Interval-valued Uncertainty Capability Indices with South African Industrial Applications

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Abstract

Since the advent of statistical quality control and process capability analysis, its study and application has gained tremendous attention both in academia and industry. This attention is due to its ability to describe the capability of a complex process adequately, simply (i.e. using a unitless index) and also in some instances to compare different manufacturing processes. However, the application of statistical quality control has come under intense criticism, notably in one car manufacturing industry where the actual number of non-conforming units considerably exceeded expectation, although probabilistic control measures were in place. This failure led to a large recall of their vehicles and also left a dent on the image of the company. One of the reasons for this unfortunate instance is that in classical quality control measures, human judgement is ignored and since in process engineering there is considerable expert intuition in decision making, this element cannot be undermined. Hence the research study applies the uncertainty theory proposed by Baoding Liu (2007) to enable us to incorporate human judgement into process capability analysis.

The major findings of the thesis is that the uncertain process capability indices under an uncertainty environment are interval-valued and their relevant characteristics. The study further developed the "sampling" uncertainty distributions and thus the "sampling" impacts on the newly defined uncertain process capability indices under Liu's uncertain normal distribution assumptions. In order to reach the main purpose of the thesis, a thoroughgoing literature review on probabilistic process capability indices is necessary. Comparison between the newly proposed (uncertainty) capability index and its probabilistic counterpart were conducted and the findings were that the uncertainty capability index also yields a realistic representation of process performance at a higher level of significance (i.e. $\alpha=0.5$). Although a higher significance level is used this helps since expert data usually exaggerates process performance. Secondly, the newly proposed uncertainty capability indices also help in describing how the engineers think about their manufacturing process relative to the actual performance of the process. Therefore these newly proposed uncertainty capability indices complement their classical capability counterparts.

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Operational Terms

PCI – Process Capability Index

$PCI's$ – Process Capability Indices

LCL – Lower Control Limit

UCL – Upper Control Limit

LSL – Lower Specification Limit

USL – Upper Specification Limit

T – Target value of a process

M – Nominal value of the process

X - Quality characteristics of interest

$\overline{X}$ - Mean value of the quality characteristic X

$\overline{\overline{x}}$ - overall mean of the quality characteristic generated from multiple rational subgroups

R – Range of the rational subgroup

μ – population mean

σ – population standard deviation

$\hat{\sigma}$, S - sample standard deviation

σ_R - Population standard deviation of the Range, R

Y- process yield

Q – quantification of the specification used

d – half specification limit/tolerance

ppm – parts per million

NC – Nonconforming unit

$P(NC)$ – Proportion of NC outcome

F – Cumulative distribution Function

Φ - standard normal distribution

MLE – Maximum Likelihood Estimators

$UMVUE$ – Unbiased Minimum Variance Uniform Estimators

$C_{pm}(C)$ - Taguchi capability index due to Chan

$C_{pm}(B)$ - Taguchi capability index due to Boyles

TCI – Taguchi Capability Index

AR-Autoregressive model

MA-Moving Average model

ARMA-Autoregressive Moving Average model

ARIMA-Autoregressive Integrated Moving Average model

WN- White Noise distribution

WSD- Weighted Standard Deviation Method

CR.Measurement -Conductor Resistance Measurement

Chapter 1. Introduction

1.1 Roles of The Capability Indices in Quality management

Process Capability analysis is an important tool in maintaining the quality of a process and also ensuring its continuous improvement. These processes may vary from manufactured goods such as automobile parts, clinical instruments, clothing, plastic products, to services such as health care, timing of bus arrivals, mean waiting and service times at banks, measuring student performance etc. Thus process capability studies are of importance in almost every industry where quality of output is of the main concern.

The main aim of process capability studies over the years has been to ensure that processes and services are continually improved to meet higher levels of customer satisfaction and also to reduce the number of non-conforming units produced. These higher levels of customer satisfaction are usually met by ensuring that the products are uniform. Uniformity in this sense is obtained by reducing variability of the process to the extent that the only cause of variation present in the system is random.

Moreover, process capability studies are usually summarised by numerical measures called process capability indices (*PCI's*). Generally, the capability of a process usually defines how well the process is able to meet specifications set by customers or product designers. Thus process capability indices can be expressed as the proportion of the actual process spread measured by the specification width to the allowable process spread usually measured by six standard deviations. The six-sigma spread in the process is the reference value by which one determines how well the process performs to requirements.

The study of classical capability indices where the quality measurement is precise (crisp) has received enormous attention due to authors like Juran (1974), Kane (1986), Chan et al. (1988), Boyles (1991), Montgomery (1991), Pearn et al. (1992), Kotz and Johnson (1993a), Spiring (1997), Pearn et al.(2001), Lin (2002) etc. Some processes in practice produce quality measurements that are imprecise or vague in nature and require human judgement, such as the amount of pollutants in a water body, the colour intensity of a garment/textile, the amount of light passing through a touch-screen, all elements with coarse scales, the lifetime of a bulb etc. Classical capability indices are based on probability theory (frequency of events) which does not allow expert judgements. The development of fuzzy set theory proposed by Zadeh (1965) assigns graded membership to events in order to deal with ambiguity and vagueness. The application of fuzzy set theory to process capability analysis studies has gained wide attention since the first publication by Yongting (1996). This attention is due to its flexibility and the ability to handle imprecision. However, fuzzy set theory has come under considerable criticism as a branch of mathematics, due to the fact that it relaxes the law of contradiction.

Due to the shortcomings of fuzzy set theory, Baoding Liu (2007) proposed uncertainty theory to incorporate expert judgement into modelling. For instance, under any classical six sigma process, there are cases where the actual number of non-conforming units discernibly exceeds the expected units, and then the calculation of the capability indices will falsely suggest a capable process. Assuming a Gaussian normally distributed process, there is an expected number of non-conforming units (i.e. 0.27% of the Yield). In practice this expected number of non-conforming units is sometimes significantly less than the actual number of non-conforming units, hence the variation in the process will be understated (i.e. smaller standard deviation) and since the process capability is a function of the standard deviation, the magnitude of the process capability will be overstated. A typical example involves the incidents which occurred in the Japan car manufacturing industry, where the observed proportion of faulty brake pads exceeded expectation (i.e. assuming Gaussian normality) and later led to a large number of recalls, although there were classical quality control measures (i.e. based on probability theory) in place. The question arises whether or not classical quality control measures failed in this case? And would this scenario better be addressed by incorporating expert advice to make judgements. This research seeks to explore answers to these important questions via the theory of uncertainty proposed by Baoding Liu (2007), to incorporate expert judgement into decision making in a process capability assessment.

1.2 Aims and Objectives

The overall objective of this thesis is:

- The application of uncertainty theory to process capability analysis.

The aims of this thesis are to:

- Provide a comprehensive review of Classical Process Capability Indices
- Provide a comprehensive review of Uncertainty Theory.

- Propose interval-valued uncertainty process capability indies. Thus, find estimators comparable with the classical capability indices C_p, C_{pk} and C_{pm} when the underlying process is assumed to be Liu's uncertain normal distribution.

1.3 Overview of Thesis

The thesis focus is on a new quality improvement tool, interval-valued uncertainty capability index. The development requires many mathematical tools and statistical methods. Therefore the contents cover a wide range of knowledge. The thesis is composed of ten chapters, detailed as follows:

In chapter 1, process capability analysis is discussed, the aims and objectives of the thesis are stated, and a general overview of the thesis is given.

In chapter 2, three Shewhart charts to monitor process stability are reviewed. These charts are $\bar{X}$ -Chart, *R*-Chart and the *S*-Chart respectively. A brief summary of these charts is also tabulated.

In chapter 3, the classical capability indices were introduced and discussed, specifically, in terms of their strengths and weaknesses. A summary of and comparison between the classical capability indices were also conducted.

In chapter 4, the statistical properties of classical capability indices were further reviewed. The discussions are focused on four aspects: process departure, process yield, the ability to deal with sampling error in estimation and finally, the issue of the capability measurement when the process distribution is asymmetric.

In chapter 5, The Uncertainty Theory proposed by Baoding Liu (2007) is discussed. The Uncertainty Theory comprises three core concepts: Uncertain Measure, Uncertain Variable and Uncertain Distribution and this chapter explores these main concepts. Moments of the Uncertainty theory are also reviewed.

In chapter 6, The uncertainty statistical approach is discussed. Uncertainty statistics provides a methodology for collecting and also interpreting expert information. In order to determine the uncertainty distribution from the expert experimental data, the Least Square method, Delphi method and Method of Moments are discussed.

In chapter 7, the justification for applying uncertainty theory to process capability analysis is given. The counterparts of classical capability indices are also investigated under Liu's uncertain normal distribution environment. The interval-valued uncertain process capability indices are emphasised, the "sampling" uncertainty distributions are defined and developed and thus the "sampling" impacts on the interval-valued uncertain process capability indices are evaluated.

In chapter 8, the methods for constructing classical and uncertainty capability analysis are discussed.

In chapter 9, the theories proposed in the chapter 8 are applied to data obtained from a local manufacturing company. The two datasets are obtained from the same wire manufacturing company, being data obtained directly from the process output and qualitative data obtained from the industrial engineers (expert advice) about how they think the process is performing. The data obtained from the process output is analyzed using the classical capability indices and the expert data is analyzed using the uncertainty capability indices. These two capability estimates are then compared.

In chapter 10, a summary of the thesis is given, examining the achievement of objectives stated in chapter 1 and stating the contribution of this thesis to process capability analysis.

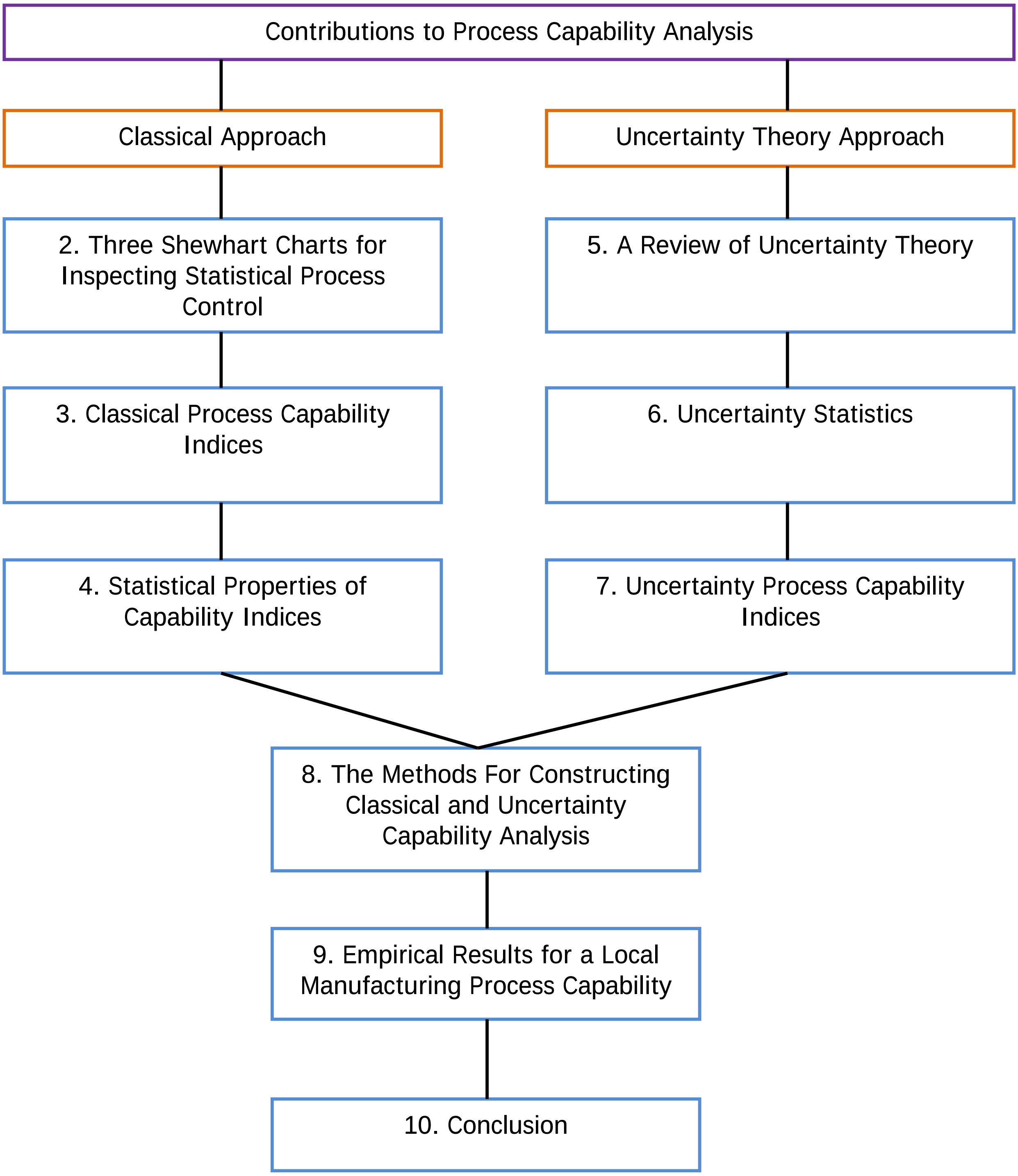

Figure 1: Overview of the Thesis.

Chapter 2. Three Shewhart Charts for Inspecting Statistical process Control

This chapter serves the aim of preparing the statistical environment for a process capability study. Hence, three statistical quality control charts proposed by Shewhart in 1924 are discussed.

In order to undertake a process capability study a fundamental requirement is to establish that the process is in statistical control. Mathematically, a process capability study could only play its role if the process is in a steady state.

Practically, it could be carried out in terms of various control charts monitoring a manufacturing process in its steady state or not. It is a well-known fact that checking if the process is in-control (i.e. the process mean, variance or range should be constant) or not, is equivalent to checking a process that is in a steady state or not.

A process is said to be in-control, when the only cause of variation present in the process is random or inherent and this variation can rarely be reduced by making any adjustments to the process (Montgomery, 2005). These inherent variations are regarded as an acceptable level of variation for the current objectives of the process. Otherwise, a process is said to be out-of-control (i.e. process influenced by external causes of variation to an unacceptable degree). An out-of-control process is deemed attributable to external causes of variation such as improper machine adjustments, operator error and defective raw materials (Montgomery, 2005).

Moreover, a process in statistical control is predictable, whereas no useful inferences about the future performance of an out-of control process can be made. The primary requirement in establishing statistical control is to assess the variability of an in-control process. If this variation is accurately established then it will be easier to detect a process that is running out of control by contrast. The most commonly used tool to monitor a process is called statistical control chart. The control chart is a graph that monitors whether a sequenced of observed data falls within the common or accepted range of variation [80]. To depict this usual range of variation a control chart consists of three horizontal lines such that the central line represents the process parameter (i.e. μ or σ) and the two other lines represent the control limits (i.e. *LCL* and *UCL*), which may be calculated by the true process parameter or the estimated one obtained by individual observations. A data point exceeding any of the limits is referred to as an out-of-control event within the process.

The most widely used control chart is the Shewhart $\overline{X}$ -control chart in combination with the $R-$ chart, and $S-$ chart or/and S^2- chart, which are attributable to Shewhart (1924) who introduced these simple but effective graphs to assess and monitor and hence assure quality. The primary uses of control charts are two- fold:

a) To collect sequenced data from a process and analyze the data, in order to establish whether or not the process is in-control for a capability study.
b) The parameters established for an in-control process in (a) above, will aid in the analysis of data sequentially into the future. (i.e. exponential weighted moving average or CUSUM charts)

However, since the main focus of this research is to deal with capability indices, the review of control charts will be limited to objective (a).

2.1 Shewhart $\overline{X}$ -Chart for Process Average Location

The $\overline{X}$ control chart is a time sequence plot of the sample averages i.e. denoted as $\overline{X}_j$ together with three horizontal lines that indicate the process centering and variation. Practically, the true parameters μ or σ of the process are unknown and necessary to be estimated from the observed data. To compute sample averages, samples of small size n (i.e. 4 or 5) called rational subgroups are to be collected. Suppose the process is normally distributed and *m* samples are available of size *n* each, let $\overline{x}_j$ be the average of j^{th} sample. Then the best estimator of μ, the overall process average is

$$\overline{\overline{x}} = \frac{1}{m}\sum_{j=1}^{m}\overline{x}_j \tag{2.1}$$

where $\overline{\overline{x}}$ represents the center line of the $\overline{X}$ chart.

In order to construct the control limits, an estimator of the standard deviation (σ) is required. The population standard deviation (σ) can be estimated via two approaches, either the standard deviation of *m* samples or the range method.

Let x_{1j}, x_{2j},..., x_{nj} be a sample of size *n*, then the range R_j of the j^{th} sample is given as

$$R_j = \max\left\{X_{(ij)}\right\} - \min\left\{X_{(ij)}\right\},\ i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, m \tag{2.2}$$

and the corresponding average range of *m* data sequences of size *n* each (i.e. $R_1,\ R_2, ...,\ R_m$) is given by

$$\overline{R} = \frac{1}{m}\sum_{j=1}^{m} R_j. \tag{2.3}$$

Hence, using the well-known relationship $\overline{R} = d_2\hat{\sigma}$ between the standard deviation and the average range $\left(\overline{R}\right)$ of the *m* sample established by Patnaik (1950), an estimator of the process standard deviation (σ) can be established as:

$$\hat{\sigma} = \frac{\overline{R}}{d_2}, \tag{2.4}$$

where d_2 is the expected value of the range and also a function of *n*. Finally, if $\overline{\overline{x}}$ represents the estimator of μ and $\overline{R}/d_2$ an estimator of σ, then the three horizontal lines of the $\overline{X}$ chart are

$$\begin{aligned} LCL &= \overline{\overline{x}} - A_2\overline{R} \\ \text{Center Line } &= \overline{\overline{x}} \\ UCL &= \overline{\overline{x}} + A_2\overline{R} \end{aligned} \tag{2.5}$$

where the coefficient A_2 in (2.5) is a constant that depends on *n*.
Actually,

$$A_2 = \frac{3}{d_2\sqrt{n}}. \tag{2.6}$$

2.2 Shewhart *R*-Chart for Process Variation (n < 10)

The Gaussian normally distributed process is characterised by the parameters μ and σ, hence it is advisable that the standard deviation of the process should also be monitored (Di-Bucchianico, 2008). A control chart can also be constructed for the standard deviation of the process. In industry, a control chart for the population standard deviation of the process can be set up by either the natural choice of the sample standard deviation or the range. Usually the range method is preferred due to its ease of calculation. However, the range method is only effective when the sample size of the subgroups are small (i.e. 4 or 5), whereas sample sizes exceeding 10 produce very inefficient estimates of the control limits for standard deviation (Bissell,1990; Montgomery, 2005).

The center line in the *R* control charts is denoted by $\overline{R}$. To construct the control limits of an *R*-chart, an estimate of σ_R (i.e. population the standard deviation of the range) is required. The estimator of σ_R is given by:

$$\widehat{\sigma}_R = \frac{d_3}{d_2}\overline{\mathrm{R}}. \tag{2.7}$$

The parameters of the control charts using the 3-sigma control limits are presented as:

$$\begin{aligned} LCL &= D_3\overline{\mathrm{R}}, \\ \text{Centre Line} &= \overline{\mathrm{R}}, \\ UCL &= D_4\overline{\mathrm{R}}, \end{aligned} \tag{2.8}$$

where the coefficients D_3 and D_4 are given by:

$$D_3 = 1 - 3\frac{d_3}{d_2}, D_4 = 1 + 3\frac{d_3}{d_2} \tag{2.9}$$

and the coefficients D_3 and D_4 are also functions of n.

2.3 Shewhart *S*-Control chart for Process Variation ($n \geq 10$)

As in section 2.2 above, when the sample size for subgroups is large (i.e. $n \geq 10$), the range method for controlling variability is unacceptable (Bissell,1990; Montgomery, 2005). It is advisable to use the standard deviation method which is a natural estimator of variability. Thus from each subgroup the sample standard deviation should be obtained. The average sample standard deviation of the *m* subgroups of equal size *n* is then obtained as follows:

$$\overline{S} = \frac{1}{m}\sum_{j=1}^{m} S_j, \tag{2.10}$$

where

$$S_j = \sqrt{\frac{\sum_{i=1}^{n}\left(x_{ij} - \overline{x}_j\right)^2}{n-1}}, j = 1, 2, \cdots, m. \tag{2.11}$$

It should be noted that the standard deviation S_j is not an unbiased estimator of σ. The standard deviation S_j is actually an unbiased estimator of $c_4\sigma$, $\mathrm{E}(S) = c_4\sigma$, where

$$c_4(n) = \frac{\sqrt{2}}{\sqrt{n-1}} \frac{\Gamma(n/2)}{\Gamma(n-1/2)}. \tag{2.12}$$

In contrast, the statistic $\overline{S}/c_4$ is an unbiased estimator of σ. The values for the *S*-control chart would be given as:

$$\begin{aligned} LCL &= B_3\overline{S}, \\ \text{Centre Line} &= \overline{S}, \\ UCL &= B_4\overline{S} \end{aligned} \tag{2.13}$$

where

$$B_3 = 1 - \frac{3}{c_4}\sqrt{1 - c_4^2},$$
$$B_4 = 1 + \frac{3}{c_4}\sqrt{1 - c_4^2}. \qquad (2.14)$$

Likewise, the coefficient c_4, B_3 and B_4 are all functions of *n*.

2.4 A Brief Summary of $\overline{X}$, R and S-Charts

The table below provides a summary of the various control charts discussed and their respective parameters:

Table 1: Formulas for Control Charts based on sample data

Chart	Center Line	Control Limits
$\overline{X}$	$\overline{\overline{x}}$	$\overline{\overline{x}} \pm A_2\overline{R}$
$\overline{X}$	$\overline{\overline{x}}$	$\overline{\overline{x}} \pm A_3\overline{S}$
R	$\overline{R}$	$LCL = D_3\overline{R}$, $UCL = D_4\overline{R}$
S	$\overline{S}$	$LCL = B_3\overline{S}$, $UCL = B_4\overline{S}$

Chapter 3. Classical Process Capability Indices

The quantification of any process performance is used as a yardstick for measuring quality in a process, thus process capability analysis has become synonymous with continuous improvement of quality and productivity (Wu, 2009). The key idea behind process capability analysis is the process capability index which tends to measure how well a process meets specification limits preset by the product designer or customer. There is no single capability index which addresses all the quality characteristic of a production process, hence an examination of all the indices holistically is essential (Nyamugure, 2011). Consequently, several Capability indices such as C_p, C_{pk} and C_{pm} have been proposed to measure different characteristics of the process, that is process consistency, process departure from the mean, process yield and process loss. This chapter presents an introduction to these Classical Capability Indices.

3.1 C_p-Index

The first and most honored capability index, C_p, also known as the potential index was introduced by Juran (1974). The C_p index tends to measure the magnitude of the overall process variability relative to tolerance (i.e. specification limits) prescribed by the customer or product designer. The potential index C_p is expressed as a ratio of the actual process spread to the allowable process spread (tolerance width), defined concisely as follows:

$$C_p = \frac{USL\ -\ LSL}{6\sigma} \tag{3.1}$$

where *USL* and *LSL* are the process upper specification limit and lower specification limit respectively, and σ is the process population standard deviation.

Alternatively, the quantification of specification used may also measure process potential, Q, thus

$$Q = \frac{1}{C_p} \times 100\% = \frac{6\sigma}{USL - LSL} \times 100\% \tag{3.2}$$

A small value of Q implies less specification width was used and hence the capability of the process is high, otherwise a high value of Q implies an unacceptable process.

The index C_p was designed to measure the magnitude of the overall process variation. For a two-sided specification limit, the yield of the process measured through the number of non-conforming units produced can also be calculated. Thus if the characteristic of interest, *X*, is governed by Gaussian distribution and if the process is centered, i.e., $\mu = T$, $T = (USL+LSL)/2$, and let $d = (USL - LSL) / 2$ be the half length of the specification interval, [*LSL, USL*]. Then the expected number of non-conforming units is $2\Phi(-d/\sigma)$, where Φ denotes the standard Gaussian distribution function. The expected number of non-conforming (*NC*) units, $\mathrm{E}[NC]$, is $P_r\{NC\} = 1 - P_r\{X \in [LSL, USL]\}$. Recall the symmetry fact of the Gaussian distribution,

$$
\begin{aligned}
&\mathrm{E}[NC] \\
&= 1 - \Pr\{LSL \le X \le USL\} \\
&= 2\Pr\{X \le LSL\} \\
&= 2\Pr\left\{\frac{X-\mu}{\sigma} \le \frac{LSL-\mu}{\sigma}\right\} \\
&= 2\Pr\left\{Z \le -\frac{USL-LSL}{2\sigma}\right\} \\
&= 2\Phi(-d/\sigma)
\end{aligned}
$$

Consequently, the expected proportion of non-conforming units can also be expressed as:

$$
\begin{aligned}
&\mathrm{E}[NC] \\
&= 2\Phi(-d/\sigma) \\
&= 2\Phi(-3(USL-LSL)/6\sigma) \\
&= 2\Phi(-3C_p)
\end{aligned}
\tag{3.3}
$$

Alternatively, the index C_p can also be expressed as:

$$C_p = \frac{d}{3\sigma} \tag{3.4}$$

Therefore, the index C_p is suitable when the process mean, μ, is symmetric and centered at the target (T) of the process (i.e. the midpoint of the process specification limits). This is displayed in Figure 2 below:

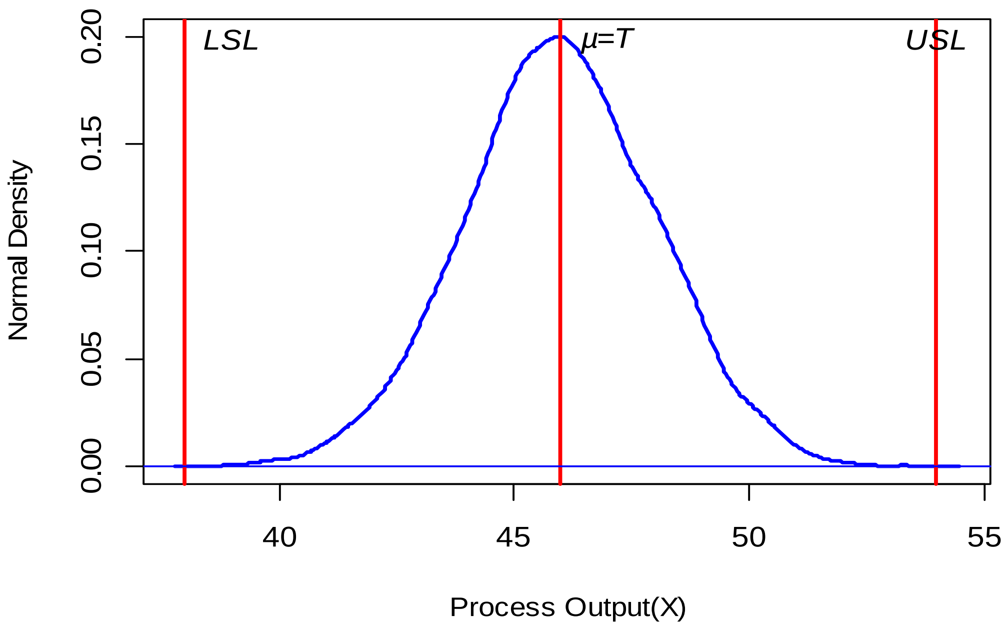

Figure 2: A Simulated Gaussian Process with μ = 46, σ = 2 and *LSL*=38, *T* =46, *USL*=54, where μ =*T*.

The drawback of the index C_p is that it does not account for the location of the process mean, hence for any departure of the process mean from its midpoint {i.e. when $\mu \neq M$ }, the index C_p will give misleading information about the process performance. For instance, it is possible to have a high percentage of non-conforming units with a high C_p value by positioning the mean close to either of the specification limits (Kane, 1986). Therefore the index C_p is not a suitable measure of process performance, but rather measures process potential (i.e. the ability of the process to perform consistently or be repeated in the future). This phenomenon is depicted in Figure 3 below:

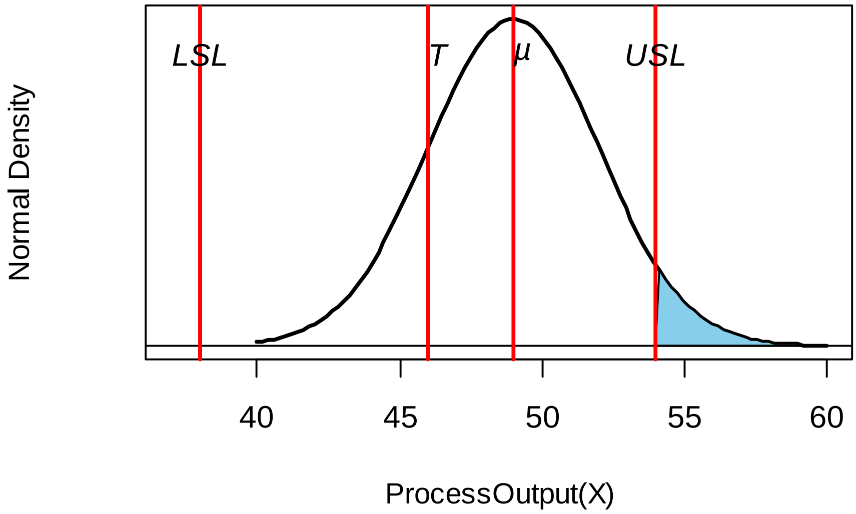

Figure 3: A Simulated Gaussian Process with μ = 49, σ =3 and *LSL*=38, *T* =46, *USL*=54 where $\mu \neq T$.

In Figure 3, a simulated process generated from a normal distribution with μ = 49 and σ = 3, it can be seen that the mean of the process is not centered at the desired level, T . However if the index C_p is used to measure process capability, it will fail to account for the shift in the process away from its tolerance (i.e. "sky blue" shaded region), hence the actual capability of the process will be overstated since the sky blue portion which indicates the number of non-conforming units is not accounted for by this index (i.e. C_p).

In case of σ is unknown, then an estimated version of C_p is

$$\hat{C}_p = \frac{USL\ -\ LSL}{6s} \tag{3.5}$$

where

$$s = \left(\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1} \right)^{1/2} . \tag{3.6}$$

3.2 C_{pk} -Index

The process capability index C_{pk} is also used to describe how well a process fits within its specification limits (Ramakrishnan et al., 2001). The index C_{pk} achieves this description by relating process variation (i.e. 3σ) to the specification limits (Zhang, 1990). However, the index C_{pk} differs from C_p since it accounts for the standardized distance between the mean and the nearest specification limit. The capability index C_{pk} is defined as:

$$C_{pk} = \min\left\{C_{pu}, C_{pl}\right\}, \tag{3.7}$$

where

$$C_{pl} = \frac{\mu - LSL}{3\sigma}, C_{pu} = \frac{USL - \mu}{3\sigma}. \tag{3.8}$$

The index C_{pk} is measured as in equation (3.1) when the process has a two-sided specification limit. However, there are some scenarios which occur in the manufacturing industry where only one-sided specification limit is necessary. For instance, a product designer may set a lower specification limit for the strength of a glass bottle produced, such that any unit that falls below this bound is deemed defective. In such a case the index C_{pl} will be preferred to C_{pk}. Likewise, an upper limit for the concentration of a particular substance, and hence the index C_{pu} may be preferred.

Alternatively, the index C_{pk} can be expressed as

$$C_{pk} = \frac{d - |\mu - M|}{3\sigma}, \tag{3.9}$$

where

$$d = \frac{USL - LSL}{2} \tag{3.10}$$

is the half of the specification length and *M* is the mid-point of the interval [*LSL*,*USL*].

However, the drawback of the index C_{pk} is that it is not an adequate measure of process centering, although that is the main reason for its existence. The index C_{pk} is inversely dependent on the process standard deviation and the magnitude of the index C_{pk} will increase as the process standard deviation approaches zero (Senvar, 2010). The index C_{pk} cannot be used as a suitable measure of process centering (Wu, 2009). Thus, a large value of C_{pk} does not necessarily indicate much about the location of the process mean within the specification limits (Wu, 2009). Therefore the indices C_p and C_{pk} are merely regarded as measures of progress for continuous quality initiatives when variability reduction and process yield are essential (Wu, 2009).

3.3 C_{pm}-Index (TCI)

Although the index C_p and C_{pk} tend to measure process performance with respect to the specification limits, they fail to consider the inability of the process to meet targets preset by the customer. Hsiang & Taguchi (1985) proposed the index C_{pm} also known as the Taguchi capability index (TCI) which was also independently proposed by Chan et al. (1988). The main idea behind the Taguchi index is the squared error loss function, designed to measure the cost of quality measured failing to meet the preset target. The index C_{pm} measures the ability of a process to cluster around its target which tends to reflect the degree of process targeting; hence the index C_{pm} provides better protection for the consumer (Lin, 2005). The C_{pm} index is defined as follows:

$$C_{pm} = \frac{d}{3\sqrt{\sigma^2 + (\mu - T)^2}} \tag{3.11}$$

where T denotes the process target. The term in the denominator

$$\sigma^2 + (\mu - T)^2 \tag{3.12}$$

which represents the expected error loss function for a measured characteristic, X, shown in the next paragraphs.

The degree or measurement that X fails to meet the target T is typically described by a loss function, i.e., loss(X). The loss function is in general very complicated but it can be reasonably approximated by the symmetric squared error loss in terms of Taylor's expansion:

$$\text{loss}(X) = (X - T)^2. \tag{3.13}$$

To avoid the random uncertainty contained in the loss function in decision-making, the expected version of the loss is taken:

$$\text{E}\left[(X - T)^2\right] \tag{3.14}$$

which gives

$$\text{E}(\text{loss}(X)) = \sigma^2 + (\mu - T)^2. \tag{3.15}$$

Therefore, Taguchi Capability Index (TCI) is sometimes referred to as a loss based capability index (Lin, 2005). Because the population mean, μ and standard deviation of the process, σ are often unknown, a TCI needs to be estimated from the observed data. Chan et al (1988) proposed the estimator of C_{pm} :

$$\hat{C}_{pm}(C) = \frac{d}{3\sqrt{S^2 + \left(\bar{X} - T\right)^2}}, \tag{3.16}$$

and Boyles (1991) suggested an estimator of C_{pm} :

$$\hat{C}_{pm}(B) = \frac{d}{3\sqrt{S_n^2 + (X - T)^2}}, \tag{3.17}$$

where

$$\begin{aligned} \bar{X} &= \frac{1}{n}\sum_{i=1}^{n} X_i, \\ S^2 &= \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2, \\ S_n^2 &= \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2. \end{aligned} \tag{3.18}$$

Boyles (1991) show that $\hat{C}_{pm}(C)$ and $\hat{C}_{pm}(B)$ are asymptotically equivalent, i.e.,

$$\lim_{n\to\infty} \hat{C}_{pm}(C) = \lim_{n\to\infty} \hat{C}_{pm}(B). \tag{3.19}$$

It should also be noted that $\overline{X}$ and S_n^2 are the maximum likelihood estimates (*MLE*) of μ and σ^2, then $\hat{C}_{pm}(B)$, which is the joint function of $\overline{X}$ and S_n^2, is the *MLE* of C_{pm}. Without any difficulty, it can be shown that $S_n^2 + \left(\overline{X} - T\right)^2$ is the maximum variance unbiased estimator (*UMVUE*) of $\mathrm{E}\left[(X - T)^2\right]$. Hence the index $\hat{C}_{pm}(B)$ due to Boyles (1991) is the preferred choice.

Finally, for examining the link between TCI C_{pm} and C_p, let us define a term called as the average process loss as:

$$\xi = \frac{\mu - T}{\sigma}. \quad (3.20)$$

Then, it is easy to establish an important functional relationship between an error based capability index and a yield based capability index:

$$C_{pm} = \frac{1}{\sqrt{1+\xi^2}} C_p. \quad (3.21)$$

which paves a way for further exposure of the properties of those capability indices.

3.4 Summary and Comparison between C_p, C_{pk} and C_{pm} indices

PCI's	Strength	Weakness
C_p	The index C_p is a yield based index assuming the process is Gaussian normally distributed and the mean is centered on the target value.	If the process is not centered, many defective units are expected although the estimated index C_p may have a high value suggesting a capable process.
	It is simple to estimate, easy to understand and interpret.	A high C_p value does not always represent a corresponding capable process.
		The index is sensitive to deviations from the Gaussian normal distribution
C_{pk}	The index C_{pk} is also a yield based index assuming the process is normally distributed	The index C_{pk} is not always an adequate measure of process capability. Thus the index C_{pk} fails to deal with the notion (i.e. off-centering) it was invented. For instance, if the mean is between the specification limits, the estimated index C_{pk} depends inversely on the process standard deviation. Thus a smaller standard deviation will result in a larger C_{pk} value and vice-versa. In summary, the C_{pk} index does not adequately take into consideration the position of the mean.
	Unlike index C_p, the index C_{pk} takes into account process centering. For instance, the index C_{pk} deals with situations where the mean is not centered on the target value and also when the mean is located outside the specification limits.	The index is sensitive to deviations from the Gaussian normal distribution
	This index is versatile, as it can be used to assess the process capability with a one-sided specification or a two-sided specification limit.	
	The index C_{pm} was designed to ensure higher quality of the produce for the consumer. The estimation of the index C_{pm} takes into account the variability	The index C_{pm} is not a yield based index. Therefore, if the intent of the process capability study is to estimate the proportion of non-conforming units then

C_{pm}	due to the process mean and deviation from the target value. Consequently, the index C_{pm} is usually smaller than both indices C_p and C_{pk} given that $\mu \neq T$.	the index C_p and C_{pk} are more suitable.
	The index C_{pm} is not sensitive to distribution function. Thus the index C_{pm} is not affected by the distribution of the process.	

Chapter 4. Statistical Properties of Capability Indices

Since Juran's pioneer publication in the early 1970's there has been extensive research conducted into process capability indices. Initially, the study of capability indices concentrated on process yield (i.e. the ability to produce a given measured characteristic within specified limits). However in later years the ability to ensure that the process clusters around a specified target has also been of priority. Hence the first two sections of this chapter, i.e., Section 4.1 and Section 4.2, will concentrate on process yield and process departure from a specified target (i.e. usually taken the midpoint of the specification limits as the specified target value) respectively. The classical capability indices are all functions of process parameters, however those parameters are usually unknown due to the fact that only rational groups (i.e. small samples) are selected from the process, hence the estimates of these indices are prone to sampling error. Finding bounds for the estimated capability indices is essential and Section 4.3 will be based on confidence intervals for classical capability indices and also hypothesis testing procedures to ascertain whether a given process is capable. Boyles (1994) has also been an advocate of asymmetry in capability studies, thus a process is said to be asymmetric when the target value, *T*, is not equivalent to the midpoint of the specification limits (i.e. $T \neq M$). In such situations the effect of asymmetry cannot be ignored, hence Section 4.4 will also review the work done on estimating capability within an asymmetric process. Finally, in Section 4.5 a comparison of the classical capability indices in terms of their own characteristics discussed in the previous sections will be presented.

4.1 Process Yield and Process Capability Indices

Process yield is a critical measure concerned by management. Further, in classical process capability study, process yield is quantified by a process capability index. This is the reason for investigating process capability index based process yield issues immediately after introducing the classical process capability indices. Hence, we will investigate the process yield based capability index at the end of the section.

Assuming a process with a Gaussian normally distributed observed characteristic $X \sim N\left(\mu,\ \sigma^2\right)$, with a given lower and upper specification limit (i.e. LSL and USL), any observation (X_i) falling outside either specification limits is characterized as non-conforming. Hence the process yield is measured as the fraction of process output conforming to specifications and is defined as:

$$Y = \int_{LSL}^{USL} dF(x) \tag{4.1}$$

where $F(x)$ is the cumulative distribution function of the measured characteristic X .

Alternatively, assuming a Gaussian normal distribution the fraction non-conforming can be expressed as:

$$\begin{aligned} &\Pr\{NC\} \\ &= 1 - \Pr\{LSL < X < USL\} \\ &= \Pr\{X < LSL\} + 1 - \Pr\{X > USL\} \\ &= \Phi\left(\frac{LSL-\mu}{\sigma}\right) + 1 - \Phi\left(\frac{USL-\mu}{\sigma}\right) \end{aligned} \tag{4.2}$$

where Φ represents the cumulative distribution function of the standard normal distribution *N*(0,1).

Notice that *USL = M + d* and *LSL = M - d*, the last term of (4.2) becomes

$$\begin{aligned} &\Pr\left\{\frac{M-d-\mu}{\sigma}\right\} + 1 - \Pr\left\{\frac{M+d-\mu}{\sigma}\right\} \\ &= \Pr\left\{-\frac{d+\mu-M}{d}\frac{d}{\sigma}\right\} + \Pr\left\{-\frac{d-\mu+M}{d}\frac{d}{\sigma}\right\} \\ &= \Pr\left\{-\frac{d+\mu-M}{d}\frac{d}{\sigma}\right\} + \Pr\left\{-\frac{d-\mu+M}{d}\frac{d}{\sigma}\right\} \\ &= \Pr\left\{-\frac{1+\delta}{\gamma}\right\} + \Pr\left\{-\frac{1-\delta}{\gamma}\right\} \end{aligned} \tag{4.3}$$

where $\delta = (\mu - M)/d$ and $\gamma = \sigma/d$.

Recall that $C_a = 1 - \delta$ and $C_p = 1/3\gamma$, then (4.3) can be expressed in terms of the indices C_a and C_p as:

$$\Pr\{NC\} = \Phi\left(-3C_pC_a\right) + \Phi\left(-3C_p\left(2-C_a\right)\right). \tag{4.4}$$

The index C_{pk} as noted above was proposed to measure process yield, by using the exact number of non-conforming units.

$$\begin{aligned} &NC\% \\ &= 1 - \left[F(USL) - F(LSL)\right] \\ &\leq 1 - \left[F(USL) - \left[1 - F(USL)\right]\right] \\ &= 1 - \left[2F(USL) - 1\right] \\ &= 2 - 2F(USL) \end{aligned} \tag{4.5}$$

where F is the cumulative distribution function of the process characteristic X . We conclude that the upper bound of the non-conforming proportion is 2-2F(USL).

Furthermore, assuming that F be a Gaussian normal distribution function, i.e., $F \triangleq \Phi$,

$$NC\% \leq \left[2 - 2\Phi\left(\frac{USL - \mu}{\sigma}\right)\right] \times 100\% = \left[2 - 2\Phi\left(3C_{pk}\right)\right] \times 100\% \tag{4.6}$$

Process yield, Y, has its lower bound and upper bound, by assuming a Gaussian normal process (Boyles, 1991):

$$2\Phi\left(3C_{pk}\right) - 1 \leq Y \leq \Phi\left(3C_{pk}\right) \tag{4.7}$$

which gives an intuitive interpretation of calling the index C_{pk} as the yield based index.

Similarly, in terms of C_{pk} and C_a :

$$\Pr\left(NC\right) = \Phi\left(-3C_{pk}\right) + \Phi\left(-3C_{pk}\,\frac{2 - C_a}{C_a}\right) \tag{4.8}$$

which offers the bounds of the proportion of non-conforming units.

However, the actual number of non-conforming units depends on the location of the mean and magnitude of process variation.

$$\Phi\left(-3C_{pk}\right) \leq NC\% \leq 2 - \Phi\left(3C_{pk}\right). \tag{4.9}$$

Wu et al (2009) expressed that the bounds in yield of (4.7) and the bounds in proportion of non-conforming units of (4.9) are equivalent for the cases of $0 \leq C_a \leq 1$ and $C_{pk} > 0$ in (4.8). Moreover, Wu (2009) noted that a process with fixed C_{pk} , reached its maximum when the process is perfectly centered (C_a = 1), and reduces asymptotically as the mean (μ) departs from the target, M.

Similarly for the error based TCI, index C_{pm}, Wu (2009) expressed the $\Pr\{NC\}$ as follows:

$$\Pr\{NC\} = \Phi\left(-\frac{2 - C_a}{\sqrt{\frac{1}{\left(3C_{pm}\right)^2} - \left(1 - C_a\right)^2}}\right) + \Phi\left(-\frac{C_a}{\sqrt{\frac{1}{\left(3C_{pm}\right)^2} - \left(1 - C_a\right)^2}}\right), \quad (4.10)$$

based on which, it can be inferred that the percentage non-conforming has an upper bound (i.e. $\Pr\{NC\} \le 2\Phi\left(-3C_{pm}\right)$) and this bound is realized if the process is centered (C_a=1).

Wu et al. (2009) also made comparisons between the capability index C_{pk} and index C_{pm}, concerning the fraction being non-conforming with a Gaussian normally distributed process. He established a fact that by assuming Gaussian normality both capability indices provide the same lower bound for yield *Y*:

$$2\Phi\left(3C_{pk}\right) - 1 = 2\Phi\left(3C_{pm}\right) - 1. \quad (4.11)$$

Moreover, assuming Gaussian normality, it has also been shown by Wu et al. (2009) that if $C_{pk} = 1$, then $\Pr\{NC\} \le 2700$ per million (ppm) and $0 \le C_a \le 1$, whereas if $C_{pm} = 1$, then $\Pr\{NC\} = 2700$ ppm and $0.67 \le C_a \le 1$. Therefore, a fixed C_{pm} index (i.e. if $C_{pk} = C_{pm}$) provides more information on the process centering, which implies better quality for the consumer. This advantage applies only if the equivalence condition holds (i.e. $C_{pk} = C_{pm}$), otherwise the index C_{pk} is a better measure of process yield than that of index C_{pm}.

Recently, Kenyon and Sale (2010) proposed two process capability indices that are based on process yield rather than the traditional process capability indices that are indirectly based on yield through measures such as the mean and standard deviation. Thus the new indices are a function of process yield and since this population parameter is accessible, hence these new indices are not liable to sampling error. These new indices are also effective measures of process yield. Thus given that $\Pr\{NC\}$ represents the proportion of non-conforming unit and *Y* the fraction conforming, the first new index is defined as:

$$C_{pky} = \frac{1}{3}\Phi^{-1}\left(Y + \Pr\left(NC\right)\right), \tag{4.12}$$

where

$$NC = \min\left\{\int_{-Y}^{LSL} dF\left(x\right), \int_{USL}^{Y} dF\left(x\right)\right\}. \tag{4.13}$$

However, the drawback of the index C_{pky} is similar to that of C_{pk} because it ignores the non-conformance in one of the tails. To address the weakness, Kenyon et al. (2010) proposed an alternative index C_{py} which differs by setting $\Pr\{NC\} = 0$ in equation (4.10) to yield. The modified index C_{py} defined as:

$$C_{py} = \frac{1}{3}\Phi^{-1}\left(Y\right) \tag{4.14}$$

As noted by Kenyon et al. (2010), the advantages of the new indices over the traditional classical capability indices are two-fold: firstly, the index C_{py} is based on the yield directly, hence it can be used to estimate the capability of the process irrespective of the distribution of the process; secondly, the index C_{py} is immune to the process characteristic measured, thus either the measured characteristic is continuous or discrete, C_{py} is flexible enough to measure process capability.

4.2 Process Departure Impacts

Initially, the main purpose of engaging process capability indices was to measure how well a measured process characteristic satisfies specification limits. Nevertheless, the departure of process mean from the midpoint $M = (LSL+USL)/2$ does generate impacts in process capability indices as mentioned in Chapter 3 as well as in Section 4.1. In the literature, Hsiang and Taguchi (1985) and also Chan et al (1988) independently studied process clustering around a given target (i.e. this target is usually the midpoint of the specification limit). Therefore, this section will further focus on the departure impact issue.

A departure measurement index is:

$$\kappa = \frac{|\mu - M|}{d}, \tag{4.15}$$

which measures the absolute departure of the process mean μ from the midpoint *M*.

A small value of κ indicates a lower degree of off-centering and vice-versa. If $\kappa = 0$, this implies that the process is centered at the midpoint (i.e. $\mu = M$), while if $\kappa = 1$ implies the mean is centered at one of the specification limits (i.e. either $\mu = LSL$ or $\mu = USL$).

Please note that index C_a measures degree of process centering and thus is known as the process accuracy index, which indicates the process performance over time. C_{pk} index can be expressed by

$$C_{pk} = (1-\kappa)C_p = C_a C_p. \tag{4.16}$$

this implies the following facts of the process:

$$C_a = \begin{cases} = 1.0, \ C_{pk} = C_p & \mu = M \\ > 0.5 & \mu \in [LSL, USL] \\ = 0.0 & \mu = LSL, \text{or } USL \\ < 0.0 & \mu < LSL, \text{or } > USL \end{cases} \tag{4.17}$$

In other words, we can say that if C_a =1 (i.e., $\kappa = 0$), then the process is perfectly centered (i.e. $\mu = M$) and $C_{pk} = C_p$, thus the upper bound of the index C_{pk} is C_p; if C_a > 0.5, (i.e., $0 < \kappa < 0.5$), this inequality indicates that the process mean μ is within the specification limits (i.e., $LSL < \mu < USL$) and, if C_a = 0, (i.e., $\kappa = 1$), this indicates the value μ is located at one of the specification limits (i.e. $\mu = USL$ or $\mu = LSL$) ; if C_a < 0, this signifies a process being out of control and requires special attention. And finally, a large value of C_{pk} does not necessarily imply the process is centered, because the index C_{pk} is influenced by the magnitude of the process standard deviation. It is therefore concluded that the process is perfectly centered and thus C_{pk} index makes sense if and only if the departure index $\kappa = 0$.

The index C_{pm} is seen as a better measure of consumer protection, as process variation is measured in two ways, process variation within the process and also the deviation of the process mean from its target value. Actually, this responsiveness makes the index C_{pm} more sensitive to process shifts (Wu, 2009). Similar to the index C_{pk}, the upper bound of the Taguchi capability index (TCI) is C_p, and this bound is achieved when $\mu = T = M$ or $\xi = 0$. Thus,

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - T)^2}} = \frac{C_P}{\sqrt{1+\xi^2}}. \tag{4.18}$$

Moreover, Boyles (1991) observed the equivalence relationship $C_p = C_{pk} = C_{pm}$ when $\mu = M$ and these indices reduced as the mean (μ) shifts from the target, *T*. In addition, the index C_{pm} is non-negative and bounded above by the index C_p, whereas $C_{pk} < 0$ for $\mu < LSL$ or $\mu > USL$. Thus the Taguchi capability index (TCI) C_{pm} approaches zero asymptotically as $|\mu - M| \to \infty$.

Kotz and Johnson (1999) observed the relationship between the indices C_p, C_{pk} and C_{pm} assuming a fixed value of κ and established the following equation:

$$\frac{C_{pk}}{C_{pm}} = (1-\kappa)\sqrt{1+\left(\frac{\mu - M}{\sigma}\right)^2} = (1-\kappa)\sqrt{1+9C_p^2\kappa^2} \tag{4.19}$$

where the value of $C_{pm} > C_{pk}$ for small values of C_p and $C_{pm} < C_{pk}$ for large values of C_p. Thus, the impacts from the departure can be summarized as:

$$\begin{cases} C_{pm} = C_{pk} & \kappa = 0 \\ C_{pm} > C_{pk} & (1-\kappa)\sqrt{1+9C_p^2\kappa^2} < 1 \\ C_{pm} < C_{pk} & (1-\kappa)\sqrt{1+9C_p^2\kappa^2} > 1 \end{cases} \tag{4.20}$$

4.3 The Impacts of Statistical Estimation

In this section, we will investigate the impacts from statistical estimation in order to engage the estimated-parameter process capability indices since usually the true-parameter-supported process capability indices are not available.

Process Capability indices are defined as a function of process parameters (e.g. mean, standard deviation). Usually the process parameters are not known, however, in order to estimate these process capability indices, estimates for the parameters have to be obtained from the observed data. The capability indices estimated from sample statistics are subject to statistical variability and this variability has an effect on the estimated indices. Thus the estimated process capability indices are different from the actual process capability indices (Senvar, 2010). Due to these problems several researchers such as Bissell (1990), Kushler (1992), Pearn (1992), Wasserman (1992), Kotz (1993a), Lin (2005), Hsu (2008), etc. proposed confidence intervals for these classical capability indices, and these intervals are dependent on the distribution of the parameter estimates invoked in the process capability indices. For instance, the index C_p is a function of the standard deviation, hence the confidence interval for the index C_p will follow a chi-square distribution providing the process is Gaussian normally distributed.

Kane (1986) was the first researcher who studied and established the distribution of C_p, assuming the process follows a Gaussian normal distribution. Recall that the random variable $(n-1)S^2/\sigma^2$ has χ^2 distribution with n -1 degree of freedom. Thus we have:

$$\Pr\left\{\chi^2_{1-\alpha/2,n-1} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{\alpha/2,n-1}\right\} = 1-\alpha, \tag{4.21}$$

where the sample variance is

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2. \tag{4.22}$$

Therefore a (1 - α)100% confidence interval for process standard deviation parameter, σ:

$$\left[\frac{\sqrt{n-1}}{\sqrt{\chi^2_{\alpha/2,n-1}}}S, \frac{\sqrt{n-1}}{\sqrt{\chi^2_{1-\alpha/2,n-1}}}S\right]. \tag{4.23}$$

Then, a precise (1 - α)100% confidence interval for process capability index C_p proposed by Kane (1986) is:

$$\left[\frac{\sqrt{\chi^2_{\alpha/2,n-1}}}{\sqrt{n-1}}\hat{C}_p, \frac{\sqrt{\chi^2_{1-\alpha/2,n-1}}}{\sqrt{n-1}}\hat{C}_p\right]. \tag{4.24}$$

The confidence interval for C_p in (4.24) has also been reviewed and used by Chou and Owen (1989), Chou et al. (1990) and Li et al. (1990). They all have arrived at the same conclusion that the confidence interval specified in (4.24) produces reliable results in practice.

In the cases of the absence of the chi-square distribution table, we can use Gaussian normal distribution approximations to Chi-square distribution proposed by Fisher (1922):

$$\Pr\left\{\chi^2_v < x\right\} \approx \Phi\left(\sqrt{2x} - \sqrt{2v-1}\right), \tag{4.25}$$

and Wilson-Hilferty (1931):

$$\Pr\left\{\chi^2_v < x\right\} \approx \Phi\left(\left(\left(\frac{x}{v}\right)^{1/3} - 1 + \frac{2}{9v}\right)\sqrt{\frac{9v}{2}}\right).$$

which lead to the approximated (1 - α)100% confidence interval for process capability index C_p

$$\left[\left(\frac{\sqrt{n-\frac{3}{2}}}{\sqrt{n-1}} - \frac{z_{1-\alpha/2}}{\sqrt{2}}\right)C_p, \left(\frac{\sqrt{n-\frac{3}{2}}}{\sqrt{n-1}} + \frac{z_{1-\alpha/2}}{\sqrt{2}}\right)C_p\right] \tag{4.26}$$

and

$$\left[\left(1 - \frac{2}{9(n-1)} - z_{1-\alpha/2}\sqrt{\frac{2}{9(n-1)}}\right)C_p, \left(1 - \frac{2}{9(n-1)} + z_{1-\alpha/2}\sqrt{\frac{2}{9(n-1)}}\right)C_p\right] \tag{4.27}$$

respectively.

Constructing 100(1-α) % confidence intervals for the index C_{pk} is relatively complicated than that of C_p, since C_{pk} is a function of two parameters (i.e. the mean and standard deviation) which follow different distribution. Bissell (1990) noted that the index C_{pk} follows a non-central *t*-distribution, but due to scarce publications of non-central *t*-tables and also the difficulty of interpreting these table's, the author also proposed simple but effective approximations for the 100(1 - α)% confidence interval of C_{pk}.

The standard error of $\hat{C}_{pk}$ is:

$$\sqrt{\frac{1}{n\left(3C_{pk}\right)^2}+\frac{1}{2v}} \tag{4.28}$$

which gives the confidence limits for C_{pk}, assuming Gaussian normality:

$$\left[3C_{pk}\left(1-z_{\alpha/2}\sqrt{\frac{1}{n\left(3C_{pk}^2\right)}+\frac{1}{2v}}\right),3C_{pk}\left(1+z_{\alpha/2}\sqrt{\frac{1}{n\left(3C_{pk}^2\right)}+\frac{1}{2v}}\right)\right] \tag{4.29}$$

where $z_{\alpha/2}$ is the Gaussian normal $(100\times\alpha/2)$ % point and v denotes the degrees of freedom associated with the sample standard deviation *S*. In order to justify the significance of these approximations, Bissell (1990) made a comparison between the non-central *t* - percentage points and the normal approximations and the findings were that the normal approximations converged towards the non-central *t*-percentage points as the sample size increased and since the sample sizes in capability studies are usually large enough (i.e. $n \geq 50$), these approximations are efficient. Zhang et al. (1990), Kushler and Hurley (1992) have also studied and proposed confidence intervals for the index C_{pk}.

Boyles (1991) and Chan (1988) provided approximated estimates of the index C_{pm} as shown in Chapter 3. However, these indices are liable to sampling error; hence Boyles (1991) provided approximated confidence intervals for the estimated capability index C_{pm} given that the process is Gaussian normally distributed.

Boyles (1991) and Pearn et al. (1992) noted that

$$
\begin{aligned}
C_{pm} \\
&= \frac{d}{3}\frac{1}{\sqrt{\sigma^2+(\mu-T)^2}} \\
&= \frac{d}{3\sigma}\frac{1}{\sqrt{1+\frac{\lambda}{n}}} \\
&= \frac{d}{3S\sqrt{n+\lambda}}\left(\frac{nS^2}{\sigma^2}\right)^{1/2}
\end{aligned} \tag{4.30}
$$

where

$$\lambda = n\left(\frac{\mu-T}{\sigma}\right)^2. \tag{4.31}$$

The quantiles of the index C_{pm} can be expressed in the following functional form:

$$\frac{USL\text{ - }LSL}{6\sigma}\sqrt{\frac{n}{\chi^2_{n,v}}}. \tag{4.32}$$

Hence, $\hat{C}_{pm}$ is distributed as:

$$C_{pm}\sqrt{1+\frac{\lambda}{n}}\sqrt{\frac{n}{\chi^2_{n,v}}}. \tag{4.33}$$

The only drawback of the formula (4.33) is that the quantile value of the non-central chi-squared distribution is rarely in use and also a bit complicated to read relative to the more user friendly chi-squared distribution. However, Zimmer and Hubele (1997) have provided tables of exact quantiles for the sampling distribution of the estimator $\hat{C}_{pm}$ .Therefore, the 100(1- α) % confidence interval for C_{pm} can be expressed as follows:

$$\left[\frac{\hat{C}_{pm}}{\sqrt{1+\lambda/n}\sqrt{n/\chi^2_{\alpha/2,n,\lambda}}},\frac{\hat{C}_{pm}}{\sqrt{1+\lambda/n}\sqrt{n/\chi^2_{1\text{-}\alpha/2,n,\lambda}}}\right]. \tag{4.34}$$

Similarly, the distribution for $\hat{C}_{pm}(C)$ proposed by Chan et al (1988) is:

$$C_{pm}\sqrt{1+\frac{\lambda}{n}}\sqrt{\left(\frac{n-1}{\chi^2_{n,\nu}}\right)}. \tag{4.35}$$

which can be used to develop a confidence interval for $C_{pm}(C)$.

4.4 Process Capability Indices for Asymmetric Processes

A process is said to be symmetric when the midpoint of the specification limits (i.e. M) is equivalent to the set target, T (i.e. $M = T$). In contrast, a process is said to be asymmetric when the equivalence does not hold (i.e. $M \neq T$). It is necessary to review those process capability indices for asymmetric process.

The pioneering capability index to deal with asymmetric processes tends to shift one of the specification limits such that the target becomes the midpoint of the new specification limits. With regards to the generalized capability index proposed by Vannman (1995), the new symmetric specification limits are $T \pm d^*$, where $d^*=\min\{D_u, D_l\}$ and $D_u = USL - T$ and $D_l = T - LSL$. Hence the classical capability indices are defined differently:

$$C_p^+(u,v) = \frac{d^* - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}, \tag{4.36}$$

where u and v are two nonnegative parameters. It is obvious that if $u = v = 1$ $C_p^+(1,1) = C_p$.

The chart displayed in Figure 4 explains how either one or both the lower or/and upper specification limit may be shifted to make the process output symmetric to the specified target or the new midpoint of the specification limits. Thus one or both of the specification limits may be shifted in order to make the specified target the new midpoint of the new specification limits (LSL^*, USL^*).

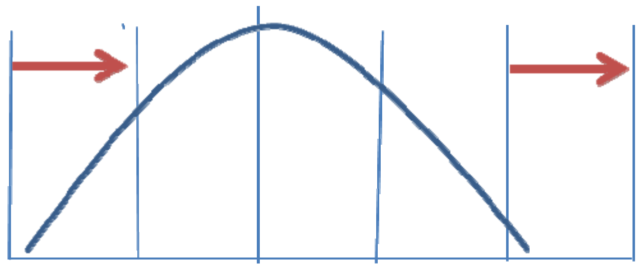

Figure 4: A shift of *LSL* and *USL* by a distance of d^* such that the new specification limits (LSL^*,USL^*) are symmetric about the process Target (*T*).

The drawback of the index in (4.36) is that it tends to under-estimate process capability by limiting the process to a proper subset of the actual specification range. If $D_u = D_l$, the process is symmetric and the formula reverts back to the original generalized capability index proposed by Vannman (1995):

$$C'_p(u,v) = \frac{d - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}}. \quad (4.37)$$

In order to deal with asymmetric capability indices proposed in equation (4.36) and (4.37), Boyles (1994) presented a smooth function defined as:

$$S(x, y) = \Phi^{-1}\{\Phi(x)/2 + \Phi(y)/2\}/3 \quad (4.38)$$

where Φ represents the cumulative distribution function of the standard normal distribution.

By applying the smooth function, Boyles (1994) proposed a new capability index S_{pk} which is very similar to the index C_{pk} and defined as:

$$S_{pk} = S\left(\left(\frac{USL - \mu}{\sigma}\right), \left(\frac{\mu - LSL}{\sigma}\right)\right) \quad (4.39)$$

Given $S_{pk} = c$, the process yield for the index S_{pk} is calculated as:

$$\text{Yield\%} = \Phi\left(\frac{USL-\mu}{\sigma}\right) - \Phi\left(\frac{LSL-\mu}{\sigma}\right) = 2\Phi(3c) - 1. \tag{4.40}$$

Hence, S_{pk} represents the actual process yield unlike C_{pk} which measures the approximate process yield.

However, Pearn et al. (1995) noted that with regard to S_{pk}, the process achieves its maximal capability when $\mu \neq T$ but $T \leq \mu \leq M$, where $T < M$ or $M \leq \mu \leq T$ when $T > M$. This implies that process yield is maximized at the expense of process centering.

Hence, Chen et al.(2001) proposed the new index, thus

$$C_p^{\bullet}(u,\ v) = \frac{d^* \text{ - } uF^*}{3\sqrt{\sigma^2 + vF}}, \tag{4.41}$$

where $F^* = \max\left\{d^*\left(\mu\text{-}T\right)/D_u, d^*\left(\mu\text{-}T\right)/D_l\right\}$ and $u,\ v \geq 0$. If $T = M$, then $F = F^* = \left|\mu\text{-}T\right|$ and generally the index $C_p^{\bullet}(u,\ v)$ reverts to the classical capability index $C_p(u,\ v)$. Hence the new index $C_p^{\bullet}(u,\ v)$ obtains its maximum when $\mu = T$, regardless of whether the tolerance is symmetric or not.

4.5 Comparisons between C_p, C_{pk} and C_{pm}

Firstly, the index C_{pm} is equivalent to C_p, if $\mu = T$, and also $C_{pk} = C_{pm} = C_p$, if $\mu = T = M$. When these conditions do not hold, then generally the process capability indices satisfy $C_p \geq C_{pk}$ and $C_p \geq C_{pm}$. The index C_p serves as an upper limit or upper bound. The index C_p and C_{pm} are always positive, however the index C_{pk} is positive when the mean, μ, falls within the specification limit interval [LSL, USL].

Secondly, comparisons are carried on in terms of index formed base. The indices C_p and C_{pk} are yield based indices, thus they are related to measuring the expected proportion of non-conforming units $\Pr(NC)$ produced. However, the index C_{pm} is not a suitable measure of process yield, if the expected number of non-conforming units is of primary concern, then the indices C_p and C_{pk} will be more adequate than C_{pm}. While the index C_{pm} is a loss based index, it measures the ability of a process to cluster around its target, and hence it is usually seen as the best index to measure quality (i.e. customer protection). The primary importance of C_{pm} is process targeting and $C_{pm} \to \infty$ as $\mu \to T$ and $\sigma \to 0$. Thus the index C_{pm} is a better quality measure and usually seen as better protection for the consumer.

Thirdly, comparisons are carried on in terms of index sensitivity to the distribution of the characteristic measured. The index C_p and C_{pk} are readily sensitive to the distribution of the characteristic measured, and usually give unreliable information if the process distribution is not considered. However, the index C_{pm} is very flexible and is not distribution sensitive. Kushler and Hurley (1992) inferred that if the process is asymmetric (i.e. $M \neq T$), then the process mean moves towards the target ($\mu \to T$), which results in an increase in the magnitude of the indices C_p and C_{pk} but also a corresponding increase in the proportion of non-conforming units (i.e. since the fraction of the distribution outside the specification limits also increases).

Fourthly, comparisons are carried out in terms of index popularity in industry. In industrial practices, the index C_{pm} is rarely utilized whereas the index C_{pk} is mostly widely accepted among practitioners (Kotz et al., 2002). The index C_{pk} is the preferred choice due to its simplicity in calculating and also gives a better representation of the process (i.e. C_{pk} only takes into consideration the variation with respect to the mean). Moreover, the selection of a capability index usually depends on the state of process performance. Thus if the proportion non-conforming is more than 5%, then the indices C_p and C_{pk} are preferred. However, if the percentage non-conforming is very small ($\Pr(NC) \leq 5\%$), then the index C_{pm} is used to assess the uniformity of the process around the target.

Chapter 5. A Review of Uncertainty Theory

Uncertainty theory is a new branch of mathematics for modelling human thinking first proposed by Liu in 2007 and refined in 2010, (Liu, 2007, 2010). Uncertainty usually arises in real world situations where data on a particular area of interest is insufficient to construct a probability distribution, hence the researchers tends to rely on expert judgment on their belief that the particular events will occur. Human thinking tends to exaggerate unlikely events (Tversky, 1986), hence the belief degree tends to significantly differ from the actual frequency and applying probability theory to such a situation may definitely produce misleading results. Such situations may be better dealt with by Liu's uncertainty theory.

Due to the drawback of probability theory, Zadeh (1965) proposed the concept of fuzzy set theory which seeks to model subjective uncertainty via a membership function. However, fuzzy measure does not invoke self-duality, and allows the possibility of an event or its complement being assigned an equal possibility measure of one each. For instance, under fuzziness, the possibility of rain today or no rain can be given an equal possibility measure of one each. This outcome is contrary to human thinking and also inconsistent with the law of contradiction. Thus probability theory and fuzzy set theory are two extremes of uncertainty measure, in which the former requires strictly complete additivity while the latter is characterized by non-additivity.

Uncertainty theory is perceived as a bridge between randomness and fuzziness. Probability theory is a branch of mathematics concerned with modelling randomness such that for a given random variable its probability distribution can easily be constructed. In order, to construct a probability distribution, historical data is needed and should be large enough (i.e. $n \geq 30$). There are usually real world situations in which the available data set is small or even no data, hence a probability distribution cannot be estimated. Moreover, probability theory is defined on axiomatic foundations that include σ-additivity, which may seem to be too restrictive and impractical under some real world situations (Liu, 2010).

Uncertainty theory is defined based on an axiomatic system which includes self-duality, thus uncertainty theory satisfies the law of contradiction that appears consistent with human thinking. Moreover, uncertainty theory is neither completely additive nor completely non-additive but has a sub-additivity property which is perceived consistent with real world applications (Guo, 2010).

Uncertainty theory is based on three core concepts namely uncertain measure, uncertain variable and uncertain distribution. Uncertain measure is a belief degree assigned to an uncertain event. Uncertain variable seeks to represent uncertain quantities arising from a common phenomenon. Finally, an uncertain distribution seeks to partially describe an uncertain variable.

5.1 Uncertain Measure

Let Γ be a non-empty set, and $\mathcal{L}$ an σ-algebra on Γ. A collection $\mathcal{L}$ of subsets Γ is called a σ-algebra if (a) $\Gamma \in \mathcal{L}$ (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, then $\Lambda_1 \cup \Lambda_2 \cup \cdots \in \mathcal{L}$. Each element Λ in $\mathcal{L}$ is called an event. For each event Λ, a number $\mathcal{M}\{\Lambda\}$ between 0 and 1.0 is assigned, which indicates the degree of belief that event $\Lambda \in \mathcal{L}$ will occur. A set function $\mathcal{M}\{\Lambda\}$ has certain mathematical properties, which are stated as follows:

Axiom 1. (**Normality Axiom**): $\mathcal{M}\{\Gamma\} = 1$ for the universal set Γ.

Axiom 2. (**Duality Axiom**): $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1,$ for any event Λ.

Axiom 3. (**Subadditivity Axiom**): For every countable sequence of events $\Lambda_1, \Lambda_2, \cdots$, we have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \leq \sum_{i=1}^{\infty}\mathcal{M}\{\Lambda_i\}. \quad (5.1)$$

Definition 5.1 (Liu, 2013**)**: A set function $\mathcal{M}: \mathcal{L} \rightarrow [0, 1]$ which satisfies normality, duality, and subadditivity axioms is called an uncertain measure. The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space.

An uncertain measure is interpreted as a degree of personal knowledge on an uncertain event. An uncertain measure should not be regarded as some frequency of uncertain event. An uncertain measure can be proved further that it satisfies monotonicity property, null-additivity property, asymptotic property, and extension property. It is also worthwhile to mention that for an empty set $\varnothing$, the uncertain measure of it is zero, i.e., $\mathcal{M}(\varnothing) = 0.$

Axiom 4. (**Product Axiom**): Let $(\Gamma_k, \mathfrak{L}_k, \lambda_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ respectively. The product uncertain measure λ is an uncertain measure on the product σ-algebra $\mathfrak{L} = \mathfrak{L}_1 \times \mathfrak{L}_2 \times \cdots$ satisfying

$$\lambda\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \lambda_k \{\Lambda_k\}, \tag{5.2}$$

where Λ_k are arbitrary uncertain event chosen from σ-algebras $\mathfrak{L}_k$ on non-empty sets Γ_k, for k = 1,2, $\cdots$, respectively.

Finally, we must emphasize that a probability measure is not a special case of an uncertain measure. A probability measure is typically interpreted as a frequency of repeated events. Although today in modern probability theory a subjective probability measure is popular, the nature of an uncertain measure and that of a subjective measure are totally different. Therefore, the mathematical treatments are different too.

5.2 Uncertain Variable and Uncertain Distributions

An uncertain variable is a real valued function that is defined on an uncertain measure space. Thus, analogous to a random variable in probability theory, an uncertain variable is used to represent uncertain quantities within an uncertain environment. The generally accepted definition is stated as follows:

Definition 5.2 (Liu, 2007**)**: An uncertain variable is a measurable function ξ from an uncertainty space (Γ, $\mathcal{L}$, λ) to the set of real numbers, i.e. for any Borel set B of real numbers, the set

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

The concept of uncertain variable is distinguished from both probability random variable and fuzzy variable because it is defined on the uncertainty space. However, an uncertain variable also has some similarities with fuzzy variable since its able to describe quantities defined imprecisely, where this imprecision may be due to knowledge imprecision or vagueness (i.e. qualitative concepts like “low” , “high”, “hot”, “dry” etc).

An uncertainty distribution is often used to characterize an uncertain variable. Unlike a probability distribution, an uncertainty distribution partially describes an uncertain variable only because an uncertain variable is supposed to be described completely by an uncertain measure (Guo, 2012).

Definition 5.3 (Liu, 2007): The uncertainty distribution of an uncertain variable ξ is defined by:

$$\Psi(x) = \lambda\{\xi \le x\}, x \in \mathbb{R} = (-\infty, +\infty) \tag{5.3}$$

where interval $(-\infty, +\infty)$ is often denoted by $\mathbb{R}$.

In order for a function to qualify as an uncertainty distribution it should satisfy the monotone axiom. Peng and Iwamura (2010) proved a necessary and sufficient condition for a distribution to qualify as uncertain. The following theorem was constructed from their derivations.

Theorem 5.1 (**Guo, Guo, and Thiart, 2009**): A non-negative real function Ψ is an uncertainty distribution if and only if it is

(1) a monotone increasing function, i.e., $\Psi(x_1) \le \Psi(x_2)$, for any x_1, x_2, $x_1 \le x_2$;

(2) a left-continuous function, i.e.,

$$\lim_{x_2 \to x_1} \Psi(x) = \Psi(x_1 - 0), \tag{5.4}$$

where $x_2 \le x_1$, and $\Psi(x_1 - 0)$ is the left-limit of the function Ψ at point x_1;

(3) a function takes values between zero and one, i.e., $\exists x_1 < x_2$, for $x \le x_1$, and $x \ge x_2$,

$$\Psi(x) \equiv 0, \text{ and } \Psi(x) \equiv 1 \tag{5.5}$$

where *x* may be $-\infty$ or $+\infty$.

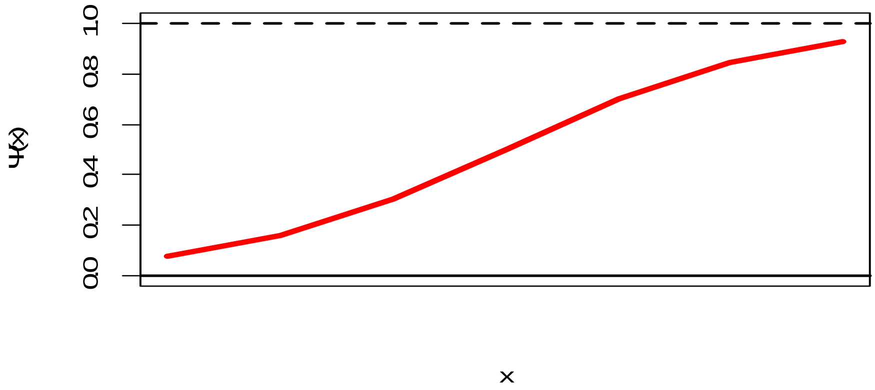

Figure 5 : Uncertainty Distribution

Having defining what is an uncertainty distribution and the necessary and sufficiency conditions for a function to qualify as an uncertainty distribution, we seek to review several established special uncertain variables and their corresponding uncertainty distributions.

5.2.1 Linear Uncertain Distribution

An uncertain variable ξ is called linear if it possesses a linear uncertainty distribution given as follows:

$$\Psi(x)=\begin{cases} 0 & \text{if} \quad x<a \\ \dfrac{x-a}{b-a} & \text{if} \quad a\le x<b \\ 1 & \text{if} \quad x\ge b \end{cases} \tag{5.6}$$

where a and b are real numbers with condition $a < b$. A linear uncertainty distribution can be denoted by $\mathfrak{L}(a,b)$.

An example of a linear uncertainty distribution $\mathfrak{L}(1.0,3.0)$ is

$$\Psi(x)=\begin{cases} 0 & \text{if} \quad x<1.0 \\ \dfrac{x-1.0}{2.0} & \text{if} \quad 1.0\le x<3.0 \\ 1 & \text{if} \quad x\ge 3.0 \end{cases} \tag{5.7}$$

which can be presented graphically by the following figure:

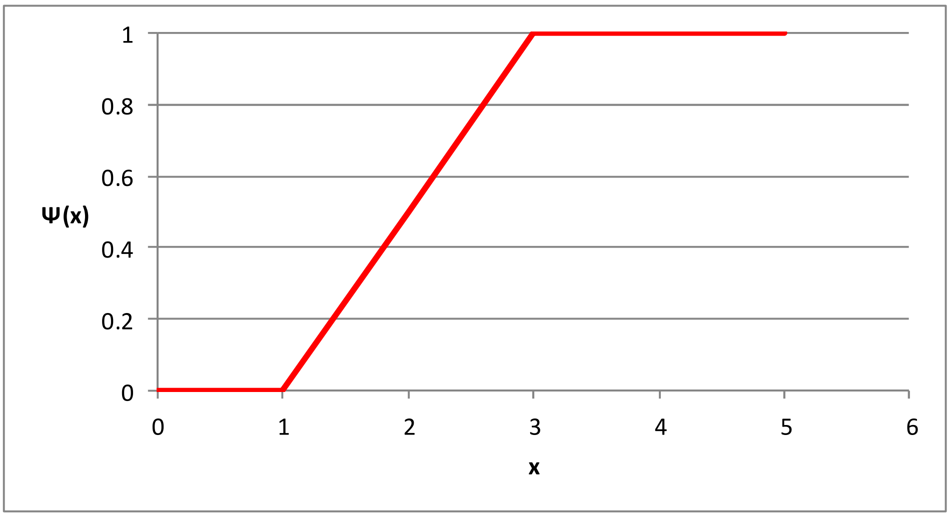

Figure 6: Linear Uncertainty Distribution

5.2.2 Normal Uncertain Distribution

An uncertain variable ξ is called normal if it possesses a normal uncertainty distribution, denoted by $\mathcal{N}(\mu, \sigma)$. The normal uncertainty distribution is defined by:

$$\Psi(x)=\left(1+\exp\left(\frac{\pi(\mu-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \ x \in \mathbb{R} \tag{5.8}$$

where is μ a real-valued parameter and $\sigma>0$ is a positive parameter. Figure 7 gives a graphical representation of a normal uncertainty distribution $\mathcal{N}(\mu, \sigma)$.

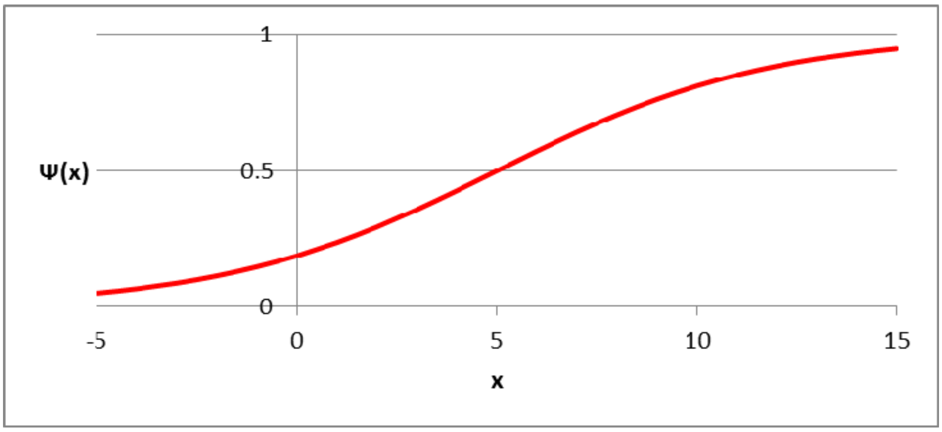

Figure 7: Normal Uncertainty Distribution

In probability theory or statistics, Gaussian distribution function is often called normal distribution function, denoted as $\Phi(x)$. In order to distinguish them, the normal uncertainty distribution function $\Psi(x)$ is called as Liu's normal distribution function, while the normal probability distribution function $\Phi(x)$ is called as Gaussian normal distribution function.

5.2.3 Lognormal Uncertain Distribution

An uncertain variable ξ is called lognormal if lnξ possesses a Liu's normal uncertainty distribution $\mathcal{N}(\mu, \sigma)$, denoted by $\mathcal{LOGN}(\mu, \sigma)$. Thus a lognormal uncertain variable has an uncertainty normal distribution defined as follows:

$$\Psi(x) = \left(1 + \exp\left(\frac{\pi(\mu - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, x \in (0, +\infty) \tag{5.9}$$

This is represented by where μ and σ are real numbers with $\sigma > 0$. The lognormal uncertainty distribution of $\mathcal{LOGN}(0, 1)$ is graphically presented in the Figure 8 below.

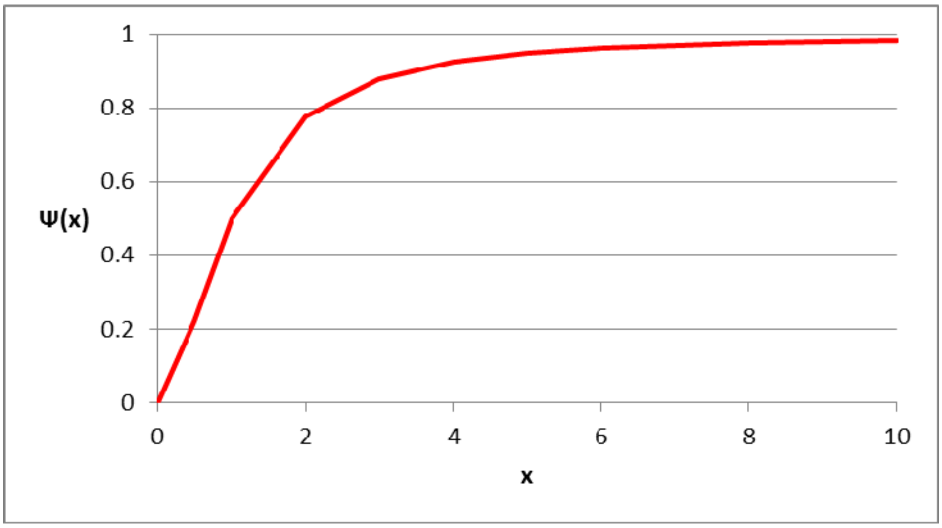

Figure 8: Lognormal Uncertainty Distribution

Theorem 5.2 (Measure Inversion Theorem (Liu, 2010)): Let ξ be an uncertain variable with continuous uncertainty distribution Ψ. Then for any real number *x*, we have

$$\lambda\{\xi \leq x\} = \Psi(x), \lambda\{\xi > x\} = 1 - \Psi(x) \quad (5.10)$$

where λ represents the belief degree that a particular uncertain event will happen.

Proof: The uncertainty distribution of any uncertain variable is defined as: $\Psi(x) = \lambda\{\xi \leq x\}$. By employing the Duality Axiom and the continuity of the uncertainty distribution, then

$$\begin{aligned} &\lambda\{\xi > x\} \\ &= \lambda\left\{\{\xi \leq x\}^c\right\} \\ &= 1 - \lambda\{\xi \leq x\} \\ &= 1 - \Psi(x) \end{aligned} \quad (5.11)$$

which gives the conclusion.

Theorem 5.3 (**Liu**, 2010) Let ξ be an uncertain variable with continuous uncertainty distribution Ψ. Then for any interval $[a, b]$, we have

$$\Psi(b)-\Psi(a)\le \lambda\{a\le \xi \le b\}\le \Psi(b)\wedge(1-\Psi(a)) \tag{5.12}$$

where λ represents the belief degree that a particular uncertain event will happen.

Proof: Notice that

$$\{\xi \le b\}=\{\xi \le a\}\cup\{a\le \xi \le b\} \tag{5.13}$$

which leads to

$$\lambda\{\xi \le b\}\le \lambda\{\xi \le a\}+\lambda\{a\le \xi \le b\} \tag{5.14}$$

in terms of subadditivity property of an uncertain measure. By the definition of an uncertainty distribution, we have:

$$\Psi(b)\le \Psi(a)+\lambda\{a\le \xi \le b\} \tag{5.15}$$

which gives $\Psi(b)-\Psi(a)\le \lambda\{a\le \xi \le b\}$. Furthermore, notice that

$$\begin{aligned}\{a\le \xi \le b\}&\subseteq\{\xi \le b\}\\ \{a\le \xi \le b\}&\subseteq\{\xi \le a\}^c\end{aligned} \tag{5.16}$$

Then, the monotonicity property of an uncertain measure gives

$$\begin{aligned}\lambda\{a\le \xi \le b\}&\le \lambda\{\xi \le b\}=\Psi(b)\\ \lambda\{a\le \xi \le b\}&\le 1-\lambda\{\xi \le a\}=1-\Psi(a)\end{aligned} \tag{5.17}$$

which imply

$$\lambda\{a\le \xi \le b\}\le \min(\Psi(b),1-\Psi(a))=\Psi(b)\wedge(1-\Psi(a)). \tag{5.18}$$

Definition 5.4 (Liu, 2010). An uncertainty distribution Ψ is said to be regular if its inverse function Ψ^{-1} exists and is unique for each $\alpha\in(0,1)$.

Actually, for a regular uncertainty distribution, any point $\alpha\in(0,1)$, if and only if $\Psi(\Psi^{-1}(\alpha))=\alpha$. Typically, the continuity of an uncertainty distribution would secure the regularity of it.

Although the term "distribution" is used in a wide sense in mathematical literature, we prefer to the narrow sense of the distribution in probability theory. In the uncertainty theory under review, we intend to use term "distribution" in a narrow sense. Therefore, we will not use term "inverse distribution function", rather, we call it the inverse function of a distribution function. For the same reason, we will not use term "inverse uncertain variable" and term "inverse uncertainty distribution function" respectively.

The three special uncertainty distributions possess their inverse functions respectively. The next three subsections will give a brief introduction of them.

5.2.4 Inverse Function for a Linear Uncertain Distribution

The inverse function for a linear uncertain distribution is given as:

$$\Psi^{-1}(\alpha) = (1-\alpha)a + \alpha b, \ \forall \alpha \in (0,1) \tag{5.19}$$

which has a graphical representation shown in Figure 9.

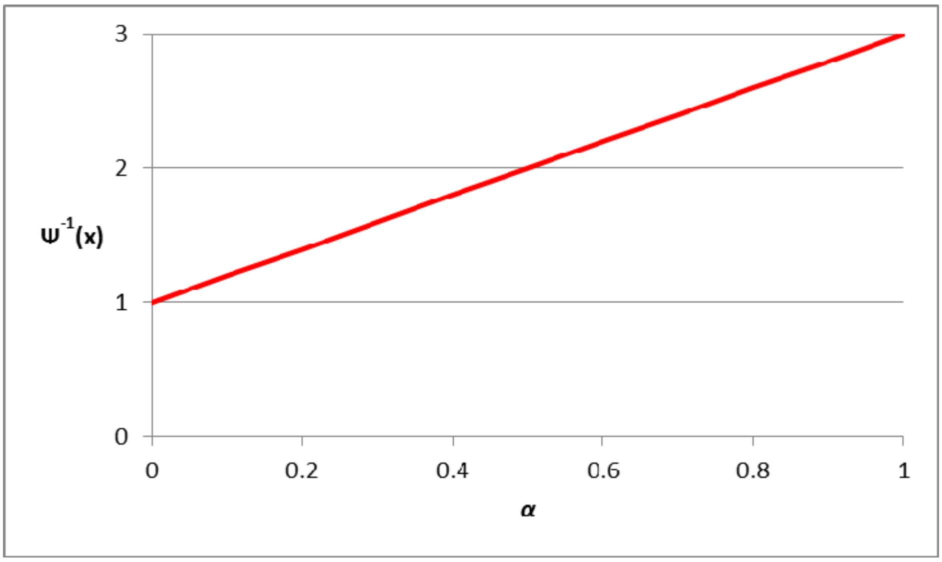

Figure 9: Inverse Function for a Linear Uncertainty Distribution

5.2.5 Inverse Function for A Normal Uncertainty Distribution

The inverse function for the uncertainty distribution of Liu's normal uncertain variable $\mathcal{N}(\mu, \sigma)$ is given as:

$$\Psi^{-1}(\alpha) = \mu + \frac{\sqrt{3}\sigma}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right), \ \forall\alpha \in (0,1) \quad (5.20)$$

which has a graphical representation shown in Figure 10.

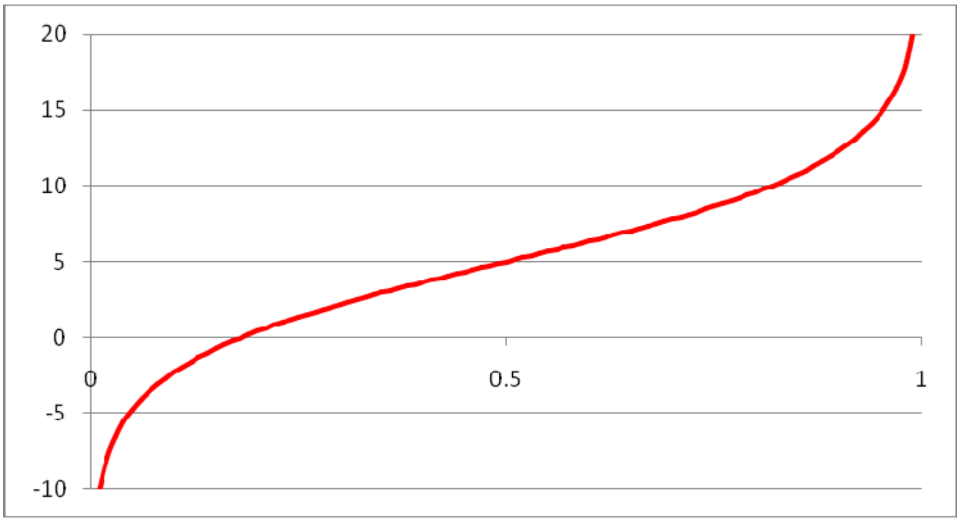

Figure 10: Inverse Normal Uncertain Distribution

5.2.6 Inverse Function for A Lognormal Uncertainty Distribution

The inverse function for the uncertainty distribution of a lognormal uncertain variable $\mathcal{LOGN}(\mu,\sigma)$ can be derived.

Setting up

$$\alpha = \Psi(x) = \left(1 + \exp\left(\frac{\pi(\mu - \ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \forall \alpha \in (0,1) \tag{5.21}$$

Then

$$\ln x = \mu - \frac{\sqrt{3}\sigma}{\pi} \ln\left(\frac{1-\alpha}{\alpha}\right) \tag{5.22}$$

which leads to

$$\Psi^{-1}(\alpha) = \exp(\mu)\left(\frac{\alpha}{1-\alpha}\right)^{\sqrt{3}\sigma/\pi}. \tag{5.23}$$

Let $\mu = 0$, $\sigma = 1$, the inverse function $\Psi^{-1}(\alpha) = (\alpha/(1-\alpha))^{\sqrt{3}/\pi}$ for the lognormal distribution is plotted in Figure 11.

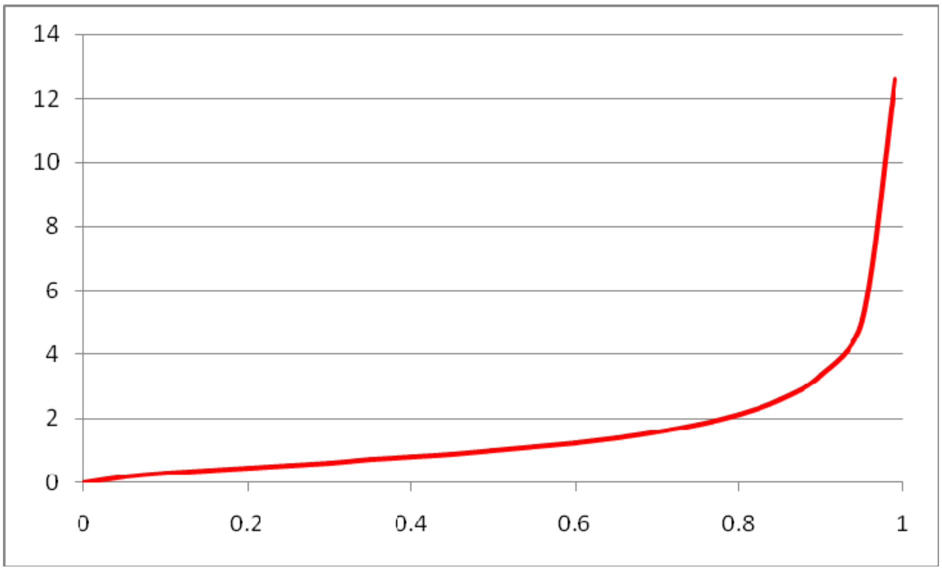

Figure 11: Inverse Function For Lognormal Uncertainty Distribution

5.3 Uncertain Mean and Variance

When presented with an uncertain variable with its corresponding uncertainty distribution, it is usually of interest to determine certain descriptive quantities for this uncertain variable, which are known as moments of the uncertain variable. These moments include but are not limited to expectation, variance, skewness, kurtosis etc. Among those moments, the expected value and the variance are the most important because the expected value describes the central tendency and the variance describes the concentration of an uncertain variable. The following sub-sections seek to present these concepts under Liu's uncertainty environment.

5.3.1 Expected Value of an Uncertain Variable

In analogy to probability theory, an expected value of an uncertain variable is the weighted average of an uncertain variable described by its uncertain measure. Technically, an expected value of an uncertain variable can also be defined as the integral of the uncertain variable with respect to its uncertain measure. In summary, the expected value of an uncertain variable measures the most typical value this uncertain variable is to take.

Definition 5.5 (Liu, 2007) Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$\mathrm{E}[\xi] = \int_{0}^{+\infty} \lambda\{\xi \geq r\}\, dr - \int_{-\infty}^{0} \lambda\{\xi \leq r\}\, dr, \tag{5.24}$$

provided that at least one of the integrals is finite.

Theorem 5.4 (Liu, 2007) Let ξ be an uncertain variable with uncertainty distribution Ψ. If the expected value exists, then

$$\mathrm{E}[\xi] = \int_{0}^{+\infty} \left(1 - \Psi(r)\right) dr - \int_{-\infty}^{0} \Psi(r)\, dr, \tag{5.25}$$

Proof: This result is very straightforward in terms of the definition of the expected value of an uncertain variable and the definition of an uncertainty distribution.

$$\begin{aligned} &\mathrm{E}[\xi] \\ &= \int_{0}^{+\infty} \lambda\{\xi \geq r\}\,dr - \int_{-\infty}^{0} \lambda\{\xi \leq r\}\,dr \\ &= \int_{0}^{+\infty} (1-\Psi(r))\,dr - \int_{-\infty}^{0} \Psi(r)\,dr. \end{aligned} \tag{5.26}$$

Theorem 5.5 (Liu, 2010) Let ξ be an uncertain variable with a regular uncertainty distribution Ψ. If the expected value exists, then

$$\mathrm{E}[\xi] = \int_{0}^{1} \Psi^{-1}(\alpha)\,d\alpha. \tag{5.27}$$

Proof: According to Theorem 5.4 just stated and proved and the inverse function definition of an uncertainty distribution,

$$\begin{aligned} &\mathrm{E}[\xi] \\ &= \int_{0}^{+\infty} (1-\Psi(r))\,dr - \int_{-\infty}^{0} \Psi(r)\,dr \\ &= \int_{\Psi(0)}^{1} (1-\alpha)\,d\Psi^{-1}(\alpha) - \int_{0}^{\Psi(0)} \alpha\,d\Psi^{-1}(\alpha) \\ &= \Psi^{-1}(\alpha)(1-\alpha)\Big|_{\Psi(0)}^{1} - \int_{\Psi(0)}^{1} \Psi^{-1}(\alpha)\,d(1-\alpha) \\ &- \left(\Psi^{-1}(\alpha)\alpha\Big|_{0}^{\Psi(0)} - \int_{0}^{\Psi(0)} \Psi^{-1}(\alpha)\,d\alpha \right) \\ &= 0 - 0 + \int_{\Psi(0)}^{1} \Psi^{-1}(\alpha)\,d\alpha - \left(0 - 0 - \int_{0}^{\Psi(0)} \Psi^{-1}(\alpha)\,d\alpha \right) \\ &= \int_{0}^{\Psi(0)} \Psi^{-1}(\alpha)\,d\alpha + \int_{\Psi(0)}^{1} \Psi^{-1}(\alpha)\,d\alpha \\ &= \int_{0}^{1} \Psi^{-1}(\alpha)\,d\alpha \end{aligned} \tag{5.28}$$

where the changing variable rule and the integration by part rule are applied. The interval limits are

$$\begin{aligned} &r = -\infty,\ \alpha = \Psi(-\infty) = 0; \\ &r = 0,\ \alpha = \Psi(0); \\ &r = +\infty,\ \alpha = \Psi(+\infty) = 1 \end{aligned} \tag{5.29}$$

Example 5.1 Assuming that ξ is a linear uncertain variable, $\xi \sim \mathcal{L}(a,b)$. Consequently, its inverse function of the uncertainty distribution is $\Psi^{-1}(\alpha) = (1-\alpha)a + \alpha b$, and its expected value is derived by following steps:

$$\begin{aligned} &\mathrm{E}[\xi] \\ &= \int_0^1 \Psi^{-1}(\alpha)\, d\alpha \\ &= \int_0^1 \left((1-\alpha)a + \alpha b\right) d\alpha \\ &= a\int_0^1 (1-\alpha)\, d\alpha + b\int_0^1 \alpha\, d\alpha \\ &= a + (b-a)\int_0^1 \alpha\, d\alpha \\ &= a + (b-a)\left.\frac{\alpha^2}{2}\right|_0^1 \\ &= a + \frac{b-a}{2} \\ &= \frac{a+b}{2}. \end{aligned} \tag{5.30}$$

Liu and Ha (2009) investigated the independent multivariable situation and proved that the expected value of a monotone function of an uncertain variable is just a Lebesgue-Stieltjes integral of the function with respect to its uncertainty distribution.

Theorem 5.6 (Liu and Ha, 2009): Assuming that $\xi_1, \xi_2, \cdots, \xi_n$ are independent uncertain variables with regular uncertainty distributions, $\Psi_1, \Psi_2, \cdots, \Psi_n$ respectively. If $f(x_1, x_2, \cdots, x_m, x_{m+1}, x_{m+2}, \cdots, x_n)$ is strictly increasing with respect to $x_1, x_2, \cdots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \cdots, x_n$**,** then the uncertain variable $\xi = f(\xi_1, \xi_2, \cdots, \xi_n)$ has an expected value

$$\mathrm{E}[\xi]=\int_0^1 f\left(\Psi_1^{-1}(\alpha),\cdots,\Psi_m^{-1}(\alpha),\Psi_{m+1}^{-1}(1-\alpha),\cdots,\Psi_n^{-1}(1-\alpha)\right)d\alpha \tag{5.31}$$

provided that E[ξ] exists.

Proof: First we notice that the inverse function for the uncertainty distribution of an uncertain variable $\xi=f(\xi_1,\xi_2,\cdots,\xi_n)$ with the assumptions: (1) $\xi_1,\xi_2,\cdots,\xi_n$ are independent uncertain variables with regular uncertainty distributions, $\Psi_1,\Psi_2,\cdots,\Psi_n$ respectively; (2) $f(x_1,x_2,\cdots,x_m,x_{m+1},x_{m+2},\cdots,x_n)$ is strictly increasing with respect to $x_1,x_2,\cdots,x_m$ and strictly decreasing with respect to $x_{m+1},x_{m+2},\cdots,x_n$ is

$$\Psi_\xi^{-1}(\alpha)=f\left(\Psi_1^{-1}(\alpha),\cdots,\Psi_m^{-1}(\alpha),\Psi_{m+1}^{-1}(1-\alpha),\cdots,\Psi_n^{-1}(1-\alpha)\right),\forall\alpha\in(0,1) \tag{5.32}$$

Then, in terms of Theorem 5.5, we have

$$\begin{aligned}&\mathrm{E}[\xi]\\&=\int_0^1\Psi_\xi^{-1}(\alpha)d\alpha\\&=\int_0^1 f\left(\Psi_1^{-1}(\alpha),\cdots,\Psi_m^{-1}(\alpha),\Psi_{m+1}^{-1}(1-\alpha),\cdots,\Psi_n^{-1}(1-\alpha)\right)d\alpha.\end{aligned} \tag{5.33}$$

As to the derivation of the inverse function $\Psi_\xi^{-1}(\alpha)$, we just derive it for $f(x_1,x_2)$ which is strictly increasing in x_1 and strictly decreasing in x_2. ξ_1,ξ_2 are independent uncertain variables with regular uncertainty distributions, Ψ_1,Ψ_2 respectively. First, we notice that

$$\lambda\left\{\xi\le\Psi_\xi^{-1}(\alpha)\right\}=\lambda\left\{f(\xi_1,\xi_2)\le f\left(\Psi_1^{-1}(\alpha),\Psi_2^{-1}(1-\alpha)\right)\right\} \tag{5.34}$$

which leads to

$$\begin{aligned}&\left\{\xi\le\Psi_\xi^{-1}(\alpha)\right\}\supset\left\{\xi_1\le\Psi_1^{-1}(\alpha)\right\}\cap\left\{\xi_2\ge\Psi_2^{-1}(1-\alpha)\right\};\\&\left\{\xi\le\Psi_\xi^{-1}(\alpha)\right\}\subset\left\{\xi_1\le\Psi_1^{-1}(\alpha)\right\}\cup\left\{\xi_2\ge\Psi_2^{-1}(1-\alpha)\right\}\end{aligned} \tag{5.35}$$

In terms of the independence of ξ_1,ξ_2, we can obtain

$$\begin{aligned}&\mathcal{M}\left\{\xi \le \Psi_{\xi}^{-1}(\alpha)\right\}\\ &\ge \mathcal{M}\left\{\xi_1 \le \Psi_1^{-1}(\alpha)\right\} \wedge \mathcal{M}\left\{\xi_2 \ge \Psi_2^{-1}(1-\alpha)\right\}\\ &= \alpha \wedge \alpha\\ &= \alpha\end{aligned} \tag{5.36}$$

as we as

$$\begin{aligned}&\mathcal{M}\left\{\xi \le \Psi_{\xi}^{-1}(\alpha)\right\}\\ &\le \mathcal{M}\left\{\xi_1 \le \Psi_1^{-1}(\alpha)\right\} \vee \mathcal{M}\left\{\xi_2 \ge \Psi_2^{-1}(1-\alpha)\right\}\\ &= \alpha \vee \alpha\\ &= \alpha\end{aligned} \tag{5.37}$$

By combining the inequality in (5.36) and (inequality in (5.37), the result for n = 2 can conclude. Definitely, for cases n > 2, we can prove the inverse function follow the form in (5.32).

Theorem 5.7 (Liu, 2010): Let ξ and η be independent uncertain variables with finite expected values. Then for any real numbers c and d, we have

$$\mathrm{E}[c\xi + d\zeta] = c\mathrm{E}[\xi] + d\mathrm{E}[\zeta]. \tag{5.38}$$

Proof: For simplicity, we assume the regular uncertainty distributions Ψ_{ξ} and Ψ_{ζ} for uncertain variables ξ and ζ respectively.

First let us show that

$$\mathrm{E}[c\xi] = c\mathrm{E}[\xi] \tag{5.39}$$

Notice that $f(x) = cx$, then the f function is strictly increasing in x if $c > 0$ or strictly decreasing in x if $c < 0$. Thus the inverse function will be

$$\Psi_{c\xi}^{-1}(\alpha) = \begin{cases} c\Psi_{\xi}^{-1}(\alpha) & \text{if} \quad c > 0 \\ c\Psi_{\xi}^{-1}(1-\alpha) & \text{if} \quad c < 0 \end{cases} \tag{5.40}$$

which gives

$$\begin{aligned}
\mathrm{E}[c\xi] &= \int_0^1 \Psi_{c\xi}^{-1}(\alpha)\,d\alpha \\
&= \begin{cases} c\int_0^1 \Psi_{\xi}^{-1}(\alpha)\,d\alpha & \text{if} \quad c>0 \\ c\int_0^1 \Psi_{\xi}^{-1}(1-\alpha)\,d\alpha & \text{if} \quad c<0 \end{cases} \\
&= \begin{cases} c\mathrm{E}[\xi] & \text{if} \quad c>0 \\ c\mathrm{E}[\xi] & \text{if} \quad c<0 \end{cases} \\
&= c\mathrm{E}[\xi].
\end{aligned} \tag{5.41}$$

Secondly, let us show that

$$\mathrm{E}[\xi+\zeta] = \mathrm{E}[\xi] + \mathrm{E}[\zeta] \tag{5.42}$$

The inverse function for uncertainty distribution $\Psi_{\xi+\zeta}$ of the sum of two independent uncertain variables $\xi+\zeta$ is

$$\Psi_{\xi+\zeta}^{-1}(\alpha) = \Psi_{\xi}^{-1}(\alpha) + \Psi_{\zeta}^{-1}(\alpha). \tag{5.43}$$

which gives the second result needed.

Finally, by combining result in (5.41) and result in (5.42), we have the result in (5.38).

We should be fully aware that the linearity of uncertain expectation operator of an uncertain variable is not the same as that of a random variable in probability theory. In uncertainty theory, it is conditional on the independence of the summand. While in probability theory, the linearity of expectation operator holds always i.e. holds unconditionally.

We should also emphasize that the expected value represents the "centre" of an uncertainty distribution in certain sense.

5.3.2 Variance

The variance of an uncertain variable presents the concentration surrounding its expected value, which helps to judge the central tendency quality of the expected value. Alternatively, the variance is a quantity describing the spread or variation around its expected value. The better prediction of the expected value as central position of an uncertain variable is, if the smaller variance is. Otherwise, the larger variance is, the poorer quality of the centre is predicted.

Definition 5.6 (Liu, 2007): Let ξ be an uncertain variable with finite expected value μ. Then the variance of ξ is

$$V[\xi] = \mathrm{E}\left[(\xi-\mu)^2\right]. \tag{5.44}$$

It is critical that different from the scale-valued variance in probability theory the variance of an uncertain variable is not necessary taking a scale-value unless the uncertain measure of an uncertain variable is given. If only the uncertainty distribution of an uncertain variable is available, the variance of an uncertain variable is an interval-valued quantity. This fact can be realized by the following expression:

$$\begin{aligned}
&V[\xi] \\
&= \mathrm{E}\left[(\xi-\mu)^2\right] \\
&= \int_0^{+\infty} ƛ\left\{(\xi-\mu)^2 > r\right\} dr - \int_{-\infty}^{0} ƛ\left\{(\xi-\mu)^2 \le r\right\} dr \\
&= \int_0^{+\infty} ƛ\left\{(\xi-\mu)^2 > r\right\} dr - \int_{-\infty}^{0} \Psi_{(\xi-\mu)^2}(r)\, dr \\
&\le \int_0^{+\infty} ƛ\left\{(\xi-\mu)^2 > r\right\} dr \\
&= \int_0^{+\infty} ƛ\left\{\left\{\xi-\mu > \sqrt{r}\right\} \cup \left\{\xi-\mu < -\sqrt{r}\right\}\right\} dr.
\end{aligned} \tag{5.45}$$

It is obvious that if the uncertain measure $ƛ$ is given then event $\left\{\xi-\mu > \sqrt{r}\right\} \cup \left\{\xi-\mu \le -\sqrt{r}\right\}$ can be calculated accurately. If the uncertainty distribution Ψ is available, then the event $\left\{\xi-\mu > \sqrt{r}\right\} \cup \left\{\xi-\mu \le -\sqrt{r}\right\}$ can be partially determined. By noticing that

$$\left\{(\xi-\mu)^2 > r\right\} = \left\{\xi-\mu > \sqrt{r}\right\} \cup \left\{\xi-\mu < -\sqrt{r}\right\}, \forall r > 0 \tag{5.46}$$

The upper limit of the integral is

$$\begin{aligned}
&\int_0^{+\infty} \lambda\left\{\left\{\xi-\mu>\sqrt{r}\right\}\cup\left\{\xi-\mu<-\sqrt{r}\right\}\right\}dr \\
&\le \int_0^{+\infty} \lambda\left\{\xi-\mu>\sqrt{r}\right\}dr+\int_0^{+\infty} \lambda\left\{\xi-\mu\le-\sqrt{r}\right\}dr \\
&= \int_0^{+\infty}\left(1-\Psi\left(\mu+\sqrt{r}\right)\right)dr+\int_0^{+\infty}\Psi\left(\mu-\sqrt{r}\right)dr \\
&= 2\int_0^{+\infty} x\left(1-\Psi\left(\mu+x\right)+\Psi\left(\mu-x\right)\right)dx.
\end{aligned} \tag{5.47}$$

Theorem 5.8: If ξ is an uncertain variable with finite expected value, then

$$V\left[a\xi+b\right]=a^2V\left[\xi\right] \tag{5.48}$$

where a and b are real numbers.

Proof: Notice that

$$\begin{aligned}
&V\left[a\xi+b\right] \\
&=\mathrm{E}\left[\left(a\xi+b-\mathrm{E}\left[\left(a\mu+b\right)\right]\right)^2\right] \\
&=\mathrm{E}\left[\left(a\left(\xi-\mu\right)\right)^2\right] \\
&=a^2\mathrm{E}\left[\left(\xi-\mu\right)^2\right] \\
&=a^2V\left[\xi\right].
\end{aligned} \tag{5.49}$$

5.4 Variance of Liu's Normal Uncertainty Distribution

The Gaussian normal random variable with its normal distribution serves a fundamental role in probability theory due to the central limit theorem and also its application in statistical modelling and inference. Likewise, it is expected that Liu’s uncertain normal variable and its distribution will play a similar key role within an uncertainty environment (Guo et al, 2010). This section reviews the variance of Liu’s uncertainty normal distribution.

In subsection 5.2.2, Liu's normal uncertainty distribution is defined in (5.8). Keep in mind, the uncertain measure of Liu's normal uncertain variable is not defined. Therefore, the variance of Liu's normal uncertainty distribution is an interval. Let us derive the interval limits. Notice that the inverse function of Liu's normal uncertainty distribution is

$$\Psi^{-1}(\alpha) = \mu + \frac{\sqrt{3}\sigma}{\pi}\left(\ln\alpha - \ln(1-\alpha)\right), \tag{5.50}$$

Then the expected value of Liu's normal uncertainty distribution is

$$\begin{aligned} &\mathrm{E}[\xi] \\ &= \int_0^1 \Psi^{-1}(\alpha)\,d\alpha \\ &= \int_0^1 \left(\mu + \frac{\sqrt{3}\sigma}{\pi}\left(\ln\alpha - \ln(1-\alpha)\right)\right)d\alpha \\ &= \mu + \frac{\sqrt{3}\sigma}{\pi}\int_0^1 \left(\ln\alpha - \ln(1-\alpha)\right)d\alpha \\ &= \mu. \end{aligned} \tag{5.51}$$

by noticing the following integral takes zero value:

$$\begin{aligned} &\int_0^1 \left(\ln\alpha - \ln(1-\alpha)\right)d\alpha \\ &= \int_0^1 \ln\alpha\, d\alpha - \int_0^1 \ln(1-\alpha)\,d\alpha \\ &= \int_0^1 \ln\alpha\, d\alpha + \int_0^1 \ln(1-\alpha)\,d(1-\alpha) \\ &= 0. \end{aligned} \tag{5.52}$$

Please note that Liu's normal uncertain variable $\xi-\mu$ has uncertainty distribution

$$\Psi_{\xi-\mu}(x) = \left(1+\exp\left(-\frac{\pi}{\sqrt{3}\sigma}x\right)\right)^{-1} \tag{5.53}$$

which gives an inverse function as

$$\Psi_{\xi-\mu}^{-1}(\alpha) = \frac{\sqrt{3}\sigma}{\pi}\left(\ln\alpha - \ln(1-\alpha)\right) \tag{5.54}$$

We can also notice that for function $f(x)=x^2$ in a strictly monotone-increasing of x on $[0,+\infty)$, thus the inverse function is

$$\Psi^{-1}_{(\xi-\mu)^2}(\alpha) = \left(\Psi^{-1}_{\xi-\mu}(\alpha)\right)^2, \forall \alpha \in (0,1). \tag{5.55}$$

Then the job of deriving the variance of an uncertain variable becomes that of deriving the expression of the expected value of the uncertain variable $(\xi-\mu)^2$ in terms of the inverse function:

$$\begin{aligned} & V[\xi] \\ & = \mathrm{E}\left[(\xi-\mu)^2\right] \\ & \le \int_0^1 \left(\Psi^{-1}_{\xi-\mu}(\alpha)\right)^2 d\alpha \\ & = \frac{3\sigma^2}{\pi^2}\int_0^1 \left(\ln\alpha - \ln(1-\alpha)\right)^2 d\alpha \\ & = \frac{3\sigma^2}{\pi^2}\left(4-2\times\left(2-\frac{\pi^2}{6}\right)\right) \\ & = \sigma^2. \end{aligned} \tag{5.56}$$

In other words, the upper bound of the variance for Liu's normal uncertainty distribution is

$$\sigma_u^2 = \max\left(V[\xi]\right) = \sigma^2. \tag{5.57}$$

The lower bound of the variance for Liu's normal uncertainty distribution is

$$\begin{aligned} & \mathrm{E}\left[(\xi-\mu)^2\right] \\ & = \int_0^{+\infty} \lambda\left\{\left\{\xi \ge \mu+\sqrt{x}\right\} \cup \left\{\xi \ge \mu-\sqrt{x}\right\}\right\} dx \\ & \ge \int_0^{+\infty} \left(\lambda\left\{\xi \ge \mu+\sqrt{x}\right\} dx + \right) dx \vee \int_0^{+\infty} \lambda\left\{\xi \ge \mu-\sqrt{x}\right\} dx \\ & = \int_0^{+\infty} \left(1-\Psi_\xi\left(\mu+\sqrt{x}\right)\right) dx \vee \int_0^{+\infty} \Psi_\xi\left(\mu-\sqrt{x}\right) dx \\ & = 2\int_0^{+\infty} r\left(1-\Psi_\xi(r+\mu)\right) dr \vee 2\int_0^{+\infty} r\Psi_\xi(\mu-r) dr \\ & = \sigma_l^2. \end{aligned} \tag{5.58}$$

Therefore, we have

$$\sigma_l^2 \leq V\left[\xi\right] \leq \sigma_u^2 = \sigma^2. \tag{5.59}$$

It is also an easy task to derive the ratio, denoted by $\tau = \sigma_l^2 / \sigma_u^2$,

$$\tau = \frac{1}{2}. \tag{5.60}$$

For Liu's normal uncertainty distribution, its variance take an interval-value, i.e.,

$$\left[\frac{\sigma^2}{2}, \sigma^2\right] \tag{5.61}$$

This fact will have huge impacts in theoretical developments or in practical applications.

Chapter 6. Uncertain Statistics

Uncertain statistics was introduced to estimate the experimental uncertainty distribution of an uncertain variable (Chen and Ralescu, 2012). Typically, in order to estimate an uncertainty distribution, expert knowledge data are required. The data with their corresponding belief degree will be collected from experts who have specialized knowledge and experience on a particular field. Slightly in details, in order to collect this expert information a questionnaire was designed by Liu (2007) to match the subject of interest. Let x_i denote experts quantity and α_i their corresponding belief degree of an uncertain event, i =1,2, $\cdots$. Assuming that we have obtained a set of n observational data points from an expert expressed as follows:

$$(x_1,\alpha_1),(x_2,\alpha_2),...,(x_n,\alpha_n), \tag{6.1}$$

where

$$x_1 < x_2 < ... < x_n \text{ and } 0 \le \alpha_1 \le \alpha_2 \le ... \le \alpha_n \le 1. \tag{6.2}$$

Furthermore, in order to determine which distribution to use in a particular situation depends on the kind of information inherent in the uncertain variable. Section 6.1 will review the uncertainty empirical distributions and their possible applications.

6.1 Empirical Uncertainty Distribution

Given an expert's experimental data as shown in (6.1) and (6.2), Liu (2010) proposed the empirical uncertainty distribution based on the linear interpolation method that is structured as follows:

$$\hat{\Psi}(x) = \begin{cases} 0 & \text{if} \quad x < x_1 \\ \alpha_i + \dfrac{x - x_i}{x_{i+1} - x_i}(\alpha_{i+1} - \alpha_i) & \text{if} \quad x_i \le x < x_{i+1},\ i = 1,2,\cdots,n \\ 1 & \text{if} \quad x \ge x_n \end{cases} \tag{6.3}$$

which has a piecewise linear inverse function:

$$\hat{\Psi}^{-1}(\alpha)=\begin{cases} x_0\,(<x_1) & \text{if} \quad \alpha<\alpha_1 \\ x_i+\dfrac{\alpha-\alpha_i}{\alpha_{i+1}-\alpha_i}(x_{i+1}-x_i) & \text{if} \quad \alpha_i\le\alpha<\alpha_{i+1} \\ x_n & \text{if} \quad \alpha\ge\alpha_n \end{cases} \tag{6.4}$$

Based on the empirical uncertainty distribution in (6.3), then the expected value is calculated as

$$\mathrm{E}_{\hat{\Psi}}[\xi]=\frac{\alpha_1+\alpha_2}{2}x_1+\sum_{i=2}^{n-1}\frac{\alpha_{i+1}-\alpha_{i-1}}{2}x_i+\left(1-\frac{\alpha_{n-1}+\alpha_n}{2}\right)x_n. \tag{6.5}$$

The result of (6.5) can be derived in terms of the piecewise linear inverse function in (6.4):

$$\mathrm{E}_{\hat{\Psi}}[\xi]=\int_0^1\hat{\Psi}^{-1}(\alpha)\,d\alpha \tag{6.6}$$

Assuming that the uncertain variable is strictly non-negative then the k^{th} empirical moments are

$$\mathrm{E}_{\hat{\Psi}}\left[\xi^k\right]=\alpha_1x_1^k+\frac{1}{k+1}\sum_{i=1}^{n-1}\sum_{j=0}^{k}(\alpha_{i+1}-\alpha_i)x_i^jx_{i+1}^{k-j}+(1-\alpha_n)x_n^k. \tag{6.7}$$

6.2 Least-Squares Method

Liu (2010) proposed the principle of least squares to estimate the uncertainty distribution of a known functional form $\Psi(x\,|\,\theta)$ (i.e. linear, quadratic etc) with an unknown parameter θ. The least squares method provides an estimate for this unknown parameter θ by minimizing the distance of the experts experimental data to the uncertainty distribution. Assuming that the data are given as in (6.1), then the unknown parameter can be found by optimizing the following:

$$\min_{\theta}\sum_{i=1}^{n}\left(\Psi\left(x_i|\theta\right)-\alpha_i\right) \tag{6.8}$$

Hence the optimal solution of (6.7) is known as the least square estimate of θ, and ξ has an estimated uncertainty distribution $\tilde{\Psi}\left(x|\hat{\theta}\right)$.

6.3 Method of Moments

Assuming that an uncertain variable of interest is strictly non-negative and is described incompletely by an uncertainty distribution $\Psi\left(x \mid \theta_1, \theta_2, \cdots, \theta_p\right)$, with unknown parameters $\theta_1, \theta_2, ..., \theta_p$. Then it is critical to obtain an estimate of the uncertainty distribution.

Wang and Peng (2010) proposed a method of moments for estimating parameters $\theta_1, \theta_2, ..., \theta_p$, by solving the following equation system

$$\begin{cases} \int_0^{+\infty} \left(1 - \Psi\left(x \middle| \theta_1, \theta_2, \cdots, \theta_p\right)\right) dx = \hat{\mathrm{E}}\left[\xi\right] \\ \int_0^{+\infty} x\left(1 - \Psi\left(x \middle| \theta_1, \theta_2, \cdots, \theta_p\right)\right) dx = \hat{\mathrm{E}}\left[\xi^2\right] \\ \vdots \\ \int_0^{+\infty} x^{p-1}\left(1 - \Psi\left(x \middle| \theta_1, \theta_2, \cdots, \theta_p\right)\right) dx = \hat{\mathrm{E}}\left[\xi^p\right] \end{cases} \tag{6.9}$$

where the estimated k^{th} moments

$$\hat{\mathrm{E}}\left[\xi^k\right] = \alpha_1 x_1^k + \frac{1}{k+1}\sum_{i=1}^{n-1}\sum_{j=0}^{k}\left(\alpha_{i+1} - \alpha_i\right) x_i^j x_{i+1}^{k-j} + \left(1 - \alpha_n\right) x_n^k. \tag{6.10}$$

6.4 The Delphi Method

A fundamental question arises, what happens when experimental data is collected with their corresponding estimated uncertainty distribution from multiple experts? For instance, assuming that data have been obtained from *m* experts and each expert produces an uncertainty distribution, i.e., $\hat{\Psi}_1(x), \hat{\Psi}_2(x), \cdots, \hat{\Psi}_m(x)$. Liu (2010) proposed a method for obtaining the overall uncertainty distribution by using the weighted average approach given as:

$$\hat{\Psi}(x) = w_1 \hat{\Psi}_1(x) + w_2 \hat{\Psi}_2(x) + \cdots + w_m \hat{\Psi}_m(x), \tag{6.11}$$

where $w_1, w_2, \cdots, w_m$ are non-negative convex combinations coefficients and they sum up to 1 (i.e. $w_1 + w_2 + \cdots + w_m = 1$). The weights attached to these uncertainty distributions in (6.11) are usually represented as equally likely. Thus

$$w_i = \frac{1}{m}, i = 1, 2, \cdots, m \tag{6.12}$$

The Delphi method is developed by the RAND corporation in the 1950’s to obtain group expert opinion which seeks to build consensus iteratively among these opinions. The Delphi method is a systematic process.

A questionnaire on a subject under investigation, designed by a facilitator, is sent out independently to experts to obtain their experience and knowledge about the subject under investigation. Upon its return, the facilitator analyzes and summarizes the group information and provides the feedback to each respondent, so they can make a revised judgment based on the group feedback. The process continues until a consistent state is reached. Thus by consistently feeding information into the process the Delphi method seeks to build consensus.

Wang, Gao and Guo (2010) modified the Delphi method as a process to determine the uncertainty distribution.

Let ξ be an uncertain variable with its corresponding distribution $\Psi(x)$ of ξ, hence in order to estimate the uncertain distribution, m experts are invited to choose n possible values (i.e. $x_{m1}, x_{m2}, \cdots, x_{mn}$), then the uncertain variable ξ is likely to take these values assuming that $x_{m1} < x_{m2} < \cdots < x_{mn}$. The steps of estimating an uncertainty distribution are given as follows:

Step 1: The m domain experts provide their experimental data (x_{ij}, α_{ij}), where x_{ij} denote the j^{th} value provided by the i^{th} expert and α_{ij} denotes the i^{th} expert's belief degree that ξ is less than x_{ij}, $j = 1, 2, \cdots, n_i$ and $i = 1, 2, \cdots, m$ respectively.

Step 2: Use the i^{th} expert's experimental data (x_{i1}, α_{i1}), (x_{i2}, α_{i2}), ...,$(x_{in_i}, \alpha_{in_i})$ to generate an uncertainty distribution $\hat{\Psi}_i$.

Step 3: Calculate the number of possible values of the uncertain variable, ξ, presented by all experts denoted by n, where the same values from different experts are considered as one. Then the possible values of ξ are $x_1 \le x_2 \le x_3 \le ... \le x_n$ and calculate

$$\bar{\alpha}_j = \frac{1}{m}\sum_{i=1}^{m}\hat{\Psi}_i\left(x_j\right),\ j = 1,2,\cdots,n \tag{6.13}$$

and

$$d_j = \frac{1}{m}\sum_{i=1}^{m}\left(\hat{\Psi}_i\left(x_j\right) - \bar{\alpha}_j\right)^2,\ j = 1,2,\cdots,n \tag{6.14}$$

Step 4: For a pre-specified level $\varepsilon > 0$ if $d_j < \varepsilon$ for all j then proceed to **Step 5**. Otherwise, the process is iterated again, so that the i^{th} domain expert will receive a summary and then provide a revised experts experimental data vector $\left(x_{i1}, \alpha_{i1}\right), \left(x_{i2}, \alpha_{i2}\right), ..., \left(x_{in_i}, \alpha_{in_i}\right)$ for $i = 1,2,\cdots,m.$ Go to **Step 2**.

Step 5: Finally use the integrated dataset $\left(x_1, \bar{\alpha}_1\right), \left(x_2, \bar{\alpha}_2\right), ..., \left(x_n, \bar{\alpha}_n\right)$ to generate an uncertain distribution $\tilde{\Psi}(x) = w_1\hat{\Psi}_1(x) + w_2\hat{\Psi}_2(x) + \cdots + w_m\hat{\Psi}_m(x)$, where w_i are convex weights adding to one.

6.5 Wang-Gao-Guo Hypothesis Testing

Hypothesis testing is a method of verifying whether statements about the characteristic(s) of a given population are valid. These statements are called hypothesis and the process of verification is based on expert's experimental data. There are two kinds of decisions that can be made under hypothesis testing, its either we reject the null hypothesis or fail to reject the null hypothesis. The null hypothesis is usually rejected when evidence from the data is not sufficient to justify the hypothesis, hence the hypothesis is rejected with some degree of confidence or otherwise, the hypothesis is rejected. This section serves to review methods of hypothesis testing under an uncertainty environment.

Let us first review the method to test the statement that two uncertainty distributions are equal. In order to test the equality, this method makes use of expert's experimental data and the corresponding uncertainty distribution (Wang et al, 2012). For instance, under an uncertainty environment data may be obtained from two experts based on their knowledge and experience, hence how does the facilitator determine whether their opinions coincide or not?

Assuming that data were collected from two experts based on the same uncertain variable then Wang-Gao-Guo (2012) proposed a hypothesis testing scheme. Let the two domain experts data be represented by A and B respectively. Hence, the experts' experimental data are given as follows:

$$\begin{aligned} &\mathrm{A}:(x_1,\alpha_1),(x_2,\alpha_2),\cdots,(x_m,\alpha_m)\\ &\mathrm{B}:(x_1,\alpha_1),(x_2,\alpha_2),\cdots,(x_n,\alpha_n) \end{aligned} \tag{6.15}$$

where the two expert's experimental data A and B meet the following conditions respectively:

$$\begin{aligned} &C_{\mathrm{A}}:x_1<x_2<\cdots<x_m,0\le\alpha_1\le\alpha_2\le\cdots\le\alpha_m\\ &C_{\mathrm{B}}:x_1<x_2<\cdots<x_n,0\le\beta_1\le\beta_2\le\cdots\le\beta_n \end{aligned} \tag{6.16}$$

The empirical uncertainty distributions are $\Psi_{\mathrm{A}}(x)$ and $\Psi_{\mathrm{B}}(x)$ respectively:

$$\begin{aligned} &\Psi_{\mathrm{A}}(x_i)=\alpha_i, i=1,2,\cdots,m\\ &\Psi_{\mathrm{B}}(x_j)=\beta_j, j=1,2,\cdots,n \end{aligned} \tag{6.17}$$

Assuming that $F_1(x)$ and $F_2(x)$ are the two theoretical uncertainty distributions from the two experts. The hypothesis testing statement is

$$\begin{aligned} &H_0:F_1(x)=F_2(x)\\ &H_1:F_1(x)\ne F_2(x) \end{aligned} \tag{6.18}$$

where H_0 represents the null hypothesis and H_1 represents the alternative hypothesis. In other words, H_0 is equivalent to the statement that the two domain experts have the same views on uncertain variable ξ, while H_1 is equivalent to the statement that the two domain experts have different same views on uncertain variable ξ.

It should be noted that the experts experimental data in (6.15) with conditions in (6.16) are ordinal. This feature will determine the characteristic of the testing scheme of the statement whether the two uncertainty distributions $F_1(x)$ and $F_2(y)$ are equal. The testing scheme is stated as following five steps.

Step 1. Select s arbitrary points $(x_1,\alpha_1),(x_2,\alpha_2),\cdots,(x_s,\alpha_s)$ from A in (6.15) and t points $(x_1,\beta_1),(x_2,\beta_2),\cdots,(x_t,\beta_t)$, from B in (6.15) such that $s>m$ and $t>n$, i.e.,

$$\begin{aligned}&\mathrm{A}_H:(x_1,\alpha_1),(x_2,\alpha_2),\cdots,(x_s,\alpha_s)\\&\mathrm{B}_H:(x_1,\beta_1),(x_2,\beta_2),\cdots,(x_t,\beta_t)\end{aligned} \tag{6.19}$$

Step 2. In ascending order, rank x_i, x_j listed from A_H and B_H in (6.19). Also in ascending order, rank α_i, β_j listed from A_H and B_H in (6.19). The two new sequences will be obtained, the first is with respect to $x_i(x_j)$ value, and the second is with respect to $\alpha_i(\beta_j)$value. The two new ranked sequences are

$$\begin{aligned}&n_{\mathrm{A}}^x n_{\mathrm{B}}^x n_{\mathrm{A}}^x n_{\mathrm{A}}^x \cdots n_{\mathrm{B}}^x\\&n_{\mathrm{B}}^\alpha n_{\mathrm{A}}^\alpha n_{\mathrm{A}}^\alpha n_{\mathrm{A}}^\alpha \cdots n_{\mathrm{B}}^\alpha\end{aligned} \tag{6.20}$$

where $n_{\mathrm{A}}^x n_{\mathrm{B}}^x n_{\mathrm{A}}^x n_{\mathrm{A}}^x \cdots n_{\mathrm{B}}^x$ and $n_{\mathrm{A}}^x n_{\mathrm{B}}^x n_{\mathrm{A}}^x n_{\mathrm{A}}^x \cdots n_{\mathrm{B}}^x$ represent numbers obtained from the uncertainty distributions $\Psi_{\mathrm{A}}(x)$ and $\Psi_{\mathrm{B}}(x)$ with respect to x and α respectively.

Step 3. Compare the two sequences $n_{\mathrm{A}}^x n_{\mathrm{B}}^x n_{\mathrm{A}}^x n_{\mathrm{A}}^x \cdots n_{\mathrm{B}}^x$ and $n_{\mathrm{A}}^x n_{\mathrm{B}}^x n_{\mathrm{A}}^x n_{\mathrm{A}}^x \cdots n_{\mathrm{B}}^x$ obtained in (6.20) and assign value 0 or 1 according to the criterion: if the numbers in the same position are equal then a value of 0 is assigned, otherwise if they are different assign a value of 1. Thus a 0 - 1 sequence of length $m+n$ can be generated as follows:

$$1100\cdots 0 \tag{6.21}$$

Step 4. Define the test statistic

$$T = \sum_{l=1}^{m+n} \vartheta\left\{ n_{\mathrm{A}}^{x} \neq n_{\mathrm{B}}^{\alpha} \right\} \qquad (6.22)$$

where the indicator function is defined by

$$\vartheta\left\{ n_{\mathrm{A}}^{x} \neq n_{\mathrm{B}}^{\alpha} \right\} = \begin{cases} 1 & \text{if} \quad \left\{ n_{\mathrm{A}}^{x} \neq n_{\mathrm{B}}^{\alpha} \right\} \text{ is true} \\ 0 & \text{if} \quad \left\{ n_{\mathrm{A}}^{x} = n_{\mathrm{B}}^{\alpha} \right\} \text{ is true} \end{cases} \qquad (6.23)$$

Step 5. For a preset criterion $0 < p < 1$ (say, p = 0.2), the null hypothesis H_0 is rejected if the testing statistic

$$T > (m+n)\,p \qquad (6.24)$$

Since if the null hypothesis H_0 holds, the testing statistic T, which is the count of 1's in the sequence in (6.21), should be too large.

6.6 Hesamian-Taheri Method

Hesamian and Taheri (2011) also proposed an uncertainty hypothesis testing to determine whether a set of experimental data fits a specific uncertainty distribution $F(x)$. This method can be regarded as some kind of goodness of fit test.

Assuming that $F_0(x)$ is a known uncertainty distribution related to an expert's view. Hence, in order to detect whether a given experimental data, $F_0(x)$, follows this specific uncertainty distribution F, the hypothesis testing is given as follows:

$$\begin{aligned} H_0 &: F(x) = F_0(x), \forall x \in \mathbb{R} \\ H_1 &: F(x) \neq F_0(x), \exists\, x \in \mathbb{R} \end{aligned} \qquad (6.25)$$

Let expert's data be

$$(x_1,\alpha_1),(x_2,\alpha_2),\cdots,(x_n,\alpha_n), \qquad (6.26)$$

where conditions

$$\begin{aligned} &x_1 < x_2 < \cdots < x_n \\ &0 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n \,(\le 1) \end{aligned} \tag{6.27}$$

Note that the empirical uncertainty distribution $\Psi(x)$ has to be estimated from expert's data. In order to enhance the testing power, the points in testing data should be increased as a natural choice. The "new" data set is

$$\left(z_1,\alpha_1^*\right),\left(z_2,\alpha_2^*\right),\left(z_3,\alpha_3^*\right)\cdots,\left(z_{2n-1},\alpha_{2n-1}^*\right), \tag{6.28}$$

where

$$\begin{aligned} &z_1 = x_1, \alpha_1^* = \alpha_1 \\ &z_2 = \frac{x_1 + x_2}{2}, \alpha_2^* = \Psi\left(z_2\right) \\ &z_3 = x_2, \alpha_3^* = \alpha_2 \\ &\quad\vdots \\ &z_{2m-1} = x_m, \alpha_{2m-1}^* = \alpha_m \\ &z_{2m} = \frac{x_m + x_{m+1}}{2}, \alpha_{2m}^* = \Psi\left(z_{2m}\right) \\ &\quad\vdots \\ &z_{2n-1} = x_n, \alpha_{2n-1}^* = \alpha_n \end{aligned} \tag{6.29}$$

i.e., data are enhanced by adding $y_i = 0.5\left(x_i + x_{i+1}\right),\ i = 1, 2, ..., n\text{-}1$ to the original dataset. , then a new sequence of size $N = 2n-1$ is obtained and each unit is represented as $\mathrm{z_i}$ such that $z_1, z_2, ..., z_{2n-1}$ and since $\Psi\left(x\right)$ and $F_0(x)$ are increasing in order then,

$$\begin{aligned} &\Psi\left(z_1\right),\Psi\left(z_2\right),\cdots,\Psi\left(z_{2n-1}\right) \\ &F_0\left(z_1\right),F_0\left(z_2\right),...,F_0\left(z_{2n-1}\right) \end{aligned} \tag{6.30}$$

Let $\varepsilon > 0$ be a very small number that measures the difference between the value obtained from the uncertainty distribution $F_0(x)$ and the empirical distribution function $\Psi\left(x\right)$. Then a new 0-1 sequence of size $N = 2n-1$ can be generated based on the following criterion:

$$\vartheta_\varepsilon\left\{\left|\Psi\left(z_l\right) - F_0\left(z_l\right)\right|\right\}, l = 1, 2, \cdots, 2n-1 \tag{6.31}$$

where the indicator function

$$\vartheta_{\varepsilon}\left\{\left|\Psi\left(z_l\right)-F_0\left(z_l\right)\right|\right\}=\begin{cases}0 & \text{if} \quad \left|\Psi\left(z_l\right)-F_0\left(z_l\right)\right|\le\varepsilon\\ 1 & \text{if} \quad \left|\Psi\left(z_l\right)-F_0\left(z_l\right)\right|>\varepsilon\end{cases}$$
$$l=1,2,\cdots,2n-1 \tag{6.32}$$

If the number of 1’s in the 0-1 sequence is small then we fail to reject the null hypothesis. The testing statistic is defined by

$$T=\sum_{l=1}^{2n-1}\vartheta_{\varepsilon}\left\{\left|\Psi\left(z_l\right)-F_0\left(z_l\right)\right|\right\} \tag{6.33}$$

The decision rule to reject the null hypothesis is given as follows

$$T>\left(2n-1\right)p \tag{6.34}$$

where p is a pre-specified criterion.

Chapter 7. Uncertainty Process Capability Indices

7.1 Justification for Applying Uncertainty Theory and Statistics

As reviewed in proceeding chapters with respect to probability theory based process capability studies and Liu's Uncertainty theory, we justify the reason for developing uncertainty theory based process capability indices from two aspects: (1) certain exposures in current industries about the contradiction from applying classical process capability indices based on probability theory; (2) the new features in uncertainty theory, particularly the interval-valued variance will definitely enhance the ability of the newly created uncertainty theory based process capability indices.

In probability theory and statistics, the law of large numbers and central limit theorem play vital roles and thus many statistical quantities can be specified by Gaussian Normal distribution, either accurately or approximately or asymptotically. As a matter of well-known fact, the probability theory based statistics is so powerful so that modern statistics facilitates extremely powerful theoretical and practical support in assessing quality, risk management and general quality management.

The uniformity of a production process, i.e., the variability of a process, is the key concern of management, typically measured by taking six-sigma spread specified by tolerance limits in the distribution of the process characteristic. In other words, the fundamental role played in process capability study is the distribution function of process characteristic, whose shape, central position, the spread about the centre, etc, are all determined by relevant parameters of the distribution function, either true or estimated values.

On the other side, the capability of a process holding the tolerance, either engineering or statistical, is one of the critical components in today's quality improvement. The popular process capability indices are certain measures by using the process tolerance over the six-sigma spread, either their interval lengths, or the process yield, or the expected loss away from the departure of the process.

We have to emphasize that the statistical analysis of a process capability involves many uncertainty aspects, for example, process tolerance level, equipment tuning level, material supply quality level, vendor competing level, sampling process level, the management decision making level, and the interactive level among those effects. The complexity of process capability study will not be explained simply by the random uncertainty facilitated by probability theory. What we may admit is random uncertainty can partially explain certain uncertainty in industrial processing.

For example, if a process characteristic *X* follows Gaussian Normal distribution $N\left(\mu,\sigma^2\right)$, six-sigma spread about the centre quantifies the probability that a given value of a Gaussian normal random variable will fall within 3 standard deviations away from the mean $\left(2\Phi_0(3)-1\right)$, where Φ_0 represents the standard Gaussian normal distribution function.

However, probabilistic six-sigma spread has come under some criticism due to problems encountered by the Japanese car industry in recent years. It is obvious that the probabilistic process capability indices and their applications are facing certain scepticism. Overall, a fundamental question arises, is probability theory still a viable option in maintaining quality in today's industrial environment?

The involvement of human thinking behaviour will be inevitable because the decision-making level in quality improvements no matter at the individual workman level, field supervisor level, or top managing director level is an interaction between human mind and the real manufacturing world.

Recall that in Liu's (2007, 2010, 2013) uncertainty theory, an uncertain variable is fully specified if the corresponding uncertain measure λ is given. In practice, there are situations where only the uncertainty distributions are available. Even the most popular uncertain variable (i.e. Liu's normal uncertain variable) only the normal distribution functions is available. Furthermore, the variance of Liu's normal uncertain variable is interval-valued, $\left[0.5\sigma^2,\sigma^2\right]$. This will add more uncertainty feature into process capability study, although interval-valued quantities have already appeared in probability based process capability studies. This fact may open a new door toward a different process capability study.

7.2 $_uC_p$,$_uC_{pk}$ and $_uC_{pm}$ Indices

Under probability theory several classical capability indices such as C_p, C_{pk} and C_{pm} have been proposed to assess how well a process satisfies customer requirements. However, classical capability indices tend to take on a strict definition, and are unable to accommodate imprecision in terms of data. Moreover, industrial processes are usually influenced by human judgment and classical process capability indices fail to capture this influence. Hence, this section proposes new capability indices under an uncertain environment.

Under the context of probability theory, the most used process capability indices are:

$$C_p = \frac{USL - LSL}{6\sigma} \tag{7.1}$$

and

$$C_{pk} = C_p(1-\kappa) \tag{7.2}$$

where

$$\kappa = \frac{\mu - M}{(USL - LSL)/2} \tag{7.3}$$

the departure from the midpoint of the process.

If the uncertainty environment is switching from classical probability theory into Liu's uncertainty environment, the process capability indices will be defined in the form similar to those in probability environment. Nevertheless, in most practical circumstances, the uncertain process capability indices will appear in interval-valued form, i.e., in the form of an interval. For further processing the interval-valued uncertain process capability indices, it is necessary to review interval arithmetic as a preparation.

Interval Arithmetic: Let two intervals [a,b] and [c,d] be subsets of real-line (-∞,+∞).

Rule 1. (Addition) The addition of the two intervals is [a, b] + [c, d] = [$a + c$, $b + d$];
Rule 2. (Subtraction) The subtraction of the two intervals is [a, b] - [c, d] = [$a - d$, $b - c$];
Rule 3. (Multiplication) The multiplication of the two intervals is [a, b] × [c, d] = [$\min(a\times c, a\times d, b\times c, a\times d)$, $\max(a\times c, a\times d, b\times c, a\times d)$];

Rule 4. (Division) The division of the two intervals is [a, b] ÷ [c, d] = [$\min(a\div c, a\div d, b\div c, a\div d)$, $\max(a\div c, a\div d, b\div c, a\div d)$], in which $0 \notin [c,d]$.

Interval arithmetic rules were first stated by R.E. Moore. In interval division operation, if $0 \in [c, d]$, the division is not defined. For convenience, we state the rule for square root operation of a positive interval as a definition.

Definition 7.1: (Square Root Rule) Let interval the $[a,b]$ be a subset of (0, +∞), the square root of the interval is defined as $\sqrt{[a,b]} = \left[\sqrt{a}, \sqrt{b}\right]$.

Now, we are in a good position to define the uncertain process capability indices precisely. Assuming that an uncertain process is under investigation, in which the characteristic of the process, denoted as *X*, is specified by an uncertainty distribution function, denoted as Ψ. Therefore, the uncertain variable *X* has its expectation μ and interval-valued variance $V[X] = \left[\sigma_L^2, \sigma_U^2\right]$ available.

Definition 7.2: The uncertain process capability index ${}_uC_p$ is defined by:

$$ {}_uC_p = \left[\frac{USL - LSL}{6\sigma_U}, \frac{USL - LSL}{6\sigma_L}\right], \tag{7.4} $$

where σ_L and σ_U are the lower limit and upper limit of the square root of variance interval $\sqrt{V[X]} = [\sigma_L, \sigma_U]$.

Parallel to the definition of C_p in probability context,

$$ \begin{aligned} {}_uC_p &\triangleq \frac{USL - LSL}{6\sqrt{V[X]}} \\ &= \frac{USL - LSL}{6[\sigma_L, \sigma_U]} \\ &= \frac{USL - LSL}{6} \frac{1}{[\sigma_L, \sigma_U]} \\ &= \frac{USL - LSL}{6}\left[\frac{1}{\sigma_U}, \frac{1}{\sigma_L}\right] \\ &= \left[\frac{USL - LSL}{6\sigma_U}, \frac{USL - LSL}{6\sigma_L}\right] \end{aligned} \tag{7.5} $$

From Eq. (7.5), it is easy to see that the uncertain process capability index ${}_uC_p$ is defined similar to that in probability context. As to the interval form of the uncertain process capability index, it is just a reflection of interval-valued variance of the uncertain process characteristic conditioning on.

As to definition of ${}_uC_{pk}$, it is noticed from Eq. (7.3) that the departure coefficient κ is merely related to the half length of specification interval, the parameter μ and the midpoint M, and hence a scalar quantity because of the availability of the true parameter μ.

Definition 7.3: The uncertain process capability index ${}_uC_{pk}$ is defined by:

$$ {}_uC_{pk} = \left[\frac{(USL-LSL)-2(\mu-M)}{6\sigma_U}, \frac{(USL-LSL)-2(\mu-M)}{6\sigma_L} \right], \tag{7.6} $$

where parameter μ is the expectation, M is the midpoint, σ_L and σ_U are the lower limit and upper limit of the square root of variance interval $\sqrt{V[X]} = [\sigma_L, \sigma_U]$.

Definition 7.4: The uncertain process capability index ${}_uC_{pm}$ is defined by:

$$ {}_uC_{pm} = \left[\frac{1}{\sqrt{1+\left(\frac{\mu-M}{\sigma_L}\right)^2}} \frac{USL-LSL}{6\sigma_U}, \frac{1}{\sqrt{1+\left(\frac{\mu-M}{\sigma_U}\right)^2}} \frac{USL-LSL}{6\sigma_L} \right], \tag{7.7} $$

Where the parameter μ is the expectation, M is the midpoint, σ_L and σ_U are the lower limit and upper limit of the square root of variance interval $\sqrt{V[X]} = [\sigma_L, \sigma_U]$.

As to an error based process capability index, it is necessary to define the interval-valued average process loss ξ_u and the interval function g_u:

$$ \xi_u = \left[\frac{\mu-T}{\sigma_U}, \frac{\mu-T}{\sigma_L} \right]. \tag{7.8} $$

and

$$g_u = \frac{1}{\sqrt{1+\xi_u^2}} = \left[\frac{1}{\sqrt{1+\left(\frac{\mu - M}{\sigma_L}\right)^2}}, \frac{1}{\sqrt{1+\left(\frac{\mu - M}{\sigma_U}\right)^2}} \right]. \qquad (7.9)$$

The expression of g_u in Eq. (7.9) is obtained in the following way:

$$\begin{aligned} g_u &= \frac{1}{\sqrt{1+\xi_u^2}} \\ &= \frac{1}{\sqrt{1+\left[\left(\frac{\mu - M}{\sigma_U}\right)^2, \left(\frac{\mu - M}{\sigma_L}\right)^2\right]}} \\ &= \frac{1}{\left[\sqrt{1+\left(\frac{\mu - M}{\sigma_U}\right)^2}, \sqrt{1+\left(\frac{\mu - M}{\sigma_L}\right)^2}\right]} \\ &= \left[\frac{1}{\sqrt{1+\left(\frac{\mu - M}{\sigma_L}\right)^2}}, \frac{1}{\sqrt{1+\left(\frac{\mu - M}{\sigma_U}\right)^2}} \right]. \end{aligned} \qquad (7.10)$$

It should be fully aware that the interval-valued uncertain process capability indices are still mutually related in the way similar to those in probabilistic context.

7.3 Uncertain Normal Process Capability Indices

In probabilistic quality control several classical capability indices such as C_p, C_{pk} and C_{pm} were investigated under Gaussian normal distribution. Parallel to probabilistic process index development, it is a necessary to investigate the basic feature of those uncertain process capability indices (i.e. ${}_uC_p$, ${}_uC_{pk}$, and ${}_uC_{pm}$) under Liu's uncertainty normal distribution. As a preparation, let us examine the variance interval of Liu's uncertainty normal distribution.

Theorem 7.1 (Guo, 2012): Given Liu's uncertainty normal distribution, then the standard deviation interval can be expressed as

$$\sqrt{V[X]} = \left[\frac{1}{\sqrt{2}}\sigma_U, \sigma_U\right], \tag{7.11}$$

where σ_U is the stipulated standard deviation parameter σ in Liu's uncertainty normal distribution function.
Proof. In general, the variance of an uncertainty distribution is

$$\begin{aligned} V[X] &\triangleq \mathrm{E}\left[(X-\mu)^2\right] \\ &= \int_0^{+\infty} ƛ\left\{(\xi-\mu)^2 \geq x\right\} dx \\ &= \int_0^{+\infty} ƛ\left\{\left\{\xi \geq \mu+\sqrt{x}\right\} \cup \left\{\xi \geq \mu-\sqrt{x}\right\}\right\} dx \end{aligned} \tag{7.12}$$

Notice that the uncertain measure is not given here. What is given is the uncertainty distribution. Therefore, the variance is an interval, denoted as $V[X] = \left[\sigma_L^2, \sigma_U^2\right]$.

From Eq. (7.12), it is ready to obtain the upper limit σ_U^2:

$$\begin{aligned} &\int_0^{+\infty} ƛ\left\{\left\{\xi \geq \mu+\sqrt{x}\right\} \cup \left\{\xi \geq \mu-\sqrt{x}\right\}\right\} dx \\ &\leq \int_0^{+\infty} ƛ\left\{\xi \geq \mu+\sqrt{x}\right\} dx + \int_0^{+\infty} ƛ\left\{\xi \geq \mu-\sqrt{x}\right\} dx \\ &= \int_0^{+\infty} \left(\left(1-\Psi\left(\mu+\sqrt{x}\right)\right) + \left(\Psi\left(\mu-\sqrt{x}\right)\right)\right) dx \\ &= 2\int_0^{+\infty} \left(r\left(1-\Psi\left(r+\mu\right)\right) + \left(\Psi\left(\mu-r\right)\right)\right) dr \triangleq \sigma_U^2 \end{aligned} \tag{7.13}$$

From Eq. (7.12), the lower limit σ_L^2:

$$\begin{aligned} &\int_0^{+\infty} ƛ\left\{\left\{\xi \geq \mu+\sqrt{x}\right\} \cup \left\{\xi \geq \mu-\sqrt{x}\right\}\right\} dx \\ &\geq \int_0^{+\infty} ƛ\left\{\xi \geq \mu+\sqrt{x}\right\} dx \wedge \int_0^{+\infty} ƛ\left\{\xi \geq \mu-\sqrt{x}\right\} dx \\ &= \int_0^{+\infty} \left(1-\Psi\left(\mu+\sqrt{x}\right)\right) dx \wedge \int_0^{+\infty} \left(1-\Psi\left(\mu+\sqrt{x}\right)\right) dx \\ &= 2\left(\int_0^{+\infty} r\left(1-\Psi\left(\mu+r\right)\right) dr \wedge \int_0^{+\infty} r\left(1-\Psi\left(\mu-r\right)\right) dr\right) \triangleq \sigma_L^2 \end{aligned} \tag{7.14}$$

Given Liu's uncertainty normal distribution function with parameters μ and σ:

$$\Psi(x) = \left(1 + \exp\left(-\frac{\pi}{\sqrt{3}\sigma}(x-\mu)\right)\right)^{-1}, x \in (-\infty, +\infty) \tag{7.15}$$

Then, the upper limit of the variance interval is:

$$\begin{aligned}
&\sigma_U^2 \\
&= \int_0^{+\infty} \left(1 - \Psi_\xi\left(\mu + \sqrt{x}\right)\right) dx + \int_0^{+\infty} \Psi_\xi\left(\mu - \sqrt{x}\right) dx \\
&= \int_0^{+\infty} \left(1 - \left(1 + \exp\left(-\frac{\pi}{\sqrt{3}}\left(\frac{\left(\mu + \sqrt{x}\right) - \mu}{\sigma}\right)\right)\right)^{-1}\right) dx \\
&+ \int_0^{+\infty} \left(1 + \exp\left(-\frac{\pi}{\sqrt{3}}\left(\frac{\left(\mu - \sqrt{x}\right) - \mu}{\sigma}\right)\right)\right)^{-1} dx \\
&= \int_0^{+\infty} \left(1 - \left(1 + \exp\left(-\frac{\pi}{\sqrt{3}}\left(\frac{\sqrt{x}}{\sigma}\right)\right)\right)^{-1}\right) dx \\
&+ \int_0^{+\infty} \left(1 - \left(1 + \exp\left(-\frac{\pi}{\sqrt{3}}\left(\frac{-\sqrt{x}}{\sigma}\right)\right)\right)^{-1}\right) dx \\
&= \int_0^{+\infty} \left(1 - \left(1 + \exp\left(-\frac{\pi}{\sqrt{3}} z\right)\right)^{-1}\right) d(\sigma z)^2 \\
&+ \int_0^{+\infty} \left(1 - \left(1 + \exp\left(-\frac{\pi}{\sqrt{3}} (-z)\right)\right)^{-1}\right) d(\sigma z)^2 \\
&= 2\sigma^2 \int_0^{+\infty} z \left(1 - \left(1 + \exp\left(-\frac{\pi}{\sqrt{3}} z\right)\right)^{-1}\right) dz \\
&+ 2\sigma^2 \int_0^{+\infty} z \left(1 - \left(1 + \exp\left(\frac{\pi}{\sqrt{3}} z\right)\right)^{-1}\right) dz
\end{aligned} \tag{7.16}$$

Similarly, the lower limit of the variance interval is:

$$\sigma_L^2 = 2\sigma^2 \int_0^{+\infty} z \left(1 - \left(1 + \exp\left(-\frac{\pi}{\sqrt{3}} z\right)\right)^{-1}\right) dz \vee 2\sigma^2 \int_0^{+\infty} z \left(1 - \left(1 + \exp\left(\frac{\pi}{\sqrt{3}} z\right)\right)^{-1}\right) dz \tag{7.17}$$

Now, we need to show that

$$\int_0^{+\infty} z\left(1-\left(1+\exp\left(-\frac{\pi}{\sqrt{3}}z\right)\right)^{-1}\right)dz = \int_0^{+\infty} z\left(1+\exp\left(\frac{\pi}{\sqrt{3}}z\right)\right)^{-1} dz \tag{7.18}$$

Notice that

$$\begin{aligned}
&\int_0^{+\infty} z\left(1-\left(1+\exp\left(-\frac{\pi}{\sqrt{3}}z\right)\right)^{-1}\right)dz \\
&= \int_0^{+\infty} z\left(1-\left(1+\frac{1}{\exp\left(\frac{\pi}{\sqrt{3}}z\right)}\right)^{-1}\right)dz \\
&= \int_0^{+\infty} z\left(1-\frac{\exp\left(\frac{\pi}{\sqrt{3}}z\right)}{\exp\left(\frac{\pi}{\sqrt{3}}z\right)+1}\right)dz \\
&= \int_0^{+\infty} z\left(1-\frac{\exp\left(\frac{\pi}{\sqrt{3}}z\right)}{\exp\left(\frac{\pi}{\sqrt{3}}z\right)+1}\right)dz \\
&= \int_0^{+\infty} z\frac{1}{\exp\left(\frac{\pi}{\sqrt{3}}z\right)+1}dz \\
&= \int_0^{+\infty} z\left(1+\exp\left(\frac{\pi}{\sqrt{3}}z\right)\right)^{-1} dz
\end{aligned} \tag{7.19}$$

Hence,

$$\begin{aligned}
\sigma_U^2 &= 2\sigma^2\int_0^{+\infty} z\left(1+\exp\left(\frac{\pi}{\sqrt{3}}z\right)\right)^{-1} dz + 2\sigma^2\int_0^{+\infty} z\left(1-\left(1+\exp\left(\frac{\pi}{\sqrt{3}}z\right)\right)^{-1}\right)dz \\
\sigma_L^2 &= 2\sigma^2\int_0^{+\infty} z\left(1+\exp\left(\frac{\pi}{\sqrt{3}}z\right)\right)^{-1} dz \wedge 2\sigma^2\int_0^{+\infty} z\left(1-\left(1+\exp\left(\frac{\pi}{\sqrt{3}}z\right)\right)^{-1}\right)dz
\end{aligned} \tag{7.20}$$

which leads to the ratio of the two limits is

$$\frac{\sigma_L^2}{\sigma_U^2} == \frac{\int_0^{+\infty}\left(1-\left(1+\exp\left(-\frac{\pi}{\sqrt{3}}z\right)\right)^{-1}\right)dz}{2\int_0^{+\infty}\left(1-\left(1+\exp\left(-\frac{\pi}{\sqrt{3}}z\right)\right)^{-1}\right)dz} = \frac{1}{2}. \tag{7.21}$$

Accordingly, we obtain the ratio of the standard deviation interval limits:

$$\frac{\sigma_L}{\sigma_U} = \sqrt{\frac{\sigma_L^2}{\sigma_U^2}} = \frac{1}{\sqrt{2}}. \tag{7.22}$$

Theorem 7.2 (Guo, 2014): Given the process characteristic with Liu's uncertain normal distribution, then the uncertain process capability index ${}_uC_p$ can be expressed by:

$${}_uC_p = \left[\frac{USL-LSL}{6\sigma_U}, \frac{USL-LSL}{3\sqrt{2}\sigma_U}\right]. \tag{7.23}$$

Proof: Under Liu's uncertain normal distribution, $\sigma_L = 1/\sqrt{2}\sigma_U$, thus

$$\begin{aligned}
&{}_uC_p \\
&= \left[\frac{USL-LSL}{6\sigma_U}, \frac{USL-LSL}{6\sigma_L}\right] \\
&= \left[\frac{USL-LSL}{6\sigma_U}, \frac{USL-LSL}{6\frac{1}{\sqrt{2}}\sigma_U}\right] \\
&= \left[\frac{USL-LSL}{6\sigma_U}, \frac{USL-LSL}{3\sqrt{2}\sigma_U}\right].
\end{aligned} \tag{7.24}$$

Theorem 7.3 (Guo, 2014): Given the process characteristic with Liu's uncertain normal distribution, then the uncertain process capability index ${}_uC_{pk}$ can be expressed by:

$$ {}_{u}C_{pk} = \left[\frac{(USL - LSL) - 2(\mu - M)}{6\sigma_U}, \frac{(USL - LSL) - 2(\mu - M)}{3\sqrt{2}\sigma_U} \right]. \tag{7.25} $$

Proof: Under Liu's uncertain normal distribution, $\sigma_L = 1/\sqrt{2}\sigma_U$, thus

$$ \begin{aligned} {}_{u}C_{pk} & \\ &= \left[\frac{(USL - LSL) - 2(\mu - M)}{6\sigma_U}, \frac{(USL - LSL) - 2(\mu - M)}{6\sigma_L} \right] \\ &= \left[\frac{(USL - LSL) - 2(\mu - M)}{6\sigma_U}, \frac{(USL - LSL) - 2(\mu - M)}{6\frac{1}{\sqrt{2}}\sigma_U} \right] \\ &= \left[\frac{(USL - LSL) - 2(\mu - M)}{6\sigma_U}, \frac{(USL - LSL) - 2(\mu - M)}{3\sqrt{2}\sigma_U} \right]. \end{aligned} \tag{7.26} $$

Theorem 7.4 (Guo, 2014): Given the process characteristic with Liu's uncertain normal distribution, then the uncertain process capability index ${}_{u}C_{pm}$ can be expressed by:

$$ {}_{u}C_{pm} = \left[\frac{USL - LSL}{6\sqrt{\sigma_U^2 + 2(\mu - M)^2}}, \frac{USL - LSL}{3\sqrt{2}\sqrt{\sigma_U^2 + (\mu - M)^2}} \right]. \tag{7.27} $$

Proof: Under Liu's uncertain normal distribution, $\sigma_L = 1/\sqrt{2}\sigma_U$, thus

$$
\begin{aligned}
&{}_uC_{pm} \\
&= \left[\frac{1}{\sqrt{1+\left(\dfrac{\mu-M}{\sigma_L}\right)^2}} \frac{USL-LSL}{6\sigma_U}, \frac{1}{\sqrt{1+\left(\dfrac{\mu-M}{\sigma_U}\right)^2}} \frac{USL-LSL}{6\sigma_L} \right] \\
&= \left[\frac{1}{\sqrt{1+\left(\dfrac{\mu-M}{\sigma_U/\sqrt{2}}\right)^2}} \frac{USL-LSL}{6\sigma_U}, \frac{1}{\sqrt{1+\left(\dfrac{\mu-M}{\sigma_U}\right)^2}} \frac{USL-LSL}{3\sqrt{2}\sigma_U} \right] \\
&= \left[\frac{USL-LSL}{6\sqrt{\sigma_U^2+2(\mu-M)^2}}, \frac{USL-LSL}{3\sqrt{2}\sqrt{\sigma_U^2+(\mu-M)^2}} \right].
\end{aligned} \tag{7.28}
$$

It should be fully aware that the interval-valued uncertain process capability indices are still mutually related in the way similar to those in probabilistic context.

7.4 Uncertainty Distribution For Mean and Variance

In probability theory, sampling statistic plays critical roles in confidence interval construction and hypothesis testing. From the review on classical process capability study literature, we can realize the roles of two important sampling statistics: sampling mean, $\bar{x}$, and sampling variance, S^2 . Similarly, once we define the three uncertain process capability indices, ${}_uC_p$, ${}_uC_{pk}$ and ${}_uC_{pm}$ under Liu's normal uncertainty distribution with true parameters μ and σ, then we have to address the situations where the true parameters μ and σ are not available but their "sampling" mean and variance are available.

It is necessary to stress here, in uncertainty statistics, term "sample" is referred to as a group of representative observation taken from a given population or uncertainty distribution. Different from probabilistic statistics, where sample is strictly defined, up to now in uncertainty statistics, term "sample" is not defined. The usage of "sample" is just for convenience. Whenever the term "sample" is used, it just tells us that a group of representative observations from a given population or an uncertainty distribution.

Uncertain "Sample" Postulate: In uncertainty statistics, it is assumed that a "sample" $x_1, x_2, \cdots, x_n$ have the properties: (1) every member of the "sample" follows the same uncertainty distribution Ψ, i.e., the population uncertainty distribution; (2) every member is independent of other members within the group. The identically, independently distributed "sampling" properties can be abbreviated as "i.i.d.".

Let us explore the uncertainty distribution for "sampling" mean, $\bar{x}$, first. Given a "sample" from an uncertain normal population, Ψ, $x_1, x_2, \cdots, x_n$,

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{7.29}$$

Theorem 7.5: (**Guo et al, 2012**) Assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Then the "sampling" mean follows an uncertain normal distribution with parameters μ and σ.

Proof: Let W denote the "sample" sum $\sum_{i=1}^{n} x_i$. Notice that W is strictly monotone increasing in x_i, then in terms of Uncertain "Sample" Postulate, and the condition $x_1, x_2, \cdots, x_n \sim \mathcal{N}(\mu, \sigma)$, then

$$\Psi_W^{-1}(\alpha) = \Psi_{x_1}^{-1}(\alpha) + \Psi_{x_2}^{-1}(\alpha) + \cdots + \Psi_{x_n}^{-1}(\alpha) = n\Psi^{-1}(\alpha) = n\mu + \frac{n\sigma\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right) \tag{7.30}$$

Further notice that the ratio between "sample" mean $\bar{x}$ and "" sum W is a constant $1/n$, given the sampling size n. Thus

$$\Psi_{\bar{x}}^{-1}(\alpha) = \frac{1}{n}\Psi_W^{-1}(\alpha) = \mu + \frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right) \tag{7.31}$$

i.e., the "sample" mean $\bar{x} \sim \mathcal{N}(\mu, \sigma)$.

Definition 7.6: (**Guo, 2014**) Assuming that the square of Liu's standard uncertain normal variable, which is called as uncertain chi-square variable with one degree of freedom, denoted by $\chi^2_{(1)}$, follows uncertainty distribution:

$$\Psi_{\chi^2_{(1)}}(z) = 1 - \exp\left(-\sqrt{z}\right). \tag{7.32}$$

It is noticed that $\Psi_{\chi^2_{(1)}}(0)=0,$ and $\Psi_{\chi^2_{(1)}}(+\infty)=1.$ Technically, given the standard uncertain normal distribution function, the distribution function for the square of an uncertain normal variable is not derivable, because only the square of a positive uncertain variable has an uncertainty distribution function. Therefore, defining an uncertain distribution function for the square of an uncertain normal variable is a feasible way to process the next distributional developments.

Theorem 7.7: (**Guo, 2014**) Assuming that a "sample" $\chi^2_{(1),1},\chi^2_{(1),2},\cdots,\chi^2_{(1),d}$, from a chi-square variable with one degree of freedom, $\chi^2_{(1)}$, then the sum, denoted by $\chi^2_{(d)}$, which is called as uncertain chi-square variable with one *d* degree of freedom. Then the sampling sum follows an uncertain distribution:

$$\Psi_{\chi^2_{(d)}}(z)=1-\exp\left(-\sqrt{\frac{z}{d}}\right). \tag{7.33}$$

Proof: First we derive the inverse function for the chi-square variable with one degree of freedom, $\chi^2_{(1)}$:

$$\alpha=1-\exp\left(-\sqrt{\Psi^{-1}_{\chi^2_{(1)}}(\alpha)}\right) \tag{7.34}$$

which gives

$$\Psi^{-1}_{\chi^2_{(1)}}(\alpha)=\ln^2\frac{1}{1-\alpha} \tag{7.35}$$

By definition,

$$\chi^2_{(d)}=\sum_{i=1}^{d}\chi^2_{(1),i} \tag{7.36}$$

Then, the inverse function for the uncertain chi-square variable with one *d* degree of freedom

$$\Psi^{-1}_{\chi^2_{(d)}}(\alpha)=\sum_{i=1}^{d}\Psi^{-1}_{\chi^2_{(1),i}}(\alpha)=\left[\sqrt{d}\ln\frac{1}{1-\alpha}\right]^2 \tag{7.37}$$

which leads to the uncertain chi-square distribution with d degree of freedom.

For further uncertainty distribution developments, let us show a simple lemma.

Lemma 7.1: Assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Then

$$\sum_{i=1}^{n}(x_i-\mu)^2=\sum_{i=1}^{n}(x_i-\overline{x})^2+n(\overline{x}-\mu)^2 \tag{7.38 }$$

Proof: It is a really algebra operation,

$$\begin{aligned}
&\sum_{i=1}^{n}(x_i-\mu)^2\\
&=\sum_{i=1}^{n}(x_i-\overline{x}+\overline{x}-\mu)^2\\
&=\sum_{i=1}^{n}(x_i-\overline{x})^2+2\sum_{i=1}^{n}(x_i-\overline{x})(\overline{x}-\mu)+\sum_{i=1}^{n}(\overline{x}-\mu)^2\\
&=\sum_{i=1}^{n}(x_i-\overline{x})^2+2(\overline{x}-\mu)\sum_{i=1}^{n}(x_i-\overline{x})+(\overline{x}-\mu)^2\sum_{i=1}^{n}1\\
&=\sum_{i=1}^{n}(x_i-\overline{x})^2+n(\overline{x}-\mu)^2
\end{aligned} \tag{7.39}$$

Lemma 7.2: (Standardization) . Let x follow Liu's uncertain normal distribution with parameters μ and σ. Then the standardized variable

$$\frac{x-\mu}{\sigma}\sim\mathcal{N}(0,1). \tag{7.40}$$

Proof: The inverse function of Liu's uncertain normal distribution with parameters μ and σ is

$$\Psi^{-1}(\alpha)=\mu+\frac{\sqrt{3}\sigma}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right) \tag{7.41}$$

Then, for constant parameters μ and σ

$$\Psi^{-1}_{\frac{x-\mu}{\sigma}}(\alpha) = \frac{1}{\sigma}\left(\mu + \frac{\sqrt{3}\sigma}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right) - \mu\right) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right) \tag{7.42}$$

which indicates

$$\frac{x-\mu}{\sigma} \sim \mathcal{N}(0,1). \tag{7.43}$$

Lemma 7.3: Assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Then the sum of standardized "sample", follows uncertain chi-square distribution with *n* degree of freedom:

$$\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 = \chi^2_{(n)} \tag{7.44}$$

Proof: According to the Uncertain "Sample" Postulate, each "standardized" term

$$\frac{x_i-\mu}{\sigma} \sim \mathcal{N}(0,1)$$

and therefore

$$\left(\frac{x_i-\mu}{\sigma}\right)^2 = \chi^2_{(1),i}$$

which leads to an uncertain chi-square variable with *n* degree of freedom

$$\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 = \chi^2_{(n)} \tag{7.45}$$

Lemma 7.4: Assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Then the square of a standardized "sample" mean, follows uncertain chi-square distribution with one degree of freedom:

$$\left(\frac{\overline{x}-\mu}{\sigma}\right)^2 = \chi^2_{(1)} \tag{7.46}$$

Proof: According to the Theorem 7.5, the "sampling" mean follows an uncertain normal distribution with parameters μ and σ. Thus, the standardized "sampling" mean

$$\frac{\overline{x}-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{7.47}$$

and therefore the square of standardized "sampling" mean follows an uncertain chi-square distribution with one degree of freedom.

It is noticed that different from the circumstance in probability theory, we have no way to show the independence between $(\overline{x}-\mu)^2$ and $\sum(x_i-\overline{x})^2$. Therefore, we have no way to show that $\sum(x_i-\overline{x})^2$ is linked to the uncertain chi-square distribution with *n* - 1 degree of freedom. That means that the accurate uncertainty distribution for $\sum(x_i-\overline{x})^2$ is not available. However, we may think of an upper bound for the uncertainty distribution of $\sum(x_i-\overline{x})^2$.

Theorem 7.8: (**Guo, 2014**) Assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Then the standardized sum of squared deviation from "sampling" mean follows an uncertainty distribution, Ψ, which is upper bounded by an uncertainty distribution of an uncertain chi-square variable with *n* degree of freedom, i.e.,

$$\Psi_{\sum_{i=1}^{n}\left(\frac{x_i-\overline{x}}{\sigma}\right)^2}(z) \leq \Psi_{\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}(z) = \Psi_{\chi^2_{(n)}}(z). \tag{7.48}$$

Proof: Based on the facts stated in Lemma 7.1 to Lemma 7.4, it is easy to obtain the following equality:

$$\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2-n\left(\frac{\overline{x}-\mu}{\sigma}\right)^2=\sum_{i=1}^{n}\left(\frac{x_i-\overline{x}}{\sigma}\right)^2 \tag{7.49}$$

Thus an important relationship is obtained:

$$\sum_{i=1}^{n}\left(\frac{x_i-\overline{x}}{\sigma}\right)^2\le\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2 \tag{7.50}$$

since term $n\left(\frac{\overline{x}-\mu}{\sigma}\right)^2$ is always positive.

Converting the inequality in Eq. (7.50) into an equivalent event expression, we have

$$\left\{\sum_{i=1}^{n}\left(\frac{x_i-\overline{x}}{\sigma}\right)^2\le z\right\}\subseteq\left\{\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2\le z\right\} \tag{7.51}$$

which leads to distribution function inequality:

$$\Psi_{\sum_{i=1}^{n}\left(\frac{x_i-\overline{x}}{\sigma}\right)^2}(z)=\lambda\left\{\sum_{i=1}^{n}\left(\frac{x_i-\overline{x}}{\sigma}\right)^2\le z\right\}\le\lambda\left\{\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2\le z\right\}=\Psi_{\sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right)^2}(z)=\Psi_{\chi^2_{(n)}}(z) \tag{7.52}$$

which concludes the proof.

Theorem 7.9: (**Guo, 2014**) Assuming that a "sample" $x_1,x_2,\cdots,x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Further, denote the standardized sum of squared deviation from "sampling" mean follows an uncertainty distribution, Υ, and the uncertainty distribution of an uncertain chi-square variable with *n* degree of freedom, i.e., $\Psi_{\chi^2_{(n)}}$. Then

$$\Psi^{-1}_{\chi^2_{(n)}}(\alpha)\le\Upsilon^{-1}(\alpha). \tag{7.53}$$

Proof: Based on the fact proved in Theorem 7.8:

$$\Upsilon(z) = \Psi_{\sum_{i=1}^{n}\left(\frac{x_i - \overline{x}}{\sigma}\right)^2}(z) \le \Psi_{\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^2}(z) = \Psi_{\chi^2_{(n)}}(z) \tag{7.54}$$

Recall that the inverse function for the uncertainty distribution of an uncertain chi-square variable with *n* degree of freedom is:

$$\Psi^{-1}_{\chi^2_{(n)}}(\alpha) = \left[\sqrt{n}\ln\left(\frac{1}{1-\alpha}\right)\right]^2. \tag{7.55}$$

Thus, for a given $\alpha, \alpha \in (0,1)$, we have

$$\Psi^{-1}_{\chi^2_{(n)}}(\alpha) = \left[\sqrt{n}\ln\left(\frac{1}{1-\alpha}\right)\right]^2 \le \Upsilon^{-1}(\alpha), \forall\alpha, \alpha \in (0,1) \tag{7.56}$$

which provides a lower bound for the inverse function of $\Upsilon(z)$.

Theorem 7.9 can facilitate the lower bound of confidence interval of the uncertain variable $\sum(x_i - \overline{x})^2$.

7.5 Sampling Impacts on Uncertain Process Capability Indices

In probabilistic quality control several classical capability indices such as C_p, C_{pk} and C_{pm} were investigated under Gaussian normal distribution when the true process parameters are not available. Then the confidence interval for process mean or process variance would be involved in the process capability study. Similarly, within an uncertainty environment, confidence intervals can be constructed for uncertain capability indices (i.e. ${}_uC_p$, ${}_uC_{pk}$ and ${}_uC_{pm}$). Let us investigate the impacts when the true process mean μ or process variance σ^2 are not available.

Theorem 7.10: (**Guo, 2014**) Assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Further, it is assumed that the true parameter σ is given. Then, for any pre-determined level α, $\alpha \in (0,1)$, the 1- α level confidence interval for μ is

$$\left[\bar{x} - \frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{2-\alpha}{\alpha}\right),\ \bar{x} + \frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{2-\alpha}{\alpha}\right)\right],\ \alpha \in (0,1). \qquad (7.57)$$

Proof: Recall that according to Theorem 7.5 the inverse function for the sample mean from Liu's uncertain normal is

$$\Psi_{\bar{x}}^{-1}(\alpha) = \mu + \frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right) \qquad (7.58)$$

Hence, the lower limit of the 1 - α confidence interval is

$$\Psi_{\bar{x}}^{-1}\left(\frac{\alpha}{2}\right) = \mu + \frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{2-\alpha}\right), \qquad (7.59)$$

and the upper limit of the 1 - α confidence interval is

$$\Psi_{\frac{\bar{x}-\mu}{\sigma}}^{-1}\left(1-\frac{\alpha}{2}\right) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{1-\frac{\alpha}{2}}{1-\left(1-\frac{\alpha}{2}\right)}\right) = \frac{\sqrt{3}}{\pi}\ln\left(\frac{2-\alpha}{\alpha}\right) \qquad (7.60)$$

which leads to the inequality:

$$-\frac{\sqrt{3}}{\pi}\ln\left(\frac{2-\alpha}{\alpha}\right) \le \frac{\bar{x}-\mu}{\sigma} \le \frac{\sqrt{3}}{\pi}\ln\left(\frac{2-\alpha}{\alpha}\right) \qquad (7.61)$$

which gives the 1 - α confidence interval for μ:

$$\left[\bar{x}-\frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{2-\alpha}{\alpha}\right),\ \bar{x}+\frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{2-\alpha}{\alpha}\right)\right]. \tag{7.62}$$

Please note that when the process variance parameter is also unknown, then we have to find out a confidence interval for σ. The next theorem will give an approximate confidence interval for parameter σ.

Theorem 7.11: (Guo, 2014) Assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Then, for any pre-determined level α, $\alpha \in (0.5,1)$, the 1 - α level confidence interval for σ is

$$\left[\frac{S}{\ln 2-\ln\alpha}, \frac{S}{\ln 2-\ln(1-\alpha)}\right],\ \alpha\in(0,1). \tag{7.63}$$

where

$$S=\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2}. \tag{7.64}$$

Proof: Recall that according to Theorem 7.9 the inverse function for the sample mean from Liu's uncertain normal is

$$\Psi_{\bar{x}}^{-1}(\alpha)=\mu+\frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{1-\alpha}\right) \tag{7.65}$$

Hence, the lower limit of the 1 - α confidence interval can be obtained from

$$\Psi_{\chi^2_{(n)}}^{-1}(\alpha)=\left[\sqrt{n}\ln\left(\frac{1}{1-\alpha}\right)\right]^2 \le \Upsilon^{-1}(\alpha), \forall\alpha, \alpha\in(0,1) \tag{7.66}$$

Thus,

$$\left[\sqrt{n}\ln\left(\frac{2}{2-\alpha}\right)\right]^2 \le \frac{\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}{\frac{1}{n}\sigma^2} = \frac{nS^2}{\sigma^2} \le \left[\sqrt{n}\ln\left(\frac{2}{\alpha}\right)\right]^2 \quad (7.67)$$

which leads to

$$\left(\ln\left(\frac{2}{\alpha}\right)\right)^{-2} \le \frac{\sigma^2}{S^2} \le \left(\ln\left(\frac{2}{1-\alpha}\right)\right)^{-2}, \forall \alpha \in (0,1) \quad (7.68)$$

Then, the lower limit of the 1- α level confidence interval for σ^2 is

$$\left[S^2\left(\ln\left(\frac{2}{\alpha}\right)\right)^{-2}, S^2\left(\ln\left(\frac{2}{1-\alpha}\right)\right)^{-2}\right], 0<\alpha<1 \quad (7.69)$$

Example 7.1: Let S^2 = 2.0, α =0.025, then 95% confidence interval for σ^2 is [0.146974, 3120.167].

Theorem 7.12: (**Guo, 2014**) Assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with unknown parameters μ and σ. Then, for any pre-determined level α, $\alpha \in (0,1)$, the 1- α level confidence interval for μ is

$$\left[\bar{x} - \frac{S\sqrt{3}}{\pi}\frac{\ln(2-\alpha)-\ln\alpha}{\ln 2-\ln(1-\alpha)},\ \bar{x} + \frac{S\sqrt{3}}{\pi}\frac{\ln(2-\alpha)-\ln\alpha}{\ln 2-\ln(1-\alpha)}\right], \forall \alpha, \alpha \in (0,1), \quad (7.70)$$

where

$$S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}. \quad (7.71)$$

Proof: Recall that according to Theorem 7.10, if the variance is given, the 1 - α confidence interval is

$$\left[\bar{x}-\frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{2-\alpha}{\alpha}\right),\ \bar{x}+\frac{\sigma\sqrt{3}}{\pi}\ln\left(\frac{2-\alpha}{\alpha}\right)\right],\forall\alpha,\alpha\in(0,1), \tag{7.72}$$

If the variance is unknown, in terms of Theorem 7.11, the 1 - α confidence interval for σ^2 is

$$\left[\frac{S}{\ln 2-\ln\alpha},\frac{S}{\ln 2-\ln(1-\alpha)}\right], \tag{7.73}$$

which gives the conclusion:

$$\left[\bar{x}-\frac{S\sqrt{3}}{\pi}\frac{\ln(2-\alpha)-\ln\alpha}{\ln 2-\ln(1-\alpha)},\ \bar{x}+\frac{S\sqrt{3}}{\pi}\frac{\ln(2-\alpha)-\ln\alpha}{\ln 2-\ln(1-\alpha)}\right],\forall\alpha,\alpha\in(0,1). \tag{7.74}$$

Theorem 7.13 (Guo, 2014): Assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Then, for any pre-determined level α, $\alpha\in(0,1)$, the 1- α level confidence interval for the uncertain process capability index ${}_uC_p$ can be expressed by:

$${}_uC_p=\left[\frac{USL-LSL}{6S\left[\frac{\sqrt{3}}{\pi}\ln\left(\frac{\alpha}{2-\alpha}\right)\right]^{-1}},\frac{USL-LSL}{3\sqrt{2}S\left[\frac{\sqrt{3}}{\pi}\ln\left(\frac{2-\alpha}{\alpha}\right)\right]^{-1}}\right],\ \alpha\in(0,1), \tag{7.75}$$

where

$$S = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}. \tag{7.76}$$

Proof: Recall Theorem 7.2,

$${}_uC_p = \left[\frac{USL - LSL}{6\sigma_U}, \frac{USL - LSL}{3\sqrt{2}\sigma_U}\right]. \tag{7.77}$$

Further, we apply Theorem 7.11, the 1- α level confidence interval for σ is

$${}_u\hat{C}_p = \left[\frac{USL - LSL}{6S}\left(\ln 2 - \ln\left(1-\alpha\right)\right), \frac{USL - LSL}{3\sqrt{2}S}\left(\ln 2 - \ln \alpha\right)\right] \tag{7.78}$$

Then in terms of interval arithmetic, the conclusion is obtained.

Theorem 7.14 (Guo, 2014): Assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Then, for any pre-determined level α, $\alpha \in (0,1)$, the 1- α level confidence interval for the uncertain process capability index ${}_uC_{pk}$ can be expressed by:

$${}_u\hat{C}_{pk} = \left[LL, UU\right],\ \alpha \in \left(0,1\right), \tag{7.79}$$

where

$$LL = \frac{USL - LSL - 2(\overline{x} + \frac{S\sqrt{3}}{\pi}\frac{\ln\left(2-\alpha\right) - \ln\alpha}{\ln 2 - \ln\left(1-\alpha\right)} - M)}{6S}\left(\ln 2 - \ln\left(1-\alpha\right)\right), \tag{7.80}$$

and

$$UU = \frac{USL - LSL - 2(\overline{x} - \frac{S\sqrt{3}}{\pi}\frac{\ln\left(2-\alpha\right) - \ln\alpha}{\ln 2 - \ln\left(1-\alpha\right)} - M)}{3\sqrt{2}S}\left(\ln 2 - \ln\alpha\right) \tag{7.81}$$

Proof: Under Liu's uncertain normal distribution, $\sigma_L = 1/\sqrt{2}\sigma_U$,

$$ {}_uC_{pk} = \left[\frac{(USL - LSL) - 2(\mu - M)}{6\sigma_U}, \frac{(USL - LSL) - 2(\mu - M)}{3\sqrt{2}\sigma_U} \right]. \tag{7.82} $$

Thus, it is necessary to do a two-step replacement of parameters μ and σ.

The first step is to replace σ by S:

$$ {}_u\tilde{C}_{pk} = \left[\frac{USL - LSL - 2(\mu - M)}{6S} \left(\ln 2 - \ln(1-\alpha) \right), \frac{USL - LSL - 2(\mu - M)}{3\sqrt{2}S} \left(\ln 2 - \ln \alpha \right) \right] \tag{7.83} $$

The second step is to replace μ by $\bar{x}$:

$$ {}_u\hat{C}_{pk} = [LL, UU] \tag{7.84} $$

where

$$ LL = \frac{USL - LSL - 2(\bar{x} + \frac{S\sqrt{3}}{\pi} \frac{\ln(2-\alpha) - \ln \alpha}{\ln 2 - \ln(1-\alpha)} - M)}{6S} \left(\ln 2 - \ln(1-\alpha) \right) \tag{7.85} $$

and

$$ UU = \frac{USL - LSL - 2(\bar{x} - \frac{S\sqrt{3}}{\pi} \frac{\ln(2-\alpha) - \ln \alpha}{\ln 2 - \ln(1-\alpha)} - M)}{3\sqrt{2}S} \left(\ln 2 - \ln \alpha \right) \tag{7.86} $$

Chapter 8. Methods for Constructing Classical and Uncertain Capability Analysis

The main idea behind this research is to compare the estimates of classical capability analysis to their corresponding equivalents under uncertainty theory. This chapter will then present:

1) The methodology of estimating the process capability using the classical approach when autocorrelation permeates the data
2) Determine a test of hypothesis to ascertain the classical capability of the process
3) The methodology for collecting expert data and also estimating process capability under uncertainty theory (i.e. when expert opinion about the process has been incorporated into capability assessment).

In order to conduct any process capability study, the underlying assumptions governing process capability analysis should be verified to ensure that the analysis of the process is reliable. These assumptions are stated as follows:

Assumption 1: (**Process Stability)** The process is in a state of statistical control, thus no special cause of variation is present and the process does not wander away from its process characteristic (i.e. such as the mean or standard deviation).

Assumption 2: (**Representative samples**) The obtained sample should be representative of the population.

Assumption 3: (**Normality**) The underlying process distribution should be Gaussian normal.

However, some process distributions are non-normal and some authors such as Kotz and Johnson (1993a) and also Bai et al. (1995) have developed procedures to deal with non-normal processes.

Assumption 4: (**Independence**) The observations should be independent and identically distributed.

The validity of the independence assumption has come under intense criticism in recent years. The advent and continuous development of new technologies has been accompanied by the ability to observe process outputs that are not far apart in time and thus these observations usually tend to co-vary (Shore, 1997). Ignoring autocorrelation in process capability analysis tends to bias upward the process capability, thus the capability of the process will be inflated (Shore, 1997). As noted when autocorrelation permeates the process data, it might cause some undesirable effects, hence the need to assess whether autocorrelation exists and how to deal with it.

The first step to dealing with autocorrelation is to establish whether there is a relationship between observations not far apart in time and to what extent (i.e. AR(1), AR(2),…, AR(∞), MA(1),…, MA(∞), ARMA, ARIMA etc). The autocorrelation between observations separated by *k* time units can be approximated as follows:

$$\rho_k = \frac{\mathrm{E}\left[\left(x_t - \mu\right)\left(x_{t+k} - \mu\right)\right]}{\left\{\mathrm{E}\left[\left(x_t - \mu\right)^2\right]\mathrm{E}\left[\left(x_{t+k} - \mu\right)^2\right]\right\}^{1/2}} = \frac{\mathrm{E}\left[\left(x_t - \mu\right)\left(x_{t+k} - \mu\right)\right]}{\sigma^2} \tag{8.1}$$

where ρ_k represents the autocorrelation function between observations separated by *k* time units, $\mathrm{E}[\;]$ denotes the expected value operator and σ^2 the process variance which is independent of *k*, assuming the process is stable.

Autocorrelation between observations separated by *k* time units can be simultaneously tested by plotting a correlogram (i.e. plot of the autocorrelation estimates as a function of *k*, such that vertical lines above the zero line signify estimates that are non-zero) or by using a sample test that uses an approximate standard error of ρ_k as a test statistic, that is:

$$\mathrm{SE}\left(\hat{\rho}_k\right) = \sqrt{\frac{1 + 2\sum_{v=1}^{q}\hat{\rho}_v}{n}} \tag{8.2}$$

where *n* denotes the number of observations used to estimate ρ_k and by assumption $\rho_k = 0$ for all $k > q$.

A two sided test can be constructed as $\mathbf{H_0}$: $\rho_k = 0$ against $\mathbf{H_1}$: $\rho_k \neq 0$ to determine whether the lag *k* autocorrelation is significantly non-zero at α significance level. The test criterion is:

$$\left|\hat{\rho}_k\right| \; > \; \left(\left[1+2\sum_{v=1}^{q}\hat{\rho}_v\right]\Big/ n\right)^{1/2} \tag{8.3}$$

As noted by Box et al. (1994) only estimates found to be significantly non-zero are included in the estimation of the standard error. Under the assumption that beyond a certain lag *q*, all autocorrelation may be non-existent, hence the above test is conducted in such a way that if for a certain *k*, ρ_k = 0 then the test may stop.

So far we have established a way to test autocorrelation, now we would have to ascertain how to correct for autocorrelation when it exists. As prescribed by Shore (1997) a process may always be reconstructed if we skipped enough observations so that those remaining may be shown to be non-autocorrelated. These observations if numerous enough will faithfully reconstruct the underlying distributions and allow for the estimation of the process capability. This strategy is only suitable in a data rich environment; fortunately in process control environment large datasets are readily available. Some authors such as Zhang (1998), Wallgren (2007), and Noorossana (2002) have dealt with autocorrelation when estimating *PCI's* by finding the underlying pattern which describes the process (i.e. establishing whether the process follows an AR, ARMA, ARIMA etc.). When the underlying process distribution is established the residuals of the model is used to assess capability such that the residuals are assumed to follow a white noise and as known, a white noise process is uncorrelated and approximately follow a normal distribution with mean zero and constant variance σ^2 (i.e. $\varepsilon_t \sim WN\left(0,\sigma^2\right)$.

This modelling dependent approach appears to be complicated and less known how to interpret your results after the white noise transformation. Shore (1997) and Vanmman et al. (2008) prefer the model free approach were the underlying distribution of the process is ignored but subsamples are formed from skipping enough observations in the original dataset such that the subsamples achieved are independent. There model free approach is achieved by using the iterative skipping strategy proposed by Vannman et al. (2008). This approach will be pursued in detail in the next section.

8.1 Iterative Skipping Strategy

Under this skipping method proposed by Vannman (2008), the autocorrelated original dataset is partitioned into subsamples such that the iterative skipping strategy is used to achieve subsamples that are approximately independent. The capability of each subsample will then be estimated and combined in such a way that it can be compared to the critical values to ascertain whether the process meets the prescribed quality (i.e. C_p = 1.0 or 1.33).

Assuming that an autocorrelated dataset of size *N* is obtained from a specified manufacturing process. The total set of observations (*N*) is then divided into subsamples each of size $m = [N/r]$, where *r* defines the number of independent subsamples required and *m* the size of each subsample. The first subsample will consist of observation numbers 1, 1+*r*, 1+2*r*,….., 1+(*m*-1)*r*, likewise the second subsample will consist of observation number 2, 2+*r*, 2+2*r*, ….., 2+(*m*-1)*r* and so forth. Each subsample will be made-up of size *m*, where *r* is chosen so that the subsamples are independent.

To find the capability of the process, several decision rules may be derived from the subsamples obtained from the skipping strategy. Let *C* denote the required capability of the process such that the distribution of the estimates can be derived under the assumption of independent observations. Let $\widehat{C}_i$ represent the estimated index from the independent subsample *i*, $i = 1, 2, ..., r$. The decision rule is then derived based on the following hypothesis: **H$_0$**: $C = k_0$ against **H$_1$**: $C > k_0$ at significance level *α*.

Let k_0 and $c_{\alpha_0(m)}$ denote the required capability of the process and the critical value of the hypothesis respectively, stated such that if $\widehat{C}_i > c_{\alpha_0(m)}$, the decision rule declares the process is capable and if $\widehat{C}_i < c_{\alpha_0(m)}$, the process is judged otherwise.

There are several decision rules that can be applied and are detailed as follows:

Rule A:

Reject **H$_0$** if $\widehat{C}_i > c_{\alpha_0(m)}$ for one randomly chosen subsample *i*.

The weakness with Rule A is the power of the test is small unless *m* is large.

Rule B:

Reject **H$_0$** if $\max\left(\widehat{C}_i\right) > c_{\alpha_0(m)}$, $i = 1, 2, ..., r$ with $\alpha_0 = \alpha/r$

$$P\left(1 \text{ or more of } \widehat{C}_i > c_{\alpha_0(m)},\ i=1,\ 2,...r \mid H_0 \text{ is true}\right) \le \alpha$$

Rule C:

Reject $\mathbf{H_0}$ if 2 or more $\widehat{C}_i > c_{\alpha_0(m)},\ i=1,2,...r$ with $\alpha_0 = (2\alpha)^{1/2} / r$.

$$P\left(2 \text{ or more of } \widehat{C}_i > c_{\alpha_0(m)},\ i=1,\ 2,...r \mid H_0 \text{ is true}\right) \le \alpha$$

Rule D:

Reject $\mathbf{H_0}$ if the third largest value of or more $\widehat{C}_i > c_{\alpha_0(m)}, i=1,2,...r$ where $\alpha_0 = (6\alpha)^{1/3} / r$.

$$P\left(3 \text{ or more of } \widehat{C}_i > c_{\alpha_0(m)},\ i=1,\ 2,...r \mid H_0 \text{ is true}\right) \le \alpha$$

Each of the Rules A-D will have a significance level of at most α under the assumption that observation in each subsample are independent. The Rules A-D does not require the subsamples to be independent (Vannman et al., 2008), hence as long as the within subsample is independent and the between subsamples are dependent the Rules are still applicable.

8.2 Test for Independence

Choosing an appropriate *r*-value such that the subsample generated using the iterative skipping strategy is independent requires assessing the level of autocorrelation existent in the data, the total sample size (*N*) and also the power of the test (Vannman et al., 2008). In view of this, Vannman et al. (2008) demonstrated using an AR (1) process the necessary *r* value which will be able to partition the original dataset into independent subsamples. The table below demonstrates the number of times the hypothesis is not rejected with possible autocorrelation ρ=0.3, 0.5, 0.7, 0.9 , total sample size (i.e. *N*=100, 200, 500, 1000) and *m* = [*N/r*]. The number in brackets indicate the size of the subsample.

	N			
r	100	200	500	1000
ρ =0.3				
5	$0.99^{(20)}$	$0.99^{(40)}$	$0.98^{(100)}$	$0.98^{(200)}$
10		$0.99^{(20)}$	$0.98^{(50)}$	$0.98^{(100)}$
20			$0.99^{(25)}$	$0.99^{(50)}$
25			$0.99^{(20)}$	$0.99^{(40)}$

$\rho = 0.5$				
5	$0.99^{(20)}$	$0.98^{(40)}$	$0.96^{(100)}$	$0.95^{(200)}$
10		$0.99^{(20)}$	$0.99^{(50)}$	$0.98^{(100)}$
20			$0.99^{(25)}$	$0.99^{(50)}$
25			$0.99^{(20)}$	$0.99^{(40)}$
$\rho = 0.7$				
5	$0.95^{(20)}$	$0.88^{(40)}$	$0.68^{(100)}$	$0.38^{(200)}$
10		$0.99^{(20)}$	$0.98^{(50)}$	$0.97^{(100)}$
20			$0.99^{(25)}$	$0.99^{(50)}$
25			$0.99^{(20)}$	$0.99^{(40)}$
$\rho = 0.9$				
5	$0.49^{(20)}$	$0.08^{(40)}$	$0.00^{(100)}$	$0.00^{(200)}$
10		$0.83^{(20)}$	$0.41^{(50)}$	$0.09^{(100)}$
20			$0.96^{(25)}$	$0.91^{(50)}$
25			$0.98^{(20)}$	$0.96^{(40)}$

Table 2: The proportion of times for which the null hypothesis of independence is not rejected in the subsamples, when N = 100, 200, 500, 1000, r = 5, 10, 20, 25 and $m = N / r \geq 20$. This table was adopted from Vannman et al. (2008).

From Table 2, it can be inferred that when autocorrelation is low (i.e. ρ = 0.3, 0.5), then r = 5 is a suitable choice to deal with this autocorrelation. However, as the autocorrelation in the dataset increases the choice of r becomes crucial. When ρ = 0.9 and r = 5, autocorrelation will still exist as 5 subsamples are not enough to correct for high autocorrelation but as r is increased to 20, this is efficient enough to correct for autocorrelation. Thus as autocorrelation increases, there should be a corresponding increase in the size of r. Finally, it can also be seen in Table 2 that when N is large enough and also r is large the subsample size m reduces but it is effective with dealing with higher levels of autocorrelation. Hence this table serves as guidance when choosing an r value for auto-correlated data.

8.3 Statistical Test to Evaluate Process Capability

In industry, some practitioners usually use the estimated capability index derived from the sample data to judge whether the process is capable. Such a procedure is unreliable because sampling error has been ignored (Pearn et al., 1999). Hence a test of hypothesis has been developed by Pearn et al. (1999) to judge the capability of the process based on the critical value usually computed via the significance level, α and the sample size, n.

In this section, the hypothesis constructed for the classical capability indices will be reviewed and applied in the analysis section where necessary. These hypothesis tests are applied given that the distribution of the data values is independent. The test of hypothesis for the indices C_{pl}, C_{pu} and C_{pk} will be reviewed and that of C_p will be ignored, as in industry process characteristic such as the mean μ are rarely centred on the process target T.

8.3.1 Hypothesis Test for C_{pk}

To determine whether a process meets the capability requirement, Pearn et al. (1999) proposed the following hypothesis $\mathbf{H_0}$: $C_{pk} \le C$ against $\mathbf{H_1}$: $C_{pk} > C$ at significance level α, where the process is deemed capable if $C_{pk} > C$ and otherwise, if the process fails to meet the required capability, $C_{pk} \le C$.

In order to estimate the capability using the index C_{pk}, three estimators are available namely; the natural estimator $\hat{C}_{pk}$, Bissell's estimator $\hat{C}'_{pk}$ and the Bayesian estimator $\hat{C}''_{pk}$ proposed by the Bissell (1990), Kotz et al. (1993) and Pearn et al. (1996b) respectively.

The estimator $\hat{C}_{pk}$ has been shown by Kotz et al. (1993b) to have a smaller variance than Bissell's estimator, but the Bayesian like estimator $\hat{C}^{//}_{pk}$ is a UMVUE of C_{pk} when the correction factor is added. Since the index $\hat{C}^{//}_{pk}$ is a UMVUE by adding the correction factor b_f to the estimated $\hat{C}^{//}_{pk}$, hence sole attention will be placed on this estimator.

In order to calculate the Bayesian like estimator $\hat{C}^{//}_{pk}$ the location of the mean is vital. Thus the following should be known either $P(\mu \ge m) = p$ or $P(\mu < m) = 1 - p$ where $0 \le p \le 1$, which can be obtained from historical information. The Bayesian like estimator is defined as:

$$\hat{C}^{//}_{pk} = \left\{ d - \left(\bar{x} - M \right) I_A(\mu) \right\} \Big/ 3S, \qquad (8.4)$$

where $I_A(\mu)=1$, if $\mu \in A$, and $I_A(\mu)=-1$ if $\mu \notin A$, where $A=\{\mu | \mu \geq M\}$.

Pearn and Chen (1996a) showed that under the assumption of normality the distribution of the estimator $3\sqrt{n}\hat{C}^{//}_{pk}$ is $t_{n-1}(\delta)$, a non-central t with n-1 degrees of freedom and non-centrality parameter $\delta = 3\sqrt{n}C_{pk}$. The probability density function can be expressed as:

$$f(x)=\frac{3n^{1/2}}{2^{n/2}\Gamma\left(\frac{n-1}{2}\right)\left(\pi(n-1)\right)^{0.5}}\int_0^{\infty} y^{(n-2)/2}\times\exp\left(-\frac{y+9n\left[xy^{0.5}(n-1)^{-0.5}-C_{pk}\right]^2}{2}\right)dy \qquad (8.5)$$

Pearn and Chen (1996a) also showed that by adding the well-known correction factor b_f to the estimator $\hat{C}^{//}_{pk}$, with $b_f=\left[2/(n-1)\right]^{0.5}\Gamma\left[(n-1)/2\right]\left\{\Gamma\left[(n-2)/2\right]\right\}^{-1}$, an unbiased estimator $\tilde{C}_{pk} = b_f\hat{C}^{//}_{pk}$ can be obtained and then $\tilde{C}_{pk}$ is a UMVUE of C_{pk}. The possible values of the correction factor b_f factor based on the sample size of the sample are stated in the Appendix.

Furthermore, the critical value C_0 is determined by:

$$C_0=\left(b_f/3\sqrt{n}\right)t_{n-1,\alpha}(\delta_c) \qquad (8.6)$$

where $t_{n-1,\alpha}(\delta_c)$ is the upper α quantile of $t_{n-1}(\delta_c)$ and $\delta = 3n^{1/2}C$. Hence if $\tilde{C}_{pk} > C_0$, then the null hypothesis (H_0) is rejected and conversely.

Similarly, the hypothesis test for the one-sided capability indices (i.e. C_{pl} and C_{pu}) can be constructed. The indices C_{pl} and C_{pu} can be estimated using the natural estimators due to Kotz et al. (1993b) as:

$$C_{pl} = \frac{USL - \bar{x}}{3S}, C_{pu} = \frac{\bar{x} - LSL}{3S} \qquad (8.7)$$

where $\bar{x}$ and S are conventional estimates of μ and σ. They may be obtained assuming the process is in statistical control. Chou and Owen (1989) showed that the estimators C_{pl} and C_{pu} are distributed as $ct_{n\text{-}1}(\delta_c)$, with $c = \left(3\sqrt{n}\right)^{-1}$ and $t_{n\text{-}1}(\delta)$ being a non-central *t*-distribution with *n* -1 degrees of freedom and non-centrality parameter $\delta = 3\sqrt{n}C_{pk}$ and $\delta = 3\sqrt{n}C_{pl}$ respectively.

Although both estimators are biased but by adding the correction factor b_f to C_{pl} and C_{pu}, we obtain unbiased estimators and also *UMVUE* of C_{pl} and C_{pu} (Pearn et al, 2002).

Identical to the hypothesis test for C_{pk}, the hypothesis test for C_{pl} and C_{pu} may also be constructed as follows:

H₀: $C_I \le C$

H₁: $C_I > C$ at significance level α

and the critical value C_0 is determined by $C_0 = \left(b_f / 3n^{1/2}\right) t_{n\text{-}1,\alpha}(\delta_c)$, where $t_{n\text{-}1,\alpha}(\delta_c)$ is the upper α quantile of $t_{n\text{-}1}(\delta_c)$ and $\delta = 3n^{1/2}C$. Hence if $C_I > C_0$, where $I = l$ or u depending on the capability required, (either C_{pl} or C_{pu}), then the null hypothesis **H_0** is rejected and vice versa.

8.3.2 Estimating Process Capability for Skewed Population Distributions

In order to conduct any process capability assessment, the normality assumption is one of the conditions that have to be tested. Moreover, many processes in industry such as the chemical process industry output tend to deviate from Gaussian normality and ignoring the process distribution may give misleading information about the process performance. Usually, for skewed populations the number of non-conforming units tend to increase depending on the degree of skewness and the traditional PCI's are insensitive to the skewness of the underlying distribution (Chang et al., 2002). Therefore the process capability estimate will exaggerate the process performance. A method to adjust the traditional capability indices to account for skewness which causes an increase in the number of defective units is required. In this light, several methods have been proposed in the past to address the issue of non-normality when estimating PCI's with reference to authors such as Clements(1989), Pearn and Kotz (1994), Franklin and Wasserman (1991; 1998), Shore (1998), Polansky (1998), Sommerville and Montgomery (1996) etc. However these methods have their own shortcomings either in terms of the fact that they are too complicated and unattractive to practitioners or the use of data transformation techniques which usually cause difficulty in interpretation of results in terms of the original data (Chang et al., 2002).

Therefore Chang et al. (2002) proposed the weighted standard deviation (WSD) method which is relatively easier in computation and also accounts for the skewness in the underlying distribution. The weighted standard deviation method divides the process standard deviation into two parts, that is the lower and upper standard deviations (i.e. σ_L^{WSD} and σ_U^{WSD}), which measure the extent of deviation of the lower and upper distribution from the overall process mean (μ). Then the standard process capability indices are adjusted for skewness, such that the estimated PCI's for non-normal process distributions are less than their standard PCI's but these non-normal PCI's revert back to the standard PCI's when the process is symmetric.

Although the distribution of the underlying population distribution may be asymmetric, this can be decomposed into two normal pdf's (Chang et al., 2002) such that;

$$f_u(x) = \frac{1}{2\sigma_u^{WSD}} \phi\left(\frac{x-\mu}{2\sigma_u^{WSD}}\right) \tag{8.8}$$

and

$$f_L(x) = \frac{1}{2\sigma_L^{WSD}} \phi\left(\frac{x-\mu}{2\sigma_L^{WSD}}\right) \tag{8.9}$$

From the equations (8.8) and (8.9) the mean is constant but the standard deviations differ and $\phi(.)$ represent the standard normal density function. The two weighted standard deviations, that is

σ_L^{WSD} and σ_U^{WSD} are expressed as $\sigma_u^{WSD} = P\sigma$ and $\sigma_L^{WSD} = (1-P)\sigma$ where $P = P(X \le \mu)$. In this instance the deviations used are $2\sigma_L^{WSD}$ and $2\sigma_U^{WSD}$, but these deviations usually depend on the extent of skewness, for instance the deviation may take a form of $3\sigma_L^{WSD}$ and $3\sigma_U^{WSD}$ etc. In summary the WSD method is able to account for skewness when estimating process capability in a simple and robust way.

8.3.2(a) C_p based on the WSD Method

The C_p based on the WSD method, C_p^{WSD} is estimated as follows:

$$\begin{aligned} C_p^{WSD} &= \min\left\{\frac{USL-LSL}{6.2\sigma_U^{WSD}}, \frac{USL-LSL}{6.2\sigma_L^{WSD}}\right\} \\ &= \min\left\{\frac{USL-LSL}{6.2P\sigma}, \frac{USL-LSL}{6.2(1-P)\sigma}\right\} \\ &= \frac{USL-LSL}{6\sigma}\min\left\{\frac{1}{2P}, \frac{1}{2(1-P)}\right\} \\ &= \frac{C_p}{D_x} \end{aligned} \quad (8.10)$$

where $D_x = 1+|1-2P|$. It should be noted that $2\sigma_L^{WSD}$ and $2\sigma_U^{WSD}$ are used in place of σ to reflect the degree of skewness. Assuming the process is symmetric, that is $P = 0.5$, then the non-normal PCI reverts back to its standard PCI (i.e. $C_p^{WSD} = C_p$). However, assuming that the population is skewed, then $D_x > 1$ and $C_p^{WSD} < C_p$.

8.3.2(b) C_{pk} based on the WSD Method

The index is usually used when the process has a one-sided specification limit. In terms of the WSD method, the standard PCI's are defined as:

$$C_{pku}^{WSD} = \frac{USL - \mu}{3.2\sigma_u^{WSD}} = \frac{USL - \mu}{6P\sigma} \tag{8.11}$$

and

$$C_{pkl}^{WSD} = \frac{\mu - LSL}{3.2\sigma_l^{WSD}} = \frac{\mu - LSL}{6(1-P)\sigma} \tag{8.12}$$

Similarly, the WSD C_{pk} can be estimated as:

$$C_{pk}^{WSD} = \min\left\{C_{pku}^{WSD}, C_{pkl}^{WSD}\right\} = \min\left\{\frac{USL - \bar{x}}{6Ps}, \frac{\bar{x} - LSL}{6(1-P)s}\right\} \tag{8.13}$$

8.3.2(c) Estimating the Parameters invoked in the Non-normal PCI's

In order to calculate the WSD PCI's, the parameters invoked in the non-normal PCI are unknown and have to be estimated. Assuming a random sample of $X_1, X_2, ..., X_n$, then the population mean μ and standard deviation σ, may be estimated using the sample mean, $\bar{x}$ and sample standard deviation, s. Since $P = P(X \le \mu)$, it can be estimated by using the number of observations less than or equal to X such that:

$\hat{P} = \frac{1}{n}\sum_{i=1}^{n} I\left(\bar{x} - x_i\right)$, where $I(x) = 1$ for $x \ge 0$ and $I(x) = 1$ for $x < 0$.

8.4 The Uncertainty Theory Approach

The previous section dealt with the classical approach to estimating process capability which entails the collection and analysis of data usually with a correlative structure (Chen et al, 2012). The classical approach depends on a large historical data in which a probability distribution is required to describe the underlying structure of the data. However, uncertainty theory was proposed to obtain intuitive opinion from several experts to broaden knowledge about a process or system. Hence, little or no historical data is required as the main objective is to model expert intuition about a system.

In uncertainty theory, an uncertain variable is used to obtain uncertain information and an uncertain distribution is then constructed to describe the range of the uncertain variable. The approach of uncertain statistics proposed by Liu (2010) was designed to collect and interpret subjective information from experts about a particular system and this expert information is obtained via a structured survey.

8.4.1 Experts Data and Uncertainty distribution

As noted in the previous section, information obtained under uncertainty theory is based on expert opinion rather than historical data. In applying uncertainty theory to process engineering, a questionnaire was designed to obtain information from experts. An example of the questionnaire method is described as follows:
Given a manufacturing process designed to produce a wire of mass: 73.1g and using the machine 21503, an engineer was asked about the likely process output. Below is an example of how an industrial engineer, with a considerable level of experience in working on the manufacturing process, might regard the process output is:

Q1. What do you think is the minimum mass measurement of the process output?
A1: 70g (an expert experimental datum (70, 0) is obtained)
Q2. What do you think is the maximum mass measurement of the process output ?
A2: 75g (an expert experimental datum (75, 1) is obtained)
Q3. From your experience with working on the process, what is the likely mean mass of the process output?
A3: 72g
Q4. What percentage of values are likely to be less than the mean mass you specified ?
A4: 50% (an expert experimental datum (72, 0.5) is obtained)
Q5. How likely is the process to yield mass measurements between (M – 3.0g) and (M + 3.0g)? where M stands for the mean you have chosen in Q2.
A5: 90%

Q6. What percentages of the mass measurements will be below M-3.0g ?

A6: 0%

Q7.a What percentages of the mass measurements will be above M+3.0g ?

A7: 5% (an expert experimental datum (75, 0.95) is obtained)

Q8. Is there any other possible values you regard the process output is likely to take?

A8: None.

Q9. Please select the belief degree (percentage) that this process output selected in Q8 is likely to take?

A9: N/A.

Moreover, to ensure the validity of information obtained under uncertainty theory, multiple expert opinion may be obtained via the Delphi method. The Delphi method is a structural survey technique used to obtain information about a process from several experts and then a group judgement is inferred from the expert opinion and the feedback presented to the experts and then asked whether they would like to revise their initial judgement about the process based on the group judgement. This process is continued until a terminal condition is met or consensus is reached. The advantages of the Delphi method as noted by Wang et al. (2010) are the collection of experts advise, independent judgement by each expert to avoid any domination of a group by an individual, iterations to help reach a consensus etc. Hence the Delphi method would also be applied in this research and is adequately described as follows:

Step 1: For the first interview, set the iteration number *k* equal to 1.

Step 2: A group of *m* experts are invited to provide their experimental data in the form $\left(x_{ij}^{(k)},\ \alpha_{ij}^{(k)}\right)$, where x_{ij} denotes the j^{th} value provided by the i^{th} expert and α_{ij} represents the i^{th} experts belief degree that ξ is less than x_{ij} , $i = 1, 2, ..., m$ and $j = 1, 2, ..., n_i$ respectively. This represents the uncertainty measure of the uncertain event $\left\{\xi \le x_{ij}^{(k)}\right\}$.

Step 3: Calculate the uncertainty distribution for the i^{th} expert based on the experimental data $\left(x_{ij}^{(k)},\ \alpha_{ij}^{(k)}\right)$, various interpolation methods such as the linear interpolation, cubic-spline method, quadratic-spline, sin *x*-spline etc., may be used to generate a continuous distribution $\Phi_i^k,\ i = 1,\ 2, ...,\ m.$ Calculate the number of possible values of the uncertain variable, ξ, presented by all experts denoted by *N*, where the same values from different experts are considered as one. Thus the possible values of ξ are $x_1 \le x_2 \le x_3 \le ... \le x_N$. Then compute the aggregated distribution for the group using the weighted mean as:

$$\Phi^{(k)}(x) = \frac{1}{n}\sum_{i=1}^{n}\Phi_i^{(k)}(x) \tag{8.14}$$

In estimating the aggregate distribution a weight may be assigned to each expert and in this case we will assume that each experts has an equal weight of $1/n$.

Step 4: Generate the feedback information for the next iteration by presenting the i^{th} domain expert a revised group feedback as well as the original information given. By examining the group judgement each participant may voluntarily adjust their judgement or leave it unchanged.

Step 5: The data is then passed through a stability test to determine whether the experts data is consistent. This test is conducted by using the sum squared differences between individual and group uncertainty distributions.

$$d_j = \frac{1}{m}\sum_{i=1}^{m}\left(\Phi_{ij}^{(k)} - \Phi_j^{(k)}(x)\right)^2, \tag{8.15}$$

Step 6: Test the stability of the Delphi process. If d_j is less than a predetermined level, say α_0 then terminate the iteration. Finally use the integrated dataset $\left(x_1, \Phi_1^{(k)}\right), \left(x_2, \Phi_2^{(k)}\right), ..., \left(x_N, \Phi_N^{(k)}\right)$ to generate an uncertain distribution.

8.5 Uncertainty Capability Indices

Under probability theory several classical indices such as C_p, C_{pk} and C_{pm} have been proposed to assess how well a process satisfies customer requirements. The classical capability indices also have their counterparts in uncertainty theory. This section will apply the new capability indices in an uncertain environment to the group expert's information obtained via the Delphi method in order to estimate the capability of the process.

Most importantly, it should be aware that uncertain process capability indices are interval valued, which are different from their real-valued scalar probabilistic counterpart. As investigated in Chapter 7, we defined those uncertain capability indices ${}_uC_p$, ${}_uC_{pk}$, and ${}_uC_{pm}$ respectively. Furthermore, we developed the expressions for those uncertain capability indices ${}_uC_p$, ${}_uC_{pk}$, and ${}_uC_{pm}$ without process parameters available.

So far the uncertainty capability indices derived in Chapter 7, are based on a process that has both lower and upper specification limits. However, the industrial data obtained from the wire manufacturing company (which will be described in detail in Chapter 9) possess only an upper specification limit. Hence, proposing an uncertainty capability index that can handle a single specification limit is necessary.

Given the process characteristic with Liu's uncertain normal distribution, then the uncertain process capability index ${}_uC_{pu}$ can be expressed as:

$$ {}_uC_{pu} = \left[\frac{USL-\mu}{3\sigma_U}, \frac{USL-\mu}{3\sigma_U / \sqrt{2}} \right]. \tag{8.16} $$

As discussed in Chapter 7, usually the process statistics which are functions of the capability indices are known, so it is always necessary to accommodate this sampling error. Likewise, confidence intervals will also be constructed for this one-sided specification uncertainty capability index.

Assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Then, for any pre-determined level α, $\alpha \in (0,1)$, the 1- α level confidence interval for the uncertain process capability index ${}_uC_{pu}$ can be expressed as:

$$ {}_u\hat{C}_{pu} = [LL, UU], \ \alpha \in (0,1), \tag{8.17} $$

where

$$LL = \hat{C}_{pu} \ln\left(\frac{2}{1-\alpha}\right) - \frac{1}{\sqrt{3}\pi} \ln\left(\frac{2-\alpha}{\alpha}\right), \tag{8.18}$$

and

$$UU = \left\{ \hat{C}_{pu} + \frac{1}{\sqrt{3}\pi} \left[\frac{\ln\left(\frac{2-\alpha}{\alpha}\right)}{\ln\left(\frac{2}{1-\alpha}\right)} \right] \right\} \ln\left(\frac{2}{\alpha}\right) \tag{8.19}$$

where $\hat{C}_{pu} = \left(\frac{USL - \bar{x}}{3s}\right)$

Proof: Under Liu's uncertain normal distribution, $\sigma_L = 1/\sqrt{2}\sigma_U$,

$$_{u}\tilde{C}_{pk(USL)} = \left[\frac{USL-\mu}{3\sigma_U}, \frac{USL-\mu}{3\sigma_U/\sqrt{2}} \right]. \tag{8.20}$$

Thus, it is necessary to do a two-step replacement of parameters μ and σ.

The first step is to replace σ by S:

$$_{u}\tilde{C}_{pk} = \left[\frac{USL-\mu}{3S}\left(\ln 2 - \ln(1-\alpha)\right), \frac{USL-\mu}{3S/\sqrt{2}}\left(\ln 2 - \ln\alpha\right) \right] \tag{8.21}$$

The second step is to replace μ by $\bar{x}$:

$$_{u}\hat{C}_{pk(USL)} = [LL, UU] \tag{8.22}$$

where

$$LL = \left\{ \frac{USL - (\bar{x} + \frac{S\sqrt{3}}{\pi} \frac{\ln(2-\alpha) - \ln\alpha}{\ln 2 - \ln(1-\alpha)})}{3S} \right\} \left(\ln 2 - \ln(1-\alpha)\right) \tag{8.23}$$

hence

$$= \left(\frac{USL-\bar{x}}{3s}\right)\ln\left(\frac{2}{1-\alpha}\right) - \frac{1}{\pi\sqrt{3}}\ln\left(\frac{2-\alpha}{\alpha}\right) \tag{8.24}$$

$$LL = \hat{C}_{pu}\ln\left(\frac{2}{1-\alpha}\right) - \frac{1}{\pi\sqrt{3}}\ln\left(\frac{2-\alpha}{\alpha}\right) \tag{8.25}$$

and using the same analogy, then;

$$UU = \left\{\frac{USL-(\bar{x}-\frac{S\sqrt{3}}{\pi}\frac{\ln(2-\alpha)-\ln\alpha}{\ln 2-\ln(1-\alpha)})}{3S/\sqrt{2}}\right\}\left(\ln 2-\ln(\alpha)\right) \tag{8.26}$$

hence

$$= \left\{\left(\frac{USL-\bar{x}}{3S}\right) + \frac{1}{\pi\sqrt{3}}\frac{\ln\left(\frac{2-\alpha}{\alpha}\right)}{\ln\left(\frac{2}{1-\alpha}\right)}\right\}\sqrt{2}\left(\ln 2-\ln(\alpha)\right) \tag{8.27}$$

$$\mathrm{UU} = \left\{\hat{C}_{pu} + \frac{1}{\pi\sqrt{3}}\frac{\ln\left(\frac{2-\alpha}{\alpha}\right)}{\ln\left(\frac{2}{1-\alpha}\right)}\right\}\sqrt{2}\left(\ln 2-\ln(\alpha)\right) \tag{8.28}$$

Chapter 9. Empirical Results for a Local Manufacturing Process Capability

This chapter describes the analysis and interpretation of data obtained from a confidential and hence anonymous wire manufacturing company in South Africa. The methodology established in Chapter 7 and 8 will be applied to assess the classical capability in terms of the process ability to meet set quality requirements. Similarly, the experimental data obtained from experts will be analyzed using the Delphi approach to establish group judgement (consensus) as well as expert process capability (using the proposed uncertain capability index). These two approaches will be compared to determine whether the uncertainty approach will be able to provide valuable information as its probabilistic counterpart.

The dataset used in the thesis was collected from a South African wire manufacturing company who prefers to remain anonymous. The wire data collected was based on three core parameters measured, namely; conductor resistance, mass and core diameter. The conductor resistance is the most essential parameter of the three parameters measured, because the core diameter and mass of the wire depend on the specified tolerance of the wire's conductor resistance. Hence, a wire with a failed conductor resistance implies a defective wire since the most essential feature of any manufactured wire is its ability to resist current. Quality of the manufacturing output will concentrate on the wire conductor resistance.

The conductor resistance of the copper wires have a one-sided tolerance limit expressed as 4.61Ω, such that any wire measurement fallen below this limit passes the capability test while any value exceeding the limit represents a failed wire. In summary, the following conditions apply:

$$if\ \left(CR.Measurement < 4.61\Omega\right) then\ \left\{Quality = Pass\right\} else\ \left\{Quality = Fail\right\}.$$

where *CR.Measurement* represents the conductor resistance measurement of a wire.

In order to conduct any process capability study, the underlying assumption governing process capability analysis should be verified to ensure the validity of the analysis (Pignatiello and Ramberg, 1993). These assumptions are stated in the Chapter 8 as follows: The observed sample should be **Representative**, **Normally distributed**, **Stable** and **Independent**. Hence the following section will test whether all these assumptions hold.

9.1 Sample Representation issue

The conductor resistance data of the wire obtained from the manufacturing process has a total sample size of 424 observations which had been collected over a 5-month manufacturing period (08/01/2013-28/05/2013), expert opinion holds this sample is representative of the entire production process. The next section will test for normality, stability and independence.

9.2 Test Process Distributional Normality

In order to assess process capability, histogram and boxplots are reliable quality control tools to visualise and assess process performance and also determine the shape, center and spread of the distribution (Senvar, 2010). If the distribution of the quality characteristic measured is fairly skewed and this is not accounted for, then the capability estimates from the process may be misleading.

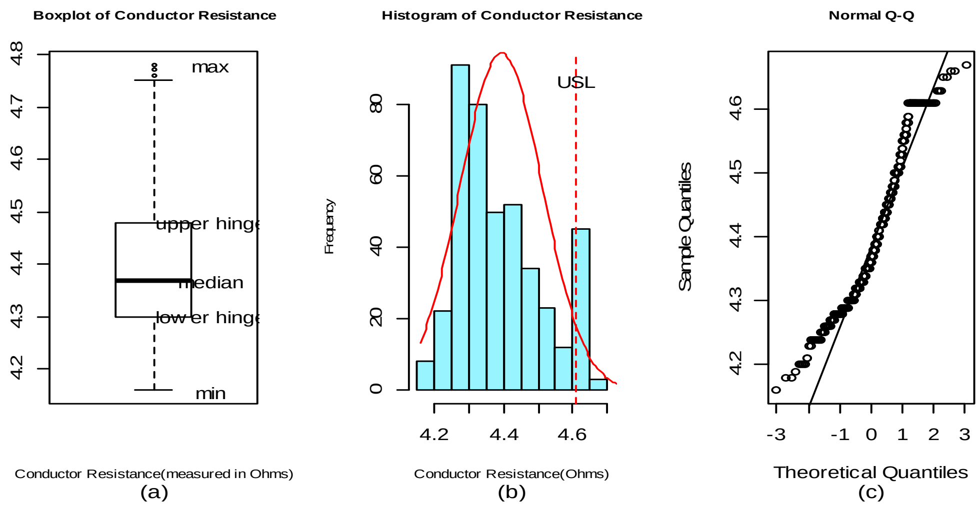

Figure 12: Boxplot, Histogram and Normal quantile-quantile plot of the Conductor Resistance of a wire with specified tolerance limit of 4.61Ω.

Fig. 12 (a) represents a boxplot of the measured wire conductor resistance with a specified tolerance limit of 4.61Ω. A box plot is usually used to determine whether there are outliers in the observed data and also the distribution of the dataset. If the mean and median are approximately equal, then the distribution of the data follows a Gaussian Normal distribution. It can be clearly seen that the dataset contains three outliers which were removed or else this may have skewed the distribution of the data and consequently, bias the analysis.

Fig.12(b) represents the distribution (shape) of the wire conductor resistance data with a specified tolerance limit of 4.61Ω, the distribution of the data appears deviated from the Gaussian normal although the outlier which was present in the boxplot (Fig. 12 (a)) has been removed. Fig.12 (b) shows that the process distribution is skewed to the right and also an interesting observation is that a proportion of observed conductor resistance measurements exceed the specified tolerance limit of 4.61Ω, which count as failed (i.e. defective) wires. Due to the one-sided specification limit, the natural choice of the classical process capability index C_{pk} is suitable and hence will be used to estimate process capability.

The probability plots in Fig. 12 (c) are also useful in determining the distribution and spread of the quality characteristic measured (i.e. conductor resistance). These plots are also used to test for the Gaussian normality assumption. The main idea of the probability plots is to assess the data plotted against the theoretical Gaussian normal distribution (i.e. plotted as a straight diagonal line in Fig. 12 (c)) and if the plotted data seriously deviates from the straight line then the normality assumption is in doubts. In this case both tails of the data tend to depart from the straight line.

Hence one can assume that the process distribution follows a non-normal distribution. The Shapiro-Wilk test will be applied to justify if there evidence of non-normality is valid.

Table 3: Shapiro-Wilk normality test

W (Wilks statistic)	p-value
0.9364	$2.08x10^{-12}$

The Shapiro Wilk test (Table 3) indicates that the wire conductor resistance data significantly deviates from normality at 5% level of significance. The validity of a non-normal distributed dataset is verified and will be adopted in this case.

9.3 Test of Process Stability

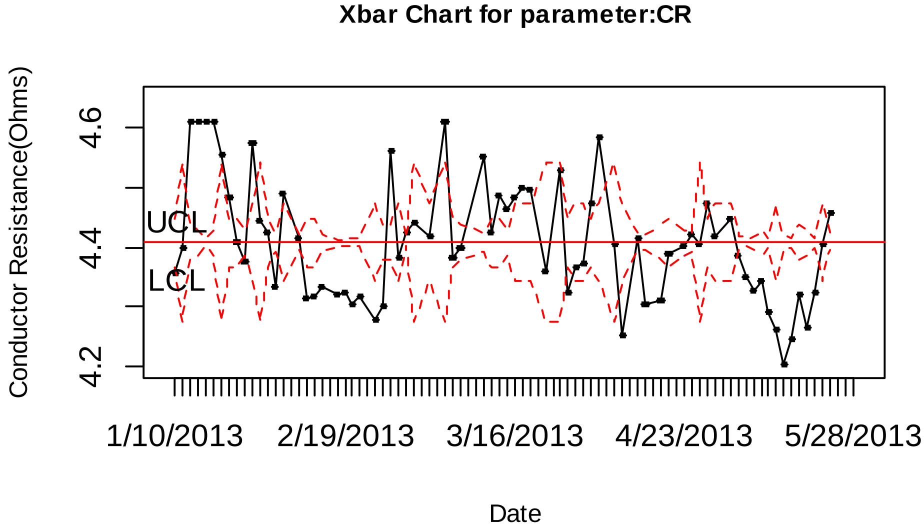

Figure 13: The Quality Control chart of the wire conductor resistance data with specified upper limit of 4.61Ω.

The quality control chart is used to assess whether the process is in a stable condition (i.e. the process does not deviate from its mean). The red broken lines represent the control limits while the **black** solid lines demonstrate the performance of the process. When the process exceeds the control limits (red broken lines) this is deemed as a point that is out of control. Thus, as depicted in Fig. 13, the process does not seem to be definitely in a stable condition (i.e. subject only to be random influences). The process takes long excursions from the mean, and sometimes does not fluctuate around the mean as expected from a stable process. The several out of control events indicate that process targeting has not been effectively pursued in this case.

9.4 Results for Testing Independence

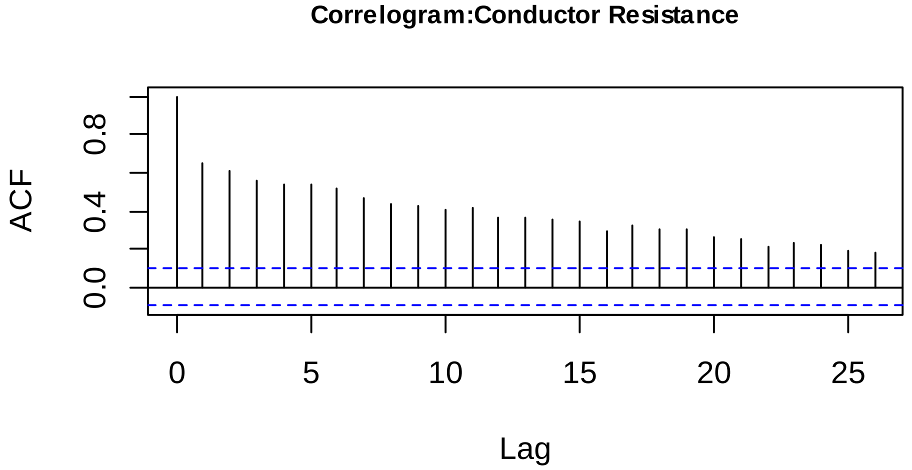

Figure 14: The Correlogram describing the Autocorrelation permeating the dataset

Fig. 14 represents the correlogram in which the vertical lines above the blue broken horizontal lines represent the type of autocorrelation that permeates the data. The blue broken lines represent the standard error in which if the vertical lines are above the blue broken line it signifies a sign of autocorrelation as a function of lag *k*. In Fig. 14, there conductor resistance data follows an autoregressive model of order 6 (i.e. AR(6)). The process is described by the following AR(6) model:

$$Z_t = 0.3396Z_{t-1} + 0.2066Z_{t-2} + 0.0621Z_{t-3} + 0.0563Z_{t-4} + 0.1195Z_{t-5} + 0.0779Z_{t-6} + \varepsilon_t \qquad (9.1)$$

where Z_t is the observed value at time *t*, and ε_t is a series of uncorrelated errors that are white noise distributed with zero mean and constant variance. This model implies that there is a certain inertia which drives the process. Thus every observation is influenced by the previous six observations. To reiterate, every current observation is a function of the past six observations and ε_t, that is a random shock usually called a white noise which has a random influence on the process. Due to this, the assumption of independent observations does not hold in this case.

As described in the Methodology section, since autocorrelation permeates the data the iterative skipping strategy will be used to obtain independent within subsamples and the process capability will be estimated from each subsample and combined in such a way that the Type I error will be controlled. We note that this test does not require independence relationships between the subsamples. Thus one can have a situation where values within each subsample is independent but amongst themselves there is some dependence and the test statistic will still be plausible (Vannman et al., 2008).

Only a one-sided specification limit for the wire conductor resistance measured was given as 4.61Ω. Any product with conductor resistance greater than this specified limit is deemed defective. However, any product with a value less than or equal to 4.61Ω has passed the capability test. Hence the one-sided index C_{pu} will be used to ascertain the capability of the process. Using the formula of $\hat{C}_{pu} = \left(USL - \bar{x}\right)/3s$ the capability of the process ignoring autocorrelation and non-normality, although both conditions exist, the capability was given as $\hat{C}_{pu} = 0.61$. A process is said to be capable if $C_{pu} \geq 1$. As noted by Shore (1997) when autocorrelation permeates the data, the process capability estimates tend to be inflated, hence we are expecting the actual performance yield of the process to be even less than 0.61.

9.5 Skipping Rules Results

The total sample size of the wire conductor resistance data is 420 observations (i.e. after removing four outliers) in which the underlying distribution of the data follows an autoregressive model of order 6. The iterative skipping strategy is then employed to partition the total sample size N into r subsamples of size m each (i.e. $m = \left[N / r\right]$). In this instance, the autocorrelation in the dataset is high (i.e. AR(6)) hence referring to Table 6 (see below), when autocorrelation is high, a large value of r will be required to deal with autocorrelation. The total dataset was partitioned into 8 subsamples (r = 8) such that observations within each subsample are independent and at least of approximately size 52 each $\left(\text{i.e. } m = \left[N / r\right] = \left[420/8\right] = 52.5\right)$.

In order to remove autocorrelation, the iterative skipping strategy was used to partition the original dataset into 8 subsamples, it can be seen from the correlogram in Fig 15. (a)-(h) that the assumption of independence holds for most of the subsamples and autocorrelation is at most of order 2 compared to the original dataset which had an autocorrelation of order 6. After using the skipping strategy, only a few correlograms of the sub samples have vertical lines that exceed the blue broken lines, this implies that autocorrelation has been removed in most instances and severely tamed down in the remaining. This justifies the use of Vannman et al.(2008) approach in tackling autocorrelation when it permeates the dataset. Since autocorrelation has been removed from the dataset then the capability of the process can now be estimated as the assumption of identically and independently distributed process holds.

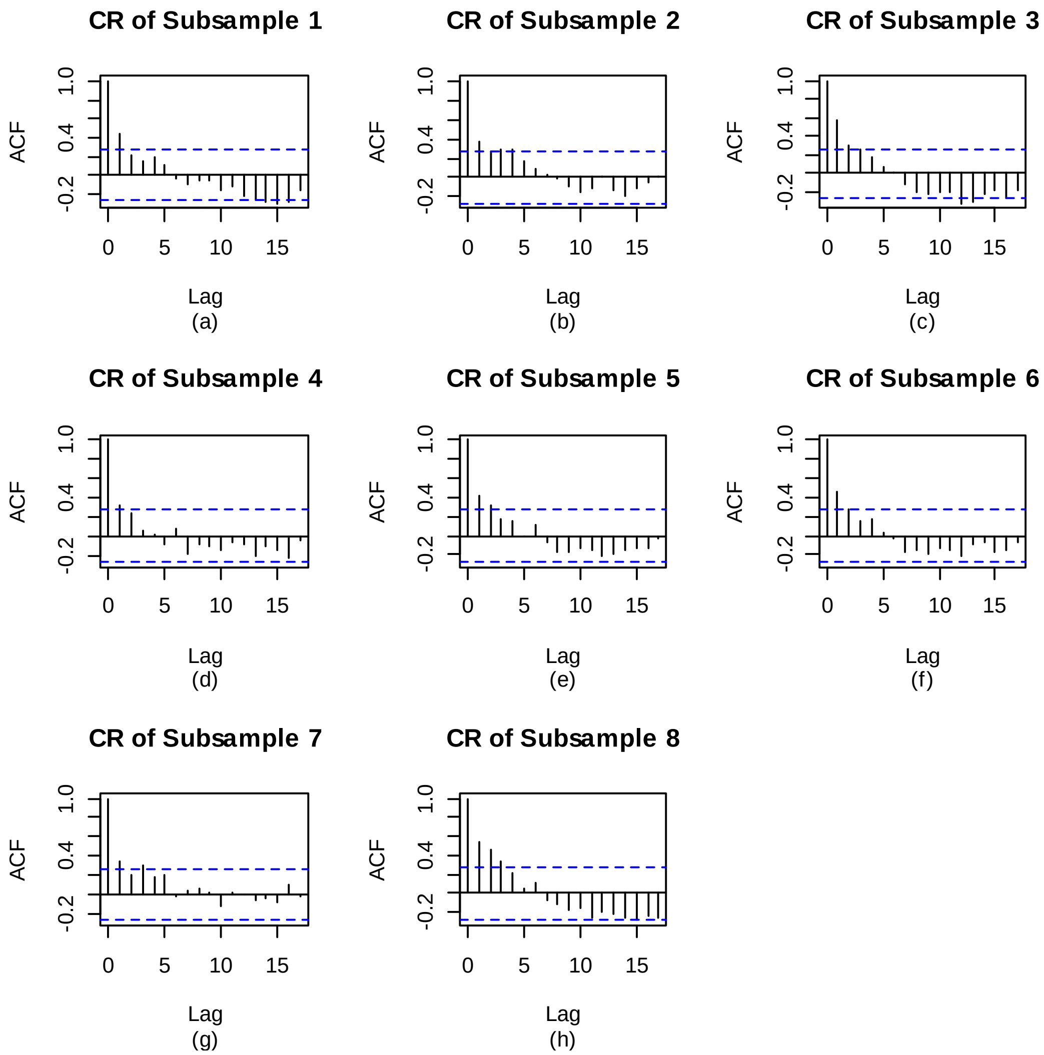

Figure 15: Assessing the level of autocorrelation in the 8 subsamples after applying the iterative skipping strategy.

9.6 Test of Process Capability

A process is described as incapable if $C_{pu} < 1.0$ and in such a case either process targeting should be aggressively pursued or process variance should be reduced (Pearn et al., 1999). A process is deemed capable if the 1.0 $\le$ $C_{pu} \le 1.33$ in this case some process control is necessary. Moreover if, 1.33 $\le$ $C_{pu} \le 1.50$, the process is regarded as satisfactory and so forth. The table below depicts the possible capability index values and their corresponding interpretation in terms of quality requirements.

Table 4: The C_{pu} value and the corresponding Quality conditions.

C_{pu} value	Quality condition
$C_{pu} \le 1.0$	Inadequate
$1.0 \le C_{pu} \le 1.33$	Capable
$1.33 \le C_{pu} \le 1.50$	Satisfactory
$1.50 \le C_{pu} \le 2.00$	Excellent
$C_{pu} \ge 2.00$	Super

To determine whether the process meets the capability, the specifications of the process should be defined. In this case the required capability $C = 1$, is chosen and the α risk is set at 5% significance (i.e. the probability of committing a Type I error). The hypothesis is stated as:

H₀: $C_{pu} < 1$

H₁: $C_{pu} \ge 1$ at significance level 5%.

Calculate the value of the estimator C^{WSD}_{pu} from the 8 subsamples. In order to estimate the process capability, the C^{WSD}_{pu} formula in Section 8.3.2(b) is employed to account for skewed distributions.

It should be noted that when the data is normally distributed then $\hat{C}^{WSD}_{pu} = \hat{C}_{pu}$. The capability of the process was also estimated using the index $\hat{C}_{pu}$ to determine the effect of the skewed population distribution. The estimates of the process capability are given in Table 5:

Table 5: Estimates of Process capability from the 8 independent subsamples.

r (subsample no.)	C_{pu}^{WSD}	C_{pu}
1	0.6168902	0.6983662
2	0.4941277	0.5780361
3	0.559204	0.6330612
4	0.4580344	0.5185295
5	0.4772929	0.5507226
6	0.5379412	0.6207014
7	0.5906069	0.6587538
8	0.5889459	0.6115976

The critical value C_0 is determined by using a proxy proposed by Pearn et al. (1999), $C_0 = \left(b_f \big/ 3\sqrt{n}\right) t_{n\text{-}1,\alpha}\left(\delta_c\right)$, where $t_{n\text{-}1,\alpha}\left(\delta_c\right)$ is the upper α quantile of $t_{n\text{-}1}\left(\delta_c\right)$ and $\delta = 3n^{1/2}C$.

From the table provided by Pearn et al. (1999) the critical value is C_0 =1.201 assuming that α=0.05 and *n*=52.

Table 6: Recommendations for the skipping strategy (adapted from Vannman et al., 2008).

	Recommendation	
	r	Rule
$\hat{\rho}_U \leq 0.5$	5	C or D
$0.5 \leq \hat{\rho}_U \leq 0.7$	10	C or D
$0.7 \leq \rho_U$	20 or 25	B, C or D

Since *r* = 8 in this case and is closer to 10 than 5, from Table 6 Rule C will apply. The Rule C is stated as follows:

Reject H_0 if 2 or more $\hat{C}_{pl(i)} >\ C_0,\ i = 1, 2, ...r$.

$$P\left(2 \text{ or more of } \hat{C}_{pl(i)} > C_o,\ i = 1,\ 2,...\text{r}|\text{H}_0 \text{ is true }\right) \leq \alpha$$

From Table 5, none of the estimated capability indices of the independent sample meet the basic capability requirement of C=1 (i.e. $\widehat{C}_{pu(i)} < C_0, \ \forall i$).

The conclusion is that we fail to reject the null hypothesis (i.e. the process is inadequate). Thus the process fails to meet the set capability requirement; hence process targeting efforts may be used to reduce process variance.

The hypothesis test and the descriptive statistics (i.e. QC chart, histogram, boxplot etc.) demonstrate that the process is sporadic with no uniformity, i.e. there is a large variation in the conductor resistance data. Although, almost all the observations are within the 4.61Ω tolerance limit (i.e. refer to Fig. 12), which signifies a process that has less defective units, the variability within the system is still very high and results in a low capability index. Therefore a change in the thinking of the way process capability analysis is conducted is required from the engineers in order to improve quality. For example, a process producing wire mass measured may be given specifications as: 48 ±0.5grams. The engineer should concentrate on producing the most likely value, that is 48grams rather than monitoring the process to produce mass measurement within the accepted range [47.5g, 48.5g].

9.7 Estimating Process Capability under Uncertainty Theory

In order to estimate process capability under uncertainty theory, the process conductor resistance measurements were defined as an uncertain variable. To obtain an uncertainty distribution for the uncertain variable, experts who have worked on the process were given a questionnaire about the possible values the uncertain variable may take and the belief degree depending on their experience in producing these wires. Experts were instructed to answer the questionnaire independent of one another, to make sure that we capture each experts opinion and also reduce bias. However, only two experts (n=2) had the convenience to answer my questionnaire due to their hectic work schedule at that particular time. The expert experimental data is given as follows:

$$E_1 : \left(4.1, 0\right), (4.2, 0.02), (4.3, 0.41), (4.4, 0.78), (4.5, 0.95), (4.6, 0.99), (4.61, 1)$$

$$E_2 : \left(4.1, 0\right), (4.2, 0.10), (4.3, 0.5), (4.4, 0.65), (4.5, 0.8), (4.6, 0.98), (4.61, 1)$$

where E_i represents the experimental data of expert i, $i = 1, 2$.

The total possible values provided by both experts is given as: 4.1, 4.2, 4.3, 4.4, 4.5, 4.6, 4.61. These values and their corresponding uncertainty distributions for each expert are summarized in the Table 7 below:

Table 7: Experimental data collected from two experts based on their experience.

x	$\Phi_1(x)$	$\Phi_2(x)$
4.1	0	0
4.2	0.02	0.1
4.3	0.41	0.5
4.4	0.78	0.65
4.5	0.95	0.8
4.6	0.99	0.98
4.61	1	1

From equations (8.14) and (8.15), let:

$$\alpha_j = \frac{1}{2}\sum_{i=1}^{2}\Phi_i\left(x_j\right), \; i = 1, 2, \; j = 1, 2, 3, 4, 5, 6, 7..$$

and

$$d_j = \frac{1}{2}\sum_{i=1}^{2}\left(\Phi_i\left(x_j\right) - \overline{\alpha}_j\right)^2, \ i = 1, 2, \ j = 1, 2, ..., 7.$$

In estimating the aggregate distribution a weight may be assigned to each expert and in this case we will assume that each experts has an equal weight of $1/2,$ since $n = 2.$ Hence the corresponding statistics $\overline{\alpha}_j$ and d_j are calculated as displayed in Table 8:

Table 8: The Integrated Uncertainty distribution and test to judge expert opinion has reached consensus.

x	$\overline{\alpha}_j$	d_j
4.1	0	0
4.2	0.06	0.0016
4.3	0.455	0.002
4.4	0.715	0.004
4.5	0.875	0.006
4.6	0.985	0.000025
4.61	1	0

Since $d_j < 0.05, \ j = 1, 2, 3, ..., 7$ therefore there is no need to keep on iterating the process as the views of the two engineers are aligned. From Table 8 we get the experts integrated observational data as follows:

$$E : \left(4.1, 0\right), (4.2, 0.06), (4.3, 0.455), (4.4, 0.715), (4.5, 0.875), (4.6, 0.985), (4.61, 1) \qquad (9.2)$$

The uncertainty distribution for the expert data can be derived as follows:

$$\Phi\left(x\right) = \begin{cases} 0 \quad , & if \ x < 4.2 \\ 3.95x - 16.53, & if \ 4.2 \le x \le 4.3 \\ 2.6x - 10.725, & if \ 4.3 \le x \le 4.4 \\ 1.6x - 6.325 \ , & if \ 4.4 \le x \le 4.5 \\ 1.1x - 4.075 \ , & if \ 4.5 \le x \le 4.6 \\ 1.5x - 5.915 \ , & if \ 4.6 \le x \le 4.61 \\ 1 \quad , & if \ x > 4.61 \end{cases} \qquad (9.3)$$

Figure 16 below depicts a graphical representation of the expert data provided. This figure resembles an uncertain normal distribution, hence we assumed that the expert data follows Liu's uncertain normal distribution.

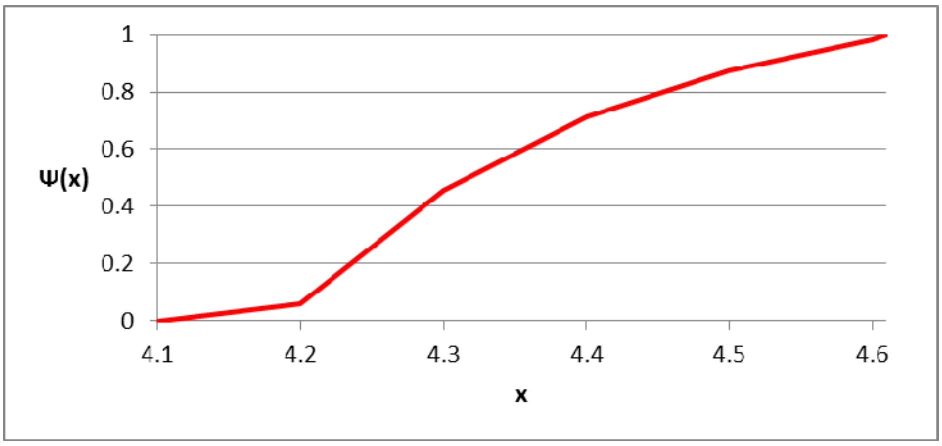

Figure 16: A graphical representation of integrated Expert data

9.8 Uncertainty Process Capability Index Computations

Since only a one-sided tolerance limit of 4.61Ω was given, the index ${}_u\widetilde{C}_{pu}$ will be used in an uncertain environment to estimate expert process capability.

Definition 9.1: The uncertain process capability index ${}_uC_{pu}$ is defined by:

$$ {}_uC_{pu} = \left[\frac{USL-\mu}{3\sigma_U}, \frac{USL-\mu}{3\sigma_L}\right], \tag{9.4} $$

where σ_L and σ_U are the lower limit and upper limit of the square root of variance interval $\sqrt{V[X]} = [\sigma_L, \sigma_U]$.

Given the uncertain process is governed by Liu's uncertain normal distribution, $\sqrt{V[X]} = \left[\sigma_U/\sqrt{2}, \sigma_U\right]$. Then, the uncertain process capability index ${}_uC_{pk(USL)}$ is expressed by:

$$ {}_uC_{pu} = \left[\frac{USL-\mu}{3\sigma_U}, \frac{USL-\mu}{3\sigma_L}\right] = \left[\frac{USL-\mu}{3\sigma_U}, \frac{(USL-\mu)}{3\sigma_U/\sqrt{2}}\right], \tag{9.5} $$

where parameter μ is the expectation, σ_L and σ_U are the lower limit and upper limit of the square root of variance interval $\sqrt{V[X]} = \left[\sigma_U / \sqrt{2}, \sigma_U\right]$.

From the uncertain capability index in (9.5), the sample mean and standard deviation need to be estimated from the expert experimental data. To estimate these unknown parameters, the sample mean and sample standard deviation were defined as:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \; s = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2} \tag{9.6}$$

Hence, replacing the parameters in in (9.5) with the sample statistics results in the following:

$$_u\tilde{C}_{pu} = \left[\frac{\left(USL - \bar{x}\right)}{3s}, \frac{\left(USL - \bar{x}\right)}{3s/\sqrt{2}}\right], \tag{9.7}$$

Similarly, confidence intervals will also be estimated for the one-sided capability index (i.e. $_uC_{pu}$). Thus, assuming that a "sample" $x_1, x_2, \cdots, x_n$, from Liu's uncertain normal distribution with parameters μ and σ. Then, for any pre-determined level α, $\alpha \in (0,1)$, the 1- α level confidence interval for the uncertain process capability index $_uC_{pu}$ can be expressed as:

$$_u\hat{C}_{pu} = [LL, UU], \; \alpha \in (0,1), \tag{9.8}$$

where

$$LL = \hat{C}_{pu}\ln\left(\frac{2}{1-\alpha}\right) - \frac{1}{\sqrt{3}\pi}\ln\left(\frac{2-\alpha}{\alpha}\right), \tag{9.9}$$

and

$$UU = \left\{\hat{C}_{pu} + \frac{1}{\sqrt{3}\pi}\left[\frac{\ln\left(\frac{2-\alpha}{\alpha}\right)}{\ln\left(\frac{2}{1-\alpha}\right)}\right]\right\}\sqrt{2}\ln\left(\frac{2}{\alpha}\right) \tag{9.10}$$

where $\widehat{C}_{pu} = \left(\frac{USL - \overline{x}}{3s} \right)$

Using the formula in (9.7), that is after the parameters are replaced with the sample mean and standard deviation statistic, the uncertain process capability of the process was estimated over an interval as ${}_u\widetilde{C}_{pu} = [0.407, 0.580]$.

As noted by Moore (1996) any number between the intervals of ${}_uC_{pu}$ qualifies as the capability of the process (i.e. under an uncertain environment). In comparison, to the estimated classical capability index of $\max_{1 \le i \le 8}\left(\widehat{C}_{pu(i)}\right) = 0.62,$ the uncertain capability index tends to give an adequate representation of the process performance at a significance level of α=0.5 (i.e. at a 50% confidence interval). The low confidence interval of 50% is justified this is due to the fact that the expert data is based on intuition, hence we cannot assign a confidence interval similar to when we are dealing with empirical data (i.e. classical approach), hence to adjust for that element of uncertainty a higher significance level was used. Using the confidence limits in (9.9) and (9.10), the confidence interval for the one-sided uncertain capability index was estimated as ${}_u\hat{C}_{pu} = [0.363, 0.770]$. The confidence interval of the index ${}_uC_{pu}$ reflects a process that is not capable which aligns with the classical capability analysis.

Since the parameters of most production processes are unknown, the uncertain capability index tends to give an idea of process performance and also the general thinking about the process amongst the process engineers, i.e. how well a process is assumed to perform in relation to the actual performance or classical process capability. Hence the uncertain capability indices are not proposed to necessarily replace the existing classical capability indices but to complement.

Chapter 10. Conclusion

The objective of the thesis were to comprehensively review the classical capability analysis to gain an in-depth knowledge, review uncertainty theory in order to apply it to process capability analysis, propose the counterparts of the classical capability indices (i.e. $C_{p,}$ C_{pk} and C_{pm}) and finally compare the selected uncertainty capability to the estimates of its classical counterpart.

Thus the thesis set out to achieve these objectives listed and fortunately all these objectives were achieved.

The essential part of the thesis was to extend process capability analysis to an uncertain environment and the findings were that under an uncertain environment the uncertainty capability indices are expressed as an interval, such that any number within the interval gives a representation of the process. This was the major highlight of the thesis. It was identified that the uncertain capability index tends to give a realistic representation of the process performance especially at a lower level of confidence, i.e. higher significance level α=0.5. The index is also informative and can be used as a yardstick for measuring the general thinking of the process performance (i.e. by the engineers who operate on it) compared to the actual performance of the process and this index serves as a guide to correct for discrepancies between what is expected and the actual. The most commonly used sampling statistics (i.e. mean μ and standard deviation, s) were also proposed under uncertain environment, leading to the constructing of confidence intervals within an uncertain environment for the newly defined uncertainty capability indices.

This research has had its own limitations in execution and this is stated as follows:

1) In terms of expert data, we only received feedback from the manager and one of the engineers working on the process due to a busy schedule at that point in time, hence a sample of two was used to represent views of about ten engineers who work on the same process. This generalisation might be bias.

For future research, a hypothesis test needs to be developed to address the situation of how to determine whether a process is capable from expert experimental data.

Finally, to the best of my knowledge there has not been any publication on the application of uncertainty theory to process capability analysis, hence this serves as a pioneer research in which others can build upon.

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Appendix

Chart for Averages			Chart for Std. Dev.			Chart for Ranges			
Factors For Control Limits									
n	A_2	A_3	c_4	B_3	B_4	d_2	d_3	D_3	D_4
2	1.880	2.659	0.7979	0	3.267	1.128	0.853	0	3.267
3	1.023	1.954	0.8862	0	2.568	1.693	0.888	0	2.574
4	0.729	1.628	0.9213	0	2.266	2.059	0.880	0	2.282
5	0.577	1.427	0.9400	0	2.089	2.326	0.864	0	2.114
6	0.483	1.287	0.9515	0.030	1.970	2.534	0.848	0	2.004
7	0.419	1.182	0.9594	0.118	1.882	2.704	0.833	0.076	1.924
8	0.373	1.099	0.9650	0.185	1.815	2.847	0.820	0.136	1.864
9	0.337	1.032	0.9693	0.239	1.761	2.970	0.808	0.184	1.816
10	0.308	0.975	0.9727	0.284	1.716	3.078	0.797	0.223	1.777
11	0.285	0.927	0.9754	0.321	1.679	3.173	0.787	0.256	1.744
12	0.266	0.886	0.9776	0.354	1.646	3.258	0.778	0.283	1.717
13	0.249	0.850	0.9794	0.382	1.618	3.336	0.770	0.307	1.693
14	0.235	0.817	0.9810	0.406	1.594	3.407	0.763	0.328	1.672
15	0.223	0.789	0.9823	0.428	1.572	3.472	0.756	0.347	1.653
16	0.212	0.763	0.9835	0.448	1.552	3.532	0.750	0.363	1.637
17	0.203	0.739	0.9845	0.466	1.534	3.588	0.744	0.378	1.622
18	0.707	0.718	0.9854	0.482	1.518	3.640	0.739	0.391	1.608
19	0.187	0.698	0.9862	0.497	1.503	3.689	0.734	0.403	1.597
20	0.180	0.680	0.9869	0.510	1.490	3.735	0.729	0.415	1.585
21	0.173	0.663	0.9876	0.523	1.477	3.778	0.724	0.425	1.575
22	0.167	0.647	0.9882	0.534	1.466	3.819	0.720	0.434	1.566
23	0.162	0.633	0.9887	0.545	1.455	3.858	0.716	0.443	1.557
24	0.157	0.619	0.9892	0.555	1.445	3.895	0.712	0.451	1.548
25	0.153	0.606	0.9896	0.565	1.435	3.931	0.708	0.459	1.541

Table I: Factors For Constructing Variable Control Charts. This table was adopted from Montgomery(1985).

n	b_f	*n*	b_f	*n*	b_f	*n*	b_f	*n*	b_f	*n*	b_f
10	0.914	45	0.983	80	0.99	115	0.993	150	0.995	185	0.996
15	0.945	50	0.985	85	0.991	120	0.994	155	0.995	190	0.996
20	0.96	55	0.986	90	0.992	125	0.994	160	0.995	195	0.996
25	0.968	60	0.987	95	0.992	130	0.994	165	0.995	200	0.996
30	0.974	65	0.988	100	0.992	135	0.994	170	0.996	205	0.996
35	0.978	70	0.989	105	0.993	140	0.995	175	0.996	210	0.996
40	0.981	75	0.99	110	0.993	145	0.995	180	0.996	215	0.996

Table II: Values of the correction factor b_f for various sample sizes.

This table was adopted from Pearn et al. (1999).

R Syntax

a) Syntax for Gaussian process that is centered (i.e. μ=T) found on pg. 14

```
set.seed(1)
x<-rnorm(10000,46,2)
plot(density(x),main="",ylab="NormalDensity",xlab="Process
Output(X)",col="blue",lwd=2,cex.lab=0.9)
abline(v=c(38,46,54),col="red",lwd=2)
text(c(38,47,54),rep(0.2,3),labels=c("LSL","μ=T","USL"),font=3)
abline(h=0, col="blue")
```

b) Syntax for Gaussian process that is centered (i.e. μ=T) found on pg. 15

```
mean=49;sd=3;LSL=38;T=46;USL=54
x<-seq(-3,3.67,length=100)*sd+mean
hx<-dnorm(x,mean,sd)
plot(x,hx,type="n",xlab="ProcessOutput(X)",ylab="Normal
Density",main="",xlim=c(37,60),yaxt="n",cex.lab=0.9)
i<-x>=USL
lines(x,hx,lwd=2)
polygon(c(USL,x[i]),c(0,hx[i]),col="skyblue")
abline(h=0)
abline(v=c(38,46,49,54),col="red",lwd=2)
text(c(38,46,49,54),rep(0.12,4),labels=c("LSL","T","μ","USL"),font=3)
```

c) Syntax for descriptive stats, QC chart, correlogram, iterative skipping strategy found page 114,116, 117 and 119

```
#reading in the CBI dataset
mydata<-
read.csv("C:\\Users\\Gyekkw01\\Desktop\\RESEARCH\\CBI_Data\\Data4.61.csv",header=TRUE)
#view the dataset variables
fix(mydata)
```

```
#replacing zeros in CR.Measurement with 4.61V
mydata$CR.Measurement[mydata$CR.Measurement==0]<-4.61
fix(mydata)
head(mydata)
attach(mydata)
#creating a 1x3 window
par(mfrow=c(1,3))
#drawing a box plot
boxplot(CR.Measurement,main="Boxplot of Conductor Resistance",cex.main=0.8,xlab="Conductor
Resistance(measured in Ohms)",cex.lab=0.8,sub="(a)",cex.sub=1.1)
#calculating 5 summary statistics
f=fivenum(CR.Measurement)
#adding 5 summary statistics to the boxplot
text(rep(1.3,5),f,labels=c("min","lower hinge","median","upper hinge","max"))#label the boxplot
#Removing the outliers
mydata<-subset(mydata,mydata$CR.Measurement<4.75)
attach(mydata)
#drawing a histogram
hist(mydata$CR.Measurement,main="HistogramofConductor
Resistance",cex.main=0.8,xlab="Conductor
Resistance(Ohms)",cex.lab=0.8,sub="(b)",cex.sub=1.1,col="cadetblue1")
#plotting a density curve over the histogram
xv<-seq(4.16,4.78,0.005)
yv<-dnorm(xv,mean=mean(CR.Measurement),sd=sd(CR.Measurement))*28
lines(xv,yv,col="red")
#adding the upper limit of CR to the histogram
abline(v=4.61,lwd=1.8,col="red",lty=2)
#label the limits
text(4.61,87,labels="USL")
# Constructing a Normal Quantile plot
qqnorm(mydata$CR.Measurement,main="Normal Q-Q",cex.main=0.8,sub="(c)",cex.sub=1.1)
qqline(mydata$CR.Measurement)
```

```
#Testing for Autocorrelation
par(mfrow=c(1,1))
acf(mydata$CR.Measurement,main="Correlogram:Conductor
Resistance",cex.main=0.8,cex.lab=0.8)
ar(mydata$CR.Measurement,aic=TRUE,order.max=NULL)
#Defining a Filter variable
Filter<-function(x){
x<-vector(mode="numeric",length=420)
for (i in 1:8){
x[seq(i,420,8)]<-i
}
mydata<-data.frame(mydata,x)
}
mydata1<-Filter(y)
names(mydata1)[17]<-"Filter"
#Plotting the correlelogram after applying iterative skipping strategy
par(mfrow=c(3,3))
acf(mydata1$CR.Measurement[mydata1$Filter==1],main="CR of Subsample 1",sub="(a)")
acf(mydata1$CR.Measurement[mydata1$Filter==2],main="CR of Subsample 2",sub="(b)")
acf(mydata1$CR.Measurement[mydata1$Filter==3],main="CR of Subsample 3",sub="(c)")
acf(mydata1$CR.Measurement[mydata1$Filter==4],main="CR of Subsample 4",sub="(d)")
acf(mydata1$CR.Measurement[mydata1$Filter==5],main="CR of Subsample 5",sub="(e)")
acf(mydata1$CR.Measurement[mydata1$Filter==6],main="CR of Subsample 6",sub="(f)")
acf(mydata1$CR.Measurement[mydata1$Filter==7],main="CR of Subsample 7",sub="(g)")
acf(mydata1$CR.Measurement[mydata1$Filter==8],main="CR of Subsample 8",sub="(h)")


#Estimating capability for each subsample using the index Cpk_WSD found on pg. 121
Cpk_W<-c()
xbar<-c()
s<-c()
p<-c()
```

```
for (i in 1:7){
SS<-subset(mydata1,mydata1$Filter==i)
CR<-SS[,7]
n=length(CR)
USL=4.61
xbar[i]=mean(CR)
s[i]=sd(CR)
p[i]=length(CR[CR<=xbar[i]])/n
Cpk_W[i]=(USL-xbar[i])/(6*p[i]*s[i])
 }
warnings()

#Estimating capability using the index Cpk found on pg. 121
Cpu<-c()
xbar<-c()
s<-c()
for (i in 1:7){
SS<-subset(mydata1,mydata1$Filter==i)
USL=4.61
xbar[i]=mean(SS[,7])
s[i]=sd(SS[,7])
Cpu[i]=(USL-xbar[i])/(3*s[i])
 }
##Plotting a control chart
#plot an Xbar Chart
z<-c(by(mydata1[,7],Date,mean))
#convert into to a time series object
z<-ts(z)
y<-table(mydata1$Date)
seriesdata<-data.frame(y,z)
#change the name of column [,1] to "Date" & column [,3] to avg (i.e. group average)
names(seriesdata)[1]<-"Date"
```

```
names(seriesdata)[3]<-"avg"
#resetting the plotting window
par(mfrow=c(1,1))
#constructing the control chart
seriesdata<-seriesdata[complete.cases(seriesdata),]
#subsetting where seriesdata$Freq>1
seriesdata<-subset(seriesdata,!(seriesdata$Freq==1))
plot(seriesdata$Date,seriesdata$avg,main="XbarChartforparameter:CR",cex.main=0.8,ylab="Condu
ctorResistance(Ohms)",xlab="Date",pch=23,ylim=c(4.2,4.65))
lines(seriesdata$Date,seriesdata$avg)
UCL=c()
LCL=c()
n=length(seriesdata$avg)
xbar=mean(seriesdata$avg)
s=sd(seriesdata$avg)
c4=c(1.2533,1.1284,1.0854,1.0638,1.0510,1.0423,1.0363,1.0317,1.0281,1.0252,1.0229,1.0210,1.01
94,1.0180)
for (i in 1:n) {
UCL[i]=xbar+(3*s*c4[seriesdata$Freq[i]-1])/(seriesdata$Freq[i]*sqrt(seriesdata$Freq[i]))
LCL[i]=xbar-(3*s*c4[seriesdata$Freq[i]-1])/(seriesdata$Freq[i]*sqrt(seriesdata$Freq[i]))
}
#adding the LCL and UCL to seriesdata
seriesdata<-data.frame(seriesdata,LCL,UCL)
#adding the mean to the QC chart
abline(h=mean(seriesdata$avg),col="red")
# adding the LCL to the QC chart
lines(seriesdata$Date,seriesdata$LCL,lty=2,lwd=1.5,col="red")
lines(seriesdata$Date,seriesdata$UCL,lty=2,lwd=1.5,col="red")
text(rep(1.1,2),c(4.35,4.45),labels=c("LCL","UCL"))
```

d) Estimating uncertainty capability index start from pg. 123-127

```
# Load in the XLConnect package to enable import and export of
```

```
# files from excel
require(XLConnect)
excelFile <- "C:\\Users\\Gyekkw01\\Desktop\\RESEARCH\\CBI Data\\exp_data.xls"
wb <- loadWorkbook(excelFile)
exampleData <- readWorksheet(wb, sheet = "Expert",header=TRUE)
attach(exampleData)
#computation for the mean
L1=X[1]*M[1]
n=length(X)
LN=(1-M[n])*X[n]
L=c()
L[1]=L1
for (i in 1:(n-1)){
L[i+1]=(M[i+1]-M[i])*(X[i+1]+X[i])
}
#computing the first moment(mean)of the expert data
mu=L1+0.5*sum(L[2:n])+LN
mu
#computation for standard deviation of expert data
S1=(X[1]^2*M[1])
SN=(X[n]^2*(1-M[n]))
S=c()
S[1]=S1
for (i in 2:n){
S[i]=(M[i]-M[i-1])*(X[i]^2+X[i]*X[i-1]+X[i-1]^2)
}
S
mu_2=S1+(1/3)*sum(S[2:n])+SN
mu_2
s=sqrt(mu_2-(mu)^2)
s
##computing the uncertainty capability index
```

```
USL=4.61
#computing the lower capability of the uncertainty capability index
Cpu_l=(USL-mu)/(3*s)
Cpu_l
#computing the upper capability of the uncertainty capability index
Cpu_u=(USL-mu)/((3/sqrt(2))*s)
Cpu_u
```

e) Syntax for Questionnaire

```
x<-c(4.1,4.3,4.4,4.5,4.61,4.65,4.7,4.8)
y<-c(0,13,30,40,65,77,88,100)
plot(x,y,pch=20,main="Conductor Resistance of a Wire with Upper Limit of 4.61Ω",ylab="Belief Degree(%)",xlab="Conductor Resistance(Ohms)",col="red",cex=2,cex.main=0.8)
lines(x,y,col="blue",lwd=2)
plot(x,y,type="n",main="Conductor Resistance of a Wire with Upper Limit of 4.61Ω",ylab="Belief Degree(%)",xlab="Conductor Resistance(measured in Ohms)")
axis(side=2,at=c(10,30,50,70,90),labels=c("10","30","50","70","90"))
abline(h=c(0,10,20,30,40,50,60,70,80,90,100))
```

Questionnaire

GENERAL THEME OF THE QUESTIONNAIRE
The main idea of the study is to determine the general performance of the production process. Please note that there are no precise answers but we seek explicit opinions based on your experience with working on the production process.
To make this study as anonymous as possible no personal information about any participant will be required. As discussed, what we seek to achieve is to capture the general thinking about the performance of the process and also to be able to assess how to enhance the production process performance.

INSTRUCTIONS TO ANSWER THE QUESTIONNAIRE

- This questionnaire should be answered independently of other engineers, the whole idea is to obtain your personal view about the performance of the process

- Please place the values you regard the process output is likely to take, on the graph.

- Each point you place on the graph should be in an increasing order from left to right, thus from your experience if the likely wire conductor Resistance (CR) are 4.2Ω and 4.3Ω respectively, then the percentage attained at (or below) 4.2Ω should be less than the percentage attained at or below 4.3Ω.

- The points chosen may be as close together or as far apart as your experience allows you to expect.

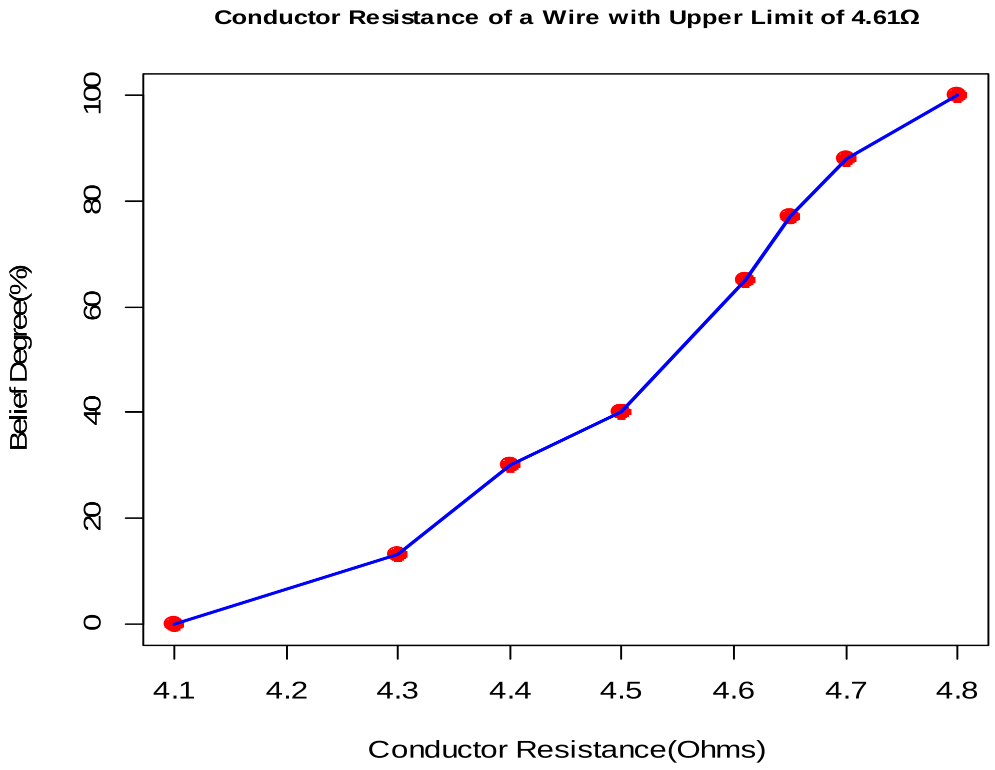
Example:

Below is an example of how a CBI industrial engineer with experience in working on the manufacturing process might regard the process output is likely to perform. Given a manufacturing process designed to produce a wire conductor resistance of 4.61Ω and using the machine 21252, an engineer might report that the process is likely to output the following values:
4.3Ω, 4.4Ω, 4.5Ω, 4.61Ω, 4.65Ω, 4.7Ω and the engineer may attach the corresponding cumulative percentages likely to fall below these values as follows: 13%, 30%, 40%, 65%, 77%, and 88%. The engineers experience is summarized as**:** (4.3Ω, 13%), (4.4Ω, 30%), (4.5Ω, 40%), (4.61Ω, 65%),

(4.65Ω, 77%), (4.7Ω, 88%). The engineer also stated that the minimum and maximum of the process output as 4.1Ω and 4.8Ω respectively.

These values presented by the engineer imply that 13% of the values observed fall below 4.3Ω, 30% of the values fall below 4.5Ω and so forth.

These points selected by the CBI industrial engineer are plotted in the chart below:



From the chart above, the engineer's view demonstrates that approximately 20% of the values are off target.

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Please follow the rational in which the engineer used but also in your own experience with the plant to answer the questions below

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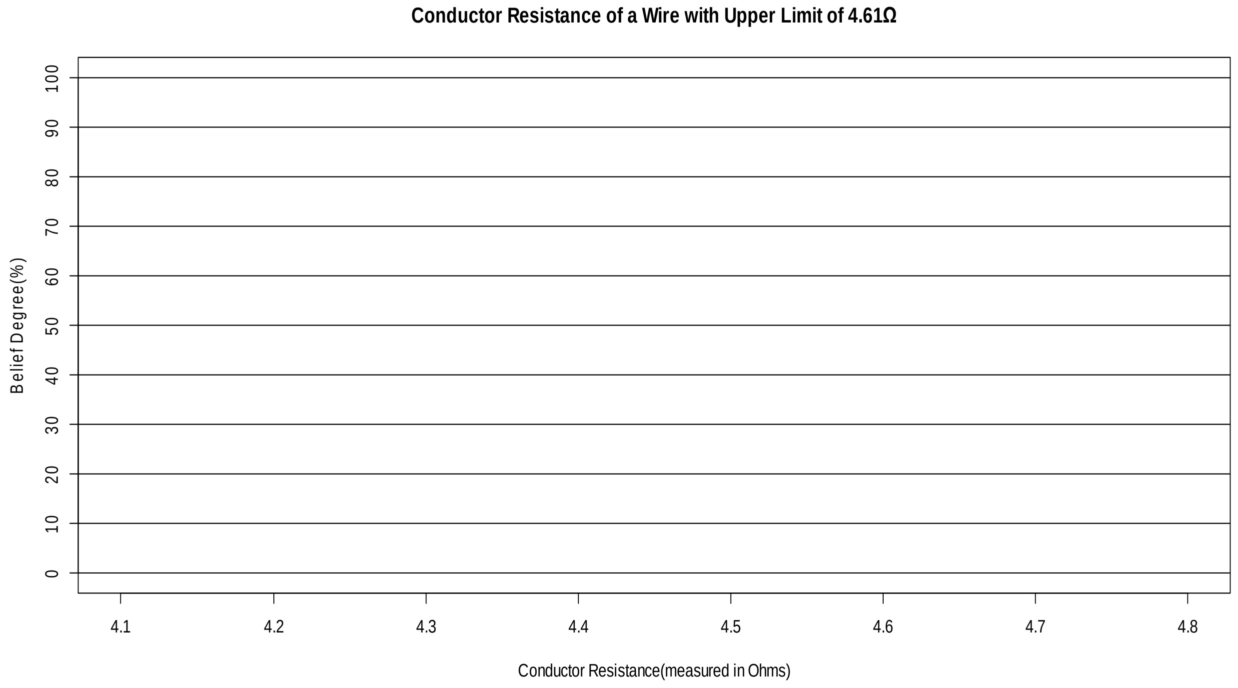

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Please place all the points and their corresponding cumulative percentages on the graph.

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The blank chart above requires you to describe a manufacturing process designed to produce a wire conductor resistance of 4.61Ω (i.e. the specified upper limit), using the machine 21252.

Q1. What do you think is the minimum Conductor Resistance of the process output? __________ Ω

Q2. What do you think is the maximum Conductor Resistance of the process output? __________ Ω

Q3. From your experience with working on the process, what is the mean Conductor Resistance of the process output with a specified Upper Limit of 4.61Ω? ____________

Q4. What percentages of values are likely to be less than the mean conductor resistance you specified? ___________%

Q5. How likely is the process to take Conductor Resistance measurements between (M-0.3 Ω) and (M + 0.3Ω)? __________%

Where M stands for the mean you have chosen in Q4,

Q6. What percentages of the Conductor Resistance measurements will be below M-0.3Ω? ______________%

Q7. What percentages of the Conductor Resistance measurements will be above M+0.3 Ω? ______________%

Q8. Is there any other possible values you regard the process output is likely to take? ____________

Q9. Please give the percentage (%) of values likely to fall below the value selected in Q8.? ____________%

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THANK YOU FOR YOUR CO-OPERATION

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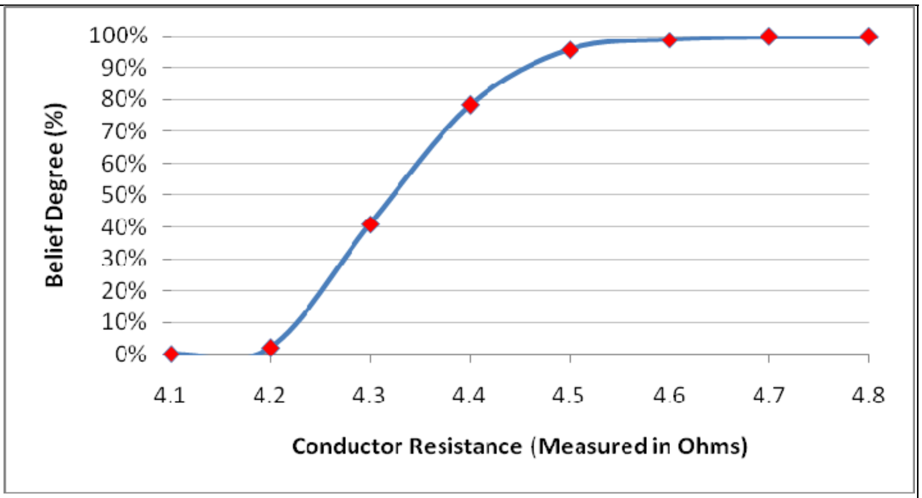
Completed Questionnaire

- ***Completed by Manager***



Ohms	%
< 4.1	0%
>= 4.1 < 4.2	2%
>= 4.2 < 4.3	39%
>= 4.3 < 4.4	37%
>= 4.4 < 4.5	17%
>= 4.5 < 4.6	3%
>= 4.6 < 4.7	1%
>= 4.7 < 4.8	0%

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Please place all the points and their corresponding cumulative percentages on the graph.

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The blank chart above requires you to describe a manufacturing process designed to produce a wire conductor resistance of 4.61Ω (i.e. the specified upper limit), using the machine 21252.

Q1. What do you think is the minimum Conductor Resistance of the process output? ____4.18______ Ω

Q2. What do you think is the maximum Conductor Resistance of the process output? _____4.61_____ Ω

Q3. From your experience with working on the process, what is the mean Conductor Resistance of the process output with a specified Upper Limit of 4.61Ω? _____4.34_______

Q4. What percentages of values are likely to be less than the mean conductor resistance you specified? ____55.57_______%

Q5. How likely is the process to take Conductor Resistance measurements between (M-0.3 Ω) and (M + 0.3Ω)? _____99.98____%

Where M stands for the mean you have chosen in Q4,

Q6. What percentages of the Conductor Resistance measurements will be below M-0.3Ω? ___0.0001136__________%

Q7. What percentages of the Conductor Resistance measurements will be above M+0.3 Ω? ____0.0001147_________%

Q8. Is there any other possible values you regard the process output is likely to take? _Sample size too small to give definite answer __

Q9. Please give the percentage (%) of values likely to fall below the value selected in Q8.? ____N/A_______%

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THANK YOU FOR YOUR CO-OPERATION

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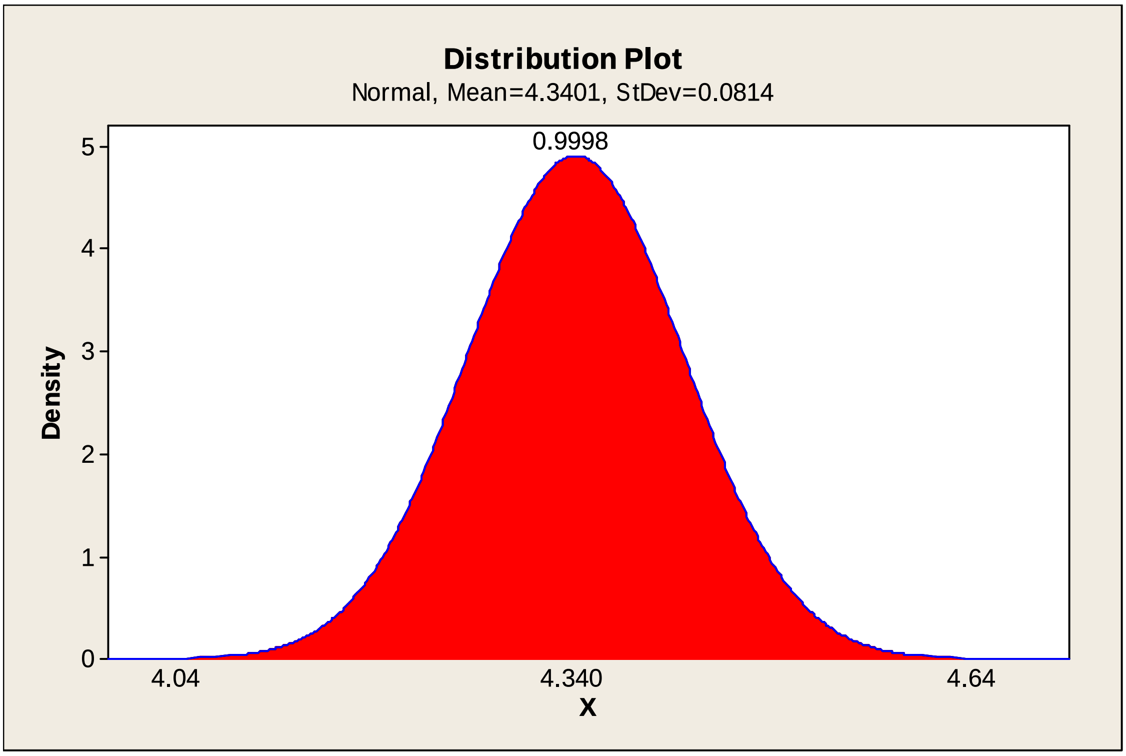
Distribution Plot
Normal, Mean=4.3401, StDev=0.0814
0.9998
Density
0
1
2
3
4
5
4.04
4.340
4.64
X