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UNBOUNDED LINEAR OPERATORS IN SEMINORMED SPACES

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Synopsis

Linear operator theory is usually studied in the setting of normed or Banach spaces. However , careful examination of proofs shows that in many cases the Hausdorff property of normed spaces is not used. Even in those cases where explicit use of the Hausdorff property is made , one can often get around this (should one wish to work in seminormed spaces) by suitable identification of elements and then working in the resulting normed space. Working in seminormed spaces rather than normed spaces is especially advantageous when dealing with quotients (which occur in linear operator theory when one considers the factorisation of an operator through its domain space quotiented by its null space) : when taking the quotient of a normed space by a subspace , one requires the subspace to be closed in order for the quotient to be a normed space ; however , in the seminormed space case the requirement that the subspace be closed is no longer necessary. Seminorms are also important in the study of certain properties of the second adjoint of an operator (for example , seminorms occur in the study of operators of the Tauberian type (see [C2]) and operators analagous to weakly compact operators (see Chapter VI)). It is the aim of this work to generalise as much of the basic theory of unbounded linear operators as possible to seminormed spaces. In Chapter I , some aspects of topological vector spaces (which will be used throughout this work) are presented , the most important parts being the Hahn–Banach theorem and the section on weak topologies. In Chapter II , we restrict our attention to seminormed spaces , the setting in which the remainder of this work takes place. The basic theory of unbounded linear operators , their adjoints and the relationship between operators and their adjoints is covered in Chapter III. Chapter IV concentrates on characterising unbounded strictly singular operators while in Chapter V operators with closed range are studied. Finally , in Chapter VI , a property corresponding to one of the equivalent conditions for a bounded operator to be weakly compact is studied for unbounded operators.

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Contents

Conventions

Throughout this work , a great deal of use is made of nets in topological arguments and the reader is referred to any of the following three sources for treatments of nets : [K] , [N] or [Will].

If (X,τ) is a topological space and $A \subset X$, then the closure of A w.r.t. τ will be denoted by $\overline{A}^{\tau}$ and the interior of A will be denoted by $\text{int } A$ or $\text{int}_{\tau}(A)$.

If $M \subset X$, then $\tau|_M = \{ U \cap M : U \in \tau \}$ (i.e. the subspace topology).

Notations

Chapter I Topological Vector Spaces

The definitions and results in this chapter can be found in [KN] , [RR] and [S].

1. Convex sets

Throughout this section E is a vector space over $\mathbb{F}$ (= $\mathbb{C}$ or $\mathbb{R}$) and A , B $\subset$ E.

Definition : A is *convex* iff $\forall\, t \in [0,1]$, $t\,A + (1-t)\,A \subset A$.

Proposition I.1.1 : Let A , B be convex , $\lambda \in \mathbb{F}$ and $\alpha , \beta \in \mathbb{R}$.

(i) λA is convex.

(ii) $A + B$ is convex.

(iii) If $\alpha , \beta > 0$, then $(\alpha + \beta)\,A = \alpha\,A + \beta\,A$.

(iv) If $\mathscr{A}$ is a family of convex subsets of E , then $\cap\mathscr{A}$ is convex.

(v) If $\mathscr{A}$ is a directed family (i.e. $\forall$ A , B $\in \mathscr{A}$, $\exists$ C $\in \mathscr{A}$ s.t. $A \cup B \subset C$) of convex subsets of E , then $\cup\mathscr{A}$ is convex.

Proof : (i) , (ii) , (iv) and (v) are immediate consequences of the definition.

(iii) Suppose $\alpha , \beta > 0$.

Clearly , $(\alpha + \beta)\,A \subset \alpha\,A + \beta\,A$.

Let $x \in \alpha\,A + \beta\,A$.

Then $\exists$ a , b $\in$ A s.t. $x = \alpha a + \beta b$.

Now , $\frac{1}{\alpha + \beta} x = \frac{\alpha}{\alpha + \beta} a + \frac{\beta}{\alpha + \beta} b = \frac{\alpha}{\alpha + \beta} a + (1 - \frac{\alpha}{\alpha + \beta}) b \in A$

since A is convex.

Thus , $x \in (\alpha + \beta)\,A$.

Thus , $\alpha\,A + \beta\,B \subset (\alpha + \beta)\,A$. $\square$

Note : If $A \subset E$ then (by (iv)) there is a smallest convex subset of E which contains A. This set is called the *convex hull* of A and is denoted by $\mathrm{co}A$.

Definition : A is *balanced* iff $(\forall \lambda \in \mathbb{F})\,[|\lambda| \leq 1 \Rightarrow \lambda A \subset A]$.

Proposition I.1.2 : The intersection of a family of balanced sets is balanced.

Definition : A is *absolutely convex* iff it is balanced and convex.

Note : By Propositions I.1.1 and I.1.2 there is a smallest absolutely convex subset of E containing A. This set is denoted by $\Gamma(A)$ and is called the *absolutely convex hull* of A.

Definition : A is *absorbent* iff $(\forall x \in E)\,(\exists \mu > 0)\,(\forall \lambda \in \mathbb{F})\,[|\lambda| \leq \mu \Rightarrow \lambda x \in A]$.

Note : If A is balanced, then A is absorbent $\Leftrightarrow \forall x \in E, \exists \mu > 0$ s.t. $\mu x \in A$.

2. Topological vector spaces

Definition : A *topological vector space* (E,τ) is a vector space E over $\mathbb{F}$ together with a topology τ s.t. the maps

$$E \times E \longrightarrow E : (x,y) \longmapsto x + y$$

$$\mathbb{F} \times E \longrightarrow E : (\lambda,x) \longmapsto \lambda x \qquad \text{are continuous.}$$

($\mathbb{F}$ has its usual topology and $E \times E$, $\mathbb{F} \times E$ have their respective product topologies.)

Notation :

(i) If τ is clear from the context then E will be written for (E,τ).

(ii) The set of all neighbourhoods of a is denoted by $\mathcal{U}_a$.

Proposition I.2.1 : Let E be a topological vector space , $A , B , U \subset E$, $a , b \in E$ and $\alpha \in \mathbb{F}\backslash\{0\}$.

(i) The map $E \longrightarrow E : x \longmapsto \alpha x + b$ is a homeomorphism.

(ii) U is open $\Leftrightarrow$ $\alpha U + b$ is open.

(iii) A is closed $\Leftrightarrow$ $\alpha A + b$ is closed.

(iv) $\overline{a + \alpha A} = a + \alpha \overline{A}$, $\text{int}(a + \alpha A) = a + \alpha\, \text{int}(A)$.

(v) $\overline{A} + \overline{B} \subset \overline{A+B}$.

(vi) U open $\Rightarrow$ A + U open.

(vii) $U \in \mathcal{U}_0 \Leftrightarrow \alpha U + a \in \mathcal{U}_a$.

(viii) $\mathcal{B}$ is a base of neighbourhoods of O $\Leftrightarrow$ $\{ a + B \mid B \in \mathcal{B} \}$ is a base of neighbourhoods of a.

(ix) A balanced (resp. convex , absolutely convex) $\Rightarrow$ $\overline{A}$, int(A) balanced (resp. convex , absolutely convex).

Proof :

(i) From the definition , the map $E \longrightarrow E : x \longmapsto \alpha x + b$ is continuous.

The map $E \longrightarrow E : x \longmapsto \frac{1}{\alpha}(x - b)$ is its inverse and from the definition is also continuous.

(ii) , (iii) , (iv) , (vii) and (viii) follow from (i).

(v) follows from the continuity of addition.

(vi) follows from (ii) and $A + U = \bigcup_{a \in A} (a + U)$.

(ix) If A is balanced and $0 < |\lambda| \leq 1$, then by (iv) $\lambda \overline{A} = \overline{\lambda A} \subset \overline{A}$.

If A is convex and $t \in [0,1]$, then $t\,\overline{A} + (1 - t)\,\overline{A} = \overline{tA + (1-t)A} \subset \overline{A}$. □

Proposition I.2.2 : Let E be a topological vector space , $\mathscr{U}$ a base of neighbourhoods of O and $U \in \mathscr{U}$.

(i) U is absorbent.

(ii) $\exists\, V \in \mathscr{U}$ s.t. $V + V \subset U$.

(iii) $\exists$ balanced $W \in \mathscr{U}_O$ s.t. $W \subset U$.

Proof :

(i) Let $a \in E$.

Then the map $f : \mathbb{F} \longrightarrow E : \lambda \longmapsto \lambda a$ is continuous at 0.

Thus , $\exists\, \mu > 0$ s.t. $\{ \lambda \in \mathbb{F} : |\lambda| \leq \mu \} \subset f^{-1}[U]$.

Thus , if $|\lambda| \leq \mu$ then $\lambda a \in U$.

(ii) Since addition is continuous at (O,O) , $\exists\, V_1 , V_2 \in \mathscr{U}$ s.t. $V_1 + V_2 \subset U$.

Since $\mathscr{U}$ is a base of neighbourhoods of O , $\exists\, V \in \mathscr{U}$ s.t. $V \subset V_1 \cap V_2$.

Clearly , $V + V \subset U$.

(iii) Since scalar multiplication is continuous at $(0,O)$, $\exists\, \mu > 0$ and $\exists\, V \in \mathcal{U}$ s.t. if $|\lambda| \leq \mu$ and $x \in V$ then $\lambda x \in U$.

Thus, $\mu V \subset \cap \{ \alpha U : \alpha \in \mathbb{F}, |\alpha| \geq 1 \}$.

Let $W = \cap \{ \alpha U : \alpha \in \mathbb{F}, |\alpha| \geq 1 \}$.

Then $W \subset U$ and $\mu V \subset W$ so W is a neighbourhood of O.

Let $x \in W$, $0 < |\alpha| \leq 1$ and $|\lambda| \geq 1$.

Then $|\frac{\lambda}{\alpha}| \geq 1$ so that $x \in \frac{\lambda}{\alpha} U$.

Thus, $\alpha x \in \lambda U$.

Since λ was arbitrary, $\alpha x \in W$.

Thus, W is balanced. □

Definition : A topological vector space E is *(locally) convex* iff E has a base of convex neighbourhoods of O.

Proposition I.2.3 : A convex space E has a base $\mathcal{U}$ of neighbourhoods of O satisfying :

(a) $\mathcal{U} \neq \emptyset$ and $U, V \in \mathcal{U} \Rightarrow \exists\, W \in \mathcal{U}$ s.t. $W \subset U \cap V$.

(b) $U \in \mathcal{U}$, $\alpha \in \mathbb{F}\backslash\{0\} \Rightarrow \alpha U \in \mathcal{U}$.

(c) $U \in \mathcal{U} \Rightarrow U$ is absolutely convex and absorbent.

Conversely, if $\mathcal{U}$ is a collection of subsets of a vector space E satisfying (a), (b) and (c), then there is a unique topology on E making E into a convex space with $\mathcal{U}$ as a base of neighbourhoods of O.

Proof :

Suppose E is a convex space.

Then E has a base $\mathcal{V}$ of convex neighbourhoods of O.

$\forall\, V \in \mathcal{V}$, let $U_V = \cap \{ \lambda V : \lambda \in \mathbb{F}, |\lambda| \geq 1 \}$.

It follows from Propositions I.1.1, I.1.2 and I.2.2 that $\{ \alpha U_V : \alpha \in \mathbb{F}\backslash\{0\}, V \in \mathcal{V} \}$ is a base of neighbourhoods of O satisfying (a), (b) and (c).

Suppose $\mathscr{U}$ is a collection of subsets of a vector space E satisfying (a) , (b)and (c).

Let $\tau = \{ V \subset E : \forall x \in V , \exists U \in \mathscr{U} \text{ s.t. } x + U \subset V \}$.

Then τ is a topology on E and $\mathscr{U}$ is a base of neighbourhoods of O for τ.

The continuity of addition and scalar multiplication will now be verified.

Let $a , b \in E$ and $U \in \mathscr{U}$.

Then $a + \frac{1}{2} U$ and $b + \frac{1}{2} U$ are neighbourhoods of a and b respectively.

Since U is convex , $\frac{1}{2} U + \frac{1}{2} U = U$.

Thus , $(a + \frac{1}{2} U) + (b + \frac{1}{2} U) = a + b + \frac{1}{2} U + \frac{1}{2} U = a + b + U$.

Thus , addition is continuous.

Let $\lambda \in \mathbb{F}$, $a \in E$ and $U \in \mathscr{U}$.

Since U is absorbent , $\exists\, \mu > 0$ s.t. $\mu a \in U$.

Let $\epsilon \in (0 , \frac{\mu}{2})$ and $\delta \in (0 , \frac{1}{2\,(|\lambda| + \epsilon)})$.

If $|\alpha - \lambda| < \epsilon$ and $x \in \delta U + a$ then , since U is balanced ,

$\alpha x - \lambda a = \alpha (x - a) + (\alpha - \lambda) a \in (|\lambda| + \epsilon)\, \delta U + \epsilon \frac{1}{\mu} U \subset \frac{1}{2} U + \frac{1}{2} U = U.$ □

3. Seminorms

Throughout this section , E is a vector space over $\mathbb{F}$.

Definition : A function $p : E \longrightarrow \mathbb{R}$ is a *seminorm* on E iff

(a) $\forall x , y \in E , p(x + y) \leq p(x) + p(y)$ and

(b) $\forall x \in E , \forall \lambda \in \mathbb{F} , p(\lambda x) = |\lambda| \, p(x)$.

Note :

(i) $p(O) = p(0\,O) = 0\,p(O) = 0$.

(ii) If $p(x) = 0$ implies $x = O$ then p is a *norm* on E.

(iii) If $x \in E$ then $0 = p(O) = p(x - x) \leq p(x) + p(-x) = 2\,p(x)$.

Thus , $p(x) \geq 0 , \forall x \in E$.

(iv) If $x , y \in E$ then $p(x) = p(x - y + y) \leq p(x - y) + p(y)$

and $p(y) = p(y - x + x) \leq p(y - x) + p(x)$.

Thus , $\forall x , y \in E , |p(x) - p(y)| \leq p(x - y)$.

Proposition I.3.1 : Let p , q be seminorms on E s.t. $(\forall x \in E) [p(x) < 1 \Rightarrow q(x) \leq 1]$. Then $\forall x \in E , q(x) \leq p(x)$.

Proof : Suppose $\exists x \in E$ s.t. $q(x) > p(x)$.

Then , $\exists \alpha \in \mathbb{R}$ s.t. $p(x) < \alpha < q(x)$.

Thus , $p(\frac{x}{\alpha}) < 1$ and $q(\frac{x}{\alpha}) > 1$, a contradiction. □

Note : If A is an absorbent subset of E then $\forall x \in E , \exists \mu > 0$ s.t. $\mu x \in A$. Thus , $\forall x \in E , \{ \mu > 0 : x \in \mu A \} \neq \emptyset$.

The properties of seminorms give the following result.

Proposition I.3.2 :

(a) If p is a seminorm on E then $\forall \alpha > 0$ the sets

$\{ x \in E : p(x) < \alpha \}$ and $\{ x \in E : p(x) \leq \alpha \}$ are absolutely convex and absorbent.

(b) If A is an absolutely convex absorbent subset of E, then $p : E \longrightarrow \mathbb{R}$ defined by

$p(x) = \inf\{ \mu > 0 : x \in \mu E \}$ is a seminorm on E s.t.

$$\{ x \in E : p(x) < 1 \} \subset A \subset \{ x \in E : p(x) \leq 1 \}.$$

Note : The seminorm corresponding to an absolutely convex, absorbent set is called the *gauge* of the set.

Let p be a (semi–) norm on E, $B_E = \{ x \in E : p(x) \leq 1 \}$ (this notation will be retained for the remainder of this work) and $\mathcal{U} = \{ \alpha B_E : \alpha > 0 \}$.

Then $\mathcal{U}$ satisfies the conditions (a), (b) and (c) of Proposition I.2.3.

Thus, there is a unique topology on E making E a convex space and having $\mathcal{U}$ as a base of neighbourhoods of O.

Note that $d : E \times E \longrightarrow \mathbb{R} : (x,y) \longmapsto p(x - y)$ is a (semi–)metric on E and the topology generated by d on E is the same as that obtained from Proposition I.2.3.

(Recall that for $x \in E$ and $r > 0$, $B(x,r) = \{ y \in E : d(x,y) < r \}$ and the topology generated by d is $\tau_d = \{ U \subset E : \forall x \in U, \exists r > 0 \text{ s.t. } B(x,r) \subset U \}$.)

4. Duality and the Hahn–Banach Theorem

Throughout this section, E and F are vector spaces over $\mathbb{F}$.

Definition : A map $f : E \longrightarrow F$ is *linear* iff
$\forall x, y \in E, \forall \lambda \in \mathbb{F}, f(\lambda x + y) = \lambda f(x) + f(y)$.
If $f : E \longrightarrow F$ is a linear map, then the *kernel* of f is $\ker(f) = \{ x \in E : f(x) = 0 \}$

Note :

(i) $\ker(f)$ is a vector subspace of E.

(ii) If $F = \mathbb{F}$ and f is linear, then f is called a *linear functional* on E.

(iii) The set of all linear functionals on E is denoted by E^*.

(iv) E^* is a vector space over $\mathbb{F}$ if addition and scalar multiplication are defined pointwise.

Proposition I.4.1 : Let E and F be topological vector spaces and $f : E \longrightarrow F$ linear. Then f is continuous $\Leftrightarrow$ f is continuous at 0.

Proof :

$(\Rightarrow)$ Clear.

$(\Rightarrow)$ Suppose f is continuous at 0.
Let $x \in E$ and let V be a neighbourhood of $f(x)$ in F.
Then $\exists W \in \mathcal{U}_0$ s.t. $f(x) + W \subset V$.
Since f is continuous at 0, $\exists U \in \mathcal{U}_0$ s.t. $U \subset f^{-1}[W]$.
If $z \in x + U$, then $f(z) \in f(x) + f[U] \subset f(x) + f[f^{-1}[W]] \subset f(x) + W \subset V$.
i.e. $x + U \subset f^{-1}[V]$.
Thus, f is continuous. □

Definition : A *hyperplane* in a vector space is a maximal proper subspace.

Note : Let M be a vector subspace of E. For $x \in E$, denote the coset $x + M$ of M by $[x]$ and let $E/M = \{ [x] : x \in E \}$.

Then E/M can be made into a vector space as follows :

For $x , y \in E$, let $[x + y] = [x] + [y]$ and

for $\lambda \in \mathbb{F}$ let $\lambda [x] = [\lambda x]$.

These operations are well–defined :

If $[x] = [x_1]$ and $[y] = [y_1]$ then $x - x_1 , y - y_1 \in M$.

Thus , $(x + y) - (x_1 + y_1) = (x - x_1) + (y - y_1) \in M$.

Thus , $[x + y] = [x_1 + y_1]$.

Similarly , scalar multiplication is well–defined.

Definition : If M is a subspace of E , then the *codimension* of M , written codim M , is the dimension of E/M.

Proposition I.4.2 : Let H be a subspace of E. Then

T.F.A.E. :

(a) H is a hyperplane in E.

(b) $\exists f \in E^* \backslash \{O\}$ s.t. $H = \ker(f)$.

(c) codim $H = 1$.

Proof :

(a) $\Rightarrow$ (b) Suppose H is a hyperplane in E.

Let $x_0 \in E \backslash H$.

Then $\operatorname{span}(\{x_0\} \cup H) = E$.

Define $f : E \longrightarrow \mathbb{F}$ by $f(\alpha x_0 + y) = \alpha$ $(\alpha \in \mathbb{F} , y \in H)$.

Then $f \in E^* \backslash \{O\}$ and $\ker(f) = H$.

(b) ⇒ (c) Suppose $\exists\, f \in E^*\backslash\{O\}$ s.t. $H = \ker(f)$.

Since $f \neq O$, $\exists\, x_0 \in E\backslash H$.

Let $x \in E$.

Then $x = (x - \frac{f(x)}{f(x_0)} x_0) + \frac{f(x)}{f(x_0)} x_0$.

But, $x - \frac{f(x)}{f(x_0)} x_0 \in \ker(f) = H$.

Thus, $[x] = [\frac{f(x)}{f(x_0)} x_0] \in \text{span}\,\{[x_0]\}$.

Thus, $\dim E/H = 1$.

(c) ⇒ (a) Suppose $\text{codim}\, H = 1$.

Let $z \in E\backslash H$ and $x \in E$.

Since $[z] \neq O$, $E/H = \text{span}\,\{[z]\}$.

Thus, $\exists\, \lambda \in \mathbb{F}$ s.t. $[x] = \lambda\,[z] = [\lambda\, z]$.

i.e. $x - \lambda\, z \in H$.

Thus, $\text{span}(\{z\} \cup H) = E$.

Thus, H is a hyperplane in E. □

Lemma I.4.3 : Let $f \in E^*\backslash\{O\}$, $H = \ker(f)$, $a \in E$ with $f(a) = 1$, $V = \{x \in E : |f(x)| < 1\}$ and U a balanced subset of E.

Then $(a + U) \cap H = \emptyset \Leftrightarrow U \subset V$.

Proof :

(⇒) Suppose $\exists\, x \in U\backslash V$.

Then $|f(x)| \geq 1$.

Put $y = -\frac{x}{f(x)} \in U$ (U is balanced).

Then $f(a + y) = 0$ so $a + y \in (a + U) \cap H$.

Thus, $(a + U) \cap H \neq \emptyset$.

(⇐) Suppose $U \subset V$.

Let $x \in U$.

Then $x \in V$ so $|f(x)| < 1$.

Thus, $|f(a + x)| = |f(a) + f(x)| \geq 1 - |f(x)| > 0$ so $a + x \notin H$.

Thus, $(a + U) \cap H = \emptyset$. □

Note : Let E be a topological vector space and $f \in E^*$. Then

f is continuous $\Leftrightarrow$ f is bounded on some neighbourhood of O :

($\Rightarrow$) If f is continuous , then f is bounded on $f^{-1}[\{ \lambda \in \mathbb{F} : |\lambda| \leq 1 \}]$ which is a neighbourhood of O.

($\Leftarrow$) Suppose U is a neighbourhood of O and $|f(x)| \leq C , \forall x \in U$ for some $C > 0$.

If $\epsilon > 0$, then $f^{-1}[\{ \lambda \in \mathbb{F} : |\lambda| \leq \epsilon \}] = \frac{\epsilon}{C} f^{-1}[\{ \lambda \in \mathbb{F} : |\lambda| \leq C \}] \supset \frac{\epsilon}{C} U$.

Thus , f is continuous at O so f is continuous.

Proposition I.4.4 : Let $f \in E^* \setminus \{O\}$. Then f is continuous $\Leftrightarrow$ ker(f) is closed.

Proof :

($\Rightarrow$) $\{0\}$ is closed in $\mathbb{F}$ and $\ker(f) = f^{-1}[\{0\}]$.

($\Leftarrow$) Suppose ker(f) is closed.

Let $H = \ker(f)$ and $V = \{ x \in E : |f(x)| < 1 \}$.

Since $f \neq O$, $\exists\, a \in E$ s.t. $f(a) = 1$.

Since $a \notin H$ and H is closed , $\exists$ balanced $U \in \mathcal{U}_O$ s.t $(a + U) \cap H = \emptyset$.

By Lemma I.4.3 , $U \subset V$.

Thus , f is bounded on U. □

Proposition I.4.5 : Let E be a topological vector space and M a vector subspace of E. Then $\overline{M}$ is also a vector subspace of E.

Proof :

Let $x , y \in \overline{M}$ and let $U \in \mathcal{U}_0$.

Then $\exists\, V \in \mathcal{U}_0$ s.t. $V + V \subset U$.

Since $x , y \in \overline{M}$, $x + V$ and $y + V$ meet M.

Thus , $(x + V) + (y + V)$ meets $M + M = M$.

Since $(x + V) + (y + V) \subset x + y + U$, this means that $x + y + U$ meets M.

Thus , $x + y \in \overline{M}$.

Similarly , if $\lambda \in \mathbb{F}$ and $x \in \overline{M}$ then $\lambda x \in \overline{M}$. □

Corollary I.4.6 : In a topological vector space a hyperplane is either closed or dense.

Proof : Let H be a hyperplane in the topological vector space E.

By Proposition I.4.5 , $\overline{H}$ is a vector subspace of E.

If H is not closed , then $H \subsetneq \overline{H} \subset E$ so that $\overline{H} = E$. □

The proof of the following result can be found in [RR ; p 27].

Proposition I.4.7 : Let A be an open convex suset of a convex space E and M a vector subspace of E with $A \cap M = \emptyset$. Then there is a closed hyperplane containing M and not meeting A.

Theorem I.4.8 (The Hahn–Banach extension theorem) : Let M be a vector subspace of E, p a seminorm on E and $f : M \longrightarrow \mathbb{F}$ a linear map s.t. $\forall\, x \in M$, $|f(x)| \le p(x)$. Then $\exists\, f_1 \in E^*$ s.t. $\forall\, x \in M$, $f_1(x) = f(x)$ and $\forall\, x \in E$, $|f_1(x)| \le p(x)$.

Proof :

Let E have the topology determined by p and let $U = \{ x \in E : p(x) < 1 \}$.

We may assume that $f \neq O$.

Thus, $\exists\, a \in M$ s.t. $f(a) = 1$.

Let $A = a + U$. Then A is open (Proposition I.2.1) and convex (Proposition I.1.1).

Now, $U \cap M \subset \{ x \in M : |f(x)| < 1 \}$ so by Lemma I.4.3, $(a + U) \cap \ker(f) = \emptyset$.

By Proposition I.4.7, there is a closed hyperplane H in E s.t $\ker(f) \subset H$ and $A \cap H = \emptyset$.

Define $f_1 : E \longrightarrow \mathbb{F}$ by $f_1(\alpha a + y) = \alpha \quad (\alpha \in \mathbb{F}, y \in H)$.

Then $f_1 \in E^*$ and $\forall\, x \in M$, $f_1(x) = f(x)$.

By Lemma I.4.3, $(\forall\, x \in E)\ [p(x) < 1 \Rightarrow |f_1(x)| < 1]$.

Thus, by Proposition I.3.1, $\forall\, x \in E$, $|f_1(x)| \le p(x)$. □

Corollary I.4.9 : Let $a \in E$ and $p : E \longrightarrow \mathbb{R}$ a seminorm. Then $\exists\, f \in E^*$ s.t. $\forall\, x \in E$, $|f(x)| \le p(x)$ and $f(a) = p(a)$.

Proof : In Theorem I.4.8 take $M = \text{span}\,\{a\}$ and define $f : M \longrightarrow \mathbb{F}$ by $f(\alpha a) = \alpha\, p(a)$.

Definition : The *dual* of a topological vector space E is $E' = \{ f \in E^* : f \text{ continuous} \}$.

Note : E' is a vector subspace of E^*.

Theorem I.4.10 (The Hahn–Banach separation theorem) : Let E be a convex space and A and B disjoint convex subsets of E with A open. Then $\exists\, f \in E'$ s.t. $f[A] \cap f[B] = \emptyset$.

Proof : $A - B$ is open , convex and $\{O\} \cap (A - B) = \emptyset$.

By Proposition I.4.7 , $\exists$ a closed hyperplane H s.t. $H \cap (A - B) = \emptyset$.

Let $a \in E \backslash H$ and define $f : E \longrightarrow \mathbb{F}$ by $f(\alpha a + y) = \alpha \quad (\alpha \in \mathbb{F}, y \in H)$.

Then $f \in E^*$ and $\ker(f) = H$ is closed so by Proposition I.4.4 , $f \in E'$.

Clearly , $f[A] \cap f[B] = \emptyset$. □

Definition : A map $f : X \longrightarrow Y$ where X and Y are topological spaces is *open* iff $(\forall A \subset X)\,[\, A \text{ open in } X \Rightarrow f[A] \text{ open in } Y \,]$.

Lemma I.4.11 : Let E be a topological vector space and $f \in E^* \backslash \{O\}$. Then f is open.

Proof : Let A be an open subset of E and $x \in A$.

Then $A - x$ is an open neighbourhood of O so is absorbent.

Since $f \neq O$, $\exists\, a \in E$ s.t. $f(a) = 1$.

Since $A - x$ is absorbent , $\exists\, \mu > 0$ s.t. if $|\lambda| < \mu$, then $\lambda a \in A - x$.

If $|\lambda| < \mu$, then $f(x) + \lambda = f(x + \lambda a) \in f[A]$.

Thus , $f[A]$ is open in $\mathbb{F}$. □

Corollary I.4.12 : Let B be a convex subset of a convex space E and $a \in E \backslash \overline{B}$. Then $\exists\, f \in E'$ s.t. $f(a) \notin \overline{f[B]}$.

Proof : Since $a \notin \overline{B}$, $\exists$ absolutely convex $U \in \mathcal{U}_O$ s.t. $(a + U) \cap B = \emptyset$.

By Theorem I.4.10 , $\exists\, f \in E'$ s.t. $f[\operatorname{int}(a + U)] \cap f[B] = \emptyset$.

But , by Lemma I.4.11 , $f[\operatorname{int}(a + U)]$ is a neighbourhood of $f(a)$.

Thus , $f(a) \notin \overline{f[B]}$. □

Corollary I.4.13 : Let B be an absolutely convex subset of a convex space E and $a \in E \backslash \overline{B}$. Then $\exists\, f \in E'$ s.t. $f(a) > 1$ and $\forall\, x \in B$, $|f(x)| \le 1$.

Proof : By Corollary I.4.12 , $\exists\, g \in E'$ s.t. $g(a) \notin \overline{g[B]}$.

Now , $g[B]$ is absolutely convex so $|g(a)| > \sup\{\, |g(x)| : x \in B \,\}$.

Put $\alpha = \sup\{\, |g(x)| : x \in B \,\}$.

If $\alpha = 0$, then $f = \frac{2}{g(a)} g$ will suffice , otherwise put $f = \frac{|g(a)|}{\alpha\, g(a)} g$. □

Corollary I.4.14 : Let M be a vector subspace of a convex space E and $a \in E \backslash \overline{M}$. Then $\exists\, f \in E'$ s.t. $f(a) \neq 0$ and $f[M] = \{O\}$.

5. Polar sets and weak topologies

Definition : Let E and F be vector spaces over $\mathbb{F}$. A *bilinear functional* on $E \times F$ is a mapping $B : E \times F \longrightarrow \mathbb{F}$ s.t. $\forall\, x , y \in E , \forall\, z , w \in F , \forall\, \alpha , \beta \in \mathbb{F}$,

$$B(\alpha x + \beta y , z) = \alpha B(x,z) + \beta B(y,z)$$

$$B(x , \alpha z + \beta w) = \alpha B(x,z) + \beta B(x,w).$$

Definition : A *pairing* is an ordered pair (E,F) of vector spaces together with a bilinear functional on $E \times F$.

Note :

(i) $B(x,y)$ will be written as $<x,y>$.

(ii) If B is clear from the context, then explicit mention of B is omitted and (E,F) is referred to as the pairing.

(iii) If (E,F) is a pairing, then usually $F \subset E^*$ (e.g. $F = E'$ when E is a topological vector space). Even if F is not a subspace of E^*, it can be mapped into E^* as follows :

$\forall\, y \in F$, define $\hat{y} : E \longrightarrow \mathbb{F}$ by $\hat{y}(x) = <x,y> \quad (x \in E)$.

Then each $\hat{y} \in E^*$ and the map $T : F \longrightarrow E^* : y \longmapsto \hat{y}$ is linear.

T is injective $\Leftrightarrow \forall\, y \in F \backslash \{0\} , \exists\, x \in E \backslash \{0\}$ s.t. $<x,y> \neq 0$.

If $B \subset F$, then $T[A]$ will be denoted by $\hat{A}$.

Definition : Let (E,F) be a pairing, $A \subset E$ and $B \subset F$. Then

(a) the *polar* of A (in F) is the set $A^{\circ} = \{ y \in F : \forall\, x \in A , |<x,y>| \leq 1 \}$

(b) the *polar* of B (in E) is the set $B_{\circ} = \{ x \in E : \forall\, y \in B , |<x,y>| \leq 1 \}$.

The following result is an immediate consequence of the definition.

Proposition I.5.1 : Let (E,F) be a pairing and A , $A_1 \subset E$.

(i) A° is absolutely convex.

(ii) $A \subset A^\circ{}_\circ$.

(iii) $A \subset A_1 \Rightarrow A_1^\circ \subset A^\circ$.

(iv) If $\lambda \in \mathbb{F}\backslash\{0\}$, then $(\lambda A)^\circ = \frac{1}{\lambda} A^\circ = \frac{1}{|\lambda|} A^\circ$.

(v) $\{0\}^\circ = F$.

(vi) If $\mathscr{A}$ is a family of subsets of E , then $(\cup \mathscr{A})^\circ = \cap \{ A^\circ : A \in \mathscr{A} \}$.

(vii) If M is a subspace of E , then M° is a subspace of F and

$$M^\circ = \{ y \in F : \forall x \in M , <x,y> = 0 \}.$$

Note : The corresponding results hold for $B \subset F$.

Notation : If M is a subspace of E , then $M^\perp$ is written for M°.

Similarly , if N is a subspace of F , then $N_\perp$ is written for $N_\circ$.

Let (E,F) be a pairing and $\mathscr{B} = \{ B_\circ : B \subset F , B \text{ finite} \}$.
Then $\mathscr{B}$ satisfies (a) , (b) and (c) of Proposition I.2.3. Thus , by Proposition I.2.3 , there is a unique topology on E making E a convex space and having $\mathscr{B}$ as a base of neighbourhoods of O. This topology is called the *weak topology* (on E) and is denoted by $\sigma(E,F)$.

Note :

(i) If (E,F) is a pairing , so is (F,E).

$\sigma(F,E)$ is then defined in a similar fashion.

(ii) If E is a topological vector space , then $\sigma(E',E)$ is called the *weak* topology* on E′.

(iii) $\sigma(E',E)$ is Hausdorff.

For the remainder of this section, (E,F) is a pairing.

Proposition I.5.2 : $\sigma(E,F)$ is the coarsest topology on E making E a topological vector space s.t. $\forall\, y \in F$, $\hat{y}$ is continuous.

Proof : If $y \in F$, then $\hat{y}$ is bounded on $\{y\}_o$ so is continuous.

Suppose τ is a topology on E s.t. (E,τ) is a topological vector space and each $\hat{y}$ is continuous w.r.t. τ.

Let $B \subset F$ be finite.

Then $B_o = \bigcap_{y \in B} \hat{y}^{-1}[\{ \lambda \in \mathbb{F} : |\lambda| \leq 1 \}]$ is a neighbourhood of O for τ.

Thus, $\sigma(E,F) \subset \tau$. □

Note : Let (x_α) be a net in E and $x \in E$. Then

$$x_\alpha \longrightarrow x \text{ w.r.t. } \sigma(E,F) \iff \forall\, y \in F,\ \langle x_\alpha, y\rangle \longrightarrow \langle x,y\rangle :$$

($\Rightarrow$) Proposition I.5.2.

($\Leftarrow$) Suppose $\forall\, y \in F$, $\langle x_\alpha, y\rangle \longrightarrow \langle x,y\rangle$.

Let $B \subset F$ be finite.

$\forall\, y \in B$, $\exists\, \alpha_y$ s.t. $\forall\, \alpha \geq \alpha_y$, $|\langle x_\alpha, y\rangle - \langle x,y\rangle| \leq 1$.

Choose β s.t. $\forall\, y \in B$, $\beta \geq \alpha_y$.

Then $\forall\, \alpha \geq \beta$, $\forall\, y \in B$, $|\langle x_\alpha, y\rangle - \langle x,y\rangle| \leq 1$.

i.e. $\forall\, \alpha \geq \beta$, $x_\alpha - x \in B_o$.

Thus, $x_\alpha \longrightarrow x$ w.r.t. $\sigma(E,F)$.

Lemma I.5.3 : Let V be a vector space and $f, f_1, \dots, f_n \in V^*$. If $\bigcap_{i=1}^{n} \ker(f_i) \subset \ker(f)$, then $\exists\, \lambda_1, \dots, \lambda_n \in \mathbb{F}$ s.t. $f = \sum_{i=1}^{n} \lambda_i f_i$.

Proof : We may assume that $f \neq O$.

First suppose $n = 1$.

Since $f \neq O$, $\exists\, z \in V \setminus \ker(f)$.

Since $\ker(f_1) \subset \ker(f)$, $z \notin \ker(f_1)$.

Let $x \in V$.

Then $x = (x - \frac{f_1(x)}{f_1(z)} z) + \frac{f_1(x)}{f_1(z)} z$.

But, $x - \frac{f_1(x)}{f_1(z)} z \in \ker(f_1) \subset \ker(f)$.

Thus, $f(x) = \frac{f(z)}{f_1(z)} f_1(x)$.

Now suppose the result holds for $n = k$ and $\bigcap_{i=1}^{k+1} \ker(f_i) \subset \ker(f)$.

Then, $\bigcap_{i=1}^{k} \ker(f_i|_{\ker(f_{k+1})}) \subset \ker(f|_{\ker(f_{k+1})})$.

By assumption, $\exists\, \lambda_1, \dots, \lambda_k \in \mathbb{F}$, s.t $f|_{\ker(f_{k+1})} = \sum_{i=1}^{k} \lambda_i f_i|_{\ker(f_{k+1})}$.

Now, $\ker(f_{k+1}) \subset \ker(f - \sum_{i=1}^{k} \lambda_i f_i)$.

Thus, $\exists\, \lambda_{k+1} \in \mathbb{F}$ s.t. $f - \sum_{i=1}^{k} \lambda_i f_i = \lambda_{k+1} f_{k+1}$. □

Proposition I.5.4 : $(E,\sigma(E,F))' = \hat{F}$.

Proof : By Proposition I.5.2, $\hat{F} \subset (E,\sigma(E,F))'$.

Let $f \in (E,\sigma(E,F))'$.

Then $\exists$ finite $B \subset F$ s.t. $B_o \subset f^{-1}[\{ \lambda \in \mathbb{F} : |\lambda| \leq 1 \}]$.

This means that $\bigcap_{y \in B} \ker(\hat{y}) \subset \ker(f)$.

By Lemma I.5.3, $f \in \operatorname{span} \hat{B} \subset \hat{F}$. □

Lemma I.5.5 : If $B \subset F$ then B_o is $\sigma(E,F)$ closed.

Proof : $B_o = \bigcap_{y \in B} \ker(\hat{y})$. □

Theorem I.5.6 (The Bipolar Theorem) : If $A \subset E$, then $A^o{}_o = \overline{\Gamma(A)}^{\sigma(E,F)}$.

Proof :

By Lemma I.5.5 and Proposition I.5.1 , $A^o{}_o$ is absolutely convex , $\sigma(E,F)$ closed and $A \subset A^o{}_o$ so $\overline{\Gamma(A)}^{\sigma(E,F)} \subset A^o{}_o$.

Suppose $z \notin \overline{\Gamma(A)}^{\sigma(E,F)}$.

By Corollary I.4.13 and Proposition I.5.4 , $\exists\, y \in F$ s.t. $\langle z,y\rangle > 1$ and $\forall\, x \in A$, $|\langle x,y\rangle| \leq 1$ so $y \in A^o$.

Thus , $z \notin A^o{}_o$. □

Corollary I.5.7 : If M and N are subspaces of E and F respectively then

$$\overline{M}^{\sigma(E,F)} = M^{\perp}{}_{\perp} \text{ and } \overline{N}^{\sigma(F,E)} = N_{\perp}{}^{\perp}.$$

Proposition I.5.8 : Let τ be a topology on E making E into a convex topological vector space with $(E,\tau)' = \hat{F}$. If A is a convex subset of E , then $\overline{A}^{\tau} = \overline{A}^{\sigma(E,F)}$.

Proof : By Proposition I.5.2 , $\sigma(E,F) \subset \tau$ so $\overline{A}^{\tau} \subset \overline{A}^{\sigma(E,F)}$.

Suppose $z \notin \overline{A}^{\tau}$.

By Corollary I.4.12 , $\exists\, y \in F$ s.t. $\hat{y}(z) \notin \overline{\hat{y}[A]}$.

Thus , $\exists\, \delta > 0$ s.t. $\forall\, a \in A$, $|\langle z - a , y\rangle| \geq \delta$.

Thus , $(z + \frac{\delta}{2}\{y\}_o) \cap A = \emptyset$ so $z \notin \overline{A}^{\sigma(E,F)}$. □

The proof of the next result can be found in [KN ; 17.4].

Theorem I.5.9 (Banach–Alaoglu) : If U is a neighbourhood of O in a convex space E, then U° is $\sigma(E',E)$ compact.

Chapter II Seminormed Spaces

Definition : A *(semi–)normed space* (X,p) is a vector space X together with a (semi–)norm p.

Usually , X will be written for (X,p) and $\|x\|$ will be written for p(x).

Recall that if X is a (semi–)normed space , then $d : X \times X \longrightarrow \mathbb{R} : (x,y) \longmapsto \|x - y\|$ defines a (semi–)metric on X and the topology generated by d makes X a convex space and a base of neighbourhoods of O for this topology is given by $\mathscr{B} = \{ \alpha B_X : \alpha > 0 \}$ where $B_X = \{ x \in X : \|x\| \leq 1 \}$. In what follows , $\tau_{\|\ \|}$ will denote this topology and any topological notion in X will be w.r.t. $\tau_{\|\ \|}$ unless otherwise stated.

Notation : $U_X = \{ x \in X : \|x\| < 1 \}$.

Note : $\mathbb{F}^n$ with coordinatewise addition and scalar multiplication and norm $\|(\lambda_1, \dots ,\lambda_n)\| = (\sum_{i=1}^{n} |\lambda_i|^2)^{\frac{1}{2}}$ is a complete normed space. (see for example [Kr])

1. Bounded Linear Operators

Definition : Let X and Y be vector spaces over $\mathbb{F}$ and T a function with domain in X and range in Y. Then T is a *linear operator* iff its domain is a subspace of X and for all x and y in the domain of T and for all $\alpha \in \mathbb{F}$, $T(\alpha x + y) = \alpha Tx + Ty$.

Notation :

(i) The domain of T is denoted by D(T).

(ii) The range of T is denoted by R(T).

(iii) The *null space* of T is the set $N(T) = \{ x \in D(T) : Tx = O \}$.

(iv) The set of all linear operators with domain in X and range in Y is denoted by L(X,Y). The set $\{ T \in L(X,Y) : D(T) = X \}$ is denoted by L[X,Y].

(v) If $T \in L(X,Y)$ and $M \subset X$, then $TM = \{ Tx : x \in D(T) \cap M \}$.

Note :

(i) Note that R(T) is a subspace of Y.

(ii) Note that N(T) is a subspace of D(T) (hence of X).

(iii) If T is a linear operator , then TO = O.

(iv) A linear operator T is injective $\Leftrightarrow$ $N(T) = \{O\}$.

For the remainder of this section , X and Y are seminormed spaces and $T \in L(X,Y)$.

Note : If (X,p) is a seminormed space and M is a subspace of X , then $p|_M$ is a seminorm on M (this will be defined more formally in the next section). Thus , it makes sense to discuss the continuity of T.

The following result includes [Wil1 ; Theorem1 , p65].

Theorem II.1.1 :

T.F.A.E.: (a) T is continuous.

(b) The set $\{ \|Tx\| : x \in B_{D(T)} \}$ is bounded.

(c) $\exists\, M > 0$ s.t. $\forall\, x \in D(T)$, $\|Tx\| \leq M \|x\|$.

Proof:

(a) ⇒ (b) Suppose $\{ \|Tx\| : x \in B_{D(T)} \}$ is not bounded.

Then $\forall\, n \in \mathbb{N}$, $\exists\, x_n \in B_{D(T)}$ s.t. $\|Tx_n\| > n^2$.

Now , $\|x_n/n\| \leq \frac{1}{n} \longrightarrow 0$ but $\|T(x_n/n)\| > n$ so T is not continuous.

(b) ⇒ (c) Suppose $\exists\, M > 0$ s.t. $\forall\, x \in B_{D(T)}$, $\|Tx\| \leq M$.

case (i) : $\|x\| = 0$.

Then $\forall\, n \in \mathbb{N}$, $n\,x \in B_{D(T)}$.

Since T is linear , $\forall\, n \in \mathbb{N}$, $n\,\|Tx\| = \|T(n\,x)\| \leq M$ so $\|Tx\| = 0$.

case (ii) : $\|x\| \neq 0$.

Then $\frac{x}{\|x\|} \in B_{D(T)}$ so $\|T\frac{x}{\|x\|}\| \leq M$.

Since T is linear , $\|Tx\| \leq M \|x\|$.

(c) ⇒ (a) Suppose $\exists\, M > 0$ s.t. $\forall\, x \in D(T)$, $\|Tx\| \leq M \|x\|$.

Then T is continuous at O.

By Proposition I.4.1 , T is continuous. □

Notation :

(i) $BL(X,Y) = \{ T \in L(X,Y) : T \text{ continuous} \}$.

(ii) $BL[X,Y] = \{ T \in BL(X,Y) : D(T) = X \}$.

If $T \in BL[X,Y]$, then T is called *bounded.*

Note : $X' = BL[X,\mathbb{F}]$.

Definition : If $T \in BL(X,Y)$, then the *norm* of T is

$$\|T\| = \sup\{ \|Tx\| : x \in D(T) , \|x\| \leq 1 \}.$$

Note : $\|T\|$ behaves like a seminorm (in fact it will turn out to be a seminorm on $BL[X,Y]$).

The following result generalises the corresponding normed space result to seminormed spaces.

Proposition II.1.2 : Let $T \in BL(X,Y)$.

(a) $\forall x \in D(T) , \|Tx\| \leq \|T\| \|x\|$.

(b) $\|T\| = \inf\{ M > 0 : \forall x \in D(T) , \|Tx\| \leq M \|x\| \}$.

(c) If $\exists x \in D(T)$ s.t. $\|x\| \neq 0$, then $\|T\| = \sup\{ \|Tx\| : x \in D(T) , \|x\| = 1 \}$.

Proof :

(a) $\exists M > 0$ s.t. $\forall x \in D(T) , \|Tx\| \leq M \|x\|$.

Let $x \in D(T)$.

case (i) : $\|x\| = 0$.

$0 \leq \|Tx\| \leq M \|x\| = 0$ so $\|Tx\| = 0$.

case (ii) : $\|x\| \neq 0$.

$\|\frac{x}{\|x\|}\| = 1 \leq 1$ so $\|T\frac{x}{\|x\|}\| \leq \|T\|$.

Thus , $\|Tx\| \leq \|T\| \|x\|$.

(b) Let $A = \{ M > 0 : \forall x \in D(T) , \|Tx\| \leq M \|x\| \}$.

If $\|T\| = 0$, then $\forall x \in D(T)$, $\|Tx\| = 0$ so $\inf A = 0$.

Assume $\|T\| > 0$.

By (a) , $\|T\| \in A$ so $\inf A \leq \|T\|$.

$\forall M \in A$, $\|T\| \leq M$ so $\|T\| \leq \inf A$.

(c) Suppose $\exists x \in D(T)$ s.t. $\|x\| \neq 0$.

Let $B = \{ \|Tx\| : x \in D(T) , \|x\| \leq 1 \}$ and $C = \{ \|Tx\| : x \in D(T) , \|x\| = 1 \}$.

Then , $\emptyset \neq C \subset B$.

Thus , $\sup C \leq \sup B = \|T\|$.

Let $x \in D(T)$ with $0 < \|x\| \leq 1$.

Then , $\|\frac{x}{\|x\|}\| = 1$ so $\|T\frac{x}{\|x\|}\| \leq \sup C$.

Thus , $\|Tx\| \leq (\sup C) \|x\| \leq \sup C$.

Since x was arbitrarily chosen , $\|T\| \leq \sup C$. □

Definition :

(i) A sequence (x_n) in X is a *Cauchy sequence* iff

$$\forall \epsilon > 0 , \exists N \in \mathbb{N} \text{ s.t. } \forall m , n \geq N , \|x_n - x_m\| < \epsilon.$$

(ii) X is *complete* iff every Cauchy sequence in X converges.

Note : Let M be a subspace of X (where M is considered as a seminormed space in its own right). Then

(i) X complete , M closed $\Rightarrow$ M complete.

(ii) X normed , M complete $\Rightarrow$ M closed.

Note that if $N = \{ x \in X : \|x\| = 0 \}$, then every subset of N is complete but no proper nonempty subset of N is closed. Thus (ii) fails if X is not normed.

Definition : Let (x_n) be a sequence in X and $x \in X$. $\forall\, n \in \mathbb{N}$, let $s_n = \sum_{i=1}^{n} x_i$.

Then the *series* $\sum_{n=1}^{\infty} x_n$ *converges* to x iff $s_n \longrightarrow x$ in X.

The series is *absolutely convergent* iff $\sum_{n=1}^{\infty} \|x_n\|$ converges in $\mathbb{R}$.

The following proposition generalises the well known characterisation of completeness for normed spaces to seminormed spaces.

Proposition II.1.3 :

T.F.A.E. :

(a) X is complete.

(b) Every absolutely convergent series in X converges in X.

Proof :

(a) ⇒ (b) Suppose X is complete.

Let $\sum_{n=1}^{\infty} x_n$ be an absolutely convergent series in X.

$\forall\, n \in \mathbb{N}$, let $s_n = \sum_{i=1}^{n} x_i$.

For $n, k \in \mathbb{N}$, $\|s_{n+k} - s_n\| = \|\sum_{i=n+1}^{n+k} x_i\| \leq \sum_{i=n+1}^{\infty} \|x_i\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Thus, (s_n) is a Cauchy sequence in X.

(b) ⇒ (a) Suppose every absolutely convergent series in X converges in X.

Let (x_n) be a Cauchy sequence in X.

$\exists\, n_1 \in \mathbb{N}$ s.t. $\forall\, m, n \geq n_1$, $\|x_m - x_n\| < \frac{1}{2}$.

In particular, $\|x_{n_1+1} - x_{n_1}\| < \frac{1}{2}$.

For $k \geq 2$, $\exists\, n_k > n_{k-1}$ s.t. $\|x_{n_k+1} - x_{n_k}\| < 2^{-k}$.

Now, $\sum_{k=1}^{\infty} \|x_{n_k+1} - x_{n_k}\| \leq \sum_{k=1}^{\infty} 2^{-k} < \infty$.

Since $x_{n_k} - x_{n_1} = \sum_{i=1}^{k-1} (x_{n_i+1} - x_{n_i})$ for $k \geq 2$, it follows from the assumption that (x_{n_k}) converges.

But, (x_{n_k}) is a subsequence of the Cauchy sequence (x_n) so (x_n) must also converge. □

Definition : A complete normed space is a *Banach space.*

Example : Let $c_o = \{ (x_n) \in \mathbb{F}^{\mathbb{N}} : x_n \to 0 \}$ and $l_1 = \{ (x_n) \in \mathbb{F}^{\mathbb{N}} : \sum_{n=1}^{\infty} |x_n| < \infty \}$.

Under coordinatewise addition and scalar multiplication, c_o and l_1 become vector spaces. If $\| \; \|_\infty$ and $\| \; \|_1$ are defined on c_o and l_1 respectively by

$$\|(x_n)\|_\infty = \sup\{ |x_n| : n \in \mathbb{N} \} \quad ((x_n) \in c_o)$$

$$\|(x_n)\|_1 = \sum_{n=1}^{\infty} |x_n| \quad ((x_n) \in l_1)$$

then c_o and l_1 become Banach spaces. (See [Kr] for details.)

Definition : Let $S \in L(X,Y)$. Then the operator $T + S$ in defined as follows :

$T + S \in L(X,Y)$, $D(T + S) = D(T) \cap D(S)$, $\forall x \in D(T + S)$, $(T + S)x = Tx + Sx$.

Proposition II.1.4 : $S, T \in BL(X,Y) \Rightarrow T + S \in BL(X,Y)$ and $\|T + S\| \leq \|T\| + \|S\|$.

Note : $BL[X,Y]$ is a vector space.

Proposition II.1.5 :

(a) $BL[X,Y]$ is a seminormed space.

(b) Y normed $\Rightarrow BL[X,Y]$ normed.

(c) Y a Banach space $\Rightarrow BL[X,Y]$ a Banach space.

Proof :

(a) $\|T\|$ defined earlier is a seminorm on $BL[X,Y]$.

(b) Suppose Y is normed.

Let $T \in BL[X,Y]$ with $\|T\| = 0$.

By Proposition II.1.2 , $\forall\, x \in X$, $\|Tx\| = 0$.

Since Y is a normed space , this means that $\forall\, x \in X$, $Tx = O$, i.e. $T = O$.

(c) Suppose Y is a Banach space.

Let (T_n) be a sequence in $BL[X,Y]$ s.t. $\sum_{n=1}^{\infty} \|T_n\| < \infty$.

$\forall\, x \in X$, $\sum_{n=1}^{\infty} \|T_n x\| \leq \left(\sum_{n=1}^{\infty} \|T_n\| \right) \|x\| < \infty$. (1)

Since Y is a Banach space , $T : X \longrightarrow Y$ can be defined by $Tx = \sum_{n=1}^{\infty} T_n x$.

Note that T is linear and by (1) , $T \in BL[X,Y]$.

Let $n \in \mathbb{N}$, $x \in B_X$.

Then $\| \sum_{i=1}^{n} T_i x - Tx\| = \| \sum_{i=n+1}^{\infty} T_i x \| \leq \sum_{i=n+1}^{\infty} \|T_i x\| \leq \sum_{i=n+1}^{\infty} \|T_i\|$.

Since x was arbitrarily chosen , $\| \sum_{i=1}^{n} T_i - T\| \leq \sum_{i=n+1}^{\infty} \|T_i\| \longrightarrow 0$ as $n \longrightarrow \infty$.

By Proposition II.1.3 , $BL[X,Y]$ is complete. □

Corollary II.1.6 : X' is a Banach space.

Definition : Let T be injective. Then the *inverse* T^{-1} of T is defined as follows : $T^{-1} \in L(Y,X)$, $D(T^{-1}) = R(T)$ and $\forall\, x \in D(T)$, $T^{-1}(Tx) = x$.

Proposition II.1.7 : Let T be injective. Then

$$T^{-1} \text{ is continuous} \Leftrightarrow \exists\, m > 0 \text{ s.t. } \forall\, x \in D(T) ,\ m \|x\| \leq \|Tx\|.$$

Proof :

($\Rightarrow$) Suppose T^{-1} is continuous.

By Theorem II.1.1 , $\exists\, M > 0$ s.t. $\forall\, x \in D(T)$, $\|T^{-1}(Tx)\| \leq M \|Tx\|$.

Put $m = \frac{1}{M}$.

Then $\forall\, x \in D(T)$, $m\,\|x\| \leq \|Tx\|$.

($\Leftarrow$) Suppose $\exists\, m > 0$ s.t. $\forall\, x \in D(T)$, $m\,\|x\| \leq \|Tx\|$.

Then $\forall\, x \in D(T)$, $\|T^{-1}(Tx)\| = \|x\| \leq \frac{1}{m}\|Tx\|$.

By Theorem II.1.1 , T^{-1} is continuous. □

Corollary II.1.8 : Let X be normed. Then

T.F.A.E. (a) T is injective and T^{-1} is continuous.

(b) $\exists\, m > 0$ s.t. $\forall\, x \in D(T)$, $m\,\|x\| \leq \|Tx\|$.

Note : The example $T = O$, with $\|x\| = 0$ for all x , shows that (b) $\Rightarrow$ (a) of Corollary II.1.8 fails if X is allowed to be a seminormed space.

Definition : Let Z be a seminormed space and $S \in L(Z,X)$. Then the operator TS is defined as follows :

$TS \in L(Z,Y)$, $D(TS) = \{\, z \in D(S) : Sz \in D(T) \,\}$ and $\forall\, z \in D(TS)$, $(TS)z = T(Sz)$.

Proposition II.1.9 : $T \in BL(X,Y)$, $S \in BL(Z,X) \Rightarrow TS \in BL(Z,Y)$ and $\|TS\| \leq \|T\|\,\|S\|$.

Proof : $\forall\, z \in D(TS)$, $\|TSz\| \leq \|T\|\,\|Sz\| \leq \|T\|\,\|S\|\,\|z\|$. □

2. Subspace , Quotient , Product

Definition : Let (X,p) be a (semi–)normed space and M a vector subspace of X. Then , $(M,p|_M)$ is a (semi–)normed space and is called a *subspace* of (X,p). The map $J_M^X : M \longrightarrow X : m \longmapsto m$ is called the *canonical injection* (of M into X). We take J_M^X to be an element of $L(M,X)$.

Note : $J_M^X \in BL[M,X]$.

Definition : Let X be a seminormed space and M a vector subspace of X. $\forall\, x \in X$, let $\|[x]\| = d(x,M) = \inf\{\ \| x - m \| : m \in M\ \}$. Then $\|\ \|$ is a seminorm on X/M and X/M together with this seminorm is called the *quotient* of X by M. The map $Q_M^X : X \longrightarrow X/M : x \longmapsto [x]$ is called the *canonical quotient map.* We take Q_M^X to be an element of $L(X,X/M)$.

Note : $\|[x]\| = \inf\{\ \|y\| : y \in [x]\ \}$

If $\|[x] - [z]\| < \epsilon\ (\epsilon > 0)$, then $\exists\ v \in [z]$ s.t. $\|x - v\| < \epsilon$.

In the next three results , X is a seminormed space and M a subspace of X.

Proposition II.2.1 : $Q_M^X \in BL[X,X/M]$ and Q_M^X is open. (cf. [KN ; 5.7])

Proof :

$\forall\, x \in X$, $\|Q_M^X x\| = \|[x]\| \leq \|x\|$ so $Q_M^X \in BL[X,X/M]$.

Let $[x] \in U_{X/M}$.

Then $\exists\ y \in [x]$ s.t. $\|y\| < 1$.

Now , $[x] = [y] = Q_M^X y \in Q_M^X U_X$.

Thus , $U_{X/M} \subset Q_M^X U_X$. □

Proposition II.2.2 : X/M is normed $\Leftrightarrow$ M is closed. (cf [KN ; 5.7])

Proof :

($\Rightarrow$) Suppose X/M is normed.

Let $x \in \overline{M}$.

Then $d(x,M) = 0$.

i.e. $\|[x]\| = 0$.

Since X/M is normed , $[x] = O$.

i.e. $x \in M$.

($\Leftarrow$) Suppose M is closed.

Let $\|[x]\| = 0$.

Then $d(x,M) = 0$ so $x \in \overline{M} = M$.

Thus , $[x] = O$. □

Proposition II.2.3 : X complete $\Rightarrow$ X/M complete.

Proof :

Suppose X is complete.

Let (x_n) be a sequence in X s.t. $\sum_{n=1}^{\infty} \|[x_n]\| < \infty$.

$\forall n \in \mathbb{N}, \exists y_n \in [x_n]$ s.t. $\|y_n\| < \|[x_n]\| + 2^{-n}$.

Now , $\sum_{n=1}^{\infty} \|y_n\| < \infty$.

Since X is complete , it follows from Proposition II.1.3 that $\exists x \in X$ s.t. $\sum_{i=1}^{n} y_i \longrightarrow x$.

$\forall n \in \mathbb{N}, \|\sum_{i=1}^{n} [x_i] - [x]\| = \|\sum_{i=1}^{n} [y_i] - [x]\| = \|[\sum_{i=1}^{n} y_i - x]\| \leq \|\sum_{i=1}^{n} y_i - x\| \longrightarrow 0.$

By Proposition II.1.3 , X/M is complete. □

Notation : Let X be a seminormed space. Write $N_X = \{ x \in X : \|x\| = 0 \}$.

Note :

(i) $N_X = \overline{\{0\}}$.

(ii) Since N_X is closed, X/N_X is a normed space.

(iii) It will follow from theorem II.4.2 and the fact that $N_X^{\perp} = X'$ that X' and $(X/N_X)'$ may be identified.

Proposition II.2.4 : Let X be a seminormed space, $(x_n) \in X^{\mathbb{N}}$ and $x \in X$. Then $x_n \longrightarrow x$ in $X \Leftrightarrow [x_n] \longrightarrow [x]$ in X/N_X.

Proof :

($\Rightarrow$) $Q^X_{N_X}$ is bounded.

($\Leftarrow$) Suppose $[x_n] \longrightarrow [x]$ in X/N_X.

$\forall\, n \in \mathbb{N}, \exists\, z_n \in N_X$ s.t. $\|x_n - x + z_n\| < \|[x_n] - [x]\| + \frac{1}{n}$.

$\|x_n - x\|$

$= \|x_n - x + z_n - z_n\|$

$\leq \|x_n - x + z_n\| + \|z_n\|$

$< \|[x_n] - [x]\| + \frac{1}{n}$

$\longrightarrow 0$.

Thus, $x_n \longrightarrow x$ in X. □

Corollary II.2.5 : For $A \subset X$, $Q^X_{N_X} \overline{A} = \overline{Q^X_{N_X} A}$.

Corollary II.2.6 : X is complete $\Leftrightarrow$ X/N_X is complete.

Definition : Let X and Y be (semi–)normed spaces.

For (x,y) , $(u,v) \in X \times Y$ and $\alpha \in \mathbb{F}$, let

$$(x,y) + (u,v) = (x + u , y + v)$$

$$\alpha (x,y) = (\alpha x , \alpha y)$$

$$\|(x,y)\| = \max\{ \|x\| , \|y\| \}.$$

Then , under these operations , $X \times Y$ is a (semi–)normed space called the *product* of X and Y.

The maps $\Pi_X : X \times Y \longrightarrow X : (x,y) \longmapsto x$ and $\Pi_Y : X \times Y \longrightarrow Y : (x,y) \longmapsto y$ are called the *projections* of $X \times Y$ onto X and Y respectively.

We take Π_X and Π_Y to be elements of $L(X{\times}Y,X)$ and $L(X{\times}Y,Y)$ respectively.

Note :

(i) X and Y complete $\Rightarrow$ $X \times Y$ complete.

(ii) $\Pi_X \in BL[X{\times}Y,X]$ and $\Pi_Y \in BL[X{\times}Y,Y]$.

Note that the definitions given here are consistent with the usual definitions of General Topology (i.e. the subspace seminorm generates the subspace topology , the quotient seminorm generates the quotient topology where the equivalence relation is given by $x \,\rho\, y$ iff $x - y \in M$ and the product seminorm generates the product topology).

3. Finite dimensional Normed Spaces

In this section , we derive some properties of finite dimensional seminormed spaces from those of finite dimensional normed spaces. Note that in [Wil1 ; pp192,193] it is shown that the topology of a finite dimensional seminormed space is uniquely determined by N_X.

Definition : A linear map from a seminormed space into a seminormed space is an *isomorphism* iff it is continuous and has a continuous inverse. Two seminormed spaces are *isomorphic* iff there is an isomorphism from the one space onto the other.

Theorem II.3.1 : Let X be an n dimensional normed space over $\mathbb{F}$. Then X is isomorphic to $\mathbb{F}^n$. (cf. for example [Gol ; I.4.2])

Proof :

Let $\{x_1 , \ldots , x_n\}$ be a basis for X.

Define $T : \mathbb{F}^n \longrightarrow X$ by $T\alpha = \sum_{i=1}^{n} \alpha_i x_i \quad (\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{F}^n)$.

Then T is linear and surjective.

$\|T\alpha\| = \|\sum_{i=1}^{n} \alpha_i x_i\| \leq \sum_{i=1}^{n} |\alpha_i| \, \|x_i\| \leq n \,(\max_{1\leq i\leq n} |\alpha_i| \,)\,(\max_{1\leq i\leq n} \|x_i\| \,) \leq n \,(\max_{1\leq i\leq n} \|x_i\| \,)\, \|\alpha\|$.

Thus , T is continuous.

Define $f : \mathbb{F}^n \longrightarrow \mathbb{R}$ by $f(\alpha) = \|T\alpha\| \quad (\alpha \in \mathbb{F}^n)$.

Then f is continuous.

Now , $K = \{ \alpha \in \mathbb{F}^n : \|\alpha\| = 1 \}$ is compact so $f[K]$ is compact.

Thus , $\exists \, \gamma \in K$ s.t. $\forall \, \alpha \in K , f(\alpha) \geq f(\gamma)$.

Also , $f(\gamma) > 0$. $\quad (f(\gamma) = 0 \Rightarrow \|T\gamma\| = 0 \Rightarrow T\gamma = O \Rightarrow \gamma = O$, a contradiction.)

For $\alpha \in \mathbb{F}\backslash\{O\}$, $\frac{\alpha}{\|\alpha\|} \in K$ so $f(\frac{\alpha}{\|\alpha\|}) \geq f(\gamma)$.

i.e. $\|T\alpha\| \geq f(\gamma) \, \|\alpha\|$.

By Corollary II.1.8 , T has a continuous inverse. □

Since isomorphisms take closed bounded sets onto closed bounded sets and since a set in $\mathbb{F}^n$ is compact if and only if it is closed and bounded, the following corollary is obtained.

Corollary II.3.2 : A closed bounded set in a finite dimensional normed space is compact.

Corollary II.3.3 : A closed bounded subset of a finite dimensional seminormed space is compact.

Proof :

Let X be a finite dimensional seminormed space and A a closed bounded subset of X.

By Corollary II.2.5, $Q^X_{N_X} A$ is a closed bounded subset of X/N_X which is a finite dimensional normed space.

By Corollary II.3.2, $Q^X_{N_X} A$ is compact in X/N_X.

Since A is closed, it follows from Proposition II.2.4 that A is compact in X. □

For the remainder of this section X and Y are seminormed spaces.

Proposition II.3.4 : If X is complete and X and Y are isomorphic, then Y is complete.

Proof :

Suppose X is complete and $T : X \longrightarrow Y$ is a surjective isomorphism.

Let (y_n) be a Cauchy sequence in Y.

Since T^{-1} is bounded, $\forall\, m, n \in \mathbb{N}$, $\|T^{-1}y_n - T^{-1}y_m\| \le \|T^{-1}\|\, \|y_n - y_m\|$.

Thus, $(T^{-1}y_n)$ is a Cauchy sequence in X.

Since X is complete, $\exists\, x \in X$ s.t. $T^{-1}y_n \longrightarrow x$.

Since T is continuous, $y_n = TT^{-1}y_n \longrightarrow Tx$. □

Corollary II.3.5 : Any two normed spaces of the same finite dimension over the same field are isomorphic.

Corollary II.3.6 : Every finite dimensional seminormed space is complete.

Corollary II.3.7 : A finite dimensional subspace of a normed space is closed.

Note : If X is not normed , then Corollary II.3.7 fails since $\{O\}$ is not closed.

Theorem II.3.8 : Let M and N be subspaces of X with M closed and N finite dimensional. Then M + N is closed. (cf. [Wil1 ; p192])

Proof :

Since M is closed , X/M is a normed space.

Since N is finite dimensional and Q_M^X is linear , $Q_M^X N$ is finite dimensional.

By Corollary II.3.6 , $Q_M^X N$ is closed in X/M.

Since Q_M^X is continuous , $M + N = Q_M^{X\,-1} Q_M^X N$ is closed in X. □

The following is a partial generalisation of the corresponding normed space result.

Proposition II.3.9 : Let X be normed and finite dimensional. Then L[X,Y] = BL[X,Y].

Proof :

Suppose X is normed and $\{e_1 , \dots , e_n\}$ is a basis for X.

Let $T \in L[X,Y]$ s.t. $\exists\, i$ with $\|Te_i\| \neq O$.

Define $\| \|_1$ on X by $\|\sum_{i=1}^{n} \alpha_i e_i\|_1 = \sum_{i=1}^{n} |\alpha_i| \max_{1 \leq i \leq n} \|Te_i\|$.

Then $\| \|_1$ is a norm on X. Let $X_1 = (X, \| \|_1)$.

By considering the map in Theorem II.3.1 , it can be seen that $id_X : X \longrightarrow X_1$ is an isomorphism.

Thus , $\exists\ C > 0$ s.t. $\forall\ x \in X$, $\|x\|_1 \leq C\ \|x\|$.

Now , $\|T(\sum_{i=1}^{n} \alpha_i e_i)\| = \|\sum_{i=1}^{n} \alpha_i Te_i\| \leq \sum_{i=1}^{n} |\alpha_i| \max_{1\leq i\leq n} \|Te_i\| = \|\sum_{i=1}^{n} \alpha_i e_i\|_1 \leq C\ \|\sum_{i=1}^{n} \alpha_i e_i\|$.

Thus , $T \in BL[X,Y]$. □

Example : Proposition II.3.9 need not hold if X is not normed.

Let X be a finite dimensional space and $x \in X$ with $\|x\| = 0$ and $\|Tx\| \neq 0$.

Then $T \notin BL[X,Y]$.

The next result is a partial generalisation to seminormed spaces of exercise 20.1 in [Jam].

Corollary II.3.10 : Let Y be normed and finite dimensional and $R(T) = Y$. Then T is open.

Proof :

Let $\{y_1,...,y_n\}$ be a basis for $R(T)$.

For $1 \leq i \leq n$, $\exists\ x_i \in D(T)$ s.t. $y_i = Tx_i$.

Define $S : R(T) \longrightarrow D(T)$ by $S(\sum_{i=1}^{n} \alpha_i y_i) = \sum_{i=1}^{n} \alpha_i x_i$.

Then S is linear and $\forall\ y \in Y$, $TSy = y$.

By Proposition II.3.9 , S is continuous.

Thus , $S^{-1}B_{D(T)}$ is a neighbourhood of O in Y.

Now , $TB_{D(T)} \supset TSS^{-1}B_{D(T)} = S^{-1}B_{D(T)}$.

Thus , T is open. □

Example : Corollary II.3.10 need not hold if Y is not normed.
Let $n \in \mathbb{N}\backslash\{1\}$ and for $1 \leq i \leq n$, define the seminorm $\| \|_i$ on $\mathbb{F}^n$ by $\|\alpha\|_i = |\alpha_i| \quad (\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{F}^n)$.
Let $1 \leq i < j \leq n$, $X = (\mathbb{F}^n, \| \|_i)$, $Y = (\mathbb{F}^n, \| \|_j)$ and $T = id_{\mathbb{F}^n}$.
Then $U_X = \{ \alpha \in \mathbb{F}^n : |\alpha_i| < 1 \}$ and $U_Y = \{ \alpha \in \mathbb{F}^n : |\alpha_j| < 1 \}$.
Since $i \neq j$, there is no $r > 0$ s.t. $r\, U_Y \subset TU_X$ so T is not open.

Definition : Let M and N be subspaces of X. Then X is the *direct sum* of M and N, written $X = M \oplus N$, iff $X = M + N$ and $M \cap N = \{0\}$.

For the remainder of this section M is a subspace of X.

Proposition II.3.11 : If $\dim X/M < \infty$, then there is a finite dimensional subspace F of X s.t. $X = M \oplus F$.

Proof : If $\{[x_1],\ldots,[x_n]\}$ is a basis for X/M, then $X = M \oplus \operatorname{span}\{x_1,\ldots,x_n\}$. □

Definition : A *projection* of X onto M is a linear map $P : X \longrightarrow X$ s.t. $R(P) = M$ and $P^2 = P$.

Proposition II.3.12 : Let X normed and $\dim M < \infty$. Then there is a bounded projection P of X onto M and $X = N(P) \oplus M$.

Proof :
Let $B = \{x_1,\ldots,x_n\}$ be a basis for M.
By the Hahn–Banach theorem, for $1 \leq i \leq n$, $\exists\, x_i' \in X'$ s.t. for $1 \leq i, j \leq n$, $x_i' x_j = \delta_{ij}$.
Then the map $P : X \longrightarrow X : x \longmapsto \sum_{i=1}^{n} x_i'(x)\, x_i$ has the required properties. □

4. *Dual Spaces*

Definition : Let X and Y be seminormed spaces and $T \in L(X,Y)$. Then T is an *isometry* iff $\forall\, x \in D(T)$, $\|Tx\| = \|x\|$.

Note :

(i) If T is an isometry, then $D(T)$ is complete $\Leftrightarrow$ $R(T)$ is complete.

(ii) Unlike in the normed space case, an isometry need not be injective.

Definition : Two normed spaces X and Y are *equivalent*, written $X \equiv Y$, iff there is a linear isometry from the one space onto the other.

(We reserve $\equiv$ for normed spaces.)

Example : $c_o' \equiv l_1$ where for $(a_n) \in l_1$ the corresponding $f \in c_o'$ is given by $f((\lambda_n)) = \sum_{n=1}^{\infty} \lambda_n a_n \quad ((\lambda_n) \in c_o)$. (See [Kr] for details.)

Note : If $X \equiv Y$, then the linear isometry from X onto Y is denoted by I_X^Y.

Note that $(I_X^Y)^{-1} = I_Y^X$.

For the remainder of this section, X is a seminormed space and M is a subspace of X.

Notation : If $x' \in X'$, then $x'|_M$ will be denoted by x_M'. Clearly, $x_M' \in M'$.

Lemma II.4.1 : Let $m' \in M'$. Then $\exists\, x' \in X'$ s.t. $x_M' = m'$ and $\|x'\| = \|m'\|$.

Proof :

Since $m' \in M'$, $\forall\, m \in M$, $|m'm| \leq \|m'\|\,\|m\|$.

Now take $f = m'$ and $p(x) = \|m'\|\,\|x\|$ in Theorem I.4.8. □

Theorem II.4.2 :

(a) $X'/M^{\perp} \equiv M'$.

(b) $(X/M)' \equiv M^{\perp}$. (cf. for example [Gol2 ; II.2.1])

Proof :

(a) If $x' \in X'$ and $y' \in x' + M^{\perp}$, then $x' - y' \in M^{\perp}$ so $\forall\, m \in M$, $x'm = y'm$.

Thus, $U : X'/M^{\perp} \longrightarrow M'$ can be defined by $U[x'] = x'_M$.

Note that U is linear.

If $m' \in M'$ then, by Lemma II.4.1, $\exists\, x' \in X'$ s.t. $x'_M = m'$.

Thus, U is surjective.

Let $x' \in X'$.

Then $\forall\, y' \in [x']$, $\|U[x']\| = \|U[y']\| = \|y'_M\| \leq \|y'\|$ so $\|U[x']\| \leq \|[x']\|$.

By Lemma II.4.1, $\exists\, v' \in X'$ s.t. $v'_M = x'_M$ and $\|v'\| = \|x'_M\|$.

Now, $v' \in [x']$ so $\|[x']\| \leq \|v'\| = \|x'_M\| = \|U[x']\|$.

Thus, U is an isometry.

(b) Let $z' \in (X/M)'$.

Define $x'_{z'} : X \longrightarrow \mathbb{F}$ by $x'_{z'}(x) = z'[x]$.

Then $x'_{z'}$ is linear and $\forall\, x \in X$, $|x'_{z'}x| = |z'[x]| \leq \|z'\|\, \|[x]\| \leq \|z'\|\, \|x\|$

so $x'_{z'} \in X'$.

Also, $\forall\, m \in M$, $x'_{z'}m = z'[m] = z'O = 0$ so $x'_{z'} \in M^{\perp}$.

Define $V : (X/M)' \longrightarrow M^{\perp}$ by $Vz' = x'_{z'}$.

Then, V is linear and $\forall\, z' \in (X/M)'$, $\|Vz'\| \leq \|z'\|$.

Let $z' \in (X/M)'$ and $x \in X$.

Then $\forall\, y \in [x]$, $|z'[x]| = |z'[y]| = |(Vz')y| \leq \|Vz'\|\, \|y\|$ so

$|z'[x]| \leq \|Vz'\|\, \|[x]\|$.

Thus, $\|z'\| \leq \|Vz'\|$ so V is an isometry.

Let $x' \in M^{\perp}$.

Define $z' : X/M \longrightarrow \mathbb{F}$ by $z'[x] = x'x$. (Note that z' is well–defined.)

Then , z' is linear and $\forall\, y \in [x]$, $|z'[x]| = |z'[y]| = |x'y| \le \|x'\|\,\|y\|$.

Thus , $z' \in (X/M)'$.

Also , $Vz' = x'$ so V is surjective. □

Theorem II.4.3 : Let M be dense in X , Y a Banach space and $A \in BL[M,Y]$.

Then $\exists!\, \overline{A} \in BL[X,Y]$ s.t. $\overline{A}|_M = A$ and $\|\overline{A}\| = \|A\|$.

Also , $M' \equiv X'$. (cf [Gol2 ; II.2.1])

Proof :

Let $x \in X$.

$\exists\, (x_n) \in M^{\mathbb{N}}$ s.t. $x_n \longrightarrow x$.

$\forall\, m , n \in \mathbb{N}$, $\|Ax_n - Ax_m\| \le \|A\|\,\|x_n - x_m\|$.

Thus , (Ax_n) is a Cauchy sequence in Y.

Since Y is a Banach space , $\exists!\, y \in Y$ s.t. $Ax_n \longrightarrow y$.

Let (z_n) be any sequence in M with $z_n \longrightarrow x$.

Then $\|Az_n - y\| \le \|A\|\,\|z_n - x\| + \|A\|\,\|x - x_n\| + \|Ax_n - y\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Thus , y does not depend on the sequence chosen.

Thus , $\overline{A}$ can be defined by $\overline{A}x = \lim_n Ax_n$.

Clearly , $\overline{A}$ is a linear extension of A to X.

$\forall\, x \in X$, $\|\overline{A}x\| = \lim_n \|Ax_n\| \le \|A\| \lim_n \|x_n\| = \|A\|\,\|x\|$ so $\|\overline{A}\| \le \|A\|$.

Since $\overline{A}$ is an extension of A , $\|A\| \le \|\overline{A}\|$.

It is easy to see that $\overline{A}$ is unique.

If $Y = \mathbb{F}$, then the above shows that $M' \longrightarrow X' : A \longmapsto \overline{A}$ is a surjective linear isometry. □

5. The Second Dual of X

Theorem II.5.1 : $\forall\, x \in X$, $\|x\| = \sup\{\, |x'x| : x' \in B_{X'} \}$.

Proof :

Let $x \in X$.

$\forall\, x' \in B_{X'}$, $|x'x| \le \|x'\|\,\|x\| \le \|x\|$ so $\sup\{\, |x'x| : x' \in B_{X'} \} \le \|x\|$.

By Corollary I.4.9 , $\exists\, x' \in B_{X'}$ s.t. $x'x = \|x\|$.

Thus , $\|x\| \le \sup\{\, |x'x| : x' \in B_{X'} \}$. □

Let $x \in X$.

Define $\hat{x} : X' \longrightarrow \mathbb{F}$ by $\hat{x}x' = x'x$.

Then $\hat{x}$ is linear and $\forall\, x' \in X'$, $|\hat{x}x'| = |x'x| \le \|x'\|\,\|x\|$ so $\hat{x} \in X''$.

By Theorem II.5.1 , $\|\hat{x}\| = \|x\|$.

Thus , the map $J_X^{X''} : X \longrightarrow X'' : x \longmapsto \hat{x}$ is an isometry.

From the remark on page 41 , this means that X is complete $\Leftrightarrow$ $J_X^{X''}X$ is complete.

Note :

(i) $J_X^{X''}$ is injective $\Leftrightarrow$ X is normed.

(ii) If $A \subset X$ then $\hat{A}$ or $A^{\wedge}$ will be written for $J_X^{X''}A$.

(iii) $\overline{\hat{X}}$ is a Banach space.

(iv) $J_X^{X''} : (X , \sigma(X,X')) \longrightarrow (X'' , \sigma(X'',X'))$ is continuous.

Definition : $\overline{\hat{X}}$ is called the *completion* of X and is denoted by $\tilde{X}$.

(We do not use the definition in [Gol2 ; p31] as $J_X^{X''}$ need not be injective.)

Lemma II.5.2 : $B_{X''}$ is $\sigma(X'',X')$ compact. In particular, $B_{X''}$ is $\sigma(X'',X')$ closed.

Proof : Immediate by the Banach–Alaoglu theorem. □

Theorem II.5.3 (Goldstine's Theorem) : $\hat{B}_X$ is $\sigma(X'',X')$ dense in $B_{X''}$. (cf for example [DS ; p424])

Proof :

Let $B_1 = \overline{\hat{B}_X}^{\sigma(X'',X')}$.

Since $J_X^{X''}$ is an isometry and $B_{X''}$ is $\sigma(X'',X')$ closed, $B_1 \subset B_{X''}$.

Let $z'' \notin B_1$.

By Corollary I.4.13, $\exists\, x' \in X'$ s.t. $|z''x'| > 1$ and $\forall\, x'' \in B_1$, $|x''x'| \leq 1$.

Since $\hat{B}_X \subset B_1$, this means that $\forall\, x \in B_X$, $|x'x| \leq 1$ so $\|x'\| \leq 1$.

Since $|z''x'| > 1$, $\|z''\| > 1$.

i.e. $z'' \notin B_{X''}$. □

Corollary II.5.4 : $\forall\, x'' \in X''$, $\exists$ a bounded net (x_α) in X s.t. $\hat{x}_\alpha \longrightarrow x''$ w.r.t. $\sigma(X'',X')$.

Definition : X is *semireflexive* iff $J_X^{X''}$ is surjective.

A semireflexive normed space is called *reflexive.*

The next result generalises a well known characterisation of reflexivity for normed spaces (see for example [DS ; p425]) to seminormed spaces.

Theorem II.5.5 : X is semireflexive $\Leftrightarrow$ B_X is $\sigma(X,X')$ compact.

Proof :

($\Rightarrow$) Suppose X is semireflexive.

Let (x_α) be a net in B_X.

Since $B_{X''}$ is $\sigma(X'',X')$ compact, there is a subnet (x_{α_β}) of (x_α) and there is an $x'' \in B_{X''}$ s.t. $\hat{x}_{\alpha_\beta} \longrightarrow x''$ w.r.t. $\sigma(X'',X')$.

Since X is semireflexive, $J_X^{X''} B_X = B_{X''}$ so $\exists\, x \in B_X$ s.t. $\hat{x} = x''$.

Clearly, $x_{\alpha_\beta} \longrightarrow x$ w.r.t. $\sigma(X,X')$.

($\Leftarrow$) Suppose B_X is $\sigma(X,X')$ compact.

Then $J_X^{X''} B_X$ is $\sigma(X'',X')$ compact, hence $\sigma(X'',X')$ closed since $\sigma(X'',X')$ is Hausdorff.

Thus, $J_X^{X''} B_X = \overline{J_X^{X''} B_X}^{\sigma(X'',X')} = B_{X''}$. □

Corollary II.5.6 : Let X be semireflexive and M be a closed subspace of X. Then M is semireflexive.

Proof :

Let (x_α) be a net in B_M.

Then (x_α) is a net in B_X.

Since X is semireflexive, (x_α) has a $\sigma(X,X')$ convergent subnet (x_{α_β}) with limit $x \in B_X$.

Now, $x \in \overline{M}^{\sigma(X,X')} = \overline{M} = M$.

Thus, $x_\alpha \longrightarrow x$ w.r.t. $\sigma(M,M')$ and $x \in B_M$. □

Note : Let $N = N_X$.

Since $N^{\perp} = X'$, it follows from [KN ; 16.11 , 17.13] that $\sigma(X/N,(X/N)') = \sigma(X/N,X')$ where for $x \in X$ and $x' \in N^{\perp}$, $<[x],x'> = x'x$. This means that if (x_α) is a net in X and $x \in X$, then $x_\alpha \longrightarrow x$ w.r.t. $\sigma(X,X') \Leftrightarrow [x_\alpha] \longrightarrow [x]$ w.r.t. $\sigma(X/N,(X/N)')$. Thus , if $A \subset X$ is closed (= $\sigma(X,X')$ closed) , then A is $\sigma(X,X')$ (sequentially) compact if and only if $Q_N^X A$ is $\sigma(X/N,(X/N)')$ (sequentially) compact. It is well known that for a Banach space X , X is reflexive if and only if X' is reflexive. Also , if two normed spaces are isomorphic , then the one space is reflexive if and only if the other space is reflexive. Thus , we obtain :

Theorem II.5.7 : Let X be complete. Then

T.F.A.E. :

(a) X is semireflexive.

(b) X/N_X is reflexive.

(c) $(X/N_X)'$ is reflexive.

(d) X' is reflexive.

Note : It has been shown (see for example [F ; 3.10]) that in a normed space , a set is weakly compact if and only if it is weakly sequentially compact. Thus , the following result is obtained :

Theorem II.5.8 : A seminormed space is semireflexive $\Leftrightarrow$ every bounded sequence has a weakly convergent subsequence.

Chapter III Linear Operators and their Adjoints

In this chapter we start off by examining closed linear operators. The adjoint of a linear operator is then defined (note that the definition given here differs from that given in [Gol2 ; II.2.2]) and the relationship between an operator and its adjoint is studied. Most of the work will concentrate on generalising results in [Gol2 ; II] to seminormed spaces. We also introduce three states for linear operators which correspond to those given in [Gol2 ; p58] when X is normed and state diagrams are produced which have the same form as those obtained in [Gol1].

Throughout this chapter , X and Y are seminormed spaces and $T \in L(X,Y)$.

1. Closed Linear Operators

Definition :

(a) The *graph* of T is $G(T) = \{ (x,Tx) : x \in D(T) \} \subset X \times Y$.

(b) T is *closed* (as an element of $L(X,Y)$) iff $G(T)$ is closed in $X \times Y$.

Note : Since T is linear, $G(T)$ is a subspace of $X \times Y$.

Proposition III.1.1 :

(i) T is closed $\Leftrightarrow$ $(\forall (x_n) \in D(T)^{\mathbb{N}})\ [x_n \to x ,\ Tx_n \to y \Rightarrow x \in D(T) ,\ Tx = y]$.

(ii) T injective, closed $\Rightarrow$ T^{-1} closed.

(iii) T closed $\Rightarrow$ $N(T)$ closed.

(iv) Y normed, $D(T)$ closed, T continuous $\Rightarrow$ T closed.

Proof :

(i) ($\Rightarrow$) Suppose T is closed.

Let (x_n) be a sequence in $D(T)$ s.t. $x_n \to x$ and $Tx_n \to y$.

Then (x_n,Tx_n) is a sequence in $G(T)$ with $(x_n,Tx_n) \to (x,y)$.

Since $G(T)$ is closed in $X \times Y$, $(x,y) \in G(T)$.

Thus, $x \in D(T)$ and $y = Tx$.

($\Leftarrow$) Suppose $(\forall (x_n) \in D(T)^{\mathbb{N}})\ [x_n \to x ,\ Tx_n \to y \Rightarrow x \in D(T) ,\ y = Tx]$.

Let $(x,y) \in \overline{G(T)}$.

Then $\exists\ (x_n) \in D(T)^{\mathbb{N}}$ s.t. $(x_n,Tx_n) \to (x,y)$.

Note that $x_n \to x$ and $Tx_n \to y$.

By assumption, $x \in D(T)$ and $y = Tx$.

i.e. $(x,y) \in G(T)$.

(ii) Suppose T is injective and closed.

Let (y_n) be a sequence in $D(T^{-1}) = R(T)$ s.t. $y_n \longrightarrow y$ and $T^{-1}y_n \longrightarrow x$.

Then , $T^{-1}y_n \longrightarrow x$ and $T(T^{-1}y_n) = y_n \longrightarrow y$.

Since T is closed , $x \in D(T)$ and $y = Tx$.

i.e. $y \in D(T^{-1})$ and $x = T^{-1}y$.

(iii) Suppose T is closed.

Let (x_n) be a sequence in $N(T)$ with $x_n \longrightarrow x$.

$\forall\, n \in \mathbb{N}$, $Tx_n = O$ so $Tx_n \longrightarrow O$.

Since T is closed , $x \in D(T)$ and $Tx = O$.

i.e. $x \in N(T)$.

(iv) Suppose Y is normed , $D(T)$ is closed and T is continuous.

Let (x_n) be a sequence in $D(T)$ s.t. $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$.

Since $D(T)$ is closed , $x \in D(T)$.

Since T is continuous , $Tx_n \longrightarrow Tx$.

Since Y is normed , $y = Tx$. □

Lemma III.1.2 : Let X be complete and T be closed , and suppose there is an $r > 0$ s.t. $r\,U_Y \subset \overline{TB}_{D(T)}$. Then $r\,U_Y \subset TB_{D(T)}$. (cf [Gol2 ; II.1.7])

Proof :

Note that it is sufficient to prove that $\forall\, \epsilon \in (0,1)$, $r\,U_Y \subset \frac{1}{1-\epsilon} TB_{D(T)}$. (1)

[Then , if $y \in r\,U_Y$, $\exists\, \epsilon \in (0,1)$ s.t. $\frac{1}{1-\epsilon} y \in r\,U_Y$.

From (1) , $\exists\, x \in B_{D(T)}$ s.t. $\frac{1}{1-\epsilon} y = \frac{1}{1-\epsilon} Tx$.

Thus , $y = Tx \in TB_{D(T)}$.]

Let $\epsilon > 0$ and $y \in r\, U_Y$.

Note that $\forall\, n \in \mathbb{N} \cup \{0\}$, $r\, \epsilon^n U_Y \subset \epsilon^n\, \overline{TB}_{D(T)} = \overline{T(\, \epsilon^n B_{D(T)})}$.

Thus , $\exists\, x_0 \in B_{D(T)}$ s.t. $\|y - Tx_0\| < r\, \epsilon$. i.e. $y - Tx_0 \in r\, \epsilon\, U_Y$.

Also , $\exists\, x_1 \in \epsilon\, B_{D(T)}$ s.t. $\|y - Tx_0 - Tx_1\| < r\, \epsilon^2$. i.e. $y - Tx_0 - Tx_1 \in r\, \epsilon^2\, U_Y$.

Continuing in this way , a sequence (x_n) is obtained s.t. $\forall\, n \in \mathbb{N}$, $x_n \in \epsilon^n\, B_{D(T)}$

and $\|y - \sum_{i=0}^{n} Tx_i\| < r\, \epsilon^{n+1}$. (2)

Now , $\sum_{n=0}^{\infty} \|x_n\| \leq \frac{1}{1-\epsilon} < \infty$.

Since X is complete , $\exists\, x \in X$ s.t. $\sum_{i=0}^{n} x_i \longrightarrow x$.

Note that $\|x\| \leq \frac{1}{1-\epsilon}$.

By (2) , $T(\sum_{i=0}^{n} x_i) \longrightarrow y$.

Since T is closed , $x \in D(T)$ and $y = Tx \in \frac{1}{1-\epsilon}\, TB_{D(T)}$. □

Definition : A topological space X is of the *second category* iff whenever (A_n) is a sequence of subsets of X s.t. $X = \bigcup_{n=1}^{\infty} A_n$, $\exists\, n$ s.t. $\operatorname{int} \overline{A_n} \neq \emptyset$.

The proof of the following result can be found in [K ; p200].

Theorem III.1.3 (Baire Category Theorem) : If X is a complete semimetric space , then X is of the second category.

Theorem III.1.4 (Open Mapping Theorem for Seminormed Spaces) : Let X be complete and Y be of the second category , T closed and $R(T) = Y$. Then T is open.

Proof :

By Lemma III.1.2 , it is sufficient to prove that $\exists\, r > 0$ s.t. $r\,U_Y \subset \overline{TB}_{D(T)}$.

Since $Y = R(T) = \bigcup_{n=1}^{\infty} n\,TB_{D(T)}$ and Y is of the second category , $\exists\, n$ s.t.

$\operatorname{int} \overline{nTB}_{D(T)} \neq \emptyset$.

Thus , $\exists\, y \in \operatorname{int} \overline{nTB}_{D(T)}$.

Thus , $\exists\, \delta > 0$ s.t. $y + \delta U_Y \subset \overline{nTB}_{D(T)}$.

Let $V = \frac{1}{2n}(y + \delta U_Y)$ and $U = V - V$.

Then , U is open and $O \in U$.

Also , $U = V - V \subset \frac{1}{2}\overline{TB}_{D(T)} + \frac{1}{2}\overline{TB}_{D(T)} = \overline{TB}_{D(T)}$.

Thus , $\exists\, r > 0$ s.t. $r\,U_Y \subset \overline{TB}_{D(T)}$. □

Theorem III.1.5 (Closed Graph Theorem for Seminormed Spaces) : Let X be normed , X and Y complete , $D(T) = X$ and T closed. Then T is continuous.

Proof :

Since X and Y are complete , $X \times Y$ is also complete.

Since $G(T)$ is a closed subspace of $X \times Y$, it is complete.

Note that $\Pi_X|_{G(T)}$ is bijective.

We now show that it is closed.

Suppose $(x_n,Tx_n) \longrightarrow (x,y)$ and $x_n = \Pi_X|_{G(T)}(x_n,Tx_n) \longrightarrow z$.

Since $G(T)$ is closed , $(x,y) \in G(T)$.

Also , $x_n \longrightarrow x$ so $z = x = \Pi_X|_{G(T)}(x,y)$ since X is normed.

By the open mapping theorem , $\Pi_X|_{G(T)}$ is open so $(\Pi_X|_{G(T)})^{-1}$ is continuous.

Thus , $T = \Pi_Y(\Pi_X|_{G(T)})^{-1}$ is continuous. □

We now consider a seminormed space which will be used in the proof of the next result.

Let $F \neq \emptyset$ be a set and $B(F,Y) = \{ f : F \longrightarrow Y : \exists\, C > 0 \text{ s.t. } \forall x \in F , \|f(x)\| \leq C \}$.

For $f , g \in B(F,Y)$ and $\lambda \in \mathbb{F}$, define

$f + g : F \longrightarrow Y$ by $(f + g)(x) = f(x) + g(x) \quad (x \in F)$

$\lambda f : F \longrightarrow Y$ by $(\lambda f)(x) = \lambda f(x) \quad (x \in F)$

$\|f\| = \sup\{ \|f(x)\| : x \in F \}$.

Under these operations , $B(F,Y)$ becomes a seminormed space.

Note :

(i) If $f_n \longrightarrow f$ in $B(F,Y)$, then $\forall x \in F$, $f_n(x) \longrightarrow f(x)$ in Y.

(ii) Y normed $\Rightarrow$ $B(F,Y)$ normed.

(iii) Y normed , complete $\Rightarrow$ $B(F,Y)$ complete.

Proof of (iii) :

Let (f_n) be a sequence in $B(F,Y)$ with $\sum_{n=1}^{\infty} \|f_n\| < \infty$.

$\forall x \in F$, $\sum_{n=1}^{\infty} \|f_n(x)\| < \infty$.

Since Y is normed and complete , $f : F \longrightarrow Y$ can be defined

by $f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in F)$.

$\forall x \in F$, $\|f(x)\| \leq \sum_{n=1}^{\infty} \|f_n\|$ so $f \in B(F,Y)$.

Also , $\forall x \in F$, $\| \sum_{i=1}^{n} f_i(x) - f(x)\| = \| \sum_{i=n+1}^{\infty} f_i(x)\| \leq \sum_{i=n+1}^{\infty} \|f_i\|$.

Thus , $\| \sum_{i=1}^{n} f_i - f\| \leq \sum_{i=n+1}^{\infty} \|f_i\| \longrightarrow 0$.

By Proposition II.1.3 , $B(F,Y)$ is complete. □

Theorem III.1.6 (Principle of Uniform Boundedness for Seminormed Spaces) : Let X be complete and $F \subset BL[X,Y]$ and suppose that $\forall x \in X$, $\sup_{T \in F} \|Tx\| < \infty$.

Then $\sup_{T \in F} \|T\| < \infty$.

Proof :

Let $x \in X$.

Define $f_x : F \longrightarrow \tilde{Y}$ by $f_x(T) = J_Y^{Y''} Tx$.

Then $\forall T \in F$, $\|f_x(T)\| = \|Tx\| \leq \sup_{T \in F} \|Tx\| < \infty$.

Thus , $f_x \in B(F,\tilde{Y})$.

Suppose $x , z \in X$ with $\|x - z\| = 0$.

$\forall T \in F$, $\|f_x(T) - f_z(T)\| = \|Tx - Tz\| \leq \|T\| \|x - z\| = 0$ so $f_x = f_z$.

Define $A : \hat{X} \longrightarrow B(F,\tilde{Y})$ by $A\hat{x} = f_x \quad (x \in X)$.

Then $A \in L[\hat{X},B(F,\tilde{Y})]$.

We now show that A is closed.

Suppose $\hat{x}_n \longrightarrow \hat{x}$ and $A\hat{x}_n \longrightarrow f$.

Then $x_n \longrightarrow x$.

Thus , $\forall T \in F$, $f(T) = \lim_n (A\hat{x}_n)(T) = \lim_n J_Y^{Y''} Tx_n = J_Y^{Y''} Tx = (A\hat{x})(T)$ so $f = A\hat{x}$.

By the closed graph theorem , A is continuous.

$\forall T \in F , \forall x \in X$, $\|Tx\| = \|(A\hat{x})(T)\| \leq \|A\hat{x}\| \leq \|A\| \|\hat{x}\| = \|A\| \|x\|$.

Thus , $\sup_{T \in F} \|T\| \leq \|A\|$. □

Corollary III.1.7 : Let $K \subset X$ and suppose that $\forall x' \in X'$, $x'[K]$ is bounded. Then K is bounded.

Proof : $\hat{K} \subset BL[X',\mathbb{F}]$ and $\forall x' \in X'$, $\sup_{k \in K} |\hat{k}x'| < \infty$.

By Theorem III.1.6 , $\sup_{k \in K} \|k\| = \sup_{k \in K} \|\hat{k}\| < \infty$. □

2. The Adjoint of a Linear Operator

Adjoints are usually defined for densely defined operators. Here we extend the notion to arbitrary linear operators.

Definition : The *adjoint* T' of T is defined as follows :
$T' \in L(Y',D(T)')$, $D(T') = \{ y' \in Y' : y'T \in D(T)' \}$ and $\forall\, y' \in D(T')$, $T'y' = y'T$.

Note : The definition given above coincides with the definition of the conjugate of $TJ^{X}_{D(T)}$ given in [Gol2 ; II.2.2].

Theorem III.2.1 [C1] : $G(T')$ is closed in $(Y',\sigma(Y',Y)) \times (D(T)',\sigma(D(T)',D(T)))$.

Proof :
Let (y'_α) be a net in $D(T')$ s.t. $y'_\alpha \longrightarrow y'$ w.r.t. $\sigma(Y',Y)$ and $T'y'_\alpha \longrightarrow x'$ w.r.t. $\sigma(D(T)',D(T))$.
$\forall\, x \in D(T)$, $y'Tx = \lim_\alpha y'_\alpha Tx = \lim_\alpha (T'y'_\alpha)x = x'x$.
Thus , $y' \in D(T')$ and $x' = T'y'$. □

Corollary III.2.2 : $N(T')$ is $\sigma(Y',Y)$ closed.

Proof :
Let (y'_α) be a net in $N(T')$ with $y'_\alpha \longrightarrow y'$ w.r.t. $\sigma(Y',Y)$.
Then , $\forall\, \alpha$, $T'y'_\alpha = 0$.
Thus , $T'y'_\alpha \longrightarrow 0$.
By Theorem III.2.1 , $y' \in D(T')$ and $T'y' = 0$.
i.e. $y' \in N(T')$. □

Since $\sigma(Y',Y) \times \sigma(D(T)',D(T))$ is weaker than the norm topology on $Y' \times D(T)'$, we obtain :

Corollary III.2.3 : T' is closed.

Theorem III.2.4 : T' is $\sigma(D(T'),Y)$–$\sigma(D(T)',D(T))$ continuous.

Proof :

Suppose $y'_\alpha \to y'$ w.r.t. $\sigma(D(T'),Y)$.

$\forall x \in D(T)$, $T'y'x = y'Tx = \lim_\alpha y'_\alpha Tx = \lim_\alpha (T'y'_\alpha)x$.

Thus, $T'y'_\alpha \to T'y'$ w.r.t. $\sigma(D(T)',D(T))$. □

Theorem III.2.5 : $D(T') = Y' \Leftrightarrow T$ is continuous, in which case T' is bounded and $\|T'\| = \|T\|$. (cf [Gol2 ; II.2.8])

Proof :

($\Rightarrow$) Suppose $D(T') = Y'$.

Then $\forall y' \in Y'$, the set $\{ |y'Tx| : x \in B_{D(T)} \}$ is bounded.

By Corollary III.1.7, the set $\{ \|Tx\| : x \in B_{D(T)} \}$ is bounded.

Thus, T is continuous.

($\Leftarrow$) Clear.

Suppose T is continuous.

Then $\forall y' \in Y'$, $\forall x \in D(T)$, $|T'y'x| = |y'Tx| \le \|y'\| \, \|T\| \, \|x\|$.

Thus, T' is bounded and $\|T'\| \le \|T\|$.

$\forall x \in D(T)$, $\|Tx\| = \sup_{y' \in B_{Y'}} |y'Tx| = \sup_{y' \in B_{Y'}} |T'y'x| \le \sup_{y' \in B_{Y'}} \|T'y'\| \, \|x\| = \|T'\| \, \|x\|$.

Thus, $\|T\| \le \|T'\|$. □

Definition : $F \subset X'$ is *total* iff $\forall\, x \in X \backslash \{O\}$, $\exists\, x' \in F$ s.t. $x'x \neq 0$.

Note : Let F be a subspace of X. Then

(i) F is total $\Leftrightarrow$ $F_{\perp} = \{O\}$.

(ii) F total $\Rightarrow$ $F_{\perp}{}^{\perp} = X'$.

(iii) If X is normed, then F is total $\Leftrightarrow$ $F_{\perp}{}^{\perp} = X'$. (see Corollary I.4.9)

Definition : T is *closable* iff $\exists$ closed $S \in L(X,Y)$ s.t. $G(T) \subset G(S)$.

Theorem III.2.6 :

T.F.A.E.

(a) T is closable.

(b) $(\forall\, y \in Y)\,[y \neq O \Rightarrow (O,y) \notin \overline{G(T)}]$.

(c) $D(T')$ is total.

(d) T has a minimal closed linear extension. (cf [Gol2 ; II.2.11])

Proof :

(a) $\Rightarrow$ (b) Suppose T is closable.

Then $\exists$ closed $S \in L(X,Y)$ s.t. $G(T) \subset G(S)$.

Since $G(S)$ is closed, $\overline{G(T)} \subset G(S)$.

Let $y \in Y \backslash \{O\}$.

Since S is linear, $(O,y) \notin G(S)$.

Thus, $(O,y) \notin \overline{G(T)}$.

(b) ⇒ (c) Suppose $(\forall\, y \in Y)\,[y \neq O \Rightarrow (O,y) \notin \overline{G(T)}]$.

Let $w \in Y\backslash\{O\}$.

Then $(O,w) \notin \overline{G(T)}$.

By Corollary I.4.14 , $\exists\, z' \in (X \times Y)'$ s.t. $z'(O,w) \neq 0$ and $z'[G(T)] = \{0\}$.

The maps $x' : X \longrightarrow \mathbb{F} : x \longmapsto z'(x,O)$ and $y' : Y \longrightarrow \mathbb{F} : y \longmapsto z'(O,y)$ are in X' and Y' respectively.

$\forall\, x \in D(T)$, $0 = z'(x,Tx) = x'x + y'Tx$ so $y' \in D(T')$.

Also , $y'w = z'(O,w) \neq 0$.

Thus , $D(T')$ is total.

(c) ⇒ (b) Suppose $D(T')$ is total.

Let $y \in Y$ with $(O,y) \in \overline{G(T)}$.

Then $\exists\, (x_n) \in D(T)^{\mathbb{N}}$ s.t. $x_n \longrightarrow O$ and $Tx_n \longrightarrow y$.

$\forall\, y' \in D(T')$, $y'y = \lim\limits_n y'Tx_n = \lim\limits_n T'y'x_n = 0$.

Since $D(T')$ is total , $y = O$.

(b) ⇒ (d) Suppose $(\forall\, y \in Y)\,[y \neq O \Rightarrow (O,y) \notin \overline{G(T)}]$.

Define $\overline{T}$ as follows :

$\overline{T} \in L(X,Y)$, $D(\overline{T}) = \{\, x \in X : \exists\, z \in Y$ s.t. $(x,z) \in \overline{G(T)}\,\}$ and $\overline{T}x = z$ where $(x,z) \in \overline{G(T)}$.

It follows from the assumption that $\overline{T}$ is well–defined.

Since $\overline{G(T)}$ is a subspace of $X \times Y$, $\overline{T}$ is linear.

Also , $G(T) \subset \overline{G(T)} = G(\overline{T})$ so $\overline{T}$ is a closed linear extension of T.

If S is any other closed linear extension of T , then $G(\overline{T}) = \overline{G(T)} \subset G(S)$ so $\overline{T}$ is the minimal closed linear extension of T.

(d) ⇒ (a) Clear. □

Corollary III.2.7 [Gol2 ; II.2.12] : Let X and Y be Banach spaces and let $D(T) = X$. Then $T \in BL[X,Y] \Leftrightarrow D(T')$ is total.

Proof :

($\Rightarrow$) Suppose $T \in BL[X,Y]$.

Then $D(T') = Y'$ which is total since Y is normed.

($\Leftarrow$) Suppose $D(T')$ is total.

By Theorem III.2.6 , T is closable.

Since $D(T) = X$, this means that T must be closed.

By the closed graph theorem , T is continuous. □

Note :

(i) X normed , complete and Y complete were not needed for ($\Rightarrow$).

(ii) Y normed was not needed for ($\Leftarrow$).

Theorem III.2.8 : T' is continuous $\Leftrightarrow D(T')$ is closed.

Proof :

($\Rightarrow$) Suppose T' is continuous.

Let $(y_n') \in D(T')^{\mathbb{N}}$ with $y_n' \longrightarrow y'$.

$\forall x \in D(T)$, $|y'Tx| = \lim_n |y_n'Tx| = \lim_n |T'y_n'x| \leq \lim_n \|T'\| \|y_n'\| \|x\| =$

$\|T'\| \|y'\| \|x\|$ so $y' \in D(T')$.

($\Leftarrow$) Closed graph theorem. □

3. States of Linear Operators

The aim of this section as well as the next section is to obtain the Taylor–Halberg–Goldberg state diagrams (see [TH] , [Gol2 ; II.3.14 , II.4.11]) in our more general setting. For the most part , the proofs are simple modifications of the corresponding results in [Gol2 ; II.3, II.4]. Since the state diagrams obtained coincide with those in [Gol2] , the examples in [Gol2 ; II.5] show that they are complete.

The following states for T will be considered :

I : $R(T) = Y$

II : $R(T) \neq Y$, $\overline{R(T)} = Y$

III : $\overline{R(T)} \neq Y$

$1'$: $\exists\, m > 0$ s.t. $\forall\, x \in D(T)$, $m\,\|x\| \leq \|Tx\|$ (T is *bounded below*)

$2'$: $N(T)^{\perp} = D(T)'$, $T \notin 1'$

$3'$: $N(T)^{\perp} \neq D(T)'$

If T is in state $1'$ then this will be written as $T \in 1'$.
If T is in state II and in state $3'$, then this will be written as $T \in II_{3'}$.
Similar notation is used for the other possible states of T.
Note that if $T \in 1'$, then $N(T)^{\perp} = D(T)'$.

In [Gol2 ; II.3] , the following states are considered instead of $1'$, $2'$ and $3'$:

1 : T injective , T^{-1} continuous

2 : T injective , T^{-1} not continuous

3 : T not injective

Note that if X is normed , then these are exactly the same as $1'$, $2'$ and $3'$.

(When X is normed , T is injective $\Leftrightarrow$ $N(T)^{\perp} = D(T)'$)

Theorem III.3.1 : $T' \in 1' \Rightarrow R(T')$ closed.

Proof :

Suppose $T' \in 1'$.

Then $\exists\, c > 0$ s.t. $\forall\, y' \in D(T')$, $c\,\|y'\| \leq \|T'y'\|$.

Let $x' \in \overline{R(T')}$.

Then $\exists\, (y'_n) \in D(T')^{\mathbb{N}}$ s.t. $T'y'_n \longrightarrow x'$.

$\forall\, m, n \in \mathbb{N}$, $c\,\|y'_n - y'_m\| \leq \|T'y'_n - T'y'_m\|$.

Thus, (y'_n) is a Cauchy sequence in Y'.

Since Y' is complete, $\exists\, y' \in Y'$ s.t. $y'_n \longrightarrow y'$.

Since T' is closed, $y' \in D(T')$ and $x' = T'y' \in R(T')$. □

Corollary III.3.2 : $T' \notin II_{1'}$.

Theorem III.3.3 :

(i) $R(T)^{\perp} = N(T')$.

(ii) $\overline{R(T)} = N(T')_{\perp}$.

In particular, T has dense range $\Leftrightarrow$ T' is injective.

(i.e. $T \in I \cup II \Leftrightarrow T' \in 1' \cup 2'$)

Proof :

(i) $y' \in R(T)^{\perp} \Leftrightarrow \forall\, x \in D(T),\ y'Tx = 0 \Leftrightarrow y' \in N(T')$.

(ii) $\overline{R(T)} = {R(T)^{\perp}}_{\perp} = N(T')_{\perp}$.

Finally, note that $N(T') = \{O\} \Leftrightarrow N(T')_{\perp} = Y$. □

Theorem III.3.4 : (Here the pairing considered is $(D(T),D(T)')$.)

(i) $N(T) \subset R(T')_{\perp}$.

(ii) $D(T')$ total $\Rightarrow N(T) = R(T')_{\perp}$.

(iii) $\overline{R(T')}^{\sigma(D(T)',D(T))} \subset N(T)^{\perp}$ (In particular, $T' \in I \cup II \Rightarrow T \in 1' \cup 2'$).

(iv) X normed, $R(T')$ total $\Rightarrow T$ injective.

Proof :

(i) $x \in N(T) \Rightarrow \forall y' \in D(T'), (T'y')x = y'(Tx) = y'O = 0 \Rightarrow x \in R(T')_{\perp}$.

(ii) Suppose $D(T')$ is total.

Then, $x \notin N(T) \Rightarrow Tx \neq O \Rightarrow \exists y' \in D(T')$ s.t. $y'Tx \neq 0 \Rightarrow x \notin R(T')_{\perp}$.

(iii) $\overline{R(T')}^{\sigma(D(T)',D(T))} = R(T')_{\perp}{}^{\perp} \subset N(T)^{\perp}$.

(iv) Immediate from (iii). □

Theorem III.3.5 : Let T and T' be injective and $T_1 = TJ^{X}_{\overline{D(T)}}$. Then $R(T)' \equiv Y'$ and $(T_1^{-1})' = I^{R(T)'}_{Y'} T'^{-1} I^{D(T)'}_{\overline{D(T)}'}$.

Proof :

By Theorem III.3.3, $\overline{R(T)} = Y$ so $R(T)' \equiv Y'$.

Let $z' \in D((T_1^{-1})')$.

Then $z'T^{-1} \in R(T)'$.

Let $y' \in Y'$ be an extension to Y of $z'T^{-1}$.

Then $\forall x \in D(T)$, $y'Tx = z'T^{-1}Tx = z'x$.

Thus, $y' \in D(T')$ and $T'y' = z'_{D(T)}$.

i.e. $I^{D(T)'}_{\overline{D(T)}'} z' \in R(T') = D(T'^{-1})$.

Let $x' \in D(T'^{-1}) = R(T')$ and let $\bar{x}' \in \overline{D(T)}'$ be an extension of x' to $\overline{D(T)}$.

Then , $\exists\, y' \in D(T')$ s.t. $x' = T'y'$.

$\forall\, x \in D(T)$, $\bar{x}'T^{-1}Tx = x'x = T'y'x = y'Tx$.

Thus , $I_{D(T)'}^{\overline{D(T)}'} x' = \bar{x}' \in D((T_1^{-1})')$.

$\forall\, x \in D(T)$, $\forall\, z' \in D((T_1^{-1})')$,

$(I_{Y'}^{R(T)'}\, T'^{-1} I_{\overline{D(T)}'}^{D(T)'} z')Tx = (T'^{-1}z'_{D(T)})Tx = (T'T'^{-1}z'_{D(T)})x = z'x = z'T^{-1}Tx$

$= ((T_1^{-1})'z')Tx.$ □

We now define an injective operator associated with T and which has a number of properties in common with T. This will be useful in what follows as a number of results are proved for injective operators and then by using this operator the correspondimg results are deduced for T.

Definition : The *induced injective operator* $\hat{T}$ is defined as follows :

$\hat{T} \in L(X/N(T),Y)$, $D(\hat{T}) = D(T)/N(T)$ and $\forall\, x \in D(T)$, $\hat{T}[x] = Tx$.

Note :

(i) $\hat{T}$ is well defined since if $[x] = [z]$ then $x - z \in N(T)$ so that $Tx = Tz$.

(ii) This generalises the definition in [Gol2 ; II.4.6] – here $N(T)$ is no longer required to be closed.

(iii) $R(\hat{T}) = R(T)$.

(iv) $T = \hat{T}Q_{N(T)}^{X}$.

(v) The induced injective operator associated with T' will be denoted by $(T')\hat{}$.

Theorem III.3.6 :

(i) T is continuous $\Leftrightarrow$ $\hat{T}$ is continuous in which case $\|T\| = \|\hat{T}\|$.

(ii) T is closed $\Leftrightarrow$ $\hat{T}$ is closed.

(iii) $T \in 1' \Rightarrow \hat{T} \in 1'$.

(iv) $T' = J^{D(T)'}_{N(T)^{\perp}} \, I^{N(T)^{\perp}}_{(D(T)/N(T))'} \, \hat{T}'$.

Proof :

(i) ($\Rightarrow$) Suppose T is continuous.

Let $x \in D(T)$.

$\forall\, y \in [x]$, $\|\hat{T}[x]\| = \|\hat{T}[y]\| = \|Ty\| \leq \|T\| \, \|y\|$.

Thus , $\|\hat{T}[x]\| \leq \|T\| \, \|[x]\|$.

Thus , $\hat{T}$ is continuous and $\|\hat{T}\| \leq \|T\|$.

($\Leftarrow$) Follows from the fact that $Q^X_{N(T)}$ is bounded.

Also , $\|T\| = \|\hat{T}Q^X_{N(T)}\| \leq \|\hat{T}\| \, \|Q^X_{N(T)}\| \leq \|\hat{T}\|$.

(ii) ($\Rightarrow$) Suppose T is closed.

Let (x_n) be a sequence in $D(T)$ s.t. $[x_n] \longrightarrow [x]$ and $Tx_n \longrightarrow y$.

$\exists\, (v_n) \in N(T)^{\mathbb{N}}$ s.t. $x_n + v_n \longrightarrow x$.

Since T is closed and $T(x_n + v_n) = Tx_n \longrightarrow y$, $x \in D(T)$ and

$y = Tx = \hat{T}[x]$.

($\Leftarrow$) Suppose $\hat{T}$ is closed.

Let (x_n) be a sequence in $D(T)$ s.t. $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$.

Then $[x_n] \longrightarrow [x]$ and $\hat{T}[x_n] \longrightarrow y$.

Since $\hat{T}$ is closed , $[x] \in D(\hat{T})$ and $y = \hat{T}[x]$.

Thus , $x \in D(T) + N(T) \subset D(T)$ and $y = Tx$.

(iii) Suppose $T \in 1'$.

Then $\exists\, m > 0$ s.t. $\forall\, x \in D(T)$, $m \, \|x\| \leq \|Tx\|$.

$\forall\, x \in D(T)$, $m \, \|[x]\| \leq m \, \|x\| \leq \|Tx\| = \|\hat{T}[x]\|$.

Thus , $\hat{T} \in 1'$.

(iv) Let $y' \in D(T')$ and $x \in D(T)$.

$\forall z \in [x]$, $|y'\hat{T}[x]| = |y'\hat{T}[z]| = |y'Tz| \leq \|y'T\| \, \|z\|$.

Thus, $y' \in D(\hat{T}')$.

Let $y' \in D(\hat{T}')$.

Then $\forall x \in D(T)$, $|y'Tx| = |y'\hat{T}[x]| \leq \|y'\hat{T}\| \, \|[x]\| \leq \|y\hat{T}\| \, \|x\|$.

Thus, $y' \in D(T')$.

If $y' \in D(T')$ and $x \in D(T)$, then $(J^{D(T)'}_{N(T)^{\perp}} I^{N(T)^{\perp}}_{(D(T)/N(T))'} \hat{T}'y')x$

$= (\hat{T}'y')[x] = y'\hat{T}[x] = y'Tx = (T'y')x$. □

Lemma III.3.7 : If $T \notin 1'$, then there is a sequence (x_n) in $D(T)$ s.t. $\|x_n\| \longrightarrow \infty$ and $\|Tx_n\| \longrightarrow 0$.

Proof : Suppose $T \notin 1'$.

Then $\forall m > 0$, $\exists x \in D(T)$ s.t. $m\|x\| > \|Tx\|$.

Thus, $\forall n \in \mathbb{N}$, $\exists z_n \in D(T)$ s.t. $\|z_n\| = 1$ and $\|Tz_n\| < \frac{1}{n}$.

$\forall n \in \mathbb{N}$, let $x_n = \|Tz_n\|^{-\frac{1}{2}} z_n$ if $\|Tz_n\| \neq 0$ and $n\,z_n$ otherwise.

Then (x_n) has the required properties. □

Theorem III.3.8 : $T' \in I \Leftrightarrow T \in 1'$.

Proof :

($\Rightarrow$) Suppose $T' \in I$ but $T \notin 1'$.

By Lemma III.3.7, $\exists (x_n) \in D(T)^{\mathbb{N}}$ s.t. $\|x_n\| \longrightarrow \infty$ and $Tx_n \longrightarrow O$.

$\forall y' \in D(T')$, $T'y'x_n = y'Tx_n \longrightarrow 0$.

Since $T' \in I$, $\forall x' \in D(T)'$, $x'x_n \longrightarrow 0$

Thus, $\forall x' \in D(T)'$, the set $\{ x'x_n : n \in \mathbb{N} \}$ is bounded.

By Corollary III.1.7, the set $\{ x_n : n \in \mathbb{N} \}$ is bounded which contradicts the fact that $\|x_n\| \longrightarrow \infty$.

(⇐) Suppose $T \in 1'$.

By Theorem III.3.6 , $\hat{T} \in 1'$.

Let $z' \in (D(T)/N(T))'$.

Then $z'\hat{T}^{-1} \in R(T)'$.

Let $y' \in Y'$ be an extension of $z'\hat{T}^{-1}$ to Y.

$\forall\, x \in D(T)$, $y'\hat{T}[x] = z'\hat{T}^{-1}\hat{T}[x] = z'[x]$.

Thus , $y' \in D(\hat{T}')$ and $\hat{T}'y' = z'$.

Thus , $R(\hat{T}') = (D(T)/N(T))'$.

By Theorem III.3.6 , $R(T') = N(T)^{\perp}$.

Since $T \in 1'$, $N(T)^{\perp} = D(T)'$.

Thus , $T' \in I$. □

Theorem III.3.9 : $T \in (I \cup II)_{1'} \Leftrightarrow T' \in I_{1'}$.

Proof :

(⇒) Suppose $T \in (I \cup II)_{1'}$.

By Theorem III.3.3 , $T' \in 1' \cup 2'$.

By Theorem III.3.8 , $T' \in I$.

Since $D(T)'$ is normed and complete , it follows from the closed graph theorem that $T' \in 1'$.

(⇐) Suppose $T' \in I_{1'}$.

By Theorem III.3.3 , $T \in I \cup II$ and by Theorem III.3.8 , $T \in 1'$. □

Theorem III.3.10 : Let Y be complete. Then $T \in I \Rightarrow T' \in 1'$.

Proof :

Suppose $T \in I$ but $T' \notin 1'$.

By Lemma III.3.7 , $\exists\, (y_n') \in D(T')^{\mathbb{N}}$ s.t. $\|y_n'\| \longrightarrow \infty$ and $T'y_n' \longrightarrow 0$.

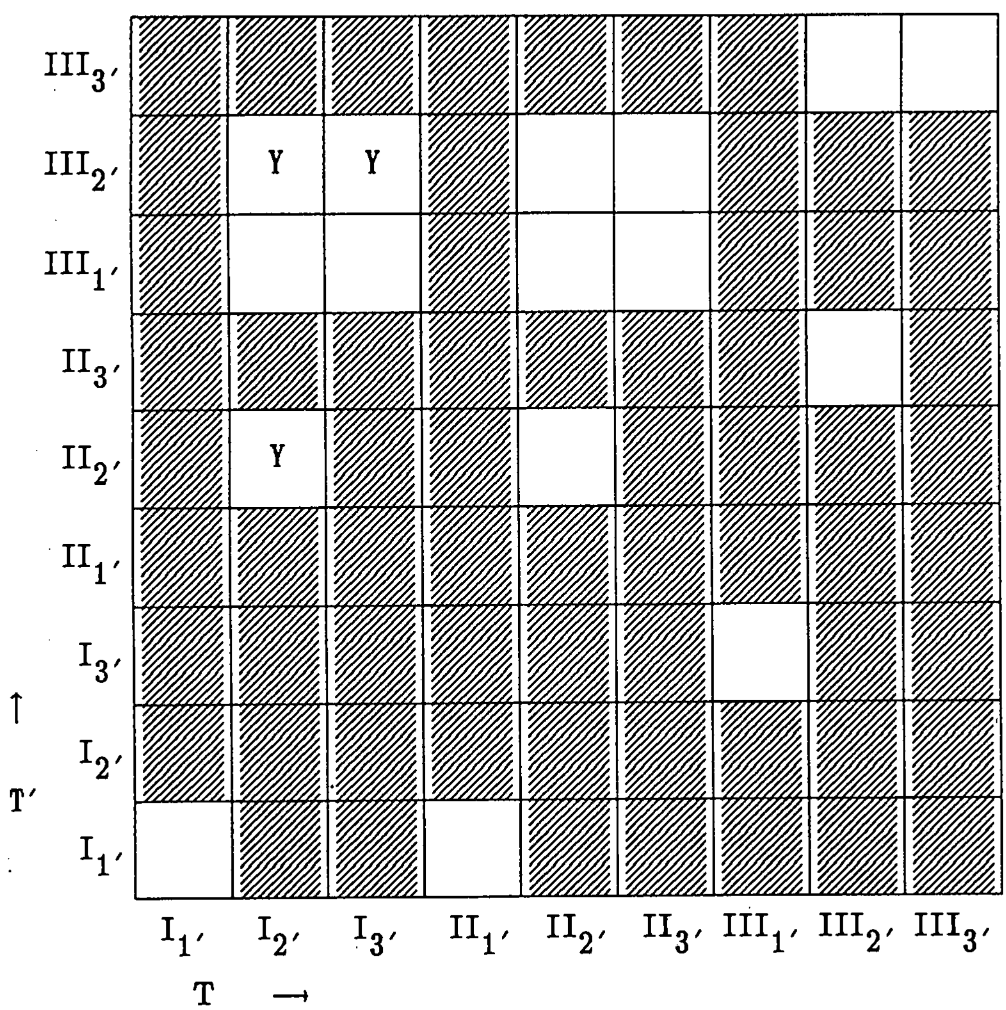
Since $T \in I$, $\forall\, y \in Y$, $y_n'y \longrightarrow 0$.

Thus , $\forall\, y \in Y$, $\sup_n |y_n'y| < \infty$.

By Theorem III.1.6 , $\sup_n \|y_n'\| < \infty$, a contradiction. □

Note : If $T \in I_{1'}$ and $T' \in II_{3'}$, then this is written as $(T,T') \in (I_{1'},II_{3'})$ and similar notation is used for the other possible states of (T,T'). The preceding theorems show that certain states for (T,T') cannot occur. These results are summarised in the *state diagram* on the next page where eliminated states are indicated by shaded squares. Additional states are eliminated if Y is assumed to be complete. These are indicated by placing a Y in the appropriate squares. That the blank spaces can occur is shown by means of examples in [Gol2 ; II.5]. Note that the diagram obtained has the same form as that obtained in [Gol1 ; II.3.14].

State diagram for linear operators



Y : Cannot occur if Y is complete

4. States of Closed Linear Operators

In this section , we obtain the analogues of the results in Section II.4 of [Gol2] necessary to establish our state diagram for closed linear operators.

Lemma III.4.1 : $T' \in 1' \Rightarrow r\,U_Y \subset \overline{TB}_{D(T)}$ where $r = \dfrac{1}{\|T'^{-1}\|}$.

Proof : Suppose $\exists\, y \in r\,U_Y \setminus \overline{TB}_{D(T)}$.

Then $\exists\, y' \in Y'$ s.t. $\forall\, x \in B_{D(T)}$, $|y'Tx| < |y'y|$ so that $y' \in D(T')$.

Now , $r\,\|y'\| \le \|T'y'\| = \sup\limits_{x \in B_{D(T)}} |y'Tx| \le |y'y| \le \|y'\|\,\|y\|$.

Thus , $\|y\| \ge r$ which contradicts $y \in r\,U_Y$. □

Lemma III.4.2 : Suppose $\exists\, r > 0$ s.t. $r\,U_Y \subset TB_{D(T)}$ and $N(T)^{\perp} = D(T)'$. Then $T \in 1'$ and $\forall\, x \in D(T)$, $r\,\|x\| \le \|Tx\|$.

Proof :

Let $x \in D(T)$.

case(i) : $\|Tx\| = 0$.

$\forall\, n \in \mathbb{N}$, $T(nx) \in r\,U_Y$.

Thus , $\forall\, n \in \mathbb{N}$, $\exists\, z_n \in B_{D(T)}$ s.t. $T(nx) = Tz_n$.

Since $N(T)^{\perp} = D(T)'$, $\forall\, n \in \mathbb{N}$, $\forall\, x' \in D(T)'$, $x'(nx) = x'z_n$.

By Proposition II.5.1 , $\forall\, n \in \mathbb{N}$, $n\,\|x\| = \|nx\| = \|z_n\| \le 1$.

Thus , $\|x\| = 0$.

case(ii) : $\|Tx\| \neq 0$.

Let $\epsilon \in (0,1)$.

Then $\dfrac{(1-\epsilon)\, r\, Tx}{\|Tx\|} \in r\, U_Y$.

Thus , $\exists\, z \in B_{D(T)}$ s.t. $\dfrac{(1-\epsilon)\, r\, Tx}{\|Tx\|} = Tz$.

Since $N(T)^{\perp} = D(T)'$, $\forall\, x' \in D(T)'$, $x'(\dfrac{(1-\epsilon)\, r\, x}{\|Tx\|}) = x'z$.

Thus , $(1-\epsilon)\, r\, \|x\| = \|Tx\|\, \|z\| \leq \|Tx\|$.

Since ϵ was arbitrary , $r\, \|x\| \leq \|Tx\|$.

Thus , $T \in 1'$ and $\forall\, x \in D(T)$, $r\, \|x\| \leq \|Tx\|$. □

Theorem III.4.3 : Let X be complete , T closed and $T' \in 1'$. Then

(a) $r\, U_Y \subset TB_X$ where $r = \dfrac{1}{\|T'^{-1}\|}$.

(b) $N(T)^{\perp} = D(T)' \Rightarrow T \in 1'$ and $\forall\, x \in D(T)$, $r\, \|x\| \leq \|Tx\|$.

Proof : Lemmas III.1.2 , III.4.1 and III.4.2 □.

Corollary III.4.4 : Under the same hypotheses as Theorem III.4.3 , $T \in I$ and T is open.

Theorem III.4.5 : Let Y be complete and T closed. If T′ continuous , then D(T) is closed. If , in addition , X is a Banach space , then

T.F.A.E.
(a) T′ is continuous.
(b) T is continuous and D(T) is closed.
(c) D(T′) is closed.

Proof :

First assume that T is injective.

Let $T_1 = (J^{Y}_{\overline{R(T)}})^{-1} T J^{X}_{\overline{D(T)}}$ and $S = T_1^{-1}$.

Then S is closed and $S \in L(\overline{R(T)}, \overline{D(T)}) = L(\overline{D(S)}, \overline{R(S)})$.

By Theorem III.3.3 , S' is injective.

By Theorem III.3.5 , $S'^{-1} = I^{\overline{D(T)}'}_{D(T)'} (S^{-1})' I^{\overline{D(S)}'}_{D(S)'} = I^{\overline{D(T)}'}_{D(T)'} T_1' I^{\overline{D(S)}'}_{D(S)'}$.

Now , T_1' is also continuous so this means that $S' \in 1'$.

By Corollary III.4.4 , $S \in I$.

Thus , $\overline{D(T)} = R(S) = D(T)$.

If T is not injective , then $\hat{T}$ is closed and by Theorem III.3.6 , $\hat{T}'$ is continuous.

By the above , $\overline{D(\hat{T})} = D(\hat{T})$

Thus , $D(T) = \overline{D(T)}$.

The rest of the theorem follows from the closed graph theorem. □

Corollary III.4.6 : T'' continuous $\Leftrightarrow$ T' continuous.

Lemma III.4.7 : If T is closed and injective , then X and Y are normed.

Proof : If T is closed and injective , then $\{O\} = N(T)$ is closed in X and $\{O\} = N(T^{-1})$ is closed in Y.

Note : If X is a reflexive normed space and F is a subspace of X , then F is total $\Leftrightarrow$ $\overline{F} = X'$ (see p 57). Thus , if Y is a reflexive normed space , then T is closable $\Leftrightarrow$ $D(T')$ is dense in Y'.

Theorem III.4.8 : Let X be semireflexive , T be closed and $N(T)^{\perp} = D(T)'$.
Then $R(T')$ is dense in $D(T)'$.

Proof :

First assume that T is injective.

By Lemma III.4.7 , X and Y are normed.

Let $T_1 = (J^{Y}_{R(T)})^{-1} T J^{X}_{\overline{D(T)}}$.

Then $R(T') = R(T_1')$ and T_1' is injective.

By Theorem III.3.5 , $(T_1^{-1})' = T_1'^{-1} I^{D(T)'}_{(\overline{D(T)})'}$.

Now $\overline{D(T)}$ is reflexive and T_1^{-1} is closed so $D((T_1^{-1})')$ is dense $(\overline{D(T)})'$.

Thus , $R(T') = R(T_1') = D(T_1'^{-1})$ is dense in $D(T)'$.

If T is not injective , then $\hat{T}$ is closed and injective.

Also , $X/N(T)$ is reflexive.

[Let $([x_n])$ be a bounded sequence in $X/N(T)$.

$\forall\, n \in \mathbb{N}\, , \exists\, z_n \in [x_n]$ s.t. $\|z_n\| < \|[x_n]\| + 1$.

Since X is semi–reflexive , (z_n) has a convergent subsequence (z_{n_r}).

Clearly , $([x_{n_r}])$ converges in $X/N(T)$.]

By the preceding argument , $R(\hat{T}')$ is dense in $(D(T)/N(T))'$.

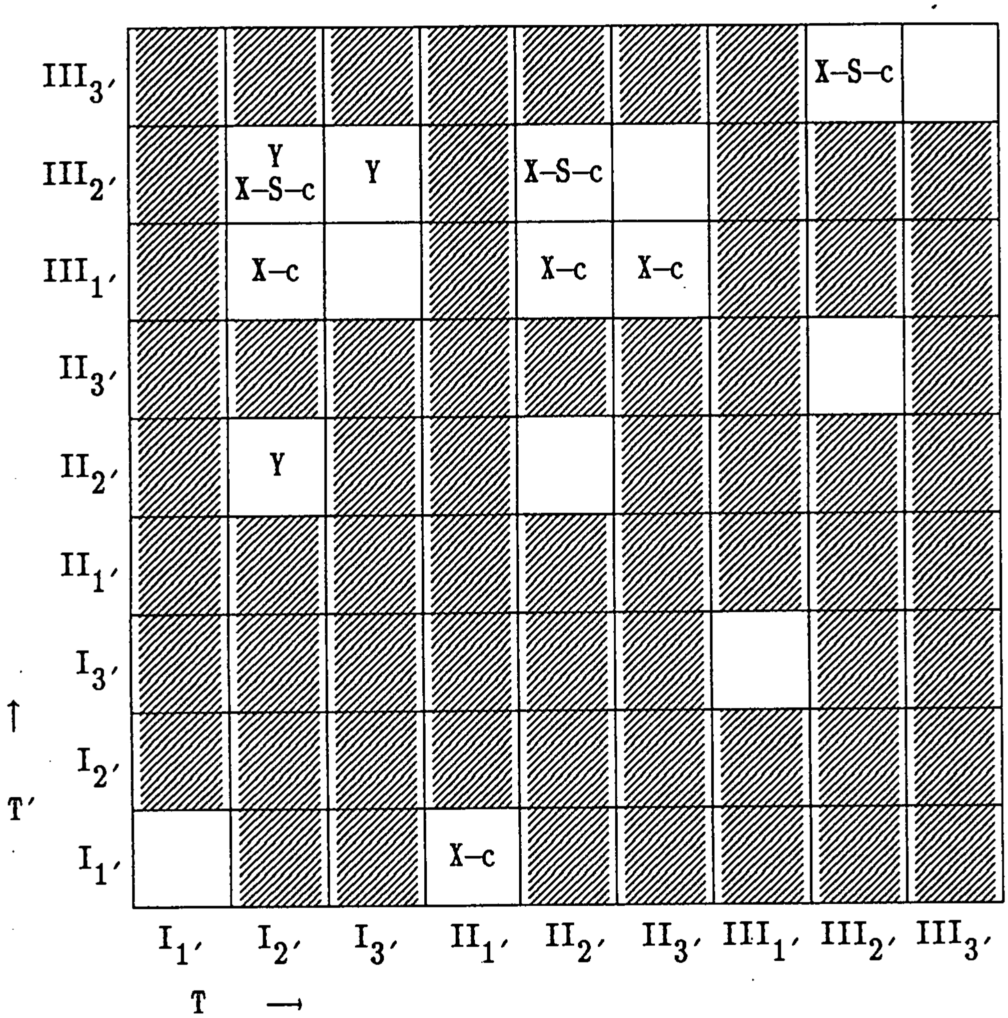
By Theorem III.3.6 , $R(T')$ is dense in $N(T)^{\perp} = D(T)'$. □

Note :

(i) The state diagram for closed linear operators appears on the following page. Note that it has the same form as that obtained in [Gol1].

(ii) For examples of states , see [Gol2 ; II.5].

State diagram for closed linear operators



Y : Cannot occur if Y is complete

X–c : Cannot occur if X is complete and T is closed

X–S–c : Cannot occur if X is semireflexive and T is closed

Chapter IV Strictly Singular Operators

In this chapter , only a brief study of strictly singular operators is made in order to make certain results available for the next chapter. For a more indepth study of continuous strictly singular operators in normed spaces and their relationship with precompact operators , see for example [Gol2 ; III]. Unbounded strictly singular operators in normed spaces are dealt with in [C1] , [C3] , [C4] , [C6] , [C7] , [CL1] and [CL2]. Here we consider strictly singular operators in seminormed spaces.

Throughout this chapter , X and Y are seminormed spaces and $T \in L(X,Y)$.

Definition : A subset A of X is *totally bounded* iff $\forall \epsilon > 0$, $\exists$ finite $F \subset A$ s.t. $A \subset F + \epsilon B_X$.

Definition : T is *precompact* iff TB_X is totally bounded in Y.

T is *compact* iff $\overline{TB_X}$ is compact in Y.

Note : T precompact $\Rightarrow$ T continuous.

We state the following result from [RR ; p60] without proof.

Theorem IV.1 : $A \subset X$ is compact $\Leftrightarrow$ A is totally bounded and A is complete.

Proposition IV.2 : $T \in BL(X,Y)$, $\dim R(T) < \infty \Rightarrow T$ precompact.

Proof :

Suppose $T \in BL(X,Y)$ and $\dim R(T) < \infty$.

By Corollary II.3.3 , $\overline{TB_X} \cap R(T)$ is compact in $R(T)$.

By Theorem IV.1.1 , $\overline{TB_X} \cap R(T)$ is totally bounded.

Since $TB_X \subset \overline{TB_X} \cap R(T)$, TB_X is totally bounded. □

Lemma IV.3 : Let (K_n) be a sequence of precompact operators with $K_n \longrightarrow K$ in $BL[X,Y]$. Then K is precompact. (cf [Gol2 ; III.1.5])

Proof :

Let $\epsilon > 0$.

Then $\exists\, N \in \mathbb{N}$ s.t. $\|K_N - K\| < \frac{\epsilon}{3}$.

Since K_N is precompact , $\exists$ finite $F \subset B_X$ s.t. $K_N B_X \subset K_N F + \frac{\epsilon}{3} B_Y$.

Let $x \in B_X$.

Then $\exists\, z \in F$ s.t. $\|K_N x - K_N z\| \leq \frac{\epsilon}{3}$.

$\|Kx - Kz\|$

$\leq \|Kx - K_N x\| + \|K_N x - K_N z\| + \|K_N z - Kz\|$

$\leq 2\,\|K - K_N\| + \|K_N x - K_N z\|$

$\leq 2\frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$ □

Notation : $\mathcal{J}(X) = \{\, M : M$ a subspace of X , $\dim M = \infty \,\}$

Theorem IV.4 : Suppose that for every closed finite codimensional subspace M of X, $T|_M \notin 1'$. Then $\forall \epsilon > 0, \exists M \in \mathcal{J}(D(T))$ s.t.

(a) $M \cap N_X = \{O\}$.

(b) $T|_M$ is precompact.

(c) $\|T|_M\| \leq \epsilon$. (cf [Gol2 ; III.1.9] , [Ka2])

Proof :

Let $\epsilon > 0$.

Since $T \notin 1'$, $\exists x_1 \in X$ s.t. $\|x_1\| = 1$ and $\|Tx_1\| < 3^{-1}\epsilon$.

$\exists x_1' \in X'$ s.t. $x_1'x_1 = \|x_1'\| = \|x_1\| = 1$.

Since codim $N(x_1') = 1$, $\exists x_2 \in N(x_1')$ s.t. $\|x_2\| = 1$ and $\|Tx_2\| < 3^{-2}\epsilon$.

$\exists x_2' \in X'$ s.t. $x_2'x_2 = \|x_2'\| = \|x_2\| = 1$.

Since codim $(N(x_1') \cap N(x_2')) < \infty$, $\exists x_3 \in N(x_1') \cap N(x_2')$ s.t. $\|x_3\| = 1$ and $\|Tx_3\| < 3^{-3}\epsilon$.

Continuing in this way, sequences (x_n) and (x_n') are obtained in X and X' respectively s.t. :

$\forall n \in \mathbb{N}$, $x_n'x_n = \|x_n'\| = \|x_n\| = 1$ and $x_n \in \bigcap_{i=1}^{n-1} N(x_i')$.

Suppose that $\|\sum_{i=1}^{n} \alpha_i x_i\| = 0$.

Then $0 = x_1'(\sum_{i=1}^{n} \alpha_i x_i) = \alpha_1$.

$0 = x_2'(\sum_{i=2}^{n} \alpha_i x_i) = \alpha_2$.

.

. .

$0 = x_n'(\alpha_n x_n) = \alpha_n$ so $\sum_{i=1}^{n} \alpha_i x_i = O$.

Thus, the set $\{x_1, x_2, \ldots\}$ is linearly independant.

Put $M = \text{span}\{x_1, x_2, \ldots\}$. Then $M \in \mathcal{J}(D(T))$.

It follows from the above argument that $M \cap N_X = \{O\}$.

We now verify that $\|T|_M\| \leq \epsilon$.

Let $x = \sum_{i=1}^{m} \alpha_i x_i \in M$.

It follows by induction that for $1 \leq k \leq m$, $|\alpha_k| \leq 2^{k-1} \|x\|$.

Thus, $\|Tx\| \leq \sum_{i=1}^{m} |\alpha_i| \, \|Tx_i\| \leq \sum_{i=1}^{m} 2^{i-1} 3^{-i} \epsilon \|x\| \leq \epsilon \|x\|$.

We now show that $T|_M$ is precompact.

$\forall n \in \mathbb{N}$, define $T_n^M : M \longrightarrow Y$ to be T on span $\{x_1,\dots,x_n\}$ and O on span $\{x_{n+1},\dots\}$.

Then each T_n^M is linear, has finite dimensional range and is bounded, hence, by Proposition IV.2, is precompact.

Also, $T_n^M \longrightarrow TJ_M^X$ in $BL[M,Y]$ so by Lemma IV.3, $T|_M$ is precompact. □

Definition : T is *strictly singular* iff for every subspace M of $D(T)$ we have $\dim M/(M \cap N_X) < \infty$ whenever $T|_M \in 1'$.

Note : This definition coincides with the classical one (see [Ka2]) when T is bounded and X and Y are Banach spaces.

The following result is exercise 8 on page 193 of [Wil1].

Lemma IV.5 : If M is a finite dimensional subspace of X, then $\overline{M} = M + N_X$.

Proof :

By Theorem II.3.8, $M + N_X$ is closed so $\overline{M} \subset M + N_X$.

Let $x \in M + N_X$. Then $\exists\, m \in M$, $\exists\, n \in N_X$ s.t. $x = m + n$.

Now, $\|x - m\| = \|n\| = 0$ so $d(x,M) = 0$.

Thus, $x \in \overline{M}$. □

Lemma IV.6 : B_X totally bounded $\Rightarrow$ $\dim X/N_X < \infty$.

Proof :

Suppose B_X is totally bounded.

Then $\exists$ finite $F \subset B_X$ s.t. $B_X \subset F + \frac{1}{2} B_X$.

Let $M = \text{span } F$.

$\forall k \in \mathbb{N}$, $B_X \subset M + \frac{1}{2} B_X \subset M + \frac{1}{2}(M + \frac{1}{2} B_X) = M + \frac{1}{4} B_X \subset \ldots \subset M + 2^{-k} B_X$.

Thus, $B_X \subset \overline{M}$ so $X = \overline{M} = M + N_X$.

Since $\dim M < \infty$, $\dim X/N_X < \infty$. □

Theorem IV.7 : T precompact $\Rightarrow$ T strictly singular. (cf [Gol ; III.1.3])

Proof :

Suppose that T is precompact.

Let M be a subspace of D(T) s.t. $T|_M \in 1'$.

Since T is precompact, TB_M is totally bounded in Y.

Since $T|_M \in 1'$, B_M is totally bounded in M.

By Lemma IV.4, $\dim M/M \cap N_X = \dim M/N_M < \infty$. □

Lemma IV.8 : Let M and E be subspaces of X with codim $E < \infty$. Then there is a finite dimensional subspace F of X s.t. $M = (M \cap E) \oplus F$.

Proof : The map $M/M \cap E \longrightarrow X/E : m + M \cap E \longmapsto m + E$ $(m \in M)$ is injective. □

Definition : T is *partially continuous* iff there is a finite codimensional subspace M of X s.t. $T|_M$ is continuous..

(See [C3], [CL2] and [La1])

Notation : $\| \|_T$ will denote the norm on $D(T)$ defined by $\|x\|_T = \|x\| + \|Tx\|$. X_T will denote $(D(T), \| \|_T)$. The operator $G \in L(X_T, X)$ is defined by $Gx = x$ $(x \in X_T)$.

Lemma IV.9 : T is partially continuous $\Leftrightarrow$ $\forall E \in \mathcal{I}(X)$, $\exists F \in \mathcal{I}(E)$ s.t. $T|_F$ is continuous. (cf [C4])

Proof :

($\Rightarrow$) Suppose T is partially continuous.

Then there is a finite codimensional subspace E of X s.t. $T|_E$ is continuous.

Let $M \in \mathcal{I}(X)$.

By Lemma IV.8, $\exists$ a finite dimensional subspace F of X s.t. $M = (M \cap E) \oplus F$.

Thus, $M \cap E \in \mathcal{I}(M)$ and $T|_{M \cap E}$ is continuous.

($\Leftarrow$) Suppose $\forall E \in \mathcal{I}(X)$, $\exists F \in \mathcal{I}(E)$ s.t. $T|_F$ is continuous.

Let $E \in \mathcal{I}(D(T))$.

Then $\exists F \in \mathcal{I}(E)$ s.t. $T|_F$ is continuous.

$\forall x \in F$, $\|G^{-1}x\| = \|x\| + \|Tx\| \leq (1 + \|T|_F\|) \|x\|$ so $G^{-1}|_F$ is an isomorphism.

Thus, G has no precompact restriction on any infinite dimensional subspace M of its domain satisfying $M \cap N_{X_T} = \{O\}$.

By Theorem IV.4, there is a closed finite codimensional subspace M of X_T s.t. $G|_M \in 1'$.

Thus, $T|_{GM}$ is continuous.

Let N be such that $X = D(T) \oplus N$.

Then $\operatorname{codim}(GM \oplus N) < \infty$ and $T|_{GM \oplus N}$ is continuous. □

We now come to the main result of this chapter in which unbounded strictly singular operators are characterised.

Theorem IV.10 [C6] : Let X be a normed space and Y a Banach space. Then T.F.A.E.

(a) T is an unbounded strictly singular operator.

(b) There is a continuous strictly singular operator A and an unbounded finite rank operator F s.t. $T = A + F$.

Proof :

(a) ⇒ (b) Suppose T is an unbounded strictly singular operator.

Then $\dim D(T) = \infty$ and by Theorem IV.4 , $\forall\, F \in \mathcal{J}(D(T))$, $\exists\, M \in \mathcal{J}(F)$ s.t. $T|_M$ is precompact.

By Lemma IV.7 , T is partially continuous so $\exists$ a finite codimensional subspace E of $D(T)$ s.t. $T|_E$ is continuous.

Since Y is complete , $T|_E$ extends to a continuous operator T_1 on $\overline{E}$.

Now , $\operatorname{codim} \overline{E} < \infty$ so there is a finite dimensional subspace N of $D(T)$ s.t. $D(T) = \overline{E} \oplus N$.

Let $B = \{x_1,\ldots,x_n\}$ be a basis for N.

For $1 \leq i \leq n$, let $N_i = \operatorname{span}(B \backslash \{x_i\})$.

By Theorem II.3.8 , $\overline{E} + N_i$ is closed.

For $1 \leq i \leq n$, $\exists\, x_i' \in D(T)'$ s.t. $x_i' x_i = 1$ and $\forall\, x \in \overline{E} + N_i$, $x_i' x = 0$.

Define $Q : D(T) \longrightarrow D(T)$ by $Qx = \sum_{i=1}^{n} x_i'(x)\, x_i$.

Then Q is bounded , $R(Q) = N$ and $N(Q) = \overline{E}$.

Let $A = T_1(I - Q)$ and $F = T - A$.

Then A and F have the required properties.

(b) ⇒ (a) Suppose $T = A + F$ where A is a bounded strictly singular operator and F is an unbounded finite rank operator.

Let M be a subspace of $D(T)$ s.t. $T|_M \in 1'$.

Then $T|_{M \cap N(F)} = A|_{M \cap N(F)}$ has a bounded inverse.

Since A is strictly singular , $\dim(M \cap N(F)) < \infty$. $(D(T) \subset D(A))$

Now , $\dim D(F)/N(F) = \dim R(F) < \infty$. $(\hat{F} : D(F)/N(F) \longrightarrow R(F)$ is bijective)

Since $D(T) \subset D(F)$, this means that $\dim M < \infty$.

Thus , T is strictly singular.

Since A is bounded and F is unbounded , T is unbounded. □

Corollary IV.11 : T strictly singular ⇒ T partially continuous.

Chapter V Operators with Closed Range

In this chapter , operators with closed range are studied and again the emphasis is on generalising results in [Gol ; IV] to seminormed spaces. At the end of the chapter a partial generalisation of an important stability result of Kato to unbounded strictly singular operators is presented.

Note that a number of results in this chapter which are proved in the setting of Banach spaces in [Gol ; IV] have been generalised to normed spaces in [La2].

Throughout this chapter , X and Y are seminormed spaces and $T \in L(X,Y)$.

1. The Minimum Modulus of T

Lemma V.1.1 : Let X and Y be complete and T closed. Then T has a continuous inverse $\Leftrightarrow$ T is injective and $R(T)$ is closed. (cf [Gol ; IV.1.1])

Proof :

($\Rightarrow$) Suppose $T \in 1$.

Then T is injective.

Let $y \in \overline{R(T)}$.

Then $\exists\, (y_n) \in R(T)^{\mathbb{N}}$ s.t. $y_n \longrightarrow y$.

$\forall\, m, n \in \mathbb{N}$, $\|T^{-1}y_n - T^{-1}y_m\| \leq \|T^{-1}\|\, \|y_n - y_m\|$.

Thus, $(T^{-1}y_n)$ is a Cauchy sequence in X.

Since X is complete, $\exists\, x \in X$ s.t. $T^{-1}y_n \longrightarrow x$.

Since T is closed, T^{-1} is also closed.

Thus, $y \in D(T^{-1}) = R(T)$.

($\Leftarrow$) Suppose T is injective and $R(T)$ is closed.

By Lemma III.4.7, Y is normed.

Since $R(T)$ is closed, $D(T^{-1}) = R(T)$ is complete.

Thus, by the closed graph theorem, T^{-1} is continuous. □

Theorem V.1.2 : Let X and Y be complete and T closed. Then

T.F.A.E. (a) $R(T)$ is closed.

(b) $R(T) = N(T')_{\perp}$.

(c) $R(T') = N(T)^{\perp}$.

(d) $R(T')$ is closed. (cf [Gol ; IV.1.2])

Proof :

(a)$\Leftrightarrow$(b) Theorem III.3.3.

(a) $\Rightarrow$ (c) Suppose $R(T)$ is closed.

Then $R(\hat{T})$ is closed so by Lemma V.1.1 , $\hat{T} \in 1$.

By Theorem III.3.8 , $\hat{T}' \in I$.

By Theorem III.3.6 , $R(T') = N(T)^{\perp}$.

(c) $\Rightarrow$ (d) Clear.

(d) $\Rightarrow$ (a) Suppose $R(T')$ is closed.

Let $T_1 = (J^{Y}_{\overline{R(T)}})^{-1}T$.

Then T_1 is closed.

We now show that T_1 is surjective.

By Theorem III.3.3 , T_1' is injective.

Now $R(T_1') = R(T')$ which is closed.

By Lemma V.1.1 , $T_1' \in 1$.

By Corollary III.4.4 , $T_1 \in I$.

i.e. $R(T) = R(T_1) = \overline{R(T)}$. □

Definition : The *minimum modulus* of T is

$$\gamma(T) = \sup\{ \gamma \geq 0 : \forall x \in D(T) , \|Tx\| \geq \gamma\, d(x,N(T)) \}.$$

Note :

(i) This generalises the definition in [Gol 2].

(ii) $\gamma(T) = \infty \Leftrightarrow \overline{N(T)} \cap D(T) = D(T)$.

(iii) $\gamma(T) = \gamma(\hat{T})$.

(iv) $\gamma(T) > 0 \Leftrightarrow \hat{T} \in 1$.

(v) $0 < \gamma(T) < \infty \Rightarrow \|\hat{T}^{-1}\| = \dfrac{1}{\gamma(T)}$.

$\hat{T} \in 1 , \|\hat{T}^{-1}\| \neq 0 \Rightarrow \gamma(T) = \dfrac{1}{\|\hat{T}^{-1}\|}$.

Definition : T is *relatively open* iff $(J^{Y}_{R(T)})^{-1}T$ is open.

Theorem V.1.3 : $\gamma(T) > 0 \Leftrightarrow$ T is relatively open. (cf [La2])

Proof :

($\Rightarrow$) Suppose $\gamma(T) > 0$.

Then $\exists\, \gamma > 0$ s.t. $\forall x \in D(T) , \|Tx\| \geq \gamma\, \||[x]\||$.

We now show that $\gamma\, U_{R(T)} \subset TB_{D(T)}$.

Let $y \in \gamma\, U_{R(T)}$.

Then $\exists\, x \in D(T)$ s.t. $y = Tx$.

If $\||[x]\|| \geq 1$, then $\|Tx\| \geq \gamma\, \||[x]\|| \geq \gamma$ a contradiction.

Thus , $\||[x]\|| < 1$ so $y \in \hat{T}U_{D(T)/N(T)}$.

Thus , $\gamma\, U_{R(T)} \subset \hat{T}U_{D(T)/N(T)} \subset \hat{T}Q^{X}_{N(T)}B_{D(T)} = TB_{D(T)}$.

($\Leftarrow$) Suppose T is relatively open.

Then $\exists\ \gamma > 0$ s.t. $\gamma\, U_{R(T)} \subset TB_{D(T)}$.

Let $x \in D(T)$.

case (i) : $\|Tx\| = 0$.

$\forall\ n \in \mathbb{N}$, $\|T(n\,x)\| = 0$ so $T(n\,x) \in \gamma\, U_{R(T)}$.

Thus, $\forall\ n \in \mathbb{N}$, $\exists\ z_n \in B_{D(T)}$ s.t. $T(n\,x) = Tz_n$.

$\forall\ n \in \mathbb{N}$, $n\,\|[x]\| = \|[n\,x]\| = \|[z_n]\| \leq \|z_n\| \leq 1$ so $\|[x]\| = 0$.

case (ii) : $\|Tx\| \neq 0$.

Let $\epsilon \in (0,1)$. Then

$\frac{(1-\epsilon)\ \gamma\ Tx}{\|Tx\|} \in \gamma\, U_{R(T)}$ so $\exists\ z \in B_{D(T)}$ s.t. $T(\frac{(1-\epsilon)\ \gamma\ x}{\|Tx\|}) = Tz$.

Thus, $\|[\frac{(1-\epsilon)\ \gamma\ x}{\|Tx\|}]\| = \|[z]\| \leq \|z\| \leq 1$ so $(1-\epsilon)\ \gamma\, \|[x]\| \leq \|Tx\|$.

Since ϵ was arbitrary, $\gamma\, \|[x]\| \leq \|Tx\|$.

Thus, $\gamma(T) \geq \gamma > 0$. □

Corollary V.1.4 : Y normed, dim $R(T) < \infty \Rightarrow \gamma(T) > 0$.

The next result generalises [Jam ; 20.2] to seminormed spaces and provides an alternative proof.

Theorem V.1.5 : $N(T)$ closed, dim $R(T) < \infty \Rightarrow$ T continuous.

Proof :

Suppose $N(T)$ is closed and dim $R(T) < \infty$.

Then dim $R(\hat{T}) < \infty$.

Since $\hat{T}$ is injective, dim $D(\hat{T}) < \infty$.

Since $N(T)$ is closed, $X/N(T)$ is normed.

By Theorem II.3.9, $\hat{T}$ is continuous.

Thus, T is continuous. □

The next result follows from Lemma V.1.1 and (iv) of the remark on page 85.

Theorem V.1.6 : Let X and Y be complete and T closed. Then

$$R(T) \text{ is closed } \Leftrightarrow \gamma(T) > 0. \quad \text{(cf [Gol2 ; IV.1.6])}$$

Theorem V.1.7 : Let $N(T)$ and $R(T)$ be closed and let $\gamma(T) > 0$. Then T is closed.

Proof :

Let (x_n) be a sequence in $D(T)$ s.t. $x_n \longrightarrow x$ and $Tx \longrightarrow y$.

Since $R(T)$ is closed , $\exists\, z \in D(T)$ s.t. $y = Tz$.

Since $\gamma(T) > 0$, $\exists\, \gamma > 0$ s.t. $\forall\, x \in D(T)$, $\|Tx\| \geq \gamma \,\|[x]\|$.

$\gamma \,\|[x - z]\| \leq \gamma \,\|[x - x_n]\| + \gamma \,\|[x_n - z]\| \leq \gamma \,\|x - x_n\| + \|Tx_n - y\| \longrightarrow 0.$

Thus , $x - z \in \overline{N(T)} = N(T)$ so $x \in D(T)$ and $y = Tz = Tx$. □

The next result generalises [Gol2 ; IV.1.8] ; in particular , $N(T)$ is not required to be closed.

Theorem V.1.8 : $\gamma(T) > 0 \Rightarrow \gamma(T) = \gamma(T')$ and $R(T')$ is closed. (cf [La2 ; 2.7])

Proof :

Suppose $\gamma(T) > 0$.

case (i) : $\gamma(T) = \infty$.

Then $\overline{N(T)} \cap D(T) = D(T)$.

$\forall\, y' \in D(T')$, $y'T[N(T)] = \{0\}$ so $y'T = O$.

Thus , $T' = O$ so $\gamma(T') = \infty$.

case (ii) : $\gamma(T) < \infty$.

Let $T_1 = (J^Y_{R(T)})^{-1} T J^X_{D(T)}$.

Then $\gamma(T_1) = \gamma(T) > 0$ so $\hat{T}_1 \in I_1$.

By Theorem III.3.9 , $\hat{T}_1' \in I_1$.

From $Y'/N(T') = Y'/R(T)^{\perp} = I^{Y'/R(T)^{\perp}}_{R(T)'} R(T)'$ it follows that

$D((T')\hat{}) \equiv D(T_1')$ where for $[y'] \in D((T')\hat{})$, the corresponding element in $D(T_1')$ is $y'_{R(T)}$.

$\forall\, y' \in D(T')$, $\forall\, x \in D(T)$, $y'\hat{T}[x] = y'_{R(T)}\hat{T}_1[x]$ so

$\forall\, y' \in D(T')$, $\|\hat{T}'y'\| = \|\hat{T}_1' y'_{R(T)}\|$.

By Theorem III.3.6 , $\forall\, y' \in D(T')$, $\|T'y'\| = \|\hat{T}_1' y'_{R(T)}\|$.

By Theorem III.3.5 , $(\hat{T}_1^{-1})' = (\hat{T}_1')^{-1}$.

Thus , $\|(\hat{T}_1')^{-1}\| = \|(\hat{T}_1^{-1})'\| = \|\hat{T}_1^{-1}\|$.

$$\gamma(T') = \sup\{\, \gamma \geq 0 : \forall\, y' \in D(T') , \|T'y'\| \geq \gamma \,\|[y']\| \,\}$$
$$= \sup\{\, \gamma \geq 0 : \forall\, y' \in D(\hat{T}_1') , \|\hat{T}_1' y'\| \geq \gamma \,\|y'\| \,\}$$
$$= \frac{1}{\|(\hat{T}_1')^{-1}\|} = \frac{1}{\|\hat{T}_1^{-1}\|} = \gamma(\hat{T}_1) = \gamma(T_1) = \gamma(T). \quad \square$$

Corollary V.1.9 : Let X and Y be complete and T closed. Then

$\gamma(T) = \gamma(T')$. (cf [Gol2 ; IV.1.9])

Proof :

If $\gamma(T) > 0$, then the result follows from the preceding theorem.

Otherwise , it follows from Theorems V.1.2 and V.1.6 that

$\gamma(T) = 0 \Leftrightarrow R(T)$ is not closed $\Leftrightarrow R(T')$ is not closed $\Leftrightarrow \gamma(T') = 0$. $\square$

Theorem V.1.10 : Let X and Y be complete and T closed. If T maps closed bounded subsets of X onto closed subsets of Y , then R(T) is closed. If $\dim N(T) < \infty$, then the converse also holds. (cf [Gol2 ; IV.1.10])

Proof :

Suppose T maps closed bounded sets onto closed sets but that R(T) is not closed.

Then $\gamma(T) = 0$ so $\exists\, (x_n) \in D(T)^{\mathbb{N}}$ s.t. $\forall\, n \in \mathbb{N}$, $\|[x_n]\| = 1$ and $Tx_n \to O$.

Let (z_n) be a sequence (in D(T)) s.t. $\forall\, n \in \mathbb{N}$, $z_n \in [x_n]$ and $\|z_n\| \leq 2$.

case (i) : (z_n) has no convergent subsequence.

Then $\{ z_n : n \in \mathbb{N} \}$ is closed and bounded.

By the assumption , $\{ Tz_n : n \in \mathbb{N} \}$ is closed.

Since $Tz_n = Tx_n \to O$, this means that $\exists\, n$ s.t. $Tz_n = O$.

Thus , $\|[x_n]\| = \|[z_n]\| = 0$ which contradicts the fact that $\|[x_n]\| = 1$.

case (ii) : (z_n) has a convergent subsequence (z_{n_r}) with limit z.

Since $Tz_n \to O$, $Tz_{n_r} \to O$.

Since T is closed , $z \in D(T)$ and $Tz = O$.

Thus , $[x_{n_r}] = [z_{n_r}] \to [O]$ which is also a contradiction.

Thus , R(T) is closed.

Now suppose $\dim N(T) < \infty$ and R(T) is closed.

Let S be a closed , bounded subset of X and let $y \in \overline{TS}$.

Then there is a sequence (x_n) in $S \cap D(T)$ s.t. $Tx_n \to y$.

Since R(T) is closed , $\exists\, x \in D(T)$ s.t. $Tx = y$.

Also , $\hat{T} \in 1$ (Lemma V.1.1) so $[x_n] = \hat{T}^{-1}Tx_n \to [x]$.

Now , $\exists\, (z_n) \in N(T)^{\mathbb{N}}$ s.t. $x_n + z_n \to x$.

Since (x_n) is bounded , this means that (z_n) is bounded.

Since $\dim N(T) < \infty$, (z_n) has a convergent subsequence (z_{n_r}) with limit $z \in N(T)$.

Now , $x_{n_r} \to x - z \in S \cap D(T)$. (S is closed.)

Thus , $y = Tx = T(x - z) \in TS$. □

2. Normally Solvable Operators

Definition : T is *normally solvable* iff T is closed and $R(T)$ is closed.

Lemma V.2.1 : Let X and Y be complete and T normally solvable. If M is a subspace of X s.t. $M + N(T)$ is closed , then TM is closed.
In particular , if M is closed and $\dim N(T) < \infty$, then TM is closed. (cf [Gol ; IV.2.9])

Proof : Suppose M is a subspace of X s.t. $M + N(T)$ is closed.
Let $T_1 = T|_{D(T) \cap (M + N(T))}$.
Then T_1 is closed and $N(T_1) = N(T)$.
Thus , $\gamma(T_1) \geq \gamma(T) > 0$ so $R(T_1)$ is closed (Theorem V.1.6).
i.e. TM is closed. □

Theorem V.2.2 : Let X and Y be complete , T normally solvable , $\dim N(T) < \infty$, Z a seminormed space and $B \in L(Z,X)$. Then
(a) B closed $\Rightarrow$ TB closed.
(b) B normally solvable $\Rightarrow$ TB normally solvable.

Proof :
(a) Suppose B is closed.
Let (z_n) be a sequence in $D(TB)$ s.t. $z_n \rightarrow z$ and $TBz_n \rightarrow y$.
Since T is normally solvable , $\gamma(T) > 0$ so $([Bz_n])$ is a Cauchy sequence in $D(T)/N(T)$.
Since $X/N(T)$ is complete , $\exists\, x \in X$ s.t. $[Bz_n] \rightarrow [x]$.
Also , $\exists\, (x_n) \in N(T)^{\mathbb{N}}$ s.t. $Bz_n + x_n \rightarrow x$.

We now show that (x_n) is bounded.

Suppose (x_n) is unbounded.

Then (x_n) has a subsequence $(x_{n'})$ s.t. $\|x_{n'}\| \longrightarrow \infty$.

Note that $\dfrac{Bz_{n'} + x_{n'}}{\|x_{n'}\|} \longrightarrow O$.

Since $\left(\dfrac{x_{n'}}{\|x_{n'}\|}\right)$ is a bounded sequence in the finite dimensional seminormed

space $N(T)$, $(x_{n'})$ has a subsequence $(x_{n''})$ s.t. $\dfrac{x_{n''}}{\|x_{n''}\|} \longrightarrow v$ for some $v \in N(T)$.

Now , $\dfrac{Bz_{n''}}{\|x_{n''}\|} \longrightarrow -v$ and $\dfrac{z_{n''}}{\|x_{n''}\|} \longrightarrow O$.

Since B is closed , $-v = BO = O$ which is impossible as $\|v\| = 1$.

Thus , (x_n) is bounded.

Since $\dim N(T) < \infty$, (x_n) has a convergent subsequence (x_{n_r}) with limit $w \in N(T)$.

Now , $Bz_{n_r} \longrightarrow x - w$.

Since B is closed , $z \in D(B)$ and $x - w = Bz$.

Since T is closed and $TBz_{n_r} \longrightarrow y$, $Bz \in D(T)$ and $TBz = y$.

Thus , TB is closed.

(b) If BZ is closed , then by Lemma V.2.1 $R(TB) = TBZ$ is closed. □

For the remainder of this chapter , X and Y are normed.

The proof of the following portion of a perturbation result due to Kato[Ka2] can be found in [Gol ; V.2.1].

Theorem V.2.3 : Let X and Y be complete and T normally solvable with $\dim N(T) < \infty$. If B is a continuous strictly singular operator with $D(T) \subset D(B)$, then $T + B$ is normally solvable and $\dim N(T + B) < \infty$.

We now generalise this partially to include unbounded strictly singular operators.

Theorem V.2.4 : Let X and Y be complete , T normally solvable with $\dim N(T) < \infty$. If B is an unbounded strictly singular operator with $D(T) \subset D(B)$, then

(a) $\dim N(T + B) < \infty$.

(b) $\exists$ a finite codimensional subspace N of $D(T)$ and a normally solvable operator S s.t. $D(S) = D(T)$ and $S|_N = (T + B)|_N$. (i.e. $T + B$ is "almost" normally solvable.)

Proof :

(a) By Theorem IV.10 , there is a continuous strictly singular operator A and an unbounded finite rank operator F s.t. $B = A + F$.

By Theorem V.2.3 , $T + A$ is normally solvable with $\dim N(T + A) < \infty$.

Now , $N(T + A + F) \subset D(F)$ and $\dim D(F)/N(F) < \infty$ so by Lemma IV.6 , there is a finite dimensional subspace M of $D(F)$ s.t.

$N(T + A + F) = N(T + A + F) \cap N(F) \oplus M = N(T + A) \oplus M$.

Thus , $\dim N(T + A + F) < \infty$.

(b) As already noted , $T + A$ is normally solvable.

Examination of the construction in Theorem IV.10 shows that $D(T) = D(F)$.

Thus , $\dim D(T)/N(F) = \dim D(F)/N(F) = \dim R(F) < \infty$.

Finally , $(T + A)|_{N(F)} = (T + B)|_{N(F)}$. □

Chapter VI Operators analagous to Weakly Compact Operators

Very little seems to be known about the second adjoint of an arbitrary linear operator. A recent paper in which the second adjoint is studied is [C2] which deals with operators of the Tauberian type. This chapter is , therefore , to be seen in the light of a broader study of the second adjoint of a linear operator.

In the classical case , if X and Y are Banach spaces and $T \in BL[X,Y]$, then T is *weakly compact* iff $\overline{TB_X}$ is $\sigma(Y,Y')$ compact. In this case , the following characterisation is obtained (see [Con] or [DS] or [HP]) :

T.F.A.E. (a) T is weakly compact.

(b) T' is weakly compact.

(c) T' is $\sigma(Y',Y)$–$\sigma(X',X'')$ continuous.

(d) $T'' X'' \subset \hat{Y}$.

The purpose of this chapter is to characterise those operators (not necessarily bounded) for which a property corresponding to (d) holds.

For the remainder of this chapter , X and Y are seminormed spaces and $T \in L(X,Y)$. The following notations will be used :

$$I = I^{D(T')'}_{Y''/D(T')^{\perp}} \qquad Q = Q^{Y''}_{D(T')^{\perp}} \qquad J_{D(T)} = J^{D(T)''}_{D(T)} \qquad J_Y = J^{Y''}_{Y} .$$

Note that $\forall\, y'' \in Y''$, $IQy'' = y'' J^{Y'}_{D(T')}$.

The author believes all the results in this chapter to be his own.

We first prove a few properties of T''.

Proposition VI.1 :

(a) $T''J_{D(T)} = IQJ_YT$.

(b) Corresponding to a fixed T, T'' is the only $\sigma(D(T''),D(T)')$–$\sigma(D(T')',D(T'))$ continuous operator from $D(T'')$ into $D(T')'$ satisfying (a).

Proof :

(a) $\forall\, x \in D(T)$, $\forall\, y' \in D(T')$, $\hat{x}T'y' = y'Tx = (Tx)\hat{\ }y' = IQ(Tx)\hat{\ }y'$.

Thus, $D(T)\hat{\ } \subset D(T'')$ and $T''J_{D(T)} = IQJ_YT$.

(b) By Theorem III.2.4, T'' is $\sigma(D(T''),D(T)')$–$\sigma(D(T')',D(T'))$ continuous.

Suppose that $S : D(T'') \longrightarrow D(T')'$ is a $\sigma(D(T''),D(T)')$–$\sigma(D(T')',D(T'))$ continuous operator satisfying $SJ_{D(T)} = IQJ_YT$.

Let $x'' \in D(T'')$.

By Goldstine's theorem, $\exists$ a net (x_α) in $D(T)$ s.t. $\hat{x}_\alpha \longrightarrow x''$ w.r.t. $\sigma(D(T)'',D(T)')$.

By assumption, $S\hat{x}_\alpha \longrightarrow Sx''$ w.r.t. $\sigma(D(T')',D(T'))$.

Thus, $Sx'' = \lim_\alpha S\hat{x}_\alpha = \lim_\alpha IQ(Tx_\alpha)\hat{\ } = \lim_\alpha T''\hat{x}_\alpha = T''x''$. □

Proposition VI.2 : $T''B_{D(T'')} \subset \overline{IQ(TB_{D(T)})\hat{\ }}^{\sigma(D(T')',D(T'))}$.

Proof : Let $x'' \in B_{D(T'')}$.

By Goldstine's theorem, $\exists$ a net (x_α) in $B_{D(T)}$ s.t. $\hat{x}_\alpha \longrightarrow x''$ w.r.t. $\sigma(D(T)'',D(T)')$.

Since T'' is $\sigma(D(T''),D(T)')$–$\sigma(D(T')',D(T'))$ continuous,

$T''x'' = \lim_\alpha T''\hat{x}_\alpha = \lim_\alpha IQ(Tx_\alpha)\hat{\ } \in \overline{IQ(TB_{D(T)})\hat{\ }}^{\sigma(D(T')',D(T'))}$. □

Proposition VI.3 : Let $\hat{Y} \subset E \subset Y''$. Then

$T''D(T'') \subset IQE \Leftrightarrow T'$ is $\sigma(D(T'),E)$–$\sigma(D(T)',D(T''))$ continuous.

Proof :

($\Rightarrow$) Suppose $T''D(T'') \subset IQE$.

Let (y'_α) be a net in $D(T')$ s.t. $y'_\alpha \to y'$ w.r.t. $\sigma(D(T'),E)$.

Let $x'' \in D(T'')$.

By assumption , $\exists\, y'' \in E$ s.t. $T''x'' = IQy''$.

$$x''T'y' = T''x''y' = y''y' = \lim_\alpha y''y'_\alpha = \lim_\alpha T''x''y'_\alpha = \lim_\alpha x''T'y'_\alpha .$$

Thus , $T'y'_\alpha \to T'y'$ w.r.t. $\sigma(D(T)',D(T''))$.

($\Leftarrow$) Suppose T' is $\sigma(D(T'),E)$–$\sigma(D(T)',D(T''))$ continuous.

Let $x'' \in B_{D(T'')}$.

By assumption , $\exists$ finite $F \subset E$ s.t. $F_o \subset T'^{-1}[\{x''\}_o]$ (see p18).

We now show that $\bigcap_{z' \in IQF} \ker(z') \subset \ker(T''x'')$.

Let $y' \in D(T')\backslash\ker(T''x'')$.

Put $w' = \dfrac{2\,y'}{T''x''y'}$.

Then $|x''T'w'| = 2 > 1$ so $T'w' \notin \{x''\}_o$.

Thus , $w' \notin T'^{-1}[\{x''\}_o] \supset F_o \supset \bigcap_{y'' \in F} \ker(y'')$.

Thus , $y' \notin \bigcap_{y'' \in F} \ker(y'')$ so $\exists\, y'' \in F$ s.t. $y''y' \neq 0$.

Since $y' \in D(T')$, this means that $IQy''y' \neq 0$.

By Lemma I.5.3 , $T''x'' \in \operatorname{span} IQF \subset IQE$. □

Definition : A subset of a topological space is *relatively compact* iff every net in the set has a convergent subnet.

Proposition VI.4 : Let $\hat{Y} \subset E \subset Y''$.

(a) $(TB_{D(T)})\hat{}$ relatively $\sigma(E,D(T'))$ compact $\Rightarrow$ $T''D(T'') \subset IQE$.

(b) If T' is continuous , then

$$(TB_{D(T)})\hat{} \text{ relatively } \sigma(E,D(T')) \text{ compact } \Leftrightarrow T''D(T'') \subset IQE.$$

Proof :

(a) Suppose $(TB_{D(T)})\hat{}$ is relatively $\sigma(E,D(T'))$ compact.

Let $x'' \in B_{D(T'')}$.

By Proposition VI.2 , $\exists$ a net (x_α) in $B_{D(T)}$ s.t. $IQ(Tx_\alpha)\hat{} \longrightarrow T''x''$ w.r.t. $\sigma(D(T')',D(T'))$.

Since $(TB_{D(T)})\hat{}$ is relatively $\sigma(E,D(T'))$ compact , (x_α) has a subnet (x_{α_β}) s.t. $(Tx_{\alpha_\beta})\hat{} \longrightarrow y''$ w.r.t. $\sigma(E,D(T'))$ for some $y'' \in E$.

Now , $IQ(Tx_{\alpha_\beta})\hat{} \longrightarrow IQy''$ w.r.t. $\sigma(D(T')',D(T'))$.

Since $IQ(Tx_{\alpha_\beta})\hat{} \longrightarrow T''x''$ w.r.t. $\sigma(D(T')',D(T'))$ and $\sigma(D(T')',D(T'))$ is Hausdorff , $T''x'' = IQy'' \in IQE$.

(b) Suppose T' is continuous and $T''D(T'') \subset IQE$.

Then $D(T'') = D(T)''$.

Let (x_α) be a net in $B_{D(T)}$.

Then $(\hat{x}_\alpha)$ is a net in $B_{D(T'')} = B_{D(T)''}$.

Since $B_{D(T)''}$ is $\sigma(D(T)'',D(T)')$ compact , (x_α) has a subnet (x_{α_β}) s.t. $\hat{x}_{\alpha_\beta} \longrightarrow x''$ w.r.t. $\sigma(D(T)'',D(T)')$ for some $x'' \in B_{D(T)''}$.

Since $T''D(T'') \subset IQE$, $\exists\, y'' \in E$ s.t. $T''x'' = IQy''$.

Now $T''\hat{x}_{\alpha_\beta} \longrightarrow T''x''$ w.r.t. $\sigma(D(T')',D(T'))$.

$\forall\, y' \in D(T')$, $y''y' = IQy''y' = \lim_\beta (T''\hat{x}_{\alpha_\beta})y' = \lim_\beta (Tx_{\alpha_\beta})\hat{}\, y'$.

Thus , $(Tx_{\alpha_\beta})\hat{} \longrightarrow y''$ w.r.t. $\sigma(E,D(T'))$. $\square$

Definition : $\| \|_{D(T')}$ is defined on Y'' by $\|y''\|_{D(T')} = \sup \{ |y''y'| : y' \in B_{D(T')} \}$.

Note : (i) $\| \|_{D(T')}$ is a seminorm on Y''.

(ii) $\forall y'' \in Y''$, $\|y''\|_{D(T')} \leq \|y''\|$

Notation : Let E be a subspace of Y'' and F a subspace of Y'.

$\overline{E}^{D(T')}$ will denote the closure of E w.r.t. the $D(T')$ seminorm.

$F\hat{\ }|_E$ will denote the set $\{ (y')\hat{\ }|_E : y' \in F \}$.

$E_{D(T')}$ will denote E equipped with the $D(T')$ seminorm restricted to E.

Proposition VI.5 : Let E be a subspace of Y''. Then $(\overline{D(T')})\hat{\ }|_E \subset (E_{D(T')})'$.

Proof :

Let $y' \in \overline{D(T')}\backslash\{0\}$.

Then $\exists\, (y'_n) \in D(T')^{\mathbb{N}}$ s.t. $y'_n \rightarrow y'$ and $\forall\, n \in \mathbb{N}$, $y'_n \neq 0$.

$\forall\, y'' \in Y''$, $\left|\frac{(y')\hat{\ }}{\|y'\|} y''\right| = \lim_n \left|y''\left(\frac{y'_n}{\|y'_n\|}\right)\right| \leq \|y''\|_{D(T')}$.

Thus , $\forall\, y'' \in E$, $|(y')\hat{\ }y''| \leq \|y'\|\, \|y''\|_{D(T')}$ so $(y')\hat{\ }|_E \in (E_{D(T')})'$. □

Proposition VI.6 : Let $\hat{Y} \subset E \subset Y''$.

(a) $D(T')\hat{\ }|_E = (E_{D(T')})' \Rightarrow T'$ is continuous.

(b) If $\hat{Y} \subset E \subset \overline{\hat{Y}}^{D(T')}$, then $D(T')\hat{\ }|_E = (E_{D(T')})' \Leftrightarrow T'$ is continuous.

Proof :

(a) Suppose $D(T')\hat{\ }|_E = (E_{D(T')})'$.

By Proposition VI.5 , $D(T')\hat{\ }|_E = \overline{D(T')}\hat{\ }|_E$.

Since $\hat{Y} \subset E$, $D(T') = \overline{D(T')}$ so T' is continuous. (Theorem III.2.8)

(b) Suppose $\hat{Y} \subset E \subset Y''$ and T' is continuous..

Let $z' \in (E_{D(T')})'$.

Define $y' : Y \longrightarrow \mathbb{F}$ by $y'y = z'\hat{y} \quad (y \in Y)$.

$\forall\, y \in Y$, $|y'y| = |z'\hat{y}| \le \|z'\|\, \|\hat{y}\|_{D(T')} \le \|z'\|\, \|\hat{y}\| = \|z'\|\, \|y\|$ so $y' \in Y'$.

$\forall\, x \in D(T)$, $|y'Tx| \le \|z'\|\, \|(Tx)^{\hat{}}\|_{D(T')} = \|z'\| \sup_{y' \in B_{D(T')}} |y'Tx| \le \|z'\|\, \|T'\|\, \|x\|$

so $y' \in D(T')$.

Let $y'' \in E$.

By assumption, $\exists\, (y_n) \in Y$ s.t. $\|\hat{y}_n - y''\|_{D(T')} \longrightarrow 0$.

Now, $y''y' = \lim_n y'y_n = \lim_n z'\hat{y}_n = z'y''$.

Thus, $z' = (y')^{\hat{}}\,|_E \in D(T')^{\hat{}}\,|_E$. □

Corollary VI.7 : Let $\hat{Y} \subset E \subset \overline{\hat{Y}}^{D(T')}$ and let T' be continuous. Then $B_{D(T')}$ is $\sigma(D(T'),E)$ compact.

Proof :

By Proposition VI.6, $(E_{D(T')})' = D(T')^{\hat{}}\,|_E$.

We first show that $B_{(E_{D(T')})'} = (B_{D(T')})^{\hat{}}\,|_E$.

Let $z' \in B_{(E_{D(T')})'}$.

Then $\|z'\| \le 1$ and $\exists\, y' \in D(T')$ s.t. $z' = (y')^{\hat{}}\,|_E$.

Let $y \in B_Y$.

Then $\|\hat{y}\|_{D(T')} \le 1$ so $|y'y| = |z'\hat{y}| \le 1$.

Thus, $y' \in B_{D(T')}$.

Let $z' \in (B_{D(T')})^{\hat{}}\,|_E$. Then $\exists\, y' \in B_{D(T')}$ s.t. $z' = (y')^{\hat{}}\,|_E$.

$\forall\, y'' \in B_{E_{D(T')}}$, $|z'y''| = |y''y'| \le 1$ so $\|z'\| \le 1$.

Thus, $B_{(E_{D(T')})'} = (B_{D(T')})^{\hat{}}\,|_E$.

By the Banach–Alaoglu theorem, $B_{(E_{D(T')})'}$ is $\sigma((E_{D(T')})', E_{D(T')})$ compact.

From the preceding discussion, $B_{D(T')}$ is $\sigma(D(T'),E)$ compact. □

Corollary VI.8 : Let $\hat{Y} \subset E \subset \overline{\hat{Y}}^{D(T')}$ and let T' be continuous (so $D(T'') = D(T)''$).
If T' is $\sigma(D(T'),E)$–$\sigma(D(T)',D(T''))$ continuous , then $T'B_{D(T')}$ is $\sigma(D(T)',D(T)'')$ compact.

Proposition VI.9 : Let $E = \overline{\hat{Y}}^{D(T')}$ and T' be continuous.
If $T'B_{D(T')}$ is $\sigma(D(T)',D(T)'')$ compact , then $(TB_{D(T)})\hat{}$ is relatively $\sigma(E,D(T'))$ compact.

Proof :

Suppose $T'B_{D(T')}$ is $\sigma(D(T)',D(T)'')$ compact.
Let $S = T'$ and $E_1 = (D(T)')\hat{}$.
By Proposition VI.4 , $S''D(S'') \subset IQE_1$, where I and Q have the appropriate meanings.
By Proposition VI.3 , S' is $\sigma(D(S'),E_1)$–$\sigma(D(S)',D(S''))$ continuous.
S is continuous so S' is continuous.
By Corollary VI.8 , $S'B_{D(S')}$ is $\sigma(D(S)',D(S)'')$ compact.
i.e. $T''B_{D(T'')}$ is $\sigma(D(T')',D(T')'')$ compact.
Let (x_α) be a net in $B_{D(T)}$.
Then $\exists$ a subnet (x_{α_β}) of (x_α) , $\exists\, x'' \in B_{D(T'')}$ s.t.
$T''\hat{x}_{\alpha_\beta} \longrightarrow T''x''$ w.r.t. $\sigma(D(T')',D(T')'')$.
Now , $T''x'' \in \overline{IQ\hat{Y}}^{\sigma(D(T')',D(T')'')} = \overline{IQ\hat{Y}}$ so $\exists\, (y_n) \in Y^{\mathbb{N}}$ s.t. $IQ\hat{y}_n \longrightarrow T''x''$.
By the Hahn–Banach theorem , $\exists\, y'' \in Y''$ s.t. $IQy'' = y''J^{Y'}_{D(T')} = T''x''$.
$\|\hat{y}_n - y''\|_{D(T')} = \sup_{y' \in B_{D(T')}} |y'y_n - y''y'| = \|IQ\hat{y}_n - IQy''\| \longrightarrow 0.$
Thus , $y'' \in E$.
$\forall\, y' \in D(T')$, $y''y' = T''x''y' = \lim_\beta (Tx_{\alpha_\beta})\hat{}\,y'$ so $(Tx_{\alpha_\beta})\hat{} \longrightarrow y''$ w.r.t. $\sigma(E,D(T'))$.
Thus , $(TB_{D(T)})\hat{}$ is relatively $\sigma(E,D(T'))$ compact. □

Proposition VI.10 : Let $E = \hat{Y}^{-D(T')}$ and let T' be continuous. Then

T.F.A.E.
(a) $(TB_{D(T)})^{\wedge}$ is relatively $\sigma(E,D(T'))$ compact.
(b) $T''D(T'') \subset IQE$.
(c) T' is $\sigma(D(T'),E)-\sigma(D(T)',D(T)'')$ continuous.
(d) $T'B_{D(T')}$ is $\sigma(D(T)',D(T)'')$ compact.

Corollary VI.11 : Let T' be continuous. If $\hat{Y} = \hat{Y}^{-D(T')}$, then

T.F.A.E.
(a) $TB_{D(T)}$ is relatively $\sigma(Y,D(T'))$ compact.
(b) $T''D(T'') \subset IQ\hat{Y}$.
(c) T' is $\sigma(D(T'),Y)-\sigma(D(T)',D(T)'')$ continuous.
(d) $T'B_{D(T')}$ is $\sigma(D(T)',D(T)'')$ compact.

Note : Condition (d) in the preceding two results is exactly the requirement for T' to be weakly compact. Thus , Theorem VI.10 and Corollary VI.11 provide characterisations of operators with weakly compact adjoints.

It is well known that in the setting of Banach spaces a continuous operator is weakly compact if either the domain space or the range space are relexive. Also , if Y is semireflexive , then $\hat{Y} = \hat{Y}^{-D(T')}$ and by Theorem II.5.7 Y' is reflexive. Thus , we obtain :

Proposition VI.12 : Let Y be semireflexive and T' continuous. Then

(a) T' is weakly compact.
(b) $TB_{D(T)}$ is relatively $\sigma(Y,D(T'))$ compact.
(c) $R(T'') \subset IQ\hat{Y}$.
(d) T' is $\sigma(D(T'),Y)-\sigma(D(T)',D(T)'')$ continuous.

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