Inductive Coupled Transfer Functions for Four Major Topologies, and their Relevant Design Equations

Hendrik D. Mouton

Abstract – **The four major topologies for inductive couplers (such as transformers with air gaps used for wireless power transfer), are discussed in many articles widely available. In this article all the relevant transfer functions and design equations for all four topologies are given and derived. It contains Bode plots showing gain and phase characteristics against frequency with different load resistances, demonstrating clearly that each topology has its own specific characteristics so that the best one for a particular application can be selected. The author used the Laplace *s*-parameter in all derivations, and only convert later to the frequency domain when required. The concept of reflected impedances due to the transformer in all the designs is used, because it simplifies the derivations significantly, though care must be taken to apply it correctly. Therefore, the basic derivation of reflected impedances is also given. A novel set of formulas is derived for calculating the capacitance of the capacitor on the secondary side of the transformer. It is demonstrated to give satisfactory results even though the resultant formulas are relatively simple.**

***Keywords* – Wireless power transfer, Transformer, Circuit topology, Voltage control, Current control, Frequency response, Simulation**

I. INTRODUCTION

Wireless power transfer, for example to charge batteries, is important in the field of autonomous robotics and is mostly done with types of transformers. This type of transfer is called inductive coupled transfer. Depending on the magnetic material, apart from the air gap, the most power transfer efficient frequency can be determined. The design of the components surrounding the transformer must be such that the transfer function at this optimal frequency must have high gain, and this gain must ideally be almost independent of the load on the secondary side of the transformer.

A transformer with its coils, and input and output components, can be represented as follows:

Hendrik D. Mouton did most of this work in 2019 while being an Associate Professor at the University of Cape Town, South Africa, and completed it after retirement. No special funding was received. E-mail: hennie.mouton@uct.ac.za

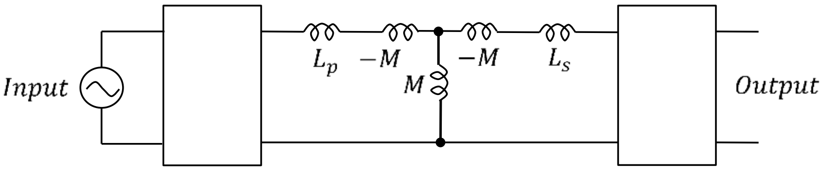

Fig. 1. Transformer with its inductances, input and output components

L_p is the primary coil inductance.
L_s is the secondary coil inductance.
M is the mutual inductance of the two coils.

Using Thevenin's theorem, the input and its components can be replaced by a voltage source V_1 and an impedance in series. The output components, including a connected load, can also be replaced by a single impedance. So, this is the relevant circuit:

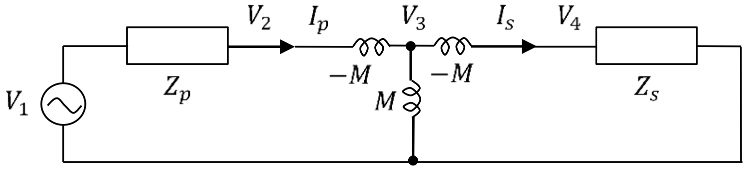

Fig. 2. Transformer with its inductances, input and output impedances

$$I_p = \frac{V_1}{Z_p - sM + \frac{1}{\frac{1}{sM} + \frac{1}{Z_s - sM}}} = \frac{V_1}{Z_p - sM + \frac{sM(Z_s - sM)}{sM + Z_s - sM}} = \frac{V_1}{Z_p - \frac{s^2M^2}{Z_s}}$$

$$V_3 = I_p \times \frac{1}{\frac{1}{sM} + \frac{1}{Z_s - sM}} = I_p \frac{sM(Z_s - sM)}{Z_s}$$

$$\therefore V_4 = \frac{Z_s}{Z_s - sM} V_3 = I_p sM \text{ and } I_s = \frac{I_p sM}{Z_s} - \frac{s^2M^2}{Z_s}$$

is the reflected impedance of the secondary impedance on the primary side, and $I_p sM$ is the voltage driving the secondary side.

The block diagram is the following, but one must be careful how components are connected at the input and the output.

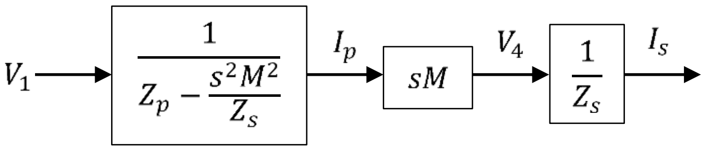

Fig. 3. Block diagram of transformer with its components

Next is shown how reflected impedances are applied to the four major topologies of inductive couplers, namely the S/S, S/P, P/S and P/P configurations [1 – 8], and how to design the primary and secondary capacitor values. The secondary capacitor values are designed in a novel way, not seen in any previous papers. It is demonstrated by simulation that it works well.

II. S/S CONFIGURATION

S/S refers to the primary and secondary capacitors both connected in series, as can be seen in the next circuit diagram.

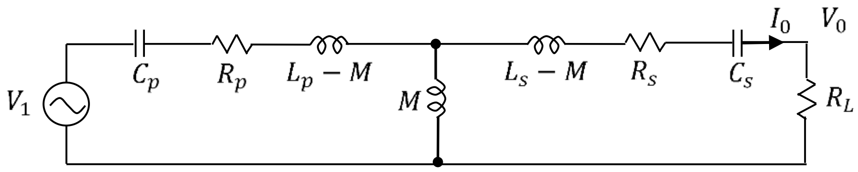

Fig. 4. Circuit diagram of S/S configuration

$Z_p = sL_p + R_p + \frac{1}{sC_p}$
$Z_s = sL_s + R_s + \frac{1}{sC_s} + R_L$
$I_0 = I_s$

Applying these to the block diagram of Fig. 3, results in:
$\frac{I_0}{V_1} = \frac{sM}{\left(Z_p - \frac{s^2M^2}{Z_s}\right)Z_s}$. Note that $\frac{V_0}{V_1}$ in this configuration would have been very dependent on R_L and is thus not considered.
$\therefore \frac{I_0}{V_1} = \frac{sM}{\left(sL_p + R_p + \frac{1}{sC_p}\right)\left[sL_s + (R_s + R_L) + \frac{1}{sC_s}\right] - s^2M^2}$

$\therefore \frac{I_0}{V_1} = \frac{N_3 s^3}{D_4 s^4 + D_3 s^3 + D_2 s^2 + D_1 s + D_0}$, with $N_3 = MC_pC_s$,
$D_4 = C_pC_s(L_pL_s - M^2)$, $D_3 = C_pC_s[R_pL_s + (R_s + R_L)L_p]$,
$D_2 = C_pL_p + C_sL_s + C_pC_sR_p(R_s + R_L)$,
$D_1 = C_pR_p + C_s(R_s + R_L)$, $D_0 = 1$
$\therefore \frac{I_0}{V_1}/_{s=i\omega} = \frac{-iN_3\omega^3}{D_4\omega^4 - iD_3\omega^3 - D_2\omega^2 + iD_1\omega + D_0}$

The best applied frequency (ω_{app}) for this system would be where the $\frac{I_0}{V_1}$ gain is maximum and it can be expected to be where either the real part or the imaginary part of the denominator is zero. So, $D_4\omega^4 - D_2\omega^2 + D_0 = 0$ or $-D_3\omega^3 + D_1\omega = 0$. Looking at the expressions of these D coefficients, the last equation is the best, because with that part zero, $\frac{I_0}{V_1}$ will be very little dependent on R_L. This is desired because R_L should ideally not change the gain.
$\therefore \omega_{app}^2 = \frac{D_1}{D_3} = \frac{C_pR_p + C_s(R_s + R_L)}{C_pC_s[R_pL_s + (R_s + R_L)L_p]}$

R_p and R_s will be as small as possible because the bigger they are, the more heat will be dissipated.

$\therefore \omega_{app}^2/_{R_p = R_s = 0} = \frac{1}{C_pL_p}$, because R_L cancels out. Since the ideal applied frequency should be known from the characteristics of the transformer, this formula should be used to determine the primary capacitance C_p. This is ideal too, in that ω_{app} is also independent of R_L.

$D_4\omega^4 - D_2\omega^2 + D_0 = 0$ provides two frequencies where the gain is potentially maximum. It makes sense to put them on both sides of ω_{app}, so that the square root of the product of the two, is equal to ω_{app}. This is sensible because it will cause the gain at the applied frequency to be less sensitive to the exact value of the frequency.
$\therefore \frac{D_2 + \sqrt{D_2^2 - 4D_4D_0}}{2D_4} \times \frac{D_2 - \sqrt{D_2^2 - 4D_4D_0}}{2D_4} = \omega_{app}^4$

$\therefore \frac{4D_4D_0}{4D_4^2} = \frac{D_0}{D_4} = \frac{1}{C_pC_s(L_pL_s - M^2)} = \frac{1}{C_p^2L_p^2}$

$\therefore C_s = \frac{C_pL_p^2}{L_pL_s - M^2}$ From this formula the secondary capacitance C_s can be determined.

$\frac{I_0}{V_1}$ at ω_{app} is important.

$$\left|\frac{I_0}{V_1}\right| /_{\omega=\omega_{app}} = \frac{|N_3\omega_{app}^3|}{|D_4\omega_{app}^4 - D_2\omega_{app}^2 + D_0|} = \frac{MC_pC_s\left(\frac{1}{C_pL_p}\right)^{1.5}}{\left|C_pC_s(L_pL_s - M^2)\left(\frac{1}{C_pL_p}\right)^2 - (C_pL_p + C_sL_s)\frac{1}{C_pL_p} + 1\right|}$$, if $R_p = R_s = 0$.

$$\therefore \left|\frac{I_0}{V_1}\right| /_{\omega=\omega_{app},\, R_p=R_s=0} = \frac{\sqrt{C_pL_p}}{M}$$

The following graphs show the gain and phase against frequency with different values of R_L. Different values of C_s will change the shape of the graphs, but even then, the gain and phase at the optimal applied frequency will stay the same.

III. S/S EXAMPLE AND GRAPHS

The values chosen are from a successful design of a wireless battery charger for a hexapod robot by a student from UCT (University of Cape Town) and were measured or calculated by using the formulas above. They are:
$\omega_{app} = 2\pi 143.0$ krad/s
$L_p = L_s = 24.0$ μH, $M = 16.1$ μH, $R_p = R_s = 0.1\ \Omega$
$\therefore C_p = 51.6$ nF and $C_s = 93.8$ nF

Graphs were plotted with:
$R_L = \frac{60.0}{18}, \frac{60.0}{12}, \frac{60.0}{8}, \frac{60.0}{4}, \frac{60.0}{2}$ and 60.0 Ω. The gain at the optimal applied frequency of 143 kHz should be close to $\frac{\sqrt{C_pL_p}}{M} = -23.2$ dB, which it is as shown in Fig. 5. If wanted, the two peaks in the gain plot can be changed to be the same height by modifying C_s a bit.

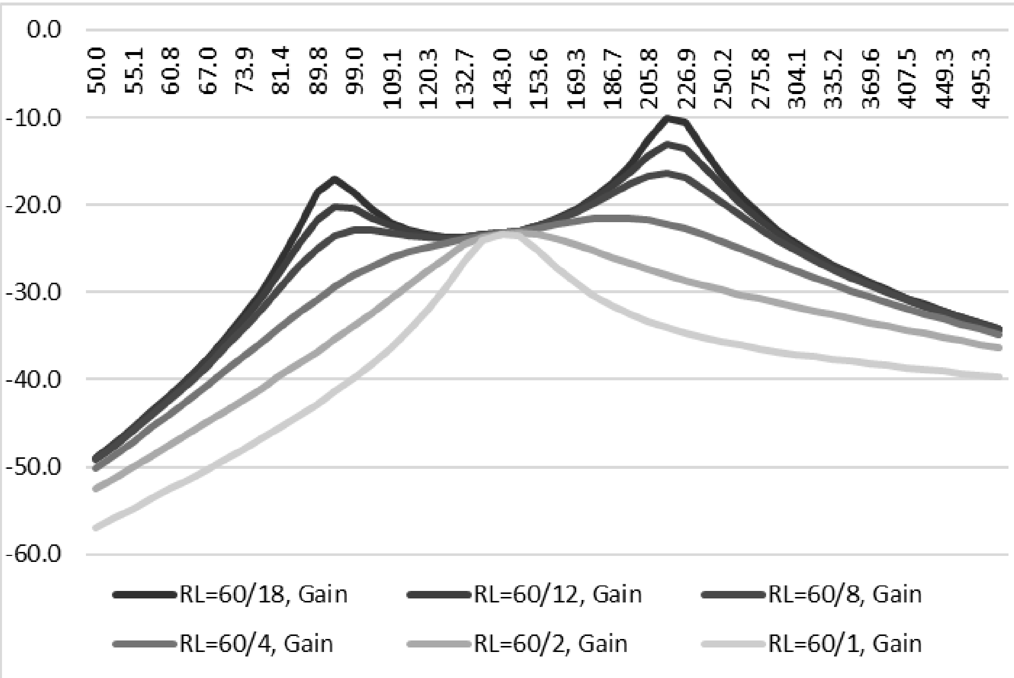

Fig. 5. Gain plot of S/S configuration with different loads

The gain stays the same at the optimal applied frequency of 143 kHz in this example.

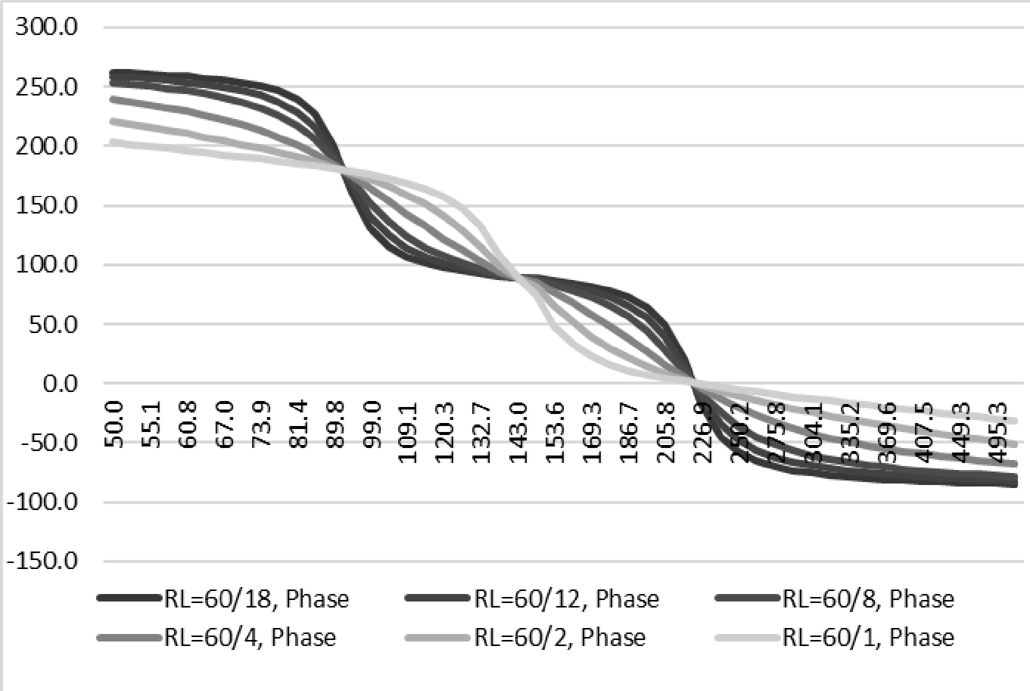

Fig. 6. Phase plot of S/S configuration with different loads

The phase stays the same (90°) at the optimal applied frequency of 143 kHz. It also stays the same (180° and 0° respectively) at two other frequencies, but there the gain did not stay the same, therefore they are not optimal frequencies.

IV. S/P CONFIGURATION

S/P refers to the primary capacitor connected in series and the secondary capacitor in parallel, as can be seen in the next circuit diagram.

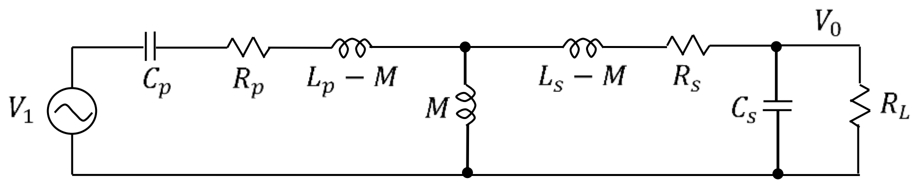

Fig. 7. Circuit diagram of S/P configuration

$Z_p = sL_p + R_p + \frac{1}{sC_p}$

$Z_s = sL_s + R_s + \frac{1}{sC_s + {}^{1}/_{R_L}} = sL_s + R_s + \frac{R_L}{sC_sR_L + 1}$

Applying these to the block diagram of Fig. 3, results in:

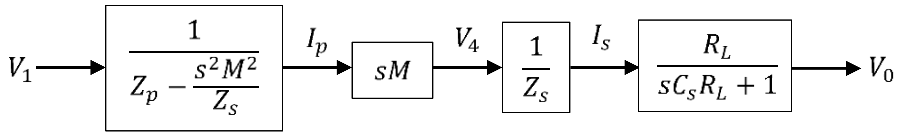

Fig. 8. Block diagram of S/P transformer

$\therefore \frac{V_0}{V_1} = \frac{sMR_L}{(Z_pZ_s - s^2M^2)(sC_sR_L+1)}$

It will be shown that $\frac{V_0}{V_1}$ in this configuration at the optimal applied frequency, is quite independent on R_L.

$\therefore \frac{V_0}{V_1} = \frac{N_2s^2}{D_4s^4+D_3s^3+D_2s^2+D_1s+D_0}$, with $N_2 = MC_p$,

$D_4 = C_pC_s(L_pL_s - M^2)$,

$D_3 = C_pC_s(L_pR_s + L_sR_p) + \frac{C_p}{R_L}(L_pL_s - M^2)$,

$D_2 = C_pR_p\left(C_sR_s + \frac{L_s}{R_L}\right) + C_sL_s + C_pL_p\left(1 + \frac{R_s}{R_L}\right)$,

$D_1 = C_sR_s + \frac{L_s}{R_L} + C_pR_p\left(1 + \frac{R_s}{R_L}\right)$, $D_0 = 1 + \frac{R_s}{R_L}$

$\therefore \frac{V_0}{V_1}/_{s=i\omega} = \frac{-N_2\omega^2}{D_4\omega^4 - iD_3\omega^3 - D_2\omega^2 + iD_1\omega + D_0}$

The best applied frequency (ω_{app}) for this system would be where the $\frac{V_0}{V_1}$ gain is maximum and it can be expected to be where either the real part or the imaginary part of the denominator is zero. So, $D_4\omega^4 - D_2\omega^2 + D_0 = 0$ or $-D_3\omega^3 + D_1\omega = 0$. Looking at the expressions of these D coefficients, the last equation is the best, because with that part zero, $\frac{V_0}{V_1}$ will be very little dependent on R_L. This is desired because R_L should ideally not change the gain.

$\therefore \omega_{app}^2 = \frac{D_1}{D_3} = \frac{C_sR_s + \frac{L_s}{R_L} + C_pR_p\left(1+\frac{R_s}{R_L}\right)}{C_pC_s(L_pR_s + L_sR_p) + \frac{C_p}{R_L}(L_pL_s - M^2)}$

R_p and R_s will be as small as possible because the bigger they are, the more heat will be dissipated.

$\therefore \omega_{app}^2/_{R_p=R_s=0} = \frac{L_s}{C_p(L_pL_s - M^2)}$, because R_L cancels out. Since the ideal applied frequency should be known from the characteristics of the transformer, this formula should be used to determine the primary capacitance C_p. This is ideal too, in that ω_{app} is also independent of R_L.

$D_4\omega^4 - D_2\omega^2 + D_0 = 0$ provides two frequencies where the gain is potentially maximum. It makes sense to put them on both sides of ω_{app}, so that the square root of the product of the two, is equal to ω_{app}. This is sensible because it will cause the gain at the applied frequency to be less sensitive to the exact value of the frequency.

$\therefore \frac{D_2 + \sqrt{D_2^2 - 4D_4D_0}}{2D_4} \times \frac{D_2 - \sqrt{D_2^2 - 4D_4D_0}}{2D_4} = \omega_{app}^4$

$\therefore \frac{4D_4D_0}{4D_4^2} = \frac{D_0}{D_4} = \frac{1}{C_pC_s(L_pL_s - M^2)} = \frac{L_s^2}{C_p^2(L_pL_s - M^2)^2}$

$\therefore C_s = \frac{C_p(L_pL_s - M^2)}{L_s^2}$ From this formula the secondary capacitance C_s can be determined.

$\frac{V_0}{V_1}$ at ω_{app} is important.

$$\left|\frac{V_0}{V_1}\right| /_{\omega=\omega_{app}} = \frac{|N_2\omega_{app}^2|}{|D_4\omega_{app}^4 - D_2\omega_{app}^2 + D_0|} = \frac{MC_p\frac{L_S}{C_p(L_pL_S-M^2)}}{\left|\frac{C_pC_S(L_pL_S-M^2)L_S^2}{C_p^2(L_pL_S-M^2)^2} - \frac{(C_pL_p+C_SL_S)L_S}{C_p(L_pL_S-M^2)}+1\right|}, \text{ if } R_p = R_s = 0.$$

$$\therefore \left|\frac{V_0}{V_1}\right| /_{\omega=\omega_{app},\, R_p=R_S=0} = \frac{L_S}{M}$$

The following graphs show the gain and phase against frequency with different values of R_L. Different values of C_s will change the shape of the graphs, but even then, the gain and phase at the optimal applied frequency will stay the same.

V. S/P EXAMPLE AND GRAPHS

The values chosen and designed by using the formulas above are:
$\omega_{app} = 2\pi 143.0$ krad/s
$L_p = L_s = 24.0$ µH, $M = 16.1$ µH, $R_p = R_s = 0.1\ \Omega$
$\therefore C_p = 93.8$ nF and $C_s = 51.6$ nF

Graphs were plotted with:
$R_L = \frac{60.0}{18}, \frac{60.0}{12}, \frac{60.0}{8}, \frac{60.0}{4}, \frac{60.0}{2}$ and 60.0 Ω. The gain at the optimal applied frequency of 143 kHz should be close to $\frac{L_S}{M} = 3.47$ dB, which it is as shown in Fig. 9. The slight difference in gain at 143 kHz for different loads are due to the R_p and R_s values in the example that are not zero. The two peaks in the gain plot are the same height with C_s as calculated by the formula above.

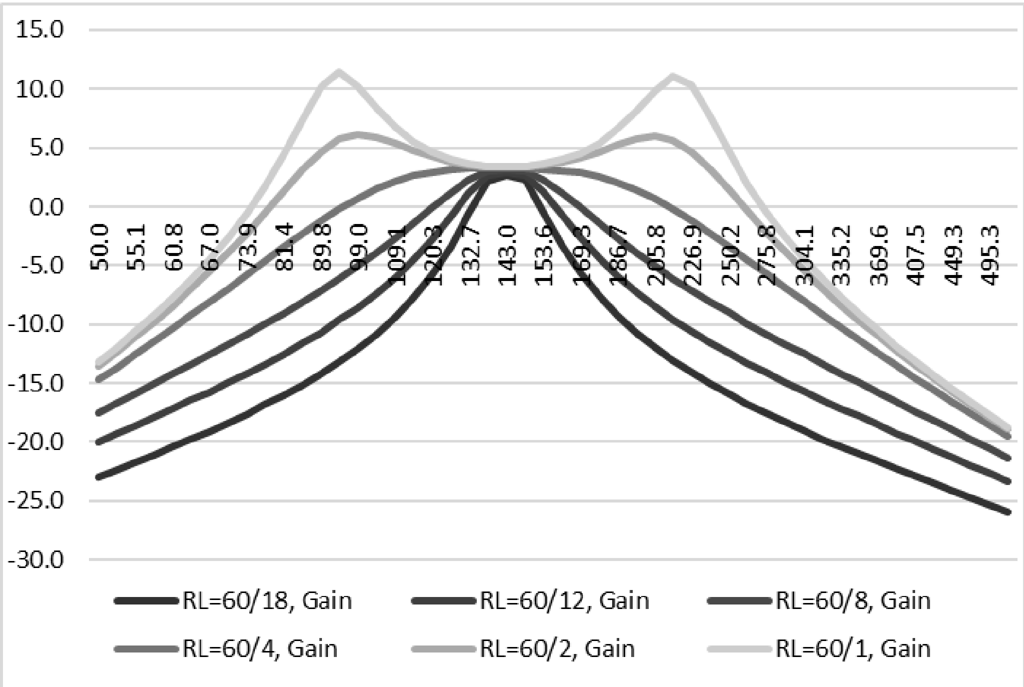

Fig. 9. Gain plot of S/P configuration with different loads

The gain stays the same at the optimal applied frequency of 143 kHz in this example.

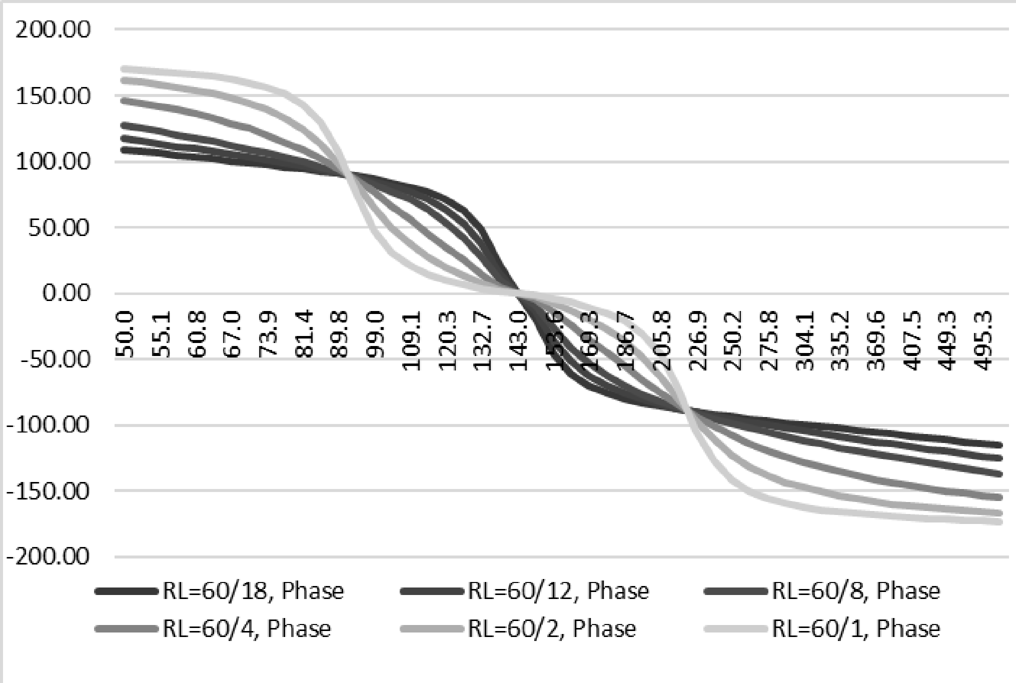

Fig. 10. Phase plot of S/P configuration with different loads

The phase stays the same (0°) at the optimal applied frequency of 143 kHz. It also stays the same (90° and -90° respectively) at two other frequencies, but there the gain did not stay the same, therefore they are not optimal frequencies.

VI. P/S CONFIGURATION

This topology is found in the literature, but this author is unsure of its practical use because it requires a current source as input.

P/S refers to the primary capacitor connected in parallel, and the secondary capacitor connected in series, as can be seen in the next circuit diagram.

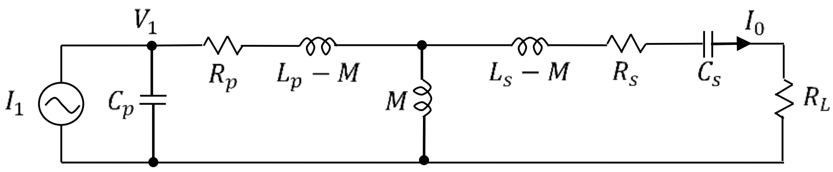

Fig. 11. Circuit diagram of P/S configuration

$$Z_p = sL_p + R_p$$
$$Z_s = sL_s + R_s + R_L + \frac{1}{sC_s} = \frac{s^2 C_s L_s + sC_s(R_s + R_L) + 1}{sC_s}$$
$$Z_{inp} = \frac{1}{sC_p + \frac{1}{Z_p - \frac{s^2 M^2}{Z_s}}} = \frac{Z_p Z_s - s^2 M^2}{sC_p(Z_p Z_s - s^2 M^2) + Z_s}$$

Applying these to the block diagram of Fig. 3, results in:

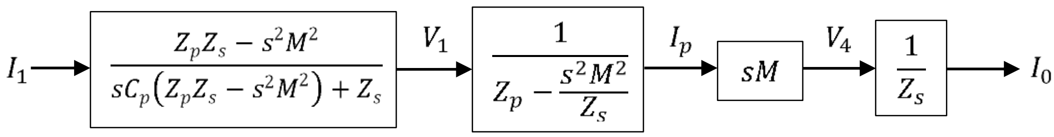

Fig. 12. First block diagram of P/S transformer

Simplifying the block diagram, results in:

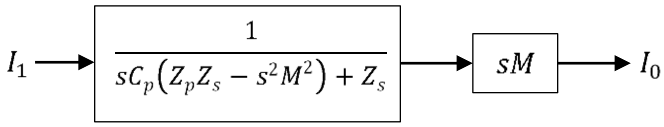

Fig. 13. Final block diagram of P/S transformer

$\therefore \frac{I_0}{I_1} = \frac{sM}{sC_p(Z_pZ_s - s^2M^2) + Z_s} = \frac{sM}{Z_s(sC_pZ_p + 1) - s^3M^2C_p}$

It will be shown that $\frac{I_0}{I_1}$ in this configuration at the optimal applied frequency, is quite independent on R_L.

$\therefore \frac{I_0}{I_1} = \frac{N_2s^2}{D_4s^4 + D_3s^3 + D_2s^2 + D_1s + D_0}$, with $N_2 = MC_s$,
$D_4 = C_pC_s(L_pL_s - M^2)$, $D_3 = C_pC_s[R_pL_s + (R_s + R_L)L_p]$,
$D_2 = C_pL_p + C_sL_s + C_pC_sR_p(R_s + R_L)$,
$D_1 = C_pR_p + C_s(R_s + R_L)$, $D_0 = 1$
$\therefore \frac{I_0}{I_1}/_{s=i\omega} = \frac{-N_2\omega^2}{D_4\omega^4 - iD_3\omega^3 - D_2\omega^2 + iD_1\omega + D_0}$

The best applied frequency (ω_{app}) for this system would be where the $\frac{I_0}{I_1}$ gain is maximum and it can be expected to be where either the real part or the imaginary part of the denominator is zero. So, $D_4\omega^4 - D_2\omega^2 + D_0 = 0$ or $-D_3\omega^3 + D_1\omega = 0$. Looking at the expressions of these D coefficients, the last equation is the best, because with that part zero, $\frac{V_0}{V_1}$ will be very little dependent on R_L. This is desired because R_L should ideally not change the gain.
$\therefore \omega_{app}^2 = \frac{D_1}{D_3} = \frac{C_pR_p + C_s(R_s + R_L)}{C_pC_s[R_pL_s + (R_s + R_L)L_p]}$

R_p and R_s will be as small as possible because the bigger they are, the more heat will be dissipated.

$\therefore \omega_{app}^2/_{R_p=R_s=0} = \frac{1}{C_pL_p}$, because R_L cancels out. Since the ideal applied frequency should be known from the characteristics of the transformer, this formula should be used to determine the primary capacitance C_p. This is ideal too, in that ω_{app} is also independent of R_L.

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$\therefore \frac{D_2 + \sqrt{D_2^2 - 4D_4D_0}}{2D_4} \times \frac{D_2 - \sqrt{D_2^2 - 4D_4D_0}}{2D_4} = \omega_{app}^4$
$\therefore \frac{4D_4D_0}{4D_4^2} = \frac{D_0}{D_4} = \frac{1}{C_pC_s(L_pL_s - M^2)} = \frac{1}{C_p^2L_p^2}$

$\therefore C_s = \frac{C_p L_p^2}{L_p L_s - M^2}$ From this formula the secondary capacitance C_s can be determined.

$\frac{I_0}{I_1}$ at ω_{app} is important.

$$\left|\frac{I_0}{I_1}\right| / _{\omega=\omega_{app}} = \frac{|N_2 \omega_{app}^2|}{|D_4 \omega_{app}^4 - D_2 \omega_{app}^2 + D_0|} = \frac{M C_s \frac{1}{C_p L_p}}{\left|\frac{C_p C_s (L_p L_s - M^2)}{C_p^2 L_p^2} - \frac{(C_p L_p + C_s L_s)}{C_p L_p} + 1\right|}, \text{ if } R_p = R_s = 0.$$

$$\therefore \left|\frac{I_0}{I_1}\right| / _{\omega=\omega_{app},\, R_p=R_s=0} = \frac{L_p}{M}$$

The following graphs show the gain and phase against frequency with different values of R_L. Different values of C_s will change the shape of the graphs, but even then, the gain and phase at the optimal applied frequency will stay the same.

VII. P/S EXAMPLE AND GRAPHS

The values chosen and designed by using the formulas above are:
$\omega_{app} = 2\pi 143.0$ krad/s
$L_p = L_s = 24.0$ μH, $M = 16.1$ μH, $R_p = R_s = 0.1$ Ω
$\therefore C_p = 51.6$ nF and $C_s = 93.8$ nF

Graphs were plotted with:
$R_L = \frac{60.0}{18}, \frac{60.0}{12}, \frac{60.0}{8}, \frac{60.0}{4}, \frac{60.0}{2}$ and 60.0 Ω. The gain at the optimal applied frequency of 143 kHz should be close to $\frac{L_p}{M} = 3.47$ dB, which it is as shown in Fig. 14. The two peaks in the gain plot are the same height with C_s as calculated by the formula above.

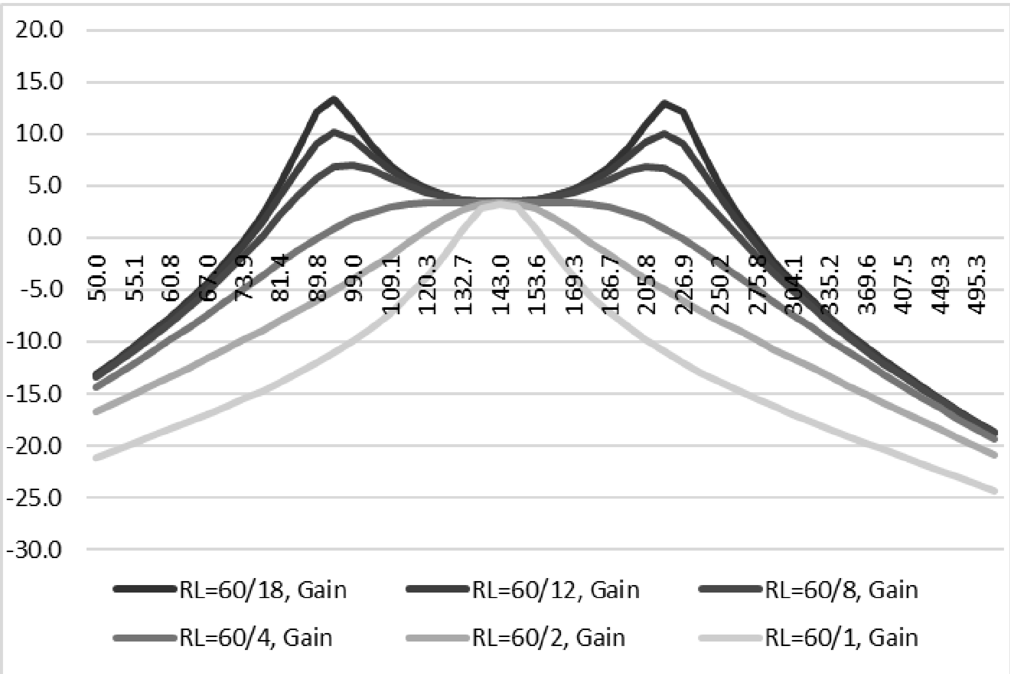

Fig. 14. Gain plot of P/S configuration with different loads

The gain stays the same at the optimal applied frequency of 143 kHz in this example.

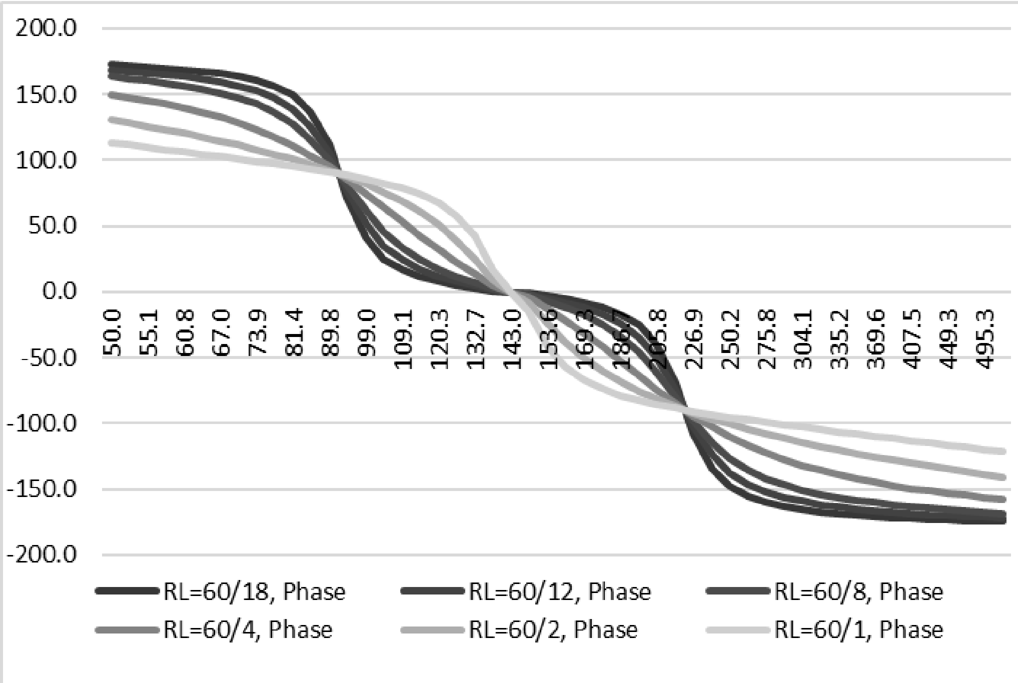

Fig. 15. Phase plot of P/S configuration with different loads

The phase stays the same (0°) at the optimal applied frequency of 143 kHz. It also stays the same (90° and -90° respectively) at two other frequencies, but there the gain did not stay the same, therefore they are not optimal frequencies.

VIII. P/P CONFIGURATION

This topology is found in the literature, but this author is unsure of its practical use because it requires a current source as input. P/P refers to the primary and secondary capacitors both connected in parallel, as can be seen in the next circuit diagram.

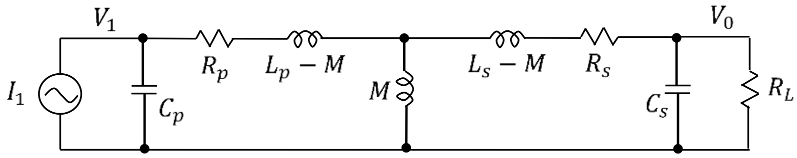

Fig. 16. Circuit diagram of P/P configuration

$$Z_p = sL_p + R_p$$

$$Z_s = sL_s + R_s + \frac{1}{sC_s + {}^{1}/_{R_L}} = sL_s + R_s + \frac{R_L}{sC_sR_L + 1}$$

$$Z_{inp} = \frac{1}{sC_p + \frac{1}{Z_p - \frac{s^2M^2}{Z_s}}} = \frac{Z_pZ_s - s^2M^2}{sC_p(Z_pZ_s - s^2M^2) + Z_s}$$

Applying these to the block diagram of Fig. 3, results in:

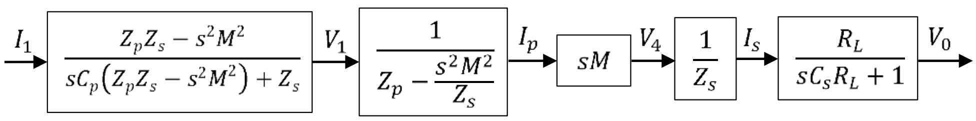

Fig. 17. First block diagram of P/P transformer

Simplifying the block diagram, results in:

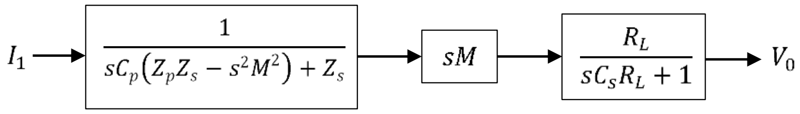

Fig. 18. Final block diagram of P/P transformer

$\therefore \frac{V_0}{I_1} = \frac{sMR_L}{[sC_p(Z_pZ_s - s^2M^2) + Z_s](sC_sR_L + 1)} = \frac{sMR_L}{[Z_s(sC_pZ_p + 1) - s^3M^2C_p](sC_sR_L + 1)}$

It will be shown that $\frac{V_0}{I_1}$ in this configuration at the optimal applied frequency, is quite independent on R_L.

$\therefore \frac{V_0}{I_1} = \frac{N_1 s}{D_4s^4 + D_3s^3 + D_2s^2 + D_1s + D_0}$, with $N_1 = M$,
$D_4 = C_pC_s(L_pL_s - M^2)$,
$D_3 = C_pC_s(L_pR_s + L_sR_p) + \frac{C_p}{R_L}(L_pL_s - M^2)$,
$D_2 = C_pR_p\left(C_sR_s + \frac{L_s}{R_L}\right) + C_sL_s + C_pL_p\left(1 + \frac{R_s}{R_L}\right)$,
$D_1 = C_sR_s + \frac{L_s}{R_L} + C_pR_p\left(1 + \frac{R_s}{R_L}\right)$, $D_0 = 1 + \frac{R_s}{R_L}$

From these, the expressions for the N coefficient and D coefficients can be found.
$\therefore \frac{V_0}{I_1}/_{s=i\omega} = \frac{iN_1\omega}{D_4\omega^4 - iD_3\omega^3 - D_2\omega^2 + iD_1\omega + D_0}$

The best applied frequency (ω_{app}) for this system would be where the $\frac{V_0}{I_1}$ gain is maximum and it can be expected to be where either the real part or the imaginary part of the denominator is zero. So, $D_4\omega^4 - D_2\omega^2 + D_0 = 0$ or $-D_3\omega^3 + D_1\omega = 0$. Looking at the expressions of these D coefficients, the last equation is the best, because with that part zero, $\frac{V_0}{I_1}$ will be very little dependent on R_L. This is desired because R_L should ideally not change the gain.

$$\therefore \omega_{app}^2 = \frac{D_1}{D_3} = \frac{C_sR_s + \frac{L_s}{R_L} + C_pR_p\left(1 + \frac{R_s}{R_L}\right)}{C_pC_s(L_pR_s + L_sR_p) + \frac{C_p}{R_L}(L_pL_s - M^2)}$$

R_p and R_s will be as small as possible because the bigger they are, the more heat will be dissipated.

$\therefore \omega_{app}^2/_{R_p=R_s=0} = \frac{L_s}{C_p(L_pL_s - M^2)}$, because R_L cancels out. Since the ideal applied frequency should be known from the characteristics of the transformer, this formula should be used to determine the primary capacitance C_p. This is ideal too, in that ω_{app} is also independent of R_L.

$D_4\omega^4 - D_2\omega^2 + D_0 = 0$ provides two frequencies where the gain is potentially maximum. It makes sense to put them on both sides of ω_{app}, so that the square root of the product of the two, is equal to ω_{app}. This is sensible because it will cause the gain at the applied frequency to be less sensitive to the exact value of the frequency.

$$\therefore \frac{D_2 + \sqrt{D_2^2 - 4D_4D_0}}{2D_4} \times \frac{D_2 - \sqrt{D_2^2 - 4D_4D_0}}{2D_4} = \omega_{app}^4$$

$$\therefore \frac{4D_4D_0}{4D_4^2} = \frac{D_0}{D_4} = \frac{1}{C_pC_s(L_pL_s - M^2)} = \frac{L_s^2}{C_p^2(L_pL_s - M^2)^2}$$

$\therefore C_s = \frac{C_p(L_pL_s - M^2)}{L_s^2}$ From this formula the secondary capacitance C_s can be determined.

$\frac{V_0}{I_1}$ at ω_{app} is important.

$$\left|\frac{V_0}{I_1}\right| /_{\omega=\omega_{app}} = \frac{|N_1\omega_{app}|}{|D_4\omega_{app}^4 - D_2\omega_{app}^2 + D_0|} = \frac{M\sqrt{\frac{L_S}{C_p(L_pL_S - M^2)}}}{\left|\frac{C_pC_S(L_pL_S - M^2)L_S^2}{C_p^2(L_pL_S - M^2)^2} - \frac{(C_pL_p + C_SL_S)L_S}{C_p(L_pL_S - M^2)} + 1\right|}, \text{ if } R_p = R_s = 0.$$

$$\therefore \left|\frac{V_0}{I_1}\right| /_{\omega=\omega_{app},\, R_p=R_s=0} = \frac{1}{M}\sqrt{\frac{L_s(L_pL_s - M^2)}{C_p}}$$

The following graphs show the gain and phase against frequency with different values of R_L. Different values of C_s will change the shape of the graphs, but even then, the gain and phase at the optimal applied frequency will stay the same.

IX. P/P EXAMPLE AND GRAPHS

The values chosen and designed by using the formulas above are: $\omega_{app} = 2\pi 143.0$ krad/s
$L_p = L_s = 24.0$ µH, $M = 16.1$ µH, $R_p = R_s = 0.1\ \Omega$
$\therefore C_p = 93.8$ nF and $C_s = 51.6$ nF

Graphs were plotted with:
$R_L = \frac{60.0}{18}, \frac{60.0}{12}, \frac{60.0}{8}, \frac{60.0}{4}, \frac{60.0}{2}$ and 60.0 Ω. The gain at the optimal applied frequency of 143 kHz should be close to $\frac{1}{M}\sqrt{\frac{L_s(L_pL_s - M^2)}{C_p}} = 25.0$ dB, which it is as shown in Fig. 19. The slight difference in gain at 143 kHz for different loads are due to the R_p and R_s values in the example that are not zero. If wanted, the two peaks in the gain plot can be changed to be the same height by modifying C_s a bit.

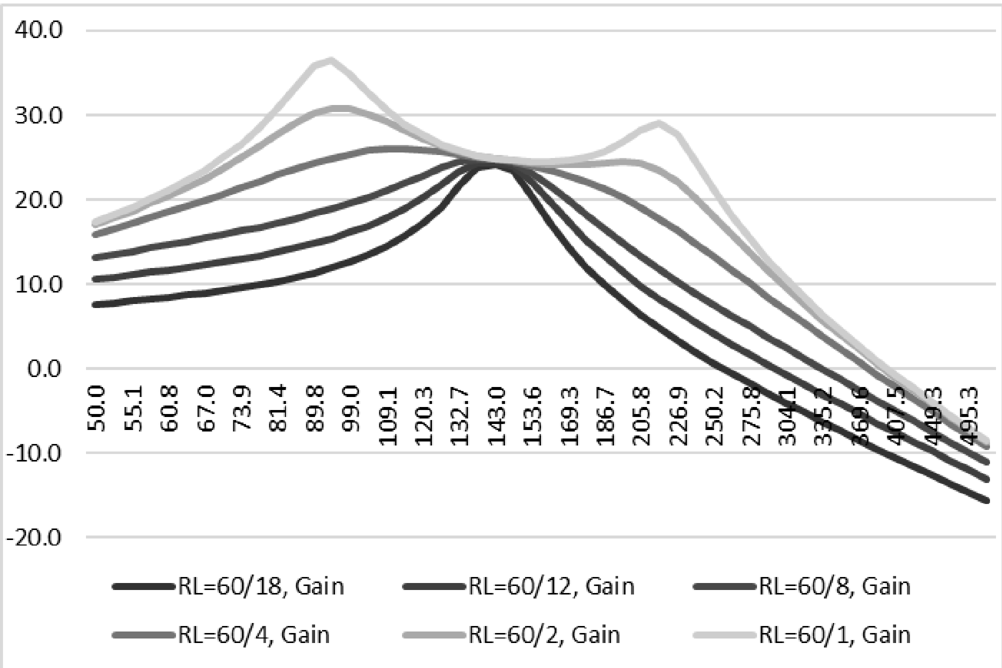

Fig. 19. Gain plot of P/P configuration with different loads

The gain stays the same at the optimal applied frequency of 143 kHz in this example.

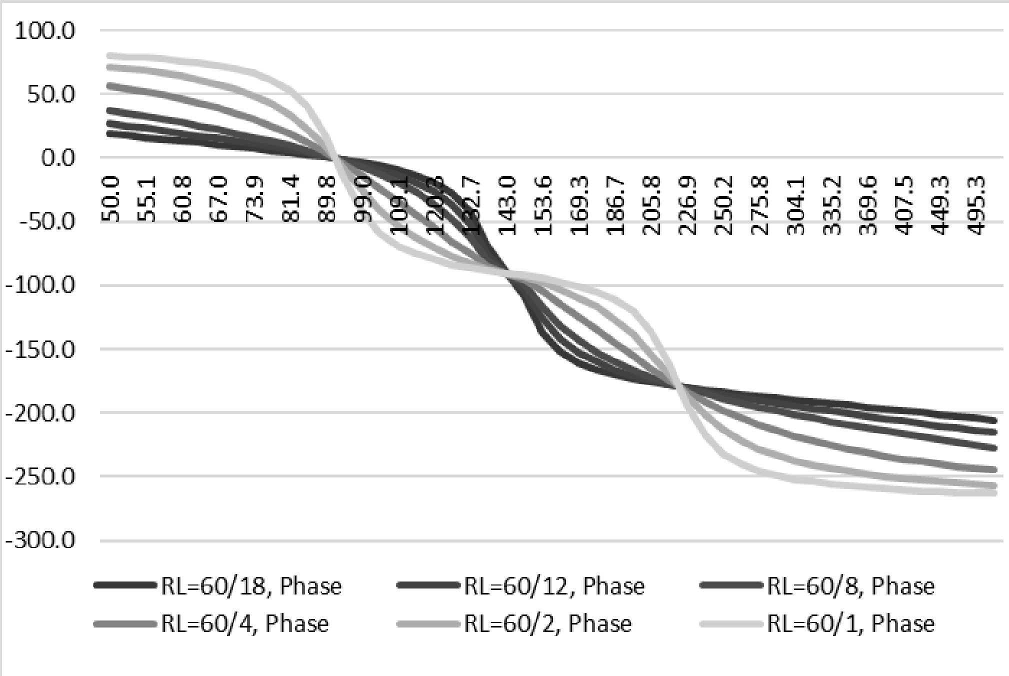

Fig. 20. Phase plot of P/P configuration with different loads

The phase stays the same (-90°) at the optimal applied frequency of 143 kHz. It also stays the same (0° and -180° respectively) at two other frequencies, but there the gain did not stay the same, therefore they are not optimal frequencies.

X. SUMMARY OF FORMULAS

TABLE I
Calculations for Cp, Cs and the Gain

Topology:	For calculating C_p $(R_p = R_s = 0)$:	C_s $(R_p = R_s = 0)$:	Gain $(\omega = \omega_{app},\ R_p = R_s = 0)$:
S/S	$\omega_{app}^2 = \frac{1}{C_p L_p}$	$C_s = \frac{C_p L_p^2}{L_p L_s - M^2}$	$\left\|\frac{I_0}{V_1}\right\| = \frac{\sqrt{C_p L_p}}{M}$
S/P	$\omega_{app}^2 = \frac{L_s}{C_p(L_p L_s - M^2)}$	$C_s = \frac{C_p(L_p L_s - M^2)}{L_s^2}$	$\left\|\frac{V_0}{V_1}\right\| = \frac{L_s}{M}$
P/S	$\omega_{app}^2 = \frac{1}{C_p L_p}$	$C_s = \frac{C_p L_p^2}{L_p L_s - M^2}$	$\left\|\frac{I_0}{I_1}\right\| = \frac{L_p}{M}$
P/P	$\omega_{app}^2 = \frac{L_s}{C_p(L_p L_s - M^2)}$	$C_s = \frac{C_p(L_p L_s - M^2)}{L_s^2}$	$\left\|\frac{V_0}{I_1}\right\| = \frac{1}{M}\sqrt{\frac{L_s(L_p L_s - M^2)}{C_p}}$

Note again that the optimal frequencies and the gains are independent of the load resistance R_L and the secondary capacitance C_s. These formulas are often adequate even if R_p and R_s are not zero because the design should be such that the latter resistances are very small relative to R_L.

The examples discussed above, demonstrated by both simulations and Excel calculations that all these formulas for the optimal frequencies and gains are correct. Although the formulas for the secondary capacitance differ from many resources [2, 5, 6], the reasoning behind the formula proposed here is good and they produce fairly symmetrical graphs around the optimal frequency.

XI. CONCLUSIONS

The four major topologies for inductive couplers (such as transformers with air gaps used for wireless power transfer), were designed and analysed, relevant formulas were derived, and all were tested by simulation. Excel was used to produce clear graphs of all the relevant transfer functions. The major topologies and their relevant transfer functions are:

1. The S/S configuration, $\frac{Output\ current}{Input\ voltage}$, the output is acting as a current source
2. The S/P configuration, $\frac{Output\ voltage}{Input\ voltage}$, the output is acting as a voltage source
3. The P/S configuration, $\frac{Output\ current}{Input\ current}$, the output is acting as a current source but needs a current source as input
4. The P/P configuration, $\frac{Output\ voltage}{Input\ current}$, the output is acting as a voltage source but needs a current source as input

Each one of these transfer functions is almost independent of the load resistance at the optimal applied frequency, but this would not have been the case if the wrong transfer function was determined for any topology. The secondary capacitance was calculated in a novel way namely with formulas derived to reduce the sensitivity of the gain of the transfer function to frequency variation, around the optimal applied frequency.

XII. REFERENCES

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