d(x_i, x_j) \leq r_i + s_j$, for $i, j \in I$, Since X is q-hyperconvex, the set

$$X_0 = \bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset.$$

Let $z \in X_0$. Then $r_i \geq d(x_i, z) \geq d(T(x_i), T(z)) = d(x_i, T(z))$ which implies $T(z) \in C_d(x_i, r_i)$, and by a similar argument $T(z) \in C_{d^{-1}}(x_i, s_i)$. Therefore $T(X_0) \subseteq X_0$.

Moreover X_0 is a bounded q-hyperconvex T_0-quasi-metric space by Proposition 4.1.1. So the first part of the proof implies that T has a fixed point in X_0, which implies

$$Fix(T) \cap [\bigcap_{i \in I} (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))] \neq \emptyset.$$

This proves that $Fix(T)$ is q-hyperconvex. □

In the next theorem we successfully show that the intersection of a descending family of q-hyperconvex spaces is as well q-hyperconvex. It should be compared with [25, Theorem 5.1].

4.1.2 Theorem. *(compare [6, Theorem 7]) Let (X, d) be a bounded T_0-quasi-metric space. Let $\{H_i\}_{i\in I}$ be a descending family of nonempty bounded q-hyperconvex subsets of X, where we suppose that I is totally ordered by $i \leq j \Leftrightarrow H_j \subseteq H_i$. Then $\bigcap_{i\in I} H_i$ is nonempty and q-hyperconvex.*

Proof. We begin by showing that $H = \bigcap_{i\in I} H_i \neq \emptyset$. Consider (X, d) a bounded T_0-quasi-metric space and let $\{(H_i, d|_{H_i})\}_{i\in I}$ be a descending family of nonempty bounded q-hyperconvex subsets of X such that $i \leq j \Leftrightarrow H_j \subseteq H_i$ whenever $i, j \in I$.

By Proposition 3.1.1 (b) $(H_i, (d|_{H_i})^s)$ is a bounded hyperconvex metric space whenever $i \in I$. By Baillon's theorem (see [6, Theorem 7]) $H = \bigcap_{i\in I} H_i \neq \emptyset$.

In order to complete the proof, we need to show that $H = \bigcap_{i\in I} H_i$ is q-hyperconvex. Let a family $(x_\alpha)_{\alpha\in\Gamma}$ of points in H and families of nonnegative real numbers $(r_\alpha)_{\alpha\in\Gamma}$ and $(s_\alpha)_{\alpha\in\Gamma}$ be given such that $d(x_\alpha, x_\beta) \leq r_\alpha + s_\beta$ whenever $\alpha, \beta \in \Gamma$.

Since H_i is a q-hyperconvex space for each $i \in I$ and $x_\alpha \in H_i$ whenever $\alpha \in \Gamma$, therefore $\mathcal{D}_i = \bigcap_{\alpha\in\Gamma}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \cap H_i \neq \emptyset$.
Since $i \leq j \Leftrightarrow H_j \subseteq H_i$ whenever $i, j \in I$ we have $\mathcal{D}_j \subseteq \mathcal{D}_i$ whenever $i \leq j$. $\{\mathcal{D}_i\}_{i\in I}$ is a decreasing family subsets of X .

Thus by the first paragraph of this proof

$$\emptyset \neq \bigcap_{i\in I} \mathcal{D}_i = \bigcap_{i\in I} [\bigcap_{\alpha\in\Gamma} (C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \cap H_i]$$

$$= \bigcap_{\alpha\in\Gamma} (C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \cap \bigcap_{i\in I} H_i,$$

since $\{\mathcal{D}_i\}_{i\in I}$ is descending. This proves that $H = \bigcap_{i\in I} H_i$ is q-hyperconvex. □

4.1.2 Corollary. *If $\{H_i\}_{i\in I}$ is a family of bounded q-hyperconvex subsets of a T_0-quasi-metric space X with the finite intersection property, then the intersection $\bigcap_{i\in I} H_i$ is nonempty and q-hyperconvex.*

Proof. Consider: $\Psi = \{\Lambda \subseteq I : \text{for any} \quad \Omega \quad \text{finite}, \Omega \subseteq I, \bigcap_{\Lambda\cup\Omega} H_i \text{ is nonempty and } q\text{-hyperconvex}\}$. Observe that $\Psi \neq \emptyset$ since $\emptyset \in \Psi$, and then Ψ satisfies the hypothesis of Zorn's Lemma by Theorem 4.1.2. Let Λ be maximal in Ψ. So $\Lambda \cup \{i\} \in \Psi$ whenever $i \in I$. By maximality of Λ, we have that $i \in \Lambda$ whenever $i \in I$. □

4.1.2 Definition. *Let (X, d) be a T_0-quasi-metric space and let a family of nonexpansive maps $\{T_i\}_{i\in I}$, with $T_i : X \longrightarrow X$ be given. We say that $\{T_i\}_{i\in I}$ is a* commuting family *if $T_i \circ T_j = T_j \circ T_i$ whenever $i, j \in I$.*

The next result is a consequence of Theorems 4.1.1 and 4.1.2. It is similar to [25, Theorem 6.2].

4.1.3 Theorem. *Let (X, d) be a bounded q-hyperconvex T_0-quasi-metric space. Any commuting family of nonexpansive maps $\{T_i\}_{i\in I}$, with $T_i : X \longrightarrow X$, has a common fixed point. Moreover, the common fixed point set $\bigcap_{i\in I} Fix(T_i)$ is q-hyperconvex.*

Proof. We know that $Fix(T_i)$ is q-hyperconvex whenever $i \in I$ by Theorem 4.1.1. By Corollary 4.1.2, it suffices to show that $\bigcap_{i\in F} Fix(T_i) \neq \emptyset$ and q-hyperconvex for any finite subset F of I. Suppose $F = \{1, 2, ..., n\}$. Then $Fix(T_i)_{i\in F} = \{Fix(T_1), ..., Fix(T_n)\}$. Since T_1 and T_2 commute, it is immediate that $T_2 : Fix(T_1) \to Fix(T_1)$. Thus $Fix(T_1) \cap Fix(T_2) \neq \emptyset$. Proceeding step by step one concludes that $\bigcap_{i\in F} Fix(T_i) \neq \emptyset$ and q-hyperconvex. □

4.2 Approximate fixed points

In this section we are investigating the approximation of fixed points of a nonexpansive self-map in a q-hyperconvex T_0-quasi-metric space by generalizing some results from metric spaces (see [25] and [34]).

The following defines an ϵ_1, ϵ_2-parallel set of a T_0-quasi-metric subspace similar to [34, p. 89] for $\epsilon_1, \epsilon_2 \geq 0$.

4.2.1 Definition. *Let (X,d) be a T_0-quasi-metric space. For a quasi-metric subspace A of X, we define for $\epsilon_1, \epsilon_2 \geq 0$ the ϵ_1, ϵ_2-*parallel *set of A as*

$$N_{\epsilon_1,\epsilon_2}(A) = \bigcup\nolimits_{a\in A}(C_d(a,\epsilon_2)\cap C_{d^{-1}}(a,\epsilon_1)).$$

(Note that for each $\epsilon > 0$ in particular $N_{\epsilon,\epsilon}(A) = \bigcup_{a\in A}(C_{d^s}(a,\epsilon))$).

Thus $x \in N_{\epsilon_1,\epsilon_2}(A)$ if and only if there exists $a \in A$ such that $d(a,x) \leq \epsilon_2$ and $d^{-1}(a,x) \leq \epsilon_1$.

We next give a characterization of $N_{\epsilon_1,\epsilon_2}(A)$ if A is a q-admissible set in a q-hyperconvex T_0-quasi-metric space.

4.2.1 Lemma. *(compare [34, Lemma 4.2]) Let (X,d) be a q-hyperconvex T_0-quasi-metric space. Let A be a q-admissible subset of X, say $\emptyset \neq A = \bigcap_{i\in I}(C_d(x_i,r_i)\cap C_{d^{-1}}(x_i,s_i))$ with $x_i \in X$ and r_i, s_i nonnegative reals whenever $i \in I \neq \emptyset$. Then for each $\epsilon_1, \epsilon_2 \geq 0$,*

$$N_{\epsilon_1,\epsilon_2}(A) = \bigcap\nolimits_{i\in I}(C_d(x_i, r_i+\epsilon_2)\cap C_{d^{-1}}(x_i, s_i+\epsilon_1)).$$

Proof. Suppose $y \in N_{\epsilon_1,\epsilon_2}(A)$. Then $d(a,y) \leq \epsilon_2$ and $d(y,a) \leq \epsilon_1$ for some $a \in A$. But for each $i \in I$,

$$d(x_i,y) \leq d(x_i,a) + d(a,y) \leq r_i + \epsilon_2$$

and

$$d(y,x_i) \leq d(y,a) + d(a,x_i) \leq \epsilon_1 + s_i.$$

Then for each $i \in I$, we have that $y \in C_d(x_i, r_i+\epsilon_2)$ and $y \in C_{d^{-1}}(x_i, s_i+\epsilon_1)$ which imply that $N_{\epsilon_1,\epsilon_2}(A) \subseteq \bigcap_{i\in I}(C_d(x_i,r_i+\epsilon_2)\cap C_{d^{-1}}(x_i,s_i+\epsilon_1))$.

Now, let us consider $y \in \bigcap_{i\in I}(C_d(x_i, r_i + \epsilon_2) \cap C_{d^{-1}}(x_i, s_i + \epsilon_1))$ and let $i \in I$. We have that

$$d(x_i, y) \leq r_i + \epsilon_2$$

and

$$d(y, x_i) \leq \epsilon_1 + s_i.$$

Since A is nonempty and by definition of A, we must have for any $i, j \in I$,

$$d(x_i, x_j) \leq d(x_i, a) + d(a, x_j) \leq r_i + s_j.$$

So by q-hyperconvexity of X

$$\begin{aligned}\emptyset \neq \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_d(y, \epsilon_1)) \cap \bigcap_{i\in I}(C_{d^{-1}}(x_i, s_i) \cap C_{d^{-1}}(y, \epsilon_2))\\ = \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \cap (C_d(y, \epsilon_1) \cap d_{d^{-1}}(y, \epsilon_2))\\ = A \cap (C_d(y, \epsilon_1) \cap C_{d^{-1}}(y, \epsilon_2)).\end{aligned}$$

Therefore, there is $a \in A$ such that $d(y, a) \leq \epsilon_1$ and $d(a, y) \leq \epsilon_2$. Hence $y \in N_{\epsilon_1,\epsilon_2}(A)$ and the proof is complete. □

We next recall the well-known definition of a retraction of a quasi-metric space to a subset.

4.2.2 Definition. *Let (X, d) be a quasi-metric space and Y subset of X. A map $f : X \to Y$ is said to be a* nonexpansive retraction *if*

1. For each $x \in Y$, $f(x) = x$; that is, f is the identity function on its image, and

2. For any $x, y \in X$, $d(f(x), f(y)) \leq d(x, y)$; that is, f is nonexpansive.

4.2.2 Lemma. *(compare [34, Lemma 4.3]) Suppose (X, d) is a q-hyperconvex T_0-quasi-metric space and let A be a nonempty q-admissible subset of X. Then for each $\epsilon_1, \epsilon_2 \geq 0$ there is a nonexpansive retraction R of $N_{\epsilon_1,\epsilon_2}(A)$ onto A which has the property $d(x, R(x)) \leq \epsilon_1$ and $d(R(x), x) \leq \epsilon_2$ for each $x \in N_{\epsilon_1,\epsilon_2}(A)$.*

Proof. Assume $\emptyset \neq A = \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$ with $I \neq \emptyset$. By Lemma 4.2.1 we know that $N_{\epsilon_1,\epsilon_2}(A)$ is q-admissible in (X, d) and so $N_{\epsilon_1,\epsilon_2}(A)$ is itself q-hyperconvex by Proposition 4.1.1. Consider the family $\mathcal{F} = \{(D, R_D) : A \subseteq D \subseteq N_{\epsilon_1,\epsilon_2}(A)$ and $R_D : D \longrightarrow A$ is a nonexpansive retraction such that $d(x, R(x)) \leq \epsilon_1$ and $d(R(x), x) \leq \epsilon_2$ for each $x \in D\}$.

Note that $(A, I_A) \in \mathcal{F}$, where I_A is the identity map on A. So $\mathcal{F} \neq \emptyset$. If one orders $\mathcal{F}$ in the usual way ($(D, R_D) \preceq (H, R_H)$ if and only if $D \subseteq H$ and R_H is an extension of R_D) then each chain in $(\mathcal{F}, \preceq)$ is bounded above, so by Zorn's Lemma $\mathcal{F}$ has a maximal element which we again denote by (D, R_D). We need to show that $D = N_{\epsilon_1,\epsilon_2}(A)$. Suppose there exists $x \in N_{\epsilon_1,\epsilon_2}(A)$ such that $x \notin D$, and consider the set

$C = [\bigcap_{w\in D} C_d(R_D(w), d(w, x)) \cap C_{d^{-1}}(R_D(w), d(x, w))] \cap [\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))] \cap [C_d(x, \epsilon_1) \cap C_{d^{-1}}(x, \epsilon_2)]$.

First we show that $C \neq \emptyset$, and in order to do this we need only to show that C has the mixed binary intersection property.

If $w_1, w_2 \in D$ then

$$d(R_D(w_1), R_D(w_2)) \leq d(w_1, w_2) \leq d(w_1, x) + d(x, w_2).$$

This proves that

$C_d(R_D(w_1), d(w_1, x)) \cap C_{d^{-1}}(R_D(w_2), d(x, w_2)) \neq \emptyset$ by metric convexity of (X, d), so C has the mixed binary intersection property for the first family. Also for each $w \in D$, $R_D(w) \in A = \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$. So the mixed binary intersection property is satisfied for the second family.

Since

$$x \in N_{\epsilon_1,\epsilon_2}(A) = \bigcap_{i\in I}(C_d(x_i, r_i + \epsilon_2) \cap C_{d^{-1}}(x_i, s_i + \epsilon_1))$$

we know that $(C_d(x, \epsilon_1) \cap C_{d^{-1}}(x, \epsilon_2)) \cap (C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \neq \emptyset$ for each

$i \in I$. Finally, if $w \in D$, then

$$d(R_D(w), x) \leq d(R_D(w), w) + d(w, x) \leq \epsilon_2 + d(w, x)$$

and

$$d(x, R_D(w)) \leq d(x, w) + d(w, R_D(w)) \leq d(x, w) + \epsilon_1.$$

Thus by metric convexity of (X, d) we have that

$$C_d(R_D(w), d(w, x)) \cap C_{d^{-1}}(x, \epsilon_2) \neq \emptyset$$

as well as

$$C_{d^{-1}}(R_D(w), d(x, w)) \cap C_d(x, \epsilon_1) \neq \emptyset.$$

Of course, $C_d(x, \epsilon_1)$ and $C_{d^{-1}}(x, \epsilon_2)$ intersect.

We have shown that the family $[C_d(R_D(w), d(w, x))_{w \in D}, (C_d(x_i, r_i))_{i \in I}, C_d(x, \epsilon_1);$ $C_{d^{-1}}(R_D(w), d(x, w))_{w \in D}, (C_{d^{-1}}(x_i, s_i))_{i \in I}, C_{d^{-1}}(x, \epsilon_2)]$ of double balls has the mixed binary intersection property.

We conclude therefore that $\emptyset \neq C \subseteq A$. Now let $u \in C$ and define $R' : D \cup \{x\} \longrightarrow A$ by setting $R'(w) = R_D(w)$ if $w \in D$ and $R'(x) = u$. Then for $w \in D$,

$$d(R'(x), R'(w)) = d(u, R(w)) \leq d(u, w)$$

and

$$d(R'(w), R'(x)) = d(R(w), u) \leq d(w, u).$$

So R' is nonexpansive. Also $d(R'(x), x) = d(u, x) \leq \epsilon_2$ and $d(x, R'(x)) = d(x, u) \leq \epsilon_1$. With this we conclude that the pair $(D \cup \{x\}, R')$ contradicts the maximality of (D, R_D) in $(\mathcal{F}, \preceq)$. Therefore, $D = N_{\epsilon_1, \epsilon_2}(A)$ and the proof is complete. □

We next define approximate fixed points of a nonexpansive self-map in a T_0-quasi-metric space.

4.2.3 Definition. *([35]) Let (X, d) be a T_0-quasi-metric space and $T : X \to X$ a nonexpansive map. We say that T has approximate fixed points if*

$$\inf\{d^s(x, T(x)) : x \in X\} = 0.$$

4.2.4 Definition. *Let (X, d) be a T_0-quasi-metric space. For a map $T : X \longrightarrow X$ we use $F_{\epsilon_1,\epsilon_2}$ to denote the set of ϵ_1, ϵ_2-fixed points of T; that is $F_{\epsilon_1,\epsilon_2}(T) = \{x \in X : d(x, T(x)) \leq \epsilon_2$ and $d(T(x), x) \leq \epsilon_1\}$, for $\epsilon_1, \epsilon_2 \geq 0$.*

4.2.1 Theorem. *(compare [34, Theorem 4.11]) Suppose (X, d) is a q-hyperconvex T_0-quasi-metric space and suppose $T : X \longrightarrow X$ is nonexpansive. Furthermore suppose that for some $\epsilon_1, \epsilon_2 \geq 0$ we have that $F_{\epsilon_1,\epsilon_2}(T)$ is nonempty. Then the set $F_{\epsilon_1,\epsilon_2}(T)$ is q-hyperconvex.*

Proof. Clearly we may suppose $F_{\epsilon_1,\epsilon_2}(T) \neq \emptyset$. For each i in some nonempty index set I, let $x_i \in F_{\epsilon_1,\epsilon_2}(T)$, and let $r_i \geq 0$ and $s_i \geq 0$ satisfy

$$d(x_i, x_j) \leq r_i + s_j.$$

We need to show that

$$[\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))] \cap F_{\epsilon_1,\epsilon_2}(T) \neq \emptyset.$$

We know that $\emptyset \neq J = \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$ is q-hyperconvex according to Proposition 4.1.1, since (X, d) is q-hyperconvex. Furthermore J is obviously bounded in (X, d).

Also if $x \in J$ then for each $i \in I$,

$$d(x_i, T(x)) \leq d(x_i, T(x_i)) + d(T(x_i), T(x)) \leq \epsilon_2 + d(x_i, x) \leq \epsilon_2 + r_i$$

and

$$d(T(x), x_i) \leq d(T(x), T(x_i)) + d(T(x_i), x_i) \leq d(x, x_i) + \epsilon_1 \leq s_i + \epsilon_1.$$

This proves that $T(x) \in N_{\epsilon_1,\epsilon_2}(J)$. Now, by Lemma 4.2.2, there is a nonexpansive retraction R of $N_{\epsilon_1,\epsilon_2}(J)$ onto J for which $d(R(x), x) \leq \epsilon_2$ and $d(x, R(x)) \leq \epsilon_1$ for each $x \in N_{\epsilon_1,\epsilon_2}(J)$. Also since $R \circ T$ is a nonexpansive map of J into J, it must have a fixed point by Theorem 4.1.1.

Suppose $(R \circ T)(x_0) = x_0$ for $x_0 \in J$. Then

$$d(x_0, T(x_0)) = d((R \circ T)(x_0), T(x_0)) \leq \epsilon_2$$

and

$$d(T(x_0), x_0) = d(T(x_0), (R \circ T)(x_0)) \leq \epsilon_1.$$

Thus the proof is complete, since $x_0 \in J \cap F_{\epsilon_1,\epsilon_2}(T)$. □

4.3 External q-hyperconvexity

In this section we introduce the notion of external q-hyperconvexity in analogy to the notion of external hyperconvexity in a metric space (see [25]).

We begin this section by defining external q-hyperconvexity similarly to [4, Defintion 3] and [25, Definition 3.5]. Note that this concept is stronger than q-hyperconvexity.

4.3.1 Definition. *(compare [34, Definition 3.5]) Let (X, d) be a quasi-pseudometric space. A quasi-pseudometric subspace E of X is said to be* externally q-hyperconvex *(relative to X) if given any family $(x_i)_{i\in I}$ of points in X and families of nonnegative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ the following condition holds:*

If $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$ and $dist(x_i, E) \leq r_i$ and $dist(E, x_i) \leq s_i$ whenever $i \in I$, where $dist(x, E) := \inf\{d(x, y) : y \in E\}$ and $dist(E, x) := \inf\{d(y, x) : y \in E\}$, then $\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)) \cap E \neq \emptyset$. (Compare Example 3.2.1 for the definition of $dist(E, x)$.)

4.3.1 Example. *(compare [4, Theorem 7]) Let E be externally q-hyperconvex in a T_0-quasi-metric space (X, d) and let x be any point of X. Set $dist(x, E) = r$ and $dist(E, x) = s$. Then by applying external q-hyperconvexity to the pair of balls $(C_d(x, r), C_{d^{-1}}(x, s))$, we conclude that there is $p \in C_d(x, r) \cap C_{d^{-1}}(x, s) \cap E$. Thus $d(x, p) = dist(x, E)$ and $d(p, x) = dist(E, x)$.*

4.3.1 Lemma. *(compare [25, Lemma 3.8]) Let (X, d) be a q-hyperconvex space and let $x \in X$. Furthermore let $\emptyset \neq A = \bigcap_{i \in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$ where $(x_i)_{i \in I}$ is a nonempty family of points in X and $(r_i)_{i \in I}$ and $(s_i)_{i \in I}$ are families of nonnegative reals. Then there is $p \in A$ such that $dist(x, A) = d(x, p)$ and $dist(A, x) = d(p, x)$.*

Proof. Evidently for $\epsilon > 0$ and $x \in X$

$$[(C_d(x_i, r_i))_{i \in I}, C_d(x, dist(x, A) + \epsilon); (C_{d^{-1}}(x_i, s_i))_{i \in I}, C_{d^{-1}}(x, dist(A, x) + \epsilon)]$$

satisfies the mixed binary intersection property.

Thus there is $p \in A \cap C_d(x, dist(A, x)) \cap C_{d^{-1}}(x, dist(A, x))$ by q-hyperconvexity of (X, d). Obviously p then satisfies the stated condition. □

The following lemma will be useful in the proof of Theorem 4.3.1. Considering the case that $E = X$, we see that Lemma 4.3.2 improves on Proposition 4.1.1.

4.3.2 Lemma. *(compare [6, Lemma 2]) Let (X, d) be a q-hyperconvex T_0-quasi-metric space. Suppose $E \subseteq X$ is externally q-hyperconvex relative to X and suppose A is a q-admissible subset of X such that $E \cap A \neq \emptyset$. Then $E \cap A$ is externally q-hyperconvex relative to X.*

Proof. Suppose the existence of a family $(x_\alpha)_{\alpha \in \Gamma}$ of points in X and families of nonnegative real numbers $(r_\alpha)_{\alpha \in \Gamma}$ and $(s_\alpha)_{\alpha \in \Gamma}$ that satisfy $d(x_\alpha, x_\beta) \leq r_\alpha + s_\beta$ and $dist(x_\alpha, E \cap A) \leq r_\alpha$ and $dist(E \cap A, x_\alpha) \leq s_\alpha$, whenever $\alpha, \beta \in \Gamma$ are given.

Since A is q-admissible, $\emptyset \neq A = \bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))$ with $x_i \in X$ and $r_i, s_i \geq 0$ whenever $i \in I$, and since $dist(x_\alpha, E \cap A) \leq r_\alpha$ and $dist(E \cap A, x_\alpha) \leq s_\alpha$ whenever $\alpha \in \Gamma$, it follows that for each $i \in I$ and for $z \in A$ chosen according to Lemma 4.3.1 we have

$$d(x_\alpha, x_i) \leq d(x_\alpha, z) + d(z, x_i) \leq r_\alpha + s_i$$

and

$$d(x_i, x_\alpha) \leq d(x_i, z) + d(z, x_\alpha) \leq r_i + s_\alpha.$$

Also, since $A \subseteq C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i)$, it follows that

$$dist(x_i, E) \leq r_i$$

and

$$dist(E, x_i) \leq s_i,$$

and that $d(x_i, x_j) \leq r_i + s_j$ whenever $i, j \in I$. Trivially we have $dist(x_\alpha, E) \leq r_\alpha$ and $dist(E, x_\alpha) \leq s_\alpha$ whenever $\alpha \in \Gamma$.

Therefore by external q-hyperconvexity of E

$$[\bigcap_{i\in I}(C_d(x_i, r_i) \cap C_{d^{-1}}(x_i, s_i))] \cap [\bigcap_{\alpha\in\Gamma}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \cap E]$$

$$= \bigcap_{\alpha\in\Gamma}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, s_\alpha)) \cap (E \cap A) \neq \emptyset.$$

Thus the proof is complete. □

We next show that the intersection of a descending family of externally q-hyperconvex nonempty subspaces of a bounded q-hyperconvex T_0-quasi-metric space is externally q-hyperconvex.

4.3.1 Theorem. *(compare [25, Theorem 5.4]) Let (X, d) be a bounded q-hyperconvex T_0-quasi-metric space. Let $\{X_i\}_{i\in I}$ be a descending family of nonempty externally q-hyperconvex subsets of X, where we suppose that I is totally ordered such that $i \leq j \Leftrightarrow X_j \subseteq X_i$. Then $\bigcap_{i\in I} X_i$ is nonempty and externally q-hyperconvex.*

Proof. Theorem 4.1.2 assures that $D = \bigcap_{i\in I} X_i \neq \emptyset$. In order to show that D is externally q-hyperconvex, let a family $(x_\alpha)_{\alpha\in\Gamma}$ of points in X and families of nonnegative real numbers $(r_\alpha)_{\alpha\in\Gamma}$ and $(s_\alpha)_{\alpha\in\Gamma}$ be given such that $d(x_\alpha, x_\beta) \leq r_\alpha + s_\beta$ and $dist(x_\alpha, D) \leq r_\alpha$, and $dist(D, x_\alpha) \leq s_\alpha$ whenever $\alpha, \beta \in \Gamma$.

Since X is q-hyperconvex, we know that $A = \bigcap_{\alpha\in\Gamma}(C_d(x_\alpha, r_\alpha) \cap C_{d^{-1}}(x_\alpha, \beta)) \neq \emptyset$. Also, since for each $\alpha \in \Gamma$ $dist(x_\alpha, D) \leq r_\alpha$ and $dist(D, x_\alpha) \leq s_\alpha$, we have $dist(x_\alpha, X_i) \leq r_\alpha$ and $dist(X_i, x_\alpha) \leq s_\alpha$ for each $i \in I$. So, by external q-hyperconvexity of X_i, we conclude $A \cap X_i \neq \emptyset$ whenever $i \in I$.

By Lemma 4.3.2 $\{A \cap X_i\}_{i\in I}$ is a descending chain of nonempty (externally) q-hyperconvex subsets of X, so that again by Theorem 4.1.2 $\bigcap_{i\in I}(A \cap X_i) = A \cap D \neq \emptyset$. □

Chapter 5

The u-injective hull of a T_0-ultra-quasi-metric space

In Chapter 3 we constructed the so-called q-hyperconvex hull of a T_0-quasi-metric space. In this chapter we continue these investigations by presenting a similar construction for T_0-ultra-quasi-metric spaces. Comparable studies in the area of ultra-metric spaces have been presented in Chapter 2. In this chapter we shall show how the investigation in Chapter 3 can be modified in order to obtain a theory that is suitable for T_0-ultra-quasi-metric spaces.

5.1 Strongly tight function pairs

In this section we are going to define strongly tight function pairs and minimal strongly tight function pairs and to provide some properties of these function pairs.

5.1.1 Definition. *Let (X, u) be an T_0-ultra-quasi-metric space and let $\mathcal{FP}(X, u)$ be the set of all pairs $f = (f_1, f_2)$ of functions where $f_i : X \longrightarrow [0, \infty)$ $(i = 1, 2)$*

(for the definition of n see Example 0.2.1).

For any such pairs (f_1, f_2) and (g_1, g_2) we set

$$N((f_1, f_2), (g_1, g_2)) = \max\{\sup_{x\in X} n(f_1(x), g_1(x)), \sup_{x\in X} n(g_2(x), f_2(x))\}.$$

It is obvious that N is an extended T_0-ultra-quasi-metric on the set $\mathcal{FP}(X, u)$ of these function pairs.

5.1.2 Definition. *Let (X, u) be a T_0-ultra-quasi-metric space. We say that a pair $f \in \mathcal{PF}(X, u)$ is* strongly tight *if for all $x, y \in X$, we have*

$$u(x, y) \leq \max\{f_2(x), f_1(y)\}.$$

The set of all strongly tight function pairs on a T_0-ultra-quasi-metric space (X, u) will be denoted by $U\mathcal{T}(X, u)$.

5.1.1 Lemma. *Let (X, u) be a T_0-ultra-quasi-metric space. For each $a \in X$, $f_a := (u(a, x), u(x, a))$ whenever $x \in X$ is a strongly tight pair in $U\mathcal{T}(X, u)$.*

Proof. Indeed $u(x, y) \leq \max\{(f_a)_2(x), (f_a)_1(y)\} = \max\{u(x, a), u(a, y)\}$. □

Let (X, u) be a T_0-ultra-quasi-metric space. We say that a function pair $f = (f_1, f_2)$ is *minimal* among the strongly tight pairs on (X, u) if it is a strongly tight pair and if $g = (g_1, g_2)$ is a strongly tight pair on (X, u) and for each $x \in X$, $g_1(x) \leq f_1(x)$ and $g_2(x) \leq f_2(x)$, then $f = g$. Minimal strongly tight function pairs are also called *extremal strongly tight function pairs.* By $\nu_q(X, u)$ (or more briefly, $\nu_q(X)$) we denote the set of all minimal strongly tight function pairs in (X, u) equipped with the restriction of N to $\nu_q(X)$, which we shall denote again by N. We note that the restriction of N to $\nu_q(X)$ is indeed a T_0-ultra-quasi-metric on $\nu_q(X)$ (see Corollary 5.2.1). In the following we shall call $\nu_q(X)$ the *u-injective hull* of (X, u).

5.1.2 Lemma. *Let (X, u) be a T_0-ultra-quasi-metric space and let $f \in \nu_q(X, u)$. For all $x, y \in X$, ($f_1(x) > f_1(y)$ implies that $f_1(x) \leq u^{-1}(x, y)$) and ($f_2(x) > f_2(y)$ implies that $f_2(x) \leq u(x, y)$).*

Proof. Let us consider the stated result for f_2. Suppose the contrary. Then there are $x_0, y_0 \in X$ such that $f_2(x_0) > u(x_0, y_0)$ and $f_2(x_0) > f_2(y_0)$. Set $g_2(x) = f_2(x)$ if $x \in X$ and $x \neq x_0$, and $g_2(x) = \max\{u(x_0, y_0), f_2(y_0)\}$ if $x = x_0$. Clearly $(f_1, g_2) < (f_1, f_2)$. Let $x, y \in X$. Then $u(x, y) \leq \max\{f_2(x), f_1(y)\} = \max\{g_2(x), f_1(y)\}$ if $x \neq x_0$. So assume that $x = x_0$ and consider any $y \in X$. Then

$$u(x, y) = u(x_0, y) \leq \max\{u(x_0, y_0), u(y_0, y)\} \leq$$

$$\max\{u(x_0, y_0), f_2(y_0), f_1(y)\} \leq \max\{g_2(x_0), f_1(y)\}.$$

It follows that (f_1, g_2) is strongly tight and we have reached a contradiction to the minimal strong tightness of (f_1, f_2).

Similarly one shows that for each $x, y \in X$ we have $f_1(x) \leq u^{-1}(x, y)$ whenever $f_1(x) > f_1(y)$. □

5.1.1 Corollary. *Let (X, u) be a T_0-ultra-quasi-metric space. If f is minimal strongly tight, then $f_1(x) \leq \max\{f_1(y), u^{-1}(x, y)\}$ whenever $x, y \in X$ and $f_2(x) \leq \max\{f_2(y), u(x, y)\}$. Thus $f_1 : (X, u) \to ([0, \infty), n^{-1})$ and $f_2 : (X, u) \to ([0, \infty), n)$ are contracting maps (see Corollary 0.2.2).* □

Our next result shows among other things that the two functions of a minimal strongly tight pair of functions on a T_0-ultra-quasi-metric space determine each other.

5.1.1 Proposition. *Let (X, u) be a T_0-ultra-quasi-metric space. Suppose that $f = (f_1, f_2)$ is an extremal strongly tight function pair on (X, u). Then*

$f_1(x) = \sup\{u^{-1}(x, y) :$ *for all* $y \in X$ *such that* $f_2(y) < u^{-1}(x, y)\}$

and

$f_2(x) = \sup\{u(x,y) : \textit{for all } y \in X \textit{ such that } f_1(y) < u(x,y)\}$ *whenever* $x \in X$.

Proof. Since (f_1, f_2) is a strongly tight function pair, $\sup\{u^{-1}(x,y)$: for all $y \in X$ such that $f_2(y) < u^{-1}(x,y)\} \leq f_1(x)$ whenever $x \in X$.

Suppose that there is $x_0 \in X$ such that $\sup\{u^{-1}(x_0,y)$: for all $y \in X$ such that $f_2(y) < u^{-1}(x_0,y)\} < f_1(x_0)$. Set $g_1(x) = f_1(x)$ if $x \in X$ and $x \neq x_0$, and $g_1(x_0) = \sup\{u^{-1}(x_0,y)$: for all $y \in X$ such that $f_2(y) < u^{-1}(x_0,y)\}$. Thus (g_1, f_2) is strongly tight : Indeed for any $y \in X$ with $f_2(y) < u^{-1}(x_0,y)$ we have $u^{-1}(x_0,y) \leq \sup\{u^{-1}(x_0,a)$: for all $a \in X$ such that $f_2(a) < u^{-1}(x_0,a)\}$. Thus

$u^{-1}(x_0,y) \leq \max\{f_2(y), \sup\{u^{-1}(x_0,a) : \text{for all } a \in X \text{ such } f_2(a) < u^{-1}(x_0,a)\}\} \leq \max\{f_2(y), g_1(x_0)\}$ whenever $y \in X$. It follows that (f_1, f_2) is strongly tight and $(g_1, f_2) < (f_1, f_2)$, but (f_1, f_2) was an extremal strongly tight function pair. We have reached a contradiction and conclude that $f_1(x) = \sup\{u^{-1}(x,y)$: for all $y \in X$ such that $f_2(y) < u^{-1}(x,y)\}$ whenever $x \in X$.

Similarly one can show that $f_2(x) = \sup\{u(x,y)$: for all $y \in X$ such $f_1(y) < u(x,y)\}$ whenever $x \in X$. □

5.1.2 Proposition. *Let* (X, u) *be a* T_0*-ultra-quasi-metric space. If* f *is a strongly tight function pair on* (X, u)*, then* $f_1(x) \leq \max\{f_1(y), u^{-1}(x,y)\}$ *and* $f_2(x) \leq \max\{f_2(y), u(x,y)\}$ *whenever* $x, y \in X$*. Furthermore suppose that there is a sequence* $(a_n)_{n\in\mathbb{N}}$ *in* X *with* $\lim_{n\to\infty} f_1(a_n) = 0$ *and* $\lim_{n\to\infty} f_2(a_n) = 0$*. Then* f *is a minimal strongly tight pair.*

Proof. Suppose otherwise. There is a strongly tight pair g such that $g < f$. Without loss of generality we assume that there is $x_0 \in X$ such that $g_2(x_0) < f_2(x_0)$. (The case for f_1 is dealt with analogously.) Therefore by our non-expansivity assumption applied to f_2, we have that $f_2(x_0) \leq \max\{u(x_0, a_n), f_2(a_n)\}$

whenever $n \in \mathbb{N}$. Thus there is $n_0 \in \mathbb{N}$ such that $0 < f_2(x_0) \leq u(x_0, a_n)$ whenever $n \in \mathbb{N}$ and $n > n_0$. By strong tightness of g, we have $u(x_0, a_n) \leq \max\{g_2(x_0), f_1(a_n)\}$ whenever $n \in \mathbb{N}$. Hence there is $n_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq n_1$ we have $u(x_0, a_n) \leq g_2(x_0)$. We conclude that $f_2(x_0) \leq g_2(x_0)$—a contradiction. Therefore we deduce that the pair f is minimal strongly tight. $\square$

Our next result describes the distance between extremal strongly tight function pairs. It shows that our definition of N was unnecessarily complicated if we are only interested in the distance between pairs of functions defined on a T_0-ultra-quasi-metric space (X, u) belonging to $\nu_q(X, u)$.

5.1.3 Lemma. *(compare Lemma 3.3.3)*
Let $(f_1, f_2), (g_1, g_2)$ be minimal strongly tight pairs of functions on a T_0-ultra-quasi-metric space (X, u). Then

$N((f_1, f_2), (g_1, g_2)) = \sup_{x \in X} n(f_1(x), g_1(x)) = \sup_{x \in X} n(g_2(x), f_2(x)).$

Proof. Suppose that for some $x \in X$, we have that $n(f_1(x), g_1(x)) > 0$. Then $f_1(x) > g_1(x)$. Consider any $\epsilon > 0$ such that $f_1(x) - g_1(x) > \epsilon$. In particular $f_1(x) > 0$. Then there is $a \in X$ with $u(a, x) > f_2(a)$ and $f_1(x) - \epsilon < u(a, x)$, since $f_1(x) = \sup\{u(a, x) : \text{for all } a \in X \text{ such that } f_2(a) < u(a, x)\}$. Therefore $f_1(x) = (f_1(x) - u(a, x)) + u(a, x) < \epsilon + u(a, x) \leq \epsilon + u(a, x) \leq \epsilon + \max\{g_2(a), g_1(x)\} \leq \epsilon + g_2(a)$, because g is strongly tight and $g_1(x) < f_1(x)$.

In order to reach a contradiction, suppose that $g_2(a) \leq f_2(a)$. Hence $g_2(a) \leq f_2(a) < u(a, x) \leq \max\{g_2(a), g_1(x)\} = g_1(x) < f_1(x) - \epsilon \leq g_2(a)$. Thus we have reached the contradiction that $g_2(a) < g_2(a)$ and conclude that $g_2(a) > f_2(a)$.

Since the inequality above holds for any sufficiently small $\epsilon > 0$ we get that

$\sup_{x \in X} n(f_1(x), g_1(x)) \leq \sup_{x \in X} n(g_2(x), f_2(x))$. Similarly $0 < \sup_{x \in X} n(g_2(x), f_2(x))$ implies that $\sup_{x \in X} n(g_2(x), f_2(x)) \leq \sup_{x \in X} n(f_1(x), g_1(x))$.

We conclude that $\sup_{x\in X} n(f_1(x), g_1(x)) = \sup_{x\in X} n(g_2(x), f_2(x))$ in any case. $\square$

The following result is a consequence of Proposition 5.1.1 and Lemma 5.1.3.

5.1.2 Corollary. *Let (X, u) be a T_0-ultra-quasi-metric space. Any minimal strongly tight function pair f on X satisfies the following conditions:*

$$f_1(x) = \sup_{y\in X} n(u^{-1}(x, y), f_2(y)) = \sup_{y\in X} n(f_1(y), u(x, y))$$

and

$$f_2(x) = \sup_{y\in X} n(u(x, y), f_1(y)) = \sup_{y\in X} n(f_2(y), u^{-1}(x, y))$$

whenever $x \in X$. $\square$

5.2 Envelopes or u-injective hulls of a T_0-ultra-quasi-metric space

In the following section we present an explicit construction of the ultra-quasi-metrically injective hull of a T_0-ultra-quasi-metric space.

The following lemma is analogous to Lemma 3.3.5.

5.2.1 Lemma. *(compare Lemma 3.3.5)*
Let (X, u) be a T_0-ultra-quasi-metric space. For each $a \in X$, the pair f_a belongs to $\nu_q(X, u)$.

Proof. Let $(g_1, g_2) \in \mathcal{UT}(X, u)$ be such that $g_1(x) \leq u(a, x)$ and $g_2(x) \leq u(x, a)$ for all $x \in X$.

Since $u(a, x) \leq \max\{g_2(a), g_1(x)\}$ and $g_2(a) = 0$ imply that $u(a, x) \leq g_1(x)$, we have $u(a, x) = g_1(x)$ for all $x \in X$.

Similarly one can show that $u(x, a) = g_2(x)$ for all $x \in X$. $\square$

5.2.2 Lemma. *Suppose that (X, u) is a T_0-ultra-quasi-metric space and $(f_1, f_2) \in \nu_q(X, u)$ such that $f_1(a) = 0 = f_2(a)$ for some $a \in X$. Then $(f_1, f_2) = e_X(a)$.*

Proof. By strong tightness of (f_1, f_2) we have $u(x, a) \leq \max\{f_2(x), 0\}$ and $u(a, x) \leq \max\{0, f_1(x)\} \leq f_2(x)$, and $u(a, x) \leq f_1(x)$ whenever $x \in X$. Thus $e_X(a) = (f_1, f_2)$. $\square$

The next theorem shows that the canonical map f_a is an isometric embedding.

5.2.1 Theorem. *(compare Lemma 3.3.4 and Lemma 3.2.1)*
Let (X, u) be a T_0-ultra-quasi-metric space.

(a) For $f \in \nu_q(X, u)$ and $a \in X$, $N(f, f_a) = f_1(a)$ and $N(f_a, f) = f_2(a)$.

(b) The map $i : (X, u) \longrightarrow (\nu_q(X, u), N)$, defined by $i(a) = f_a$ whenever $a \in X$ is an isometric embedding.

Proof. (a) Consider the case that $f_1(a) \neq 0$. We have that $f_1(a) \leq \sup\{f_1(x) : u(a, x) < f_1(x)\}$, since $u(a, a) = 0$. The latter inequality holds if $f_1(a) = 0$. We have $f_1(a) \leq \sup_{x \in X} n(f_1(x), u(a, x))$.

Furthermore $\sup_{x \in X} n(f_1(x), u(a, x)) = \sup\{f_1(x)$: for all $x \in X$ such that $u(a, x) < f_1(x)\} \leq f_1(a)$, since the existence of an $x \in X$ with $f_1(x) > f_1(a)$ and $f_1(x) > u(a, x)$ contradicts that $f_1(x) \leq \max\{u(a, x), f_1(a)\}$ because $f_1(x) \leq u(a, x)$ if $f_1(x) > f_1(a)$ whenever $x \in X$.

Thus $f_1(a) = \sup_{x \in X} n(f_1(x), u(a, x))$.

Similarly, $\sup\{u(x, a)$: for all $x \in X$ such that $f_2(x) < u(x, a)\} \leq f_1(a)$, since for each $x \in X$, by strong tightness of f, we have $u(x, a) \leq \max\{f_2(x), f_1(a)\}$

and since therefore the existence of an $x \in X$ with $f_2(x) < u(x,a)$ implies that $u(x,a) \leq f_1(a)$. In fact we have equality according to the definition of N, so certainly $N(f, f_a) = f_1(a)$.

Similarly one verifies that $N(f_a, f) = f_2(a)$.

(b) Obviously $\sup\{u(a,x) : u(b,x) < u(a,x), x \in X\} = u(a,b)$, as we see by setting $x = b$. Similarly $\sup\{u(x,b) : u(x,a) < u(x,b), x \in X\} = u(a,b)$ whenever $a, b \in X$. Hence i is an isometric map. If for $a, b \in X$ we have that $i(a) = i(b)$, then $0 = u(a,a) = u(a,b)$ and $0 = u(b,b) = u(b,a)$. Consequently $a = b$ by the T_0-property. □

5.2.1 Corollary. *Let (X,u) be a T_0-ultra-quasi-metric space. Then N is indeed a T_0-ultra-quasi-metric on $\nu_q(X,u)$.*

Proof. By Definition 5.1.1 we only have to prove that N is not attaining infinity. Let $f, g \in \nu_q(X,u)$ and $x \in X$. Then $N(f,g) \leq \max\{N(f,f_x), N(f_x,g)\} = \max\{f_1(x), g_2(x)\}$. □

Our next lemma gives another distance formula between two extremal strongly tight function pairs.

5.2.3 Lemma. *Let (X,u) be a T_0-ultra-quasi-metric space. Then for any $f, g \in \nu_q(X,u)$ we have that*

$$N(f,g) = \sup\{u(x_1,x_2) : x_1, x_2 \in X, f_2(x_1) < u(x_1,x_2) \text{ and } g_1(x_2) < u(x_1,x_2)\}.$$

Proof. Assume first that for some $f, g \in \nu_q(X,u)$ we have $N(f,g) > 0$. So $N(f,g) = \sup\{g_2(x) : x \in X \text{ and } g_2(x) > f_2(x)\}$ by Lemma 5.1.3. Then for each $\epsilon > 0$ there is $x_1 \in X$ such that $g_2(x_1) > f_1(x_1)$ and $N(f,g) - \epsilon < g_2(x_1)$. Set $\epsilon_2 = \min\{g_2(x_1) - f_2(x_1), \epsilon\} > 0$. Since $g_2(x_1) > 0$, by Lemma 5.1.1 there is $x_2 \in X$ with $u(x_1,x_2) > g_1(x_2)$ and $g_2(x_1) - \epsilon_2 < u(x_1,x_2)$. Thus $N(f,g) - \epsilon - \epsilon_2 < u(x_1,x_2) > g_1(x_2)$ and $u(x_1,x_2) > f_2(x_1)$.

Since ϵ was arbitrary, we have shown that $N(f,g) \leq \sup\{u(x_1,x_2) : x_1, x_2 \in X, u(x_1,x_2) > f_2(x_1) \text{ and } u(x_1,x_2) > g_1(x_2)\}$, which also holds in the remaining case where $f, g \in \nu_q(X)$ and we have $N(f,g) = 0$.

By the strong triangle inequality, we have

$N(f_{x_1}, f_{x_2}) \leq \max\{N(f_{x_1}, f), N(f,g), N(g, f_{x_2})\} = \max\{f_2(x_1), N(f,g), g_1(x_2)\}$ whenever $f, g \in \nu_q(X)$ and $x_1, x_2 \in X$. Thus $N(f_{x_1}, f_{x_2}) \leq N(f,g)$ whenever $N(f_{x_1}, f_{x_2}) > f_2(x_1)$ and $N(f_{x_1}, f_{x_2}) > g_1(x_2)$. We have shown that $\sup_{x_1,x_2}\{u(x_1,x_2) : x_1, x_2 \in X, f_2(x_1) < u(x_1,x_2) \text{ and } g_1(x_2) < u(x_1,x_2)\} \leq N(f,g)$. This establishes the claimed equality $N(f,g) = \sup\{u(x_1,x_2) : x_1,x_2 \in X, f_2(x_1) < u(x_1,x_2) \text{ and } g_1(x_2) < u(x_1,x_2)\}$ whenever $f, g \in \nu_q(X,u)$. $\square$

5.2.1 Remark. *(compare Remark 3.3.2)*
It follows from this formula that the isometry $e_X : (X,u) \to (\nu_q(X), N)$ has the following tightness property (compare Theorem 1.3.1 and Remark 3.3.2): If q is an ultra-quasi-pseudometric on $\nu_q(X,u)$ such that $q \leq N$ and $q(e_X(x), e_X(y)) = N(e_X(x), e_X(y))$ whenever $x, y \in X$, then $N(f,g) = q(f,g)$ whenever $f, g \in \nu_q(X,u)$:

Indeed let q be any ultra-quasi-pseudometric on $\nu_q(X,u)$ such that $q \leq N$. We have $N(f,g) = \sup\{N(f_{x_1}, f_{x_2}) : x_1, x_2 \in X, q(f_{x_1}, f_{x_2}) > q(f_{x_1}, f) \text{ and } q(f_{x_1}, f_{x_2}) > q(g, f_{x_2})\} \leq \sup\{q(f_{x_1}, f_{x_2}) : x_1, x_2 \in X, q(f_{x_1}, f_{x_2}) > q(f_{x_1}, f) \text{ and } q(f_{x_1}, f_{x_2}) > q(g, f_{x_1})\} \leq q(f,g)$ in the light of the strong triangle inequality $q(f_{x_1}, f_{x_2}) \leq \max\{q(f_{x_1}, f), q(f,g), q(g, f_{x_2})\} \quad (x_1, x_2 \in X)$. Thus $q = N$.

5.3 q-spherical completeness

In this section we introduce the notion of q-spherical completeness and we characterize q-spherically complete T_0-ultra-quasi-metric spaces.

5.3.1 Lemma. *Let (X,u) be a T_0-ultra-quasi-metric space. Let $x, y \in X$ and $r, s \geq 0$. Then $C_u(x,r) \cap C_{u^{-1}}(y,s) \neq \emptyset$ if and only if $u(x,y) \leq \max\{r,s\}$.*

Proof. Suppose that $C_u(x,r) \cap C_{u^{-1}}(y,s) \neq \emptyset$. Then there is $z \in X$ such that $u(x,z) \leq r$ and $u(z,y) \leq s$. Thus $u(x,y) \leq \max\{r,s\}$. In order to prove the converse, suppose that $u(x,y) \leq \max\{r,s\} = s$, that is $x \in C_u(x,r) \cap C_{u^{-1}}(y,s)$. If $\max\{r,s\} = r$, then $y \in C_u(x,r) \cap C_{u^{-1}}(y,s)$. Therefore $C_u(x,r) \cap C_{u^{-1}}(y,s) \neq \emptyset$ in either case. □

5.3.1 Definition. *(compare Definition 3.1.1) Let (X,u) be an ultra-quasi-pseudometric space. Let $(x_i)_{i\in I}$ be a family of points in X and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be families of nonnegative reals numbers.*
We say that $(C_u(x_i,r_i), C_{u^{-1}}(x_i,s_i))_{i\in I}$ has the mixed binary intersection property *provided that $u(x_i,x_j) \leq \max\{r_i,s_j\}$ whenever $i,j \in I$ (compare to Lemma 5.3.1). We say that (X,u) is q-*spherically complete *provided that each family $(C_u(x_i,r_i), C_{u^{-1}}(x_i,s_i))_{i\in I}$ possessing the mixed binary intersection property satisfies $\bigcap_{i\in I}(C_u(x_i,r_i) \cap C_{u^{-1}}(x_i,s_i)) \neq \emptyset$.*

The following useful remark should be compared with Remark 3.1.1.

5.3.1 Remark. *We can assume without loss of generality that the points x_i $(i \in I)$ are pairwise distinct in Definition 5.3.1: Indeed if this is not the case, then for each $x \in X$, set $T(x) = \{i \in I : x_i = x\}$ and consider only those points x of X that satisfy $T(x) \neq \emptyset$. Furthermore set $r(x) = \inf\{r_i : i \in T(x)\}$ and $s(x) = \inf\{s_i : i \in T(x)\}$. Then we have $u(x,y) \leq \max\{r_i,s_j\}$ whenever $i \in T(x)$ and $j \in T(y)$. Thus $u(x,y) \leq \max\{r(x),s_j\}$ whenever $j \in T(y)$, and consequently $u(x,y) \leq \max\{r(x),s(y)\}$. Applying the definition of q-spherical completeness to the family $(x)_{T(x)\neq\emptyset}$ of pairwise distinct points of X and the families $(r(x))_{T(x)\neq\emptyset}$ and $(s(x))_{T(x)\neq\emptyset}$ of nonnegative reals we find that $\emptyset \neq \bigcap_{T(x)\neq\emptyset}(C_u(x,r(x)) \cap C_{u^{-1}}(x,s(x))) \subseteq \bigcap_{i\in I}(C_u(x_i,r_i) \cap C_{u^{-1}}(x_i,s_i))$. Hence the apparently weaker condition is indeed equivalent to our definition.*

We next give a useful example of a q-spherically complete space.

5.3.1 Example. *The T_0-ultra-quasi-metric space $([0,\infty), n)$ is q-spherically complete.*

Proof. Let $x \in [0,\infty)$ and $\epsilon > 0$. Then $C_n(x,\epsilon) = [x,\infty)$ if $x > \epsilon$ and $C_n(x,\epsilon) = [0,\infty)$ if $x \leq \epsilon$.

Furthermore $C_{n^{-1}}(x,\epsilon) = [0,x]$ if $x > \epsilon$ and $C_{n^{-1}}(x,\epsilon) = [0,\epsilon]$ if $x \leq \epsilon$. Note that for any $x, s \in [0,\infty)$, therefore $C_{n^{-1}}(x,s) = [0, x \vee s)$.

Let $(x_i)_{i\in I}$ be a family of points in X and let $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be two families of nonnegative reals such that for each $i, j \in I$ we have $C_n(x_i,r_i) \cap C_{n^{-1}}(x_j,s_j) \neq \emptyset$. Then it is readily checked that $\emptyset = \bigcap_{i\in I}(C_n(x_i,r_i) \cap C_{n^{-1}}(x_i,s_i))$ implies that there are $j,k \in I$ such that $[0, x_k \vee s_k] \cap [x_j,\infty) = \emptyset$—a contradiction. We have shown that $([0,\infty), n)$ is q-spherically complete. □

Let $\nu_s(X)$ be the set of minimal strongly tight functions on an ultra-metric space (X,m) equipped with $E(f,g) = \sup_{x\in X} n^s(f(x),g(x))$ whenever $f,g \in \nu_s(X)$. It is known that the ultra-metric space $(\nu_s(X),E)$ yields the ultra-metrically injective hull of (X,m) (see Theorem 2.3.1) with the isometric embedding $x \mapsto m(x,\cdot)$ where $x \in X$.

5.3.2 Lemma. *Let (X,m) be an ultra-metric space and let $f,g \in \nu_m(X)$. Furthermore let $x,y \in X$ be such that $f(x) \neq g(x)$. Then* $\max\{f(x),g(x)\} \leq \max\{f(y),g(y)\}$.

Proof. In case (1) suppose that $f(x) > g(x)$. Then using nonexpansivity of f and strong tightness of g we get

$$\max\{f(x),g(x)\} = f(x) \leq \max\{f(y),m(x,y)\} \leq$$

$$\max\{f(y),g(x),g(y)\} = \max\{g(x),g(y)\}.$$

Analogously, if in case (2), $g(x) \geq f(x)$, then using nonexpansivity of g and strong tightness of g we get

$$\max\{f(x), g(x)\} = g(x) \leq \max\{g(y), m(x, y)\} \leq$$

$$\max\{g(y), f(x), f(y)\} = \max\{f(y), g(y)\}.$$

Note that in particular if also $g(x) \neq g(y)$, then we get $\max\{f(x), g(x)\} = \max\{f(y), g(y)\}$. □

5.3.1 Corollary. *Let (X, m) be an ultra-metric space. Let $f, g \in \nu_s(X)$ with $f \neq g$. Then*

$$\inf_{x \in X} \max\{f(x), g(x)\} = \sup_{x \in X} n^s(f(x), g(x)).$$

Proof. Since $f \neq g$, there is $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. Hence by definition of n^s and Lemma 5.3.2 $\sup_{x \in X} n^s(f(x), g(x)) = \max\{f(x_0), g(x_0)\}$.

Suppose that there is $x \in X$ such that $f(x) = g(x)$. By Lemma 5.3.2 $f(x) = g(x) \geq \max\{f(x_0), g(x_0)\}$ and it follows that

$$\inf_{x \in X} \max\{f(x), g(x)\} = \max\{f(x_0), g(x_0)\}.$$

Therefore the stated equality is established. □

Hence the two approaches to the distance E presented in Theorem 2.3.1 and the distance N presented in Definition 5.1.1 are equivalent.

We next establish a connection between a q-spherically complete ultra-quasi-pseudometric space and its conjugate.

5.3.1 Proposition. *(compare Proposition 3.1.1) (a) Let (X, u) be an ultra-quasi-pseudometric space. Then (X, d) is q-spherically complete if (X, u^{-1}) is q-spherically complete.*

(b) Let (X,u) be a T_0-ultra-quasi-metric space. If (X,u) is q-spherically complete, then (X,u^s) is spherically complete.

Proof. (a) The statement immediately follows from the definition.

(b) Let the family of balls $(C_{u^s}(x_i,r_i))_{i\in I}$ (where $x_i \in X$ and $r_i \in [0,\infty)$ whenever $i \in I$) have the mixed binary intersection property. Then $(C_u(x_i,r_i), C_{u^{-1}}(x_i,r_i))_{i\in I}$ has the mixed binary intersection property. Consequently $\emptyset \neq \bigcap_{i\in I} C_{u^s}(x_i,r_i)$. Thus (X,u^s) is spherically complete. □

5.3.2 Proposition. *(compare Corollary 3.1.3) Each q-spherically complete T_0-ultra-quasi-metric space (X,u) is bicomplete.*

Proof. By Proposition 5.3.1 u^s is spherically complete. Since spherically complete ultra-metric spaces are complete (see [23, Corollary 3]), we conclude that the T_0-ultra-quasi-metric space (X,u) is bicomplete. □

We next define the concept of an u-injective space.

5.3.2 Definition. *(compare Definition 1.1.2 and Definition 3.3.2) A T_0-ultra-quasi-metric space (X,u) is said to be* u-injective *if it has the following extension property: Whenever Y is a subspace of a T_0-ultra-quasi-metric-space Z and $f : Y \longrightarrow X$ is nonexpansive, then f has a nonexpansive extension $\bar{f} : Z \longrightarrow X$.*

The next result states that q-spherical completeness is analogous to u-injectivity and is similar to Theorem 3.3.1.

5.3.1 Theorem. *Let (X,u) be a T_0-ultra-quasi-metric space.Then the following properties are equivalent:*

1. X is q-spherically complete.

2. X is u-injective.

Proof. (1) $\Longrightarrow$ (2). First assume that X is a q-spherically complete T_0-ultra-quasi-metric space. Let A be a T_0-ultra-quasi-metric space and $T : A \longrightarrow X$ be a non-expansive map. Let (M, q) be a T_0-ultra-quasi-metric space containing A as a subspace. Consider the following set $C = \{(T_F, F) : T_F : F \longrightarrow X, A \subseteq F \subseteq M,$ F is a subspace of M and T_F is a non-expansive extension of $T\}$. We have $(T, A) \in C$. Therefore, C is nonempty.

On the other hand, one can partially order C by $(T_F, F) \leq (T_G, G)$ if and only if $F \subseteq G$ and the restriction of T_G to F is T_F. It is easy to see that C satisfies the hypothesis of Zorn's Lemma. Therefore, C has a maximal element. Let (T_1, F_1) be a maximal element of C.

Let us show that $F_1 = M$. Assume not. Let $z \in M$ but $z \notin F_1$ and set $F = F_1 \cup \{z_1\}$. Let us extend T_1 to F. Consider the family of closed balls

$$(C_u(T_1(x), q(x, z)), C_{u^{-1}}(T_1(x), q^{-1}(x, z)))_{x \in F_1}.$$

Note that $u(T_1(x), T_1(y)) \leq q(x, y) \leq \max\{q(x, z), q(z, y)\}$ for all $x, y \in F_1$. Thus $(C_u(T_1(x), q(x, z)), C_{u^{-1}}(T_1(x), q^{-1}(x, z)))_{x \in X}$ has the mixed binary intersection property. The q-spherical completeness of X implies that

$\bigcap_{x \in F_1}[C_u(T_1(x), q(x, z)) \cap C_{u^{-1}}(T_1(x), q^{-1}(x, z))] \neq \emptyset.$

Let z_1 be any point in this intersection. Set $T^* : F \longrightarrow X$ as follows: $T^*(x) = T_1(x)$ if $x \in F$ and $x \neq z$, $T^*(x) = z_1$ if $x = z$.

For all $x, y \in F$ we have

$u(T^*(x), T^*(y)) = u(z_1, T_1(y))$ if $x = z, y \neq z$; furthermore $u(T^*(x), T^*(y)) = u(T_1(y), T_1(x))$ if $x, y \neq z$, and $u(T^*(y), T^*(x)) = u(T_1(y), z_1)$ if $x = z, y \neq z$.

Then, we can see that for arbitrary $x, y \in F$, $u(T^*(x), T^*(y)) \leq q(x, y)$. Therefore, T^* is a nonexpansive extension of T, thus (T^*, F) belongs to C, hence

$(T_1, F_1) \leq (T^*, F)$ and $(T_1, F_1) \neq (T^*, F)$. This contradicts the maximality of (T_1, F_1). Therefore $F_1 = M$ and hence T has a nonexpansive extension to M. Consequently, (X, u) is u-injective.

(2) $\Longrightarrow$ (1) Assume that (X, u) is u-injective. We want to prove that X is q-spherically complete. Suppose that there is given a family $(x_i)_{i \in I}$ of pairwise distinct points in X and two families of nonnegative real numbers $(r_i)_{i \in I}$ and $(s_j)_{j \in I}$ such that $u(x_i, x_j) \leq \max\{r_i, s_j\}$ for any $i, j \in I$. We want to show that $\bigcap_{i \in I}(C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \neq \emptyset$.

Similarly as before, consider the set $\mathcal{PF}(A, u)$ of all nonnegative real-valued pair functions $f = (f_1, f_2)$ defined on the set $A = \{x_i : i \in I\}$ such that $u(x_i, x_j) \leq \max\{f_2(x_i), f_1(x_j)\}$ whenever $i, j \in I$. The distance u between the elements of A is the one inherited from X. As before, here we write u for the restriction of u to A. By assumption the pair (s, r) of functions such that:

$r : A \longrightarrow [0, \infty), x_i \mapsto r(x_i)$

$s : A \longrightarrow [0, \infty), s_i \mapsto s(x_i)$

belongs to $\mathcal{PF}(A, u)$.

The set $\mathcal{PF}(A, u)$ is partially ordered by the afore-mentioned pointwise order on the pair functions. Obviously, any descending chain of elements of $\mathcal{PF}(A, u)$ has a lower bound. Hence, Zorn's Lemma implies the existence of a minimal strongly tight element $(f_1, f_2) \in \mathcal{PF}(A, u)$ smaller than (r, s), i.e. $f_1(x_i) \leq s(x_i) = s_i$, $f_2(x_i) \leq r(x_i) = r_i$ whenever $i \in I$.

Case 1: There is $a \in A$ such that $(e_X(a))(x) = (u(a, x), u(x, a)) = (f_1, f_2) = (u(a, x), u(x, a))$ whenever $x \in X$. Then

$$a \in \cap_{i \in I}(C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \neq \emptyset.$$

Case 2: $(f_1, f_2) \neq e_X(a)$ whenever $a \in A$. Using the strong minimality of (f_1, f_2), it follows from Proposition 5.1.2 that for any $i, j \in I$, $f_1(x_i) \leq \max\{f_1(x_j), u(x_j, x_i)\}$ and $f_2(x_i) \leq \max\{u(x_i, x_j), f_2(x_j)\}$. Let w be a point from X not in the set A. Consider $A^* = A \cup \{w\}$. For the new point w, set $u(w, x_i) = f_1(x_i)$ and $u(x_i, w) = f_2(x_i)$ whenever $i \in I$, as well as $u(w, w) = 0$. Making use of minimal strong tightness of f, it is readily checked that u satisfies the strong triangle inequality on A^*. Because for any $a \in A$, $f_1(a)$ or $f_2(a)$ is positive by Lemma 5.2.2, (A^*, u) is a T_0-ultra-quasi-metric space which contains A as a subspace.

According to u-injectivity of (X, u), there exists a nonexpansive extension $R : A^* \to X$ of the inclusion map (defined from A to X). It is clear that $u(R(x_i), R(w)) = u(x_i, R(w)) \leq u(x_i, w) = f_2(x_i) \leq r_i$, whenever $i \in I$ and $u(R(w), R(x_i)) = u(R(w), x_i) \leq u(w, x_i) = f_1(x_i) \leq s_i$, whenever $i \in I$.

Consequently $R(w) \in \bigcap_{i \in I}(C_u(x_i, r_i) \cap C_{u^{-1}}(x_i, s_i)) \neq \emptyset$. We have shown that X is q-spherically complete. □

The result shows that the u-injective hull of a T_0-ultra-quasi-metric space is isometric to the conjugate space of the u-injective hull of its conjugate.

5.3.3 Proposition. *Let (X, u) be an ultra-quasi-metric space. Then $(f_1, f_2) \in \nu_q(X, u)$ implies that $(f_2, f_1) \in \nu_q(X, u^{-1})$. It follows that*

$$s : (\nu_q(X, u), N) \to (\nu_q(X, u^{-1}), N^{-1})$$

where s is defined by $s((f, g)) = (g, f)$ whenever $(f, g) \in \nu_q(X, u)$ is a bijective isometry. (Hence the u-injective hull of (X, u) is isometric to the conjugate space of the u-injective hull of (X, u^{-1}).)

Proof. Suppose that $(f_1, f_2) \in \nu_q(X, u)$. Let $x, y \in X$. Then $u(x, y) \leq \max\{f_2(x), f_1(y)\}$. Therefore $u^{-1}(x, y) \leq \max\{f_1(x), f_2(y)\}$ whenever $x, y \in X$. Obviously (f_2, f_1) is an extremal strongly tight function pair on (X, u^{-1}), since (f_1, f_2) is an extremal strongly tight function pair on (X, u). By this observation and $(u^{-1})^{-1} = u$, it is evident that $s : (\nu_q(X,u), N) \to (\nu_q(X,u^{-1}), N^{-1})$ is a bijection. Therefore s is an isometry, since

$$N^{-1}(s(f_1,f_2),(g_1,g_2)) = N((g_2,g_1),(f_2,f_1)) = N((f_1,f_2),(g_1,g_2))$$

whenever $(f_1,f_2),(g_1,g_2) \in \nu_q(X,u)$. □

For an ultra-metric space, the connection between injective hull and u-injective hull is established in the following result.

5.3.4 Proposition. *(compare Proposition 3.3.4) Let (X,m) be an ultra-metric space. Then $h(f) = (f,f)$ defines an isometric embedding of $(\nu_m(X,m),E)$ into $(\nu_q(X,m),N)$.*

Proof. Given $f \in \nu_m(X,m)$, it is evident that the pair (f,f) is u-tight on (X,m). Suppose that (g_1,g_2) is a strongly tight function pair on (X,m) and $g_1 \leq f$ and $g_2 \leq f$. Assume also that there is $x_0 \in X$ such that $g_1(x_0) < f(x_0)$. (The second case is similar.) Set $h(x) = g_1(x)$ if $x = x_0$ and $h(x) = f(x)$ otherwise. Let us check that $m(x,y) \leq \max\{h(x),h(y)\}$ whenever $x,y \in X$. Since $m(x,y) \leq \max\{f(x),f(y)\}$ whenever $x,y \in X$ and $m(x_0,x_0) = 0$, the only interesting cases are (1) $x = x_0$ and $y \neq x_0$, and (2) $x \neq x_0$ and $y = x_0$. In case (1) we have $m(x_0,y) = m(y,x_0) \leq \max\{g_2(y),g_1(x_0)\} \leq \max\{f(y),g_1(x_0)\} = \max\{h(x_0),h(y)\}$. In case (2) $m(y,x_0) \leq \max\{f(y),g_1(x_0)\} \leq \max\{h(y),h(x_0)\}$. We conclude that h is strongly tight on (X,m) and $h < f$ —a contradiction to $f \in \nu_m(X,m)$. Therefore (g_1,g_2) cannot exist and (f,f) is an extremal strongly tight function pair on (X,m). We have shown that $(f,f) \in \nu_q(X,m)$.

Let $f,g \in \nu_m(X,m)$. Then $E(f,g) = \sup_{x\in X} n^s(f(x),g(x)) = \sup_{x\in X} n(f(x),g(x)) \vee \sup_{x\in X} n(g(x),f(x)) = N((f,f),(g,g))$. Hence $p : (\nu_m(X,m),E) \to (\nu_q(X,m),N)$ is an isometric embedding, where $p(f) = (f,f)$ whenever $f \in \nu_m(X,m)$. □

5.3.3 Lemma. *(compare Lemma 3.3.7) Let A be a nonempty subset of a T_0-ultra-quasi-metric space (X,u) and let $(r_1,r_2) : A \to [0,\infty)$ be such that for* *all* $x, y \in A, u(x, y) \leq \max\{r_2(x), r_1(y)\}$. *Then there exists* $(R_1, R_2) : X \to [0, \infty)$ *which extends the pair* (r_1, r_2) *such that for all* $x, y \in X$, $u(x, y) \leq \max\{R_2(x), R_1(y)\}$. *Moreover, there exists a minimal strongly tight pair of functions* (f_1, f_2) *defined on* X *such that for all* $x \in X$, $f_1(x) \leq R_1(x)$ *and* $f_2(x) \leq R_2(x)$.

Proof. Choose $x_0 \in A$ fixed. Define $(R_1, R_2) : X \longrightarrow [0, \infty)$ by setting

$R_1(x) = r_1(x)$ if $x \in A$ and $R_1(x) = \max\{u(x_0, x), r_1(x_0)\}$ if $x \notin A$

and

$R_2(x) = r_2(x)$ if $x \in A$ and $R_2(x) = \max\{u(x, x_0), r_2(x_0)\}$ if $x \notin A$. We next check our claim that (R_1, R_2) is tight. Let $x, y \in X$: We consider four cases.

Case 1: $x, y \in A \Rightarrow u(x, y) \leq \max\{R_2(x), R_1(y)\}$ by assumption.

Case 2: $x \notin A, y \in A \Rightarrow u(x, y) \leq \max\{u(x, x_0), u(x_0, y)\}$ by the strong triangle inequality

$\Rightarrow u(x, y) \leq \max\{u(x, x_0), r_2(x_0), r_1(y)\}$ by assumption

$\Rightarrow u(x, y) \leq \max\{R_2(x), R_1(y)\}$ by definitions.

Case 3: $x \in A, y \notin A \Rightarrow u(x, y) \leq \max\{u(x, x_0), u(x_0, y)\}$

$\Rightarrow u(x, y) \leq \max\{r_2(x), r_1(x_0), u(x_0, y)\}$ as in Case 2.

$\Rightarrow u(x, y) \leq \max\{R_2(x), R_1(y)\}$

Case 4: $x \notin A, y \notin A \Rightarrow u(x, y) \leq \max\{u(x, x_0), u(x_0, y)\}$ by the strong triangle inequality

$\Rightarrow u(x, y) \leq \max\{u(x, x_0), r_2(x_0), u(x_0, y), r_1(x_0)\}$ by assumption

$\Rightarrow u(x,y) \le \max\{R_2(x), R_1(y)\}$ by definitions.

Then $(R_1, R_2) : X \longrightarrow [0, \infty)$ extends (r_1, r_2) such that for all

$$x, y \in X, u(x,y) \le \max\{R_2(x), R_1(y)\}.$$

So (R_1, R_2) is a strongly tight pair on X. Since $\mathcal{FP}(X,u)$ is partially ordered by the pointwise order on function pairs, Zorn's Lemma implies the existence of a minimal strongly tight pair $f = (f_1, f_2)$ defined on X such that $f_1(x) \le R_1(x)$ and $f_2(x) \le R_2(x)$ for any $x \in X$. $\square$

5.3.5 Proposition. *(compare Proposition 3.3.5) Let (X,u) be a T_0-ultra-quasi-metric space. If $s = (s_1, s_2)$ is a minimal strongly tight pair of functions on the T_0-ultra-quasi-metric space $\nu_q(X)$, then $s \circ e_X$ is a minimal strongly tight pair of functions on (X,u).*

Proof. Let $s = (s_1, s_2)$ be a minimal strongly tight pair of functions on the T_0-ultra-quasi-metric space $\nu_q(X)$. Note that for any $x, y \in X$, we have $u(x,y) = N(f_x, f_y) = N(e_X(x), e_X(y)) \le \max\{s_2(e_X(x)), s_1(e_X(y))\}$, because (s_1, s_2) is strongly tight on $\nu_q(X)$. Assume that $s \circ e_X$ is not a minimal strongly tight pair of functions on X. Then there exists a pair $(h_1, h_2) \in \nu_q(X)$ such that $h_1(x) \le s_1(e_X(x))$ and $h_2(x) \le s_2(e_X(x))$ whenever $x \in X$, and one of the inequalities is strict at some point $x_0 \in X$.

In the following we shall consider the case that $h_1(x_0) \le s_1(e_X(x_0))$. (The case $h_1(x_0) \le s_1(e_X(x_0))$ can be dealt with similarly.) Define the function pair (t_1, t_2) on $\nu_q(X)$ where $t_2(f) = s_2(f)$ whenever $f \in \nu_q(X)$, and for any $f \in \nu_q(X)$, $t_1(f) = s_1(f)$ if $f \neq e_X(x_0)$ and $t_1(f) = s_1(f)$ if $f = e_X(x_0)$.

Let us show that (t_1, t_2) satisfies the inequality $N(f,g) \le \max\{t_2(f), t_1(g)\}$ whenever $f, g \in \nu_q(X)$, which will contradict the fact that (s_1, s_2) is a minimal strongly tight pair of functions on $\nu_q(X)$ so that our assumption that (h_1, h_2) exists is false.

Since (t_1, t_2) and (s_1, s_2) coincide almost everywhere and (s_1, s_2) is a strongly tight pair of functions, we only need to prove the above inequality for $g = e_X(x_0)$ and $f \neq e_X(x_0)$, i.e. $N(f, e_X(x_0)) \leq \max\{t_2(f), t_1(e_X(x_0))\}$.

We distinguish two cases: Case 1: $f_1(x_0) = 0$. Then $N(f, e_X(x_0)) = f_1(x_0) = 0$ by Theorem 5.2.1 and our claim is obviously satisfied.

Case 2: $f_1(x_0) > 0$. By Proposition 5.1.1 $f_1(x) = \sup\{u^{-1}(x_0, y) : y \in X$ and $f_2(y) < u^{-1}(x_0, y)\}$. Therefore for any $\delta > 0$ there exists $y \in X$ such that $f_1(x_0) - \delta \leq u(y, x_0)$ and $f_2(y) < u(y, x_0)$.

Then $N(f, e_X(x_0)) - \delta = f_1(x_0) - \delta \leq \max\{s_2(e_X(y)), t_1(e_X(x_0))\}$.

Since (s_1, s_2) is a minimal strongly tight pair of functions on $\nu_q(X)$, then by Corollary 5.1.1 $t_2(e_X(y)) = s_2(e_X(y)) \leq \max\{s_2(f), N(e_X(y), f)\} = \max\{t_2(f), f_2(y)\}$ whenever $f \in \nu_q(X)$. So we have the two inequalities

$$N(f, e_X(x_0)) - \delta < \max\{s_2(e_X(y)), t_1(e_X(x_0))\}$$

and

$$s_2(e_X(y)) \leq \max\{t_2(f), f_2(y)\}.$$

We then get $N(f, e_X(x_0)) - \delta \leq \max\{t_2(f), f_2(y), t_1(e_X(x_0))\} = \max\{t_2(f), t_1(e_X(x_0))\}$, since $f_2(y) < u(y, x_0) \leq \max\{t_2(f), f_2(y), t_1(e_X(x_0)\}$, which implies

$$N(f, e_X(x_0)) - \delta \leq \max\{t_2(f), t_1(e_X(x_0))\}.$$

Since δ is arbitrary, we get the desired inequality

$$N(f, e_X(x_0)) \leq \max\{t_2(f), t_1(e_X(x_0)).$$

We conclude that $s \circ e_X \in \nu_q(X)$. □

The following proposition shows that the u-injective hull is unique up to isometries.

5.3.6 Proposition. *(compare Proposition 3.3.6) The following statements are true for any T_0-ultra-quasi-metric space (X, u).*

(a) $\nu_q(X)$ is q-spherically complete.

(b) $\nu_q(X)$ is an ultra-quasi-metrically injective hull of X, i.e. no proper subset of $\nu_q(X)$ which contains X as a subspace is q-spherically complete. The ultra-quasi-metrically injective hull of the T_0-ultra-quasi-metric space (X, u) is unique up to isometries.

Proof. (a) In order to prove that $\nu_q(X)$ is q-spherically complete, let $(f)_{i\in I}$ be a family of pairwise distinct points $f_i = ((f_i)_1, (f_i)_2) \in \nu_q(X)$ and $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ be two families of nonnegative real numbers such that $N(f_i, f_j) \leq \max\{r_i, r_j\}$ whenever $i, j \in I$. Set $Y = \{f_i : i \in I\}$. Define a map $s : Y \longrightarrow [0, \infty)$ by $s(f_i) = s_i$ and $r : Y \longrightarrow [0, \infty)$ by $r(f_i) = r_i$ whenever $i \in I$. By Lemma 5.3.3, we extend r and s to (R, S) onto the entire set $\nu_q(X)$ such that $N(f, g) \leq \max\{R(f), S(g)\}$ whenever $f, g \in \nu_q(X)$.

Using Lemma 5.3.3, there exists an extremal strongly tight pair $h = (h_1, h_2)$ of functions on $\epsilon_q(X)$ such that $h_2 \leq R$ and $h_1 \leq S$ where R and S are the extensions of r and s, respectively.

Using the property established in Proposition 5.3.5 for extremal strongly tight pairs of functions on $\nu_q(X)$, we know that $h \circ e_X \in \nu_q(X)$. It is then easy to see that

$$h \circ e_X \in \bigcap\nolimits_{f\in\nu_q(X)} (C_N(f, R(f)) \cap C_{N^{-1}}(f, S(f)))$$
$$\subseteq \bigcap\nolimits_{i\in I} (C_N(f_i, r_i) \cap C_{N^{-1}}(f_i, s_i)) :$$

Indeed, the distance D between $h \circ e_X$ and $f = (f_1, f_2) \in \nu_q(X)$ is defined by

$$N(h \circ e_X, f) = \sup_{x\in X}[n(h_1(e_X(x)), f_1(x)) \vee n(f_2(x), h_2(e_X(x)))].$$

Using Theorem 5.2.1, we can write $f_1(x) = N(f, e_X(x))$ and $f_2(x) = N(e_X(x), f)$ whenever $x \in X$.

Moreover, since $h = (h_1, h_2)$ is an extremal strongly tight pair of functions on $\nu_q(X)$ and using Corollary 5.1.1, we have that for each $x \in X$,

$$h_1(e_X(x)) \leq \max\{h_1(f), N(f, e_X(x))\} = \max\{h_1(f), f_1(x)\}.$$

By strong tightness of (h_1, h_2), we see that for each $x \in X$,

$$f_2(x) = N(e_X(x), f) \leq \max\{h_2(e_X(x)), h_1(f)\}.$$

Furthermore by the choice of $h = (h_1, h_2)$, we have

$$h_1(f) \leq S(f).$$

Therefore we get that $N(h \circ e_X, f) = \sup\{h_1(e_X(x)) : x \in X \text{ and } f_1(x) < h_1(e_X(x))\} \vee \sup\{f_2(x) : x \in X \text{ and } f_2(x) > h_2(e_X(x))\} \leq h_1(f) \leq S(f)$ whenever $f = (f_1, f_2) \in \nu_q(X)$. Similarly we see that

$$N(f, h \circ e_X) \leq h_2(f) \leq R(f)$$

whenever $f = (f_1, f_2) \in \nu_q(X)$. The proof is therefore complete.

(b) Let H be a subset of $\nu_q(X)$ such that $X \subseteq H$. Assume that H is q-spherically complete, hence u-injective by Theorem 5.3.1. There exists a nonexpansive map (R_1, R_2) extending the inclusion map $i : X \longrightarrow H$ such that

$$R = (R_1, R_2) : \nu_q(X) \longrightarrow H$$

$$f = (f_1, f_2) \longmapsto (R_1(f), R_2(f)).$$

Using Theorem 5.2.1 and nonexpansivity of $R = (R_1, R_2)$, we have

$$(R_1(f))(x) = N(R(f), f_x) = N(R(f), R(f_x)) \leq N(f, f_x) = f_1(x)$$

whenever $x \in X$. Similarly $(R_2(f))(x) \leq f_2(x)$ whenever $x \in X$.

Since $f = (f_1, f_2)$ is an extremal strongly tight pair of functions on X, we must have $R_1(f) = f_1$ and $R_2(f) = f_2$. This implies that R is the identity map and $H = \nu_q(X)$. Consequently, no proper subset of $\nu_q(X)$ which contains X is q-spherically complete.

Let H be a q-spherically complete T_0-ultra-quasi-metric space which contains X as a subspace such that no proper subset of H which contains X as subset is q-spherically complete. Consider a nonexpansive map $\phi : \nu_q(X) \longrightarrow H$ extending the isometric map $i : e_X(X) \longrightarrow H$ defined by $i(e_X(x)) = x$ whenever $x \in X$. Furthermore consider a nonexpansive map $\varphi : H \longrightarrow \nu_q(X)$ extending the map $i^{-1} : X \longrightarrow \nu_q(X)$. Then the nonexpansive $\varphi \circ \phi : \nu_q(X) \longrightarrow \nu_q(X)$ extends the identity map on $e_X(X)$. The argument described in the preceding step of the proof implies that $\varphi \circ \phi$ is the identity map on $\nu_q(X)$. This implies that φ is injective isometry, because ϕ and φ are nonexpansive. Therefore $\phi(\nu_q(X))$ is a q-spherically complete subspace of H containing X. We deduce that $\phi(\nu_q(X)) = H$, and thus ϕ is bijective and $\phi^{-1} = \varphi$. We have shown that $\nu_q(X)$ and H are isometric T_0-ultra-quasi-metric spaces. □

5.3.2 Corollary. *The following statements are true for any T_0-ultra-quasi-metric space (X, u).*

(a) $\nu_q(X)$ is q-spherically complete.

(b) If $f \in \nu_q(X)$ then there is $x \in X$ such that $f_1 = (f_x)_1$ and $f_2 = (f_x)_2$.

(c) If $f \in \nu_q(X)$ then there is $x \in X$ such that $f_1(x) = 0 = f_2(x)$.

Proof. This is a consequence of Proposition 5.3.6 and Lemma 5.2.2. □

5.3.2 Remark. *Let (X, u) be a T_0-ultra-quasi-metric space and let $\nu_q(X, u)$ be bicomplete. By Proposition 5.3.2 the $\tau(N^s)$-closure of $e_X(X)$ in $\nu_q(X, u)$ yields a subspace of $\nu_q(X, u)$ that is isometric to the (quasi-metric bicompletion) of (X, u). Of course, $f \in \nu_q(X, u)$ belongs to the $\tau(N^s)$-closure of $e_X(X)$ if and only if there is a sequence $(a_n)_{n \in \mathbb{N}}$ in X such that $\lim_{n \to \infty} N^s(f_{a_n}, f) = 0$. In the light of the distance formula proved above (see Theorem 5.2.1), this formula is* *equivalent to the existence of a sequence* $(a_n)_{n\in\mathbb{N}}$ *in* X *such that* $\lim_{n\to\infty} f_1(a_n) = 0$ *and* $\lim_{n\to\infty} f_2(a_n) = 0$.

5.4 Total boundedness in ultra-quasi-metric spaces

In this last section of Chapter 5, we are going to investigate total boundedness of the u-injective hull of a totally bounded T_0-ultra-quasi-metric space.

The following lemma is useful for showing that an u-injective hull is totally bounded.

5.4.1 Lemma. *Let* (X, u) *be a* T_0*-ultra-quasi-metric space that is totally bounded and let* $\epsilon > 0$. *Then there is a finite subset* E *of* X *such that*

$$\{f_1(x) : f \in \nu_q(X), x \in X, f_1(x) > \epsilon\} \cup \{f_2(x) : f \in \nu_q(X), x \in X, f_2(x) > \epsilon\} =$$

$$\{u(e, e^{'}) : e, e^{'} \in E, u(e, e^{'}) > \epsilon\}.$$

Proof. By total boundedness of (X, u) we can find a finite set $E \subseteq X$ such that for all $a \in X$, there is $e \in E$ such that $u^s(e, a) < \epsilon$. Let us show first that $\{u(x, y) : x, y \in X, u(x, y) > \epsilon\} = \{u(e, e^{'}) : e \in E, e^{'} \in E, u(e, e^{'}) > \epsilon\}$.

Consider any $x, y \in X$ such that $u(x, y) > \epsilon$. First note that there are $e_x \in X$ and $e_y \in E$ such that $u^s(e_x, x) \leq \epsilon$ and $u^s(e_y, x) \leq \epsilon$. Then $\epsilon < u(x, y) \leq \max\{u(x, e_x), u(e_x, e_y), u(e_y, x)\}$. Thus $\epsilon < u(x, y) \leq u(e_x, e_y)$. Moreover $\epsilon < u(e_x, e_y) \leq \max\{u(e_x, x), u(x, y), u(y, e_y)\}$. Therefore $u(e_x, e_y) \leq u(x, y)$, and thus $u(x, y) = u(e_x, e_y)$. Hence the first assertion is verified.

Fix $f \in \nu_q(X)$. Recall that by Lemma 5.1.1 $f_2(x) = \sup\{u(x, y) : y \in X$ and $u(x, y) > f_1(y)\}$ and $f_1(x) = \sup\{u^{-1}(x, y) : y \in X$ and $u^{-1}(x, y) > f_2(y)\}$ whenever $x \in X$. Let $x \in X$ such that $\epsilon < f_2(x)$. We can represent $f_2(x)$ as the supremum of an increasing sequence of values $u(x, y_n)$ all of which are larger than ϵ. By the result just proved that sequence must be eventually constant. Hence $f_2(x) = u(x, y)$ for some $x, y \in X$. The analogous result holds for the values $f_1(x)$ with $x \in X$ and $f_1(x) > \epsilon$. Hence $\{f_1(x) : f \in \nu_q(X), x \in X, f_1(x) > \epsilon\} \cup \{f_2(x) : f \in \nu_q(X), x \in X, f_2(x) > \epsilon\} \subseteq \{u(x, y) : x, y \in X, u(x, y) > \epsilon\} = \{(f_x)_1(y) : x \in X, y \in X, (f_x)_1(y) > \epsilon\}$, where the latter set is finite by the previous paragraph.□

The next result shows that total boundedness is preserved by the ultra-quasi-metrically injective hull of a T_0-ultra-quasi-metric space.

5.4.1 Proposition. *(compare Proposition 3.3.7) If (X, u) is a totally bounded T_0-ultra-quasi-metric space, then the T_0-ultra-quasi-metric space $(\nu_q(X, u), N)$ is totally bounded, too.*

Proof. Let $\epsilon > 0$. There is a finite set $E \subseteq X$ such that for each $x \in X$ there is $e_x \in E$ with $u^s(e_x, x) < \epsilon$. By the proof of Lemma 5.4.1 $D = \{f_1(e) : f \in \nu_q(X), e \in E, f_1(e) > \epsilon\} \cup \{f_2(e) : f \in \nu_q(X), e \in E, f_2(e) > \epsilon\} \subseteq \{u(x, y) : x, y \in E, u(x, y) > \epsilon\}$. Set $V = D \cup \{0\}$.

Consider any pair $(\psi_1, \psi_2) \in V^E \times V^E$. Define

$$\lambda_{(\psi_1,\psi_2)} = \{(f_1, f_2) \in \nu_q(X, u) : \sup_{e \in E}(n^s(f_1(e), \psi_1(e)) \vee n^s(f_2(e), \psi_2(e)) \leq \epsilon\}.$$

Note that $\nu_q(X, u)$ is the union of the finitely many sets $\lambda_{(\psi_1,\psi_2)}$. Indeed given $f \in \nu_q(X)$ and $i \in \{1, 2\}$ we set $\psi_i(e) = f_i(e)$ if $f_i(e) > \epsilon$, and $\psi_i(e) = 0$ if $f_i(e) \leq \epsilon$. Hence $(\psi_1, \psi_2) \in V^E \times V^E$ and $f \in \lambda_{(\psi_1,\psi_2)}$.

We show that

$$\bigcup_{(\psi_1,\psi_2) \in V^E \times V^E} {\lambda_{(\psi_1,\psi_2)}}^2 \subseteq C_{N,\epsilon},$$

where

$$C_{N,\epsilon} := \{(f, g) \in \nu_q(X, u) \times \nu_q(X, u) : N(f, g) \leq \epsilon\}.$$

For $(\psi_1, \psi_2) \in V^E \times V^E$ let $(f_1, f_2), (g_1, g_2) \in \lambda_{(\psi_1,\psi_2)}$. Given $x \in X$, there exists $e \in E$ such that $u^s(e, x) \leq \epsilon$. Then by the strong triangle inequality we have $n(f_1(x), g_1(x)) \leq n(f_1(x), f_1(e)) \vee n(f_1(e), \psi_1(e)) \vee n(\psi_1(e), g_1(e)) \vee n(g_1(e), g_1(x))$ which implies by Corollary 5.1.1 that $n(f_1(x), g_1(x)) \leq \epsilon$. Thus $\sup_{x\in X} n(f_1(x), g_1(x)) \leq \epsilon$. Analogously one can show that

$$\sup_{x\in X} n(g_2(x), f_2(x)) \leq \epsilon$$

(or use Lemma 5.1.3). By definition of N we conclude that $N((f_1, f_2), (g_1, g_2)) \leq \epsilon$. Hence we are done. □

5.4.1 Corollary. *If (X, u) is a T_0-ultra-quasi-metric space such that $\tau(u^s)$ is compact, then N^s induces a compact topology on $\nu_q(X, u)$.*

Proof. If $\tau(u^s)$ is compact, then the T_0-quasi-metric d is totally bounded. By Proposition 5.4.1 $(\nu_q(X, d), N)$ is totally bounded. The result follows, since the ultra-metric N^s is always complete on $\nu_q(X, u)$, because $(\nu_q(X, d), N)$ is q-spherically complete and thus bicomplete by Proposition 5.3.2. □

5.4.2 Corollary. *Let (X, m) be an ultra-metric totally bounded space. Then the completion of (X, m) is isometric to $(\nu_q(X, u), E)$.*

Proof. The completion of (X, m) sits inside its u-injective hull $(\nu_s(X), E)$ and is joincompact and hence spherically complete. Thus the u-injective hull of X must be equal to the completion of (X, m). □

5.4.1 Example. *Let $X = \{0, 1\}$ be equipped with the discrete metric u, Then (X, u) is not q-spherically complete, although it is spherically complete.*

Proof. Indeed (X, u) is a compact ultra-metric space, so spherically complete. Consider the family

$$((C_u(0, \frac{1}{2}), C_u(1, \frac{1}{2})); (C_{u^{-1}}(0, 1), C_{u^{-1}}(1, 1))).$$

Obviously it has the mixed binary intersection property, but $C_u(0, \frac{1}{2}) \cap C_u(1, \frac{1}{2}) = \emptyset$. Hence (X, u) is not q-spherically complete.

We now compute the ultra-quasi-metrically injective hull of (X, u). If $f = (f_1, f_2) \in \nu_q(X)$ is strongly tight, then we have $1 = u(0, 1) \leq \max\{f_2(0), f_1(1)\}$ and $1 = u(1, 0) \leq \max\{f_2(1), f_1(0)\}$. If f is also minimal strongly tight, then we find four point pairs $((f_1(0), f_1(1)), (f_2(0), f_2(1)))$ determined as follows:

$$((0,1),(0,1)), ((1,1),(0,0)), ((0,0),(1,1)), ((1,0),(1,0)).$$

Identifying these points $f = (f_1, f_2)$ according to $(f_1(0), f_1(1)) = (\alpha, \beta)$ with $\alpha, \beta \in \{0, 1\}$ we obtain $N((\alpha, \beta), (\alpha', \beta')) = 1$ if ($\alpha = 1$ and $\alpha' = 0$) or ($\beta = 1$ and $\beta' = 0$), and $N((\alpha, \beta), (\alpha', \beta')) = 0$ otherwise.

5.4.2 Lemma. *Let (X, u) be a T_0-ultra-quasi-metric space. Let $f \in \nu_q(X)$ be such that there is $a \in X$ with $f_1(a) \leq \inf_{x \in X} f_2(x)$. Then $f_1(a) = 0$. (Note that result remains true if f_1 and f_2 are interchanged in the statement.)*

Proof. Set $h_1(x) = f_1(x)$ if $x \in X$ and $x \neq a$, and $h_1(a) = 0$. Then (h_1, f_2) is a strongly tight pair: Indeed for any $x, y \in X$ we have to show $u(x, y) \leq \max\{f_2(x), h_1(y)\}$. The inequality holds for any $x \in X$ and any $y \in X$ with $y \neq a$, but it holds also for any $x \in X$ and $y = a$, since $u(x, a) \leq \max\{f_2(x), f_1(a)\} = f_2(x) = \max\{f_2(x), 0\}$. Hence $(h_1, f_2) < (f_1, f_2)$ is strongly tight, and since (f_1, f_2) is minimal strongly tight therefore $h_1 = f_1$. □

5.4.3 Lemma. *Let (X, u) be a joincompact T_0-ultra-quasi-metric space and let $f \in \nu_q(X)$. Then there is $x \in X$ such that $f_1(x) = 0$ or $f_2(x) = 0$.*

Proof. Let us first show that $a = \inf_{x \in X} f_1(x)$ is attained. If $a > 0$ this is obvious by Lemma 5.4.1 because of total boundedness of u. If $a = 0$, then there is a sequence $(x_n)_{n \in \mathbb{N}}$ in X such that $(f_1(x_n))_{n \in \mathbb{N}}$ converges to 0 (with respect to the usual topology on $\mathbb{R}$). By joincompactness of (X, u) there are a subsequence

$(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ and $y \in X$ such that $u^s(y, x_{n_k}) \to 0$. Then by Corollary 5.1.1 $f_1(y) \leq \max\{u(x_{n_k}, y), f_1(x_{n_k})\}$ whenever $k \in \mathbb{N}$ and thus $f_1(y) = 0$. Similarly we see that f_2 attains $\inf_{x\in X} f_2(x)$.

It remains to exclude the case that both the infima of f_1 and f_2 on $[0, \infty)$ are positive. But then we have reached a contradiction, since Lemma 5.4.3 implies that one of the attained infima must be zero. So this case does not occur. □

Chapter 6

Conclusion

In this thesis many aspects of the q-hyperconvex hull of a T_0-quasi-metric space and the u-injective hull of a T_0-ultra-quasi-metric space have been described. In the last chapter of this work, we want to present the conclusion of our investigation and underline some open problems encountered throughout the present work that may constitute the topics of further research.

Firstly in this thesis, we have achieved the task of presenting an explicit method to construct the q-hyperconvex hull of a T_0-quasi-metric space. In the second part of the work, we provided some interesting properties of the q-hyperconvex subsets of a T_0-quasi-metric space and we discussed the new concept of external q-hyperconvex subsets. In the third part of the thesis, we presented an explicit construction of the u-injective hull of a T_0-ultra-quasi-metric space in parallel with the q-hyperconvex hull of a T_0-quasi-metric space.

In the following we give a summary of the work which we studied in each chapter of the thesis and then suggest two important areas of future research which are related to our q-hyperconvexity theory.

6.1 Summary of the achieved work

The first three chapters recalled well-known results from the literature. In Chapter 0, we presented some preliminaries and an overview of certain well-known definitions from the theory of quasi-metric spaces. We also summarized facts about ultra-quasi-metric spaces and gave some interesting examples related to ultra-quasi-metric spaces.

In Chapter 1, we presented an overview of certain results dealing with the concept of hyperconvexity. We presented the well-known Isbell hyperconvex hull of a metric space and recalled the T-theory introduced by Dress. We also summarized the following results: A metric space is hyperconvex if and only if it is metrically convex and hypercomplete, and any hyperconvex metric hull is injective. We pointed out the connection between the injective hull, the hyperconvex hull, and the tight span or the metric envelope.

In Chapter 2, we summarized the concept of ultra-metric injectivity. We started by recalling the definition of spherical completeness in ultra-metric spaces. We also discussed the construction of the ultra-metrically injective hull of an ultra-metric space.

Chapter 3, was the main chapter of the thesis where we showed that a quasi-pseudometric space is q-hyperconvex if and only if it is metrically convex and q-hypercomplete. We also proved that any q-hyperconvex space is bicomplete. "q-hyperconvex" has been characterized as "q-injective". We constructed the q-hyperconvex hull of a T_0-quasi-metric space. We have proved that the q-hyperconvex hull of a T_0-quasi-metric space is unique up to isometries and for any totally bounded T_0-quasi-metric space its q-hyperconvex hull is totally bounded too.

In Chapter 4, we continued our investigations of the q-hyperconvex subsets of a

T_0-quasi-metric space. This chapter was devoted to some interesting properties of q-hyperconvexity. We introduced the q-admissibility concept. We have shown that a q-admissible subset of a q-hyperconvex space is itself q-hyperconvex. We also showed that the intersection of any descending family of nonempty bounded q-hyperconvex subsets of a q-hyperconvex space is q-hyperconvex. The notion of the ϵ_1, ϵ_2-approximate fixed points of a nonexpansive self-map on a q-hyperconvex space has been investigated and we have introduced the concept of externally q-hyperconvex subspaces.

In Chapter 5, we constructed the u-injective hull of a T_0-ultra-quasi-metric space. We introduced the concept of q-spherical completeness. We showed that a T_0-ultra-quasi-metric space is u-injective if and only if it is q-spherically complete. Furthermore we have proved that any q-spherically complete T_0-ultra-quasi-metric space is bicomplete. We also showed that the u-injective hull of a T_0-ultra-quasi-metric space is unique up to isometries and for any totally bounded T_0-ultra-quasi-metric space its u-injective hull is totally bounded, too.

In Chapter 6, the last chapter of this thesis, we concluded this work by reflecting on the main results of the thesis and highlighted some connections of this current work with older work in the literature, which we believe also provides a rich mine for future exploration.

The following problem is related to our investigations in Chapter 5.

6.1.1 Problem. *Investigate the ultra-quasi-metrically tight extensions of an ultra-quasi-metric space (see Section 2.2)?*

The theory of q-hyperconvexity may have some interesting applications in other structures of mathematics. In the following we point out two areas where the theory of q-hyperconvexity can lead to reasonable applications, namely in asymmetrically normed linear spaces and oriented graph theory.

6.2 Two possible areas for future work

6.2.1 The q-hyperconvex hull of asymmetrically normed linear spaces

The injective envelope or hyperconvex hull theory of a real Banach space was studied by several authors: In [16], H. Cohen constructed the injective envelope of Banach spaces and showed that the injective hull is unique up to a linear isometry.

In [44], N. V. Rao showed that the injective hull of a Banach space X in the category of metric spaces with contractions as morphisms coincides with the injective hull of X in the category of real normed spaces with linear contractions as morphisms.

The theory of the q-hyperconvex hull may have some applications in asymmetrically normed linear spaces, too. However, we point out that the linearly q-hyperconvex hull of an asymmetrically real normed space may be different from the q-hyperconvex hull of an asymmetrically normed linear space since the product of a q-tight function pair by a real scalar or the sum of two q-tight function pairs is not necessarily a q-tight function pair. But note that similar difficulties already occur in the symmetric case (see [44]). So we have the following question:

6.2.1 Problem. *Let (X, q) be an asymmetrically normed linear space. Does there exist an injective hull of (X, q) in the category of asymmetrically normed linear spaces and linear contractions? Furthermore does $\epsilon_q(X, d)$ coincide with the linearly injective hull of (X, q)?*

6.2.2 T-theory in quasi-metric setting

We recall (see [22] and [p. 5] of this thesis) that T-theory is the name that A. Dress, V. Moulton and W. Terhalle adopted for the theory of trees, injective envelopes of metric spaces, and all of the areas that are connected with these topics. Its motivation was originally and still is today to a large extent the development of mathematical tools for reconstructing phylogenetic trees. Then we have the following question:

6.2.2 Problem. *Is it possible to develop a kind of T-theory for trees or similar structures in the asymmetric setting?*

We finally list all the references that we have consulted when writing this dissertation.

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