AN ELECTRONIC ANALOGUE COMPUTER FOR THE SOLUTION OF NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS ENCOUNTERED IN THE STUDY OF THE SELF-HEATING OF FISHMEAL

by

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INCLUSION OF PUBLISHED MATERIAL

This thesis includes material from a paper by the author with the same title as the thesis, which was read at the First Symposium on Automation and Computation held at the University of the Witwatersrand on the 25th September, 1963. The Symposium was convened by the South African Council for Automation and Computation.

This paper, together with the discussion which followed, will be published in the Proceedings of the First Symposium on Automation and Computation.

The paper contained a summary of the principle of the computer described in this thesis.

This material is included in the thesis with the permission of the Supervisors :

J.L.N. Besseling

N.C. de V. Enslin
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SUMMARY

A discussion of the non-linear, partial differential equations encountered in the study of the self-heating of fishmeal and a review of existing computer methods for their solution are given. A new electronic analogue computer is described which was designed to overcome certain disadvantages of the existing instruments when applied to these equations.

The computer uses an explicit finite difference method of solution to minimize the amount of equipment required. Using analogue storage and computation techniques, the advantages of the analogue method of ease of non-linear function generation and graphical solution display are obtained.

The experimental computer consists of a single computing section which is used to perform all the calculations required on data held in a capacitor store. A compensation technique is used to compensate for charge leakage from the capacitor store and contributes to the accuracy of the instrument.

Solutions of some linear and non-linear partial differential equations are calculated to illustrate the operation of the computer.

The possibility of extending these computing techniques to other partial differential equations is discussed.

Page

OMPUTER

onal Amplifier

Generator

Circuit

itions Circuit

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_4 e^{\varphi}$$

1A 4A 4A 1B 1B 3B 8B 10B 12B 14B 16B 16B 18B 1C

LIST OF SYMBOLS

MATHEMATICAL SYMBOLS

φ	=	reduced temperature, non-dimensional
t	=	reduced time, non-dimensional
x	=	reduced space dimension, non-dimensional
Δt	=	reduced time interval.
Δx	=	reduced space interval
m	=	number of space intervals, integer
n	=	number of time intervals, integer
K	=	coefficient
δ	=	fractional error
$O()$	=	notation indicating error of the order of the bracketed quantity.

PRINCIPAL COMPONENT SYMBOLS

C_a	=	auxiliary store capacitance
C_m	=	main store capacitance
C_t	=	integrator transfer capacitance
G_c	=	common-mode gain of the differential amplifier
G_d	=	differential gain of the differential amplifier
R_a	=	input resistance to the negative gain input of the amplifier
R_b	=	feedback resistance of the amplifier
R_c	=	leakage compensation resistance
R_d, R_e & R_f	=	voltage divider resistances connected to the positive gain input of the amplifier.
V	=	voltage, a suffix indicates the component across which the voltage is measured.

1 INTRODUCTION

1.1 MATHEMATICAL PROBLEMS IN THE STUDY OF SELF-HEATING

The analogue computer described in this thesis was built to solve non-linear partial differential equations, encountered in the study of the self-heating of fishmeal.

Self-heating or spontaneous combustion can occur in large stacks of fishmeal which has a low thermal conductivity and generates heat within itself by oxidation of the residual oil.[1]

A study of the self-heating of fishmeal has been attempted using a development of a method successfully used for coal by van Doornum.[2] With this method the problem is divided into two parts.

1 A study of the heat generation characteristics of the material under idealized conditions. Here, the rate of heat generation under fixed and abruptly changing temperature conditions are measured using an isothermal calorimeter,[3,4] and a single non-linear mathematical expression giving the rate of heat generation as a function of temperature and time is deduced.[5]

2 A study of the physical environment of a particle in a stack where it is apparent that at least three physical mechanisms are involved. These are :-

a The diffusion of heat generated by the material from the inside to the surface of the stack.

b The diffusion of water vapour, associated with the constants are incorporated in the constants relating the original and reduced variables. The simplest diffusion equation encountered in the present study takes the form :

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + f(\varphi,t) \qquad \ldots \qquad 1.1$$

This equation describes the temperature of a slab with internal heat generation given by the reduced function $f(\varphi,t)$, where φ = reduced temperature rise, t = reduced time and x = reduced space dimension.

The solutions required are the variation of temperature with time in the stack for a specified heat generation function and initial and boundary conditions. Such solutions are best presented graphically as sets of curves giving :-

a The spacial temperature distribution at successive instants of time.

b The variation of temperature with time at different points in the stack.

An accuracy of solution of the order of 5% is adequate as the accuracy of the experimental data is limited to a similar figure.

The time required for the solution of the non-linear differential equations by manual analytical or numerical methods may be considerable. Replacing the non-linear functions by simpler expressions, to simplify the problem of solving the equations, is not desirable as this may completely alter the character of the solution[6]. As the research progresses and more insight into the non-linear problems is obtained, the form of the equations being studied may have to be modified and new solutions obtained as a guide to further research. It is evident that a method for the rapid solution of partial differential equations involving arbitrary non-linear functions is required.

1.2 METHODS AVAILABLE FOR THE SOLUTION OF QUADRATIC PARTIAL DIFFERENTIAL EQUATIONS

The diffusion equation is a special case of the general class of quadratic partial differential equations. General methods developed for the solution of quadratic partial differential equations can be used to solve equations encountered in the study of self-heating. For the purpose of this thesis, the various methods will be discussed with reference to equation 1.1.

Analytical methods exist for the solution of linear and certain non-linear quadratic partial differential equations. Reviews of these methods are given by Carslaw and Jaeger[6] and Morse and Feshbach[7]. Unfortunately most non-linear partial differential equations cannot be solved using analytical methods.

Finite difference methods have been developed for the solution of non-linear partial differential equations.[8,9,10] In the finite difference method one or more of the independent variables are split into a finite number of discrete intervals. In solving the problem attention is limited to the points defining the intervals and solutions are only obtained at these points. These methods may be classified into cases where :

1 <u>One variable is differenced while the other is continuous</u>

If the second order space derivative $\frac{\partial^2 \varphi}{\partial x^2}$ is replaced by finite differences, the partial differential equation is reduced to a set of simultaneous, first order differential equations which are functions of φ and t only. These equations may be solved using a parallel method, where a separate computer is used for each equation and the solution is produced by all the computers suitably coupled and acting simultaneously. Alternatively, a serial method of solution may be used where a single computer is used to solve the equations in succession using previously computed information.[11]

If the first time derivative, $\frac{\partial \varphi}{\partial t}$ is replaced by finite differences, the partial differential equation is reduced to a set of simultaneous second order differential equations which are functions of φ and x only. These equations may similarly be solved using either parallel or serial methodsof calculation.

An analysis of these four methods by McKay and Fisher[11] has shown that errors due to noise tend to build up in solutions obtained using the three last mentioned methods, so that the first method, in which the second space derivative is differenced and the equations solved in parallel, is preferable.

The resistive-reactive analogues and differential analyser techniques are based on this method (see section 1.4)

2 <u>Both variables are differenced.</u>

Using an explicit method, the value φ at the next time instant $t + \Delta t$ is calculated directly from the known values at instant t.

The disadvantage of the explicit method is that the calculations may be unstable because errors build up unless the choice of time and space intervals is restricted.

Using an implicit method, such as the Crank and Nicholson method[3], the values of the function at time $t + \Delta t$ for all nodes in the x interval are calculated from the values of the function at time t, by solving a set of simultaneous equations. Although this method involves a great deal of work, it has the advantage of being stable for all values of the time and space intervals so that large time intervals can be used and consequently less steps are required.

The implicit method is used in the resistance network analogue computer (see section 1.4), while both implicit and explict methods are used for digital calculations.

Manual solution of partial differential equations by finite difference methods is laborious and becomes impractical if more than one or two solutions are required so that a digital or analogue computer must be used.

1.3 COMPARISON OF DIGITAL AND ANALOGUE COMPUTERS

In the digital computer, information is handled as discrete quantities. The instrument can perform simple arithmetic and logical operations with great rapidity. Provision is made for storing large amounts of information, comprising the data being operated on and the programme of arithmetical operations to be performed. As a large amount of basic equipment must be provided these instruments are necessarily large and expensive and only large organisations can justify these instruments.

The digital computer has the advantages of versatility and almost unlimited accuracy. Any problem can be solved if it can be broken down into a succession of simple arithmetical steps, while the accuracy is determined by the number of digits carried. However, the speed of solution decreases as the accuracy or complexity of a problem increases.

The digital computer is ideally suited for the solution of problems which require the processing of large amounts of data involving relatively simple operations. For the solution of partial differential equations, such as the diffusion problems encountered here, it is necessary to discretize the problem using finite difference techniques. A large number of arithmetical operations are required so that although the individual machine operations are rapid the solution may take a long time. This disadvantage is aggravated when non-linear functions are encountered, as these must be calculated from tabulated values by interpolation, or approximated by mathematical functions, which are usually calculated from series expansions. In addition to the machine time required considerable time may be required to write out the programme of the operations to be performed. This aspect presents a special problem when the nature of the non-linear functions are liable to change. A further disadvantage is that the solutions are produced in numerical form. If a graphical display is required either additional equipment or labour is needed.

It is apparent that the digital computer is not ideally suited for the solution of non-linear partial differential equations.

In the analogue computer[12,13] information is handled in a continuous manner. This automatically limits the accuracy of the solution to the accuracy of the components and measuring instruments used. The maximum accuracy which can be achieved in practice is 0.01% while the majority of instruments have a solution accuracy of the order of 1.0%.

Analogue computers may be classified into two classes according to their basic principle of operation.

In the first class, termed direct analogues, use is made of the existence of a direct physical analogue between the analogue and the system being studied. Such an analogy may be recognised by comparing the equations describing the two systems.

In the second class, termed indirect analogues, the computer is used to solve the mathematical equations of the system being studied by interconnecting components, each performing a particular mathematical operation in such a manner as to produce a solution of the equation. Where the equations solved are differential equations, the instrument is known as a differential analyser. The electronic differential analyser, popularly called the electronic analogue computer, is the most widely used and versatile instrument in this class.[14,15]

One of the advantages of the analogue techniques is that comparatively simple devices can solve quite complex problems. While these devices are simple and cheap, the range of problems solvable by a particular type of instrument is usually limited to a particular equation or class of equations. The versatile general purpose electronic analogue computer loses the advantage of simplicity as the size of the installation is increased. Large installations of this type approach the complexity and cost of digital systems.

Analogue computers are inherently fast as each mathematical operation is performed by a separate component or unit, with all units operating simultaneously. The non-linear functions and other parameters of the equation are readily adjusted. The introduction of non-linearities, using non-linear function generators, presentsno fundamental problems although the complexity and cost of the instrument may be increased. As the information is handled in continuous form within the computer, the solutions are easily presented in the required graphical form.

A third type of computer, the digital differential analyser, combines the ease of problem solution of the general purpose analogue computer with the accuracy of the digital computer. This instrument is designed to solve algebraic and differential equations by interconnecting components, each performing a particular mathematical operation, to produce a solution as in the indirect analogue computer. However, within these components the variables are handled by digital and not analogue methods so that greater accuracy may be obtained. The principle disadvantage of this type of instrument has been that, even when working to the same accuracy (0.1%) as the electronic analogue computer, these instruments were much slower in operation. Recently, instruments have been reported which can exceed the performance of the electronic analogue computer.[16] At present these instruments, which are comparable in size to a digital computer, are not widely used. At present there is no such instrument in South Africa.

The choice between digital and analogue computers is seen to be determined by the type of problem to be solved. If the problem involves the processing of large amount of data, the solution of a number of completely different types of equations or a solution accuracy of greater than 0.1%, the digital computer must be used. On the other hand, if a solution accuracy of 1% is sufficient and a large number of solutions for different parameter values of the same type of equation are required, solutions can possibly be obtained using a special purpose analogue computer. As the capital and running cost of such an analogue computer can be of the order of 1/100th to 1/1000th of the digital computer, it is possible to justify the cost of a special purpose computer for use with a particular major research project.

The self-heating problem falls into the second category, and this led to the investigation into the application of analogue techniques to the solution of this type of problem.

1.4 ANALOGUE TECHNIQUES

This section is a review of electrical analogues devloped for the solution of the diffusion or the more general parabolic partial differential equation. All these instruments use finite difference techniques and may conveniently be classified according to the particular finite difference method used (see section 1.2). For convenience the discussion will deal only with equations in one space dimension such as equation 1.1.

The first main class of instruments employ the technique where one independant variable is continuous and the other is differenced. As mentioned in section 1.2, the only method which is not subject to a build-up of error is that in which the space variable is differenced and a parallel method of solution is used. The resulting equations are similar to those describing an electrical line consisting of series resistors and shunt capacitors, where temperature is analogous to voltage and heat flow is analogous to current. The simple R.C. line analogue has been widely used in the study of transient heat conduction problems.[12,13,17] The equation may also be set up on a general purpose differential analyser, using summing integrators employing operational amplifiers, as described by Karplus[12] and Harbert.[18] When the heat diffusion equation with internal heat generation is solved using these methods, currents must be injected into each node. If the heat generation is a non-linear function of temperature and time, as in the present problems, a suitable non-linear function generator must be connected to each node to inject a current which is analogous to the heat generation at the corresponding point in material described by the equation. The serial method with the time variable differenced was used successfully by Hartree in the 1930's in his classical work using a mechanical differential analyser. He minimized the build up of error by selecting the best solution after each step. One of Hartree's solutions[19] has been recalculated as an example using the computer described in this thesis.

The second class of computers are characterized by having both the time and space variables differenced. Here, the most important recent contribution is the resistance network analyser proposed by Leibman.[21]

The analyser uses the implicit method. It consists of a simple resistance network to which potentials are applied corresponding to the values of dependant variable for a given instant of time. The network solves the set of simultaneous equations and the required values for the next time instant appear as voltages at the nodes of the network. These voltages are recorded and introduced at the inputs to the network so that the calculation proceeds in a stepwise fashion. Liebman subsequently extended this treatment to include equations describing situations with non-linear conductivity or internal sources.[22] The technique was further developed by Karplus,[23] who added active circuits based on operational amplifiers, so that negative as well as positive resistance values could be used, making possible the solution of more complicated forms of diffusion, wave and biharmonic equations. Forms including first space derivatives are also handled by this instrument.

In the solution of diffusion problems involving sources or sinks, currents must be injected into each node of the network corresponding to the source function. In the self-heating problem these currents would be non-linear functions of the node potential. Because an implicit method is used the node potentials are not known so that the node currents cannot immediately be adjusted to their correct values. The currents into each node must be re-adjusted sequentially and the process repeated until the node currents do not change. This tedious iterative process was eliminated by Karplus,[24] who connected an analogue function generator to each node, so that the apparatus adjusts itself to a solution instantaneously.

To reduce the number of function generators required to one, Hutcheon[25] and Altes[26] built computers employing an automatic iterative process. The single function generator is connected to successive nodes setting the node currents to the required values. After the function generator has moved on, the current at each node is maintained by a separate analogue storage device. The iteration process is continued until a steady solution is produced. The only equations studied by Hutcheon and Altes were of the elliptic type (Poissons equation), where the solution is obtained after one iteration process. It seems that this system could be applied to the Liebman resistive network computer and used for diffusion equations, where an iteration process would be required at each step corresponding to the time intervals. Such a combination of resistive network, based on the implicit Crank-Nicholson method, and a single shared function generator with some form of analogue storage for the solution of diffusions equations, similar to equation 1.1 was suggested by Dekker.[21] No details or results have yet been published.

Although analogue computers for the solution of the diffusion equation with internal sources have been devised, none of these satisfy the requirements of simplicity, rapid operation and that the solutions are produced in the graphical form required. As the non-linear function generator is often the most complex part of the analogue computer the duplication of these items, required by the continuous-discrete and some discrete-discrete methods, must add considerably to the cost and complexity of the instrument. The instruments using time shared function generators appear to be unnecessarily time-consuming when applied to the diffusion equations which is borne out by the fact that there appears to be no published record of such an application.

1.5 PROPOSED ANALOGUE COMPUTER

A new, special purpose, analogue computer is proposed for the solution of non-linear partial differential equations such as the diffusion equation with internal sources encountered in the study of self-heating. It is an attempt to meet the requirements of a relatively simple computer able to handle non-linear functions, and produce solutions directly in graphical form.

The proposed computer should use a finite difference method in which both time and space variables are differenced. It should consist of a single computing section which performs all the calculations required on data held in a store. The computing section can be time shared between all the store positions while the non-shared equipment should be kept as simple as possible to reduce the amount of equipment required. By using electronic analogue storage and calculation techniques the advantages inherent in the analogue method of ease of non-linear function generation and graphical solution display would be obtained.

It is proposed to use an explicit method in preference to an implicit method of calculation, so that the amount of information read from the store at any instant and hence the amount of computing equipment is minimized. The attendant problem of computational instability can be controlled by a proper choice of the space and time intervals used. With this method the node voltages would be stored in the analogue store and read when required. The node voltage for the following time instant would be calculated in the computing section and the modified value written back into the store. The computing section would then be switched to the next set of nodes and the process repeated so that the solution proceeds in a step-wise fashion. The required graphical solution display can be obtained by recording the store voltage against time.

The computing section can be based on conventional operational amplifier techniques, but because the calculations involve differences it appears logical to use computing circuits based on differential rather than single ended operational amplifiers.

These ideas form the basis of this thesis.

2 EXPLICIT FINITE DIFFERENCE METHOD

Before the partial differential equations can be handled on the computer, these equations must be rewritten in terms of finite differences. This section contains a brief review of the finite difference method sufficient to indicate the factors which must be taken into account in the design of a computer using the explicit finite difference method. The finite difference expressions required when setting up partial differential equations on the computer are quoted.

2.1 EXPRESSING PARTIAL DIFFERENTIAL EQUATIONS IN TERMS OF FINITE DIFFERENCES

If the values φ_m of a function $\varphi(x)$ are known at regular intervals of its argument $x = m\Delta x$, where $m = 0,1,2,3.....$, then the finite difference method is based on the principle that a polynomial can be found which passes through any specified number of these points. The coefficients of this polynomial can be expressed in terms of the known values of φ_m or their differences.[6] Any operations such as differentiation are then performed on this polynomial and the resulting expressions will be in terms of the known function or its differences. Thus the first derivative $(\frac{d\varphi}{dx})_m$ implies the first derivative of the interpolating polynomial at the point $m\Delta x$.

If the values of the function are known at the points $m\Delta x$, a large number of formulae are available which express its derivatives in terms of the known function or its differences. The form of these formulae depends on the type of difference expression (e.g forward, backward, central or mean differences[6]) used for the interpolating polynomial. The formulae consist of an infinite series of terms. If only the first one or two terms are used and the rest neglected a truncation error is introduced. The truncation error is discussed in Section 2.2.

Typical expressions for the first and second derivatives are given below. In these expressions only one or two terms have been used, but the order of magnitude of the truncation error introduced is indicated by the notation $O(\Delta x^2)$ which implies that the error is of the order of Δx^2.

Finite difference expressions for the first derivative

$$\left(\frac{d\varphi}{dx}\right)_{x=m\Delta x} = \left(\frac{d\varphi}{dx}\right)_m = \frac{1}{\Delta x}(\varphi_{m+1} - \varphi_m) + O(\Delta x) \quad \ldots \quad 2.1.1$$

$$= \frac{1}{2\Delta x}(-\varphi_{m+2} + 4\varphi_{m+1} - 3\varphi_m) + O(\Delta x^2) \quad \ldots \quad 2.1.1$$

$$= \frac{1}{2\Delta x}(\varphi_{m+1} - \varphi_{m-1}) + O(\Delta x^2) \quad \ldots \quad 2.1.3$$

Finite difference expressions for the second derivative

$$\left(\frac{d^2\varphi}{dx^2}\right)_{x=m\Delta x} = \left(\frac{d^2\varphi}{dx^2}\right)_m = \frac{1}{\Delta x^2}(\varphi_{m+2} - 2\varphi_{m+1} + \varphi_m) + O(\Delta x) \quad \ldots \quad 2.1.4$$

$$= \frac{1}{\Delta x^2}(\varphi_{m-1} - 2\varphi_m + \varphi_{m+1}) + O(\Delta x^2) \quad \ldots \quad 2.1.5$$

If the function $\varphi(x,t)$ is a function of two variables, partial differences may be defined in the same way. If we choose a time interval Δt in t and a space interval Δx in x and write $\varphi_{m,n}$ for the value of $\varphi(x,t)$ at the time $t = n\Delta t$ at the point $x = m\Delta x$, the space and time derivatives in equation 1.1 may be replaced using equations 2.1.5 and 2.1.1 giving :-

$$\left(\frac{\partial^2\varphi_n}{\partial x^2}\right)_{x=m\Delta x} = \frac{1}{\Delta x^2}(\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n}) + O(\Delta x^2) \quad \ldots \quad 2.1.6$$

and

$$\left(\frac{\partial\varphi_m}{\partial t}\right)_{t=n\Delta t} = \frac{1}{\Delta t}(\varphi_{m,n+1} - \varphi_{m,n}) + O(\Delta t) \quad \ldots \quad 2.1.7$$

If equations 2.1.6 and 2.1.7 are substituted in equation 1.1 which was

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + f(\varphi,t) \quad \ldots \quad 1.1$$

this equation becomes (considering non-linear functions of φ only)

$$\varphi_{m,n+1} - \varphi_{m,n} = \Delta\varphi_{m,n} = \frac{\Delta t}{\Delta x^2}(\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n}) + \Delta t\, f(\varphi_{m,n}) \quad \ldots \quad 2.1.8$$

If an equation including a 1st space derivative is considered such as

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K\frac{\partial\varphi}{\partial x} + f(\varphi) \quad \ldots \quad 2.1.9$$

The first space derivative can be represented by finite differences to the same order of accuracy as the second space derivative by using equation 2.1.3. Using equations 2.1.3 and 2.1.8 equation 2.1.9 becomes :-

$$\varphi_{m,n+1} - \varphi_{m,n} = \frac{\Delta t}{\Delta x^2}(\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n}) + \frac{\Delta t}{2\Delta x} K(\varphi_{m+1,n} - \varphi_{m-1,n})$$

$$+ \Delta t\ f(\varphi_{m,n}) \qquad \ldots \qquad 2.1\text{-}10$$

which may be simplified to

$$\Delta\varphi_{m,n} = \frac{\Delta t}{\Delta x^2}\left[(1 + \frac{K}{2}\Delta x)\ \varphi_{m+1,n} - 2\varphi_{m,n} + (1 - \frac{K}{2}\Delta x)\ \varphi_{m-1,n}\right]$$

$$+ \Delta t\ f(\varphi_{m,n}) \qquad \ldots \qquad 2.1\text{-}11$$

Before these equations can be solved the initial and boundary conditions must be specified. For the problems in this thesis only constant boundary conditions are considered corresponding to fixed surface temperatures φ_{b1} and φ_{b2} at the surfaces defined by $x = 0$ and $x = 2$. In problems where the temperature distribution is symmetrical about the central plane only half the slab need be considered. At the central plane defined by $x = m_c\Delta x = 1$, $\frac{d\varphi}{dx} = 0$ so that the centre temperature is $\varphi_{m_c,n}$ and the symmetry condition obtained using equation 2.1-3, is $\varphi_{m_c-1,n} = \varphi_{m_c+1,n}$.

The initial temperature distribution is assumed to be uniform so that $\varphi_{m,o} = \varphi_i$ at $t = 0$ where $m = 1,2,3.....$

2.2 TRUNCATION ERRORS

Truncation errors arise when higher order terms of finite difference expressions are neglected. The truncation error depends on the grid spacing chosen and on the nature of the equation being approximated. A knowledge of the nature of the truncation errors is essential so that an appropriate choice of grid spacing and number of terms of the finite difference expressions required to produce a solution of a certain accuracy may be determined. The manner in which the grid spacing and nature of the equation effect the truncation error may be found from a Taylor series expansion of the function at a point.[13]

A Taylor series expression for a function at a point x in terms of the function and its derivatives at a neighbouring point x_m is given by :-

$$\varphi = \varphi_m + (x - x_m)\left(\frac{\partial\varphi}{\partial x}\right)_m + \frac{(x - x_m)^2}{2!}\left(\frac{\partial^2\varphi}{\partial x^2}\right)_m + \frac{(x - x_m)^3}{3!}\left(\frac{\partial^3\varphi}{\partial x^3}\right)_m + \cdots\cdots \qquad \ldots \qquad 2.2\text{-}1$$

Calculating the functions φ_{m+1} and φ_{m-1} at points $x_{m+1} = x_m + \Delta x$ and $x_{m-1} = x_m - \Delta x$ in terms of the function and its derivatives at point x_m gives :-

$$\varphi_{m+1} = \varphi_m + \Delta x\left(\frac{\partial\varphi}{\partial x}\right)_m + \frac{\Delta x^2}{2!}\left(\frac{\partial^2\varphi}{\partial x^2}\right)_m + \frac{\Delta x^3}{3!}\left(\frac{\partial^3\varphi}{\partial x^3}\right)_m + \cdots\cdots \qquad \ldots \qquad 2.2\text{-}2$$

and

$$\varphi_{m-1} = \varphi_m - \Delta x\left(\frac{\partial\varphi}{\partial x}\right)_m + \frac{\Delta x^2}{2!}\left(\frac{\partial^2\varphi}{\partial x^2}\right)_m - \frac{\Delta x^3}{3!}\left(\frac{\partial^3\varphi}{\partial x^3}\right)_m + \cdots\cdots \qquad \ldots \qquad 2.2\text{-}3$$

Rewriting equation 2.2-2 to bring it into the same form as equation 2.1-1 gives :-

$$\left(\frac{\partial\varphi}{\partial x}\right)_m = \frac{1}{\Delta x}(\varphi_{m+1} - \varphi_m) - \frac{\Delta x}{2!}\left(\frac{\partial^2\varphi}{\partial x^2}\right)_m - \frac{\Delta x^2}{3!}\left(\frac{\partial^3\varphi}{\partial x^3}\right)_m + \cdots\cdots \qquad \ldots \qquad 2.2\text{-}4$$

which shows that the truncation error is proportional to Δx and the second derivative of the function at point $x = m\Delta x$.

Similarly equation 2.1-5 may be obtained by adding equations 2.2-2 and 2.2-3 giving :-

$$\left(\frac{\partial^2\varphi}{\partial x^2}\right)_m = \frac{1}{\Delta x^2}\left(\varphi_{m+1} - 2\varphi_m + \varphi_{m-1}\right) - \frac{\Delta x^2}{12}\left(\frac{\partial^4\varphi}{\partial x^4}\right)_m + \text{higher order terms} \quad \ldots \quad 2.2\text{-}5$$

Here the truncation error is proportional to Δx^2 and the fourth derivative. The errors in other finite difference expressions may be similarly estimated.

As the nature of function $\varphi(x)$ is not generally known beforehand, exact calculation of the truncation error is not possible. The order of magnitude of the truncation error can be estimated from the coefficients of the neglected terms when $\varphi(x)$ is a continuous function.

In the special case where $\varphi(x)$ is a continuous function with a shape such that all derivatives higher than third order are zero then equation 2.1-5 is completely accurate. In the example of a uniform slab with internal heat generation given in section 4.2.1, the steady state temperature distribution is a quadratic function[6] of x and hence $\frac{\partial^4\varphi}{\partial x^4} = 0$ in the steady state. As equation 2.1-5 was used to obtain this solution the steady state solution has no truncation error and may be used as a check on the computer accuracy.

For a given solution, the truncation error may be reduced by reducing the grid spacing or by employing higher order difference expressions where the order of the terms neglected are much greater than those neglected in equations 2.1-1 to 2.1-5. A disadvantage of employing higher order difference expressions is that when the order of the difference expression exceeds the order of the differential equation being solved, spurious solutions may be produced.[15, p. 188] To avoid any difficulty it is usual to restrict the order of the difference expression used to the order of the differential equation being considered. Recently it has been shown by Fischer,[11] that by appropriate choice of the higher order difference equations and the additional boundary conditions required by their use, that solutions may be obtained where the spurious solutions are reduced to the same order of magnitude as the inherent truncation errors of the method used.

To calculate the higher order differences it is necessary to use the values of four or more nodes. The reading of this additional information requires additional switch and computing section capacity. As the object of the present computer is to keep the equipment as simple as possible, it appears preferable to use the simpler difference expression and decrease the grid spacing (increase the number of nodes) if an increased accuracy is required.

In addition to errors in the solution due to truncation errors, additional errors also arise due to component tolerances in the computer. These errors are discussed in section 2.7.

2.3 STABILITY

When both the time and space intervals are differenced and the value of temperature for the next time interval is obtained by extrapolation, situations may arise where small errors tend to build up rapidly till the error terms completely overshadow the desired solution.

This is termed computational instability. While some finite difference expansions are stable, critical values of the ratio $\frac{\Delta t}{\Delta x}$ exist in the case of other expansions. The simple explicit finite difference method used for this computer is subject to such instability.

The stability of linear equations, such as equations 2.1-8 and 2.1-11 when $f(\varphi,t)$ is a constant, has been extensively studied.[6,9,10,13] Using general mathematical methods, such as the matrix method described by Fox[10] it is shown that the stability is influenced by the nature of the initial and boundary conditions. However when constant initial and boundary values are used, as in the present study, the simple method developed by Karplus may be used.[13,28] Karplus uses the fact that the finite difference expression may be represented by an electrical network. Using electric circuit theory developed by Bode, criteria for the stability of the network and hence the finite difference expression may be derived. Using these criteria, the conditions for stability of equations 2.1-8 and 2.1-11 when $f(\varphi,t)$ is a constant is given by $\frac{\Delta t}{\Delta x^2} < \frac{1}{2}$.

Although the criteria for the stability of linear equations are known, little is known in the case of non-linear equations in spite of intensive study.[6,10] Even in a recent Summer School held in Oxford it was concluded that the problem is largely unsolved.[10] Here it was also reported that in the study of a particular equation the general stability criterion was replaced by a 'local' stability criterion. This indicated that the stability depends on the solution and hence $\frac{\Delta t}{\Delta x^2}$ may have to be varied in different regions.

Even though a general answer to the stability problem has not been obtained the method can still be used. If instability is encountered the mesh ratio $\frac{\Delta t}{\Delta x^2}$ may be changed to determine whether the instability is due to the finite difference method or whether it is inherent in the solution itself.

2.4 CONVERGENCE

A finite difference expression is said to converge if the approximate solution obtained using finite difference methods approaches the exact solution of the partial differential equation as the grid spacing is refined, while the ratio of the grid spacing along the various co-ordinates is kept constant.

Karplus[13] states that no general criteria for convergence have yet been developed but is is generally believed that if a finite difference equation is stable it also converges. However, in a recent analysis by Fox[10] it is pointed out that convergence requires that equations be both stable and compatible. A finite difference equation is compatible with the differential equation if the truncation errors tend to zero as the grid spacing is reduced.* On this basis all the finite difference methods mentioned in this thesis, including the simple explicit method used for the computer, will converge if they are stable.

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*In particular an explicit expression suggested by Du Fort & Frankel[10] which has the advantage of being unconditionally stable but is not compatible and hence may be non-convergent if special precautions in the choice of Δx and Δt are not taken.

2.5 CHOICE OF CONSTANTS

It has been shown in sections 2.1, 2.2 and 2.3 that the choice of the space and time intervals Δx and Δt are restricted by accuracy and stability considerations. In the present study a solution accuracy of 5% is required so that it is necessary to restrict individual errors to the order of 1% (O(0.01)). Choosing $\Delta x = 0.2$ and $\Delta t = 0.01$ the corresponding errors in x and t are O(0.04) and O(0.01) respectively. As the curve of φ against x tends to be parabolic in shape in many of the solutions considered the actual error in x will be smaller and even zero as mentioned in section 2.2.

The above choice of Δx and Δt satisfies the stability condition as $\frac{\Delta t}{\Delta x^2} = \frac{1}{4} < \frac{1}{2}$.

Using these constants, the equations to be solved on the computer become :-

From equation 2.1-8

$$\Delta\varphi_{m,n} = \varphi_{m,n+1} - \varphi_{m,n} = \frac{1}{4}(\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n}) + 0.01\ f(\varphi_{m,n}) \quad \ldots \quad 2.5\text{-}1$$

From equation 2.1-11

$$\Delta\varphi_{m,n} = \frac{1}{4}\left[(1 + 0.1\ K)\varphi_{m+1,n} - 2\varphi_{m,n} + (1 - 0.1\ K)\varphi_{m-1,n}\right] + 0.01\ f(\varphi_{m,n}) \quad \ldots \quad 2.5\text{-}2$$

When $\Delta x = 0.2$, the interval $0 < x < 1$ must be divided into 5 intervals so that the boundary conditions for a symmetrical solution become (see also section 2.1)

$$\varphi_{0,n} = \varphi_{b1}$$
$$\varphi_{4,n} = \varphi_{6,n} \quad \ldots \quad 2.5\text{-}3$$

and initial conditions become :

$$\varphi_{1,0} = \varphi_{2,0} = \varphi_{3,0} = \varphi_{4,0} = \varphi_{5,0} = \varphi_i \quad \ldots \quad 2.5\text{-}4$$

A different set of grid spacings were used for one problem discussed in section 4.2.2. but as the treatment is similar it will not be repeated.

2.6 FORM OF THE DIFFERENCE EQUATION

The computer must calculate the value of $\varphi_{m,n+1}$ from the known values $\varphi_{m+1,n}$, $\varphi_{m,n}$ and $\varphi_{m-1,n}$. This calculation may be done in two ways.

1 By calculating $\varphi_{m,n+1}$ directly in the computer using the relationship obtained from equation 2.5-1.

$$\varphi_{m,n+1} = \frac{1}{4}(\varphi_{m+1,n} + 2\varphi_{m,n} + \varphi_{m-1,n}) + 0.01\ f(\varphi_{m,n})$$

If the computer introduces an error δ, the computed value becomes instead :-

$$\varphi'_{m,n+1} = \left[\frac{1}{4}(\varphi_{m+1,n} + 2\varphi_{m,n} + \varphi_{m-1,n}) + 0.01\ f(\varphi_{m,n})\right](1 + \delta)$$

If δ is small the total error after n calculations will be $O(n\delta)$.

2 By calculating $\Delta\varphi_{m,n}$ and adding it to the stored value of $\varphi_{m,n}$ to obtain $\varphi_{m,n+1}$. If the error in each calculation of $\Delta\varphi_{m,n}$ is δ the total error in $\varphi_{m,n}$ after n calculations will be $O(n\delta\frac{\Delta\varphi_{m,n}}{\varphi_{m,n}})$ and as $\varphi_{m,n} = O(n\Delta\varphi_{m,n})$ the total error will be $O(\delta)$.

It is apparent that it is more accurate to arrange the computer to calculate the increment given by equation 2.5-1 and to add it to the stored value of $\varphi_{m,n}$ to obtain $\varphi_{m,n+1}$ rather than calculating $\varphi_{m,n+1}$ directly.

2.7 ERRORS INTRODUCED BY COMPUTING ELEMENTS

In addition to the truncation errors inherent in the finite difference method additional errors are introduced by imperfections in the computing elements. Two forms of error can be distinguished.

1 Coefficient Errors

The finite gain of the operational amplifiers and the finite tolerances of the resistors and potentiometers used in each computing element cause small departures from the desired transfer function for the element. This has the effect of producing slight alterations in the coefficients of the equation being solved. The solutions produced are those of the desired equation with slightly altered coefficients. These coefficient errors are easily corrected as the actual value of the coefficients can be measured and corrected.

2 Spurious Term Errors.

Store leakage and certain combinations of component tolerance errors effectively introduce additional spurious terms into the equation being solved.

The simplest form of spurious term arises due to amplifier drift which takes the form of an additional constant term. However as this type of error is readily detected and corrected it will not be discussed in detail.

A general study of error is outside the scope of this thesis. Instead the errors introduced into the basic equation $\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2}$ by component and store imperfections are briefly considered to indicate the precautions necessary in the design of the computer.

A Error due to store leakage

Expressing $\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2}$ in terms of finite differences using equations 2.1-1 and 2.1-5 gives :-

$$\Delta\varphi_{m,n} = \frac{1}{\Delta x^2}(\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n})\Delta t \quad \ldots \quad 2.7\text{-}1$$

If the store holding the value φ loses an amount $-K_s\varphi$ due to leakage in the time required to calculate one increment in n, the actual equation solved becomes :-

$$\Delta\varphi_{m,n} = \frac{1}{\Delta x^2}(\varphi_{m-1,n} - 2\varphi_{m,n} + \varphi_{m+1,n})\Delta t + K_s\varphi_{m,n} \quad \ldots \quad 2.7\text{-}2$$

which corresponds to the equation

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + \left(\frac{K_s}{\Delta t}\right)\varphi \qquad \ldots \qquad 2.7\text{-}3$$

Equation 2.7-3 shows that a spurious term proportional to φ has been introduced due to store leakage.

B <u>Error due to component tolerances in the circuit generating the difference</u> $(\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n})$

Due to component tolerances the actual value of the difference $\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n}$ obtained in the computer will be modified so that equation 2.7-1 becomes :-

$$\Delta\varphi_{m,n} = \frac{1}{\Delta x^2}\Big[(1 + \delta_1)\varphi_{m+1,n} - (1 + \delta_2)2\varphi_{m,n} + (1 + \delta_3)\varphi_{m-1,n}\Big]\Delta t \qquad \ldots \qquad 2.7\text{-}4$$

Where δ_1, δ_2 and δ_3 are fractional error terms which have the same order of magnitude as the component tolerance of the circuit used to generate the difference. Equation 2.7-4 may be rewritten as :-

$$\begin{aligned}\Delta\varphi_{m,n} = \Big[&\left(1 + \frac{\delta_1}{2} + \frac{\delta_2}{2}\right)\left(\frac{1}{\Delta x^2}\right)(\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n}) \\ &+ \left(\frac{\delta_2}{2} - \frac{\delta_1}{2}\right)\left(\frac{1}{\Delta x^2}\right)(\varphi_{m+1,n} - \varphi_{m-1,n}) \\ &+ (\delta_1 - 2\delta_2 + \delta_3)\left(\frac{1}{\Delta x^2}\right)\varphi_{m,n}\Big]\Delta t\end{aligned} \qquad \ldots \qquad 2.7\text{-}5$$

Using equations 2.1-4 and 2.1-3 it can be shown that equation 2.7-5 corresponds to the equation :-

$$\frac{\partial\varphi}{\partial t} = \left(1 + \frac{\delta_1}{2} + \frac{\delta_2}{2}\right)\frac{\partial^2\varphi}{\partial x^2} + \left(\frac{\delta_3 - \delta_1}{\Delta x}\right)\frac{\partial\varphi}{\partial x} + \left(\frac{\delta_1 - 2\delta_2 + \delta_3}{\Delta x^2}\right)\varphi \quad \ldots \qquad 2.7\text{-}6$$

It is evident that the component errors have introduced a coefficient error and two spurious terms involving $\frac{\partial\varphi}{\partial x}$ and φ respectively.

The co-efficients of the spurious terms introduced by store leakage and component tolerance shown in equations 2.7-3 and 2.7-6 increase as the grid spacings are decreased. As the truncation error decreases with decreasing grid spacing it is evident that, for a given computer and finite difference expansion, an optimum value of grid spacing exists which gives the minimum solution error. Fischer[11] has shown that a similar optimum error exists when partial difference equations are solved on conventional analogue computers using finite difference methods.

The permissable store leakage and component tolerance which will produce a 1% error in the solution due to the spurious terms may be estimated using equations 2.7-3 and 2.7-6 and the solutions of equations 2.7-1 given in section 4.2 which show the effect of the spurious terms.

Fig. 4.2-6 shows that the error due to the $\frac{\partial\varphi}{\partial x}$ term when $\frac{1}{\Delta x}(\delta_3 - \delta_1) = 1$ is of the order of 20%. For 1% error with $\Delta x = 0.2$, $(\delta_3 - \delta_1) = 1\%$, so that components with 1% tolerance can be used.

Fig. 4.2-5 shows that the error due to the φ term when

$\frac{1}{\Delta x^2}(\delta_1 - 2\delta_2 + \delta_3) = 0.5$ is of the order of 30%. For 1% error with $\Delta x = 0.2$, $(\delta_1 - 2\delta_2 + \delta_3) = 0.2\%$. As only 1% tolerance resistors are readily available special precautions are required to reduce the $\delta_1 - 2\delta_2 - \delta_3$ error in the circuit generating the difference expression if the spurious term error is to be limited to 1%.

Similarly the maximum allowable store leakage for 1% error due to the φ term is $K_s = 1.5 \times 10^{-4}$ or 1.5% for 100 calculations (n=100).

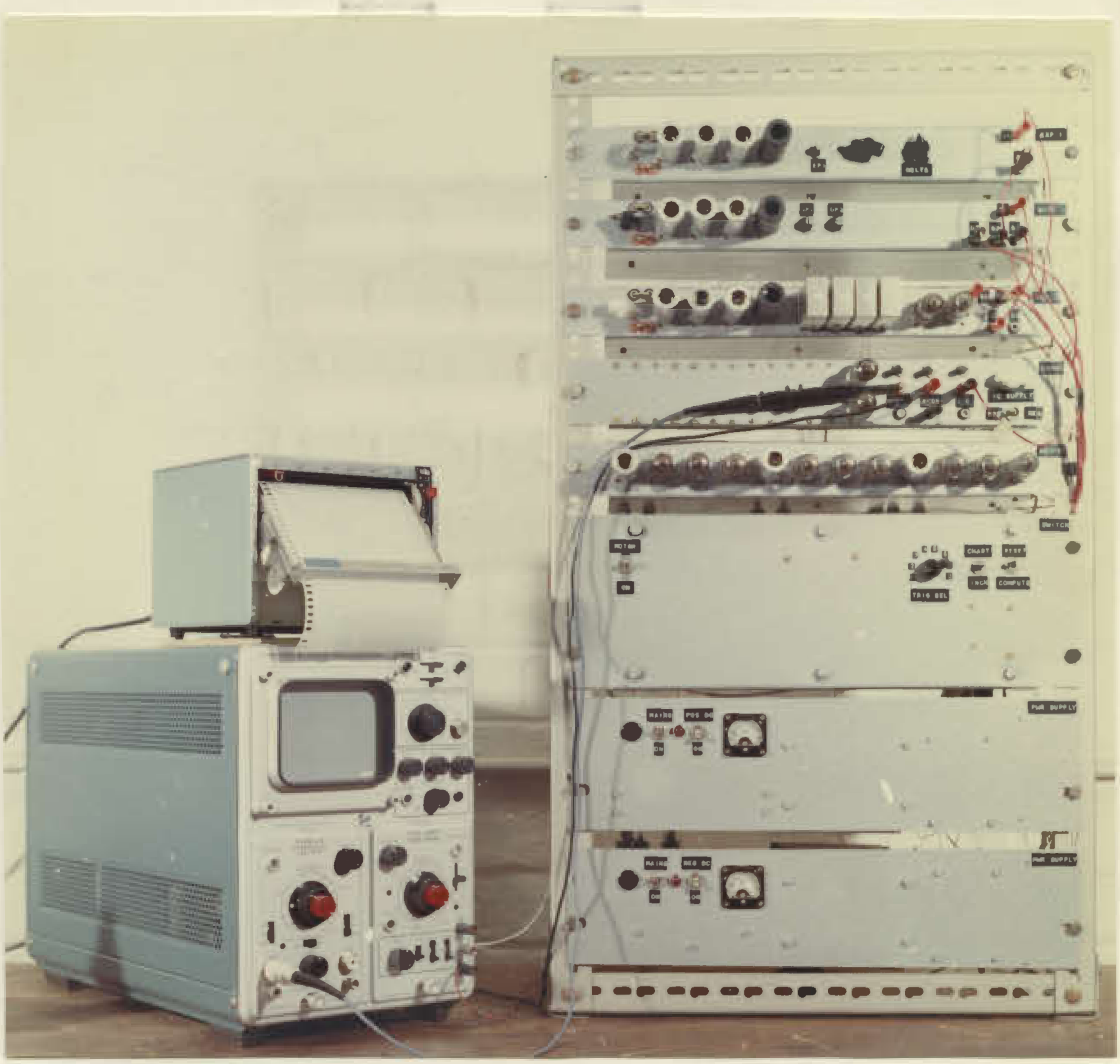

Fig. 3.1-1 Experimental Analogue Computer

3 EXPERIMENTAL ANALOGUE COMPUTER

3.1 COMPUTER OUTLINE

The analogue computer to solve partial differential equations by the finite difference method discussed in section 2 consists of two principal sections.

1 THE STORE which stores the values of $\varphi_{m,n}$

2 THE COMPUTING SECTION which

a Reads the stored values of $\varphi_{m+1,n}$, $\varphi_{m,n}$ and $\varphi_{m-1,n}$

b Calculates the increment $\Delta\varphi_{m,n}$ from $\varphi_{m+1,n}$, $\varphi_{m,n}$ and $\varphi_{m-1,n}$.

c Adds the increment $\Delta\varphi_{m,n}$ to the stored value $\varphi_{m,n}$ producing the new stored value $\varphi_{m,n+1}$.

The experimental analogue computer uses a passive capacitor store and an active computing section based on differential operational amplifiers. A photograph of this instrument appears in fig. 3.1-1.

The block diagram of the experimental computer is shown in fig. 3.1-2. The principal functions of each block are :-

1 THE CAPACITOR STORE stores the values of the variables φ_1, φ_2, φ_3 ..... in analogue form as voltages on a set of capacitors.

2 THE SWITCH connects the computing section buffer and integrator circuits to the required store capacitors. The buffers read the values of $\varphi_{m+1,n}$, $\varphi_{m,n}$ and $\varphi_{m-1,n}$ and the

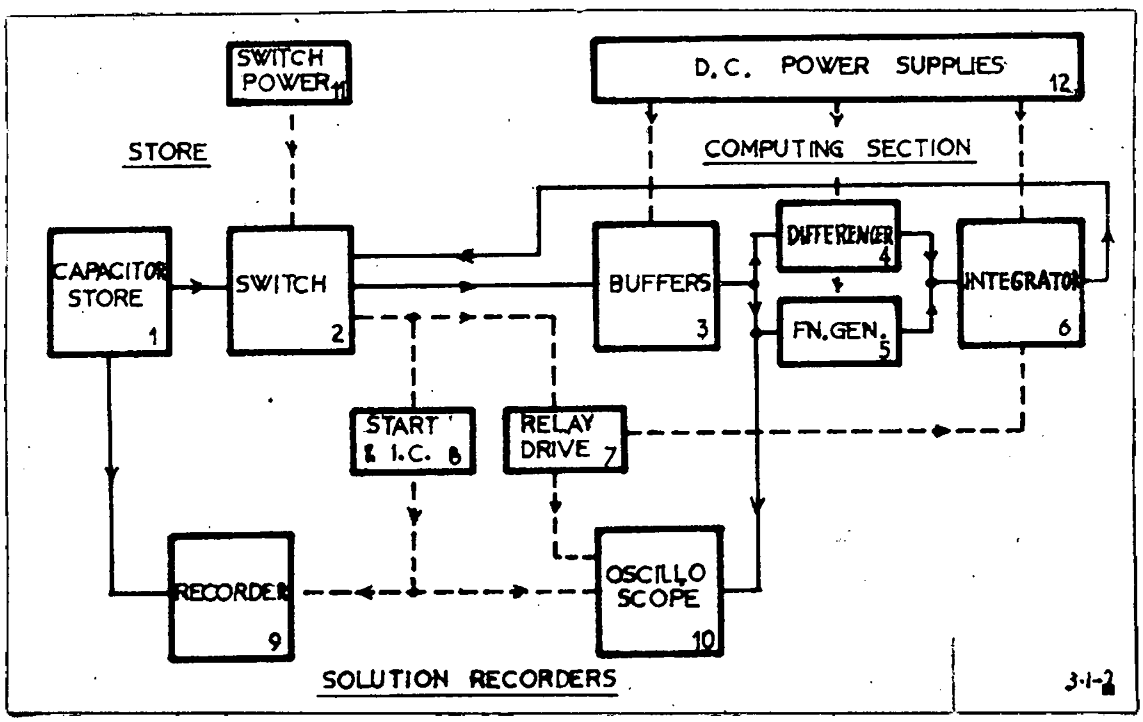

Fig. 3.1-2 Block diagram of the analogue computer

integrator writes the calculated value of $\Delta\varphi_{m,n}$ into the store.

The switch also supplies synchronising signals for the relay-drive and start circuits.

3 THE BUFFERS are unity gain isolating amplifiers which prevent the discharge of the store capacitors by the computing circuits.

4 THE DIFFERENCER generates the difference $\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n}$ corresponding to the second space derivative.

5 THE FUNCTION GENERATOR generates the function $f(\varphi,t)$. (Only an exponential function generator giving Ke^{φ} was studied.)

6 THE INTEGRATOR adds $\Delta\varphi_{m,n}$ to the contents of the $\varphi_{m,n}$ store via the switch.

7 THE RELAY DRIVE CIRCUITS operate the relays in the integrator required to affect the addition of $\Delta\varphi_{m,n}$ in synchronism with the switch.

8 THE START AND INITIAL CONDITIONS CIRCUITS set the store and computer circuits to the initial and boundary values required and start the computation and the recording devices at the correct instant.

9 THE POTENTIOMETRIC RECORDER records the potential of a selected store φ_m producing a solution of φ_m against t.

10 THE STORAGE OSCILLOSCOPE displays the output of the φ_m buffer and using a linear sweep and intensity modulation produces a solution of φ, for all values of m, against x and t.

11 THE SWITCH POWER SUPPLY operates the switch motor synchronously at mains frequency.

12 THE COMPUTING CIRCUIT POWER SUPPLIES supply the regulated d.c. voltages required by the computing amplifier and relay circuits.

The modification of the stored value of $\varphi_{m,n}$ to $\varphi_{m,n+1}$ is carried out in two steps as the switch wiper passes over the fixed contact connected to the φ_m store capacitor.

1 During the first half of the period during which the switch wipers are on the fixed contacts, termed the compute period, the store capacitors holding the values of $\varphi_{m+1,n}$, $\varphi_{m,n}$ and $\varphi_{m-1,n}$ are connected to the computing section which calculates $\Delta\varphi_{m,n}$ and stores the result in a transfer capacitor in the integrator.

2 During the second half of the period during which the switch wipers are on the fixed contacts, termed the write period, the calculated increment of charge corresponding to $\Delta\varphi_{m,n}$ stored in the transfer capacitor is added to the capacitor holding $\varphi_{m,n}$.

The switch wipers then move on to connect the store capacitors holding $\varphi_{m,n}$, $\varphi_{m+1,n}$ and $\varphi_{m+2,n}$ so that $\varphi_{m+1,n+1}$ is produced. This process is repeated until all the stored values are modified to correspond to time $(n + 1)\Delta t$, when the switch will arrive back at the capacitors holding $\varphi_{1,n+1}$ and proceed to calculate the values for time $(n+2)\Delta t$. The calculation continues in this fashion until sufficient of the required solution is obtained, corresponding typically to $n = 100$ to 200.

3.2 STORE

3.2.1 ANALOGUE STORAGE

Storage entails the sampling and holding of the instantaneous value of a function from one calculation to the next. A number of electrostatic and magnetic storage devices have been developed for the storage of analogue computer data.[30] The choice of device most suited for a particular application is determined by the accuracy (linearity and leakage) and the sampling speed required.

Electrostatic storage of charge on a capacitor has the advantage of being inherently linear, but suffers from charge leakage. The simplest storage device consists of a bank of capacitors connected to a switch. The capacitors are charged to definite voltage levels and the voltage read off when required. Kozak[36] using a rotating wheel of capacitors with mechanical switching has achieved an accuracy of 1% at a sampling rate of 30 capacitors per second. A similar type of store but using solid state switching described by Wager[29] offers a potentially higher sampling rate. An ultra-high speed store using charge storage on the face of a cathode ray tube developed by Bergman[37] has a sampling rate of the order of 1 micro second.

Magnetic storage devices utilize the high remanence and low coercive force of certain modern magnetic materials[38] and have the advantage of excellent permanence but are inherently non linear. Applications may be divided into continuous and two state devices.

The storage of analogue data as flux in multiaperture core devices such as the Transfluxor[30] have been considered and dismissed because of low accuracy.[29] The magnetic tape recorder has been proposed as an analogue storage device by Wainwright[20] and attempted by Fujino[32] who found that the computer accuracy was limited to the order of 10% by the inherently unreliable amplitude characteristic of the tape.

Accurate storage with magnetic devices is possible using two state techniques. A quasi-analogue store using two-state cores has been described by Watanbe.[33] Similarly, accurate storage using tape recorders may be obtained by employing frequency or pulse modulation techniques. Belck[38] claims an accuracy of 0.1% from d.c. to 0.5 c/s for such a technique. Although accurate, these techniques have the disadvantage of digital methods of requiring complicated auxiliary equipment.

In the experimental analogue computer a solution time of the order of 50 seconds was chosen so that a high speed potentiometric recorder, with a response time of 0.5 seconds, could be used to record solutions with an accuracy of the order of 1%. For a calculation involving 5 space and 200 time intervals a sampling rate of 20 store positions per second is required. As modern, motor driven, multipole switches capable of scanning 200 contacts per second are available,[40] the simple capacitor store and mechanical switch was chosen.

3.2.2 CAPACITOR STORE

The computer uses a store consisting of 5 polystyrene-dielectric capacitors connected to a multipole, motor driven rotary switch. Ideally each capacitor should hold the charge placed on it without change between successive calculations and should be unaffected by the charge on adjacent store capacitors. The practical store departs from these ideals because of leakage, dielectric storage and cross-talk effects. The experimental approach consists of an attempt to eliminate dielectric storage and cross-talk effects and to make all forms of leakage proportional to the store voltage and then to use a compensation technique to reduce the effective store leakage to zero. The various causes of store leakage are discussed in section 3.2.3 and a method of store leakage compensation is described in section 3.3.2.

Dielectric storage effects cause a capacitor to act as if it is in parallel with a number of series resistance-capacitance combinations with time constants ranging from the order of seconds to hundreds of seconds.[34] As the integrator adds a calculated charge to the store capacitor, the dielectric storage effect introduces an uncertainty in the corresponding voltage increment. This uncertainty is equal to the ratio of the additional 'dielectric storage' shunt capacitance to the total capacitance of the store capacitor. The storage effect depends on the type of dielectric employed. The ratio of 'dielectric storage' to total capacitance is approximately 1% for paper as against 0.1% for polystyrene.[34]

Dielectric storage also causes a form of leakage due to the charge on the capacitor 'leaking' into the shunt 'dielectric storage' capacitors. As this leakage is not proportional to the capacitor voltage simple leakage compensation is impossible. These effects which were troublesome when paper dielectric capacitors were tried, were negligible when polystyrene dielectric capacitors were used.

Cross talk occurs between adjacent store capacitors due to charge transferred on the switch wiper stray capacitance, C_w. When the switch wiper passes successive store capacitors charged to voltages V_{m-1}, V_m and V_{m+1} the voltage increment at the middle capacitor ΔV_m is:

$$\Delta V_m = \frac{C_w}{C_m}(V_{m+1} - 2V_m + V_{m-1}) \quad \ldots \quad 3.2\text{-}2$$

where C_m is the store capacitance. This charge transfer introduces a spurious term error, called cross-talk error, into the equation being solved of the form $\frac{\partial^2\varphi}{\partial x^2}$.

The cross-talk error can be eliminated by connecting the store capacitors to alternate contacts on the switch and connecting discharge resistors to the intermediate contacts. As the stray capacitance is completely discharged after being connected to each store capacitor charge leakage proportional to the store capacitor voltage results.

Measurements showed that the cross-talk was not completely eliminated in the experimental computer because of the action of the integrator which becomes unstable on the intermediate contacts, charging the wiper stray capacitance and also reducing the discharge time available. The measured coefficient in equation 3.2-2 was 2.5×10^{-4}. This introduces a negligible error into the solution of the diffusion equation and could be eliminated if necessary by slightly modifying the integrator circuit.

3.2.3 CAPACITOR STORE LEAKAGE

Leakage of stored charge from the store capacitors occurs at several points in the computer. It was shown in section 2.7 that this leakage introduces a spurious term into the equation being solved. It was estimated that, to limit the solution error due to this spurious term to 1%, the leakage error must be limited to 1.5% in the time required to calculate 100 time intervals. This time corresponds to approximately 15 seconds in the experimental computer.

The principal sources of leakage are :-

1 Capacitor Leakage

Capacitor leakage occurs due to dielectric and surface effects. Using paper capacitors with a decay time constant of 10^4 seconds the leakage after 15 seconds is 0.15%. For polystyrene capacitors with a time constant of 5×10^6 seconds the corresponding leakage is less than 0.001%.

2 Circuit Insulation Leakage

Leakage occurs due to the imperfect insulation of the switch and computing circuits. In the experimental computer, each capacitor is connected to 16 fixed contacts on the switch bank having a total leakage resistance of 1,500 megohms which gives a leakage of 2% in 15 seconds. Each capacitor is also connected via the switch wipers to the buffer, integrator and initial condition circuits where additional leakage will occur.

4 Leakage due to Switch Wiper Contact Overlap

The store capacitors are momentarily connected to earth via the switch wiper discharge resistors if bridging switch wipers are used. Bridging wipers were used in the experimental computer to obtain bounce free contact operation. The leakage depends on the time that the capacitor contact is bridged to an intermediate contact and the discharge resistance. The measured leakage due to contact overlap after 15 seconds was 4%.

5 Leakage due to Finite Integrator Amplifier Gain

It will be shown in section 3.3.2 that a loss of store. charge, proportional to the store voltage, occurs each time the integrator is switched from the 'compute' to the 'write' state, due to the finite gain of the operational amplifier. The leakage error for typical values of the constants used in the computer will be of the order of 1%.

The total measured leakage after 15 seconds is of the order of 10% which is considerably in excess of the 1.5% required. Means for reducing all the leakage errors mentioned exist but were not required as it is possible to reduce the effective leakage by a factor of 10 using a simple compensation circuit on the integrator. The leakage compensation circuit is described in section 3.3.2. The results of tests on the capacitor store used with the integrator employing leakage compensation are given.

3.2.4 AUXILIARY STORE

The auxiliary store is a short term store which holds the value of the variable required for the $\varphi_{m-1,n}$ input to the computing section for two calculation periods. The necessity for the additional store is best explained with reference to fig. 3.2.4. Notice that the φ_m buffer BUF 2, and the φ_{m+1} buffer BUF 3, are shown connected to the main store capacitors C_{m2} and C_{m3}. Assume that the computer is calculating the increments for the nth time interval, so that BUF 2 and BUF 3 are reading the values of $\varphi_{2,n}$ and $\varphi_{3,n}$ from C_{m2} and C_{m3}. Assume also that $\varphi_{1,n}$ is available at the output of BUF 1. The computing section calculates the increment $\Delta\varphi_{m,n} = \Delta\,\varphi_{2,n}$ which is added to $\varphi_{2,n}$ so that the value stored in C_{m2} becomes $\varphi_{2,n+1}$.

The switch now moves on to the next contact position. BUF 2 reads $\varphi_{3,n}$ and BUF 3 reads $\varphi_{4,n}$. Also required is $\varphi_{m-1,n} = \varphi_{2,n}$. However, $\varphi_{2,n}$ no longer exists in the main store as the value in the φ_2 store (C_{m2}) was increased to $\varphi_{2,n+1}$ by the previous calculation. It is thus necessary to store each value in the main store for use in the calculation which follows that in which the main store value is increased. This is the function of the auxiliary store.

Using the auxiliary store, BUF 3 now writes the φ_{m+1} value into an auxiliary capacitor via the D1 relay. This φ_{m+1} value is stored until the switch moves on two steps when it is picked up by BUF 1. In the situation described above, the $\varphi_{2,n}$ value in the main store capacitor C_{m2} would have been written into the auxiliary store capacitor C_{a2} directly below it while BUF 3 was connected to C_{m2}.

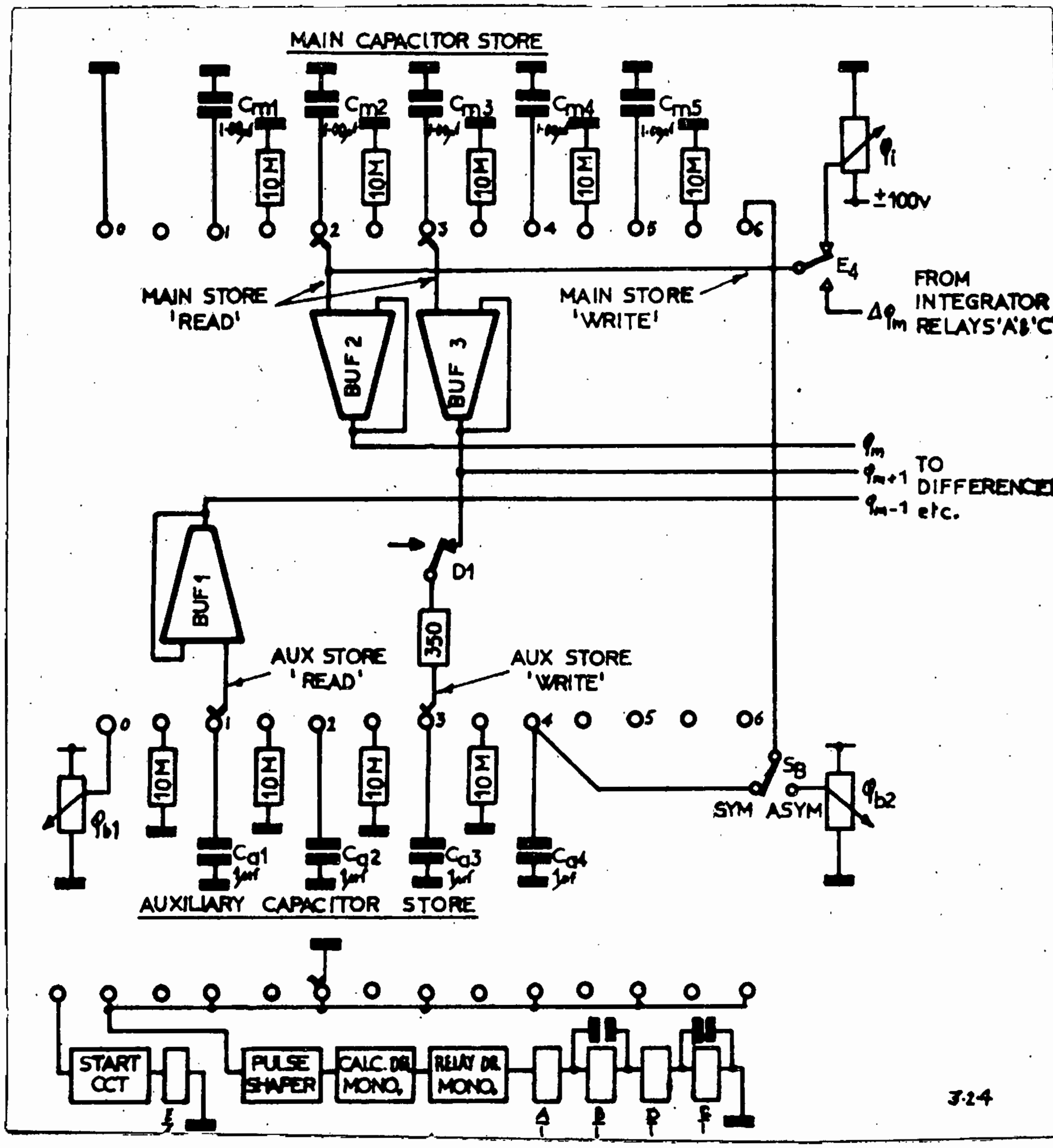

Fig. 3.2.4 Schematic diagram showing:-

a. Interconnection of the main and auxiliary stores, the switch and the buffers.

b. Initial and boundary conditions circuits.

c. Start and integrator relay circuits.

adjacent store capacitors. The practical store departs from these ideals because of leakage, dielectric storage and cross-talk effects. The experimental approach consists of an attempt to eliminate dielectric storage and cross-talk effects and to make all forms of leakage proportional to the store voltage and then to use a compensation technique to reduce the effective store leakage to zero. The various causes of store leakage are discussed in section 3.2.3 and a method of store leakage compensation is described in section 3.3.2.

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Dielectric storage also causes a form of leakage due to the charge on the capacitor 'leaking' into the shunt 'dielectric storage' capacitors. As this leakage is not proportional to the capacitor voltage simple leakage compensation is impossible. These effects which were troublesome when paper dielectric capacitors were tried, were negligible when polystyrene dielectric capacitors were used.

Cross talk occurs between adjacent store capacitors due to charge transferred on the switch wiper stray capacitance, C_w. When the switch wiper passes successive store capacitors charged to voltages V_{m-1}, V_m and V_{m+1} the voltage increment at the middle capacitor ΔV_m is:

$$\Delta V_m = \frac{C_w}{C_m}(V_{m+1} - 2V_m + V_{m-1}) \qquad \ldots \qquad 3.2\text{-}2$$

where C_m is the store capacitance. This charge transfer introduces a spurious term error, called cross-talk error, into the equation being solved of the form $\frac{\partial^2 \varphi}{\partial x^2}$.

The cross-talk error can be eliminated by connecting the store capacitors to alternate contacts on the switch and connecting discharge resistors to the intermediate contacts. As the stray capacitance is completely discharged after being connected to each store capacitor charge leakage proportional to the store capacitor voltage results.

Measurements showed that the cross-talk was not completely eliminated in the experimental computer because of the action of the integrator which becomes unstable on the intermediate contacts, charging the wiper stray capacitance and also reducing the discharge time available. The measured coefficient in equation 3.2-2 was 2.5×10^{-4}. This introduces a negligible error into the solution of the diffusion equation and could be eliminated if necessary by slightly modifying the integrator circuit.

3.2.3 CAPACITOR STORE LEAKAGE

Leakage of stored charge from the store capacitors occurs at several points in the computer. It was shown in section 2.7 that this leakage introduces a spurious term into the equation being solved. It was estimated that, to limit the solution error due to this spurious term to 1%, the leakage error must be limited to 1.5% in the time required to calculate 100 time intervals. This time corresponds to approximately 15 seconds in the experimental computer.

The principal sources of leakage are :-

1 Capacitor Leakage

Capacitor leakage occurs due to dielectric and surface effects. Using paper capacitors with a decay time constant of 10^4 seconds the leakage after 15 seconds is 0.15%. For polystyrene capacitors with a time constant of 5×10^6 seconds the corresponding leakage is less than 0.001%.

2 Circuit Insulation Leakage

Leakage occurs due to the imperfect insulation of the switch and computing circuits. In the experimental computer, each capacitor is connected to 16 fixed contacts on the switch bank having a total leakage resistance of 1,500 megohms which gives a leakage of 2% in 15 seconds. Each capacitor is also connected via the switch wipers to the buffer, integrator and initial condition circuits where additional leakage will occur.

4 Leakage due to Switch Wiper Contact Overlap

The store capacitors are momentarily connected to earth via the switch wiper discharge resistors if bridging switch wipers are used. Bridging wipers were used in the experimental computer to obtain bounce free contact operation. The leakage depends on the time that the capacitor contact is bridged to an intermediate contact and the discharge resistance. The measured leakage due to contact overlap after 15 seconds was 4%.

5 Leakage due to Finite Integrator Amplifier Gain

It will be shown in section 3.3.2 that a loss of store charge, proportional to the store voltage, occurs each time the integrator is switched from the 'compute' to the 'write' state, due to the finite gain of the operational amplifier. The leakage error for typical values of the constants used in the computer will be of the order of 1%.

The total measured leakage after 15 seconds is of the order of 10% which is considerably in excess of the 1.5% required. Means for reducing all the leakage errors mentioned exist but were not required as it is possible to reduce the effective leakage by a factor of 10 using a simple compensation circuit on the integrator. The leakage compensation circuit is described in section 3.3.2. The results of tests on the capacitor store used with the integrator employing leakage compensation are given.

3.2.4 AUXILIARY STORE

The auxiliary store is a short term store which holds the value of the variable required for the $\varphi_{m-1,n}$ input to the computing section for two calculation periods. The necessity for the additional store is best explained with reference to fig. 3.2.4. Notice that the φ_m buffer BUF 2, and the φ_{m+1} buffer BUF 3, are shown connected to the main store capacitors C_{m2} and C_{m3}. Assume that the computer is calculating the increments for the nth time interval, so that BUF 2 and BUF 3 are reading the values of $\varphi_{2,n}$ and $\varphi_{3,n}$ from C_{m2} and C_{m3}. Assume also that $\varphi_{1,n}$ is available at the output of BUF 1. The computing section calculates the increment $\Delta\varphi_{m,n} = \Delta\,\varphi_{2,n}$ which is added to $\varphi_{2,n}$ so that the value stored in C_{m2} becomes $\varphi_{2,n+1}$.

The switch now moves on to the next contact position. BUF 2 reads $\varphi_{3,n}$ and BUF 3 reads $\varphi_{4,n}$. Also required is $\varphi_{m-1,n} = \varphi_{2,n}$. However, $\varphi_{2,n}$ no longer exists in the main store as the value in the φ_2 store (C_{m2}) was increased to $\varphi_{2,n+1}$ by the previous calculation. It is thus necessary to store each value in the main store for use in the calculation which follows that in which the main store value is increased. This is the function of the auxiliary store.

Using the auxiliary store, BUF 3 now writes the φ_{m+1} value into an auxiliary capacitor via the D1 relay. This φ_{m+1} value is stored until the switch moves on two steps when it is picked up by BUF 1. In the situation described above, the $\varphi_{2,n}$ value in the main store capacitor C_{m2} would have been written into the auxiliary store capacitor C_{a2} directly below it while BUF 3 was connected to C_{m2}.

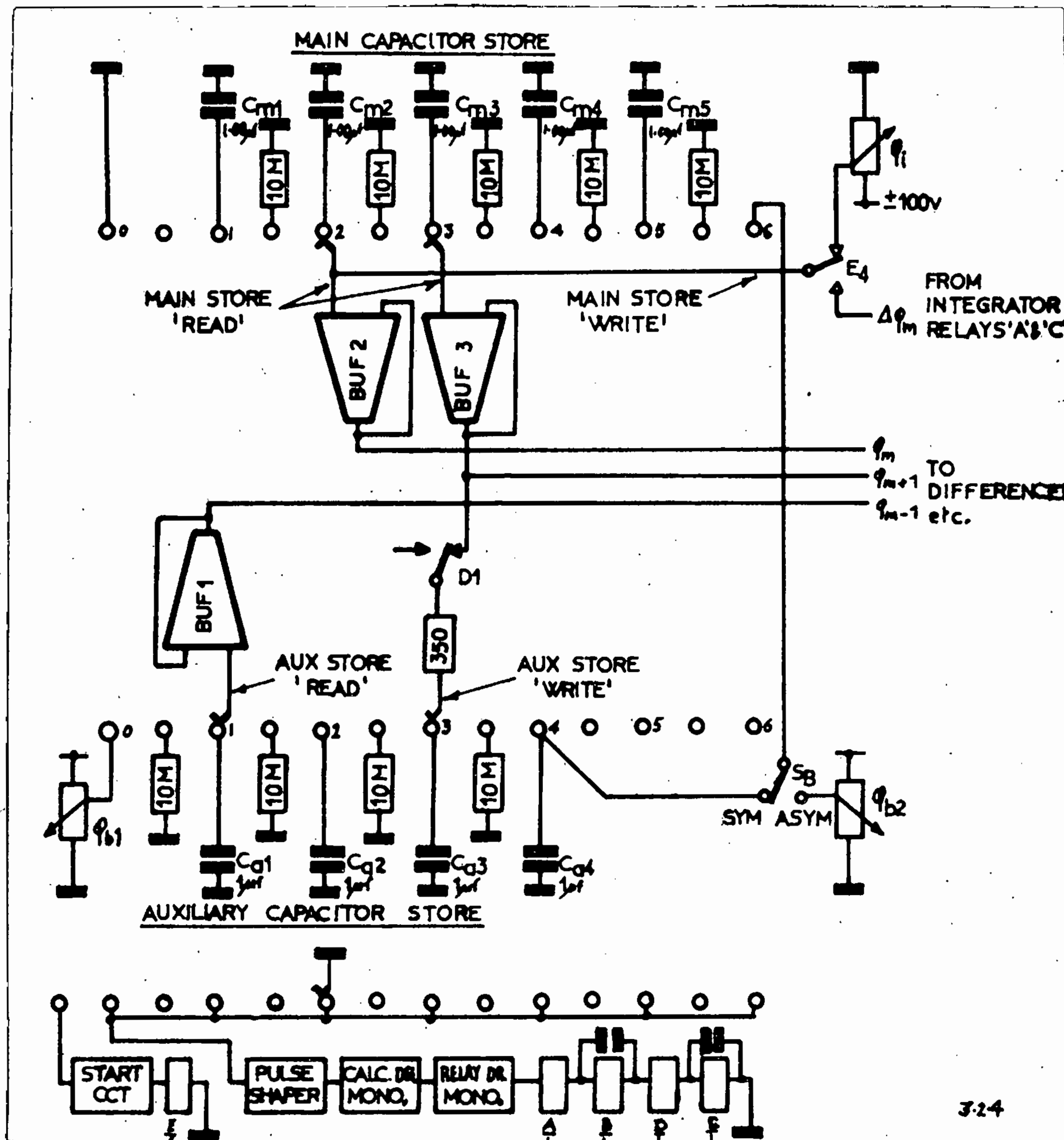

Fig. 3.2.4 Schematic diagram showing:-

a. Interconnection of the main and auxiliary stores, the switch and the buffers.

b. Initial and boundary conditions circuits.

c. Start and integrator relay circuits.

The value of $\varphi_{2,n}$ will remain stored in this auxiliary capacitor after $\varphi_{2,n}$ has been modified to $\varphi_{2,n+1}$ in the main store (C_{m2}) and will thus be available when required to calculate $\Delta\varphi_{3,n}$. The value of $\varphi_{2,n}$ will remain in the auxiliary store until it is eventually overwritten.

Only 3 auxiliary store capacitors are required regardless of the number of main store capacitors in use. However because of the small number of main store capacitors used, 5, the wiring of the auxiliary store was simplified if 4 auxiliary store capacitors are used as shown in fig. 3.2.4. The requirements for auxiliary store capacitors are not stringent. The exact capacitance is unimportant as BUF 3 charges the capacitor to the same voltage as the corresponding main store capacitor. Because of the short storage time of approximately 40 milli-seconds and because errors are not cumulative, unlike the main store, the errors introduced by the factors considered in section 3.2.3 are relatively unimportant in the case of the auxiliary store.

The 350 ohm resistor placed between the output of BUF 3 and the auxiliary store 'write' wiper is required to prevent high frequency instability of the BUF 3 amplifier which results if the output is shunted by a 1μfd capacitor. It is important to keep the auxiliary store charging time constant (= 0.35 milliseconds) small compared to the 'write' period (5 milli-seconds) otherwise the auxiliary store does not exactly duplicate the main store values when the latter are changing rapidly which gives rise to transient errors in the computed solutions.

Relay D is provided to break the auxiliary store write-path before either of the wipers leave the contacts to prevent errors arising from the partial discharge of the auxiliary capacitor which would occur if the φ_{m+1} contact is broken before the auxiliary store write contact.

3.2.5 INITIAL AND BOUNDARY CONDITIONS

The voltages on the main store capacitors at the start of the computation must be set to values corresponding to the assumed initial conditions (see 2.1).

Only uniform initial conditions are considered in the experimental computer corresponding to :-

$$\varphi_{1,0} = \varphi_{2,0} = \varphi_{3,0} = \varphi_{4,0} = \varphi_{5,0} = \varphi_i$$

These initial voltages are set by breaking the 'write' path between the switch and the integrator and connecting it to the initial conditions supply. This operation is performed by relay E in figs. 3.2.4 and B6. The calculation is started by operating relay E which connects the integrator to the switch. Relay E must be operated in synchronism with the switch so that the computation commences with the first store capacitor.

Boundary conditions are set by connecting the contacts on the switch corresponding to the boundaries to boundary condition supplies. In the experimental computer only two boundary conditions were used corresponding to :-

1 A slab divided into 6 intervals with fixed boundary conditions φ_{b1} and φ_{b2}. Referring to fig. 3.2.4 it is seen that this is achieved by connecting contact 0 on the auxiliary store bank

to the φ_{b1} supply and contact 6 on the main store bank to the φ_{b2} supply via the boundary condition switch S_B (S_B in the ASYMmetrical position.)

2 A half-slab divided into 5 intervals with a fixed boundary condition φ_{b1} and with symmetry about the central plane at $\varphi_{5,n}$. This symmetry condition is imposed by connecting the auxiliary store capacitor C_{a4} holding $\varphi_{4,n}$ to contact 6 on the main store bank. (S_B in SYMmetrical position.)

3.2.6 SWITCH

The switch used is a high-speed motor-driven uniselector designed for automatic telephone applications.[40] It is connected as an 8 bank 104 contact switch. The switch connections are indicated schematically in fig. 3.2.4 and described in appendix B1.

Three banks equipped with bridging wipers, to ensure bounce free operation, are used to connect the main and auxiliary store capacitors to the computing section. A further two banks are used to provide the synchronising signals for the relay drive and start circuits.

Only alternate contacts are used to avoid short circuits between stores by the bridging wipers. Stray capacitance discharge resistors are connected to the intermediate contacts where required. Twelve contact positions are required to accommodate 5 space intervals (see fig. B1/2) and all the connections are repeated 8 times. In one revolution of the switch the computer can thus calculate the 5 values corresponding to the 5 space intervals for 8 time intervals making 40 calculations in all.

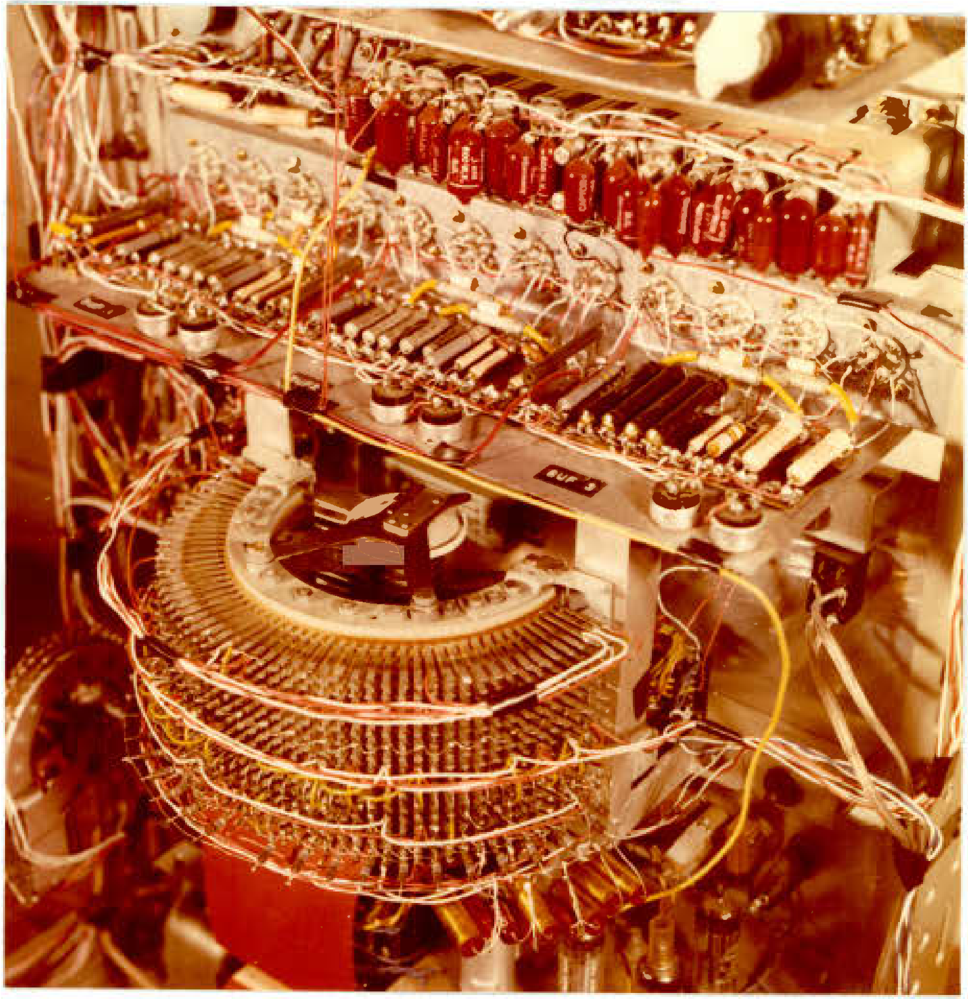

Fig. 3.2.6 Rear View of the experimental computer showing :

From top to bottom

a Auxiliary capacitor store
b Buffer amplifiers
c Switch
(Stray capacitance discharge resistances and main store capacitors are connected to the switch.)

$$G_d = \frac{V_o}{V_p - V_n} \quad \ldots \quad 3.3.1\text{-}1$$

Note that the output voltage is in phase with the input terminal marked (+) and in antiphase with input terminal marked (-).

2 The common-mode gain is zero. The common-mode gain G_c is defined by :-

$$G_c = \frac{2V_o}{V_p + V_n} \quad \ldots \quad 3.3.1\text{-}2$$

3 The input terminals are open circuits so that no current flows into these terminals.

Practical differential operational amplifiers depart from the ideal by having finite differential and common mode gains and measurable grid current.

The action of circuits employing the differential operational amplifier are reedily understood if the following property is recognized. The differential gain of the amplifier is very large so that the full output voltage can be produced by a very small voltage difference between the input terminals. If a negative feedback path is provided between the output and the (-) input terminal the amplifier will act to reduce the voltage difference between the input terminals to a minimum. As the amplifier gain is increased to infinity this voltage difference will tend to zero. The differential amplifier which has such a negative feedback path thus acts to produce a 'virtual short circuit' between the input terminals. This virtual short circuit is analogous to the 'virtual earth' produced in the conventional operational amplifier circuit.

A description of the differential operational amplifier designed for the experimental computer is given in appendix B2. This amplifier has a differential gain of 2,000 a common-mode gain of less than unity and negligible grid current. The effect of the finite differential and common-mode gains on the integrator, buffer and differencer circuits are considered in Appendix A.

3.3.2 INTEGRATOR

A. Action of the integrator

The integrator must add the calculated increment of charge corresponding to $\Delta\varphi_{m,n}$ to the φ_m store capacitor regardless of the value of the charge, corresponding to $\varphi_{m,n}$, already on the capacitor. The basic circuit of the integrator based on the differential operational amplifier is shown in fig. 3.3.2-1.

To simplify the explanation of the integrator action assume that the differential operational amplifier is ideal. The integrator injects the calculated charge into the store capacitor C_m in two steps called the 'compute' and 'write' periods.

During the compute period the integrator A,B and C relays are in position 'a' as shown. The differential amplifier is now connected as a normal single input operational amplifier with the transfer capacitor C_t connected from the amplifier output to earth. The amplifier charges C_t to the calculated voltage $V_{Ct(a)} = -\frac{R_b}{R_a} V_i$. thus storing a certain charge. The bracketed suffix (a) indicates that this is the voltage on capacitor C_t while the relays are in position 'a'.

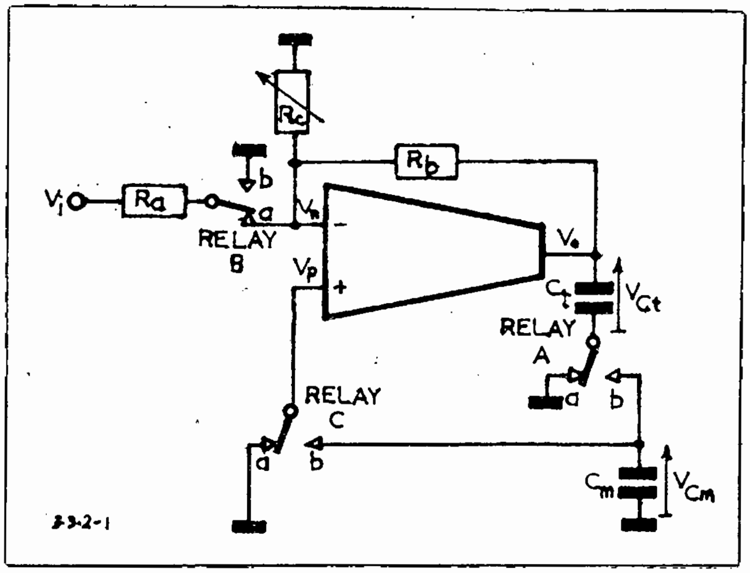

Fig. 3.3.2-1 Basic circuit of the integrator with leakage compensation

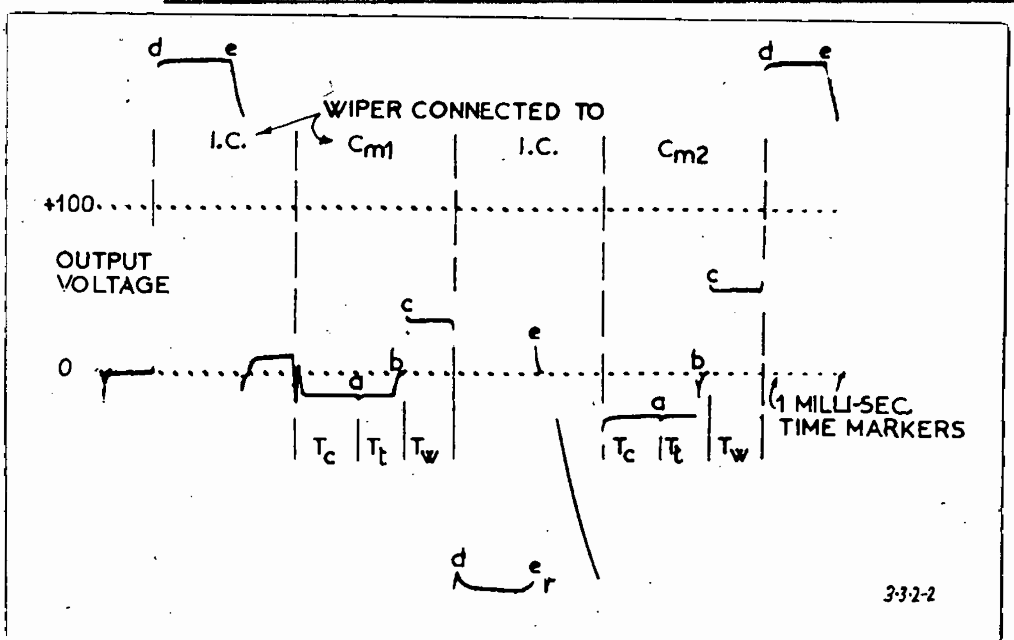

Fig. 3.3.2-2 Waveform at the output of the integrator amplifier during normal computer operation.

where

T_c = compute period

T_t = relay transition period

T_w = write period

I.C = intermediate contact

a = relay A, 'a' contact breaks

b = relay B & C 'a' contacts break

c = relay B & C 'b' contacts make

d = Switch wiper leaves store capacitor contact. Note resulting instability.

e = relays return to position 'a'

During the write period, the relays are switched to position 'b'. The transfer capacitor C_t and the store capacitor C_m are connected in series by relay A. If the voltage across C_t could be reduced to zero all the charge on C_t would be transferred to C_m which is the desired result. This is achieved using the differential amplifier in the following manner. Ignoring the presence of R_c, it can be seen that the (-) input of the differential operational amplifier is connected directly to the output via R_b making $V_n = V_o$, R_a having been disconnected by relay B. The (+) input is connected to the junction between C_t and C_m by relay C. As the differential amplifier acts to produce a virtual short circuit between the input terminals it acts to make $V_p = V_n = V_o$ thus reducing the voltage across the transfer capacitor $V_{Ct} = V_o - V_p$ to zero. As no current flows into the input terminals of the amplifier all the charge on C_t is transferred to C_m.

An expression for the increment in store voltage ΔV_{Cm} taking into account the finite differential gain G_d and common mode gain G_c of the amplifier is derived in appendix A1 and is :

$$\Delta V_{Cm} = \frac{C_t}{C_m}\frac{R_b}{R_a}V_i \left[1 - \frac{1}{G_d}\left(1 + \frac{R_b}{R_a} + \frac{C_t}{C_m} - \frac{C_t}{C_m}G_c\right) + \frac{C_t}{C_m}\frac{R_b}{R_c}\right]$$

error term

$$- \frac{1}{G_d}\frac{C_t}{C_m}(1 - G_c)\, V_{Cm(a)} \qquad + \frac{C_t}{C_m}\frac{R_b}{R_c}V_{Cm(a)} \qquad \ldots \qquad 3.3.2\text{-}1$$

leakage term — compensation term

The first term of the expression shows the normal reduction in the voltage increment due to the finite gain of the amplifier. The error introduced is of the order $\frac{1}{G_d}$.

The second term, indicates that due to the finite gain a negative voltage increment results which is proportional to the voltage on the store capacitor and occurs every time the integrator circuit is switched from compute to write. This is the effect which resembles simple charge leakage and was mentioned in section 3.2.3 dealing with charge leakage from the capacitor store.

The third term will be discussed later.

Reliable operation of the integrator circuit requires the A relay to operate before the B and C relays. This is necessary to prevent an alteration in the charge on the transfer capacitor which may occur if the B or C relays operate first, altering conditions at the inputs to the amplifier. Prior operation of the A relay is achieved by delaying the operation of the B and C relays by shunting the relays coils with small capacitors as shown in figs. 3.2.4 and B-6. Additional practical details of the integrator are given in appendix B3.

A typical waveform of the voltage occuring at the output of integrator amplifier during normal computer operation is shown in fig. 3.3.2-2. The waveform shows the compute, relay transition and write periods which occur while the switch connects the integrator to a store capacitor. When the switch moves off a store capacitor contact on to an intermediate contact the integrator which is still in the write state becomes unstable. The instability does not affect the store voltage as it only occurs when the integrator is not connected to a store capacitor. This instability occurs because a direct positive feedback path from the output to the (+) input of the amplifier via the transfer capacitor occurs when the store capacitor is disconnected by the switch. This instability causes the amplifier output voltage to increase until it is limited by the voltage swing limits of the operational amplifier at approximately $\pm$ 150 volts and persists until the relays are returned to their compute state (position a). The amplifier is d.c. coupled and recovers almost instantaneously when the positive feedback path is removed.

B. Compensation for charge leakage

The third term in equation 3.3.2-1 for the store voltage increment arises when R_c is connected from the (-) input of the amplifier to earth. This has the effect of producing a positive voltage increment, which is proportional to the voltage across the store capacitor, each time the integrator circuit is switched from 'compute' to 'write'. As all the stored charge leakage effects, discussed in section 3.2.3, produce negative voltage increments which are proportional to the store capacitor voltage, an appropriate choice of R_c provides a means of compensating for all forms of charge leakage.

The reduction in charge leakage achieved using leakage compensation on the integrator is shown in fig. 3.3.2-3. This chart record shows the reduction in leakage from a capacitor initially charged to a fixed voltage which results when the leakage compensation resistor R_c is connected. It is evident that the leakage is reduced from the order of 10% for n = 100 to the order of 1.0%.

The remaining charge leakage on the 5 store capacitors C_{m1} to C_{m5}, connected as shown in the circuit given in fig. 3.2.4, after adjusting the leakage compensation to reduce the leakage on C_{m5} to zero is shown in fig. 3.3.2-4. Unselected components were used. These curves

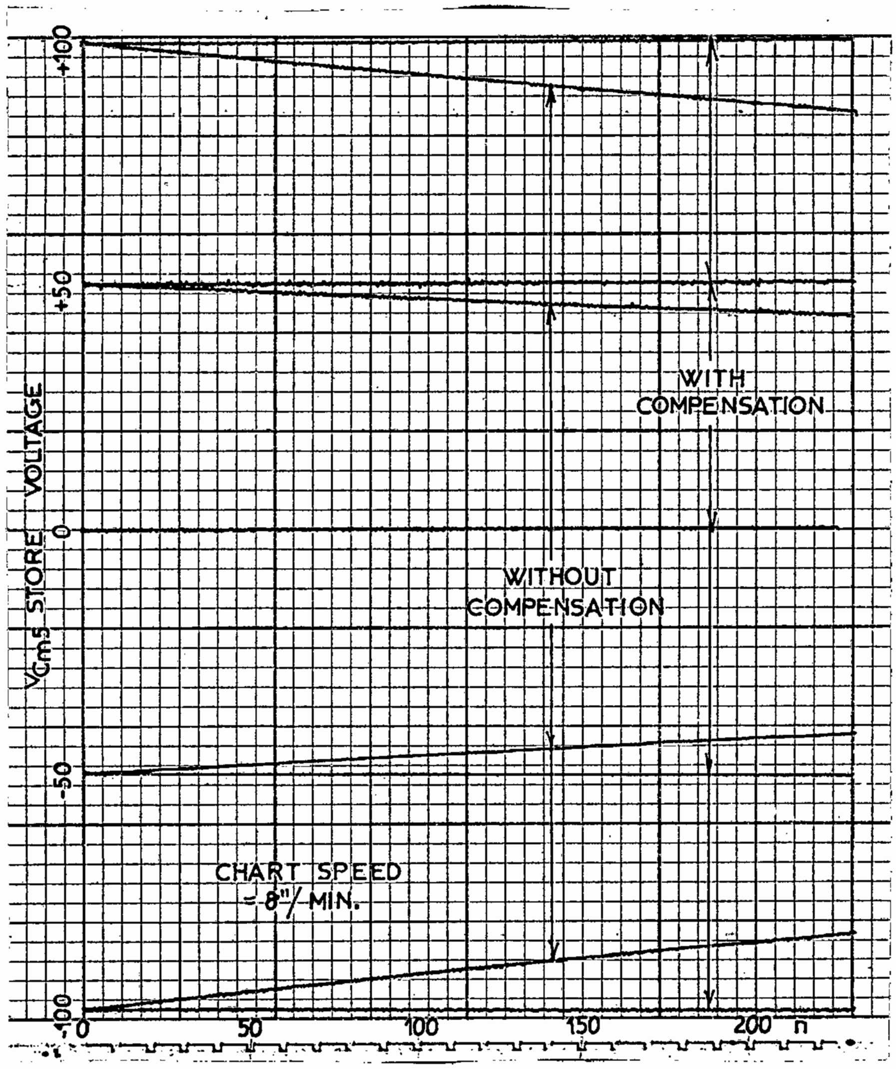

Fig. 3.3.2-3 Reduction in charge leakage using leakage compensation

PARTICULARS

The figure is an original chart recording showing the variation with time of the voltage across store capacitor C_{m5} which is connected to the integrator via the switch, with both switch and integrator operating. Each curve was produced by connecting a d.c. supply to the capacitor, charging it to the supply voltage, before disconnecting the supply at time $t = n = 0$ and allowing the capacitor to discharge due to leakage. The chart was reset and a second curve obtained but with the leakage compensation resistor R_c connected. The process was repeated for initial store voltages of approximately $\pm$ 100, $\pm$ 50 and 0 volts.

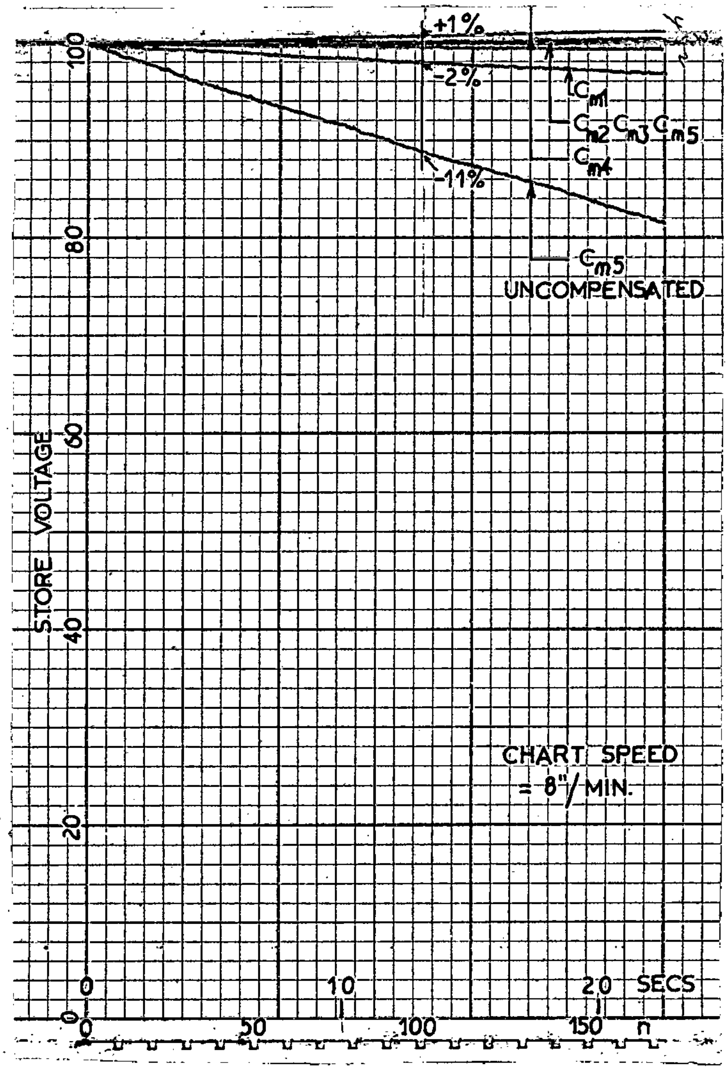

Fig. 3.3.2-4 <u>Reduction of charge leakage using leakage compensation</u>

<u>PARTICULARS</u>

The figure shows the variation with time and number of calculations of the voltage across the 5 main store capacitors, C_{m1} to C_{m5} which were initially charged to + 100 V. The curves were produced using the method described under Fig. 3.3.2-3. The compensation was adjusted to reduce the charge leakage on C_{m5} to zero.

show that the leakage corresponding to n = 100 has been reduced from the order of 11% to less than 1% for capacitors C_{m2}, C_{m3}, C_{m4} and C_{m5} and to 2% for C_{m1}. Inability to compensate perfectly for the leakage occuring on all the store capacitors is due to small variations of the leakage in the components associated with each store capacitor. Perfect compensation is possible if all the components are selected to have identical leakage, or alternatively, additional leakage may be introduced and adjusted to make the leakage from each store capacitor identical.

The accuracy and linearity of the integrator can be verified by comparing the actual and calculated performance. Equation 3.3.2-1 gives the voltage increment per calculation of an ideal integrator as :-

$$\Delta V_{Cm} = \frac{C_t}{C_m} \frac{R_b}{R_a} V_i$$

For a fixed input voltage V_i and zero initial stored charge ($V_{Cm} = 0$) at n = 0, after n calculations the store voltage V_{Cm} should be :-

$$V_{Cm} = n \frac{C_t}{C_m} \frac{R_b}{R_a} V_i \qquad \ldots \quad 3.3.2\text{-}2$$

A chart record of the variation of V_{Cm} with various fixed input voltages is shown in fig. 3.3.2-5. The step nature of the store voltage is readily observed on the curves corresponding to $V_i = \pm 10$ and $\pm$ 30 volts. Points calculated using equation 3.3.2-2 are plotted in fig. 3.3.2-5 and show that the overall error of the integrator and potentiometric recorder is of the order of 1%. The recorder accuracy is essentially eliminated as the recorder was used to set the input

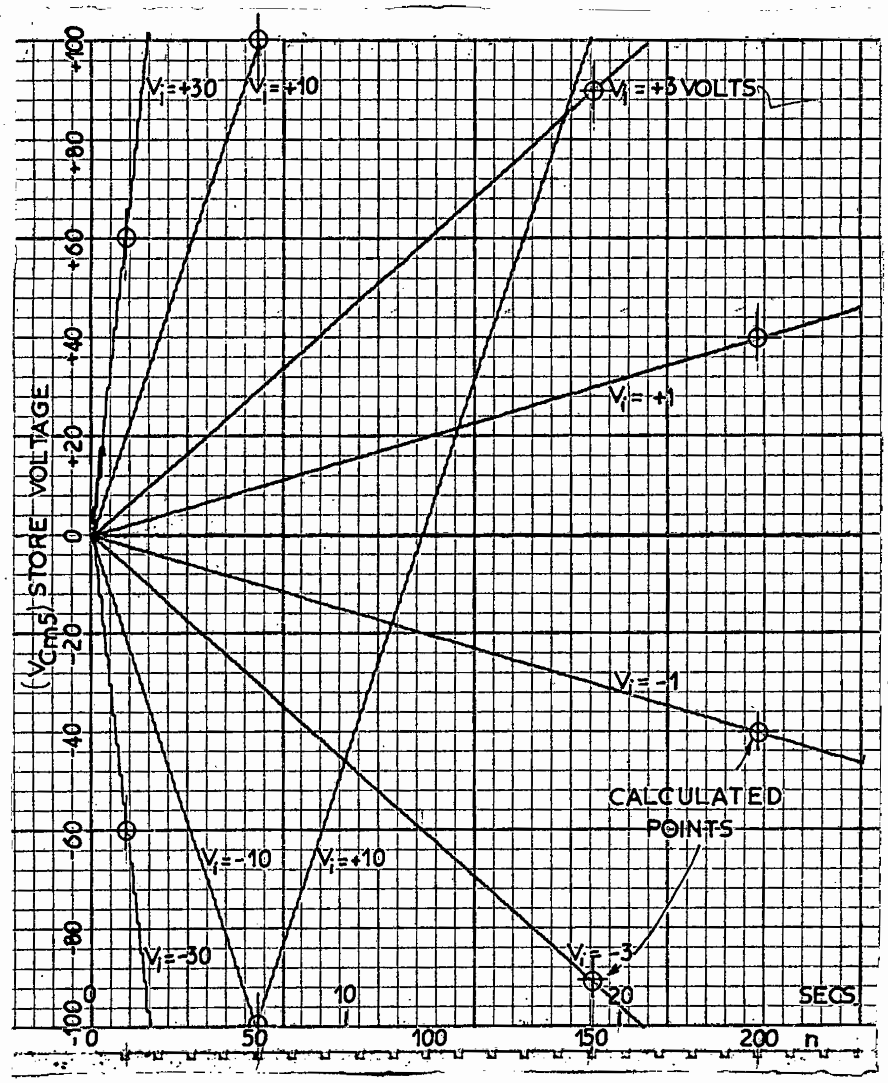

Fig. 3.3.2-5 Chart record showing variation with time of store capacitor voltage V_{cm5}, with various constant voltages V_i applied to the input of the integrator (using leakage compensation).

PARTICULARS

The curves were obtained by connecting the integrator with a fixed input voltage to the switch and store capacitor at a time corresponding to $n = 0$. The set of curves were obtained by resetting the chart, discharging the store capacitor and repeating the process for the different input voltages.

Here, $C_{m5} = 1.00$ µfd, $C_t = 0.080$ µfd, $R_a = 300$ K, $R_b = 750$ K
All components values and voltages measured to ± 1% giving ideally

$$V_{cm} = 0.2\, n\, V_i$$

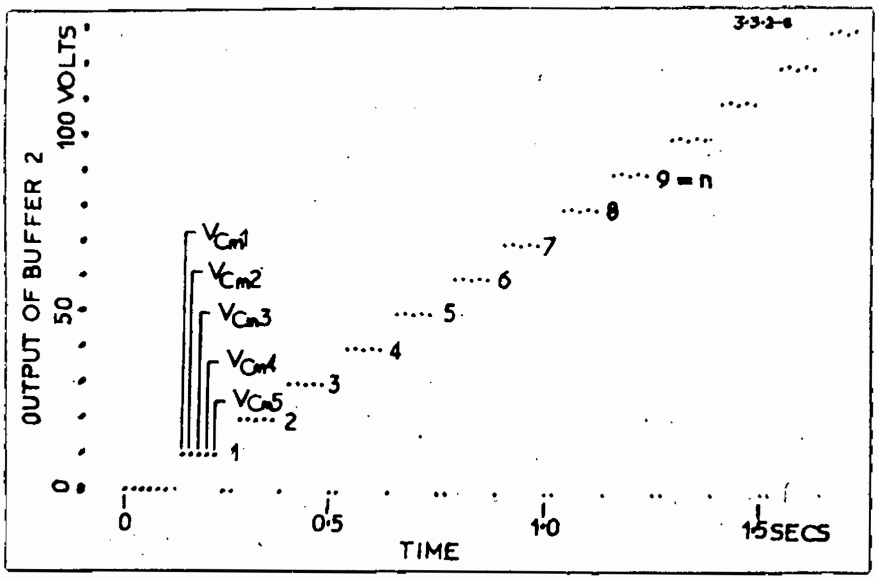

Fig. 3.3.2-6 Oscillograph showing the increase of the 5 store voltages with time with a constant voltage applied to the integrator

PARTICULARS

This photographic record was produced using the intensity modulation technique for recording the store voltages described in section 3.5.2. The 5 dots represent the voltages of the 5 store capacitors. The step ramp is produced by adding a fixed voltage increment ΔV_{cm} to the store capacitors in turn, so that the steps of the ramp correspond to the values $n\Delta V_{cm}$.

(C_m = 1.00 µfd, C_t = 0.08 µfd, R_a = 300K, R_b = 750K, and V_i = 50V so that ΔV_{cm} = 10V.)

voltages V_i.

The accuracy of the integrator with store capacitors other than that used to obtain fig. 3.3.2-5, depends on the capacitance of these capacitors. In the computer the store capacitors were trimmed using small fixed capacitors to 1.00 ± 0.01 μfd. Fig. 3.3.2-6 shows the increase of the 5 store voltages with a fixed voltage applied to the input of the integrator. The voltage differences between the stores are of the order of 1% of the store voltage indicating that the integrator accuracy is essentially the same for all the stores.

3.3.3 BUFFER

The function of the buffer is to isolate the store capacitors from the computing circuits to prevent the discharge of the stored charge. Ideally the buffer should have an infinite input resistance, zero input current and should reproduce the stored voltage exactly at its output.

The differential operational amplifier may be used as an accurate buffer by connecting the output to the negative gain input terminal and applying the input signal to the positive gain input terminal as shown in fig. 3.3.3.

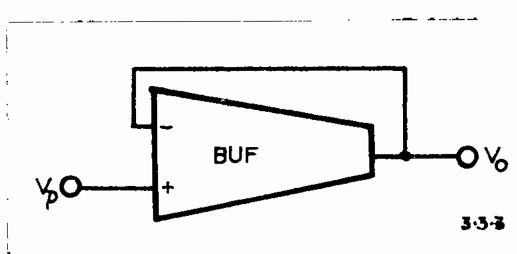

Fig. 3.3.3. Buffer based on the differential operational amplifier

The input to the buffer is applied directly to the grid of the input valve which has been designed to have negligible grid current so that the buffer effectively has an infinite input resistance.

An expression for the output voltage of the buffer giving the error introduced by the finite differential and common-mode gains, G_d and G_c, of the differential operational amplifier is derived in Appendix A2 and is :-

$$V_o = V_p \left[1 - \frac{1}{G_d}(1 - G_c)\right] \qquad \ldots \qquad 3.3.3\text{-}1$$

This expression indicates that the buffer repeats the input voltage V_p with an error of the order of $1/G_d$ when the common-mode gain G_c is less than unity. For $G_d = 2000$ the error is of the order of 0.1%.

3.3.4 <u>DIFFERENCER</u>

The differencer is the circuit used to generate the voltage difference corresponding to the first or second space derivatives. When $\frac{\partial^2 \varphi}{\partial x^2}$ and $\frac{\partial \varphi}{\partial x}$ are replaced by the finite difference expressions $(1/\Delta x^2)(\varphi_{m+1} - 2\varphi_m + \varphi_{m-1})$ and $(1/2\Delta x)(\varphi_{m+1} - \varphi_{m-1})$, the differencer must generate the corresponding voltage differences $K_1(V_{m+1} - 2V_m + V_{m-1})$ and $K_2(V_{m+1} - V_{m-1})$, where K_1, K_2 and V are related to the original variables by scale factors as explained in section 4.1. If both first and second space derivatives are present the differencer must produce an output voltage V_o given by :-

$$V_o = (K_1 + K_2)V_{m+1} - 2K_2 V_m + (K_1 - K_2)V_{m-1} \qquad \ldots \qquad 3.3.4\text{-}1$$

A single differential amplifier may be used to obtain the sum or difference of a number of inputs by using the two input terminals available.

The circuit of a differencer based on the differential operational amplifier is given in fig. 3.3.4.

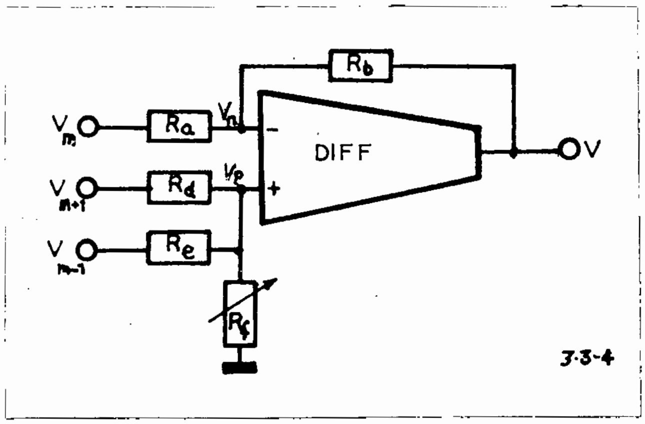

Fig. 3.3.4 Differencer based on the differential operational amplifier

The expression for the output voltage in terms of the input voltage V_m applied to R_a and the voltage V_p applied to the positive gain input is derived in the appendix A3 and is :-

$$V_o = (1 + \frac{R_b}{R_a})\, V_p \left[1 - \frac{1}{G_d}(1 + \frac{R_b}{R_a} - G_c)\right] - \frac{R_b}{R_a} V_m \left[1 - \frac{1}{G_d}(1 + \frac{R_b}{R_a})\right] \quad \ldots \quad 3.3.4\text{-}2$$

This expression shows that the output voltage V_o due to V_m applied to R_a is $-(R_b/R_a)V_m$ as in a normal operational amplifier and is independent of the voltage applied to the positive gain input.

The component of the output voltage due to V_p applied to the positive gain input is in phase with the input but the gain is determined by R_a and R_b. This gain may be adjusted by inserting a resistive divider between the signal and the positive gain input terminal. Using the resistive divider circuit shown in fig. 3.3.4 and neglecting the error terms the output voltage becomes :-

$$V_o = \left(1 + \frac{R_b}{R_a}\right) \left(\frac{R_f}{R_dR_e+R_dR_f+R_eR_f}\right)(R_eV_{m+1} + R_dV_{m-1}) - \frac{R_b}{R_a} V_m \quad \ldots \qquad 3.3.4\text{-}3$$

The values of the resistors required were found using the following method. Equating the coefficients of equations 3.3.4-1 and 3.3.4-3 gives :-

$$\frac{R_b}{R_a} = 2K_1 \quad \text{and} \quad \frac{R_d}{R_e} = \frac{K_1 - K_2}{K_1 + K_2}$$

Having determined these ratios the value of R_f may be found experimentally by noting from equation 3.3.4-1 that applying a fixed voltage V to all three inputs should produce no output voltage. The correct value of R_f is then found by applying + 100V to all three inputs and adjusting R_f to give zero output voltage.

This method of adjusting R_f has the important advantage that it eliminates the spurious φ term error introduced by component tolerances which was discussed in section 2.7. This may readily be verified by applying the criterion of zero output voltage for identical non-zero input voltages to equation 2.7-4

3.3.5 EXPONENTIAL FUNCTION GENERATOR

A diode function generator[14,15] based on a differential operational amplifier was built. It has an output voltage which is an exponential function of the input voltage. The basic circuit is shown in fig. 3.3.5 and additional details are given in Appendix B4.

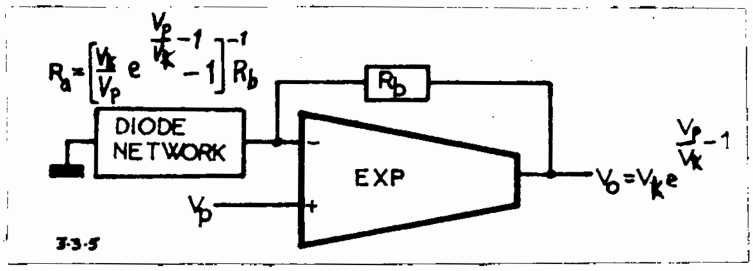

Fig. 3.3.5 Exponential function generator based on the differential operational amplifier

The function generator has a non-linear resistor connected from the negative gain input of the amplifier to ground. The input signal is applied directly to the positive gain input so that the output of the function generator is in phase with the input.

A simple shunt diode network was used for the non-linear resistor. Four straight line segments were used to represent the exponential function with a root-mean-square error of 2.0% over the output voltage range of + 10 to +100 volts.

3.4 AUXILIARY CIRCUITS

The auxiliary circuits provide the power and switching signals necessary for proper operation of the computing elements. These circuits are briefly discussed below while additional details and circuit diagrams appear in the appendices.

3.4.1 INTEGRATOR RELAY DRIVE CIRCUITS

The integrator relay drive circuits control the switching of the integrator A,B and C relays from the 'compute' to the 'write' states, and also relay D in the auxiliary store circuit.

The relay and switch operations are synchronized by pulses obtained from a separate bank of contacts on the switch. These pulses are shaped by a pulse-shaping circuit to remove spurious signals caused by contact bounce. The shaped pulse triggers the compute-delay-monostable circuit which determines the 'compute' period. This circuit in turn triggers the relay-drive monostable circuit which operates the relays and determines the 'write' period.

A pulse obtained from the relay-drive-monostable circuit is used to intensity modulate the oscilloscope (see section 3.5.2) Additional details are given in Appendix B5.

3.4.2 START AND INITIAL CONDITIONS CIRCUIT

The start circuit starts the calculation by disconnecting the initial conditions circuit and connecting the integrator to the switch. This circuit also starts the solution recorders.

These operations are performed by relay E which is interlocked with the switch to ensure that the calculation always commences at a specified boundary point.

Additional details are given in Appendix B6.

3.4.3. D.C. POWER SUPPLIES

Two electronically stabilized power supplies described and constructed by Powell[42] supply ± 300 volts d.c. at up to ± 200 mA for the computing and auxiliary circuits. The circuit is given in appendix B7.

3.4.4. SWITCH MOTOR POWER SUPPLY

The switch motor power supply energises the existing d.c. impulse motor of the switch forcing it to run as a synchronous motor. As the solution recorder uses a synchronous motor for the chart drive, the fixed relationship between switch and chart speed allows the superposition of different solutions against the same time scale on the recorder.

Additional details are given in appendix B1.

3.5 SOLUTION RECORDERS

Graphical solutions to the problems which involve 3 variables (φ, $t = n\Delta t$, $x = m\Delta x$) were recorded by plotting :-

a φ against n for various values of m (Corresponds to the temperature against time at various distances from the surface of a slab.)

b φ against m for various n (Corresponds to the temperature distribution in the slab at successive instants of time)

The third combination of n against m for various φ cannot be obtained as n and m are integers with only φ continuous.

Two forms of display are used :-

3.5.1 CHART RECORDER

A high-speed potentiometric chart recorder may be connected to any of the store capacitors, via a buffer amplifier or cathode follower, to plot the variation of the store voltage against time. The solution time scale corresponding to fixed values of n is simultaneously recorded using an event marker. The time scale accuracy is then independent of the switch and chart speeds.

Using synchronous switch and chart drive motors, a number of different solutions may be superimposed against the same time scale. Fig. 4.2.4 is an example of this form of display.

The chart recorder used has an accuracy of 0.5% of full scale with a full scale response time of 0.5 seconds. With a solution time of 30 seconds the recording accuracy will be of the order of 1%.

3.5.2 OSCILLOSCOPE

The variation of all the store voltages with time may be displayed using an oscilloscope. This is achieved by displaying the waveform at the output of the φ_m buffer using a linear sweep. The waveform consists of a fixed voltage portion when the wiper is connected to a store capacitor and a random voltage portion when the wiper is on an intermediate contact. The undesired portions of the waveform may be suppressed by applying brightness modulation to brighten up the trace momentarily each time the buffer is connected to a store capacitor. The store capacitor voltages then appear as a series of dots.

If the sweep period is chosen equal to the solution time a dotted outline of the solution φ against t for the various values of n is produced. Fig. 4.2-2b is an example of such a display. If the sweep period is chosen equal to the time taken to scan all the capacitors, the dots trace the outline of the φ against x profile for successive time intervals. Fig. 4.2-2a is an example of this display. Using a storage oscilloscope the amplitude and time scale markers may be superimposed and the complete display photographed for future reference. The amplitude markers are produced by integrating a fixed voltage as shown in fig. 3.3.2-6 while the time markers are obtained from the event marker signal.

4.0 COMPUTER SOLUTIONS OF SOME DIFFUSION EQUATIONS

The analogue computer was used to calculate some solutions of the diffusion equations :-

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_1\varphi + K_2 \qquad \ldots \qquad 4.0\text{-}1$$

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_3\frac{\partial\varphi}{\partial x} + K_2 \qquad \ldots \qquad 4.0\text{-}2$$

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_4 e^{\varphi} \qquad \ldots \qquad 4.0\text{-}3$$

Equations 4.0-1 and 4.0-2 are linear while equation 4.0-3 is non-linear. All the equations include terms which represent internal heat generation in a slab and are typical of equations encountered in the study of self-heating.

The linear equations 4.0-1 and 4.0-2 have known analytical solutions and are used to check the accuracy of the complete computer with the exception of non-linear elements which are checked separately. These equations include terms similar to the spurious terms introduced into the equation due to store leakage and component tolerances as discussed in section 2.7 so that these solutions may be used to estimate the error due to these spurious terms.

The non-linear equation 4.0-3 has no analytical solution. A few solutions, calculated by Copple using a differential analyser[19], are used to indicate the relative accuracy of the methods. A further set of solutions are given for different boundary conditions typical of situations encountered in practical self-heating problems.

4.1 SETTING-UP EQUATIONS ON THE COMPUTER

Before an equation can be solved on the computer it is necessary to :-

1 Rewrite equation in terms of finite differences using the methods described in section 2.

2 Introduce scaling factors into the difference equation

The finite difference equations must be written so that a signal voltage in the computer representing a term in the equation does not exceed the maximum voltage limits of the equipment. This manipulation known as scaling consists of inserting coefficients known as scale factors, which relate the voltages in the computer with the terms they represent.* Methods of scaling equations for solution on general purpose analogue computers have been described[14,15] and may be applied to this computer.

The scaling calculation for equation 4.0-3 is given in Appendix C. The same scaling is used for equations 4.0-1 and 4.0-2 where these represented simple cases of equations 4.0-3 and are used as preliminary computer checks. Equation 4.0-2 was also solved for different boundary conditions and was rescaled and the scaled equation and component values are given in table 4.2.

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* The dimensions of the scale factor are volts for a non-dimensional equation such as is considered here. If the equation is written in terms of physical variables e.g. temperature (°C) the scale factor will have the dimension volts/°C.

4.2 SOLUTIONS

Several solutions of equations 4.0-1, 4.0-2 and 4.0-3 were computed for the various coefficient values and boundary conditions detailed in table 4.2. The corresponding scaled equations, and computer set-ups and solutions are given in table 4.2, fig. 4.2-1 and figs.4.2-2 to 4.2-9 respectively.

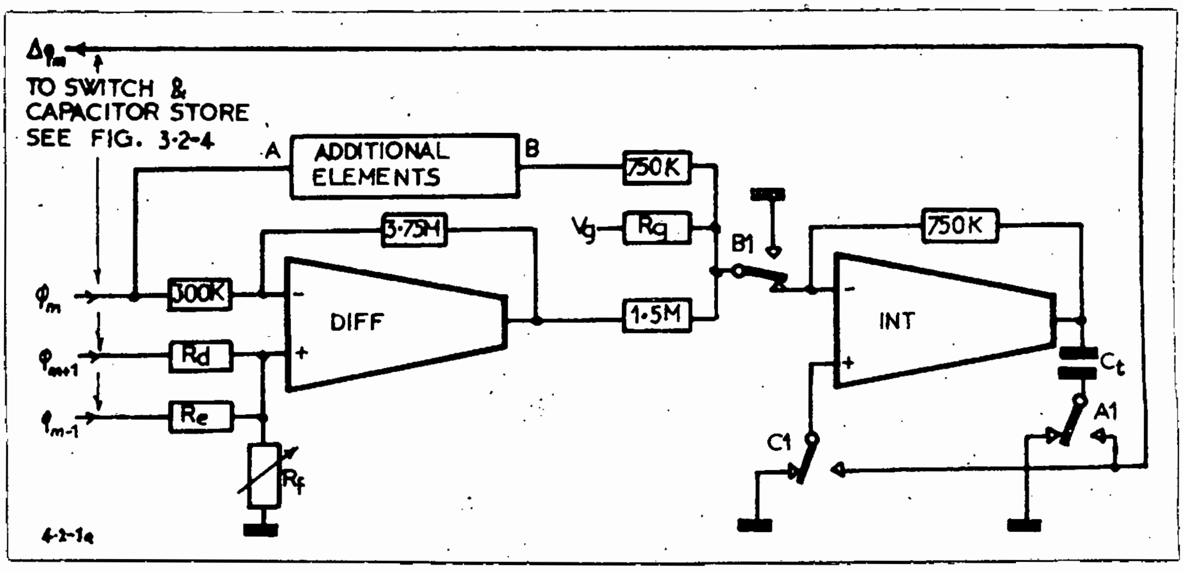
The solutions are primarily intended to illustrate the operation of the computer and to allow an assessment of overall accuracy of the instrument to be made. However these equations also have practical applications and a brief discussion of their physical significance in the study of self-heating is given in the following paragraphs.

Unless specifically mentioned in the following discussions, it is assumed that the boundary conditions define one half of a slab which has a symmetrical temperature distribution, zero surface temperature and a uniform zero initial temperature distribution.

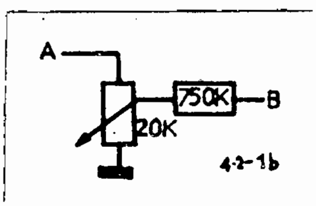
CASE	PARTIAL DIFFERENTIAL EQN	CO-EFFICIENT	CONDITIONS: BOUNDARY	CONDITIONS: INITIAL	EXISTING SOLUTION Ref & Pg	GRID Δx	GRID Δt	SCALED EQUATION	CONDITIONS: BOUNDARY volts	CONDITIONS: INITIAL volts	CIRCUIT Fig 4.2-1	R_d[2] K	R_e[2] K	R_g M	V_g volt	C_t μfd	SOLUTION in Fig
Eqn 4.0-1 $K_1 = 0$ $K_2 = 0$	$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2}$	-	$\varphi_{b1} = 0$ at $x = 0$ $\frac{d\varphi}{dx} = 0$ at $x = 1$	$\varphi_i = 2.5$	6, 101	0.2	0.01	$40\Delta\varphi_m = \{[(40\varphi_{m+1} - 80\varphi_m + 40\varphi_{m-1})6.25]0.5\}0.08$	S_B[3] = SYM $\varphi_{b1} = 0$	$\varphi_i = 100$	a	300	300	-	-	0.03	4.2-2
$K_1 = 0$ $K_2 = 5$	$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + 5$	5	do	$\varphi_i = 0$	6, 130	do	do	$40\Delta\varphi_m = \{[(40\varphi_{m+1} - 80\varphi_m + 40\varphi_{m-1})6.25]0.5 + [25]1.0\}0.08$	do	$\varphi_i = 0$	a	do	do	0.75	25	do	4.2-4
$K_1 \neq 0$[1] $K_2 = 1$	$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_1\varphi + 1$	0, 0.5 1.0, 1.5 2.0, 4.0	do	do	6, 404	do	do	$40\Delta\varphi_m = \{[(40\varphi_{m+1} - 80\varphi_m + 40\varphi_{m-1})6.25]0.5 + [(40\varphi_m)K_1/4]0.5 + [10]0.5\}0.08$	do	do	a + b	do	do	1.50	10	do	4.2-5
Eqn 4.0-2 $K_2 = 4$ $K_3 \neq 0$	$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_3\frac{\partial\varphi}{\partial x} + 4$	+1 0 -1	do	do		do	do	$40\Delta\varphi_m = \{[(40\varphi_{m+1} - 80\varphi_m + 40\varphi_{m-1})6.25 + (40\varphi_{m+1} - 40\varphi_{m-1})0.625K_3]0.5 + [40]0.5\}0.08$	do	do	a	300 300 367	367 300 300	1.50	40	do	4.2-6
$K_2 = 4$ $K_3 \neq 0$	$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_3\frac{\partial\varphi}{\partial x} + 4$	+2 0	$\varphi_{b1} = 0$ at $x = 0$ $\varphi_{b2} = 0$ at $x = +2$	do		0.33	0.005	$50\Delta\varphi_m = \{[(50\varphi_{m+1} - 100\varphi_m + 50\varphi_{m-1})6.25 + (50\varphi_{m+1} - 50\varphi_{m-1})1.04K_3]0.5 + [69.5]1.0\}0.0144$	S_B = ASYM $\varphi_{b1} = 0$ $\varphi_{b2} = 0$	do	a	300 300	600 300	0.75	69.5	0.0144	4.2-3
Eqn 4.0-3 $K_4 \neq 0$	$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_4 e^{\varphi}$	0.50, 0.75 0.95, 1.00 1.05, 1.50 2.00	$\varphi_{b1} = 0$ at $x = 0$ $\frac{d\varphi}{dx} = 0$ at $x = 1$	do	19	0.2	0.01	$40\Delta\varphi_m = \{[(40\varphi_{m+1} - 80\varphi_m + 40\varphi_{m-1})6.25]0.5 + [(10e^{40\varphi_m/40})0.5K_4]1.0\}0.08$	S_B = SYM $\varphi_{b1} = 0$	do	a + c	300	300	-	-	0.08	4.2-7 4.2-9
$K_4 \neq 0$	$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_4 e^{\varphi}$	0, 0.5 0.75, 1.0 2.0	do	$\varphi_i = 1$	-	do	do	do	do	$\varphi_i = 40$	a + c	do	do	-	-	do	4.2-8 4.2-9

Note 1 : Values of the coefficient tabulated in the third column.
2 : R_f adjusted to produce zero output voltage from the difference amplifier with +100 volts applied to the three input terminals.
3 : Refers to the position of the boundary condition switch S_B shown in fig. 3.2-4.

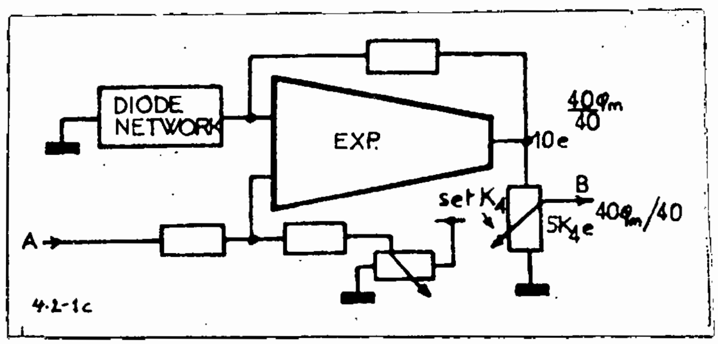
TABLE 4.2 DETAILS OF PARTIAL DIFFERENTIAL EQUATIONS SOLVED ON THE COMPUTER



(a) Differencer and integrator circuit.



(b) $K_1\varphi$ term circuit



(c) Exponential function generator

Fig. 4.2-1 Computer set-up used to obtain solutions

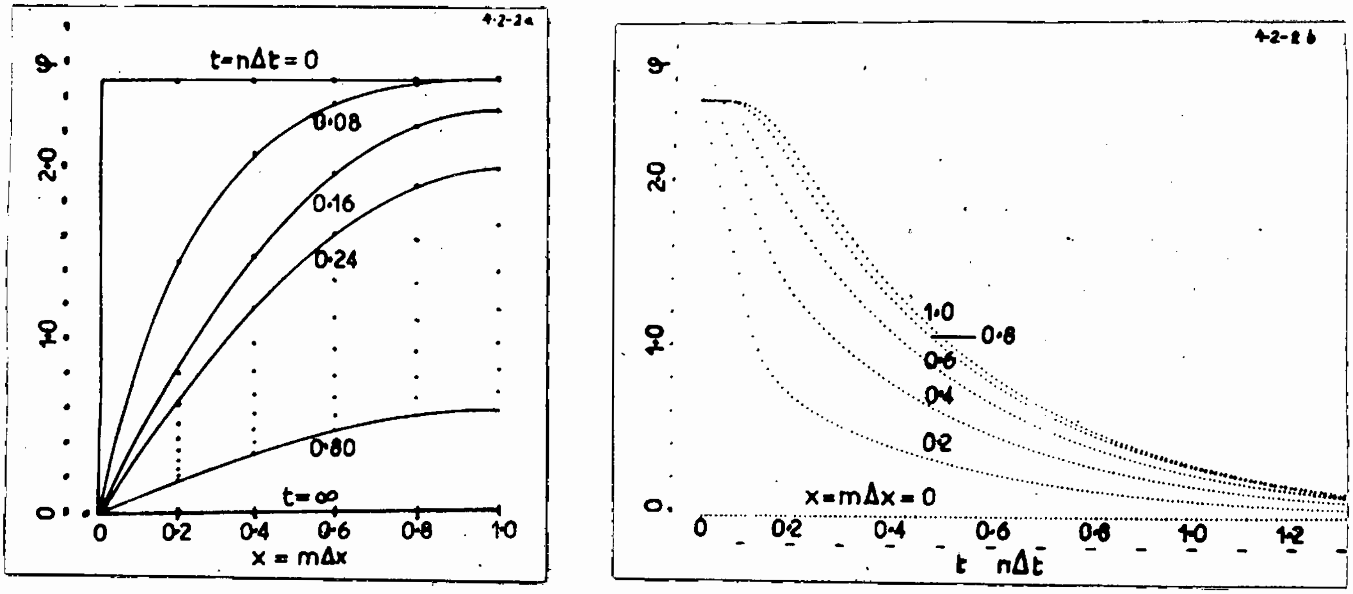
(Interconnections and component values given in table 4.2.)

4.2.1 $\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial t^2} + K_1\varphi + K_2$

When $K_1 = K_2 = 0$ the solution of the equation for a non-zero uniform initial temperature shown in fig. 4.2-2 gives the cooling curves of a slab with a fixed surface temperature. The solution when $K_1 = 0$ but $K_2 \neq 0$ given in fig. 4.2-4 describes the variation of temperature with time in a slab with uniform internal heat generation and a fixed surface temperature. A comparison of the computed curves and published values provide a check on the accuracy of the complete computer with the exception of the non-linear units. The calculated points superimposed on fig. 4.2-4 indicate that the solution accuracy is of the order of 1%.

When $K_1 \neq 0$, and $K_2 \neq 0$ the solution given in fig. 4.2-5 describes the temperature in the centre of a slab with internal heat generation which increases linearly with temperature. The case $K_1 = K_2 = 1$ represents a linear approximation of $K_4 e^{\varphi}$ when $K_4 = 1$. Comparison of the corresponding curves in figs. 4.2-5 and 4.2-7 shows that solutions are appreciably different indicating the danger of using linear approximations to non-linear equations.

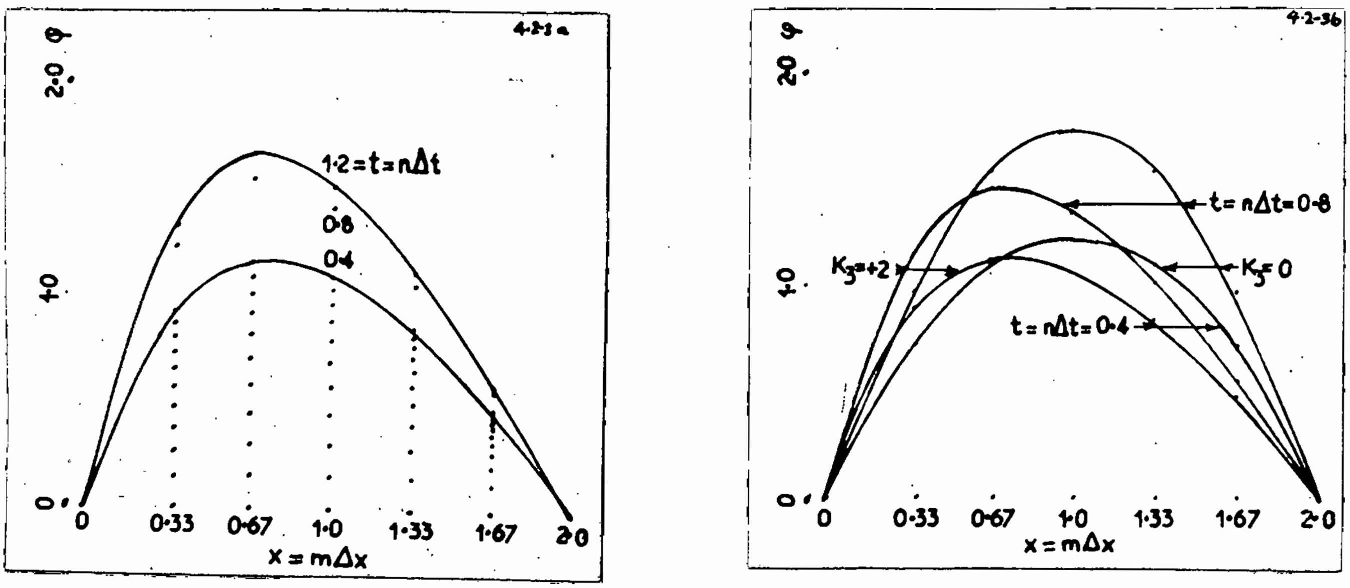
As explained in section 2, store leakage and component tolerances give rise to spurious terms of the form $K\varphi$. From fig. 4.2-5 it appears that $K = 0.5$ gives an error of the order of 30%. This figure was used to estimate the permissable store leakage and component tolerances for a specified error in section 2.7.



(a) φ vs x for various t

(b) φ vs t for various x

Fig. 4.2-2 Solution of $\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2}$ with boundary conditions $\varphi = 0$ at $x = 0$ $\frac{d\varphi}{dx} = 0$ at $x = 1$ and initial condition $\varphi = 2.5$ at $t = 0$
(Photograph of oscillographic display)



(a) φ vs x for various t for $K_3 = +2$

(b) φ vs x for t = 0.4, 0.8 and $K_3 = 0, + 2$.

Fig. 4.2-3 Solution of $\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial t^2} + K_3 \frac{\partial\varphi}{\partial x} + 4$ for $K_3 = 0, + 2$ with boundary conditions $\varphi = 0$ at $x = 0$ and $x = 2$ and initial condition $\varphi = 0$ at $t = 0$.
(Photograph of oscillographic display.)

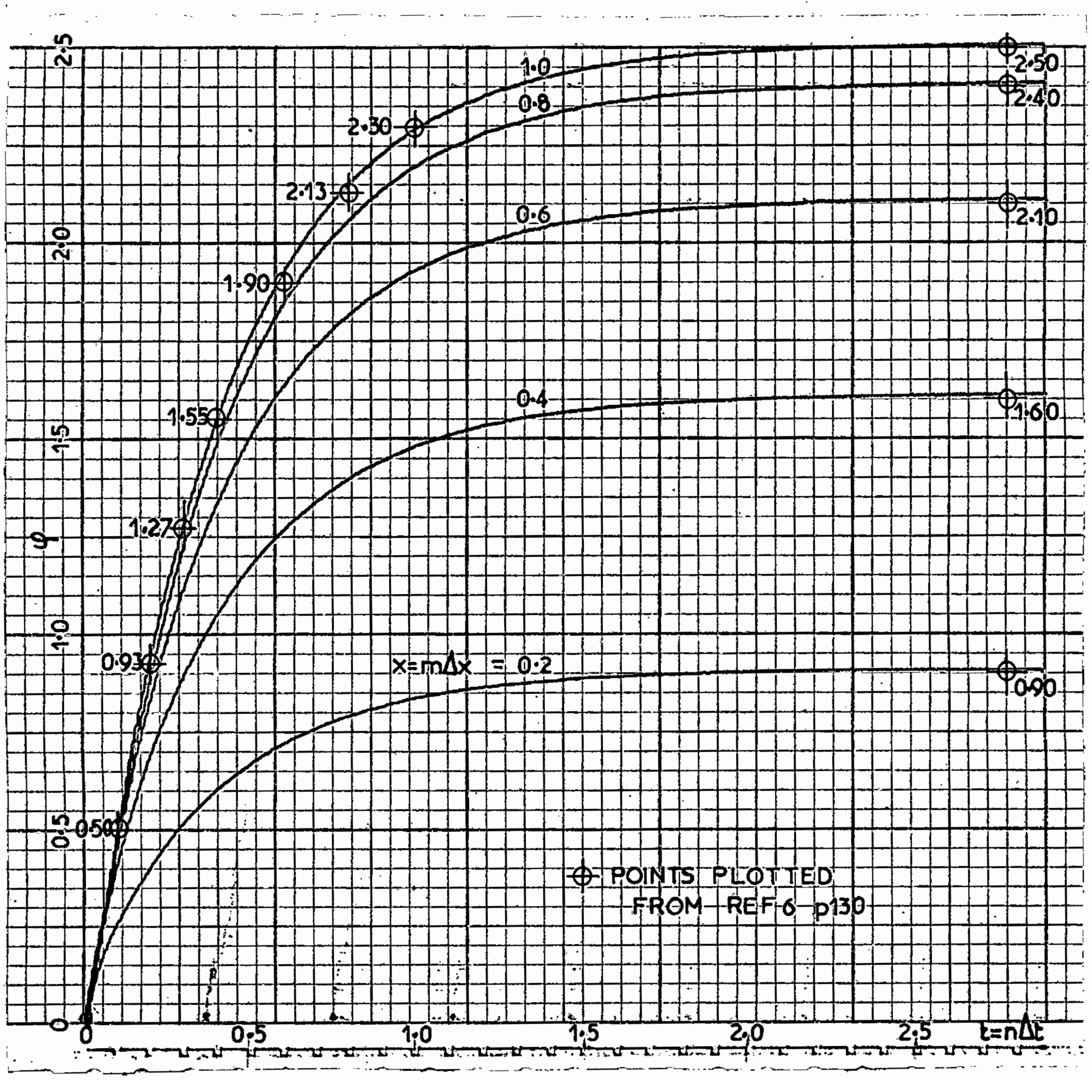


Fig. 4.2-4 Solution of $\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + 5$ for $\varphi = 0$ at $x = 0$, $\frac{d\varphi}{dx} = 0$ at $x = 1$ and $\varphi = 0$ at $t = 0$ (Original chart recording) Also some calculated values plotted against same axes.

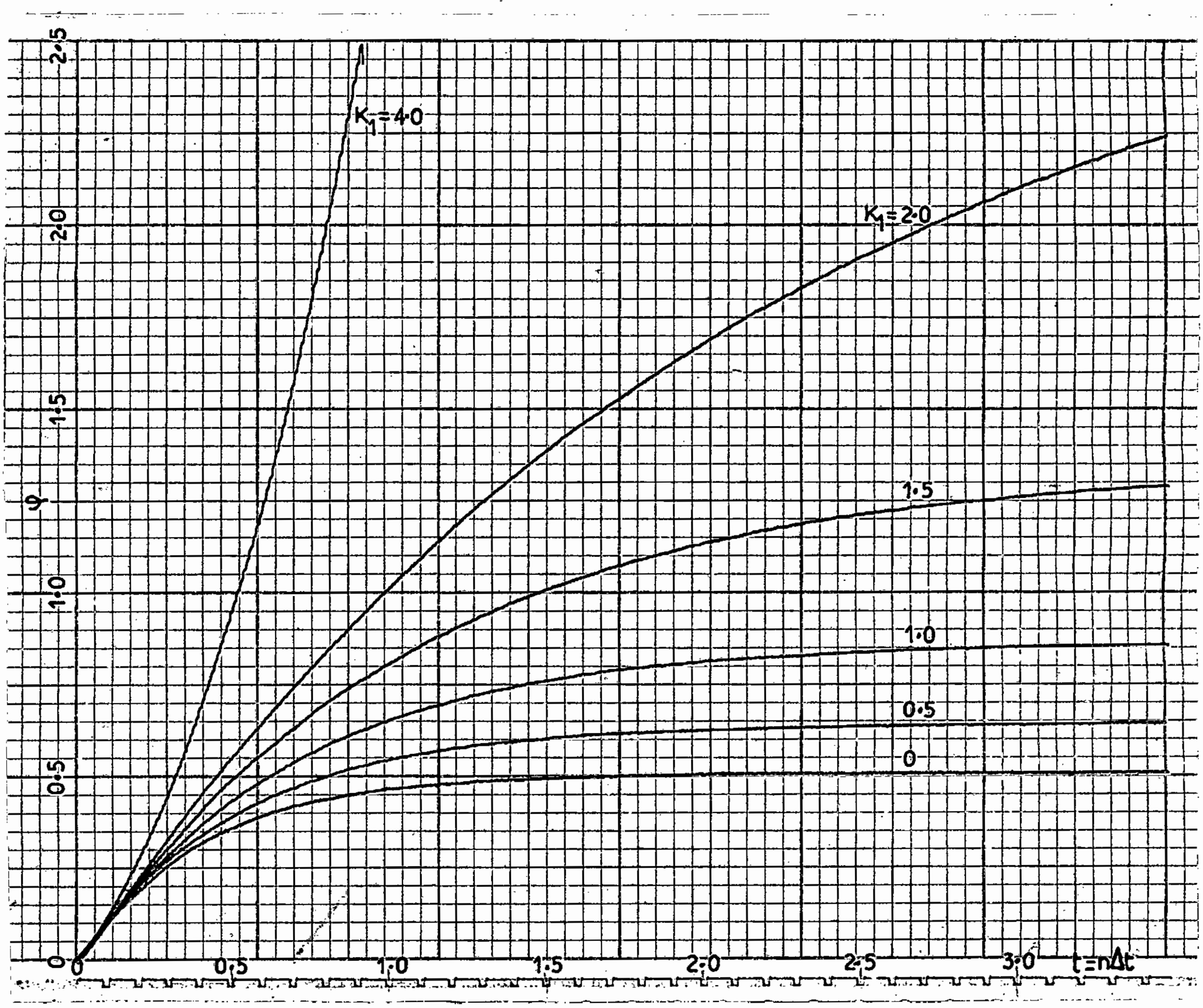


Fig. 4.2-5 Solution of $\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_1\varphi + 1$ at point $x = 0$ for $K_1 = 0$, 0.5, 1.0, 1.5, 2.0, 4.0 with $\varphi = 0$ at $x = 0$, $\frac{d\varphi}{dx} = 0$ at $x = 1$ and $\varphi = 0$ at $t = 0$. (Original chart recording.)

4.2.2 $\underline{\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_3 \frac{\partial\varphi}{\partial x} + K_2}$

When $K_2 \neq 0$, $K_3 \neq 0$ these solutions describe the temperature within a slab with uniform internal heat generation with additional heat transfer due to wind blowing through the slab. (Assumes thermal equilibrium between gas and solid and negligible density changes in gas.) This causes the temperature distribution to become unsymmetrical as shown in figs. 4.2-3 (a) and (b). These solutions demonstrate the ease with which first order space derivative may be included.

The first derivative term is introduced by modifying the adder input resistor values. Small modifications due to resistor tolerances similarly introduce unwanted derivative terms as explained in section 2.7. Fig. 4.2-6 shows that an unwanted term of 1.0 $\frac{\partial\varphi}{\partial x}$ would introduce an error into the solution given in fig. 4.2-4 of the order of 20%. This figure was used to estimate the permissable component tolerance for a specified error in section 2.7.

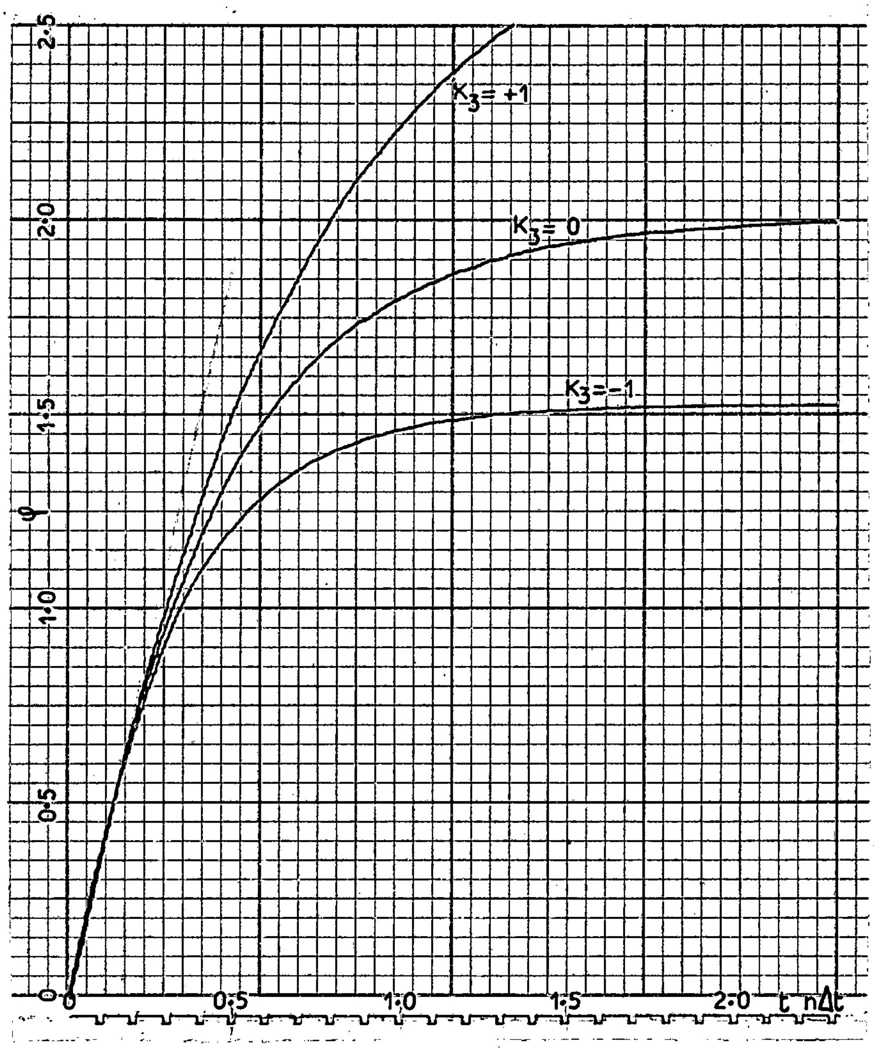
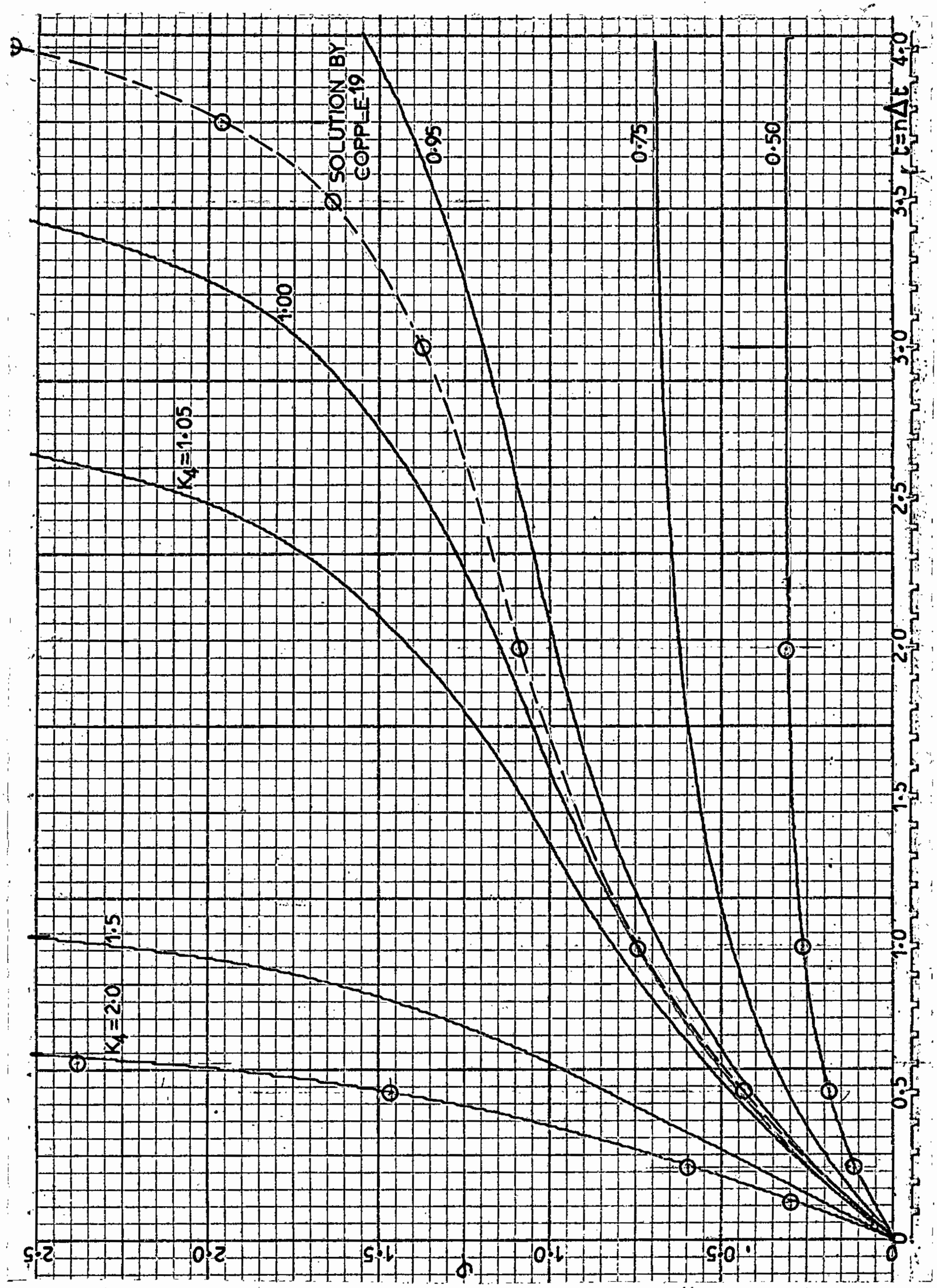


Fig. 4.2-6 Solution of $\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2} + K_3 \frac{\partial \varphi}{\partial x} + 4$ at point $x = 1$, for $K_3 = -1, 0 +1$ with $\varphi = 0$ at $x = 0$ $\frac{d\varphi}{dx} = 0$ at $x = 1$ and $\varphi = 0$ at $t = 0$ (Original chart recording.)

4.2.3 $\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_4 e^{\varphi}$

The solutions given in figs. 4.2-7 and 4.2-9 (a) and (b) describe the temperature in the centre of a slab of material with internal heat generation which increases exponentially with temperature. Earlier solutions, obtained by Copple using a differential analyser[19] for the case of zero initial temperature are plotted in fig. 4.2-7. Correspondence is good except for the $K_4 = 1$ case where some difference is apparent. However also shown in Fig. 4.2-7 are the curves for $K_4 = 0.95$ and 1.05. These curves indicate that the $K_4 = 1.00$ curve falls in a critical region and that a change of the order of 3% in K_4 is required to explain the difference between the earlier and present results.

A second set of solutions for a non-zero initial temperature case are shown in figs. 4.2-8 and 4.2-9 (c) and (d). These solutions are typical examples of solutions required in self-heating work. They represent a stack built with material which has a temperature above ambient. A point of interest is that although shape of curves are different the critical case still occurs between $K_4 = 0.75$ and $K_4 = 1.00$.



<u>Fig. 4.2-7</u> Solution of $\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_4 e^{\varphi}$ at point $x = 1$, for $K_4 = 0.5$, 0.75, 0.95, 1.00, 1.05, 1.5 and 2.0 for $\varphi = 0$ at $x = 0$, $\frac{d\varphi}{dx} = 0$ at $x = 1$ and $\varphi = 0$ at $t = 0$ (Original chart recording.) Earlier solutions by Copple[19] included for comparison.

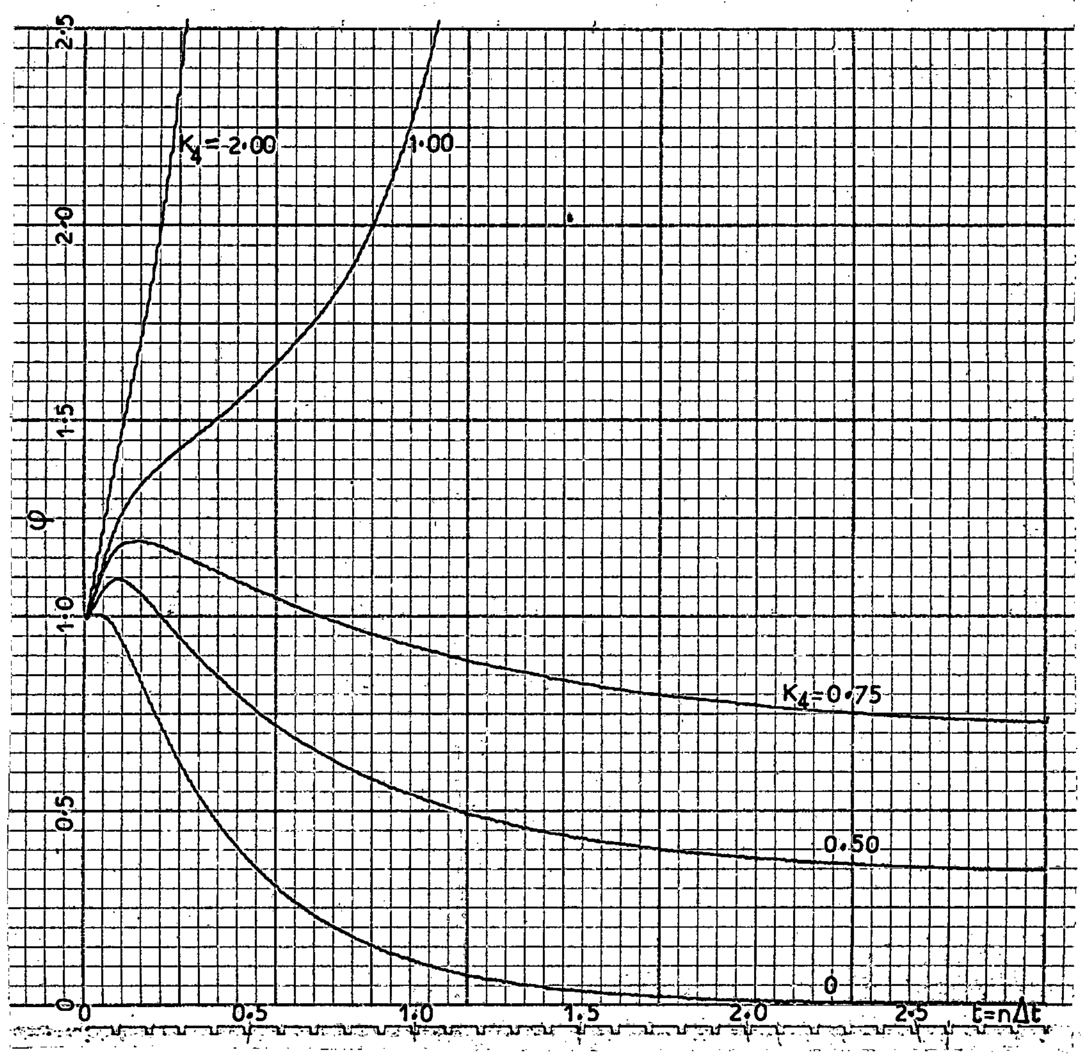
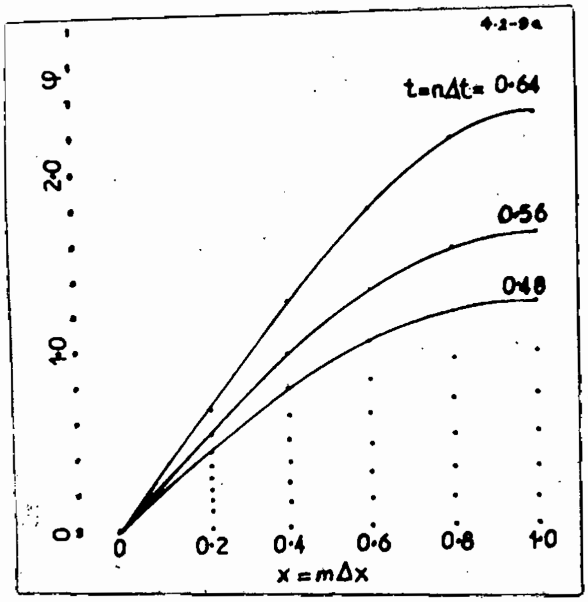
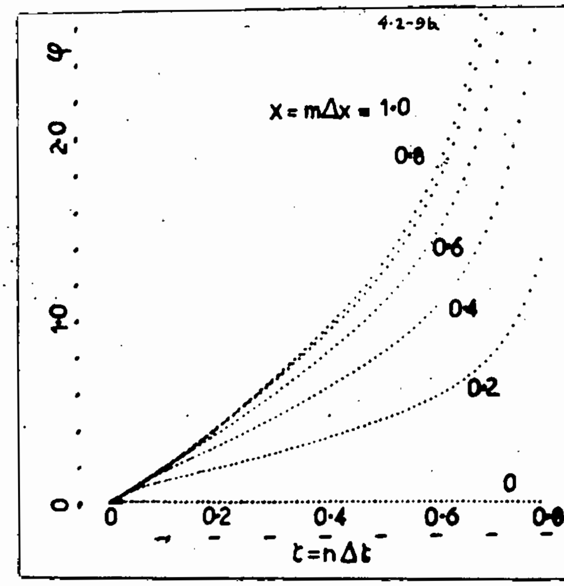


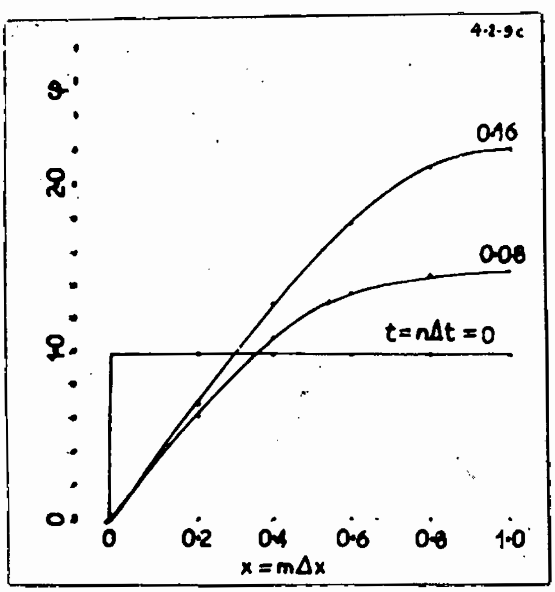
Fig. 4.2-8 Solution of $\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_4 e^{\varphi}$ at point $x = 1$ for various values of K_4 with $\varphi = 0$ at $x = 0$, $\frac{d\varphi}{dx} = 0$ at $x = 1$ and $\varphi = 1$ at $t = 0$ (Original chart recording.)

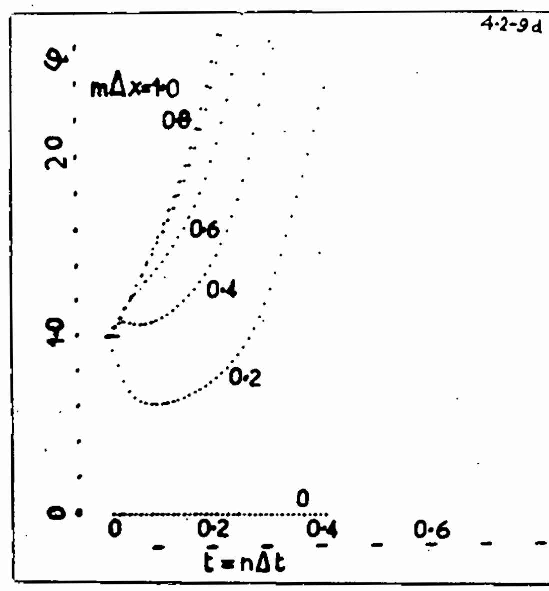




(a) <u>φ vs x for various values of t</u> (b) <u>φ vs t for various values of x</u>

<u>φ = 0 at t = 0</u>





(c) <u>φ vs x for t = 0,0.08 and 0.16</u> (d) <u>φ vs t for various values of x</u>

<u>φ = 1 at t = 0</u>

<u>Fig. 4.2-9</u> Solutions of $\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + 2e^{\varphi}$ with $\varphi = 0$ at $x = 0$, $\frac{d\varphi}{dx} = 0$ at $x = 1$. (Photographs of oscillographic display.)

5 DISCUSSION

An analogue computer for the solution of certain partial differential equations based on the principles outlined in the introduction has been successfully constructed and tested. The main features of this instrument are that (i) it utilises the finite difference method of solution ; (ii) all computations are performed in a single computing block which is time shared with all the nodes, resulting in an economical use of the equipment; (iii) all computations are performed using analogue techniques to retain the advantages of ease of non-linear function generation and graphical solution display.

By basing the computing elements on a differential operational amplifier all the computing elements could be realized using relatively simple circuits. In particular the choice of this type of amplifier for the integrator circuit made possible the use of a simple compensation circuit which compensates for the loss of charge, due to leakage, from the capacitor store. This compensation is an important factor contributing to the accuracy of the computer.

It is shown that the error in the solution passes through a minimum as the finite difference grid spacing is varied. This occurs because the truncation errors decrease while the errors due to spurious terms introduced into the equation by component tolerances increase as the grid spacing is decreased. A similar property has been observed when partial differential equations are solved using finite difference methods on the electronic differential analyser. Spurious terms are introduced by store leakage and component tolerances in the differencer circuit. Using store leakage increases because the amount of information read from the store at any instant is increased so that each store capacitor must be connected to more switch contacts. Also a greater number of points must be calculated the time taken to compute a time interval is increased.

It also appears practical to use the computer to solve certain elliptic partial differential equations in addition to the quadratic partial differential equations considered here. This could be done by replacing first time derivative by a second time derivative in which case a second auxiliary store will be required.

The speed of the computer is limited by the electro-mechanical components, switch, integrator relays and solution recorders, so that a single solution takes approximately 30 seconds.

During the course of this investigation, a similar equation with the non-linear function given in reference 5 was solved, using a similar grid spacing and to a similar accuracy, on a Ferranti Perseus digital computer.[31] The solution time was of the order of 10 minutes for solutions in the form of tables of co-ordinates. A large amount of the solution time was occupied by the calculation, using series expansions, of the relatively complicated non-linear function. In the analogue computer the solution time is not affected by the complexity of the non-linear function as all the terms are calculated by separate computing elements working in parallel. It is apparent that when working to the same accuracy the speed of the analogue computer compares favourably with available digital computers with the further advantage that the solutions are produced directly in graphical form.

A characteristic of the computer is that the type of partial differential equation, geometry and grid spacing used, determine the pattern of switch connections and are not readily changed, while the non-linear functions, coefficients, initial and boundary conditions may be easily adjusted. Using the convenient storage oscilloscope form of solution display the computer is well suited for rapid investigations of the effect of parameter variations in a given type of equation. As such the computer supplements existing analogue techniques for the solution of non-linear partial differential equations and should find applications in fields other than that for which it was originally designed - the solution of equations encountered in the self-heating of fishmeal.

APPENDIX A : ERROR DUE TO THE FINITE DIFFERENTIAL AND COMMON-MODE GAINS OF THE DIFFERENTIAL AMPLIFIER

A1. INTEGRATOR ERROR

Consider the integrator circuit shown in fig. 3.3.2-1.

1. Relays in position 'a'

The sum of the currents flowing into (-) terminal of amplifier must equal zero. As ideally grid current is zero we get :

$$\frac{V_i - V_n}{R_a} + \frac{V_o - V_n}{R_b} - \frac{V_n}{R_c} = 0 \qquad \ldots \quad \text{A1-1}$$

The output voltage V_o due to input voltages V_n and V_p may be expressed in terms of the differential gain G_d and the common-mode gain G_c as :

$$V_o = G_d(V_p - V_n) + G_c \frac{V_n + V_p}{2} \qquad \ldots \quad \text{A1-2}$$

combining equations A1.1 and A1.2 and eliminating V_n gives :

$$V_o = \frac{(1 + \frac{G_c}{2G_d}) \frac{R_b}{R_p} V_p - (1 - \frac{G_c}{2G_d}) \frac{R_b}{R_a} V_i}{1 + \frac{1}{G_d}(\frac{R_b}{R_p} - \frac{G_c}{2})} \qquad \ldots \quad \text{A1-3(a)}$$

where $R_p = \frac{R_a R_b R_c}{R_a R_b + R_a R_c + R_b R_c}$

If $G_d \gg 1$ equation A1-3(a) may be written by neglecting second order terms as

$$V_o = \left[(1 + \frac{G_c}{2G_d}) \frac{R_b}{R_p} V_p - (1 - \frac{G_c}{2G_d}) \frac{R_b}{R_a} V_i\right]\left[1 - \frac{1}{G_d}(\frac{R_b}{R_p} - \frac{G_c}{2})\right]$$

or

$$V_o = \left[1 + \frac{1}{G_d}(G_c - \frac{R_b}{R_p})\right] \frac{R_b}{R_p} V_p - (1 - \frac{1}{G_d}\frac{R_b}{R_p}) \frac{R_b}{R_a} V_i \qquad \ldots \quad \text{A1-3b}$$

With relays in position 'a' $V_2 = 0$ and $V_o = V_{o(a)}$ which is voltage applied across C_t so that :

$$V_{o(a)} = V_{C_{t(a)}} = -\frac{R_b}{R_a} V_i \; (1 - \frac{1}{G_d}\; \frac{R_b}{R_p}) \qquad \ldots \quad \text{A1-3c}$$

where the suffix (a) indicates that this is the voltage with the relays in position 'a'

2. Relays in position 'b'

Capacitors C_t and C_m are now connected in series by relay A so that

$$V_{C_{m(b)}} = V_{C_{m(a)}} + \frac{Q}{C_m} \qquad \ldots \quad \text{A1-4}$$

$$V_{o(b)} = V_{C_{m(a)}} + V_{C_{t(a)}} + \frac{Q/C_tC_m}{C_t + C_m} \qquad \ldots \quad \text{A1-5}$$

where Q is the charge flowing into C_t and C_m when the relays go from positions 'a' to 'b'.

Eliminating φ from equations A1-4 and A1-5 and solving for $V_{C_{m(b)}}$ gives

$$V_{C_{m(b)}} = \frac{C_m V_{C_{m(a)}} + C_t V_{o(b)} - C_t V_{C_{t(a)}}}{C_t + C_m} \qquad \ldots \quad \text{A1-6}$$

With the relays in position 'b' $V_p = V_{C_{m(b)}}$ and $R_a = \infty$ (effectively as relay B is open.) Substituting these values in equation A1-3b gives :

$$V_{o(b)} = (1 + \frac{R_b}{R_c})V_{C_{m(b)}} \left[1 - \frac{1}{G_d}(1 + \frac{R_b}{R_c} - G_c)\right] \qquad \ldots \quad A1\text{-}7$$

Substituting equations A1-3 and A1-7 in equation A1-6 gives

$$V_{C_{m(b)}}(1 + X) - V_{C_{m(a)}} = \frac{C_t}{C_m} \frac{R_t}{R_c} V_i \left(1 - \frac{1}{G_d} \frac{R_b}{R_p}\right) \qquad \ldots \quad A1\text{-}8$$

where

$$X = \frac{C_t}{C_m} \frac{1}{G_d}(1 - G_c + \frac{R_b}{R_c}) - \frac{C_t}{C_m} \frac{R_b}{R_c} \left[1 - \frac{1}{G_d}(1 - G_c + \frac{R_b}{R_c})\right] \qquad \ldots \quad A1\text{-}9$$

If $G_d \gg 1$ and $R_c \gg R_b$ then $X \ll 1$ and $\frac{1}{1 + X}$ may be replaced by $1 - X$ with an error of the order of $1/G_d$ or R_b/R_c so that equation A1-8 may be written as

$$V_{C_{m(b)}} - V_{C_{m(a)}}(1 - X) = \frac{C_t}{C_m} \frac{R_b}{R_a} V_i \left(1 - \frac{1}{G_d} \frac{R_a + R_b}{R_a}\right)(1 - X)$$

or solving for the change in voltage on C_m, ΔV_{C_m}

$$\Delta V_{C_m} = V_{C_{m(b)}} - V_{C_{m(a)}} = \frac{C_t}{C_m} \frac{R_b}{R_c} V_i \left(1 - \frac{1}{G_d} \frac{R_a + R_b}{R_a}\right)(1 - X) + X V_{C_{m(a)}} \qquad \ldots \quad A1\text{-}10$$

Substituting equation A1-9 in A1-10 and neglecting second order terms gives

$$\Delta V_{C_m} = \frac{C_t}{C_m} \frac{R_b}{R_a} V_i \left[1 - \underbrace{\frac{1}{G_d}(1 + \frac{R_b}{R_a} + \frac{C_t}{C_m} - \frac{C_t}{C_m} G_c)}_{\text{error term}} + \frac{C_t}{C_m} \frac{R_b}{R_c}\right]$$

$$\underbrace{- \frac{1}{G_d} \frac{C_t}{C_m}(1 - G_c) V_{C_{m(a)}}}_{\text{leakage term}} + \underbrace{\frac{C_t}{C_m} \frac{R_b}{R_c} V_{C_{m(a)}}}_{\text{compensation term}} \qquad \ldots \quad A1\text{-}11$$

A2 BUFFER ERROR

The error in the buffer circuit shown in fig. 3.3.3 may be obtained using equation A1-3b in Appendix A1 by noting that for the buffer $R_b = 0$ and $R_b/R_p = 1$ giving :-

$$V_o = V_i \left[1 - \frac{1}{G_d}(1 - G_c)\right] \qquad \ldots \quad A2\text{-}1$$

A3 DIFFERENCER ERROR

The error in the differencer circuit shown in fig. 3.3.4 may be obtained using equation A1-3b in appendix A1 by noting that for the differencer $R_c = \infty$ so that $R_b/R_p = (R_a + R_b)/R_a$ giving :-

$$V_o = \frac{R_a + R_b}{R_a} V_p \left[1 - \frac{1}{G_d}\left(1 + \frac{R_b}{R_a} - G_c\right)\right] - \frac{R_b}{R_a} V_m \left[1 - \frac{1}{G_d}\left(1 + \frac{R_b}{R_a}\right)\right] \qquad \ldots \quad A3\text{-}1$$

APPENDIX B DETAILS OF EXPERIMENTAL COMPUTER

B1 SWITCH

The switch used in an A.E.I. motor driven uniselector.[40] It has 16 banks of 52 contacts and is equipped with 16 single ended wipers arranged so that 8 are in, while 8 are out of the bank, producing in effect an 8 bank 104 contact switch.

The switch connections are shown in fig. B1-1.

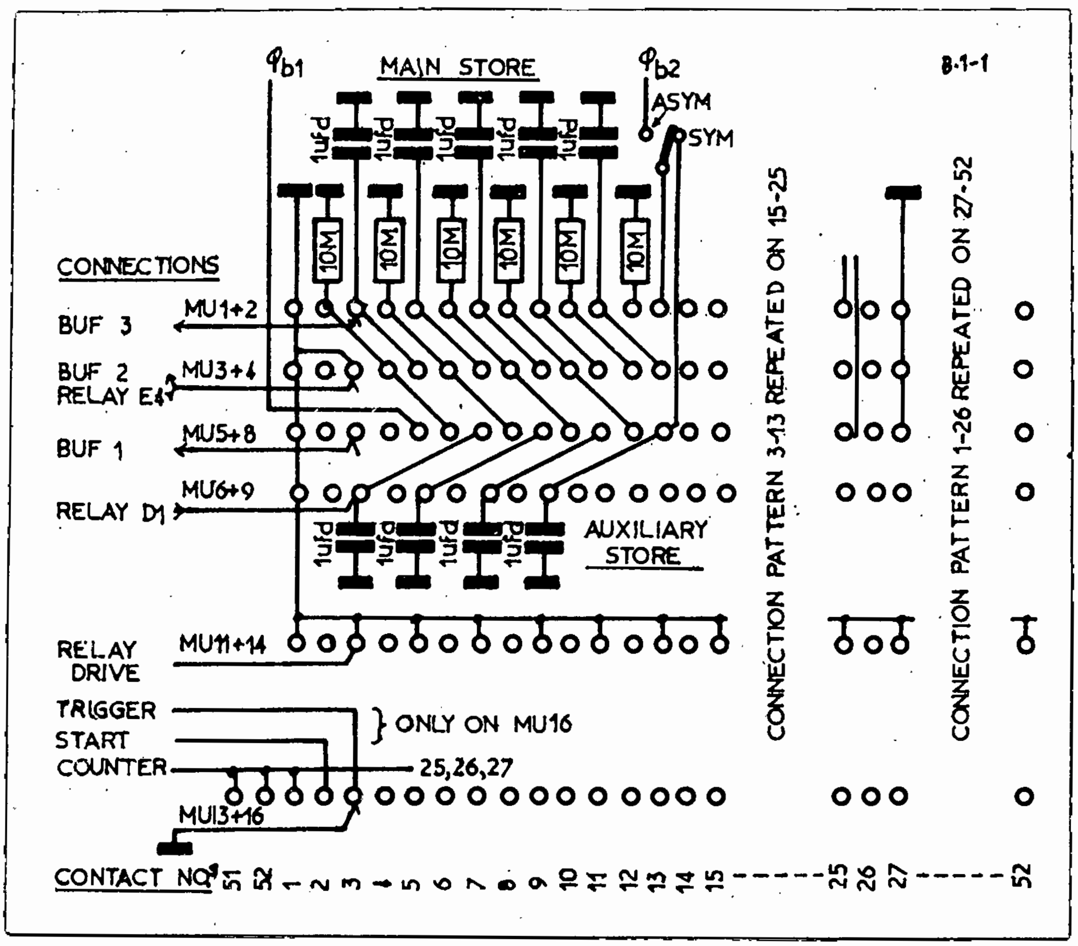


Fig. B 1-1 Diagram of switch connections

Note:
1. Banks numbered MU1 to MU16 from bottom upwards.
2. Contacts numbered 1 to 52 in direction of switch rotation.
3. Wipers of banks MU 1,2,3,4,5,8,13 and 16 are bridging type and MU 11 and 14 are non-bridging.

The wipers are driven by a d.c. motor which consists of an unwound rotor mounted between two coils disposed at 90° to each other, and a set of contacts operated by a cam on the rotor to control the energisation of the coils. The motor is geared to the wiper so that a 90° movement of the rotor causes the wiper to advance by one step. When the motor runs at 1500 r.p.m, the wipers scan 100 contacts per second, and the motor coils are effectively energized by two 50 c/s pulsating d.c. supplies which are 90° out of phase.

The motor was synchronised to the mains frequency by using a special power supply shown in fig. B 1-2 which supplies d.c. to one coil via its contact and half wave rectified d.c. to the other coil via its contact.

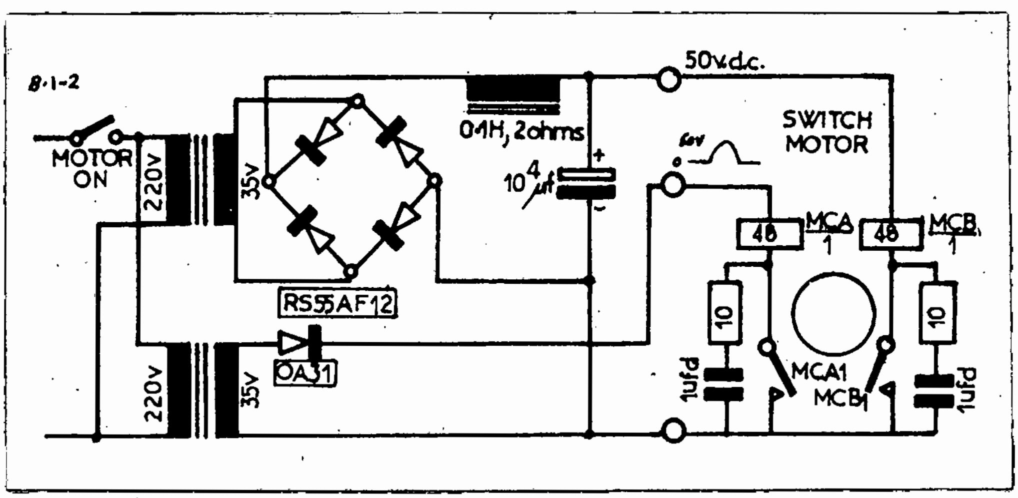


Fig. B 1-2 Switch Motor Power Supply.

The switch contacts were treated with a contact lubricating and cleaning agent* to obtain clean and reliable contact operation.

* Corrosion Reaction Consultants - CRC Formula 2.26.

B2 DIFFERENTIAL OPERATIONAL AMPLIFIER

The differential operational amplifier required for the experimental computer was designed to meet the following requirements :

1 Differential gain $G_d > 1000$ Calculated using equation A1-11 to give 1% error with a voltage gain of 10 ×

2 Common-mode gain $G_c \leq 1$ Equation A2-1 shows that reducing G_c below unity does not reduce the error appreciably.

3 Cut-off frequency > 30 kc/s The cut-off frequency is that at which $G_d = 1$. This ensures that the calculation is within 0.01% of the steady state value after a compute period of 3 milli-seconds with a stage gain of 10 ×.

4 Stability. The amplifier must be stable with its output connected to (-) gain input terminal.

5 Output Voltage = $\pm$ 100 V at $\pm$ 5 mA.

6 Common-mode input voltage swing = $\pm$ 100 V

7 Grid current $< 10^{-8}$ A This current produces a 1 volt change in voltage across a 1 μfd capacitor in 100 seconds and a 10 mV d.c. offset in an amplifier with 1 M ohm input resistance.

The differential operational amplifier resembles a conventional operational amplifier[14,15,20] with a special differential input stage.[35,43] The ratio of the differential to the common-mode gain, known as the common-mode rejection factor[35,44] is a measure of quality of the differential stage. Zaalberg[45] has shown that a double-triode long-tail-pair stage with an infinite cathode resistor has a rejection factor of approximately $2\mu/\delta$ where μ is the amplification factor of the triodes and δ is the fractional difference between the valve parameters. Assuming $\delta = 0.1$ and a rejection factor of 1000 the required amplification factor for the input stage is 50. Thus the present specification can be realized using a simple double triode input stage. Although not required in this application, rejection factors as high as 10^6 can be obtained.[46]

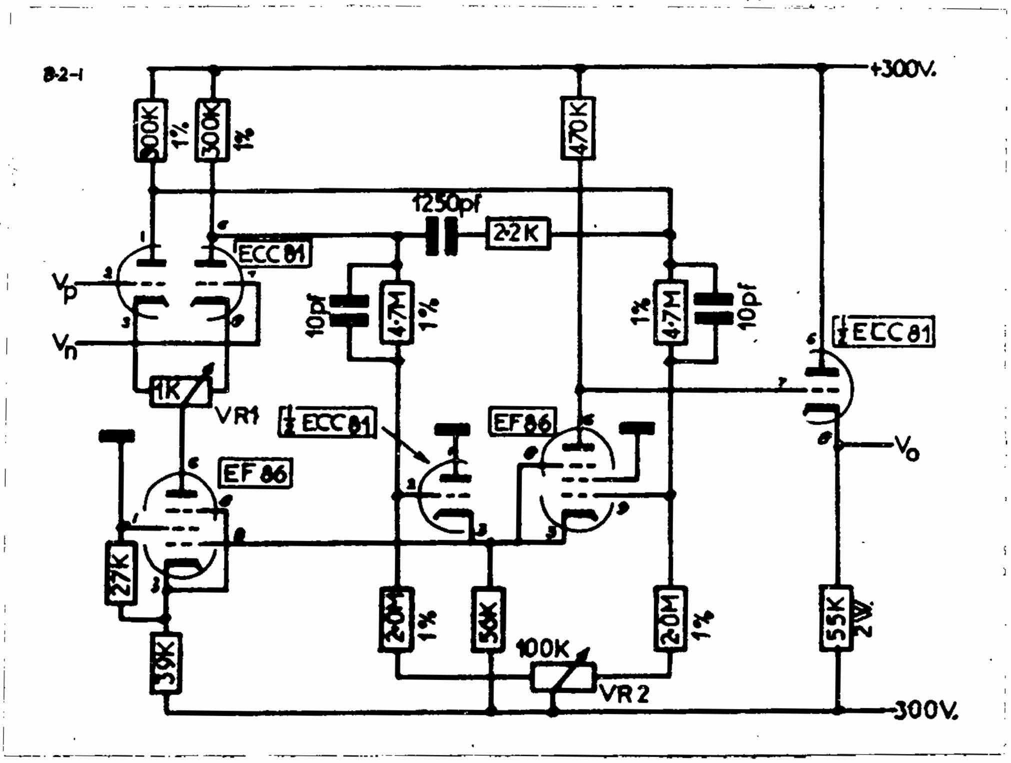


Fig. B 2-1 Circuit of differential operational amplifier

The circuit of the d.c. differential operational amplifier is shown in Fig. B 2-1. The amplifier has an ECC81 double-triode long-tail-pair input stage. Balanced push-pull coupling is used between the first and second stages. The second stage consists of a triode-pentode long-tail-pair combination consisting of an EF86 and ½ ECC81 with a single ended output. The output from the EF86 second stage is applied to the grid of amplifier stages and the divider network. The breakpoint due to the divider network is eliminated by by-passing the 4.7 M.ohm divider chain resistors giving the response shown in curve 'b'. The addition of the 2.2 K ohm. and 1250 pfd. lag-lead stabilizing network modifies the frequency response to that shown in curve 'c'. where the gain drops at a constant rate of 20 dB/decade. The amplifier has a cutoff frequency of 2 mc/s.

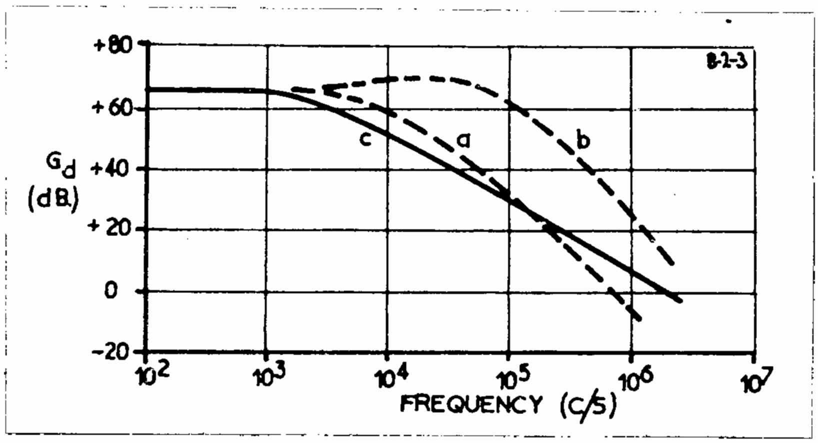


Fig. B 2-3 Measured frequency response of differential operational amplifier

a) without additional capacitors or stabilizing networks.
b) with divider network compensated for 2nd stage input capacitance.
c) complete amplifier with stabilizing network.

When the amplifier is used with a feedback resistor a small bypass capacitor must be connected in parallel to provide a high frequency path. Also when the amplifier is used to drive capacitive loads (e.g. integrator, buffer 2) a resistor ($>$ 200 ohms) must be placed in series with the capacitors. These precautions prevent the introduction of additional phase shifts at high frequencies which can cause instability.

The grid current was measured with common-mode voltages of 0 and $\pm$ 100 volts applied to the amplifier. The current was measured by connecting the amplifier as a buffer with its input connected to a 0.1 μfd capacitor and observing the charging rate of the capacitor. The grid current was found to be less than 5×10^{-10} Amps.

D.C. offset due to drift was measured by recording the output voltage of the amplifier with both input terminals grounded. The referred drift consisted of random variations of $\pm$ 10 mV with long term drift rates of less than 50 mV/hour. Before using the computer all amplifiers were allowed to warm up for a half hour before the amplifiers were zero balanced. During a set of solutions small drift errors were corrected by adjusting the differencer circuit balance only.

B3 INTEGRATOR

The complete circuit diagram of the practical integrator circuit used in the experimental computer is shown in fig. B3-1

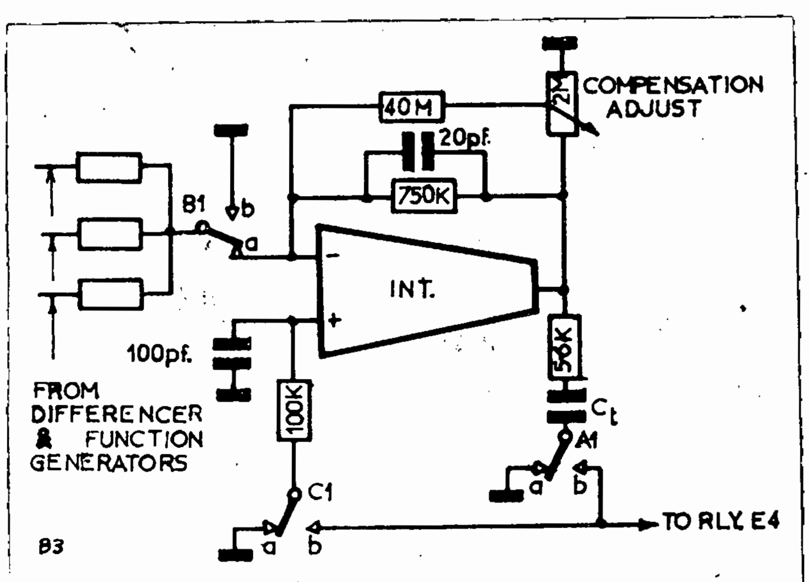


Fig. B 3-1 Practical Integrator Circuit

The following additions and alterations have been made to the basic circuit given in fig. 3.3.2-1.

1 A 100 pfd capacitor is connected from the positive gain input terminal to ground to hold the input voltage during the transition period of relay C. The 100 K resistor is included to limit the charging current through the relay contacts.

2 A 5.6 K resistor is placed in series with the transfer capacitor to prevent high frequency instability of the amplifier which may result when the output is shunted by a large capacitance. This resistor also limits the capacitor charging current to avoid overloading the amplifier. The charging time constant of transfer capacitor (0.5 millisecs.) must be small compared to the compute and write periods (3.5 millisecs each) so that the charging and discharging of the transfer capacitor will be completed within these periods.

3 Because of the small amount of leakage compensation required a compensating resistance (R_c in Fig. 3.3.2-1) of approximately 100 M.ohm is required. As a stable adjustable 100 M.ohm resistor is difficult to realize the simple resistor was replaced by a 40 M.ohm resistor and a 2 M.ohm compensation potentiometer from the amplifier output to earth. With the wiper at the output-end of the potentiometer no leakage compensation is produced while with the wiper at the ground-end full compensation results. The leakage compensation potentiometer is adjusted so that leakage is eliminated as described in fig. 3.32-5.

4 A 20 pfd capacitor is placed in parallel with the feedback network to prevent high frequency instability of the amplifier.

B4 EXPONENTIAL FUNCTION GENERATOR

The circuit of an exponential function generator based on a differential operational amplifier is shown in fig. B 4-1. The function generator has a non-linear resistance approximated by a biased diode network from the negative gain input to earth. The input is applied to the positive gain input.

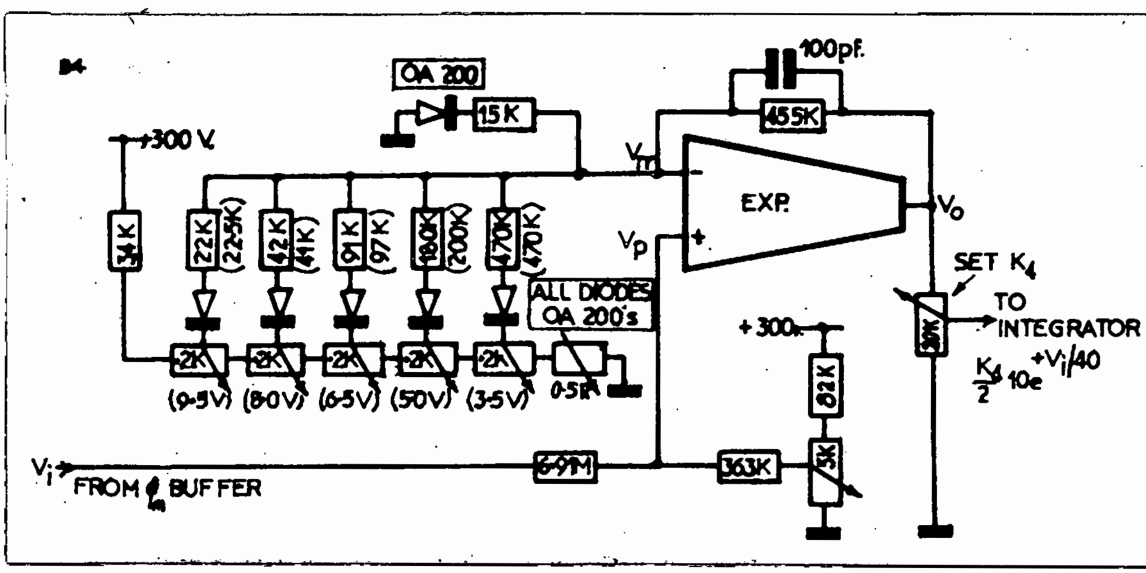


Fig. B 4-1 Circuit of diode exponential function generator.

The exponential function generator must generate an output $V_o = V_k e^{kV_i}$ where V_k is output voltage when $V_i = 0$ and k is a scale factor. If we choose the resistance of the non-linear resistor $R_a = \infty$ when $V_i = 0$ it can be shown using equation A3-1 that :-

(assuming $G_d = \infty$ so that $V_n = V_p$)

$$R_a = \left[\frac{V_k}{V_n} e^{\frac{V_n}{V_k} - 1} - 1 \right]^{-1} R_b \qquad \ldots \qquad \text{B4-1}$$

The output voltage as a function of the voltage V_p then becomes.

$$V_o = V_k e^{\frac{V_p}{V_k} - 1} \qquad \ldots \qquad \text{B4-2}$$

The exponential function generator was built to generate an output voltage $V_o = 2e^{kV_i}$ where $kV_i = 0$ to $+ 3.9$. The non-linear resistor was generated using 6 segments equally spaced along the input axis. Equal spacing was chosen as little improvement results from using the optimum unequal spacing.[47] The resistance values were calculated to correspond to equation B4-1 at the midpoints of the segments. The calculated values are shown in brackets in fig. B4-1.

The exponential function generator required for the solution of equation 4.0-3 must generate :-

$$V_o = 10\, e^{\frac{V_i}{40}} \qquad \ldots \qquad \text{B4-3}$$

Equating equations B4-3 and B4-2 gives $V_p = 5.222 + V_i/20$. This is generated by the voltage divider and d.c. supply shown connected to the positive gain input in Fig. B4-1. The calculated and measured performance of the function generator is given in table B4.

V_i	$V_o = 10\ e^{\frac{V_i}{40}}$	Measured V_o	Error	% Error	
0	10.0	10.0	0.0	0.0	
10	12.8	12.5	- 0.3	- 2.4	
20	16.5	17.1	+ 0.6	+ 3.5	
30	21.2	21.8	+ 0.6	+ 2.8	Root-mean-
40	27.2	26.7	- 0.5	- 1.8	square error
50	34.9	35.3	+ 0.4	+ 1.1	= 2.0%
60	44.8	45.0	+ 0.2	+ 0.4	
70	57.5	56.7	- 0.8	- 1.4	
80	74.0	74.5	+ 0.5	+ 0.7	
90	94.9	92.5	- 2.4	- 2.5	

Table B4 - Calculated and measured performance of the exponential function generator

The root-mean-square error of the function generator is 2%. This low accuracy results mainly because the full dynamic range of the diode network is not used as the curve is generated by only 4 of the 6 available segments.

B5 INTEGRATOR RELAY DRIVE CIRCUIT

The integrator relay drive circuit is shown in Fig. B5.

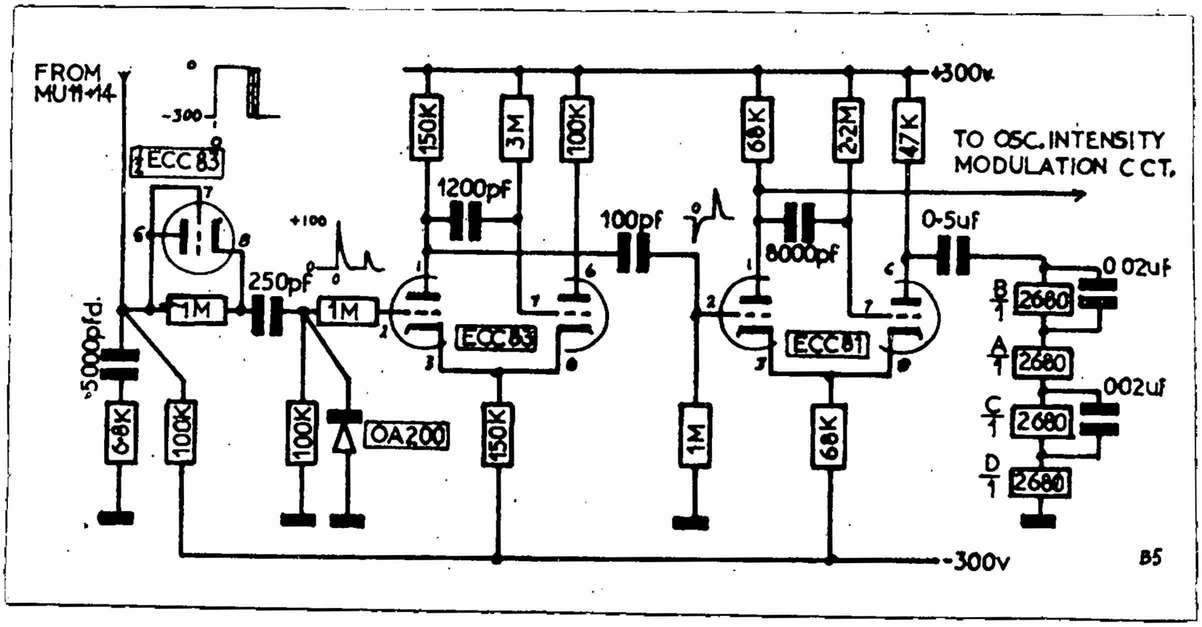


Fig. B5 <u>Integrator-relay drive circuit.</u>

The circuit operates as follows :-

1 A negative synchronizing pulse is obtained by applying - 300 V via a 100 K resistor to the wipers of banks MU 11 & 14 of the switch. Alternate contacts on these banks are grounded.

2 This pulse is pased through a pulse shaper which differentiates the square pulse from the switch producing a sharp positive going pulse which triggers the calculation-delay monostable. Spurious pulses produced by contact bounce occuring as the wipers leave the contact are suppressed by the diode (½ ECC83) and 1 M resistor which are inserted in series with the differentiating circuit to prevent the 250 pfd capacitor recharging in the interval between the main and 'bounce' pulse.

3 The calculation delay monostable returns to its stable state after 3.5 millisecs. The output is differentiated and used to trigger the relay-drive monostable.

4 The relay-drive monostable has two functions. It provides :

a) Switch pulses to the integrator and auxiliary store A,B,C & D relays.

b) A pulse for intensity modulating the oscillograph. (see appendix B9).

This monostable returns to its stable state after approximately 8 millisecs by which time the switch wipers have left the store capacitor contacts.

5 Polarized high-speed 'Carpenter' relays were used for the A,B,C & D relays (Telephone Manufacturing Co. Type 5M32A). The relays were bi-stable. The output of the relay-drive monostable was differentiated and used to change the state of these relays. The contact spacing and friction damping on each relay was adjusted to ensure bounce free operation which is absolutely essential for reliable computer operation. These relays required re-adjustment after approximately a million operations.

B6 START AND INITIAL CONDITIONS CIRCUIT

The circuit diagram of the start and initial conditions circuit is shown in Fig. B6. This circuit consists of :-

1 Counter which is an auxiliary stepping switch used to count the

number of calculations and produce marker pulses corresponding to every 10th time interval. The counter is stepped by pulses obtained from banks MU 13 & 16 which operate relay F which in turn energizes the solenoid US/2 of the stepping switch.

2 Start Relay E The calculation is started by operating relay E which disconnects the initial conditions supply and connects the integrator to the switch wipers MU 3 and 4 via the relay E4 contacts. Relay E is interlocked via the counter (bank US 2) and the switch (MU 16 contact 2) so that calculation can only start with both the switch and counter in the correct positions.

3 Initial conditions supply. The initial conditions supply is obtained from a potentiometer via an isolating cathode follower. A 22K ohm resistor is placed in series with the supply to prevent short circuits when the switch wiper passes over contacts which are grounded.

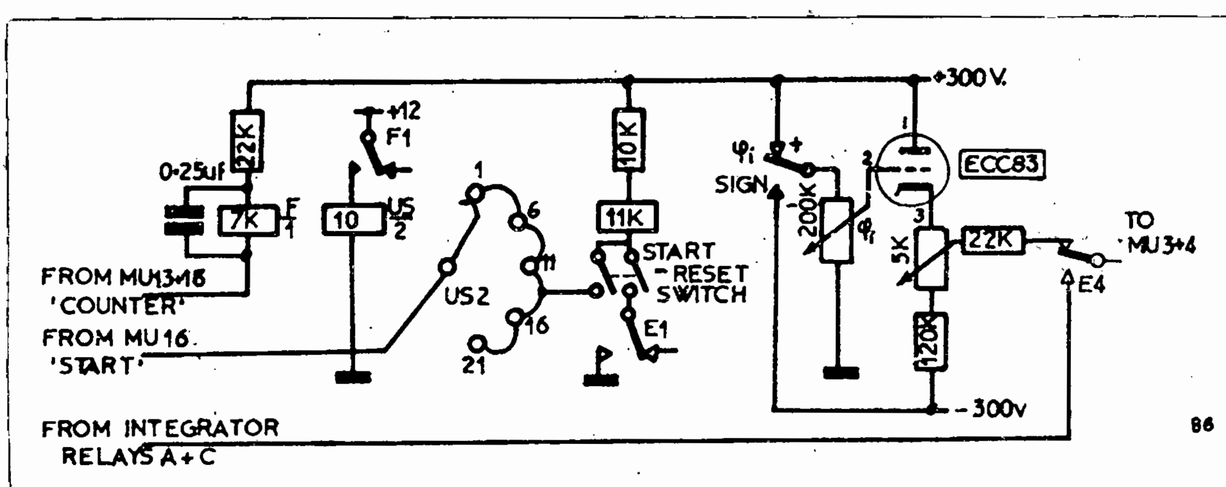


Fig. B6. Start and initial conditions circuit

B7 D.C. POWER SUPPLIES

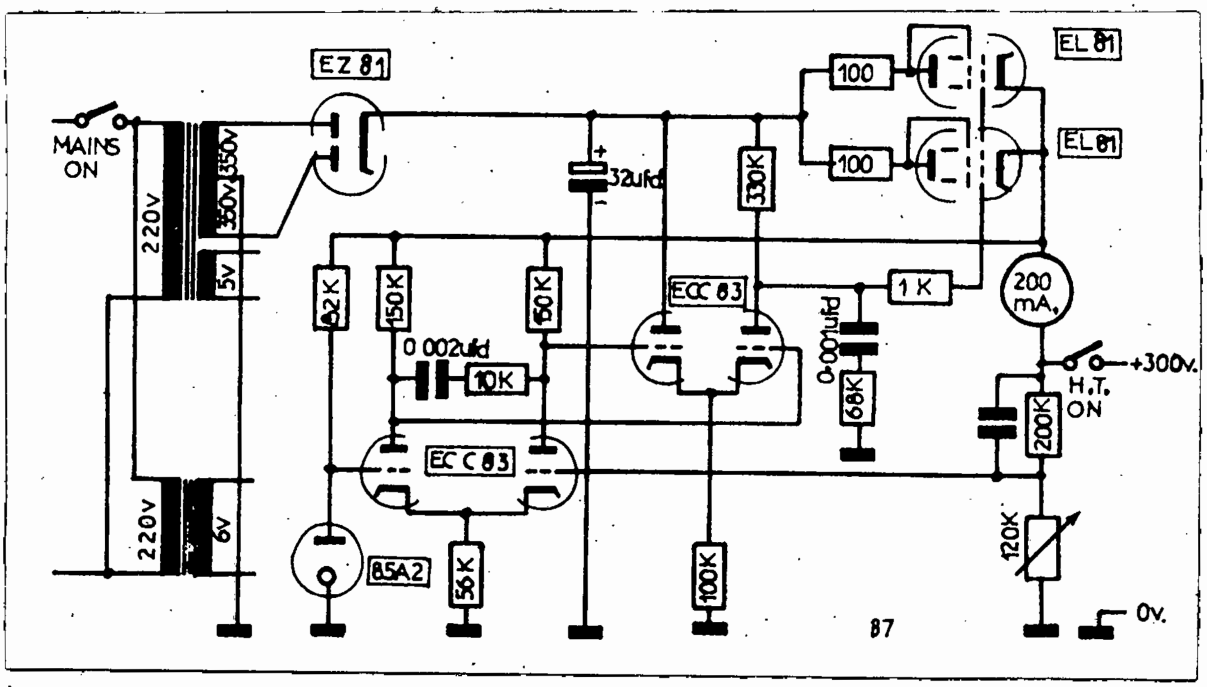


Fig. B7 Circuit of 300 V, 200 mA stabilized d.c. power supply[42]

B8 CHART RECORDER

All chart recordings were made using a high-speed potentiometric recorder. The main specifications were :

1 Make: Moseley Autograph - Model 680

2 Pen: Servo actuated with remote pen lift

3 Balance time: ½ second.

4 Range: adjustable 0-1 mV to 0-100 V. (100 V range used for solution records.)

5 Accuracy and Resolution: 0.2% F.S with 0.1% resetability.

6 Event marker: Solenoid operated.

7 Chart drive: Synchronous motor giving chart speeds of 1"

2",4",8" per minute. (8"/minute speed used for solution records.)

For the present application the recorder was equipped with a relay (relay G) to allow the motor to be started remotely. The motor reaches full speed in 2 cycles. A diagram of circuits associated with the recorder is given in Fig. B8.

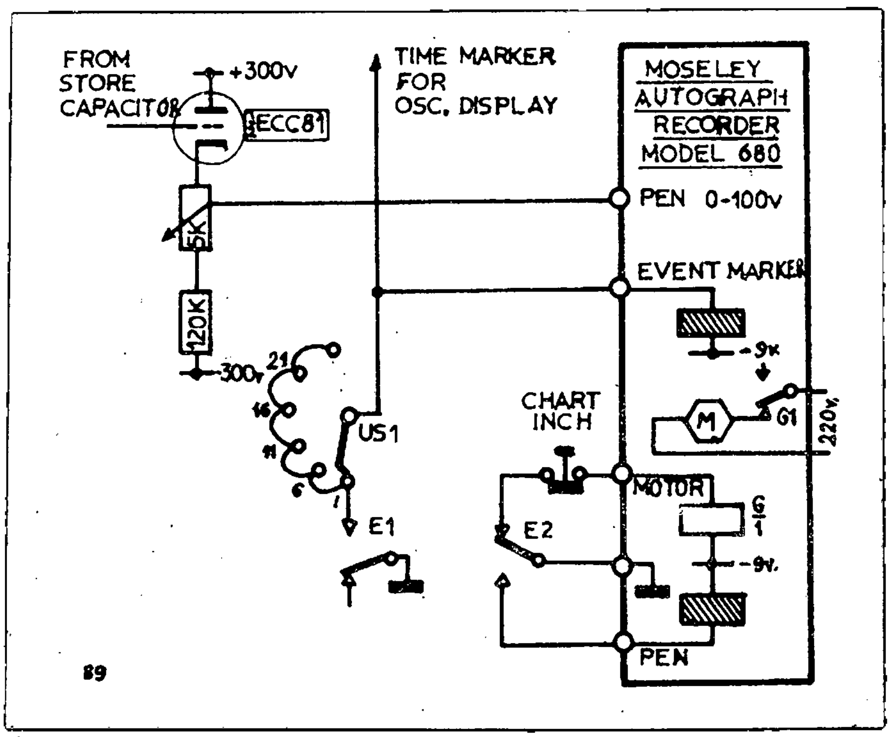


Fig. B8 <u>Chart recorder circuits</u>

Four separate circuits are used :

1 <u>Pen circuit</u> As the recorder had an input resistance of 2 M.ohm a cathode follower was used to isolate the recorder from the test circuit. The cathode follower had a gain of 0.96 but as this recorder was used for all voltage measurements made on the computer

the exact voltage unit employed is unimportant.

2 Time marker The event marker solenoid was actuated by marker step-switch contacts US1 via the E1 relay contact. A mark was produced corresponding every 10th time interval.

3 Chart Drive The chart drive was energized at the start of solution by relay contact E2 which de-energizes relay G which in turn energizes the chart drive motor. A chart 'inch' switch was installed to allow the chart to be moved to the required position before starting the solution.

4 Pen Solenoid The E2 relay also energizes the pen solenoid, dropping the pens onto the chart, at the start of the solution.

B9 OSCILLOSCOPE

Several oscilloscopes were used during the course of the investigation but it was found that the storage oscilloscope described below was superior from the standpoint of quality of display and convenience.

Main Specifications of Oscilloscope

1 Make: Tektronix Type 564, Storage Oscilloscope
with Type 2B67 Time base
and Type 63 Differential Amplifier

2 Display: Storage time approximately 1 hour.
writing speed 40 μsec./cm.
intensity modulation requires 75V negative pulse.

3 Accuracy: ± 3%

Settings

1 Y Amplifier

All measurements using 10x probe

a) All solutions recorded on 2V/div position.

b) Time markers recorded on 5V/div position.

2 Time Base

a) φ against t sweep speed used 2 secs/div (trigger: single shot)

b) φ against x sweep speed used .02 secs/div (trigger: repetitive)

Note: 1 div = $\frac{1}{4}$ inch approximately.

The associated circuits are shown in fig. B9. Three separate circuits are involved :-

1 Y deflection The Y deflection input is connected to the φ_m buffer, BUF 2, via a diode clipping network. This network clips the applied waveform at - 10 and + 110 volts. Such processing does not affect the solution being recorded but removes the large voltages which occur when the buffer is on the intermediate contacts which are not displayed but can overload the Y amplifier.

2 Intensity Modulation The display is produced by brightening the trace only when BUF 2 is connected to a store capacitor. The required intensity-modulation pulse is obtained by differentiating the waveform from the anode of the relay-drive monostable. The intensity modulation is suppressed until the start of a solution by the E1 relay contacts.

3 Time-base Trigger The time base is triggered by a pulse obtained from the MU16 bank of the uniselector. A trigger selector switch is

provided so that the oscilloscope can be triggered ahead of any group of connections to allow the operation of any bank contact to be examined. All oscillographic solution records were produced with the trigger pulse obtained from the MU16 contact 3 connection

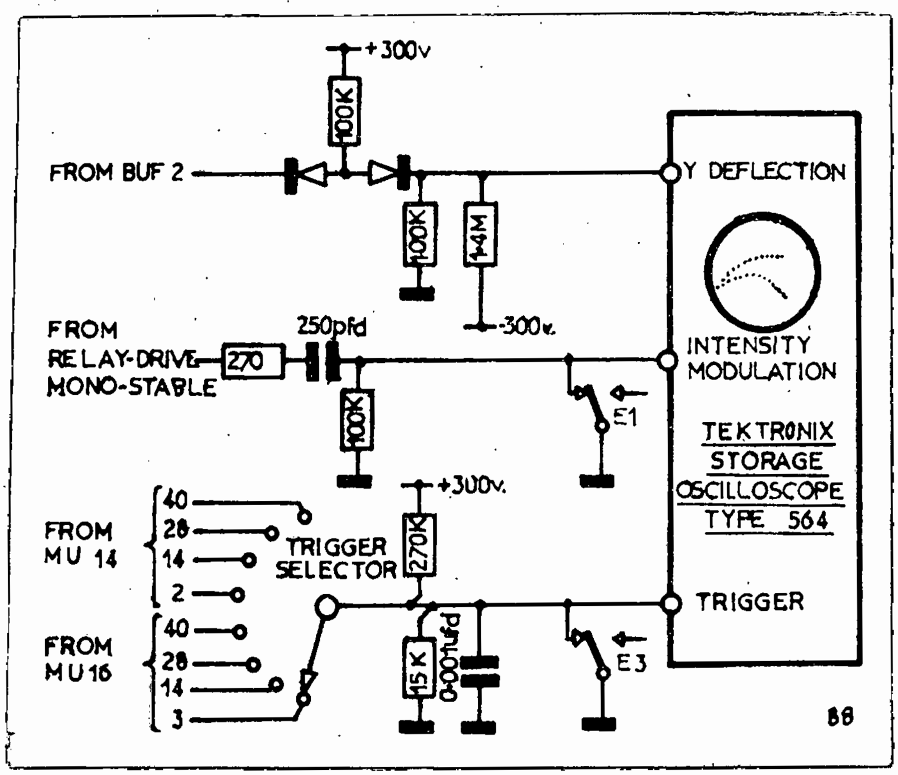


Fig. B9 Auxiliary Oscilloscope Circuits.

APPENDIX C

COMPUTER SCALING FOR EQUATION: $\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2} + K_4 e^{\varphi}$

Scaling the computer involves the following operations :

1 Estimating the maximum value attained by each term of the equation during the solution.

2 Choosing a scale factor for each term so that the corresponding voltage in the computer is as close to maximum designed machine voltage ($\pm$ 100 V) as possible.

3 Multiplying the difference equation through by constant factors so that each term has a coefficient corresponding to the scale factor chosen.

4 Choosing the computer components to correspond to the coefficients of the scaled difference equation.

The scaled equations in this thesis are written in a form suggested by Rogers and Connolly.[15] They are written using the original variables, rather than introducing new variables so that actual value of the variables in the equation are readily apparent.

The maximum value of the various terms were measured on the computer which had been scaled using estimated values. These measured maximum values, the points at which they occur and the corresponding scaled variable are given in table C.

Term	Measured maximum value	Occurs at output of	Scaled variable
φ	2.5	store and buffer	40φ
$\frac{\partial\varphi}{\partial t}$	15	integrator	$5\frac{\partial\varphi}{\partial t}$
$\frac{\partial^2\varphi}{\partial x^2}$	7.5	differencer	$10\frac{\partial^2\varphi}{\partial x^2}$
K_4e^{φ}	20	exponential function generator	$5\ K_4e^{\varphi}$

Table C - Determination of scaled variables

The difference equation to be computed is :-

$$\Delta\varphi_{m,n} = \left[\frac{1}{\Delta x^2}(\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n}) + K_4e^{\varphi_{m,n}}\right]\Delta t \qquad \ldots \qquad C-1$$

Substituting $\Delta x = 0.2$ and $\Delta t = 0.01$ and rewriting equation C-1 to adjust the coefficients of the terms to correspond to the values in the table we get :-

$$40\Delta\varphi_{m,n} = \left\{\left[6.25(40\varphi_{m+1,n} - 80\varphi_{m,n} + 40_{m-1,n})\right]0.5 + \left[10\ K_4e^{\frac{40\varphi_{m,n}}{40}}\right]1.00\right\}0.08 \qquad \ldots \qquad C-2$$

Writing equation C-2 in terms of voltages and component ratios gives :- (The symbols refer to components shown in fig. C)

$$40\Delta\varphi_{m,n} = \left\{\left[10\frac{\partial^2\varphi}{\partial x^2}\right]\frac{R_b}{R_{aDIFF}} + \left[5\ K_4e^{\varphi_{m,n}}\right]\frac{R_b}{R_{aEXP}}\right\}\frac{C_t}{C_m} \qquad \ldots \qquad C-3$$

The corresponding computer circuit showing only the relevant component ratios is given in Fig. C.

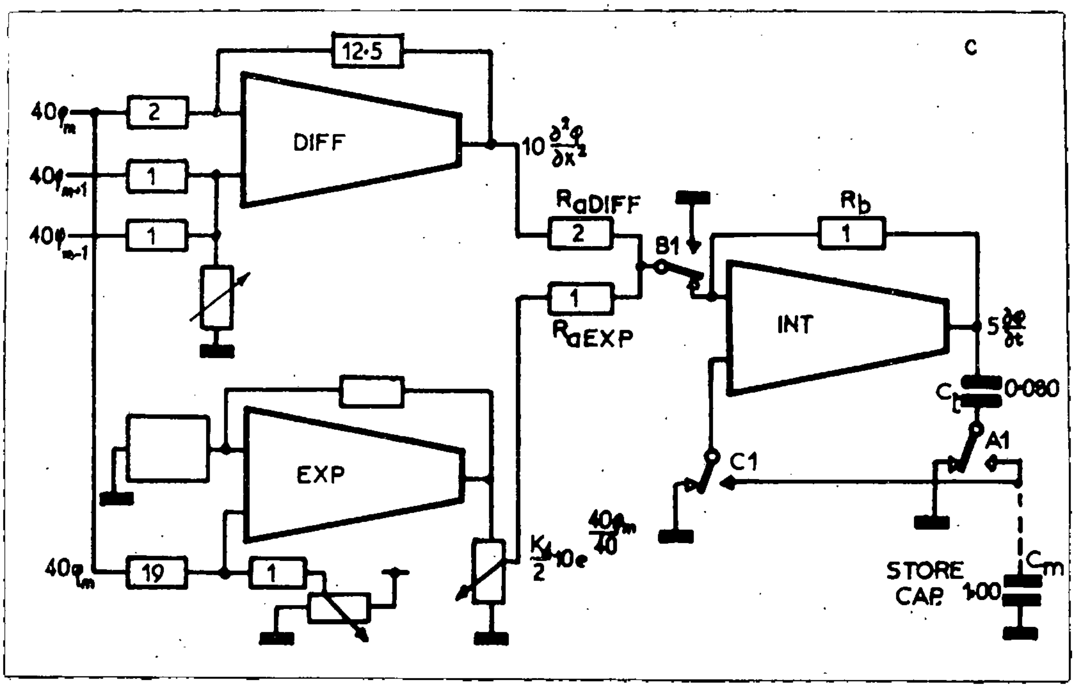


Fig. C Component ratios required for the solution of

$$\frac{\partial\varphi}{\partial t} = \frac{\partial^2\varphi}{\partial x^2} + K_4 e^{\varphi}$$

The scaled equations for the other expressions given in table 4.2 are written in the same form as equation C-2

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