Quantitative Methods for Economics

Tutorial 2

Katherine Eyal

TUTORIAL 2

2nd-6th August 2010

ECO3021S

PART 1

1. Use Gaussian elimination to solve the system $\mathbf{Ax} = \mathbf{b}$, where
$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$

2. Use Gaussian elimination to invert the matrices

$$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 5 \\ 1 & 7 & 5 \\ 5 & 10 & 15 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 4 \\ 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix}$$

   Hence solve the systems of equations:

   (a) $3x_1 - x_2 = 5, \;\; 2x_1 + x_2 = 15$

   (b) $3x_2 + 4x_3 = 5, \;\; 2x_1 + 3x_3 = -1, \;\; -x_1 + x_2 = 2$

3. Given the following augmented matrix

$$\mathbf{A} = \left[\begin{array}{ccc|c} 6 & 8 & 6 & 6 \\ 8 & -2 & 5 & 3 \\ 4 & 4 & 4 & 0 \\ 5 & 6 & 5 & k \end{array}\right]$$

   (a) Find the row echelon form of $\mathbf{A}$.

   (b) Explain the concept of rank (briefly) and give the ranks of the coefficient and augmented matrices.

   (c) Determine which value of k will make this system of equations consistent.

   (d) Using this value of k, solve for the values of x_1, x_2 and x_3.

4. Solve the following system of equations, using the inverse method:

$$\begin{aligned} 6x + 5y &= 2 \\ x + y &= -3 \end{aligned}$$

5. Solve the following system of equations, using Cramer's Rule:

$$\begin{aligned}
w + 2y + z &= 4 \\
w - x + 2z &= 12 \\
2w + x + z &= 12 \\
w + 2x + y + z &= 12
\end{aligned}$$

6. The system of equations below describes the market for Blah Blah shoes:

$$\begin{aligned}
Q_d &= \alpha - \beta P + \gamma G \\
Q_s &= -\delta + \theta P - \lambda N \\
Q_d &= Q_s \qquad\qquad \alpha, \beta, \gamma, \delta, \theta, \lambda > 0
\end{aligned}$$

where G is the price of substitutes (e.g. Blah More shoes) and N is the price of inputs.

(a) Identify which variables in this model are endogenous, and which are exogenous.

(b) Use the inverse method to solve for the equilibrium price and quantity.

PART 2

7. Use Gaussian elimination to solve the system $\mathbf{Ax} = \mathbf{b}$ in each of the following cases.

(a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 4 & 1 \\ 4 & 3 & 3 & 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

8. Solve the following system of equations, using the inverse method:

$$\begin{aligned}
x + 3y + 3z &= 7 \\
2x + y + z &= 4 \\
x + y + z &= 4
\end{aligned}$$

9. Solve the following system of equations, using Cramer's Rule:

$$\begin{aligned} 2x + y &= 5 \\ 3x - y &= 0 \end{aligned}$$

ADDITIONAL QUESTIONS

10. All three kinds of elementary row operation (see p.12 of Section 2) can be represented as matrix multiplication. For the 3x3 case, show how:

    - Multiplying the second row by λ
    - Adding the second and third rows
    - Swopping the first and second rows

    can be represented by matrix multiplication.

11. You know now that: i. The rank of a matrix is the number of pivots in row echelon form, ii. An nxn square matrix is singular if it's determinant is equal to zero, iii. A matrix is singular if its rows are not linearly independent (i.e. one or more rows can be written as a linear combination of the others). Explain, using these facts and what you know about calculating the determinant, why matrices that look like the following example in row echelon form (see p.21 of Section 2) must be singular.

$$\begin{bmatrix} 1 & 0 & -\frac{1}{11} \\ 0 & 1 & \frac{4}{11} \\ 0 & 0 & 0 \end{bmatrix}$$

TUTORIAL 2 SOLUTIONS

2010

ECO3021S

PART 1

1. Use Gaussian elimination to solve the system $\mathbf{Ax} = \mathbf{b}$, and discuss the ranks of $\mathbf{A}$ and the augmented matrix.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 2 \\ 8 & 9 & 10 & 4 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -\frac{3}{2} & -3 & -\frac{1}{2} \\ 0 & -3 & -6 & 0 \end{array}\right] \quad \begin{array}{l} (2) - \frac{5}{2} \times (1) \\ (3) - 4 \times (1) \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -\frac{3}{2} & -3 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{array}\right] \quad (3) - 2 \times (2)$$

2. Use Gaussian elimination to invert the matrices

$$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 5 \\ 1 & 7 & 5 \\ 5 & 10 & 15 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 4 \\ 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix}$$

Hence solve the systems of equations:

(a) $3x_1 - x_2 = 5, \; 2x_1 + x_2 = 1$

(b) $3x_2 + 4x_3 = 5, \; 2x_1 + 3x_3 = -1, \; -x_1 + x_2 = 2$

Let $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 7 & 5 \\ 5 & 10 & 15 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & 3 & 4 \\ 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix}$.,

To find $\mathbf{A}^{-1}$:

$$\left[\begin{array}{cc|cc} 3 & -1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{array}\right]$$
$$\vdots$$
$$\rightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{3}{5} \end{array}\right]$$
$$\therefore \quad \mathbf{A}^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$$

To find $\mathbf{B}^{-1}$:

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 1 & 7 & 5 & 0 & 1 & 0 \\ 5 & 10 & 15 & 0 & 0 & 1 \end{array}\right]$$
$$\vdots$$
$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{11}{8} & -\frac{1}{8} & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{5}{8} & -\frac{1}{8} & -\frac{1}{10} \end{array}\right]$$
$$\therefore \quad \mathbf{B}^{-1} = \begin{bmatrix} -\frac{11}{8} & -\frac{1}{8} & \frac{1}{2} \\ -\frac{1}{4} & \frac{1}{4} & 0 \\ \frac{5}{8} & -\frac{1}{8} & -\frac{1}{10} \end{bmatrix}$$

To find $\mathbf{C}^{-1}$:

$$\left[\begin{array}{ccc|ccc} 0 & 3 & 4 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$
$$\vdots$$
$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & -9 \\ 0 & 1 & 0 & 3 & -4 & -8 \\ 0 & 0 & 1 & -2 & 3 & 6 \end{array}\right]$$
$$\therefore \quad \mathbf{C}^{-1} = \begin{bmatrix} 3 & -4 & -9 \\ 3 & -4 & -8 \\ -2 & 3 & 6 \end{bmatrix}$$

(a) In matrix form

$$\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

The coefficient matrix is $\mathbf{A}$ and we have calculated $\mathbf{A}^{-1}$. So

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \mathbf{A}^{-1} \begin{bmatrix} 5 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 5 \\ 15 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 7 \end{bmatrix} \end{aligned}$$

(b) In matrix form

$$\begin{bmatrix} 0 & 3 & 4 \\ 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

The coefficient matrix is $\mathbf{C}$ and we have calculated $\mathbf{C}^{-1}$. So

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \mathbf{C}^{-1} \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -4 & -9 \\ 3 & -4 & -8 \\ -2 & 3 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} \end{aligned}$$

3. Given the following augmented matrix

$$\mathbf{A} = \left[\begin{array}{ccc|c} 6 & 8 & 6 & 6 \\ 8 & -2 & 5 & 3 \\ 4 & 4 & 4 & 0 \\ 5 & 6 & 5 & k \end{array}\right]$$

(a) Find the row echelon form of $\mathbf{A}$.

(b) Explain the concept of rank (briefly) and give the ranks of the coefficient and augmented matrices.

(c) Determine which value of k will make this system of equations consistent.

(d) Using this value of k, solve for the values of x_1, x_2 and x_3.

(a) Row echelon form:

$$\mathbf{A} = \left[\begin{array}{ccc|c} 6 & 8 & 6 & 6 \\ 0 & -\frac{4}{3} & 0 & -4 \\ 0 & 0 & -3 & 33 \\ 0 & 0 & 0 & k-3 \end{array}\right]$$

Row echelon form (fraction-free):

$$\mathbf{A} = \left[\begin{array}{ccc|c} 6 & 8 & 6 & 6 \\ 0 & -76 & -18 & -30 \\ 0 & 0 & -24 & 264 \\ 0 & 0 & 0 & -24k+72 \end{array}\right]$$

(b) The rank of a matrix is the maximal number of linearly independent columns of the matrix. The rank = number of pivots = number of non-zero rows = number of basic columns.

The rank of the coefficient matrix is 3.

The rank of the augmented matrix is 4.

(c) The system of equations is consistent for $k = 3$.

(d)

$$\mathbf{x} = \begin{bmatrix} 8 \\ 3 \\ -11 \end{bmatrix}$$

4. Solve the following system of equations, using the inverse method:

$$\begin{aligned} 6x + 5y &= 2 \\ x + y &= -3 \end{aligned}$$

In matrix form

$$\mathbf{A} = \begin{bmatrix} 6 & 5 \\ 1 & 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Calculate $\mathbf{A}^{-1}$ (using either Gaussian elimination or "adj-over-det" formula):

$$\mathbf{A}^{-1} = \begin{bmatrix} 1 & -5 \\ -1 & 6 \end{bmatrix}$$

Then

$$\begin{aligned} \mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & -5 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 17 \\ -20 \end{bmatrix} \end{aligned}$$

5. Solve the following systems of equations, using Cramer's Rule:

$$\begin{aligned} w + 2y + z &= 4 \\ w - x + 2z &= 12 \\ 2w + x + z &= 12 \\ w + 2x + y + z &= 12 \end{aligned}$$

In matrix form

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & -1 & 0 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 12 \\ 12 \end{bmatrix}$$

$$\begin{aligned}
|\mathbf{A}| &= 1\begin{vmatrix}-1 & 0 & 2\\ 1 & 0 & 1\\ 2 & 1 & 1\end{vmatrix} - 0 + 2\begin{vmatrix}1 & -1 & 2\\ 2 & 1 & 1\\ 1 & 2 & 1\end{vmatrix} - 1\begin{vmatrix}1 & -1 & 0\\ 2 & 1 & 0\\ 1 & 2 & 1\end{vmatrix}\\
&= 1\left[-0 + 0 - 1\begin{vmatrix}-1 & 2\\ 1 & 1\end{vmatrix}\right] + 2\left[1\begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix} + 1\begin{vmatrix}2 & 1\\ 1 & 1\end{vmatrix} + 2\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}\right] - 1\left[0 - 0 + 1\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix}\right]\\
&= 1\,[-1\,(-3)] + 2\,[1\,(-1) + 1\,(1) + 2\,(3)] - 1\,[1\,(3)]\\
&= 3 + 2\,(6) - 3\\
&= 12 \neq 0 \therefore \text{ unique solution exists}
\end{aligned}$$

Solve for w :

$$\begin{aligned}
|\mathbf{A}_1| &= \begin{vmatrix}4 & 0 & 2 & 1\\ 12 & -1 & 0 & 2\\ 12 & 1 & 0 & 1\\ 12 & 2 & 1 & 1\end{vmatrix}\\
&= 2\begin{vmatrix}12 & -1 & 2\\ 12 & 1 & 1\\ 12 & 2 & 1\end{vmatrix} - 0 + 0 - 1\begin{vmatrix}4 & 0 & 1\\ 12 & -1 & 2\\ 12 & 1 & 1\end{vmatrix}\\
&= 2\,[12\,(-1) + 1\,(0) + 2\,(12)] - 1\,[4\,(-3) - 0 + 1\,(24)]\\
&= 2\,(12) - 1\,(12)\\
&= 12
\end{aligned}$$

$$w = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{12}{12} = 1$$

Solve for x :

$$\begin{aligned}
|\mathbf{A}_2| &= \begin{vmatrix}1 & 4 & 2 & 1\\ 1 & 12 & 0 & 2\\ 2 & 12 & 0 & 1\\ 1 & 12 & 1 & 1\end{vmatrix}\\
&= 2\begin{vmatrix}1 & 12 & 2\\ 2 & 12 & 1\\ 1 & 12 & 1\end{vmatrix} - 0 + 0 - 1\begin{vmatrix}1 & 4 & 1\\ 1 & 12 & 2\\ 2 & 12 & 1\end{vmatrix}\\
&= 2\,[1\,(0) - 2\,(-12) + 1\,(-12)] - 1\,[1\,(-12) - 1\,(-8) + 2\,(-4)]\\
&= 2\,(12) - 1\,(-12)\\
&= 36
\end{aligned}$$

$$x = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{36}{12} = 3$$

Solve for y :

$$\begin{aligned}
|\mathbf{A}_3| &= \begin{vmatrix} 1 & 0 & 4 & 1 \\ 1 & -1 & 12 & 2 \\ 2 & 1 & 12 & 1 \\ 1 & 2 & 12 & 1 \end{vmatrix} \\
&= 1 \begin{vmatrix} -1 & 12 & 2 \\ 1 & 12 & 1 \\ 2 & 12 & 1 \end{vmatrix} - 0 + 4 \begin{vmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 1 & -1 & 12 \\ 2 & 1 & 12 \\ 1 & 2 & 12 \end{vmatrix} \\
&= 1\left[-1\left(0\right) - 1\left(-12\right) + 2\left(-12\right)\right] + 4\left[1\left(-1\right) + 1\left(1\right) + 2\left(3\right)\right] - 1\left[1\left(-12\right) + 1\left(12\right) + 12\left(3\right)\right] \\
&= 1\left(-12\right) + 4\left(6\right) - 1\left(36\right) \\
&= -24
\end{aligned}$$

$$y = \frac{|\mathbf{A}_3|}{|\mathbf{A}|} = \frac{-24}{12} = -2$$

Solve for z :

$$\begin{aligned}
|\mathbf{A}_4| &= \begin{bmatrix} 1 & 0 & 2 & 4 \\ 1 & -1 & 0 & 12 \\ 2 & 1 & 0 & 12 \\ 1 & 2 & 1 & 12 \end{bmatrix} \\
&= 2 \begin{vmatrix} 1 & -1 & 12 \\ 2 & 1 & 12 \\ 1 & 2 & 12 \end{vmatrix} - 0 + 0 - 1 \begin{vmatrix} 1 & 0 & 4 \\ 1 & -1 & 12 \\ 2 & 1 & 12 \end{vmatrix} \\
&= 2\left[1\left(-12\right) + 1\left(12\right) + 12\left(3\right)\right] - 1\left[1\left(-24\right) - 0 + 4\left(3\right)\right] \\
&= 2\left(36\right) - 1\left(-12\right) \\
&= 84
\end{aligned}$$

$$z = \frac{|\mathbf{A}_4|}{|\mathbf{A}|} = \frac{84}{12} = 7$$

Therefore

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \\ 7 \end{bmatrix}$$

6. The system of equations below describes the market for Blah Blah shoes:

$$\begin{aligned} Q_d &= \alpha - \beta P + \gamma G \\ Q_s &= -\delta + \theta P - \lambda N \\ Q_d &= Q_s \qquad\qquad\qquad \alpha, \beta, \gamma, \delta, \theta, \lambda > 0 \end{aligned}$$

where G is the price of substitutes (e.g. Blah More shoes) and N is the price of inputs.

(a) Identify which variables in this model are endogenous, and which are exogenous.

(b) Use the inverse method to solve for the equilibrium price and quantity.

(a) Endogenous: Q_d, Q_s, P
Exogenous: G, N

(b) Re-arrange equations:

$$\begin{aligned} Q^d + \beta P &= \alpha + \gamma G \\ Q^s - \theta P &= -\delta - \lambda N \\ Q^d - Q^s &= 0 \end{aligned}$$

Put into matrix form:

$$\mathbf{Ax} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 0 & \beta \\ 0 & 1 & -\theta \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix} = \begin{bmatrix} \alpha + \gamma G \\ -\delta - \lambda N \\ 0 \end{bmatrix}$$

Find $\mathbf{A}^{-1}$:

$$\begin{aligned} |\mathbf{A}| &= 1(0 - \theta) - 0 + 1(-\beta) \\ &= -(\beta + \theta) \\ &\neq 0 \text{ as long as } \beta \neq -\theta \text{ (It is important that you specify this restriction!)} \end{aligned}$$

$$
\begin{aligned}
\mathbf{C} &= \begin{bmatrix} +\begin{vmatrix} 1 & -\theta \\ -1 & 0 \end{vmatrix} & -\begin{vmatrix} 0 & -\theta \\ 1 & 0 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} \\ -\begin{vmatrix} 0 & \beta \\ -1 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & \beta \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 0 & \beta \\ 1 & -\theta \end{vmatrix} & -\begin{vmatrix} 1 & \beta \\ 0 & -\theta \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \end{bmatrix} \\
&= \begin{bmatrix} -\theta & -\theta & -1 \\ -\beta & -\beta & 1 \\ -\beta & \theta & 1 \end{bmatrix} \\
adj\,\mathbf{A} &= \mathbf{C}' \\
&= \begin{bmatrix} -\theta & -\beta & -\beta \\ -\theta & -\beta & \theta \\ -1 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{A}^{-1} &= \frac{1}{|\mathbf{A}|} adj\,\mathbf{A} \\
&= \frac{1}{-(\beta+\theta)} \begin{bmatrix} -\theta & -\beta & -\beta \\ -\theta & -\beta & \theta \\ -1 & 1 & 1 \end{bmatrix} \\
&= \frac{1}{(\beta+\theta)} \begin{bmatrix} \theta & \beta & \beta \\ \theta & \beta & -\theta \\ 1 & -1 & -1 \end{bmatrix}
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbf{x} &= \mathbf{A}^{-1}\mathbf{b} \\
\begin{bmatrix} Q^d \\ Q^s \\ P \end{bmatrix} &= \frac{1}{(\beta+\theta)} \begin{bmatrix} \theta & \beta & \beta \\ \theta & \beta & -\theta \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} \alpha+\gamma G \\ -\delta-\lambda N \\ 0 \end{bmatrix} \\
&= \frac{1}{(\beta+\theta)} \begin{bmatrix} \theta(\alpha+\gamma G) - \beta(\delta+\lambda N) \\ \theta(\alpha+\gamma G) - \beta(\delta+\lambda N) \\ \delta+\lambda N+\alpha+\gamma G \end{bmatrix}
\end{aligned}
$$

PART 2

7. Use Gaussian elimination to solve the system $\mathbf{Ax} = \mathbf{b}$ in each of the following cases, and discuss the ranks of $\mathbf{A}$ and the augmented matrix.

   (a) $\mathbf{A} = \begin{bmatrix} 2 & -1 & 4 & 1 \\ 4 & 3 & 3 & 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$$\begin{aligned} & \left[\begin{array}{cccc|c} 2 & -1 & 4 & 1 & 0 \\ 4 & 3 & 3 & 7 & -1 \end{array}\right] \\ \rightarrow & \left[\begin{array}{cccc|c} 2 & -1 & 4 & 1 & 0 \\ 0 & 5 & -5 & 5 & -1 \end{array}\right] \quad (2) - 2 \times (1) \end{aligned}$$

Columns 1 and 2 are basic, so we assign arbitrary values to the non-basic columns, $x_3 = \lambda, x_4 = \mu$. Then

$$\begin{aligned} 5x_2 - 5\lambda + 5\mu &= -1 \\ \Rightarrow x_2 &= -\frac{1}{5} + \lambda - \mu \end{aligned}$$

$$\begin{aligned} 2x_1 - x_2 + 4\lambda + \mu &= 0 \\ \Rightarrow x_1 &= \frac{1}{2}\left(-\frac{1}{5} + \lambda - \mu - 4\lambda - \mu\right) \\ &= \frac{1}{2}\left(-\frac{1}{5} - 3\lambda - 2\mu\right) \\ &= -\frac{1}{10} - \frac{3}{2}\lambda - \mu \end{aligned}$$

The complete solution in vector form is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{10} \\ -\frac{1}{5} \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 1 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(b) $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 2 \\ 8 & 9 & 10 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -\frac{3}{2} & -3 & -\frac{1}{2} \\ 0 & -3 & -6 & -1 \end{array}\right] \quad \begin{array}{l} (2) - \frac{5}{2} \times (1) \\ (3) - 4 \times (1) \end{array}$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -\frac{3}{2} & -3 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (3) - 2 \times (2)$$

We can ignore the last row of zeros and solve the system above it. The first and second columns are basic, so we let $x_3 = \lambda$. Then

$$\begin{aligned} -\frac{3}{2}x_2 - 3\lambda &= -\frac{1}{2} \\ \Rightarrow x_2 &= -\frac{2}{3}\left(-\frac{1}{2} + 3\lambda\right) \\ &= \frac{1}{3} - 2\lambda \end{aligned}$$

$$\begin{aligned} 2x_1 + 3x_2 + 4\lambda &= 1 \\ \Rightarrow x_1 &= \frac{1}{2}\left(1 - 3\left(\frac{1}{3} - 2\lambda\right) - 4\lambda\right) \\ &= \frac{1}{2}(1 - 1 + 2\lambda) \\ &= \lambda \end{aligned}$$

The complete solution in vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

8. Solve the following system of equations, using the inverse method:

$$
\begin{aligned}
x + 3y + 3z &= 7 \\
2x + y + z &= 4 \\
x + y + z &= 4
\end{aligned}
$$

No solution. Equations inconsistent (compare the second and third equations).

9. Solve the following system of equations, using Cramer's Rule:

$$
\begin{aligned}
2x + y &= 5 \\
3x - y &= 0
\end{aligned}
$$

In matrix form

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

$$|\mathbf{A}| = -5 \neq 0 \therefore \text{ unique solution exists}$$

Solve for x :

$$|\mathbf{A}_1| = \begin{vmatrix} 5 & 1 \\ 0 & -1 \end{vmatrix} = -5$$

$$x = \frac{|\mathbf{A}_1|}{|\mathbf{A}|} = \frac{-5}{-5} = 1$$

Solve for y :

$$|\mathbf{A}_2| = \begin{vmatrix} 2 & 5 \\ 3 & 0 \end{vmatrix} = -15$$

$$y = \frac{|\mathbf{A}_2|}{|\mathbf{A}|} = \frac{-15}{-5} = 3$$