



Finite energy chiral sum rules in QCD [☆]

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Abstract

The saturation of QCD chiral sum rules of the Weinberg-type is analyzed using ALEPH and OPAL experimental data on the difference between vector and axial-vector correlators (V–A). The sum rules exhibit poor saturation up to current energies below the tau-lepton mass. A remarkable improvement is achieved by introducing integral kernels that vanish at the upper limit of integration. The method is used to determine the value of the finite remainder of the V–A correlator, and its first derivative, at zero momentum: $\bar{\Pi}(0) = -4\bar{L}_{10} = 0.0257 \pm 0.0003$, and $\bar{\Pi}'(0) = 0.065 \pm 0.007 \text{ GeV}^{-2}$. The dimension $d = 6$ and $d = 8$ vacuum condensates in the operator product expansion are also determined: $\langle \mathcal{O}_6 \rangle = -(0.004 \pm 0.001) \text{ GeV}^6$, and $\langle \mathcal{O}_8 \rangle = -(0.001 \pm 0.006) \text{ GeV}^8$.

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Since the pioneering work of Shifman, Vainshtein and Zakharov [1], a few thousand papers have been published on applications of the QCD sum rule method in all corners of low energy hadronic physics. Unavoidably, results from different collaborations were not always consistent [2]. The main reason for these inconsistencies was frequently the impossibility of estimating reliably the errors in the method. With the advent of precise measurements of the vector (V) and axial-vector (A) spectral functions, obtained from tau-lepton decay [3,4], an opportunity was opened to check the precision of the QCD sum rules in the light-quark sector of QCD. In this Letter we would like to present a critical and conservative appraisal of chiral

sum rules of the Weinberg type [5], as they are confronted with experimental data for the spectral functions. This kind of sum rules involve the difference between the vector and the axial-vector correlators V–A, which vanishes identically to all orders in perturbative QCD in the chiral limit. In fact, neglecting the light quark masses, the V–A two-point function vanishes like $1/q^6$ in the space-like region, where the scale $\mathcal{O}(300 \text{ MeV})$ is set by the quark and gluon condensates. In the time-like region the chiral spectral function $\rho_{V-A}(q^2)$ should also vanish for large $Q^2 \equiv -q^2$, but judging from the ALEPH data [3], the asymptotic regime of local duality may not have been reached in τ -decay. Under less stringent assumptions one expects global duality to hold in the time like region; in particular, this should be the case for the Weinberg-type sum rules. Surprisingly, these sum rules also appear to be poorly convergent. A possible source of duality vi-

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olation could be some non-perturbative contribution to the correlator (e.g., due to instantons) which falls off exponentially in the space-like region but oscillates in the time-like region. If the duality violations were due to this source, then there would be a simple recipe (introduced 30 years ago [6]) to improve convergence.

In a previous publication [7] we studied some QCD chiral sum rules of the Weinberg type, and their saturation by the ALEPH data. In particular, we showed that a remarkable improvement of this saturation can be achieved by introducing a polynomial integration kernel which vanishes at the upper limit of integration. However, no detailed quantitative error analysis was performed in [7]. In this Letter we reexamine the saturation of several QCD chiral sum rules using the ALEPH [3], as well as the OPAL data [4], and paying particular attention to the error analysis. We obtain an updated determination of \bar{L}_{10} , the scale-independent part of the coupling constant of the relevant operator in the $\mathcal{O}(p^4)$ counterterms in the Lagrangian of chiral perturbation theory [8]. This quantity is related to the finite remainder of the V–A correlator at zero momentum. We also determine the finite remainder of the first derivative of the V–A correlator at zero momentum, which is related to the $\mathcal{O}(p^6)$ counterterms. Finally, we introduce combinations of QCD chiral sum rules which allow for a determination of the V–A dimension $d = 6$ and $d = 8$ vacuum condensates. The former can be extracted with reasonable precision, while the latter is affected by much larger uncertainties.

We begin by defining the vector and axial-vector current correlators

$$\begin{aligned} \Pi_{\mu\nu}^{\text{VV}}(q^2) &= i \int d^4x e^{iqx} \langle 0 | T (V_\mu(x) V_\nu^\dagger(0)) | 0 \rangle \\ &= (-g_{\mu\nu} q^2 + q_\mu q_\nu) \Pi_V(q^2), \end{aligned} \quad (1)$$

$$\begin{aligned} \Pi_{\mu\nu}^{\text{AA}}(q^2) &= i \int d^4x e^{iqx} \langle 0 | T (A_\mu(x) A_\nu^\dagger(0)) | 0 \rangle \\ &= (-g_{\mu\nu} q^2 + q_\mu q_\nu) \Pi_A(q^2) \\ &\quad - q_\mu q_\nu \Pi_0(q^2), \end{aligned} \quad (2)$$

where $V_\mu(x) = :\bar{q}(x)\gamma_\mu q(x):$, $A_\mu(x) = :\bar{q}(x)\gamma_\mu \times \gamma_5 q(x):$, and $q = (u, d)$. Here we shall concentrate on the chiral correlator $\Pi_{\text{V-A}} \equiv \Pi_V - \Pi_A$. This correlator vanishes identically in the chiral limit ($m_q = 0$), to all orders in QCD perturbation theory. Renormalon ambiguities are thus avoided. Non-perturbative contributions due to vacuum condensates contribute to

this two-point function starting with dimension $d = 6$ and involving the four-quark condensate. The Operator Product Expansion (OPE) of the chiral correlator can be written as

$$\begin{aligned} \Pi(Q^2)|_{\text{V-A}} &= \sum_{N=1}^{\infty} \frac{1}{Q^{2N+4}} C_{2N+4}(Q^2, \mu^2) \\ &\quad \times \langle \mathcal{O}_{2N+4}(\mu^2) \rangle, \end{aligned} \quad (3)$$

with $Q^2 \equiv -q^2$. It is valid away from the positive real axis for complex q^2 , and $|q^2|$ large. Radiative corrections to the $d = 6$ contribution are known [9]. They depend on the regularization scheme, implying that the value of the condensate itself is a scheme-dependent quantity. Explicitly,

$$\begin{aligned} \Pi(Q^2)|_{\text{V-A}} &= -\frac{32\pi}{9} \frac{\alpha_s(\bar{q}q)^2}{Q^6} \\ &\quad \times \left\{ 1 + \frac{\alpha_s(Q^2)}{4\pi} \left[\frac{247}{12} + \ln\left(\frac{\mu^2}{Q^2}\right) \right] \right\} \\ &\quad + \mathcal{O}(1/Q^8), \end{aligned} \quad (4)$$

in the anti-commuting γ_5 scheme, and assuming vacuum saturation of the four-quark condensate. Radiative corrections for $d \geq 8$ are not known. To facilitate comparison with current conventions in the literature it will be convenient to absorb the Wilson coefficients, including radiative corrections, into the operators, and rewrite Eq. (3) as

$$\Pi(Q^2) = \sum_{N=1}^{\infty} \frac{1}{Q^{2N+4}} \langle \mathcal{O}_{2N+4} \rangle, \quad (5)$$

where we have dropped the subscript V–A for simplicity. We will be concerned with Finite Energy Sum Rules of the type

$$W(s_0) \equiv \int_0^{s_0} ds f(s) \rho(s), \quad (6)$$

where $f(s)$ is a weight function, and the hadronic spectral function $\rho(s) \equiv \rho_V(s) - \rho_A(s)$, with $\rho_{\text{V,A}}(s) = \frac{1}{\pi} \text{Im} \Pi_{\text{V,A}}(s)$ (pion pole excluded from $\rho_A(s)$). For instance, if $f(s) = s^N$ ($N = 0, 1, 2, \dots$),

then one obtains

$$\int_0^{s_0} ds s^N \rho(s) = f_\pi^2 \delta_{N0} + (-)^N \langle \mathcal{O}_{2N+2} \rangle$$

$$(N = 0, 1, 2, \dots), \quad (7)$$

where $f_\pi = 92.4 \pm 0.26$ MeV [10]. For $N = 0, 1$ Eq. (8) leads to the first two (finite energy) Weinberg sum rules, while for $N = 2, 3$ the sum rules project the $d = 6, 8$ vacuum condensates, respectively; notice that in the chiral limit $\langle \mathcal{O}_2 \rangle = \langle \mathcal{O}_4 \rangle = 0$. To first order in α_s , radiative corrections to the vacuum condensates do not induce mixing of condensates of different dimension in a given FESR [11]. We shall also consider the chiral correlator, and its first derivative, at zero momentum; the finite remainder of these being given by the sum rules

$$\bar{\Pi}(0) = \int_0^{s_0} \frac{ds}{s} \rho(s), \quad (8)$$

$$\bar{\Pi}'(0) = \int_0^{s_0} \frac{ds}{s^2} \rho(s), \quad (9)$$

where $\rho(s)$ does not contain the pion pole. Eq. (8) is the Das–Mathur–Okubo (finite energy) sum rule [5]. The finite remainder $\bar{\Pi}(0) = -4\bar{L}_{10}$, where \bar{L}_{10} is a counter term of the $\mathcal{O}(p^4)$ Lagrangian of chiral perturbation theory, can be expressed as

$$\bar{\Pi}(0) = -4\bar{L}_{10} = \left[\frac{1}{3} f_\pi^2 \langle r_\pi^2 \rangle - F_A \right] = 0.026 \pm 0.001, \quad (10)$$

where $\langle r_\pi^2 \rangle$ is the electromagnetic mean squared radius of the pion, $\langle r_\pi^2 \rangle = 0.439 \pm 0.008$ fm² [12], and F_A is the axial-vector coupling measured in radiative pion decay, $F_A = 0.0058 \pm 0.0008$ [10]. Similarly, $\bar{\Pi}'(0)$ is related to the $\mathcal{O}(p^6)$ counterterms.

As mentioned earlier, the saturation of the various chiral sum rules can be considerably improved by introducing an integration kernel that vanishes at the upper limit of integration ($s = s_0$). We have tested a variety of such kernels searching for optimal saturation. The following results have been obtained using the ALEPH data for $\rho(s)$, with the errors at each energy bin calculated from the error correlation matrix. Use of the OPAL data [4] data leads to similar

results, albeit with much larger error bands. Starting with the first Weinberg sum rule, Fig. 1 shows the left-hand side of Eq. (7) for $N = 0$ (curve (a)), together with the right-hand side, i.e., f_π^2 (straight line (c)), as well as the modified sum rule (curve (b))

$$W_1(s_0) = \int_0^{s_0} ds \left(1 - \frac{s}{s_0} \right) \rho(s). \quad (11)$$

On account of the second Weinberg sum rule, curves (a) and (b) should be identical; the improved saturation achieved with Eq. (11) being remarkable. Fig. 1 can be used to present our criterion to judge the reliability of a QCD sum rule. The sum rule must be presented explicitly as a function of the upper integration limit s_0 . If the left-hand side is a constant, then the spectral integral must also be approximately a constant, starting from 1 to 2 GeV² up to the maximum s_0 of the data. From Fig. 1 we would extract

$$f_\pi^2 = 0.008 \pm 0.004 \text{ GeV}^2, \quad (12)$$

for curve (a), and

$$f_\pi^2 = 0.0084 \pm 0.0004 \text{ GeV}^2, \quad (13)$$

for curve (b), to be compared with the experimental value $f_\pi^2|_{\text{EXP}} = 0.00854 \pm 0.00005$ GeV². Curve (a) demonstrates the fact that if the spectral integral is not a constant then the experimental errors are quite irrelevant in a test of duality. It is very dangerous to pick up a small stability region to obtain a prediction (here one could choose the region around 2 GeV²).

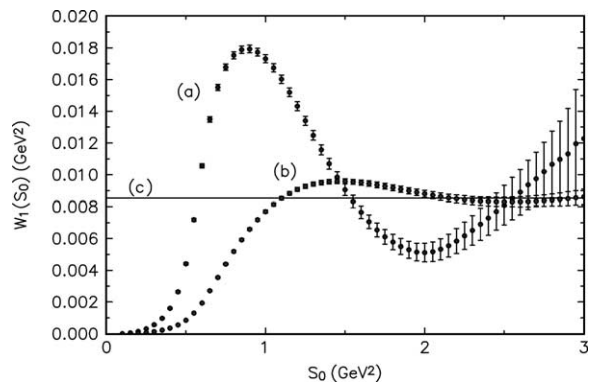


Fig. 1. Curve (a) is the standard first Weinberg sum rule, Eq. (7) with $N = 0$, curve (b) is the modified sum rule Eq. (11), and curve (c) is the experimental value of f_π^2 .

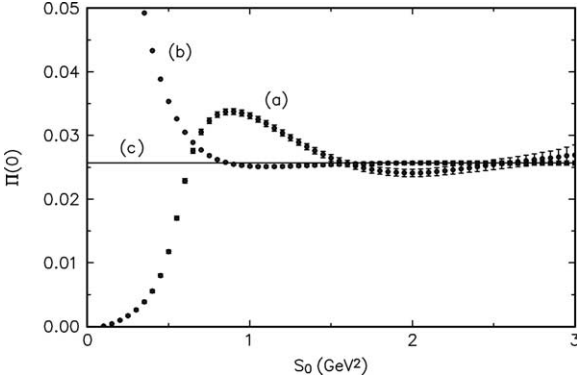


Fig. 2. The chiral correlator at zero momentum, $\bar{\Pi}(0)$, from the standard sum rule Eq. (8) (curve (a)), and from the modified sum rule Eq. (14) (curve (b)), the latter leading to the prediction Eq. (15) (curve (c)).

In Fig. 2 we show Eq. (8) (curve (a)) together with the modified sum rule (curve (b))

$$\bar{\Pi}(0) = 2 \frac{f_\pi^2}{s_0} + \int_0^{s_0} \frac{ds}{s} \left(1 - \frac{s}{s_0}\right)^2 \rho(s). \quad (14)$$

From the optimized sum rule (14) we obtain the value (straight line (c))

$$\bar{\Pi}(0) = -4\bar{L}_{10} = 0.0257 \pm 0.0003, \quad (15)$$

which is considerably more accurate than the leading-order chiral perturbation theory result, Eq. (10). The agreement between Eqs. (10) and (15) may be an indication that higher-order chiral corrections to the Das–Mathur–Okubo sum rule are indeed very small. Fig. 3 shows Eq. (9) (curve (a)) together with the optimized sum rule (curve (b))

$$\bar{\Pi}'(0) = \frac{3}{s_0} \bar{\Pi}(0) - 3 \frac{f_\pi^2}{s_0^2} + \int_0^{s_0} \frac{ds}{s^2} \left(1 - \frac{s}{s_0}\right)^3 \rho(s), \quad (16)$$

the latter giving (curve (c))

$$\bar{\Pi}'(0) = 0.065 \pm 0.001 \text{ GeV}^{-2}. \quad (17)$$

We turn now to the determination of the $d = 6$ and $d = 8$ vacuum condensates. In Fig. 4 we show $\langle \mathcal{O}_6 \rangle$ as obtained from Eq. (7) with $N = 2$ (curve (a)), together with the result from the improved sum rule (curve (b))

$$\langle \mathcal{O}_6 \rangle = -f_\pi^2 s_0^2 + s_0^2 \int_0^{s_0} ds \left(1 - \frac{s}{s_0}\right)^2 \rho(s), \quad (18)$$

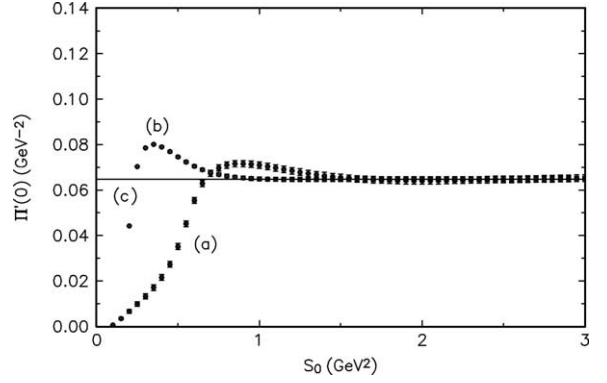


Fig. 3. The first derivative of the chiral correlator at zero momentum, $\bar{\Pi}'(0)$, from the standard sum rule Eq. (9) (curve (a)), and from the modified sum rule Eq. (16) (curve (b)), the latter leading to the prediction Eq. (17) (curve (c)).

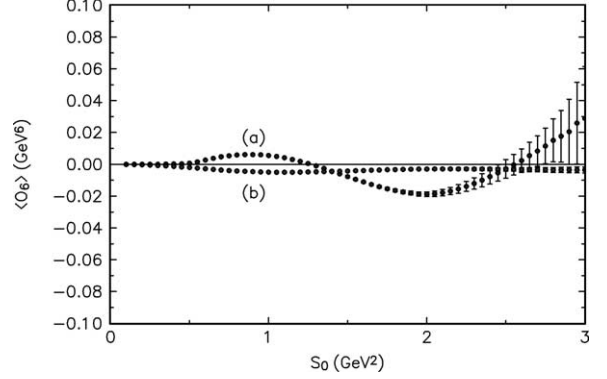


Fig. 4. The dimension-six vacuum condensate from the standard sum rule, Eq. (7) with $N = 2$ (curve (a)), and from the modified sum rule Eq. (18) (curve (b)).

which gives

$$\langle \mathcal{O}_6 \rangle = -(0.004 \pm 0.001) \text{ GeV}^6. \quad (19)$$

This result can be compared with the vacuum saturation expression

$$\langle \mathcal{O}_6 \rangle|_{\text{VS}} = -\frac{32}{9} \pi \bar{\alpha}_s |\langle \bar{q}q \rangle|^2 \simeq -1.1 \times 10^{-3} \text{ GeV}^6, \quad (20)$$

to leading order in α_s , and where we used $\langle \bar{q}q \rangle = -0.014 \text{ GeV}^3$, and $\bar{\alpha}_s = 0.5$, at a scale of 1 GeV. Radiative corrections increase this estimate by a factor of two. The result Eq. (19) confirms pioneer determinations from e^+e^- , as well as tau-lepton decay data [13, 14] indicating that the vacuum saturation approxima-

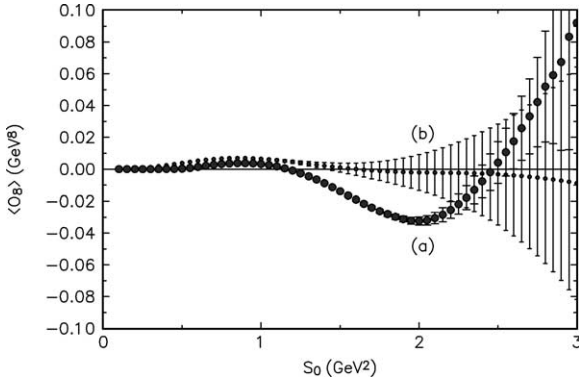


Fig. 5. The dimension-eight vacuum condensate from the standard sum rule, Eq. (7) with $N = 3$ (curve (a)), and from the modified sum rule Eq. (21) (curve (b)).

tion underestimates the $d = 6$ condensate roughly by a factor of 2–3.

Finally, for $\langle \mathcal{O}_8 \rangle$ Fig. 5 (curve (a)) shows the result from Eq. (7) with $N = 3$, together with the improved determination from the sum rule (curve (b))

$$\langle \mathcal{O}_8 \rangle = 8s_0^3 f_\pi^2 - 3s_0^4 \bar{\Pi}(0) + s_0^3 \int_0^{s_0} \frac{ds}{s} \left(1 - \frac{s}{s_0}\right)^3 (s + 3s_0) \rho(s), \quad (21)$$

which gives

$$\langle \mathcal{O}_8 \rangle = -(0.001 \pm 0.006) \text{ GeV}^8, \quad (22)$$

in the region where the condensate is approximately constant ($s_0 \simeq 1.75\text{--}2.5 \text{ GeV}^2$), and assuming, optimistically, that the stability region has been reached beyond $s_0 \simeq 2.5 \text{ GeV}^2$. It should be clear from Fig. 5 that no meaningful determination of $\langle \mathcal{O}_8 \rangle$ is possible using the standard FESR, Eq. (7). As expected, with increasing dimensionality, i.e., higher powers of s in the dispersive integrals, the accuracy of the determination of the vacuum condensates deteriorates considerably. It should be noticed that the results (19) and (22) do not rely on the vacuum saturation approximation. They also include all radiative corrections, and are correct to first order in α_s . At order α_s^2 and beyond, there is no longer decoupling of condensates of different dimensionality in a given FESR [11]. However, one expects these higher-order radiative corrections to the Wilson coefficients in the OPE to be small. There seems to be general agreement in the literature

on the size of the $d = 6$ condensate, but there exists a number of inconsistent QCD sum rule determinations of the value of the $d = 8$ condensate. The results range from $\langle \mathcal{O}_8 \rangle = -(3.5 \pm 2.0) \times 10^{-3} \text{ GeV}^8$ [15] to $\langle \mathcal{O}_8 \rangle = (4.4 \pm 1.2) \times 10^{-3} \text{ GeV}^8$ [16]. Our result is consistent, within the large errors, with a recent determination [17] (this reference contains a detailed comparative study of the literature).

The poor convergence of ordinary QCD chiral sum rules is rather intriguing, as one would have expected good saturation at relatively low energies, given the very rapid fall-off of the chiral V–A correlator (see Eq. (4)). However, extrapolating the chiral correlator from the space-like to the time-like region can produce strong changes close to the real-axis. In fact, violations of local duality at the 100% level have been shown to be possible using realistic models of the heavy quark chiral correlator [18]. The remarkable improved saturation achieved by introducing weight functions that vanish on the real axis at $s = s_0$ could be taken as an indication that although perturbative QCD works well in the space-like region, this may not be the case in the time-like region, or near the cut, at least at energies below $s_0 \simeq 3.5 \text{ GeV}^2$. Finally, by using the chiral V–A correlator we have been able to extract the value of the $d = 6$ vacuum condensate with reasonable accuracy; for the $d = 8$ condensate the result is affected by a large uncertainty. In contrast, were one to attempt a determination from the vector correlator, and separately from the axial-vector one, the results would be quite inconclusive. This is due to the very large current value of Λ_{QCD} ($\Lambda_{\text{QCD}} \simeq 400 \text{ MeV}$) which makes the perturbative QCD term in the OPE so big that it overwhelms the power corrections. We estimate that if $\Lambda_{\text{QCD}} \gtrsim 330 \text{ MeV}$ then the FESR, and even Laplace transform sum rules, will be unable to provide a conclusive determination of the vacuum condensates. Earlier standard extractions of these condensates from electron–positron annihilation [13] and tau-lepton decay [14] relied on past values $\Lambda_{\text{QCD}} \simeq 100\text{--}200 \text{ MeV}$. With these values of Λ_{QCD} the perturbative QCD term in the OPE is dominant but not overwhelming, and the power corrections can be clearly discerned.

We would like to conclude with a general comment. Mathematically, the extraction of QCD parameters from experiment via sum rules constitutes a so-called ill posed inverse problem (analytic continuation

of an imprecisely known function). Small changes in the input data lead to large changes in the output. The problem is stabilized by extracting only a small number of parameters. Given the present accuracy of the τ -decay data, we conclude from our analysis that only the condensate $\langle \mathcal{O}_6 \rangle$ can be extracted with some degree of confidence, and only a rough idea of the order of magnitude of $\langle \mathcal{O}_8 \rangle$ can be obtained. This situation cannot be remedied by mathematical tricks like employing Laplace or Gaussian integration kernels. Only with forthcoming more accurate data, can one expect to extract higher-dimensional condensates.

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