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Addressing dualism in mathematical abstraction: An argument for the role of Construal Level Theory in mathematics education

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Learners of mathematics often struggle to balance the apparently conflicting demands for abstract thinking as well as (often simultaneous) concrete cognitive engagement. Conflicting demands of successful mathematical engagement have been addressed in the literature pertaining to procedural versus conceptual approaches to mathematical learning as well as in the literature on cognitive and meta-cognitive mathematical demands. Construal Level Theory offers an opportunity to understand both these dualities as aspects of the same psychological response to contextual priming. In addition, Construal Level Theory can be understood to illuminate student difficulties with heuristic strategies in mathematical problem-solving. The focus of Construal Level Theory on abstract and concrete cognitive construals as a consequence of psychological distance provides a useful lens for teaching and learning opportunities. We argue that Construal Level Theory offers an opportunity to draw together several strands of mathematics education theory and to help educators address learning challenges in the classroom.

Keywords: construal level theory; priming; metacognition; procedural; conceptual; heuristic strategies; classroom communication

1. Introduction

Mathematics is often regarded by students as a difficult subject to study. One among many reasons for this difficulty is the necessity for practitioners of mathematics to simultaneously engage with the work abstractly, bringing conceptual knowledge to bear, and to engage with the work at a more concrete level, using algorithms or algebraic procedures. Novice mathematicians, a role students of mathematics are necessarily filling, often struggle to bring an appropriate level of abstraction to bear on mathematical problems. In order to help students to adopt the appropriate level of abstraction it is important for teachers to first be conscious of what the appropriate level is and then communicate effectively with students to help them adopt it.

There exists an innate duality in mathematical engagement which has been addressed in multiple forms in the mathematical education literature. One example of an attempt to give expression to this duality is apparent in the discussion on procedural and conceptual knowledge and another, closely related, distinction is the cognition and metacognition contrast. In the case of both of these imposed structures, it is not possible to fully separate the different modes of thinking and both are required in varying degrees for effective problem solving. As a general matter, increased procedural proficiency benefits conceptual understanding and improved conceptual understanding helps in the development of procedural fluency. This does not imply, however, that these crossover effects are symmetrical in their importance and there is debate about where the emphasis in teaching should lie [1]. In addition, being conscious of the desired level of abstraction does not necessarily mean it is a simple matter to instill this awareness in students. Construal Level Theory (CLT) resonates with both of these dualisms.

A third substantial section of the mathematics education literature which can be viewed through the lens of Construal Level Theory is the use of heuristic strategies in mathematical problem-solving. The use of heuristics or “rules of thumb” in mathematical problem-solving has seen a great deal of support yet the classroom evidence suggests that students often find the simply-expressed strategies challenging to use in mathematical contexts [2,3]. Considering the archetypal heuristic strategies from the viewpoint of Construal Level Theory it seems apparent that different strategies require different construals, potentially at odds with the construal called for by the problem within which the heuristic strategy is being used.

In this article we shall argue that Construal Level Theory not only sheds light on the challenging nature of mathematics as a subject of study (prompting suggestions for successful pedagogy), but that it knits together and deepens insight into several major areas of the mathematics education literature. The duality apparent in much of the mathematics education literature on problem-solving is echoed in CLT’s notion of near and far psychological distance.

2. Construal Level Theory

A theory which is emerging from the psychosocial work on "priming" promises to shed new light on the sometimes uneasy truce between abstraction and concreteness in mathematics. Construal Level Theory [4,5] is a theory in social psychology that suggests a relation between psychological distance and the level of mental abstraction a person will adopt. Greater psychological distance typically primes people to adopt an abstract mental mindset while psychological closeness causes people to adopt a concrete mental mindset. The converse is also true and the level of abstraction adopted has a similar effect on perception of psychological distance. In general people tend to operate in one of two mental modes, "near mode" or "far mode" [6], with closeness and concrete thinking comprising near mode and distance and abstract thinking comprising far mode.

People only ever experience the here and now. All events and objects removed from our direct experience can be said to be psychologically distant. There are four basic dimensions of psychological distance; these include the common sense dimensions of time and space but also the more abstract notions of social distance and hypotheticality [5]. Social distance can refer to how familiar people are with each other but also to other social differences. For example, employers and their employees are often socially distant from one another even if they spend a great deal of time together. The dimension of hypotheticality involves the ability of people to form counterfactuals of past events, imagine hypothetical future scenarios and imagine different ways the world could be. The more improbable an imagined event, the greater the psychological distance involved. Psychological distance is thus egocentric and subjective; it is measured by perceived distance from the self.

Construal Level Theory suggests that we cope with psychological distance by forming abstract construals of distant events and objects and the more distant the object the more abstract the construal [4]. Abstraction necessarily involves omitting details which are considered irrelevant and focusing on the core important features. Psychologically distant objects are typically lacking in context-specific detail but nonetheless often guide action based on the core abstract features of the object. For example, people are often required to interact with socially distant strangers where the only information we have about them is the social space they occupy which plays the dominant role in determining appropriate behaviour. Our actions will be different if the stranger is our new boss, a shop assistant or our doctor. Our behaviour will largely be determined by the relatively stable, abstract norms governing the social role the stranger is fulfilling and not on idiosyncratic details of the individual.

The different distance dimensions and level of mental construal are all interrelated [4]. An event distant in one dimension is more likely to be perceived as distant on the others as well. People thinking about an event in the distant future are more likely to think it will be far away and are more likely to consider more improbable scenarios. Similarly, adopting an abstract mental mindset increases the likelihood of viewing other objects abstractly. Importantly for our purposes in this paper, psychological distance impacts on the level of mental construal people will adopt. In general, greater psychological distance primes more abstract thinking and psychological proximity primes more concrete thinking. The converse is also true so thinking of an object abstractly causes it to be viewed as more psychologically distant.

To the extent that encouraging students to adopt the appropriate level of abstraction is important, CLT has significant implications for mathematics education. Teachers have a substantial level of control over the classroom environment and CLT implies that there are many ways of altering the environment to induce students to feel increased or decreased psychological distance.

3. Dual-Process Theory

A cognitive theory with superficially similar structure to Construal Level Theory, and predating it by several years, is dual-process theory (DPT) [7]. We discuss DPT here as it is a useful theory with close ties to CLT, yet it is beyond the scope of this paper to go into much detail. There are several versions of the DPT but common to all of them is the distinction between slow, effortful and analytic reasoning on the one hand and more automatic intuitive and emotional cognition on the other. In addition to the on-going theoretical work, there is now a large body of empirical evidence that there are two distinct cognitive systems, each originating in different parts of the brain and with distinct evolutionary origins. The terminology most commonly used to describe the two modes of thinking classifies them system 1 and system 2 cognition [8,9].

Using the definitions adopted by Daniel Kahneman, System 1 type cognition "operates automatically and quickly, with little or no effort and no sense of voluntary control." [9,p.20]. Despite the terminology, system 1 is not a single unified system, but rather a collection of sub-subsystems, some instinctive and not requiring conscious control (for example breathing and blinking), others comprising of domain -specific knowledge acquired through training. System 2 on the other hand, "allocates attention to the effortful mental activities that demand it, including complex computations. The operations of System 2 are often associated with the subjective experience of agency, choice, and concentration." [9,p.21]. Compared to system 1, system 2 is slow, effortful and has a much lower capacity. Due to the demanding nature of system 2 thinking operates serially. System 2 is responsible for rule based, systematic, step by step reasoning and is capable of running hypothetical simulations. System 2 thinking is responsible for playing an inhibiting role where appropriate on the quickly generated reactions of system 1. Some researchers have suggested that system 2 thinking is primarily concerned with abstract reasoning (and thereby associating DPT with CLT), but others have warned against this [7]. While it is true that some types of abstract thinking can only occur in system 2, much effortful, sequential reasoning is not especially abstract in nature. Additionally, depending on how the term is used, much of the heuristic reasoning of system 1 could be considered to involve abstraction. Since abstraction is central to the present study this term is considered in more detail.

4. Abstraction

Abstraction is used in a philosophical and mathematical sense, as a topic of metaphysics or ontology and it is also used in a psychological sense, as something that humans, and to a lesser degree other animals, do naturally and automatically. The philosophical and psychological senses of abstraction are related because humans have to use their powers of abstraction to think, however imperfectly, about abstract mathematical objects and if we are to be able to solve mathematical problems.

Mathematics deals with abstract objects and processes. Abstract objects, as opposed to concrete objects, have no tangible existence in the natural world. Philosophers debate whether abstract objects can meaningfully be said to exist but mathematicians take their existence for granted and so will we in this article. While all mathematical objects are equally abstract in the sense that they have no tangible existence, there are superordinate and subordinate objects.

As a psychological process, we classify physical objects and actions with varying degrees of abstraction and levels of abstraction form a hierarchy so it makes sense to speak of higher and lower levels of abstraction; a physical object can be considered a concrete representation of a purely mathematical object . Much of mathematics and science involves developing and systematising our natural ability to conceptualise things abstractly. This enables us to separate powerful ideas from their concrete context in the here and now and apply them in novel circumstances and far away in space and time.

Actions and events also can be viewed more or less abstractly. Viewing an action abstractly typically involves considering the high level goal of the action. A concrete view of the action focuses more on how the action is performed. Broadly speaking, high-level abstract construals consider why an action is performed and a low-level construal considers how it is performed [4]. Often, evidence that a person has developed an abstract conception of a mathematical notion is that she can call to mind an appropriate concrete representation of that notion [10]

Finally it is important to note that thinking about abstract objects does not necessarily mean thinking in a highly abstract way. The numbers 3 and *i* are purely abstract objects, but once we have developed proficiency with using and manipulating these objects, they will be viewed at a level low level of cognitive abstraction because they are merely building blocks for thinking about higher order objects or solving problems which requires thinking about relationship at a much higher level of abstraction.

5. Procedural and conceptual approaches

Most, if not all, of the articles in the edited volume of [11] stress simultaneously the distinction between procedural and conceptual knowledge, the reliance of the development of one on the development of the other, and the importance of links between them. Hiebert and Lefevre [1] define conceptual knowledge as "knowledge that is rich in relationships" [1,p.3] and procedural knowledge as "the formal language, or symbol system, of mathematics" as well as the "algorithms, or rules, for completing mathematical tasks" [1,p.6]. Both types of knowledge are necessary for competent mathematical engagement, neither can categorically be considered more important than the other, and both types of knowledge are often inextricably entwined in a mathematical task. Sinclair and Sinclair [12] similarly draw a distinction between 'knowing-how-to" and understanding. Carpenter [13] argues that "distinctions between levels of problem solving reflect differences in how problems are represented and how these representations incorporate different kinds of relations [between conceptual and procedural knowledge]" [13,p.121]. Hiebert and Wearne [14] argue that competence is characterised by the links and connections between procedural and conceptual knowledge.

Rittle-Johnson and Alibali have argued that procedural and conceptual knowledge influence each other in an iterative fashion, with improvements in one domain allowing and spurring developments in the other [15] but they have also provided some evidence that at least in the case of developing proficiency in arithmetic, conceptual understanding is a greater aide to spuring development in procedural knowledge than vice versa [16]. On the other hand, Jon Star has argued that conceptual knowledge has received too much attention in the mathematics education research literature and argues for a reconceptualization of procedural knowledge that should receive more research focus [17]. He argues that procedural knowledge can be of a more flexible and robust nature than that implied by the definitions typically used without deserving the classification of conceptual knowledge. Typically procedural is taken to mean superficial and conceptual is taken to mean deep but this is not always the case. Star argues for classifications that would include deep procedural knowledge and shallow conceptual knowledge. In a similar way to Star, de Jong and Ferguson-Hessler suggest distinctions between types of knowledge on the one hand and qualities of knowledge on the other. In this classification, types of knowledge are either procedural or conceptual and qualities are deep or shallow [18].

Construal Level Theory displays obvious parallels with the debate around procedural versus conceptual approaches to mathematical engagement yet adds depth by arguing that such approaches are innate to the human brain's construal as influenced by contextual priming. Mathematics education always takes place within a broader (non-mathematical) context over which the teacher has some control. We are primed to think abstractly or concretely in surprising and unintentional ways; for example, thinking about distant times, places and socially distant individuals primes abstract thinking. These non-mathematical cues can be used along with explicit mathematical instruction to aid students. The relationship between distance and construal level means that communication can unintentionally be at odds with the appropriate level of abstraction helping to explain how easily communication can break down in the classroom.

Research investigating student performance in procedural and conceptual mathematics involving first year calculus students [19] suggests that while students do not necessarily perform better on either procedural or conceptual problems they typically are more confident of their ability to handle conceptual problems. Relative to their ability to cope with procedural and conceptual problems, students tend to be more accurate in their assessment of their procedural abilities and overconfident of their conceptual understand.

Construal Level Theory provides a potential theoretical explanation of this empirical finding. In other research Construal Level Theory has been invoked to explain a phenomenon known as the illusion of explanatory depth (IOED). IOEDs occur when people believe they understand a concept better than they do in reality. Alter et al [20] argue that IOEDs occur when people adopt an inappropriately abstract construal when assessing their own understanding of concrete concepts. Studies have shown that people who naturally adopt a concrete construal style and people who are primed to adopt a more concrete construal style experience diminished IOEDs.

We argue that Construal Level Theory provides an alternate language for addressing the distinction between procedural and conceptual knowledge as well as providing a theoretical framework for understanding existing empirical work in this area.

6. Cognition and meta-cognition

Metacognition, distinct from cognition itself, has seen a variety of definitions. When cognition is defined explicitly in contrast to metacognition (rather than being used as a catch-all for anything memory or problem-solving related) it invites descriptions such as applying knowledge or using skills; in mathematics, cognition is the manipulation of symbols, perceptions of physical space, application of learned algorithms. In bald, simplistic terms – easily challenged – cognition is thinking and metacognition is thinking about thinking. Schoenfeld [21] divides metacognitive processes into three categories: "(1) individuals' declarative knowledge about their cognitive processes, (2) self-regulatory procedures, including monitoring and "on-line" decision making, and (3) beliefs and affects and their effects on performance" (p. 347). Often it is only the first two categories, that is, individuals' awareness of their own cognitive processes, and the monitoring and regulation of these processes which are referred to in discussions on metacognition [22,23,24]. It is these definitions of metacognition as distinct from cognition which embody dualism and are amenable to analysis through Construal Level Theory.

It has been argued that successful mathematical problem-solving involves interplay between both cognitive and metacognitive strategies [23,25], although it has been observed that telling them apart can be difficult. Artzt and Armour-Thomas [25], acknowledging this difficulty of separating cognitive from metacognitive, divide the problem-solving process into episodes characterised as either cognitive or metacognitive or both. In the (perhaps somewhat artificial) distinction between cognition and metacognition, one can see the utility of Construal Level Theory, shedding illumination on the reasons why it is challenging to practice successful metacognition when in a construal mode suited to more concrete cognitive action.

7. Problem-solving heuristics

The term *heuristic* is usually used as an adjective (Oxford English Dictionary), such as in heuristic process, heuristic technique, and so on, to the point where the adjective form is used to define the noun form. *Heuristic (adjective): serving to find out or discover. Heuristic (noun): A heuristic process or method for attempting the solution of a problem; a rule or item of information used in such a process* (Oxford English Dictionary). Georg Pólya's [26] widely cited heuristic strategies include *draw a diagram, solve a related problem, think of a theorem with a similar conclusion*, and several more. Heuristics can be understood as being rules of thumb, or rough guides to how to respond to particular situations.

Those who have made a study of heuristic strategies agree on at least two important features: they are useful problem-solving tools and they are difficult to master [2,27] . Scrutiny of specific heuristic strategies suggests that there is a duality inherent in them. The heuristic strategy of *draw a diagram* requires attention to the specific requirements of the problem, such as shapes, variables, quantities, relative positions or motions. In contrast, the heuristic strategy of *think of a theorem with a similar conclusion* requires the student to consider clusters of theorems, to explicitly back away from the minutiae of the problem and consider a more general area of mathematics within which the problem is located. Further, that strategy requires the student to already have recognized links between theorems, a demand for abstraction.

Schoenfeld [2] has made major contributions to the study of heuristics in problem-solving by breaking many of Pólya's strategies into sub-strategies, specific to particular types of problems, and experienced considerable success in teaching a problem-solving course based on those sub-strategies. At the broadest level the strategy suggests students approach each new problem by attempting to progress through a series of stages: Analysis, design, exploration, implementation and verification.

Design is not a distinct stage to be implemented in sequence but is rather intended to remind students to maintain a global, goal oriented mind set. In terms of Construal Level Theory, this can be viewed as an explicit suggestion to begin the solution process by adopting an abstract construal and then proceeding to more concrete complex, detailed calculation.

Schoenfeld's description of design also suggests that no matter how demanding the concrete details become it is important to maintain a global perspective so that unproductive lines of calculation can be terminated and other avenues explored. This suggests that even though focus on concrete details may be required it is always necessary to maintain higher level awareness, in other words it is essential that a problem solver continuously shift between both abstract and concrete construals while working on a problem.

Schoenfeld elaborates more fully the analysis, exploration and verification stages. The first two steps in the analysis stage involve drawing a diagram or rough sketch and examining special cases. Both of these steps involve attempts to make the problem take a more concrete form. The third suggested step in the analysis stage involves simplifying the problem by exploiting symmetry, scaling or using "without loss of generality" arguments before getting immersed in details. This involves the use of abstract mathematical techniques and involves an abstract mind set.

The exploration stage involves three steps to be attempted sequentially and represents a progression from more concrete to more abstract and speculative. In terms of Construal Level Theory, the exploration stage moves progressively more abstract and distant on the hypotheticality dimension.

The implementation stage primarily involves procedural proficiency and is the only stage to primarily invoke a concrete mind set. The final stage of verification is broken into two steps; specific tests and general tests. The specific tests involve a relatively low level of construal, but Schoenfeld stresses that at the general level, verifying the problem solution often suggests alternate solutions and connections and can promote conscious awareness of which parts of the process were useful and can be used in future. In other words, global verification helps to develop deeper conceptual understanding of the principles involved.

Most stages of implementing this heuristic strategy involve both high and low level construals. Construal Level Theory indicates that high and low level construals involve adopting fundamentally different mind sets and thus suggests that this continuous shifting between different modes of thinking is intrinsically unintuitive and challenging.

This duality, which we argue is inherent within the lists of heuristic strategies we have encountered, constitutes a challenge for the student who attempts to use such a list as a problem-solving tool. Certain strategies require cognitive engagement with detailed problem features whereas others require disengagement from problem features and engagement with abstract concepts, theorems and higher level theoretical networks.

8. Conclusions

Procedural and conceptual knowledge distinctions lend themselves easily to analysis through the lens of Construal Level Theory. Procedural knowledge is clearly related to the concrete construal level associated with "near" psychological distance and conceptual knowledge is related to the abstract construal level associated with "far" psychological distance. While a switch of language used to refer to these two phenomena might not be seen as valuable, what is valuable is CLT's insistence that the student's mind is primed to work in one of those two modes by many other contextual factors associated with a specific mathematical problem. Simply the wording of the problem can prime the student to construe the problem in different ways. A teacher who is aware of this potential priming effect and one who wishes the student to operate at a certain construal level, can prime the student in useful and productive ways.

Cognition and metacognition, we argue, can similarly be addressed using CLT. Cognition is associated with concrete construal and metacognition with abstract construal. No doubt there are subtleties which can be argued and definitions of the two phenomena which are hard to categorise through CLT, however there is certainly a high degree of agreement between the two definition sets. Successful problem-solving usually requires the two processes of cognition and metacognition to occur for a sustained period simultaneously, or at least with frequent shifting between them. Construal Level Theory suggests that this is more easily demanded of the student than can be enacted with ease.

Interpreting heuristic strategies for problem-solving through the lens of CLT suggests that one reason for students' difficulty in using heuristic strategies is the near-far dichotomy between strategies often regarded as otherwise similar. Many environmental and contextual characteristics contribute to a student's priming for construal level, not least of which is the context and working of the mathematical problem itself. To require students to cognitively draw upon each of a list of heuristic strategies with equal emphasis is unrealistic when viewed in this light. A question for further enquiry would be to investigate whether heuristic strategies chosen for construal level similar to one another and to the specific problem construal requirements might achieve greater problem-solving success than "mixed" lists such as found in [26] and [2].

In this article we suggest that CLT provides a useful framework for analyzing students' approach to and interpretation of problems. The innate duality underpinning CLT echoes the duality inherent in procedural/conceptual approaches, cognitive/metacognitive monitoring and near/far heuristic demands.

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