



UNIVERSITY OF CAPE TOWN

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**COSMIC STRING CUSPS AND THEIR APPLICATION TO FAST  
RADIO BURSTS**

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# Abstract

This thesis concerns observational characteristics of two theoretical aspects of cosmic strings.

The first is relativistic modification of cusps. Nambu-Goto strings generically develop cusps, regions of the string which emit coherent electromagnetic radiation when they decay. We point out that consideration of relativistic effects in the rest frame of the string cusp substantially reduces the cusp length, and therefore modifies the normally assumed power, rate, and time scale of any radiation bursts. The second is consideration of wiggly cosmic strings. Simulations imply a distribution of strings in an expanding universe develop small-scale structure called wiggles. We extend on a wiggly Polyakov formalism and show that wiggles prohibit cusp formation (barring *ad-hoc* fine tuning of initial conditions).

We discuss these theoretical results in the context of using strings to explain fast radio bursts (FRBs). Cusp decay is a possible mechanism for sourcing FRBs. We show, however, that (1) consideration of relativistic effect leads to incompatibility with FRB data, and (2) the absence of cusps from “realistic” cosmic strings casts further doubt on the possibility of detecting cosmic strings via electromagnetic signatures.

# Acknowledgements

A huge thank you goes to Amanda. Many supervisors want to make sure new students are worth the investment, but you were very welcoming and accepting from day one. I'm sure others wouldn't be as forgiving of my inexperience or poor work habits. You always made time to help out with any and all concerns, academic or tangential. My primary goal was to learn more: about physics, research, how to "do science". I feel I have achieved that, in no small part due to your guidance.

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# Declaration

I, Jake Eli Blake Gordin, declare that this thesis the work presented in it are my own.

I confirm that –

- This work was done wholly while in candidature for a research degree at the University of Cape Town.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.

Signed:

Signed by candidate

Some portions of this work are based on academic papers done by the author and collaborators. These are:

- *A Living Catalogue for Fast Radio Bursts* — E. Platts, A. Weltman, A. Walters, S. P. Tendulkar, **J. E. B. Gordin**, and S. Kandhai [<https://arxiv.org/abs/1810.05836>]
- Are fast radio bursts generated by cosmic string cusps? — Renato Costa, **Jake E. B. Gordin**, and Amanda Weltman [<https://arxiv.org/abs/1807.01976>]

# Notations, Abbreviations, and (the shirking of) Conventions

In this thesis, we shall use the following notational shorthands and conventions:

- Greek indices ( $\mu, \nu$ , e.g.  $x^\mu$ ) denote 4-vectors. The indices range from 0 to 3.
- Baby Latin indices ( $a, b$ , e.g.  $\xi^a$ ) denote worldsheet quantities on a (1+1)-dimensional surface. The indices range from 0 to 1.
- Big Latin indices ( $M, N$ , e.g.  $q^M$ ) denote higher dimensional embedding. The indices range from 0 to 1, such that the embedding for the 5-vector is  $q^A = (x^\mu, \phi)$ .
- The 3-vector part of the 4-vectors in the (3+1)-dimensional spacetime will be denoted by boldface, e.g.  $x^\mu = (t, \mathbf{x})$ . Here  $\mathbf{x} = (x^1, x^2, x^3)$ .
- A time-positive metric signature will be employed, such that the Minkowski metric is given by  $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ .
- Einstein summation convention is assumed for repeated contra- and covariant indices (of any type). As such,  $x^\mu x_\mu = x^0 x_0 - x^1 x_1 - x^2 x_2 - x^3 x_3$ .
- Scalar products will be alternatively denoted by  $(x^\mu)^2 = x^\mu \cdot x_\mu$ .
- Partial differentiation will be denoted by  $\partial_x$ , where the kind of subscript (Greek, baby Latin, big Latin) denotes the coordinate with respect to which we are differentiating (e.g.  $\partial_\mu = \partial/\partial x^\mu$ ,  $\partial_a = \partial/\partial x^a$ , etc). There is room for ambiguity here once the index number is chosen: for e.g.  $\partial_0$  can refer to the 4-vector index or the 2-vector one. Typically, we will differentiate with respect to worldsheet coordinates if an index number is chosen, i.e.  $\partial_0 = \partial/\partial \xi^0$ , where  $\xi^0$  is a worldsheet coordinate. We will assume this unless otherwise stated.
- Differentiation with respect to the worldsheet timelike coordinate (i.e.  $\xi^0$ ) will be denoted by an overdot,  $\partial_0 Z = \dot{Z}$ . Differentiation with respect to the spacelike coordinate (i.e.  $\xi^1$ ) will be denoted by a prime,  $\partial_1 Z = Z'$ .
- As often as is possible, the generic 4-vector coordinate symbol will be  $x$ , e.g.  $x^\mu, x^\nu$  etc, the worldsheet symbol  $\xi^a$ , and the brane symbol  $q^A$ . When other coordinate symbols are used, their meaning will be made clear in the text.
- Various d'Alembertians:
  - $\square_4 = \partial_\mu \partial^\mu = \partial_0 \partial^0 - \partial_1 \partial^1 - \partial_2 \partial^2 - \partial_3 \partial^3$ , where these are all with respect to the 4-vector coordinate components.
  - $\square_2 = \partial_a \partial^a = \partial_0 \partial^0 - \partial_1 \partial^1$ , where these are all with respect to worldsheet coordinate components.
- The negative of the determinant of a matrix will be denoted by  $||A||$ , such that if  $A_{ij}$  is a matrix,  $||A||^{1/2} = \sqrt{-A}$ , where  $\det(A_{ij}) = A$ .

The following abbreviations are used (sometimes commonly, sometimes rarely) in the main text:

- AH - Abelian-Higgs
- GUT - Grand unified scale
- KZM - Kibble-Zurek mechanism
- NO - Nielsen-Olesen
- SSB - Spontaneous symmetry breaking
- VEV - Vacuum expectation value

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# Chapter 1

## Introduction

One of the perennial problems in cosmology is the question “What provided the initial seeds for structure formation?” The theory of cosmological perturbations (for a thorough reference, see [1]) aims to describe the evolution of the universe *given* some collection of structures, but it does not account for the generation of the density fluctuation required to produce the observed galaxy clusters. Inflation, coupled with quantum fluctuations (see ref. [2]), is currently the most promising contender. Before inflationary theory took center stage, the 1980s and 90s saw a new idea emerge: cosmic strings. These are linear topological defects, were hypothetically formed from symmetry breaking as the universe cooled down, and may be the seeds of structure formation ([3, 4]). A massive string would attract gas around it gravitationally, providing the impetus for galaxies to coalesce.

This idea, however, turned out to be insufficient to explain structure formation. In the early 90s, when cosmic microwave background (CMB) measurements were first released, the data placed constraints on the maximum possible tension parameter for a cosmic string. While the observations don’t rule out the existence of cosmic strings, the CMB does seem to imply that strings aren’t the major player in the structure formation. Indeed, the most up-to-date string tension measurements [5–7] place the upper limit on the string tension as

$$G\mu \leq 10^{-7} \tag{1.1}$$

where we’ve expressed the tension as a dimensionless quantity ( $G$  is Newton’s constant in natural units). This value is too small to account for galaxy formation. However, the physical mechanism which produces cosmic strings - spontaneous symmetry breaking (SSB) - is itself generic, and is involved in producing other physical objects. Most notably, the Higgs boson forms from SSB. When the Higgs was detected in 2012, interest somewhat revived in cosmic strings, since if SSB produces a now confirmed-to-exist particle, perhaps the same is true of cosmic strings.

Quite parallel to this, astrophysics has several open problems of its own. Among the more recent ones are the origin, or source, of objects known as fast radio bursts (FRBs), small time-scale radio pulses. Difficulties include the low number of FRBs (only about 30 have been observed so far [8]) and the lack of localisation of the bursts (only one so far has an associated host galaxy, the so-called repeater [9]). There exist a panoply of theoretical models which purport to explain FRBs (for a review of them, see [10] and [11]). Among them, the authors of [12] put forward the conjecture that decaying cosmic strings might cause FRBs.

Electromagnetic signatures from cosmic strings is not a recent (post-2010) idea; in the cosmic string heyday of the 90s, the possible radiation backgrounds produced by cosmic strings were considered [13–16]. These include a background of gamma rays or neutrinos, produced by a cosmic string. Strings produce radiation either through a coupling to electromagnetism (this string is known as a superconducting string [17]) or through cusp decay. Cusps are overlapping portions of the string, akin to the part of a whip which ‘cracks’ to produce a sound. For a cosmic string, the cusp decays, producing particles [4] instead of sound waves. The goal for utilising cosmic strings has now shifted somewhat: instead of seeding structures, it’s possible that they seed other astrophysical phenomena - possibly FRBs. This possibility was considered by ref. [12] for the case of cusp decay, and by refs [18–20] for the case of a superconducting string.

Apart from this, there is also the question of how a network of strings evolve in an expanding universe if they do exist. While there are no observational data, simulations imply that a string will, as a result of backreaction with itself and the others in a large collection, develop structure called *wiggles* [21]. These are small-scale<sup>1</sup> structures, that can be thought of as perturbations to the Nambu-Goto string. The formalism for a wiggly cosmic string has a collection of rather interesting properties [22, 23] (for an overview,

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<sup>1</sup>An observer far away cannot resolve anything.

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see [24, 25]).

The question central to this thesis is the following: from a theoretical perspective, are cosmic strings a good model for FRBs? The avenues explored are twofold.

(1) We consider cusp decay for a neutral, structureless cosmic string. The purported model laid out in ref. [12] does not account for relativistic corrections to the cusp length. This work was done first in [26], but it has not been applied in detail to the case of the radiation cusps might produce, nor to an FRB model specifically.

(2) What becomes of cusps, and radiative signatures more generally, in a wiggly string framework? To answer this question, we develop for the first time the cusp solution for a wiggly cosmic string, although we use the formalism developed in [23].

In part I, an overview of cosmic strings is presented. Chapter 2 reviews cosmic string formation viz. SSB. Chapter 3 lays out the dynamics of a Nambu-Goto string which are needed for both cusp and wiggly analysis. Part II is an application of both cusps and wiggly strings to FRBs. Chapter 4 reviews the formation of cusps, and the relativistic corrections to them - before applying the corrections to the cosmic string FRB model. Chapter 5 develops wiggly string dynamics, and we extend the model by developing its cusp solution within the Polyakov formalism and examine cusps in the wiggly context.

Ultimately, we will show that the reduced cusp length leads to an incompatibility between cusp decay and existing FRB data. We furthermore show that radiative signatures from wiggly cosmic strings will be severely suppressed since the wiggles prevent cusp formation.

## Part I

# Overview of Cosmic Strings Fundamentals

## Chapter 2

# Formation of Cosmic Strings

Topological defects are formed generally as a result of *spontaneous symmetry breaking* (SSB). In this chapter we review the generalities of SSB in field theories, using some simple models as illustrative examples. We show how defects arise. We then show how strings form in the simple Abelian-Higgs model. Finally, we review how the topology of the manifold defines the properties of the defect which forms, specifically for the topologies relevant to cosmic strings.

### 2.1 Spontaneous symmetry breaking

Cosmic strings are an example of topological defects. Their existence is admitted by the breaking of a symmetry in the overarching theory. In this section we examine the three types of spontaneous symmetry breaking: the breaking of discrete, globally continuous, and gauged continuous symmetries. The discussion here closely follows reference [27].

#### 2.1.1 Discrete symmetry breaking – the $\phi^4$ model

Consider a general scalar field theory with the Lagrangian<sup>1</sup>

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda\phi^4 \quad (2.1)$$

$\phi$  is a real scalar field;  $\lambda$  is the self-interaction coupling, and  $\mu$  is the scalar coupling. Here we demand  $\lambda > 0$ , otherwise the potential energy will be unbounded from below. This model is commonly called the  $\phi^4$  model, after the leading term in the potential. The symmetry obeyed by this theory is:

$$\phi \rightarrow -\phi, \quad (2.2)$$

and is called a discrete transformation because it acts at one point on the field and “flips” it to the mirror point. This symmetry in particular is a  $Z_2$  symmetry mapping [27]. One can see that the theory (2.1) is  $Z_2$  symmetry invariant: swapping  $\phi$  for  $-\phi$  doesn’t change the Lagrangian. However, the symmetry can be broken by tuning the coupling constants. In particular, the potential is qualitatively different for the case of  $\mu^2 > 0$  vs.  $\mu^2 < 0$ . In the former case, the potential is the standard parabola potential:

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4 \quad (2.3)$$

The minimum of this potential is at  $\phi = 0$ , since if

$$V'(\phi) = \phi(\mu^2 + \lambda\phi^2) = 0, \quad (2.4)$$

$\phi = 0$  is a solution (where a prime denotes differentiation with respect to the argument). The other solution to eq. (2.4) is disallowed by the conditions that  $\phi$  is real and that  $\mu^2 > 0$ . The value of  $\phi$  which minimises the potential is known as the vacuum expectation value (VEV) of the field [27]. In this case it is zero:

$$\langle\phi\rangle_0 = \langle 0|\phi|0\rangle = 0 \quad (2.5)$$

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<sup>1</sup>We employ the regular abuse of terminology: by Lagrangian, we mean Lagrangian density. Rather “an abused terminology to cluttered notation and unbearable pedantry” [28].

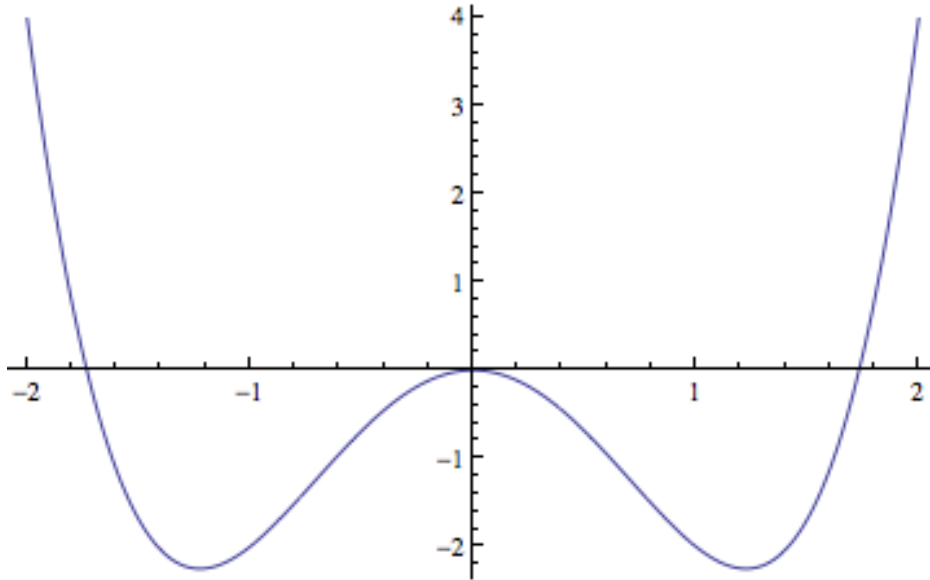


Figure 2.1: The double well potential for eq. (2.7).

The field  $\phi$  is a series of operators which act on the ground state of the system,  $|0\rangle$ , and says that for  $\mu^2 > 0$  we expect the potential to have a value of zero.

The  $Z_2$  symmetry mapping which leaves (2.1) invariant also leaves the VEV (2.5) invariant - positive 0 mapped to negative 0 is still 0. This kind of symmetry is known as a *Wigner symmetry mode*, which results from a theory obeying a discrete symmetry law. If we imagine that our model describes physics, perturbations of the scalar field correspond to particles. If we perturb about the vacuum at the origin to second order, by writing  $\phi = \xi + \delta\phi$ , with  $\delta\phi = 0$  (since the VEV = 0), eq. (2.1) then reads:

$$\mathcal{L} \approx \frac{1}{2} \partial_\mu \xi \partial^\mu \xi - \frac{1}{2} \mu^2 \xi^2 \quad (2.6)$$

The particle spectrum of the theory becomes manifest: it is one of a free scalar field with mass  $\mu$ . Hence when  $\mu^2 > 0$ , small oscillations of the field about the origin of the potential correspond to massive particles, with mass of order  $\mu$ , and nothing seems pathological since the original symmetry is respected.

Now, if  $\mu^2 < 0$ , then the potential has a “double-well” shape, not a parabolic one:

$$V(\phi) = -\frac{1}{2} |\mu^2| \phi^2 + \frac{1}{4} \lambda \phi^4 \quad (2.7)$$

Eq. (2.7) has three extrema, and the global minima - and therefore the associated VEVs are:

$$\langle \phi \rangle_0 = \pm \sqrt{\frac{-\mu^2}{\lambda}} \equiv \pm v \quad (2.8)$$

The field  $\phi$  now is able to take on this value since the square root won't return an imaginary value. The two VEVs have the same energy, and so are two degenerate vacuum states.

The change of the VEV from 0 to  $\pm v$  is induced by a phase transition: in this case, the transition from the  $\mu^2 > 0$  phase to the  $\mu^2 < 0$  phase. When  $\mu^2 > 0$ , the field settles into fluctuations about  $\phi = 0$ ; but when  $\mu^2$  becomes less than 0, the field settles into fluctuations about  $\pm v$ . Once it does so, something interesting happens - there is a case of symmetry breaking. Eq. (2.1) remains invariant under the transformation (2.2), but the vacuum state for  $\mu^2 < 0$  does not. If the field settled into  $\langle \phi \rangle_0 = v$ , then applying the  $Z_2$  transformation would give  $\langle \phi \rangle_0 = -v$ . While  $v$  and  $-v$  have the same energy, they are not the same vacuum *configuration*. This is a case of *spontaneous symmetry breaking*: the Lagrangian is invariant under some symmetry operation, but the vacuum is not - that symmetry is broken.

Let's say we pick  $\langle \phi \rangle_0 = +v$  as the state into which the field settled. We want to construct our theory by expanding about the points  $+v$ , not  $\phi = 0$ , as  $+v$  is now a minimum. Indeed,  $\phi = 0$  is now an unstable configuration of the field - any perturbation about that point will force the system to one of the two

vacuum states anyway, as it will immediately roll down the potential hill.

We expand about  $\langle\phi\rangle_0 = +v$  by first defining  $\xi(x) = \phi(x) - v$  (i.e. we assume fluctuations about the field's minimum is of order  $\xi(x)$ ). We recast the original Lagrangian (2.1) in terms of  $\xi(x)$ , remembering that  $v$  was constructed from constants:

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) - \frac{1}{2}\mu^2(\xi^2 + v^2 + 2\xi v) - \frac{1}{4}\lambda(\xi^2 + v^2 + 2\xi v)^2 \\ &= \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) - \frac{1}{2}\mu^2(\xi^2 + v^2 + 2\xi v) - \frac{1}{4}\lambda(\xi^4 + 4v\xi^3 + 6v^2\xi^2 + 4v^3\xi + v^4)\end{aligned}$$

After neglecting the constant terms,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) - \frac{1}{2}\mu^2(\xi^2 + 2\xi v) - \frac{1}{4}\lambda(\xi^4 + 4v\xi^3 + 6v^2\xi^2 + 4v^3\xi)$$

Using the relation  $v^2\lambda = -\mu^2$ ,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + \frac{1}{2}v^2\lambda(\xi^2 + 2\xi v) - \frac{1}{4}\lambda(\xi^4 + 4v\xi^3 + 6v^2\xi^2 + 4v^3\xi) \\ &= \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + \frac{1}{2}v^2\lambda\xi^2 + \xi v^3\lambda - \frac{1}{4}\lambda\xi^4 - \lambda v\xi^3 - \frac{3}{2}\lambda v^2\xi^2 - \lambda v^3\xi \\ &= \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) + \frac{1}{2}v^2\lambda\xi^2 - \frac{1}{4}\lambda\xi^4 - \lambda v\xi^3 - \frac{3}{2}\lambda v^2\xi^2 \\ &= \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) - \lambda v^2\xi^2 - \lambda v\xi^3 - \frac{1}{4}\lambda\xi^4\end{aligned}$$

The redefined Lagrangian, as one perturbed about the non-zero vacuum state, has no obvious symmetry: it is hidden by its “breaking” from the phase transition, but it is still there. If we retain now only second order in  $\xi$  terms,

$$\mathcal{L} \approx \frac{1}{2}(\partial_\mu\xi)(\partial^\mu\xi) - \lambda v^2\xi^2, \tag{2.9}$$

this again is a theory describing a free scalar field but with mass of  $m = \sqrt{-2\mu^2}$  ( $\mu^2 < 0$ ). Note that this is different from the mass of the field in the  $\mu^2 > 0$  case.

In terms of discrete symmetries as given by (2.2), spontaneous symmetry breaking is characterised by (1) a non-zero VEV for  $\phi$ ; (2) degenerate vacua, the choice of which is arbitrary, and the vacua do not exhibit the same symmetry as the theory's Lagrangian; (3) transition from symmetric vacua to degenerate ones result from varying some order parameter,  $\mu^2$ , and (4) a Lagrangian density, with degenerate vacuum states, and no obvious symmetry, but the vacua are all related by the original symmetry transformation – i.e. a “hidden” symmetry.

Two further aspects are not present in the  $\phi^4$  model, because the theory has discrete, but not *continuous* symmetries. We shall discuss these next: there are two types of continuous symmetries, global and gauge (or local) ones.

### 2.1.2 Global symmetries - Goldstone Bosons

Consider a Lagrangian:

$$\mathcal{L} = \partial_\mu\bar{\phi}\partial^\mu\phi - \mu^2\bar{\phi}\phi - \lambda(\bar{\phi}\phi)^2 \tag{2.10}$$

$\phi$  is now a complex scalar field; an overbar denotes the complex conjugate.  $\lambda$  and  $\mu$  are the same coupling constants. The potential is the “Mexican Hat” potential (fig. 2.2). Again we demand that  $\lambda > 0$ .

The Lagrangian (2.10) is called the Goldstone model, and it is invariant under the global group of U(1) transformations [3, 27] (i.e. rotations about the complex plane):

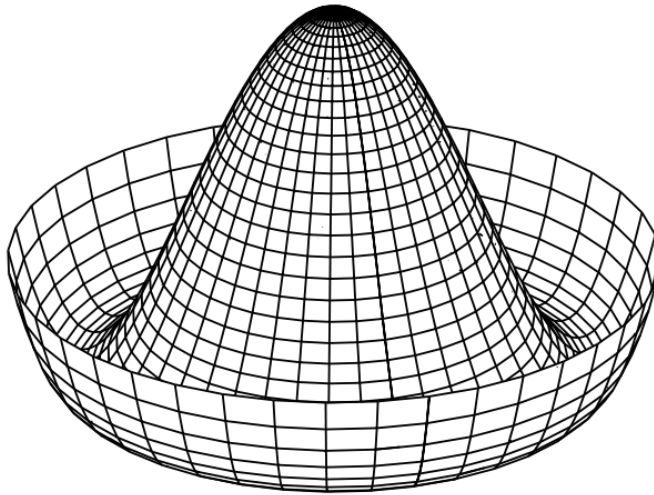


Figure 2.2: Graph of potential  $V(\phi) = \mu^2 \bar{\phi} \phi + \lambda (\bar{\phi} \phi)^2$ . The z-direction shows values of  $V$ ; the x and y directions show values of  $\phi$  and  $\bar{\phi}$  (taken from [27])

$$\phi(x) \rightarrow \phi(x) e^{i\alpha} \quad (2.11)$$

The transformation is *global* because the phase offset  $\alpha$  is independent of position  $x$ . The invariance is manifest by looking at the terms in the Lagrangian: each term is of order  $|\phi|$ , and so the exponential factor will cancel out. As before, we distinguish between the  $\mu^2 > 0$  case and the  $\mu^2 < 0$  case, since this causes a change in the shape of the potential. The  $\mu^2 > 0$  proceeds case as before but with a complex field  $\phi$ , since the VEV of the Mexican Hat potential is again  $\phi = 0$ , and that is symmetric under U(1) rotations.

For  $\mu^2 < 0$ , the VEVs for  $V'(\phi) = 0$  are now:

$$\begin{aligned} \mu^2 \bar{\phi} + 2\lambda (\bar{\phi} \phi) \bar{\phi} &= 0 \\ \mu^2 + 2\lambda (\bar{\phi} \phi) &= 0 \\ |\phi| &= \sqrt{\frac{-\mu^2}{2\lambda}} \\ |\phi| &= \frac{v}{\sqrt{2}} \end{aligned}$$

The potential is thus minimised along a circle of infinite field values, of radius  $|\phi| := \frac{v}{\sqrt{2}}$ . This can be seen graphically in fig. 2.2. We see again a case of SSB: the vacuum state is not the same under U(1) rotations, though they are all the same degenerate states. We can pick one of the vacua, as they are all mapped out by U(1), and expand about it - as for the  $\phi^4$  model. We pick

$$\text{Re}(\phi) = \frac{v}{\sqrt{2}} \quad (2.12)$$

and express  $\phi(x)$  as

$$\phi(x) = \frac{1}{\sqrt{2}} (v + \xi(x) + i\chi(x)) \quad (2.13)$$

where we have broken up the  $\phi(x)$  fluctuation into the real and complex parts (“Fluctuation” =  $\xi(x) + i\chi(x)$ ). We calculate each term in turn. The first:

$$\begin{aligned} \partial_\mu \bar{\phi} \partial^\mu \phi &= \frac{1}{2} (\partial_\mu \xi - i\partial_\mu \chi) (\partial^\mu \xi + i\partial^\mu \chi) \\ &= \frac{1}{2} (\partial_\mu \xi)^2 + \frac{1}{2} (\partial_\mu \chi)^2 \end{aligned}$$

The second term:

$$\begin{aligned}
-\frac{1}{2}\mu^2\phi^2 &= \frac{1}{2}v^2\lambda\phi^2 \\
&= \frac{1}{2}v^2\lambda(v + \xi + i\chi)(v + \xi - i\chi) \\
&= \frac{1}{2}v^2\lambda(v^2 + v\xi + i\chi v + v\xi + \xi^2 + i\chi\xi - i\chi v - i\chi\xi + \chi^2) \\
&= \frac{1}{2}v^2\lambda(v^2 + \chi^2 + \xi^2 + 2v\xi)
\end{aligned}$$

And the third, using the calculation of  $\phi^2$  above:

$$\begin{aligned}
-\frac{1}{4}\lambda(\bar{\phi}\phi)^2 &= -\frac{1}{4}\lambda(v^2 + \chi^2 + \xi^2 + 2v\xi)^2 \\
&= -\frac{1}{4}(v^4 + 2v^2\chi^2 + \chi^4 + 4v^3\xi + 4v\chi^2\xi + 6v^2\xi^2 + 2\chi^2\xi^2 + 4v\xi^3 + \xi^4)
\end{aligned}$$

Putting it all together, and dropping the constants, (2.10) simplifies to:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\xi)^2 + \frac{1}{2}(\partial_\mu\chi)^2 - \lambda v^2\xi^2 - \lambda v\xi(\xi^2 + \chi^2) - \frac{1}{4}\lambda(\xi^2 + \chi^2)^2, \quad (2.14)$$

which, to second order, takes the form:

$$\mathcal{L} \approx \frac{1}{2}(\partial_\mu\xi)^2 + \frac{1}{2}(\partial_\mu\chi)^2 - \lambda v^2\xi^2 \quad (2.15)$$

This theory (2.15) is now one of two scalar fields.  $\chi$  is massless (there are kinetic terms only at second order), and the  $\xi$  field has a mass  $m_\xi = \sqrt{2\lambda v^2}$ . The massive  $\xi$  field corresponds to radial oscillations up and down the potential hill;  $\chi$  field moves in a circular/angular motion in the vacuum state.

The appearance of this new massless field  $\chi$ , as a result of the phase transition, is a central feature of continuous global SSB. The quanta of the massless scalar field corresponds to a so-called Goldstone boson. The appearance of a new massless field via SSB is generally true for globally broken symmetries [27]:

*Goldstone theorem:* for each global symmetry broken, there is a massless particle for each group generator. U(1) has one generator: hence there is one boson.

The hidden symmetry is, in some informal sense, encoded into the boson. The  $\chi$  field rotates about the vacua - this is the same motion as the now broken symmetry. The hidden U(1) symmetry (rotations in the complex plane) (eq. 2.11).

### 2.1.3 Gauge symmetry breaking - the Higgs Mechanism

Goldstone bosons don't readily appear in nature, and it seems that in order to avoid violating the Goldstone theorem one must refrain from the application of SSB to real physics. However, gauge theories do not obey the Goldstone theorem; the theorem assumes Lorentz invariance, locality, and a positive definite norm for the Hilbert space. Gauge theories cannot possess all 3 traits. For example - Maxwell's laws do not have a positive definite norm when expressed in covariant form; similarly, keeping the norm, as is done in QED, sacrifices Lorentz invariance [27].

A crucial Lagrangian, which will be used as a toy model for many of the future discussions of cosmic strings, is the Abelian-Higgs model (AH model), a theory of scalar electrodynamics [3, 27]:

$$\mathcal{L} = (\bar{D}_\mu\bar{\phi})(D^\mu\phi) - \mu^2\bar{\phi}\phi - \lambda|\phi|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (2.16)$$

where we have defined:

$$\begin{aligned}\phi &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ D_\mu &= \partial_\mu + iqA_\mu \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\end{aligned}$$

The fields  $\phi_1$  and  $\phi_2$  are real, and have a VEV of 0.  $A_\mu$  is the gauge field.  $q$  is the charge, and  $F^{\mu\nu}$  is the Maxwell electromagnetic tensor. The transformations obeyed by this theory are local U(1) rotations and also translations of the gauge field:

$$\begin{aligned}\phi(x) &\rightarrow e^{iq\alpha(x)}\phi(x) \\ A_\mu(x) &\rightarrow A_\mu(x) - \partial_\mu\alpha(x)\end{aligned}\tag{2.17}$$

The spacial dependence of the phase  $\alpha$  indicates locality: these are continuous *gauge* symmetries.

For the  $\mu^2 > 0$ , the theory has a unique vacuum state as before ( $\phi = 0$ ), and describes a theory of two fields,  $\phi$  and  $\bar{\phi}$ , with mass  $\mu$ , as well as a massless particle,  $A_\mu$ , which we canonically identify as the photon [27].

For  $\mu^2 < 0$ , the local symmetries (eq. 2.17) are broken. The infinite degenerate vacua again occur at

$$|\phi| = \pm \frac{v}{\sqrt{2}}\tag{2.18}$$

We follow the procedure as for the Goldstone model: choose positive and real valued vacua, at  $V(\phi) = 0$ , expand about it such that:

$$\phi(x) = \frac{1}{\sqrt{2}}(v + \xi(x) + i\chi(x) + \dots)\tag{2.19}$$

This means the first term in eq. (2.16) becomes

$$\begin{aligned}(\bar{D}_\mu\bar{\phi})(D^\mu\phi) &= \frac{1}{2}(\partial_\mu - iqA_\mu)(v + \xi(x) - i\chi(x))(\partial^\mu + iqA^\mu)(v + \xi(x) + i\chi(x)) \\ &= \frac{1}{2}[\partial_\mu\xi(x) - i\partial_\mu\chi(x) - iqvA_\mu - iqA_\mu\xi(x) - qA_\mu\chi(x)] \\ &\quad [\partial^\mu\xi(x) + i\partial^\mu\chi(x) + iqvA^\mu + iqA^\mu\xi(x) - qA^\mu\chi(x)] \\ &= \frac{1}{2}[(\partial_\mu\xi(x))^2 + (\partial_\mu\chi(x))^2 + 2vqA^\mu\partial_\mu\chi(x) + v^2q^2A_\mu A^\mu] + \mathcal{O}(3)\end{aligned}$$

Here  $\mathcal{O}(3)$  are all third order terms in either  $\xi(x)$ ,  $\chi(x)$ ,  $A_\mu$ , or any product of the fields. The second term in (2.16) becomes

$$\begin{aligned}-\mu^2(\bar{\phi}\phi) &= -\mu^2(v + \xi(x) - i\chi(x))(v + \xi(x) + i\chi(x)) \\ &= -\mu^2(v^2 + 2v\xi(x) + (\xi(x))^2 + (\chi(x))^2)\end{aligned}$$

The third term becomes

$$\begin{aligned}-\lambda(\bar{\phi}\phi)^2 &= -\lambda(v^2 + 2v\xi(x) + (\xi(x))^2 + (\chi(x))^2)^2 \\ &= \frac{\mu^2}{v^2}(v^2 + 2v\xi(x) + (\xi(x))^2 + (\chi(x))^2)^2\end{aligned}$$

After dropping constants and 3rd order terms, this reduces to the AH Lagrangian to

$$\mathcal{L} \approx \frac{1}{2}(\partial_\mu\xi)^2 + \mu^2\xi^2 + \frac{1}{2}(\partial_\mu\chi)^2 + qvA_\mu\partial^\mu\chi + \frac{1}{2}q^2v^2A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \dots\tag{2.20}$$

Our quantum field theory of the vacuum state has three fields:  $\chi(x)$ ,  $\xi(x)$ , and  $A_\mu(x)$ .  $\xi$  and  $A_\mu$  are massive, and only  $\chi$  is massless. We might associate  $\chi$  with the Goldstone boson as before, but there is a subtlety one should not overlook: the photon gained mass, of order  $m_A = qv$ . For any sensible quantum field theory, we expect this to be zero. What might have happened? It's worth counting the number of degrees of freedom: before SSB, the complex field had 2 degrees of freedom, and the transverse polarisations of the field for  $A_\mu$  account for 2, which gives 4. After SSB, the field  $\xi$  and  $\chi$  have 1 each, and the now *massive* field  $A_\mu$  has 3 degrees of freedom, giving 5 in total. There is an extra degree of freedom. Since this wasn't present in the original theory, this one is unphysical. We can "use it up", as it were, by applying the following gauge transformations:

$$\phi(x) \rightarrow \frac{v + \xi(x)}{\sqrt{2}} \quad (2.21)$$

$$A_\mu(x) \rightarrow A_\mu(x) + \frac{1}{qv} \partial_\mu \chi(x) \quad (2.22)$$

Applying these to our original AH Lagrangian (2.16) leaves it in the form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \xi)^2 + \mu^2 \xi^2 + \frac{1}{2}q^2 v^2 A_\mu A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (2.23)$$

There are now two massive fields:  $\xi(x)$  and  $A_\mu$ . The  $\chi(x)$  field, the Goldstone boson, has been "eaten", or gauged away. The massless gauge field in the original theory has, in some informal sense, gotten rid of the the Goldstone boson. This allows for Goldstone's theorem to hold while explaining the lack of observable Goldstone bosons.

This process by which  $\xi(x)$  and the photon gain mass is the Higgs mechanism. The massless field is eaten by the photon and adds a longitudinal polarisation, which gives it mass.  $\xi(x)$  is the Higgs field, and the gauge choice made above is known as the unitary gauge [27]. The *Higgs mode* then is when gauge bosons acquire mass at the expense of Goldstone bosons.

## 2.2 Nielsen-Olesen String

One dimensional topological defects - strings, or vortices - can form when the first fundamental group of the vacuum manifold associated with SSB is non-trivial (this will be discussed in more detail in section 2.3). The first string solutions were found in Nielsen and Olesen in 1973 [29]. The Abelian-Higgs model admits such topological defects as solutions. To start, recall the Abelian-Higgs (AH) Lagrangian, given by eq. (2.16)

$$L = (\bar{D}_\mu \bar{\phi})(D^\mu \phi) - \mu^2 \bar{\phi}\phi - \lambda|\phi|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

and SSB occurs when  $\mu^2 < 0$ . The vacuum state has Higgs field take on a value which minimises the potential:

$$\phi_0 = \pm \frac{v}{\sqrt{2}} \quad (2.24)$$

Since this corresponds to an infinite number of degenerate values, the vacuum manifold is a circle - i.e.  $\mathcal{M} = S^1$  [16]. The U(1) transformation encodes the hidden or broken symmetry by allowing a mapping between all the points on  $\mathcal{M}$ :

$$\phi(x) \rightarrow e^{iq\alpha(x)}\phi(x) \quad (2.25)$$

This allows us to construct the string solution: See figure 2.3. The vacuum field  $\phi$  lives in a 2D plane, and is a function of  $(r, \theta)$ . Absorbing the constant  $q$  into  $\alpha(x)$ , and relabeling  $\alpha \rightarrow \theta$  and  $\frac{v}{\sqrt{2}} \rightarrow \eta$ , the

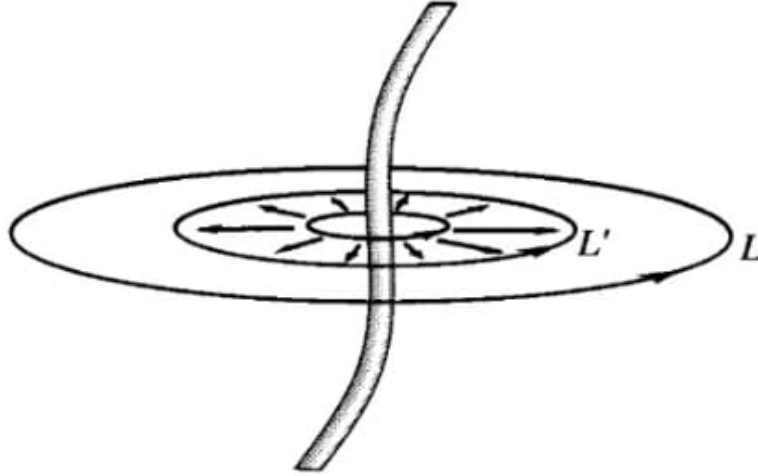


Figure 2.3: As  $\theta$  goes from 0 to  $2\pi$ ,  $\phi$  will pass through all points in  $\mathcal{M}$ . This gives the field (2.26) a winding number of 1. As we decrease the value of  $\eta$ , the manifold will approach the point  $\phi = 0$  [16]. By translational symmetry, the locus of all such points along the vertical line  $V(\phi = 0)$  will form a line with a non-zero energy density: that is, a string [29]. Image taken from [3].

allowed values for the vacuum field in  $\mathcal{M}$  are

$$\phi(r, \theta) = \eta e^{i\theta} \quad (2.26)$$

Since the energy configuration of  $\mathcal{M}$  is such to minimise the energy of the field, the width of the string is finite, and is on the order of [16]

$$w \sim \lambda^{-1/2} \eta^{-1} \quad (2.27)$$

which implies, by dimensional analysis, that the mass per unit length of the string (i.e. the energy density, i.e. the tension), is

$$\mu_0 \sim \eta^2 \quad (2.28)$$

These defects are stable. They are characterised by a topological winding number,  $n$ , which is quantised (only integer values are allowed), and the potential energy is bounded from below.

## 2.3 Homotopy theory and topological criteria

The formation of the NO string is due to the breaking of the  $U(1)$  symmetry group. The AH theory admits specific vortex solutions. It is possible to classify topological defects more generally. The key piece of information one requires is the homotopy group of the vacuum manifold. For a theory described by a Lagrangian

$$\mathcal{S} = \int d^d x \mathcal{L}, \quad (2.29)$$

the theory can obey some symmetry operation  $\hat{U}$ . This leaves the theory invariant, i.e.  $\hat{U}(\mathcal{S}) = \mathcal{S}$ . The symmetry operator(s) are part of an algebraic group  $G$ . If a phase transition occurs, then some level of the symmetry operation will be broken. Only a subset of the operators will leave the theory invariant. This subset now belongs to a group  $H$ , and is the symmetry group of the now phase transitioned theory.

When a phase transition occurs, the fields (i.e. the main dynamical variables) in the theory will fall into a new minimum configuration,  $\phi_0$  (this will, as we have seen in section 2.1, be different from the minimum field configuration prior to the phase transition). Applying  $H$  to  $\phi_0$  leaves  $\phi_0$  invariant:  $H(\phi_0) \rightarrow \phi_0$ .

Moreover, the possible confirmations for  $\phi_0$  are all related by the broken generators of the larger set  $G$ .

We can define a coset space of operators,

$$\mathcal{M} = G/H. \quad (2.30)$$

This is a collection of all the field points which are invariant under  $H$  but are mapped to each other by  $G$ .  $\mathcal{M}$  is called the *vacuum manifold*. The homotopy group of the vacuum manifold determines which type of defect may form [3, 16]. Let's apply this rather abstract discussion to a specific example.

The AH theory (2.16) obeys the symmetry group  $G$ , where the elements are rotations about the plane  $e^{i\theta}$ . We identified  $G$  as the group  $U(1)$ . Any point can be mapped to itself by rotating around by an angle of  $2\pi$ . Hence, elements of the group  $H$  are  $e^{i2\pi} = 1$ . Hence the vacuum manifold is a circle of values, mapped to each other by rotations in the complex  $\phi$  plane (the group  $G$ ), and each value is mapped to itself by a full rotation (the group  $H$ ). Hence, topologically,  $\mathcal{M} = \mathcal{S}^1$ , a circle.

This topology tells us which defects form, and how. We can contract a circle down to (almost) a point. We cannot shrink it down to an actual point, for then the manifold would not be simply connected, something we require [3]. Since we cannot do this, the first homotopy group<sup>2</sup> is non-trivial:  $\pi_1(\mathcal{S}^1) \neq I$ . This implies a non-zero winding number, and so energy can be trapped in the middle of the circle, and a line defect (cosmic string) will form (cf. section 2.2).

## 2.4 Cosmological Phase Transitions

Given some model, say the AH model (eq. (2.16)), and knowledge of the vacuum manifold of the model, the question still remains: how might defect formation be realised in the physical world? If we change some parameter to induce a phase transition, how might this take place in a cosmological setting?

This question was asked by Kibble in the 70s [30]. He and Zurek independently put forward the same mechanism, the Kibble-Zurek mechanism (KZM), which form topological defects in cosmological contexts. Zurek's motivation is particularly noteworthy given that he considers how defects *are known to form in terrestrial experiments* [16, 31]. Superfluid helium, when cooled below a critical temperature, forms defects [31]. This peculiar fact gives credence - although perhaps not in a philosophically satisfying way - that cosmological defects, in the form of cosmic strings, might exist.

The intuitive way to think about the KZM is to consider a lake that freezes over. Before the lake freezes, the water's temperature is not exactly homogeneous. Some parts of the water's surface will be slightly hotter than other parts. When the lake freezes over, all parts of the surface drop below the freezing point of water, but some parts freeze slightly before others. Since the water begins to freeze in many places independently, the sheets of ice will spread out from many starting points and connect, leaving different "coloured" boundaries between them (cf. figure 2.4). The cracks in the ice can be thought of as the trapped tubes of energy when the universe cools from expansion [3].

With this informal picture in mind, we can unpack the details of the KZM a bit more. Consider for definiteness the AH model. The KZM [30] describes the formation of CSs in a system driven by a continuous phase transition at a finite rate. For the AH model, this corresponds to the transition from the  $\mu^2 > 0$  phase to the  $\mu^2 < 0$  phase. Since  $\mu^2 \sim \eta^4$ , this corresponds to the energy of the universe dropping below some threshold scale,  $\eta^4$ . This scale, for most string models, is the GUT energy scale [16]. The KZM mechanism says that if the rate of change of the order parameter  $\mu^2$  is finite, then any system will cease to be adiabatic, allowing the formation of defects<sup>3</sup> as a result of a non-trivial homotopy group describing the region's topological space.

The gist of the argument is based on uncorrelated regions of differing temperature (*a la* the frozen lake). At a time when the temperature of some region is  $T > T_c$  ( $T_c$  is the critical temperature, below which SSB will occur), the VEV of the Higgs field is  $\phi_0 = 0$ . After SSB, when  $T < T_c$ , the Higgs field will acquire a VEV of  $\phi_0 = v/\sqrt{2}$  (cf. section 2.1). The exact value of some collection of  $\phi$ s, i.e. the value of  $\langle \phi(\mathbf{x}) \rangle$ , will be correlated with the values of  $\langle \phi(\mathbf{x}) \rangle$  in a physically close region of space. This is due to nearby points in space's being in thermal contact. These regions are related by the correlation function (in spherical coordinates and natural units):

<sup>2</sup>Recall: homotopic functions or curves are those which can be physically deformed to look like the other without "tearing" the curve.

<sup>3</sup>cf. figure 2.3



Figure 2.4: A frozen lake. The white cracks in the ice are analogous to defect formation as the universe cools and cosmic strings are ‘frozen in’. The cracks are all formed, initially, in casually separated regions.

$$\begin{aligned}
 G(r) &= \langle \phi(r_1)\phi(r_2) \rangle \\
 &= \begin{cases} \frac{T_c}{4\pi r} e^{-r/\xi}, & r \gg \xi \\ \frac{T_c^2}{2\pi}, & r \ll \xi. \end{cases} \quad (2.31)
 \end{aligned}$$

The variable  $\xi$  defines the correlation length between regions. Clearly, if the radial distance between two field values is too large (and we already have  $r \gg \xi$ ), then eq. (2.31) tends to 0. The configurations of the Higgs fields in those regions, region 1 and region 2, will likely be different. The reason they will likely be different is because when the SSB occurs, and the field settles into a new VEV, it can settle into any one of an infinite number of degenerate vacua, and the point it rolls into in the potential hill (cf. figure 2.2) is itself random. And so when the two *uncorrelated* points reconnect spatially as the universe expands, the differing values will form the defect - much like how the differing configurations of two ice pieces freezing over on a lake’s surface connect to form the “crack” between them. The different field configurations corresponds to the homotopy group’s being non-trivial, which is the topological condition allowing a line defect’s - i.e. a cosmic string’s - formation.

## Chapter 3

# Nambu-Goto Dynamics

We review the dynamics of cosmic strings. Starting with the AH model, we derive the Nambu-Goto (NG) action, used for describing zero-thickness gauge strings (independent of the original model, AH or otherwise). We derive the equations of motion and examine the solutions.

We will discuss in detail the Nambu-Goto action. This describes a one-dimensional object, which will trace out a worldsheet - a 2D surface - in a larger background 4D spacetime. The background (or ambient) spacetime has a metric  $g_{\mu\nu}$ , with coordinates  $x^\mu$ . The worldsheet will have coordinates  $\xi^a$ , where  $a = (0,1)$ . Thus, for paths traced out by the string, the spacetime coordinates will be functions of the worldsheet coordinates,  $x^\mu = x^\mu(\xi^a)$ . For definiteness, we'll say  $\xi^0$  is the timelike vector and  $\xi^1$  is the spacelike vector; the latter will label points along the string.

The worldsheet metric is constructed from the background one, and is defined as the pullback of  $g_{\mu\nu}$ :

$$\gamma_{ab} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \xi^a} \frac{\partial x^\nu}{\partial \xi^b} = g_{\mu\nu} \partial_a x^\mu \partial_b x^\nu \quad (3.1)$$

The worldsheet metric can contract and raise/lower indices, and therefore shares the property with the background metric that:

$$\gamma_{ab} \gamma^{bc} = \delta^a_c \quad (3.2)$$

### 3.1 Nielsen-Olsen String Dynamics

Using our relabeling of  $\frac{v}{\sqrt{2}} \rightarrow \eta$ , the AH Lagrangian can be written as

$$\mathcal{L} = (\bar{D}_\mu \bar{\phi})(D^\mu \phi) - \frac{1}{4} \lambda (|\phi|^2 - \eta^2)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad (3.3)$$

This admits two equations of motion, one for  $\phi$  and one for  $A_\mu$  (technically three, with one for  $\bar{\phi}$  as well, but because of the square terms, it will be identical to the  $\phi$  one).

We can find the equation of motion for  $\phi$  by applying the Euler-Lagrange equations with respect to  $\bar{\phi}$ :

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\phi})} \right) = \frac{\partial \mathcal{L}}{\partial \bar{\phi}} \quad (3.4)$$

We expand the covariant derivative term:

$$\begin{aligned} \bar{D}_\mu \bar{\phi} D^\mu \phi &= (\partial_\mu + iqA_\mu) \bar{\phi} (\partial^\mu - iqA^\mu) \phi \\ &= (\partial_\mu \bar{\phi} + iqA_\mu \bar{\phi}) (\partial^\mu \phi - iqA^\mu \phi) \\ &= \partial_\mu \bar{\phi} \partial^\mu \phi - iqA^\mu \bar{\phi} \partial_\mu \phi + iqA_\mu \bar{\phi} \partial^\mu \phi + q^2 A_\mu A^\mu \bar{\phi} \phi \end{aligned} \quad (3.5)$$

Since this term is the only one in which a  $\partial_\mu \bar{\phi}$  appears, the LHS of the Euler-Lagrange equation becomes:

$$\begin{aligned} \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} \right) &= \partial_\mu (\partial^\mu \phi - iqA^\mu \phi) \\ &= \partial_\mu \partial^\mu \phi - iq\phi \partial_\mu A^\mu - iqA^\mu \partial_\mu \phi \end{aligned} \quad (3.6)$$

The RHS is a differentiation with respect to  $\bar{\phi}$ . The field tensor term has no scalar field coupling, which means the  $\phi$ -dependent Lagrangian is:

$$\begin{aligned} \mathcal{L}[\partial_\mu \phi, \phi] &= \partial_\mu \bar{\phi} \partial^\mu \phi - iqA^\mu \phi \partial_\mu \bar{\phi} + iqA_\mu \bar{\phi} \partial^\mu \phi + q^2 A_\mu A^\mu \bar{\phi} \phi - \frac{1}{4} \lambda (\phi \bar{\phi} - \eta^2)^2 \\ \therefore \frac{\partial \mathcal{L}}{\partial \bar{\phi}} &= iqA_\mu \partial^\mu \phi + q^2 A_\mu A^\mu \phi - \frac{1}{2} \lambda \phi (\phi \bar{\phi} - \eta^2) \end{aligned} \quad (3.7)$$

Putting them together,

$$\partial_\mu \partial^\mu \phi - iq\phi \partial_\mu A^\mu - iqA^\mu \partial_\mu \phi = iqA_\mu \partial^\mu \phi + q^2 A_\mu A^\mu \phi - \frac{1}{2} \lambda \phi (\phi \bar{\phi} - \eta^2)$$

Rearranging,

$$\partial_\mu \partial^\mu \phi - iq\phi \partial_\mu A^\mu - iqA^\mu \partial_\mu \phi - iqA_\mu \partial^\mu \phi - q^2 A_\mu A^\mu \phi = -\frac{1}{2} \lambda \phi (\phi \bar{\phi} - \eta^2)$$

The terms on the LHS group together to give  $D_\mu D^\mu \phi$ . Hence the equation of motion for  $\phi$  is

$$D_\mu D^\mu \phi + \frac{1}{2} \lambda \phi (\phi \bar{\phi} - \eta^2) = 0 \quad (3.8)$$

The second equation of motion is in terms of the field  $A_\mu$

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = \frac{\partial \mathcal{L}}{\partial A_\nu} \quad (3.9)$$

We first note that the potential term has no derivatives of the gauge fields, nor do the covariant terms. Hence the inner term on the LHS becomes

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= \frac{\partial}{\partial(\partial_\mu A_\nu)} \left( -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} \right) \\ &= -\frac{1}{4} \frac{\partial}{\partial(\partial_\mu A_\nu)} \left( (\partial_\alpha A_\beta - \partial_\beta A_\alpha) (\partial^\alpha A^\beta - \partial^\beta A^\alpha) \right) \\ &= -\frac{1}{4} \frac{\partial}{\partial(\partial_\mu A_\nu)} \left( \partial_\alpha A_\beta \partial^\alpha A^\beta - \partial_\alpha A_\beta \partial^\beta A^\alpha - \partial_\beta A_\alpha \partial^\alpha A^\beta + \partial_\beta A_\alpha \partial^\beta A^\alpha \right) \\ &= -\frac{1}{4} \frac{\partial}{\partial(\partial_\mu A_\nu)} \left( 2\partial_\alpha A_\beta \partial^\alpha A^\beta - 2\partial_\alpha A_\beta \partial^\beta A^\alpha \right) \\ &= -\frac{1}{2} \frac{\partial}{\partial(\partial_\mu A_\nu)} \left( \partial_\alpha A_\beta (\partial^\alpha A^\beta - \partial^\beta A^\alpha) \right) \end{aligned}$$

We swapped dummy indices of the second and fourth term in the third line to get the second last line. We now have:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\frac{1}{2} \left( (\partial^\alpha A^\beta - \partial^\beta A^\alpha) \frac{\partial(\partial_\alpha A_\beta)}{\partial(\partial_\mu A_\nu)} + (\partial_\alpha A_\beta) \frac{\partial}{\partial(\partial_\mu A_\nu)} (\partial^\alpha A^\beta - \partial^\beta A^\alpha) \right) \\
&= -\frac{1}{2} \left( \delta^\mu_\alpha \delta^\nu_\beta (\partial^\alpha A^\beta - \partial^\beta A^\alpha) + (\partial_\alpha A_\beta) \frac{\partial(\partial^\alpha A^\beta)}{\partial(\partial_\mu A_\nu)} - (\partial_\alpha A_\beta) \frac{\partial(\partial^\beta A^\alpha)}{\partial(\partial_\mu A_\nu)} \right) \\
&= -\frac{1}{2} \left( \delta^\mu_\alpha \delta^\nu_\beta (\partial^\alpha A^\beta - \partial^\beta A^\alpha) + (\partial_\alpha A_\beta) g^{\alpha\sigma} g^{\beta\lambda} \frac{\partial(\partial_\sigma A_\lambda)}{\partial(\partial_\mu A_\nu)} - (\partial_\alpha A_\beta) g^{\beta\sigma} g^{\alpha\lambda} \frac{\partial(\partial_\sigma A_\lambda)}{\partial(\partial_\mu A_\nu)} \right) \\
&= -\frac{1}{2} \left( \delta^\mu_\alpha \delta^\nu_\beta (\partial^\alpha A^\beta - \partial^\beta A^\alpha) + \partial_\alpha A_\beta g^{\alpha\sigma} g^{\beta\lambda} \delta^\mu_\sigma \delta^\nu_\lambda - \partial_\alpha A_\beta g^{\beta\sigma} g^{\alpha\lambda} \delta^\mu_\sigma \delta^\nu_\lambda \right) \\
&= -\frac{1}{2} \left( \delta^\mu_\alpha \delta^\nu_\beta (\partial^\alpha A^\beta - \partial^\beta A^\alpha) + \partial_\alpha A_\beta g^{\alpha\mu} g^{\beta\nu} - \partial_\alpha A_\beta g^{\beta\mu} g^{\alpha\nu} \right) \\
&= -\frac{1}{2} \left( \delta^\mu_\alpha \delta^\nu_\beta (\partial^\alpha A^\beta - \partial^\beta A^\alpha) + \partial_\alpha A_\beta (g^{\alpha\mu} g^{\beta\nu} - g^{\beta\mu} g^{\alpha\nu}) \right) \\
&= -\frac{1}{2} (\partial^\mu A^\nu - \partial^\nu A^\mu + \partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -\frac{1}{2} (2\partial^\mu A^\nu - 2\partial^\nu A^\mu) \\
&= -(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -F^{\mu\nu}
\end{aligned}$$

Inserting this into the Euler-Lagrange equation,

$$\therefore \partial_\mu F^{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial A_\nu} \quad (3.10)$$

We now compute the RHS. The field tensor term and potential both have no dependency on  $A_\mu$ . So we are left with

$$\frac{\partial \mathcal{L}}{\partial A_\nu} = \frac{\partial}{\partial A_\nu} (\partial_\mu \bar{\phi} \partial^\mu \phi - iq A^\mu \phi \partial_\mu \bar{\phi} + iq A_\mu \bar{\phi} \partial^\mu \phi + q^2 A_\mu A^\mu \bar{\phi} \phi) \quad (3.11)$$

The first term is dies and we have

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial A_\nu} &= -iq \phi \partial_\mu \bar{\phi} \frac{\partial A^\mu}{\partial A_\nu} + iq \bar{\phi} \partial^\mu \phi \frac{\partial A_\mu}{\partial A_\nu} + q^2 A^\mu \bar{\phi} \phi \frac{\partial A_\mu}{\partial A_\nu} + q^2 A_\mu \bar{\phi} \phi \frac{\partial A^\mu}{\partial A_\nu} \\
&= -iq \phi \partial_\mu \bar{\phi} \delta^\nu_\lambda g^{\lambda\mu} + iq \bar{\phi} \partial^\mu \phi \delta^\nu_\mu + q^2 A^\mu \bar{\phi} \phi \delta^\nu_\mu + q^2 A_\mu \bar{\phi} \phi \delta^\nu_\lambda g^{\mu\lambda} \\
&= -iq \phi \partial^\nu \bar{\phi} + iq \bar{\phi} \partial^\nu \phi + 2q^2 A^\nu \bar{\phi} \phi \\
&= -q \text{Im}(\bar{\phi} D^\nu \phi) \\
&= j^\nu
\end{aligned} \quad (3.12)$$

The equation of motion for  $A_\mu$  is thus

$$\partial_\mu F^{\mu\nu} = j^\nu \quad (3.13)$$

## 3.2 An effective string action

The NO string is an example of a gauge string. Here we will show that one can apply effective field theory techniques to the AH model and derive a general action for string dynamics. The procedure here closely follows that done in [3]. At low temperatures, massive degrees of freedom can be integrated out, since they will have negligible impact on the dynamics [3]. We can do this to the AH action and recover the so-called Nambu-Goto (NG) action. Crucially the result of the NG is model-independent [3] provided the original string model (in our case, the AH) is gauged.

The AH action is given in somewhat generic form by

$$\mathcal{S} = \int d^4 y ||g||^{1/2} \left\{ |D_\mu \phi|^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - V(\phi) \right\} \quad (3.14)$$

This action's equations of motion are given by eqs (3.8) and (3.13). The static solutions to the equation of motions are, in cylindrical coordinates,

$$\phi_s(\mathbf{r}) = e^{in\theta} f(r) \quad (3.15)$$

$$A_{s,a}(\mathbf{r}) = -\epsilon_{ab} x_b \frac{n}{er^2} \alpha(r) \quad (3.16)$$

The subscript  $s$  means 'static'. These equations are an ansatz for such a field configuration. The indices (a, b) in (3.16) range from 1 to 2, which refer to either the  $r$  coordinate or  $\theta$  coordinate solutions. Therefore

$$A_{s,r}(\mathbf{r}) = \theta \frac{n}{er^2} \alpha(r)$$

$$A_{s,\theta}(\mathbf{r}) = \frac{-n}{er} \alpha(r)$$

On the string worldsheet, there will exist a 4-vector  $X^\mu(\xi^a)$ , that will be in the direction from the origin to a point on the string. The worldsheet can be arbitrarily curved. We can construct a solution to eqs (3.15) and (3.16) by defining a vector  $n_\mu^A$ , ( $A = 1, 2$ ) to be a spacelike vector on the worldsheet such that

$$n_\mu^A \partial_a X^\mu = 0 \quad (3.17)$$

This vector must also be orthonormal,

$$g^{\mu\nu} n_\mu^A n_\nu^B = -\delta^{AB} \quad (3.18)$$

With these choices, we can then write any worldsheet vector  $Y^\mu$  as

$$Y^\mu(\xi^a) = X^\mu + \rho^A n_A^\mu \quad (3.19)$$

The reparametrisation can be seen in figure 3.1. The solutions (3.15) and (3.16) in terms of  $Y^\mu(\xi^a)$  now become  $Y^\mu$ :

$$\phi_s(Y^\mu) = \phi_s(\mathbf{r}) \quad (3.20)$$

$$A^\mu(Y^\mu) = n_B^\mu A_{s,B}(\mathbf{r}) \quad (3.21)$$

Since we've changed coordinate system, we must now find the Jacobian from  $\xi^\mu \rightarrow Y^\mu$ :

$$\|g\|^{1/2} \det \left( \frac{\partial Y^\mu}{\partial \xi^\alpha} \right) = \|M\|^{1/2} \quad (3.22)$$

where

$$M_{\alpha\beta} = g_{\mu\nu} \frac{\partial Y^\mu}{\partial \xi^\alpha} \frac{\partial Y^\nu}{\partial \xi^\beta} \quad (3.23)$$

This is the induced metric in the  $Y^\mu$  coordinate system. Using the definition (3.19), this becomes

$$M_{\alpha\beta} = \text{diag}(\gamma_{ab}, -\delta_{AB}) + \mathcal{O}(r/R) \quad (3.24)$$

Here  $\gamma_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$  is the induced worldsheet metric in terms of  $\xi^a$ , and  $(r/R)$  are higher-order curvature terms. Since  $\det(\delta_{AB}) = 0$ ,  $\det(M) \approx \det(\gamma)$ , we have

$$\|M\|^{1/2} \approx \|\gamma\|^{1/2} \quad (3.25)$$

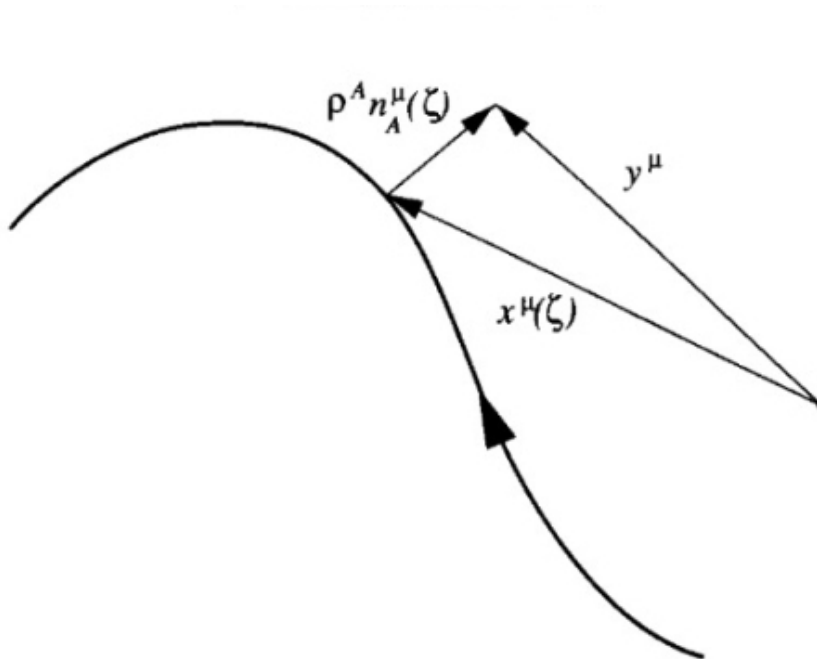


Figure 3.1: A diagram of how we have constructed (3.19). Taken from [3].

Since we are taking the approximation to lowest order, the terms in the AH action (3.14) become

$$\mathcal{S} \approx \int d^4x \|\gamma\|^{1/2} \left\{ |D_\mu \phi_s|^2 - \frac{1}{4} F_s^2 + V(\phi) \right\} \quad (3.26)$$

Crucially, the 4-volume element is given by  $d^4x = d^2\xi \wedge d^2\rho$ . Integration over  $d\rho^A d\rho^B$ , which we identified earlier as the radial coordinate, gives the integral

$$\mathcal{S} \approx \int d^2\xi \|\gamma\|^{1/2} \int d^2r \left\{ |D_\mu \phi_s|^2 - \frac{1}{4} F_s^2 + V(\phi) \right\} \quad (3.27)$$

since the worldsheet metric only depends on  $\xi^a$ . We identify the second integral as the negative of the energy per unit length of the string,  $-\mu_0$ . Therefore the action reduces to

$$\mathcal{S}_{\mathcal{NG}} = -\mu_0 \int d^2\xi \|\gamma\|^{1/2} \quad (3.28)$$

which is the Nambu-Goto action, a general action for an infinitely thin string.

### 3.3 Nambu-Goto dynamics

The equations of motion are considered here for a Minkowski spacetime:  $g_{\mu\nu} = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ . In this case, the worldsheet metric (3.1) becomes:

$$\gamma_{ab} = \begin{pmatrix} (\dot{x})^2 & \dot{x} \cdot x' \\ \dot{x} \cdot x' & (x')^2 \end{pmatrix} \quad (3.29)$$

a dot indicates a derivative with respect to  $\xi^0$ , a prime with respect to  $\xi^1$ . The determinant is then

$$\gamma = (\dot{x})^2 (x')^2 - (\dot{x} \cdot x')^2 \quad (3.30)$$

Our Nambu-Goto action (3.28) becomes

$$\mathcal{S}_{\mathcal{NG}} = -m^2 \int d^2\xi \sqrt{(\dot{x} \cdot x')^2 - (\dot{x})^2 (x')^2} \quad (3.31)$$

Here  $m^2$  is the Kibble mass, and is effectively a normalisation constant, which is of the same order as  $\mu_0$ . Varying this action is quite messy, but there is a useful calculation trick to find the equations of motion without pages of algebra. The Euler-Lagrange equations of motion are

$$\partial_a \left( \frac{\partial \mathcal{L}}{\partial(\partial_a x^\mu)} \right) = \frac{\partial \mathcal{L}}{\partial x^\mu} \quad (3.32)$$

The determinant has no dependency on  $x^\mu$ , so we have

$$\partial_a \left( \frac{\partial \mathcal{L}}{\partial(\partial_a X^\mu)} \right) = 0 \quad (3.33)$$

We note that

$$\delta \sqrt{-\gamma} = -\frac{1}{2} \sqrt{-\gamma} \gamma^{ab} \delta \gamma_{ab} \quad (3.34)$$

We then can calculate:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial(\partial_a x^\mu)} &= \frac{\partial \mathcal{L}}{\partial \gamma_{ab}} \frac{\partial \gamma_{ab}}{\partial(\partial_a x^\mu)} \\ &= \frac{-1}{2} \|\gamma\|^{1/2} \gamma^{ab} \eta_{\mu\nu} \partial_b x^\nu \end{aligned}$$

Contracting over  $\nu$ , and substituting the result into (3.33), we find

$$\partial_a \left( \|\gamma\|^{1/2} \gamma^{ab} \partial_b x_\mu \right) = 0 \quad (3.35)$$

This is the equation of motion in flat space, in terms of the worldsheet metric.

### 3.4 Gauge fixing from constraints

Gauge fixing will become very important when we consider cusp prevention in wiggly strings, and so here we present a discussion of the process for NG strings, in order to apply the procedure to wiggly ones. In order to fix a gauge sensibly, it helps to know how the dynamics are constrained. To do this, we can compute the energy-momentum tensor from the Lagrangian in eq. (3.28). The energy-momentum tensor is defined in general as

$$T_{\mu\nu} = \frac{-2}{m^2} \frac{1}{\|g\|^{1/2}} \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \quad (3.36)$$

Since the background metric is buried in the square root of the NG action, it is computationally easier to use the Polyakov action, an alternate formulation of the NG action (which produces identical dynamics at the classical level - see appendix A). The Polyakov action is given by

$$\mathcal{S}_p = \frac{-m^2}{2} \int d^2 \xi \|h\|^{1/2} h^{ab} \partial_a x^\mu \partial_b x_\mu \quad (3.37)$$

The metric  $h_{ab}$  is an auxiliary worldsheet metric, and is not the same as the one in eq. (3.1) (i.e. it is not the pullback of the ambient metric). With eq. (3.37) we can calculate the energy-momentum tensor of the NG string :

$$\frac{\delta \mathcal{S}}{\delta h^{ab}} := T_{ab} = \frac{-m^2}{2} \left( \|h\|^{1/2} \partial_a x^\mu \partial_b x_\mu - \frac{1}{2} \|h\|^{1/2} h_{ab} h^{ij} \partial_i x^\mu \partial_j x_\mu \right). \quad (3.38)$$

From the principle of least action, the constraints on the physical dynamics require that  $\delta \mathcal{S} / \delta h^{ab} = 0$ , and therefore

$$T_{ab} = \partial_a x^\mu \partial_b x_\mu - \frac{1}{2} h_{ab} h^{ij} \partial_i x^\mu \partial_j x_\mu = 0 \quad (3.39)$$

The NG action (3.28) obeys various symmetries: diffeomorphism invariance and Poincare invariance. This symmetry allows us to redefine worldsheet coordinates as  $\xi^a \rightarrow \tilde{\xi}^a(\xi)$  (diffeomorphism) and also that the

laws of special relativity hold for worldsheet objects (Poincare). Furthermore, we can set the auxiliary metric  $h_{ab} = \eta_{ab}$ , since the Polyakov actions also has an additional symmetry: Weyl invariance. Setting  $h_{ab} = \eta_{ab}$  in (3.39) gives us an equation of motion which reads

$$T_{ab} = 0 \implies \begin{cases} T_{01} = \dot{x} \cdot x' = 0 \\ T_{00} = T_{11} = (\dot{x})^2 + (x')^2 = 0 \end{cases} \quad (3.40)$$

The choice of parametrisation for the auxiliary metric gives rise to the set of constraints in eq. (3.40). This choice is the *conformal gauge*, since the worldsheet metric (3.1) now becomes  $\gamma_{ab} = \sqrt{-\gamma} \eta_{ab}$ , which is conformally flat (hence the name “conformal” gauge). The inverse metric is therefore

$$\gamma^{ab} = \frac{1}{\sqrt{-\gamma}} \eta^{ab} \quad (3.41)$$

One can simply pick the conditions (3.40) without needing the energy-momentum tensor at all, as done in ref. [3]. Doing it in the above way is useful, since it only requires using symmetries of the action, and functions as a derivation of the gauge conditions as constraints rather than something put in “by hand”. Additionally, this procedure will prove fruitful when considering more complicated string actions when the allowed gauge choices are not so obvious.

With this identification, the equation of motion (3.35) becomes

$$\begin{aligned} \partial_a \left( \sqrt{-\gamma} \gamma^{ab} \partial_b x_\mu \right) &= 0 \\ \partial_a \left( \eta^{ab} \partial_b x_\mu \right) &= 0 \\ \partial_a \partial^a x_\mu &= 0 \end{aligned}$$

we can express this using  $\square_2$ , the 2-dimensional d’Alembertian, defined by

$$\square_2 = \frac{\partial^2}{(\partial \xi^0)^2} - \frac{\partial^2}{(\partial \xi^1)^2} := \partial^2_0 - \partial^2_1 \quad (3.42)$$

and so the equation of motion is

$$\square_2 x_\mu = 0 \quad (3.43)$$

## Part II

# Application to Astrophysics

## Chapter 4

# A Fast Primer on Radio Bursts

A little over a decade after their discovery [32], Fast Radio Bursts (FRBs) remain an enigmatic class of radio transients. They are characterised by a very bright ( $\sim$  Jy) and very brief ( $\sim$  ms) burst of radio photons, and have been detected at frequencies ranging between 400 MHz – 8 GHz by a number of ground-based radio telescopes. Another important characteristic is that the arrival time of the frequency components is dispersed, precisely going as  $\Delta t \sim \nu^{-2}$ , which is consistent with the propagation of a radio wave through cold plasma [32–34] (see below).

While some early speculation considered FRBs to be of galactic origin [36–39], current consensus is that their excessively large dispersion measure, high galactic latitude and apparent isotropy over the sky [40] indicate an extragalactic or cosmological origin. This consensus is supported by the identification of the host galaxy of FRB 121102 at redshift  $z=0.1932$  [41, 42]. Lending further support to such origins are the very high Faraday rotation measures recently detected in FRB 121102 [43, 44] and FRB 160102 [45].

One of the central challenges facing theoretical model builders is finding a physical mechanism with which one can explain the vast amount of energy radiated over such short timescales. If one assumes isotropic emission, the large distances over which these bursts propagate seems to imply super-Eddington luminosities ( $\sim 10^{43}$  erg s $^{-1}$ ). This indicates some beamed, coherent emission process is required [46], and the brevity of the signals suggests the source is extremely compact [46]. Compounding the model building challenges, the characteristic properties of FRBs appear to vary. Where measurements have been possible, FRBs have been observed to have circular [45, 47, 48] and/or linear [43, 44, 49, 50] polarizations as well as some that seem unpolarized (although this may be due to extremely high Faraday rotation [43]). The pulse profiles of FRBs also differ: two have double or triple peaks [40, 51], while the rest have only single peaks. Many FRBs have now shown complex microstructure and features at timescales of 10-s of microseconds [51] (and Hessels et al, *in preparation*). Even more baffling is that only one FRB has been observed to repeat [44, 52], with modulating pulse shapes and no apparent periodicity. It is hard to confirm that the other FRBs don't repeat; few have had follow up observations, but some have been monitored for up to hundreds of hours with no indication of repetition [32, 48, 50, 53, 54]. This presents the possibility that there are (at least) two different classes of FRB [43, 55]. The repetition need not be intrinsic to the source of the FRB though: interstellar scintillations [56] or plasma lensing [52, 57, 58] may be accountable. FRB 121102 was found to have a rotation measure 400 times larger than any other known FRB [43, 44], but previous FRBs may have had high rotation measures that were simply not detectable [48, 59]. There is currently no consensus on the matter.

There have only been 36 FRB detections subsequent to the Lorimer burst in 2007<sup>1</sup>, but thankfully the non-detection of FRBs can provide some insight. For example, one can constrain event rates [60–69], spectral indices [64–66], and surface densities of transients [67]. Arguably, the most illuminating detection to date is (the repeating) FRB 121102 [70]. This is the only event to have been successfully associated with a host galaxy: a low-metallicity star-forming dwarf galaxy at a redshift of  $z = 0.193$  [41, 42, 71]. This confirms that (at least some) FRBs propagate over cosmological distances.

Apart from being interesting in and of themselves, FRBs could prove to be very useful cosmological and astrophysical probes. There are already many proposals that use FRBs to study fundamental cosmological parameters [72–78], extragalactic magnetic fields [79], properties of the intergalactic medium (IGM) [80–86], dark matter [87–89], photon mass [90–95], the Equivalence Principle [96–101], the cosmic web [50], cosmic microwave background (CMB) optical length [102], and superconducting cosmic strings [19]. As such, understanding FRBs is incredibly worth while.

The sparsity of data has led to the publication of a plethora of progenitor theories in recent years [11]. These can be broadly broken into two groups: those which appeal to astrophysical mechanisms for which there is some empirical evidence (such as plasma physics); and those which appeal to aspects of (mathematical) physics. The latter are more speculative and/or lack any empirical evidence, but nonetheless

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<sup>1</sup>See the online FRB catalogue [8], found at [www.frbcatalog.org](http://www.frbcatalog.org), for an up-to-date list

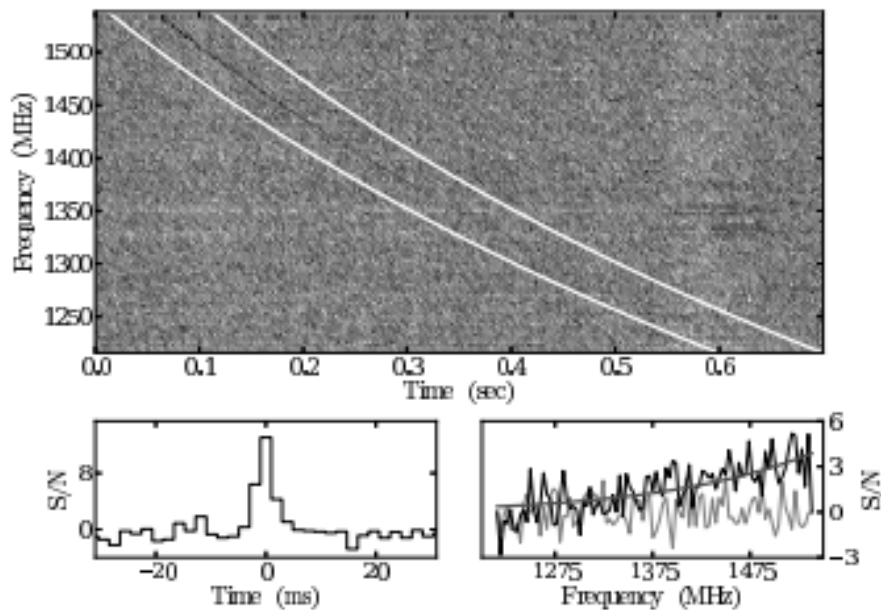


Figure 4.1: Characteristic plots of FRB 121102. The two white curves show the expected sweep for a  $\nu^2$  dispersed signal. Taken from [35].

may be well-motivated from a theoretical standpoint. While the former class may be more appealing to some, here we examine one class of theoretical models in particular: radiation signatures from cosmic strings.

Cosmic strings can emit a variety of electromagnetic signatures through two principle means: cusp decay from neutral strings, and structure collisions from superconducting strings. NG strings generically form cusps, portions of the string which fold back onto themselves and move at the speed of light. The cusps decay, emitting a beam of coherent radiation [4]. The decay particle can ostensibly be of any energy and frequency range, and so should extend down into radio bursts. Cusp decay from cosmic strings has been put forward to explain FRBs [12]. The event rate, timescale, and flux emitted are shown to be consistent with FRB data, however the relativistic effects on the cusp shape was not factored in. As will be seen later, by taking this into account, cusp decay is in fact incompatible with current FRB data [103]. Cosmic strings are not ruled out by observations, and would necessarily include counterparts of other electromagnetic frequencies—specifically, GRBs, cosmic rays [14] and neutrinos [13]—and gravitational waves.

Superconducting cosmic strings (or SCSs) are cosmic strings coupled to electromagnetism. This results from the addition of an electromagnetic gauge field in the string action - the string then exhibits superconducting properties. This can be achieved through the unbroken symmetry of an extra Higgs field in the formation of the string [17]. The first SCS was considered as an FRB model was done in [18]. The main mechanism by which SCSs emit is through oscillation.

Like neutral strings, they SCSs oscillate, and due to their carrying a current, this induces electromagnetic radiation. The radiation is beamed coherently at cusp points [18]. Unlike neutral strings, however, the cusp is a point which beams the radiation; cusp decay itself does not induce the radiation emission. Cosmic strings generically have cusps and kinks present. However, only cusp decay produces coherent radiation in the neutral case. SCSs have richer set of dynamics, and a corresponding increase in ways the superconducting string can emit radiation, other than the typical oscillatory mechanism. The structures on the string can collide (cusp-cusp collisions, kink-kink collisions, and cusp-kink collisions) [20], the string can act as a current carrying loop in a magnetic field (supplied by galaxies) [19], or can emit radiation as a result of oscillatory harmonics [18]. Cusp-cusp collisions are thought to dominate the event rate and flux for SCS emission [20].

The emission from SCSs carries a linear polarisation signature, intrinsic to the emission [104]. This is thus independent of frequency, and is not affected by polarisation via the IGM. This can also be used to distinguish it from other possible astrophysical sources. A linearly polarised signature also can function well with a possible gravitational wave counterpart [105]. The radio frequencies will also extended to higher frequency regimes, including gamma ray emissions as counterparts [20]. There are constraints on the size of SCS loops. As noted in [19], fitting the event rate and emission power to the extant FRB data shows that the loops are likely to ones which have formed in the radiation era - the loop size is constrained to be formed at that time after symmetry breaking. Constraints on the size the loops in future observations can also constrain the loops distributions [12, 103], and therefore how well the emission fit the FRB data.

## Chapter 5

# Cusps as Observational Signatures

In this chapter we review how cusp decay in NG strings produce electromagnetic radiation and can be used as observational signatures. We derive an expression for the cusp solution, and deduce relations used to estimate the power emitted from the cusp region. We then apply this to coherent emission of particles, and review some recent attempts made to explain FRBs using cusp decay. We shall then introduce a modification to the cusp decay model. Since cusps, as we shall see, move at the speed of light, Lorentz boosts will affect the cusp size. This has substantial impact on the characteristics of the radiation emitted. We calculate a collection of observables - distance criterion, event rate, time scale - after factoring in Lorentz boosts of the cusp region and show that cusp decay is incompatible with FRB data.

### 5.1 Cusp Solution

An aspect of NG dynamics that are of particular interest to us are portions of the string known as *cusps*. One may informally think of them as the part of a whip which forms when “cracking” a whip. Cusp formation occurs when certain regions of the string reach the speed of light. This may function as the mathematical definition of a cusp [3]. As we shall see later, this definition requires the NG description of a cosmic string. In this way, cusps may be thought of as artefacts from the zero-thickness approximation [15].

We shall define a cusp as a point on the string at which  $\dot{x}^\mu = 1$  (i.e. speed of light). In this section we derive the expression for a cusp “solution” to (3.35) in the temporal conformal gauge. This can be used to compute power emission estimates from the cusp region, since the relativistic velocity will cause particle emission through cusp decay [15, 16]. The discussion here closely follows ref. [15]. The temporal conformal gauge is the same conformal choice as before, but supplemented by fixing a worldsheet reparameterisation such that  $\xi^0 = x^0 = t$ . We also, from here onwards, define  $\xi^1 = \sigma$ . Our conformal gauge conditions (3.40) take the form

$$\begin{aligned} (\mathbf{x}')^2 + (\dot{\mathbf{x}})^2 &= 1 \\ \mathbf{x}' \cdot \dot{\mathbf{x}} &= 0 \end{aligned} \tag{5.1}$$

where these are now conditions on the 3-vector part of  $x^\mu$ , since now  $\dot{x}^0 = 1$  and  $x^{0'} = 0$ . The criteria for cusps are now

$$\begin{aligned} \dot{\mathbf{x}} &= 1 \\ \mathbf{x}' &= 0 \end{aligned} \tag{5.2}$$

We pick this point to occur at  $(t, \sigma) = (0, 0)$ . We now Taylor expand both  $\dot{\mathbf{x}}$  and  $\mathbf{x}'$  about  $(0, 0)$  up to third order:

$$\begin{aligned} \dot{\mathbf{x}}(\sigma, t) &= \dot{\mathbf{x}}(0, 0) + t\ddot{\mathbf{x}}(0, 0) + \sigma\dot{\mathbf{x}}'(0, 0) + t^2\ddot{\mathbf{x}}(0, 0) + 2\sigma t\ddot{\mathbf{x}}'(0, 0) + \sigma^2\dot{\mathbf{x}}''(0, 0) \\ \mathbf{x}'(\sigma, t) &= \mathbf{x}'(0, 0) + t\dot{\mathbf{x}}'(0, 0) + \sigma\mathbf{x}''(0, 0) + t^2\ddot{\mathbf{x}}'(0, 0) + 2\sigma t\ddot{\mathbf{x}}'(0, 0) + \sigma\mathbf{x}'''(0, 0) \end{aligned}$$

By the cusp definition,  $\dot{\mathbf{x}}(0, 0) = 1$  and  $\mathbf{x}'(0, 0) = 0$ ; therefore

$$\begin{aligned} \dot{\mathbf{x}}(\sigma, t) &= 1 + t\ddot{\mathbf{x}}(0, 0) + \sigma\dot{\mathbf{x}}'(0, 0) + t^2\ddot{\mathbf{x}}(0, 0) + 2\sigma t\ddot{\mathbf{x}}'(0, 0) + \sigma^2\dot{\mathbf{x}}''(0, 0) \\ \mathbf{x}'(\sigma, t) &= t\dot{\mathbf{x}}'(0, 0) + \sigma\mathbf{x}''(0, 0) + t^2\ddot{\mathbf{x}}'(0, 0) + 2\sigma t\ddot{\mathbf{x}}'(0, 0) + \sigma\mathbf{x}'''(0, 0) \end{aligned}$$

To tidy up the notation, we can relabel the first term in the  $\mathbf{x}'(\sigma, t)$  expansion  $\mathbf{a}_0$ , the next one  $\mathbf{a}_1$ , etc, and for the  $\mathbf{x}(\sigma, t)$  expansion, the first term will be  $\mathbf{b}_1$ ,  $\mathbf{b}_2$ , etc. (Since  $b_0 = 0$ ). Thus

$$\dot{\mathbf{x}}(\sigma, t) = \mathbf{a}_0 + t\mathbf{a}_1 + \sigma\mathbf{a}_2 + t^2\mathbf{a}_3 + 2\sigma t\mathbf{a}_4 + \sigma^2\mathbf{a}_5 \quad (5.3)$$

$$\mathbf{x}'(\sigma, t) = t\mathbf{b}_1 + \sigma\mathbf{b}_2 + t^2\mathbf{b}_3 + 2\mathbf{b}_4 + \sigma\mathbf{b}_5 \quad (5.4)$$

We now match up coefficients, since  $\partial^2\mathbf{x}/\partial\sigma^2 = \partial^2\mathbf{x}/\partial t^2$

$$\begin{aligned} \mathbf{a}_2 &= \mathbf{b}_1 \\ \mathbf{a}_4 &= \mathbf{b}_3 \\ \mathbf{a}_5 &= \mathbf{b}_4 \end{aligned}$$

Using the equation of motion (3.43),  $\ddot{x} = x''$ , we can match up more coefficients:

$$\begin{aligned} \mathbf{a}_1 &= \mathbf{b}_2 \\ \mathbf{a}_3 = \mathbf{b}_4 &\implies \mathbf{a}_3 = \mathbf{a}_5 \\ \mathbf{a}_4 &= \mathbf{b}_5 \end{aligned}$$

The gauge condition  $\dot{\mathbf{x}} \cdot \mathbf{x}' = 0$  implies more relations. Up to second order in  $t$  and  $\sigma$  (this means discarding terms like  $\sigma t^2$ ),

$$\begin{aligned} t(\mathbf{a}_0 \cdot \mathbf{a}_2) + \sigma(\mathbf{a}_0 \cdot \mathbf{a}_1) + t^2(\mathbf{a}_0 \cdot \mathbf{a}_4) + 2(\mathbf{a}_0 \cdot \mathbf{a}_3) + \sigma^2(\mathbf{a}_0 \cdot \mathbf{a}_4) + t^2(\mathbf{a}_1 \cdot \mathbf{a}_2) \\ + t\sigma(\mathbf{a}_1 \cdot \mathbf{a}_1) + (\mathbf{a}_2 \cdot \mathbf{a}_2) + \sigma^2(\mathbf{a}_1 \cdot \mathbf{a}_2) = 0 \end{aligned}$$

Grouping the like factors of  $\sigma$  and  $t$ ,

$$t(\mathbf{a}_0 \cdot \mathbf{a}_2) + t^2(\mathbf{a}_0 \cdot \mathbf{a}_4 + \mathbf{a}_1 \cdot \mathbf{a}_2) + \sigma(\mathbf{a}_0 \cdot \mathbf{a}_1) + \sigma^2(\mathbf{a}_0 \cdot \mathbf{a}_4 + \mathbf{a}_1 \cdot \mathbf{a}_2) + (\mathbf{a}_1 \cdot \mathbf{a}_1 + \mathbf{a}_2 \cdot \mathbf{a}_2 + 2(\mathbf{a}_0 \cdot \mathbf{a}_3)) = 0$$

All the terms must individually equal 0, which generates our final set of constraints:

$$\begin{aligned} a_0 \cdot a_1 &= 0 \\ a_0 \cdot a_2 &= 0 \\ a_0 \cdot a_3 &= -\frac{1}{2}(a_1 \cdot a_1 + a_2 \cdot a_2) \\ a_0 \cdot a_4 &= -a_1 \cdot a_2 \end{aligned}$$

Let's now consider expansion (5.3). We can express these vectors in an orthonormal basis ( $\hat{e}_0, \hat{e}_1, \hat{e}_2$ ). The vector  $a_0$  is constant and is of zero order in  $\sigma$  or  $t$ . Therefore.

$$\mathbf{a}_0 = \hat{e}_0. \quad (5.5)$$

The vectors to first order in  $\sigma$  and  $t$  are  $\mathbf{a}_1$  and  $\mathbf{a}_2$ .  $\mathbf{a}_1$  contains only time derivatives and is to first order, and so needs only one extra component to characterise its direction. Let's pick the base  $\hat{e}_1$

$$\mathbf{a}_1 = \alpha\hat{e}_1 \quad (5.6)$$

But  $a_2$  is first order with both time and space derivatives, and thus needs two directions to characterise it. As such, it's in the basis  $\hat{e}_1, \hat{e}_2$ .

$$\mathbf{a}_2 = \beta\hat{e}_1 + \gamma\hat{e}_2 \quad (5.7)$$

Both  $\mathbf{a}_1$  and  $\mathbf{a}_2$  do not have an  $\hat{e}_0$  component because of the constraints: they're orthogonal to  $a_0$ .

Being of second order, the vectors  $\mathbf{a}_3$  and  $\mathbf{a}_4$  would need three directions to characterise them. The  $\hat{e}_0$  component can be obtained from the constraints, since the inner product of  $\mathbf{a}_3$  or  $\mathbf{a}_4$  with  $\mathbf{a}_0$  is non-zero. Hence

$$\begin{aligned}\mathbf{a}_3 &= \frac{-1}{2}(\alpha^2 + \beta^2 + \gamma^2) + \delta\hat{e}_1 + \epsilon\hat{e}_2 \\ \mathbf{a}_4 &= -(\alpha\beta)\hat{e}_0 + \zeta\hat{e}_1 + \sigma\hat{e}_2\end{aligned}$$

We now have the form of all the terms in the Taylor expansion of  $\dot{\mathbf{x}}$  (and  $\mathbf{x}'$ , since the  $\mathbf{a}$ s are related to the  $\mathbf{b}$ s). We can now calculate the form of  $x^\mu$  at the cusp by integrating  $\dot{x}^\mu$  and substituting the Taylor expansion terms in terms of the coefficients,  $\alpha, \beta, \gamma$  etc. This gives, since  $x^0 = t$  by our gauge choice,  $x^\mu = (t, A\hat{e}_0, B\hat{e}_1, C\hat{e}_2)$ , where the coefficients A, B, C are the coefficients matched to each basis vector, after integrating. Therefore,

$$x^\mu_c = \begin{bmatrix} t \\ t - \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)(\frac{1}{3}t^3 + \sigma^2t) - \alpha\beta(t^2\sigma + \frac{1}{3}\sigma^3) \\ \alpha(\frac{1}{2}(\alpha + \beta)) + \beta + \delta(\frac{1}{3}t^3 + \sigma^2t) + \zeta(t^2\sigma + \frac{1}{3}\sigma^3) \\ \gamma + \epsilon(\frac{1}{3}t^3 + \sigma^2t) + \sigma(t^2\sigma + \frac{1}{3}\sigma^3) \end{bmatrix} \quad (5.8)$$

This is the cusp solution to the wave equation up to second order about the point  $(0, 0)$ .

## 5.2 Cusp Annihilation

The cusp solution can be used to calculate an order of magnitude estimate for the power emitted by a process called cusp annihilation. This section follows closely the work done in [4].

The coefficients  $\alpha, \beta, \gamma$  are on the order  $\sim 1/R$ , and  $\delta, \epsilon, \zeta, \sigma$  are on the order  $\sim 1/R^2$  [26]. The cusp region develops a finite width, and is defined as the region for which

$$|x(0, \sigma) - x(0, -\sigma)| \leq w \quad (5.9)$$

where  $w$  is the width, and scales as  $w \sim \eta^{-1}$ . Setting  $t = 0$ , we get

$$\begin{bmatrix} -\frac{1}{3}\alpha\beta\sigma^3 \\ \frac{1}{2}\alpha\sigma^2 + \frac{1}{3}\zeta\sigma^3 \\ \frac{1}{3}\sigma\sigma^3 \end{bmatrix} - \begin{bmatrix} \frac{1}{3}\alpha\beta\sigma^3 \\ \frac{1}{2}\alpha\sigma^2 + \frac{1}{3}\zeta\sigma^3 \\ \frac{1}{3}\sigma\sigma^3 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}\alpha\beta\sigma^3 \\ \frac{2}{3}\zeta\sigma^3 \\ \frac{2}{3}\sigma\sigma^3 \end{bmatrix} \quad (5.10)$$

Taking the norm gives

$$\begin{aligned}|x(0, \sigma) - x(0, -\sigma)| &= \sqrt{\sigma^6 \left( \frac{4}{9}\alpha^2\beta^2 + \frac{4}{3}\zeta^2 + \frac{4}{9}\sigma^2 \right)} \\ &= \frac{2}{3}\sigma^3 \sqrt{\alpha^2\beta^2 + \zeta^2 + \sigma^2}\end{aligned} \quad (5.11)$$

We assume the cusp region is of the same order as the defined maximum region,

$$w \sim \frac{2}{3}\sigma^3 \sqrt{\alpha^2\beta^2 + \zeta^2 + \sigma^2} \quad (5.12)$$

The size of the cusp, i.e. the length along the string,  $l_c := \sigma$  is therefore

$$l_c \sim w^{1/3} (\alpha^2\beta^2 + \zeta^2 + \sigma^2)^{-1/6} \quad (5.13)$$

Since

$$\begin{aligned}\alpha^2\beta^2 &\sim 1/R^4 \\ \zeta^2 &\sim 1/R^4 \\ \sigma &\sim 1/R^4\end{aligned} \quad (5.14)$$

the size of the cusp scales as

$$l_c \sim w^{1/3} R^{2/3} \implies l_c \sim \eta^{-1/3} R^{2/3}. \quad (5.15)$$

Each oscillation of a string loop forms one cusp [16]. The radiated power can be calculated from the ratio of the cusp length to the radius of the loop, times the string tension (for natural units, power is dimensionless). Assuming the entire cusp region decays,

$$P_c \sim \frac{\mu_0 l_c}{R} \sim \eta^2 (\eta R)^{-1/3} \quad (5.16)$$

This is an upper bound of emitted radiation from cusp annihilation.

### 5.3 Energy scaling of the decay process

Here we fully derive the energy scaling relation for the particles which are produced in cusp decay. We explicate the calculations done in ref. [13].

The number distribution for quark jets - and we assume that the cusps will produce quarks and fermions which will decay into photons - is given, as a zeroth order moment, by some complex expression which scales as  $\sqrt{E}$ . An identically scaled but much simpler distribution, which they use as a proxy, is given by

$$\frac{dN}{dx} = \frac{15}{16} x^{-3/2} (1-x)^2 \quad (5.17)$$

Here  $x = E/E_f$ , where  $E$  is the variable energy after fragmentation and  $E_f$  is the initial energy of the quarks released by the cusps. Multiplying by the invariant measure  $dx/x$ , we have

$$\begin{aligned} \frac{dN}{x} &= \frac{15}{16} x^{-5/2} (1-x)^2 dx \\ \frac{1}{x} \int_0^N dN' &= \frac{15}{16} \int_{E/E_f}^1 \frac{15}{16} x^{-5/2} (1-x)^2 dx \\ \frac{1}{x} \int_0^N dN' &= \frac{15}{16} \int_{E/E_f}^1 \frac{15}{16} x^{-5/2} (1-x)^2 dx \\ \frac{N}{x} &= -\frac{5}{8} \left( \frac{E}{E_f} \right)^{-3/2} \left( 3 \left( \frac{E}{E_f} \right)^2 + 6 \left( \frac{E}{E_f} \right) - 1 \right) - 5 \\ \frac{N}{x} &= \frac{15}{16} \left( \frac{16}{3} - 2 \left( \frac{E}{E_f} \right)^{1/2} - 4 \left( \frac{E}{E_f} \right)^{-1/2} + \frac{2}{3} \left( \frac{E}{E_f} \right)^{-3/2} \right) \\ N(E) &= \frac{E}{E_f} \frac{15}{16} \left( \frac{16}{3} - 2 \left( \frac{E}{E_f} \right)^{1/2} - 4 \left( \frac{E}{E_f} \right)^{-1/2} + \frac{2}{3} \left( \frac{E}{E_f} \right)^{-3/2} \right) \end{aligned} \quad (5.18)$$

Since the energy of a loop goes as  $E \sim \frac{\mu_0 l_c}{E_f}$ , we have

$$N(E) \propto \frac{\mu_0 l_c}{E_f^2} \left[ 1 - \left( \frac{E}{E_f} \right)^{1/2} - \left( \frac{E}{E_f} \right)^{-1/2} + \left( \frac{E}{E_f} \right)^{-3/2} \right] \quad (5.19)$$

We now assume that the energy which the photons have will be far less than the initial fragmentation energy, i.e.  $E \ll E_f$ . This means that the  $E^{1/2}$  term is negligible, and furthermore that

$$\left( \frac{E_f}{E} \right)^{3/2} \gg 1 - \left( \frac{E_f}{E} \right)^{1/2} \quad (5.20)$$

and therefore

$$N(E) \sim \frac{\mu_0 l_c}{E_f^2} \left( \frac{E_f}{E} \right)^{3/2} \quad (5.21)$$

The number of photons that strike the detector a distance  $d$  from Earth, per solid angle beam, is

$$N(E) \sim \frac{1}{d^2 \Theta^2} \frac{\mu_0 l_c}{E_f^2} \left( \frac{E_f}{E} \right)^{3/2} \quad (5.22)$$

The energy distribution hence scales as  $N \propto E^{-3/2}$ .

## 5.4 Lorentz boosts and generic scaling of cusps

The energy distribution (5.22) is derived from both an assumed decay distribution and is crucially dependent on the length of the cusp region. This is the region that effectively contributes to particle decay. The cusp length (5.15) is calculated by expanding about the cusp point. However, when imposing the cusp criteria to derive an expression for that point (cf. eq. (5.8)), we implicitly assumed that the cusp's length is unaffected by Lorentz contractions. Lengths and times are, however, affected by near-luminal velocities, and so a question emerges: what does a cusp look like - specifically, how does its length change - when one factors in relativistic corrections?

In this section we consider how cusp expansions are modified by Lorentz boosts, and show that this leads to a different cusp length than eq. (5.15). The discussion here closely follows [26].

To begin, we first unpack what a cusp is geometrically. The string lives on the worldsheet, and for our choice of metric signature, all points on the string are timelike, except at the cusp point. Here, the point(s) are null. Thus, we can construct any (null) vector and it will characterise the worldsheet surface at the cusp. The general solution to the wave equation (3.43) is

$$\mathbf{x} = \frac{1}{2}(\mathbf{a}(\sigma - t) + \mathbf{b}(\sigma + t)) \quad (5.23)$$

where we have used a worldsheet reparameterisation to set  $\xi^0 = t$ . The vectors  $\mathbf{a}$  and  $\mathbf{b}$  are left and right movers, respectively, along the string. Consider the time and space derivatives of a generic vector  $x^\mu = (t, \mathbf{x})$ . We can compute  $x^{\mu'} = (0, \mathbf{x}')$  and  $\dot{x}^\mu = (0, \dot{\mathbf{x}})$ . These two vectors are tangent to the string, and as such the additive/subtractive combination of them is null:

$$\begin{aligned} A^\mu &= (1, \dot{\mathbf{x}} - \mathbf{x}') = (1, -\mathbf{a}') \\ B^\mu &= (1, \dot{\mathbf{x}} + \mathbf{x}') = (1, -\mathbf{b}') \end{aligned} \quad (5.24)$$

Any vector can characterise worldsheet cusp points provided it is null; hence eqs (5.24) characterise the worldsheet. We can then apply Lorentz transformations to these vectors; the way they and their derivatives transform will tell us how the cusp region transforms.

We transform eqs. (5.24) into a new, arbitrary coordinate system, which we will call the tilde system:

$$\begin{aligned} A^{\tilde{\mu}} &= (A^t, \mathbf{A}) \\ B^{\tilde{\mu}} &= (B^t, \mathbf{B}) \end{aligned}$$

We rescale these to normalise them in the same way, and express them in terms of our now tilde'd movers ( $\mathbf{a} \rightarrow \tilde{\mathbf{a}}$  and  $\mathbf{b} \rightarrow \tilde{\mathbf{b}}$ ):

$$\begin{aligned} \tilde{A}^{\tilde{\mu}} &= A^{\tilde{\mu}}/A^t = (1, -\tilde{\mathbf{a}}') \\ \tilde{B}^{\tilde{\mu}} &= B^{\tilde{\mu}}/B^t = (1, \tilde{\mathbf{b}}') \end{aligned}$$

We now have our tilde system move with some velocity  $\beta$ , such that a particle that stays in the un-tilde frame is moving with a velocity  $\beta$  while is in the tilde frame. Canonical Lorentz transformations gives

$$\begin{aligned} A^{\tilde{t}} &= \gamma(1 - \mathbf{a}' \cdot \beta) \\ B^{\tilde{t}} &= \gamma(1 + \mathbf{b}' \cdot \beta) \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad (5.25)$$

is the Lorentz factor. We also define

$$\begin{aligned} f_A &= \frac{1}{A^{\bar{t}}} \\ f_B &= \frac{1}{B^{\bar{t}}} \end{aligned} \quad (5.26)$$

Let  $h$  be any function in the neighbourhood of the worldsheet. We can then differentiate in the direction of  $A^\mu$  or  $B^\mu$  as follows

$$\begin{aligned} A^\mu \partial_\mu h &= \frac{\partial h}{\partial t} \Big|_x - \mathbf{a}' \cdot \nabla h \\ &= \frac{\partial h}{\partial t} \Big|_x + \dot{\mathbf{x}} \cdot \nabla h - \mathbf{x}' \cdot \nabla h \\ &= \frac{\partial h}{\partial t} \Big|_\sigma - \frac{\partial h}{\partial t} \Big|_t \end{aligned} \quad (5.27)$$

The first line comes from index contractions, the second from  $\mathbf{a}(\sigma-t)$ 's argument, and the third from the definition of partial derivatives. Similarly for  $B^\mu$ :

$$B^\mu \partial_\mu h = \frac{\partial h}{\partial t} \Big|_\sigma + \frac{\partial h}{\partial t} \Big|_t \quad (5.28)$$

We have setup the machinery to differentiate in the direction of vectors. We need to perform two calculations: Lorentz boosts of second order derivative terms and of third order derivative terms in the cusp expansion.

#### 5.4.1 Cusp expansion equations

Before doing this, we first need to collect several results which will be used later. We use [3, 26] as a reference guide. Earlier, we expanded the general solution  $x^\mu(\sigma, t)$ ; now, however, we consider the Taylor series of the left and right movers (up to third order). We expand about the point  $(\sigma, 0)$ :

$$\mathbf{a}(\sigma) = \mathbf{a}'_c \sigma + \mathbf{a}''_c \frac{\sigma^2}{2} + \mathbf{a}'''_c \frac{\sigma^3}{6} \quad (5.29)$$

$$\mathbf{b}(\sigma) = \mathbf{b}'_c \sigma + \mathbf{b}''_c \frac{\sigma^2}{2} + \mathbf{b}'''_c \frac{\sigma^3}{6} \quad (5.30)$$

where the subscript  $c$  denotes quantities at the cusp point. The cusp condition is (5.2), and gives the following additional cusp constraint [3, 26]

$$\mathbf{a}'_c = -\mathbf{b}'_c$$

The conditions on the left/right movers from the temporal conformal gauge (cf. section 4.1) is that

$$|\mathbf{a}'(\sigma)| = |\mathbf{b}'(\sigma)| = 1, \quad (5.31)$$

and this gives [3]

$$\begin{aligned} \mathbf{a}'' \cdot \mathbf{a}' &= 0 \\ \mathbf{a}''' \cdot \mathbf{a}' &= -|\mathbf{a}''|^2 \end{aligned}$$

(similarly for  $\mathbf{b}'$ ). Combining these gives the following constraints for the cusp derivatives

$$\begin{aligned}\mathbf{x}_c'' \cdot \dot{\mathbf{x}}_c &= 0 \\ \mathbf{x}_c''' \cdot \dot{\mathbf{x}}_c &= \frac{1}{2} (|\mathbf{a}_c''|^2 - |\mathbf{b}_c''|^2)\end{aligned}$$

### 5.4.2 Lorentz boosts of second order derivatives

We differentiate in the direction of the cusp (in the neighbourhood near it) at second order, by using our characteristic null vectors (5.24):

$$\begin{aligned}A_2^\nu &= A^\mu \partial_\mu A^\nu \\ &= (0, -\dot{\mathbf{a}}' + \mathbf{a}'') \\ &= (0, 2\mathbf{a}'')\end{aligned}$$

$$\begin{aligned}B_2^\nu &= B^\mu \partial_\mu B^\nu \\ &= (0, \dot{\mathbf{b}}' + \mathbf{b}'') \\ &= (0, 2\mathbf{b}'')\end{aligned}$$

The second line (in each case) is from the definition of  $A^\mu$  (or  $B^\mu$ ), and the last line follows from eq. (5.24). The second derivative quantities then become - in a new, tilde frame -

$$\tilde{A}_2^{\tilde{\nu}} = (0, 2\tilde{\mathbf{a}}'') \quad (5.32)$$

The tilde frame projection is thus

$$\begin{aligned}\tilde{A}_2 &= \tilde{A}_2^{\tilde{\nu}} \partial_{\tilde{\nu}} \tilde{A} \\ &= f_A A^\mu \partial_\mu (f_A A) \\ &= (f_A)^2 A^\mu \partial_\mu A + f_A f_{A,A} A \\ &= (f_A)^2 A_2 + f_A f_{A,A} A,\end{aligned} \quad (5.33)$$

where the second line is from the definition of  $f_A$ , the third line is from the chain rule, and in the last line we have applied the projection of  $A^\mu$  in the direction of  $A$ , i.e.  $A_2 = A^\mu \partial_\mu A$ . We have also defined  $f_{A,A} = A^\mu \partial_\mu f_A$ . The same calculation holds for  $\tilde{B}_2$ , to wit

$$\tilde{B}_2 = (f_B)^2 B_2 + f_B f_{B,B} B. \quad (5.34)$$

Now that we know how the null vectors transform, how are the lengths and angles of the second derivative terms affected? First, the length. To calculate the length of  $\tilde{A}_2^{\tilde{\nu}}$ , we find the inner product:

$$|\tilde{A}_2^{\tilde{\nu}}|^2 = 4|\tilde{\mathbf{a}}''|^2 \quad (5.35)$$

If we express  $4|\tilde{\mathbf{a}}''|^2$  in terms of the transformed  $\tilde{A}_2$ , then we will see how the length changes with a Lorentz boost (since  $A_2$ , by construction, characterises the second derivative at the cusp). Hence

$$\begin{aligned}4|\tilde{\mathbf{a}}''|^2 &= ((f_A)^2 A_2 + f_A f_{A,A} A)^2 \\ &= ((f_A)^2 ((f_{A,A})^2 g(A, A) + 2((f_A)^3 f_{A,A} g(A, A_2) + ((f_A)^4 g(A_2, A_2))\end{aligned}$$

$A$  is null by construction, hence  $g(A, A) = 0$ . From the condition  $\mathbf{a}'' \cdot \mathbf{a}' = 0$ , it follows that  $g(A, A_2) = 0$ . Hence only the last terms contributes, and we have

$$\begin{aligned}|\tilde{\mathbf{a}}''| &= (f_A)^2 |\mathbf{a}''| \\ |\tilde{\mathbf{b}}''| &= (f_B)^2 |\mathbf{b}''|\end{aligned} \quad (5.36)$$

(This whole procedure is the same for  $B^\mu$ ). Eq. (5.36) imply that the lengths for second derivatives are only scaled by a factor of  $f^2$  when Lorentz contracted.

How do the angles change? We compute  $g(A, B)$ , noting that at the cusp  $f_A = f_B = f$ . And so,

$$g(A, B) = 0 \implies \tilde{\mathbf{a}}'' \cdot \tilde{\mathbf{b}}'' = (f)^4 \mathbf{a}'' \cdot \mathbf{b}'' \quad (5.37)$$

This shows that at the second derivative level, Lorentz boosts serve only to scale the lengths and angles in a new frame by some scaling factor, which can of course be normalised to be 1.

### 5.4.3 Lorentz boosts of third order derivatives

We follow an identical procedure for the third order terms. It will be a bit more hairy algebraically - but is conceptually the same. The third order projections are

$$\begin{aligned} A_3^\rho &= A^\mu \partial_\mu (A^\nu \partial_\nu A^\rho) = (0, -4\mathbf{a}''') \\ B_3^\rho &= B^\mu \partial_\mu (B^\nu \partial_\nu B^\rho) = (0, 4\mathbf{b}''') \end{aligned}$$

and the associated tilde terms are

$$\begin{aligned} \tilde{A}_3 &= f_A A^\mu \partial_\mu (f_A A^\nu \partial_\nu (f_A A)) = (f_A)^3 A_3 + 3(f_A)^2 f_{A,A} A_2 \\ &\quad + (f_A (f_{A,A})^2 + (f_A)^2 f_{A,AA}) A \end{aligned}$$

and similarly for  $\tilde{B}_3$ . (Where  $f_{A,AA} = A^\mu \partial_\mu (A^\nu \partial_\nu f_A)$ ). We use our second left/right mover constraint,  $\mathbf{a}''' \cdot \mathbf{a}' = -|\mathbf{a}''|^2$ , and note it is in the direction of  $\hat{\mathbf{x}}_c$  (see ref. [26]). Since that's fixed (i.e. the  $A_3$  term), we want to know how the directions of  $\mathbf{a}''$  and  $\mathbf{b}''$  change with Lorentz boosts. To that end, we calculate the inner product of third order terms with second order ones. Hence,

$$\begin{aligned} -8\tilde{\mathbf{a}}''' \cdot \tilde{\mathbf{a}}'' &= g(\tilde{A}_3, \tilde{A}_2) = (f_A)^5 g(A_3, A_2) + (f_A)^4 f_{A,A} g(A_3, A) \\ &\quad + 3(f_A)^4 f_{A,A} g(A_2, A_2) \\ &= -8(f_A)^5 \mathbf{a}''' \cdot \mathbf{a}'' + 4(f_A)^4 f_{A,A} \mathbf{a}''' \cdot \mathbf{a}' + 12(f_A)^4 f_{A,A} |\mathbf{a}''|^2 \end{aligned}$$

Using the constraint  $\mathbf{a}''' \cdot \mathbf{a}' = -|\mathbf{a}''|^2$ , the middle term becomes  $-4(f_A)^4 f_{A,A} |\mathbf{a}''|^2$ , and hence

$$\tilde{\mathbf{a}}''' \cdot \tilde{\mathbf{a}}'' = (f_A)^5 \mathbf{a}''' \cdot \mathbf{a}'' - (f_A)^4 f_{A,A} |\mathbf{a}''|^2 \quad (5.38)$$

We can simplify this further by expressing the  $f_A$  factors in terms of  $\mathbf{a}$  derivatives. From (5.26), we can calculate  $f_{A,A}$ :

$$\begin{aligned} A^\mu \partial_\mu f_A &= \frac{\beta}{\gamma(1 - \mathbf{a}' \cdot \beta)^2} \cdot (\dot{\mathbf{a}}' - \mathbf{a}'') \\ &= -f_A \frac{\beta}{(1 - \mathbf{a}' \cdot \beta)} \cdot 2\mathbf{a}'' \\ &= -2\gamma(f_A)^2 \beta \cdot \mathbf{a}'' \end{aligned}$$

where the second and third line came from using the definition of  $f_A$  (5.26). Substituting this into (5.38), we get

$$\tilde{\mathbf{a}}''' \cdot \tilde{\mathbf{a}}'' = (f_A)^5 (\mathbf{a}''' \cdot \mathbf{a}'' + 2\gamma f_A |\mathbf{a}''|^2 \beta \cdot \mathbf{a}'') \quad (5.39)$$

The same expression holds for  $\tilde{\mathbf{b}}''' \cdot \tilde{\mathbf{b}}''$ , barring the symbol swap. Eq. (5.39) tells us how the the lengths for third derivatives change.

As for the angles: at the cusp,  $f_A = f_B = f$ , and so

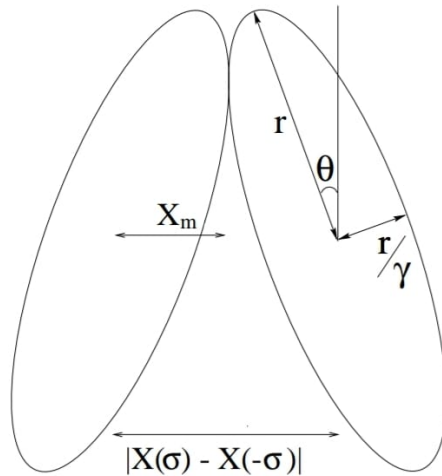


Figure 5.1: Cross-section of a string at cusp formation. Taken from ref. [26].

$$\begin{aligned}\tilde{\mathbf{a}}_c''' \cdot \tilde{\mathbf{b}}_c''' &= f^5(\mathbf{a}_c''' \cdot \mathbf{b}_c''' + \gamma f(3(\mathbf{a}_c'' \cdot \mathbf{b}_c'')\beta\mathbf{a}_c'' - |\mathbf{a}_c''|^2\beta\mathbf{b}_c'')) \\ \tilde{\mathbf{b}}_c''' \cdot \tilde{\mathbf{a}}_c''' &= f^5(\mathbf{b}_c''' \cdot \mathbf{a}_c''' - \gamma f(3(\mathbf{a}_c'' \cdot \mathbf{b}_c'')\beta\mathbf{b}_c'' + |\mathbf{b}_c''|^2\beta\mathbf{a}_c''))\end{aligned}\quad (5.40)$$

It can be seen then that at the third derivative level, Lorentz boosts do add in correction factors to both the length of the vectors and the angles between them.

#### 5.4.4 Overlap scaling

Cusp annihilation is when two portions of the cusp overlap and decay, emitting beamed radiation. We can now answer the question of what the cusp region looks like when subjected to Lorentz contractions. Call the radius of the cross-section of the string core (i.e. the cusp width)  $w$ , and the length scale of the entire string  $R$  (that is, the whole string's radius). Then, as done in section 4.2, the cusp region is one for which

$$|x(0, \sigma) - x(0, -\sigma)| := \text{cusp length} \quad (5.41)$$

We now consider the Taylor expansion for  $t = 0$ , that is, eq. (5.4). This would imply that, at the cusp,

$$|x(0, \sigma_c) - x(0, -\sigma_c)| = \frac{|\mathbf{x}_c'''}{6}\sigma^3 \quad (5.42)$$

(Since  $\mathbf{x}' = 0$  at the cusp and the  $\sigma^2$  terms cancel). Using our gauge condition (5.1),

$$\dot{\mathbf{x}}^2 = 1 - (\mathbf{x}')^2 \quad (5.43)$$

At the cusp, to lowest order, this gives the velocity of the string cusp. Using (5.4) to lowest order at the cusp for  $(\mathbf{x}')^2$ ,

$$|\dot{\mathbf{x}}_c|^2 = 1 - \sigma^2|\mathbf{x}_c''|^2 \quad (5.44)$$

Since this functions as our cusp speed, the Lorentz factor then becomes

$$\gamma = \frac{1}{\sqrt{1 - |\dot{\mathbf{x}}_c|^2}} = \frac{1}{\sigma|\mathbf{x}_c''|} \quad (5.45)$$

The cusp is contracted in the direction of motion of the velocity. If one took a cross-section of the string core, the circle with radius  $w$  would be contracted into an ellipse with semimajor axis  $w$  and semiminor axis  $w/\gamma$ . When a cusp forms, this will be when these ellipses are tiled such that they intersect - and the length between the base of the cross-sections is cusp length (See figure 5.1).

The maximum separation distance between two rotated ellipses is, in general [26],

$$d_{max} = \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \quad (5.46)$$

where  $a$  is the semimajor axis;  $b$  is the semiminor axis. For the cusp,

$$\begin{aligned} d_{max} &= \sqrt{w^2 \sin^2 \theta + \left(\frac{w}{\gamma}\right)^2 \cos^2 \theta} \\ &\approx w \sqrt{\theta^2 + 1/\gamma^2} \\ &\approx w \sqrt{\theta^2 + \sigma^2 |\mathbf{x}_c''|^2} \end{aligned} \quad (5.47)$$

In the second line we used the small angle approximation. Since the maximum separation distance can at most be the string width, this implies

$$|x(0, \sigma_c) - x(0, -\sigma_c)| = w \sqrt{\theta^2 + \sigma^2 |\mathbf{x}_c''|^2} \quad (5.48)$$

and therefore

$$\frac{|\mathbf{x}_c'''}{6} \sigma^3 = w \sqrt{\theta^2 + \sigma^2 |\mathbf{x}_c''|^2} \quad (5.49)$$

The angle between the string width and the contracted direction will be on the order  $\theta \sim \sigma \dot{\mathbf{x}}_c$  [26]. As such,

$$\frac{|\mathbf{x}_c'''}{6} \sigma^2 = w \sqrt{|\dot{\mathbf{x}}_c|^2 + \sigma^2 |\mathbf{x}_c''|^2} \quad (5.50)$$

The value of  $\sigma$  is effectively the cusp length -  $\sigma = l_c$  (at the cusp)

$$l_c \sim \left( w \frac{\sqrt{|\dot{\mathbf{x}}_c|^2 + |\mathbf{x}_c''|^2}}{|\mathbf{x}_c'''} \right)^{1/2} \quad (5.51)$$

We expect, by dimensional arguments, that derivative terms scale as  $R^{1-n_1}$  [4,26], since the first derivatives are constant at the cusp. Hence, we finally get, to lowest order,

$$l_c \sim \sqrt{wR} + \text{higher order terms} \quad (5.52)$$

The key difference between this cusp length and the one in section 4.2 is that the Lorentz factor modifies the scaling relation. The cusp length then, scales as  $l_c \sim \sqrt{wR}$ , not as  $l_c \sim w^{1/3} R^{2/3}$  (in eq. (5.15)). In the final section of this chapter, we apply this reconsidered relation to the particle emission parameters from cusp decay and consider how this effects potential cosmic string-FRB models.

## 5.5 Cusp reanalysis for fitting FRB data

Cusp decay may provide observational signatures, owing to the cusp's emission of electromagnetic radiation. FRBs could, in principle, provide a possible observational testing ground for cosmic strings' signatures. The two essential ingredients of a model purported to explain a phenomenon are (1) matching or retrodicting the observed feature and (2) explaining such features and predicting new ones. Since the number of observed FRBs is low ( $n \sim 30$ ), there is some theoretic degeneracy involved in proposed explanations. One can contrive any number of theories that explain the (limited) observational data and predict a panoply of different consequences. At the very least a model should be in agreement with available data, and it's this aspect of the work done in [12] that we will revisit in this section. What follows here is an elaboration of the author's (and collaborators') work, found in [103].

There are two assumptions made by the authors of [12] which, if dispensed with, alter their results. The first assumption is that the length of the cusp region is unaffected by relativity. As seen in the previous section, Lorentz contractions in the rest frame of the string alter the cusp length estimation significantly. The second assumption is that, when a cusp decays, the number density of particle emission scales as

<sup>1</sup>This is why the cusp coefficients scale as they do in (5.8).

$N(E) \propto E^{-3/2}$ . This relation comes from a simply QCD multiplicity function [13] and is shown to hold well in high energy experiments [12]. It is not clear, however, if this holds all the way down to radio energies. As such, we also consider a more general scaling law,  $N(E) \propto E^{-m}$ . This allows somewhat more flexibility in fitting a string with a now smaller cusp to the FRB data.

We start by considering the relativistic effects on the cusp length and a different scaling for the energy of decay particles. The number density of photons from the cusp region is given by (5.22)

$$N(E) \sim \frac{1}{\Theta^2 d^2} \frac{\mu_0 l_c}{E_f^2} \left(\frac{E_f}{E}\right)^m, \quad (5.53)$$

where  $E_f$  is the fragmentation energy of the particles produced in the decay process,  $d$  is the distance from the cusp at which an observer would receive such a number density of photons and  $\Theta$  is the beaming angle. We take  $E_f$  to be on the order of the symmetry breaking scale associated with the formation of the cosmic string, *i.e.*  $E_f \sim \eta$ . The difference here is that we have changed the scaling power to a more general one.

$\mu$  is the mass per unit length of the string, and scales as  $\mu \sim \eta^2$ . The beaming angle is given by

$$\Theta \sim \frac{1}{\ln(E_f/\text{GeV})}. \quad (5.54)$$

We now follow the same procedure as [12]. The flux is defined by

$$S = \int_0^E dE' (N(E') E') \quad (5.55)$$

Using eq. (5.53), this becomes

$$\begin{aligned} S &\sim \int_0^E \frac{\mu_0 l_c}{\Theta^2 d^2} E_f^{m-2} E'^{1-m} \\ &\sim \frac{\mu_0 l_c}{(2-m)\Theta^2 d^2} \left(\frac{E_f}{E}\right)^{m-2}. \end{aligned} \quad (5.56)$$

As a side point: this implies  $m < 2$  in order to have a positive, and non-divergent, flux.

We now assume  $E_f \sim \eta$ , and that the relativistic corrected cusp length is  $l_c \sim w^{1/2} R^{1/2}$ . Along with  $w \sim \eta^{-1}$  (cf. section 2.2), this means that  $l_c \sim \eta^{-1/2} R^{1/2}$ . We also have  $\mu \sim \eta^2$ . This allows us to express the flux as

$$S \sim \frac{(R\eta)^{-3/2}}{(2-m)\Theta^2} \left(\frac{E}{\eta}\right)^{2-m} \left(\frac{R}{d}\right)^2 \eta^3. \quad (5.57)$$

Here we have split up the powers of  $R$  and  $\eta$  so as to have an  $R^2$  and a  $\eta^{m-2}$  term. It is useful to express  $\eta$  in terms of  $m_p$  and  $t_0$  (the Planck mass and present time, *i.e.* time since the big bang), which are fixed scales (unlike  $\eta$ , which can change depending on what one takes to be the GUT scale). As such, we can write

$$\begin{aligned} \eta &\sim (t_0 m_p) \frac{\eta}{t_0 m_p} \sim 10^{60} \frac{1}{t_0} (G\mu)^{1/2} \\ &\sim 10^{15} \text{GeV}, \end{aligned} \quad (5.58)$$

where in the last line we used  $G\mu \sim 10^{-7}$ ,  $t_0 \sim 10^{42} \text{GeV}^{-1}$ . This value for  $G\mu$  is the upper bound on the tension of cosmic strings given by CMB measurements of the angular power spectrum [5–7]. The energy flux of FRBs is observed to be of order  $S_0 \sim 10^{-48} (\text{GeV})^3$  [12], so we must have

$$S(E, R, d(R)) = S_0. \quad (5.59)$$

If we then set (5.57) equal to  $S_0$ , we can solve for the distance of the source from the detector:

$$d(R) \sim \frac{R^{1/4}}{\sqrt{(2-m)S_0\Theta}} \left(\frac{E}{\eta}\right)^{\frac{2-m}{2}} \eta^{3/4}. \quad (5.60)$$

where we've recombined the powers for  $R$  and  $\eta$ .

In order for a theoretical model to sufficiently account for FRBs, the event rate (number of bursts in the sky) and the burst time-scale (for how long the pulse lasts) must be consistent with the extant FRB data. Additionally, since strings will typically exist at cosmological distances, the flux emitted by cusp decay must be sufficient - given the typical distance of a cosmic string - to account for the measured FRB flux. We unpack these three criteria below.

### 5.5.1 Distance criterion

The distance (5.60) must be bigger than the expected distance of a cosmic string from us. If it is, the photons will be (in principle) observable. The expected mean separation distance for a distribution of cosmic strings is given by

$$d_R^3 n(R, t) = R^{-1} \quad (5.61)$$

where  $n(R, t)$  is the number density of string in some volume  $d_R^3$ . The number density changes as the universe evolves; for the matter-dominated era, the function is given by [3, 12, 30]

$$n(R, t) \sim R^{-5/2} t_{eq}^{1/2} t^{-2} \quad (5.62)$$

We take  $t_{eq} \sim t_0 \times 10^{-6}$ , and so  $n(R, t) \sim 10^{-6} R^{-5/2} t_0^{-3/2}$ . The expected distance is then

$$d_R \sim 10 R^{1/2} t_0^{1/2}. \quad (5.63)$$

The distance criterion condition  $d_R < d(R)$  implies

$$R^{1/2} < \frac{1 \times 10^{-2}}{t_0(2-m)\Theta^2 S_0} \left(\frac{E}{\eta}\right)^{2-m} \eta^{3/2}. \quad (5.64)$$

The radio frequencies to which the telescopes are most sensitive have energies on the the order of  $E \sim 10^{-15}$  GeV. For such an  $E$ , the values for  $R$  at different  $m$  and  $\eta$  are shown in Table 5.1.

According to the scaling model for the distribution of cosmic string loops the radius, in a matter dominated universe, lies in the interval  $\gamma G\mu t_0 < R < t_{eq}$ . It is in this interval that the loop distribution (5.62) holds. Strings with a radius smaller than the lower bound decay in less than one Hubble time [12].

From Table 5.1 we see that, considering the same scaling for the energy as [12], *i.e.*  $m = 3/2$ , Lorentz contraction implies a smaller radius for the string loop, which lies outside the allowed region<sup>2</sup>. The only way to have a consistent string loop size is if the scaling of the energy is given by  $m \sim 2$ . A scaling below  $m = 3/2$  makes the situation even worse. We also compute  $R$  for  $G\mu \sim 10^{-12}$  and for  $G\mu \sim 10^{-18}$ , which implies  $\eta \sim 10^{12}$  and  $\eta \sim 10^9$  respectively, to check the impact of string tension on the radius of the loop. Changing the string tension will also change the time scale for the emitted radio burst and the event rate as we will see in the next sections.

### 5.5.2 Time scale

The power of cusp decay is given in, as we recall from section 4.2, by eq. (5.16)

$$P_c \sim \frac{\mu l_c}{R}. \quad (5.65)$$

<sup>2</sup>In [12], one gets  $R \sim 10^{51}(\text{GeV})^{-1}$  by considering  $G\mu \sim 10^{-7}$ , which would be inside the allowed region.

$\eta \backslash m$	$\sim 2$	$3/2$	$1$	$1/2$
$10^{15}$ GeV	$\frac{10^{59}}{(2-m)^2}$	$10^{29}$	$10^{-1}$	$10^{-31}$
$10^{12}$ GeV	$\frac{10^{50}}{(2-m)^2}$	$10^{23}$	$10^{-4}$	$10^{-31}$
$10^9$ GeV	$\frac{10^{42}}{(2-m)^2}$	$10^{18}$	$10^{-8}$	$10^{-32}$

Table 5.1: Values of  $R$  in  $(\text{GeV})^{-1}$  for different  $\eta$  and  $m$ . These values have to be compared to  $R_{min} \sim \gamma G\mu t_0$  for differing values of  $\eta$ . For  $\eta \sim 10^{15}\text{GeV}$ ,  $R_{min} \sim 10^{36}\text{GeV}^{-1}$ ; for  $\eta \sim 10^{12}\text{GeV}$ ,  $R_{min} \sim 10^{31}\text{GeV}^{-1}$  and for  $\eta \sim 10^9\text{GeV}$ ,  $R_{min} \sim 10^{25}\text{GeV}^{-1}$ .

$\eta \backslash m$	$\sim 2$	$3/2$	$1$
$10^{15}$ GeV	$\frac{10^{-2}}{(2-m)}$	$10^{-17}$	$10^{-32}$
$10^{12}$ GeV	$\frac{10^{-5}}{(2-m)}$	$10^{-18}$	$10^{-32}$
$10^9$ GeV	$\frac{10^{-8}}{(2-m)}$	$10^{-20}$	$10^{-32}$

Table 5.2: Time scale of a burst, in seconds, for different values of  $m$  and  $\eta$ .

The time scale for a burst can be estimated as the time taken to go from one ‘side’ of the cusp to the other, *i.e.* when two strands of the string at the cusp have moved apart by a distance more than the string width  $w$ . Since power is energy per unit time, and the string width is inverse of the energy, the time scale for the cusp decay is

$$T = \frac{1}{P_c w} \sim R^{1/2} w^{1/2}. \quad (5.66)$$

This is different to the time scale in [12] by a factor of  $(R/w)^{1/6}$ . Since the radius of the string loop  $R$  varies with the scaling of the energy we will also have varying values for the time scale. These are shown in Table 5.2. FRBs have a time scale of order  $10^{-3}s$  at the moment of the detection.

Due to the dispersion of the burst through the interstellar medium (ISM) the wave frequency will spread out in time, so its emission time scale must be smaller than that. To be consistent with the distance criterion and time scale of the burst, we must have  $m \sim 2$  and  $\eta$  of order  $10^{12}$  GeV, as one can check from Table 5.1 and Table 5.2. It is interesting to note that consistency with FRBs data could set another upper bound for the string tension. From the numbers analysed here it would be of order  $G\mu \sim 10^{-12}$ , which implies an energy scale for the formation of cosmic strings of  $\eta \sim 10^{12}\text{GeV}$ .

### 5.5.3 Event rate

The event rate of FRBs is given by [12]

$$\mathcal{R} \sim \int_{\gamma G\mu t_0}^{t_0} dR \left( d^3(R) n(R, t_0) \frac{1}{R} P(R) \right), \quad (5.67)$$

where  $P(R)$  is the probability of having a beamed jet of photons coming from the cusp in the line of sight. Since we expect one cusp per oscillation, we divide the integrand by  $R$ . The number density is given again by (5.62). Assuming  $P(R) = P_0 \sim \Theta$  as approximately constant, let’s calculate each piece in turn.

From (5.64),

$$d^3(R) = \frac{R^{3/4}}{[(2-m)S_0]^{3/2} \Theta^3} \left( \frac{E}{\eta} \right)^{\frac{3(2-m)}{2}} \eta^{9/4} \quad (5.68)$$

Plugging (5.62) and (5.60) into (5.67), we get

$$\mathcal{R} \sim \frac{t_0^{-3/2}}{[(2-m)S_0]^{3/2} \Theta^2} \left( \frac{E}{\eta} \right)^{\frac{3(2-m)}{2}} \eta^{9/4} \int_{\gamma G\mu t_0}^{t_0} dR R^{-11/4} \quad (5.69)$$

$\eta \backslash m$	$\sim 2$	$3/2$	$1$
$10^{15}$ GeV	$\frac{10^{12}}{(2-m)^{3/2}}$	$10^{-11}$	$10^{-18}$
$10^{12}$ GeV	$\frac{10^{14}}{(2-m)^{3/2}}$	$10^{-6}$	$10^{-27}$
$10^9$ GeV	$\frac{10^{15}}{(2-m)^{3/2}}$	$10^{-3}$	$10^{-21}$

Table 5.3: Number of expected FRBs per year for different values of  $m$  and  $\eta$ .

The integral is dominated by the lower integral limit, and therefore we can throw away the first term in the evaluation, and we're left with

$$\mathcal{R} \sim \frac{P_0(\gamma G\mu)^{-7/4}}{(2-m)^{3/2}\Theta^3} \frac{1}{S_0^{3/2}} \left(\frac{E}{\eta}\right)^{\frac{3(2-m)}{2}} \eta^{9/4} t_0^{-13/4} 10^{-3}. \quad (5.70)$$

The values of  $\mathcal{R}$ , for different values of  $\eta$  and  $m$ , are shown in Table 5.3. Those numbers should be compared to the estimated number of FRBs per day<sup>3</sup>, which is of order  $10^4$ . We can see that if we tune  $m$  to get a distance relation and time scale compatible with the data, the number of expected FRBs emitted would be much bigger than the estimated rate. On the other hand, if the scaling for the energy is of order  $3/2$  or smaller, the distance relation and event rate are much smaller than the consistency check allow. This analysis explicitly shows that, unfortunately, FRBs cannot be interpreted as a signature of structureless cosmic strings.

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<sup>3</sup>Although few FRBs have been detected so far, the expected rate is relevant here.

## Chapter 6

# Wiggly Cosmic Strings

In this chapter, we introduce the idea of the *wiggly cosmic string*, and review how a less elegant formulation of cosmic string dynamics is compensated for by a richer phenomenology and by a more physically realistic description of radiative processes. We review the argument for a wiggly equation of state (EOS) and the extension of the NG action to a wiggly NG string. We further show that one can recast the standard wiggly NG string action in terms of a wiggly Polyakov analogue. We extend on the dynamics of the wiggly string by analysing the cusp solution, and show that cusps cannot form in wiggly strings. We discuss the implications this has for the possibility of cosmic strings acting as a source of FRBs and electromagnetic signatures more generally.

### 6.1 Nambu perturbations

Nambu-Goto strings require a zero-thickness approximation [3], and can be described as a “point-like” string. GUT strings, if they exist, are expected to have small-scale structure [21]. “Small” meaning on scales less than the correlation length of a string network [25]. Simulations of strings have all indicated that strings build up small-scale structure [21], and that many properties of string networks resulting from the simulations cannot be explained by the number of degrees of freedom in the Nambu-Goto string description [21, 25]. The key feature which defines an NG string is its EOS: the tension is exactly equal to the energy (or mass per unit length),  $\mu_0 = T$ . Yet a wiggly string would require a variable equation of state. However, the EOS for a wiggly NG string (on small-scales) is rather special and will be discussed in this section. Much of section 6.1 closely follows ref. [3].

Recall the general solution to the NG string is

$$\mathbf{x} = \frac{1}{2}(\mathbf{a}(\sigma - t) + \mathbf{b}(\sigma + t)) \quad (6.1)$$

The vectors  $\mathbf{a}$  and  $\mathbf{b}$  are left and right movers, respectively, along the string. These NG solutions can be perturbed, i.e. one can add in wiggles. We write  $\zeta_{\pm} = \sigma \pm t$ , and perturb the string as so

$$\begin{aligned} \mathbf{a} &= k_1 \zeta \hat{n} + \delta \mathbf{a}(\zeta) \\ \mathbf{b} &= k_2 \zeta \hat{n} + \delta \mathbf{b}(\zeta) \end{aligned} \quad (6.2)$$

Here,  $k_1, k_2$  are constant. The constraints (3.40) coupled with the static gauge choice imply two things:  $|\mathbf{a}'|^2 = |\mathbf{b}'|^2 = 1$ , where  $\prime$  denotes differentiation with respect to  $\zeta$ . We can pick transverse wiggles such that

$$\hat{n} \cdot \delta \mathbf{a}(\zeta) = \hat{n} \cdot \delta \mathbf{b}(\zeta) = 0. \quad (6.3)$$

Let's calculate  $\mathbf{a}'(\zeta)$  and  $\mathbf{b}'(\zeta)$ :

$$\begin{aligned} \mathbf{a}'(\zeta) &= k_1 \hat{n} + \delta \mathbf{a}'(\zeta) \\ \mathbf{b}'(\zeta) &= k_2 \hat{n} + \delta \mathbf{b}'(\zeta) \end{aligned} \quad (6.4)$$

The constraints imply

$$|\mathbf{a}'|^2 = (k_1)^2 (\hat{n} \cdot \hat{n}) + 2k_1 \hat{n} \cdot \delta \mathbf{a}'(\zeta) + (\delta \mathbf{a}'(\zeta))^2 = 1 \quad (6.5)$$

Since  $(\hat{n} \cdot \hat{n}) = 1$  and  $\hat{n} \cdot \delta \mathbf{a}'(\zeta) = 0$ , we have

$$|\mathbf{a}'|^2 = (k_1)^2 + (\delta\mathbf{a}'(\zeta))^2 = 1 \quad (6.6)$$

Similarly, the equation for  $|\mathbf{b}'|^2$  is

$$|\mathbf{b}'|^2 = (k_2)^2 + (\delta\mathbf{b}'(\zeta))^2 = 1 \quad (6.7)$$

This allows us to write

$$(\delta\mathbf{a}'(\zeta))^2 = 1 - (k_1)^2 \quad (6.8)$$

and similarly for  $|\mathbf{b}'|^2$ . This means that  $-1 \leq k_1, k_2 \leq 1$ . Since we've taken  $k_{1,2}$  to be constant, we can average out the wiggles on large scales - since far away from the string, it will appear as though the macroscopic physical characteristics are uniform along it. Let's align the string along the x-axis. As such,  $\hat{n} = (1, 0, 0)$ . This implies  $\delta\mathbf{a} = \delta\mathbf{b} = 0$  in order for  $\hat{n} \cdot \delta\mathbf{a}/\mathbf{b}$  to be true. We can thus write down an effective energy-momentum tensor for the string,

$$T_{\text{eff}}^{\mu\nu} = \Theta^{\mu\nu} \delta(y) \delta(x) \quad (6.9)$$

The  $\Theta^{\mu\nu}$  term is the average over the microscopic energy, i.e.

$$\Theta^{\mu\nu} = \frac{1}{td} \int T^{\mu\nu} d^4x \quad (6.10)$$

where the  $td$  is the average time and distance for which a wiggle will propagate. The non-zero components are  $\Theta^{00}, \Theta^{11}, \Theta^{10}, \Theta^{01}$  [3]. The  $\Theta^{00}$  component is

$$\Theta^{00} = \frac{1}{td} \int T^{00} d^4x = \frac{\mu_0}{td} \int d^3x = \frac{\mu_0}{td} \Delta\zeta = \frac{\mu_0}{d} \langle \Delta\zeta \rangle \quad (6.11)$$

If we define the average wiggly scale as

$$d = \frac{1}{2}(k_1 + k_2) \langle \Delta\zeta \rangle \quad (6.12)$$

(which is the average point between  $\mathbf{a}(\zeta)$  and  $\mathbf{b}(\zeta)$ ), then

$$\Theta^{00} = 2\mu_0(k_1 + k_2)^{-1} \quad (6.13)$$

Similarly,

$$\begin{aligned} \Theta^{01} = \Theta^{10} &= -2\mu_0 k_1 k_2 (k_1 + k_2)^{-1} \\ \Theta^{11} &= \mu_0 (k_1 - k_2) (k_1 + k_2)^{-1} \end{aligned} \quad (6.14)$$

If the wiggles are observed from far away, and the string appears to be smooth, then  $k_1 = k_2$ , and  $\Theta^{00} = \mu_0 k^{-1}$ ,  $\Theta^{11} = -\mu_0 k$ . Call  $\Theta^{00} = \mu$  and  $\Theta^{11} = T$ . This means,

$$\Theta^{00} \Theta_{11} = \mu T \quad (6.15)$$

which gives the wiggly equation of state as

$$\mu T = \mu_0^2 \quad (6.16)$$

This equation of state is a scalar invariant of the wiggly string, since we can alternatively [3] define  $\mu, T$  and eigenvalues of  $\Theta^{\mu\nu}$ . The determinant is Lorentz invariant, and is

$$\Theta^{00} \Theta^{11} - (\Theta^{01})^2 = \mu_0^2 \quad (6.17)$$

Thus we have now characterised the wiggly string in terms of an NG string with transverse perturbations.

This EOS for wiggly strings is specific to NG strings in a certain gauge, and is only for small-scale perturbations. However, one can consider perturbations of the string's dynamical equations more generally, and show that this EOS (6.16) is a unique one for small-scale wiggles. Additionally, this EOS is a first approximation for the generalised wiggly EOS. This result is detailed in appendix B.

## 6.2 A model of wiggly cosmic strings

In order to add small-scale structure to cosmic strings - and in particular to do so to gauge strings - one requires a dynamical formalism which has the EOS (6.16). Such a model was first proposed in [24]. The idea was that a cosmic string of a permanent transonic type, i.e. a string which had identical transverse and longitudinal velocities, would yield the EOS given by eq. (6.16).

The procedure in both section 6.2 and 6.3 both treat wiggles as an average quantity. An observer far away enough from the string will not be able to resolve the structure, but will observe a different energy and tension than the bare tension,  $\mu_0$ . The EOS is the same in either case; indeed, in ref. [106] the same relation was using a now third independent method.

What we'd like then is a theory which describes wiggly strings, strings with small-scale structure. Such an action was proposed in [24] and developed for cosmic string networks in [21]. The action can be written as

$$\mathcal{S} = -\mu_0 \int d^2\xi \|\gamma\|^{1/2} \sqrt{1 - \chi(\varphi)} \quad (6.18)$$

here  $\chi = \gamma^{ab} \partial_a \phi \partial_b \phi$ , and  $\phi$  is the associated mass current of the wiggle propagating along the string, modeled as a single scalar field. The Lagrangian contains all the fields with which the string interacts excluding the gravitational one; in this case, we assume the only field is the internal field denoted by  $\Lambda$ , which contains all the internal fields: in this case, the scalar field corresponding to the wiggles. Incidentally, the action

$$\mathcal{S} = -\mu_0 \int d^2\xi \|\gamma\|^{1/2} \sqrt{1 - \gamma^{ab} \partial_a \varphi \partial_b \varphi} \quad (6.19)$$

is of the same form as the Born-Infeld (BI) action [107]. This is unsurprising given that the dynamics of the world volume have the BI action as an effective description [108]. This choice of action can be shown to be suitable for wiggly cosmic strings. Its associated energy-momentum tensor is given by

$$T_{ab} = \frac{2}{\|\gamma\|^{1/2}} \frac{\partial \mathcal{S}}{\partial \gamma^{ab}} \quad (6.20)$$

Using eq. (3.34), and the product rule, we get

$$\begin{aligned} T_{ab} &= \frac{-2\mu_0}{\|\gamma\|^{1/2}} \left( \frac{-\frac{1}{2} \|\gamma\|^{1/2} \partial_a \varphi \partial_b \varphi}{\sqrt{1 - \gamma^{ab} \partial_a \varphi \partial_b \varphi}} - \frac{1}{2} \gamma_{ab} \|\gamma\|^{1/2} \sqrt{1 - \gamma^{ab} \partial_a \varphi \partial_b \varphi} \right) \\ &= \mu_0 \left( \frac{\partial_a \varphi \partial_b \varphi}{\sqrt{1 - \gamma^{ab} \partial_a \varphi \partial_b \varphi}} + \gamma_{ab} \sqrt{1 - \gamma^{ab} \partial_a \varphi \partial_b \varphi} \right) \end{aligned}$$

One can make the following identifications [107] with the general form of a perfect fluid,

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}, \quad (6.21)$$

and set the fluid velocity to

$$u_a = \frac{\partial_a \varphi}{\sqrt{\gamma_{ab} \partial_a \varphi \partial_b \varphi}}$$

and the energy density and pressure to

$$\begin{aligned} p &= -\mu_0 \sqrt{1 - \gamma^{ab} \partial_a \varphi \partial_b \varphi} \\ \rho &= \mu_0 \frac{1}{\sqrt{1 - \gamma^{ab} \partial_a \varphi \partial_b \varphi}} \end{aligned}$$

If we now string tension to be equal in magnitude to its pressure,  $p = -T$  (and relabel  $\rho = \mu$ ), then we find

$$-\mu p = \mu T = \mu_0^2 \quad (6.22)$$

which is, of course, the required EOS.

### 6.3 Wiggly Strings Dynamics

The action for the wiggly NG action is given by (6.19), if we rewrite the mass per unit length:

$$\mathcal{S}_{\text{wiggly NG}} = \frac{-m^2}{2} \int d^2\xi \left( \|\gamma\|^{1/2} \sqrt{1 - \gamma^{ab} \partial_a \varphi \partial_b \varphi} \right) \quad (6.23)$$

Finding the wiggly equations of motion is technically difficult for a very simple reason: the square root. It would be ideal to have a “wiggly Polyakov” action: one which has no square root, but penalises such convenience with the introduction of an auxiliary field. Such an action can be derived by using Dirac’s Hamiltonian formalism for constrained systems, and was done in [23]. The wiggly Polyakov action is given by

$$\mathcal{S}_{\text{P}_w} = \frac{-m^2}{2} \int d^2\xi \|\!|h|\!|^{1/2} h^{ab} \left( \partial_a x^\mu \partial_b x_\mu + \frac{\psi^2}{m^2} \partial_a \varphi \partial_b \varphi \right) \quad (6.24)$$

where  $h_{ab}$  is now the auxiliary field which acts as a metric. To find the equations of motion, we vary the action with respect to the auxiliary field, which is by definition the energy-momentum tensor ( $T^{ab}$ ) of the theory, and use  $T^{ab}$  in conjunction with gauge choices to relate the auxiliary field to the original and ‘physical’ induced string worldsheet,  $\gamma_{ab}$ . The action can then be varied with respect to the coordinates  $x^\mu$  and  $\varphi$ , with the equation of motion for  $h_{ab}$  acting as a constraint.

Before proceeding, it will be later useful to introduce a third action, one which provides an equivalent description of wiggly string dynamics [22, 23]. A Kaluza-Klein projection of the regular NG action in one higher dimension has an action (the “Kaluza-Klein” (KK) action) given by

$$\mathcal{S}_{kk} = -m^2 \int d^2\xi \|\Gamma\|^{1/2} \quad (6.25)$$

(The equivalence of eq. (6.25) with eq. (6.23) will be proved in appendix A). This projection to (d+2)-dimensions automatically produces a scalar field coupling of the kind which describes wiggly strings, provided we make the following choice for the embedding of the brane coordinates:

$$q^M = \begin{pmatrix} x^\mu(\xi^a) \\ \varphi(\xi^a) \end{pmatrix} \quad (6.26)$$

such that the brane metric can be written as

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \psi^2/m^2 \end{pmatrix} \quad (6.27)$$

We can then construct the induced metric on the now embedded worldsheet as, by definition,

$$\Gamma_{ab} = G_{MN} \partial_a q^M \partial_b q^N = \partial_a x^\mu \partial_b x_\mu + \frac{\psi^2}{m^2} \partial_a \varphi \partial_b \varphi \quad (6.28)$$

#### 6.3.1 Constraints from the auxiliary field

With all the machinery in place, we start by calculating the energy-momentum tensor by varying eq. (6.24) with respect to  $h^{ab}$

$$\begin{aligned} \delta \mathcal{S}_{\text{P}_w} &= \frac{-m^2}{2} \int d^2\xi \|\!|h|\!|^{1/2} h^{ab} \left( \partial_a x^\mu \partial_b x_\mu + \frac{\psi^2}{m^2} \partial_a \varphi \partial_b \varphi \right) \delta h^{ab} \\ &= \frac{-m^2}{2} \int d^2\xi \left( \|\!|h|\!|^{1/2} \partial_a x^\mu \partial_b x_\mu \right) \delta h^{ab} - \left( \frac{1}{2} \|\!|h|\!|^{1/2} h_{ab} h^{ij} \partial_i x^\mu \partial_j x_\mu \right) \delta h^{ab} \\ &\quad + \left( \|\!|h|\!|^{1/2} \frac{\psi^2}{m^2} \partial_a \varphi \partial_b \varphi \right) \delta h^{ab} - \left( \frac{1}{2} \frac{\psi^2}{m^2} \|\!|h|\!|^{1/2} h_{ab} h^{ij} \partial_i \varphi \partial_j \varphi \right) \delta h^{ab} \end{aligned}$$

Each term has a common factor of  $\|h\|^{1/2}$ , and since  $\delta S = 0$  the  $\delta h^{ab}$  variations must all vanish independently. We thus have

$$\begin{aligned} \partial_a x^\mu \partial_b x_\mu - \frac{1}{2} h_{ab} h^{ij} \partial_i X^\mu \partial_j X_\mu + \frac{\psi^2}{m^2} \partial_a \phi \partial_b \phi - \frac{\psi^2}{2m^2} h_{ab} h^{ij} \partial_i \phi \partial_j \phi &= 0 \\ \partial_a x^\mu \partial_b x_\mu + \frac{\psi^2}{m^2} \partial_a \phi \partial_b \phi &= \frac{1}{2} h_{ab} \left( \frac{\psi^2}{m^2} h_{ab} h^{ij} \partial_i \phi \partial_j \phi + h^{ij} \partial_i x^\mu \partial_j x_\mu \right) \end{aligned}$$

We therefore can solve for  $h_{ab}$ :

$$h_{ab} = 2f \left( \partial_a x^\mu \partial_b x_\mu + \frac{\psi^2}{m^2} \partial_a \phi \partial_b \phi \right) \quad (6.29)$$

where

$$f^{-1} = \frac{\psi^2}{m^2} h_{ab} h^{ij} \partial_i \phi \partial_j \phi + h^{ij} \partial_i x^\mu \partial_j x_\mu \quad (6.30)$$

This acts as a constraint for the dynamics of both  $x^\mu$  and  $\phi$ . The solution of  $h_{ab}$  given by eq. (6.29) is remarkable if we note that the terms in the brackets are the exact definition of the induced Kaluza-Klein brane metric, eq. (6.28). We can thus write

$$h_{ab} = 2f \Gamma_{ab} \quad (6.31)$$

We now have a constraint for the wiggly Polyakov action (6.24). From eq. (6.31):

$$\begin{aligned} h^{ab} &\propto f^{-1} \text{ (indices raised)} \\ \|h\|^{1/2} &\propto f \end{aligned}$$

Using these, in conjunction with eq. (6.28), the action (6.24) becomes

$$\mathcal{S}_{\text{P}_w} = \frac{-m^2}{2} \int d^2 \xi \|\Gamma\|^{1/2} \Gamma^{ab} \left( \partial_a x^\mu \partial_b x_\mu + \frac{\psi^2}{m^2} \partial_a \varphi \partial_b \varphi \right) \quad (6.32)$$

### 6.3.2 Equations of motion

We can now vary this action with respect to  $x_\mu$ ,

$$\begin{aligned} \delta \mathcal{S}_{\text{P}_w} &= \frac{-m^2}{2} \int d^2 \xi \|\Gamma\|^{1/2} \Gamma^{ab} \partial_a x^\mu \delta(\partial_b x_\mu) \\ &= 0 \end{aligned}$$

Standard procedure of integration by parts and throwing away the boundary terms, the equation of motion is thus

$$\partial_b \left( \|\Gamma\|^{1/2} \Gamma^{ab} \partial_a x^\mu \right) = 0 \quad (6.33)$$

We obtain an equation of the same form for  $\varphi$ :

$$\partial_b \left( \|\Gamma\|^{1/2} \Gamma^{ab} \partial_a \varphi \right) = 0 \quad (6.34)$$

We can combine equations (6.33) and (6.34) using the embedding (6.26) and write the single equation of motion as

$$\partial_b \left( \|\Gamma\|^{1/2} \Gamma^{ab} \partial_a q^M \right) = 0; \quad M = 1, 2 \quad (6.35)$$

Owing to the diffeomorphism and Weyl invariance of the Polyakov action (6.24), it is possible to choose various possible coordinate systems and worldsheet parametrisations. Two useful (and oft used) ones are the light-cone gauge and conformal gauge [22]. These gauges typically rescale 4-vectors, but the 5-vector analogues are discussed in [22], and the proof that the extended conformal gauge holds with an extra degree of freedom is given in [23]. We can thus choose

$$||\Gamma||^{1/2} \Gamma^{ab} = \eta^{ab} \quad (6.36)$$

This reduces the equations of motion to

$$\square_2 q^M = 0 \quad (6.37)$$

## 6.4 Cusp prevention in wiggly strings

There is an assumption made in all analyses of electromagnetic radiation from cusps [4, 13, 14]: back-reaction when cusps form is nominally omitted. In NG strings this can be ignored, since with a zero thickness approximation there is no back-reaction. However, wiggles are dynamical fields coupled to the strings, and so *should* affect the formation of cusps. We show explicitly here that cusp formation is unnatural when a cosmic string is wiggly. This section is based on the author's (and collaborators') work done in [103].

The general solution to (6.37) is given by [22]

$$x^M = \frac{1}{2} (x_+^M(\sigma + t) + x_-^M(\sigma - t)) \quad (6.38)$$

where we now have embedded left and right movers,  $q^M_{\pm}$ . From our generalised conformal gauge, our constraints read

$$\dot{x}^M \cdot x'_M = 0 \quad (6.39)$$

$$(\dot{x}^M)^2 + (x'^M)^2 = 0 \quad (6.40)$$

The constraints are, of course, comprised of both the  $x^\mu$  terms and the  $\varphi$  term. In full, constraint (6.39) is

$$\dot{x}^\mu \cdot x'_\mu + \frac{\psi^2}{m^2} \dot{\varphi} \varphi' = 0$$

As before, we fix worldsheet diffeomorphisms and Weyl invariance by choosing the extra condition of the static gauge,  $x^0 = t$ . This has our full constraint become

$$\dot{\mathbf{x}} \cdot \mathbf{x}' + \frac{\psi^2}{m^2} \dot{\varphi} \varphi' = 0$$

From the definition of a cusp,  $|\dot{\mathbf{x}}| = 1$  and  $\mathbf{x}' = 0$ , we now have

$$\dot{\varphi} \varphi' = 0 \quad (6.41)$$

Using constraint (6.40), we have

$$\begin{aligned} \dot{x}^\mu \cdot \dot{x}_\mu + \frac{\psi^2}{m^2} (\dot{\varphi})^2 + x'^\mu \cdot x'_\mu + \frac{\psi^2}{m^2} \varphi'^2 &= 0 \\ 1 - \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} - \mathbf{x}' \cdot \mathbf{x}' + \frac{\psi^2}{m^2} ((\dot{\varphi})^2 + \varphi'^2) &= 0 \\ (\dot{\varphi})^2 + \varphi'^2 &= 0 \end{aligned} \quad (6.42)$$

where in the second line we used the static gauge condition, and in the last line we applied the definition of the cusp. Therefore, from (6.41), we have

$$\dot{\varphi}(t_c, \sigma_c) = 0 \text{ or } \varphi'(t_c, \sigma_c) = 0 \quad (6.43)$$

where  $(t_c, \sigma_c)$  are the points at which the cusp condition is true. Now, from (6.42), when combined with (6.41), gives

$$\begin{aligned} \text{If } \dot{\varphi}(t_c, \sigma_c) = 0, \text{ then } \varphi'(t_c, \sigma_c) = 0; \\ \text{or if } \varphi'(t_c, \sigma_c) = 0, \text{ then } \dot{\varphi}(t_c, \sigma_c) = 0. \end{aligned} \quad (6.44)$$

i.e.

$$\dot{\varphi} = \varphi' = 0 \text{ at the cusp} \quad (6.45)$$

The result (6.45) is key, if one examines its physical content. The general solution of the equation of motion for  $\varphi$  has the form

$$\varphi(t, \sigma) = \varphi_+(\sigma - t) + \varphi_-(\sigma + t). \quad (6.46)$$

At the cusp, the constraints require

$$\partial_- f(\sigma_-) + \partial_+ g(\sigma_+) = 0, \quad (6.47)$$

and

$$\partial_- f(\sigma_-) - \partial_+ g(\sigma_+) = 0. \quad (6.48)$$

where  $\sigma_{\pm} = \sigma \pm t$  and  $\partial_{\pm} = \frac{\partial \sigma \pm \partial t}{2}$ . It implies  $f(\sigma_-) = c_1$  and  $g(\sigma_+) = c_2$ , where  $c_1$  and  $c_2$  are constants. Note that this is not a requirement neither from the equation of motion, nor from the constraints. Therefore, *setting wiggles to zero, in order to allow cusps to form, would require a fine-tuning of the initial conditions*. In the words of [15], formation of cusps in this scenario would demand a ‘conspiracy’.

# Chapter 7

## Conclusions

In this thesis, we have reviewed how topological defects in field theories give rise to the possible existence of cosmic strings in the early universe. We have studied the dynamics of these cosmic strings and have extended that formalism in two ways. In the first place, we have examined cusp solutions to the string equations of motion, and have considered how relativistic effects modify the conventionally calculated scaling law for the size of the cusp region. This has consequences primarily for the cosmic string model for FRBs.

In chapter 5.2, we developed the cusp solution and computed the cusp length sans relativity. We used this to estimate the power emitted when the cusp region decays. In chapter 5.4, we showed that by factoring in Lorentz contractions, the cusp length is different. We applied this cusp length to the equations for power and for the energy distribution of particles (assuming a simple multiplicity function, as fully calculated in chapter 5.3) to show that a FRB cosmic string model is *inconsistent* with the data.

In the second place, we studied wiggly strings and their dynamical properties. In chapter 6, we examined the actions for wiggly cosmic strings. In chapter 6.3 we used the Polyakov formulation in of wiggly strings to derive the equations of motion for a string coupled to an extra scalar field. representing the wiggle. In chapter 6.4, by applying cusp conditions, we showed for the first time using direct - and dynamical - arguments that wiggles prevent cusp formation, and vice-versa. This is not a no-go theorem as such, as a cusp could still form if the initial conditions for the wiggly field are fine-tuned. Naturally, however, cusps should not form. This has dire consequences for the cosmic string FRB model, since the primary emission mechanism should not be possible. This extrapolates further to all possible electromagnetic signatures from cosmic string cusps: gamma rays, radio bursts, neutrino fluxes, and so on.

In conclusion, we must take the sombre view that cosmic strings do not explain FRBs. Furthermore, the possibility of detecting these objects via electromagnetic radiation seems far less likely than previously thought if the strings which formed are indeed wiggly. However, it should be noted that we have excluded superconducting cosmic strings from our analyses. The coupling to a charged scalar field would prevent cusp formation much like a neutral wiggly field would, but superconducting strings have an additional gauge field. This may act as a mechanism with which coherent radiation is emitted. A wiggly superconducting Polyakov action, which makes the business of cusp analyses far more analytically tractable, is a possible subject of future work.

# Appendix A

## Proof of the equivalence of various stringy actions

Here we give a detailed demonstration that (1) the regular Polyakov action is classically equivalent to the NG action, and (2) the three wiggly string actions are equivalent.

### A.1 The classical Polyakov and NG strings

The Polyakov action is given by

$$\mathcal{S}_p = \frac{-m^2}{2} \int d^2\xi ||h||^{1/2} h^{ab} (\partial_a x^\mu \partial_b x_\mu) \quad (\text{A.1})$$

We find the energy-momentum tensor for this theory by varying the action with respect to the auxiliary metric,  $h^{ab}$ , and setting the variation to zero.

$$T_{ab} = \frac{\delta \mathcal{S}_p}{\delta h^{ab}} = 0 \quad (\text{A.2})$$

Using  $\delta ||h||^{1/2} = -1/2 ||h||^{1/2} h_{ab} \delta h^{ab}$  and the chain rule, we find

$$T_{ab} = \left( ||h||^{1/2} \partial_a x^\mu \partial_b x_\mu \right) \delta h^{ab} - \left( \frac{1}{2} ||h||^{1/2} h_{ab} h^{ij} \partial_i x^\mu \partial_j x_\mu \right) \delta h^{ab} = 0 \quad (\text{A.3})$$

Dividing through by the common factors of  $||h||^{1/2}$ , and solving for  $h_{ab}$ , we get

$$h_{ab} = 2f \partial_a x^\mu \partial_b x_\mu \quad (\text{A.4})$$

where  $f^{-1} = h^{ij} \partial_i x^\mu \partial_j x_\mu$ . Substituting (A.4) into (A.1), we note (1)  $h^{ab}$  scales as  $f^{-1}$  and  $||h||^{1/2}$  scales as  $f$ , and (2) that the inverse metric  $h^{ab}$  will also have a  $(\partial_a x^\mu \partial_b x_\mu)^{-1}$  term to cancel that same in (A.1). Hence we are left with

$$\mathcal{S}_p = -m^2 \int d^2\xi \sqrt{-\partial_a x^\mu \partial_b x_\mu} \quad (\text{A.5})$$

which is in fact the physical induced metric,  $\gamma_{ab} = \eta_{\mu\nu} \partial_a x^\mu \partial_b x^\nu$ . Hence,

$$\mathcal{S}_p = -m^2 \int d^2 ||\gamma||^{1/2} \quad (\text{A.6})$$

which is in fact the NG action.

## A.2 Wiggly actions

The claim is that the canonical wiggly NG action from [24] is equivalent to both the Kaluza-Klein action for a  $(d+2)$ -embedded string (discussed in [22]) and the wiggly Polyakov action [23]. It was already shown in chapter 5 that the wiggly Polyakov action is equivalent to the wiggly NG action. Now, we consider the Kaluza-Klein (KK) action:

$$\mathcal{S}_{kk} = -m^2 \int d^2\xi \|\Gamma\|^{1/2} \quad (\text{A.7})$$

Recall that  $\Gamma_{ij}$  is the induced embedded metric,

$$\Gamma_{ab} = \partial_a x^\mu \partial_b x_\mu + \frac{\psi^2}{m^2} \partial_a \varphi \partial_b \varphi \quad (\text{A.8})$$

We set the chiral normalisation term  $\psi^2 = m^2$  (since we don't consider a case for which that term will be non-constant, and may as well set it to 1 then), and compute the determinant of (A.8). To simplify the notation, we will call  $\partial_a \varphi \partial_b \varphi := \Lambda_{ab}$ . Notice also that the first term in (A.8) is the induced metric in the absence of embedding, i.e. the normal  $\gamma_{ab}$ . We must then compute the determinant of

$$\Gamma_{ab} = \gamma_{ab} + \Lambda_{ab} \quad (\text{A.9})$$

This is a 2x2 matrix, and as such the determinant is

$$\Gamma = (\gamma_{00} + \Lambda_{00})(\gamma_{11} + \Lambda_{11}) - (\gamma_{01} + \Lambda_{01})(\gamma_{10} + \Lambda_{10}) \quad (\text{A.10})$$

Since  $\gamma_{01} = \gamma_{10}$  (partial derivatives commute) and trivially  $\Lambda_{01} = \Lambda_{10}$ , this is in fact

$$\begin{aligned} \Gamma &= (\gamma_{00} + \Lambda_{00})(\gamma_{11} + \Lambda_{11}) - (\gamma_{01} + \Lambda_{01})^2 \\ &= \gamma_{00}\gamma_{11} + \gamma_{00}\Lambda_{11} + \gamma_{11}\Lambda_{00} + \Lambda_{00}\Lambda_{11} - (\gamma_{01})^2 - 2\gamma_{01}\Lambda_{01} - (\Lambda_{01})^2 \\ &= \gamma_{00}\gamma_{11} + \gamma_{00}\Lambda_{11} + \gamma_{11}\Lambda_{00} - (\gamma_{01})^2 - 2\gamma_{01}\Lambda_{01} \\ &= \gamma + \gamma_{00}\Lambda_{11} + \gamma_{11}\Lambda_{00} - 2\gamma_{01}\Lambda_{01} \end{aligned} \quad (\text{A.11})$$

In the third line we used the fact that  $\Lambda_{00}\Lambda_{11} = (\Lambda_{01})^2$ , since  $\Lambda_{00} = (\dot{\varphi})^2$ ,  $\Lambda_{11} = (\varphi')^2$ , and  $\Lambda_{01} = \dot{\varphi}\varphi'$ . In the last line we used the fact that  $\gamma = \gamma_{00}\gamma_{11} - (\gamma_{01})^2$  (cf. chapter 3).

Now, we're going to expand the wiggly NG action

$$\mathcal{S} = -\mu_0 \int \|\gamma\|^{1/2} \sqrt{1 - \gamma^{ab} \partial_a \varphi \partial_b \varphi} \quad (\text{A.12})$$

As per our redefinitions, want to expand

$$\|\gamma\|^{1/2} \sqrt{1 - \gamma^{ab} \partial_a \varphi \partial_b \varphi} = (\gamma \gamma^{ab} \Lambda_{ab} - \gamma)^{1/2} \quad (\text{A.13})$$

Ignoring the square root for now, this becomes

$$\gamma \gamma^{ab} \Lambda_{ab} - \gamma = \gamma (\gamma^{00} \Lambda_{00} + 2\gamma^{01} \Lambda_{01} + \gamma^{11} \Lambda_{11}) - \gamma \quad (\text{A.14})$$

We now note that the inverse components of  $\gamma_{ab}$  have the following form:  $\gamma^{00} = \gamma_{11}/\gamma$ ,  $\gamma^{11} = \gamma_{00}/\gamma$ ,  $\gamma^{01} = -\gamma_{10}/\gamma = -\gamma_{01}/\gamma$ , and so

$$\gamma \gamma^{ab} \Lambda_{ab} - \gamma = \gamma_{11} \Lambda_{00} - 2\gamma_{01} \Lambda_{01} + \gamma_{00} \Lambda_{11} - \gamma \quad (\text{A.15})$$

The negative of (A.11) is  $-\Gamma = \gamma \gamma^{ab} \Lambda_{ab} - \gamma$ , and indeed then by square rooting that

$$\|\Gamma\|^{1/2} = \|\gamma\|^{1/2} \sqrt{1 - \gamma^{ab} \partial_a \varphi \partial_b \varphi} \quad (\text{A.16})$$

Hence the KK action is identical the wiggly NG formulation; and since the NG formulation is shown to be equivalent to the Polyakov action, it follows that all three wiggly string formulations are equivalent.

## Appendix B

# General equation of state for a wiggly string

Since we know what an equation of state is for a perturbed NG string, it follows that any action we write down to describe the dynamics of a so-called wiggly string must match this equation of state. However, the procedure of perturbing the NG string is not at all general. Here we'll show that the wiggly equation of state is merely a first order approximation to the general equation of state describing variable tension and energy parameters. We'll also show that for small-scales (which are the scales in which we're interested) the wiggly equation of state is a unique fixed point from a dynamical systems point of view, giving a more robust motivation for trusting the wiggly equation of state. The discussion in this section closely follows ref. [109]. Here we will use big  $U$  and not  $\mu$  for variable energy of the string, and  $\mu_0$  for the 'bare' quantity.

First, we will derive a general equation of motion for a cosmic string by considering that the EM tensor encodes, in principle, a full set of dynamical equations [110]. From [109], the induced EM tensor (akin to the induced metric) is

$$t^{ab}(\xi) = (U - T) u^a u^b + T \gamma^{ab} \quad (\text{B.1})$$

Here the energy and tension are functions of the worldsheet coordinate,  $U = U(\xi)$ ,  $T = T(\xi)$ . The fluid velocity is given by  $u^a(\xi)$ . The ambient energy-momentum tensor can then be constructed

$$T^{\mu\nu}(x) = \int d^2\xi \left( \|\gamma\|^{1/2} t^{ab}(\xi) \partial_a X^\mu \partial_b X^\nu \delta^{(4)}(x - X(\xi)) \right) \quad (\text{B.2})$$

In Minkowski space, the dynamics of a string described by (B.1) are given completely by [24, 110]

$$\partial_\mu T^{\mu\nu} = 0 \quad (\text{B.3})$$

when supplemented with an equation of state,  $T = T(U)$ . (For a nice review of dynamics as described by the EM tensor in a coordinate independent formalism, see [110]). Therefore

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= 0 \\ \partial_\mu \int d^2\xi \left( \|\gamma\|^{1/2} t^{ab}(\xi) \partial_a X^\mu \partial_b X^\nu \delta^{(4)}(x - X(\xi)) \right) &= 0 \\ \partial_\mu \left( \|\gamma\|^{1/2} t^{ab}(\xi) X^\mu \partial_b X^\nu \delta^{(4)}(x - X(\xi)) \right) & \\ - \partial_\mu \int d^2\xi \partial_a \left( \|\gamma\|^{1/2} t^{ab}(\xi) X^\mu \partial_b X^\nu \delta^{(4)}(x - X(\xi)) \right) &= 0 \\ \partial_a \left( \|\gamma\|^{1/2} t^{ab} \partial_b X^\nu \right) &= 0 \end{aligned} \quad (\text{B.4})$$

Line three followed from integration by parts, after which we threw away boundary terms and the Kronecker delta picked out one of the  $X^\mu$  terms. This is the generalised equation of motion for a string from an energy point of view. One can see that the NG string is included in this description. If we set  $U = T$ , then (B.1) becomes  $t^{ab} = T \gamma^{ab}$ , and (B.4) is

$$\partial_a (\|\gamma\|^{1/2} \gamma^{ab} T \partial_b X^\nu) = 0 \quad (\text{B.5})$$

which becomes  $\partial_a (T \partial_b X^\nu) = 0$  in the conformal gauge. For the temporal parametrisation,  $\xi^0 = t$ , it follows that  $\partial_a T = 0$  for  $\nu = 0$ , which means the tension/energy for an NG string is constant, and so must be for all  $\nu$  to have a consistent equation of state [109]. This means the formulation of the general string

allows one to “derive” what is otherwise taken to be the definition of the NG string (i.e. a string such that  $U = T = \mu_0$ ).

We have formulated a general description of string dynamics. The description would be complete if we had an explicit equation of state, from which we could find solutions. However, if we define first what kind of solution we want (a wiggly one, for some *a priori* definition of what a wiggle is), we can then work out what kind of equation of state gives rise to such solutions. The equation of state describes the microscopic configurations of the string by considering a given macrophysical setup and averaging over the micro-interactions. As we perturb the string, either transversely or longitudinally, the values of  $U$  and  $T$  change too. The relation between the modified energy and tension will give an appropriate equation of state.

First we’ll consider transverse wiggles, perturbations perpendicular to the string (like when a whip is whipped). If we orientate our string along the x-axis, then  $X^\nu = (t, \xi, y(\xi, t), z(\xi, t))$ . Eq. (B.4) becomes a set of four equations which read

$$\partial_a(|\gamma|^{1/2}\gamma^{ab}T\partial_b X^\nu) = \begin{cases} \partial_a(|\gamma|^{1/2}t^{a0}) = 0, & \nu = 0 \\ \partial_a(|\gamma|^{1/2}t^{a1}) = 0, & \nu = 1 \\ \partial_a(|\gamma|^{1/2}t^{ab}\partial_b y) = 0, & \nu = 2 \\ \partial_a(|\gamma|^{1/2}t^{ab}\partial_b z) = 0, & \nu = 3. \end{cases} \quad (\text{B.6})$$

Since  $y(\xi, t)$  and  $z(\xi, t)$  are wiggles, essentially, their evolution is independent from  $t^{ab}$ . There we can combine the first two and second two equations in (B.6) to get

$$\partial_a(|\gamma|^{1/2}) = 0 \quad (\text{B.7})$$

$$t^{ab}\partial_a\partial_b y(\xi, t) = t^{ab}\partial_a\partial_b z(\xi, t) = 0 \quad (\text{B.8})$$

We can now solve these evolution equations using appropriate initial conditions. We define the velocity of the string as  $\beta = u^1/u^0$ , and take initial conditions to be  $\beta = y = z = 0$ , and  $U = U_0$ ,  $T = T_0$ . (Subscript zero indicates the bare, or NG, energy and string tension). We then give the string a transverse wiggle, i.e. an initial velocity  $\dot{y}(\xi, 0) = \beta_T \sin(kx)$  ( $k$  is the wave number which defines the scale of the wiggles). For  $\beta \ll 1$ , we can expand in powers of  $\beta_T$  and find three relations [109]

$$\begin{aligned} U_0\ddot{y} - T_0y'' &= 0 \\ \ddot{U}_1 - v_L^2\ddot{U} &= (-U_0\partial_t^2 + T_0\partial_x^2) \left( \frac{1}{2}\dot{y}^2 + \frac{1}{2}y'^2 \right) \\ (U_0 - T_0)\dot{\beta} &= T_0y'(\ddot{y} - y'') + v_L^2 \end{aligned}$$

Here we’ve defined

$$v_L = \left. \frac{-dT}{dU} \right|_0. \quad (\text{B.9})$$

We now have transverse wiggly equations of motion. These will be the same for the longitudinal wiggles, but with  $z$  and not  $y$ . With the wiggly equations in hand, we can derive the second order averaged energy momentum tensor, (B.2). For transverse wiggles, this generates the second order energy and tension,

$$\begin{aligned} \bar{U}_T &= U_0 + U_0(\langle \dot{y}^2 \rangle + \langle \dot{z}^2 \rangle) \\ \bar{T}_T &= T_0 - \frac{1}{2}(\langle \dot{y}^2 \rangle + \langle \dot{z}^2 \rangle) \left[ T_0 + U_0 + v_L^2 U_0 \left( 1 - \frac{U_0}{T_0} \right) \right] \end{aligned}$$

The longitudinal wiggles yield

$$\begin{aligned} \bar{U}_T &= U_0 + 2(U_0 - T_0)\langle \beta^2 \rangle \\ \bar{T}_T &= T_0 - (U_0 - T_0)\langle \beta^2 \rangle \left[ 1 + v_L^2 + (U_0 - T_0) \frac{d \ln(v_L)}{dU} \Big|_0 \right] \end{aligned}$$

These are the normalised energy and string tension, i.e. average out over all the wiggles (via the averaging of eq. (B.2), which has in it the wiggly information from (B.1)). The combined renormalisation equations are therefore

$$\frac{-dU}{d\ln(k)} = W_T(k)U + 2W_L(k)(U - T)$$

$$\frac{-dT}{d\ln(k)} = \frac{-1}{2}W_T(k) \left[ T + U + v_L^2 U \left( 1 - \frac{U}{T} \right) \right] - W_L(k)(U - T) \left[ 1 + v_L^2 + (U - T) \frac{d\ln(v_L)}{dU} \right]$$

$W_T$  and  $W_L$  are the transverse and longitudinal spectral densities of the normalised energy and tension, but on a  $\ln k$  scale. Combining and integrating these equations yields, to second order

$$\begin{aligned} U(k) &= \mu (1 + W_T(k_0) \ln(k_0/k)) \\ T(k) &= \mu (1 - W_T(k_0) \ln(k_0/k)) \end{aligned} \tag{B.10}$$

where  $k_0$  is an integration constant. In the neighbourhood of  $k = k_0$  (i.e. on specified *small scales*), we get

$$UT = \mu^2 \tag{B.11}$$

which is indeed the equation of state of a linearly perturbed NG string

## Appendix C

# A (brief) proof of the Goldstone Theorem

Here we give a short proof of the Goldstone theorem. Many versions of the proof exist. We follow the outline given in [28] and [111]. We begin by collecting the requirements of the Goldstone theorem. For some symmetry group  $G$ , there exists an order parameter  $\mu(x)$  that characterises the breaking of the symmetry. A broken symmetry is one for which the VEV

$$\langle 0 | \phi_0 | 0 \rangle \neq 0 \quad (\text{C.1})$$

The symmetry has an associated Noether current, which is conserved:

$$\partial_\mu j^\mu = 0, \quad (\text{C.2})$$

and therefore the generator of the group  $G$  may be defined as

$$Q(t) = \int d^3x j^0 \quad (\text{C.3})$$

Let's consider a theory with  $N$  fields  $\phi(x)$  which obeys a generic symmetry transformation,

$$\phi^i \rightarrow \phi^i + i\epsilon T^i_j \phi^j \quad (\text{C.4})$$

$T^i_j$  is a finite, imaginary matrix. We define an effective action in terms of  $\phi$ , one which encodes the generation of 1PI correlation functions.

$$\Gamma[\phi] = -\log Z[J] - \int d^d x j(x) \phi(x) \quad (\text{C.5})$$

Since the action is invariant under (C.4), we can write

$$\begin{aligned} \delta\Gamma[\phi] &= -\log Z[J] \delta\phi^m - \left( \int d^d x j(x) \phi(x) \right) \delta\phi^m \\ &= 0 - \delta\phi^m \int d^d x j(x) \\ &= 0 \end{aligned}$$

The second line follows from the fact that the partition function and Noether current are independent of the potentials, and that

$$\int d^d x \phi(x) \delta\phi^m = \delta\phi^m \quad (\text{C.6})$$

Dividing through by constants, we thus have that

$$\left( \frac{\delta\Gamma[\phi]}{\delta\phi^n} \right) T^n_m \phi^m = 0 \quad (\text{C.7})$$

We differentiate again, and evaluate at the minimum field value,

$$\left(\frac{\delta^2\Gamma[\phi]}{\delta\phi^l\delta\phi^n}\right)_{\phi=\phi_0} T^n{}_m\phi^m + \left(\frac{\delta\Gamma[\phi]}{\delta\phi^n}\right) T^n{}_m = 0 \quad (\text{C.8})$$

Since  $\phi_0$  is the minimum, the first derivative at that point vanishes. The second derivative is the definition of the mass matrix squared [28], and so

$$[M^2]_{ln} T^n{}_m\phi^m = 0 \quad (\text{C.9})$$

This is in fact an eigen-equation; as such,  $T^n{}_m\phi^m$  are eigenvectors of the matrix  $M$  with eigenvalues  $\lambda = 0$ . If, however, we have a broken symmetry such that the field goes from  $\phi_0 = 0$  to  $\phi_0 \neq 0$ , then it follows that the mass matrix  $M^2$  must have zero-valued eigenvalues, i.e. massless modes. These modes are generated by the VEV's going from  $\phi_0 = 0$  to  $\phi_0 \neq 0$ . If we have  $N$  fields, we must have a corresponding number of zero modes for each non-value field, i.e. for each broken generator. This indeed what the Goldstone theorem states: for each broken generator, there exists an equal number of massless propagation modes.

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