HIGH-ENERGY APPROXIMATIONS TO NUCLEAR SCATTERING

by

Bernd Schürmann

Thesis submitted for the degree of Doctor of Philosophy

in the Faculty of Science, University of Cape Town

September 1973

Promoter : Professor W.E. Frahn

To Katrin, Katja and Melanie

Acknowledgements

I wish to express my thanks to:

The German Academic Exchange Service and the South African Council for Scientific and Industrial Research for financial support

Mrs. E.B. Prosser for typing the manuscript

Mr. J. de la Cruz for assistance with the duplication

My wife for her help and encouragement

Abstract*

We study high-energy approximations to nuclear scattering. These are based on expansions of the free-particle propagator. We distinguish the eikonal expansion and the Fresnel expansion and interpret their physical meaning in optical terms.
A "Fresnel approximation" is defined as a partial sum of the eikonal expansion which describes Fresnel diffraction effects. In Fresnel approximation we derive, by means of a unitary transformation, a closed representation of the scattering amplitude which is formally similar to the corresponding expression in Glauber's high-energy scattering theory.
The corrections to the Glauber model of first order in the reciprocal wave number are given by the second-order eikonal approximation. These are evaluated explicitly for high-energy elastic scattering of protons from light nuclei (^{4}He, ^{12}C, ^{16}O).

*This thesis is a combined version of three papers which have been separately prepared and submitted for publication:

I. "High-energy approximations to nuclear scattering", by W.E. Frahn and B. Schürmann, submitted to Annals of Physics (N.Y.);

II. "Fresnel diffraction in high-energy multiple scattering", by B. Schürmann and W.E. Frahn, to be published in Nuclear Physics B;

III. "Eikonal and Fresnel corrections to the Glauber theory of nuclear multiple scattering", by B. Schürmann, submitted to Nuclear Physics B.

Contents

I. Introduction

Over the past few years there has been great interest in the theoretical study of hadron-nucleus scattering at high energies (several hundred MeV and above). This was initiated by the 1 GeV proton-nucleus scattering experiments performed at the Brookhaven National Laboratory in 1967 [1] with the aim of investigating the structure of light nuclei. High-energy hadron-nucleus scattering is a useful tool for such studies because the incident particle interacts strongly with the target nucleons, and because it has a wavelength ($\simeq$ 1 fm for 1 GeV protons) comparable to the internucleonic distances ($\approx$ 1.5 fm). Due to the strong interaction it is not sufficient to take account of single scattering processes only. Double scattering, triple scattering etc. play an important role. Whereas the single scattering contribution involves the one-particle nuclear distribution only, the higher-order terms depend on the two-particle density, the three-particle density etc., and thus contain information about nuclear structure.

There is another, more general aspect of high-energy nuclear scattering, however, with which we shall mainly be concerned in the present work. This is its usefulness in studying multiple scattering theories. The results obtained from such studies may be applied also to problems in branches other than nuclear physics, for which high-energy conditions hold. The expansion of the hadron-nucleus transition operator in a multiple scattering series is suggested by the fact that the projectile may collide with the individual target nucleons an arbitrary number of times. Each term in the series corresponds to a specific scattering sequence involving free hadron-nucleon effective interactions, which represent the projectile-target nucleon collisions, and free-space Green functions, which describe the propagation of the scattered wave between successive collisions. The transition operator taken between the initial and final target states then yields the hadron-nucleus scattering amplitude. For elastic scattering from not too light nuclei, an equivalent procedure consists in considering a multiple scattering series for the optical potential. Such an expansion is essentially the target ground state average of the transition T-matrix [2]. Inserted in the Schrödinger-equation for the projectile, the optical potential reproduces the exact amplitude for elastic scattering. The essential approximations in the multiple scattering approach outlined above are i) the replacement of (A+1)-body dynamics and kinematics by two-body dynamics and kinematics, ii) regarding the energy levels of the excited intermediate target states to be degenerate with the ground state energy. The latter approximation allows for a summation over all the intermediate states. Almost all of the various multiple scattering approaches discussed in the literature make use of the above assumptions in one way or another. Some of the more recent works will now briefly be discussed.

Watson's multiple scattering expansion [2-5] has been used extensively in the study of the optical potential up to second order [6-11]. A different method based on the work by Kerman, McManus and Thaler [12] has recently been developed by Feshbach and collaborators [13-15]. These authors formulate the scattering problem in terms of an infinite set of coupled equations, an approach which is familiar in low energy scattering (coupled channels method). The calculation of the optical potential including second-order corrections is then equivalent to solving a pair of coupled equations which contain the elastic channel and an effective one which accounts for all the inelastic channels. Another approach, also based on ref. [12], consists in replacing the second-order (non-local) optical potential by an equivalent local one [16]. Foldy and Walecka have obtained an explicit solution for the scattering amplitude assuming the projectile-target nucleon interaction potentials to be non-local and separable [17].

Although the above-mentioned methods all yield reasonably accurate results for elastic nuclear scattering, far more frequently used in practice is the multiple scattering theory proposed by Glauber [18,19]. It has the advantage over other models of being both remarkably simple and versatile, and is thus of considerable use also in branches other than nuclear physics [20-22].

In addition to the basic assumptions i) and ii), in the Glauber model the free-space Green functions are replaced by the so-called eikonal propagators [23,24]. These represent, at the same time, the asymptotic energy limit $k \to \infty$, where k is the wave number of the projectile, and propagation according to geometrical optics. Thus the Glauber (eikonal) approximation is the better the higher the energy. If the theory is applied at energies below the multi-GeV region, corrections due to deviations from the two aspects of the eikonal propagator are expected to become significant. In the last few years, there have been many efforts to improve the Glauber theory by taking account of these deviations [25-44]. Sugar and Blankenbecler [29] were the first to study corrections to the Glauber model in a systematic way by expanding the full free-space Green propagator in terms of a simplified propagator. A similar method will be employed in the present work where the expansion is made in terms of the eikonal and Fresnel propagators, respectively. The relationship between certain corrections to the Glauber model and Fresnel diffraction effects was first pointed out by Gottfried [36], who employed the Fresnel-Kirchhoff diffraction theory in a study of high-energy scattering from deuterium.

In the present work we study the effects of corrections to the eikonal propagator in a systematic fashion. Like in the article by Gottfried these are interpreted in wave-optical terms. In section II we develop approximation methods for the free-space Green propagators in the framework of scattering from a potential in the Schrödinger equation, which are based on expansions in terms of the eikonal and Fresnel propagators, respectively. Using results obtained in section II we study Fresnel diffraction effects for potential scattering in section III, and Fresnel and eikonal-type first-order corrections to the Glauber potential scattering amplitude in section IV. In section V, Fresnel diffraction in multiple scattering from complex nuclei is discussed. The influence of Fresnel and eikonal-type first-order corrections to the Glauber multiple scattering amplitude is studied in section VI, and some numerical results for elastic scattering from light nuclei are presented in the same section.

A short summary and conclusions are given in section VII. In appendices A and B we discuss the effects of the rescattering terms and the centre-of-mass motion in the Fresnel approximation to a Watson-type multiple scattering series. In appendix C an expression is derived which takes account of the first-order Fresnel correction to the Glauber multiple scattering amplitude for elastic scattering from a nucleus represented by an independent particle model. The derivation of the corresponding expression for potential scattering is given in appendix D. We confine ourselves to a non-relativistic treatment. Relativistic kinematics can, however, be included in the results for potential scattering and for the optical limit $A \to \infty$ (where A is the mass number of the target nucleus) by using the method described on p. 340 of ref. [2].

II. High-energy approximations to the free-space Green propagator

In the present section we develop systematic approximation procedures by suitable expansions of the full free-particle propagator. These are first formulated in momentum representation; by transformation to configuration space we obtain expressions for the various approximate Green functions which allow an interpretation of their physical meaning in wave-optical terms, much in the spirit of Gottfried's lucid article [36].

For simplicity, our approximation methods will be formulated for potential scattering, even though the potential concept has little significance at the energies for which they are valid. Application to multiple scattering of nucleons in complex nuclei will be presented in sections V and VI.

1. First-order eikonal approximation

The wave function for nonrelativistic scattering of a particle of mass m and wave number $\underline{k}$ by a local potential $V(\underline{r})$ satisfies the integral equation

$$\psi_s(\underline{r}) = e^{i\underline{k}\underline{r}} - \int d^3r' \, G(\underline{r},\underline{r}') \, V(\underline{r}') \, \psi_s(\underline{r}') \quad , \qquad \text{(II.1)}$$

where

$$G(\underline{r},\underline{r}') = \frac{m}{2\pi\hbar^2} \, \frac{e^{ik|\underline{r}-\underline{r}'|}}{|\underline{r}-\underline{r}'|} \qquad \text{(II.2)}$$

is the free-particle Green function for outgoing-wave boundary conditions. In momentum representation

$$G(\underset{\sim}{r},\underset{\sim}{r}') = \frac{2m}{\hbar^2}\int\frac{d^3p}{(2\pi)^3}\, e^{i\underset{\sim}{p}(\underset{\sim}{r}-\underset{\sim}{r}')}\,(p^2-k^2-i\epsilon)^{-1}$$

$$= \frac{2k}{\hbar v}\, e^{ik(z-z')}\int\frac{d^3\Delta}{(2\pi)^3}\, e^{i\underset{\sim}{\Delta}(\underset{\sim}{r}-\underset{\sim}{r}')}\,(\Delta^2+2\underset{\sim}{k}\underset{\sim}{\Delta}-i\epsilon)^{-1} \quad , \tag{II.3}$$

where we have introduced $\underset{\sim}{\Delta} = \underset{\sim}{p}-\underset{\sim}{k}$, taking $\underset{\sim}{k}$ in z-direction, and the particle speed $v = \hbar k/m$.

Now we split all momenta into transverse (index $\perp$) and longitudinal (index $\parallel$) components with respect to the z-direction; a similar split in coordinate space introduces the two-dimensional impact parameter vectors $\underset{\sim}{b}$. Then we may write

$$G(\underset{\sim}{b}-\underset{\sim}{b}', z-z')$$
$$= \frac{1}{\hbar v}\, e^{ik(z-z')}\int\frac{d^2\Delta^\perp}{(2\pi)^2}\int\frac{d\Delta^\parallel}{2\pi}\, e^{i\underset{\sim}{\Delta}^\perp(\underset{\sim}{b}-\underset{\sim}{b}')}\, e^{i\Delta^\parallel(z-z')}\,\tilde{G}(\underset{\sim}{\Delta}^\perp,\Delta^\parallel) \quad , \tag{II.4}$$

where

$$\tilde{G}(\underset{\sim}{\Delta}^\perp,\Delta^\parallel) = (h+\Delta^\parallel-i\epsilon)^{-1} \quad , \tag{II.5}$$

and where we have defined

$$h = h^\perp + h^\parallel = \frac{(\underset{\sim}{\Delta}^\perp)^2}{2k} + \frac{(\Delta^\parallel)^2}{2k} \quad . \tag{II.6}$$

The eikonal approximation employed in Glauber's theory consists in neglecting h compared with $\Delta^{\parallel}$, i.e., in replacing the propagator (II.5) by the "eikonal propagator"

$$\tilde{G}_e^{(1)} = (\Delta^{\parallel} - i\epsilon)^{-1} \quad . \tag{II.7}$$

The index (1) indicates that we shall regard this as the "first-order eikonal approximation", as distinct from higher-order eikonal approximations to be defined in subsect. 4.

By inserting eq. (II.7) in eq. (II.4) and noting that

$$\int \frac{d\Delta^{\parallel}}{2\pi} e^{i\Delta^{\parallel}(z-z')} \tilde{G}_e^{(1)}(\Delta^{\parallel}) = i\,\theta(z-z')$$

it is seen that in coordinate space the eikonal Green function has the form

$$G_e^{(1)}(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = \frac{i}{\hbar v}\,\delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}')\,\theta(z-z')\,e^{ik(z-z')} \quad , \tag{II.7a}$$

where $\theta(z)$ is the unit step function. The delta function of impact parameters means that the propagating wave suffers no transverse displacement; thus the eikonal approximation corresponds to propagation according to geometrical optics.

To solve the integral equation (II.1), the wave function is first written in the form of the incoming plane wave multiplied by a modulating function $\phi_s(\underset{\sim}{r})$,

$$\psi_s(\underset{\sim}{r}) = e^{i\underset{\sim}{k}\underset{\sim}{r}}\,\phi_s(\underset{\sim}{r}) \quad , \tag{II.8}$$

where $\phi_s(r)$ satisfies

$$\phi_s(\underset{\sim}{r}) = 1 - \int d^3r' \, e^{-ik(\underset{\sim}{r}-\underset{\sim}{r}')} G(\underset{\sim}{r},\underset{\sim}{r}') \, V(\underset{\sim}{r}') \, \phi_s(\underset{\sim}{r}') \quad . \tag{II.9}$$

By inserting (II.7a) in place of $G(\underset{\sim}{r},\underset{\sim}{r}')$ we have

$$\phi_e^{(1)}(\underset{\sim}{b},z) = 1 - \frac{i}{\hbar v}\int_{-\infty}^{\infty} dz' \, \theta(z-z') \, V(\underset{\sim}{b},z') \, \phi_e^{(1)}(\underset{\sim}{b},z') \quad . \tag{II.10}$$

This can be converted into a differential equation by differentiation with respect to z,

$$\frac{\partial}{\partial z}\phi_e^{(1)}(\underset{\sim}{b},z) = -\frac{i}{\hbar v} V(\underset{\sim}{b},z) \, \phi_e^{(1)}(\underset{\sim}{b},z) \quad . \tag{II.10a}$$

It is, of course, just the first-order eikonal approximation to the Schrödinger equation (compare eq. (III.6)). With the initial condition $\phi_e^{(1)}(\underset{\sim}{b},-\infty) = 1$, the solution of eq. (II.10a) and thus of eq. (II.10) is

$$\phi_e^{(1)}(\underset{\sim}{b},z) = \exp\left[-\frac{i}{\hbar v}\int_{-\infty}^{\infty} \theta(z-z') \, V(\underset{\sim}{b},z') \, dz'\right] \quad . \tag{II.11}$$

It means that the only effect of the interaction is a modulation of the incoming plane wave, by a phase shift in case of a real potential, and by a reduction in amplitude as well, in case of a complex potential.

The scattering amplitude has the general form

$$\begin{aligned} f_s(\underset{\sim}{q}) &= -\frac{m}{2\pi\hbar^2}\int e^{i\underset{\sim}{q}\underset{\sim}{r}} \, V(\underset{\sim}{r}) \, \psi_s(\underset{\sim}{r}) \, d^3r \\ &= -\frac{k}{2\pi\hbar v}\int d^2b \, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \int dz \, e^{iq^{\parallel}z} \, V(\underset{\sim}{b},z) \, \phi_s(\underset{\sim}{b},z) \quad , \end{aligned} \tag{II.12}$$

where $\hbar \underset{\sim}{q} = \hbar(\underset{\sim}{k}-\underset{\sim}{k}')$ is the momentum transfer. Since, for fixed momentum transfer,

$$q^{\parallel} = \frac{q^2}{2k} = \frac{(\underset{\sim}{q}^{\perp})^2}{2k} + \frac{(q^{\parallel})^2}{2k} = \frac{(\underset{\sim}{q}^{\perp})^2}{2k} + \frac{(\underset{\sim}{q}^{\perp})^4}{4k^2} + \cdots \tag{II.13}$$

is of order k^{-1} compared to $\underset{\sim}{q}^{\perp}$, and on the other hand $\phi_e^{(1)}$ represents the asymptotic energy limit $k \to \infty$, in the first-order eikonal approximation to eq. (II.12) we must replace the factor $\exp(iq^{\parallel} z)$ by unity. Hence

$$\begin{aligned} f_e^{(1)}(\underset{\sim}{q}^{\perp}) &= -\frac{k}{2\pi\hbar v}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \int dz\, V(\underset{\sim}{b},z)\, \phi_e^{(1)}(\underset{\sim}{b},z) \\ &= \frac{ik}{2\pi}\int d^2b\, e^{iq^{\perp}b}\,[1-e^{i\chi(\underset{\sim}{b})}] \quad , \end{aligned} \tag{II.14}$$

where $\frac{1}{2}\chi(\underset{\sim}{b})$ plays the role of the phase shift in impact parameter representation and is given by

$$\chi(\underset{\sim}{b}) = -\frac{i}{\hbar v}\int_{-\infty}^{\infty} V(\underset{\sim}{b},z)\, dz \quad . \tag{II.15}$$

Eqs. (II.14) and (II.15) are the well-known results of Glauber.

2. Second-order eikonal approximation

An alternative derivation of the eikonal result (II.11), which emphasizes the high-energy character of the eikonal approximation, proceeds through partial integration in eq. (II.9) in spherical coordinates over the cosine of the scattering angle, and dropping the resulting terms of order $(ka)^{-1}$ and higher, where a is a characteristic dimension of the scattering potential. This shows that the first-order eikonal approximation holds in the high-energy limit ka $\to \infty$.

To obtain corrections for lower energies it seems natural to carry the same procedure to first order in $(ka)^{-1}$. This has been done recently by Baker [43] who, after a somewhat lengthy calculation, arrives at a result which in our notation may be written in the form

$$\phi_e^{(2)}(\underset{\sim}{b},z) = \phi_e^{(1)}(\underset{\sim}{b},z) + \frac{1}{2\hbar k v} V(\underset{\sim}{b},z)\,\phi_e^{(2)}(\underset{\sim}{b},z)$$

$$+ \frac{1}{2\hbar k v}\int_{-\infty}^{\infty} dz'(z-z')\theta(z-z')\nabla_{\underset{\sim}{b}}^2 V(\underset{\sim}{b},z')\,\phi_e^{(2)}(\underset{\sim}{b},z') , \tag{II.16}$$

where $\nabla_{\underset{\sim}{b}}^2$ is the two-dimensional Laplacian in impact parameter space. Baker also derives a solution of eq. (II.16) for spherically symmetric potentials which we do not reproduce here. It shows that the modulating function $\phi_e^{(2)}$ differs from $\phi_e^{(1)}$ both in phase and in amplitude by terms of order $(ka)^{-1}$ involving the potential and its derivatives with respect to $\underset{\sim}{b}$ as well as terms proportional to z.

In subsect. 4 we shall rederive Baker's equation (II.16) in a very simple fashion as the second order of a systematic approximation method, and also give a physical interpretation of it in "optical" terms.

3. Fresnel approximation

In view of the fact that the (first-order) eikonal Green function (II.7a) represents propagation according to geometrical optics, one may look for corrections which describe wave-optical modifications of the scattered wave. This has been done by Gottfried [36] for high-energy scattering by deuterium. Let us forget about potential scattering for a moment and assume that the scatterer (nucleon within a nucleus) may be regarded as perfectly absorbing. The eikonal Green function (II.7a) then describes the formation of a cylindrical "shadow" extending to infinity behind the scatterer. In actual fact, diffraction at the scatterer causes a distortion of the geometric-optical wave field which destroys the shadow at distances of the order of and larger than the Rayleigh distance $d_R = ka^2$. Thus the eikonal Green function (II.7a) approximates the propagation only at distances $z \ll d_R$, and the Glauber approximation for multiple scattering will only be valid if the Rayleigh distance is large compared with the mean free path of the projectile in the target nucleus. Since d_R is proportional to k, this condition can always be satisfied if the energy is made high enough. At lower energies, however, the diffractive distortion of the geometric-optical shadow may become appreciable.

Gottfried takes these effects into account by employing the Fresnel-Kirchhoff formulation of Huygens' principle in representing the scattered wave function. This amounts to replacing the eikonal Green function (II.7a) by the "Fresnel Green function"

$$G_F^{(1)}(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = \frac{k}{2\pi\hbar v}\,\theta(z-z')\,e^{ik(z-z')}\,\frac{\exp\left[\frac{1}{2}ik(\underset{\sim}{b}-\underset{\sim}{b}')^2/(z-z')\right]}{z-z'} . \qquad \text{(II.17a)}$$

The index (1) again indicates that we shall regard this as the first order of a systematic "Fresnel approximation" method to be defined in the following subsection.

In momentum representation the Fresnel propagator has the form

$$G_F^{(1)} = (h^{\perp} + \Delta^{\parallel} - i\epsilon)^{-1} \quad , \tag{II.17}$$

i.e., it arises from the full propagator (II.5) by neglecting $h^{\parallel}$. The relation between (II.17a) and (II.17) was noticed by Gottfried (see footnote 9 of ref. [36].)

4. Systematic approximation methods

We shall now develop systematic approximation methods for improving the eikonal propagator (II.7) which proceed in different directions indicated by the "second-order approximation" and the "first-order Fresnel approximation", respectively.

We start with the full propagator (II.5) in momentum representation,

$$\tilde{G}(\underset{\sim}{\Delta}^{\perp}, \Delta^{\parallel}) = (h + \Delta^{\parallel} - i\epsilon)^{-1} \quad ,$$

and regard h as a "perturbation" of the eikonal term $\Delta^{\parallel}$. Then we may write the relation between $\tilde{G}$ and $\tilde{G}_e^{(1)}$ in the form

$$\tilde{G} = \tilde{G}_e^{(1)} - \tilde{G}_e^{(1)} h \tilde{G} \quad . \tag{II.18}$$

Alternatively, we may regard $h^{\parallel}$ as a "perturbation" of the Fresnel term $h^{\perp} + \Delta^{\parallel}$, and write the relation between $\tilde{G}$ and $\tilde{G}_F^{(1)}$ in the form

$$\tilde{G} = \tilde{G}_F^{(1)} - \tilde{G}_F^{(1)} h^{\parallel} \tilde{G} \quad . \tag{II.19}$$

Finally we have a relation between $\tilde{G}_F^{(1)}$ and $\tilde{G}_e^{(1)}$,

$$\tilde{G}_F^{(1)} = \tilde{G}_e^{(1)} - \tilde{G}_e^{(1)} h^{\perp} \tilde{G}_F^{(1)} \quad . \tag{II.20}$$

Our approximation methods now consist in iterating in eqs. (II.18) and (II.19) with respect to $\tilde{G}_e^{(1)}$ and $\tilde{G}_F^{(1)}$, respectively:

$$\tilde{G} = \tilde{G}_e^{(1)} - \tilde{G}_e^{(1)} h \tilde{G}_e^{(1)} + \tilde{G}_e^{(1)} h \tilde{G}_e^{(1)} h \tilde{G}_e^{(1)} \mp \cdots \tag{II.21}$$

and

$$\tilde{G} = \tilde{G}_F^{(1)} - \tilde{G}_F^{(1)} h^{\parallel} \tilde{G}_F^{(1)} + \tilde{G}_F^{(1)} h^{\parallel} \tilde{G}_F^{(1)} h^{\parallel} \tilde{G}_F^{(1)} \mp \cdots \quad . \tag{II.22}$$

We call (II.21) the "eikonal expansion" and (II.22) the "Fresnel expansion" of the free-particle propagator.

Going beyond the first-order eikonal (or Glauber) approximation $\tilde{G} \approx \tilde{G}_e^{(1)}$ to the next-higher order, we obtain

$$\tilde{G} \approx \tilde{G}_e^{(1)} - \tilde{G}_e^{(1)} (h^{\perp} + h^{\parallel}) \tilde{G}_e^{(1)} \equiv \tilde{G}_e^{(2)} \quad . \tag{II.23}$$

The physical meaning of this second-order eikonal approximation is best discussed in coordinate representation. The result is

$$G_e^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = e^{ik(z-z')}\Big[\frac{i}{\hbar v}\,\delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}')\,\theta(z-z')$$

$$-\frac{1}{2\hbar k v}\,\delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}')\,\delta(z-z') - \frac{1}{2\hbar k v}\,\nabla_{\underset{\sim}{b}}^2\,\delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}')(z-z')\,\theta(z-z')\Big] . \tag{II.23a}$$

Inserting (II.23a) in place of $G(\underset{\sim}{r}-\underset{\sim}{r}')$ in eq. (II.9) yields precisely Baker's equation (II.16).

It may be useful to indicate briefly how eq. (II.23a) is derived. When $\tilde{G}(\underset{\sim}{\Delta}^{\perp},\Delta^{\parallel})$ in eq. (II.4) is replaced by $\tilde{G}_e^{(2)}$ we obtain three contributions. The first, from $\tilde{G}_e^{(1)}$, yields $G_e^{(1)}(\underset{\sim}{b}-\underset{\sim}{b}', z-z')$. The third, from the term with $h^{\perp}$, involves the integral

$$\int \frac{d\Delta^{\parallel}}{2\pi}\, e^{i\Delta^{\parallel} z}\, \tilde{G}_e^{(1)} h^{\perp} \tilde{G}_e^{(1)} = \frac{(\underset{\sim}{\Delta}^{\perp})^2}{2k}\int \frac{d\Delta^{\parallel}}{2\pi}\, e^{i\Delta^{\parallel} z}\,(\Delta^{\parallel}-i\epsilon)^{-2}$$

$$= -\frac{(\underset{\sim}{\Delta}^{\perp})^2}{2k}\, z\,\theta(z) .$$

The second contribution, from the term with $h^{\parallel}$, involves

$$\int \frac{d\Delta^{\parallel}}{2\pi}\, e^{i\Delta^{\parallel} z}\, \tilde{G}_e^{(1)} h^{\parallel} \tilde{G}_e^{(1)} = \frac{1}{2k}\int \frac{d\Delta^{\parallel}}{2\pi}\, e^{i\Delta^{\parallel} z}\,(\Delta^{\parallel})^2(\Delta^{\parallel}-i\epsilon)^{-2}$$

$$= \frac{1}{2k}\,\delta(z) .$$

To obtain the last equality we have used $\tilde{G}_e^{(1)} h^{\parallel}\, \tilde{G}_e^{(1)} = 1/2k$. That this holds true can be seen by writing

$$(h+\Delta^{\parallel}-i\epsilon)^{-1} = i\pi\,\delta(h+\Delta^{\parallel}) + \mathcal{P}\,\frac{1}{h+\Delta^{\parallel}}$$

and expanding both terms in powers of h.

Going beyond the first-order Fresnel approximation $\tilde{G} \approx \tilde{G}_F^{(1)}$ to the next-higher order, we have

$$\tilde{G} \approx \tilde{G}_F^{(1)} - \tilde{G}_F^{(1)} h^{\parallel} \tilde{G}_F^{(1)} \equiv \tilde{G}_F^{(2)} \quad . \qquad \text{(II.24)}$$

In coordinate representation we find

$$G_F^{(2)}(\underline{b}-\underline{b}', z-z')$$

$$= \left[1 - \frac{1}{2}\frac{(\underline{b}-\underline{b}')^2}{(z-z')^2} - \frac{1}{4} i k \frac{(\underline{b}-\underline{b}')^4}{(z-z')^3}\right] G_F^{(1)}(\underline{b}-\underline{b}', z-z') \qquad \text{(II.24a)}$$

$$+ \frac{i}{4\pi\hbar v}\,\delta(z-z')\, e^{ik(z-z')}\, \frac{\exp\left[\frac{1}{2} i k (\underline{b}-\underline{b}')^2/(z-z')\right]}{z-z'} \quad .$$

Again we give a brief outline of how (II.24a) is derived. When $\tilde{G}(\underline{\Delta}^{\perp}, \Delta^{\parallel})$ in eq. (II.4) is replaced by $\tilde{G}_F^{(2)}$, the first contribution, from $\tilde{G}_F^{(1)}$, yields $G_F^{(1)}(\underline{b}-\underline{b}', z-z')$. The second contribution, from the term with $h^{\parallel}$, is

$$G_F^{(2)} - G_F^{(1)}$$

$$= -\frac{1}{2\hbar k v}\, e^{ik(z-z')} \int \frac{d^2\Delta^{\perp}}{(2\pi)^2} \int \frac{d\Delta^{\parallel}}{2\pi}\, e^{i\underline{\Delta}^{\perp}(\underline{b}-\underline{b}')}\, e^{i\Delta^{\parallel}(z-z')} (\Delta^{\parallel})^2 \left[\frac{(\underline{\Delta}^{\perp})^2}{2k} + \Delta^{\parallel} - i\epsilon\right]^{-2} .$$

Transforming from $\Delta^{\parallel}$ to $(\Delta^{\parallel})' = (\underset{\sim}{\Delta}^{\perp})^2/2k + \Delta^{\parallel}$ and thereafter dropping the prime yields

$$G_F^{(2)} - G_F^{(1)} = -\frac{1}{2\hbar k v} e^{ik(z-z')} \int \frac{d^2\Delta^{\perp}}{(2\pi)^2} e^{i\underset{\sim}{\Delta}^{\perp}(\underset{\sim}{b}-\underset{\sim}{b}')} e^{-\frac{1}{2}i\frac{(\underset{\sim}{\Delta}^{\perp})^2}{k}(z-z')}$$

$$\times \int \frac{d\Delta^{\parallel}}{2\pi} e^{i\Delta^{\parallel}(z-z')} \left[(\Delta^{\parallel})^2 - \frac{(\underset{\sim}{\Delta}^{\perp})^2}{k}\Delta^{\parallel} + \frac{(\underset{\sim}{\Delta}^{\perp})^4}{4k^2} \right] (\Delta^{\parallel} - i\epsilon)^{-2} .$$

With

$$\left[(\Delta^{\parallel})^2 - \frac{(\underset{\sim}{\Delta}^{\perp})^2}{k}\Delta^{\parallel} + \frac{(\underset{\sim}{\Delta}^{\perp})^4}{4k^2} \right] (\Delta^{\parallel} - i\epsilon)^{-2} = 1 - \frac{(\underset{\sim}{\Delta}^{\perp})^2}{k}(\Delta^{\parallel}-i\epsilon)^{-1} + \frac{(\underset{\sim}{\Delta}^{\perp})^4}{4k^2}(\Delta^{\parallel}-i\epsilon)^{-2}$$

the integration over $\Delta^{\parallel}$ can be carried out with the result

$$G_F^{(2)} - G_F^{(1)} = -\frac{1}{2\hbar k v} e^{ik(z-z')} \int \frac{d^2\Delta^{\perp}}{(2\pi)^2} e^{i\underset{\sim}{\Delta}^{\perp}(\underset{\sim}{b}-\underset{\sim}{b}')} e^{-\frac{1}{2}i\frac{(\underset{\sim}{\Delta}^{\perp})^2}{k}(z-z')}$$

$$\times \left[\delta(z-z') - i\frac{(\underset{\sim}{\Delta}^{\perp})^2}{k}\theta(z-z') - \frac{(\underset{\sim}{\Delta}^{\perp})^4}{4k^2}(z-z')\theta(z-z') \right] .$$

The integration over $\underset{\sim}{\Delta}^{\perp}$ is then straightforward.

5. Partial summation of the eikonal expansion

After writing eq. (II.18) in the form

$$\tilde{G} = \tilde{G}_e^{(1)} - \tilde{G}_e^{(1)} h^{\perp} \tilde{G} - \tilde{G}_e^{(1)} h^{\parallel} \tilde{G} , \tag{II.25}$$

the eikonal expansion (II.21) may be regarded as composed of three contributions,

$$\tilde{G} = \tilde{G}_1 + \tilde{G}_2 + \tilde{G}_3 , \tag{II.26}$$

where $\tilde{G}_1$ contains $\tilde{G}_e^{(1)}$ plus the terms involving $h^{\perp}$ only, $\tilde{G}_2$ those involving $h^{\parallel}$ only, and $\tilde{G}_3$ the "mixed" terms involving products of $h^{\perp}$ and $h^{\parallel}$.

Since all the factors in the expansion are ordinary functions and commute, we have

$$\tilde{G}_1 = \tilde{G}_e^{(1)} - \tilde{G}_e^{(1)2} h^{\perp} + \tilde{G}_e^{(1)3} (h^{\perp})^2 \mp \cdots = \tilde{G}_e^{(1)} \sum_{n=0}^{\infty} (-1)^n (\tilde{G}_e^{(1)} h^{\perp})^n = \tilde{G}_F^{(1)} \;, \quad \text{(II.27)}$$

where the last equality follows from eq. (II.20). Thus the sum total of the terms involving $h^{\perp}$ in the eikonal expansion is just the first-order Fresnel approximation. Hence the coordinate representation $G_1(\underset{\sim}{b}-\underset{\sim}{b}', z-z')$ is given by eq. (II.17a).

The second contribution can also be summed in closed form,

$$\tilde{G}_2 = -\tilde{G}_e^{(1)2} h^{\parallel} + \tilde{G}_e^{(1)3} (h^{\parallel})^2 \mp \cdots = \tilde{G}_e^{(1)} \sum_{n=1}^{\infty} (-1)^n (\tilde{G}_e^{(1)} h^{\parallel})^n \;. \quad \text{(II.28)}$$

With $\tilde{G}_e^{(1)} h^{\parallel} = \Delta^{\parallel}/2k$ and $\tilde{G}_e^{(1)2} h^{\parallel} = (2k)^{-1}$ we have

$$\tilde{G}_2 = -\frac{1}{2k} \sum_{n=0}^{\infty} (-1)^n (\Delta^{\parallel}/2k)^n \;. \quad \text{(II.29)}$$

In coordinate representation this yields

$$G_2(\underset{\sim}{b}-\underset{\sim}{b}', z-z')$$

$$= -\frac{1}{2\hbar k v} \delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}') e^{ik(z-z')} \left[\sum_{n=0}^{\infty} \left(-\frac{1}{2k}\right)^n \frac{\partial^n}{\partial z^n} \right] \delta(z-z') \quad \text{(II.30)}$$

$$= -\frac{1}{2\hbar k v} \delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}') e^{ik(z-z')} \left(1 + \frac{1}{2k}\frac{\partial}{\partial z}\right)^{-1} \delta(z-z') \;.$$

Since we have closed expressions for $G_1 = G_F^{(1)}$ and G_2, and of course for the full propagator G, the mixed contribution G_3 may also be regarded as known in closed form,

$$G_3 = G - G_F^{(1)} - G_2 \quad . \tag{II.31}$$

6. Physical interpretation

The two expansions of the full free-particle propagator defined in subsect. 4, eqs. (II.21) and (II.22), yield successive approximations subject to the conditions

$$h^{\parallel} + h^{\perp} \ll \Delta^{\parallel} \quad , \qquad \text{eikonal approximation,} \tag{II.32}$$

$$h^{\parallel} \ll h^{\perp} \approx \Delta^{\parallel} \quad , \qquad \text{Fresnel approximation.} \tag{II.33}$$

Thus the Fresnel approximation only requires that $h^{\parallel} = \Delta^{\parallel 2}/2k$ is small compared with the eikonal term $\Delta^{\parallel}$, whereas in the eikonal approximation the sum $h = (\Delta^{\perp 2} + \Delta^{\parallel 2})/2k$ is small compared to $\Delta^{\parallel}$. Indeed, as we have shown in the preceding subsection, the first-order Fresnel approximation $\tilde{G}_F^{(1)}$ already represents a partial sum of the eikonal expansion: it comprises to all orders the terms involving $h^{\perp}$ only. Thus the two expansions are physically distinct, emphasizing different aspects of the scattering process.

The Fresnel expansion concerns the wave-optical (diffractive) deviations, in the transverse direction, from geometric-optical propagation. As we have discussed earlier, following Gottfried's interpretation, the first-order Fresnel approximation accounts for the diffractive distortion of the geometrical shadow beyond the Rayleigh distance ka^2. This is caused by a distortion of the scattered wave front due to diffraction at the boundary of the scatterer. The Fresnel propagator $G_F^{(1)}$ describes this distortion to first-order as a transverse curvature of the wave front, by retaining the term proportional to $(\underset{\sim}{b}-\underset{\sim}{b}')^2$ in the phase of the Green function. This is best seen by an alternative derivation of $G_F^{(1)}$ from the full free-particle Green function in the form

$$G(\underset{\sim}{b}-\underset{\sim}{b}', z-z')$$

$$= \frac{k}{2\pi\hbar v} \frac{e^{ik[(\underset{\sim}{b}-\underset{\sim}{b}')^2+(z-z')^2]^{1/2}}}{[(\underset{\sim}{b}-\underset{\sim}{b}')^2+(z-z')^2]^{1/2}} = \frac{k}{2\pi\hbar v} \frac{e^{ik(z-z')(1+\beta)^{1/2}}}{(z-z')(1+\beta)^{1/2}} \quad . \tag{II.34}$$

Expansion in powers of $\beta = (\underset{\sim}{b}-\underset{\sim}{b}')^2/(z-z')^2$ and retaining the leading term in the phase only, yields eq. (II.17a) except for the factor $\theta(z-z')$ which restricts propagation to the forward direction.

The integral equation (II.1) or (II.9) may be regarded as an exact formulation of (the quantal analogue of) Huygens' principle: the scattered wave at observation point $\underset{\sim}{r}$ is formed by interference of freely propagating spherical waves $G(\underset{\sim}{r}-\underset{\sim}{r}')$ originating with amplitude $V\psi$ from each source point $\underset{\sim}{r}'$ in the interaction region. The first-order Fresnel approximation represents an approximate form of Huygens' principle (corresponding to the Fresnel-Kirchhoff approximation in physical optics), in which only the leading distortion of the wave front in transverse direction (i.e., over the $\underset{\sim}{b}$-plane) is retained and the longitudinal (z) propagation is restricted to the forward direction. Higher-order terms in the Fresnel expansion correspond to higher powers of β in the expansion of both amplitude and phase of G in eq.(II.34). This may be seen from the form (II.24a) of the second-order Fresnel propagator.

Each term of the Fresnel expansion contains contributions of all orders in k^{-1}. If such a term is further expanded in powers of k^{-1}, the transverse curvature of the wave front is represented by powers of the transverse Laplacian $\nabla_{\underset{\sim}{b}}^2$. For instance, the expansion of the Fresnel propagator $G_F^{(1)}$ with respect to k^{-1} is

$$G_F^{(1)}(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = e^{\frac{i}{2k}(z-z')\nabla_{\underset{\sim}{b}}^2}\; G_e^{(1)}(\underset{\sim}{b}-\underset{\sim}{b}', z-z')$$

$$= \theta(z-z')\, e^{ik(z-z')}\, \frac{1}{\hbar v}\left[i\,\delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}') - \frac{z-z'}{2k}\nabla_{\underset{\sim}{b}}^2\, \delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}') + \cdots \right], \tag{II.35}$$

which is of course nothing else than the coordinate representation of the expansion obtained by iteration of eq. (II.20),

$$\tilde{G}_F^{(1)} = \tilde{G}_e^{(1)} - \tilde{G}_e^{(1)} h^{\perp} \tilde{G}_e^{(1)} + \tilde{G}_e^{(1)} h^{\perp} \tilde{G}_e^{(1)} h^{\perp} \tilde{G}_e^{(1)} \mp \cdots \quad . \tag{II.36}$$

Comparison with eq. (II.21) shows that this forms part $(\tilde{G}_1)$ of the eikonal expansion which we consider next.

The eikonal expansion represents a systematic expansion in increasing orders of k^{-1}, or rather in powers of $(ka)^{-1}$ where a is the characteristic dimension of the scatterer. It therefore concerns the deviations from the asymptotic energy limit $ka \to \infty$. In each order beyond $G_e^{(1)}$ it collects contributions describing both longitudinal (eikonal type) and transverse (Fresnel type) distortions of the geometric-optical pattern represented by $G_e^{(1)}$. For instance, in the second-order eikonal approximation $G_e^{(2)}$ these are given by the second and third terms, respectively, in eq. (II.23a). The partial summations given in subsect. 5 show that the transverse distortions, represented by the derivatives of $\delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}')$, result mainly from the contribution $G_1 = G_F^{(1)}$ determined by $h^{\perp}$, and that the longitudinal distortions, described by the derivatives of $\theta(z-z')$, are mainly due to the contribution G_2 determined by $h^{\parallel}$. However, the presence of the "mixed" contribution G_3 shows that there is no clear-cut separation between transverse and longitudinal distortions in the eikonal expansion.

III. Fresnel approximation in potential scattering

In the preceding section we have seen that the (first-order) Fresnel and eikonal Green functions are related by a unitary operator,

$$G_F^{(1)}(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = e^{-\frac{i}{\hbar v}\hat{K}_\perp (z-z')} G_e^{(1)}(\underset{\sim}{b}-\underset{\sim}{b}', z-z') , \tag{III.1}$$

where

$$\hat{K}_\perp = -\frac{\hbar^2}{2m}\nabla^2_{\underset{\sim}{b}} \tag{III.2}$$

is the transverse kinetic energy operator. This remarkable connection allows us to represent the Fresnel approximation of the scattering amplitude by a closed expression which is formally analogous to the eikonal (Glauber) amplitude. This is achieved by means of a unitary transformation of the Fresnel solution which is a precise formal analog of the transformation from the Schrödinger picture to the interaction picture in time-dependent perturbation theory.

Starting from the Schrödinger equation for the modulating function $\phi_s(\underset{\sim}{r}) = \phi_s(\underset{\sim}{b}, z)$ defined by eq. (II.8),

$$\left(i\hbar v \frac{\partial}{\partial z} - \hat{K}\right)\phi_s(\underset{\sim}{b}, z) = V(\underset{\sim}{b}, z)\phi_s(\underset{\sim}{b}, z) , \tag{III.3}$$

where

$$\hat{K} = \hat{K}_\perp + \hat{K}_\parallel = -\frac{\hbar^2}{2m}\nabla^2_{\underset{\sim}{b}} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} \tag{III.4}$$

is the full kinetic energy operator, the Fresnel and eikonal wave functions ϕ_F and ϕ_e satisfy the wave equations

$$\left(i\hbar v \frac{\partial}{\partial z} - \hat{K}_\perp\right)\phi_F(\underset{\sim}{b}, z) = V(\underset{\sim}{b}, z)\, \phi_F(\underset{\sim}{b}, z) \tag{III.5}$$

and

$$i\hbar v \frac{\partial}{\partial z} \phi_e(\underset{\sim}{b}, z) = V(\underset{\sim}{b}, z)\phi_e(\underset{\sim}{b}, z) \,, \tag{III.6}$$

respectively. These equations are formally analogous to the time-dependent Schrödinger equation with a "time parameter" $\tau = z/v$ and a time-dependent interaction Hamiltonian. (According to ref. [13] , this analogy for the eikonal equation (III.6) was previously noticed by Reading.)

By a unitary transformation from the "Fresnel picture" $|\phi_F\rangle$ to an "interaction picture" $|\phi\rangle$ with respect to the interaction $V(\underset{\sim}{b}, z)$,

$$\phi_F(\underset{\sim}{b}, z) = e^{-\frac{i}{\hbar v}\hat{K}_\perp z}\, \phi(\underset{\sim}{b}, z) \quad , \tag{III.7}$$

eq. (III.5) transforms to an equation of "eikonal" type,

$$i\hbar v \frac{\partial}{\partial z}\phi(\underset{\sim}{b}, z) = \hat{V}(\underset{\sim}{b}, z)\, \phi(\underset{\sim}{b}, z) \quad , \tag{III.8}$$

with an interaction operator

$$\hat{V}(\underset{\sim}{b}, z) = e^{\frac{i}{\hbar v}\hat{K}_\perp z}\, V(\underset{\sim}{b}, z)\, e^{-\frac{i}{\hbar v}\hat{K}_\perp z} \quad . \tag{III.9}$$

The "z-translation operator" $\hat{U}_{\underset{\sim}{b}}(z,z_0)$ defined by

$$\phi(\underset{\sim}{b},z) = \hat{U}_{\underset{\sim}{b}}(z,z_0)\,\phi(\underset{\sim}{b},z_0) \tag{III.10}$$

satisfies the same equation (III.8); in integrated form

$$\hat{U}_{\underset{\sim}{b}}(z,z_0) = 1 - \frac{i}{\hbar v}\int_{z_0}^{z} dz'\,\hat{V}(\underset{\sim}{b},z')\,\hat{U}_{\underset{\sim}{b}}(z',z_0)\ . \tag{III.11}$$

Equation (III.11) is formally solved by a Dyson-type perturbation expansion [45]

$$\hat{U}_{\underset{\sim}{b}}(z,z_0) = 1 + \sum_{n=1}^{\infty}\left(-\frac{i}{\hbar v}\right)^n \frac{1}{n!}\int_{z_0}^{z} dz_n \cdots \int_{z_0}^{z} dz_1\, Z\left[\hat{V}(\underset{\sim}{b},z_n)\cdots\hat{V}(\underset{\sim}{b},z_1)\right], \tag{III.12}$$

where the z-ordering operator Z arranges the noncommuting interaction operators $\hat{V}$ in the order of increasing z-values from right to left. Writing symbolically

$$\hat{U}_{\underset{\sim}{b}}(z,z_0) = Z\exp\left[-\frac{i}{\hbar v}\int_{z_0}^{z} dz'\,\hat{V}(\underset{\sim}{b},z')\right]\ , \tag{III.13}$$

the total wave function in the "eikonal interaction picture" becomes

$$\psi(\underset{\sim}{b},z) = e^{ikz}\phi(\underset{\sim}{b},z) = e^{ikz}\,\hat{U}_{\underset{\sim}{b}}(z,-\infty)\,\phi(\underset{\sim}{b},-\infty)\ . \tag{III.14}$$

Its asymptotic form for $z \to \infty$,

$$\lim_{z\to\infty}\psi(\underset{\sim}{b},z) = e^{ikz}\phi(\underset{\sim}{b},\infty) = e^{ikz}\,S(\underset{\sim}{b}) \tag{III.15}$$

defines the function $S(\underset{\sim}{b})$ by the action of the operator $\hat{U}_{\underset{\sim}{b}}(\infty,-\infty) \equiv \hat{S}_{\underset{\sim}{b}}$ on the constant function $\phi(\underset{\sim}{b},-\infty) = 1$,

$$S(\underset{\sim}{b}) = \hat{U}_{\underset{\sim}{b}}(\infty,-\infty)\,\phi(\underset{\sim}{b},-\infty) \equiv \langle \hat{S}_{\underset{\sim}{b}} \rangle \; . \tag{III.16}$$

Thus $S(\underset{\sim}{b})$ is the expectation value, in the constant state $\phi(\underset{\sim}{b},-\infty) = 1$, of the scattering operator

$$\begin{aligned} \hat{S}_{\underset{\sim}{b}} &= \hat{U}_{\underset{\sim}{b}}(\infty,-\infty) \\ &= 1 + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar v}\right)^n \frac{1}{n!} \int_{-\infty}^{\infty} dz_n \cdots \int_{-\infty}^{\infty} dz_1 \, Z\left[\hat{V}(\underset{\sim}{b},z_n)\cdots\hat{V}(\underset{\sim}{b},z_1)\right] \\ &= Z\, e^{i\hat{\chi}(\underset{\sim}{b})} \quad , \end{aligned} \tag{III.17}$$

where

$$\hat{\chi}(\underset{\sim}{b}) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} dz\, \hat{V}(\underset{\sim}{b},z) \tag{III.18}$$

is the "phase shift operator" in the interaction picture.

With the profile function

$$\Gamma(\underset{\sim}{b}) = 1 - S(\underset{\sim}{b}) \tag{III.19}$$

the scattering amplitude (as a function of momentum transfer $\underset{\sim}{q}^{\perp}$) takes the form

$$f(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}\,\Gamma(\underset{\sim}{b}) = \frac{ik}{2\pi}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}\left[1 - e^{i\chi(\underset{\sim}{b})}\right] \tag{III.20}$$

where

$$e^{i\chi(\underset{\sim}{b})} \equiv \langle Z\, e^{i\hat{\chi}(\underset{\sim}{b})} \rangle \; . \qquad \text{(III.21)}$$

Thus the Fresnel approximation has been cast into a form which resembles Glauber's expression for the eikonal approximation to potential scattering. In section V it will be shown that eq. (III.20) is identical with the amplitude obtained in the optical limit of the Fresnel approximation to multiple scattering.

IV. Eikonal and Fresnel corrections to the Glauber theory of nuclear potential scattering

Although eq. (III.20) shows that Fresnel diffraction in high-energy scattering can be described by a closed "Glauber-type" expression of the scattering amplitude, this result may appear to be merely of formal interest and of no immediate practical use. Of course, there would be no point in trying to evaluate the perturbation expansion in powers of the potential operator $\hat{V}$; thereby one would lose the decisive advantage of Glauber theory (over, say, the Born approximation) of incorporating the effects of the interaction potential to all orders.

However, the result (III.20) is useful as a starting point of a different approximation method, by expanding the interaction operator $\hat{V}$ (defined by eq. (III.9)) in powers of $(z/\hbar v)\hat{K}_{\perp} = -(z/2k)\nabla_{\underset{\sim}{b}}^{2}$ which because of its proportionality to k^{-1} is the appropriate expansion operator at high energy. In fact, this approximation procedure is equivalent to the expansion (II.35) or (II.36) of the Fresnel propagator in powers of the eikonal propagator.

To first order in this expansion, $\hat{V} = V$, and eq. (III.20) reduces to the Glauber formula (II.14). To second order,

$$\hat{V} = V + i\frac{z}{2k}\left[V, \nabla_{\underset{\sim}{b}}^{2}\right] \quad . \tag{IV.1}$$

Inserting (IV.1) in (III.17) and retaining only contributions of order k^{-1} in each term of the perturbation series yields

$$S(\underset{\sim}{b}) = S_G(\underset{\sim}{b})\left[1 - \frac{i}{k}\, d_F(\underset{\sim}{b})\right] . \tag{IV.2}$$

Here

$$S_G(\underset{\sim}{b}) = e^{i\chi_G(\underset{\sim}{b})} \tag{IV.3}$$

is the Glauber contribution with the Glauber phase shift (II.15),

$$\chi_G(\underset{\sim}{b}) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} dz\, V(\underset{\sim}{b}, z) .$$

The function

$$d_F(\underset{\sim}{b}) = d_{F1}(\underset{\sim}{b}) + d_{F2}(\underset{\sim}{b}) \tag{IV.4}$$

defines the Fresnel correction, with

$$d_{F1}(\underset{\sim}{b}) = -\frac{i}{2\hbar v}\nabla_{\underset{\sim}{b}}^2 \int_{-\infty}^{\infty} dz\, z\, V(\underset{\sim}{b}, z) \tag{IV.5}$$

and

$$\begin{aligned} d_{F2}(\underset{\sim}{b}) = {} & \left(\frac{1}{\hbar v}\right)^2 \int_{-\infty}^{\infty} dz \left[\nabla_{\underset{\sim}{b}} V(\underset{\sim}{b}, z)\right]\left[\nabla_{\underset{\sim}{b}} \int_{-\infty}^{\infty} du\, u\, \theta(z-u) V(\underset{\sim}{b}, u)\right] \\ & - \left(\frac{1}{\hbar v}\right)^2 \left[\nabla_{\underset{\sim}{b}} \int_{-\infty}^{\infty} dz\, V(\underset{\sim}{b}, z)\right]\left[\nabla_{\underset{\sim}{b}} \int_{-\infty}^{\infty} dz\, z\, V(\underset{\sim}{b}, z)\right] . \end{aligned} \tag{IV.6}$$

The derivation of these results will be discussed in appendix D. For spherically symmetrical potentials the Fresnel correction simplifies to

$$d_F(b) = d_{F2}(b) = \left(\frac{1}{\hbar v}\right)^2 b \frac{\partial}{\partial b} \int_0^\infty dz \left[V(b,z)\right]^2 . \qquad \text{(IV.4a)}$$

Here we have used

$$\begin{aligned} \underset{\sim}{\nabla}_b V(r) &= \frac{1}{b} \underset{\sim}{b} \frac{\partial}{\partial b} V(r) \quad , \\ z \frac{\partial}{\partial b} V(r) &= b \frac{\partial}{\partial z} V(r) \quad . \end{aligned} \qquad \text{(IV.7)}$$

In addition to the Fresnel correction there is an eikonal-type contribution which originates from the first-order longitudinal modification of the eikonal propagator (cf. eq. (II.23)). Including this correction, the function S of eq. (IV.2) changes to

$$S(\underset{\sim}{b}) = S_G(\underset{\sim}{b}) \left\{ 1 - \frac{i}{k} \left[d_e(\underset{\sim}{b}) + d_F(\underset{\sim}{b}) \right] \right\} \quad , \qquad \text{(IV.8)}$$

where

$$d_e(\underset{\sim}{b}) = \frac{1}{2} \left(\frac{1}{\hbar v}\right)^2 \int_{-\infty}^{\infty} dz \left[V(\underset{\sim}{b}, z)\right]^2 \qquad \text{(IV.9)}$$

defines the eikonal correction. Its derivation will also be discussed in appendix D. From eqs. (IV.8) and (IV.3) we see that to first order in k^{-1} the Glauber phase shift (II.15) is modified by the Fresnel and eikonal corrections to

$$\chi(\underset{\sim}{b}) = \chi_G(\underset{\sim}{b}) - \frac{1}{k} \left[d_e(\underset{\sim}{b}) + d_F(\underset{\sim}{b}) \right] \quad . \qquad \text{(IV.10)}$$

It is instructive to study this equation for spherically symmetrical potentials. Defining $V(r) = -V_0 U(r)$ and introducing the energy $E = \frac{1}{2}mv^2$ of the projectile, we obtain

$$\chi(b) = \frac{kV_0}{E}\left\{\int_0^\infty dz\, U(b,z) - \frac{1}{4}\frac{V_0}{E}\int_0^\infty dz\left[U(b,z)\right]^2 - \frac{1}{4}\frac{V_0}{E}\, b\frac{\partial}{\partial b}\int_0^\infty dz\left[U(b,z)\right]^2\right\}. \tag{IV.10a}$$

This clearly exhibits the distinct physical nature of the Glauber contribution and the eikonal correction on one hand and the Fresnel correction on the other. Whereas the former essentially depend on the ratio V_0/E and the range a of the potential ($\int_0^\infty dz U(b,z) \approx \int_0^\infty dz[U(b,z)]^2 \approx a$), the latter in addition depends strongly on the form of the potential because of the derivative $\partial/\partial b$. For a Woods-Saxon form (corresponding to a heavy nucleus) the main contribution to the function d_F comes from the surface region of the potential, in accordance with the physical picture of Fresnel diffraction.

It should be mentioned that eq. (IV.10a) agrees with the result obtained by Wallace [44] , but disagrees with the earlier mentioned one by Baker [43]. This is due to the fact that the longitudinal momentum transfer $q^{\parallel}$ appears to have been neglected in ref. [43].

It will be shown in section VI that the inclusion of eikonal and Fresnel corrections in multiple scattering leads in the optical limit to a result similar to eq. (IV.10). There are additional eikonal contributions, however, due to rescattering on the same nucleon.

V. Fresnel diffraction in high-energy multiple scattering

In section II we have discussed systematic approximation methods for high-energy nuclear scattering which improve upon Glauber's eikonal approximation.

These methods were formulated, for simplicity, in the framework of potential scattering. Since the potential concept has probably little significance for high-energy nuclear interactions, however, it is more appropriate to describe hadron-nucleus scattering as a multiple scattering process. In the present section we study the Fresnel approximation to multiple hadron-nucleus scattering. This consists in approximating the free-space propagator between successive scatterings by the Fresnel propagator rather than by the eikonal propagator as in Glauber's multiple scattering theory. The eikonal approximation employed in the latter means, in optical terms, that the projectile propagates between successive interactions with the target nucleons according to geometrical optics; the Fresnel approximation takes account of the diffractive deviations from straight-line propagation. As we have shown in section II, the connexion between these two approximations can be expressed in terms of a unitary transformation relating the Fresnel and eikonal propagators which is generated by the transverse kinetic energy operator.

As has been mentioned earlier the Fresnel approximation was first investigated for hadron-deuteron scattering by Gottfried [36]. Other corrections to the Glauber theory, such as those arising from the recoil of the target nucleons[37,38, 40,41], have also been studied mainly in scattering from the deuteron. In fact it has been shown by Fäldt [41] that in this case both recoil and Fresnel corrections to the Glauber model can be obtained from a careful treatment of the Green function in the double scattering contribution. However, for target nuclei with A > 2 it becomes increasingly difficult to include nucleon recoil corrections. We therefore retain the frozen nucleus approximation of Glauber theory and confine ourselves to the effects of Fresnel diffraction which we study for an arbitrary number A of target nucleons.

As in earlier investigations of multiple nuclear scattering [27,31,46] we start from a Watson-type multiple scattering series to be discussed in subsect. 1. We first derive the Glauber theory from the eikonal approximation in subsect. 2. We then deal in subsect. 3 with the Fresnel approximation for which we derive closed expressions of the scattering amplitude that are formally similar to the corresponding Glauber formulae. In subsect. 4 the general expressions are applied to elastic scattering by a target nucleus represented by an independent particle model. As an example, the amplitude for nucleon-deuteron scattering is explicitly evaluated in subsect. 5 and compared with an expression previously derived by Gottfried [36]. The results of this section are briefly summarized and discussed in subsect. 6.

1. Multiple scattering series

We consider the scattering of a single hadron by a nucleus composed of A nucleons. Our description is based on a Watson-type multiple scattering series for the hadron-nucleus transition operator $\hat{T}$,

$$\hat{T} = \sum_{j=1}^{A} \hat{t}_j + \sum_{j=1}^{A} \sum_{l \neq j} \hat{t}_j \hat{G} \hat{t}_l + \sum_{j=1}^{A} \sum_{l \neq j} \sum_{m \neq l} \hat{t}_j \hat{G} \hat{t}_l \hat{G} \hat{t}_m$$
$$+ \text{A-3 scattering terms} + \hat{T}_r \quad . \tag{V.1}$$

The operators $\hat{t}_j$ describe effective hadron-nucleon interactions and $\hat{G}$ denotes the free-space propagator

$$\hat{G} = (E + i\epsilon - \hat{K})^{-1} \quad , \tag{V.2}$$

where E is the energy and $\hat{K}$ the kinetic energy operator of the projectile. The series (V.1) consists of a finite sum of A terms that describe scatterings in which the projectile collides with each target nucleon only once (referred to as "direct" terms), plus an infinite remainder $\hat{T}_r$ which incorporates all processes where the same nucleon is hit more than once except in consecutive scatterings (referred to as "rescattering" terms).

Expression (V.1) is a good approximation to the exact Watson series [2-5] provided that the binding energy of the target nucleons is small compared to the incident energy (adiabatic approximation) and the target nucleons are stationary during a collision (frozen nucleus approximation). These conditions are well satisfied at high energies (for a thorough discussion see ref.[17]).

There has been considerable discussion, for potential scattering in the eikonal (Glauber) approximation [47-52], whether in addition the rescattering contributions $\hat{T}_r$ are negligible. Here we offer arguments that $\hat{T}_r$ may in fact be discarded in the context of our approximations. Firstly, as the potential concept appears to have little meaning for high-energy hadronic interactions, we regard the hadron-nucleon transition operators $\hat{t}_j$ as given by the experimental hadron-nucleon elastic scattering amplitudes rather than as constructed from a two-particle potential via the Lippmann-Schwinger equation. (This is essentially equivalent to the assumption of non-overlapping potentials in potential scattering [53].) Such a "phenomenological" treatment of the hadron-nucleon interaction is in accord with the spirit of Glauber's original theory [19] and its very successful applications to high-energy nucleon-nucleus scattering. (For comprehensive surveys of these applications see refs. [20,21] ; recent discussions of the phenomenological viewpoint can be found in refs. [48,49,42].) Since the observed hadron-nucleus elastic scattering amplitudes are sharply forward-peaked and predominantly imaginary, indicating a strongly absorptive interaction, the contributions $\hat{T}_r$ which mostly involve large momentum transfers are expected to be negligible. Secondly, it turns out (as shown in appendix A) that, granted our assumption on the $\hat{t}_j$, the rescattering terms in fact vanish exactly in the eikonal and Fresnel approximations with which we are concerned in this section. We therefore disregard $\hat{T}_r$ in what follows and are thus left with a finite multiple scattering series of A terms.

The hadron-nucleus scattering amplitude $F(\underset{\sim}{k}',\underset{\sim}{k})$ is then obtained from $\hat{T}$ by

$$F(\underset{\sim}{k}',\underset{\sim}{k}) = -\frac{m}{2\pi\hbar^2}\langle \underset{\sim}{k}',n|\hat{T}|o,\underset{\sim}{k}\rangle$$
$$= \sum_{j=1}^{A} F^{(j)}(\underset{\sim}{k}',\underset{\sim}{k}) \quad , \tag{V.3}$$

where $|o,\underset{\sim}{k}\rangle$ and $|n,\underset{\sim}{k}'\rangle$ are the unperturbed initial and final states, respectively, of the system projectile + target, and $F^{(1)}$ is the contribution from single scattering, $F^{(2)}$ from double scattering, etc.

In terms of the hadron-nucleon scattering amplitudes

$$f(\underset{\sim}{p}'-\underset{\sim}{p}) = -\frac{m}{2\pi\hbar^2}\langle \underset{\sim}{p}'|\hat{t}|\underset{\sim}{p}\rangle \tag{V.4}$$

and the propagators

$$\tilde{G}'(p,k)\delta(\underset{\sim}{p}'-\underset{\sim}{p}) = (p^2-k^2-i\epsilon)^{-1}\delta(\underset{\sim}{p}'-\underset{\sim}{p}) = -\frac{\hbar^2}{2m}\langle \underset{\sim}{p}'|\hat{G}|\underset{\sim}{p}\rangle \quad , \tag{V.5}$$

the scattering contributions $F^{(j)}$ become

$$F^{(1)}(\underset{\sim}{k}',\underset{\sim}{k}) = A\, f(\underset{\sim}{k}'-\underset{\sim}{k})\langle n|\, e^{-i(\underset{\sim}{k}'-\underset{\sim}{k})\underset{\sim}{r}}\,|o\rangle \tag{V.6}$$

and

$$F^{(j)}(\underset{\sim}{k}',\underset{\sim}{k}) = (4\pi)^{j-1}\binom{A}{j}\langle n|\sum_{P(1\cdots j)}\int\frac{d^3p_1}{(2\pi)^3}\cdots\int\frac{d^3p_{j-1}}{(2\pi)^3}\,\tilde{G}'(k,p_{j-1})\cdots\tilde{G}'(k,p_1)$$
$$\times f(\underset{\sim}{k}'-\underset{\sim}{p}_{j-1})\cdots f(\underset{\sim}{p}_2-\underset{\sim}{p}_1)f(\underset{\sim}{p}_1-\underset{\sim}{k})\,e^{-i(\underset{\sim}{k}'-\underset{\sim}{p}_{j-1})\underset{\sim}{r}_j}\cdots e^{-i(\underset{\sim}{p}_2-\underset{\sim}{p}_1)\underset{\sim}{r}_2}\,e^{-i(\underset{\sim}{p}_1-\underset{\sim}{k})\underset{\sim}{r}_1}|o\rangle ,$$
$$j>1 . \tag{V.7}$$

The coordinates $\underset{\sim}{r}_1, \ldots, \underset{\sim}{r}_j$ are the instantaneous positions of the target nucleons, and the summation index P(1...j) means that the sum is taken over all permutations of the indices 1, ..., j. For simplicity we neglect spin and charge exchange effects, and replace the hadron-proton and hadron-neutron amplitudes by an average amplitude f.

To proceed further, we transform the variables $\underset{\sim}{p}_\nu$ in eq. (V.7) to $\underset{\sim}{\Delta}_\nu = \underset{\sim}{p}_\nu - \underset{\sim}{k}$, denote by $\underset{\sim}{\Delta}_\nu^\perp$ and $\Delta_\nu^\parallel$ the components of $\underset{\sim}{\Delta}_\nu$ transverse and parallel to $\underset{\sim}{k}$, respectively, and choose as the z-axis the direction of $\underset{\sim}{k}$. In addition, we define as in section II

$$\tilde{G}(\underset{\sim}{\Delta}^\perp, \Delta^\parallel) = (h + \Delta^\parallel - i\epsilon)^{-1} = 2k\,\tilde{G}'(p,k) \tag{V.8}$$

with

$$h = h^\perp + h^\parallel = \frac{(\underset{\sim}{\Delta}^\perp)^2}{2k} + \frac{(\Delta^\parallel)^2}{2k} \quad . \tag{V.9}$$

Then eq. (V.7) becomes

$$F^{(j)}(\underset{\sim}{q}) = \left(\frac{2\pi}{k}\right)^{j-1} \binom{A}{j} \langle n | \sum_{P(1\cdots j)} \int \frac{d^3\Delta_1}{(2\pi)^3} \cdots \int \frac{d^3\Delta_{j-1}}{(2\pi)^3}\, \tilde{G}(\underset{\sim}{\Delta}_{j-1}^\perp, \Delta_{j-1}^\parallel) \cdots \tilde{G}(\underset{\sim}{\Delta}_1^\perp, \Delta_1^\parallel) \tag{V.10}$$

$$\times f(-\underset{\sim}{q} - \underset{\sim}{\Delta}_{j-1}) \cdots f(\underset{\sim}{\Delta}_2 - \underset{\sim}{\Delta}_1) f(\underset{\sim}{\Delta}_1) e^{i(\underset{\sim}{\Delta}_{j-1} + \underset{\sim}{q})\underset{\sim}{r}_j} \cdots e^{i(\underset{\sim}{\Delta}_1 - \underset{\sim}{\Delta}_2)\underset{\sim}{r}_2} e^{-i\underset{\sim}{\Delta}_1 \underset{\sim}{r}_1} |0\rangle,$$

$$j > 1,$$

where $\underset{\sim}{q} = \underset{\sim}{k} - \underset{\sim}{k}'$ is the momentum transfer.

We now use our pragmatic assumption that in the high-energy regime the amplitude f depends on the transverse momentum transfer only and thus defines a hadron-nucleon profile function $\gamma(\underset{\sim}{b})$ by

$$f(\underset{\sim}{\Delta}^\perp) = \frac{ik}{2\pi} \int d^2b\, e^{-i\underset{\sim}{\Delta}^\perp \underset{\sim}{b}}\, \gamma(\underset{\sim}{b}) \quad . \tag{V.11}$$

We insert this in eq. (V.6) and (V.10), write $\underset{\sim}{r}_\nu = (\underset{\sim}{s}_\nu, z_\nu)$ and introduce the Fourier transforms of the propagators $\tilde{G}$,

$$G(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = \frac{1}{\hbar v} e^{ik(z-z')} \int \frac{d^2\Delta^\perp}{(2\pi)^2} e^{i\underset{\sim}{\Delta}^\perp(\underset{\sim}{b}-\underset{\sim}{b}')} \int \frac{d\Delta^\parallel}{2\pi} e^{i\Delta^\parallel(z-z')} \tilde{G}(\underset{\sim}{\Delta}^\perp, \Delta^\parallel), \tag{V.12}$$

where v = ħk/m is the speed of the projectile. This yields

$$F^{(1)}(\underset{\sim}{q}) = \frac{ik}{2\pi} A \langle n | \int d^2b_1\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}_1} e^{iq^\parallel z_1} \gamma(\underset{\sim}{b}_1 - \underset{\sim}{s}_1) | 0 \rangle, \tag{V.13}$$

$$F^{(j)}(\underset{\sim}{q}) = \frac{ik}{2\pi} (i\hbar v)^{j-1} \binom{A}{j} \langle n | \sum_{P(1\cdots j)} \int d^2b_1 \cdots \int d^2b_j\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}_j} e^{iq^\parallel z_j} e^{ik(z_1-z_j)} \tag{V.14}$$

$$\times \gamma(\underset{\sim}{b}_j - \underset{\sim}{s}_j) G(\underset{\sim}{b}_j - \underset{\sim}{b}_{j-1}, z_j - z_{j-1}) \gamma(\underset{\sim}{b}_{j-1} - \underset{\sim}{s}_{j-1}) \cdots G(\underset{\sim}{b}_2 - \underset{\sim}{b}_1, z_2 - z_1) \gamma(\underset{\sim}{b}_1 - \underset{\sim}{s}_1) | 0 \rangle,$$

$$j > 1 .$$

These expressions have an obvious physical interpretation. The incident wave with impact parameter $\underset{\sim}{b}_1$ is distorted by scattering at nucleon 1 situated at $(\underset{\sim}{s}_1, z_1)$ and described by the profile $\gamma(\underset{\sim}{b}_1 - \underset{\sim}{s}_1)$, then in case of $j > 1$ propagates with impact parameter $\underset{\sim}{b}_2$ to nucleon 2 at $(\underset{\sim}{s}_2, z_2)$ and with profile $\gamma(\underset{\sim}{b}_2 - \underset{\sim}{s}_2)$, undergoes a second scattering, and so on for $j > 2$ collisions.

Our main purpose is to simplify expression (V.14) by suitable approximations to the Green functions G. This will be discussed next.

2. Eikonal approximation

This approximation, which consists in replacing the Green functions G in eq. (V.14) by the eikonal Green function (II.7a),

$$G_e(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = \frac{i}{\hbar v}\,\delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}')\,\theta(z-z')\,e^{ik(z-z')} \quad ,$$

is equivalent to Glauber's theory of multiple scattering [19] as has been shown previously by Remler [46].

In evaluating eq. (V.14) with the eikonal Green function we note that $q^{\parallel}$ is of order k^{-1} compared to $q^{\perp}$ (cf. eq. (II.13), and the factor $\exp(iq^{\parallel} z_j)$ may be replaced by unity in the limit $k \to \infty$. Then we use

$$\sum_{P(1\cdots j)} \theta(z_j - z_{j-1}) \cdots \theta(z_2 - z_1) = 1 \quad , \tag{V.15}$$

integrate over $\underset{\sim}{b}_1, \ldots, \underset{\sim}{b}_{j-1}$ and write $\underset{\sim}{b}_j = \underset{\sim}{b}$, with the result

$$F_G^{(j)}(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \langle n|\gamma(\underset{\sim}{b}-\underset{\sim}{s}_j)\cdots\gamma(\underset{\sim}{b}-\underset{\sim}{s}_1)|0\rangle\, , \quad j \geq 1 . \tag{V.16}$$

The total amplitude becomes

$$F_G(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \langle n|\hat{\Gamma}(\underset{\sim}{b}, \underset{\sim}{s}_1, \ldots, \underset{\sim}{s}_A)|0\rangle \tag{V.17}$$

with

$$\hat{\Gamma}(\underset{\sim}{b}, \underset{\sim}{s}_1 \cdots \underset{\sim}{s}_A) = 1 - \prod_{j=1}^{A} [1 - \gamma(\underset{\sim}{b} - \underset{\sim}{s}_j)] \quad . \tag{V.18}$$

These are the well-known formulae of Glauber's multiple scattering theory.

3. Fresnel approximation

In this approximation we replace the Green functions G in eq. (V.14) by the Fresnel Green functions (II.17a),

$$G_F(\underset{\sim}{b} - \underset{\sim}{b}', z - z') = \frac{k}{2\pi\hbar v} \; \frac{\exp[\frac{1}{2} ik(\underset{\sim}{b} - \underset{\sim}{b}')^2/(z-z')]}{z - z'} \, \theta(z - z') \, e^{ik(z-z')} .$$

In evaluating eq. (V.14) with the Fresnel Green function we now have to use a better approximation for $q^{\parallel}$; according to eq. (II.13) we put $q^{\parallel} \approx (q^{\perp})^2/2k$. This is consistent with the fact that the Fresnel approximation involves $h^{\perp}$ only (on the energy shell we have $h^{\perp} = (\underset{\sim}{q}^{\perp})^2/2k$). Equation (V.14) then becomes

$$F_F^{(j)}(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi} (i\hbar v)^{j-1} \binom{A}{j} \langle n | \sum_{P(1\cdots j)} \int d^2 b_1 \cdots \int d^2 b_j \, e^{i\underset{\sim}{q}^{\perp} \underset{\sim}{b}_j} \exp[i z_j (\underset{\sim}{q}^{\perp})^2/2k]$$

$$\times e^{ik(z_1 - z_j)} \gamma(\underset{\sim}{b}_j - \underset{\sim}{s}_j) G_F(\underset{\sim}{b}_j - \underset{\sim}{b}_{j-1}, z_j - z_{j-1}) \gamma(\underset{\sim}{b}_{j-1} - \underset{\sim}{s}_{j-1}) \cdots G_F(\underset{\sim}{b}_2 - \underset{\sim}{b}_1, z_2 - z_1) \gamma(\underset{\sim}{b}_1 - \underset{\sim}{s}_1) | 0 \rangle , \tag{V.19}$$

$$j > 1 .$$

It would be desirable to carry out the integrations over $\underset{\sim}{b}_1$, ..., $\underset{\sim}{b}_{j-1}$ to arrive at results analogous to eqs. (V.16)-(V.18). This can be done in a formal fashion by using the fact, derived in section II (eq. (II.35)), that the Fresnel and eikonal Green functions are related by a unitary transformation,

$$G_F(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = e^{-\frac{i}{\hbar v}\hat{K}_\perp (z-z')} G_e(\underset{\sim}{b}-\underset{\sim}{b}', z-z') \quad ,$$

where

$$\hat{K}_\perp = -\frac{\hbar^2}{2m}\nabla_{\underset{\sim}{b}}^2 = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial b_x^2} + \frac{\partial^2}{\partial b_y^2}\right) \quad .$$

Then the integrations over $\underset{\sim}{b}_1$, ..., $\underset{\sim}{b}_{j-1}$, and a partial integration over $\underset{\sim}{b}_j$ using

$$\exp[iz_j(\underset{\sim}{q}^\perp)^2/2k]\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}_j} = e^{\frac{i}{\hbar v}\hat{K}_\perp z_j}\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}_j} \quad , \tag{V.20}$$

yield (again setting $\underset{\sim}{b}_j = \underset{\sim}{b}$)

$$F_F^{(j)}(\underset{\sim}{q}^\perp) = \frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}\int d^2b\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}} \langle n| \langle \sum_{P(1\cdots j)} \theta(z_j - z_{j-1})\cdots\theta(z_2-z_1)$$
$$\times\, \hat{\gamma}(\underset{\sim}{b}-\underset{\sim}{s}_j, z_j)\cdots\hat{\gamma}(\underset{\sim}{b}-\underset{\sim}{s}_1, z_1)\rangle|0\rangle \quad , \qquad j>1 \quad , \tag{V.21}$$

where

$$\hat{\gamma}(\underset{\sim}{b}-\underset{\sim}{s}, z) = e^{\frac{i}{\hbar v}\hat{K}_\perp z}\, \gamma(\underset{\sim}{b}-\underset{\sim}{s})\, e^{-\frac{i}{\hbar v}\hat{K}_\perp z} \tag{V.22}$$

is an operator in impact parameter $\underset{\sim}{b}$-space. As in section III, the symbol $\langle \hat{O}_{\underset{\sim}{b}} \rangle$ denotes the expectation value of a $\underset{\sim}{b}$-space operator $\hat{O}_{\underset{\sim}{b}}$ in a constant state of value unity.

The combination

$$Z\{\cdots\} \equiv \sum_{P(1\cdots j)} \theta(z_j - z_{j-1}) \cdots \theta(z_2 - z_1)\{\cdots\} \quad , \quad j > 1, \tag{V.23}$$

is a "z-ordering operator" which, acting on noncommuting operators $\hat{\gamma}$, arranges them in the order of increasing z-values from right to left. Thus we may write (defining Z as the identity operator for j = 1)

$$\begin{aligned} F_F^{(j)}(\underset{\sim}{q}^{\perp}) &= \frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \\ &\quad \times \langle n | \langle Z\{\hat{\gamma}(\underset{\sim}{b}-\underset{\sim}{s}_j, z_j)\cdots\hat{\gamma}(\underset{\sim}{b}-\underset{\sim}{s}_1, z_1)\}\rangle | 0\rangle \quad , \quad j \geq 1, \end{aligned} \tag{V.24}$$

and the total amplitude becomes

$$F_F(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \langle n | \hat{\Gamma}_F(\underset{\sim}{b}, \underset{\sim}{s}_1\cdots\underset{\sim}{s}_A, z_1\cdots z_A) | 0\rangle \quad , \tag{V.25}$$

where

$$\hat{\Gamma}_F(\underset{\sim}{b}, \underset{\sim}{s}_1\cdots\underset{\sim}{s}_A, z_1\cdots z_A) = 1 - \langle Z\{\prod_{j=1}^{A}[1-\hat{\gamma}(\underset{\sim}{b}-\underset{\sim}{s}_j, z_j)]\}\rangle. \tag{V.26}$$

Thus the Fresnel approximation leads to results which are formally closely similar to the Glauber expression. However, because of the noncommutability of the profile operators $\hat{\gamma}$ their explicit evaluation is much more difficult.

4. Elastic scattering and independent particle model

We now consider elastic scattering, $|n\rangle = |o\rangle$ and $k' = k$, and describe the target nucleus by an independent particle model. For comparison again we first deal with the eikonal approximation. Then we have in eq. (V.16)

$$\langle o|\gamma(\underset{\sim}{b}-\underset{\sim}{s}_j)\cdots\gamma(\underset{\sim}{b}-\underset{\sim}{s}_1)|o\rangle = \int d^3r_1\cdots\int d^3r_j\, \rho^{(j)}(\underset{\sim}{r}_1\cdots\underset{\sim}{r}_j)\gamma(\underset{\sim}{b}-\underset{\sim}{s}_j)\cdots\gamma(\underset{\sim}{b}-\underset{\sim}{s}_1)$$
$$= \left[-\frac{i}{A}\chi_G(\underset{\sim}{b})\right]^j \quad , \tag{V.27}$$

with $\chi_G(\underset{\sim}{b})$ defined by (compare eq. (II.15))

$$\chi_G(\underset{\sim}{b}) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} dz\, V(\underset{\sim}{b},z) \tag{V.28}$$

and

$$V(\underset{\sim}{b},z) = -i\hbar v A\int d^2s\, \rho(\underset{\sim}{s},z)\gamma(\underset{\sim}{b}-\underset{\sim}{s}) \tag{V.29}$$

or

$$V(\underset{\sim}{r}) = -\frac{2\pi\hbar^2}{m}A\int\frac{d^3\Delta}{(2\pi)^3}e^{i\underset{\sim}{\Delta}\underset{\sim}{r}}f(\underset{\sim}{\Delta})\tilde{\rho}(\underset{\sim}{\Delta}) \quad , \tag{V.30}$$

where $\tilde{\rho}(\underset{\sim}{\Delta})$ is the Fourier transform of the single-particle density $\rho(\underset{\sim}{r}) = \rho(\underset{\sim}{b},z)$.

Then the elastic scattering amplitude in the eikonal approximation becomes

$$F_G(\underset{\sim}{q}^\perp) = \frac{ik}{2\pi}\int d^2b\, e^{i\underset{\sim}{q}^\perp\underset{\sim}{b}}\,\Gamma(\underset{\sim}{b}) \quad , \tag{V.31}$$

where

$$\Gamma(\underset{\sim}{b}) = \langle 0|\hat{\Gamma}(\underset{\sim}{b}, \underset{\sim}{s}_1 \cdots \underset{\sim}{s}_A)|0\rangle = 1 - [1 + \tfrac{i}{A}\chi_G(\underset{\sim}{b})]^A \quad . \tag{V.32}$$

In the optical limit $A \to \infty$ we have

$$\Gamma_{opt}(\underset{\sim}{b}) = 1 - e^{i\chi_G(\underset{\sim}{b})} \tag{V.33}$$

and $\frac{1}{2}\chi_G(\underset{\sim}{b})$ plays the role of a phase shift for scattering by an optical potential $V(\underset{\sim}{r}) = V(\underset{\sim}{b}, z)$ defined by eqs. (V.29), (V.30).

In the Fresnel approximation we have in eq. (V.24)

$$\begin{aligned} &\langle 0|\langle Z\{\hat{\gamma}(\underset{\sim}{b}-\underset{\sim}{s}_j, z_j)\cdots\hat{\gamma}(\underset{\sim}{b}-\underset{\sim}{s}_1, z_1)\}\rangle|0\rangle \\ &= \left(\tfrac{i}{\hbar v A}\right)^j \int_{-\infty}^{\infty} dz_j \cdots \int_{-\infty}^{\infty} dz_1 \langle Z\{\hat{V}(\underset{\sim}{b}, z_j)\cdots\hat{V}(\underset{\sim}{b}, z_1)\}\rangle , \end{aligned} \tag{V.34}$$

where we have defined the operator

$$\hat{V}(\underset{\sim}{b}, z) = e^{\frac{i}{\hbar v}\hat{K}_\perp z}\, V(\underset{\sim}{b}, z)\, e^{-\frac{i}{\hbar v}\hat{K}_\perp z} \tag{V.35}$$

with $V(\underset{\sim}{b}, z)$ given by eq. (V.29).

Introducing the $\underset{\sim}{b}$-space operators

$$\hat{\chi}(\underset{\sim}{b}) = -\frac{1}{\hbar v}\int_{-\infty}^{\infty} dz\, \hat{V}(\underset{\sim}{b}, z) \quad , \tag{V.36}$$

and with the symbolic definition

$$Z[\hat{\chi}(\underset{\sim}{b})]^j \equiv \left(-\frac{1}{\hbar v}\right)^j \int_{-\infty}^{\infty} dz_j \cdots \int_{-\infty}^{\infty} dz_1 \, Z[\hat{V}(\underset{\sim}{b}, z_j) \cdots \hat{V}(\underset{\sim}{b}, z_1)] \,, \tag{V.37}$$

the elastic scattering amplitude in the Fresnel approximation may be written as

$$F_F(\underset{\sim}{q}^\perp) = \frac{ik}{2\pi} \int d^2b \, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}} \, \Gamma_F(\underset{\sim}{b}) \,, \tag{V.38}$$

where

$$\Gamma_F(\underset{\sim}{b}) = \langle 0 | \hat{\Gamma}_F(\underset{\sim}{b}, \underset{\sim}{s}_1 \cdots \underset{\sim}{s}_A, z_1 \cdots z_A) | 0 \rangle = 1 - \langle Z[1 + \tfrac{i}{A}\hat{\chi}(\underset{\sim}{b})]^A \rangle . \tag{V.39}$$

In the optical limit $A \to \infty$ we have

$$\Gamma_{F,opt}(\underset{\sim}{b}) = 1 - e^{i\chi_F(\underset{\sim}{b})} \equiv 1 - \langle Z \, e^{i\hat{\chi}(\underset{\sim}{b})} \rangle \tag{V.40}$$

in formal analogy with eq. (V.33).

Comparison with the results of section III shows that the scattering amplitude in the optical limit of the multiple scattering series is identical to that of potential scattering, if the optical potential is defined in terms of the hadron-nucleon amplitude and the nuclear form factor according to eq. (V.30). Although this identity holds only in the optical limit, there is a structural analogy between the j-th term of the (finite) multiple scattering series and the j-th term of the (infinite) perturbation expansion with respect to the interaction operator $\hat{V}$; the former differs from the latter only by a factor $A!/A^j(A-j)!$.

5. Application to the deuteron

It will be instructive to apply our general expressions to the simplest example, the deuteron, for comparison with the results previously obtained by Gottfried [36]. Again we first briefly consider the eikonal approximation.

Assume the two target nucleons in positions symmetrical about the origin, at $\underset{\sim}{r}_1 = (\underset{\sim}{s}_1, z_1) \equiv (-\frac{1}{2}\underset{\sim}{s}, -\frac{1}{2}z)$ and $\underset{\sim}{r}_2 = (\underset{\sim}{s}_2, z_2) \equiv (\frac{1}{2}\underset{\sim}{s}, \frac{1}{2}z)$. We confine ourselves to the (elastic) double scattering term which, from eq. (V.16), becomes

$$F_G^{(2)}(\underset{\sim}{q}^\perp) = \langle 0 | \hat{F}_G^{(2)}(\underset{\sim}{q}^\perp, \underset{\sim}{s}) | 0 \rangle \tag{V.41}$$

with

$$\hat{F}_G^{(2)}(\underset{\sim}{q}^\perp, \underset{\sim}{s}) = -\frac{ik}{2\pi} \int d^2b \, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}} \gamma(\underset{\sim}{b} - \tfrac{1}{2}\underset{\sim}{s}) \gamma(\underset{\sim}{b} + \tfrac{1}{2}\underset{\sim}{s}) \quad . \tag{V.42}$$

Reintroducing the amplitudes f by

$$\gamma(\underset{\sim}{b}) = -i \frac{2\pi}{k} \int \frac{d^2\Delta^\perp}{(2\pi)^2} e^{i\underset{\sim}{\Delta}^\perp \underset{\sim}{b}} f(\underset{\sim}{\Delta}^\perp) \tag{V.43}$$

yields the well-known Glauber expression [19]

$$\hat{F}_G^{(2)}(\underset{\sim}{q}^\perp, \underset{\sim}{s}) = \hat{F}_G^{(2)}(\underset{\sim}{q}^\perp, -\underset{\sim}{s}) = i\frac{2\pi}{k} e^{-\frac{1}{2}i\underset{\sim}{q}^\perp \underset{\sim}{s}} \int \frac{d^2\Delta^\perp}{(2\pi)^2} e^{i\underset{\sim}{\Delta}^\perp \underset{\sim}{s}} f(\underset{\sim}{\Delta}^\perp) f(\underset{\sim}{q}^\perp - \underset{\sim}{\Delta}^\perp) \quad . \tag{V.44}$$

For an amplitude of the form, suggested by experiment,

$$f(\underset{\sim}{\Delta}^\perp) = i\frac{k\sigma}{4\pi}(1 - i\alpha) \exp\left[-\tfrac{1}{2} B (\underset{\sim}{\Delta}^\perp)^2\right] \tag{V.45}$$

(where σ is the average of the p-n and p-p total cross sections), this becomes

$$\hat{F}_G^{(2)}(\underset{\sim}{q}^\perp,\underset{\sim}{s}) = -i\frac{k}{2B}\left(\frac{\sigma}{4\pi}\right)^2(1-i\alpha)^2 \exp\left[-\frac{B}{4}(\underset{\sim}{q}^\perp)^2 - \frac{1}{4B}\underset{\sim}{s}^2\right] . \qquad (V.46)$$

In the Fresnel approximation we have, from eq. (V.24),

$$F_F^{(2)}(\underset{\sim}{q}^\perp) = \langle 0|\,\hat{F}_F^{(2)}(\underset{\sim}{q}^\perp,\underset{\sim}{s},z)\,|0\rangle \qquad (V.47)$$

where

$$\hat{F}_F^{(2)}(\underset{\sim}{q}^\perp,\underset{\sim}{s},z)$$
$$= -\frac{ik}{2\pi}\int d^2b\, e^{i\underset{\sim}{q}^\perp\underset{\sim}{b}}\langle Z\{\hat{\gamma}(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{s},\tfrac{1}{2}z)\hat{\gamma}(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{s},-\tfrac{1}{2}z)\}\rangle . \qquad (V.48)$$

With the definition (V.22) of the profile operators and (V.23) of the z-ordering operator this can be written as

$$\hat{F}_F^{(2)}(\underset{\sim}{q}^\perp,\underset{\sim}{s},z) \equiv \hat{F}_F^{(+)}(\underset{\sim}{q}^\perp,\underset{\sim}{s},z) + \hat{F}_F^{(+)}(\underset{\sim}{q}^\perp,-\underset{\sim}{s},-z) \quad , \qquad (V.49)$$

where

$$\hat{F}_F^{(+)}(\underset{\sim}{q}^\perp,\underset{\sim}{s},z)$$
$$= -\frac{ik}{2\pi}\theta(z)\int d^2b\, e^{i\underset{\sim}{q}^\perp\underset{\sim}{b}}\, e^{-\frac{iz}{4k}\nabla^2_{\underset{\sim}{b}}}\gamma(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{s})\, e^{\frac{iz}{2k}\nabla^2_{\underset{\sim}{b}}}\gamma(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{s}) . \qquad (V.50)$$

By using eq. (V.43) and noting that $\exp(i\tau\nabla^2_{\underset{\sim}{b}})\exp(i\underset{\sim}{\Delta}^\perp\underset{\sim}{b})$ $= \exp(-i\tau\Delta^{\perp 2} + i\underset{\sim}{\Delta}^\perp\underset{\sim}{b})$, eq. (V.50) becomes

$$\hat{F}_F^{(+)}(\underset{\sim}{q}^\perp, \underset{\sim}{s}, z)$$

$$= i\,\frac{2\pi}{k}\,\theta(z)\, e^{-\frac{1}{2} i \underset{\sim}{q}^\perp \underset{\sim}{s}}\, e^{\frac{iz}{4k}(\underset{\sim}{q}^\perp)^2} \int \frac{d^2\Delta^\perp}{(2\pi)^2}\, e^{i \underset{\sim}{\Delta}^\perp \underset{\sim}{s}}\, f(\underset{\sim}{\Delta}^\perp) f(\underset{\sim}{q}^\perp - \underset{\sim}{\Delta}^\perp)\, e^{-\frac{iz}{2k}(\underset{\sim}{\Delta}^\perp)^2} . \tag{V.51}$$

If the eikonal amplitude is similarly split into +z and -z contributions,

$$\hat{F}_G^{(2)}(\underset{\sim}{q}^\perp, \underset{\sim}{s}) = \left[\theta(z) + \theta(-z)\right] \hat{F}_G^{(2)}(\underset{\sim}{q}^\perp, \underset{\sim}{s})$$
$$\equiv \hat{F}_G^{(+)}(\underset{\sim}{q}^\perp, \underset{\sim}{s}, z) + \hat{F}_G^{(+)}(\underset{\sim}{q}^\perp, -\underset{\sim}{s}, -z) \quad , \tag{V.52}$$

we see that the Fresnel components $\hat{F}_F^{(+)}(\underset{\sim}{q}^\perp, \underset{\sim}{s}, z)$ and $\hat{F}_F^{(+)}(\underset{\sim}{q}^\perp, -\underset{\sim}{s}, -z)$ differ from the corresponding eikonal terms by the factors $\exp[i\xi(\underset{\sim}{q}^\perp)^2/4k]\exp(-i\xi\underset{\sim}{\Delta}^{\perp 2}/2k)$ with $\xi = +z$ and $\xi = -z$, respectively. The first exponential arises from retaining the longitudinal momentum transfer to the order k^{-1}, $\left(q^{\parallel} \approx (\underset{\sim}{q}^\perp)^2/2k\right.$ from eq. $\left.(II.13)\right)$, while the second exponential originates from the replacement of the eikonal by the Fresnel propagator.

Using the form (V.45) of the hadron-nucleon scattering amplitude, the integration in eq. (V.51) can be carried out with the result

$$\hat{F}_F^{(+)}(\underset{\sim}{q}^\perp, \underset{\sim}{s}, z) = -\frac{ik}{2B}\left(\frac{\sigma}{4\pi}\right)^2 (1-i\alpha)^2\, e^{\frac{iz}{4k}(\underset{\sim}{q}^\perp)^2}\, \theta(z)$$
$$\times \left(1+\tfrac{1}{2} i\zeta\right)^{-1} \exp\left\{\tfrac{1}{4}\left(1+\tfrac{1}{2}i\zeta\right)^{-1}\left[\zeta \underset{\sim}{q}^\perp \underset{\sim}{s} - (1+i\zeta)B(\underset{\sim}{q}^\perp)^2 - \underset{\sim}{s}^2/B\right]\right\} , \tag{V.53}$$

where $\zeta = z/kB$. The total double scattering amplitude $\hat{F}_F^{(2)}$ is then given by the sum (V.49) of the contributions (V.53) with arguments $(\underset{\sim}{s}, z)$ and $(-\underset{\sim}{s}, -z)$. The form of this result agrees with Gottfried's expression (eq. (22) of ref. [36]), except that the latter lacks the z-ordering. This is because in Gottfried's definition of the Fresnel propagator (II. 17a) the step function $\Theta(z-z')$ is omitted. It should be mentioned that for spherically symmetrical nuclear wave functions the z-ordering in eq. (V.53) is immaterial, so that in this case we would obtain the same result as Gottfried for the averaged double scattering amplitude $F_F^{(2)}(\underset{\sim}{q}^{\perp})$ from eq. (V.47). For higher-order terms, however, the step function in G_F (as well as in G_e) and the resulting z-ordering is essential.

6. Summary and discussion

We have derived a closed expression for the hadron-nucleus multiple scattering amplitude at high energies which includes the effects of Fresnel diffraction on the propagation of the projectile between successive collisions. Although this expression is formally very similar to the corresponding result in eikonal (Glauber) approximation, it differs from the latter by the appearance of noncommuting profile operators which occur in a characteristic Dyson-type ordering with respect to the z-coordinates of the target nucleons. This feature makes the explicit evaluation very difficult for all but the lightest target nuclei. However, a significant physical aspect of our result is that it shows how the equivalence or interchangeability of the scatterers with respect to their longitudinal positions, which makes Glauber theory so simple, is destroyed by the diffractive distortion in transverse directions of the wave front of the projectile as it propagates through the target.

As far as explicit evaluation of our expressions for $A > 2$ is concerned it would be possible to develop systematic procedures for the expansion of the matrix elements of the z-ordered products of the profile operators $\hat{\gamma}$ which are analogous to the methods used in time-dependent perturbation theory or in the cluster expansion of the partition function in statistical mechanics. There is, however, not much point in carrying the full form (V.22) of $\hat{\gamma}$ if we are interested in Fresnel diffraction effects of a given order in the reciprocal wave number. This is because the Fresnel approximation, as pointed out in subsect.II.6, contains contributions to all orders of k^{-1}; it represents a particular partial summation of the eikonal expansion which has a well-defined physical meaning in terms of the "optics" of the scattering process. If we confine ourselves to high-energy Fresnel corrections to the Glauber amplitude, to first order in k^{-1}, a different procedure is appropriate. This consists in expanding the profile operators (V.22) with respect to $z/\hbar v \propto k^{-1}$ and retaining only the contributions up to second order. In this way we have derived in section IV Fresnel corrections of order k^{-1} to the Glauber phase shift for potential scattering. The same method applied to eqs. (V.24)-(V.26) yields Fresnel corrections of order k^{-1} to the Glauber amplitude for multiple scattering in target nuclei of any mass number A. These, as well as the eikonal-type corrections of order k^{-1}, will be studied in the following section.

VI. Eikonal and Fresnel corrections to the Glauber theory of nuclear multiple scattering

In this section we study first-order corrections to the Glauber high-energy multiple collision model. These are obtained by replacing in a Watson-type multiple scattering series the free-space Green function between successive scatterings by the second-order eikonal propagator (II.23a),

$$G_e^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = G_e(\underset{\sim}{b}-\underset{\sim}{b}', z-z') + g_e(\underset{\sim}{b}-\underset{\sim}{b}', z-z') + g_F(\underset{\sim}{b}-\underset{\sim}{b}', z-z')$$

where G_e is the eikonal Green function (II.7a),

$$g_e(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = -\frac{1}{2\hbar k v}\,\delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}')\,\delta(z-z')\,e^{ik(z-z')} \qquad \text{(VI.1)}$$

and

$$g_F(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = -\frac{1}{2\hbar k v}\nabla^2_{\underset{\sim}{b}}\,\delta^{(2)}(\underset{\sim}{b}-\underset{\sim}{b}')(z-z')\,\theta(z-z')\,e^{ik(z-z')} \qquad \text{(VI.2)}$$

are the eikonal and Fresnel corrections, respectively. Like in the case for potential scattering (section IV) these corrections lead to two physically distinct types of modifications of the Glauber multiple scattering amplitude. Since the Glauber theory is already a very good zero-order approximation to high-energy scattering, it might be sufficient for practical applications to confine explicit calculations to the second-order eikonal approximation.

As an introduction the elastic scattering from the deuteron in the second-order eikonal approximation is studied in subsect. 1. The expressions obtained are generalised in subsect. 2 to scattering from nuclei consisting of an arbitrary number of nucleons. In subsect. 3 the general formulae are applied to elastic scattering from a nucleus represented by an independent particle model. Numerical results for elastic scattering by light nuclei are presented in subsect. 4, and a short summary and discussion of results are given in subsect. 5.

1. Scattering from the deuteron

The second-order eikonal approximation to the multiple scattering series (V.1), applied to scattering from the deuteron, yields very simple and instructive results which already contain the main features of the more general expressions for $A > 2$. Thus considering the deuteron as a target nucleus first will bring out the essential points more clearly, and will also considerably shorten the discussion for a target nucleus with arbitrary A. The single scattering contribution (V.13) which does not contain any propagator is disregarded in the following discussion. (There is, however, a correction of order k^{-1} to the Glauber single scattering term arising from the factor $\exp(iq^{\parallel} z_1)$).

1.a) Fresnel term

As shown in appendix A the rescattering contributions $\hat{T}_r$ of eq. (V.1) vanish for propagators with a factor $\theta(z)$ in the framework of our approximations. Thus we are concerned with the double scattering term only. In the Fresnel approximation, this becomes

$$F_F^{(2)}(\underset{\sim}{q}^\perp) = -\frac{ik}{2\pi}\int d^2b\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}} \langle 0|\langle \sum_{P(1,2)} \theta(z_2 - z_1)\hat{\gamma}(\underset{\sim}{b} - \underset{\sim}{s}_2, z_2)\hat{\gamma}(\underset{\sim}{b} - \underset{\sim}{s}_1, z_1)\rangle|0\rangle ,$$

(VI.3)

where

$$\hat{\gamma}(\underset{\sim}{b} - \underset{\sim}{s}, z) = e^{-\frac{1}{2}\frac{iz}{k}\nabla_{\underset{\sim}{b}}^2}\, \gamma(\underset{\sim}{b} - \underset{\sim}{s})\, e^{\frac{1}{2}\frac{iz}{k}\nabla_{\underset{\sim}{b}}^2} \tag{VI.4}$$

(cf. eqs. (V.21) and (V.22)). Expansion of eq. (VI.4) to first order in k^{-1} yields

$$\hat{\gamma}(\underset{\sim}{b} - \underset{\sim}{s}, z) = \gamma(\underset{\sim}{b} - \underset{\sim}{s}) + \frac{1}{2}\frac{iz}{k}\left[\gamma(\underset{\sim}{b} - \underset{\sim}{s}), \nabla_{\underset{\sim}{b}}^2\right] . \tag{VI.5}$$

Noting that

$$\left[\gamma(\underset{\sim}{b} - \underset{\sim}{s}), \nabla_{\underset{\sim}{b}}^2\right] g(\underset{\sim}{b}) = -g(\underset{\sim}{b})\nabla_{\underset{\sim}{b}}^2 \gamma(\underset{\sim}{b} - \underset{\sim}{s}) - 2\left[\underset{\sim}{\nabla}_{\underset{\sim}{b}}\gamma(\underset{\sim}{b} - \underset{\sim}{s})\right]\underset{\sim}{\nabla}_{\underset{\sim}{b}}\, g(\underset{\sim}{b}) \tag{VI.6}$$

for an arbitrary function g, and assuming the two nucleons to be in positions symmetrical about the origin at $\underset{\sim}{r}_1 = (-\frac{1}{2}\underset{\sim}{s}, -\frac{1}{2}z)$ and $\underset{\sim}{r}_2 = (\frac{1}{2}\underset{\sim}{s}, \frac{1}{2}z)$, eq. (VI.3) becomes to first order in k^{-1}

$$F_F^{(2)}(\underset{\sim}{q}^\perp) = F_G^{(2)}(\underset{\sim}{q}^\perp) + D_F^{(2)}(\underset{\sim}{q}^\perp) . \tag{VI.7}$$

Here

$$F_G^{(2)}(\underset{\sim}{q}^\perp) = -\frac{ik}{2\pi}\int d^2b\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}} \langle 0|\gamma(\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{s})\gamma(\underset{\sim}{b} - \frac{1}{2}\underset{\sim}{s})|0\rangle \tag{VI.8}$$

is the Glauber result, and the Fresnel correction is given by

$$D_F^{(2)}(\underset{\sim}{q}^{\perp}) = D_{F1}^{(2)}(\underset{\sim}{q}^{\perp}) + D_{F2}^{(2)}(\underset{\sim}{q}^{\perp}) \quad , \tag{VI.9}$$

where

$$D_{F1}^{(2)}(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi}\frac{i}{2k}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \langle 0|\tfrac{1}{2}z\{\gamma(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{s})\nabla^2_{\underset{\sim}{b}}\gamma(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{s}) - [\nabla^2_{\underset{\sim}{b}}\gamma(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{s})]\,\gamma(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{s})\}|0\rangle \tag{VI.9a}$$

and

$$D_{F2}^{(2)}(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi}\frac{i}{2k}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \langle 0|[\theta(z)-\theta(-z)]\,z\,[\nabla_{\underset{\sim}{b}}\gamma(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{s})] \times [\nabla_{\underset{\sim}{b}}\gamma(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{s})]|0\rangle \quad . \tag{VI.9b}$$

We note that the contribution $D_{F1}^{(2)}$ vanishes for spherically symmetrical wave functions.

1.b) Eikonal terms

Replacing the Green function G in eq. (V.14) by the eikonal correction (VI.1) and integrating over one impact parameter $\underset{\sim}{b}$, yields the direct eikonal term

$$D_{eo}^{(2)}(\underset{\sim}{q}^{\perp}) = -\frac{ik}{2\pi}\frac{i}{k}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \langle 0|\gamma(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{s})\,\gamma(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{s})\,\delta(z)|0\rangle . \tag{VI.10}$$

However, this is not the only eikonal-type contribution. A triple scattering sequence consisting of scattering from nucleon 1 to nucleon 2 and back to nucleon 1 ("rescattering process 1") also yields a non-vanishing contribution. The rescattering term 1 in its general form is (cf. eq. (A.1))

$$F_{r1}^{(3)}(\underset{\sim}{q}) = \frac{ik}{2\pi}(i\hbar v)^2 \langle 0 | \sum_{P(1,2)} \int d^2b_1 \int d^2b_2 \int d^2b_3 \, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}_3} \, e^{-\frac{1}{2}iq^{\parallel}z} \tag{VI.11}$$

$$\times \gamma(\underset{\sim}{b}_3+\tfrac{1}{2}\underset{\sim}{s}) G(\underset{\sim}{b}_3-\underset{\sim}{b}_2,-z) \gamma(\underset{\sim}{b}_2-\tfrac{1}{2}\underset{\sim}{s}) G(\underset{\sim}{b}_2-\underset{\sim}{b}_1,z) \gamma(\underset{\sim}{b}_1+\tfrac{1}{2}\underset{\sim}{s}) | 0 \rangle .$$

Applying the second-order eikonal approximation in eq. (VI.11) and using $\theta(z)\theta(-z) = o$, $\theta(z) + \theta(-z) = 1$, the product of η_{e1} of the propagators becomes to first order in k^{-1}

$$\eta_{e1} = -\frac{1}{2}\frac{i}{k}\left(\frac{1}{\hbar v}\right)^2 \delta^{(2)}(\underset{\sim}{b}_3-\underset{\sim}{b}_2)\,\delta^{(2)}(\underset{\sim}{b}_2-\underset{\sim}{b}_1)\,\delta(z) \quad . \tag{VI.12}$$

With eq. (VI.12), $F_{r1}^{(3)}$ reduces to the rescattering eikonal term of type 1,

$$D_{e1}^{(3)}(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi}\frac{i}{2k}\int d^2b \, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \langle 0 | \{ [\gamma(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{s})]^2 \gamma(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{s}) \tag{VI.13}$$

$$+ \gamma(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{s})[\gamma(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{s})]^2 \} \delta(z) | 0 \rangle \quad .$$

A further contribution originates from the quadruple scattering sequence $1 \to 2 \to 1 \to 2$ ("rescattering process 2"). Its general form is

$$F_{r2}^{(4)}(\underset{\sim}{q}) = \frac{ik}{2\pi}(i\hbar v)^3 \langle 0| \sum_{P(1,2)} \int d^2b_1 \cdots \int d^2b_4 \, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}_4} e^{\frac{1}{2}iq^\parallel z} e^{-ikz}$$

$$\times \gamma(\underset{\sim}{b}_4 - \tfrac{1}{2}\underset{\sim}{s}) G(\underset{\sim}{b}_4 - \underset{\sim}{b}_3, z) \gamma(\underset{\sim}{b}_3 + \tfrac{1}{2}\underset{\sim}{s}) G(\underset{\sim}{b}_3 - \underset{\sim}{b}_2, -z) \qquad \text{(VI.14)}$$

$$\times \gamma(\underset{\sim}{b}_3 - \tfrac{1}{2}\underset{\sim}{s}) G(\underset{\sim}{b}_2 - \underset{\sim}{b}_1, z) \gamma(\underset{\sim}{b}_1 + \tfrac{1}{2}\underset{\sim}{s}) |0\rangle \quad .$$

The non-vanishing part in the product of the propagators to order k^{-1} is

$$\eta_{e2} = \frac{1}{2k}\left(\frac{1}{\hbar v}\right)^3 \delta^{(2)}(\underset{\sim}{b}_4 - \underset{\sim}{b}_3) \cdots \delta^{(2)}(\underset{\sim}{b}_2 - \underset{\sim}{b}_1) \delta(z) \Theta(z) \ . \qquad \text{(VI.15)}$$

With eq. (VI.15) $F_{r2}^{(4)}$ reduces to the rescattering eikonal term of type 2

$$D_{e2}^{(4)}(\underset{\sim}{q}^\perp) = -\frac{ik}{2\pi}\frac{i}{2k}\int d^2b \, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}} \langle 0| [\gamma(\underset{\sim}{b} - \tfrac{1}{2}\underset{\sim}{s})]^2 [\gamma(\underset{\sim}{b} + \tfrac{1}{2}\underset{\sim}{s})]^2 \delta(z) |0\rangle . \qquad \text{(VI.16)}$$

All higher-order rescattering eikonal terms vanish because of a factor $\theta(z)\theta(-z)$.

Comparison of eqs. (VI.10), (VI.13) and (VI.16) shows that the eikonal terms have similar structure. Because of the delta-function they are non-zero only for those positions of the target nucleons which involve the same z-coordinate. The factors γ^2 in $D_{e1}^{(3)}$ and $D_{e2}^{(4)}$ appear because of double scattering on the same nucleon. Thus the size of the eikonal terms relative to one another depends on the form of the effective interaction γ. If $\gamma \approx \gamma^2$, $D_{eo}^{(2)}$ will tend to cancel against $D_{e1}^{(3)}$ because of the opposite signs involved. This tendency of cancellation and the resulting smallness of the eikonal contribution with respect to the Fresnel contribution can be demonstrated for a simple example.

Assume the two-particle scattering amplitude to be given by eq. (V.45),

$$f(\underset{\sim}{\Delta}^{\perp}) = i\frac{k\sigma}{4\pi}(1-i\alpha)\exp\left[-\frac{1}{2}B(\underset{\sim}{\Delta}^{\perp})^2\right] \quad .$$

This yields for the profile function (V.43)

$$\gamma(\underset{\sim}{b}-\underset{\sim}{s}) = \frac{\sigma}{4\pi B}(1-i\alpha)\exp\left[-\frac{1}{2B}(\underset{\sim}{b}-\underset{\sim}{s})^2\right] \quad . \tag{VI.17}$$

In evaluating the various scattering contributions we must use the deuteron internal wave function (cf. appendix B). We assume this function to be Gaussian,

$$\varphi_D(r) = (2\pi C)^{-\frac{3}{4}}\exp\left(-\frac{1}{4C}r^2\right) \tag{VI.18}$$

with C = 64 $(\mathrm{GeV}/c)^{-2}$ taken from ref. [54]. For simplicity we neglect in the correction terms the imaginary part of the profile γ and terms of order B/C. We thus obtain

$$F_G^{(2)}(\underset{\sim}{q}^{\perp}) = -ik\left[\frac{\sigma}{4\pi}(1-i\alpha)\right]^2 C^{-1}\exp\left[-\frac{1}{4}B(\underset{\sim}{q}^{\perp})^2\right] \quad , \tag{VI.8a}$$

$$D_F^{(2)}(\underset{\sim}{q}^{\perp}) = F_G^{(2)}(\underset{\sim}{q}^{\perp})\frac{i}{\sqrt{2\pi}}\frac{1}{k\sqrt{C}}\left[\frac{1}{4}C(\underset{\sim}{q}^{\perp})^2-2\right] \quad , \tag{VI.9c}$$

$$D_{e0}^{(2)}(\underset{\sim}{q}^{\perp}) = F_G^{(2)}(\underset{\sim}{q}^{\perp})\frac{i}{\sqrt{2\pi}}\frac{1}{k\sqrt{C}} \quad , \tag{VI.10a}$$

$$D_{e1}^{(3)}(\underset{\sim}{q}^{\perp}) = -F_G^{(2)}(\underset{\sim}{q}^{\perp})\frac{i}{\sqrt{2\pi}}\frac{1}{k\sqrt{C}}\frac{\sigma}{4\pi B}\exp\left[\frac{1}{4}\left(B-\frac{1}{12}\right)(\underset{\sim}{q}^{\perp})^2\right] , \tag{VI.13a}$$

$$D^{(4)}_{e2}(\underset{\sim}{q}^{\perp}) = F^{(2)}_{G}(\underset{\sim}{q}^{\perp})\,\frac{i}{\sqrt{2\pi}}\,\frac{1}{k\sqrt{C}}\;\frac{1}{8}\left(\frac{\sigma}{4\pi B}\right)^{2}\exp\left[\tfrac{1}{8}B(\underset{\sim}{q}^{\perp})^{2}\right]. \qquad \text{(VI.16a)}$$

Table 1 shows the relative sizes of the various correction terms in dependence on the momentum transfer squared. The parameter μ is defined as $\mu = F^{(2)}_{G}(\underset{\sim}{q}^{\perp})(i/k)(2\pi c)^{-\frac{1}{2}}$. The last row gives the sum $D^{(2)}_{e}$ of the eikonal terms in percentages of the Fresnel contribution. There is a complete cancellation between $D^{(2)}_{eo}$ and $D^{(3)}_{e1}$ in the forward direction. At the diffraction minimum (which is near 0.3 $(\mathrm{GeV}/c)^{2}$ at high energies), this cancellation is still appreciable, but becomes less pronounced at larger momenta. Due to a rapid increase with increasing angle, the Fresnel term still remains the dominant contribution, however. The calculations have been performed for $\sigma/4\pi B = 1$, $B = 8(\mathrm{GeV}/c)^{-2}$, as in ref. [36]. We also have employed the quite different parameters $\sigma/4\pi B = 1.6$ and $B = 5.45\ (\mathrm{GeV}/c)^{-2}$, used by Bassel and Wilkin [55], and found that the sum of the eikonal terms is about 15% of the Fresnel contribution throughout. At least to order k^{-1}, these results lend support to the procedure of truncating the Watson series after the double scattering term and taking only the Fresnel corrections to the Glauber model into account.

The main effect of Fresnel diffraction (and of corrections to the Glauber double scattering term in general [27,31]) is to fill in the diffraction minimum predicted by Glauber theory. Especially at lower energies, the Fresnel effects also reduce the Glauber differential cross section at large angles [36]. This feature is not reproduced by our first-order correction and thus appears to be due to the influence of the higher-order terms in k^{-1}. We note that in the forward direction the Fresnel correction (VI.9c) agrees with Gottfried's expression (39). This means that at zero momentum transfer all higher-order Fresnel contributions vanish.

Table 1 : Fresnel and eikonal corrections to the Glauber double scattering term $F_G^{(2)}$

$q^2\,(\mathrm{GeV}/c)^2$	0	0.3 (min.)	0.6 (max.)	1.0
$D_F^{(2)}/\mu$	-2	2.88	7.75	14.25
$D_{eo}^{(2)}/\mu$	1	1	1	1
$D_{e1}^{(3)}/\mu$	-1	-1.56	-2.43	-4.39
$D_{e2}^{(4)}/\mu$	0.13	0.17	0.23	0.34
$D_e^{(2)}/\mu$	6%	14%	16%	21%

2. Scattering from complex nuclei

Like for scattering by the deuteron, we get a Fresnel term and three eikonal-type contributions. We again first discuss the Fresnel term.

2.a) Fresnel term

The j-th multiple scattering contribution including Fresnel diffraction, is (cf. eq.(V.24))

$$F_F^{(j)}(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \langle n|Z\{\hat{\gamma}(\underset{\sim}{b}-\underset{\sim}{s}_j, z_j)\cdots\hat{\gamma}(\underset{\sim}{b}-\underset{\sim}{s}_1, z_1)\}|0\rangle,$$

$$j \geq 1 .$$

Using eqs. (VI.5) and (VI.6), we obtain to first order in k^{-1}

$$F_F^{(j)}(\underset{\sim}{q}^{\perp}) = F_G^{(j)}(\underset{\sim}{q}^{\perp}) + D_F^{(j)}(\underset{\sim}{q}^{\perp}) \quad , \quad j \geq 1 \quad , \tag{VI.19}$$

where

$$F_G^{(j)}(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}\langle n|\gamma(\underset{\sim}{b}-\underset{\sim}{s}_j)\cdots\gamma(\underset{\sim}{b}-\underset{\sim}{s}_1)|0\rangle$$

is the Glauber result (V.16), and

$$D_F^{(j)}(\underset{\sim}{q}^{\perp}) = D_{F1}^{(j)}(\underset{\sim}{q}^{\perp}) + D_{F2}^{(j)}(\underset{\sim}{q}^{\perp}) \tag{VI.20}$$

is the Fresnel correction, with

$$D_{F1}^{(j)}(\underset{\sim}{q}^{\perp}) = -\frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}\frac{i}{2k}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}\langle n|\sum_{\lambda=1}^{j}\gamma(\underset{\sim}{b}-\underset{\sim}{s}_j)\cdots \\ \times z_\lambda[\nabla_{\underset{\sim}{b}}^2\gamma(\underset{\sim}{b}-\underset{\sim}{s}_\lambda)]\cdots\gamma(\underset{\sim}{b}-\underset{\sim}{s}_1)|0\rangle \quad , \quad j \geq 1 \quad , \tag{VI.20a}$$

$$D_{F2}^{(j)}(\underset{\sim}{q}^{\perp}) = -\frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}\frac{i}{k}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}\langle n|Z\Big\{\sum_{\lambda=1}^{j-1}\sum_{\nu=1}^{\lambda}\gamma(\underset{\sim}{b}-\underset{\sim}{s}_j)\cdots \\ \times z_{\lambda+1}[\nabla_{\underset{\sim}{b}}\gamma(\underset{\sim}{b}-\underset{\sim}{s}_{\lambda+1})]\gamma(\underset{\sim}{b}-\underset{\sim}{s}_\lambda)\cdots[\nabla_{\underset{\sim}{b}}\gamma(\underset{\sim}{b}-\underset{\sim}{s}_\nu)]\cdots\gamma(\underset{\sim}{b}-\underset{\sim}{s}_1)\Big\}|0\rangle, \\ j > 1 \, . \tag{VI.20b}$$

Again, the expression for $D_{F1}^{(j)}$ vanishes for spherically symmetrical target wave functions. (We note that it is formally justified to call the first-order single scattering correction $D_{F1}^{(1)}$ a "Fresnel" correction, although it originates not from a propagator but from the phase factor $\exp[i(q^{\perp 2}/2k)z_1]$.

The reason is that this phase is closely connected to the Fresnel propagator for on-shell values of $h^{\perp}$ ($h^{\perp}$ o.sh. = $q^{\perp 2}/2k$),

$$G_F = e^{-\frac{i}{2k} q^{\perp 2} z} \quad G_e \quad . \,)$$

2.b) Eikonal terms

We replace the full propagators in eq. (V.14) by their second-order eikonal approximations (II.23a), and extract the eikonal correction of order k^{-1}. This yields for the direct eikonal term after summation over the permutations

$$D^{(j)}_{eo}(\underline{q}^{\perp}) = \frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}\frac{i}{k}\int d^2b\, e^{i\underline{q}^{\perp}\underline{b}}\langle n|\gamma(\underline{b}-\underline{s}_j)\cdots\gamma(\underline{b}-\underline{s}_1)$$
$$\times\sum_{\nu>\lambda}^{j}\sum_{\lambda=1}^{j-1}\delta(z_\nu - z_\lambda)|0\rangle \quad , \quad j>1 \quad , \tag{VI.21}$$

and for j = 2 reduces to eq. (VI.10). Next we discuss the rescattering terms. To order k^{-1} the only non-vanishing contributions to the rescattering processes l involving j + l collisions come from sequences $\alpha \to \alpha + 1 \to \alpha$, $\alpha = 1, \ldots, j-1$. The sequences $\alpha \to \ldots\, \alpha + \mu \to \alpha$, $\mu > 1$, all contain products of step functions which give rise to zero. As a generalisation of eq. (VI.13) we obtain

$$D^{(j+1)}_{e1}(\underline{q}^{\perp}) = -\frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}\frac{i}{2k}\int d^2b\, e^{i\underline{q}^{\perp}\underline{b}}\langle n|\sum_{\nu>\lambda}^{j}\sum_{\lambda=1}^{j-1}\{\gamma(\underline{b}-\underline{s}_1)\cdots$$
$$\times[\gamma(\underline{b}-\underline{s}_\lambda)]^2\cdots\gamma(\underline{b}-\underline{s}_j)+\gamma(\underline{b}-\underline{s}_1)\cdots[\gamma(\underline{b}-\underline{s}_\nu)]^2\cdots\gamma(\underline{b}-\underline{s}_j)\}\delta(z_\nu - z_\lambda)|0\rangle, \tag{VI.22}$$
$$j>1.$$

For the rescattering processes 2 involving j + 2 collisions, only sequences $\alpha \to \alpha + 1 \to \alpha \to \alpha + 1$, $\alpha = 1,\ldots,j-1$, yield nonzero contributions. In analogy to eq. (VI.16), the rescattering eikonal correction of type 2 is

$$D_{e2}^{(j+2)}(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}\frac{i}{2k}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}\langle n|\sum_{\nu>\lambda}^{j}\sum_{\lambda=1}^{j-1}\gamma(\underset{\sim}{b}-\underset{\sim}{s}_1)\cdots \tag{VI.23}$$

$$\times[\gamma(\underset{\sim}{b}-\underset{\sim}{s}_\lambda)]^2\cdots[\gamma(\underset{\sim}{b}-\underset{\sim}{s}_\nu)]^2\cdots\gamma(\underset{\sim}{b}-\underset{\sim}{s}_j)\,\delta(z_\nu-z_\lambda)|0\rangle \quad , \; j>1 .$$

Like for the deuteron target, higher-order rescattering processes do not contribute. The sum of all eikonal-type corrections to order j (where j denotes the number of different target nucleons involved in the collisions, and not the number of collisions!) is

$$D_{e}^{(j)}(\underset{\sim}{q}^{\perp}) \equiv D_{e0}^{(j)}(\underset{\sim}{q}^{\perp}) + D_{e1}^{(j+1)}(\underset{\sim}{q}^{\perp}) + D_{e2}^{(j+2)}(\underset{\sim}{q}^{\perp}) \quad . \tag{VI.24}$$

3. Elastic scattering, independent particle model

We now consider elastic scattering from a nucleus represented by an independent particle model. The definitions and notations are the same as in subsect. V.4.

The jth contribution to the Glauber amplitude is given by (cf. eq.(V.27))

$$F_G^{(j)}(\underset{\sim}{q}^{\perp}) = \frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}\left[-\frac{i}{A}\chi_G(\underset{\sim}{b})\right]^j \quad . \tag{VI.25}$$

For the Fresnel correction we obtain

$$D_F^{(j)}(\underset{\sim}{q}^{\perp}) = -\frac{ik}{2\pi}(-1)^{j-1}\frac{i}{kA^2}\binom{A}{j} j \int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}} \left[-\frac{i}{A}\chi_G(\underset{\sim}{b})\right]^{j-2}$$
$$\times \left[i\chi_G(\underset{\sim}{b})\, d_{F1}(\underset{\sim}{b}) + (j-1)\, d_{F2}(\underset{\sim}{b})\right] \quad , \quad j \geq 1 \quad , \tag{V.26}$$

with

$$d_{F1}(\underset{\sim}{b}) = -\frac{i}{2\hbar v}\nabla^2_{\underset{\sim}{b}} \int_{-\infty}^{\infty} dz\, z\, V(\underset{\sim}{b}, z) \tag{V.27}$$

and

$$d_{F2}(\underset{\sim}{b}) = \left(\frac{1}{\hbar v}\right)^2 \int_{-\infty}^{\infty} dz \left[\nabla_{\underset{\sim}{b}} V(\underset{\sim}{b}, z)\right]\left[\nabla_{\underset{\sim}{b}} \int_{-\infty}^{\infty} du\, u\, \Theta(z-u) V(\underset{\sim}{b}, u)\right]$$
$$-\left(\frac{1}{\hbar v}\right)^2 \left[\nabla_{\underset{\sim}{b}} \int_{-\infty}^{\infty} dz\, V(\underset{\sim}{b}, z)\right]\left[\nabla_{\underset{\sim}{b}} \int_{-\infty}^{\infty} dz\, z\, V(\underset{\sim}{b}, z)\right] . \tag{VI.28}$$

The derivation of eq. (VI.28) is not trivial and will be given in appendix C. We note that for spherically symmetrical functions V eqs. (VI.27) and (VI.28) simplify to

$$d_{F1}(b) = 0 \quad , \tag{VI.27a}$$

$$d_{F2}(b) = \left(\frac{1}{\hbar v}\right)^2 b \frac{\partial}{\partial b} \int_0^{\infty} dz \left[V(b, z)\right]^2 \quad . \tag{VI.28a}$$

The sum $D_e^{(j)}$ of the eikonal-type contributions becomes

$$D_e^{(j)}(\underset{\sim}{q}^{\perp}) = -\frac{ik}{2\pi}(-1)^{j-1}\frac{i}{kA^2}\binom{A}{j}(j-1) j \int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}\left[-\frac{i}{A}\chi_G(\underset{\sim}{b})\right]^{j-2} d_e(\underset{\sim}{b}), \tag{VI.29}$$
$$j \geq 1 \quad ,$$

where

$$d_e(\underset{\sim}{b}) = d_{eo}(\underset{\sim}{b}) + d_{e1}(\underset{\sim}{b}) + d_{e2}(\underset{\sim}{b}) \qquad , \tag{VI.30}$$

and d_{eo} , d_{e1} , d_{e2} are the direct eikonal contribution, the eikonal contribution from rescattering 1, and the eikonal contribution from rescattering 2, respectively,

$$d_{eo}(\underset{\sim}{b}) = \frac{1}{2}\left(\frac{i}{\hbar v}\right)^2 \int_{-\infty}^{\infty} dz \left[V(\underset{\sim}{b},z)\right]^2 \qquad , \tag{VI.31}$$

$$d_{e1}(\underset{\sim}{b}) = -\frac{1}{2}\left(\frac{i}{\hbar v}\right)^2 \int_{-\infty}^{\infty} dz\, V(\underset{\sim}{b},z)\, V'(\underset{\sim}{b},z) \qquad , \tag{VI.32}$$

$$d_{e2}(\underset{\sim}{b}) = \frac{1}{4}\left(\frac{1}{\hbar v}\right)^2 \int_{-\infty}^{\infty} dz \left[V'(\underset{\sim}{b},z)\right]^2 \qquad . \tag{VI.33}$$

Here

$$V'(\underset{\sim}{b},z) = V'(\underset{\sim}{r}) = -\frac{2\pi\hbar^2}{m} A \int \frac{d^3\Delta}{(2\pi)^3}\, e^{i\underset{\sim}{\Delta}\underset{\sim}{r}}\, f'(\underset{\sim}{\Delta})\,\tilde{\rho}(\underset{\sim}{\Delta}) \tag{VI.34}$$

with

$$f'(\underset{\sim}{\Delta}) = f'(\underset{\sim}{\Delta}^{\perp}) = \frac{ik}{2\pi} \int d^2b\, e^{-i\underset{\sim}{\Delta}^{\perp}\underset{\sim}{b}} \left[\gamma(\underset{\sim}{b})\right]^2 \qquad . \tag{VI.35}$$

The total jth scattering contribution is

$$F^{(j)}(\underset{\sim}{q}^{\perp}) = F_G^{(j)}(\underset{\sim}{q}^{\perp}) + D_F^{(j)}(\underset{\sim}{q}^{\perp}) + D_e^{(j)}(\underset{\sim}{q}^{\perp}) \qquad . \tag{VI.36}$$

Summation over j = 1,..., A yields the total scattering amplitude

$$F(\underset{\sim}{q}^{\perp}) = F_G(\underset{\sim}{q}^{\perp}) + D_F(\underset{\sim}{q}^{\perp}) + D_e(\underset{\sim}{q}^{\perp})$$
$$= \frac{ik}{2\pi}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}\,[1 - S(\underset{\sim}{b})] \quad , \qquad \text{(VI.37)}$$

with

$$S(\underset{\sim}{b}) = \left[1+\frac{i}{A}\chi_G(\underset{\sim}{b})\right]^A \left\{1 - \frac{i}{k}\left[1+\frac{i}{A}\chi_G(\underset{\sim}{b})\right]^{-1} d_{F1}(\underset{\sim}{b}) - \frac{A-1}{A}\,\frac{i}{k}\left[1+\frac{i}{A}\chi_G(\underset{\sim}{b})\right]^{-2}\right.$$
$$\left.\times\left[d_{F2}(\underset{\sim}{b}) + d_e(\underset{\sim}{b})\right]\right\} \quad . \qquad \text{(VI.38)}$$

In the optical limit A → ∞ this becomes

$$S_{opt}(\underset{\sim}{b}) = e^{i\chi_G(\underset{\sim}{b})}\left\{1 - \frac{i}{k}\left[d_{F1}(\underset{\sim}{b}) + d_{F2}(\underset{\sim}{b}) + d_e(\underset{\sim}{b})\right]\right\} . \qquad \text{(VI.39)}$$

To first order in k^{-1}, the Glauber phase shift χ_G will thus be modified to

$$\chi(\underset{\sim}{b}) = \chi_G(\underset{\sim}{b}) - \frac{1}{k}\left[d_{F1}(\underset{\sim}{b}) + d_{F2}(\underset{\sim}{b}) + d_e(\underset{\sim}{b})\right] \quad . \qquad \text{(VI.40)}$$

The differential cross section is given by

$$d\sigma/d\Omega = |F(\underset{\sim}{q}^{\perp})|^2 \quad . \qquad \text{(VI.41)}$$

The total cross section is (using the optical theorem)

$$\sigma_T = \frac{4\pi}{k} \operatorname{Im} F(0)$$

$$= 2\int d^2b\,[1 - \operatorname{Re} S(\underset{\sim}{b})] \quad . \qquad \text{(VI.42)}$$

If the Glauber phase shift is pure imaginary ($\alpha = o$ in eq. (VI.17)), we obtain the remarkable result that the total cross section is independent of the first-order corrections. This can be concluded from eq. (VI.38) by noting that d_F and d_e are real for $\alpha = o$. Since σ_T is not very sensitive to α, one therefore expects the corrections of order k^{-1} to the total cross section to be quite small.

We note that if we replace the function V by a real potential and d_e by its direct contribution d_{eo}, eqs. (VI.39) and (VI.40) become identical to the result obtained in section IV for potential scattering. There are no analogues in section IV to the rescattering terms d_{e1} and d_{e2}, however.

4. Application to elastic scattering from light nuclei

In this section we study the elastic scattering of high-energy protons from ^{4}He, ^{12}C and ^{16}O. We assume the single particle density in momentum space to be given by

$$\tilde{\rho}(\underset{\sim}{\Delta}) = [1 - \tau\,(\underset{\sim}{\Delta})^2]\, \exp[-\tfrac{1}{4} C\,(\underset{\sim}{\Delta})^2] \quad . \qquad \text{(VI.43)}$$

Such a Gaussian-type function is consistent with the harmonic-oscillator nuclear model and fits the electron elastic scattering data fairly well at small scattering angles [56]. Moreover, it allows to take the c.m. motion of the nucleus correctly into account by employing the Gartenhaus-Schwartz transformation [57].

This means that instead of having to use expressions which contain a delta function constraint (cf. appendix B) and therefore are difficult to handle, we can employ the formulae of subsect. 3 derived for an independent particle nuclear model. We only have to multiply the total scattering amplitude by a factor $\exp\left(\frac{1}{4}\,\frac{C}{A}\,q^2\right)$. It is instructive to study our first-order corrections for a simple Gaussian first. With the explicit form (VI.17) for the profile γ, all integrations can be done analytically. The results are

$$F_G^{(j)}(\underset{\sim}{q}^\perp) = \frac{ik}{2}(-1)^{j-1}\binom{A}{j}\frac{2B+C}{j}\left[\frac{\sigma(1-i\alpha)}{2\pi(2B+C)}\right]^j \exp\left[-\frac{2B+C}{4j}(\underset{\sim}{q}^\perp)^2\right] , \tag{VI.44}$$

$$D_F^{(j)}(\underset{\sim}{q}^\perp) = \mu^{(j)}(\underset{\sim}{q}^\perp)\left[\frac{2B+C}{j^2}(\underset{\sim}{q}^\perp)^2 - \frac{4}{j}\right] , \tag{VI.45}$$

where

$$\mu^{(j)}(\underset{\sim}{q}^\perp) = F_G^{(j)}(\underset{\sim}{q}^\perp)\binom{j}{2}\frac{i}{\sqrt{2\pi}}\,\frac{1}{k\sqrt{2B+C}} , \tag{VI.46}$$

$$D_{eo}^{(j)}(\underset{\sim}{q}^\perp) = \mu^{(j)}(\underset{\sim}{q}^\perp) , \tag{VI.47}$$

$$D_{e1}^{(j+1)}(\underset{\sim}{q}^\perp) = -\mu^{(j)}(\underset{\sim}{q}^\perp)\left(\frac{2B+C}{\frac{3}{2}B+C}\right)^{\frac{1}{2}}\frac{2B+C}{\frac{j+1}{j}B+C}\,\frac{\sigma}{8\pi B}\exp\left[\frac{B(2B+C)}{4j^2}\,\frac{1}{\frac{j+1}{j}B+C}(\underset{\sim}{q}^\perp)^2\right] , \tag{VI.48}$$

$$D_{e2}^{(j+2)}(\underset{\sim}{q}^\perp) = \mu^{(j)}(\underset{\sim}{q}^\perp)\left(\frac{2B+C}{B+C}\right)^{\frac{1}{2}}\frac{(2B+C)^2}{(B+C)(\frac{j+2}{j}B+C)}\,\frac{1}{2}\left(\frac{\sigma}{8\pi B}\right)^2 \times \exp\left[\frac{B(2B+C)}{2j^2}\,\frac{1}{\frac{j+2}{j}B+C}(\underset{\sim}{q}^\perp)^2\right] . \tag{VI.49}$$

For simplicity the imaginary part of the profile function (VI.17) has again been neglected in the correction terms.

Fig. 1 shows the influence of the corrections on the Glauber differential cross section for p-^{4}He elastic scattering at 1 GeV incident energy, with experimental points from ref. [1]. The Fresnel contribution is seen to be the dominant correction. It has a sizeable effect on the diffraction minimum regions. The eikonal-type corrections are negligible everywhere except at the second diffraction minimum where they have a marginal effect. A more detailed analysis shows that this is due to an almost complete cancellation between $D_{eo}^{(j)}$ and $D_{e1}^{(j+1)}$ at all momentum transfers, and to the smallness of $D_{e2}^{(j+2)}$. Similar results are obtained for scattering by ^{12}C and ^{16}O at 1 GeV. However,the Gaussian density reproduces only the gross features of the angular distributions for these nuclei,and we therefore have to use the more realistic density (VI.43). ($\tau = 6.93$ (GeV/c)$^{-2}$ for ^{12}C and $\tau = 9.15$ (GeV/c)$^{-2}$ for ^{16}O as in ref.[55]). As discussed earlier, the size of the eikonal-type corrections mainly depends on the form of the effective interaction γ and less on the specific type of wave function employed. We therefore conclude that the eikonal corrections will be small with respect to the Fresnel corrections also for densities other than Gaussian, and thus disregard them. Figs. 2 and 3 show results for p-^{12}C and p-^{16}O elastic scattering at 1 GeV incident energy, with experimental data from refs. [58] and [1], respectively. Again the Fresnel corrections mainly influence the diffraction minimum regions. Their effect is of the same size for ^{12}C and ^{16}O. Though still significant it is smaller throughout than for p-^{4}He scattering at the same energy, due to the different densities employed. The prediction of too large values for the second and third maxima in the case of the

^{12}C-target possibly originates from the deformation of the ^{12}C-nucleus [59].

We also have studied the influence of the Fresnel corrections on the total cross sections for scattering of protons by ^{4}He, ^{12}C, and ^{16}O in the energy range 200 MeV to 1 GeV. It is found to be very small; less than two percent of the Glauber result for incident energies > 300 MeV (compare figs. 4-6). This was to be expected from the discussion at the end of subsect. 3. We should mention that our results for ^{12}C and ^{16}O closely agree with the Glauber model calculations of ref. [60] where a simple Gaussian was used. This indicates that the total cross sections are largely independent of the nuclear model employed. Thus σ_T-data appear to be suitable for studying corrections to the Glauber model which are of a different nature than those discussed here, such as spin and correlation effects, Fermi motion, recoil corrections etc.

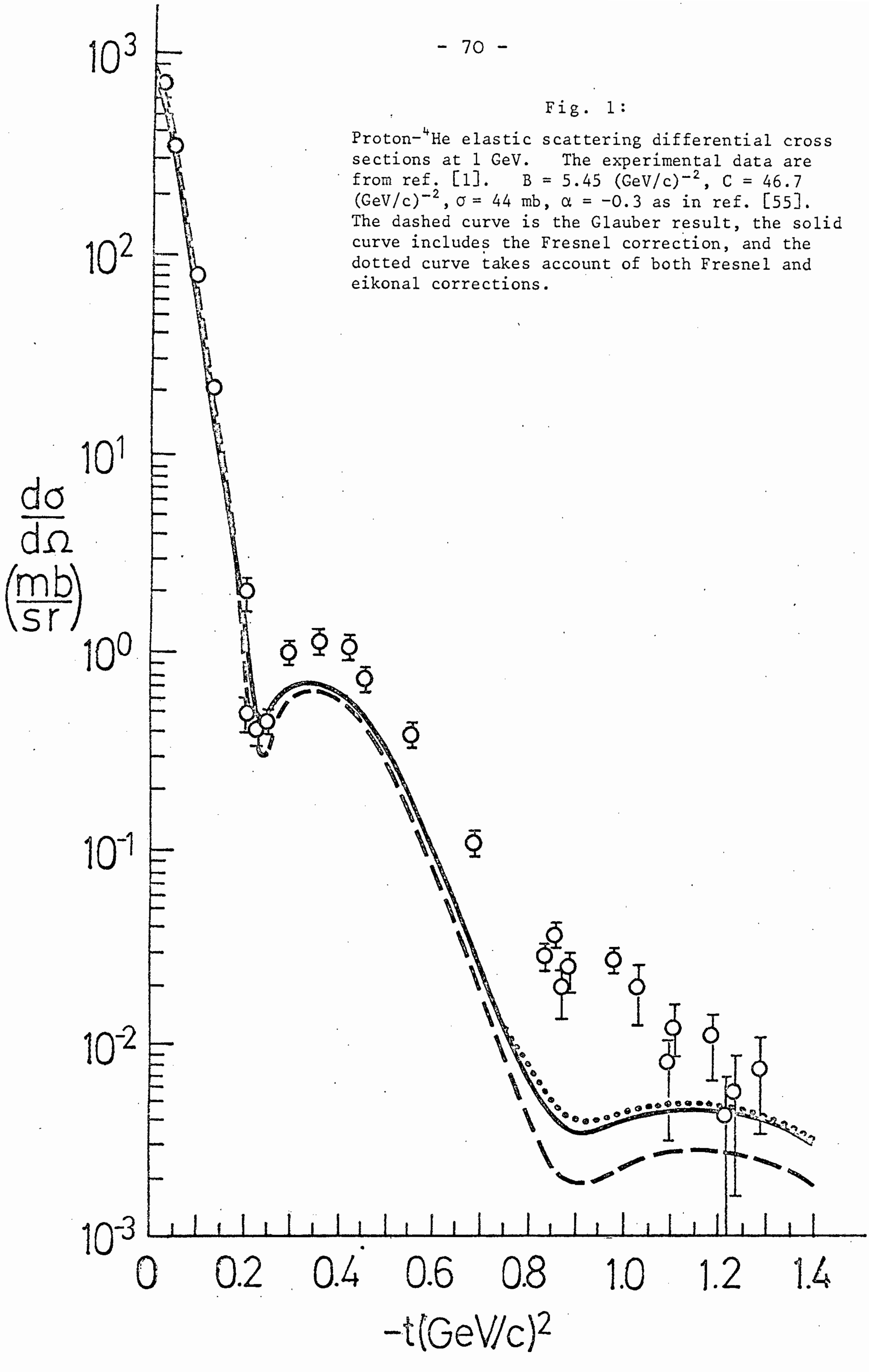


Fig. 1:

Proton-^{4}He elastic scattering differential cross sections at 1 GeV. The experimental data are from ref. [1]. $B = 5.45\ (GeV/c)^{-2}$, $C = 46.7\ (GeV/c)^{-2}$, $\sigma = 44$ mb, $\alpha = -0.3$ as in ref. [55]. The dashed curve is the Glauber result, the solid curve includes the Fresnel correction, and the dotted curve takes account of both Fresnel and eikonal corrections.

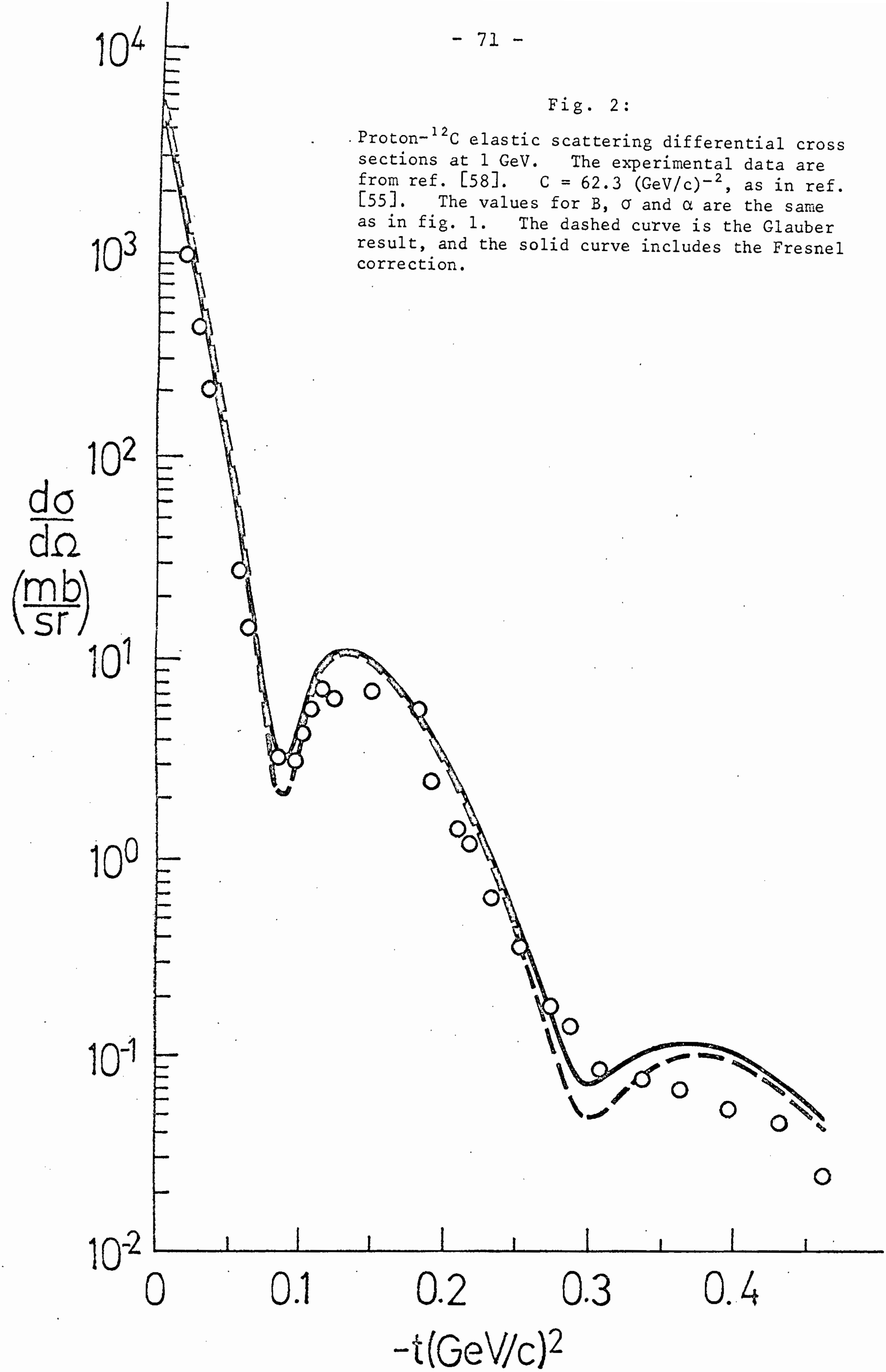


Fig. 2:

Proton-^{12}C elastic scattering differential cross sections at 1 GeV. The experimental data are from ref. [58]. $C = 62.3\ (\text{GeV/c})^{-2}$, as in ref. [55]. The values for B, σ and α are the same as in fig. 1. The dashed curve is the Glauber result, and the solid curve includes the Fresnel correction.

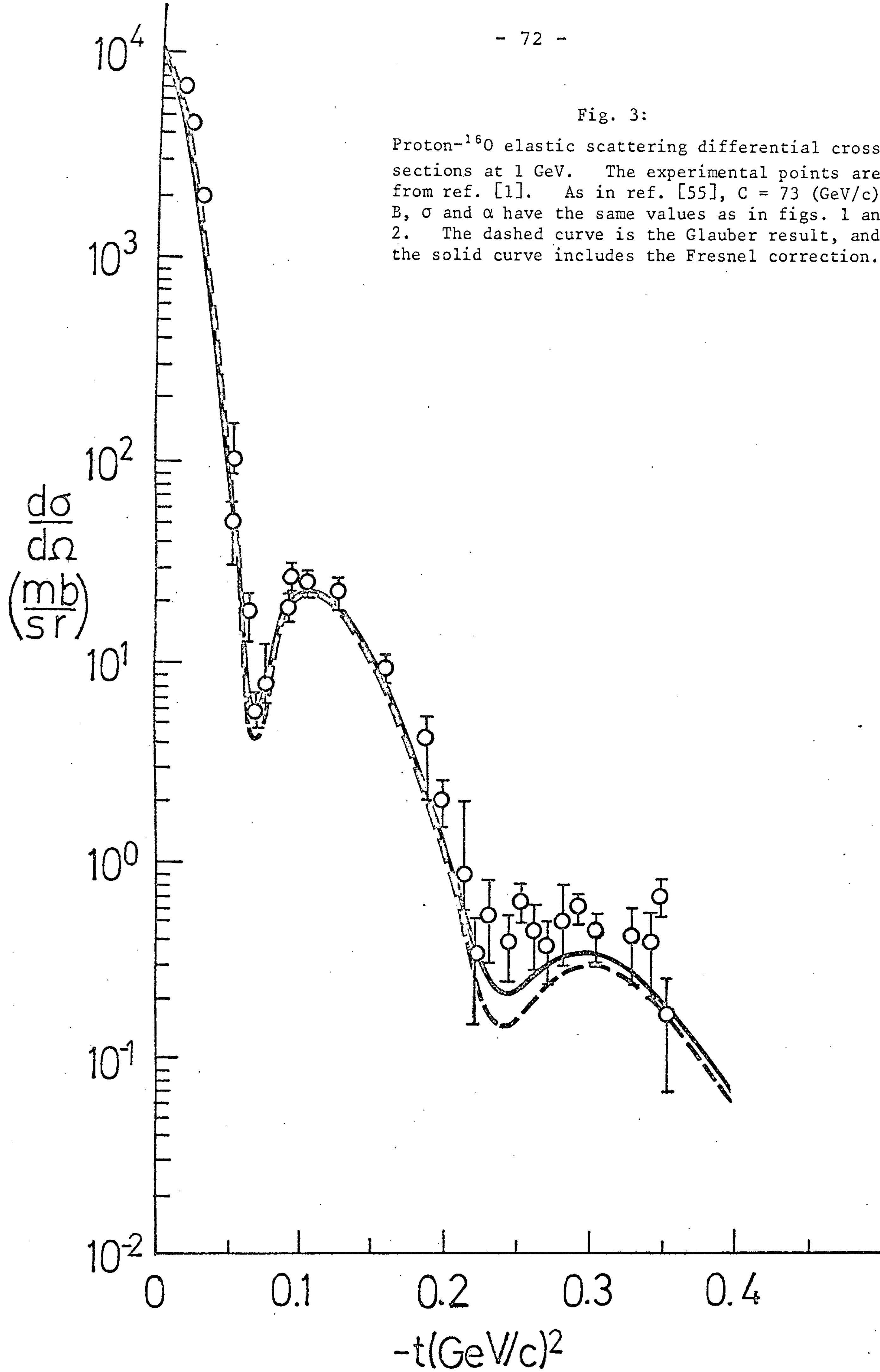


Fig. 3:

Proton-^{16}O elastic scattering differential cross sections at 1 GeV. The experimental points are from ref. [1]. As in ref. [55], C = 73 (GeV/c)$^{-2}$. B, σ and α have the same values as in figs. 1 and 2. The dashed curve is the Glauber result, and the solid curve includes the Fresnel correction.

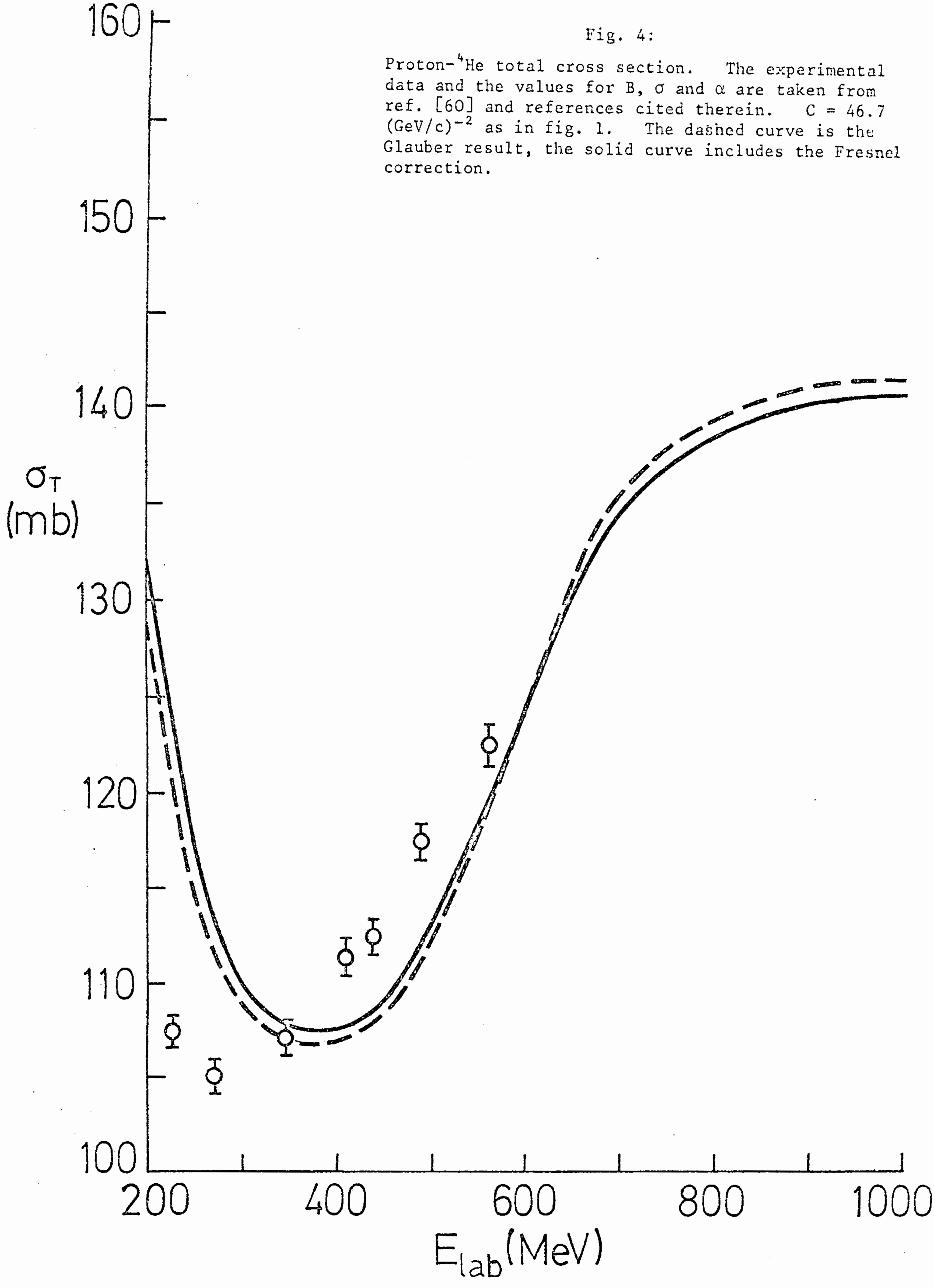


Fig. 4:

Proton-^{4}He total cross section. The experimental data and the values for B, σ and α are taken from ref. [60] and references cited therein. C = 46.7 $(GeV/c)^{-2}$ as in fig. 1. The dashed curve is the Glauber result, the solid curve includes the Fresnel correction.

Fig. 5:

Proton-^{12}C total cross section. The experimental data and the values for B, σ and α are taken from ref. [60] and references cited therein. C = 62.3 $(GeV/c)^{-2}$ as in fig. 2. The dashed curve is the Glauber result, the solid curve includes the Fresnel correction.

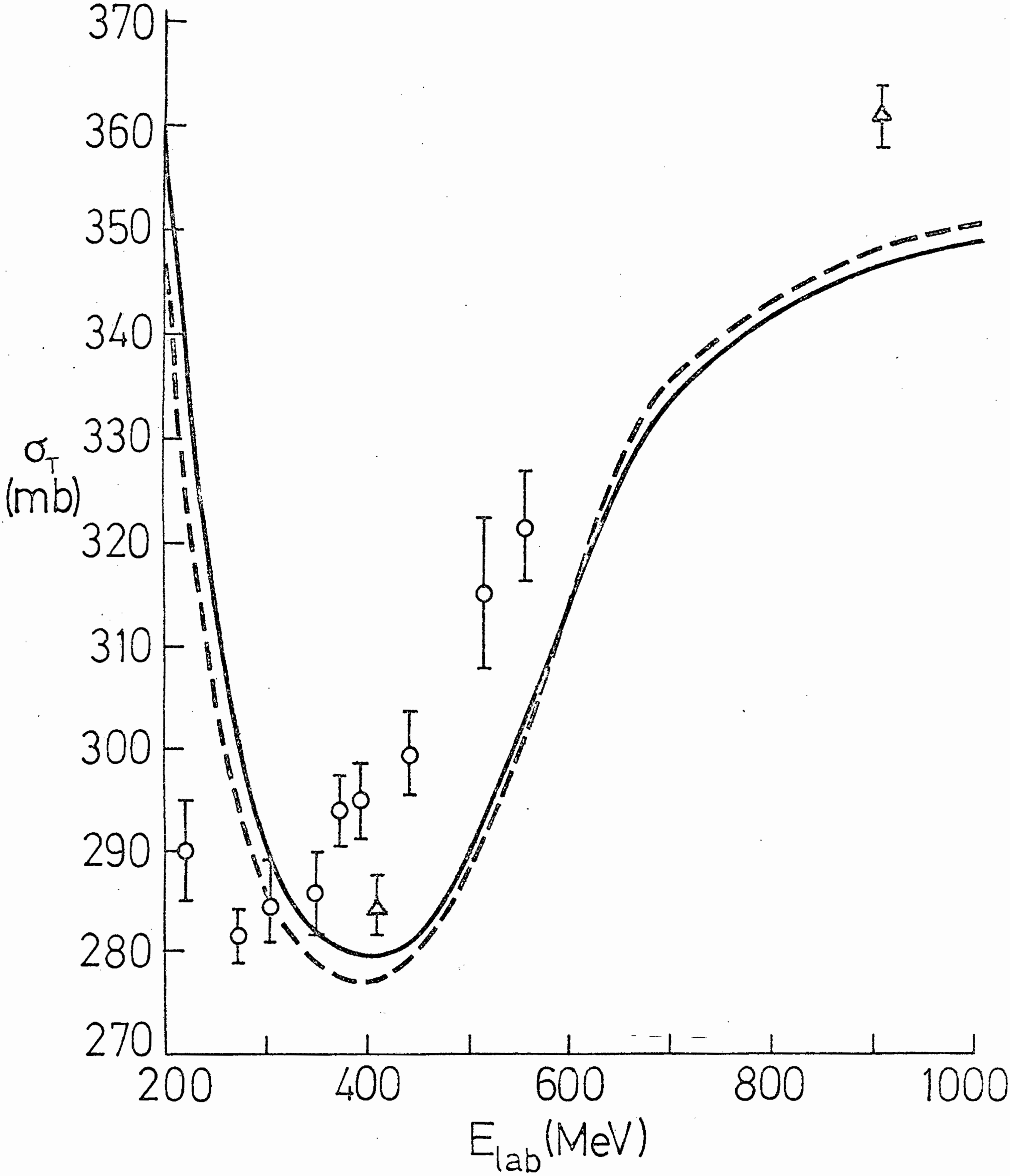
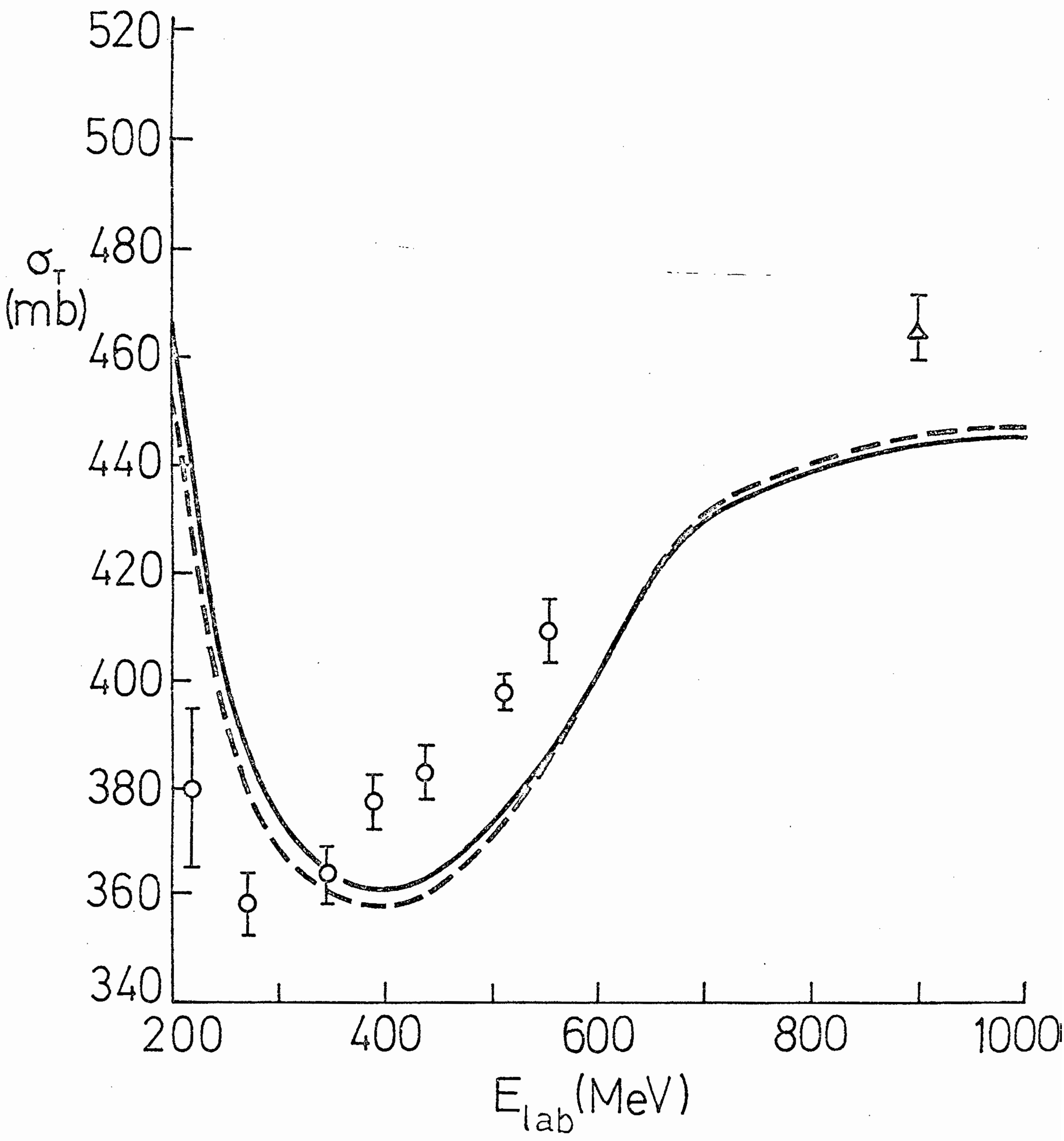


Fig. 6:

Proton-^{16}O total cross section. The experimental data and the values for B, σ and α are taken from ref. [60] and references cited therein. C = 73 $(\mathrm{GeV}/c)^{-2}$ as in fig. 3. The dashed curve is the Glauber result, the solid curve includes the Fresnel correction.



5. Discussion of results

We have obtained the first-order corrections to Glauber's multiple scattering model which originate from the replacement of the free-space Green function between successive scatterings by the second-order eikonal propagator. These corrections consist of Fresnel and eikonal-type contributions. Approximating the projectile-target nucleon scattering amplitudes in the multiple scattering series by their on-shell values (which corresponds essentially to the assumption of non-overlapping potentials in potential scattering), the Fresnel contribution arises from collision processes where the same target nucleon is hit only once. Its size strongly depends on the form of the nuclear density. For heavy nuclei it originates mainly from the nuclear surface, in accordance with the physical picture of Fresnel diffraction.

The eikonal contribution consists of three terms. These arise from single scattering on the same nucleon, double scattering on the same nucleon, and two double scatterings on two different nucleons, respectively. Due to large cancellation effects the total eikonal contribution is found to be very small with respect to the Fresnel correction. This result underlines the usefulness of the Fresnel approximation to multiple scattering.

We have made a comparison with Glauber model results for high-energy elastic scattering of protons from ^{4}He, ^{12}C and ^{16}O. While the Fresnel corrections have a negligible influence on the total cross sections, they have a sizeable effect on the diffraction minimum regions of the angular distributions.

VII. Summary and conclusions

We have studied approximations to high-energy nuclear scattering which go beyond the Glauber theory. Our investigations have been based on expansions of the free-space Green propagator in terms of simplified propagators (eikonal expansion and Fresnel expansion). For simplicity, these methods have first been formulated for scattering from a potential in the Schrödinger-equation.

Two approximations obtained from our expansions, the Fresnel approximation and the second-order eikonal approximation, have been employed in an analysis of nuclear multiple scattering formulated in the framework of the frozen-nucleus assumption and the on-shell approximation for the hadron-nucleon scattering amplitudes. We have shown that the Fresnel approximation leads, as in the corresponding Glauber approximation, to a finite multiple scattering series. The resulting formal expressions for the multiple scattering amplitude are, however, difficult to evaluate for target nuclei with mass number $A > 2$.

The second-order eikonal approximation also yields a finite scattering series. This, however, includes contributions from rescattering processes. Due to cancellations between these rescattering contributions and the eikonal-type correction from the direct terms, the only significant first-order correction to the Glauber results originates from the leading term of the Fresnel approximation. Our results therefore indicate that the Fresnel contributions constitute the dominant corrections to first order in the reciprocal wave number. These corrections have a sizeable effect on the diffraction minima of the elastic scattering differential cross sections, comparable to the influence of nuclear correlation and spin effects [15,61]. For this reason they must be taken into account in realistic calculations of high-energy scattering differential cross sections.

Appendix A : Rescattering terms

In this appendix we show that the infinite remainder $\hat{T}_r$ of the multiple scattering series (V.1) vanishes in the approximation considered here.

The simplest contribution to $\hat{T}_r$ represents a triple scattering event in which the projectile first collides with nucleon 1, then with nucleon 2, and again with nucleon 1. This is of the form

$$F_r^{(3)}(\underset{\sim}{q}) = \frac{ik}{2\pi}(i\hbar v)^2 \binom{A}{2} \langle n | \sum_{P(1,2)} \int d^2b_1 \int d^2b_2 \int d^2b_3 \, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}_3} e^{iq^{\parallel}z_1}$$
$$\times \gamma(\underset{\sim}{b}_3-\underset{\sim}{s}_1) G(\underset{\sim}{b}_3-\underset{\sim}{b}_2, z_1-z_2) \gamma(\underset{\sim}{b}_2-\underset{\sim}{s}_2) G(\underset{\sim}{b}_2-\underset{\sim}{b}_1, z_2-z_1) \gamma(\underset{\sim}{b}_1-\underset{\sim}{s}_1) |0\rangle . \qquad (A.1)$$

Now it is clear that any approximation to the free-space Green function G of the form

$$G_1(\underset{\sim}{b}-\underset{\sim}{b}', z-z') = g(\underset{\sim}{b}-\underset{\sim}{b}', z-z')\,\theta(z-z') , \qquad (A.2)$$

where $g(\underset{\sim}{b}-\underset{\sim}{b}', z-z')$ is continuous at $z = z'$, yields $F_r^{(3)}(\underset{\sim}{q}) = 0$ because of the factor $\theta(z_1 - z_2)\theta(z_2 - z_1)$. The same argument is valid for any of the contributions to $\hat{T}_r$. These are defined as containing "closed loops" of the form $j \to j+1 \to \ldots \to j+\ell \to j$, giving rise to a factor

$$\theta(z_j - z_{j+\ell})\,\theta(z_{j+\ell} - z_{j+\ell-1}) \cdots \theta(z_{j+1} - z_j) , \qquad (A.3)$$

so that the corresponding term in $\hat{T}_r$ vanishes.

As can be seen from eqs. (II.7a) and (II.17a), the eikonal and Fresnel Green functions are of the form (A.2). For the eikonal approximation it has been noticed by Eisenberg [50] that the contributions from $\hat{T}_r$ vanish if in the multiple scattering series (V.1) the transition operators $\hat{t}_j$ are approximated by their on-shell values. The latter approximation is equivalent to our assumption that these operators are given by the observed two-particle scattering amplitudes. The above considerations show that, under the same assumption on the $\hat{t}_j$, the rescattering terms vanish also in the Fresnel approximation.

Moreover, it appears that the Fresnel approximation is in fact the closest approximation to G for which this holds true. The only contributions to G which are of the form (A.2) are those whose Fourier transforms involve only $\tilde{G}_e$ and $h^{\perp}$ as defined by eq. (II.6); as shown in subsect. II.5 these add up precisely to the Fresnel propagator $\tilde{G}_F$.

Appendix B : Centre-of-mass motion of target nucleus

Here we briefly discuss how to take account of the centre-of-mass motion of the target nucleus in the Fresnel approximation. To bring out the point, we first deal with the eikonal case. The amplitude (V.17) may be written explicitly as [19]

$$F_e(\underset{\sim}{q}^\perp) = \frac{ik}{2\pi}\int d^2b'\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}'} \int d^3R\, e^{i(\underset{\sim}{P}_0 - \underset{\sim}{P}_n)\underset{\sim}{R}}\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{S}}$$

$$\times \int d^3r_1' \cdots \int d^3r_A'\, \delta\Big(A^{-1}\sum_{j=1}^{A} \underset{\sim}{r}_j'\Big)\, \varphi_n^*(\underset{\sim}{r}_1' \cdots \underset{\sim}{r}_A')\, \varphi_0(\underset{\sim}{r}_1' \cdots \underset{\sim}{r}_A')\, \hat{\Gamma}(\underset{\sim}{b}', \underset{\sim}{s}_1' \cdots \underset{\sim}{s}_A'). \qquad \text{(B.1)}$$

Here, $\hbar \underset{\sim}{P}_0$ and $\hbar \underset{\sim}{P}_n$ are the total initial and final momenta, φ_0 and φ_n the initial and final internal wave functions of the nucleus, respectively; $\underset{\sim}{R} = (\underset{\sim}{S}, Z)$ is its centre-of-mass coordinate, and $\underset{\sim}{r}_j' = (\underset{\sim}{s}_j', z_j') = (\underset{\sim}{s}_j - \underset{\sim}{S}, z_j - Z)$, $\underset{\sim}{b}' = \underset{\sim}{b} - \underset{\sim}{S}$. In (B.1) use has been made of the fact that the internal wave functions and the profile function $\hat{\Gamma}$ are translation invariant. The integration over $\underset{\sim}{R}$ yields

$$F_e(\underset{\sim}{q}^\perp) = \delta^{(2)}(\underset{\sim}{q}^\perp + \underset{\sim}{P}_0^\perp - \underset{\sim}{P}_n^\perp)\, \delta(P_0^{\parallel} - P_n^{\parallel})\, \frac{ik}{2\pi}\int d^2b'\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}'} \int d^3r_1' \cdots \int d^3r_A'$$

$$\times\, \delta\Big(A^{-1}\sum_{j=1}^{A} \underset{\sim}{r}_j'\Big)\, \varphi_n^*(\underset{\sim}{r}_1' \cdots \underset{\sim}{r}_A')\, \varphi_0(\underset{\sim}{r}_1' \cdots \underset{\sim}{r}_A')\, \hat{\Gamma}(\underset{\sim}{b}', \underset{\sim}{s}_1' \cdots \underset{\sim}{s}_A'). \qquad \text{(B.2)}$$

Aside from the delta functions expressing overall momentum conservation, eq. (B.2) is of the form (V.17) if it is understood that the internal target wave functions are to be used together with the delta function $\delta(A^{-1} \sum_{j=1}^{A} \underset{\sim}{r}_j')$.

The Fresnel amplitude (V.25) could be treated in the same way, were it not for the fact that the profile function $\hat{\Gamma}_F$ defined by eq. (V.26) is not translation invariant as it stands. However, by a partial integration over $\underset{\sim}{b}$ (similar to the one over $\underset{\sim}{b}_j$ used in deriving eq. (V.21)) it can be shown that

$$\int d^2b\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}} \hat{\Gamma}_F(\underset{\sim}{b}, \underset{\sim}{s}_1 \cdots \underset{\sim}{s}_A, z_1 \cdots z_A)$$
$$= \int d^2b'\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}'} e^{i\underset{\sim}{q}^\perp \underset{\sim}{S}} e^{iZ(\underset{\sim}{q}^\perp)^2/2k} \hat{\Gamma}_F(\underset{\sim}{b}', \underset{\sim}{s}'_1 \cdots \underset{\sim}{s}'_A, z'_1 \cdots z'_A). \tag{B.3}$$

Then we obtain in analogy to (B.2)

$$F_F(\underset{\sim}{q}^\perp) = \delta^{(2)}(\underset{\sim}{q}^\perp + \underset{\sim}{P}_0^\perp - \underset{\sim}{P}_n^\perp)\, \delta\left[\frac{(\underset{\sim}{q}^\perp)^2}{2k} + P_0^\parallel - P_n^\parallel\right] \frac{ik}{2\pi} \int d^2b'\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}'} \int d^3r'_1 \cdots \int d^3r'_A$$
$$\times\, \delta(A^{-1}\sum_{j=1}^{A} \underset{\sim}{r}'_j)\, \varphi_n^*(\underset{\sim}{r}'_1 \cdots \underset{\sim}{r}'_A)\, \varphi_0(\underset{\sim}{r}'_1 \cdots \underset{\sim}{r}'_A)\, \hat{\Gamma}_F(\underset{\sim}{b}', \underset{\sim}{s}'_1 \cdots \underset{\sim}{s}'_A, z'_1 \cdots z'_A) \tag{B.4}$$

which is, again up to the appropriate momentum-conserving delta function, of the same form as eq. (V.25). Similar results are obtained in the second-order eikonal approximation.

Appendix C : Fresnel correction for elastic scattering in an independent-particle nuclear model

The derivation of the Fresnel contribution d_{F1} (eq. VI.27)) from the general expression $D_{F1}^{(j)}$ given in eq.(VI. 20a) is straightforward. We therefore only deal with the second contribution d_{F2}. For an independent particle model of the nucleus, the general expression $D_{F2}^{(j)}$ from eq. (VI.20b) becomes

$$D_{F2}^{(j)}(\underset{\sim}{q}^{\perp}) = -\frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j} j!\,\frac{i}{k}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}\int d^3r_1\cdots\int d^3r_j$$
$$\times\,\Theta(z_j-z_{j-1})\cdots\Theta(z_2-z_1)\times\sum_{\lambda=1}^{j-1}\sum_{\nu=1}^{\lambda}\gamma(\underset{\sim}{b}-\underset{\sim}{s}_j)\rho(\underset{\sim}{r}_j)\cdots z_{\lambda+1} \tag{C.1}$$
$$\times[\underset{\sim}{\nabla}_{\underset{\sim}{b}}\gamma(\underset{\sim}{b}-\underset{\sim}{s}_{\lambda+1})]\rho(\underset{\sim}{r}_{\lambda+1})\gamma(\underset{\sim}{b}-\underset{\sim}{s}_{\lambda})\rho(\underset{\sim}{r}_{\lambda})\cdots[\underset{\sim}{\nabla}_{\underset{\sim}{b}}\gamma(\underset{\sim}{b}-\underset{\sim}{s}_{\nu})]\rho(\underset{\sim}{r}_{\nu})\cdots\gamma(\underset{\sim}{b}-\underset{\sim}{s}_1)\rho(\underset{\sim}{r}_1).$$

Since each term in the sum over the j! permutations yields the same contribution, we have chosen one specific sequence of indices 1,...,j. Introducing the functions $V(\underset{\sim}{r})$ defined in eq. (V.30) we obtain from eq. (C.1)

$$D_{F2}^{(j)}(\underset{\sim}{q}^{\perp}) = -\frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j} j!\left(\frac{i}{\hbar vA}\right)^{j}\frac{i}{k}\int d^2b\, e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}$$
$$\times\int dz_1\cdots\int dz_j\;\Theta(z_j-z_{j-1})\cdots\Theta(z_2-z_1)\sum_{\lambda=1}^{j-1}\sum_{\nu=1}^{\lambda}V(\underset{\sim}{b},z_j)\cdots$$
$$\times z_{\lambda+1}[\underset{\sim}{\nabla}_{\underset{\sim}{b}}V(\underset{\sim}{b},z_{\lambda+1})]\cdots[\underset{\sim}{\nabla}_{\underset{\sim}{b}}V(\underset{\sim}{b},z_{\nu})]\cdots V(\underset{\sim}{b},z_1)\;. \tag{C.2}$$

We now define

$$W(\underset{\sim}{b},z) \equiv \int^z du\, u\, V(\underset{\sim}{b},u) \quad . \tag{C.3}$$

By noting that $d\theta/dz = \delta(z)$, partial integration yields

$$\int dz_j\, \theta(z_j - z_{j-1})\, z_j\, V(\underset{\sim}{b}, z_j) = W(\underset{\sim}{b},\infty) - W(\underset{\sim}{b}, z_{j-1}) \quad , \tag{C.4}$$

$$\begin{aligned}&\int dz_{\lambda+1}\, \theta(z_{\lambda+2} - z_{\lambda+1})\, z_{\lambda+1}\, \theta(z_{\lambda+1} - z_\lambda)\, V(\underset{\sim}{b}, z_{\lambda+1}) \\ &= [W(\underset{\sim}{b}, z_{\lambda+2}) - W(\underset{\sim}{b}, z_\lambda)]\, \theta(z_{\lambda+2} - z_\lambda) \quad , \qquad 1 \le \lambda \le j-2 \; .\end{aligned} \tag{C.4a}$$

Using these results in eq. (C.2) and changing indices $\lambda+2,\ldots,j$ into $\lambda+1,\ldots,j-1$, many terms cancel, with the result

$$\begin{aligned}D^{(j)}_{F2}(\underset{\sim}{q}^\perp) = &\frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j} j!\left(\frac{i}{\hbar v A}\right)^j \frac{i}{k}\int d^2b\, e^{i\underset{\sim}{q}^\perp \underset{\sim}{b}} \\ &\times \int dz_1 \cdots \int dz_{j-1}\, \theta(z_{j-1} - z_{j-2}) \cdots \theta(z_2 - z_1) \sum_{\lambda=1}^{j-1} V(\underset{\sim}{b}, z_1) \cdots \\ &\times [\underset{\sim}{\nabla}_{\underset{\sim}{b}} V(\underset{\sim}{b}, z_\lambda)]\, [\underset{\sim}{\nabla}_{\underset{\sim}{b}} W(\underset{\sim}{b}, z_\lambda) - \underset{\sim}{\nabla}_{\underset{\sim}{b}} W(\underset{\sim}{b},\infty)] \cdots V(\underset{\sim}{b}, z_{j-1}) \, .\end{aligned} \tag{C.5}$$

The terms summed over λ represent a function symmetric in $z_1,\ldots,z_{j-1}$. We thus can eliminate the step functions by introducing the sum over all permutations of the indices $1,\ldots,j-1$ and using

$$\sum_{P(1\cdots j-1)} \theta(z_{j-1} - z_{j-2}) \cdots \theta(z_2 - z_1) = 1 \quad . \tag{C.6}$$

We note further that

$$W(\underset{\sim}{b},z)-W(\underset{\sim}{b},\infty)=\int_{-\infty}^{\infty}[\theta(z-u)-1]\,u\,V(\underset{\sim}{b},u)\,du\ . \tag{C.7}$$

Thus we finally obtain

$$D_{F2}^{(j)}(\underset{\sim}{q}^{\perp})=-\frac{ik}{2\pi}(-1)^{j-1}\binom{A}{j}(j-1)j\,\frac{i}{kA^2}\int d^2b\,e^{i\underset{\sim}{q}^{\perp}\underset{\sim}{b}}\left[-\frac{i}{A}\chi_G(\underset{\sim}{b})\right]^{j-2}d_{F2}(\underset{\sim}{b})\ , \tag{C.8}$$

with $\chi_G(\underset{\sim}{b})$ defined in eq. (V.28) and $d_{F2}(\underset{\sim}{b})$ given by eq. (VI.28).

Appendix D : Fresnel correction for potential scattering

The amplitude for scattering from a potential is

$$f(\underset{\sim}{q}) = -\frac{m}{2\pi\hbar^2}\langle \underset{\sim}{k}'|\hat{t}|\underset{\sim}{k}\rangle \quad . \tag{D.1}$$

Here $\hat{t}$ is defined as

$$\hat{t} = \hat{V} + \hat{V}\hat{G}\hat{t} \quad , \tag{D.2}$$

with a potential operator $\hat{V}$ and the free-space Green propagator defined in eq. (V.2). We now expand (D.2) in a Born series and insert this in eq. (D.1). The resulting expression is very similar in its form to the sum of the direct terms of the multiple scattering series (V.1) for a target nucleus represented by an independent particle model. The former differs from the latter by a factor $A^j(A-j)!/A!$, and by the fact that it consists of an infinite number of terms. Both differences are unimportant for the present consideration, and thus the derivation of the Fresnel correction (IV.4) proceeds along the same lines as in appendix C. Due to the above-discussed analogy the eikonal contribution (IV.9) is also obtained in the same way as the corresponding expression (VI.31) for multiple scattering.

References

[1] H. Palevski, J.L. Friedes, R.J. Sutter, G.W. Bennett, G.J. Igo, W.D. Simpson, G.C. Phillips, D.M. Corley, N.S. Wall, R.L. Stearns, and B. Gottschalk, Phys. Rev. Lett. 18 (1967) 1200

[2] M.L. Goldberger and K.M. Watson, Collision theory (Wiley, New York, 1964) p.749

[3] K.M. Watson, Phys. Rev. 89 (1953) 575

[4] N.C. Francis and K.M. Watson, Phys. Rev. 92 (1953) 291

[5] K.M. Watson, Phys. Rev. 105 (1957) 1388

[6] R.R. Johnston and K.M. Watson, Nucl. Phys. 28 (1961) 583

[7] R.R. Johnston, Nucl. Phys. 36 (1962) 368

[8] F.A. McDonald and M.H. Hull Jr., Phys. Rev. 143 (1966) 838

[9] J.S. Chalmers and A.M. Saperstein, Phys. Rev. 156 (1967) 1099

[10] J.F. Reading and A.D. MacKellar, Phys. Rev. 173 (1968) 1026

[11] J.S. Chalmers and A.M. Saperstein, Phys. Rev. 168 (1968) 1145

[12] A.K. Kerman, H. McManus, R.M. Thaler, Ann. of Phys. 8 (1959) 551

[13] H. Feshbach and J. Hüfner, Ann. of Phys. 56 (1970) 268

[14] H. Feshbach, A. Gal, and J. Hüfner, Ann. of Phys. 66 (1971) 20

[15] E. Lambert and H. Feshbach, Phys. Letters 38B (1972) 487; Ann. of Phys. 76 (1973) 80

[16] E. Kujawski, Phys. Rev. C1 (1970) 1651

[17] L.L. Foldy and J.D. Walecka, Ann. of Phys. 54 (1969) 447

[18] R.J. Glauber, Lectures in theoretical physics, vol. 1, ed. W.E. Brittin and L.G. Dunham (Interscience, New York, 1959) p. 315

[19] R.J. Glauber, High-energy physics and nuclear structure, ed. G. Alexander (North Holland, Amsterdam, 1967) p.311

[20] R.J. Glauber, High-energy physics and nuclear structure, ed. S. Devons (Plenum Press, New York, 1970) p. 207

[21] W. Czyż, Advances in Nuclear Physics 4 (1971) 61
(The above two references are comprehensive surveys of applications of the Glauber model in nuclear and high-energy physics)

[22] H. Tai, R.H. Bassel, E. Gerjuoy, and V. Franco, Phys. Rev. A1 (1970) 1819
(Cites many references where Glauber theory has been used in atomic physics)

[23] R.J. Glauber, Phys. Rev. 91 (1953) 459

[24] B.J. Malenka, Phys. Rev. 95 (1954) 522

[25] D.S. Saxon and L.I. Schiff, Nuovo Cimento 6 (1957) 614

[26] T.T. Wu, Phys. Rev. 108 (1957) 466

[27] J. Pumplin, Phys. Rev. 173 (1968) 1651

[28] D.K. Ross, Phys. Rev. 173 (1968) 1695

[29] R.L. Sugar and R. Blankenbecler, Phys. Rev. 183 (1969) 1387

[30] Y. Hahn, Phys. Rev. 184 (1969) 1022

[31] V.S. Bashin and V.S. Varma, Phys. Rev. 184 (1969) 1338

[32] H.D.I. Abarbanel and C. Itzykson, Phys. Rev. Lett. 23 (1969) 53

[33] Y. Hahn, Phys. Rev. C2 (1970) 775

[34] R.J. Moore, Phys. Rev. D2 (1970) 313

[35] S.J. Wallace, Phys. Rev. Letters 27 (1971) 622

[36] K. Gottfried, Ann. of Phys. 66 (1971) 868; Optical concepts in high-energy physics, CERN lectures 1971, CERN 72-20, (Geneva, 1972)

[37] J.F. Gunion and R. Blankenbecler, Phys. Rev. D3 (1971) 2125

[38] G. Fäldt, Nucl. Phys. B29 (1971) 16

[39] E. Kujawski, Phys. Rev. D4 (1971) 2573

[40] J.M. Namyslowski, Nuovo Cimento Letters 4 (1972) 517

[41] G. Fäldt, Nucl. Phys. B46 (1972) 460

[42] E. Kujawski, Ann. of Phys. 74 (1972) 567

[43] A. Baker, Phys. Rev. D6 (1972) 3462

[44] S.J. Wallace, Ann. of Phys. 78 (1973) 190

[45] F.J. Dyson, Phys. Rev. 75 (1949) 486

[46] E.A. Remler, Phys. Rev. 176 (1968) 2108

[47] D.R. Harrington, Phys. Rev. 184 (1969) 1745

[48] T.A. Osborn, Ann. of Phys. 58 (1970) 417

[49] T.A. Osborn, Nucl. Phys. B46 (1972) 538

[50] J.M. Eisenberg, Ann. of Phys. 71 (1972) 542

[51] M. Yamazaki, Progr. Theor. Phys. 47 (1972) 1593

[52] J.M. Namyslowski, Nuovo Cimento 12A (1972) 321

[53] D. Agassi and A. Gal, Ann. of Phys. 75 (1973) 56

[54] V. Franco and R.J. Glauber, Phys. Rev. 142 (1966) 1195

[55] R.H. Bassel and C. Wilkin, Phys. Rev. 174 (1966) 1195

[56] P. Goldhammer, Rev. Mod. Phys. 35 (1963) 40

[57] S. Gartenhaus and C. Schwartz, Phys. Rev. 108 (1957) 482

[58] R. Bertini, R. Beurtey, F. Brochard, G. Bruge, H. Catz, A. Chaumeaux, J.M. Durand, J.C. Faivre, J.M. Fontaine, D. Garreta, C. Gustafsson, D. Hendrie, F. Hibou, D. Legrand, J. Saudinos, and J. Thiron, Phys. Lett. 45B (1973) 119

[59] H. Leśniak and L. Leśniak, Nucl. Phys. B25 (1971) 525

[60] P. Schwaller, B. Favier, D.F. Measday, M. Pepin, P.U. Renberg, and C. Serre, CERN Report 72-13, Geneva 1972

[61] E.J. Moniz and G.D. Nixon, Ann. of Phys. 67 (1971) 58