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DEPARTMENT OF MATHEMATICS

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On Coincidence of Algebras of Functions with C(X)

by

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A thesis prepared under the supervision of Dr. S. de O. Salbany, in fulfilment of the requirements of the degree of Master of Science in Mathematics.

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CONTENTS

INTRODUCTION

The study of approximation of continuous functions has always evoked great interest. The classical result in the theory of approximation is Weierstrass's theorem on the uniform approximation of continuous functions on a bounded closed interval by polynomials. This result was later extended to compact Hausdorff topological spaces by M. H. Stone.

Approximation theorems were later also established for almost compact spaces by Hewitt, and for Lindelöf and almost Lindelöf spaces by Hager and Frolik. All the results above were originally proved for real-valued functions. Indeed, they do not generally extend to complex-valued functions; one has to consider different types of algebras and employ different techniques. The theory of complex measures plays an important part in this case.

In this thesis we discuss approximation theorems of the Stone-Weierstrass type. We found it convenient to divide the thesis into two parts. Part I concerns real-valued functions and Part II dicusses the complex case. At the end of each part we have included short bibliographical notes.

Summary of contents

Part I - In Section 1 we introduce preliminary results and definitions. In Section 2 we discuss approximation theorems on compact spaces and prove a converse to the Stone-Weierstrass theorem. We restrict ourselves to the topology of uniform convergence on $C_R(X)$.

In Section 3 we introduce the concept of a S-topology and give examples. A topology t on $C_R(X)$ is a S-topology for a class S of spaces X if the following condition holds: X belongs to the class S if and only if every subalgebra of $C_R(X)$ which contains the constants and separates the points in X is t-dense in $C_R(X)$.

In Section 4 we investigate S-topologies for algebras which separate points and closed sets. The major portion of the work is devoted to developing the machinery necessary to prove the following result of Hager and Johnson: the only Algebra on X is $C_R(X)$ itself if and only if X is either Lindelöf or almost compact, where 'Algebra' denotes a subalgebra of $C_R(X)$ which is uniformly closed, separates points and closed sets, contains the constants, and is closed under inversion. It is also shown that the topology of pointwise convergence of sequences in $C_R(X)$ is a S-topology for the class of almost Lindelöf spaces.

Part II - In Section 1 we list preliminary results and definitions, and outline our aims for Part II.

In Section 2 we discuss restriction algebras: if the restrictions of an algebra A to certain compact subsets of X satisfy suitable conditions, then $A = C(X)$.

In Section 3 we investigate the approximation properties of normal and approximately normal algebras. These are algebras which, besides separating points or points and closed sets, distinguish between disjoint closed sets.

In Section 4 the notion of ε-normality is introduced. We establish the following result: if A is a uniformly closed locally bounded ε-normal subalgebra of C(X), then $A = C(X)$.

Throughout Part II we restrict ourselves to compact spaces.

Notation

Throughout the thesis all topological spaces are assumed to be completely regular Hausdorff.

$C_R(X)$ (resp. $C_R^*(X)$) denotes the space of all real-valued continuous (resp. bounded continuous) functions on the space X.

C(X) denotes the space of all complex-valued continuous functions on X.

If A is an algebra of functions, then clA denotes the uniform closure of A.

For a Banach space B, $[B]^*$ denotes the space of all bounded linear functionals on B.

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PART I

ALGEBRAS OF REAL-VALUED CONTINUOUS FUNCTIONS

1.1 Preliminaries.

Let X be a completely regular Hausdorff space. Recall ([11]) that each x in X generates a fixed maximal ideal M_x in $C_R(X)$, viz. $M_x = \{f \in C_R(X) : f(x) = 0\}$. It can be shown that the quotient algebra $A/_{M_x}$ is isomorphic to the real field $\mathbb{R}$. Furthermore, X is compact if and only if the only maximal ideals are the fixed maximal ideals. If X is not compact, then the maximal ideal M is called real if $A/_M$ is isomorphic to $\mathbb{R}$. Thus every fixed maximal ideal is real.

If $C_R(X)$ contains unbounded functions, then it is possible to find maximal ideals M such that $A/_M$ is isomorphic to a field properly containing $\mathbb{R}$. Such ideals are called hyper-real. For example, if f is an unbounded function in $C_R(X)$, let $M = (f)$, the ideal generated by f. Then $A/_M$ is a hyper-real field.

We also note that every completely regular Hausdorff space X can be homeomorphically embedded in a compact space Y. Clearly $C_R^*(Y) \subset C_R^*(X)$. We denote by βX the compact space Y which contains X densely and such that every $f \in C_R^*(X)$ can be uniquely extended to a continuous function on Y (see [11], Ch. 6). Thus $C_R(\beta X) = C_R^*(X)$, and βX is known as the Stone-Cech compactification of X. If $f \in C_R^*(X)$, then f^β denotes its extension to βX.

A space X is realcompact if every real maximal ideal M in $C_R(X)$ is fixed, i.e. $M = M_x$ for some $x \in X$. As above, we denote by νX the realcompact space Y such that every function f in $C_R(X)$ can be uniquely extended to a function f^ν on Y, where $X \subset Y \subset \beta X$. We have that $C_R(X) = C_R(\nu X)$ and νX is known as the Hewitt realcompactification of the space X. νX is the smallest realcompact space between X and βX ([11], Ch.8).

We shall generally be concerned with subalgebras A of $C_R(X)$ or $C_R^*(X)$ where the operations of addition, multiplication and scalar multiplication are defined pointwise. We will primarily investigate under what conditions, either on the space X or on the algebra A, A coincides with $C_R(X)$ or $C_R^*(X)$.

1.1.1 Definition. By a function algebra A on X we mean a subalgebra A of $C_R(X)$ or $C_R^*(X)$ satisfying the following conditions:

(1) A separates the points in X, i.e. given $x, y \in X$ with $x \neq y$, there exists a function $f \in A$ such that $f(x) \neq f(y)$

(2) A contains the constant functions

(3) A is uniformly closed, i.e. A is closed with respect to the supremum norm.

We say that a subalgebra A strongly separates the points in X, if for all $x, y \in X$ with $x \neq y$, and all $\alpha, \beta \in \mathbb{R}$ there exists a function $f \in A$ such $f(x) = \alpha$ and $f(y) = \beta$.

It follows immediately that a sufficient condition for strong separability is that the algebra contains constants and separates points.

1.2 Stone-Weierstrass theorems.

We briefly mention the well-known Stone-Weierstrass theorems for algebras of real- and complex-valued functions and also discuss a converse to one of the results (see 1.2.4). It is this characterization that will form the basis of our discussions in Part I.

1.2.1 Theorem. If A is a function algebra on a compact Hausdorff space X, then $A = C_R(X)$.

Proof. See e.g. [25], page 36

This theorem can be formulated slightly differently: if a subalgebra A of $C_R(X)$ satisfies conditions (1) and (2) above, then A is uniformly dense in $C_R(X)$.

Using this formulation, condition (2) is no longer a necessary one, but Nel [32] proved the following result.

1.2.2 Theorem. For a subalgebra A of $C_R(X)$ to be uniformly dense in $C_R(X)$, X compact, it is necessary and sufficient that

(1) A separates points

(2) A contains a function u bounded away from zero.

Proof. Observe that, if X is compact, then every function u which does not vanish on X is bounded away from zero. In the proof of theorem 1.2.1 condition (1) of 1.1.1 is used essentially only once, namely to show that A separates the points in X in the strong sense. But if (1) and (2) hold, there is a $g \in A$ with $g(x_1) \neq g(x_2)$ for every pair of distinct points $x_1, x_2 \in X$, and u in A with $u \neq 0$ on X. Choose $\alpha, \beta \in \mathbb{R}$ and let

$$f(x) = \frac{\alpha u(x)}{u(x_1)} + \left[\frac{\beta u(x)}{u(x_2)} - \frac{\alpha u(x)}{u(x_1)} \right] \cdot \frac{g(x) - g(x_1)}{g(x_2) - g(x_1)} \quad (x \in X)$$

Then $f \in A$, $f(x_1) = \alpha$, and $f(x_2) = \beta$. Hence A strongly separates the points in X.

Clearly conditions (1) and (2) are necessary.

If we consider complex-valued functions then Theorem 1.2.1 no longer holds. However, we have

1.2.3 Theorem. Let A be a function algebra on a compact space X, where $A \subset C(X)$. If A is closed under complex conjugation, then $A = C(X)$.

Proof. See [25], page 36

If we restrict ourselves to bounded functions, then a converse to Theorem 1.2.1 can be given.

1.2.4 Theorem. If $C_R^*(X)$ is the only function algebra on X, then X is compact.

Proof. Suppose X is not compact. Let p be any fixed point of $\beta X - X$ and q any fixed point of X.
Let $A = \{f \in C_R^*(X) : f(q) = f^\beta(p)\}$. Then A is clearly a uniformly closed subalgebra of $C_R^*(X)$ which contains the constants. Since $C_R(\beta X)$ separates the points of βX, there exists a function $g \in C_R(\beta X)$ for which $g(q) \neq g(p)$. Hence $A \neq C_R^*(X)$. Finally, given any two distinct points $x, y \in X$, there is a function $h \in C_R^*(X)$ such that $h(x) = 0$ and $h(y) = h(q) = h^\beta(p) = 1$ (this follows by applying Urysohn's lemma to βX). Now h belongs to A and thus we have shown that A is a proper function algebra on X. This completes the proof.

1.3 The notion of S-topology and its application to algebras which separate points.

Let S denote a class of topological spaces. The above characterization for compact spaces leads us to the notion of S-topologies on function spaces, in the sense that appropriate conditions on subalgebras of such spaces characterize the space X as a member of the class S. Formally, we have

1.3.1 Definition. A topology t on $C^*_R(X)$ or $C_R(X)$ is called a S-topology if the following condition holds:

> X belongs to the class S if and only if every 'separating' subalgebra A of $C^*_R(X)$ (or $C_R(X)$) is t-dense in $C^*_R(X)$ (or $C_R(X)$).

In this section 'separating' subalgebras will mean subalgebras which strongly separate the points in X.

Remark. For the class C of compact spaces, the C-topology is the SW-topology introduced by Meyer in [28].

Examples of C-topologies.

(1) From 1.2.2 and 1.2.4 it is immediate that the topology of uniform convergence (the u-topology) is a C-topology

(2) p^β (resp. p) : the topology on $C^*_R(X)$ of pointwise convergence on βX (resp. X)

(3) m : the topology on $C_R(X)$ which has as its neighbourhood base at 0 sets of the form

$$\{f \in C_R(X): |f(x)| \leq u(x) \text{ for all } x \in X\}$$

where u is a positive unit of $C_R(X)$

(4) u_m : This denotes the topology on $C_R^*(X)$ of uniform convergence on X of monotone sequences in $C_R^*(X)$

It is clear that the following order relations hold:

(1) $p^\nu \leqslant p^\beta \leqslant u \leqslant u_m$ and

(2) $u \leqslant m$

Furthermore, $u = m$ if and only if X is <u>pseudocompact</u>, i.e. if and only if $C_R(X) = C_R^*(X)$ ([11],2N). We also observe that the 'p^β-topology' is meaningful as a topology on $C_R^*(X)$ since every function in $C_R^*(X)$ can be extended to a unique function on βX.

<u>1.3.2 Proposition.</u> The following are C-topologies:

(1) p^β on $C_R^*(X)$

(2) m on $C_R(X)$

(3) u_m on $C_R^*(X)$.

<u>Proof.</u> (1) If X is compact then $p^\beta = p$, and the result follows from Theorem 1.2.2, since $p\text{-cl}\, A = u\text{-cl}\, A = C_R(X)$.

If X is not compact, we can choose $p \in \beta X - X$. Let $A = \{f \in C_R^*(X) : f^\beta(p) = 0\}$. Then A is a proper subalgebra of $C_R^*(X)$ which strongly separates the points in X. We now show that A is p^β-closed. Let $f \in C_R^*(X)$ and assume that every p^β-neighbourhood of f meets A. We show that f is in A. By assumption,

$$U(f,\{p\},\tfrac{1}{n}) = \{g \in C_R^*(X) : |g^\beta(p) - f^\beta(p)| < \tfrac{1}{n}\}$$

meets A for all n, and for each n there exists a $g_n \in A$ such that $|g_n^\beta(p) - f^\beta(p)| < \frac{1}{n}$. But $g_n^\beta(p) = 0$ for all n, and hence $|f^\beta(p)| < \frac{1}{n}$, all n. Therefore $f^\beta(p) = 0$ and hence f belongs to A.

(2) If X is compact, $C_R^*(X) = C_R(X)$ and hence $u = m$. Result now follows by 1.2.2.

Conversely, if X is not compact, the result is immediate since $u \leq m$ and u is a C-topology.

(3) If X is not compact, the result is trivial, since $u \leq u_m$ and u is a C-topology.

Suppose now that X is compact. If A is a strongly separating subalgebra of $C_R^*(X)$, then u_m-cl A is a sublattice of $C_R^*(X)$ (by 1.3.3(1) below). Hence, by part (2) of 1.3.3, u_m-cl A is u-closed. The conclusion now follows by the usual Stone-Weierstrass theorem.

<u>1.3.3 Proposition</u>. (1) If A is a subalgebra of $C_R^*(X)$, then u_m-cl A is a sublattice of $C_R^*(X)$.

(2) If X is compact, A a sublattice of $C_R^*(X)$, and t any topology on $C_R^*(X)$ such that $p \leq t \leq u_m$, then t-cl A = u_m-cl A. Here closures are understood to be in $C_R^*(X)$.

<u>Proof</u>. For completeness we give the proof of (2). The proof of (1) is technical and we refer the reader to the paper by Meyer ([28], Prop. 3.7).

To prove (2), it suffices to show that p-cl A $\subset$ u_m-cl A (clearly u_m-cl A $\subset$ p-cl A since $p \leq u_m$). Since X is compact, p-cl A = u-cl A. Assume f is in u-cl A. Then there exists a sequence $\{g_n\}$ in A such that $\{g_n\}$ converges uniformly to f on X. Put $g_{nk} = g_n \vee \dots \vee g_{n+k}$. Then $g_{nk} \in A$ for all k (A is a lattice). Clearly $\{g_{nk}\}_k$ is a uniformly bounded monotone sequence which is uniformly Cauchy. Hence there exists, for each n, $h_n \in C_R^*(X)$ such that h_n is the uniform limit of $\{g_{nk}\}_k$. Thus $h_n \in u_m$-cl A. But $\{h_n\}$ converges uniformly and monotonically to f. Hence $f \in u_m$-cl A. This completes the proof.

Further examples of S-topologies.

We will now show that S-topologies exist where S denotes either the class of realcompact spaces or the class of pseudocompact spaces.

Let p^{ν} (resp. p') denote the topology on $C_R^*(X)$ of pointwise convergence on νX (resp. $X \cup (\beta X - \nu X)$). Note that p^{ν} is also meaningful as a topology on $C_R(X)$ since every function in $C_R(X)$ can be extended to a function on νX.

We will now prove results similar to those in 1.3.2.

1.3.4 Theorem. Let S denote the class of realcompact spaces. Then p^{ν} is a S-topology on $C_R^*(X)$ and on $C_R(X)$.

Proof. If X is realcompact, $\nu X = X$ and hence $p^{\nu} = p$. We show that every strongly separating subalgebra A of $C_R(X)$ is p-dense in $C_R(X)$. The same argument holds also for subalgebras of $C_R^*(X)$. Choose f in $C_R(X)$. Let

$$U = U(f,F,\varepsilon) = \{g \in C_R(X) : |g(x) - f(x)| < \varepsilon;\ \forall\, x \in F\}$$

be a basic p-neighbourhood of f, where F is a finite subset of X. Suppose $F = \{x_1, \ldots, x_n\}$. For $1 \leq i \leq n$ construct $g_i \in A$ such that $g_i(x_i) = f(x_i)$ and $g_i(x_j) = 0$ for $i \neq j$. (The construction of each g_i requires n applications of the strong separation property and the fact that A is closed under multiplication.) Let $g = g_1 + \ldots + g_n$. Then $g \in A$, and since $g|_F = f|_F$, g also belongs to U. Hence A is p-dense in $C_R(X)$.

The converse is proved along the lines of the proof of 1.3.2(1).

Remark. We have in fact shown that every strongly separating subalgebra A of $C_R(X)$, X a completely regular space, is p-dense in $C_R(X)$.

1.3.5 Theorem. Now let S denote the class of pseudocompact spaces. Then p' is a S-topology on $C_R^*(X)$.

Proof. Analogous to proof of 1.3.4 above.

1.4 S-topologies for algebras which separate points and closed sets.

In discussing S-topologies above we restricted ourselves to algebras which strongly separate the points in X. In this section we consider algebras which separate points and closed sets, i.e. given any closed set $F \subset X$ and any point $x \notin F$, there is a function f in the algebra such that $f(x) = 1$ and $f(F) = 0$.

1.4.1 Definition. A topological space X is almost compact if βX can be obtained from X by adjoining at most one point, i.e. $\text{card}(\beta X - X) \leq 1$.

A space X is said to be Lindelöf if every open cover admits a countable subcover.

1.4.2 Proposition. (1) A Lindelöf space is realcompact.

(2) An almost compact space is pseudocompact.

Proof. (1) See [11], Theorem 8.2

$f(Z) = 0$. But then $f^{\beta}(q) = 1$ and $f^{\beta}(p) = 0$ since $p \in cl_{\beta X} Z$. Hence A^{β} separates the points in βX. Since u is bounded away from zero, u^{β} does not vanish on βX and $u^{\beta} \in A$.

<u>Necessity</u>: Suppose $\beta X-X$ contains two distinct points p_1 and p_2. Let $A = \{f \in C_R^*(X) : f^{\beta}(p_1) = f^{\beta}(p_2)\}$. Then A contains the constants and separates points and closed sets. To see this, let $q \in X$ and $F \subset X$ be such that $q \notin F$. Then $q \notin cl_{\beta X} F$. Hence there exists a function $g \in C_R(\beta X)$ with $g(q) = 1$ and $g(F) = g(p_1) = g(p_2) = 0$. Then $g|_X$ belongs to A and separates q and F.
But A is not uniformly dense in $C_R^*(X)$. For let $h \in C_R^*(X)$ be such that $h^{\beta}(p_1) = 1$ and $h^{\beta}(p_2) = 0$. Then clearly h does not belong to the uniform closure of A. Thus clA is a proper subalgebra of $C_R^*(X)$ which satisfies (1) and (2) above.

<u>1.4.4 Definition</u>. Let A be a subalgebra of $C_R(X)$. If A

(1) is closed under uniform convergence

(2) contains the constants

(3) separates points and closed sets

(4) is closed under inversion, i.e. $f \in A$ and $Z(f) = \emptyset$ imply $\frac{1}{f} \in A$

then A is called an <u>Algebra</u> on X.

Denote by A^* the space of all bounded functions in A. If A is an Algebra on X, then clearly A^* is uniformly closed, contains the constants and is closed under bounded inversion, i.e. $f \in A^*$ and $f \geq a > 0$ imply $\frac{1}{f} \in A^*$. Furthermore, A^* separates points and closed sets: let F be a closed set in X and choose $x \notin F$. Then there is a function $f \in A$ such that $f(x) = 1$ and $f(F) = 0$. Put $g = 2f^2/f^2+1$. Then $g \in A^*$ and $g(x) = 1$, $g(F) = 0$.

For unbounded functions Hager and Johnson [16] proved the following interesting characterization.

<u>1.4.5 Theorem</u>. For a completely regular space X the following are equivalent:

(1) The only Algebra on X is $C_R(X)$ itself

(2) The only realcompact space in which X is G_δ-dense is νX

(3) νX is Lindelöf, $\text{card}(\nu X - X) \leq 1$, and νX is the only space in which X is G_δ-dense with these two properties

(4) Either X is Lindelöf or X is almost compact

(5) Every embedding is a z-embedding.

Before we can prove this result we need some further machinery. We note, however, that for almost compact spaces (1) follows immediately from 1.4.3 since almost compact spaces are pseudocompact.

<u>1.4.6 Preliminary results</u>. Let $f \in C_R(X)$. The <u>zero-set</u> of f is given by $Z(f) = \{x \in X : f(x) = 0\}$, and the <u>cozero-set</u> of f is defined by $\text{coz}(f) = X - Z(f)$. Now let X be a subspace of Y. We say that X is <u>C-embedded</u> (resp. C^*-embedded) in Y if, given $f \in C_R(X)$ (resp. $C^*_R(X)$), there is a g in $C_R(Y)$ such that $g|_X = f$.

If $f \in C^*_R(X)$ has an extension g to Y, then it has a bounded extension: let n be the bound of $|f|$. Then

$(n \wedge g) \vee (-n)$ is a bounded function on Y which agrees with f on X.

X is said to be z-embedded in Y if, given any zero-set Z in X, there is a zero-set Z' in Y such that $Z = X \cap Z'$.

(1) Every C^*-embedding is a z-embedding:-

Let Z be a zero-set in X. Hence $Z = Z(f)$ for some f in $C^*_R(X)$. Now let f^* be the extension of f to Y. Then $Z(f) = Z = Z(f^*) \cap X$.

(2) If X is dense in Y, then X is C^*-embedded in Y if and only if any two disjoint zero-sets in X have disjoint closures in Y:-

For a proof of this result see [11], Theorem 6.4

A subset G of X is a G_δ-set if $G = \bigcap_{i=1}^{\infty} U_i$, where each U_i is an open subset of X. Similarly, a subset F of X is a F_σ-set if $F = \bigcup_{i=1}^{\infty} F_i$, where each F_i is closed in X.

Let Y be an extension of X. Then $C_R(Y)$ may be regarded as a subalgebra of $C_R(X)$, consisting of those f in $C_R(X)$ with continuous extension over Y. Since X is dense in Y, these extensions are unique.

We say X is G_δ-dense in Y if each non-void G_δ-set in Y meets X. (This notion was first introduced by Mrowka [30] and he called it Q-dense.)

(3) $C_R(Y)$ is closed under inversion in $C_R(X)$ if and only if X is G_δ-dense in Y:-

Suppose $C_R(Y)$ is not closed under inversion in $C_R(X)$. Thus there is a function $f \in C_R(Y)$ such that $Z(f) \cap X = \emptyset$ and $Z(f) \neq \emptyset$. But $Z(f) = \bigcap_{n=1}^{\infty} \{y \in Y : |f(y)| < \frac{1}{n}\}$ and hence

$Z(f)$ is a G_δ-set which does not meet X.

Conversely, suppose $G \subset Y$ is a non-void G_δ-set such that $G \cap X = \emptyset$. Then G contains a non-void zero-set, say $Z(g)$([11], 3.11(b)). Thus $g \neq 0$ on X but $\frac{1}{g} \notin C_R(Y)$. Hence $C_R(Y)$ is not closed under inversion.

It follows that $C_R(Y)$ is an Algebra on X if and only if X is G_δ-dense in Y (the other conditions prevailing automatically).

(4) X is G_δ-dense in νX:-

This follows from (3), since $C_R(\nu X)$ is closed under inversion in $C_R(X)$ $(= C_R(\nu X))$.

The spaces $H(A^*)$ and $H(A)$.

Let A be a subalgebra of $C_R(X)$ which contains the constants. We define a partial order on A by $f \geq g$ provided $f(x) \geq g(x)$ for all $x \in X$. If a homomorphism m preserves order, i.e. if $f \geq g$ then $m(f) \geq m(g)$, we say that m is isotone. Now each $x \in X$ generates an isotone homomorphism m_x, defined by

$$m_x(f) = f(x).$$

Denote by H(A) the space of all isotone homomorphisms from A into $\mathbb{R}$ (thus every element in H(A) generates a real maximal ideal), and define the map $\varphi: X \to H(A)$ by

$$\varphi(x) = m_x.$$

Now suppose that A separates points and closed sets in X. Then φ is one-one: let x_1 and x_2 be two distinct points in X. Thus there is a function $f \in A$ such that $f(x_1) \neq f(x_2)$. But then $m_{x_2}(f) \neq m_{x_1}(f)$ and hence $\varphi(x_1) \neq \varphi(x_2)$.

Each f in A defines a function $\hat{f}$ on H(A) by

$$\hat{f}(m) = m(f).$$

Identifying $\hat{f}$ with f, A generates a weak topology on H(A) in which X is a <u>dense subspace</u> (since A separates points and closed sets).

We note that A separates the points in H(A): suppose $m_1 \neq m_2$ where $m_1, m_2 \in H(A)$. Thus there is a $f \in A$ such that $m_1(f) \neq m_2(f)$. Hence $\hat{f}(m_1) \neq \hat{f}(m_2)$.

The above also holds for $H(A^*)$, the space of all isotone homomorphisms from A^* into $\mathbb{R}$. We recall that, if A is closed under inversion and separates points and closed sets, then A^* separates points and closed sets.

<u>Proposition 1.</u> If A^* is uniformly closed and separates points and closed sets in X, then $H(A^*)$ is a compact Hausdorff space containing X as a dense subspace and $A^* = C_R(H(A^*))$.

<u>Proof.</u> The weak topology may be defined by embedding $H(A^*)$ in a product of real lines over A^*. Since each function in A^* is bounded, the image of $H(A^*)$ is contained in a product of closed bounded intervals, which is compact Hausdorff. Since $H(A^*)$ is weakly closed, $H(A^*)$ is itself compact Hausdorff.

Clearly A^* separates the points in $H(A^*)$ and contains the constants. Thus by the Stone-Weierstrass theorem $A^* = C_R(H(A^*))$.

<u>Lemma 2.</u> Every homomorphism from A^* onto $\mathbb{R}$ is isotone.

<u>Proof.</u> Suppose $f \in A^*$ is such that $f \geq 0$ and let m be a homomorphism into $\mathbb{R}$. Since $A^* = C_R(H(A^*))$, we can write f as

g^2 where $g \in A^*$. Then $m(f) = m(g^2) = m(g).m(g) = [m(g)]^2 \geqslant 0$. Hence m is isotone.

It now follows from Lemma 2 that $H(A^*)$ is the space of all maximal ideals in A^*, denoted by M_{A^*} (every maximal ideal in A^* is real).

Proposition 3. Let A be a subalgebra of $C_R(X)$ which is closed under inversion. Then every homomorphism of A onto $\mathbb{R}$ is isotone, i.e. $H(A)$ is the space of all real maximal ideals in A.

Proof. Suppose $m: A \to \mathbb{R}$ is a non-isotone homomorphism. Choose $f, g \in A$ with $f \geqslant g$ and $m(f) < m(g)$. Thus $m(f - g) = c < 0$ and $m(f - g - c) = 0$.
Since $f - g - c \geqslant -c > 0$, $f - g - c$ is invertible. But then m cannot be real-valued at $\frac{1}{f - g - c}$ - contradiction.

Let A be an Algebra on X. Then A determines two extensions of X: a compact one, $H(A^*)$, characterized by the property $C_R(H(A^*)) = A^*$, and a realcompact one, $H(A)$. [The former is a consequence of Proposition 1; the latter follows from Proposition 3.]

Clearly $H(C_R^*(X)) = \beta X$ and $H(C_R(X)) = \nu X$.

As was pointed out before, $H(A^*)$ can be described as the space of all maximal ideals in A^* and $H(A)$ as the space of all real maximal ideals in A. In either case, the one-one correspondence is defined by assigning to each non-zero homomorphism its kernel, which is a maximal ideal. Now

(5) $A = C_R(H(A))$ if and only if $H(A^*) = \beta(H(A))$:-

Suppose $H(A^*) = \beta(H(A))$. Then $A^* = C_R(H(A^*)) = C_R^*(H(A))$. Thus $A = C_R(H(A))$.

Conversely, $A = C_R(H(A))$ implies $A^* = C_R^*(H(A)) = C_R(H(A^*))$. Thus $\beta(H(A)) = H(A^*)$.

Hence it follows that $H(A^*) = \beta X$ if and only if $A = C_R(X)$.

We denote by $coz(H(A^*),X)$ the family of cozero-sets in $H(A^*)$ that contain X. We then have

(6) Every $S \in coz(H(A^*),X)$ is C^*-embedded in $H(A^*)$, and hence $\beta S = H(A^*)$:-

This can be verified by appealing to results in Isbell's paper on uniform algebras [22], but Hager states in [15] that this can be proved directly. We were able to do so. See p. 31

<u>Lemma 4</u>. A space X is Lindelöf if and only if it is G_δ-dense in no proper superspace.

<u>Proof</u>. Suppose Y is a proper superspace of X and X is Lindelöf. Choose $y_0 \in Y-X$. For each $x \in X$ there is a neighbourhood U_x of x such that $y_0 \notin \overline{U}_x$ (closure in Y). Since $\{U_x\}_{x \in X}$ is an open cover of X, there is a countable subcover, say $U_{x_1}, \ldots, U_{x_n}, \ldots$ Put

$$U = \bigcap_{i=1}^{\infty} \{Y-\overline{U}_{x_i}\}$$

Then $y_0 \in U$ and U is a G_δ-set in Y which is disjoint from X. Thus X is not G_δ-dense in Y.

Conversely, suppose X is not a Lindelöf space. Then there is a family $\{U_\alpha\}_{\alpha \in A}$ of sets in βX which covers X but no

countable subfamily covers X. Let $H = \beta X - \cup\{U_\alpha : \alpha \in A\}$ and let bX be the compactification of X which is obtained from βX by identifying H with a single point x_0.
Suppose that there is a G_δ-set $G \subset bX$ which contains x_0 and is disjoint from X. Put $F = bX-G$.
Then $X \subset F \subset \cup\{U_\alpha : \alpha \in A\}$. But F is a F_σ-set in a compact space bX, and hence can be covered by countably many sets U_α. This leads to a contradiction. Hence X is G_δ-dense in bX.

We are now able to prove Theorem 1.4.5.

Proof of Theorem 1.4.5. We have to prove that the following are equivalent:

(1) The only Algebra on X is $C_R(X)$

(2) The only realcompact space in which X is G_δ-dense is νX

(3) νX is Lindelöf, $\text{card}(\nu X-X) \leq 1$, and νX is the only space in which X is G_δ-dense with these two properties

(4) Either X is Lindelöf or X is almost compact

(5) Every embedding is a z-embedding

(1) ⇒ (2): Suppose D is a realcompact space in which X is G_δ-dense and $D \neq \nu X$. Then by 1.4.6(3) $C_R(D)$ is an Algebra on X that is different from $C_R(X)$ - contradiction.

(2) ⇒ (3): Assume (2). If νX is not Lindelöf, then there is a space D, properly containing νX, in which νX is G_δ-dense (by Lemma 4 above). Since X is G_δ-dense in νX, it is also G_δ-dense in νD. Thus νX and νD are two different realcompact extensions of X in which X is G_δ-dense. Hence (2) is contradicted and thus νX is Lindelöf.

If card(νX-X) > 1, choose two distinct points of νX-X, and let D denote the quotient of νX obtained by identifying them. Since each regular Lindelöf space is normal ([23], page 112), νX is normal and by a result in [6], VII 3.5, D is normal. The quotient mapping is not one-one, so that D and νX are different. Since D is the continuous image of a Lindelöf space, it is itself Lindelöf and contains X G_δ-densely (since νX does). Again (2) is contradicted, so card(νX-X) $\leq$ 1. Finally, if X is Lindelöf, then the remaining statement of (2) follows from Lemma 4 (for then $X = \nu X$ by 1.4.2(1)). If X is not Lindelöf, it is G_δ-dense in any Lindelöf superspace D with card(D-X) = 1 (otherwise X would be a F_σ-set in D and hence Lindelöf). By (2), $\nu X = D$.

<u>(3) ⇒ (4)</u>: Suppose X is not Lindelöf. By assumption, νX-X is a singleton, say $\{p_0\}$. So $\beta X = \nu X$ if and only if X is almost compact. Suppose, therefore, that there is $p_1 \in \beta X - \nu X$. Consider the subspace $X \cup \{p_0, p_1\}$ of βX, and let D denote the quotient of this space obtained by identifying p_0 and p_1. Since νX is Lindelöf, so is $X \cup \{p_0, p_1\}$. Thus D is Lindelöf, and since X is not Lindelöf, it is G_δ-dense in D by the same argument as above. Now νX and D are different extensions of X: e.g. X is clearly not C-embedded in D. Thus (3) is contradicted.

<u>(4) ⇒ (5)</u>: (i) Suppose X is Lindelöf and that X is embedded in Y. Let Z(f) be a zero-set in X. For each $x \in X - Z(f)$ there is a $g_x \in C_R(Y)$ such that $g_x(x) = 1$, $g_x(Z(f)) = 0$ and $\|g_x\| \leq 1$. ($x \notin \overline{Z(f)}$, closure in Y) Let $U_x = \{y \in X : g_x(y) > \frac{1}{2}\}$. Then $\{U_x : x \in X - Z(f)\}$ is an open cover of X-Z(f). Since X-Z(f) is a F_σ-subset of the

Lindelöf space X it is itself a Lindelöf space. So there exist countably many elements $x_1, \dots, x_n, \dots$ of X such that $X - Z(f) \subset \bigcup_{i=1}^{\infty} \{U_{x_i}\}$. Put $g = \sum_{i=1}^{\infty} \frac{1}{2^i} |g_{x_i}|$. Then $g \in C_R(Y)$ and $Z(f) = Z(g) \cap X$.

(ii) Suppose X is almost compact and embedded in Y. We show that $\beta X \subset \beta Y$, for then the result follows.
Let $f \in C^*_R(X)$. Since βX is compact, f^β has an extension $\bar{f}^\beta$ to βY. But then $\bar{f}^\beta|_Y$ will be an extension of f to Y. Hence X is C^*-embedded in Y and by 1.4.6(1) X is z-embedded in Y.

Without loss of generality we assume that Y is compact. To show that $\beta X \subset Y$, let $i: X \to Y$ be defined by $i(x) = x$. Denote by j the extension of i to $\beta X = X \cup \{x_\infty\}$. We show that j is one-one.
So suppose there is a $x_0 \in X$ such that $j(x_\infty) = j(x_0) = x_0$.
Let V be a neighbourhood of x_0 in X such that $\bar{V}$ is compact (in βX), and $\bar{V} \subset X$. Choose a neighbourhood W of x_0 in Y such that $W \cap X = V$.
Since j is continuous, there exists a neighbourhood U of x_∞, $U \subset \beta X - \bar{V}$, so that $j(U) \subset W$. Clearly $U \cap X \neq \emptyset$ and $U \cap V = \emptyset$. But $j(U \cap X) = i(U \cap X) = U \cap X \subset W \cap X = V$ - contradiction.
Thus j is a homeomorphism from βX into Y, and hence $\beta X \subset Y$.

<u>(5) ⇒ (1)</u>: Let A be an Algebra on X. We show that X is C^*-embedded in $H(A^*)$. Thus $\beta X = H(A^*)$ and by 1.4.6(5) it follows that $A = C_R(X)$.
By 1.4.6(2) it suffices to show that disjoint zero-sets in X

have disjoint closures in $H(A^*)$. Let Z_1, Z_2 be two disjoint zero-sets in X. Since X is z-embedded in $H(A^*)$ (by hypothesis) there exist $f_i \in C_R(H(A^*)) = A^*$ with $X \cap Z(f_i) = Z_i$ for $i = 1,2$.

Now $Z(f_1^2 + f_2^2) = Z(f_1) \cap Z(f_2)$ is disjoint from X, so by 1.4.6(6) $\mathrm{coz}(f_1^2 + f_2^2)$ is C^*-embedded in $H(A^*)$. Let Z_i' denote the zero-set of $f_i^2 / f_1^2 + f_2^2$ in $\mathrm{coz}(f_1^2 + f_2^2)$. Since $Z_1' \cap Z_2' = \emptyset$, $\overline{Z}_1' \cap \overline{Z}_2' = \emptyset$ by 1.4.6(2) (closures are taken in $H(A^*)$).

But $Z_i \subset Z_i'$ for $i = 1,2$, so $\overline{Z}_1 \cap \overline{Z}_2 = \emptyset$ as well.

Pointwise convergence of sequences of functions.

Finally we will show that the topology of pointwise convergence of sequences of functions is a S-topology for the class of almost Lindelöf spaces.

1.4.7 Definition. A completely regular Hausdorff space X will be called almost Lindelöf if and only if of any two disjoint zero-sets Z_1 and Z_2, at least one is Lindelöf.

On replacing "Lindelöf" with "compact", Hewitt's definition of almost compact is obtained ([19]). Thus an almost compact space is almost Lindelöf. Furthermore, it can be shown that the condition in 1.4.7 above is equivalent to (see [16], 4.3):

(c) νX is Lindelöf and $\mathrm{card}(\nu X - X) \leq 1$.

Condition (c) does not imply condition (3) of Theorem 1.4.5, since an almost Lindelöf space need not be Lindelöf nor almost compact: for example, remove one cluster point from the space of countable ordinals.

In a more general setting, denote by $F(X)$ the algebra of all real-valued functions on X. For each $B \subset F(X)$ let uB be the set of all pointwise limits of sequences in B. It may happen that $u(uB) \neq uB$, but in any case we let sB denote the smallest set $P \supset B$ for which $uP = P$. Then s is a topological closure operator, and the corresponding topology is called the topology of <u>pointwise sequential convergence</u>, which we will denote by p_s.

The following result is due to Hager [14]

<u>1.4.8 Proposition.</u> Let X be a Lindelöf space and A a subalgebra of $C_R^*(X)$ which contains the constants and separates points and closed sets. Then $uA \supset C_R(X)$ (i.e. A is p_s-dense in $C_R(X)$).

<u>Proof.</u> We will only give an outline of Hager's proof.

Suppose X is an arbitrary space and let A_1 be the smallest subalgebra of $C_R(X)$ which contains A and is closed under uniform convergence and inversion. Thus A_1 is an Algebra on X. If clA denotes the collection of uniform limits of sequences in A, then $A_1 = cl\{f/g : f,g \in clA \text{ and } g \text{ has no zeros}\}$. It can be shown that $uA \supset A_1$ ([14], Proposition 1.1). For Lindelöf spaces the result now follows immediately from Theorem 1.4.5: for then $A_1 = C_R(X)$ and hence $uA \supset C_R(X)$.

One can, however, follow another line of proof: by a result due to Isbell [22], $A_1 = C_R(X)$ if the zero-sets of functions in $C_R(X)$ and in A_1 coincide. Since zero-sets of functions in A_1 and in clA coincide, one need only consider clA and $C_R(X)$. For Lindelöf spaces the following explains why $A_1 = C_R(X)$.

Proposition. Let X be a Lindelöf space, and B a uniformly closed subalgebra of $C^*_R(X)$ containing constants and separating points and closed sets. Then zero-sets of functions in $C_R(X)$ are zero-sets of functions in B.

Clearly clA is such an algebra. I thought that this result could be weakened to include almost Lindelöf spaces, but this is, however, not possible. For suppose the Proposition were true for almost Lindelöf spaces, and suppose A is any Algebra on X. Then clearly A^* would be a candidate for an algebra B, and hence $A = C_R(X)$ (since zero-sets in A and A^* coincide). But then Theorem 1.4.5 would imply that almost Lindelöf spaces are either Lindelöf or almost compact, which contradicts the remark after 1.4.7.

The proof of 1.4.8 depends on the rather strong condition that $A_1 = C_R(X)$, thus it seems possible that one could get a class of topological spaces X such that $uA \supset C_R(X)$, yet where A_1 need not necessarily be the whole of $C_R(X)$. Indeed, such a result was obtained by Frolik [8]. In fact, he proved the following characterization for almost Lindelöf spaces.

1.4.9 Theorem. The following are equivalent:

(1) X is almost Lindelöf

(2) If an algebra A of bounded continuous functions contains the constants and separates points and closed sets, then A is p_s-dense in $C_R(X)$ (in fact $uA \supset C_R(X)$)

(3) With the assumptions in (2), if A is a lattice, then every non-negative function f in $C_R(X)$ is the pointwise limit of an increasing sequence of

functions in A.

Proof. We need the notion of a 'measurable' function and the lemmas below.

Let M be a σ-algebra of subsets of X and let $f \in F(X)$. Then $f: (X,M) \to \mathbb{R}$ is said to be measurable if $f^{-1}(U) \in M$ for every open subset U of $\mathbb{R}$ (equivalently, $f^{-1}(F) \in M$ for every closed subset F of $\mathbb{R}$ since M is closed under the taking of complements).

We can now state the first lemma.

Lemma 1. Let A be a subset of F(X) and let M be the smallest σ-algebra of subsets of X such that every $f: (X,M) \to \mathbb{R}$, $f \in$ A, is measurable. (Thus M contains all zero-sets and cozero-sets of functions in A.) Then every pointwise limit of a sequence of functions in A is measurable. Furthermore, if $P \in M$ then there exists a countable subset $B \subset A$ such that if $x \in P$ and $y \notin P$ then $f(x) \neq f(y)$ for some $f \in B$ (we shall say that B distinguishes P).

Proof. Suppose $\{g_n\}$ is a sequence in A which converges pointwise to g. We want to show that g is measurable. So let F be a closed subset of $\mathbb{R}$ and let $U_m = \{\alpha \in \mathbb{R}: \ d(\alpha, F) < \frac{1}{m}\}$. Then $g_k^{-1}(U_m)$ belongs to M for every k and every m. We claim

$$g^{-1}(F) = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} g_k^{-1}(U_m) ,$$

for then $g^{-1}(F) \in M$ since M is a σ-algebra.

"$\supset$": Choose $x \in \bigcap_m \bigcup_n \bigcap_{k \geqslant n} g_k^{-1}(U_m)$ and fix m. Then there exists an integer N such that $x \in \bigcap_{k=N}^{\infty} g_k^{-1}(U_m)$, so $g_k(x) \in U_m$

for every $k \geq N$. Since $g_k(x) \to g(x)$ and U_m is open, we have that $g(x) \in \overline{U}_m$. This holds for every m, so $g(x) \in \bigcap_{m=1}^{\infty} \overline{U}_m = F$ and thus $x \in g^{-1}(F)$.

"$\subset$": Choose $x \in g^{-1}(F)$ and fix m. So $g(x) \in F \subset U_m$. Since $g_n(x) \to g(x)$ there exists an integer N such that $g_k(x) \in U_m$ for $k \geq N$ and hence $x \in \bigcap_{k=N}^{\infty} g_k^{-1}(U_m) \subset \bigcup_n \bigcap_{k \geq n} g_k^{-1}(U_m)$. This holds for every m, so $x \in \bigcap_m \bigcup_n \bigcap_{k \geq n} g_k^{-1}(U_m)$.

To prove the second statement, observe that the collection M^* of all 'distinguishable' sets is a σ-algebra. We will show that every $f: (X, M^*) \to \mathbb{R}$, $f \in A$, is measurable. For then $M \subset M^*$ by the minimality of M.

So let F be a closed set in $\mathbb{R}$ and choose $f \in A$. Then clearly $\{f\}$ distinguishes the set $f^{-1}(F)$ and so $f^{-1}(F) \in M^*$. This completes the proof of the lemma.

Lemma 2. A space X is almost Lindelöf if and only if for each continuous function f on X there exists a real number r such that the set $E = \{x : f(x) \neq r\}$ is Lindelöf.

Proof. If: Let Z(g), Z(f) be two disjoint zero-sets in X. By assumption, there is a number r such that $E = \{x: g(x) \neq r\}$ is Lindelöf.

If $r = 0$, then $E = X - Z(g)$ and hence Z(f) is Lindelöf (being a closed subset of a Lindelöf set).

If $r \neq 0$, then $Z(g) \subset E$ and hence Z(g) is Lindelöf. In either case, at least one of Z(g), Z(f) is Lindelöf and so X is almost Lindelöf.

Only if: Let S be the set of numbers s such that $\{x : f(x) < s\}$ is Lindelöf. Let r be the supremum of S. Suppose that r is finite. Now $E_n = \{x: f(x) \leq r - \frac{1}{n}\}$ is Lindelöf for each n. Hence

$$\bigcup_{n=1}^{\infty} E_n = \{x: f(x) < r\}$$

is Lindelöf. Similarly, $E_n' = \{x: f(x) \geq r + \frac{1}{n}\}$ is Lindelöf for each n since it is disjoint from the zero-set $\{x: f(x) \leq r + \frac{1}{2n}\}$ and the latter is not Lindelöf (by definition of r).

Hence $\bigcup_{n=1}^{\infty} E_n' = \{x: f(x) > r\}$ is Lindelöf and it follows that $E = \{x: f(x) \neq r\}$ is Lindelöf.

If r is not finite, then X is Lindelöf.

We are now in a position to prove the theorem.

(3) ⇒ (2): Obvious

(2) ⇒ (1): Suppose X is not almost Lindelöf. Thus there exist two disjoint zero-sets Z_1 and Z_2 such that neither Z_1 nor Z_2 is Lindelöf. Take a countably additive open cover $\mathcal{U}$ of X such that neither Z_1 nor Z_2 is contained in an element of $\mathcal{U}$. Thus $\mathcal{U}$ is uncountable. (That such a cover exists can be seen as follows: Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be covers of Z_1 and Z_2 respectively which do not admit countable subcovers. Also ensure that no element of $\mathcal{U}_1$ intersects with any element of $\mathcal{U}_2$. For each point $x \in X-(Z_1 \cup Z_2)$ let U_x be the collection of all neighbourhoods N_x of x which are subsets of $X-(Z_1 \cup Z_2)$. Now let $\mathcal{U}$ be the cover generated by $\mathcal{U}_1$, $\mathcal{U}_2$ and $\{U_x : x \in X-(Z_1 \cup Z_2)\}$, ensuring that, whenever x belongs to some element of $\mathcal{U}_1$ or $\mathcal{U}_2$, all neighbourhoods of x which are contained in

this element also belong to $\mathcal{U}$.)

Let **A** be the space of all functions in $C_R^*(X)$ which are constant on the complement of an element of $\mathcal{U}$. Clearly **A** is uniformly closed, contains the constants and separates points and closed sets. Let M be defined as in Lemma 1 above. Then Z_1 does not belong to M. For suppose $Z_1 \in M$, then there would exist a countable subset $F = \{f_1, \ldots, f_n, \ldots\}$ of **A** which distinguishes Z_1. Now each $f_i \in F$ is constant on $X-U_i$ for some $U_i \in \mathcal{U}$. Hence every $f_i \in F$ is constant on

$$\bigcap_{i=1}^{\infty} (X-U_i) = X-\{\bigcup_{i=1}^{\infty} U_i\} = X-U$$

where $U \in \mathcal{U}$.

Now X-U is non-empty and meets both Z_1 and Z_2 (by the construction of $\mathcal{U}$). So if we choose $x \in (X-U) \cap Z_1$ and $y \in (X-U) \cap Z_2$ then $x \in Z_1$, $y \notin Z_1$, but F does not distinguish x and y - contradiction. Thus $Z_1 \notin M$.

Consider the function $f \in C_R(X)$ which has $Z(f) = Z_1$. Thus $Z(f) \notin M$ and by Lemma 1, f is not in **uA**.

<u>(1) ⇒ (3)</u>: Assume condition (1). Let **A** be an algebra of bounded functions in $C_R^*(X)$ which contains the constants and separates points and closed sets. Denote by B the smallest lattice algebra containing A. Since the smallest uniformly closed algebra C containing **A** is a lattice (see [11], 16.2), we have $\mathbf{A} \subset \mathbf{B} \subset \mathbf{C} \subset \mathbf{uA}$, and hence it is obvious that it is enough to prove that every nonnegative continuous function is the pointwise limit of an increasing sequence of nonnegative functions in B.

<u>Lemma 3</u>: If U is an open Lindelöf subspace of X, then there exists an increasing sequence $\{g_n\}$ of nonnegative functions in

B which converges pointwise to the characteristic function of U.

Proof. Following the argument in the proof of Theorem 1.4.5 [(4)⇒(5), (i)], there exists a sequence $\{h_n\}$ of nonnegative functions in B such that $h_n(X-U) = 0$ for all n, and if $x \in U$, then $h_n(x) > 0$ for some n. Put

$$g_n = \min(1, n(h_1 + \dots + h_n)).$$

Then $\{g_n\}$ satisfies the condition of the Lemma.

Now let f be a nonnegative continuous function on X. First assume that X is Lindelöf. Then all cozero-sets are Lindelöf (being F_σ-subsets), and we can proceed as follows. Given an $\varepsilon > 0$, put $E_n = \{x \in X: f(x) > n\varepsilon\}$ and let h_n be the characteristic function of E_n, for $n = 1,2,\dots$. Define

$$h = \varepsilon\left(\sum_{i=1}^{\infty} h_i\right)$$

Then $h \leq f$ and $\|f - h\| \leq \varepsilon$.

By Lemma 3 above, each h_n is the pointwise limit of an increasing sequence $\{g_{nm}\}$ of nonnegative functions in B. Now, for $\varepsilon = \frac{1}{k}$ $k = 1,2,\dots$, let f_i be the maximum of all g_{nm}; $n \leq i$, $m \leq i$, g_{nm} corresponding to $\varepsilon = \frac{1}{k}$ for $k \leq i$. Clearly $\{f_i\}$ is an increasing sequence of nonnegative functions in B, and $\{f_i\}$ converges pointwise to f.

This shows that conditions (2) and (3) hold if X is Lindelöf. It should be remarked that this is Proposition 1.4.8 above.

Finally assume that X is almost Lindelöf. This case is reduced to the previous one as follows. By Lemma 2 there is a real number r such that $E = \{x: f(x) \neq r\}$ is Lindelöf. We may assume that $r = 0$. Apply the previous case to the restriction of f to E, and the restriction of B to E. We thus obtain a sequence $\{h_n\}$ in B such that $\{h_n|_E\}$ is an increasing sequence of nonnegative functions, which converges to $f|_E$. By Lemma 3 we can choose $\{g_n\}$ in B such that $\{g_n\}$ converges pointwise to the characteristic function of E. Put

$$f_n = g_n . h_n$$

Then $\{f_n\}$ is an increasing sequence of nonnegative functions in B which converges pointwise to f. This completes the proof.

We observe that the p_s-topology is a S-topology for the class of almost Lindelöf spaces, where we require the algebra to separate points and closed sets. This condition cannot be replaced with the hypothesis that A separates points. An example is given in [14] (1.5(ii)). We note, however, that any subalgebra which srongly separates the points in X is p-dense in $C_R(X)$ (see proof of 1.3.4(1)).

We conclude this part with a final observation. Let c denote the topology of uniform convergence on compact sets. We have

<u>1.4.10 Proposition</u>: Let X be an almost Lindelöf space. If A is a lattice subalgebra of $C_R^*(X)$ which contains constants and separates points and closed sets, then A is sequentially dense in $C_R(X)$ with the c-topology.

<u>Proof</u>. We will need Dini's theorem, which states that a

pointwise convergent increasing sequence of continuous functions with continuous limit converges uniformly on compact sets ([23], 7E)

Let $f \in C_R(X)$ and write $f = f \vee 0 - (-f) \vee 0 = f^+ - f^-$. By Theorem 1.4.9(3) and the remark above, there are sequences $\{g_n\}$ and $\{h_n\}$ in A such that $\{g_n\}$ converges to f^+ and $\{h_n\}$ converges to f^-, uniformly on compact sets.
Then $\{g_n - h_n\}$ converges to f, uniformly on compact sets, and hence in $C_R(X)$ with the c-topology.

Appendix.

We will now give the direct proof of 1.4.6(6) promised above. (It turned out to be "rather easy", as Hager in fact suggested.) Recall that we have to show the following:

Every $S \in \operatorname{coz}(H(A^*),X)$ is C^*-embedded in $H(A^*)$.

So suppose $S = H(A^*) - Z(f)$, where $f \in C_R(H(A^*))$ and $Z(f) \cap X = \emptyset$. Let $g \in C_R^*(S)$. We want to show that g has an extension to $H(A^*)$. Define k by

$$\begin{aligned} k &= g.f \quad \text{on } S \\ &= 0 \quad \text{on } Z(f) \end{aligned}$$

Since g is bounded, k is continuous on $H(A^*)$. Now $C_R(H(A^*)) = A^*$, so $k|_X$ and $f|_X$ both belong to A^*. Also $f \neq 0$ on X and since A is closed under inversion we have that $(\frac{1}{f})|_X \in A$. Since A is an algebra, $(\frac{1}{f}\cdot k)|_X$ $(= (\frac{1}{f})|_X \cdot k|_X)$ belongs to A. In fact it belongs to A^*, since g is bounded and

$$(\tfrac{1}{f}\cdot k)|_X = (\tfrac{1}{f}(g.f))|_X = g|_X \,.$$

Again $C_R(H(A^*)) = A^*$, so $(\frac{1}{f}\cdot k)|_X$ has an extension h to $H(A^*)$ with $h|_X = g|_X$. But X is dense in S, so $h|_S = g$. Thus g has an extension to $H(A^*)$.

Notes

The study of approximation of continuous functions originated with Weierstrass's well-known paper [35]. Theorem 1.2.1 is a generalization of this result to compact spaces and is due to Stone (see Stone's expository article in [5]). Theorem 1.2.4 is due to Hewitt [19], but the proof presented here appears in [3].

The notion of S-topology is based on Meyer's definition of a SW-topology in [28]. Propositions 1.3.2 and 1.3.3 are due to him and we have given his proofs of these results. Theorem 1.3.4 and its proof is also due to Meyer. ([27], [28])

The proof of 1.4.2(2) is our own, and is based on an exercise in [11]. Theorem 1.4.3 is due to Hewitt [19]. The proof presented here is our own. The sufficiency part uses an idea of Nel [32]; the necessity is based on an idea of Banaschewski [3]. Most of the proofs of the results under 1.4.6 are our own, except Propositions 1 and 3, which appear in Isbell's paper [22], and Lemma 4, the proof of which is in [29]. Theorem 1.4.5 is due to Hager and Johnson [16], and the proof presented here is essentially theirs. The proof of part (1) of the implication (4) ⇒ (5) appears in [17]; the proof of part (2) is our own.

Theorem 1.4.9 is due to Frolik. We have in places elaborated on his arguments and simplified his proof. The proofs of Lemmas 1 and 2 are our own.

Finally, the proof of the result in the Appendix is our own.

PART II

ALGEBRAS OF COMPLEX-VALUED CONTINUOUS FUNCTIONS

2.1 Preliminaries.

In Part I we developed results concerning the approximation properties of algebras of real-valued functions. These results are no longer true if we consider algebras of complex-valued functions. In this chapter we will investigate such algebras. Henceforth X will denote, unless otherwise stated, a compact Hausdorff topological space, and C(X) will be endowed with the topology of uniform convergence. We furthermore remark that the definition of a function algebra as given in 1.1.1 can clearly be extended to complex-valued functions.

Before we proceed, however, it is necessary to collect some information on Borel measures. A Borel set B in X is a member of the σ-ring $\mathcal{B}$ generated by the closed subsets of X. A complex Borel measure μ on X is a countably additive complex-valued set-function on $\mathcal{B}$ such that $\mu(\emptyset) = 0$. For every complex Borel measure μ we define a positive measure $|\mu|$ by:

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)|$$

where the supremum is taken over all representations of E as a countable union of disjoint Borel sets E_i. $|\mu|$ is called the total variation of μ. An important property of every complex measure μ is that $|\mu|(X) < \infty$. ([33], Theorem 6.4) This immediately implies that every complex measure μ on any σ-algebra is bounded. A positive Borel measure ν on X is regular if and only if

$$\nu(E) = \inf \{ \nu(U) : U \text{ open and } U \supset E\}$$
$$= \sup \{ \nu(F) : F \text{ closed and } F \subset E\}$$

for every $E \in \mathcal{B}$.

We denote by $M(X)$ the space of all regular complex Borel measures on X. For each $\mu \in M(X)$ we define a norm for μ by $\|\mu\| = |\mu|(X)$. The map

$$f \to \int_X f\, d\mu$$

is a bounded linear functional on $C(X)$ and the <u>Riesz Representation Theorem</u> asserts that these are all the continuous linear functionals on $C(X)$. (For a detailed account of the above the reader is referred to [33], Ch. 6)

Furthermore, if $\mu \in M(X)$ and F is a Borel set in X, then μ is said to be <u>concentrated</u> on F if and only if $|\mu|(X-F) = 0$. If $\mu \in M(X)$, then there exists a smallest closed subset on which μ is concentrated. This can be shown as follows: Let $U = \bigcup_\alpha U_\alpha$ where $|\mu|(U_\alpha) = 0$ for each α, and every U_α is an open subset of X. We will show that $|\mu|(U) = 0$. So suppose that $|\mu|(U) > 0$. Since $|\mu|$ is regular, there is a closed set $F \subset U$ such that $|\mu|(F) > 0$. Since F is compact there exist $\alpha_1, \ldots, \alpha_n$ such that $F \subset \bigcup_{i=1}^{n} U_{\alpha_i}$. But then $|\mu|(F) \leq \sum_{i=1}^{n} |\mu|(U_{\alpha_i}) = 0$ - contradiction. Thus $X-U$ is the desired set. We call this set the <u>support</u> of μ and denote it by supp μ. Finally, a measure μ is <u>multiplicative</u> on a subalgebra A of $C(X)$ if and only if $\int fg\, d\mu = (\int f\, d\mu)(\int g\, d\mu)$ for all f,g in A.

We endow $M(X)$ with the w*-topology. This is the weakest topology on $M(X)$ which makes all the maps

$$\mu \to \int f \, d\mu \qquad (f \in C(X))$$

continuous.

<u>2.1.1 Proposition</u>. Let W be a w^*-closed linear subspace of M(X), and let $b(W) = \{\mu \in W : \|\mu\| \leq 1\}$ (the unit sphere). Then b(W) is the w^*-closed hull of its set of extreme points, denoted by $b(W)^e$ (see [25], p. 9).

This follows from the Krein-Milman theorem ([24], p. 131), which states that if A is a convex compact subset of a locally convex linear topological Hausdorff space E, then A is the closed convex extension of the set of all its extreme points, and the fact that b(W) is w^*-closed and convex.

Thus any w^*-continuous linear functional that vanishes on $b(W)^e$ vanishes on b(W).

<u>2.1.2 Definition</u>. Let $A \subset C(X)$. The <u>annihilator</u> of A is the set $A^{\perp} = \{\mu \in M(X) : \int f \, d\mu = 0 \, , \, \forall f \in A\}$.

Combining the Hahn-Banach theorem with Proposition 2.1.1 we obtain

<u>2.1.3 Lemma</u>. Let $f \in C(X)$ and $V \subset C(X)$. Then f belongs to the closed subspace of C(X) spanned by V if and only if $\int f \, d\mu = 0$ for every $\mu \in b(V^{\perp})^e$.

The following lemma is basic in establishing several of the later results.

<u>2.1.4 Lemma</u>. Let $f \in C(X)$ and $V \subset C(X)$. Suppose that

(1) $V^{\perp} \neq \{0\}$

(2) $\mu \in b(V^{\perp})^e$

$$(3) \quad \int f.g \, d\mu = 0 \quad \text{for every} \quad g \in V.$$

If f is real-valued on supp μ, then f is constant on supp μ.

Proof. E.g. [25], p. 35

As mentioned earlier in Part I (see 1.2.3), if A is a function algebra on a compact Hausdorff space X which is closed under complex conjugation, then $A = C(X)$. It is our aim in this chapter to indicate other properties that ensure that $A = C(X)$.

Let Re(A) denote the set of all $u \in C_R(X)$ such that $(u + iv) \in A$ for some $v \in C_R(X)$, i.e. $Re(A) = \{Re(f) : f \in A\}$ where Re(f) denotes the real part of f(z).

2.1.5 Theorem. Let A be a function algebra on X. If either Re(A) is uniformly closed in $C_R(X)$ or a subalgebra of $C_R(X)$, then $A = C(X)$.

Proof. Both results are well-known and proofs can be found in the papers by Wermer [36] and Hoffman and Wermer [21].

2.2 Restriction algebras.

Many results assert that $A = C(X)$ if the restrictions of the algebra A to certain subsets of X have suitable properties. We will now discuss some of these results in greater detail.

The following theorem is a famous result due to Glicksberg

2.2.1 Theorem. Let A be a function algebra on X. Suppose $A|_F$ is closed in C(F) for every closed subset F of X.

Then $A = C(X)$.

Proof. See [13]

2.2.2 Definition. Let X be a compact space and A a non-void subset of C(X). A set K in X is a set of antisymmetry for A if and only if K is non-empty and if $f \in A$ is real-valued on K, then f is constant on K.

If X is a set of antisymmetry for A, then A is called antisymmetric.

Clearly every singleton set $\{x\}$ is a set of antisymmetry. If K is a set of antisymmetry, so is its closure $\overline{K}$: for if $f \in A$ and $f|_{\overline{K}}$ is real-valued, then f is constant on K and hence, by continuity, f is constant on $\overline{K}$. Furthermore, if two sets of antisymmetry overlap, then their union is again a set of antisymmetry, since the values assigned to any real-valued function must be the same on these two sets.

2.2.3 Lemma. Let A be a non-void subset of C(X).

(1) If K is a set of antisymmetry for A and if $\tilde{K}$ is the union of all sets of antisymmetry for A which contain K, then $\tilde{K}$ is also an antisymmetric set for A

(2) Every set of antisymmetry is contained in a maximal set of antisymmetry

(3) Let K_A be the family of all maximal sets of antisymmetry for A. Then K_A decomposes X into a union of pairwise disjoint closed sets.

Proof. (1) Let $f \in A$ be real-valued on $\tilde{K}$. Choose $x_1, x_2 \in \tilde{K}$ and antisymmetric sets K_1, K_2 containing K such that $x_1 \in K_1$ and $x_2 \in K_2$. Since $K_1 \cap K_2 \supset K \neq \emptyset$, $K_1 \cup K_2$

is a set of antisymmetry for A. Since $f|_{K_1 \cup K_2}$ is real-valued, $f|_{K_1 \cup K_2}$ is constant; hence $f(x_1) = f(x_2)$. This shows that f is constant on $\tilde{K}$, as required.

(2) $\tilde{K}$ is clearly a maximal antisymmetric set containing K. (We note that no use is made of the axiom of choice.)

(3) $\{x\}$ is a set of antisymmetry for A, and $x \in \{\tilde{x}\}$. So X is the union of the members of K_A. Each $K \in K_A$ is closed since $\bar{K}$ is a set of antisymmetry. Distinct maximal antisymmetric sets are disjoint, otherwise their union would again be such a set properly containing both of them. Thus X is the union of pairwise disjoint closed subsets.

<u>2.2.4 Proposition</u>. If A is a subalgebra of C(X), then A is antisymmetric on the closed support of every extreme point of the unit ball of $A^{\perp}$.

<u>Proof</u>. Follows immediately from Lemma 2.1.4. Indeed, we only require that A be closed under multiplication.

We are now able to state and prove Bishop's theorem.

<u>2.2.5 Theorem</u>. Let A be a <u>closed</u> subalgebra of C(X) and let K_A be the family of maximal sets of antisymmetry for A. Let $f \in C(X)$. If $f|_K \in A|_K$ for every $K \in K_A$, then $f \in A$.

<u>Proof</u>. Suppose that $f \notin A$. Then by Lemma 2.1.3 there exists $\mu \in b(A^{\perp})^e$ with $\int f\,d\mu \neq 0$. Since supp μ is a set of antisymmetry for A, we have that supp $\mu \subset K$ for some $K \in K_A$ (by Lemma 2.2.3). By hypothesis, $f|_K = g|_K$ for some $g \in A$.

Now $0 \neq \int f\,d\mu = \int_{\text{supp}\ \mu} f\,d\mu = \int_K f\,d\mu = \int_K g\,d\mu = \int g\,d\mu = 0$

since $\mu \in A^{\perp}$. This gives the desired contradiction.

As an immediate corollary we have the following Stone-Weierstrass type result.

<u>2.2.6 Corollary</u>. Let A be a closed subalgebra of C(X). If $A|_K = C(K)$ for every $K \in K_A$, then $A = C(X)$.

Before we can prove our next result, we need the following

<u>2.2.7 Lemma</u>. Let A be a subset of C(X) that separates the points in X. Suppose that F is a closed subset of X such that $A|_F = C(F)$. If $p \in X-F$, then either $f(p) = 0$ for every f in A, or $f(p) \neq 0$ for some $f \in A$ which vanishes on F.

<u>Proof</u>. Suppose that $f \in A$ and $f(F) = 0$ imply that $f(p) = 0$. If $f, g \in A$ and $f|_F = g|_F$, then $f(p) = g(p)$. So if we set $h(f|_F) = f(p)$ for $f \in A$, then h is a well-defined homomorphism from $A|_F$ $(=C(F))$ to $\mathbb{C}$. Either $f(p) = 0$ for all f in A, or h is non-zero. In the second case there is a point $q \in F$ with $h(\varphi) = \varphi(q)$ for all $\varphi \in C(F)$. But then $f(p) = h(f|_F) = (f|_F)(q) = f(q)$ for all $f \in A$. Since p, q are distinct and A separates points this case cannot arise.

<u>2.2.8 Theorem</u>. Suppose that $X = \bigcup_{i=1}^{n} F_i$ where all F_i are closed subsets of X. Let A be a closed subalgebra of C(X) which separates the points in X. If $A|_{F_i} = C(F_i)$ for each

$i = 1,2,\ldots,n$, then $A = C(X)$.

Proof. Let $K \in K_A$. We shall show that K is contained in some F_i, for then $A|_{F_i} = C(F_i)$ implies that $A|_K = C(K)$. The result then follows by Corollary 2.2.6.

Let m be the smallest integer such that $K \subset F_1 \cup \ldots \cup F_m$ where $1 \leq m \leq n$. If $m = 1$, then $K \subset F_1$ and this would give us the result. So suppose $m > 1$. Clearly $K \not\subset F_1 \cup \ldots \cup F_{m-1} = F$. Choose $p \in K - F$. Then $p \notin F_i$ for $i = 1,\ldots,m-1$, but $p \in F_m$. Since $A|_{F_m} = C(F_m)$, $f(p) \neq 0$ for some f in A. Hence by Lemma 2.2.7 above there exist $f_i \in A$ such that $f_i(F_i) = 0$ and $f_i(p) = 1$ for $i = 1,\ldots,$ $m-1$. Put $f = f_1 . f_2 . \ldots . f_{m-1}$. Then $f \in A$ and $f(F) = 0$, $f(p) = 1$.

There exists a function $h \in A$ such that $h = 0$ on F, $h \geq 0$ on K, and $h(p) = 1$: since $A|_{F_m} = C(F_m)$ there is a g in A with $g|_{F_m} = \overline{f}|_{F_m}$ (complex conjugate of f). Now set $h = g.f \in A$. Since h is real-valued on K it is constant on K. In fact $h(K) = h(p) = 1$. Hence $F \cap K = \emptyset$, and since $K \subset F \cup F_m$ we have that $K \subset F_m$.

The corollary below follows immediately from above and the compactness of X.

2.2.9 Corollary. Let A be a closed subalgebra of C(X) which separates points. Suppose that for each $x \in X$ there is a closed neighbourhood V_x of x such that $A|_{V_x} = C(V_x)$. Then $A = C(X)$.

Remark. Theorem 2.2.8 generalizes a result in [7] (Th. 4.10) in two ways :

(i) our space X need not be a metric space

(ii) A need not contain the constants, i.e. A need not be a function algebra.

Suppose now that the sets F_i in 2.2.8 are pairwise disjoint. Then A need not be closed for the theorem to hold. Specifically, we have

2.2.10 Theorem. Let A be a subalgebra of C(X) which separates the points in X. Suppose that $X = F_1 \cup \ldots \cup F_n$ where the F_i are disjoint closed subsets of X. If $A|_{F_i} = C(F_i)$ for $i = 1, \ldots, n$, then $A = C(X)$.

Proof. We prove it for the case $n = 2$. The general case can then be proved as follows : suppose $X = F_1 \cup \ldots \cup F_n$ and that $A|_{F_i} = C(F_i)$ $i = 1, \ldots, n$. Let $X_r = \bigcup_{i=1}^{r} F_i$. It follows that $A|_{X_r} = C(X_r)$ for each $r = 1, \ldots, n$.

So let $X = F_1 \cup F_2$ with $F_1 \cap F_2 = \emptyset$. By Lemma 2.2.7, if $p \in F_2$, then there exists $f_p \in A$ with $f_p(F_1) = 0$ and $f_p(p) \neq 0$. Now for each $p \in F_2$ find $h_p \in A$ such that $h_p|_{F_2} = \bar{f}_p|_{F_2}$. Put $g_p = h_p . f_p$. Then g_p belongs to A, $g_p(F_1) = 0$ and $g_p > 0$ in a neighbourhood V_p of p.

Cover F_2 by a finite number of these neighbourhoods, say $F_2 \subset \bigcup_{i=1}^{n} V_{p_i}$. Put $g = g_{p_1} + \ldots + g_{p_n}$. Then $g(F_1) = 0$ and $g > 0$ on F_2. Clearly g belongs to A. Since $\frac{1}{g}$ is continuous on F_2, there exists a function $g' \in A$ such that $g.g'$ is the characteristic function of F_2. So by symmetry there exist χ_1, χ_2 in A with $\chi_1(F_1) = 1$, $\chi_1(F_2) = 0$ and $\chi_2(F_1) = 0$,

$\chi_2(F_2) = 1$. Given any φ in $C(X)$, we choose $\varphi_i \in A$ such that $\varphi_i|_{F_i} = \varphi|_{F_i}$ for $i = 1,2$. But $\varphi = \varphi_1\chi_1 + \varphi_2\chi_2$, so φ belongs to A. Hence $A = C(X)$.

2.2.11 Note. If A fails to separate points then the above results need no longer be true. For example, let $X = [-1,1]$ and let A be the set of all even continuous functions on X, i.e. $f \in A$ if and only if $f(-x) = f(x)$, $0 \leq x \leq 1$. Let $X = F_1 \cup F_2$ where $F_1 = [-1,0]$ and $F_2 = [0,1]$. Clearly $A \neq C(X)$, but $A|_{F_i} = C(F_i)$ for $i = 1,2$.

For the case that F_1 and F_2 are disjoint, consider $X = [-2,-1] \cup [1,2]$ and let A be all the even functions on X. Again $A|_{F_i} = C(F_i)$ for $i = 1,2$ but $A \neq C(X)$.

2.3 Normal and approximately normal function algebras

We now turn our attention to separation properties of function algebras. We will mainly consider normal and approximately normal algebras. Besides separating points or points and closed sets, such algebras distinguish between disjoint closed sets.

2.3.1 Definition. (1) A function algebra A is *approximately normal* if, given any two disjoint closed sets F_1 and F_2, and any $\varepsilon > 0$, there is a function $f \in A$ such that $|f(F_1)| < \varepsilon$ and $|f(F_2) - 1| < \varepsilon$.

(2) A function algebra A is *normal* if, given any two disjoint closed sets F_1 and F_2, there is a function $f \in A$ with $f(F_1) = 1$ and $f(F_2) = 0$.

It was conjectured for some time that the only normal function algebra on a compact space X is $C(X)$ itself, but McKissick ([26]) succeeded in constructing a counter-example. Briefly: let $R(X)$ denote the uniform closure of the set of rational functions on X. In McKissick's example, X is a compact subset of the unit disc in the complex plane such that $\text{int } X = \emptyset$. It is proved that $R(X)$ is normal, but that $R(X) \neq C(X)$.

Normal function algebras are difficult to construct, but approximately normal function algebras exist in abundance. For instance, the algebras $R(X)$ where X is a compact set in the plane with empty interior are approximately normal. Another class of approximately normal function algebras are the Dirichlet algebras. (A function algebra A is a Dirichlet algebra if and only if $\mathrm{Re}(A)$ is uniformly dense in $C_R(X)$.)

We say a subset A of $C(X)$ is <u>boundedly</u> approximately normal if there exists a constant M such that for each $\varepsilon > 0$ and each pair of disjoint closed sets F_1, F_2, there is a function f in A such that $\|f\| \leq M$, $|f(F_1)| < \varepsilon$ and $|f(F_2) - 1| < \varepsilon$.

The following result shows that bounded approximate normality is possible only for $C(X)$.

<u>2.3.2 Theorem</u>. Let A be a closed boundedly approximately normal subspace of $C(X)$. Then $A = C(X)$.

<u>Proof</u>. We show that if $\mu \in A^{\perp}$, then $\mu = 0$. It then follows from Lemma 2.1.3 that, since A is closed, $A = C(X)$.

So choose $\varepsilon > 0$ and let F be any compact subset of X. Choose V open in X with $F \subset V$ and $|\mu|(V-F) < \varepsilon$. By hypothesis there exists $f \in A$ with $\|f\| \leq M$, $|f - 1| < \varepsilon$ on F and $|f| < \varepsilon$ on X-V. Since $f \in A$ and $\mu \in A^{\perp}$, $\int f\, d\mu = 0$. We have

$$\left|\int_F f\, d\mu\right| = \left|\int_{X-F} f\, d\mu\right| \leqslant \left|\int_{X-V} f\, d\mu\right| + \left|\int_{V-F} f\, d\mu\right|$$

$$\leqslant \varepsilon|\mu|(X-V) + M|\mu|(V-F)$$

$$\leqslant \varepsilon(M + \|\mu\|)$$

Now
$$\mu(F) = \int_F f\, d\mu + \int_F (1 - f)d\mu$$

hence
$$|\mu(F)| \leqslant \varepsilon(M + \|\mu\|) + \varepsilon\|\mu\|$$

$$= \varepsilon(M + 2\|\mu\|).$$

Since ε is arbitrary, $\mu(F) = 0$. Since F is arbitrary, $\mu = 0$. Thus $A^{\perp} = \{0\}$ and the proof is complete.

<u>2.3.3 Proposition</u>. Let A be an approximately normal function algebra on X and let $\{C_\alpha\}$ be the family of components of X. If $f \in C(X)$ satisfies $f|_{C_\alpha} \in A|_{C_\alpha}$ for each α, then f belongs to A.

<u>Proof</u>. For each α choose $g_\alpha \in A$ such that $f|_{C_\alpha} = g_\alpha|_{C_\alpha}$. Let $\varepsilon > 0$ and put

$$U_\alpha = \{x \in X : |f(x) - g_\alpha(x)| < \varepsilon\} \quad \text{for all } \alpha.$$

Since U_α is an open neighbourhood of C_α and X is compact Hausdorff, there exists an open-closed subset F_α of X with $C_\alpha \subset F_\alpha \subset U_\alpha$ (see [20], Theorem 2.15). Since X is compact, a finite number of the F_α's cover X. Call them $F_1, \ldots, F_n$ and the corresponding U_α's $U_1, \ldots, U_n$. By taking finite unions and complements, one can ensure that the F_i's, $1 \leqslant i \leqslant n$, are pairwise disjoint. (Put $F_1' = F_1$, $F_2' = F_2 - F_1$, $F_3' = F_3 - (F_1 \cup F_2)$

and so on.)

Since A is approximately normal and the F_i's are open-closed, there exists $h_i \in A$ with $h_i(F_i) = 1$ and $h_i(X-F_i) = 0$ for all $i = 1,\ldots,n$: let $m \in \mathbb{N}$, then there exists $f_m \in A$ with $|f_m(F_i) - 1| < \frac{1}{m}$ and $|f_m(X-F_i)| < \frac{1}{m}$. Thus the sequence $\{f_m\}$ converges uniformly to h_i on X and since A is closed, $h_i \in A$. Put

$$g = \sum_{i=1}^{n} h_i g_i$$

where $f|_{C_i} = g_i|_{C_i}$ and $F_i \subset U_i$. Then $g \in A$. We claim that $\|f - g\| < \varepsilon$, for then, since ε is arbitrary, f belongs to A.

So let $x \in X$. Then there is a unique i_0 such that $x \in F_{i_0}$. Hence $h_{i_0}(x) = 1$ and $h_i(x) = 0$ for $i \neq i_0$. Thus $g(x) = g_{i_0}(x)$ and since $x \in F_{i_0} \subset U_{i_0}$ we have $|g(x) - f(x)| = |g_{i_0}(x) - f(x)| < \varepsilon$.

We immediately obtain

<u>2.3.4 Corollary</u>. If X is compact totally disconnected, then C(X) is the only approximately normal function algebra on X.

<u>Proof</u>. The components $\{C_\alpha\}$ of X are points, hence $f|_{C_\alpha} \in A|_{C_\alpha} = C(C_\alpha)$ for each α. Thus $f \in A$ for every f in C(X) (A as in 2.3.3 above).

Now let B be any Banach algebra. By a <u>complex homomorphism</u> m of B we mean a map

$$m : B \to \mathbb{C}$$

such that (a) m is a linear non-zero functional and

(b) $m(xy) = m(x).m(y)$ for all $x,y \in B$.

The pertinent information in 1.4.6 regarding real-valued linear functionals also applies to complex homomorphisms. The necessary details for such homomorphisms are contained in the following theorem.

2.3.5 Theorem. Let A be a sup norm algebra. Define $\varphi: X \to M_A$ by $\varphi(x) = m_x$ and put $\hat{A} = \{\hat{f}: f \in A\}$. Then

(1) φ is a continuous mapping from X onto a compact subset of M_A

(2) Each $\hat{f} \in \hat{A}$ assumes its maximum modulus on the image of X in M_A

(3) The Gelfand map $f \to \hat{f}$ is an isometric algebra isomorphism

(4) $\hat{A}$ is a function algebra on M_A

(5) If A separates points in X (i.e. if A is a function algebra), then the mapping φ is a homomorphism onto a compact subset of M_A .

Proof. [25], p. 46

2.3.6 Definition. (1) Let $m \in M_A$. A representing measure for m is a positive measure μ satisfying

$$\hat{f}(m) = \int_X f \, d\mu \qquad (\forall\ f \in A)$$

By the Riesz Representation Theorem (see also 2.1) every complex homomorphism of A has a (not necessarily unique) representing measure.

(2) Let A be a sup norm algebra. By a boundary ∂ of A we mean a subset of M_A such that every $\hat{f} \in \hat{A}$ attains its

maximum modulus at a point of ∂.

By virtue of the Axiom of Choice there exists a unique minimal boundary for A, called the <u>Šilov boundary</u> and denoted by ∂_A.

If A is a function algebra on X, then it follows from Theorem 2.3.5 that $\partial_A \subset \partial_{\hat{A}} \subset X \subset M_A$.

<u>2.3.7 Lemma.</u> Let A be an approximately normal function algebra on X. Then X is the Šilov boundary of A.

<u>Proof.</u> Let F be any proper closed subset of X and choose $x_0 \in X-F$. By hypothesis there exists a function f in A with $|f(F)| < \frac{1}{4}$ and $|1 - f(x_0)| < \frac{1}{4}$. Hence

$$|f(x_0)| > \frac{3}{4} > \|f|_F\|$$

so F is not a boundary for A.

For normal algebras we have

<u>2.3.8 Proposition.</u> Suppose A is a normal function algebra on X. Then $X = M_A$ and hence $A = \hat{A}$.

<u>Proof.</u> [25], p. 183

We need the following result from the thesis by Sacks.

<u>2.3.9 Proposition.</u> Let $A = [g]$ be a singly generated function algebra on X. If $\partial_A = M_A$, then $A = C(X)$.

<u>Proof.</u> See [34], Proposition 3.22

We are now able to prove

<u>2.3.10 Theorem.</u> If A is a singly generated normal function

algebra on X, then $A = C(X)$.

Proof. By 2.3.7 and 2.3.8 we have that $\partial_A = X = M_A$. Hence, by 2.3.9, $A = C(X)$.

Remark. We note that McKissick's example (see remark after 2.3.1) has two generators. This follows from a result in [25] (Ch. 5, Th. 7), which states that, if X is a compact subset of $\mathbb{C}^n$, then there is a function $f \in R(X)$ such that $z_1, \ldots, z_n, f$ generate R(X). In McKissick's example X is a compact subset of the plane, so R(X) has two generators.

We now introduce another separation property for function algebras (see [38])

2.3.11 Definition. A function algebra A on X is strongly regular if for each f in A, each point x in X, and every $\varepsilon > 0$, there is a neighbourhood U of x and a function g in A with $g(y) = f(x)$ for all $y \in U$ and $|f(y) - g(y)| < \varepsilon$ for all $y \in X$.

Equivalently, each function $f \in A$ is uniformly approximable on X by functions in A which are constant near any point of X. Clearly strong regularity implies regularity.

Before proceeding, we need the notion of a Jensen measure. Let $m \in M_A$ where A is a function algebra. Then a Jensen measure for m is a positive measure $\mu \in M(X)$ such that

$$\log|\hat{f}(m)| \leq \int \log|f|\, d\mu \quad \text{for all } f \text{ in } A.$$

Lemma. Every Jensen measure is a representing measure.

<u>Proof</u>. See e.g. [25], p. 76

Leibowitz also shows that every $m \in M_A$ has a Jensen measure μ on X ([25]). This was first proved by Bishop.

<u>2.3.12 Lemma</u>. If A is a strongly regular function algebra on X, then $M_A = X$.

<u>Proof</u>. Let $m \in M_A$ and let μ be a Jensen measure for m on X. Suppose that $x \in \operatorname{supp} \mu$, with $x \neq m$. Let $\hat{f} \in \hat{A}$ with $\hat{f}(m) = 1$ and $\hat{f}(x) = 0$. Choose g in A satisfying $g(y) = f(x) = 0$ if y is in a neighbourhood U of x, and $|g(y) - f(y)| < \varepsilon$ for all y in X. Then

$$|\hat{g}(m) - \hat{f}(m)| = \left|\int g\, d\mu - \int f\, d\mu\right| \leq \int |g - f|\, d\mu$$

$$< \varepsilon \ . \ (\text{Assume } |\mu|(X) = 1)$$

Thus $|\hat{g}(m)| > 1 - \varepsilon$ (assume $\varepsilon < 1$) and hence

$$-\infty < \log |\hat{g}(m)| \leq \int \log |g|\, d\mu = -\infty$$

– which is impossible. The last equality follows from the fact that g is identically zero in a neighbourhood of a point in supp μ which has positive μ-measure. It follows that $\operatorname{supp} \mu = \{m\}$ and hence $m \in X$.

<u>2.3.13 Corollary</u>. If A is strongly regular on X, then A is normal on X.

<u>Proof</u>. By 2.3.12 $X = M_A$. Let F,S be disjoint closed sets in X, and let $I = kF$, $J = kS$ be their kernels (e.g. I is the set of functions in A that vanish on F)

Now let $x \in X$. Then either $f(x) \neq 0$ for some $f \in I$ or $g(x) \neq 0$ for some $g \in J$: for if $x \in F$, then $x \notin S$, so there is $g \in A$ with $g(x) \neq 0$, $g(S) = 0$ (by regularity) while if $x \notin F$, then there is f in A with $f(x) \neq 0$, $f(F) = 0$ So $I \cup J$ is contained in no maximal ideal of A, which are all of the form M_x, some $x \in X = M_A$. Hence the ideal generated by $I \cup J$ is A, i.e. $A = I + J$. So there is $f \in I$ such that $(1 - f) \in J$. Thus $f(F) = 0$ and $f(S) = 1$. So A is normal.

<u>2.3.14 Definition.</u> We say that a point x_0 in X is a <u>peak point</u> for the function algebra A if there is f in A with $f(x_0) = 1$ and $|f(x)| < 1$ for $x \neq x_0$. Similarly, a <u>peak set</u> for A is a non-void subset F of X such that there exists a function f in A with $f(F) = 1$ and $|f| < 1$ on X-F. We note that finite unions and countable intersections of peak sets are again peak sets.

<u>2.3.15 Lemma.</u> If F is an intersection of peak sets and A is a function algebra on X, then $A|_F$ is closed in C(F).

<u>Proof.</u> Let $kF = \{f \in A: f(F) = 0\}$. Then the quotient algebra $A/_{kF}$ is a Banach algebra under the norm

$$\|f + kF\| = \inf\{\|f + g\| : g \in kF\}$$

while $\|f|_F\| = \sup\{|f(x)| : x \in F\}$ will denote the norm of $f|_F$ in $A|_F$. Since multiplicative linear functionals on $A/_{kF}$ are of norm $\leqslant 1$, $|f(x)| \leqslant \|f + kF\|$ for $x \in F$, and so $\|f|_F\| \leqslant \|f + kF\|$.

Let $\varepsilon > 0$ and let V be an open neighbourhood of F on which $|f| \leqslant \|f|_F\| + \varepsilon$. By compactness of X-V, there exists a finite intersection S of peak sets containing F such that

$F \subset S \subset V$. So S is a peak set for A and hence there is $g \in A$ with $g(S) = 1$ and $|g(x)| < 1$ for $x \in X-S$.

But $g^n f \in f + kF$ and

$$\limsup \|g^n f\| \leq \sup |f(V)| \leq \|f|_F\| + \varepsilon .$$

Consequently $\|f + kF\| = \|f|_F\|$. Thus $A|_F$, as a subalgebra of $C(F)$, is isometrically isomorphic with the complete algebra $A/_{kF}$, and hence $A|_F$ is closed in $C(F)$.

Let A be a function algebra on X. A function $f \in C(X)$ is said to be <u>locally in</u> A if there is a finite open cover $\{U_i\}_{i=1}^n$ of X such that for each i, there exists a function $g_i \in A$ with $f|_{U_i} = g_i|_{U_i}$. A function algebra A is called <u>local</u> if every function locally belonging to A belongs to A.

For normal algebras we have

<u>2.3.16 Proposition</u>. If A is normal, then A is a local function algebra.

<u>Proof</u>. E.g. Ewer's thesis [7], Theorem 7.4

We also need a result due to Bishop ([4]). The proof can be found in the same paper.

<u>2.3.17 Proposition</u>. If A is a function algebra on a compact metrizable space X, then the set of peak points for A is dense in the Šilov boundary ∂_A .

We are now able to prove the following theorem.

<u>2.3.18 Theorem</u>. If A is a strongly regular function algebra on a closed Jordan arc Γ, then $A = C(\Gamma)$.

Proof. By 2.3.13 above, $M_A = \Gamma$. Since A is normal (2.3.14), $\partial_A = M_A = \Gamma$. Let j be a homeomorphism from [0,1] onto Γ, and let x be a peak point of A, $x \neq j(0), j(1)$. (By 2.3.17 the set of peak points of A is dense in Γ.) Let f be a function in A peaking at x. Thus $f(x) = 1$ and $|f(y)| < 1$ for $y \neq x$.

Let $\{(a_n, b_n)\}$ be a sequence of open subarcs about x and $\{f_n\}$ a sequence of functions in A such that $f_n = 1$ on (a_n, b_n) for each n and $f_n \to f$ uniformly on Γ (by strong regularity). Define a sequence of functions $\{g_n\}$ by

$$g_n(t) = \begin{cases} 1 & \text{for } t \text{ such that } j^{-1}(t) > j^{-1}(a_n) \\ f_n(t) & \text{for } t \text{ such that } j^{-1}(t) < j^{-1}(b_n) \end{cases}$$

Thus $\{g_n\} \subset A$: Fix n. Then $g_n|_{[j(0),b_n)} = f_n$ and $g_n|_{(a_n, j(1)]} = 1$. Thus g belongs locally to A. But A is local (by 2.3.14 and 2.3.16), so g_n belongs to A for each n.

Now $g_n \to g$ uniformly on Γ where

$$g(t) = \begin{cases} 1 & \text{for } t \text{ such that } j^{-1}(t) \geq j^{-1}(x) \\ f(t) & \text{for } t \text{ such that } j^{-1}(t) \leq j^{-1}(x) \end{cases}$$

Clearly $g \in A$. Hence $[j(0),x]$, and similarly $[x,j(1)]$, are peak sets of A. Thus every closed subset F of Γ is a finite union or an intersection of peak sets. Hence $A|_F$ is closed in C(F) by Lemma 2.3.15 and so by Theorem 2.2.1 $A = C(\Gamma)$.

We end this section with

Remark. Using Jensen measures one can obtain a generalization of Theorem 2.2.8 to countable unions. Formally, we have:

Let A be a function algebra on X. Suppose that $X = \bigcup_{i=1}^{\infty} F_i$, where F_i is closed in X and $A|_{F_i} = C(F_i)$ for each i.
Then $A = C(X)$.

For a proof of this result the reader is referred to [25], p. 185. A detailed discussion would be too lengthy, as several auxiliary results are needed. The technique, however, is similar to previous ones. Essentially, it is shown that A is an approximately normal algebra on X and that the essential set of A is empty. (The essential set of A is the smallest closed subset E of X such that A contains all the continuous functions that vanish on E. Equivalently, E is the zero-set of the largest ideal of C(X) contained in A.)

2.4 ε-Normal function algebras

In this section we mainly consider the work done by Curtis and Bade in their papers [1] and [2]. We have adapted some of their results to the case of uniformly closed subalgebras of C(X). The methods used in this section differ markedly from those employed in the preceeding ones. Throughout this section, X denotes a compact Hausdorff space.

We have the following easy result.

2.4.1 Proposition. Let A be a subspace of C(X). Suppose there exists a constant $k < 1$ such that for each $f \in C(X)$ with $\|f\| \leq 1$, there is a g in A with $\|f - g\| \leq k$.
Then $\mathrm{cl}A = C(X)$.

Proof. Suppose $\mathrm{cl}A \neq C(X)$. Since clA is a closed proper subspace of the Banach space C(X), there exists a continuous linear functional φ, $\varphi \neq 0$, with $\varphi(\text{cl}A) = 0$. Let $\|f\| \leqslant 1$. Choose $g \in A$ with $\|f - g\| \leqslant k$. Then

$$\begin{aligned} |\varphi(f)| &\leqslant |\varphi(f - g)| + |\varphi(g)| \\ &= |\varphi(f - g)| \\ &\leqslant \|\varphi\| . \|f - g\| \\ &\leqslant k\|\varphi\| \end{aligned}$$

But $\|\varphi\| = \sup\limits_{f \neq 0} \dfrac{|\varphi(f)|}{\|f\|} = \sup\limits_{\substack{\|f\| \leqslant 1 \\ f \neq 0}} |\varphi(f)|$ and hence

$\|\varphi\| \leqslant k\|\varphi\| \lneq \|\varphi\|$ - contradiction.

Denote by $S_{C(X)}$ the unit ball $\{f \in C(X) : \|f\| \leqslant 1\}$.

<u>2.4.2 Theorem</u>. Let A be a uniformly closed subspace of C(X) and $T: A \to A|_F$ be the linear map defined by $T(f) = f|_F$, for some compact subset F of X and all $f \in A$. Suppose there exist constants $k < 1$ and M such that for each $f \in S_{C(F)}$, there is a g in A with $\|Tg - f\| \leqslant k$ and $\|g\| \leqslant M$. Then $TA = A|_F = C(F)$.

<u>Proof</u>. By 2.4.1 it is immediate that $\text{cl}(A|_F) = C(F)$. Hence we need only show that $A|_F$ is in fact a closed subspace of C(F). We will show that the adjoint T^* of T has closed range, from which it follows that T has closed range (e.g. [25], p. 16).

Clearly T is continuous. Let φ be a continuous linear functional on C(F) and let $f \in S_{C(F)}$. Choose $g \in A$ with $\|Tg - f\| \leqslant k$. Then

$$\begin{aligned} \|\varphi(f)\| &= \|\varphi(Tg - f) + \varphi(Tg)\| \\ &\leqslant \|\varphi(Tg - f)\| + \|\varphi(Tg)\| \end{aligned}$$

$$\leq \|\varphi\| . k + M . \|T^{*}\varphi\|$$

Taking sups over $S_{C(F)}$, we get that

$$\|\varphi\| \leq k . \|\varphi\| + M . \|T^{*}\varphi\|$$

Hence $\|T^{*}\varphi\| \geq M^{-1}(1-k)\|\varphi\|$, all $\varphi \in [C(F)]^{*}$. Now let $\{T^{*}\varphi_n\}$ be a Cauchy sequence in $[A]^{*}$. Since $\|T^{*}\varphi_n - T^{*}\varphi_m\| \geq M^{-1}(1-k)\|\varphi_n - \varphi_m\|$, $\{\varphi_n\}$ forms a Cauchy sequence in $[C(F)]^{*}$. As the latter space is complete, $\{\varphi_n\}$ tends to a limit φ. But then $\{T^{*}\varphi_n\} \to T^{*}\varphi$ as well. Thus T^{*} has closed range and so also has T.

<u>2.4.3 Note</u>. As we pointed out above, $cl(A|_F) = C(F)$ trivially. Indeed, this is true whether A is uniformly closed or not. It is also clear that the boundedness condition is not a necessary one for $cl(A|_F)$ to coincide with C(F). In 2.4.13, however, we require that $A|_F = C(F)$ for certain compact subsets F of X. It is for this reason that we have stated Theorem 2.4.2 in this particular form. (See also 2.4.14)

We are already familiar with the notion of approximate normality. Bade and Curtis introduced a (apparently) weaker form of this concept.

<u>2.4.4 Definition</u>. A family A of functions in C(X) is <u>ε-normal</u> for some fixed $\varepsilon > 0$, if, given any two disjoint compact subsets F_1 and F_2 of X, there exists $f \in A$ with $|f(F_1)| \leq \varepsilon$ and $|f(F_2) - 1| \leq \varepsilon$.

We will show below that for subalgebras of C(X) ε-normality (for $\varepsilon < \frac{1}{2}$) and approximate normality are equivalent. However, if we add a boundedness condition as outlined in 2.4.2, we are able to obtain a generalization of an earlier result. We will need the following

2.4.5 Theorem. (Mergelyan) Suppose $f \in C(X)$ is analytic on a compact set X in $\mathbb{C}$. If $\mathbb{C}-X$ is connected, then f can be uniformly approximated on X by polynomials in z.

Proof. See e.g. [33]

2.4.6 Proposition. If A is an ε-normal subalgebra of $C(X)$ for some $\varepsilon < \frac{1}{2}$, then A is ε-normal for all such ε (i.e. A is approximately normal).

Proof. Let $0 < \eta < \varepsilon$. Suppose F_1 and F_2 are two disjoint compact subsets of X. We need to find an $h \in A$ with $|h(F_1)| \leq \eta$ and $|h(F_2) - 1| \leq \eta$.

By ε-normality of A there is $f \in A$ with $|f(F_1)| \leq \varepsilon$ and $|f(F_2) - 1| \leq \varepsilon$. Consider the two disjoint discs $D_0 = \{ z : |z| \leq \varepsilon \}$ and $D_1 = \{ z : |z - 1| \leq \varepsilon \}$. There exists a continuous function g such that

$$g(z) = \begin{cases} 0 & z \in D_0 \\ 1 & z \in D_1 \end{cases}$$

By Mergelyan's theorem (2.4.5) g can be uniformly approximated on $D_0 \cup D_1$ by polynomials in z. Let $p(z) = a_1 z + \ldots + a_n z^n$ be a polynomial without constant term such that $|g(z) - p(z)| < \eta$ on $D_0 \cup D_1$, i.e. $|p(z)| < \eta$ on D_0 and $|p(z) - 1| < \eta$ on D_1.

In fact, $|p(z)| < \eta$ on $f(F_1) \subset D_0$

and $|1 - p(z)| < \eta$ on $f(F_2) \subset D_1$.

Thus $|a_1 f(w) + \dots + a_n[f(w)]^n| < \eta$ on F_1

and $|a_1 f(w) + \dots + a_n[f(w)]^n - 1| < \eta$ on F_2.

Now let $h = a_1 f + a_2 f^2 + \dots + a_n f^n$. Then h belongs to A and satisfies the required separation property.

2.4.7 Remark. (1) It is clear that the polynomial p(z) depends on η only.

(2) We say that a subalgebra A of C(X) is boundedly ε-normal if there exist constants $\varepsilon > 0$ and M such that, given two disjoint compact subsets F_1 and F_2 of X, there is a $f \in A$ with $|f(F_1)| \leq \varepsilon$, $|f(F_2) - 1| \leq \varepsilon$ and $\|f\| \leq M$.

(3) Suppose A is boundedly ε-normal for some $\varepsilon < \frac{1}{2}$. Then A is boundedly ε-normal for all such ε, where the constant M^* depends on ε. For choose $0 < \eta < \varepsilon$ and let $h = a_1 f + \dots + a_n f^n$, where the function f depends on the pair of compact sets under consideration. Since any such f will have $\|f\| \leq M$, we get that

$$\|h\| \leq |a_1|M + |a_2|M^2 + \dots + |a_n|M^n = M^*.$$

(4) We note, however, that we cannot say that A will be boundedly approximately normal as we cannot determine a bound for $|a_1|M + \dots + |a_n|M^n$ (n and the coefficients $a_1, \dots, a_n$ depend on η - see (1) above).

Before we proceed we need some further definitions and results. A subset B of a Banach space Y is balanced if

$y \in B$ and $|\alpha| \leqslant 1$ imply $\alpha y \in B$. Denote by co(B) the smallest convex set containing B and by coe(B) the smallest convex balanced set containing B. Their respective closures are denoted by $\overline{co}(B)$ and $\overline{coe}(B)$. Note that coe(B) consists of all sums $\sum_{i=1}^{n} \alpha_i y_i$ where $y_i \in B$ and $\sum_{i=1}^{n} |\alpha_i| \leqslant 1$.

Also S_+ denotes the real-valued nonnegative functions in $S_{C(X)}$. We say that a family E of functions is bounded if $\sup \{ \|f\| : f \in E\} < \infty$. We now come to

2.4.8 Lemma. If E is a bounded normal family in C(X), then $\overline{co}(E)$ contains S_+ and hence $\overline{coe}(4E)$ contains $S_{C(X)}$.

Proof. See [2], Lemma 1.4

The following results will lead us to a generalization of Theorem 2.3.2. We will show that 2.3.2 holds if the uniformly closed algebra A is boundedly ε-normal for some $\varepsilon < \frac{1}{2}$ (see Corollary 2.4.11). In fact it holds if A is ε-normal locally bounded. The meaning of 'locally bounded' will be made clear later.

2.4.9 Theorem. Let F be a compact subset of X, A a uniformly closed algebra on X and $T: A \to A|_F$ the restriction map. Suppose there exist constants M and $0 < \varepsilon < \frac{1}{2}$ such that, whenever F_1 and F_2 are disjoint compact subsets of F, there is a g in A with $|g(F_1)| \leqslant \varepsilon$, $|g(F_2) - 1| \leqslant \varepsilon$ and $\|g\|_X \leqslant M$. Then $A|_F = C(F)$.

Proof. By 2.4.7(3) A is boundedly ε-normal on F for some $\varepsilon < \frac{1}{4}$. Let $\varepsilon < \varepsilon^* < \frac{1}{4}$. If F_1 and F_2 are two disjoint compact subsets of F, choose $g \in A$ with $|g(F_1)| \leq \varepsilon$, $|g(F_2) - 1| \leq \varepsilon$ and $\|g\|_X \leq M^*$. Also choose $f \in S_{C(X)}$ with $f(F_1) = 0$ and $f(F_2) = 1$. Thus $\|f - g\|_{F_1 \cup F_2} \leq \varepsilon$.

Since $C(F_1 \cup F_2)$ is isometrically isomorphic with the quotient of C(F) by the closed ideal of all functions vanishing on $F_1 \cup F_2$ (i.e. $C(F_1 \cup F_2) \cong C(F)/_{k(F_1 \cup F_2)}$), it follows that

$$\begin{aligned} \|f - g\|_{F_1 \cup F_2} &= \|(f - g) + k(F_1 \cup F_2)\|_F \\ &= \inf\{ \|(f - g) + h\|_F : h \in k(F_1 \cup F_2)\} \\ &\leq \varepsilon \end{aligned}$$

Hence we can select $g_f \in C(F)$ with

$$g_f(F_1 \cup F_2) = 0 \quad \text{and} \quad \|g - (f + g_f)\|_F < \varepsilon^* \qquad \ldots \text{ I}$$

Let $\tilde{S} = \{f + g_f : f \in S_{C(F)}\} = \{\tilde{f}\}$. Clearly $\tilde{S}$ is normal on F. Also $\tilde{S}$ is bounded: Let $\tilde{f} \in \tilde{S}$. Then there exists $g \in A$ with $\|g - \tilde{f}\|_F \leq \varepsilon^*$ (by I). Thus $\|\tilde{f}\|_F \leq \varepsilon^* + \|g\|_F \leq \frac{1}{4} + M^*$. By Lemma 2.4.8 it follows that $\overline{co}\,(\tilde{S}) \supset S_+$, and hence $\overline{coe}(4\tilde{S}) \supset S_{C(F)}$.

Now choose $f \in S_{C(F)}$. By above there exists $\tilde{f} = \sum_{i=1}^{n} \alpha_i \tilde{f}_i$ with $\tilde{f}_i \in 4\tilde{S}$ and $\sum_{i=1}^{n} |\alpha_i| \leq 1$ such that

$$\|\tilde{f} - f\|_F < \frac{1}{4} - \varepsilon^* = \eta.$$

For each i there exists $g_i \in A$ with $\|g_i - \tilde{f}_i\|_F \leq 4\varepsilon^*$ and $\|g_i\|_X \leq 4M^*$. Let $g = \sum_{i=1}^{n} \alpha_i g_i$. Then $g \in A$ (since A is an algebra) and $\|g\|_X \leq \sum_{i=1}^{n} |\alpha_i| . 4M^* \leq 4M^*$.

Now
$$\|g - f\|_F = \|g - \tilde{f} + \tilde{f} - f\|_F$$
$$\leqslant \|g - \tilde{f}\|_F + \|\tilde{f} - f\|_F$$
$$< \sum_{i=1}^{n} |\alpha_i| . \|g_i - \tilde{f}_i\|_F + \eta$$
$$\leqslant \sum_{i=1}^{n} |\alpha_i| . 4\varepsilon^* + \eta$$
$$\leqslant 4\varepsilon^* + \eta < 1 .$$

Thus we can find $k < 1$ such that $\|f - g\|_F \leqslant k$ and $\|g\|_X \leqslant 4M^*$. Since $f \in S_{C(F)}$ was chosen arbitrary, result follows by Theorem 2.4.2.

<u>2.4.10 Note</u>. (1) If we only assume that $\|g\|_F$ is bounded by M, then one can show that $cl(A|_F) = C(F)$ (see Note 2.4.3) Indeed, in the proof above it would still be true that $\tilde{S}$ is bounded and that the constant $k < 1$ satisfying the condition in Theorem 2.4.2 exists.

(2) However, the assumption that $\|g\|_X \leqslant M$ guarantees that $A|_F$ is closed in C(F) if A is closed in C(X).

<u>2.4.11 Corollary</u>. If A is a boundedly ε-normal subalgebra of C(X) for some $\varepsilon < \frac{1}{2}$, then $clA = C(X)$.

This corollary is a generalization of Theorem 2.3.2. We note however that we do require A to be a sub<u>algebra</u> of C(X).

<u>2.4.12 Corollary</u>. Let X be a compact totally disconnected space. Suppose B is an ε-normal subalgebra of C(X) for some $\varepsilon < \frac{1}{2}$. Then B is uniformly dense in C(X).

<u>Proof</u>. We show that B is boundedly ε-normal. So let F_1 and

F_2 be two disjoint compact subsets of X. Since X is totally disconnected, there exist closed-open disjoint sets $\overline{F}_1$ and $\overline{F}_2$ containing F_1 and F_2 respectively such that $\overline{F}_1 \cup \overline{F}_2 = X$. Since B is ε-normal there is a function f in B with $|f(\overline{F}_1)| \leq \varepsilon$ and $|f(\overline{F}_2) - 1| \leq \varepsilon$, i.e. $\|f\| \leq 1 + \varepsilon$. Choose $M = 1 + \varepsilon$. By 2.4.11 the uniform closure of B coincides with C(X).

We now turn our attention to locally bounded algebras. We say that an ε-normal family B in C(X) is <u>locally bounded</u> if for each point $x \in X$ there exist a compact neighbourhood N_x of x and a constant K_x such that the conditions in Theorem 2.4.9 are satisfied with $F = N_x$ and $M = K_x$.

For Banach algebras we have

<u>2.4.13 Theorem</u>. Let A be an ε-normal locally bounded uniformly closed algebra on X ($\varepsilon<\frac{1}{2}$). Then $A = C(X)$.

<u>Proof</u>. For each $x \in X$ find a compact neighbourhood N_x of x and a constant K_x satisfying the conditions in 2.4.9 with $F = N_x$ and $M = K_x$. By Theorem 2.4.9 it is immediate that

$$A|_{N_x} = C(N_x)$$

for each $x \in X$. Since A is ε-normal on X, A separates the points in X (in the weak sense). Hence the result follows by Corollary 2.2.9.

<u>2.4.14 Note</u>. (1) We have thus shown that the condition of boundedness in Corollary 2.4.11 can be weakened to that of local boundedness, although we must then ensure that the

algebra A is uniformly closed. (This latter condition guarantees that $A|_{N_x} = C(N_x)$ for each $x \in X$ - see 2.4.3)

(2) We will show that neither the condition that A be uniformly closed nor the condition that A be locally bounded is superfluous.

Example. We say that a function algebra A is <u>pervasive</u> if $cl(A|_F) = C(F)$ for every proper closed subset F of X. We will now show that proper pervasive algebras exist.

Let Δ be the unit disc and $b\Delta$ the unit circle in the complex plane ($b\Delta$ is the boundary of Δ). Let $A = \{f \in C(\Delta)$: f is holomorphic in Δ and continuous on $\Delta\}$ and let $A^1 = A|_{b\Delta}$ be the restriction algebra of A to $b\Delta$. Then A is a uniformly closed (proper) subalgebra of $C(\Delta)$ and A^1 is a uniformly closed (proper) subalgebra of $C(b\Delta)$.

Now A, and hence also A^1, is essential (i.e. A contains no non-trivial closed ideal of $C(\Delta)$): Let D be a proper closed set in Δ. Then there exists $f \in C(\Delta)$ which vanishes on D but is not holomorphic in Δ. Thus $\{0\}$ is the only ideal contained in A.

Also A^1 is maximal (i.e. if a closed subalgebra B is such that $A^1 \subset B \subset C(b\Delta)$, then either $A^1 = B$ or $B = C(b\Delta)$): This follows from Wermer's maximality theorem. For a detailed proof the reader is referred to Gamelin's book [10], p. 38.

By a result in [25] (Ch. 7, Th. 25) A^1 is pervasive on $b\Delta$. It is clear however that A is not pervasive on Δ.

Now A and A^1 are also both approximately normal algebras. In the case of the algebra A this follows from Mergelyan's theorem - see 2.4.5 and the proof of 2.4.6. In the case of the algebra A^1 this can be seen as follows: Let F_1 and F_2 be two disjoint closed sets in $b\Delta$. Since $b\Delta$ is connected, $F_1 \cup F_2$ is a proper closed set in $b\Delta$. Thus, since A^1 is pervasive, $cl(A^1|_{F_1 \cup F_2}) = C(F_1 \cup F_2)$. So, given any $\varepsilon > 0$, there is a function f in A with $|f(F_1)| < \varepsilon$ and $|f(F_2) - 1| < \varepsilon$. Thus A^1 is approximately normal on $b\Delta$.

To illustrate remark (2) above suppose that

(i) A is not uniformly closed in C(X), but satisfies the local boundedness condition. Then it is true that $cl(A|_{N_x}) = C(N_x)$ for each $x \in X$, but the example shows that this condition is <u>not</u> sufficient to ensure that $cl A = C(X)$. (A^1 is pervasive but $A^1 \neq C(b\Delta)$.)

(ii) Suppose now that A is uniformly closed but fails to be locally bounded. Let F be any compact subset of X. Then the example shows that $A|_F$ need not be closed in C(F), nor need it be uniformly dense in C(F). (See Note 2.4.3) (The algebra A as defined in the example is a proper closed subalgebra of $C(\Delta)$ which is not pervasive on Δ.)

From the discussion above it is apparent that some sort of boundedness condition is required to ensure that ε-normal closed subalgebras of C(X) coincide with C(X).

Finally, we will briefly outline a class of spaces which will ensure that ε-normal subalgebras ($\varepsilon<\frac{1}{2}$) do in fact satisfy the local boundedness property.

F-spaces

2.4.15 Definition. A compact Hausdorff space X is called a F-space if disjoint F_σ-sets in X have disjoint closures.

This class of spaces was introduced by Gillman and Henriksen in [39]. It includes stonian and σ-stonian spaces. Connected examples exist, such as $\beta R_+ - R_+$ where R_+ denotes the nonnegative reals ([11], 6.10).

Seever in [40] has shown that the support of any measure on a compact F-space is totally disconnected. This result allows us to prove the following theorem.

2.4.16 Theorem. Let X be a F-space and A an ε-normal subalgebra of C(X), for some $\varepsilon < \frac{1}{2}$. Then A is uniformly dense in C(X).

Proof. It suffices to show that A restricted to the support of any measure on X is uniformly dense in the space of all continuous functions on this support. But by the remark above every support is totally disconnected. Thus by Corollary 2.4.12 $A|_{\text{supp}\,\mu}$ is uniformly dense in C(supp μ), where μ is any Borel measure on X.

Notes

To obtain Lemma 2.1.3 we have used the following version of the Hahn-Banach theorem: if F is a closed subspace of a locally convex vector space E and if f is in E, then $f \notin F$ if and only if there is a continuous linear functional φ on E such that $\varphi(f) = 1$ and $\varphi(F) = 0$. Lemmas 2.1.3 and 2.1.4 both appear in [25] (Ch. 3, 3.2). Lemma 2.2.3 is also known as the Bishop Antisymmetric Decomposition theorem. The proof given here is in [10]. Theorem 2.2.5 appears in [25], as does Lemma 2.2.7 and Theorem 2.2.8. In the proof of 2.2.8 we have simplified the argument as given in [25]. Theorem 2.2.10 is due to Leibowitz [25].

The proof of Theorem 2.3.2 also appears in [25]. Proposition 2.3.3 is in Wilken's paper [37]. In our opinion Wilken's proof is not complete. We have given a corrected version. Theorem 2.3.10 appears as a problem in [25]. Lemma 2.3.13 is due to Wilken [38] and we have used his proof. The proof of Corollary 2.3.14 is in [25], as is the proof of Lemma 2.3.15. Theorem 2.3.18 is an extension of the main result of [38], the proof being our own.

The results in Section 2.4 are based on a paper by Bade and Curtis [2]. The proof of Proposition 2.4.6 is our own. Regarding this, I wish to thank Dr. Walker for his effort in trying to find an approximation to

$$f(z) = \begin{cases} 0 & \text{on } D_0 = \{z: |z| \leq r\} \\ 1 & \text{on } D_1 = \{z: |z-1| \leq r\} \end{cases}$$

for $r < \frac{1}{2}$, by a function analytic in a region containing $D_0 \cup D_1$. Our efforts together finally succeeded. However, the mapping

$$f_n(z) = \frac{z^n}{z^n - (z - 1)^n}$$

did not find a place in this thesis as we found later that the desired approximation was a consequence of Mergelyan's theorem as given in Rudin. The observations in Remarks 2.4.7 and 2.4.14 are our own. In particular, Bade and Curtis restrict $\varepsilon < \frac{1}{4}$ in Theorem 2.4.9. This is unnecessary, in view of 2.4.7(3). Theorem 2.4.13 is also our own.

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