KVA in Black Scholes Pricing

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy to the University of Cape Town. It has not before been submitted for any degree or examination.

Signed by candidate

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August 9, 2019

Abstract

The post 2007-financial crisis era has led to renewed zeal in quantifying market incompleteness when pricing contingent claims. This quantification exercise is necessary in maintaining a stable and sustainable banking operation and thus the XVAs have emerged as the metrics for market incompleteness. This dissertation focuses solely on the capital valuation adjustment (KVA) and aims to use the definition of KVA as set out by Albanese *et al.* (2016) in an investigation of different numerical techniques for calculating KVA. A single equity forward is considered first, followed by an equity option and then portfolios of options on two underlying assets, with the dissertation ending by considering a practical example on discrete delta and vega-delta hedging an index option. The numerical approaches explored are the binomial tree method and a combination of the crude and quasi-Monte Carlo method.

Contents

Chapter 1

Introduction

The post 2007-crisis banking reforms necessitated the quantification of market incompleteness and its effect on contingent claim pricing due to the increase in regulation costs. The response of regulators was to reform banking by raising capital and collateral requirements to buffer the financial system. These regulations are enshrined by the Basel III Accord and many other follow-up regulatory reforms such as the Fundamental Review of the Trading Book (BCBS, 2018). Market incompleteness is now being quantified by several x- valuation adjustments (xVA) metrics with the purpose of reflecting fairer valuations and making obsolete the need for banks to enter into Ponzi scheme-like behaviour. With regard to capital costs, they must be considered over the entire life of a trade in order for shareholders to be remunerated at their required hurdle rate over the life of their investment. The capital valuation adjustment (KVA) is the quantification of these capital costs. The definition of KVA has not been ensconced by regulation and the definition that forms the basis of this report is that of Albanese *et al.* (2016)

This dissertation takes the mathematical definition of KVA as given by Albanese *et al.* (2016) and explores different numerical schemes, namely binomial tree and quasi and crude Monte-Carlo techniques, for computing KVA. KVA is considered first in the case of a single forward, then a single option. Since KVA is ultimately calculated at a portfolio level, the dissertation concludes by considering option portfolios and a practical example on an index option that is vega-delta hedged using an index constituent versus hedging with an option on an asset correlated to the index. In the single forward and option case, hedging is ignored completely. Only market risk is considered in the loss process and other XVAs are completely ignored. Constructing the toy portfolios serves to build intuition on KVA and how it is affected by correlation between assets.

1.1 Cost of Capital

In complete markets all cash flows are replicable, therefore no capital would be at risk and remuneration on capital is unjustified. Incompleteness puts shareholder capital at risk and this risk appetite needs to be remunerated at a hurdle rate (return on investment the shareholders expect) by sourcing cash from clients in proportion to the measured capital consumption. Shareholder capital is integral in banking activities and its level is specified by Basel III in an attempt to keep the financial sector robust. The primary loss absorbing barrier for banks is the reserve capital (RC) account. Additionally, banks are required to hold economic capital (EC) which is defined in Basel II Pillar II as the quantification of shareholder capital at risk (SCR). This acts as a buffer against exceptional losses (losses beyond RC coverage).

Another important term is common equity Tier 1 capital (CET1), which is the metric reflecting the fair value of shareholder capital and is equal to assets less reserve capital and liabilities. Economic capital is defined in the Fundamental Review of the Trading Book (FRTB) as the 97.5% confidence expected shortfall (ES) with regard to market risk while Basel II Pillar II defines it as 99% VaR(-CET1) year-on-year (Albanese *et al.*, 2016). Here VaR stands for value-at-risk, the stalwart statistical measure in traditional risk management practices. It is reviewed thoroughly by (Duffie and Pan, 1997). The total regulatory capital requirements of a bank are made up of various sub-categories of capital-requiring activities. As is summarised by Green *et al.* (2014), such capital necessitating sub-categories are market risk, CVA capital and counterparty credit risk capital.

Albanese *et al.* (2016) define the KVA as the ”cost of remunerating shareholder capital at a sustainable hurdle rate” and must reflect the lifetime cost of capital. Essentially, the KVA is the market price of risky capital. They propose an accounting framework for KVA where it is treated as a retained earning that leads to a sustainable dividend policy throughout the life of the portfolio, even in the case where no new deals are booked until maturity of the portfolio. This treatment of KVA allows it to be a loss absorbing account, forming part of EC, and thus EC corresponds to the sum of SCR and KVA. As an example for why KVA (and FVA for that matter) is important, consider the case where a bank does not account for the lifetime cost of capital of their derivatives book. How will they make future payments to their shareholders? The only way is to initiate Ponzi-scheme like behaviour where an exponential increase in leverage is required. The bank is forced to book new trades at an exponential rate in order to pay shareholders their next dividend. KVA must be incorporated into pricing if a healthy return-on-equity is to be maintained.

Albanese *et al.* (2016) note that the cost of capital has been ignored in derivative pricing by the International Financial Reporting Standards (IFRS) 9, GAAP and tax codes which treat capital costs (KVA) sourced from clients as day-one profits, implying their immediate release as dividends (unless directors decide otherwise). Such a dividend policy is inefficient for the same reason that KVA must be included in pricing: it leads to Ponzi-scheme behaviour. A more careful accounting consideration and policy framework is required when dealing with KVA so as to introduce greater stability into derivative markets. Admati *et al.* (2018) give more extensive motivation for regulating retained earnings policies for banks.

1.2 Cost of Funding

Albanese *et al.* (2016) define the expectation of the discounted cost of obtaining funds for posting variation margin that is re-hypothecable as the funding valuation adjustment (FVA). Re-hypothecation is the practice of using collateral received for one trade as the collateral posted for another trade (usualy a hedge) Gregory (2015). Often banks will engage in an exchange of variation margin with their counterparties to mitigate counterparty default risk. This variation margin is cash that is free to be used by the receiver for other funding purposes and is remunerated at maturity at the overnight interest rate (OIS) to the issuer. When engaging in unsecured funding, lenders will ask for a credit spread to compensate for the defaultable nature of the bank. In Albanese, Andersen and Iabichino (2015) the FVA is defined as the metric that quantifies paying the spread to carry assets.

Albanese, Caenazzo and Crépey (2015) observe that the FVA is a logical adjustment and metric for intrinsic market incompleteness considering the leveraged nature of banks: banks cannot completely de-leverage, which is necessary to prevent wealth transfer to creditors, and the FVA compensates shareholders for this wealth transfer. The economic capital of a bank is a viable source of funding, as noted by Albanese and Andersen (2015), and as such can be used to post variation margin that would otherwise be sourced from an external party, thus decreasing FVA. So, there will be a coupling between the EC and FVA account. As a visual guide to the high level summary of the balance sheet relevant to this study, consider the Figure 1.1. The positivity condition on shareholder capital at risk is discussed further in Section 1.3.

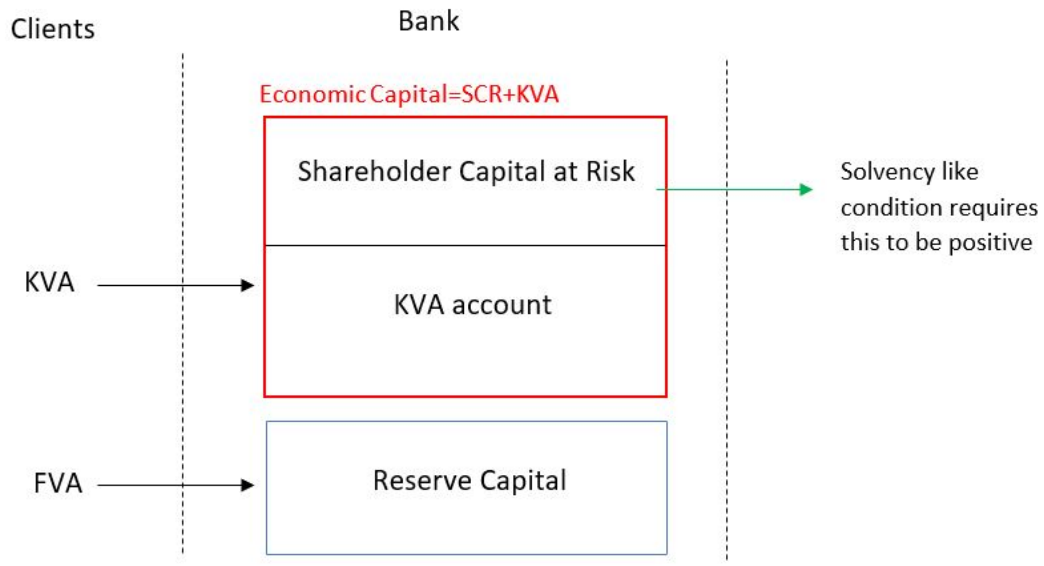

Fig. 1.1: Stylised balance sheet of a bank showing structure of the ES account and its relationship with KVA and FVA.

1.3 Motivation for Approach to KVA

This dissertation is based on the work of Albanese *et al.* (2016). These authors have defined the KVA as a risk premium that is charged to clients as remuneration to shareholders for placing capital at risk. However, other literature such as Green *et al.* (2014) and Salzmann and Wüthrich (2010) compute KVA via risk-neutral valuation of cash flows using semi-replication techniques as first formalised by Burgard and Kjaer (2013) with respect to CVA and FVA. Albanese *et al.* (2016) note that this approach is conceptually flawed and presents numerical difficulties which require simplifying approximations to be implemented in practice.

Kenyon and Kenyon (2016) conclude that the fair value ideology of IFRS 13 and IFRS 9 see (IFRS (2014) and IFRS (2012)) should be applied to compute and treat KVA. Although this approach seems natural and direct, it appears to be ill-suited to defining KVA in the most economically sensible way. Basel III defines CET1 as the fair valuation of shareholder capital while these authors treat KVA as a CET1 deductible which transforms this definition into a 'risk-adjusted' CET1. Albanese, Caenazzo and Crépey (2015) note that this also contradicts the Solvency II definition of risk margin as a loss buffer and not as a CET1 deductible. We note that

Solvency II Solvency II (2004) and IFRS 4 Phase II IFRS (2013) are standards meant for the insurance industry. However, as Albanese *et al.* (2016) do, we leverage the definition of KVA off these standards since their rigorous definition of and framework for KVA-like risk margins has no analogue in Basel III and IFRS 9, which do not provide a method for dealing with these entries in the banking realm.

1.4 A Word on Portfolio and Marginal KVA

Since expected shortfall and regulatory capital required by a bank is calculated across all positions held by the bank, the true KVA to be charged on a new deal needs to come from computations performed on the portfolio as a whole. This immediately exposes the natural difficulty of a KVA calculation, which is that the whole portfolio needs to be subjected to a new simulation rather than just the new trade. This quickly becomes a massive calculation, especially when considering all the XVAs which are often coupled. Green *et al.* (2014) note that calculations of capital requirements are done at multiple levels and combined. Table 1.1 from their paper nicely illustrates the most frequent capital regulations as specified by Basel III.

With regards to Table 1.1, CS01 is the securitie's change in value for a single basis-point increase on the credit spread. Albanese and London (2015) note that there is a worry among regulators due to the overlap of many of the capital charges that define the total capital requirements and the lack of agreement between Pillar I and Pillar II capital regulations. This points to the necessity of a complete EC calculation and its high computational cost. Although, as the literature suggests, the incremental KVA on a new trade should be computed taking into account the entire portfolio composition of the bank, in this work we explore the KVA on a new trade as a marginal value. It seems reasonable to assume that KVA would be calculated on a stand-alone basis for fairly small trades while significantly large trades would require a full portfolio calculation.

<table>
<tr><th>Source of Capital Requirement</th><th>Specified Calculations</th><th>Calculation Type</th></tr>
<tr><td rowspan="8">Counterparty Credit Risk</td><td>Exposure at Default (EAD) Calculations</td><td></td></tr>
<tr><td>Current exposure method (CEM)</td><td>Netting set value</td></tr>
<tr><td>Standardized</td><td>Netting set value</td></tr>
<tr><td>Banks model</td><td>Exposure profile</td></tr>
<tr><td>Weight Calculations</td><td></td></tr>
<tr><td>Standardized</td><td>External ratings</td></tr>
<tr><td>Foundation internal ratings-based (FIRB)</td><td>Internal and external ratings</td></tr>
<tr><td>Advanced internal ratings-based (AIRB)</td><td>Internal and external ratings, internal LGDs (loss given default)</td></tr>
<tr><td rowspan="2">CVA Capital</td><td>Standardized</td><td>EAD</td></tr>
<tr><td>Advanced</td><td>VAR/SVAR (stressed VAR) on CVA or CS01</td></tr>
<tr><td rowspan="2">Market Risk</td><td>Standardized</td><td>Deterministic</td></tr>
<tr><td>Banks model</td><td>Historically VAR and SVAR but ES in 2016</td></tr>
</table>

Tab. 1.1: Capital sources as specified by Basel III and calculations required. Taken from Green *et al.* (2014)

1.5 The Lack of a ZAR OIS Curve and Dividends

The analysis in this paper is a general venture into exploring KVA and as such the risk free-rate r used in the rest of the dissertation is assumed to be a relevant OIS rate. The South African market lacks a liquid OIS market and therefore an OIS curve. Sender (2017) shows that a co-integration relationship holds between the 3-month Johannesburg Interbank Agreed Rate (JIBAR) and the South African Futures Exchange (SAFEX) overnight rate. Using a dual bootstrapping algorithm constrained by arbitrage arguments, the ZAR OIS curve is estimated. Until the ZAR OIS market is developed, the ZAR OIS curve will have to be estimated. If such a method is adopted, the frequent updating of co-integration parameters by the bank will need to be incorporated in its operations.

In order to make the exploration of KVA tractable for a master's level dissertation, many simplifications are made to the financial and mathematical processes that are required for KVA computation. One of these is the exclusion of dividends in the asset process S_t and a very simple loss process L_t introduced in section 2.1. A stock paying dividends can be incorporated easily, one should only be diligent in specifying the gains process for completeness. Green *et al.* (2014) and Armenti *et al.* (2016) both make use of a dividend paying stock. If a dividend yield curve is available this should obviously be used from a practitioner point of view but the academic would not worry about this aspect too much.

1.6 Research Objectives

The objectives of this study are to shed light on the nature of KVA by:

1. Investigating binomial tree and quasi-Monte Carlo pricing in calculating the KVA on a single *equity forward*.

2. Performing the same investigation of a single *equity call option*.

3. Investigating how the KVA calculation is affected when incorporating a *correlation structure* by considering portfolios of options on 2 underlying assets and exploring vega-delta hedging an index option consisting of two assets using the exact constituents or a correlated asset.

Chapter 2

Analysis for Single Equity Forward

Since a forward can be perfectly hedged, KVA reduces to zero if we use perfect replication to hedge the forward. However, in order to explore KVA in the simplified setting of a forward, we do not allow ourselves to hedge and will thus generate an expected shortfall for the KVA calculation.

2.1 Mathematical Framework

The work in this dissertation will take place in the elementary Black-Scholes framework with filtered probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ that satisfies the usual conditions. The original process is the following for a non-dividend paying asset

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma dW_t^{\mathbb{P}}. \tag{2.1}$$

This is geomteric brownian motion, detailed in Duffie (2010). It is important to note the two different measure that will feature in this dissertation. VaR and ES calculations are performed under the *real-world measure*, denoted $\mathbb{P}$ and since the KVA itself aims to quantify risk, it should also be computed under this measure. However, it is the valuation of the financial instruments that will generate the profit and loss process and this valuation should be done under the *risk-neutral measure*, denoted $\mathbb{Q}$. Throughout μ will denote the return on the stock S, σ its volatility and T the maturity of the contract. The following are the key assumptions that hold throughout.

1. The underlying pays no dividends.

2. There are no hedge costs, margin fees or any other market frictions.

3. There are no arbitrages.

4. The hurdle rate h and OIS rate r are constants.

5. EC is not fully depleted by exceptional losses during the life of the portfolio.

6. All XVAs other than the KVA are ignored.

7. Only market risk as a result of no hedging is considered as a driver of economic capital requirements. Credit risk is completely ignored.

8. Volatility is constant.

We use the set-up of Albanese *et al.* (2016) in calculating KVA. The KVA process must be sustainable, meaning that the KVA account must be able to sustain the dividend stream even in the absence of any new trades added to the portfolio between the current time and the final maturity of the portfolio. In the absence of a ubiquitous regulatory mechanism for treating KVA, Albanese *et al.* (2016) consider the KVA to be a risk premium: a cost charged to clients as compensation for bank shareholders placing capital at risk. Albanese *et al.* (2016) seek a KVA process K_t where the initial $K_t(0)$ charged to the client is deposited in an account, accrues at the overnight indexed swap (OIS) rate r and remunerates shareholders with anticipated dividends $h(\text{EC} - K)dt$ where h is the hurdle rate as a percentage. Consider the economic capital process in light of Section 1.1, calculating it as the 97.5 % one-year-ahead expected shortfall for the loss process (L_t) on the derivative portfolio

$$\text{EC}_t = \text{ES}_t(L)) = \frac{\mathbb{E}^{\mathbb{P}}[\,(L_{t+1} - L_t)\,\mathbb{I}_{L_{t+1}-L_t \geq \text{VaR}_t(L)}\;|\mathcal{F}_t\,]}{\mathbb{P}\,[L_{t+1} - L_t \geq \text{VaR}_t(L)\,|\mathcal{F}_t\,]}, \tag{2.2}$$

where $\mathbb{P}$ is the real-world measure and the 97.5% value-at-risk (VaR) is given by

$$\text{VaR}_t(L) = \inf\{x : \mathbb{P}(L_{t+1} - L_t \geq x\,|\mathcal{F}_t) \leq 2.5\%\}. \tag{2.3}$$

Recall from Section 1.1 that, under the approach of treating the KVA as a loss-absorbing and EC augmenting retained earning, EC is given by EC = KVA + SCR. However, should the KVA be larger than the EC at a particular time t, and one recognises that it does not make sense for SCR to be negative, one arrives at the updated definition of EC,

$$\text{EC}_t^* = \max(\text{ES}_t, \text{K}_t). \tag{2.4}$$

In order to compute our KVA, we desire a process $K_t^* = K_t(\text{EC}^*)$ that accrues at the OIS rate r. Furthermore, this process should lead to an expected remuneration of shareholder capital at risk (SCR) at the rate h. These specifications lead to the non-linear backward stochastic differential equation (BSDE)

$$\begin{aligned} dK_t^* + h(\text{EC}_t^* - K_t^*)dt - rK_t^*\,dt, \\ \text{with } K_T^* = 0. \end{aligned} \tag{2.5}$$

Note again that SCR = EC - KVA since we are treating KVA as a loss-absorbing account that diminishes SCR in either definition (2.2) or (2.4) of EC. The integrated form of (2.5) is a non-linear BSDE with a Lipschitz coefficient

$$K_t^* = \mathbb{E}^{\mathbb{P}}\left[\int_t^T (h\max(\mathrm{ES}_s, K_s^*) - (r_s + h)K_s^*)\, ds\, |\mathcal{F}_t\right]. \tag{2.6}$$

However if we are working with the economic capital process (2.2) instead of (2.4) then the equivalent form of (2.6) is

$$K_t = \mathbb{E}^{\mathbb{P}}\left[\int_t^T (h\mathrm{EC}_s - (r_s + h)K_s)\, ds\, |\mathcal{F}_t\right]. \tag{2.7}$$

This has an explicit solution

$$K_t = h\mathbb{E}^{\mathbb{P}}\left[\int_t^T e^{\int_t^s -(r_u+h)du}\mathrm{EC}_s\, ds\, |\mathcal{F}_t\right]. \tag{2.8}$$

When r is constant the KVA is given by

$$K_t = h\mathbb{E}^{\mathbb{P}}\left[\int_t^T e^{-(r+h)(s-t)}\mathrm{EC}_s\, ds\, |\mathcal{F}_t\right]. \tag{2.9}$$

In Armenti *et al.* (2016) economic capital is defined in terms of a stylised value-at-risk (VaR)

$$\mathrm{EC}_t(L) = f\sqrt{\frac{d\langle L\rangle}{dt}}, \tag{2.10}$$

where f denotes a suitable quantile level and $\langle L\rangle$ the quadratic variation of the loss process. This parametric specification of EC allows for the authors to define Black-Scholes likes PDEs which include KVA and FVA. Albanese *et al.* (2016) note that in most cases EC = ES and so KVA is given by the linear formula (2.9). They found that the equality stops holding when there is a high hurdle rate and the EC term structure (they use a deterministic formula for ES) peaks after a low starting level.

2.2 An Analytical Expected Shortfall for the Forward

In the absence of a static hedge, the cumulative loss on the forward between two times t_1 and t_2 measured in monetary value at t_1 is given by

$$L(t_1, t_2) = V_{forward}(t_2)\, e^{-r(t_2-t_1)} - V_{forward}(t_1). \tag{2.11}$$

The value at any time $t \in [0, T]$ where T is the maturity of the forward is

$$V_{forward}(t) = [S_t\, e^{r(T-t)} - f_0]\, e^{-r(T-t)}, \tag{2.12}$$

where $f_0 = S_0 e^{rT}$ is the original fair forward price. We are interested in the 1-year ahead loss process in our ES calculation and anchor all losses to the maturity of the forward, i.e, we work with $L(t,T)$. From (2.12) and (2.11) we have that the 1-year ahead loss is

$$L(t,T) = V_{forward}(T)\, e^{-r(T-t)} - V_{forward}(t) \;=\; S_T\, e^{-r(T-t)} - S_t. \tag{2.13}$$

Note that L represents a loss when positive and profit when negative. We can find an explicit representation for the 97.5% ES at time t

$$\text{ES}_t = \frac{1}{0.025}\, e^{-(T-t)r+m(t,T)+\frac{1}{2}s(t,T)^2} \left[1 - N\left(\frac{a - m(t,T) - s(t,T)^2}{s(t,T)}\right)\right] - S_t, \tag{2.14}$$

where

$$m(t,T) = \ln S_t + (\mu - \frac{1}{2}\sigma^2)(T-t) \tag{2.15}$$

and

$$s(t,T) = \sigma\sqrt{T-t}. \tag{2.16}$$

Detailed derivations can be found in Appendix A.2.

2.3 The Binomial Pricing Model

This section presents the calculation of KVA using a binomial tree to simulate the progression of the stock such that log normality is maintained and the mean and variance of returns are recovered. The binomial tree is a well-known first approach to the numerical simulation of the stock process and although it is a discrete-time model, it can be shown to converge to the continuous time Black-Scholes-Merton model as step-length goes to zero. It is appealing in that it is simple mathematically yet rigorous. See Hull (2012) for an introduction. Essentially, the model specifies the movement of the stock S_t from its current level to one of two possible levels at the next time point, where each of the two states are determined by carefully chosen magnitude parameters to which is associated a probability.

We are particularly interested in nesting these binomial trees in order to calculate ES. The binomial model accommodates this fairly easily, one has only to generate a secondary tree starting from a stock level in one's primary tree. Of course, the parameters that will determine this secondary tree will be different to those specifying the primary tree.

2.3.1 Numerical Implementation

First, a primary tree of stock prices must be created. Consider the time period to be $[0, T]$ and discretise this into N intervals of size $\Delta_1 t = \frac{T}{N}$ such that $t = [t_0, t_1, t_2, ..., t_N]$ where $t_k = k\Delta_1 t$ for $k = 0, ..., N$. A binomial tree approximates the continuous process by allowing the underlying stock to move to one of two states from its current state. Chance (2007) reviews 11 specifications of the model. These all perform similarly in valuing simple european derivatives so here the formulation of Jarrow and Turnbull (1999) is chosen in specifying these movements, where at some time $t \in [t0, ..., t_{N-1}]$ the asset price $S(t)$ moves to $S(t + \Delta_1 t)$ via

$$S(t + \Delta_1 t) = \begin{cases} S(t)u, & \text{probability} = 1/2 \\ S(t)d, & \text{probability} = 1/2. \end{cases} \tag{2.17}$$

The up and down factors in the above representation are,

$$\begin{aligned} u &= \exp\left(\mu\Delta_1 t + \sigma\sqrt{\Delta_1 t}\right) \\ d &= \exp\left(\mu\Delta_1 t - \sigma\sqrt{\Delta_1 t}\right). \end{aligned} \tag{2.18}$$

This allows the approximation of a log-normal stock price *under the real-world measure* $\mathbb{P}$. The general approach to generating the primary tree is visualised in Figure 2.1. We generate the primary tree of stock prices using lattice parameters given by (2.18). The i^{th} node of time t_k is denoted by the stock level S_k^i where $k \in [0, N-1]$ and is given by

$$S_k^i = S_0 u^{k+1-i} d^{i-1} \tag{2.19}$$

for $i = 1, ..., k + 1$ where S_0 is the stock price at t_0.

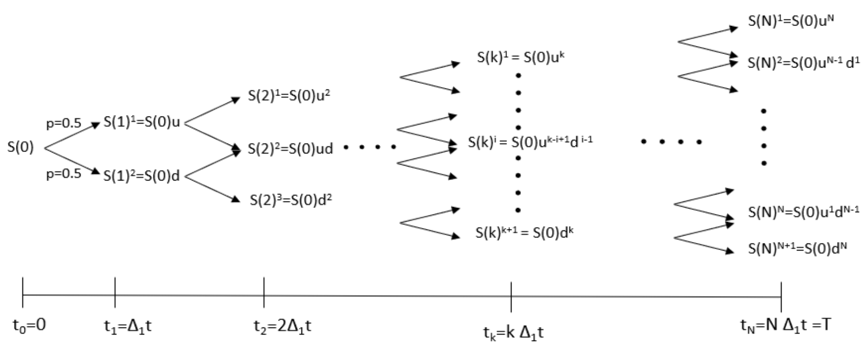

Fig. 2.1: General set-up of binomial primary tree with lattice parameters.

This primary tree forms the basis of what will be our KVA integral. We must now take this primary tree and compute expected shortfalls at each of its nodes by generating a secondary tree at each node. For some t_k where $k \in [0, N-1]$ the time period of the secondary tree becomes $[t_k, T]$. Discretise this into N_2 intervals of size $\Delta_2 t = \frac{T-t_k}{N_2}$ such that $\tau = [\tau_0, \tau_1, ..., \tau_n]$ where $\tau_k = k\Delta_2 t + t_k$ for $k = 0, ..., N_2$. The up down parameters for this secondary are again given by (2.18) but with $\Delta_2 t$ instead of $\Delta_1 t$. Note that we are still working under the $\mathbb{P}$-measure for ES computation. From here on we use Δt for $\Delta_2 t$ to simplify notation.

To solve for the KVA we start at time point N-1 and work backwards, since at $t_N = T$, KVA=0. We have the stock prices S^i_{N-1} for i=1,2,...,N from the primary tree. For each node i the year ahead loss is generated by a secondary tree starting from S^i_{N-1}. The year ahead loss is a 1 x (N_2+1) vector with entries given by (2.13)

$$L^i_n(t_{N-1}, t_N) = S^n_T \, e^{-r\Delta t} - S^i_{N-1} \qquad \text{for}\, n = 1, 2, ..., N_2 + 1. \tag{2.20}$$

Since when one starts from the top of the sub-tree and moves down one finds that the losses are already ordered, the ES at time t_{N-1} node i is computed as

$$\text{ES}^i_{N-1} = \frac{\sum_{n=1}^{x} p_n \, L^i_n(t_{N-1}, t_N)}{\sum_{n=1}^{x} p_n}, \tag{2.21}$$

where the probabilities p_n of each loss are given by

$$p_n = \binom{N_2}{n-1} p^{N_2}. \tag{2.22}$$

Here *x* is the 2.5% quantile, i.e, the 2.5% VaR at time t_{N-1}, node i satisfying

$$\sup x \sum_{n=1}^{x} p_n \leq 2.5\%. \tag{2.23}$$

In order to compute the solutions KVA=K we can use simple quadrature on (2.6). The kth integral at node i of time t_{N-1} is given by

$$I^{i,k}_{N-1} = \sum_{j=N}^{N} (h \max(\text{ES}^k_j, K^k_j) - (r+h) K^k_j) \, \Delta_1 t, \tag{2.24}$$

where k=1,2 and k=1 indicates an up move to the next state and k=2 a down move. We store the integrals in a vector

$$\mathbf{I}_{N-1} = \begin{bmatrix} I^{1,1}_{N-1} & I^{1,2}_{N-1} \\ I^{2,1}_{N-1} & I^{2,2}_{N-1} \\ \vdots & \vdots \\ I^{N,1}_{N-1} & I^{N,2}_{N-1} \end{bmatrix}. \tag{2.25}$$

The i^{th} entry contains the integrals associated with node i. The solutions KVA at each of the i nodes of time t_{N-1} are given by

$$\mathbf{K}_{N-1} = \mathbf{P}_{N-1} \, . \, \mathbf{I}_{N-1}^{'} \Delta t = [K_{N-1}^{1}, ..., K_{N-1}^{N}]. \tag{2.26}$$

Here P_{N-1} is the vector of probabilities at time t_{N-1} associated with each integral

$$\mathbf{P}_{N-1} = [0.5, 0.5,]. \tag{2.27}$$

But recall that at maturity ES and KVA both are 0 since there are no more trades involved. Thus at t=N-1, using right hand sums then ES_N, K_N = 0 so the integrals are all zero and K_{N-1}^{i} will also be zero for all i. We now step back to t_{N-2} and generate the year ahead losses with a secondary tree as before. The kth integral at node i of time t_{N-2} is given by

$$I_{N-2}^{i,k} = \sum_{j=N-1}^{N} \left(h \max(\text{ES}_j^k, K_j^k) - (r+h)K_j^k\right) \Delta_1 t, \tag{2.28}$$

where k=1,2,3,4. However we break up these integrals as follows. Let **B** be the matrix of increments to the integral (2.28) from t_{N-2} to t_{N-1}, i.e,

$$\mathbf{B}_{N-2} = \begin{bmatrix} B_{N-2}^{1,+} & B_{N-2}^{1,-} \\ B_{N-2}^{2,+} & B_{N-2}^{2,-} \\ \vdots & \vdots \\ B_{N-2}^{N-1,+} & B_{N-2}^{N-1,-} \end{bmatrix}, \tag{2.29}$$

where

$$\begin{aligned} B_{N-2}^{i,+} &= h \max(\text{ES}_{N-1}^{i}, K_{N-1}^{i}) - (r+h)K_{N-1}^{i}, \\ B_{N-2}^{i,-} &= h \max(\text{ES}_{N-1}^{i+1}, K_{N-1}^{i+1}) - (r+h)K_{N-1}^{i+1}. \end{aligned} \tag{2.30}$$

The integrals associated with each node at time t_{N-2} are given by the matrix

$$\mathbf{I}_{N-2} = \begin{bmatrix} B_{N-2}^{1,+} + I_{N-1}^{1,1} & B_{N-2}^{1,+} + I_{N-1}^{1,2} & B_{N-2}^{1,-} + I_{N-1}^{2,1} & B_{N-2}^{1,-} + I_{N-1}^{2,2} \\ B_{N-2}^{2,+} + I_{N-1}^{2,1} & B_{N-2}^{2,+} + I_{N-1}^{2,2} & B_{N-2}^{2,-} + I_{N-1}^{3,1} & B_{N-2}^{2,-} + I_{N-1}^{3,2} \\ \vdots & \vdots & \vdots & \vdots \\ B_{N-2}^{N-1,+} + I_{N-1}^{N-1,1} & B_{N-2}^{N-1,+} + I_{N-1}^{N-1,2} & B_{N-2}^{N-1,-} + I_{N-1}^{N,1} & B_{N-2}^{N-1,-} + I_{N-1}^{N,2} \end{bmatrix}. \tag{2.31}$$

More simply

$$\mathbf{I}_{N-2} = \mathbf{B}_{N-2}(:,1) + \mathbf{I}_{N-1}(1:N-1,:) + \mathbf{B}_{N-2}(:,2) + \mathbf{I}_{N-1}(2:N,:), \tag{2.32}$$

where the notation $\mathbf{X}(i:j,k:l)$ indicates the sub-matrix of $\mathbf{X}$ formed by rows i to j and columns k to l. The KVA solutions are now

$$\mathbf{K}_{N-2} = \mathbf{P}_{N-2}\,.\,\mathbf{I}^{'}_{N-2}\,\Delta_1 t, \tag{2.33}$$

where P_{N-2} is a 1x4 vector with entries 0.5. We continue doing this backwards in time until t=0. So generally, at some time t_k where $k \in [0, N-1]$

$$\begin{aligned} \mathbf{I}_k &= \mathbf{B}_k(:,1) + \mathbf{I}_{k+1}(1:k+1,:) + \mathbf{B}_k(:,2) + \mathbf{I}_{k+1}(2:k+2,:), \\ \mathbf{K}_k &= \mathbf{P}_k\,.\,\mathbf{I}^{'}_k\,\Delta_1 t. \end{aligned} \tag{2.34}$$

The KVA to be charged at $t = 0$ is then given by K_0. If we use the linear KVA equation (2.9), we can use trapezoidal quadrature in a similar algorithm to the one described above. See Sueli and Mayers (2006) for trapezoidal quadrature.

2.3.2 Numerical Results

All simulations in this dissertation are carried out in MATLAB on an Intel Core i7-7700HQ CPU with 8GB of RAM. The parameters of the forward are $S_0 = 100$, $\mu = 0.15$, $\sigma = 0.2$, $r = 0.06$, $T = 30$ days. We are working from the banks perspective having sold this forward to a client and the hurdle rate h is 10%. The number of steps in the primary tree ranged from $N = 1$ to $N = 30$. The secondary tree has $N_2 = 1020$ intervals, which gives 1021 year-ahead losses for each stock level S_k^i in the primary tree. The reason why only 1021 points are used for the loss distribution is because the binomial coefficients $\binom{N_2}{n}$ give infinity for some values of $n \in [0, N_2]$ when $N_2 > 1020$ and a general loss of accuracy occurs when calculating coefficients. Throughout this chapter we refer to the following as the '**analytic KVA**': The KVA on the forward computed using the analytic ES (2.14) based on a crude Monte Carlo estimate of (2.9) with 30 time steps and 1 000 000 primary scenarios. The methodology will be further discussed in Section 2.4.1.

Note in Figure 2.2 that the linear and non-linear KVA formulae converge for all quadrature methods, indicating that $\mathrm{EC}_t = \mathrm{ES}_t$ rather than $\mathrm{EC}_t = \max(\mathrm{ES}_t, \mathrm{KVA}_t)$. Although it seems that the convergence to the analytic KVA is good, this is misleading. The reason for this is detailed in Section 3.1.2. In essence, the calculation of the binomial coefficients for generating the probability density function in (2.22) loses accuracy for values of N_2 above about 170. We only go up to $N = 30$ quadrature points because the number of unique paths to maturity in a binomial tree with

$N = 30$ are 2^{30}. This translates to a number of matrix entries that causes the machine to run into memory difficulties for numbers larger than this. Optimisation routines might be helpful in solving this issues and should be explored in further research.

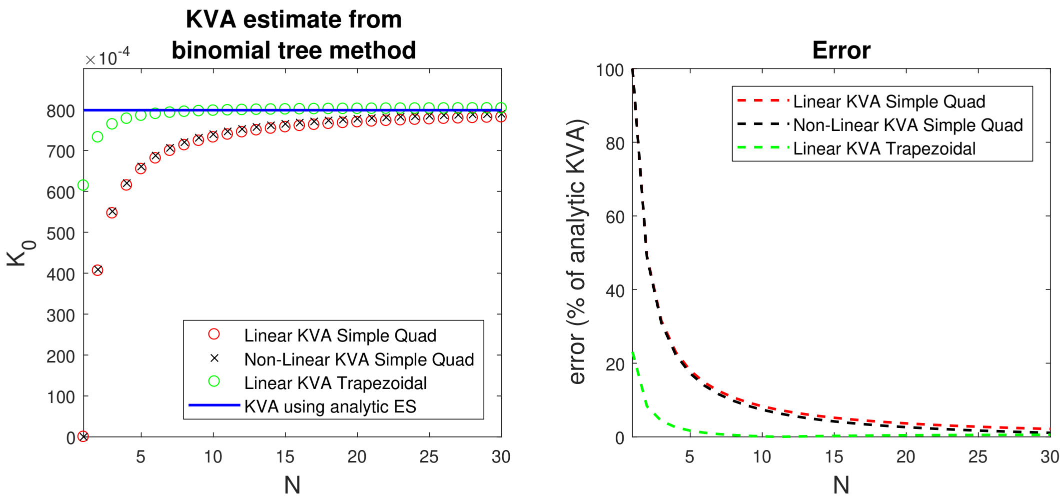

Fig. 2.2: Linear and non-linear KVA estimations on the forward for a primary tree with steps ranging from N=1 to 30.

2.4 Monte-Carlo Pricing

2.4.1 Numerical Implementation

Pseudo- and quasi-random sequences are used to simulate paths to maturity (primary scenarios) and expected shortfalls (secondary scenarios) respectively. Primary scenarios are denoted by N_1, secondary scenarios by N_2 and the number of time intervals by N. The notation for other input parameters remains the same as in the previous section. The primary scenarios are generated using $N_1 \times N$ variates $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I})$ with the stock price formula

$$S_i^k = S_0^k \exp\left(\sum_{j=1}^{i}\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\, Z_j\right) \tag{2.35}$$

for $k = 1, 2, ..., N_1$ and $i = 1, 2, ..., N$ where $\Delta t = T/N$. The formula (2.35) generates stock paths. To generate terminal values (secondary scenarios) for ES estimation a

1 x N_2 van der Corput sequence **Y** with base 3 and the stock price formula

$$S_T = S_t \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}\,Y\right) \tag{2.36}$$

are used, where $\tau = T - t$ is time-to-maturity. See van der Corput (1935) for details on the sequence. We use the same quasi-random sequence for all ES estimations so as to insure the ES is calculated with less variability between primary scenario nodes. Albanese *et al.* (2016) state that rarely is there need to use the non-linear formula (2.6). The linear KVA formula (2.9) is used as a slightly easier alternative and the KVA is checked post-calculation to ensure it is always below the ES in order to satisfy (2.4). The following trapezoidal quadrature is used to get the KVA of the i^{th} path

$$\text{KVA}_i = \frac{h}{2}\sum_{j=1}^{N}(\text{ES}^i_{j-1}\, e^{-(r+h)\,(j-1)\Delta t} + \text{ES}^i_j\, e^{-(r+h)\,j\Delta t})\,\Delta t \tag{2.37}$$

for $i = 1, 2, ..., N_1$. The Monte Carlo estimate of the KVA qnd its standard deviation are given respectively by

$$\begin{aligned} K_{MC} &= \frac{1}{N_1}\sum_{i=1}^{N_1}\text{KVA}_i, \\ \text{Var}_{MC} &= \frac{\sqrt{\sum_{i=1}^{N_1}\text{KVA}_i}}{\sqrt{N_1 - 1}}. \end{aligned} \tag{2.38}$$

2.4.2 Numerical Results

The same forward parameters as in Section 2.3.2 are used with $N = 30$ time-steps. The Figure 2.5 shows that the KVA integral converges quite rapidly in the case of the trapezoidal estimate, staying relatively unchanged after 20 time increments. The number of secondary scenarios has a much stronger impact on the convergence of the KVA, as seen in Figure 2.4. The KVA does not show improved convergence for primary scenarios up to 10 000 but shows notable convergence up to 91 000 secondary scenarios. Note that only 50 000 secondary scenarios were used in Figure 2.3 and these dictate the 'quality'of the KVA estimate. Given this estimation of ES, convergence will be to a KVA number different from the analytic KVA, which was given by the analytic ES at each node.

These figures highlight how computationally intensive secondary scenarios must be if one wishes to get closer to the analytic KVA. They also indicate that, given ES approximations, the estimation of the KVA integral is improved less with increasing primary scenarios than it is with increasing secondary scenarios. In other words, more realizations in the primary direction are less effective than better ES estimates.

The right-hand plot of Figure 2.6 is presented as evidence that the use of the linear KVA equation (2.9) does not violate the condition (2.4). This agrees with Remark 6.1 in the work of Albanese *et al.* (2016) as mentioned in the last paragraph of Section 2.1. The ES computed in this figure is the analytical ES (2.14). The KVA and ES values shown on this graph are only averages of those that were computed at each time. A more careful check will reveal that the KVA along each path is always below the ES along that same path, for all paths. This graph is evidence that in this simple case one can be confident that the KVA will always be less than the ES.

The left-hand plot of Figure 2.6 shows the difference between the binomial tree and Monte Carlo estimates with the binomial tree being within the error bounds of the Monte Carlo KVA when $N_2 = 1020$. However, as discussed in the results of the binomial tree method for the option in Section 3.1.2, many of these binomial coefficients will be inaccurate. The KVA when using $N_2 = 170$ is also plotted since this has very few coefficients that are above 9.007199 x 10^{15}, which is when accuracy is lost.

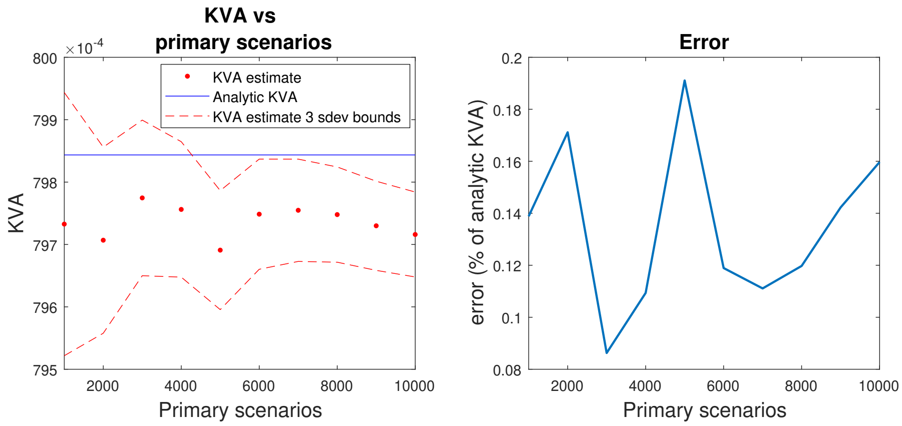

Fig. 2.3: Linear KVA estimations for 1 000 to 10 000 primary scenarios. 50 000 Secondary scenarios and 30 time increments.

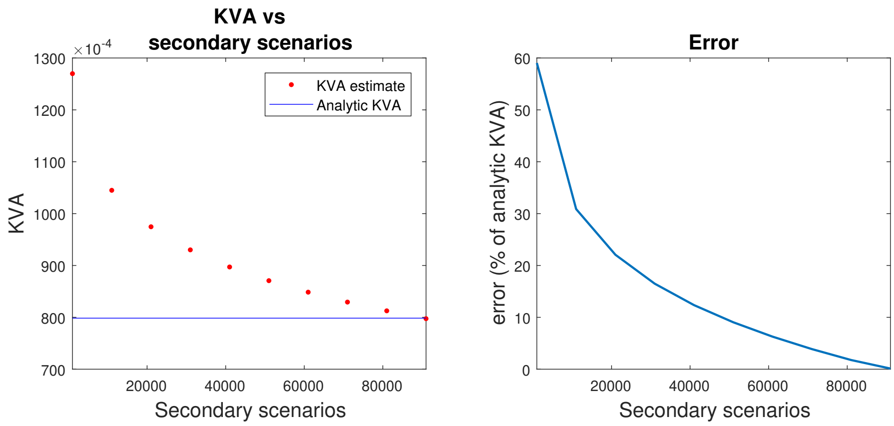

Fig. 2.4: Linear KVA estimations for 1 000 to 91 000 secondary scenarios. 5 000 Primary and 30 time increments used.

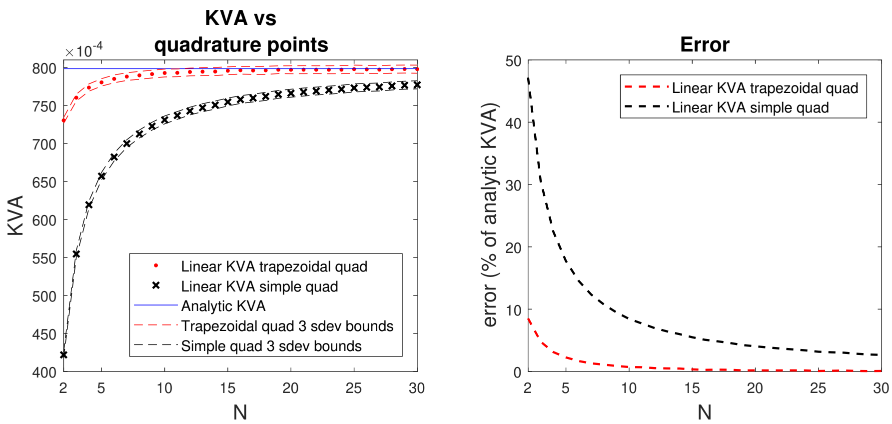

Fig. 2.5: Linear KVA estimations on the forward for 2 to 30 time increments. 5 000 Primary and 80 000 secondary scenarios were used.

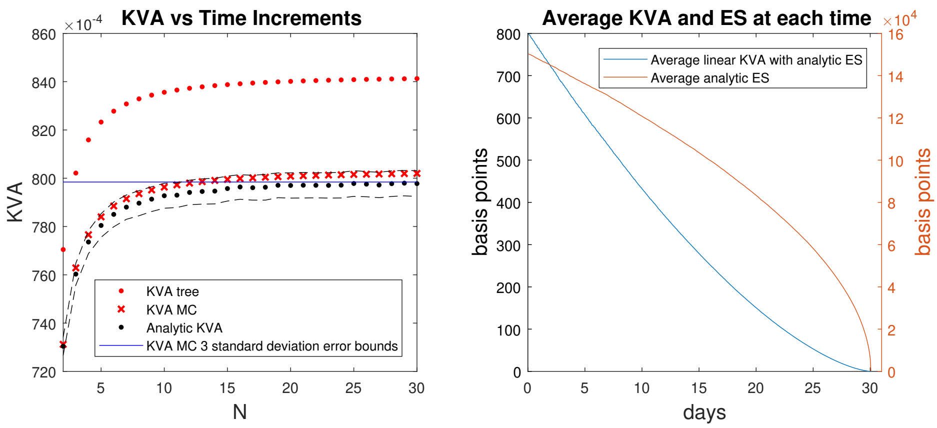

Fig. 2.6: *(Left)* Convergence of KVA for binomial tree and Monte Carlo methods for N=2 to 30 time steps.*(Right)* Average ES and KVA for forward introduced in Chapter 2.

Chapter 3

Analysis for Single Equity Option

The analysis follows for a short position in a European call option. As in the case of the forward, we implement no hedging strategy. The KVA on this option is calculated using binomial tree and Monte Carlo pricing.

3.1 Binomial Tree Pricing

The same tree methodology as in Section 2.3.1 applies but now a new set of parameters is required. Valuation takes place under the $\mathbb{Q}$-measure and new parameters are introduced for this purpose. Losses and KVA are still defined using the same $\mathbb{P}$-measure lattice parameters introduced in Section 2.3. Under the $\mathbb{Q}$-measure the following parameters apply

$$\begin{aligned} u_{\mathbb{Q}} &= \exp\left((r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\right), \\ d_{\mathbb{Q}} &= \exp\left((r - \frac{1}{2}\sigma^2)\Delta t - \sigma\sqrt{\Delta t}\right). \end{aligned} \tag{3.1}$$

The progression of the stock S through the tree is given by

$$S(t + \Delta t) = \begin{cases} S(t)u, & \text{probability} = q, \\ S(t)d, & \text{probability} = 1 - q\,. \end{cases} \tag{3.2}$$

The risk-neutral probability q is given by

$$q = \frac{e^{r\Delta t} - d}{u - d}. \tag{3.3}$$

These parameters are derived using the *method of moments* such that they yield the correct drift and volatility for the stock. More details can be found in Appendix A.1.

3.1.1 Numerical implementation

Once again T is maturity, N is the number of steps in the primary tree, $\Delta_1 t = \frac{T}{N}$ is the step size of the primary tree, N_2 is the number of steps in the secondary tree and $\Delta_2 t = \frac{T-t}{N_2}$ where t is the current time and N_2 is the step size of the secondary tree. From here on we use Δt for $\Delta_2 t$ to simplify notation. Exactly the same approach as in Section 2.3.1 is followed only now the year-ahead loss given by (2.11) changes to ,

$$L(t, T) = V_{option}(T)\, e^{-r(T-t)} - V_{option}(t)\,. \tag{3.4}$$

Accordingly, valuation for loss generation must be done by generating a secondary tree with *risk-neutral* parameters given by (3.1) and (3.3). We first generate a primary tree of stock prices using the real world lattice parameters. Say we are at S_k^i the i^{th} node of time t_k where $k \in [0, N-1]$. Then we use the risk-neutral parameters to generate a secondary tree starting at S_k^i. The possible terminal values of the option written on S_k^i are,

$$V_n^{i,k}(T) = \omega \max(S_T^{i,k,n} - K, 0) \quad n = 1, 2, ..., N_2 + 1\,, \tag{3.5}$$

where K is the trike of the option and ω = 1 (-1) if the option is a call (put). The value of the option at node i time t_k is given by,

$$V^i(k) = e^{-rN_2\Delta t} \left[\sum_{n=1}^{N_2+1} \binom{N_2}{n-1} q^{N_2+1-n}\,(1-q)^{n-1}\, V_n^{i,k}(T) \right]. \tag{3.6}$$

3.1.2 Numerical results

The parameters of the option are $S_0 = 100$, $\mu = 0.15$, $\sigma = 0.2$, $r = 0.06$, $T = 30$ days, N ranges from 1 to 30, $K = 100$ and the position is short. The secondary tree has $N_2 = 1020$ steps (for the same reason as discussed in Section 2.3.2) giving 1021 year-ahead losses for each node. In a similar fashion to Section 2.3.2, we refer to the following as the '**analytic KVA**': The KVA on the call option computed using the Monte Carlo methodology (described in the next section) and analytic ES for a call option with 30 time steps and 1 000 000 primary scenarios. The analytic ES for an option along with the derivation can be found in Appendix A.3.

Figure 3.1 shows that the KVA obtained from the binomial tree method appears to converge to the analytical value fairly quickly. As mentioned in Section 2.3.2, this is misleading. Figure 3.2 shows how the KVA is almost always over-estimated compared to the 'analytic' KVA. This must mean that the ES is over-estimated at the majority or all of the nodes. Up to 1020 steps in the secondary tree the KVA has a very hack-saw like convergence to the analytic KVA value given by Monte Carlo and analytic ES calculations. This is because MATLAB's binomial coefficient function *nchoosek* suffers from inaccuracy for values of N_2 even above about 170. This function is only accurate when calculating coefficients up to 9.007199 x 10^{15} so is only accurate to 15 digits. Constructing the pdf of the binomial tree thus becomes tricky with such large coefficients and beyond about 1020 one gets 'infinity' for some coefficients in (2.22). Note that the linear and non-linear KVA formulae converge, indicating that $\mathrm{EC}_t = \mathrm{ES}_t$ rather than $\mathrm{EC}_t = \max(\mathrm{ES}_t, \mathrm{KVA}_t)$. Also note that in Figure 3.1 there are no error bounds around the'analytic 'MC KVA because they are imperceptible on this plot.

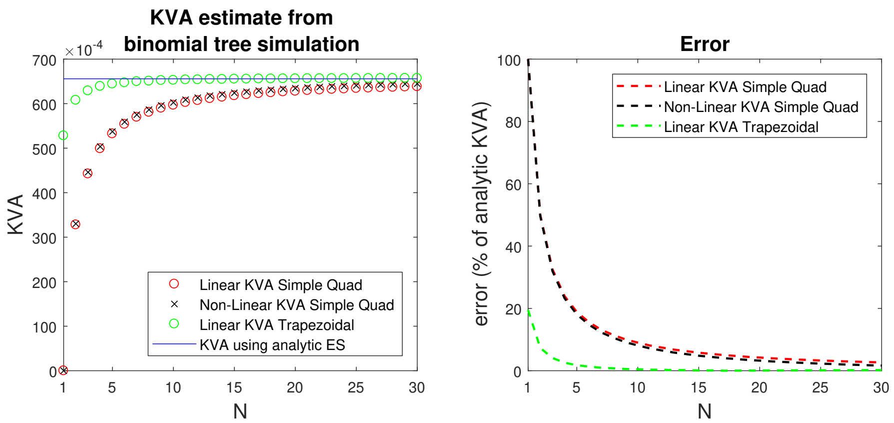

Fig. 3.1: Linear and non-linear KVA estimations on the option for a primary tree with steps ranging from 1 to 30, $N_2 = 1020$.

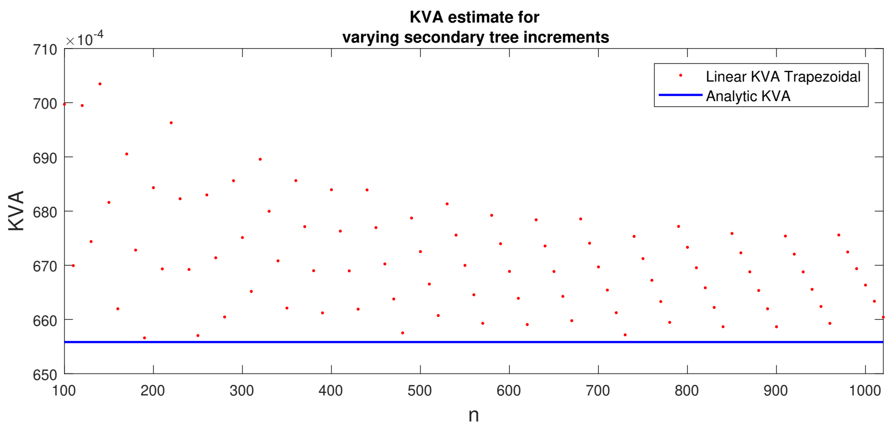

Fig. 3.2: *(Left)*Linear KVA estimations on the option using the binomila tree method for N=30 and $N_2 \in [100, 1020]$.

3.2 Monte Carlo Pricing

3.2.1 Numerical implementation

The implementation and notation follows almost exactly as in Section 2.4.1 except that valuation changes. Under GBM we have an analytical solution for the prices of simple European options. The price of a call/put at time t written on S with mean μ, volatility σ, for maturity T and strike K is given by

$$V(S_t\,, t\,, T\,, K\,, \mu\,, \sigma) = \omega[S_t\,\Phi(\,\omega d_1\,) - K\,e^{-r(T-t)}\Phi(\,\omega d_2\,)]. \tag{3.7}$$

as presented by Black and Scholes (1973). Here Φ is the standard normal distribution function, r is the risk-free rate, ω = 1 (-1) for call (put) and d_1, d_2 are given by

$$\begin{aligned} d_1 &= \frac{\ln\frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\ln\frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}. \end{aligned} \tag{3.8}$$

The $\mathbb{P}$-dynamics (2.1) are used to generate primary and secondary stock price scenarios with option prices given by (3.7). End-point stratification is employed along with Brownian bridging in an attempt to reduce the variance of the KVA calculation. This is performed by generating $d = 50$ strata of the terminal stock price and bridging stock prices between the initial S_0 and terminal S_T prices. Antithetic variates are also employed and we require in total $N_1 \times N$ variates $\mathbf{Z} \sim N(\mathbf{0},\, \mathbf{I})$ to generate our primary stock scenarios. A $\frac{N_1}{2} \times N$ vector $\mathbf{Z} \sim N(\mathbf{0},\, \mathbf{I})$ is generated to then form $\mathbf{X} = [\mathbf{Z}, -\mathbf{Z}]$ for use with (2.35). The reader is referred to Glasserman (2013) for an excellent description of stratification, Brownian bridging and antithetic variates. Due to the antithetic sample, the estimator for KVA now becomes

$$K_{MC} = \frac{2}{N_1}\,\frac{\sum_{i=1}^{\frac{1}{2}N_1} \mathrm{KVA}_i \;+\; \sum_{i=\frac{1}{2}N_1+1}^{N_1} \mathrm{KVA}_i}{2}. \tag{3.9}$$

3.2.2 Numerical results

We use the same option parameters as in Section 3.1.2. Figure 3.3 shows that when using trapezoidal quadrature one gets rapid convergence in the sense of time increments N, with relatively accurate results after $N = 10$. Note that in this figure antithetic variates have been used for KVA estimation. Figures 3.4 and 3.5 show that increasing the number of secondary scenarios play a more prominent role in obtaining an accurate KVA estimate than the primary scenarios. This is seen in the larger error for the secondary scenarios convergence. As discussed in Section 2.4.2, an increasing number of primary scenarios will converge to a KVA number that is dictated by the quality of the ES estimates. This is why in Figure 3.4, where 50 000 scenarios were used for ES estimation, the KVA seems to converge to a value that is different to the analytic KVA, where the analytic ES formula was used.

Figure 3.6 shows more clearly the role of the primary scenarios when the analytic ES is used, thus eliminating the need for secondary scenarios. The end-point stratification with Brownian bridging and antithetic variates improves convergence significantly. This highlights the importance of variance reduction in ES estimation. Further investigation into similar techniques, such as importance sampling, might benefit the implementation of KVA calculations in practice when using full revaluation for ES generation.

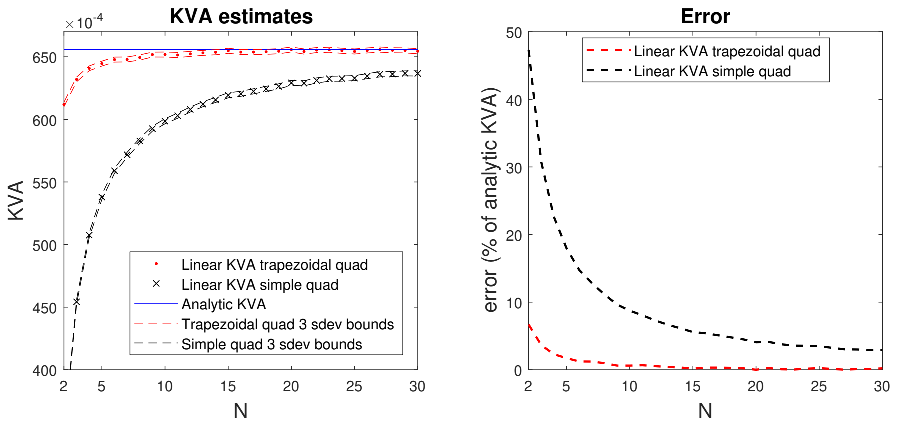

Fig. 3.3: Linear KVA estimateson the options for 2 to 30 time increments with 3 standard deviation error bounds. 5 000 Primary and 80 000 secondary scenarios were used.

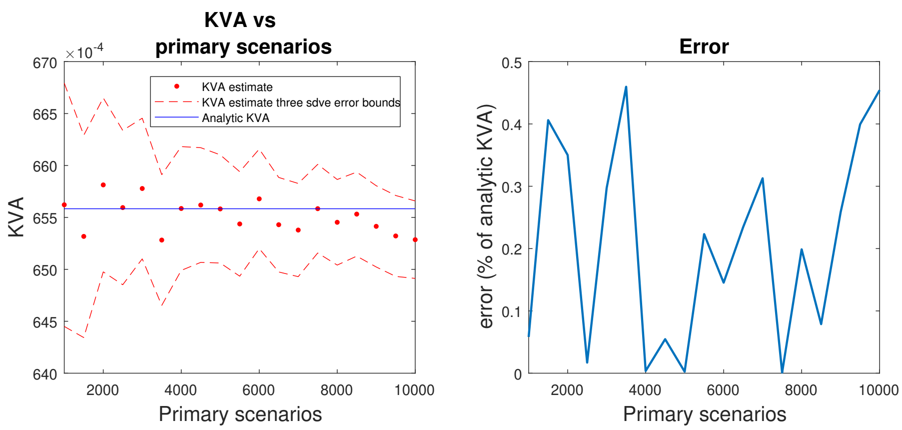

Fig. 3.4: Option linear KVA estimations for 1 000 to 10 000 primary scenarios. 50 000 Secondary scenarios with 30 time increments.

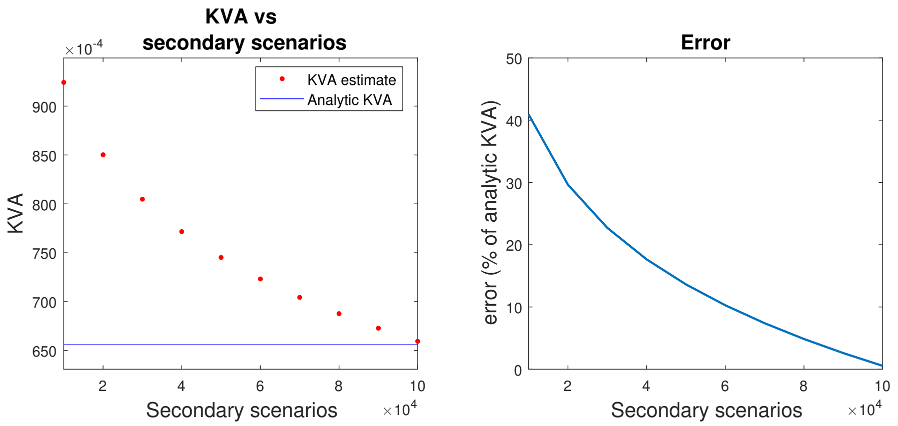

Fig. 3.5: Option linear KVA estimations 10 000 to 100 000 secondary scenarios. 5 000 Primary and 30 time increments used.

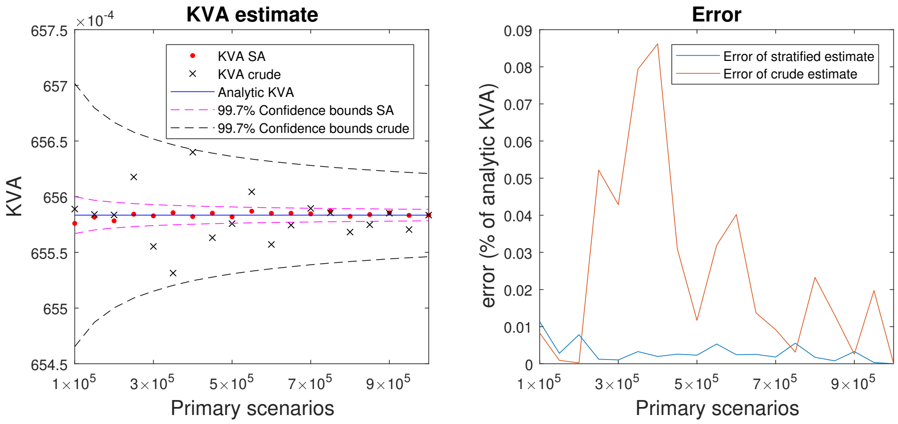

Fig. 3.6: Linear KVA estimations on the option for 100 000 to 1 000 000 primary scenarios with analytic ES. SA stands for stratified-antithetic. 30 time steps were used.

These results indicate that, given ES approximations, the estimation of the KVA integral is improved more with increasing secondary scenarios than with increasing primary scenarios. In other words, more realizations in the primary direction are less effective than better ES estimates.

In the right plot of Figure 3.7 the linear KVA formula (2.9) is validated by observing that average KVA is always below average ES. As in Section 2.4.2, this relationship actually holds for all paths and suggests that very special circumstances would be required for the KVA to be greater than the ES at any point in time, which is in line with the remark 6.1 in the paper of Albanese *et al.* (2016).

The left plot of Figure 3.7 compares the KVA obtained from the binomial tree and

Monte Carlo approach. The binomial tree overshoots the analytic KVA for both $N_2 = 1020$ and $N_2 = 170$ because of the reason already discussed. Generating the pdf in (2.22) for ES calculation becomes inaccurate fairly quickly. The results of $N_2 = 170$ are presented because up to this point the binomial coefficients are still accurate. This demonstrates the power of the Monte Carlo method which, via the central-limit theorem, is able to deal with a very large number of random simulations just by taking averages as opposed to the binomial tree method which needs some reconstruction of a terminal distribution.

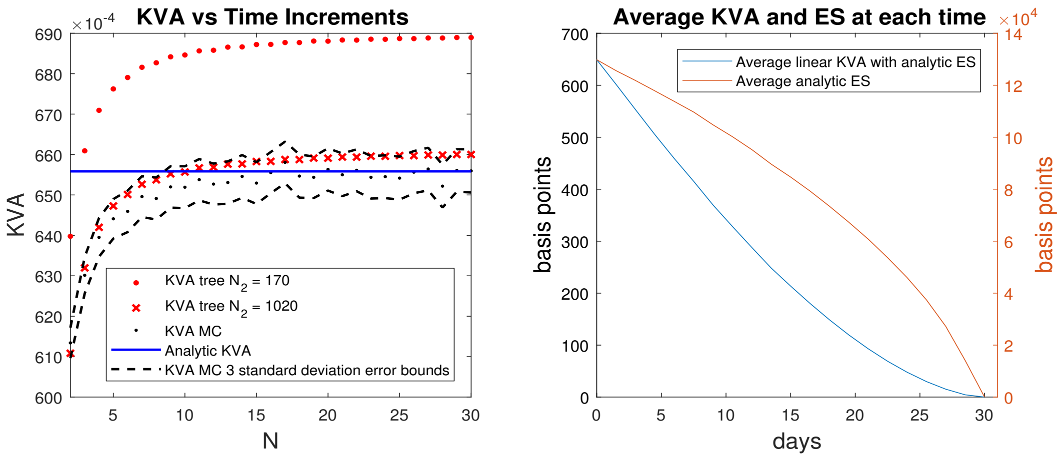

Fig. 3.7: *(Left)* Convergence of KVA when using binomial tree and Monte Carlo methods for N=2 to 30 time steps. *(Right)* Average ES and KVA for option introduced in Chapter 3.

Chapter 4

Analysis for Portfolio of Options

To this point KVA has been considered in isolation to a single trade. Of course, KVA should reflect the capital cost of an entire portfolio of trades, with correlations playing an important role in the specification of the portfolio KVA. Of particular interest is the case of a bank which is considering trading a 2-asset index option (on S_1 and S_2) which will subsequently be delta hedged in addition to a static gamma or vega hedge. These gamma or vega hedges can be formed using options on the exact underlings or even an option on a third asset S_3 that is correlated to the index. One would expect the hedge to reduce the KVA and qualify an unfavourable trade to a favourable one from a KVA perspective.

4.1 Numerical results for Single Asset portfolios

We first investigate specific examples of single asset portfolios namely a collar, fence and butterfly on the asset S from Chapter 2. These are common option strategies detailed in Hull (2012). A collar is constructed by holding the stock and out-the-money puts with strike K_1 while shorting out-the-money calls with strike K_2, all with the same expiry and contract size. Its payoff is given by

$$P_{collar}(T) = \alpha \left[K_1 \, I_{\{S_T < K_1\}} + S_T \, I_{\{K_1 \le S_T < K_2\}} + K_2 \, I_{\{S_T \ge K_2\}} \right]. \tag{4.1}$$

where α is the contract size. Note we will be looking at the case $\alpha = 1$. A fence is constructed by selling far-out-the-money calls with strike K_3 and puts with strike K_1 and buying at-the-money puts with strike K_2 while holding the underlying. The payoff is

$$P_{fence}(T) = \alpha \begin{cases} K_2 - K_1 + S_T & S_T < K_1 \\ K_2 & K_1 \le S_T < K_2 \\ S_T & K_2 \le S_T < K_3 \\ K_3 & S_T \ge K_3. \end{cases} \tag{4.2}$$

The long call butterfly is constructed by buying an in-the-money call with strike K_1, shorting two at-the-money calls with strike K_2 and buying an out-the-money call with strike K_3 such that $K_2 = \frac{1}{2}(K_3 - K_1)$. The payoff is

$$P_{butterly}(T) = \alpha\left[(S_T - K_1)\, I_{\{K_1 \leq S_T < K_2\}} + (K_3 - S_T)\, I_{\{K_2 \leq S_T < K_3\}}\right]. \quad (4.3)$$

Figures 4.1 to 4.3 show surfaces of the linear KVA (2.9) for short and long positions in the 3 strategies described above, all with maturity $T = 30$ days. The surfaces are constructed using 2 000 primary scenarios, 10 000 secondary scenarios and 30 time-steps. Theses have an intuitive shape to them, with the KVA peaking around and between the various strikes. Since the portfolios are 'at-the-money 'in some sense when the underlying S_t lies on one of the strikes, the portfolio value is relatively high. However, in projecting S_t to T for ES calculation, many realisations of S_T may end below this strike level and thus a year-ahead loss will be realised. Note that $S_0 = 100$. Interestingly, the KVA is always positive, indicating that these strategies always incur lifetime capital costs. The surfaces of the KVA are similar for long and short positions in each strategy but one might be tempted to think that opposite positions would mean opposite KVA. However, the 97.5% VaR of the year-ahead loss (change in value) process, which drives the KVA calculation, is almost always a loss for any position one takes in simple European options. This is emphasised because it is possible for a portfolio to give positive expected shortfalls on the 97.5% VaR, which would ultimately mean a negative KVA ,i.e, the portfolio if KVA profitable to the holder. However, this is not explored in this dissertation.

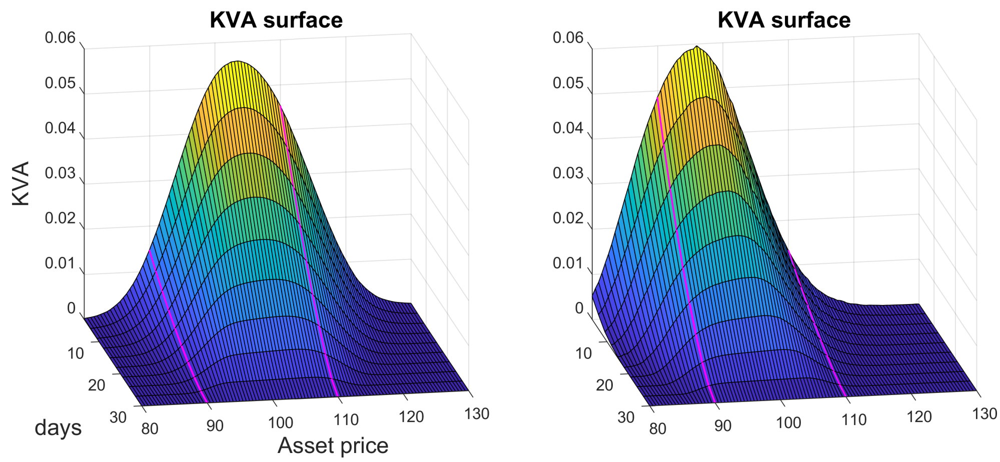

Fig. 4.1: The linear KVA on a portfolio consisting of a collar strategy on S. $K_1 = 90$, $K_2 = 110$. Left (right) long (short). Purple lines indicate strikes.

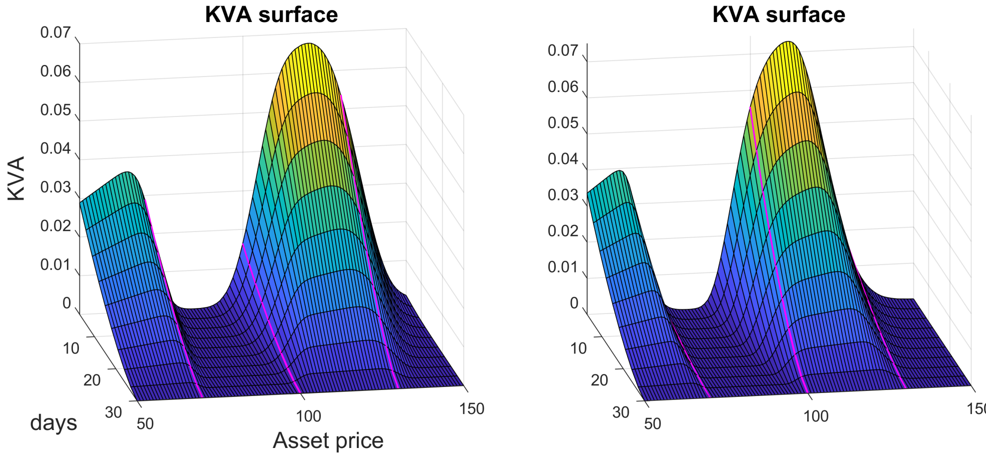

Fig. 4.2: The linear KVA on a portfolio consisting of a fence strategy on S. $K_1 = 70$, $K_2 = 100$, $K_3 = 130$. Left (right) long (short).Purple lines indicate strikes.

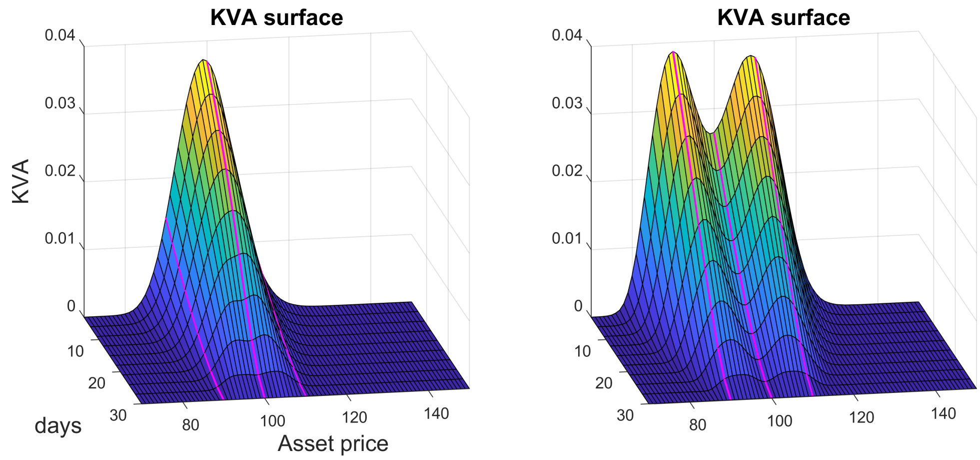

Fig. 4.3: The linear KVA on a portfolio consisting of a butterfly strategy on S. $K_1 = 90$, $K_2 = 100$, $K_3 = 110$. Left (right) long (short). Purple lines indicate strikes.

4.2 Numerical implementation for Two Asset case

Let S_1 and S_2 represent the price of Asset 1 and 2 respectively, with correlation between them given by ρ_{12}. A very similar approach to Section 3.2.1 is used but now we generate a two-dimensional van der Corput sequence (Halton sequence) using bases 3 and 5. This is again used in ES estimation with formula (2.36). To generate the primary scenarios of each asset, the Cholesky decomposition **L** of the correlation matrix

$$\mathbf{\Sigma} = \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix}$$

is used Duffy and Kienitz (2011). The decomposition is

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ \rho_{12} & \sqrt{1 - \rho_{12}{}^2} \end{bmatrix} \tag{4.4}$$

such that $\mathbf{\Sigma} = \mathbf{L}\,\mathbf{L}^T$. Let s represents the total number of random numbers required i.e $s = N_1 N$ where notation follows as in Section 2.4.1. A 2 x $\frac{s}{2}$ matrix $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I})$ is generated to form $\mathbf{X}$ by using $\mathbf{Z}$ and an antithetic sample i.e $\mathbf{X} = [\mathbf{Z}, -\mathbf{Z}]$. We multiply $\mathbf{X}$ by $\mathbf{L}$ in order to get a $2 \times s$ matrix of correlated normal random numbers

$$\mathbf{Y} = \mathbf{L}\,\mathbf{X} \sim N(\mathbf{0}, \mathbf{\Sigma})$$

The first row of $\mathbf{Y}$ is reshaped into an $N_1 \times N$ matrix where the bottom half of the matrix corresponds to the antithetic sample. This is used to generate S_1 primary scenarios with the formula (2.35) while the second row of $\mathbf{Y}$ is used similarly for S_2. This methodology is easily extended to an n-asset case.

4.3 Numerical Results: Two Asset Portfolios

Here S_1 has the same parameters as in Chapter 2. The second asset S_2 has mean 0.2 and volatility 0.3. For the second asset $S_2(0) = 160$ in all cases. The KVA is computes using 10 000 primary and secondary scenarios and 30 time steps. The following portfolios are investigated, with tenor always being $T = 30$ days.

Portfolio 1: A collar on S_1 with $K_1 = 90$, $K_2 = 110$ and a fence on S_2 with $K_1 = 130$, $K_2 = 160$, $K_3 = 190$.

Portfolio 2: A fence on S_1 with $K_1 = 70$, $K_2 = 100$, $K_3 = 130$ and a butterfly on S_2 with $K_1 = 150$, $K_2 = 160$, $K_3 = 170$.

Portfolio 3: A collar on S_1 with $K_1 = 90$, $K_2 = 110$ and a butterfly on S_2 with $K_1 = 150$, $K_2 = 160$, $K_3 = 170$.

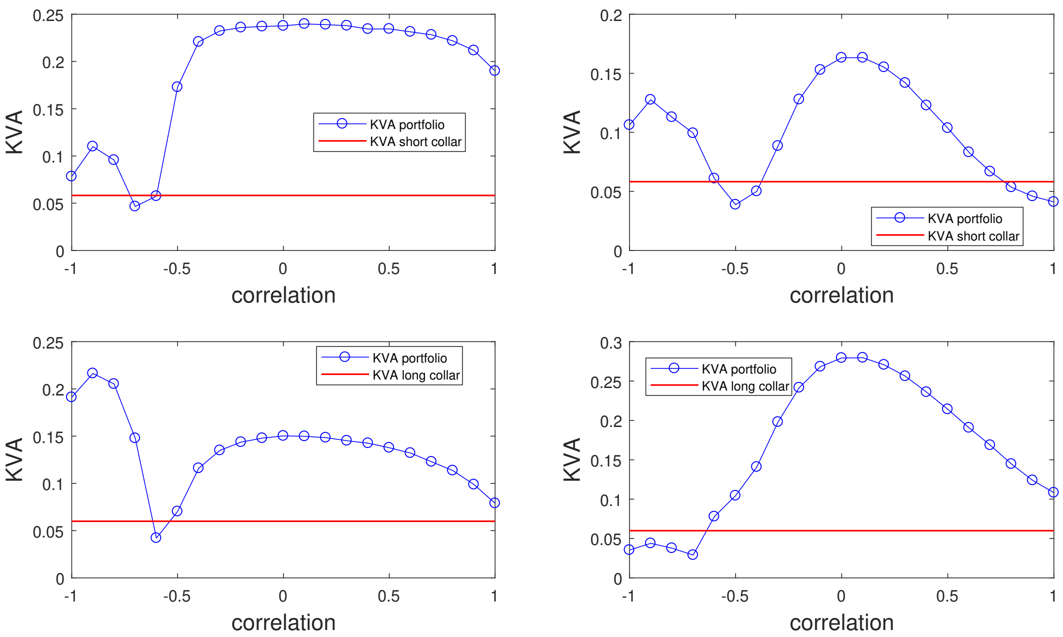

Fig. 4.4: KVA for different correlations for Portfolio 1. *(Top Left)* Short the collar, short the fence. *(Top Right)* Short the collar, long the fence. *(Bottom Left)* Long the collar, short the fence. *(Bottom Right)* Long the collar and the fence

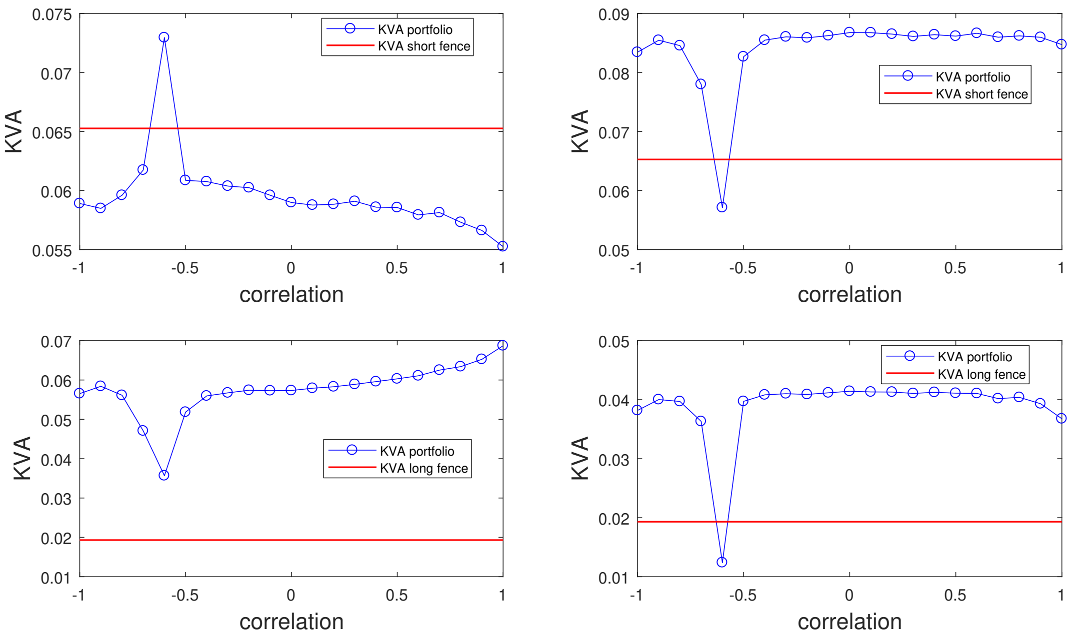

Fig. 4.5: KVA for different correlations for Portfolio 2. *(Top Left)* Short the fence, short the butterfly. *(Top Right)* Short the fence, long the butterfly. *(Bottom Left)* Long the fence, short the butterfly. *(Bottom Right)* Long the fence and the butterfly.

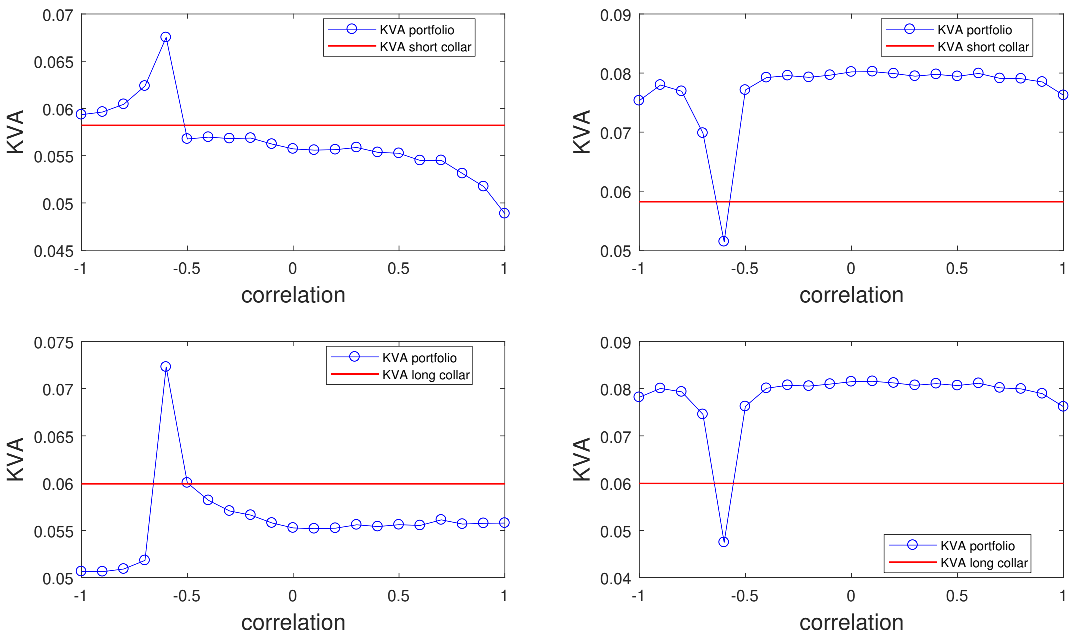

Fig. 4.6: KVA for different correlations for Portfolio 3. *(Top Left)* Short the collar, short the butterfly. *(Top Right)* Short the collar, long the butterfly. *(Bottom Left)* Long the collar, short the butterfly. *(Bottom Right)* Long the collar and the butterfly.

Figures 4.4 to 4.6 show that for 10 of the above portfolios, a correlation in the negative range gives the lowest portfolio KVA. Intuitively, one would think the diversification would reduce year-ahead losses and KVA. However, it is not possible to say whether this is a general trend that applies to all portfolios. A particular curiosity is that the maximum/minimum KVA always occurs at a correlation between -0.5 and -0.6 for these very simple portfolios. Ultimately, it seems plausible to hypothesis that diversification should decrease KVA as expected shortfall is sub additive.

4.4 Practical Example: Index Option

This is the focus of this chapter. A feature of any bank's equity portfolio is an index option. In order to hedge, the bank may seek option exposure on the individual constituents of the index. Consider a hypothetical market index made up of asset S_1 and S_2 with correlation ρ_{12}. The option market on the individual constituents may be illiquid. In this case the bank can instead find options on another, correlated asset S_3 with $\rho_{13} = \rho_{23}$ for vega hedging. Having a large range of potential assets to use for hedging, with varying degrees of correlation with the index as a whole, we try get a sense of which choice of S_3 will qualify a KVA unattractive deal to one that achieves some KVA benefit. In the analysis that follows, the majority of the vega will derive from the asset S_2, which will have a significantly higher volatility than S_1. To simplify things, the index option is priced as a basket option with constant weightings.

4.4.1 Discrete Delta Hedge

The index option has value $X(t)$ with payoff

$$X(T) = (w_1\, S_1(T) + w_2\, S_2(T) - K_{\text{indx}})^+ \tag{4.5}$$

which is estimated using Monte Carlo methods. The base case KVA is that calculated when the index is discretely delta hedged and in this case the portfolio value is

$$\Pi(t) = X(t) + \delta_1(t)\, S_1(t) + \delta_2(t)\, S_2(t) + m(t). \tag{4.6}$$

Here δ_1, δ_2 are the position taken in S_1 and S_2 and m is the money-market account which is

$$m(t) = -X(t) - \delta_1(t)\, S_1(t) - \delta_2(t)\, S_2(t), \tag{4.7}$$

such that $\Pi(t) = 0$. The stock positions are given by

$$\delta_i(t) = -\frac{\partial X(t)}{\partial S_i(t)} \tag{4.8}$$

for $i = 1, 2$. The partial deltas of the index are approximated using the central difference formula

$$\frac{\partial X(t)}{\partial S_i(t)} \approx \frac{X(S_i(t) + \Delta S) - X(S_i(t) - \Delta S)}{2\,\Delta S}. \tag{4.9}$$

The method of common random numbers, where the same random variate are used to estimate $X(S_i(t) + \Delta S)$ and $X(S_i(t) - \Delta S)$ (Glasserman (2013)), is used in all finite difference approximations in this chapter specifically with $\Delta S = 0.1$ above. The year-ahead losses are now given by

$$L(t, T) = (\Pi(T) \mid S_1 = x, S_2 = y) \exp(-r(T - t)). \tag{4.10}$$

Since $\Pi(t) = 0$. The KVA is then calculated using the 2-asset Monte Carlo methodology.

4.4.2 Direct Vega Hedge

We now consider a discretely delta-hedged portfolio with a static vega hedge at $t = 0$ using an option on S_2. Consider placing an initial vega hedge using an at-the-money option on S_2, C^*. The portfolio is

$$\Pi(t) = X(t) + \alpha\, C^*(t) + \delta_1(t)\, S_1(t) + \delta_2(t)\, S_2(t) + m(t). \tag{4.11}$$

For vega-neutrality at $t = 0$ we require

$$\alpha = -\frac{\frac{\partial X(0)}{\partial \sigma_2}}{\frac{\partial C^*(0)}{\partial \sigma_2}}. \tag{4.12}$$

The Black-Scholes vega and a central difference estimate of the index vega is given respectively by

$$\frac{\partial C^*(t)}{\partial \sigma} = S\sqrt{T-t}\,\phi(d1) \tag{4.13}$$

$$\text{and } \frac{\partial X(t)}{\partial \sigma} \approx \frac{X(\sigma+\Delta\sigma) - X(\sigma-\Delta\sigma)}{2\,\Delta\sigma}, \tag{4.14}$$

where ϕ is the standard normal density and $\Delta\sigma = 0.01$. Delta neutrality with respect to S_2 is now given by

$$\delta_2(t) = -\frac{\partial X(t)}{\partial S_2(t)} - \alpha\frac{\partial C^*(t)}{\partial S_2(t)} \tag{4.15}$$

and δ_1 is given by (4.8). The delta of a vanilla option with no dividend yield is

$$\frac{\partial C(t)}{\partial S(t)} = \omega\,\Phi(\omega\, d_1), \tag{4.16}$$

where $\omega = 1(-1)$ for a call (put). The year ahead losses are still given by (4.10).

4.4.3 Indirect Vega Hedge

We now use an option on S_3 with $\rho_{13} = \rho_{23}$ for the static-vega hedge. The option on S_3, $\widehat{C}$ is at-the-money and the portfolio value is now given by

$$\Pi(t) = X(t) + \alpha\,\widehat{C}(t) + \delta_1(t)\,S_1(t) + \delta_2(t)\,S_2(t) + \delta_3(t)\,S_3(t) + m(t). \tag{4.17}$$

A naive vega-hedge is used where the position in the option $widehatC$ is chosen such that

$$\alpha = -\frac{\frac{\partial X(0)}{\partial \sigma_2}}{\frac{\partial \widehat{C}(0)}{\partial \sigma_3}}. \tag{4.18}$$

The position in the asset S_3 is

$$\delta_3(t) = -\alpha\,\frac{\partial \widehat{C}(t)}{\partial S_3(t)}. \tag{4.19}$$

The rest follows as above. The parameters applicable to the different cases are as follows: S_1 and S_2 are the same as previously described except that $\sigma_2 = 0.5$, $\rho_{12} = 0.9$, $w_1 = 0.6$, $w_2 = 0.4$, $K_{indx} = 124$, $K_2 = 160$, $\mu_3 = 0.22$, $\sigma_3 = 0.3$, $S(0)_3 = 130$, $K_3 = 130$, number of time increments/re-balancing times is $N = 30$, primary and secondary scenarios are 10 000. For comparison we also calculate the KVA of the index option with no hedge. The figure below shows the KVA for the different strategies. Note than no standard deviation error bounds are shown as these are too narrow to see on the plot.

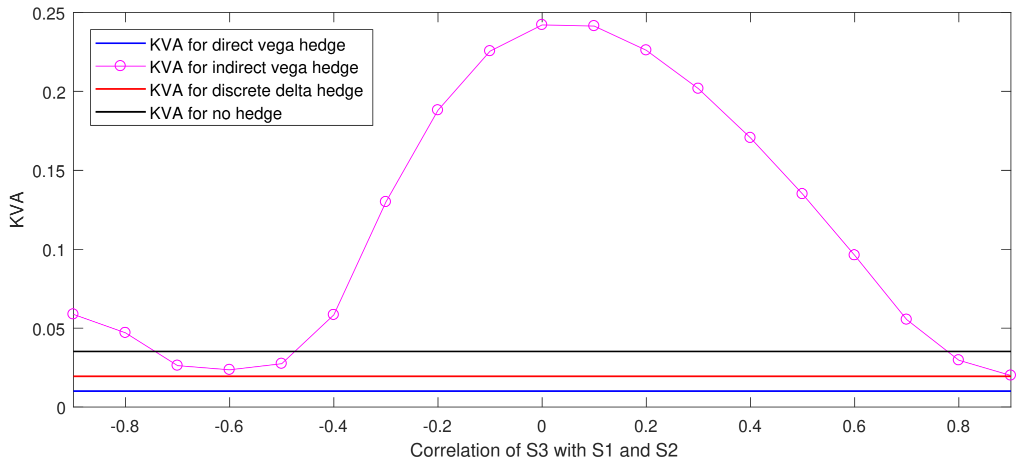

Fig. 4.7: KVA for different correlations between S_3 and S_1, S_2 in indirect vega, direct vega, discrete delta hedged portfolios.

If the bank was unable to hedge using an option on S_2 and they were forced to instead use an option on S_3, then finding such an asset with correlation 0.9 would be ideal, as this offer the lowest KVA with correlations between -0.5 and -0.7 also offering low KVA. However, the KVA is never better than when vega hedging with an option on S_2 or discrete delta hedging. If the only asset available to the bank was one with $\rho_{31} = \rho_{32} = 0$ then it might not suit the bank to enter the trade without charging the KVA and the KVA might in turn be too high for the client to pay. So the choice of S_3 has a big impact on the KVA of a potential trade with the KVA going from a minimum of about 0.02 to a maximum of about 0.24, a twelve-fold change. As expected, the KVA was reduced progressively from no hedge to discrete delta hedge to discrete delta hedge with a static vega hedge.

It must be highlighted that in this dissertation very simple portfolios were considered. For a bank with thousands of trades and the requirements to model more than one risk factor, this methodology of computing KVA will surely be intractable. A solution could be to use a projection for economical capital based on forward rates, instead of a Monte Carlo approach as noted by Gregory (2015). A further consideration could be to calculate KVA on the entire portfolio level only when large, capital intensive trades are added to the portfolio and calculate KVA on a single trade level for smaller trades.

Chapter 5

Conclusion

Computing KVA is made numerically intractable when using full revaluation to estimate expected shortfalls. Even in the simple portfolios considered in this paper, codes to compute KVA took up to 6 hours to run. One can imagine that for a portfolio that is representative of a real bank's portfolio, consisting of complex instruments along with all necessary curves required for re-valuations, plus the other XVAs which add nested levels to the computation, this exercise becomes quite impossible. An alternative is to use a deterministic term structure as Albanese *et al.* (2016) do or explore the use of regression techniques to estimate expected shortfalls. As a first step the binomial tree approach is a faster method than the Monte Carlo approach but the probability distributions are limited by computer precision for secondary steps beyond $N_2 = 1020$. On the other hand, the Monte Carlo method is easier to code and ultimately augment for more realistic cases.

The power of the Monte Carlo method is its ability to deal with a very large number of random simulations just by taking averages as opposed to the binomial tree method which needs some reconstruction of a terminal distribution. In both Chapters 2 and 3 the amount of secondary scenarios used in ES estimation was shown to be the overwhelming factor in converging to an analytical KVA. End point stratification with Brownian bridging and the use of antithetic variates were found to increase convergence, especially in the lower range of primary scenarios used in this study. It seems unlikely that in practice more than 100 000 primary scenarios could be used, given the vastness of a real banks portfolio, and variance reduction will undoubtedly be key. In terms of further improvements on Monte Carlo estimation of expected shortfalls, importance sampling may be a worthwhile enterprise.

It is not always the case that negative correlations will decrease the portfolio KVA as ES and therefore KVA also depend on the position one takes in the European options explored here. A financial institution wishing to hedge an index option without using the exact underlyings would have to give special consideration to the correlation of the asset to be used for hedging. The hypothesis that equal positions in two options on uderlyings will generate decreasing KVAs with decreasing correlation could be tested mathematically by approximating the distribution of the sum of the underlying lognormal distributions. Alternatively, a numerical study can be performed on a wide range of parameters in order to indicate a general trend on KVA versus correlation.

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Appendix A

Appendix A

A.1 Binomial Model Lattice Parameters

We choose the formulation of Jarrow and Turnbull (1999)

$$\ln\left(\frac{S_{t+\Delta t}}{S_t}\right) = \begin{cases} u, & \text{probability} = 1/2 \\ d, & \text{probability} = 1/2. \end{cases} \tag{A.1}$$

Here the up and down factors are given by (2.18). Under GBM we know that the risk-neutral dynamics of the stock are given by the SDE

$$\begin{gathered} \frac{dS_t}{S_t} = rdt + \sigma dW_t^{\mathbb{Q}}, \\ \text{where } W_t^{\mathbb{Q}} \sim N(0,t). \end{gathered} \tag{A.2}$$

By Ito we have that

$$d\ln(S_t) = (r - \frac{1}{2}\sigma^2)dt + \sigma dW_t^{\mathbb{Q}}. \tag{A.3}$$

Integration gives

$$\ln\left(\frac{S_t}{S_0}\right) = (r - \frac{1}{2}\sigma^2)t + \sigma W_t^{\mathbb{Q}}, \tag{A.4}$$

so that

$$\ln\left(\frac{S_t}{S_0}\right) \sim N\left(\left(r - \frac{1}{2}\sigma^2\right)t\,,\, \sigma^2\, t\right).$$

Thus the expected value of the log-return under $\mathbb{Q}$ is

$$\mu_{\mathbb{Q}} = \left(r - \frac{1}{2}\sigma^2\right). \tag{A.5}$$

Substituting (A.5) into (2.18) we arrive at the lattice parameters under $\mathbb{Q}$

$$\ln\left(\frac{S_{t+\Delta t}}{S_t}\right) = \begin{cases} \left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}, & \text{probability} = q \\ \left(r - \frac{1}{2}\sigma^2\right)\Delta t - \sigma\sqrt{\Delta t}, & \text{probability} = 1 - q, \end{cases} \tag{A.6}$$

where the risk-neutral probability q is given by

$$q = \frac{e^{r\Delta t} - u}{u - d}. \tag{A.7}$$

A.2 Analytic VaR, ES for forward

If S_T is described by GBM then from (A.4) but under the real world measure $\mathbb{P}$

$$\ln\left(\frac{S_T}{S_t}\right) = (\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma\int_t^T dW_s^{\mathbb{P}}.$$

We have that $\ln S_T$ is normally distributed $\mathrm{N}(\,m(t,T),\, s^2(t,T)\,)$ where $m(t,T)$ and $s(t,T)$ are given by (2.15) and (2.16). The 97.5% VaR at time t is given by $\mathrm{VaR}_t = x_{0.975}$ where

$$\mathbb{P}(L(t,T) \le x_{0.975}\,|\mathcal{F}_t) = 97.5\%. \tag{A.8}$$

Substituting for $L(t,T)$ and dropping the conditioning for brevity we have

$$\begin{aligned}
&\mathbb{P}\,[V_T e^{-(T-t)r} - V_t \le x_{0.975}] = 97.5\%,\\
&\mathbb{P}\,[(S_T - f_0)e^{-(T-t)r} - (S_t e^{(T-t)r} - f_0)e^{-(T-t)r} \le x_{0.975}] = 97.5\%,\\
&\mathbb{P}\,[S_T e^{-(T-t)r} \le x_{0.975} + S_t] = 97.5\%,\\
&\mathbb{P}\,[\ln(S_T) \le \ln(\,x_{0.975} + S_t\,) + (T-t)r\,] = 97.5\%.
\end{aligned}$$

Standardising yields

$$\mathbb{P}\,[Z \le \frac{\ln(\,x_{0.975} + S_t\,) + (T-t)r - m(t,T)}{s(t,T)}\,] = 97.5\% \tag{A.9}$$

where $Z \sim N(0,1)$. So the 97.5% VaR is given by

$$x_{0.975} = \exp\,[\,z_{0.975}\,s(t,T) + m(t,T) - (T,t)r\,] - S_t. \tag{A.10}$$

The ES of the 1-year ahead loss process is given by

$$\begin{aligned}
\mathrm{ES}_t(L(t,T)) &= \mathbb{E}^{\mathbb{P}}[V_T\,e^{-(T-t)r} - V_t\,|\,V_T\,e^{-r(T-t)} - V_t \ge x_{0.975}\,],\\
&= \mathbb{E}^{\mathbb{P}}\,[S_T\,e^{-(T-t)r} - S_t\,|\,S_T\,e^{-(T-t)r} - S_t \ge x_{0.975}\,],\\
&= \mathbb{E}^{\mathbb{P}}\,[S_T\,e^{-(T-t)r}\,|\,S_T e^{-(T-t)r} - S_t \ge x_{0.975}\,] - S_t,\\
&= e^{-(T-t)r}\,\mathbb{E}^{\mathbb{P}}[S_T\,|\,S_T\,e^{-(T-t)r} \ge x_{0.975} + S_t\,] - S_t.
\end{aligned}$$

Letting y = $\ln S_T$ where $y \sim N(\,m(t,T), s^2(t,T)\,)$ and $m(t,T)$ and $s(t,T)$ are given by (2.15) and (2.16) we have

$$\begin{aligned}
\mathrm{ES}_t &= e^{-(T-t)r}\,\mathbb{E}^{\mathbb{P}}\,[e^y\,|\,y \ge \ln(x_{0.975} + S_t) + (T-t)r\,] - S_t,\\
&= \frac{1}{0.025}\,e^{-(T-t)r}\,\mathbb{E}^{\mathbb{P}}\,[e^y\,;\,\{y \ge \ln(x_{0.975} + S_t) + (T-t)r\}\,] - S_t.
\end{aligned}$$

Now let $a = \ln(VaR_t + S_t) + (T-t)r$ and set $u = \frac{y - m(t,T)}{s(t,T)}$. We also drop the explicit dependence of m and s on t, T

$$\mathrm{ES}_t = \frac{1}{0.025}\,e^{-(T-t)r}\,\mathbb{E}^{\mathbb{P}}\left[e^{su+m}\,;\,\{u \ge \frac{a-m}{s}\}\right] - S_t,$$

$$\text{ES}_t = \frac{1}{0.025}\, e^{-(T-t)r+m} \int_{\frac{a-m}{s}}^{\infty} e^{su} \frac{1}{\sqrt{2\pi}} e^{\frac{-u^2}{2}}\, du - S_t.$$

Completing the square yields

$$\text{ES}_t = \frac{1}{0.025}\, e^{-(T-t)r+m} \int_{\frac{a-m}{s}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{((u-s)^2-s^2)}{2}}\, du - S_t\,,$$

$$\text{ES}_t = \frac{1}{0.025}\, e^{-(T-t)r+m+\frac{1}{2}s^2} \int_{\frac{a-m}{s}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(u-s)^2}{2}}\, du - S_t.$$

Letting v = u - s

$$\text{ES}_t = \frac{1}{0.025}\, e^{-(T-t)r+m+\frac{1}{2}s^2} \int_{\frac{a-m-s^2}{s}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\, dv - S_t.$$

Finally,

$$\text{ES}_t = \frac{1}{0.025}\, e^{-(T-t)r+m+\frac{1}{2}s^2} \left[1 - N\left(\frac{a-m-s^2}{s}\right)\right] - S_t. \tag{A.11}$$

A.3 Analytic VaR, ES for option

The year-ahead 97.5% VaR at time t is obtained using the analytic Black-Scholes price (3.7). **All probabilities are conditioned to time** t. We solve

$$\begin{aligned}
&\mathbb{P}[V(T)e^{-(T-t)r} - V(t) \geq x_{0.975}] = 2.5\%,\\
&\mathbb{P}[\,(S_T - K)^+\, e^{-(T-t)r} - V_{BS}(t) \geq x_{0.975}] = 2.5\%,\\
&\mathbb{P}[\,(S_T - K)^+ \,\geq\, e^{(T-t)r}\,(x_{0.975} + V_{BS}(t))\,] = 2.5\%,\\
&\mathbb{P}[\,S_T - K \,\geq\, e^{(T-t)r}\,(x_{0.975} + V_{BS}(t))\,, S_T \,\geq\, K] = 2.5\%.
\end{aligned}$$

Note that a lower bound on the 97.5% VaR is $x_{0.975} = -V(t)$ when the terminal pay-off is zero so the 1-year ahead process yields a profit. In the usual case we will have $e^{(T-t)r}\,(x_{0.975} + V_{BS}(t)) > 0$. In this case

$$\{S_T - K \,\geq\, e^{(T-t)r}\,(x_{0.975} + V_{BS}(t))\} \subset \{S_T \,\geq\, K\}. \qquad (*)$$

In some cases ,when the option is far-out-the-money, the probability of getting a non-zero pay-off $V(T)$ may be less than 2.5% and in this case we set the VaR to is lower bound i.e $x_{0.975} = -V(t)$. Continuing and applying (*)

$$\begin{aligned}
&\mathbb{P}[\,S_T - K \,\geq\, e^{(T-t)r}\,(x_{0.975} + V_{BS}(t))\,, S_T \,\geq\, K] = 2.5\%,\\
&\mathbb{P}[\,S_T - K \,\geq\, e^{(T-t)r}\,(x_{0.975} + V_{BS}(t))] = 2.5\%.
\end{aligned}$$

Let $a = e^{(T-t)r}\,(x_{0.975} + V_{BS}(t)) + K$ so that we have

$$\mathbb{P}[S_T \,\geq\, a] = 2.5\%.$$

Now take logs and standardise

$$\mathbb{P}\left[Z \geq \frac{\ln a - m(t,T)}{s(t,T)}\right] = 2.5\%$$

where $m(t,T)$ and $s(t,T)$ are given by (2.15) and (2.16). Now set

$$\frac{\ln a - m}{s} = z_{0.975}.$$

Note that we drop the explicit dependence of m and s on t and T. Solving for $x_{0.975}$ in a one arrives at

$$x_{0.975} = \left[\exp\left(z_{0.975}\, s + m\right)\right) - K\,]\; e^{-(t-t)r} - V_{BS}(t). \tag{A.12}$$

The ES of the year-ahead loss is given by the $\mathbb{P}$-expectation

$$\text{ES}_t = \frac{1}{0.025}\,\mathbb{E}\left[V_{BS}(T)e^{-(T-t)r} - V_{BS}(t)\,;\,\{V_{BS}(T)e^{-(T-t)r} - V_{BS}(t) \geq x_{0.975}\}\right],$$

$$\text{ES}_t = \frac{1}{0.025}\,e^{-(T-t)r}\,\mathbb{E}\left[(S_T-K)^+\,;\,\{\,(S_T-K)^+ \geq e^{(T-t)r}\left(V_{BS}(t) + x_{0.975}\right)\}\right] - V_{BS}(t).$$

Using the relation in (*) we can write

$$\text{ES}_t = \frac{1}{0.025}\,e^{-(T-t)r}\,\mathbb{E}\left[S_T - K\,;\,\{\,S_T - K \geq e^{(T-t)r}\left(V_{BS}(t) + x_{0.975}\right)\}\right] - V_{BS}(t),$$

$$\text{ES}_t = \frac{1}{0.025}\,e^{-(T-t)r}\,\left(\mathbb{E}\left[S_T\,;\,\{\,S_T \geq a\}\right] - K\;\mathbb{P}\left[S_T \geq a\right]\right) - V_{BS}(t).$$

Following the same steps as in Appendix A.2 for the ES of the forward we can solve for the first term in the brackets (the expectation) and the second term (the probability) can be transformed into a standard normal cdf by using the fact that the log stock is normally distributed. We arrive at

$$\text{ES}_t = \frac{1}{0.025}\,e^{-(T-t)r}\,\left(e^{m+\frac{1}{2}s^2}\left[1 - N\left(\frac{a-m-s^2}{s}\right)\right] - KN\left(\frac{m-\ln a}{s}\right)\right) - V_{BS}(t). \tag{A.13}$$