Pricing discretely monitored barrier options under exponential-Lévy processes

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy to the University of Cape Town. It has not before been submitted for any degree or examination.

Signed by candidate

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Abstract

One of the main factors in pricing barrier options is deciding whether to monitor the underlying asset price in continuous time or for a fixed set of time points. Most actively traded barrier options are monitored in discrete time due to reasons such as regulation and practical implementation. This dissertation presents transform methods for pricing discretely monitored barrier options under exponential-Lévy processes. Single-barrier knock-out options are evaluated under the Black-Scholes framework, the normal inverse Gaussian model and the Variance Gamma model. These models are widely implemented when dealing with pricing options sensitive to jumps. A diffusion component is included in the Variance Gamma model for comparison purposes. We focus on the COS method using Fourier-cosine series expansions and the Hilbert transform method to obtain prices fast and accurately. These option pricing approaches are suitable for Lévy processes where the analytical form of their characteristic function is available. Furthermore, standard Monte Carlo pricing is used as a reference and an outline of the pricing algorithms is presented. Both methods are easy to implement across the different asset price dynamics. In particular, the COS method produces results faster than the Hilbert transform method, however, the truncation assumptions under the COS method derived in (Fang and Oosterlee, 2009) prove to be unreliable. We observe the truncation range requires adjustment under the different asset price dynamics, as well as the different types of knock-out barrier options.

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Chapter 1

Introduction

Discretely monitored path-dependent options have payoffs which are contingent on the underlying asset price at a fixed set of time points. On the other hand, the payoffs of continuously monitored path-dependent options depend on the underlying asset price at any instant during the life of the option. Almost all path-dependent options actively traded in the market are discretely monitored.

Barrier options are amongst the most popular discrete path-dependent options (Jeannin and Pistorius, 2010). The payoff of a barrier option is conditional on the price of the underlying asset satisfying fixed boundary constraints (barrier levels) before the maturity date (Cont and Tankov, 2004). These path-dependent options possess similar characteristics to vanilla options in that the holder of such an option has the right to buy or sell the underlying asset at a predetermined price and date.

Single-barrier options are classified as knock-ins (the option is activated when the underlying asset price reaches the barrier level) or knock-outs (the option is deactivated when the underlying asset price passes the barrier level) (Jeannin and Pistorius, 2010). We consider knock-out options, defined by an expiry date $T > 0$, a barrier level $B > 0$ and a terminal payoff Λ_T. A down-and-out option has terminal payoff $\Lambda_T \geq 0$ if $S_t > B$ for all $t \leq T$ and an up-and-out option has $\Lambda_T \geq 0$ if $S_t < B$ for all $t \leq T$.

One of the main factors in pricing barrier options is deciding whether to monitor the underlying asset price relative to the barrier level in continuous time or for a fixed set of time points. In fact, discrete and continuous monitoring present a distinct difference in the price of barrier options. (Heynen and Kat, 1995) were among the first to notice the importance of these differences. Furthermore, we note that analytical solutions are available for continuously monitored barrier options under the Black-Scholes (BS) model assumptions (Merton, 1973).

Most barrier options which are actively traded are monitored in discrete time. Regulations and practical implementation are some of the reasons why barrier options are priced under a discrete time setting. However, pricing discretely monitored barrier options is not as simple as pricing barrier options in continuous time for several reasons. (Kou, 2007) outlines the motivation for pricing discretely monitored barrier options. Firstly, there exists no closed form solutions, as under continuous monitoring, with the exception of implementing M-dimensional Gaussian distribution functions (M is the monitoring frequency). A problem with this implementation, however, arises when dealing with a large number of monitoring points, i.e. $M > 5$. It is difficult to obtain an accurate approximation of a high-dimensional Gaussian distribution. Secondly, a standard Monte Carlo (MC) or binomial tree approach can be challenging and time-consuming in producing accurate results (Broadie *et al.*, 1999). Lastly, the central limit theorem suggests the price difference between continuous and discrete monitoring to be small as $M \to \infty$. However, we observe the price difference to be quite significant for large values of M. (Broadie *et al.*, 1997) present numerical results of barrier option prices for various values of M, illustrating the difference between discrete and continuous monitoring. More specifically, we note that the price of discretely monitored knock-out (in) options decrease (increase) as M increases (Fusai *et al.*, 2006). Due to these difficulties, several numerical methods have been studied to price discretely monitored barrier options (Kou, 2007).

By presenting barrier options using a change of numéraire, pricing can concentrate on evaluating either the joint probabilities of the first barrier crossing time and the terminal value for a random walk process, or the marginal distribution of the first barrier crossing time. In the last two decades, several pricing methods have been proposed for discretely monitored barrier options. Of the most popular approaches are those based on transform methods, and convolution methods for evaluating the joint probabilities (Kou, 2007).

In this dissertation, we focus on Fourier and transform methods for pricing knock-out call and put barrier options, where the asset price dynamics are given by exponential-Lévy processes. The asset dynamics chosen for this dissertation are the normal inverse Gaussian (NIG) model and the Variance Gamma (VG) model, including a diffusion-extended VG (DEVG) model. These models are widely implemented when dealing with pricing options sensitive to jumps. In addition, we implement these pricing methods under the BS framework. We note that there exist analytical forms of the characteristic function for a range of Lévy processes. Therefore, pricing methods based on the characteristic function of the underlying Lévy process are considered practical for their tractability. This is due to the Lévy-Khinchin representation which provides an analytical form for the characteristic function of all Lévy processes (Cont and Tankov, 2004).

This dissertation is organised as follows. Chapter 2 outlines the literature on pricing methods for discretely monitored barrier options under Lévy processes. We present several methods that have been developed for pricing single-barrier options. Background on Lévy processes, and asset price dynamics modelled as exponential-Lévy processes is presented in Chapter 3, as well as mathematical background for the Fourier- and transform methods. More specifically, we focus on numerical methods which utilise the characteristic function of the underlying Lévy process. In addition, a summary of the implementation of the pricing algorithms is presented in Chapter 4. This includes the standard MC approach under the BS, NIG and VG models. Results from the COS and the Hilbert transform method are then presented in Chapter 5. We use MC pricing to compare the results from these methods. Chapter 6 concludes.

Chapter 2

Literature review

2.1 Continuity correction

(Broadie *et al.*, 1997) introduced a correction to the continuous price of barrier options leading to an approximation for the price of discretely monitored barriers. The method involves implementing the available continuous solution and replacing the barrier level in the pricing formula by a corrected barrier level.

Closed form solutions for continuously monitored barrier options under the Brownian assumptions are found in (Merton, 1973). (Broadie *et al.*, 1997) developed the adjusted pricing method under the Brownian model specifications. A simple continuity correction is applied to the barrier which adjusts the distance between the barrier and the underlying asset by a factor of $e^{(\pm\beta\sigma\sqrt{\Delta t})}$. This is detailed by the following theorem in (Broadie *et al.*, 1997) and where a full sketch of the argument can be found.

Theorem 1. *Let $V_M(B)$ be the price of a discretely monitored single-barrier option with barrier level B. Let $V(B)$ be the price of the corresponding continuously monitored barrier option. Then*

$$V_M(B) = V\left(Be^{\pm\beta\sigma\sqrt{\frac{T}{M}}}\right) + o\left(\frac{1}{\sqrt{M}}\right),$$

where σ is the volatility of the underlying asset, T is the maturity of the option, M is the frequency of monitoring dates and $\beta = -\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}} \approx 0.5826$, with ζ representing the Riemann zeta function . If $B > S_0$, we have $(+)$ in the exponent and a $(-)$ if $B < S_0$.

Numerical results in (Broadie *et al.*, 1997) provide evidence for the continuity correction as an accurate approximation for pricing discretely monitored barrier options. As $M \to \infty$, the discrete barrier price converges to the closed form continuous price. We note, however, that the continuity correction for discrete barrier option pricing is restricted to the BS model, and is only applicable to single-barrier options. The discrete pricing problem of barrier options remains for non-Gaussian Lévy processes.

2.2 COS method

Numerical methods based on the Fourier transform for pricing options are considered efficient due to the implementation of the fast Fourier transform, and can be easily adjusted for a range of asset price processes given that their characteristic function is mathematically tractable (Fang and Oosterlee, 2009). This includes exponential-Lévy models.

(Fang and Oosterlee, 2009) introduced a transform method for early-exercise and discretely monitored barrier options. The COS method for barrier options is an extension of the Fourier-cosine series based method for European options presented in (Fang and Oosterlee, 2008). Under this pricing approach, the conditional probability function is substituted by its Fourier-cosine series expansion, which is linked to to the characteristic function of the underlying Lévy process.

Results presented in (Fang and Oosterlee, 2009) prove the method to be fast, accurate and easy to implement for exponential-Lévy processes. Furthermore, the pricing of discretely monitored barrier options can be reduced to two steps. First, the Fourier-cosine series at the first monitoring date ($V_k(t_1)$) needs to be recovered, followed by implementation of the COS pricing formula ($\hat{v}(x, t_0)$), given by

$$\hat{v}(x, t_0) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' Re \left\{ \varphi_{L\acute{e}vy} \left(\frac{k\pi}{b-a} \right) e^{ik\pi \frac{x-a}{b-a}} \right\} V_k(t_1).$$

The Fourier-cosine series at t_1 is obtained via a backward recursion algorithm. Further details are presented in Chapter 3 which provides the mathematical background for the derivation of the COS pricing formula.

2.3 Laplace transform

(Petrella and Kou, 2004) introduced pricing discrete lookback and barrier options using an inverse Laplace transform. The method can be applied to various asset price dynamics including jump-diffusion models, since the method only requires the underlying asset price process to be a Lévy process. We summarise the approach of pricing a discretely monitored up-and-out put option (with strike K and barrier level B) as outlined in (Petrella and Kou, 2004).

The Laplace transform $\hat{f}(\xi, \zeta)$ for an up-and-out put is given by

$$\hat{f}(\xi, \zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\xi\kappa - \zeta h} f(\kappa, h; S_t) \, d\kappa \, dh,$$

where $f(\kappa, h; S_t)$ is the conditional expectation of the payoff function, and $\kappa = \ln(K)$, $h = \ln(B)$. Furthermore, (Petrella and Kou, 2004) describe the Laplace transform of such an option as

$$\hat{f}(\xi,\zeta) = (S_t)^{-(\xi+\zeta-1)} \cdot \frac{C(-\zeta, 1-\xi; t)}{\xi(\xi-1)\zeta},$$

where the function C is computed via a recursion approach and derived from an application of Spitzer's formula (Spitzer, 1956). The implementation of the recursion is reduced to using only closed form solutions of European options.

More specifically, by inverting the Laplace transform of $f(\kappa, h; S_t)$, we obtain the value of the up-and-out option:

$$V(t,T) = e^{-r(T-t)} \mathcal{L}^{-1}_{\xi,\zeta} \left(\frac{S_t^{-(\xi+\zeta-1)}}{\xi(\xi-1)\zeta} C(-\zeta, 1-\xi; t) \right),$$

evaluated at ξ = ln(K), ζ = ln(B). (Petrella and Kou, 2004) use a two-sided Euler algorithm for the inverse Laplace transform. A similar approach is extended to price other single-barrier options using symmetry. Due to the explicit formula obtained for the price of a barrier option, one could easily derive the hedging parameters such has the delta, gamma and vega of the option.

The Laplace transform method includes the advantage of being able to evaluate prices at different time points, not only at monitoring points or the inception point. However, (Feng and Linetsky, 2008) highlight the method to be computationally complex for barrier options. In addition, a two-dimensional Laplace inversion for each sample point is required for the implementation and the method is known to be more computationally intensive for infinite activity Lévy processes, in contrast to the application for jump diffusion models.

2.4 Convolution

The fast Gaussian transform method (Broadie and Yamamoto, 2005) and the Hilbert transform (Feng and Linetsky, 2008) are pricing methods for discrete barrier options based on convolution. (Kou, 2007) illustrates the general idea of pricing discrete barrier options, which involves evaluating joint probabilities of the first time the barrier level is reached (τ) and the terminal value of a random walk process. These joint probabilities can be evaluated as M-dimensional convolution (M is the monitoring frequency).

More specifically, under the geometric Brownian motion framework (independent $Z_i \sim N(0,1)$ and drift $\mu = r - \frac{\sigma^2}{2}$), the underlying asset price at each monitoring point ($m = 1, 2, ..., M$ and $\Delta t = \frac{T}{M}$) is defined as

$$S_m = S_0 e^{\mu m \Delta t + \sigma\sqrt{\Delta t}\sum_{i=1}^{m} Z_i} = S_0 e^{W_m \sigma \sqrt{\Delta t}},$$

where W_m is a random walk such that $W_m = \sum_{i=1}^{m} Z_i + m\frac{\mu}{\sigma}\sqrt{\Delta t}$. It follows, the discrete option price of an up-and-out call option to be given by

$$V_M(B) = E[e^{-rT}(S_M - K)^+ \mathbf{1}_{\{\tau(B,S)>M\}}],$$

where $\tau = \inf\{m \geq 1 : S_m > B\}$. Alternatively, the indicator function can be represented as $\mathbf{1}_{\{\tau(\frac{a}{\sigma\sqrt{T}},W)>M\}}$ where $\tau = \inf\{m \geq 1 : W_m \geq \frac{a}{\sigma\sqrt{T}}\sqrt{M}\}$ and $a = \ln(\frac{B}{S_0})$. A change of numéraire argument can be applied to the price of a discrete barrier option, whereby application of Girsanov theorem describes the price as a difference of two joint probabilities (involving τ and a random walk) under different measures.

Since we can write the barrier option price involving joint probabilities of τ and random walks, the solution may be formulated using M-dimensional normal distribution functions. The fast Gaussian transform and the Hilbert transform method provide powerful approaches in computing the convolution in the M-dimensional normal distribution (Kou, 2007).

The main feature of the fast Gaussian transform method in (Broadie and Yamamoto, 2005) is that if the integrals only consist of normal distributions, they can be computed time-efficiently in convolution. A discrete sum of normal random variables can be evaluated faster using the Hermite expansion. We consider the sum

$$\sum_{n=1}^{N} w_n e^{-\frac{(x_m - y_n)^2}{\delta}},$$

for $m = 1, ..., M$ which would require $O(NM)$ computations. However, applying the Hermite expansion

$$e^{-\frac{(x_m-y_n)^2}{\delta}} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{1}{i!j!}\left(\frac{y_n - y_0}{\sqrt{\delta}}\right)^j \left(\frac{x_m - x_0}{\sqrt{\delta}}\right)^i H_{i+j}\left(\frac{x_0 - y_0}{\sqrt{\delta}}\right),$$

with H representing the Hermite function, and replacing the upper bound of the summation with $\alpha_{max} < 8$, we can approximate the Gaussian sum (Kou, 2007).

Numerical results in (Broadie and Yamamoto, 2005) prove how fast the Gaussian transform performs. Pricing a down-and-out call under the BS model with 50 monitoring dates yielded a computation time of 0.5 CPU second and an accuracy of 10^{-10}, where the true price was obtained via a trinomial lattice method.

On the other hand, the Hilbert transform method can be applied to Lévy processes outside of the Gaussian framework, as well as single- and double-barrier options. The method allows for valuation of options with non-equidistant monitoring points and valuation at non-monitoring points. The Hilbert transform method can be seen as an extension of the (Carr and Madan, 1999) fast Fourier transform application on European options to discrete barrier options. Furthermore, the method is an extension of the fast Fourier transform method introduced in (Eydeland, 1994) under Gaussian assumptions to Lévy process (Kou, 2007).

(Feng and Linetsky, 2008) evaluate the convolution by noting that the product of a function with the indicator function relates to the Hilbert transform in the Fourier space such that

$$\mathcal{F}(f \cdot I_{(0,\infty)}(\theta) = (\tfrac{1}{2}\mathcal{F}(f) + \tfrac{i}{2}\mathcal{H}(f))(\theta),$$

where the Hilbert transform is defined by the Cauchy principal value integral as

$$\mathcal{H}(f)(\theta) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(u)}{\theta - u}\, du.$$

To price barrier options, the method is reduced to evaluating Hilbert transforms at the monitoring points and concludes with an application of the inverse Fourier transform. The method involves recursive computation of Hilbert transforms of the product of two functions - the Fourier transform of the option value at the previous monitoring point and the characteristic function of the underlying Lévy process. Furthermore, the method derives the equivalent martingale measure (EMM) from application of the Esscher transform. In addition, a sinc approximation in a Hardy space is used for the Hilbert transform (Feng and Linetsky, 2008). We provide further details about this method in Section 3.3, and discuss its implementation in Section 4.3.

In addition to the literature on the Hilbert transform method, (Zeng and Kwok, 2014) extended the Hilbert transform method of (Feng and Linetsky, 2008) to price discretely monitored barrier options under time-changed Lévy process. The main implementation of the method involves application of a quadrature based rule to the log-activity rate of stochastic time change dimension. The Hilbert transform approach is applied to the log-stock return dimension. (Zeng and Kwok, 2014) provide a numerical algorithm for pricing barrier options which proves to be efficient and accurate. More specifically, numerical results are presented for pricing the dividend-ruin model (with dividend barrier specifications) under the Heston stochastic volatility model and the NIG model time-changed by the Cox-Ingersoll-Ross process. Furthermore, (Zeng and Kwok, 2014) note the Hilbert transform method to be significantly more accurate and efficient than other existing pricing algorithms.

2.5 Further pricing methods

There exists several methods that deal with the problem of pricing discretely monitored barrier options, however not all can be extended to general Lévy models. Numerical methods based on the evaluation of differential equations have been developed, for example (Howison and Steinberg, 2005). More general methods that are applicable to a wider range of path-dependent options (e.g. barriers, Americans, lookbacks) have been developed such as lattice methods, finite difference methods and numerical integration methods. Lattice methods have been introduced in (Broadie *et al.*, 1999), however, remain to be time consuming and there is uncertainty on extending the method outside the Brownian framework (Kou, 2007). The reader is referred to (Aït-Sahalia and Lai, 1998) and (Sullivan, 2000) for applications of numerical integration and (Boyle and Tian, 1998) and (Zvan *et al.*, 2000) for finite difference methods.

Chapter 3

Mathematical background

We outline the general pricing problem for barrier options, as a risk-neutral expectation of a certain discounted payoff. For barrier level B and time-t asset price S_t, we classify down options by the condition $\{B < S_0\}$ and up options by $\{B > S_0\}$. In addition, we define τ as the first time the underlying asset price knocks the barrier level, such that

$$\tau = \inf\{t > 0 : S_t = B\}.$$

In the event of $\{\tau > T\}$, a knock-out call option will pay $(S_T - K)^+$ while a knock-out put will pay $(K - S_T)^+$. It follows, the price of a knock-out call option (with maturity T and strike K) to be given by the risk-neutral expectation:

$$e^{-rT} E^{\mathbb{Q}}[(S_T - K)^+; \tau > T].$$

In the case of knock-in options, we substitute the event $\{\tau > T\}$ with $\{\tau \leq T\}$. Extending this notion to discretely monitored barrier options, we consider the set of time points $\{t_i, i = 0, 1, ..., M\}$ and define $\tilde{S}_{t_i}$ as the underlying asset price at those monitoring points. Furthermore, we define

$$\tilde{\tau} = \begin{cases} \inf\{m > 0 : \tilde{S}_{t_m} > B\} & \text{for} \quad S_0 < B \\ \inf\{m > 0 : \tilde{S}_{t_m} < B\} & \text{for} \quad S_0 > B \end{cases},$$

such that the discretely monitored price of a knock-out call option is given by the risk-neutral expectation:

$$e^{-rT} E^{\mathbb{Q}}[(\tilde{S}_{t_M} - K)^+; \tilde{\tau} > M].$$

We apply the same adjustment as before for pricing knock-in options and replace $\{\tilde{\tau} > M\}$ with its complement (Broadie *et al.*, 1997).

More specifically, the in-out barrier parity allows for the pricing of knock-in options using the result of the knock-out price, without needing to evaluate its risk-neutral expectation (Carr and Chou, 1997). We summarise the in-out barrier parity, for knock-in and knock-out options (with the same barrier level and the same payoff):

$$\text{knock-in call (put)} + \text{knock-out call (put)} = \text{vanilla call (put)}.$$

3.1 Asset price dynamics

In the last two decades, Lévy processes have become more popular in modelling market fluctuations for the purpose of pricing options, as well as managing risk. These are stochastic processes in continuous time with the properties of stationary and independent increments. In addition, Lévy processes are defined by stochastic continuity. However, this condition does not define the sample paths of Lévy processes as continuous. In fact, Brownian motion with drift is the only Lévy process with continuous paths (Cont and Tankov, 2004).

Pricing methods based on the characteristic function of the underlying Lévy process are considered practical for its tractability. This is due to the Lévy-Khinchin representation which provides an analytical form for the characteristic function of all Lévy processes (Cont and Tankov, 2004). The Lévy-Khinchin representation defines the characteristic function of a Lévy process $(X_t)_{t\geq 0}$ as

$$\phi_t(u) = E[e^{iuX_t}] = e^{-t\psi(u)},$$

where the characteristic exponent $\psi(u)$ is given by

$$\psi(u) = \frac{1}{2}\sigma^2 u^2 - i\mu u + \int_{\mathbb{R}} (1 - e^{iux} + iux\mathbf{1}_{\{|x|\leq 1\}})\,\nu(dx).$$

The parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ represent the drift and diffusion component of the Lévy process with Lévy measure ν (Feng and Linetsky, 2008). A Lévy measure ν of a Lévy process $(X_t)_{t\geq 0}$ on $\mathbb{R}^d$ is described as

$$\nu(A) = E[\#\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\}], A \in \mathcal{B}(\mathbb{R}^d),$$

where $\nu(A)$ is the expected number of jumps per unit time and with jump size contained in A (Cont and Tankov, 2004).

Exponential-Lévy processes provide an extension of the BS framework by accounting for market price jumps while maintaining the independent and stationary increments property (Cont and Tankov, 2004). The asset price under an exponential-Lévy process is modelled as $S_t = S_0 e^{X_t},\ \ t \geq 0$, where X_t represents the Lévy process. Under a given EMM and where the scaled asset price dynamics is modelled by $S_t = Ke^{X_t}$, with $X_0 = \ln(\frac{S_0}{K}) \in \mathbb{R}$, it is typical to have $K = S_0$. We require the discounted gains process to be a martingale under the EMM, such that

$$E[S_t] = S_0 e^{(r-q)t}, \tag{3.1}$$

for $t, r, q \geq 0$, where r and q are the risk-free rate and the constant dividend yield of S_t, respectively. It follows, that the drift parameter (μ) of the underlying Lévy process is given by

$$\mu = r - q - \frac{\sigma^2}{2} + \omega,$$

with

$$\omega = \int_{-\infty}^{\infty} (1 - e^x + x\mathbf{1}_{|x|\leq 1})\,\nu(dx),$$

where ω is determined by evaluating the exponent of the underlying characteristic function (of the jump element) at $-i = -\sqrt{-1}$ (Feng and Linetsky, 2008).

Furthermore, exponential-Lévy models can be characterised by finite or infinite activity, specifically jumps (Tankov, 2010).

3.1.1 Finite activity Lévy processes

The first category of exponential-Lévy process is the class of jump-diffusion models. Price movements are defined by a diffusion process with jumps occurring at random time intervals. The jumps are regarded as sporadic financial events, such as market crashes. The financial model of these price movements constitute a Lévy process representing the log-price, alongside a Gaussian component and jump component. Consequently, a finite activity Lévy process is represented as:

$$X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i.$$

The jump part is given by a compound Poisson process ($\sum_{i=1}^{N_t} Y_i$) with each time interval constituting a finite number of jumps. (Y_i) are i.i.d. variables representing the jump size and (N_t) is a standard Poisson process, interpreted as the number of jumps by time t. It follows, under exponential-Lévy process dynamics, that the asset prices in jump-diffusion models is given by

$$S_t = S_0 e^{\gamma t + \sigma W_t + \sum_{i=1}^{N_t} Y_i}.$$

In order to complete the definition of jump-diffusion models, the distribution of the jump sizes needs to specified. Examples of finite activity Lévy processes are the widely known Merton model which characterises the Y_i's to be normally distributed and the Kou model with jumps distributed as an asymmetric double-exponential (Cont and Tankov, 2004).

3.1.2 Infinite activity Lévy processes

The second category of exponential-Lévy processes is models with an infinite number of jumps in each time interval. Many believe this to be a realistic representation of market prices (Cont and Tankov, 2004). These models do not necessarily require a Brownian component as with finite activity Lévy processes. The jump dynamics have the advantage of being able to generate non-trivial small time behaviour. Furthermore, a range of models in this category can be defined using Brownian subordination. This is an advantage of analytical tractability over jump-diffusion models. Examples of asset price drivers in this category are the VG process and the NIG process (Cont and Tankov, 2004).

Under the NIG model specifications, the characteristic function is given by

$$\phi_t(u) = E[e^{iuX_t}] = e^{t\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+iu)^2}\right)},$$

with parameters $\delta, \alpha, \beta \in \mathbb{R}$ (Fang and Oosterlee, 2009). The density function is defined as

$$f_t(x) = \frac{\alpha}{\pi}\frac{K_1\left(\alpha\delta t\sqrt{1+(\frac{x-\mu t}{\delta t})^2}\right)}{\sqrt{1+(\frac{x-\mu t}{\delta t})^2}}e^{(\delta t(\gamma+\beta(\frac{x-\mu t}{\delta t})))},$$

where $\gamma = \sqrt{\alpha^2-\beta^2}$, $\mu \in \mathbb{R}$ and $K_\lambda(z)$ is a modified Bessel function of the second kind:

$$K_\lambda(z) = \frac{1}{2}\int_0^\infty y^{\lambda-1}e^{-\frac{1}{2}z(y+y^{-1})}\,dy, \tag{3.2}$$

(Ribeiro and Webber, 2003). In addition, Lévy processes can be defined by subordinating a Brownian motion with an independent increasing Lévy process (Cont and Tankov, 2004). The NIG process can be described as a subordinated Brownian motion $X_t = \mu t + w_{h(t)}$, where w_t represents Brownian motion with drift β and variance parameter equal to 1. The $h(t)$ component follows an inverse Gaussian distribution, such that $h_t \sim IG(\delta t, \gamma)$ or as an alternative parameterisation $h_t \sim IG(\frac{\delta t}{\gamma}, (\delta t)^2)$ (Ribeiro and Webber, 2003).

A VG model is established by time-changing a Brownian motion, with non-zero drift, with a gamma process (an increasing Lévy process). This model is of finite variation and is the difference of two increasing functions, whereas the NIG has infinite variation (Cont and Tankov, 2004). The form of density function $f_t(x)$ and the characteristic function $\phi_t(u)$ of the VG process with parameters $\theta \in \mathbb{R}, s, \nu > 0$ is given by

$$f_t(x) = \frac{2e^{\frac{\theta x}{s^2}}}{\nu^{\frac{t}{\nu}}\sqrt{2\pi s}\Gamma(\frac{t}{\nu})}\left(\frac{x^2}{\frac{2s^2}{\nu}+\theta^2}\right)^{\frac{t}{2\nu}-\frac{1}{4}} K_{\frac{t}{\nu}-\frac{1}{2}}\left(\frac{1}{s^2}\sqrt{x^2\left(\frac{2s^2}{\nu}+\theta^2\right)}\right),$$

$$\phi_t(u) = E[e^{iuX_t}] = (1 - i\theta\nu u + \frac{1}{2}s^2\nu u^2)^{-\frac{t}{\nu}},$$

where $K_\nu(z)$ is the modified Bessel function of the second kind as defined in equation (3.2) (Ribeiro and Webber, 2004). The VG process is a pure-jump model with no diffusion component, however, we note that the the model can be extended to include a diffusion component, i.e. the DEVG process. Furthermore, the VG process can be described as a subordinated Brownian motion $w_{h(t)}$ under which w_t represents Brownian motion with parameters θ and s^2 and $h(t)$ is a gamma process $h_t \sim G(\frac{t}{\nu}, \nu)$. A subordination involves a time change, where the subordinator is not standard time, but a gamma process regarded as "business time" (Cont and Tankov, 2004). Therefore, for the VG process, we have $X_t = w_{h(t)}$.

Under these models, the stock price can be represented as

$$S_t = S_0 e^{rt+X_t+\omega t},$$

where r is the constant risk-free rate . It follows, in order to ensure the martingale condition of $S_t e^{-rt}$ is met, we require

$$\omega = \frac{1}{\nu}\ln(1 - \theta\nu - \frac{1}{2}s^2\nu),$$

for the VG model and

$$\omega = \delta(\sqrt{\alpha^2 - (\beta+1)^2} - \sqrt{\alpha^2 - \beta^2}),$$

for the NIG process, both of which are obtained by using the definition of the characteristic function (Ribeiro and Webber, 2004).

3.2 COS method

Discrete barrier options can be priced by Fourier-cosine series expansions, similarly to the COS method applied to European options as introduced in (Fang and Oosterlee, 2008). (Fang and Oosterlee, 2009) outline how the COS method for European options can be extended to discretely monitored barrier options. We consider a zero rebate up-and-out option with payoff:

$$\Lambda(x, T) = \max(\alpha(S_T - K), 0)\mathbf{1}_{\{S_{t_i} < B\}}, \quad 0 \le i \le M,$$

where $\alpha = 1$ for a call option, $\alpha = -1$ for a put option. A discretely monitored up-and-out option with M monitoring points ($t_1 < t_2 < ... < t_{M-1} < t_M = T$) follows the recursive pricing formula

$$c(x, t_{m-1}) = e^{-r(t_m - t_{m-1})} \int_{-\infty}^{\infty} \Lambda(x, t_m) f(y|x)\, dy \tag{3.3}$$

and

$$\Lambda(x, t_{m-1}) = \begin{cases} c(x, t_{m-1}) & x < \ln(\frac{B}{K}) \\ 0 & \text{otherwise} \end{cases},$$

where $x = \ln(\frac{S_{t_{m-1}}}{K})$, $y = \ln(\frac{S_{t_m}}{K})$, $\Lambda(x, t)$ represents the option value, and $c(x, t)$ is the continuation value for $m = M, M-1, ..., 2$.

Since $f(y|x)$ decreases to 0 as $y \to \pm\infty$, the integration range of Λ can be truncated while remaining an accurate result. It follows, for a given tolerance level l and interval $[a, b] \subset \mathbb{R}$ such that

$$\int_{\mathbb{R}\setminus[a,b]} f(y|x)\, dy < l,$$

the continuation value can be approximated as

$$\hat{c}(x, t_{m-1}) = e^{-r\Delta t} \int_a^b \Lambda(y, t_m) f(y|x)\, dy.$$

The conditional probability function can be substituted by its Fourier-cosine series expansion

$$f(y|x) = \sum_{k=0}^{\infty}{}' A_k(x) \cos\left(k\pi \frac{y-a}{b-a}\right),$$

where

$$A_k(x) = \frac{2}{b-a} \int_a^b f(y|x) \cos\left(k\pi \frac{y-a}{b-a}\right)\, dy,$$

for $k = 0, ..., \infty$ and where the $\sum{}'$ symbol requires the first summation term to be multiplied by $\frac{1}{2}$ [1]. We define the Fourier-cosine series coefficients $V_k(t_m)$ of option value $\Lambda(y, t_m)$ as

$$V_k(t_m) = \frac{2}{b-a} \int_a^b \Lambda(y, t_m) \cos\left(k\pi \frac{y-a}{b-a}\right)\, dy.$$

This yields the approximated continuation value as

$$\hat{c}(x, t_{m-1}) = \frac{1}{2}(b-a) e^{-r\Delta t} \sum_{k=0}^{\infty}{}' A_k(x) V_k(t_m),$$

[1] $\sum_{k=0}^{\infty}{}' x_{k=\frac{1}{2}x_0+x_1+x_2+x_3+...}$

and with truncation of the summation range we can further approximate the continuation value as

$$\hat{c}(x, t_{m-1}) = \frac{1}{2}(b-a)e^{-r\Delta t} \sum_{k=0}^{N-1}{}' A_k(x) V_k(t_m).$$

Using the characteristic function $\phi(\theta; x) = \int_{-\infty}^{\infty} f(y|x) e^{i\theta y}\, dy$ of $f(y|x)$, we can re-write the $A_k(x)$ coefficients as

$$A_k(x) = \frac{2}{b-a} Re\left\{ e^{-ik\pi \frac{a}{b-a}} \int_a^b e^{i\frac{k\pi}{b-a}y} f(y|x)\, dy \right\},$$

and by approximating the finite integral such that $\int_{-\infty}^{\infty} e^{i\frac{k\pi}{b-a}y} f(y|x)\, dy = \phi(\frac{k\pi}{b-a}; x)$, the coefficients can be defined as

$$A_k(x) = \frac{2}{b-a} Re\left\{ \phi\left(\frac{k\pi}{b-a}; x\right) e^{-ik\pi \frac{a}{b-a}} \right\}, \tag{3.4}$$

where $Re\{x\}$ represents the real part of x. Subsequently, we replace the coefficients in the continuation value expression with its approximation. The COS formula for pricing European options for an exponential-Lévy process is then given by

$$\hat{c}(x, t_{m-1}) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' Re\left\{ \varphi\left(\frac{k\pi}{b-a}\right) e^{ik\pi \frac{x-a}{b-a}} \right\} V_k(t_m), \tag{3.5}$$

where $\varphi(\theta) = \phi(\theta, 0)$. The option value at time t_0 can then be approximated as

$$\hat{\Lambda}(x, t_0) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' Re\left\{ \varphi\left(\frac{k\pi}{b-a}\right) e^{ik\pi \frac{x-a}{b-a}} \right\} V_k(t_1). \tag{3.6}$$

(Fang and Oosterlee, 2009) outline the computation of $V_k(t_m)$ for $k = 0, 1, ..., N-1$ via a backward recursion using $V_k(t_{m+1})$. From the COS formula for pricing European options, discretely monitored barrier options are priced by first computing the Fourier-cosine series coefficients (V_k) of the barrier option at the first monitoring point t_1 and then by applying the COS formula for European options, $\Lambda(x, t_0)$.

The initial Fourier-cosine series coefficients for up-and-out options are given by:

For $h = \ln(\frac{B}{K}) < 0$, if $h < 0$

$$V_k(t_M) = \begin{cases} 0 & \text{call} \\ G_k(a, h) & \text{put} \end{cases} \tag{3.7}$$

and if $h \geq 0$,

$$V_k(t_M) = \begin{cases} G_k(0, h) & \text{call} \\ G_k(a, 0) & \text{put} \end{cases}. \tag{3.8}$$

The $G_k(x_1, x_2)$ terms can be solved analytically by the exact approach as under the COS method for European options in (Fang and Oosterlee, 2008). We recap the analytical expressions for $G_k(x_1, x_2)$:

$$G_k(x_1, x_2) = \frac{2}{b-a}\alpha K \left[\chi_k(x_1, x_2) - \Psi_k(x_1, x_2)\right],$$

where

$$\Psi_k(x_1, x_2) = \begin{cases} \sin\left(k\pi\frac{x_2-a}{b-a}\right) - \sin\left(k\pi\frac{x_1-a}{b-a}\right) & k \neq 0 \\ (x_2 - x_1) & k = 0 \end{cases},$$

and

$$\chi_k(x_1, x_2) = \frac{1}{1 + \left(\frac{k\pi}{b-a}\right)^2}\left[\cos\left(k\pi\frac{x_2-a}{b-a}\right)e^{x_2} - \cos\left(k\pi\frac{x_1-a}{b-a}\right)e^{x_1}\right.$$
$$\left. + \frac{k\pi}{b-a}\sin\left(k\pi\frac{x_2-a}{b-a}\right)e^{x_2} - \frac{k\pi}{b-a}\sin\left(k\pi\frac{x_1-a}{b-a}\right)e^{x_1}\right].$$

The consequent $V_k(t_m)$ coefficients are found by the following backward recursion

$$\hat{V}_k(t_m) = \frac{e^{-r\Delta t}}{\pi} Im\left\{\left(\mathcal{M}_c(x_1, x_2) + \mathcal{M}_s(x_1, x_2)\right)\mathbf{u}\right\},$$

where the vector $\mathbf{u}$ is defined as

$$u_0 = \frac{1}{2}\varphi(0)V_0(t_{m+1}) \tag{3.9}$$

$$u_{j=1,\ldots,N-1} = \varphi\left(\frac{j\pi}{b-a}\right)V_j(t_{m+1}) \tag{3.10}$$

and $\mathcal{M}_c$ is an $(N\text{x}N)$ *Hankel* matrix and $\mathcal{M}_s$ an $(N\text{x}N)$ *Toeplitz* matrix with matrix elements

$$m_j = \begin{cases} \frac{(x_2-x_1)\pi i}{b-a} & j = 0 \\ \frac{e^{\frac{ij(x_2-a)\pi}{b-a}} - e^{\frac{ij(x_1-a)\pi}{b-a}}}{j} & j \neq 0 \end{cases}. \tag{3.11}$$

We note the structure of the *Hankel* and *Toeplitz* matrices to be defined as

$$\mathcal{M}_c = \begin{bmatrix} m_0 & m_1 & m_2 & \ldots & m_{N-1} \\ m_1 & m_2 & \ldots & \ldots & m_N \\ . & & & & \\ . & & & & \\ . & & & & \\ m_{N-2} & m_{N-1} & \ldots & & m_{2N-3} \\ m_{N-1} & \ldots & & m_{2N-3} & m_{2N-2} \end{bmatrix}$$

$$\mathcal{M}_s = \begin{bmatrix} m_0 & m_1 & \dots & m_{N-2} & m_{N-1} \\ m_{-1} & m_0 & m_1 & \dots & m_{N-2} \\ . & & & & \\ . & & & & \\ . & & & & \\ m_{2-N} & \dots & m_{-1} & m_0 & m_1 \\ m_{1-N} & m_{2-N} & \dots & m_{-1} & m_0 \end{bmatrix},$$

respectively. Furthermore, $x_1 = a$, $x_2 = h$ for the up-and-out options.

We proceed to illustrate a more detailed motivation of the backward recursion implementation for knock-out options. We define $\hat{V}_k(t_m) = C_k(x_1, x_2, t_m)$ where

$$C_k(x_1, x_2, t_m) = \frac{2}{b-a}\int_{x_1}^{x_2} c(x, t_m)\cos\left(k\pi\frac{x-a}{b-a}\right)dx.$$

From the COS formula for European options, we can derive the approximated continuation value at t_{M-1} and substitute the expression into $C_k(x_1, x_2, t_m)$ such that we obtain an approximated coefficient

$$\hat{C}_k(x_1, x_2, t_{M-1}) = e^{-r\Delta t} Re\left\{\sum_{k=0}^{N-1}{}' \varphi\left(\frac{k\pi}{b-a}\right) V_k(t_M)\cdot\mathcal{M}_{j,k}(x_1, x_2)\right\},$$

with

$$M_{j,k}(x_1, x_2) = \frac{2}{b-a}\int_{x_1}^{x_2} e^{ik\pi\frac{x-a}{b-a}}\cos\left(j\pi\frac{x-a}{b-a}\right)dx.$$

For the consequent terms in the backward recursion, we substitute the result of $V_k(t_{m+1})$ by its approximation $\hat{V}_k(t_{m+1})$ and specify the $\hat{C}_k(x_1, x_2, t_m)$ terms as

$$\hat{C}_k(x_1, x_2, t_m) = e^{-r\Delta t} Re\left\{\sum_{k=0}^{N-1}{}' \varphi\left(\frac{k\pi}{b-a}\right) \hat{V}_k(t_{m+1})\cdot\mathcal{M}_{j,k}(x_1, x_2)\right\}.$$

Lastly, theorem 2.1 in (Fang and Oosterlee, 2009) allow $\hat{C}_k(x_1, x_2, t_m)$ to be computed more efficiently. We outline the proof:

Substituting $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha)$ in $\mathcal{M}_{j,k}$, results in the expression

$$\mathcal{M}_{j,k}(x_1, x_2) = \frac{-i}{\pi}\left(\mathcal{M}^c_{j,k}(x_1, x_2) + \mathcal{M}^s_{j,k}(x_1, x_2)\right).$$

Where $\mathcal{M}_c = \mathcal{M}^c_{j,k}(x_1, x_2)$ and $\mathcal{M}_s = \mathcal{M}^s_{j,k}(x_1, x_2)$, for $j, k = 0, 1, ..., N-1$, we obtain the representation

$$\hat{V}_k(t_m) = \frac{e^{-r\Delta t}}{\pi} Im\left\{\left(\mathcal{M}_c(x_1, x_2) + \mathcal{M}_s(x_1, x_2)\right)\mathbf{u}\right\}, \tag{3.12}$$

by replacing the alternative representation of $\mathcal{M}_{j,k}(x_1, x_2)$ in $\hat{C}_k(x_1, x_2, t_m)$. We note $Im\{x\}$ to represent the imaginary part of x.

3.3 Hilbert transform method

To summarise the application of the Hilbert transform approach, we outline the method for pricing a discretely monitored down-and-out put option as in (Feng and Linetsky, 2008). The value of the option with barrier level B, strike K and Lévy process with asset price dynamics $S_t = Ke^{X_t}$ is given by

$$V_{t_0} = E[e^{-rT}\mathbf{1}_{(l,\infty)}X_{\Delta t} \cdot \mathbf{1}_{(l,\infty)}X_{2\Delta t} \cdots \mathbf{1}_{(l,\infty)}X_{M\Delta t} \cdot K(1 - e^{X_{M\Delta t}})^+],$$

where $l = \ln(B/K)$, M is the number of monitoring points and Δt is the time between monitoring points. The price of the option can be evaluated by backward recursion of:

$$v^M(x) = K(1 - e^x)^+\mathbf{1}_{l,\infty}(x),$$
$$v^{j-1}(x) = 1_{(l,\infty)}(x) \cdot P_{\Delta t}v^j(x),$$
$$v^0(x) = P_{\Delta t}(x),$$

for $j = M, M-1, ..., 2$ and where $x = \ln(\frac{S_0}{K})$ and $P_t v(x) = E_x[f(X_t)]$. It follows, the price of the option to be given by $V_{t_0} = e^{-rT}v^0\left(\ln\left(\frac{S_0}{K}\right)\right)$. To implement the backward recursion in the Fourier space we note the following:

- $\mathcal{F}(P_t v^j)(\theta) = \phi_t(-\theta)\hat{v}^j(\theta)$, where ϕ_t is the characteristic function of the Lévy process
- An indicator function can alternatively be defined as

$$\mathbf{1}_{(0,\infty)}(x) = \frac{1}{2}(1 + \text{sgn}(x))$$

- The signum function has the Fourier relationship

$$\mathcal{F}(\text{sgn} \cdot f)(\theta) = i\mathcal{H}\hat{f}(\theta)$$

- Following the definition of the Hilbert transform, we have

$$\mathcal{F}(I_{(0,\infty)} \cdot f)(\theta) = \frac{1}{2}\hat{f}(\theta) + \frac{i}{2}e^{i\theta l}\mathcal{H}(e^{-iul}\hat{f}(u))(\theta)$$

It follows:

$$\hat{v}^M(\theta) = \frac{K(1 - e^{i\theta l})}{i\theta} - \frac{K(1 - e^{(1+i\theta)l})}{1 + i\theta},$$
$$\hat{v}^{j-1}(\theta) = \frac{1}{2}\phi_{\Delta t}(-\theta)\hat{v}^j(\theta) + \frac{i}{2}e^{i\theta l}\mathcal{H}(e^{-iul}\phi_{\Delta t}(-u)\hat{v}^j(u))(\theta),$$
$$v^0(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ix\theta}\phi_{\Delta t}(-\theta)\hat{v}^1(\theta)\, d\theta,$$

for $j = M, M-1, ..., 2$. Therefore, we require $M-1$ Hilbert transforms and one inverse Fourier transform to implement v^0. The following discretisation (with step size h and truncation degree N) is used to approximate the Hilbert transform:

$$\mathcal{H}_{h,N} f(\theta) = \sum_{i=-N}^{N} f(ih) \frac{1 - \cos\left(\frac{\pi(\theta - ih)}{h}\right)}{\frac{\pi(\theta - ih)}{h}}.$$

Computation of the approximation involves implementing Toeplitz matrix-vector multiplication using fast Fourier transform (proof in (Feng and Linetsky, 2008)).

Chapter 4

Option pricing implementation

4.1 Monte Carlo method

The price of a barrier option can be represented as the risk-neutral expectation of the discounted terminal payoff. Since we can express this as an integral, we can compute a MC estimate of the price by taking a sample mean of N simulations (Cont and Tankov, 2004).

Under the BS framework, the stock price follows geometric Brownian motion satisfying

$$S_t = S_0 e^{(r-q-\frac{1}{2}\sigma^2)t+\sigma W_t}.$$

We summarise the MC algorithm under these assumptions, for pricing a single-barrier option and for N discrete sample paths:

For each $t = 1, ..., M$ (M = number of monitoring points and $\Delta t = \frac{T}{M}$):

1. Generate an (Nx1) vector of standard normal random variables, **Z**
2. Generate stock price paths S_t according to

$$S_t = S_{t-1} e^{(r-q-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\mathbf{Z}},$$

3. Approximate the value of the barrier option by

$$\hat{V} = \frac{1}{N}\sum_{n=1}^{N} V_n$$

(In step 3 and throughout this chapter, V_n represents the discounted payoff).

For infinite activity models, we note that standard MC option pricing can be implemented for barrier options by using a subordinator MC method (Ribeiro and Webber, 2003). A Lévy process such as the NIG and VG model can be represented as a subordinated Brownian motion. We follow the MC algorithm outlined in (Ribeiro and Webber, 2003) and (Ribeiro and Webber, 2004) for pricing barrier options in the NIG and VG model, respectively.

For each $t = 1, ..., M$:

1. Generate an $(N\text{x}1)$ vector of standard normal random variables, $\mathbf{Z}$
2. If the underlying asset price process is driven by the NIG model:
   (a) Generate an $(N\text{x}1)$ vector, **ig**, of inverse Gaussian random variables using the alternative parameterisation, $ig \sim IG(\frac{\delta\Delta t}{\sqrt{\alpha^2-\beta^2}}, (\delta\Delta t)^2)$
   (b) Generate stock price paths S_t according to
   $$S_t = S_{t-1}e^{(r-q)\Delta t+\bar{\omega}\Delta t+\beta(\mathbf{ig})+(1)\sqrt{\mathbf{ig}}\mathbf{Z}},$$
   where q represents the constant dividend yield, $\bar{\omega} = -\delta(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta+1)^2})$
3. If the underlying asset price process is driven by the VG model:
   (a) Generate an $(N\text{x}1)$ vector, **g**, of gamma random variables as $g \sim G(\frac{\delta t}{\nu}, \nu)$
   (b) Generate stock price paths S_t according to
   $$S_t = S_{t-1}e^{(r-q)\Delta t+\bar{\omega}\Delta t+\theta(\mathbf{g})+s\sqrt{\mathbf{g}}\mathbf{Z}},$$
   where q represents the constant dividend yield, $\bar{\omega} = \frac{1}{\nu}\ln(1-\theta\nu-\frac{1}{2}s^2\nu)$
   (c) For a DEVG model (with diffusion component $\sigma > 0$), stock price paths are generated according to:
   $$S_t = S_{t-1}e^{(r-q-\frac{1}{2}\sigma^2)\Delta t+\sigma\sqrt{\Delta t}\mathbf{Z}+\bar{\omega}\Delta t+\theta(g)+s\sqrt{\mathbf{g}}\mathbf{Z}}$$
4. Approximate the value of the barrier option by
   $$\hat{V} = \frac{1}{N}\sum_{n=1}^{N} V_n$$

We illustrate the final step of the MC algorithm for an up-an-out call option with barrier level B:

$$\hat{V}(t_0, T) = e^{-rT}\frac{1}{N}\sum_{n=1}^{N}(S_T-K)^+\mathbf{1}_{\{\sum_{m=0}^{M}\mathbf{1}_{\{S_{t_m}<B\}}=M+1\}}.$$

Here, the truncated payoff is computed using an indicator function, noting that we require all stock prices from time t_0 till maturity of the option (time $t_M = T$) to be below the barrier level B. This is equivalent to $\sum_{m=1}^{M} 1_{(S_{t_m}<B)} = M+1$. Similar indicator expressions can be formed for the other types of single-barrier options.

4.2 COS method

Pricing discrete barrier options via Fourier-cosine expansions is a suitable method for asset price dynamics following an exponential-Lévy process. We outline the algorithm in (Fang and Oosterlee, 2009) used to price knock-out options. Knock-in options result from the in-out parity and where European options can be priced as in (Fang and Oosterlee, 2008). Noting that $x_1 = a, x_2 = h$ for up-and-out options and $x_1 = h, x_2 = b$ for down-and-out options:

1. Compute $V_k(t_M)$ (3.7) (3.8)

2. Compute ($2N$x1) vectors $\mathbf{m}_s$ and $\mathbf{m}_c$ using the following properties:

$$\mathbf{m}_s = [m_0, m_{-1}, ..., m_{1-N}, 0, m_{N-1}, m_{N-2}, ..., m_1]^T$$

$$\mathbf{m}_c = [m_{2N-1}, m_{2N-2}, ..., m_1, m_0]^T.$$

This formulates from $\mathcal{M}_s\mathbf{u}$ resulting as the first N elements of the circular convolution of vectors $\mathbf{m}_s$ and $\mathbf{u}_s$, where

$$\mathbf{u}_s = [u_0, u_1, ..., u_{N-1}, 0, ..., 0]^T_{(2N\text{x}1)}.$$

Similarly, $\mathcal{M}_c\mathbf{u}$ is the first N elements (in reversed order) of the circular convolution of vectors $\mathbf{m}_c$ and $\mathbf{u}_c$, where

$$\mathbf{u}_c = [0, ..., 0, u_0, u_1, ..., u_{N-1}]^T_{(2N\text{x}1)}.$$

Both $\mathbf{u}_s$ and $\mathbf{u}_c$ involve adding on N zeros to $\mathbf{u}$. Discrete Fourier transform ($\mathcal{D}$) and its inverse ($\mathcal{D}^{-1}$) can be used to compute the circular convolution of two vectors, i.e.:

$$\mathbf{m}_s \circledast \mathbf{u}_s = (\mathcal{D}^{-1})\{\mathcal{D}(\mathbf{m}_s) \cdot \mathcal{D}(\mathbf{u}_s)\}$$

and similarly for $\mathbf{m}_c \circledast \mathbf{u}_c$.

3. Compute

$$d_1 = \mathcal{D}(\mathbf{m}_s(x_1, x_2)),$$

$$d_2 = \mathbf{sgn} \cdot \mathcal{D}(\mathbf{m}_c(x_1, x_2)),$$

where

$$\mathbf{sgn} = [1, -1, 1, -1, 1, ...]_{(N\text{x}1)}$$

4. For monitoring points $M, M-1, ..., 3, 2$:

   (a) Formulate $\mathbf{u}_s$ by concatenating N zeros to $\mathbf{u}$

(b) Compute $\mathcal{M}_s\mathbf{u}$ = first N elements of $(\mathcal{D}^{-1})\{d_1 \cdot \mathcal{D}(\mathbf{u}_s)\}$

(c) Compute $\mathcal{M}_c\mathbf{u}$ = first N elements (in reverse order) of $(\mathcal{D}^{-1})\{d_2 \cdot \mathcal{D}(\mathbf{u}_s)\}$

(d) Compute $\hat{V}(t_{m-1})$ (3.12)

5. Price the barrier option using the COS formula evaluated at t_0: $\hat{\Lambda}(x, t_0)$ (3.6)

The matrices $\mathcal{M}_s$ and $\mathcal{M}_c$ are known in advance, therefore the fast Fourier transform is only applied three times, and reason to formulate d_1 and d_2 is due to the shift property of discrete Fourier transforms, such that $\mathcal{D}(\mathbf{u}_c) = \mathbf{sgn} \cdot \mathcal{D}(\mathbf{u}_s)$ (Fang and Oosterlee, 2009).

Alternatively, discrete barrier options could be priced more explicitly, without use of the fast Fourier transform algorithm.

1. Compute $V_k(t_M)$ (3.7) (3.8)

2. Compute the m_j's in order to construct $\mathcal{M}_s$ and $\mathcal{M}_c$ (**??**)

3. For monitoring points $M, M-1, ..., 3, 2$:

   (a) Construct vector $\mathbf{u}_{[N\text{x}1]}$ (3.9) (3.11)

   (b) Compute $\hat{V}(t_{m-1}) = \frac{e^{-r\Delta t}}{\pi} Im\{(\mathcal{M}_c + \mathcal{M}_s)\mathbf{u}\}$

4. Price the barrier option using the COS formula evaluated at t_0: $\hat{\Lambda}(x, t_0)$ (3.6)

For implementation of the COS pricing formula, the characteristic function of the logarithm of the strike scaled stock $\left(\ln\left(\frac{S_t}{K}\right)\right)$ for the BS, NIG and VG models are given by:

- BS

$$\varphi_{BS}(u) = e^{iu(r-\frac{1}{2}\sigma^2)\Delta t - \frac{1}{2}\sigma^2\Delta t u^2},$$

  with cumulants

$$c_1 = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t,$$

$$c_2 = \sigma^2\Delta t,$$

$$c_4 = 0,$$

  and $\mu = r - \frac{1}{2}\sigma^2$ (Fang and Oosterlee, 2009).

- NIG

$$\varphi_{NIG}(u) = e^{iu\mu\Delta t - \frac{1}{2}\sigma^2\Delta t u^2} e^{\delta\Delta t\left(\sqrt{\alpha 2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right)},$$

  with cumulants

$$c_1 = \left(\mu - \frac{1}{2}\sigma^2 + \omega\right)\Delta t + \frac{\delta\Delta t\beta}{\sqrt{\alpha^2 - \beta^2}},$$

$$c_2 = \delta\Delta t\alpha^2(\alpha^2-\beta^2)^{-\frac{3}{2}},$$
$$c_4 = 3\delta\Delta t\alpha^2(\alpha^2+4\beta^2)(\alpha^2-\beta^2)^{-\frac{7}{2}}$$

and $\mu = r-q+w$, where $w = -\delta\left(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+1)^2}\right)$ (Fang and Oosterlee, 2009).

- VG

$$\varphi_{VG}(u) = e^{iu\mu\Delta t}e^{(1-iu\theta\nu+\frac{1}{2}\sigma^2\nu u^2)^{-\frac{\Delta t}{\nu}}},$$
$$\varphi_{DEVG}(u) = e^{iu\mu\Delta t-\frac{1}{2}\sigma^2\Delta tu^2}e^{(1-iu\theta\nu+\frac{1}{2}\sigma^2\nu u^2)^{-\frac{\Delta t}{\nu}}},$$

with cumulants

$$c_1 = (\mu+\theta)\Delta t,$$
$$c_2 = (\sigma^2+\nu\theta^2)\Delta t,$$
$$c_4 = 3\left(\nu\sigma^4+2\theta^4\nu^3+4(\sigma\theta\nu)^2\right)\Delta_t$$

and $\mu = r-q+w$, where $w = \frac{1}{\nu}\ln(1-\theta\nu-\frac{1}{2}\sigma^2\nu)$ (Fang and Oosterlee, 2009).

We follow the error analysis presented in (Fang and Oosterlee, 2009) for the truncation interval $[a,b]$:

$$[a,b] = \left[c_1+x_0-L\sqrt{c_2+\sqrt{c_4}}, c_1+x_0+L\sqrt{c_2+\sqrt{c_4}}\right],$$

with $x_0 = \ln(\frac{S_0}{K})$ and $L=8$.

4.3 Hilbert transform

To price discretely monitored barrier options under Lévy processes, we require a recursive implementation of Hilbert transforms of the product of Fourier transformed value functions and the Esscher transformed characteristic function of the Lévy process. We summarise the approach in (Feng and Linetsky, 2008), alongside some preliminaries to evaluate barrier options under this method.

We recall the definition of P_tf in Section 3.3: $P_tf(x) = E_x[f(X_t)]$. In order to invert the Fourier transform of $\mathcal{F}(P_tf)(\theta) = \phi_t(-\theta)\hat{f}(\theta)$, for $f \in L^1(\mathbb{R})$ and $t>0$, we require the condition

$$|\phi_t(\theta)| = e^{-t\Re\psi(\theta)} \leq \kappa e^{-tc|\theta|^v}, \tag{4.1}$$

to hold for $c,\kappa>0$ and $v\in(0,2]$, where $\Re$ represents the real part. Consequently, we have the Fourier expression of P_t given by

$$P_tf(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta x}\phi_t(-\theta)\hat{f}(\theta)\,d\theta.$$

For Lévy processes which do not satisfy the condition (4.1), the Fourier expression for P_t holds if for every $t > 0, f \in L^1(\mathbb{R})$, we have $\hat{f} \in L^1(\mathbb{R}, \mathbb{C})$. In other words, where

$$\int_{-\infty}^{\infty} |\phi_t(-\theta)\hat{f}(\theta)|\, d\theta < \infty.$$

In particular, the condition does not hold under the VG process, however introducing a diffusion with $\sigma > 0$ allows the condition to be met.

Furthermore, we extend $L^1(\mathbb{R})$ to $L^1_\alpha(\mathbb{R}) = L^1(\mathbb{R}, e^{\alpha x}dx)$ via an Esscher transform, for some $\alpha \in \mathbb{R}$, to account for $f(x) \notin L^1(\mathbb{R})$. Call options with payoff $f(x) = K(e^x - 1)^+$ and $\alpha < -1$, as well as put options with payoff $f(x) = K(1 - e^x)^+$ and $\alpha > 0$ will lie in $L^1_\alpha(\mathbb{R})$. We specify the set $\mathcal{A}_f = \{\alpha \in \mathcal{R} : f \in L^1_\alpha(\mathbb{R})\}$ such that $\mathcal{A}_{\text{call}} = (-\infty, -1)$ and $\mathcal{A}_{\text{put}} = (0, \infty)$.

For a Lévy process with one or more of the bounds of A_f not equal to 0, we define the characteristics of the Esscher-transformed Lévy process (X^α) as:

- $\psi^{(\alpha)}(\theta) = \psi(\theta + i\alpha) - \psi(i\alpha)$,
- $\phi^\alpha_t(\theta) = e^{-t\psi^{(\alpha)}(\theta)} = \frac{\phi_t(\theta + i\alpha)}{\phi_t(i\alpha)} = e^{-t(\psi(\theta+i\alpha) - \psi(i\alpha))}$,
- $\mu^{(\alpha)} = \mu - \sigma^2\alpha + \int_{-1}^{1} x(e^{-\alpha x} - 1)\, \nu(dx)$.

The proof of the application of the theorem defining these characteristics is found in (Feng and Linetsky, 2008). We note the Carr-Madan option pricing method applied an exponential damping factor to the payoff functions in order to derive the Fourier transform, in terms of log-strike (Carr and Madan, 1999). In contrast, (Feng and Linetsky, 2008) have applied a Fourier transform to the log-price allowing for the Esscher transform.

To price vanilla options with payoff $F(S_T)$, the function $f(x) = F(Ke^x)$, $f_\alpha(x) = e^\alpha f(x) \in L^1(\mathbb{R})$ and forward value function $v(x,t) = e^{r(T-t)}V(Ke^x, t)$ is introduced, such that $v(x,t) = P_{T-t}f(x)$. For call options with

$$f(x) = K(e^x - 1)^+ \in L^1_\alpha(\mathbb{R}), \quad \alpha < -1,$$

$$f_\alpha(x) = K(e^{(\alpha+1)x} - e^{\alpha x})^+ \in L^1(\mathbb{R}),$$

and for put options with

$$f(x) = K(1 - e^x)^+ \in L^1_\alpha(\mathbb{R}), \quad \alpha > 0,$$

$$f_\alpha(x) = K(e^{\alpha x} - e^{(\alpha+1)x})^+ \in L^1(\mathbb{R}),$$

the Fourier transform of f_α is given by

$$\hat{f}_\alpha(\theta) = -\frac{K}{(\theta - i\alpha)(\theta - i(\alpha + 1))} \in L^1(\mathbb{R}, \mathbb{C}). \tag{4.2}$$

Truncated vanilla option payoffs are required for barrier options. It follows for

$$f(x) = K(e^x - 1)^+ \mathbf{1}_{(x<u)} \in L^1_\alpha(\mathbb{R}),$$

and

$$f(x) = K(1 - e^x)^+ \mathbf{1}_{(x>l)} \in L^1_\alpha(\mathbb{R}),$$

the Fourier transform is given by

$$\hat{f}_\alpha(\theta) = K\left(\frac{1 - e^{(i\theta+\alpha)b}}{i\theta + \alpha} - \frac{1 - e^{(1+i\theta+\alpha)b}}{1 + i\theta + \alpha}\right), \tag{4.3}$$

for $L < K < U$ and where $u = \ln(U)$, $l = \ln(L)$. The b parameter is defined separately for truncated call and put payoffs, with $b = u > 0$ and $b = l < 0$, respectively. Here L represents the lower barrier level and U the upper barrier level.

Assuming equal distance between monitoring points such that $\Delta t = \frac{T}{M}$, discrete barrier options can be priced following the backward recursion expression. The $v^j(x)$ terms are defined as $v(x, t_j)$ for $j = 0, 1, ..., M$. The Esscher transform can be applied such that the recursion is given by:

$$v^M_\alpha(x) = f_\alpha(x) = e^{\alpha x} f(x),$$

$$v^{j-1}_\alpha(x) = e^{\alpha x} v^{j-1}(x) = e^{-\Delta t \psi(i\alpha)} \mathbf{1}_I(x) \cdot P^\alpha_{\Delta t} v^j_\alpha(x),$$

$$v^0_\alpha(x) = e^{-\Delta t \psi(i\alpha)} P^\alpha_{\Delta t} v^1_\alpha(x),$$

for $j = M, M-1, ..., 2$. More specifically, $I = (l, \infty)$ for down-and-out options and $I = (-\infty, u)$ for up-and-out options. The same properties as before (3.3) are applied to the backward recursion for implementation in the Fourier space such that

$$\hat{v}^{j-1}(\theta) = \frac{1}{2} e^{-\Delta t \psi(i\alpha)} \left[\phi^{(\alpha)}_{\Delta t}(-\theta) \hat{v}^j_\alpha(\theta) + i e^{i\theta b} \mathcal{H}(e^{-i\eta b} \phi^{(\alpha)}_{\Delta t}(-\eta) \hat{v}^j_\alpha(\eta))(\theta)\right],$$

$$v^0_\alpha(x) = \frac{1}{2\pi} e^{-\Delta t \psi(i\alpha)} \int_{-\infty}^{\infty} e^{-i\theta x} \phi^{(\alpha)}_{\Delta t}(-\theta) \hat{v}^1_\alpha(\theta)\, d\theta,$$

where $b = l$ for down-and-out options and $b = u$ for up-and-out options. Once again $M - 1$ Hilbert transforms are required in the recursion and to compute these expression, numerical evaluation is implemented. The following operators are introduced:

- $\mathcal{P}^{\Delta t} f(\theta) = \frac{1}{2}\phi_{\Delta t}(-\theta) f(\theta) + \frac{1}{2} i \lambda e^{i\theta b} \mathcal{H}(e^{-i\eta b} \phi_{\Delta t}(-\eta) f(\eta))(\theta)$

  ($\lambda = 1$ for down-and-out options and $\lambda = -1$ for up-and-out options)

- $\mathcal{R}^{\Delta t} f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\theta x} \phi_{\Delta t}(-\theta) f(\theta)\, d\theta.$

These operators would need to be computed several times for single-barrier options. Discrete approximations are presented for $\mathcal{P}$ and $\mathcal{R}$, with step size $h > 0$ and truncation degree $N > 0$.

- $\mathcal{P}^{\Delta t}_{h,N} f(\theta) = \frac{1}{2}\phi_{\Delta t}(-\theta)f(\theta) + \frac{1}{2} i\lambda e^{i\theta b} \sum_{n=-N}^{N} e^{-inhb}\phi_{\Delta t}(-nh)f(nh)\frac{1-\cos\left(\frac{\pi(\theta-nh)}{h}\right)}{\frac{\pi(\theta-nh)}{h}}$
- $\mathcal{R}^{\Delta t}_{h,N} f(\theta) = \frac{1}{2\pi} h \sum_{n=-N}^{N} e^{-ixnh}\phi_{\Delta t}(-nh)f(nh)$

The step size h, which is a function of N, is chosen such that for $[-d, d] \subset (\beta_-, \beta_+)$,

$$h = \left(\frac{\pi d}{(\Delta t)c}\right)^{\frac{1}{1+v}} N^{-\frac{v}{1+v}}. \tag{4.4}$$

However, under the BS assumptions $h = \left(\frac{\pi^2}{(\Delta t)\sigma^4 T}\right)^{\frac{1}{4}} N^{-\frac{1}{2}}$. The theorem and proof is outlined in (Feng and Linetsky, 2008).

Under the Lévy process with payoff f and $\alpha \in (\beta_-, \beta_+)$, we define $\phi^{(\alpha)}_{\Delta t}(-z)\hat{f}_\alpha(z)$ to be analytical in the strip $\{z \in \mathbb{C} : \mathfrak{F}(z) \in (d_-, d_+)\}$. Then, α is selected such that the strip is symmetric about the real axis ($-d_- = d_+ = d$). We have that:

- For vanilla call option: $\alpha = \frac{\beta_- - 1}{2}, \quad d = -\frac{\beta_- + 1}{2}$
- For vanilla put option: $\alpha = \frac{\beta_+}{2}, \quad d = \frac{\beta_+}{2}$
- For truncated vanilla options: $\alpha = \frac{\beta_+ + \beta_-}{2}, \quad d = \frac{\beta_+ - \beta_-}{2}$

Single-barrier options can then be priced according to:

$$\hat{v}^M_{\alpha,N}(kh) = \hat{f}_\alpha(kh), \tag{4.5}$$

$$\hat{v}^{j-1}_{\alpha,N}(kh) = \frac{1}{2} e^{-\Delta t\psi(i\alpha)}\phi^{(\alpha)}_{\Delta t}(-kh)\hat{v}^j_{\alpha,N}(kh) \tag{4.6}$$

$$+\frac{i\lambda}{2\pi} e^{-\Delta t\psi(i\alpha)} e^{ikhb} \sum_{n=-N, n\neq k}^{N} e^{-inhb}\phi^{(\alpha)}_{\Delta t}(-nh)\hat{v}^j_{\alpha,N}(nh)\frac{1-(-1)^{k-n}}{k-n},$$

$$v^0_{\alpha,N}(x) = \frac{1}{2\pi} e^{-\Delta t\psi(i\alpha)} \sum_{n=-N}^{N} e^{-inhx}\phi^{(\alpha)}_{\Delta t}(-nh)\hat{v}^1_{\alpha,N}(nh)h, \tag{4.7}$$

for $j = M, M-1, ..., 2$ and $k = -N, ..., N$. The final step of the method follows as

$$V_{t_0} = e^{-rT}\left(\frac{S_0}{K}\right)^{-\alpha} v^0_{\alpha,N}\left(\ln\left(\frac{S_0}{K}\right)\right). \tag{4.8}$$

The implementation of the discrete Hilbert transform in the backward recursion applies the following property.

$$\frac{1-\cos\left(\frac{\pi(kh-nh)}{h}\right)}{\frac{\pi(kh-nh)}{h}} = \begin{cases} \frac{1-(-1)^{k-n}}{\pi(k-n)} & n \neq k \\ 0 & n = k \end{cases}.$$

Consequently, vector-matrix multiplication can be used in the backward recursion for ease of computation. A $(2N+1)$-square matrix is created with elements $\frac{1-(-1)^{k-n}}{k-n}$, for $k = -N, ..., N$ and $n = -N, ..., N$.

We summarize the approach to price discretely monitored barrier options for a given Lévy process and payoff $f(x)$ of the scaled-log stock price.

1. Select $\alpha \in (\beta_-, \beta_+)$ such that $f \in L^1_\alpha(\mathbb{R})$ and $-d_- = d_+ = d$
2. Determine $\hat{f}_\alpha(\theta)$ (4.2) (4.3)
3. Select $N > 0$ and compute $h = h(N)$ (4.4)
4. Evaluate and store $\left\{\phi^{(\alpha)}_{\Delta t}(-nh)\right\}_{n=-N}^{N}$ and $\left\{\hat{v}^M_\alpha(nh) = \hat{f}_\alpha(nh)\right\}_{n=-N}^{N}$
5. For each $j = M, M-1, ..., 2,$ evaluate $\left\{\hat{v}^{j-1}_{\alpha,N}(nh)\right\}_{n=-N}^{N}$ via backward recursion (4.6)
6. Compute $v^0_{\alpha,N}(x)$, for $x = \ln\left(\frac{S_0}{K}\right)$ (4.7)
7. Price the barrier option by V_{t_0} (4.8)

Knock-in options result from the in-out parity and where European options can be priced using the Hilbert transform methodology for $M = 1$. The backward recursion is not needed for pricing European options since in the formula for $v^0_{\alpha,N}(x)$ (4.7), we have $\hat{v}^1_{\alpha,N}$ (4.6) as $\hat{v}^M_{\alpha,N}$ (4.5).

Chapter 5

Numerical results

We present numerical results for discretely monitored single-barrier options for the BS, NIG and VG processes. More specifically, we price knock-out barrier options and rely on in-out parity to provide results for knock-in options. In-out parity requires numerical results of European options under the underlying asset price dynamics. Under the COS method and the Hilbert transform method, solutions for European options are obtained by having the barrier option monitored at one time point, the expiry date T. This is easily altered in the algorithm for both methods since it only requires adjusting one parameter at the outset of the computation. Furthermore, we note that both methods can be implemented without the use of the fast Fourier transform.

In the following, we present results for down-and-out calls (DOC), down-and-out puts (DOP), up-and-out calls (UOC) and up-and-out puts (UOP). We focus on at-the-money ($S_0 = K = 100$), one-year knock-out options ($T = 1$). Furthermore, the risk-free rate and the constant dividend rate are consistent across all underlying asset processes with $r = 0.06$ and $q = 0.02$, respectively. The pricing methods are evaluated at an upper barrier level of $B = 120$ for up-and-out options and a lower barrier level of $B = 80$ for down-and out options.

We consider the COS method and the Hilbert transform method for daily and monthly monitoring and assume a year to consist of 252 business days. A standard MC algorithm is implemented as a reference value to the transform methods. We perform 1,000,000 MC simulations to obtain an approximation for the value of the barrier options under the three processes. All computations were coded in Matlab (version R2017b) and performed on a computer with Intel core i3-6100 CPU, 3.7GHz with 8GB RAM.

We implement the COS method with a discretisation value of $N = 2^7$. In order to improve the level of accuracy, N needs to be larger for smaller frequencies of monitoring points. This is due to several Lévy processes having notably peaked density functions for small values of Δt.

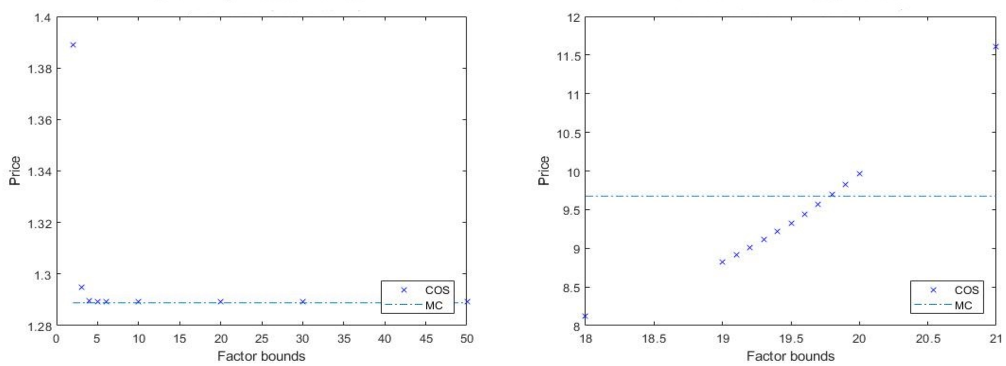
		DOC	DOP	UOC	UOP
	COS	9.691910182766151	2.244565817285296	1.793028184914674	5.792778962932250
M = 12	**Hilbert**	9.693661529229246	2.244534036022162	1.793028184914697	5.793769670934831
	MC	9.675636607377228	2.077435896973490	1.789948343415559	5.784929905148385
	COS	9.634131780388998	1.798852428451851	1.289351309210394	5.702509490045212
M = 252	**Hilbert**	9.651474099336273	1.799045540982172	1.289351309210408	5.702491209560818
	MC	9.631196263149468	1.683191694378375	1.288852901409627	5.717335275289832

Tab. 5.1: Knock-out options under BS

Knock-out options under the BS process are priced with parameters $\sigma = 0.2$ and $\mu = r - q - \frac{1}{2}\sigma^2$.

Under the COS method, we find the general setup for the truncation range ($[a, b]$) in Chapter 4 to be problematic for the BS process. In particular, we develop the notion that the generic truncation range is increasingly problematic for a large frequency of monitoring points. The price of a UOC explodes as monitoring increases from monthly to daily. However, when the truncation bounds are increased by a constant factor ($x \in \mathbb{R}$), the price of the option begins to replicate that of the MC estimate. A sketch of the argument depicting the effect of the truncation range on the price of options is shown. Figure 5.1a illustrates the convergence of a daily monitored UOC evaluated under the COS method for truncation bounds increased by a factor of $x \in \mathbb{R}$ such that the new bounds are $[xa, xb]$. Similar convergence was noted for DOP and UOP. For this reason, in Table 5.1 we evaluate knock-out options under the BS process with truncation range $[10a, 10b]$. For DOC, however, we note a different behaviour of convergence for various factor bounds and for different frequencies of monitoring points. We illustrate the convergence for monthly monitored DOC in figure 5.1b. The COS option value oscillates around the MC estimate in the truncation range of $x \in (19, 20)$. For this reason, we evaluate monthly monitored DOC under the BS process in Table 5.1 with an adjusted truncation range of $[19.8a, 19.8b]$. For daily monitored DOC, we set $x = 108.38$. This is selected in reference to the MC estimate.

The error analysis in (Fang and Oosterlee, 2009) of the convergence for European options describes the main conclusions for the discrepancies in the truncation range. We summarise the error for the derived COS formula $\hat{c}(x)$ (3.5) in Chapter 3 into three parts. Firstly, we adjusted the integration range of $c(x)$ (3.3) to $[a, b] \in \mathbb{R}$ in reference to an appropriate tolerance level. A larger range of $[b-a]$ decreases this error. Secondly, there is a series truncation error due to truncation of the summation range to further approximate the continuation value $\hat{c}(x)$. Lastly, there exists error related to the approximation of $A_k(x)$ (3.4) using the definition of the conditional characteristic function. Numerical evidence for the error analysis is provided



(a) Daily monitored UOC **(b)** Monthly monitored DOC

Fig. 5.1: COS truncation bounds under BS

in (Fang and Oosterlee, 2009).

The Hilbert transform method under the BS process was evaluated using a truncation degree of $N = 55^2$. The larger value of N resulted in an improved running time, although not largely significant. Examples for α in (Feng and Linetsky, 2008) was used with $\alpha = -10.5$ for knock-out calls and $\alpha = -5$ for UOC. The same $\alpha = -5$ was not suitable for the UOP. We observe the Hilbert transform price converge towards the MC estimate using an $\alpha = 20$. Figure 5.2 illustrates the analysis of α for a daily monitored UOP.

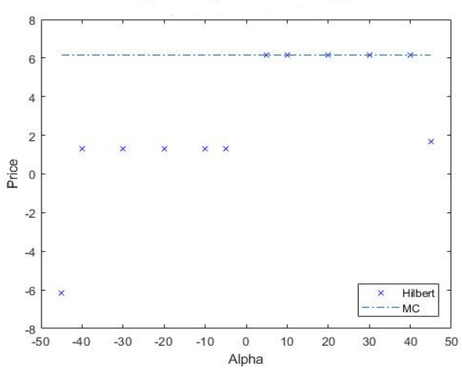


Fig. 5.2: Daily monitored UOP under BS

Knock-out options under the NIG process are priced with parameters $\alpha = 15, \beta = -5$ and $\delta = 0.5$ with $\sigma = 0$. In contrast to evaluating the options using the COS method under the NIG process, we observe the (Fang and Oosterlee, 2009) suggested truncation range to be appropriate for both monthly and daily monitored barrier options. The truncation range is suitable for all frequencies of monitoring points. We do not observe the same problematic result between monitoring points and truncation range as under the BS process. Table 5.2 summarises the results for the COS method using the truncation range outlined in Chapter 4. However, the VG process demonstrates a similar issue as the BS process with the

		DOC	DOP	UOC	UOP
M = 12	COS	9.507993504532383	2.015599049547718	2.299076945261663	5.572568991875873
	Hilbert	9.508092093383807	2.015597763900978	2.299077047128221	5.575205158037961
	MC	9.492335026483557	2.016424143606935	2.300131932743886	5.586765472063232
M = 252	COS	8.965002130094952	1.775516801045112	1.953063823921049	4.959744834486944
	Hilbert	9.491130699227348	1.770855776790779	1.949777201315867	5.512522738197332
	MC	9.493468524156887	1.769611749122926	1.946179738751348	5.512482827544194

Tab. 5.2: Knock-out options under NIG

truncation range. The discrepancy is not as large as under BS process and does not differ for varying frequencies of monitoring points. To remain consistent with the truncation range, we select bounds of $[4a, 4b]$ under all COS method option valuations for the VG process in Table 5.3. The motivation for the truncation choice is in reference to the MC estimate. Knock-out options under the VG process are priced with parameters $\theta = -0.2, s = 0.2$ and $\nu = 0.1$.

		DOC	DOP	UOC	UOP
M = 12	COS	9.925300503702488	1.968411972410469	2.049417297723391	5.977505499489055
	MC	9.914735715437098	1.972103663777236	2.051863060796189	5.978946729084339
M = 252	COS	9.898739864732775	1.699560889674073	1.718389869954377	5.900780193725662
	MC	9.920967951985631	1.720550738323831	1.755065167552881	5.890010973729999

Tab. 5.3: Knock-out options under VG

The Hilbert transform method under the NIG process selects α in the range $[\beta_{NIG} - \alpha_{NIG}, \beta_{NIG} + \alpha_{NIG}]$. Barrier options with a truncated payoff (DOP and UOC) were evaluated at a truncation degree of $N = 55^2$ in contrast to the non-truncated payoff barrier options with $N = 74^2$ for the DOC and $N = 65^2$ for the UOP. Knock-out call options were implemented using $\alpha = -10.5$ and $d = 9.5$, while the DOP used $\alpha = -5$ and $d = 15$ for the computation of $h(N)$. The same choice of parameters of the DOP was not suitable for the UOP under the NIG process. We observed the price of the UOP converge towards the MC estimate for $\alpha = 5$ and $d = 15$. For the computation of $h(N)$, we set $v = 1$ and $c = \delta_{NIG}$.

In order to apply the Hilbert transform to the VG process, a diffusion component ($\sigma > 0$) needs to be included. We use a diffusion component of $\sigma = 0.1$ in the diffusion-extended VG (DEVG) process. The characteristic exponent of the DEVG is defined as $\psi(u) = \frac{1}{\nu} \ln(1 - i\nu\theta u + \frac{1}{2}\nu s^2 u^2)(\frac{1}{2}\sigma^2 u^2 - i\mu u)$ with $\mu = r - q - \frac{1}{2}\sigma^2 + w$

to satisfy the martingale condition (3.1) outlined in Chapter 3. For the computation of $h(N)$, we set $v = 2$ and $c = \frac{\sigma^2}{2}$. The truncation degree remains constant for all options as $N = 70^2$. We observe higher accuracy for a higher truncation degree for all options under the DEVG process. The same values for α and d as in the NIG implementation was selected. We note the same inconsistency with the UOP and the value of α. Figure 5.3 demonstrates the price for the UOP converging towards the MC estimate. We select $\alpha = 5$ for the UOP valuation.

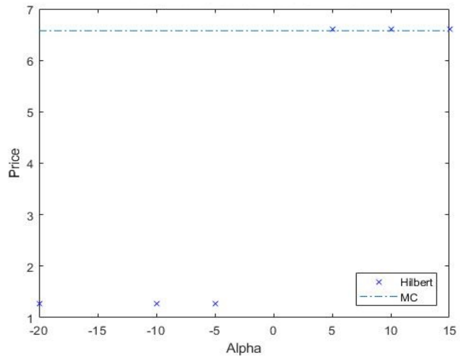


Fig. 5.3: Daily monitored UOP under DEVG

		DOC	DOP	UOC	UOP
	COS	10.741300800649100	1.839238129905228	1.632622638909011	6.746681301401555
M = 12	**Hilbert**	10.741066392225541	1.839095931106008	1.632713507682775	6.746730835074233
	MC	10.704804519755116	1.836328930609364	1.636348043193171	6.751641432584295
	COS	10.677759725126858	1.490088620454136	1.248323198897396	6.600788461440770
M = 252	**Hilbert**	10.678298081400973	1.493483041531937	1.252275605870726	6.601777192138917
	MC	10.701175736603021	1.492421118763509	1.251308493708888	6.575531464751678

Tab. 5.4: Knock-out options under DEVG

We compare the Hilbert method with the COS method in Table 5.4 since it is easily implemented for the DEVG process by including a diffusion component in the VG characteristic function. The same truncation bounds in the VG implementation of the COS method is used for the DEVG process since the same pricing discrepancy in the (Fang and Oosterlee, 2009) truncation range was observed for the DEVG process. Table 5.4 and Table 5.3 are evaluated with the same model parameters and a difference in the diffusion component of $\sigma = 0.2$ and $\sigma = 0$, respectively. We can compare the price under the COS method when a diffusion component is included in the VG process. Knock-out options with truncated payoffs exhibit a decrease in price when a diffusion of $\sigma = 0.2$ is added, while the options with non-truncated payoff increase.

The COS method and the Hilbert transform method yield fairly identical results for monthly and daily monitored knock-out options. The computation time, however, differs significantly. The time efficiency in pricing knock-out options using the COS method outweighs the Hilbert transform method, particularly as the number of monitoring points increase. Under the BS process, monthly monitored knock-out options had an average running time of 0.1s (seconds) for the COS method and 0.97s for the Hilbert transform. Daily monitoring, however, increased the average running time to 0.6s for the COS method and 6.6s for the Hilbert transform.

We observe a similar behaviour for running time under the exponential-Lévy processes. Under these processes, the average running time increases with the number of monitoring points. Table 5.5 summarises the average running time of

		NIG	**DEVG**
M=12	**COS**	0.03s	1.4s
	Hilbert	0.03s	2s
M=252	**COS**	0.05s	11.6s
	Hilbert	0.05s	16.3s

Tab. 5.5: Average running time of exponential-Lévy processes

pricing discretely monitored barrier options under the NIG and DEVG process for monthly and daily monitoring. The average running time under the NIG process does not show a significant difference between the COS and the Hilbert transform method. For the DEVG process, however, the Hilbert transform method increased the running time. Furthermore, the DEVG process has a higher average running time under both methods compared to the NIG process, even more so when the number of monitoring points increase.

Both methods of pricing have the advantage of ease of computation, which only require adjusting the characteristic function/exponent and frequency of monitoring points as the main changes across models. The Hilbert transform method, however, requires fewer input arguments, leaving little room for computation error. Despite this, the COS method remains faster than the Hilbert transform. Both methods provide straightforward computation for European options to price knock-in options by applying in-out parity.

Chapter 6

Conclusions

In this dissertation, we present transform methods for pricing discretely monitored single-barrier options where the asset price process is driven by an exponential-Lévy process. The transform pricing approaches, namely the COS method and the Hilbert transform method, are useful for Lévy process where the characteristic function is known.

The COS method is based on Fourier-cosine series. Furthermore, the COS pricing approach for European options can be extended to barrier options. To price barrier options, the Fourier-cosine coefficients of the option value at the first monitoring point is required. This is computed via a backward recursion from the coefficients of the payoff function. The COS method provides a significantly fast computation speed for various monitoring frequencies. The main insight of the Hilbert transform method is that the product of two functions in the state space, where one such function is an indicator, is consistent with the Hilbert transform in the Fourier space. Pricing barrier options under this method requires recovering a series of Hilbert transforms of the the product of two functions - the characteristic function of the underlying Lévy process and the Fourier transformed option value at the preceding monitoring point.

We observe the COS method to have an impressive running time, however, the results may be unstable for certain Lévy process due to the expression of the truncation range derived under the COS method. This deems the method unreliable as it remains inconsistent with the bounds of the truncation across different Lévy processes, particularly the BS model at high monitoring frequencies. The COS pricing approach provided ease of computation due to only adjusting the characteristic function, however this is not true since the truncation range would need to be considered for accurate results. In comparison to the Hilbert transform method, which requires few adjustments in computation for different Lévy process and barrier options, the results are reliable. The choice of α of the Esscher transformation needs to be selected accordingly to provide a sufficient degree of accuracy. The implementation of the Hilbert transform method is more readable than the COS method since fewer expressions are required in the setup. However, the COS method is notably faster for high monitoring frequencies. The Hilbert method has the advantage of being able to evaluate barrier options at time points outside of the predetermined monitoring points.

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