

Volatility derivatives in the Heston framework

Abstract

A volatility derivative is a financial contract where the payoff depends on the realized variance of a specified asset's returns. As volatility is in reality a stochastic variable, not deterministic as assumed in the Black-Scholes model, market participants may surely find volatility derivatives to be useful for hedging and speculation purposes. This study explores the construction and calibration of the Heston stochastic volatility model and the pricing of some volatility derivatives within this framework.

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1. Introduction

Together with a mandate to maximize returns, comes the necessity to minimize and manage risk. Since Markowitz in 1955, volatility is an established indicator of financial risk. **Volatility** is the average deviation from the expected return, in the context of the notion that asset returns are normally distributed. Conceptually volatility is a measure of uncertainty, unpredictability, instability or risk without direction. There are two observable volatility types, namely historical and implied volatility. **Implied volatility**¹ represents the future market expectation of volatility, as reflected by vanilla option trading. **Historical or realized volatility** is the statistical standard deviation of realized asset price returns.

In the Black-Scholes framework (1973), the volatility δ of the underlying asset S_t is assumed to be a constant² over the lifetime of the vanilla option. Under the risk-neutral measure, asset S_t has its dynamics described by the stochastic differential equation $dS_t = rS_t dt + \delta S_t dW_t$, where W_t is a Brownian motion and r the risk-free rate. The log-returns then have a normal distribution, and in the risk-neutral pricing framework a convenient closed-form solution follows. With only one stochastic variable S_t we can dynamically delta-hedge the option until maturity. Regrettably observable implied and historical volatilities contradict the Black-Scholes model's deterministic volatility assumption.

The option trading community perceives volatility to be stochastic: vanilla options on the same underlying, with the same expiry date, have different Black-Scholes implied volatilities for a given range of strike levels, often called the **volatility skew** or **smile**³. For the Black-Scholes model to hold,

¹ **Implied volatility** is the quoted volatility for a given vanilla option contract available in the market such that the theoretical (Black-Scholes) price equals the market price. A higher volatility implies a higher vanilla option price and vice versa; when the implied volatility is high, the option is expected to be worth more when it matures in-the-money. The higher implied volatility does not necessarily increase the chances that the option would end up in-the-money, as the volatility measure is not directional. In the option market the implied volatility is quoted, not the option price. This is the market practice because for a given option contract specification, it is difficult to interpret the price in the given market conditions, while the volatility is a measure of uncertainty, and subsequently easier for traders to interpret and negotiate on. Implied volatility can be seen as a weighted average of all volatility outcome scenarios.

² The volatility can be extended to be a deterministic function of time and/or the state variable δ_{t,S_t} .

³ The **volatility smile** refers to a shape where the implied volatilities are higher for lower and higher strikes and a minimum for the at-the-money strike (the strike that equals the underlying asset's forward price). The **volatility skew** refers to a shape where the implied volatilities are higher for lower strikes.

the volatility has to be the same deterministic function of time for all possible strike levels. When various option maturities are combined with the skew we have a **volatility surface**⁴. Traders who use the Black-Scholes have to adjust the volatility surface frequently to match market prices. As the volatility surface changes, the delta hedge proportions also change erratically. Options that are path-dependent and/or have early-exercise opportunities are wrongly priced in the Black-Scholes framework with the deterministic volatility assumption [1].

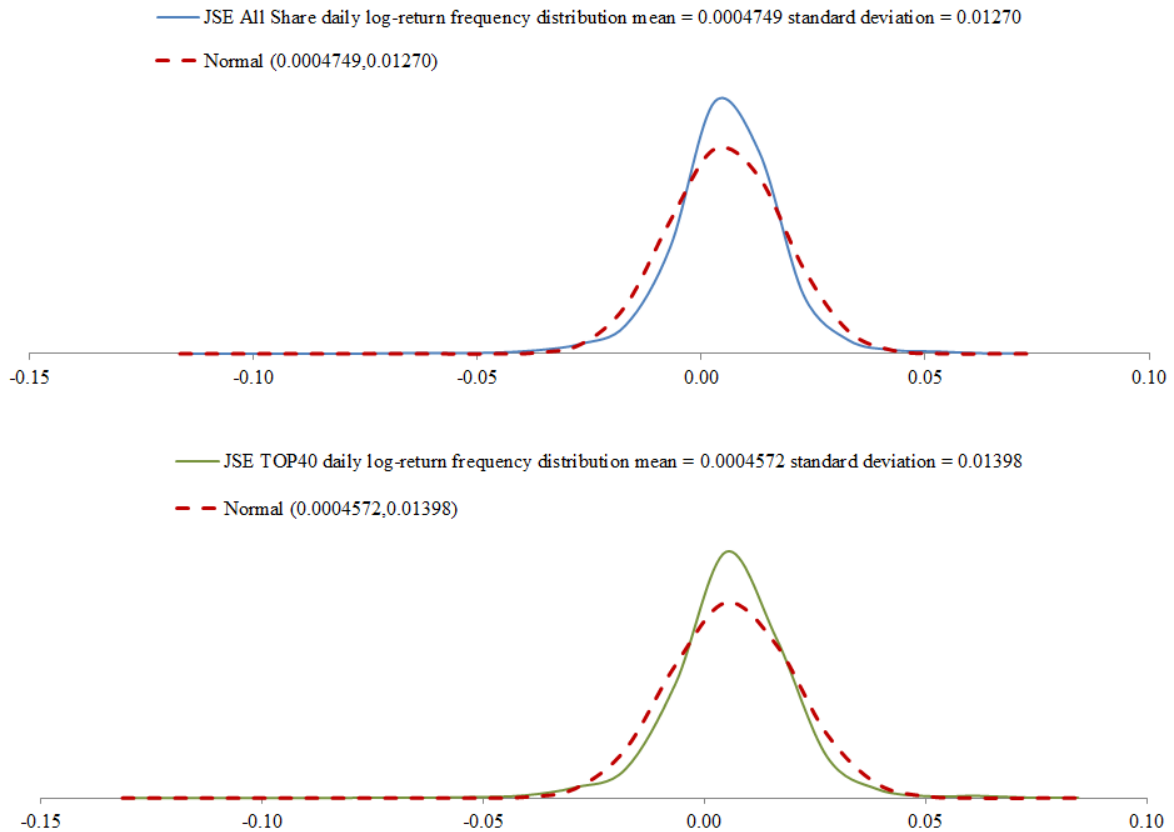
Empirical studies show that the historical frequency distribution of equity index log-returns is notably more fat-tailed and peaked (leptokurtic) compared to the normal distribution. This is characteristic of combinations of distributions with different variances (volatility-squared), indicating that the variances of equity index returns are stochastic, according to Gatheral [1]. In figure 1 the daily log-return frequency distributions of the JSE/FTSE Top40 and JSE/FTSE All Share indices are compared to the normal distribution. The frequency distributions of South African equity indices are similarly peaked and fat-tailed with a kurtosis above 6 (see table 1) and can therefore also be described as leptokurtic.

Table 1 – Daily log-return statistics, using the (business) daily closing prices from 30 June 1995 to 25 March 2013, sourced from Bloomberg.

Moment	JSE ALL SHARE	JSE TOP40	Normal Distribution
Daily Mean	0.0475%	0.0457%	Same
Annual Mean	11.9684%	11.5164%	Same
Daily Standard Deviation	1.2707%	1.3989%	Same
Annual Standard Deviation	20.1716%	22.2072%	Same
Skew	-0.4699	-0.3978	0
Kurtosis	6.1112	6.1626	3

⁴ A **volatility surface** is a two dimensional grid of implied volatilities extracted from vanilla options on the same underlying asset. Each point (implied volatility) on the grid has an associated strike and maturity date. When the implied volatility is entered into the Black-Scholes formula for a given strike and maturity, the resulting price matches the option's market price. As out-of-the-money options are the most liquid, the one side of the surface will utilize calls and the other side puts.

Figure 1 – FTSE/JSE ALL SHARE log-return frequency distribution compared to the normal distribution, using the (business) daily closing prices from 30 June 1995 to 25 March 2013, sourced from Bloomberg. The second graph shows the FTSE/JSE TOP40 log-return frequency distribution compared to the normal distribution, using the (business) daily closing prices from 30 June 1995 to 25 March 2013, sourced from Bloomberg.



As volatility is indeed a stochastic variable, also evidently in the South African market, market participants are exposed to volatility as a risk factor and may wish to trade volatility to speculate or hedge. Volatility derivatives facilitate the trading in volatility, as volatility itself is not a tradable asset. One can gain volatility exposure via a long at-the-money straddle, although the dominant volatility exposure is lost when the underlying price moves away from the initial at-the-money level. One can also assemble a static replicating portfolio of options to mimic the variance exposure of a forward contract on the logarithm of the forward price divided by the initial price [14]. The static replicating portfolio of options is based on the twice differentiable payoff relation, see section §5.4. This portfolio of options retains the volatility exposure as the price moves away from the initial level, but price exposure then becomes a side effect, obligating the pure volatility exposure seeking investor to dynamically delta hedge. Volatility derivatives provide pure exposure to volatility.

This study is focused on the pricing of some volatility derivatives in the Heston stochastic volatility framework. Stochastic volatility models aim to give structure to the random fashion in which volatility evolves. The Heston model provides a quasi-closed-form solution for call option prices, and can be viewed as an extension of the Black-Scholes model. Once the stochastic variance (volatility-squared) parameters are calibrated such that the Heston prices nearly equate the market option prices, then theoretically the stochastic processes dictate how variance and the corresponding asset price evolves through time in a risk-neutral world, and consequently enables us to price various volatility derivatives.

2. The Heston stochastic volatility model

2.1 Theory

A stochastic volatility model is a theoretical construction that describes an unobservable random variable: instantaneous variance. The models however still aim to reflect reality. Examples are, amongst others, the SABR (Stochastic Alpha Beta Rho) model, the CEV (Constant Elasticity of Variance) model and the GARCH (Generalized Autoregressive Conditional Heteroskedasticity) model. We study the Heston model with constant parameters $\Psi = [v_0 \ \kappa \ \theta \ \sigma \ \rho]$. The risk-neutral dynamics of an asset price S_t is given by

$$dS_t = S_t(r - q)dt + S_t\sqrt{v_t} dW_t^1 \quad dv_t = \kappa(\theta - v_t) dt + \sigma\sqrt{v_t} dW_t^0 \quad dW_t^1 dW_t^0 = \rho dt \quad (1)$$

The asset price process looks like a Geometric Brownian motion, but the volatility $\sqrt{v_t}$ is now stochastic instead of deterministic. The stochastic variance v_t is representative of the empirical evidence provided by implied and realized volatility. The model assumes an Ornstein-Ohlenbeck process for the volatility, then Itô's formula is applied to volatility squared, and the resulting variance process is then re-written to be the well-known Cox-Ingersoll-Ross process [8]. The risk-free rate r and the dividend yield q are assumed to be constant. When the Feller condition, $2\kappa\theta - \sigma^2 \geq 0$ holds, the variance process never reaches zero almost surely [11]. Alternatively, the origin is accessible and strongly reflective. As the volatility of variance is often quite high for stochastic volatility models, the Feller condition often does not hold. See references [21] and [22] for more on the Feller condition.

The variance process is assumed to return to its **long-term mean** θ at a **rate** κ , called mean-reversion. Volatility clustering, where large asset price return moves tend to follow large moves and small moves follow small moves, indicates that variance time-series data exhibit serial correlation. This is

echoed by the mean-reversion property of Heston's variance process and prevents the instantaneous variance from reaching unrealistic levels [1]. The $\kappa(\theta - v_t) dt$ term is asymptotically stable when the mean-reversion speed is positive ($\kappa > 0$) [11].

Volatility is typically negatively correlated with the corresponding asset price; when the market is in distress and asset prices decline rapidly, volatility symptomatically increases. When an equity price drops the underlying issuing company is expected to become more leveraged with the debt to equity ratio rising. The issuing company is therefore taking on more risk and the equity price's volatility subsequently increases. The phenomenon is often referred to as the leverage effect. The leverage effect is reflected by the two Brownian motions W_t^1 and W_t^0 that are **correlated by a factor ρ** . The correlation also partly explains the slight skew in a historical return distribution [8].

When the **volatility of variance** σ tends to zero, the variance becomes a deterministic function of time, and the Heston model then becomes the Black-Scholes model; the Black-Scholes model can be regarded as a special case of the Heston model.

By Itô's formula and a no-arbitrage argument we arrive at a partial differential equation for the option price Λ (where both the asset price and variance are stochastic state variables), see the derivation in Annex 5.1. We suppress notation: $\Lambda = \Lambda(t, S_t, v_t) = \Lambda(t, S, v)$.

$$\frac{\partial \Lambda}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 \Lambda}{\partial S^2} + \rho \sigma v S \frac{\partial^2 \Lambda}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 \Lambda}{\partial v^2} + (r - q) S \frac{\partial \Lambda}{\partial S} - (r - q) \Lambda = -\kappa(\theta - v) \frac{\partial \Lambda}{\partial v} \quad (2)$$

We are indifferent to the initial statistical measure in equation (1), but state the stochastic differential equations in risk-neutral terms as we recover the risk-neutral measure by fitting the Heston parameters to market option prices; we assume a market where no arbitrage opportunities exist. We want to derive an option price within the Heston framework, as we can then generate the risk-neutral measure by calibrating the Heston parameters such that the Heston prices nearly equate the market option prices. Heston derived a quasi-closed form solution for option prices, which is a big advantage when one implements the calibration in practice.

Below is a brief summary of the Heston formula derivation in Annex 5.2. The two references are [1] and [11]. Let t be the current time in years, T be the maturity in years, S_t the current price, X the strike price and F the forward price, with $x = \ln(F_{t,T}/X)$, $F_{t,T} = S_t e^{(r-q)\tau}$, $\tau = T - t$. We have $\Lambda(t, T, S_t, v_t) = \Lambda(\tau, x_t, v_t)$. The PDE is now written as

$$-\Lambda_\tau + \left((r - q) - \frac{1}{2}v \right) \Lambda_x + \frac{1}{2}v \Lambda_{xx} + \rho\sigma v \Lambda_{xv} + \frac{1}{2}\sigma^2 v \Lambda_{vv} - (r - q)\Lambda - \kappa(v - \theta)\Lambda_v = 0 \quad (3)$$

Assume a solution of the form below, where P^1 is the delta of the option and P^2 is the pseudo-probability of exercise, similar to the Black-Scholes model for a call.

$$\Lambda(\tau, x, v_t) = X e^{-r\tau} (e^x P^1(\tau, x, v_t) - P^2(\tau, x, v_t)) \quad x = \ln \left(\frac{S_t e^{(r-q)\tau}}{X} \right) = \ln \left(\frac{F_{t,T}}{X} \right) \quad (4)$$

The PDE can be solved with 2 new PDEs where $j = 1, 2$

$$-P_\tau^j + \left((r - q) + u^j v \right) P_x^j + \frac{1}{2}v P_{xx}^j + \rho\sigma v P_{xv}^j + \frac{1}{2}\sigma^2 v P_{vv}^j + (a - vb^j) P_v^j = 0 \quad (5)$$

$$a = \kappa\theta \quad b^1 = \kappa - \rho\sigma \quad b^2 = \kappa \quad u^1 = \frac{1}{2} \quad u^2 = -\frac{1}{2} \quad (6)$$

Since P^j resembles a probability, as we approach maturity the probability of exercise becomes one if the option is in-the-money $x = \ln(F_{t,T}/X) > 0$, and zero otherwise. The terminal condition for P^j is

$$I(x) := \lim_{\tau \rightarrow 0} P^j(\tau, x, v_t) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} \quad (7)$$

The PDE for P^j is solved via a Fourier transform technique. Using some Fourier transform results⁵, the Fourier transform of P^j is defined as

$$\mathfrak{F} \left(P^j(\tau, x, v_t) \right) = \tilde{P}^j(\tau, w, v_t) = \int_{-\infty}^{\infty} e^{-iwx} P^j(\tau, x, v_t) dx \quad i = \sqrt{-1} \quad (8)$$

$$\tilde{P}^j(0, w, v_t) = \int_{-\infty}^{\infty} e^{-iwx} I(x) dx = \frac{1}{-iw} \int_0^{\infty} -iw \cdot e^{-iwx} dx = \frac{1}{-iw} [e^{-iwx}]_0^{\infty} = \frac{[0 - 1]}{-iw} = \frac{1}{iw} \quad (9)$$

The Fourier transform of the PDE becomes

⁵ [13] Kreyzig: $\mathfrak{F}(f'(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iux} f'(x) dx = \frac{1}{\sqrt{2\pi}} \left([f(x)e^{-iux}]_{-\infty}^{\infty} + iu \int_{-\infty}^{\infty} e^{-iux} f(x) dx \right) = iu \mathfrak{F}(f(x))$
 $|x| \rightarrow \infty, f(x)e^{-iux} \rightarrow 0$
 $\mathfrak{F}(f''(x)) = iu \mathfrak{F}(f'(x)) = (iu)^2 \mathfrak{F}(f(x)) = u^2 \mathfrak{F}(f(x)) \quad \mathfrak{F}(af(x)) = a\mathfrak{F}(f(x))$

$$-\tilde{P}_\tau^j + (r - q)iw\tilde{P}^j + a\tilde{P}_v^j + v(\alpha\tilde{P}^j + \gamma\tilde{P}_{vv}^j - \beta\tilde{P}_v^j) = 0 \quad (10)$$

$$\alpha = \left(u^j iw - \frac{1}{2}w^2\right) \quad -\beta = (b^j - iw\rho\sigma) \quad \gamma = \frac{1}{2}\sigma^2 \quad (11)$$

The proposed solution for the Fourier transform PDE is given below and is satisfied with the given values for $C^j(w, \tau)$ and $D^j(w, \tau)$.

$$\tilde{P}^j(\tau, w, v_t) = \tilde{P}^j(\mathbf{0}, w, v_t) \exp(C^j(w, \tau)\theta + D^j(w, \tau)v_t) = \frac{1}{iw} \exp(C^j(w, \tau) + D^j(w, \tau)v_t) \quad (12)$$

$$C^j(\tau, w) = \frac{\kappa\theta}{\sigma^2} \left(\tau(b^j - iw\rho\sigma - d) - 2 \ln \frac{(1 - ge^{-\tau d})}{(1 - g)} \right) + (r - q)iw\tau \quad (13)$$

$$D^j(\tau, w) = \left(\frac{b^j - iw\rho\sigma - d}{\sigma^2} \right) \frac{(1 - e^{-\tau d})}{(1 - ge^{-\tau d})} \quad (14)$$

$$d = \sqrt{(iw\rho\sigma - b^j)^2 - \sigma^2(2u^j iw - w^2)} \quad (15)$$

$$g = \frac{b^j - iw\rho\sigma - d}{b^j - iw\rho\sigma + d} \quad (16)$$

Now taking the inverse transform and performing complex integration, we find a solution for $P^j(\tau, x, v_t)$.

$$\begin{aligned} \Im(\tilde{P}^j(\tau, w, v_t)) &= P^j(\tau, x, v_t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwx} \tilde{P}^j(\tau, w, v_t) du \\ &= \int_{-\infty}^{\infty} \frac{e^{iwx}}{2\pi \cdot iw} \exp(C^j(\tau, w)\theta + D_j(\tau, w)v_t) dw \end{aligned} \quad (17)$$

$$P^j(\tau, x, v_t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{\exp(C^j(\tau, w) + D^j(\tau, w)v_t + iwx)}{iw} dw \quad (18)$$

Integration only over the real and not imaginary part of the integrand gives an integral that we can compute with numerical integration techniques in a computer program. Note that v_t is the initial

instantaneous variance at time t , which is not observable, and we will estimate this parameter via the calibration procedure together with $\theta, \kappa, \sigma, \rho$.

$$P^j(\tau, x, v_t) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left\{ \frac{\exp(C^j(\tau, w) + D^j(\tau, w)v_t + iw x)}{iw} \right\} dw \quad (19)$$

The equations for the Heston call price correspond with [1]. The integrand in equation (19) is one of two available formulas (there are two possible solutions to the Riccati ODE). Looking at equation (13), when the imaginary part of the argument of the logarithm is zero, the real part is positive, preventing the argument of the logarithm to cross the negative real axis, so that standard numerical integration techniques can be used (we use the Adaptive Lobatto quadrature method in the code) [1]. The other specification can result in unstable vanilla call option prices, for nearly all Heston parameter choices, for longer dated time-to-maturities, see [20] for detail.

We can assume certain parameters for the Heston process, and then investigate the effect on the implied volatility skew. We do this by calculating the Heston call prices for various strike levels given a set of parameters, and then we solve the implied volatilities $\bar{\sigma}_i$ using a Newton–Raphson iterative procedure. We assume the Heston price is the Black (market model) price, and then solve the implied volatility $\bar{\sigma}_i$ iteratively. For $\phi(\cdot)$ the standard normal cumulative distribution function, according to the Black formula⁶ the price of a call C_i^{Black} at time $t \leq T$ is given as

$$C_i^{Black}(t, \bar{\sigma}_i, X_i) = e^{-r\tau} [F_{t,T} \cdot \phi(d_{i1}) - X_i \cdot \phi(d_{i2})] \quad (20)$$

$$\tau = T - t \quad F_{t,T} = S_t e^{(r-q)\tau} \quad d_{i1} = \frac{\ln\left(\frac{F_{t,T}}{X_i}\right) + \frac{1}{2}\bar{\sigma}_i^2\tau}{\bar{\sigma}_i\sqrt{\tau}} \quad d_{i2} = d_{i1} - \bar{\sigma}_i\sqrt{\tau} \quad (21)$$

$$\frac{\partial C_i^{Black}(t, \bar{\sigma}_i, X_i)}{\partial \bar{\sigma}_i} = S_t e^{-q\tau} \phi'(d_{i1})\sqrt{\tau} \quad \phi'(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \quad (22)$$

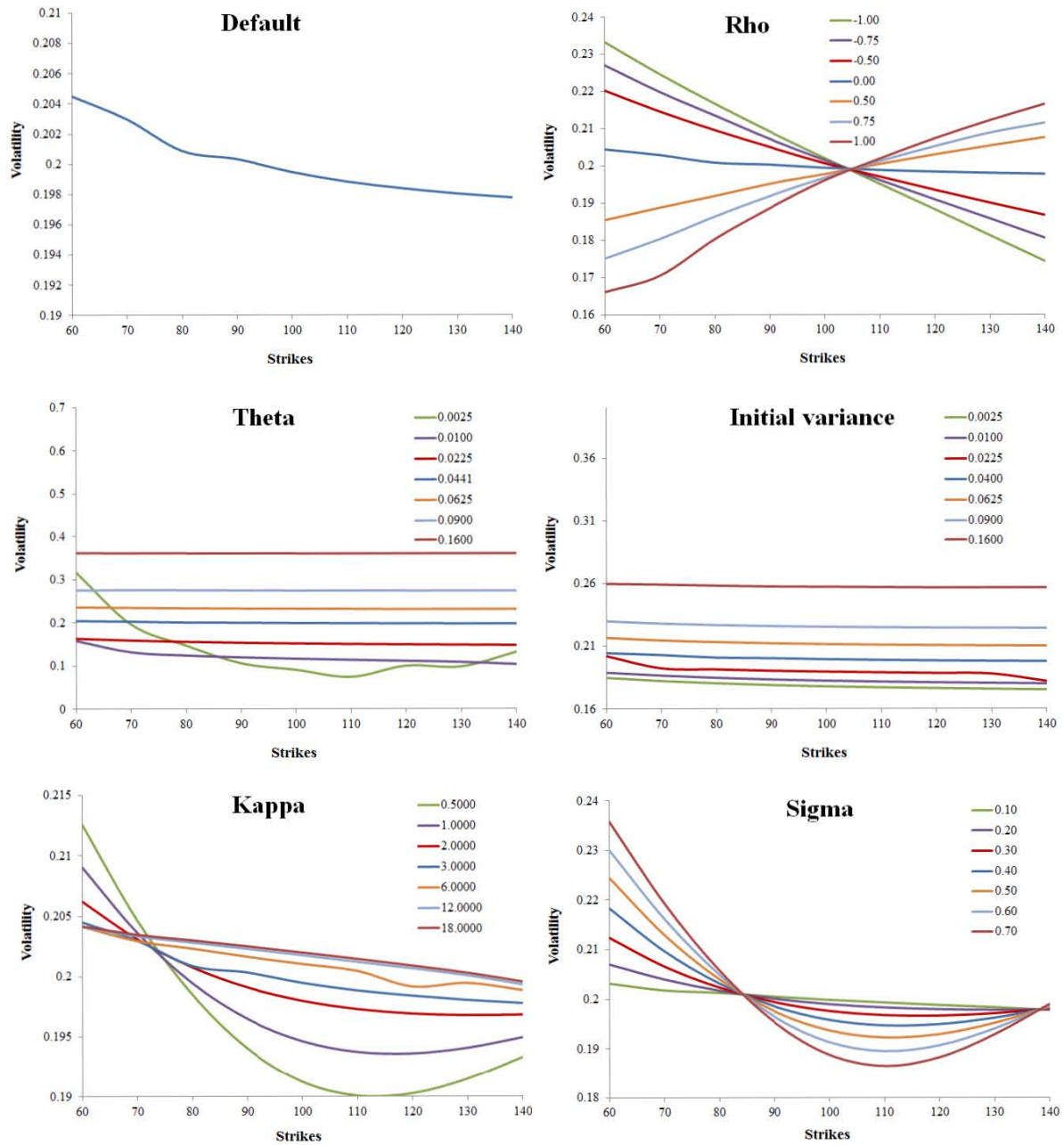
In the application the Newton–Raphson method below is used to iteratively solve for the flat implied volatility $\bar{\sigma}_i$ for each Heston option price.

⁶ We assume the option is European and OTC. Note that the JSE index future options are American and the exchange's model is the undiscounted Black model, that is in equation (20) $e^{-r\tau} = 1$, and $F_{t,T}$ is a futures price not a forward price.

$$\bar{\sigma}_i^k = \bar{\sigma}_i^{k-1} - \frac{|C_i^{Black}(t, \bar{\sigma}_i^{k-1}, X_i) - C_i^{Heston}(t, X_i, \Psi)|}{\frac{\partial C_i^{Black}(t, \bar{\sigma}_i^{k-1}, X_i)}{\partial \bar{\sigma}_i^{k-1}}} \quad \Psi = [v_0 \ \kappa \ \theta \ \sigma \ \rho] \quad (23)$$

By varying one Heston parameter at a time and keeping the other default parameters constant, we observe the results in figure 2. The effect of sigma is relatively mild, but we observe that the higher the volatility of variance, the more pronounced the corresponding implied volatility smile is. The smile is a flat horizontal line when sigma equals zero, so we expect this parameter to add curvature to an otherwise flat line. Rho has the most prominent effect on the shape of the smile/skew. A negative correlation suggests that asset prices are expected to move in the opposite direction to the variance. When market players fear future market turmoil (where asset prices tumble), currently out-of-the-money put prices are expected to be have greater payoffs when in-the-money, i.e. these volatilities are significantly higher than for the corresponding out-of-the-money calls. For theta the skew's shape stays roughly the same as the default, but the higher theta is, the more elevated the corresponding smile. Theta is the long-term mean variance and this parameter is therefore expected to affect the average level of implied volatilities. The lower the mean-reversion speed (kappa), the more convex the volatility smile. The initial variance also mainly affects how elevated the skew is. When a line is not smooth, the given set of Heston parameters generates Heston option prices which do not comfortably fit the market model. We observe that the wobbly lines occur with some extreme parameters. The very small long term mean (theta) of 0.0025 is unrealistic when initial variance is 0.04 and the mean reversion speed is 3. Understanding how each Heston parameter affects the level and shape of the embedded implied volatility smile or skew may assist us in choosing better initial guesses for the calibration procedure.

Figure 2 - The at-the-money strike price is 100, the risk-free interest rate is 5%, the dividend-yield is 0.22%, the time-to-maturity is 1.5 years. The default Heston parameters are respectively $v_0 = 0.2^2$, $\kappa = 3$, $\theta = 0.21^2$, $\sigma = 0.15$, $\rho = 0$. The bright blue line is throughout the default representative skew. Note the vertical axis scale differences.



2.2 Calibration

As we are interested in recovering the risk-neutral measure from market quoted option prices; we will calibrate the Heston parameter vector $\Psi = [v_0 \ \kappa \ \theta \ \sigma \ \rho]$ such that the Heston prices nearly equate the observable market option prices. We do this by defining an optimization problem where we want to minimize the sum of the (weighted) absolute relative errors to the power of an even number, between the market prices and the Heston prices for each point $\bar{\sigma}_{jm}$ on the (implied) volatility surface. Note that we ignore the bid-ask spread and assume mid closing volatilities. The market price is calculated using the Black model $C_{jm}^{Black}(\bar{\sigma}_{jm}, T_j, X_m)$. The optimization solves for the Heston parameters such that the total error E is minimized. J is the number of expiry dates on the volatility surface, and M is the number of strike prices. The subjective weight for each error point is ω_{jm} . One can utilize this weight to assign more or less weight to a point that the optimization favours or discriminates against to get a better fit. One has to ‘play’ with these weights to find a good fit. One can also try to start off the calibration with ‘better’ initial guesses. The power α is usually set equal to two, and then analogue to a least-squares method. The asset’s spot price, risk-free rate and dividend yield is assumed to be constant throughout the calibration.

$$\Psi = [v_0 \ \kappa \ \theta \ \sigma \ \rho] \quad \alpha = 2g \quad g \in \mathbb{N}^+$$

$$E = \sum_{j=1}^J \sum_{m=1}^M \omega_{jm} \left| \frac{C_{jm}^{Black}(\bar{\sigma}_{jm}, T_j, X_m) - C_{jm}^{Heston}(\Psi, T_j, X_m)}{C_{jm}^{Black}(\bar{\sigma}_{jm}, T_j, X_m)} \right|^\alpha \quad (24)$$

The Feller condition, in the Heston case $2\kappa\theta/\sigma^2 > 1$, ensures that the variance process never reaches zero. One can add the Feller condition as a constraint to the calibration’s optimization problem, but the parameter fit might end up being unsatisfactory. The constraint was removed in the calibration for this study after the resulting calibrated parameter fit was poor. In practice this condition is often not satisfied because the calibrated σ is relatively too high [21]. In section 2.3 we advise how to regulate the simulated variance process trajectories such that the origin is not crossed.

A volatility surface representative of the ALSI implied volatility surface is the quasi market data we use for the calibration. Referring to section 3 paragraph 3, many ALSI surface points are illiquid and for this study we use a surface that is representative of the usual overall shape observed. Some familiarity with how the parameters relate to the shape and level of a skew can help towards improved initial guess parameters that in turn can speed up the calibration and may result in a better fit. Looking at the 2-dimensional view of the volatility skews for various time-to-maturities in figure 3, we can see that the skew is significant and resembles a strong negative correlation, we guess $\rho =$

-0.55. The convexity is evident and we guess $\kappa = 0.8$. Since the line is nowhere close to horizontal we guess $\sigma = 0.6$. We guess approximately the at-the-money market implied volatility squared for the remainder of the parameters. After each calibration trial we assess the calibration success by comparing the market skew with the implied volatilities extracted from the calibrated Heston call prices, using the Newton-Raphson procedure given in equation (23).

Figure 3 - An implied volatility surface that is representative of the ALSI volatility surface shape, in a 2-dimensional view. Each line represents the skew for a time-to-maturity in years.

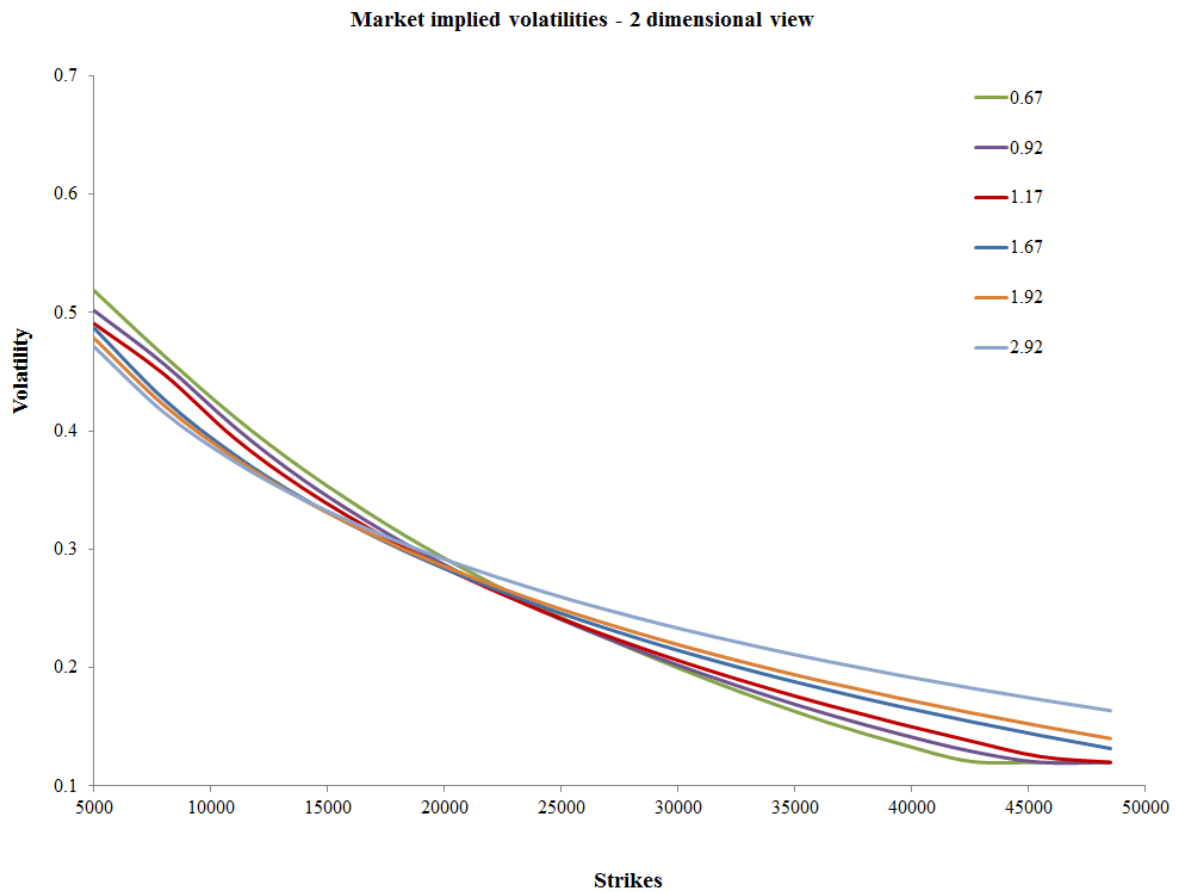
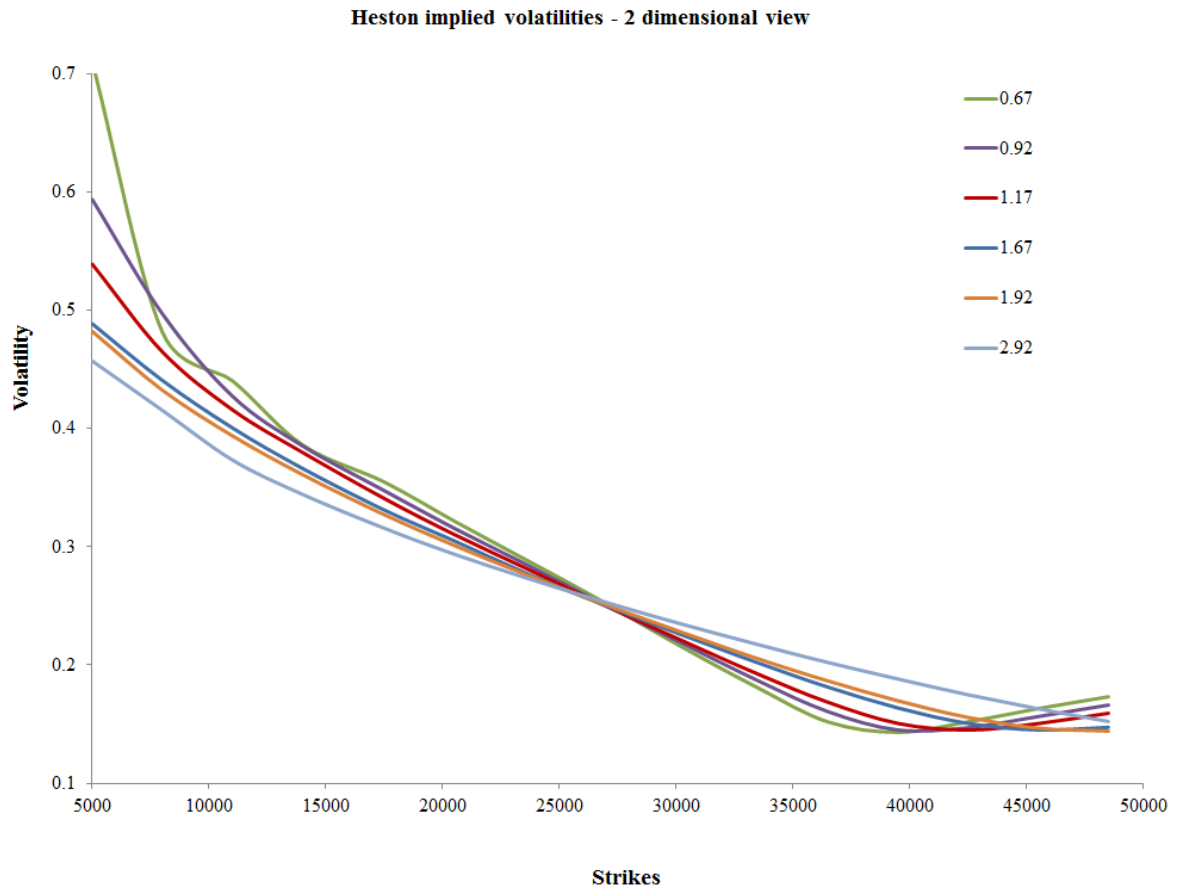


Figure 4 – The ALSI volatilities implied by the calibrated Heston prices, in a 2-dimensional view, each line represents the skew for a time-to-maturity in years.



The Heston model in our example clearly cannot fit to all the skew lines equally well, see figures 3 and 4. The 0.67 year maturity suffers the most, while the 1.67 and 1.92 maturities appear to fit in well with the calibrated parameters. The Heston fit is also notably more curved at the very high strike levels, and much higher for the 0.67 and 0.92 maturities at very low strike levels. The calibrated parameters do not satisfy the Feller condition; the market data suggests a sigma that is too high and a kappa and theta that is too low, see table 2. For a developed market volatility surface, like the SPX, the Heston fit to the long-end of the volatility surface is often quite good; however it never fits the short-end of the surface well. All stochastic volatility models generate more or less the same shape of a volatility surface, so this problem is not Heston specific [1]. To improve the fit one often has to turn to jump diffusion models or make the Heston parameters time dependent, which is out of scope for this study.

Table 2 – The calibrated Heston parameters. The initial market implied volatility surface is representative of the ALSI volatility surface shape.

Heston Parameter	Calibrated
v_0	0.027855
θ	0.080057
σ	0.642540
κ	0.865306
ρ	-0.552339
$2\kappa\theta - \sigma^2$	-0.274309

Figure 5 - An implied volatility surface that is representative of the ALSI volatility surface shape.

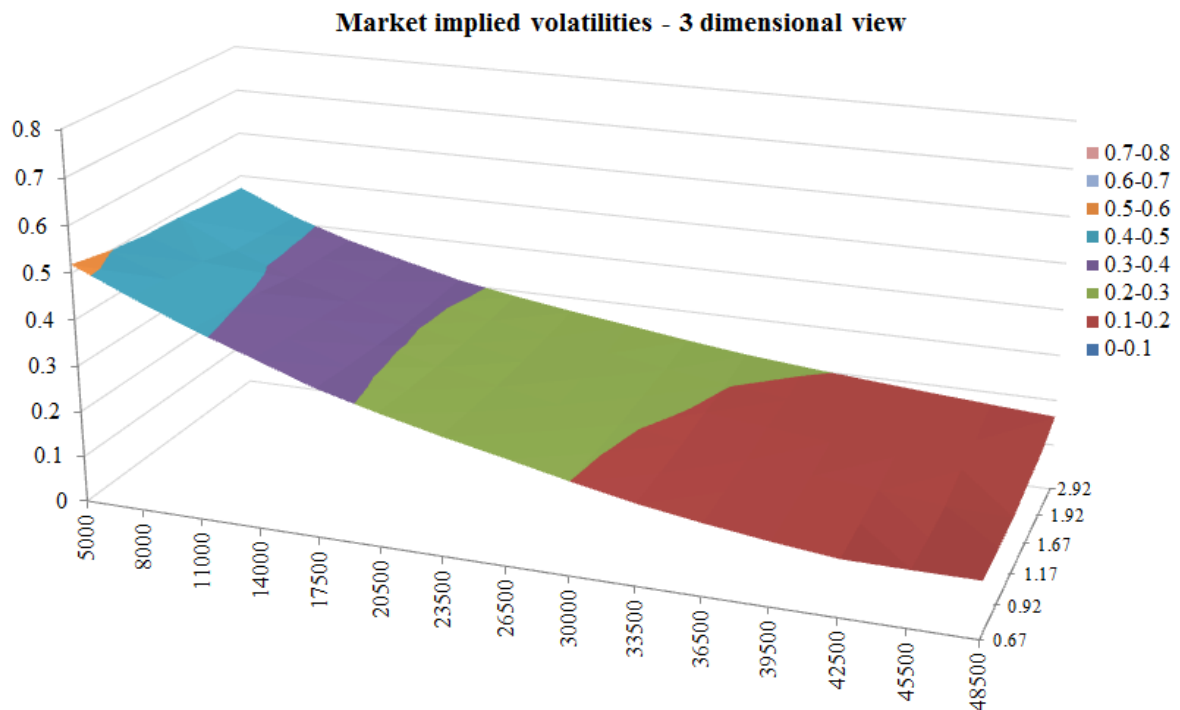
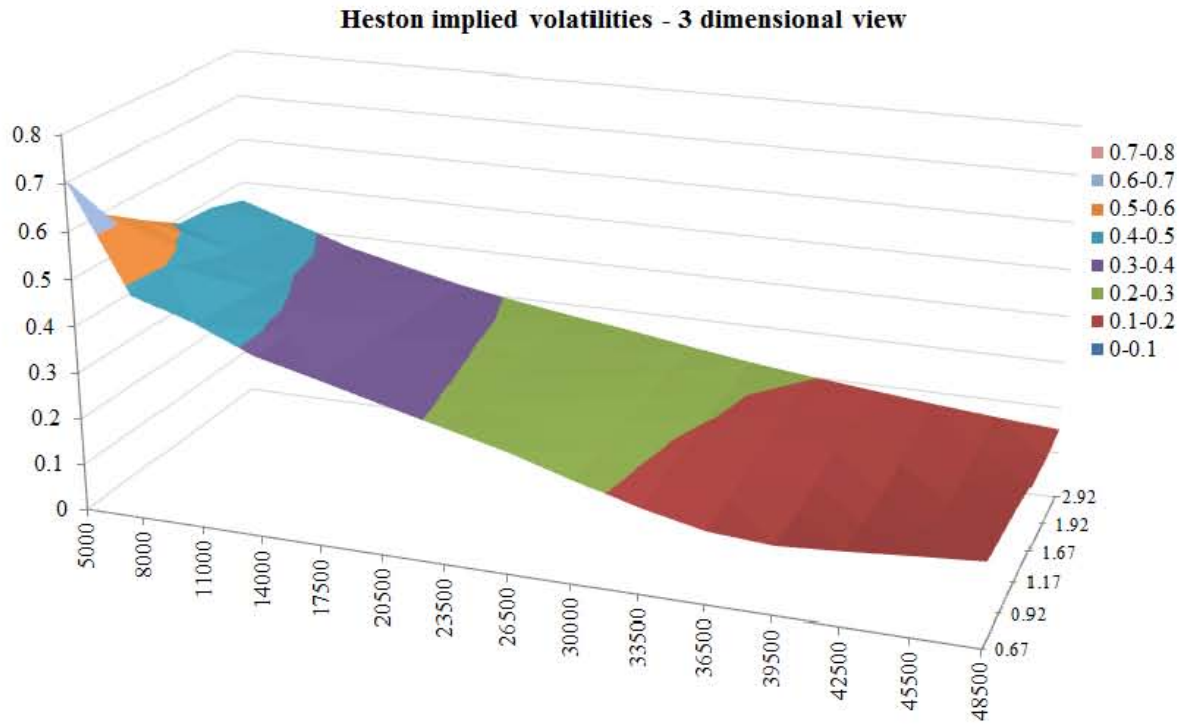


Figure 6 - The ALSI volatilities implied by the calibrated Heston prices, in a 3-dimensional view



2.3 Simulation

We now consider how we would simulate the Heston asset and variance process with a Monte Carlo simulation. It can be shown that if the asset return process is considered independent, it has a normal distribution and the variance process has a non-central Chi-squared distribution [21]. Since the Heston model's stochastic differential equations are unfortunately too complicated to allow for an analytical solution (as the Brownian motions are correlated) we have to resort to a discretization method like the Euler scheme. The Euler discretization for the Heston variance process is, when we are moving in time steps of size Δt ,

$$v_{i+1} = v_i + \kappa(\theta - v_i) \Delta t + \sigma\sqrt{v_i} \sqrt{\Delta t} Z \quad Z \sim N(0,1) \quad (25)$$

This discretization can give rise to negative v_i 's. In practice one can overwrite a negative variance with zero (absorbing assumption) or multiply it by -1 (reflective assumption), but given that this crude first-order discretization requires the step size to be very small to achieve convergence, we rather turn to a higher order discretization method, i.e. the Milstein discretization scheme. The Milstein scheme significantly reduces the occurrence of negative variance during the simulation. For a general SDE with a Riemann integral and a stochastic integral

$$dX_t = a(X_t) dt + b(X_t) dW_t \quad (26)$$

The Milstein discretization is given as

$$X_{i+1} = X_i + a(X_i)\Delta t + b(X_i)W_{i+1} + \frac{1}{2}b(X_i)b'(X_i)(W_{i+1}^2 - \Delta t) \quad W_{i+1} \sim N(0, \Delta t) \quad (27)$$

See [12] for the derivation. For the Heston process this is

$$v_{i+1} = v_i + \kappa(\theta - v_i) \Delta t + \sigma\sqrt{v_i} W_{i+1}^0 + \frac{1}{4}\sigma^2 \left((W_{i+1}^0)^2 - \Delta t \right) \quad (28)$$

$$S_{i+1} = S_i + (r - q) S_i \Delta t + \sqrt{v_i} S_i W_{i+1}^1 + \frac{1}{2} v_i S_i \left((W_{i+1}^1)^2 - \Delta t \right) \quad (29)$$

If instead we derive it for $\ln S_{i+1}$, then the higher order term falls away as $b'(X_i) = 0$. Apply Itô's formula to $\ln S_t$

$$\ln S_t = \frac{1}{S_t} dS_t + \frac{1}{2} \left(\frac{-1}{S_t^2} \right) (dS_t)^2 = \left(r - q - \frac{1}{2} v_i \right) dt + \sqrt{v_t} dW_t^1 \quad (30)$$

$$\ln S_{i+1} = \ln S_i + \left(r - q - \frac{1}{2} v_i \right) \Delta t + \sqrt{v_t} W_{i+1}^1 \quad (31)$$

The two Brownian motion processes are correlated $dW_t^1 dW_t^0 = \rho dt$. We use the Cholesky decomposition of the covariance matrix Σ , and $Z_t^0 \sim N(0,1)$, $Z_t^1 \sim N(0,1)$.

$$\Sigma = t \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = \sqrt{t} \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \cdot \sqrt{t} \begin{pmatrix} 1 & \rho \\ 0 & \sqrt{1-\rho^2} \end{pmatrix} \quad (32)$$

$$\begin{bmatrix} W_t^0 \\ W_t^1 \end{bmatrix} = \sqrt{t} \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{bmatrix} Z_t^0 \\ Z_t^1 \end{bmatrix} = \begin{bmatrix} \sqrt{t} Z_t^0 \\ \sqrt{t} \rho Z_t^0 + \sqrt{t} \sqrt{1-\rho^2} Z_t^1 \end{bmatrix} \quad (33)$$

So we can write

$$v_{i+1} = v_i + \kappa(\theta - v_i) \Delta t + \sigma \sqrt{v_i} \sqrt{\Delta t} Z_{i+1}^0 + \frac{1}{4} \sigma^2 \Delta t \left((Z_{i+1}^0)^2 - 1 \right) \quad (34)$$

$$\ln S_{i+1} = \ln S_i + \left(r - q - \frac{1}{2} v_i \right) \Delta t + \sqrt{v_i} \sqrt{\Delta t} \left(\rho Z_{i+1}^0 + \sqrt{1-\rho^2} Z_{i+1}^1 \right) \quad (35)$$

We observe that if $v_i = 0$, we want $\kappa\theta\Delta t - \frac{1}{4}\sigma^2\Delta t > 0$, i.e. $4\kappa\theta > \sigma^2$, such that $\frac{1}{4}\sigma^2\Delta t (Z_{i+1}^0)^2$ is not reduced and the state variable v_{i+1} 's chances of becoming negative is seriously reduced [1], [21]. The code makes the reflective assumption when a simulated v_{i+1} is less than zero, which may introduce bias. See reference [22] for more on Heston discretization schemes on efficiency, bias and the possibility of negativity. See also reference [21] for more detail on discretization methods' bias and convergence for stochastic volatility models. Regrettably discretization methods are computationally slow because the time step should be kept small, even for a European derivative that is not path-dependent [21].

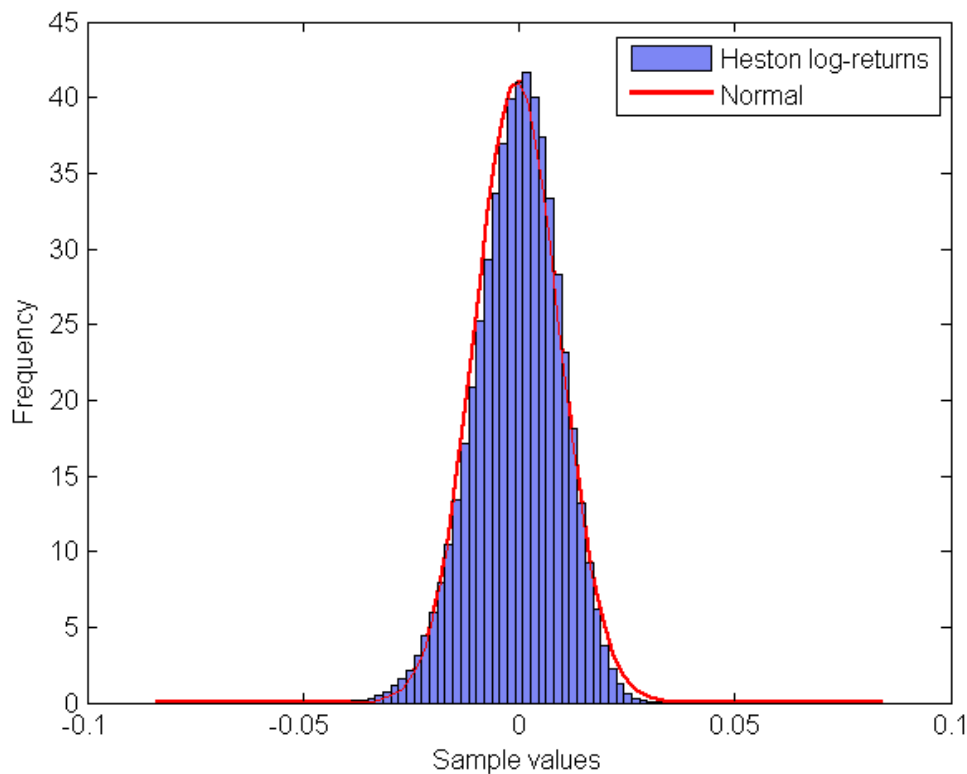
Table 3 - Daily log-return statistics for the Heston model using the calibrated Heston parameters in table 2.

Moment	Heston log-returns distribution	Normal Distribution
Daily Mean	-0.039366%	Same
Annual Mean	-14.368749%	Same
Daily Standard Deviation	0.877819%	Same
Annual Standard Deviation	16.770714%	Same
Skew	-0.333836	0
Kurtosis	3.223689	3

We can now plot the frequency distribution of the log-returns generated by the discretization, and can use the simulated paths to value various derivatives. In table 3 the moments of the Heston distribution using the calibrated parameters are displayed, and the returns distribution is plotted in figure 7. The

daily log-return distribution generated by the Heston parameters is not the same as the ALSI historical distribution, the daily return is negative and the kurtosis is almost half. The market view, the benchmark used for the Heston calibration, usually differs with history, also in this example.

Figure 7 – Heston daily log-return frequency distribution using the calibrated parameters in table 2.



3. Volatility derivatives

A volatility derivative is a financial contract where the payoff depends on the realized variance of an underlying asset's returns. In this study we consider variance and volatility swaps, variance options and mileage options. The variance swap, a contract to once-off swap realized variance (volatility-squared) for a strike variance, is the most popular volatility derivative as the contract can be hedged with vanilla options, and volatility-squared (quadratic variation) has some desirable theoretical properties. It is also the most basic volatility derivative, and can therefore be utilised to hedge other volatility derivatives.

The variance swap was already widely traded since the end of the 20th century in developed markets. In 1998 implied volatilities were significantly higher than econometric forecasts of realized volatility. Hedge funds found it desirable to be short a variance swap (receive the fair strike variance derived

from implied volatilities, and pay realised variance). Banks took a long position and hedged their position by selling a strip of options (determined by the log-contract relation below) and delta-hedging them. As long as the total initial premiums earned on selling the options was higher than the fair strike variance, banks were able to make some profit [7]. When the market is very volatile, the tracking error becomes high and following the benchmark may be costly, therefore portfolio managers may want to sell realized variance via a variance swap contract [2].

Variance swaps on single names became extinct after the 2008/9 market turmoil as options for various strikes became very illiquid [7]. In addition, when a single name defaults, the realized variance cannot be calculated. Both issues are inconsequential for major indices like the SPX. Consequently we assume that the underlying is an index for all volatility derivatives. In the South African market index options for several strikes are regrettably illiquid. Missing points are filled using a polling method, but as these implied volatilities are not derived from traded options, ‘traders are not putting their money where their mouths are’, as noted by West [3]. In addition many options are traded OTC and this real information is not always published. Interpolation and extrapolation of the volatility surface for points that are not available is not advised and can cause serious pricing error [1]. This predicament is not the focus in this study, and unless otherwise stated we assume that an adequate number of options with various strikes and maturities are liquidly traded. We assume that the underlying asset and variance evolves without jumps and that there are no discrete cash flows (like discrete dividends) on the asset.

Some volatility derivatives are exchange-traded. The most well-known index is the CBOT VIX. The SAVI Top40 (South African Volatility Index) is a forecast of equity market risk in South Africa, published by the Johannesburg Stock Exchange. A variance future is the exchange traded version of an over-the-counter variance swap. The SAVI Squared is a variance future, and contract details are determined by the exchange. A Can-do variance future is customizable; an investor can negotiate the underlying index (ALSI, TOP40 or SWIX) and expiry date. As exchange traded volatility derivatives are following specific valuation methods specified by an exchange, it is excluded from this minor study.

3.1 Variance swap

A **variance swap** is a forward contract on the realized annualized variance of the specified underlying’s price returns. The OTC contract specifies the observation period and observation frequency (which is typically every business day when the market closes). At expiry, the party that is

long the variance swap receives N currency units for every percentage point that the annualized realized variance $V_{0,T}^R$, accrued over the interval $(0, T)$, is above the strike variance V^S :

$$N(V_{0,T}^R - V^S) \quad (36)$$

At inception the strike variance is chosen such that the variance swap value X_0 is zero, and the strike variance V^S then equals the fair variance (the risk-neutral expectation of $V_{0,T}^R$).

$$X_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT} N(V_{0,T}^R - V^S)] = 0 \Rightarrow \mathbb{E}^{\mathbb{Q}}(V_{0,T}^R) = V^S \quad (37)$$

At time $t \leq T$ before expiry T the variance swap value X_t is displayed in equation (38) with $V_{0,t}^R$ the realized volatility over the interval $(0, t)$ and $\mathbb{E}^{\mathbb{Q}}(V_{t,T}^R) = V_{t,T}^{fair}$ the (fair) strike variance for a variance swap effective over the interval (t, T) . The vega of X_t is given in equation (39).

$$X_t = N \cdot e^{-r(T-t)} \left(\frac{t}{T} V_{0,t}^R + \frac{T-t}{T} V_{t,T}^{fair} - V^S \right) \quad (38)$$

$$\frac{\partial X_t}{\partial \sqrt{V_{t,T}^{fair}}} = N \cdot e^{-r(T-t)} \frac{T-t}{T} 2 \cdot \sqrt{V_{t,T}^{fair}} \quad \frac{\partial X_0}{\partial \sqrt{V^S}} = Vega\ notional = N \cdot e^{-rT} \cdot 2 \cdot \sqrt{V^S} \quad (39)$$

As the market prefers to think in terms of volatility - they are familiar dealing with implied volatility, and not variance - the market convention is to express the variance notional N in volatility terms, and therefore at inception the notional is calculated as $N = Vega\ notional / (2\sqrt{V^S})$. The realized variance is normally calculated as the annualized variance of log-returns for an ordered series of sampled asset prices [4]:

$$V_{0,T}^R = 100^2 \cdot A \cdot \sum_{i=0}^{I-1} \left[\ln \left(\frac{S_{i+1}}{S_i} \right) \right]^2 \quad A = \frac{B}{I} \quad (40)$$

The number of observation dates is denoted by I . The number of assumed observations per annum is B (business days per annum is around 252). B converts the variance to an annualized variance. S_i is the observed price for observation i . We notice that the realized variance is in effect a path-dependent derivative of the asset price.

Note that the formula for realized variance $V_{0,T}^R$ assumes that the sample mean log-return is zero. The daily mean log-return is typically almost zero (see table 1) and the effect on $V_{0,T}^R$ will be insignificant. The benefit is that $V_{0,T}^R$ is now conveniently additive, i.e. we are able to add the weighted realized and fair future variance as in equation (38). This definition also relates closely to the log-contract relation for continuously sampled variance.

A variance swap contract specifies a cap on $V_{0,T}^R$ to protect the variance seller against ridiculously high levels of realized volatility. Before 2007 the embedded cap optionality was specific to single equities and often regarded to be negligible in the pricing of the fair variance, but became significant after the realized variance for some equity returns actually exceeded the cap level. When the closing reference asset price approaches zero at some point during the sample period, the realized variance can become infinite:

$$\lim_{S_{i+1} \rightarrow 0} \left[\ln \left(\frac{S_{i+1}}{S_i} \right) \right]^2 \rightarrow \infty \quad (41)$$

Even though the chances of an index price approaching zero is low, a cap is specified as a safety measure. The cap introduces optionality into the pricing problem, and is typically set to $2.5^2 V^S$. The simulation pricing approach includes this optionality component in the pricing of the fair variance, while the Heston formula and log-contract relation below ignores the cap.

The Heston model assumes that the underlying asset price as well as the variance evolves continuously. In practice only discrete sampling is possible and introduces discretization error which increases as the time interval between sampling points deviates from continuous sampling. The theoretical definition of continuously sampled realized variance $V_{0,T}^C$ is the following integral and according to [2] a reasonable estimate for the daily sampled variance.

$$V_{0,T}^C = \frac{1}{T} \int_0^T v_s ds \quad (42)$$

The log-contract relation to continuously sampled variance is the most common approach used to price variance swaps, see section 3.1.3. The discretization error is of the third-order in daily returns when one replicates the discretely sampled variance swap with delta hedging a log-contract, see Annex 5.5. The approach is model-free in the sense that no assumptions are made about the drift and diffusion coefficient as long as the semi-martingale asset price process evolves without jumps. The fair variance then relies on a continuum of co-terminal market option prices. In 2008 some large moves in the underlying sampled prices accentuated cubed (third-order) and higher-order returns. In

addition not being able to trade in deep out-of-the money options spoils the otherwise robust hedging strategy [7].

In the sub-sections below we estimate the fair variance $V_{t,T}^{fair} = \mathbb{E}^{\mathbb{Q}}(V_{t,T}^C)$, and then substitute into equation (38) for the risk-neutral valuation for a variance swap.

3.1.1 The Heston formula

In Annex 5.3 we derive a formula for the fair continuously accrued average variance. We note that only the parameters in the drift term of the Heston variance semi-martingale process appears in the formula (long-term mean θ , the mean-reversion rate κ and the initial variance v_0). We do not expect the volatility of variance σ to affect the weighted average. Also, if the Heston formula depends on the volatility of variance, it does not make any sense that we can estimate the fair variance by replicating a portfolio of vanilla options, see section 3.1.3.

$$V_{0,T}^{Heston\ formula} = \mathbb{E}^{\mathbb{Q}}(V_{0,T}^C) = \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{T} \int_0^T v_t dt\right) = \theta + \frac{(v_0 - \theta)(1 - e^{-\kappa T})}{\kappa T} \quad (43)$$

3.1.2 Simulation

We can use a Monte Carlo simulation with the Milstein discretization of the Heston stochastic process to find the fair variance. The number of time steps is I , and the step size is Δt . In the code the step size is set to be daily, u the number of days in a year.

$$\Delta t = 1/u \quad I = u \cdot T \quad i = 1, \dots, I$$

$$\ln \frac{S_{i+1,j}}{S_{i,j}} = \left(r - q - \frac{1}{2} v_{i,j}\right) \Delta t + \sqrt{v_{i,j}} \sqrt{\Delta t} \left(\rho Z_{i+1,j}^0 + \sqrt{1 - \rho^2} Z_{i+1,j}^1\right) \quad (44)$$

For every simulation path $j = 1, \dots, J$ we calculate the realised variance $V_{0,T}^{C,j}$. When the realised variance for a given path exceeds the cap level, the realised variance is set equal to the cap level. We use a variance reduction technique, the control variate technique, to speed up the convergence. The Heston formula $V_{0,T}^{Heston\ formula}$ for the fair variance is an excellent control variate for the capped variance simulation; the simulated uncapped variance is highly correlated, by a factor λ , with the capped variance, as the cap is always set very high.

$$V_{0,T}^{C,j} = \sum_{i=0}^{I-1} \left[\ln \left(\frac{S_{i+1,j}}{S_{i,j}} \right) \right]^2 \quad \hat{V}_{0,T}^{C,j} = \min\{V_{0,T}^{C,j}, cap\} \quad cap = 2.5^2 V^S \quad (45)$$

$$V_{0,T}^{MC\ fair} = \frac{1}{J} \sum_{j=1}^J \{ \hat{V}_{0,T}^{C,j} - \lambda (V_{0,T}^{C,j} - V_{0,T}^{Heston\ formula}) \} \quad \lambda = corr(\hat{V}_{0,T}^C, V_{0,T}^C) \quad (46)$$

3.1.3 The log-contract relation

The log-contract relation, introduced by Neuberger [16], states that a continuously sampled realised variance can be replicated with a static short position in a log-contract, and a continuously rebalanced position in the underlying asset. The log-contract can be replicated by a static position in a portfolio of weighted co-terminal options on a continuous strike range from zero to infinity. The replication of the continuously sampled variance is not model dependent (no assumptions are required for the drift and diffusion term). The sole requirement is that the asset price semi-martingale process evolves continuously. As only discretely monitored variance swaps are possible in practice, we look at the discretization error in Annex 5.5. We assume that the asset price follows an Itô-process with W_t a Brownian motion with an associated filtration \mathcal{F} and where $\mu(t, S_t, \dots)$ and $\sqrt{v(t, S_t, \dots)}$ are adapted stochastic processes.

$$\frac{dS_t}{S_t} = \mu(t, S_t, \dots)dt + \sqrt{v(t, S_t, \dots)}dW_t \quad (47)$$

The theoretical definition of continuously sampled realized variance is the following integral.

$$V_{0,T}^C = \frac{1}{T} \int_0^T v(t, S_t, \dots)dt \quad (48)$$

Applying Itô's formula to $\ln S_t$, we obtain

$$d \ln S_t = \left(\mu(t, S_t, \dots) - \frac{1}{2} v(t, S_t, \dots) \right) dt + \sqrt{v(t, S_t, \dots)} dW_t \quad (49)$$

The quadratic variation of this Itô-process is

$$[\ln S, \ln S]_T = \int_0^T v(t, S_t, \dots)dt \quad (50)$$

We notice that realized variance is the reciprocal of time, times the quadratic variation accumulated by the natural logarithm of the asset. Subtracting equation (49) from (47)

$$\frac{dS_t}{S_t} - d \ln S_t = \frac{1}{2} v(t, S_t, \dots) dt \quad (51)$$

We write this in integral format, and substitute the right hand side with equation (48)

$$\int_0^T \frac{dS_t}{S_t} - \int_0^T d \ln S_t = \frac{1}{2} \int_0^T v(t, S_t, \dots) dt = \frac{T}{2} V_{0,T}^C \quad (52)$$

$$V_{0,T}^C = \frac{2}{T} \left[\int_0^T \frac{dS_t}{S_t} - \int_0^T d \ln S_t \right] \quad (53)$$

To replicate $V_{0,T}^C$ one can take a static short position in a log-contract, and a continuously rebalanced position (depending on one over the current asset price) in the underlying asset from inception until maturity.

To estimate the fair variance, we take the risk-neutral expectation. At this step we introduce model dependency; substitute the risk-neutral dynamics assumed in (1) for $\frac{dS_t}{S_t}$. We assume that r , the risk-free rate of the money market account is deterministic.

$$\mathbb{E}^{\mathbb{Q}}(V_{0,T}^C) = \frac{2}{T} \mathbb{E}^{\mathbb{Q}} \left[\int_0^T (r dt + \sqrt{v(t, S_t, \dots)} dW_t^{\mathbb{Q}}) - (\ln S_T - \ln S_0) \right] \quad (54)$$

$$\mathbb{E}^{\mathbb{Q}}(V_{0,T}^C) = \frac{2}{T} \left[rT - \mathbb{E}^{\mathbb{Q}} \left(\ln \frac{S_T}{S_*} + \ln \frac{S_*}{S_0} \right) \right] = \frac{2}{T} \left[rT + \mathbb{E}^{\mathbb{Q}} \left(- \ln \frac{S_T}{S_*} \right) + \ln \frac{S_*}{S_0} \right] \quad (55)$$

Where S_* is an arbitrary constant. As log-contracts are rarely traded, we have to piecewise replicate the log-contract payoff with linear (forward contract) and curved (option) parts. For some arbitrary constant $F_T > 0$, which we assign to be the forward price, a twice differential payoff function $f(S_T)$ can be written as, see Annex 5.4,

$$f(S_T) - f(F_T) = f'(F_T)(S_T - F_T) + \int_0^{F_T} f''(x)(x - S_T)^+ dx + \int_{F_T}^{\infty} f''(x)(S_T - x)^+ dx \quad (56)$$

If we let $f(S_T) = - \ln \frac{S_T}{F_T} = - \ln S_T + \ln F_T$ then

$$-\ln \frac{S_T}{F_T} - \left(-\ln \frac{F_T}{F_T}\right) = \frac{-1}{F_T}(S_T - F_T) + \int_0^{F_T} \frac{1}{x^2}(x - S_T)^+ dx + \int_{F_T}^{\infty} \frac{1}{x^2}(S_T - x)^+ dx \quad (57)$$

We choose S_* to be the forward price F_T , so that S_* is the at-the-money strike that divides the out-of-the-money put and call strike prices⁷.

$$-\ln \frac{S_T}{S_*} = \frac{-(S_T - S_*)}{S_*} + \int_0^{S_*} \frac{1}{x^2}(x - S_T)^+ dx + \int_{S_*}^{\infty} \frac{1}{x^2}(S_T - x)^+ dx \quad (58)$$

The first term is a $\frac{1}{S_*}$ holding in a short forward contract on the underlying with strike S_* , the second a long position in $\frac{1}{x^2}$ put options for a continuum of strikes between 0 and S_* , the third a long position in $\frac{1}{x^2}$ call options for a continuum of strikes between S_* and ∞ . Using equation (55) and substituting equation (58)

$$\mathbb{E}^{\mathbb{Q}}(V_{0,T}^C) = \frac{2}{T} \left[rT + \ln \frac{S_*}{S_0} + \mathbb{E}^{\mathbb{Q}} \left(\frac{-(S_T - S_*)}{S_*} + \int_0^{S_*} \frac{(x - S_T)^+}{x^2} dx + \int_{S_*}^{\infty} \frac{(S_T - x)^+}{x^2} dx \right) \right] \quad (59)$$

$$\mathbb{E}^{\mathbb{Q}}(V_{0,T}^C) = \frac{2}{T} \left[rT + \ln \frac{S_*}{S_0} - \frac{(S_0 e^{rT} - S_*)}{S_*} + \mathbb{E}^{\mathbb{Q}} \left(\int_0^{S_*} \frac{(x - S_T)^+}{x^2} dx + \int_{S_*}^{\infty} \frac{(S_T - x)^+}{x^2} dx \right) \right] \quad (60)$$

In theory we have established a neat connection between the market prices of options and the variance swap payoff. We have not assumed that instantaneous volatility is deterministic, so this relationship holds in the presence of the volatility skew and so in a stochastic volatility framework, including the Heston model. As options for all possible strike levels are unattainable in the market, we have to discretize the integral and cut off the ‘infinite sum’ at realistic outer strike levels. Using (58) to substitute into (60) to obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}(V_{0,T}^C) &= \frac{2}{T} \left[rT + \ln \frac{S_*}{S_0} - \frac{(S_0 e^{rT} - S_*)}{S_*} + \mathbb{E}^{\mathbb{Q}} \left(-\ln \frac{S_T}{S_*} + \frac{(S_T - S_*)}{S_*} \right) \right] \\ &= \frac{2}{T} \left[rT + \ln \frac{S_*}{S_0} - \frac{(S_0 e^{rT} - S_*)}{S_*} \right] + \mathbb{E}^{\mathbb{Q}} g(S_T) \end{aligned} \quad (61)$$

⁷ We want to replicate the log-contract with the most liquid options traded, i.e. the out-of-the-money puts and calls, and these two groups are divided by the at-the-money strike, i.e. the forward price.

$$\mathbb{E}^{\mathbb{Q}} g(S_T) = \mathbb{E}^{\mathbb{Q}} \frac{2}{T} \left[-\ln \frac{S_T}{S_*} + \frac{(S_T - S_*)}{S_*} \right]$$

We will replicate $g(S_T)$ with a discrete set of tradable option strikes, i.e. L number of puts $S_* = x_{0P} > x_{1P} > \dots > x_{LP}$ and Y calls $S_* = x_{0C} < x_{1C} < \dots < x_{YC}$. For the first call strike partition from $[x_{0C}, x_{1C}]$, we approximate $g(S_T)$ with a call option with strike x_{0C} , with a position size equal to the gradient

$$w_C(x_{0C}) = \frac{g(x_{1C}) - g(x_{0C})}{x_{1C} - x_{0C}} \quad (62)$$

Since we already hold $w_C(x_{0C})$ call options, for the second partition from $[x_{1C}, x_{2C}]$ we hold

$$\begin{aligned} w_C(x_{1C}) &= \frac{g(x_{2C}) - g(x_{1C})}{x_{2C} - x_{1C}} - w_C(x_{0C}) \\ w_C(x_{nC}) &= \frac{g(x_{(n+1)C}) - g(x_{nC})}{x_{(n+1)C} - x_{nC}} - \sum_{i=0}^{n-1} w_C(x_{iC}) \end{aligned} \quad (63)$$

Reasoning in the same way the put option weights are

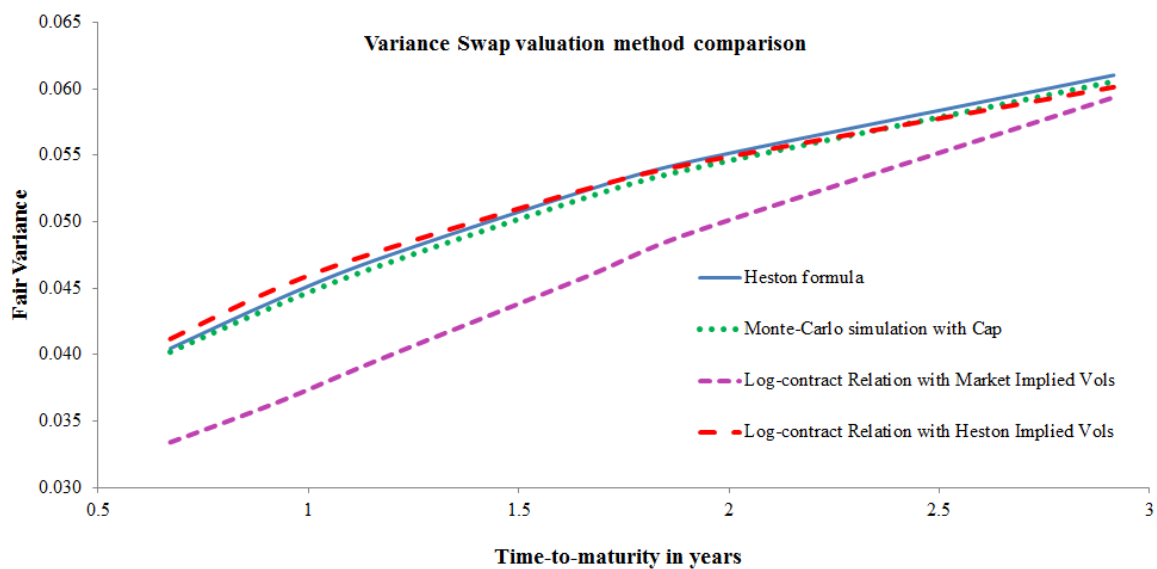
$$w_P(x_{mP}) = \frac{g(x_{(n+1)P}) - g(x_{nP})}{x_{nP} - x_{(n+1)P}} - \sum_{i=0}^{n-1} w_P(x_{iP}) \quad (64)$$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[g(S_T)] &= \sum_{i=0}^Y w_C(x_{iC}) \mathbb{E}^{\mathbb{Q}}[(S_T - x_{iC})^+] + \sum_{i=0}^L w_P(x_{iP}) \mathbb{E}^{\mathbb{Q}}[(x_{iP} - S_T)^+] \\ \mathbb{E}^{\mathbb{Q}}[g(S_T)] &= e^{rT} \left[\sum_{i=0}^Y w_C(x_{iC}) \mathbb{E}^{\mathbb{Q}}[e^{-rT}(S_T - x_{iC})^+] + \sum_{i=0}^L w_P(x_{iP}) \mathbb{E}^{\mathbb{Q}}[e^{-rT}(x_{iP} - S_T)^+] \right] \\ \mathbb{E}^{\mathbb{Q}}[g(S_T)] &= e^{rT} \left[\sum_{i=0}^Y w_C(x_{iC}) C_M(x_{iC}) + \sum_{i=0}^L w_P(x_{iP}) P_M(x_{iP}) \right] \end{aligned} \quad (65)$$

Where $C_M(x_{iC})$ and $P_M(x_{iP})$ are the market prices for the given strike levels (usually the Black-Scholes prices consistent with the market's implied volatilities for the given strikes). So the fair variance relies on the cost in the market of the approximate hedging strategy of a log-contract.

In figure 8 the three methods described above are compared. In the limit the Heston formula should agree very closely with the uncapped simulation result. The Heston formula does not incorporate the cap effect, and is therefore slightly higher than the simulated capped fair variance. The log-contract relation method using market implied volatilities differs significantly with the rest of the valuation approaches. To determine if the reason for this difference is related to the calibration error, we apply the same method to the Heston implied volatility surface which we calculated after the calibration to access the calibration success. The log-contract relation methodology applied to the Heston surface is significantly closer to the Heston formula, and we deduce that the difference is indeed mainly due to the calibration error; for the methods to concur, we need a very good calibration outcome.

Figure 8 – Variance swap valuation method comparison with Heston parameters as given in table 2. $S_0 = 33,740$, $r = 5.19\%$ and $q = 0.22\%$. The number of simulation paths is 100,000. Cap multiplier is 2.5



3.2 Volatility swap

A **volatility swap** is a forward contract on the realized annualized volatility of an underlying asset's price returns. At expiry, the volatility swap long position holder receives N_W currency units for every percentage point that the annualized realized volatility $W_{0,T}^R = \sqrt{V_{0,T}^R}$, accrued from $(0, T)$ is above the strike volatility W^S :

$$N_W(W_{0,T}^R - W^S) \tag{66}$$

At inception the strike volatility is chosen such that the volatility swap value \tilde{X}_0 is zero, and the strike variance then equals the fair variance (risk-neutral expectation of realized volatility).

$$\tilde{X}_0 = \mathbb{E}^{\mathbb{Q}}[e^{-rT} N_W(W_{0,T}^R - W^S)] = 0 \Rightarrow \mathbb{E}^{\mathbb{Q}}(W_{0,T}^R) = W^S \quad (67)$$

Although realised volatility is simply the square root of realised variance, the expectation of realised volatility is always lower than or equal to the square root of the expectation of realised variance. The square root function is a concave function, and by Jensen's inequality we have

$$\mathbb{E}(W_{0,T}^R) = \mathbb{E}\left(\sqrt{V_{0,T}^R}\right) \leq \sqrt{\mathbb{E}(V_{0,T}^R)} \quad (68)$$

In other words, we always expect the fair strike volatility for a volatility swap to be lower than the square root of the fair variance over the same future time period and underlying. The difference between the square root of the fair variance strike and fair volatility strike is called the convexity correction. Some formulas exist to approximate the convexity value, but as the approximation in the Heston framework is not very accurate [9] we rather turn to other valuation methods.

3.2.1 Simulation

We can use a Monte Carlo simulation with the Milstein discretization of the Heston stochastic process to find the fair volatility. The number of time steps is I , and the step size is Δt . In the code we choose I such that the step size is daily, u is the number of days in a year, so that even when T is large the time step size is controlled and stays small.

$$\Delta t = 1/u \quad I = u \cdot T \quad i = 1, \dots, I$$

$$\ln \frac{S_{i+1,j}}{S_{i,j}} = \left(r - q - \frac{1}{2} v_{i,j}\right) \Delta t + \sqrt{v_{i,j}} \sqrt{\Delta t} \left(\rho Z_{i+1,j}^0 + \sqrt{1 - \rho^2} Z_{i+1,j}^1\right) \quad (69)$$

For every simulation path $j = 1, \dots, J$ we calculate the realised volatility $W_{0,T}^{C,j}$. Again we use $V_{0,T}^{Heston \text{ formula}}$ as a control variate. The valuation of the volatility swap is then given by \tilde{X}_0^{MC} .

$$\widehat{W}_{0,T}^{C,j} = \min\{W_{0,T}^{C,j}, cap^{vol}\} \quad cap^{vol} = 2.5 W^S$$

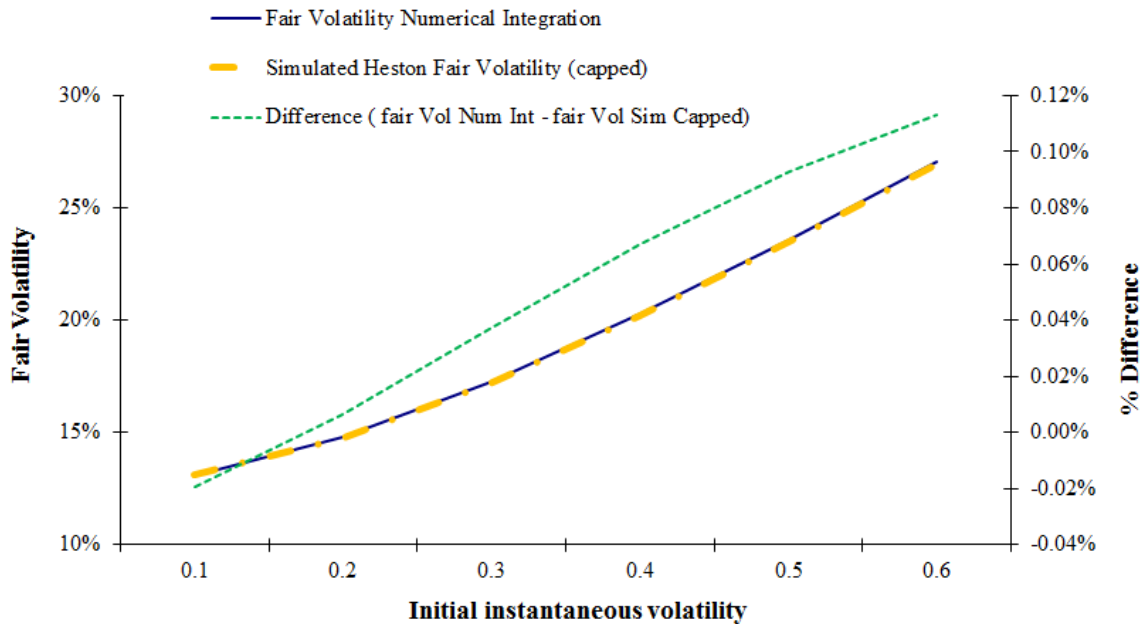
$$W_{0,T}^{C,j} = \sqrt{V_{0,T}^{C,j}} = \sqrt{100^2 \cdot A \cdot \sum_{i=0}^{I-1} \left[\ln \left(\frac{S_{i+1,j}}{S_{i,j}} \right) \right]^2} \quad (70)$$

$$W_{0,T}^{MC\ fair} = \frac{1}{J} \sum_{j=1}^J \{ \widehat{W}_{0,T}^{C,j} - \lambda (V_{0,T}^{C,j} - V_{0,T}^{Heston\ formula}) \} \quad \lambda = corr(\widehat{W}_{0,T}^{C,j}, V_{0,T}^{C,j}) \quad (71)$$

$$\tilde{X}_0^{MC} = e^{-rT} N_W(W_{0,T}^{MC\ fair} - W^S) \quad (72)$$

In figure 9 we compare the simulated fair volatility that includes the cap with the fair volatility calculated with numerical integration. Note that we use Heston parameters for the SPX from [9]. The difference is always below 0.2%. We expect capped fair volatility to always be below the uncapped fair volatility with numerical integration. This is not the case for $\sqrt{v_0} = 0.1$, and is possibly a result of the difference in numerical procedures which seems to dominate the cap effect. The simulated fair volatility has converged: the 100k simulations differs less than a third of a basis point compared to 1000k simulations. The cap effect becomes more prominent with a higher initial variance. The fair volatility is an increasing function of initial instantaneous volatility.

Figure 9 - Fair volatility calculated with numerical integration, compared to the capped fair volatility calculated with simulation (100k simulation paths). Heston parameters are as in [9] $\kappa = 6.21$, $\theta = 0.019$, $\sigma = 0.31$, $\rho = -0.7$. We have $r = 3.19\%$, $q = 0\%$, $T = 1$.



3.2.2 The Heston fair volatility with numerical integration

The Heston fair volatility can be calculated via numerical integration using a relationship between the expected continuously accumulated volatility and an integral containing the Laplace transform of the continuously accumulated instantaneous variance [1]:

$$\mathbb{E}^{\mathbb{Q}}(W_{0,T}^C) = \mathbb{E}^{\mathbb{Q}}\left(\sqrt{V_{0,T}^C}\right) = \frac{1}{2\sqrt{\pi}} \int_0^{\infty} \frac{1 - \mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\varphi \int_0^T v_t dt\right)\right)}{\varphi^{3/2}} d\varphi \quad (73)$$

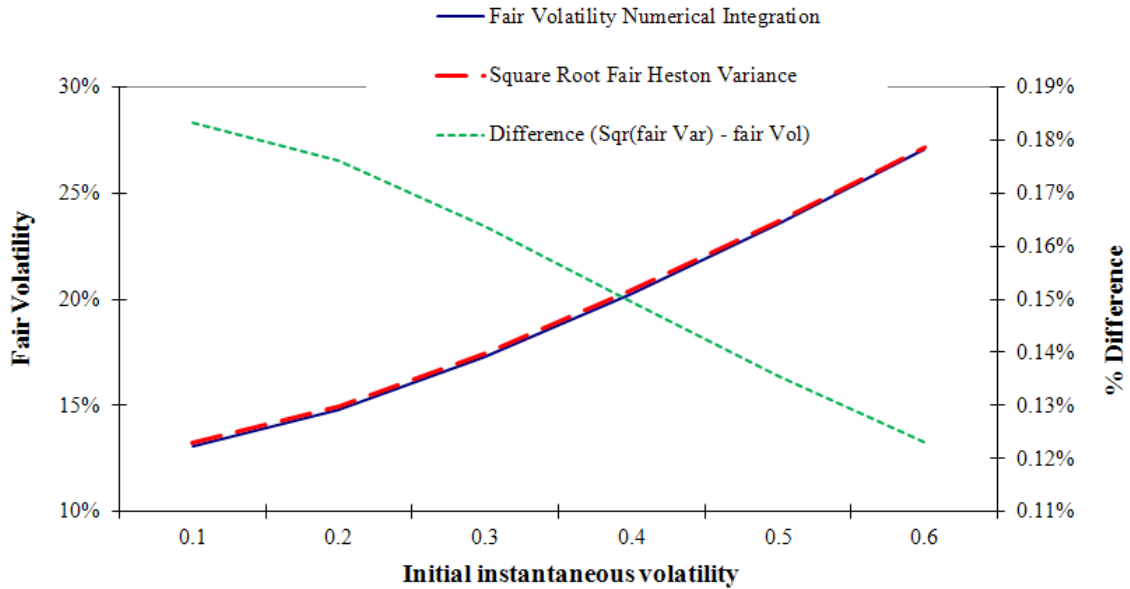
Knowing that the Laplace transform of the continuously accumulated variance is identical to the value of a zero-coupon bond in the CIR interest rate model when the short rate is now instantaneous variance v_t , we have the following

$$\mathbb{E}^{\mathbb{Q}}\left(\exp\left(-\varphi \int_0^T v_t dt\right)\right) = A^* e^{-\varphi v_0 B^*} \quad (74)$$

$$A^* = \left\{ \frac{2\varnothing e^{(\varnothing+\kappa)T/2}}{(\varnothing+\kappa)(e^{\varnothing T} - 1) + 2\varnothing} \right\}^{2\kappa\vartheta/\sigma^2} \quad B^* = \frac{2(e^{\varnothing T} - 1)}{(\varnothing+\kappa)(e^{\varnothing T} - 1) + 2\varnothing} \quad \varnothing = \sqrt{\kappa^2 + 2\varphi\sigma^2} \quad (75)$$

In figure 10 the difference between the square root of the fair variance and the fair volatility is a decreasing function of initial instantaneous volatility, and demonstrates Jensen's inequality.

Figure 10 - Fair volatility calculated with numerical integration, compared to the square root of the fair Heston variance, to demonstrate Jensen's inequality. Heston parameters are as in [9] $\kappa = 6.21$, $\theta = 0.019$, $\sigma = 0.31$, $\rho = -0.7$. We have $r = 3.19\%$, $q = 0\%$, $T = 1$.



3.3 Variance option

At maturity, the variance call option holder receives the greater of zero and the difference between realised variance and the strike variance, and vice versa for a variance put.

$$call\ payoff = N(V_{0,T}^C - V^S)^+ \quad put\ payoff = N(V^S - V_{0,T}^C)^+ \quad (76)$$

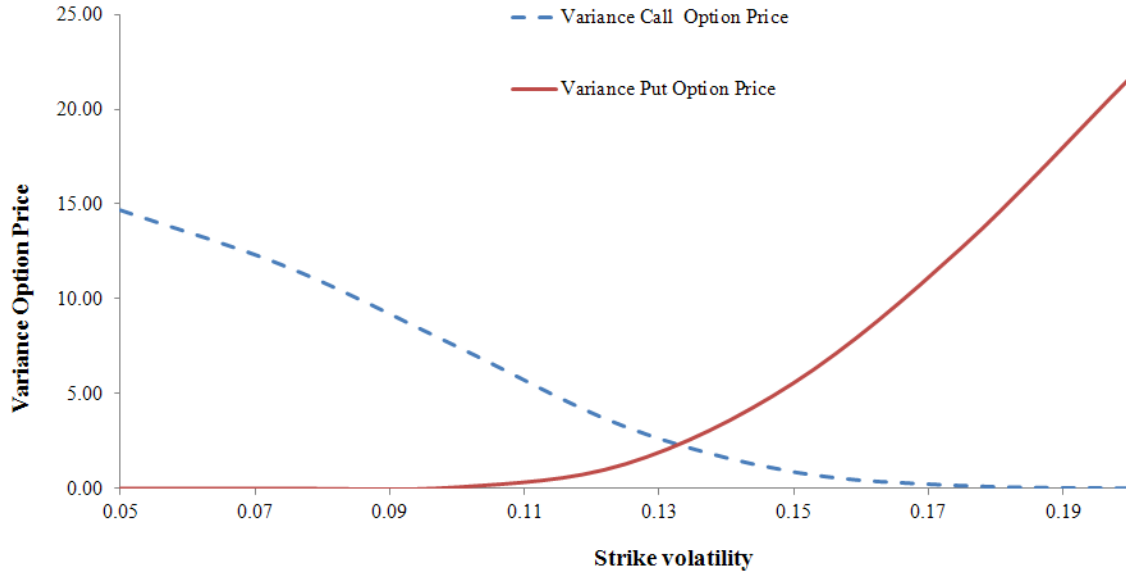
3.3.1 Simulation

In exactly the same way as for the variance swap, we can calculate for every simulation path $j = 1, \dots, J$ the simulated realised variance $V_{0,T}^{C,j}$. We then calculate the average of all the simulated payoffs for a variance option with strike V^S . The valuation is then given by Y_0^{MC} .

$$Y_0^{MC}(call) = e^{-rT} N \left(\frac{1}{J} \sum_{j=1}^J (V_{0,T}^{C,j} - V^S)^+ \right) \quad Y_0^{MC}(put) = e^{-rT} N \left(\frac{1}{J} \sum_{j=1}^J (V^S - V_{0,T}^{C,j})^+ \right) \quad (76)$$

Figure 11- Variance option prices for various strike variances. Heston parameters are as in [9]

$v_0 = 0.101^2$, $\kappa = 6.21$, $\theta = 0.019$, $\sigma = 0.31$, $\rho = -0.7$. We have $r = 3.19\%$, $q = 0\%$, $T = 1.5$.



3.4 Timer option

A timer option allows an option buyer to specify the volatility input for an option premium, while he accepts that the option maturity date is now random. When a buyer speculates that volatility will realize well below current implied volatility levels, he can pay less for an option by specifying the volatility, for a target maturity, below the current implied volatility level. The payoff for a timer (or mileage) option is identical to a vanilla option, but instead of a contract specified fixed maturity date, the payoff date is random, and depends on when the variance budget is consumed. The variance budget \mathbb{V} is the target volatility squared times the target time-to-maturity in years, specified by the option buyer. When the realized variance equals (or for the first time exceeds) the variance budget, the timer option matures. Within the continuous time Heston framework the random stopping time τ^* is the first time the quadratic variation equals the variance budget. The option payoff occurs at the stopping time.

$$\tau^* = \inf \left\{ s > 0, \int_0^s v_t dt = \mathbb{V} \right\} \quad (77)$$

$$call^{timer} = \mathbb{E}^{\mathbb{Q}}(e^{-r\tau^*} (S_{\tau^*} - K)^+) \quad put^{timer} = \mathbb{E}^{\mathbb{Q}}(e^{-r\tau^*} (K - S_{\tau^*})^+) \quad (78)$$

3.4.1 Simulation

Within the Heston framework the joint distribution of (τ^*, v_{τ^*}) is given by, where X_s is a Bessel process and W_s a standard Brownian motion [18],

$$(\tau^*, v_{\tau^*}) \sim \left(\int_0^{\mathbb{V}} \frac{1}{X_s} ds, X_{\mathbb{V}} \right) \quad dX_s = \left(\frac{\kappa\theta}{X_s} - \kappa \right) ds + \sigma dW_s \quad X_0 = v_0 \quad (79)$$

The joint distribution can be approximated, and the Bessel process can be simulated using an Euler scheme, where n is the number of steps until the variance budget has been consumed

$$(\tau^*, v_{\tau^*}) \approx \left(\sum_{i=1}^n \frac{\Delta s}{X_{i\Delta s}}, X_{n\Delta s} \right) \quad (80)$$

$$X_{i\Delta s} = X_{(i-1)\Delta s} + \left(\frac{\kappa\theta}{X_{(i-1)\Delta s}} - \kappa \right) \Delta s + \sigma W_{i\Delta s} \quad W_{i\Delta s} \sim N(0, \Delta s) \quad \Delta s = \frac{\mathbb{V}}{n} \quad (81)$$

We can then simulate the joint distribution.

$$C^{timer MC} = S_0 \mathbb{E}^{\mathbb{Q}} \left(e^{\rho c(\tau^*, v_{\tau^*}) - \frac{1}{2} \rho^2 \mathbb{V}} \phi(d_1(\tau^*, v_{\tau^*})) \right) - K \mathbb{E}^{\mathbb{Q}} \left(e^{-r\tau^*} \phi(d_2(\tau^*, v_{\tau^*})) \right) \quad (82)$$

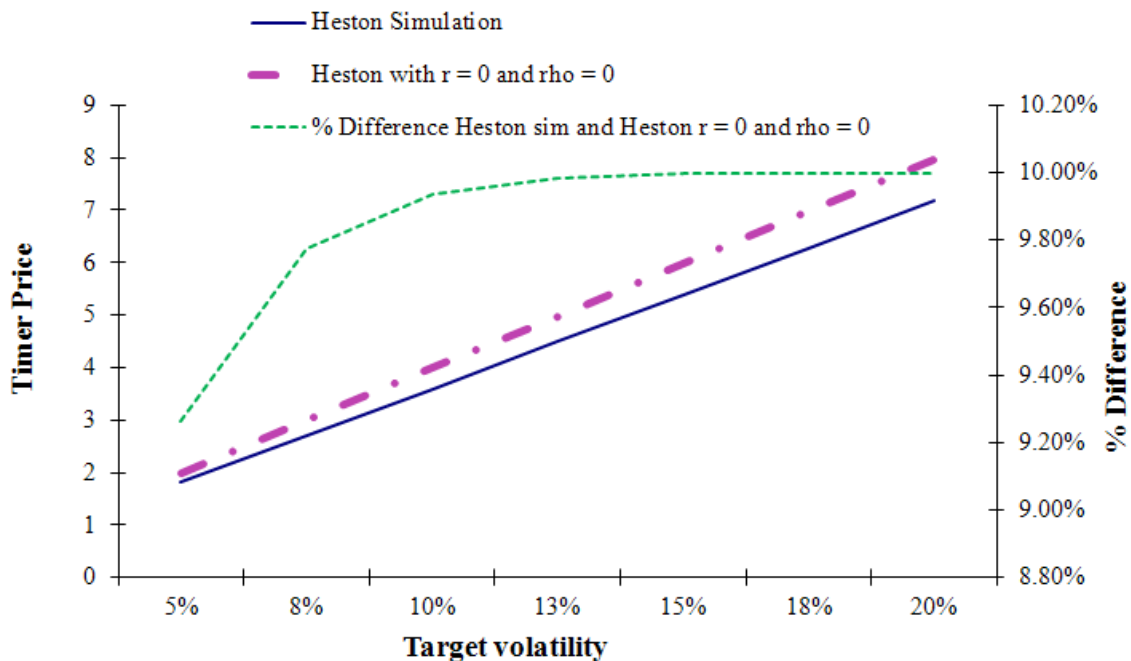
$$d_1(\tau^*, v_{\tau^*}) = \frac{\ln \frac{S_0}{K} + r\tau^* + \rho c(\tau^*, v_{\tau^*}) + \left(\frac{1}{2} - \rho^2 \right) \mathbb{V}}{\sqrt{(1 - \rho^2) \mathbb{V}}} \quad (83)$$

$$d_2(\tau^*, v_{\tau^*}) = d_1(\tau^*, v_{\tau^*}) - \sqrt{(1 - \rho^2) \mathbb{V}} \quad c(\tau^*, v_{\tau^*}) = \frac{v_{\tau^*} - v_0 - \kappa\theta\tau^* + \kappa\mathbb{V}}{\sigma} \quad (84)$$

When $\rho = 0$ and $r = 0$, the timer price does not depend on any stochastic variable. This value can be used as a control variate to acquire variance reduction. Specify $\lambda = \text{corr}(C^{timer MC}, C_{r=0, \rho=0}^{timer MC})$:

$$C_{variance\ reduction}^{timer} = C^{timer MC} - \lambda (C_{r=0, \rho=0}^{timer MC} - C_{r=0, \rho=0}^{timer\ closed\ form}) \quad (85)$$

Figure 12 - Timer option prices comparison with increasing target volatility. The target time-to-maturity is throughout 1 year and $S_0 = 100$. For the solid blue line the Heston parameters are as in [9] $v_0 = 0.101^2$, $\kappa = 6.21$, $\theta = 0.019$, $\sigma = 0.31$, $\rho = -0.7$, with $r = 3.19$, , and the number of simulation paths is 100,000. For the purple line: $\rho = 0, r = 0$.



In figure 12 we see that the timer option is an increasing function of target volatility. When the stopping time occurs, a bigger target volatility means that the option is expected to be more in-the-money when in-the-money. A longer target time-to-maturity means the option is expected to have more time value.

4. Conclusion

In the preceding sections we explored how volatility derivatives can be valued in the Heston model. We started with the rationale behind volatility derivatives; volatility is stochastic and market players may want to protect against (or bet on) future realized variance levels. A model is needed where both instantaneous variance and the underlying asset price are stochastic state variables, for example the Heston model. The quasi-closed form Heston call price formula is more straightforward to calibrate compared to a model where the call price can only result from a simulation or another numerical procedure. The Heston model only requires numerical integration to arrive at the pseudo-probabilities for a call price. The Heston model resembles the Black-Scholes model looking at the SDE for the

asset price and the call price pseudo-probabilities. This is desirable because the Black-Scholes is the market model for vanilla options, and it therefore supports the intention to reconcile a theoretical model with market practice. On the downside the shape of the Heston embedded implied volatility surface is often not very realistic, especially on the short-end [1]. Calibration can be tough for certain volatility surface shapes and levels, and a good fit of Heston parameters requires some art of optimization. To improve the fit of Heston parameters one can consider looking into time-dependent parameters or jump-diffusion models.

After calibration the risk-neutral valuation of some volatility derivatives within the Heston framework is then possible. It is ideal if one can find a closed-form solution for any derivative's price that can be statically or dynamically hedged by liquid tradable assets. Variance is not a tradable asset but the simplest volatility derivative, the variance swap, has a closed-form approximation via the log-contract relation. The log-contract has a twice differentiable payoff that can be replicated with a strip of static positions in vanilla call and put options and a dynamic position in the underlying. The variance swap in the volatility derivative world plays a similar role as the zero-coupon bond in the interest rate derivative market; it is the simplest derivative of an untradeable market variable that can be used to hedge more complicated derivatives.

Within the Heston framework the fair variance has a formula in terms of Heston parameters. The fair variance can also be computed via a Monte Carlo simulation of the log-returns using a Milstein discretization method. In a similar way the fair volatility can be estimated with a simulation, but keeping in mind that we calculate the simulated realised volatility for each simulated log-return path before averaging. The fair volatility can alternatively be calculated using a numerical integration utilising the Heston parameters. A variance call or put price follows from recording the simulated realised variance for each path and calculating the option payoff, and then averaging over the number of simulation paths. The timer option value is calculated by simulating a Bessel process. Within the Heston model partial differential equations for volatility derivatives can with relative ease be derived and solved by 3-dimensional finite difference methods (omitted in this study) [9].

In theory, a stochastic volatility model like the Heston model can be used to price almost any volatility derivative, but we have to remember that models have various assumptions in the construction that may be a bit lacking compared to realism. In practice when an asset price is known to jump, the variance will certainly be higher than when jumps do not occur. When a stochastic process allows for jumps, equation (1) will only represent the continuous contribution to the total variance. This study does not incorporate the effect of discrete dividends or when jumps are assumed to occur in the underlying asset price evolution. Discretization methods used to simulate stochastic variables introduce some error in approximating the continuous stochastic variable evolution.

Numerical procedures in the complex plane can often introduce glitches [21]. Numerical integration approximates an integral that does not have an analytical solution. The replicating options portfolio of a twice differential payoff is an integral over an infinite number of a continuous range of strikes; only partially attainable in practice. Hedging can also be problematic. Being exposed to sizable derivatives that cannot be hedged by tradable and preferably liquid market instruments is daring. Presently we only know how a variance swap can be approximately hedged in a manner consistent with the relevant option market. Variance swaps can in theory be used to hedge other volatility derivatives, but South African index variance swaps are not currently considered particularly liquid instruments. To accurately price volatility derivatives we preferably need a dense volatility surface for a relatively wide range of strike levels, while in reality we often have only a few bid-offer spreads per maturity [1]. Without a representative volatility surface we cannot accurately calibrate the Heston parameters, or we have insufficient strike levels to calculate the fair variance value via the log-contract replicating portfolio method.

Volatility derivatives surely make sense, and the pricing thereof is theoretically possible, and often quite elegantly in the Heston framework.

5. Annex

5.1 Garman's PDE

We setup a portfolio π with a one unit position in an option Λ on an underlying asset S , a $-\Delta$ position in S and $-\nabla$ position in a volatility derivative Γ [1]:

$$\pi = \Lambda - \Delta S - \nabla \Gamma \quad (5.1)$$

We apply Ito's formula to find the change in the portfolio value an infinitesimal time later.

$$\begin{aligned} d\pi = & \left[\frac{\partial \Lambda}{\partial t} + \frac{\partial \Lambda}{\partial S} dS + \frac{\partial \Lambda}{\partial v} dv + \left(\frac{1}{2} v S^2 \frac{\partial^2 \Lambda}{\partial S^2} + \rho \sigma v S \frac{\partial^2 \Lambda}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 \Lambda}{\partial v^2} \right) dt \right] - \Delta [dS] \\ & - \nabla \left[\frac{\partial \Gamma}{\partial t} + \frac{\partial \Gamma}{\partial v} dv + \frac{\partial \Gamma}{\partial S} dS \right. \\ & \left. + \left(\frac{1}{2} v S^2 \frac{\partial^2 \Gamma}{\partial S^2} + \rho \sigma v S \frac{\partial^2 \Gamma}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 \Gamma}{\partial v^2} \right) dt \right] \end{aligned}$$

$$\begin{aligned}
d\pi &= \left[\frac{\partial \Lambda}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 \Lambda}{\partial S^2} + \rho \sigma v S \frac{\partial^2 \Lambda}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 \Lambda}{\partial v^2} \right] dt \\
&\quad - \nabla \left[\frac{\partial \Gamma}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 \Gamma}{\partial S^2} + \rho \sigma v S \frac{\partial^2 \Gamma}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 \Gamma}{\partial v^2} \right] dt + \left[\frac{\partial \Lambda}{\partial S} - \nabla \frac{\partial \Gamma}{\partial S} - \Delta \right] dS \\
&\quad + \left[\frac{\partial \Lambda}{\partial v} - \nabla \frac{\partial \Gamma}{\partial v} \right] dv \\
d\pi &= \left[\frac{\partial \Lambda}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 \Lambda}{\partial S^2} + \rho \sigma v S \frac{\partial^2 \Lambda}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 \Lambda}{\partial v^2} \right] dt \\
&\quad - \nabla \left[\frac{\partial \Gamma}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 \Gamma}{\partial S^2} + \rho \sigma v S \frac{\partial^2 \Gamma}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 \Gamma}{\partial v^2} \right] dt \\
&\quad + \left[\frac{\partial \Lambda}{\partial S} - \nabla \frac{\partial \Gamma}{\partial S} - \Delta \right] (S_t(r - q)dt + S_t \sqrt{v_t} dW_t^1) \\
&\quad + \left[\frac{\partial \Lambda}{\partial v} - \nabla \frac{\partial \Gamma}{\partial v} \right] (\kappa(\theta - v_t) dt + \sigma \sqrt{v_t} dW_t^0) \tag{5.2}
\end{aligned}$$

To make the portfolio instantaneously risk-free, we choose Δ and ∇ such that the random term coefficients become zero. A side-effect is that the drift terms $S_t(r - q)dt$ and $\kappa(\theta - v_t) dt$ also vanishes. By making this choice for Δ and ∇ the portfolio is instantaneously delta-hedged and vega-hedged.

$$\left[\frac{\partial \Lambda}{\partial S} - \nabla \frac{\partial \Gamma}{\partial S} - \Delta \right] = 0 \quad \left[\frac{\partial \Lambda}{\partial v} - \nabla \frac{\partial \Gamma}{\partial v} \right] = 0 \quad \nabla = \frac{\left(\frac{\partial \Lambda}{\partial v} \right)}{\left(\frac{\partial \Gamma}{\partial v} \right)} \quad \Delta = \frac{\partial \Lambda}{\partial S} - \frac{\left(\frac{\partial \Lambda}{\partial v} \right)}{\left(\frac{\partial \Gamma}{\partial v} \right)} \frac{\partial \Gamma}{\partial S} \tag{5.3}$$

As the portfolio is instantaneously risk-free, it has to earn the risk-free rate of return to prevent arbitrage

$$\begin{aligned}
r\pi dt &= r(\Lambda - \Delta S - \nabla \Gamma) dt = \\
&= \left[\frac{\partial \Lambda}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 \Lambda}{\partial S^2} + \rho \sigma v S \frac{\partial^2 \Lambda}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 \Lambda}{\partial v^2} \right] dt \\
&\quad - \nabla \left[\frac{\partial \Gamma}{\partial t} + \frac{1}{2} v S^2 \frac{\partial^2 \Gamma}{\partial S^2} + \rho \sigma v S \frac{\partial^2 \Gamma}{\partial v \partial S} + \frac{1}{2} \sigma^2 v \frac{\partial^2 \Gamma}{\partial v^2} \right] dt
\end{aligned}$$

$$\begin{aligned}
& \left(r\Lambda - \left(\frac{\partial\Lambda}{\partial S} - \frac{\left(\frac{\partial\Lambda}{\partial v}\right)\frac{\partial\Gamma}{\partial S}}{\left(\frac{\partial\Gamma}{\partial v}\right)} \right) rS - \frac{\left(\frac{\partial\Lambda}{\partial v}\right)}{\left(\frac{\partial\Gamma}{\partial v}\right)} r\Gamma \right) dt = \\
& = \left[\frac{\partial\Lambda}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2\Lambda}{\partial S^2} + \rho\sigma vS\frac{\partial^2\Lambda}{\partial v\partial S} + \frac{1}{2}\sigma^2v\frac{\partial^2\Lambda}{\partial v^2} \right] dt \\
& - \frac{\left(\frac{\partial\Lambda}{\partial v}\right)}{\left(\frac{\partial\Gamma}{\partial v}\right)} \left[\frac{\partial\Gamma}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2\Gamma}{\partial S^2} + \rho\sigma vS\frac{\partial^2\Gamma}{\partial v\partial S} + \frac{1}{2}\sigma^2v\frac{\partial^2\Gamma}{\partial v^2} \right] dt
\end{aligned}$$

We divide each term by $\left(\frac{\partial\Lambda}{\partial v}\right)$ and re-arrange

$$\begin{aligned}
& \frac{1}{\left(\frac{\partial\Lambda}{\partial v}\right)} \left[\frac{\partial\Gamma}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2\Gamma}{\partial S^2} + \rho\sigma vS\frac{\partial^2\Gamma}{\partial v\partial S} + \frac{1}{2}\sigma^2v\frac{\partial^2\Gamma}{\partial v^2} + rS\frac{\partial\Gamma}{\partial S} - r\Gamma \right] \\
& = \frac{1}{\left(\frac{\partial\Lambda}{\partial v}\right)} \left[\frac{\partial\Lambda}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2\Lambda}{\partial S^2} + \rho\sigma vS\frac{\partial^2\Lambda}{\partial v\partial S} + \frac{1}{2}\sigma^2v\frac{\partial^2\Lambda}{\partial v^2} + rS\frac{\partial\Lambda}{\partial S} - r\Lambda \right]
\end{aligned}$$

The right hand side is a function of the volatility derivative Γ exclusively, and the right hand side a function of Λ only. This means that both sides should be some arbitrary function f of the independent variables $f(t, S, v) = -(\kappa(\theta - v_t) - \Phi(t, S, v) \cdot \sigma\sqrt{v_t})$. We obtain the PDE for an option Λ where we assume stochastic volatility:

$$\frac{\partial\Lambda}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2\Lambda}{\partial S^2} + \rho\sigma vS\frac{\partial^2\Lambda}{\partial v\partial S} + \frac{1}{2}\sigma^2v\frac{\partial^2\Lambda}{\partial v^2} + rS\frac{\partial\Lambda}{\partial S} - r\Lambda = -\frac{\partial\Lambda}{\partial v} [\kappa(\theta - v_t) - \Phi(t, S, v)\sqrt{v_t}] \quad (5.4)$$

We construct another portfolio, but this portfolio is only delta-hedged, not vega-hedged, but instantaneous variance is still a state variable.

$$\tilde{\pi} = \Lambda - \frac{\partial\Lambda}{\partial S}S \quad (5.5)$$

Apply Itô

$$\begin{aligned}
d\tilde{\pi} &= \left[\frac{\partial\Lambda}{\partial t} + \frac{\partial\Lambda}{\partial S}dS + \frac{\partial\Lambda}{\partial v}dv + \left(\frac{1}{2}vS^2\frac{\partial^2\Lambda}{\partial S^2} + \rho\sigma vS\frac{\partial^2\Lambda}{\partial v\partial S} + \frac{1}{2}\sigma^2v\frac{\partial^2\Lambda}{\partial v^2} \right) dt \right] - \frac{\partial\Lambda}{\partial S}dS \\
d\tilde{\pi} &= \left[\frac{\partial\Lambda}{\partial t} + \frac{1}{2}vS^2\frac{\partial^2\Lambda}{\partial S^2} + \rho\sigma vS\frac{\partial^2\Lambda}{\partial v\partial S} + \frac{1}{2}\sigma^2v\frac{\partial^2\Lambda}{\partial v^2} \right] dt + \frac{\partial\Lambda}{\partial v}dv \quad (5.6)
\end{aligned}$$

The difference between this portfolio's return (volatility risk present) and a risk-free portfolio that earns the risk-free rate:

$$d\tilde{\pi} - r\tilde{\pi}dt = \left[\frac{\partial\Lambda}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2\Lambda}{\partial S^2} + \rho\sigma vS \frac{\partial^2\Lambda}{\partial v\partial S} + \frac{1}{2}\sigma^2 v \frac{\partial^2\Lambda}{\partial v^2} - r\Lambda - \frac{\partial\Lambda}{\partial S}rS \right] dt + \frac{\partial\Lambda}{\partial v} dv \quad (5.7)$$

We now substitute equation (5.4)

$$\begin{aligned} d\tilde{\pi} - r\tilde{\pi}dt &= -\left(\frac{\partial\Lambda}{\partial v}\right) [\kappa(\theta - v_t) - \phi(t, S, v)\sqrt{v_t}]dt + \frac{\partial\Lambda}{\partial v} dv \\ d\tilde{\pi} - r\tilde{\pi}dt &= -\frac{\partial\Lambda}{\partial v} [\kappa(\theta - v_t) - \phi(t, S, v)\sqrt{v_t}]dt + \frac{\partial\Lambda}{\partial v} [\kappa(\theta - v_t) dt + \sigma\sqrt{v_t} dW_t^0] \\ d\tilde{\pi} - r\tilde{\pi}dt &= \frac{\partial\Lambda}{\partial v} \sqrt{v_t} [\phi(t, S, v)dt + \sigma dW_t^0] \end{aligned} \quad (5.8)$$

The access return per unit of volatility of variance risk σdW_t^0 is $\phi(t, S, v)$, and called the market price of volatility risk. To have a market price of volatility risk equal to zero, we have to initially define the risk-neutral drift as

$$\kappa_1(\theta_1 - v_t) = \kappa(\theta - v_t) - \sqrt{v_t} \cdot \phi(t, S, v) \quad (5.9)$$

When we re-do this derivation with the risk-neutral drift term, we would end up with the same results only without $\phi(t, S, v)$. So equation (5.4) becomes

$$\frac{\partial\Lambda}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2\Lambda}{\partial S^2} + \rho\sigma vS \frac{\partial^2\Lambda}{\partial v\partial S} + \frac{1}{2}\sigma^2 v \frac{\partial^2\Lambda}{\partial v^2} + rS \frac{\partial\Lambda}{\partial S} - r\Lambda = -\kappa(\theta - v) \frac{\partial\Lambda}{\partial v} \quad (5.10)$$

We can subtract an existing dividend yield and the new risk-free rate becomes $(r - q)$.

5.2 Heston call price formula

We derive the Heston call price formula using by solving the PDE derived in section 5.1.

$$\frac{\partial\Lambda}{\partial t} + \frac{1}{2}vS^2 \frac{\partial^2\Lambda}{\partial S^2} + \rho\sigma vS \frac{\partial^2\Lambda}{\partial v\partial S} + \frac{1}{2}\sigma^2 v \frac{\partial^2\Lambda}{\partial v^2} + (r - q)S \frac{\partial\Lambda}{\partial S} - (r - q)\Lambda = \kappa(v - \theta) \frac{\partial\Lambda}{\partial v} \quad (6.1)$$

A reduction in dimension will make calculations more elegant. Let t be the current time, T be the time to maturity in years, S_t the current price and F the forward price, with $x = \log(F/X)$, $F = S_t e^{(r-q)\tau}$, $\tau = T - t$. We have $\Lambda(t, T, S_t, v_t) = \Lambda(\tau, x, v_t)$.

$$\begin{aligned}\frac{\partial \Lambda}{\partial t} &= \frac{\partial \Lambda}{\partial \tau} \frac{\partial \tau}{\partial t} = \frac{\partial \Lambda}{\partial \tau} (-1) = -\frac{\partial \Lambda}{\partial \tau} = -\Lambda_\tau \\ \frac{\partial \Lambda}{\partial S} &= \frac{\partial \Lambda}{\partial x} \frac{\partial x}{\partial S} = \frac{\partial \Lambda}{\partial x} \left(\frac{1}{S}\right) = \frac{\Lambda_x}{S} \quad \frac{\partial^2 \Lambda}{\partial S^2} = \frac{\partial}{\partial S} \left(\frac{\partial \Lambda}{\partial x} \frac{1}{S}\right) = \frac{-1}{S^2} \Lambda_x + \frac{1}{S^2} \Lambda_{xx}\end{aligned}$$

$$\frac{\partial^2 \Lambda}{\partial v \partial S} = \frac{\partial}{\partial v} \left(\frac{\partial \Lambda}{\partial S}\right) = \frac{\Lambda_{xv}}{S}$$

$$\begin{aligned}-\Lambda_\tau + \frac{1}{2} v S^2 \left(-\frac{1}{S^2} \Lambda_x + \frac{1}{S^2} \Lambda_{xx}\right) + \rho \sigma v S \frac{1}{S} \Lambda_{xv} + \frac{1}{2} \sigma^2 v \Lambda_{vv} + (r - q) S \frac{1}{S} \Lambda_x - (r - q) \Lambda \\ = \kappa(v - \theta) \Lambda_v\end{aligned} \quad (6.2)$$

$$-\Lambda_\tau + \left((r - q) - \frac{1}{2} v\right) \Lambda_x + \frac{1}{2} v \Lambda_{xx} + \rho \sigma v \Lambda_{xv} + \frac{1}{2} \sigma^2 v \Lambda_{vv} - (r - q) \Lambda - \kappa(v - \theta) \Lambda_v = 0 \quad (6.3)$$

We assume a solution of the form below, where X is the strike price of the call option Λ , and P^1 is the delta of the option and P^2 is the pseudo-probability of exercise, similar to the Black-Scholes model for a call.

$$\begin{aligned}\Lambda(\tau, x, v_t) &= (e^{-q\tau} S_t P^1(\tau, x, v_t) - X e^{-r\tau} P^2(\tau, x, v_t)) \\ &= X e^{-r\tau} \left(\frac{S_t e^{(r-q)\tau}}{X} P^1(\tau, x, v_t) - P^2(\tau, x, v_t) \right)\end{aligned}$$

$$\Lambda(\tau, x, v_t) = X e^{-r\tau} (e^x P^1(\tau, x, v_t) - P^2(\tau, x, v_t))$$

$$\Lambda_\tau = X e^{-r\tau} e^x P_\tau^1 - q X e^{-r\tau} e^x P^1 - (X e^{-r\tau} P_\tau^2 - r X e^{-r\tau} P^2) \quad (6.4)$$

$X e^{-r\tau}$ will cancel out by dividing each term by it in the PDE as we start substituting: so we already chuck it away now to avoid clutter.

$$\Lambda_x = e^x P_x^1 + e^x P^1 - P_x^2 \quad \Lambda_{xx} = (e^x P_{xx}^1 + e^x P_x^1) + (e^x P_x^1 + e^x P^1) - P_{xx}^2$$

$$\Lambda_{xv} = e^x P_{xv}^1 + e^x P_v^1 - P_{xv}^2$$

$$\Lambda_v = e^x P_v^1 - P_v^2 \quad \Lambda_{vv} = e^x P_{vv}^1 - P_{vv}^2 \quad \Lambda_\tau = e^x P_\tau^1 - q e^x P^1 - P_\tau^2 + r P^2 \quad (6.5)$$

Substitute

$$\begin{aligned}
& -(e^x P_\tau^1 - q e^x P^1 - P_\tau^2 + r P^2) + \left((r - q) - \frac{1}{2} v \right) (e^x P_x^1 + e^x P^1 - P_x^2) \\
& \quad + \frac{1}{2} v \left((e^x P_{xx}^1 + e^x P_x^1) + (e^x P_x^1 + e^x P^1) - P_{xx}^2 \right) + \rho \sigma v (e^x P_{xv}^1 + e^x P_v^1 - P_{xv}^2) \\
& \quad + \frac{1}{2} \sigma^2 v (e^x P_{vv}^1 - P_{vv}^2) - r (e^x P^1 - P^2) - \kappa (v - \theta) (e^x P_v^1 - P_v^2) = 0 \\
e^x & \left[-P_\tau^1 + (r - q) P_x^1 + (r - q) P^1 - \frac{1}{2} v P_x^1 - \frac{1}{2} v P^1 + \frac{1}{2} v P_{xx}^1 + v P_x^1 + \frac{1}{2} v P^1 + \rho \sigma v P_{xv}^1 + \rho \sigma v P_v^1 \right. \\
& \quad \left. + \frac{1}{2} \sigma^2 v P_{vv}^1 - (r - q) P^1 - \kappa (v - \theta) P_v^1 \right] + P_\tau^2 - r P^2 - (r - q) P_x^2 + \frac{1}{2} v P_x^2 - \frac{1}{2} v P_{xx}^2 \\
& \quad - \rho \sigma v P_{xv}^2 - \frac{1}{2} \sigma^2 v P_{vv}^2 + r P^2 + \kappa (v - \theta) P_v^2 = 0 \\
e^x & \left[-P_\tau^1 + (r - q) P_x^1 + (r - q) P^1 - \frac{1}{2} v P_x^1 - \frac{1}{2} v P^1 + \frac{1}{2} v P_{xx}^1 + v P_x^1 + \frac{1}{2} v P^1 + \rho \sigma v P_{xv}^1 + \rho \sigma v P_v^1 \right. \\
& \quad \left. + \frac{1}{2} \sigma^2 v P_{vv}^1 - (r - q) P^1 - \kappa (v - \theta) P_v^1 \right] \\
& \quad + \left[P_\tau^2 - r P^2 - (r - q) P_x^2 + \frac{1}{2} v P_x^2 - \frac{1}{2} v P_{xx}^2 - \rho \sigma v P_{xv}^2 - \frac{1}{2} \sigma^2 v P_{vv}^2 + r P^2 \right. \\
& \quad \left. + \kappa (v - \theta) P_v^2 \right] = 0
\end{aligned}$$

As the first square bracket is a function of P^1 only and the second square bracket a function of P^2 only, both square brackets should equal zero for the sum to be equal to zero.

$$\begin{aligned}
& -P_\tau^1 + (r - q) P_x^1 - \frac{1}{2} v P_x^1 + \frac{1}{2} v P_{xx}^1 + v P_x^1 + \rho \sigma v P_{xv}^1 + \rho \sigma v P_v^1 + \frac{1}{2} \sigma^2 v P_{vv}^1 - \kappa (v - \theta) P_v^1 = 0 \\
& -P_\tau^1 + \left((r - q) + \frac{1}{2} v \right) P_x^1 + \frac{1}{2} v P_{xx}^1 + \rho \sigma v P_{xv}^1 + \frac{1}{2} \sigma^2 v P_{vv}^1 + (\kappa \theta - v(\rho \sigma + \kappa)) P_v^1 = 0 \quad (6.6)
\end{aligned}$$

$$\begin{aligned}
& -P_\tau^2 + (r - q) P_x^2 - \frac{1}{2} v P_x^2 + \frac{1}{2} v P_{xx}^2 + \rho \sigma v P_{xv}^2 + \frac{1}{2} \sigma^2 v P_{vv}^2 - \kappa (v - \theta) P_v^2 = 0 \\
& -P_\tau^2 + \left((r - q) - \frac{1}{2} v \right) P_x^2 + \frac{1}{2} v P_{xx}^2 + \rho \sigma v P_{xv}^2 + \frac{1}{2} \sigma^2 v P_{vv}^2 + (\kappa \theta - \kappa v) P_v^2 = 0 \quad (6.7)
\end{aligned}$$

$$-P_\tau^j + \left((r - q) + u^j v \right) P_x^j + \frac{1}{2} v P_{xx}^j + \rho \sigma v P_{xv}^j + \frac{1}{2} \sigma^2 v P_{vv}^j + (a - v b^j) P_v^j = 0 \quad (6.8)$$

$$a = \kappa \theta \quad b^1 = \kappa - \rho \sigma \quad b^2 = \kappa \quad u^1 = \frac{1}{2} \quad u^2 = -\frac{1}{2} \quad (6.9)$$

The terminal condition for the PDE for P^j is

$$I(x) := \lim_{\tau \rightarrow \infty} P^j(\tau, x, v_t) = \begin{cases} 1 & x > 0 \\ 0 & x \leq 0 \end{cases} \quad (6.10)$$

The PDE for P^j is solved via a Fourier transform technique. Using some Fourier transform results⁸, the Fourier transform of P^j is defined as

$$\mathfrak{F}(P^j(\tau, x, v_t)) = \tilde{P}^j(\tau, w, v_t) = \int_{-\infty}^{\infty} e^{-iwx} P^j(\tau, x, v_t) dx \quad i = \sqrt{-1} \quad (6.11)$$

$$\tilde{P}^j(0, w, v_t) = \int_{-\infty}^{\infty} e^{-iwx} I(x) dx = \frac{1}{-iw} \int_0^{\infty} -iw \cdot e^{-iwx} dx = \frac{[e^{-iwx}]_0^{\infty}}{-iw} = \frac{[0 - 1]}{-iw} = \frac{1}{iw} \quad (6.12)$$

The Fourier transform of the PDE becomes

$$\begin{aligned} -\tilde{P}_\tau^j + ((r - q) + u^j v)(iw\tilde{P}^j) + \frac{1}{2}v(-w^2\tilde{P}^j) + \rho\sigma v(iw\tilde{P}_v^j) + \frac{1}{2}\sigma^2 v\tilde{P}_{vv}^j + (a - vb^j)\tilde{P}_v^j &= 0 \\ -\tilde{P}_\tau^j + \left((r - q)iw + \left(u^j iw - \frac{1}{2}w^2 \right) v \right) \tilde{P}^j + \frac{1}{2}\sigma^2 v\tilde{P}_{vv}^j + a\tilde{P}_v^j - v(b^j - iw\rho\sigma)\tilde{P}_v^j &= 0 \end{aligned} \quad (6.13)$$

$$\alpha = \left(u^j iw - \frac{1}{2}w^2 \right) \quad -\beta = (b^j - iw\rho\sigma) \quad \gamma = \frac{1}{2}\sigma^2 \quad (6.14)$$

$$\begin{aligned} -\tilde{P}_\tau^j + \alpha v\tilde{P}^j + \gamma v\tilde{P}_{vv}^j + (r - q)iw\tilde{P}^j + a\tilde{P}_v^j - v\beta\tilde{P}_v^j &= 0 \\ -\tilde{P}_\tau^j + (r - q)iw\tilde{P}^j + a\tilde{P}_v^j + v(\alpha\tilde{P}^j + \gamma\tilde{P}_{vv}^j - \beta\tilde{P}_v^j) &= 0 \end{aligned} \quad (6.15)$$

We now propose a solution of the form

$$\tilde{P}^j(\tau, w, v_t) = \tilde{P}^j(\mathbf{0}, w, v_0) \exp(C_j(\tau, w) + D_j(\tau, w)v_t) = \frac{1}{iw} \exp(C_j(\tau, w) + D_j(\tau, w)v) \quad (6.16)$$

Taking derivatives and substitute into the Fourier transform PDE

⁸ [13] Kreyzig $\mathfrak{F}(f'(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iux} f'(x) dx = \frac{1}{\sqrt{2\pi}} \left([f(x)e^{-iux}]_{-\infty}^{\infty} + iu \int_{-\infty}^{\infty} e^{-iux} f(x) dx \right) = iu \mathfrak{F}(f(x))$
 $|x| \rightarrow \infty, f(x)e^{-iux} \rightarrow 0$
 $\mathfrak{F}(f''(x)) = iu \mathfrak{F}(f'(x)) = (iu)^2 \mathfrak{F}(f(x)) = u^2 \mathfrak{F}(f(x)) \quad \mathfrak{F}(af(x)) = a\mathfrak{F}(f(x))$

$$\begin{aligned}
\tilde{P}_\tau^j &= \tilde{P}^j \left(\frac{\partial C}{\partial \tau} + \frac{\partial D}{\partial \tau} v \right) & \tilde{P}_v^j &= D \tilde{P}^j & \tilde{P}_{vv}^j &= D^2 \tilde{P}^j \\
-\tilde{P}^j \left(\frac{\partial C}{\partial \tau} + \frac{\partial D}{\partial \tau} v \right) &+ (r - q) i w \tilde{P}^j + a D \tilde{P}^j + v \tilde{P}^j (\alpha + \gamma D^2 - \beta D) &= 0 \\
-\tilde{P}^j \left(\frac{\partial C}{\partial \tau} - \kappa \theta D - (r - q) i w \right) &+ v \tilde{P}^j \left(\alpha - \beta D + \gamma D^2 - \frac{\partial D}{\partial \tau} \right) &= 0 \\
\tilde{P}^j \left(\frac{\partial C}{\partial \tau} - \kappa \theta D - (r - q) i w \right) &+ v \tilde{P}^j \left(- \left(\alpha - \beta D + \gamma D^2 - \frac{\partial D}{\partial \tau} \right) \right) &= 0
\end{aligned}$$

This equation would be satisfied when

$$\frac{\partial C}{\partial \tau} - \kappa \theta D - (r - q) i w = 0 \quad - \left(\alpha - \beta D + \gamma D^2 - \frac{\partial D}{\partial \tau} \right) = 0 \quad (6.17)$$

We recognize that the second equation is a Ricatti ODE, and we use the formulas from Wikipedia. (We add the minus sign again at the end after we solved D).

$$\frac{\partial D}{\partial \tau} = \alpha - \beta D + \gamma D^2 \quad (6.18)$$

$$D(\tau) = q_0 + q_1 D(\tau) + q_2 D(\tau)^2 \quad D(\tau) = \frac{-u'}{q_2 u} \quad q_2 = \gamma \quad (6.20)$$

$$u'' - R u' + S u = 0 \quad R = q_1 + \frac{q_2'}{q_2} = -\beta \quad S = q_0 q_2 = \alpha \gamma \quad (6.21)$$

$$u'' + \beta u' + \alpha \gamma u = 0 \quad (6.22)$$

This is a homogeneous equation with constant coefficients. The characteristic equation is $z^2 + \beta z + \alpha \gamma = 0$, so $z_{\mp} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2}$, thus the solution basis is $\{u_1, u_2\}$ is $\{e^{z_+ \tau}, e^{z_- \tau}\}$. We check one:

$$\begin{aligned}
u &= e^{z_+ \tau} \quad u' = z_+ e^{z_+ \tau} \quad u'' = z_+^2 e^{z_+ \tau} \quad u'' + \beta u' + \alpha \gamma u = z_+^2 e^{z_+ \tau} + \beta z_+ e^{z_+ \tau} + \alpha \gamma e^{z_+ \tau} \\
&\frac{1}{4} \left(\beta^2 - 2\beta \sqrt{\beta^2 - 4\alpha\gamma} + \beta^2 - 4\alpha\gamma \right) e^{z_+ \tau} + \beta \left(\frac{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}}{2} \right) e^{z_+ \tau} + \alpha \gamma e^{z_+ \tau} = 0 \\
&\left(\frac{1}{2} \beta^2 - \frac{1}{2} \beta \sqrt{\beta^2 - 4\alpha\gamma} - \alpha \gamma \right) e^{z_+ \tau} + \left(-\frac{1}{2} \beta^2 + \frac{1}{2} \beta \sqrt{\beta^2 - 4\alpha\gamma} \right) e^{z_+ \tau} + \alpha \gamma e^{z_+ \tau} = 0
\end{aligned}$$

Now this is a solution if and only if for some $c_1, c_2 \in \mathbb{C}$

$$u = c_1 u_1 + c_2 u_2 \quad u = c_1 e^{z_- \tau} + c_2 e^{z_+ \tau} \quad \Rightarrow \quad u' = c_1 z_- e^{z_- \tau} + c_2 z_+ e^{z_+ \tau} \quad (6.26)$$

Substitute

$$D(\tau) = \frac{-u'}{q_2 u} = \frac{-u'}{\gamma u} = \frac{-(c_1 z_- e^{z_- \tau} + c_2 z_+ e^{z_+ \tau})}{\gamma(c_1 e^{z_- \tau} + c_2 e^{z_+ \tau})} \quad (6.27)$$

We use the terminal condition to find the values of c_1, c_2 :

$$D(0) = 0 = \frac{-(c_1 z_- + c_2 z_+)}{\gamma(c_1 + c_2)} \quad c_1 z_- + c_2 z_+ = 0 \quad c_1 = -c_2 \left(\frac{z_+}{z_-} \right) \quad (6.28)$$

Substitute c_1 into the equation for, and noting that each term then has a c_2 which we can cancel out:

$$\begin{aligned} D(\tau) &= \frac{-\left(-c_2 \left(\frac{z_+}{z_-}\right) z_- e^{z_- \tau} + c_2 z_+ e^{z_+ \tau}\right)}{\gamma \left(-c_2 \left(\frac{z_+}{z_-}\right) e^{z_- \tau} + c_2 e^{z_+ \tau}\right)} = \frac{\left(\left(\frac{z_+}{z_-}\right) z_- e^{z_- \tau} - z_+ e^{z_+ \tau}\right)}{\gamma \left(e^{z_+ \tau} - \left(\frac{z_+}{z_-}\right) e^{z_- \tau}\right)} = \frac{(e^{z_- \tau} - e^{z_+ \tau})}{\frac{\gamma}{z_-} \left(\frac{z_-}{z_+} e^{z_+ \tau} - e^{z_- \tau}\right)} \\ &= \frac{(1 - e^{z_+ \tau - z_- \tau})}{\frac{-\gamma}{z_-} \left(1 - \frac{z_-}{z_+} e^{z_+ \tau - z_- \tau}\right)} = \frac{(1 - e^{-(z_- \tau - z_+ \tau)})}{\frac{-\gamma}{z_-} \left(1 - \frac{z_-}{z_+} e^{-(z_- \tau - z_+ \tau)}\right)} = \frac{(1 - e^{-d\tau})}{\frac{-\gamma}{z_-} (1 - g e^{-d\tau})} \end{aligned}$$

$$\begin{aligned} d &= z_- - z_+ = \sqrt{\beta^2 - 4\alpha\gamma} = \sqrt{(i\omega\rho\sigma - b^j)^2 - 4\left(u^j i\omega - \frac{1}{2}w^2\right)\frac{1}{2}\sigma^2} \\ &= \sqrt{(i\omega\rho\sigma - b^j)^2 - \sigma^2(2u^j i\omega - w^2)} \end{aligned}$$

$$g = \frac{z_-}{z_+} = \frac{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}}{-\beta + \sqrt{\beta^2 - 4\alpha\gamma}} = \frac{b^j - i\omega\rho\sigma - d}{b^j - i\omega\rho\sigma + d}$$

$$\left(\frac{-\gamma}{z_-}\right) = \frac{-\sigma^2}{-\beta - \sqrt{\beta^2 - 4\alpha\gamma}} = \frac{-\sigma^2}{b^j - i\omega\rho\sigma - d} \quad (6.28)$$

$$D(\tau) = \left(\frac{b^j - i\omega\rho\sigma - d}{-\sigma^2}\right) \frac{(1 - e^{-\tau d})}{(1 - g e^{-\tau d})} + c \quad (6.29)$$

$$D(0) = 0 = \left(\frac{b^j - i\omega\rho\sigma - d}{-\sigma^2}\right) \frac{(1 - e^{0d})}{(1 - g e^{0d})} + c$$

The constant of integration c is zero. Add the minus sign back.

$$D(\tau) = \left(\frac{b^j - i\omega\rho\sigma - d}{\sigma^2}\right) \frac{(1 - e^{-\tau d})}{(1 - g e^{-\tau d})} \quad (6.30)$$

To solve the next ODE we use partial fractions to integrate

$$\begin{aligned}
\frac{\partial C}{\partial \tau} - \kappa \theta D - (r - q)iw &= 0 \\
\frac{\partial C}{\partial \tau} &= \kappa \theta D + (r - q)iw \\
C(\tau) &= \kappa \theta \int D(\tau) + \int (r - q)iw + c \\
\int D(\tau) d\tau &= \left(\frac{b^j - iw\rho\sigma - d}{\sigma^2} \right) \int \frac{(1 - e^{-\tau d})}{(1 - ge^{-\tau d})} d\tau \\
\int \frac{(1 - e^{-\tau d})}{(1 - ge^{-\tau d})} d\tau &= \frac{1}{-d} \int \frac{(1 - u)}{u(1 - gu)} du \quad e^{-\tau d} = u \quad -de^{-\tau d}d\tau = du \\
\frac{(1 - u)}{u(1 - gu)} &= \frac{A}{u} + \frac{B}{(1 - gu)} \quad \Rightarrow \quad (1 - u) = A(1 - gu) + Bu \\
u = 0 &\quad \Rightarrow \quad (1 - 0) = A(1 - g0) + B0 \quad \Rightarrow \quad A = 1 \\
u = 1 &\quad \Rightarrow \quad (1 - 1) = 1(1 - g1) + B1 \quad \Rightarrow \quad B = -(1 - g) \\
\frac{(1 - u)}{u(1 - gu)} &= \frac{1}{u} - \frac{(1 - g)}{(1 - gu)} \\
\frac{1}{-d} \int \left(\frac{1}{u} - \frac{(1 - g)}{(1 - gu)} \right) du &= \frac{1}{-d} \int \left(\frac{1}{u} + \frac{1}{g} \frac{-g}{(1 - gu)} - \frac{-g}{(1 - gu)} \right) du \\
&= \frac{1}{-d} \left(\ln u + \frac{1}{g} \ln(1 - gu) - \ln(1 - gu) \right) = \left(\frac{1}{-d} \ln u - \frac{1 - g}{dg} \ln(1 - gu) \right) \\
\frac{1 - g}{dg} &= \frac{\left(\frac{b^j - iw\rho\sigma + d - (b^j - iw\rho\sigma - d)}{b^j - iw\rho\sigma + d} \right)}{d \left(\frac{b^j - iw\rho\sigma - d}{b^j - iw\rho\sigma + d} \right)} = \frac{2d}{d(b^j - iw\rho\sigma - d)} = \frac{2}{(b^j - iw\rho\sigma - d)} \\
C(\tau, w) &= \kappa \theta \left(\frac{b^j - iw\rho\sigma - d}{\sigma^2} \right) \left(\frac{\ln e^{-\tau d}}{-d} - \frac{2 \ln(1 - ge^{-\tau d})}{(b^j - iw\rho\sigma - d)} \right) + (r - q)iw\tau + c \quad (6.31)
\end{aligned}$$

Use the initial condition to solve the constant of integration c :

$$\begin{aligned}
C(0, w) &= \kappa \theta \left(\frac{b^j - iw\rho\sigma - d}{\sigma^2} \right) \left(0 - \frac{2 \ln(1 - g)}{(b^j - iw\rho\sigma - d)} \right) + 0 + c = 0 \\
c &= \kappa \theta \left(\frac{b^j - iw\rho\sigma - d}{\sigma^2} \right) \left[\frac{-2 \ln(1 - g)}{(b^j - iw\rho\sigma - d)} \right] \\
C(\tau, w) &= \kappa \theta \left(\frac{b^j - iw\rho\sigma - d}{\sigma^2} \right) \left(\frac{\ln e^{-\tau d}}{-d} - \frac{2 \ln(1 - ge^{-\tau d})}{(b^j - iw\rho\sigma - d)} \right) + (r - q)iw\tau + c
\end{aligned}$$

$$\begin{aligned}
C(\tau, w) &= \kappa\theta \left(\frac{b^j - iw\rho\sigma - d}{\sigma^2} \right) \left(\tau + \frac{-2}{(b^j - iw\rho\sigma - d)} \ln \frac{(1 - ge^{-\tau d})}{(1 - g)} \right) + (r - q)iw\tau \\
&= \left(\frac{\kappa\theta}{\sigma^2} \right) \left(\tau(b^j - iw\rho\sigma - d) - 2 \ln \frac{(1 - ge^{-\tau d})}{(1 - g)} \right) + (r - q)iw\tau \\
C(\tau, w) &= \left(\frac{\kappa\theta}{\sigma^2} \right) \left(\tau(b^j - iw\rho\sigma - d) - 2 \ln \frac{(1 - ge^{-\tau d})}{(1 - g)} \right) + (r - q)iw\tau \tag{6.32}
\end{aligned}$$

Now taking the inverse transform

$$\begin{aligned}
\Im \left(\tilde{P}^j(\tau, w, v_t) \right) &= P^j(\tau, x, v_t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{iwx} \tilde{P}^j(\tau, w, v_t) du \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iwx}}{iw} \exp(C_j(\tau, w) + D_j(\tau, w)v_t) dw \\
P^j(\tau, x, v_t) &= \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{e^{iwx}}{iw} \exp(C_j(\tau, w) + D_j(\tau, w)v_t) dw \tag{6.33}
\end{aligned}$$

5.3 The fair variance in Heston parameters

We start with the Heston stochastic variance process, and choose an integrating factor.

$$dv_t = \kappa(\theta - v_t) dt + \sigma\sqrt{v_t} dW_t^0 \quad e^{-\int_0^t -\kappa ds} = e^{\kappa t} \tag{7.1}$$

Using the product rule or equivalently Ito's lemma:

$$\begin{aligned}
d(v_t e^{\kappa t}) &= e^{\kappa t} dv_t + \kappa e^{\kappa t} v_t dt \\
d(v_t e^{\kappa t}) &= e^{\kappa t} \kappa(\theta - v_t) dt + e^{\kappa t} \sigma\sqrt{v_t} dW_t^0 + \kappa e^{\kappa t} v_t dt \\
d(v_t e^{\kappa t}) &= e^{\kappa t} \kappa\theta dt - e^{\kappa t} \kappa v_t dt + e^{\kappa t} \sigma\sqrt{v_t} dW_t^0 + \kappa e^{\kappa t} v_t dt \\
d(v_t e^{\kappa t}) &= e^{\kappa t} \kappa\theta dt + e^{\kappa t} \sigma\sqrt{v_t} dW_t^0 \tag{7.2}
\end{aligned}$$

$$\begin{aligned}
\int_0^T d(v_t e^{\kappa t}) &= \theta \int_0^T \kappa e^{\kappa t} dt + \int_0^T e^{\kappa t} \sigma\sqrt{v_t} dW_t^0 \\
v_T e^{\kappa T} - v_0 &= \theta(e^{\kappa T} - 1) + \int_0^T e^{\kappa t} \sigma\sqrt{v_t} dW_t^0
\end{aligned}$$

$$\begin{aligned}
v_T &= v_0 e^{-\kappa T} + \theta(1 - e^{-\kappa T}) + \int_0^T e^{-\kappa(T-t)} \sigma \sqrt{v_t} dW_t^0 \\
\mathbb{E}^{\mathbb{Q}}(v_T) &= v_0 e^{-\kappa T} + \theta(1 - e^{-\kappa T}) + \mathbb{E}^{\mathbb{Q}}\left(\int_0^T e^{-\kappa(T-t)} \sigma \sqrt{v_t} dW_t^0\right) \\
\mathbb{E}^{\mathbb{Q}}(v_T) &= v_0 e^{-\kappa T} + \theta(1 - e^{-\kappa T})
\end{aligned} \tag{7.3}$$

The integral format of the Heston variance process

$$\begin{aligned}
v_T &= v_0 + \int_0^T \kappa \theta dt - \kappa \int_0^T v_t dt + \int_0^T \sigma \sqrt{v_t} dW_t^0 \\
\mathbb{E}^{\mathbb{Q}}(v_T) &= v_0 + \kappa \theta T - \kappa \mathbb{E}^{\mathbb{Q}}\left(\int_0^T v_t dt\right)
\end{aligned} \tag{7.4}$$

We now substitute $\mathbb{E}^{\mathbb{Q}}(v_T)$:

$$\begin{aligned}
v_0 e^{-\kappa T} + \theta(1 - e^{-\kappa T}) &= v_0 + \kappa \theta T - \kappa \mathbb{E}^{\mathbb{Q}}\left(\int_0^T v_t dt\right) \\
\frac{1}{\kappa T}(\theta - v_0)(1 - e^{-\kappa T}) - \theta &= -\mathbb{E}^{\mathbb{Q}}\left(\frac{1}{T} \int_0^T v_t dt\right) \\
\mathbb{E}^{\mathbb{Q}}\left(\frac{1}{T} \int_0^T v_t dt\right) &= \theta + \frac{(v_0 - \theta)(1 - e^{-\kappa T})}{\kappa T}
\end{aligned} \tag{7.5}$$

We now have a neat formula for $\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T} \int_0^T v_t dt\right]$, the risk-neutral expectation of continuously accrued realised variance.

5.4 Replication of European twice differentiable payoffs

If the market prices of vanilla options represent the risk-neutral prices, then it should also be possible to extract the risk-neutral probability density function from these market prices. This is what Breeden and Litzenberger formalised in 1978, and was used by Neuberger for the log-contract replication in section 3.1.3. Let $\varphi(S_0, S_T, T)$ be the probability density function for an asset price S_T , then we can write the undiscounted option prices \tilde{C} and \tilde{P} with a strike price K as

$$\tilde{C}(S_0, K, T) = \int_K^{\infty} (S_T - K) \varphi(S_0, S_T, T) dS_T \quad \tilde{P}(S_0, K, T) = \int_0^K (K - S_T) \varphi(S_0, S_T, T) dS_T \tag{8.1}$$

Using differentiation under the integral sign⁹

⁹ $\frac{d}{dx} \left(\int_{a(x)}^{b(x)} f(x, t) dt \right) = f(x, b(x))b'(x) - f(x, a(x))a'(x) + \int_{a(x)}^{b(x)} f_x(x, t) dt$

$$\begin{aligned}\frac{\partial \tilde{C}(S_0, K, T)}{\partial K} &= \int_K^\infty (-1)\varphi(S_0, S_T, T) dS_T & \frac{\partial \tilde{P}(S_0, K, T)}{\partial K} &= \int_0^K (1)\varphi(S_0, S_T, T) dS_T \\ \frac{\partial^2 \tilde{C}(S_0, K, T)}{\partial K^2} &= \varphi(S_0, K, T) & \frac{\partial^2 \tilde{P}(S_0, K, T)}{\partial K^2} &= \varphi(S_0, K, T)\end{aligned}\quad (8.2)$$

Given some payoff $f(S_T)$ that depends on the terminal asset price only, we write the undiscounted price as

$$\begin{aligned}\mathbb{E}[f(S_T)] &= \int_0^\infty f(K)\varphi(S_0, K, T) dK \\ &= \int_0^F f(K)\frac{\partial^2 \tilde{P}(S_0, K, T)}{\partial K^2} dK + \int_F^\infty f(K)\frac{\partial^2 \tilde{C}(S_0, K, T)}{\partial K^2} dK\end{aligned}\quad (8.3)$$

Where $F = S_0 e^{(r-q)T}$, the forward price. We divide the integral because out-of-the-money options are the most liquid, the one left-side of the skew will utilize puts and the other side calls. We now proceed with integration by parts¹⁰ and using put-call parity:

$$\begin{aligned}v'(K) &= \frac{\partial^2 \tilde{P}}{\partial K^2} & v(K) &= \frac{\partial \tilde{P}}{\partial K} & u(K) &= f(K) & u'(K) &= f'(K) \\ \mathbb{E}[f(S_T)] &= \left[\frac{\partial \tilde{P}}{\partial K} f(K) \right]_0^F - \int_0^F f'(K) \frac{\partial \tilde{P}}{\partial K} dK + \left[\frac{\partial \tilde{C}}{\partial K} f(K) \right]_F^\infty - \int_F^\infty f'(K) \frac{\partial \tilde{C}}{\partial K} dK \\ \frac{\partial \tilde{P}(S_0, F, T)}{\partial K} f(F) &= f(F) \int_0^F (F)\varphi(S_0, S_T, T) dS_T & \frac{\partial \tilde{C}(S_0, F, T)}{\partial K} f(F) \\ &= f(F) \int_F^\infty (-F)\varphi(S_0, S_T, T) dS_T\end{aligned}\quad (8.4)$$

$$\begin{aligned}\mathbb{E}[f(S_T)] &= \left[f(F) \int_0^F (F)\varphi(S_0, S_T, T) dS_T \right] - \int_0^F f'(K) \frac{\partial \tilde{P}}{\partial K} dK + \left[-f(F) \int_F^\infty (-F)\varphi(S_0, S_T, T) dS_T \right] \\ &\quad - \int_F^\infty f'(K) \frac{\partial \tilde{C}}{\partial K} dK \\ \mathbb{E}[f(S_T)] &= \left[f(F) \int_0^\infty (F)\varphi(S_0, S_T, T) dS_T \right] - \int_0^F f'(K) \frac{\partial \tilde{P}}{\partial K} dK - \int_F^\infty f'(K) \frac{\partial \tilde{C}}{\partial K} dK \\ &= f(F) - \int_0^F f'(K) \frac{\partial \tilde{P}}{\partial K} dK - \int_F^\infty f'(K) \frac{\partial \tilde{C}}{\partial K} dK \\ v'(K) &= \frac{\partial \tilde{P}}{\partial K} & v(K) &= \tilde{P} & u(K) &= f'(K) & u'(K) &= f''(K)\end{aligned}$$

¹⁰ $\int u(x)v'(x) dx = u(x)v(x) - \int u'(x)v(x) dx$

$$\begin{aligned} \mathbb{E}[f(S_T)] &= f(F) - \left[[f'(K)\tilde{P}]_0^F - \int_0^F f''(K) \tilde{P} dK \right] - \left[[f'(K)\tilde{C}]_F^\infty - \int_F^\infty f''(K) \tilde{C} dK \right] \\ C(K) + e^{-rT}K &= P(K) + Se^{-qT} \quad e^{rT}C(K) + K = e^{rT}P(K) + Se^{(r-q)T} \\ K = F = Se^{(r-q)T} &\Rightarrow \tilde{C}(F) + F = \tilde{P}(F) + F \Rightarrow \tilde{C}(F) = \tilde{P}(F) \\ \mathbb{E}[f(S_T)] &= f(F) + \int_0^F f''(K) \tilde{P}(S_0, K, T) dK + \int_F^\infty f''(K) \tilde{C}(S_0, K, T) dK \end{aligned} \quad (8.6)$$

In theory, any twice differentiable European payoff function can be statically hedged by a portfolio of put options with strikes ranging from zero to the forward price, and call options with strikes ranging from the forward price to infinity. The position in each call and put price is the second derivative of the payoff function w.r.t the strike, where $S_T = K$. As the strike of a vanilla option does not change, and the position size depends only K , the hedge is conveniently static.

We want a result where the expectation is not included, we follow [15]. The Dirac delta function is defined as

$$\delta(K) = \begin{cases} \infty, & K = 0 \\ 0, & \text{otherwise} \end{cases} \quad \int_{-\infty}^{\infty} \delta(K) dK = 1 \quad (8.7)$$

The sifting property for the Dirac delta function is, for $\varepsilon > 0$

$$f(a) = \int_{a-\varepsilon}^{a+\varepsilon} f(K) \delta(a - K) dK \quad (8.8)$$

The Heaviside function is defined as

$$H(K) = \mathbb{1}_{K>0} \quad \frac{d}{dK} H(K) = \delta(K) \quad \frac{d}{dK} (K)^+ = H(K) \quad (K)^+ := \max(0, K) \quad (8.9)$$

So we can write

$$\delta(S - K) = \delta(K - S) = \begin{cases} \infty, & K = S \\ 0, & \text{otherwise} \end{cases} \quad (8.10)$$

$$H(S - K) = \mathbb{1}_{S-K>0} = \mathbb{1}_{S>K} \quad H(K - S) = \mathbb{1}_{K-S>0} = \mathbb{1}_{S<K} \quad (8.11)$$

$$\frac{d}{dK} H(S - K) = \delta(S - K) \quad \frac{d}{dK} H(K - S) = \delta(K - S) \quad (8.12)$$

$$\frac{d}{dK} (S - K)^+ = (-1). H(S - K) = -\mathbb{1}_{S>K} \quad \frac{d}{dK} (K - S)^+ = H(K - S) = \mathbb{1}_{S<K} \quad (8.13)$$

Use the sifting property of the Dirac delta function and picking a threshold $\alpha > 0$

$$f(S) = \int_0^\infty f(K) \delta(S - K) dK = \int_0^\alpha f(K) \delta(K - S) dK + \int_\alpha^\infty f(K) \delta(S - K) dK \quad (8.14)$$

Using integration by parts, for the first integral $u = f(K)$ $u' = f'(K)$ $v' = \delta(K - S)$ $v = \mathbb{1}_{S < K}$

$$\int_0^\alpha f(K) \delta(K - S) dK = f(K) \mathbb{1}_{S < K} \Big|_0^\alpha - \int_0^\alpha f'(K) \mathbb{1}_{S < K} dK = f(\alpha) \mathbb{1}_{S < K} - \int_0^\alpha f'(K) \mathbb{1}_{S < K} dK \quad (8.15)$$

For the second integral $u = f(K)$ $u' = f'(K)$ $v' = \delta(S - K)$ $v = -\mathbb{1}_{S > K}$

$$\begin{aligned} \int_\alpha^\infty f(K) \delta(S - K) dK &= -\mathbb{1}_{S > K} f(K) \Big|_\alpha^\infty - \int_\alpha^\infty -\mathbb{1}_{S > K} f'(K) dK \\ \int_\alpha^\infty f(K) \delta(S - K) dK &= \mathbb{1}_{S > K} f(\alpha) + \int_\alpha^\infty \mathbb{1}_{S > K} f'(K) dK \end{aligned} \quad (8.16)$$

So

$$f(S) = f(\alpha) + \int_\alpha^\infty \mathbb{1}_{S > K} f'(K) dK - \int_0^\alpha f'(K) \mathbb{1}_{S < K} dK \quad (8.17)$$

Then for the first integral $u = f'(K)$ $u' = f''(K)$ $v' = \mathbb{1}_{S > K}$ $v = -(S - K)^+$

$$\begin{aligned} \int_\alpha^\infty \mathbb{1}_{S > K} f'(K) dK &= -(S - K)^+ f'(K) \Big|_\alpha^\infty - \int_\alpha^\infty -(S - K)^+ f''(K) dK \\ &= (S - \alpha)^+ f'(\alpha) + \int_\alpha^\infty (S - K)^+ f''(K) dK \end{aligned}$$

for the second integral $u = f'(K)$ $u' = f''(K)$ $v' = \mathbb{1}_{S < K}$ $v = (K - S)^+$

$$\begin{aligned} \int_0^\alpha f'(K) \mathbb{1}_{S < K} dK &= (K - S)^+ f'(K) \Big|_0^\alpha - \int_0^\alpha f''(K) (K - S)^+ dK \\ &= (\alpha - S)^+ f'(\alpha) - \int_0^\alpha f''(K) (K - S)^+ dK \\ f(S) &= f(\alpha) + (S - \alpha)^+ f'(\alpha) + \int_\alpha^\infty (S - K)^+ f''(K) dK - (\alpha - S)^+ f'(\alpha) \\ &\quad + \int_0^\alpha f''(K) (K - S)^+ dK \end{aligned} \quad (8.18)$$

If we use put-call parity where the risk-free rate is equal to zero, α is the strike price and S the spot price

$$(S - \alpha)^+ + \alpha = (\alpha - S)^+ + S \quad (S - \alpha)^+ - (\alpha - S)^+ = S - \alpha \quad (8.19)$$

$$f(S) = f(\alpha) + (S - \alpha)f'(\alpha) + \int_{\alpha}^{\infty} (S - K)^+ f''(K) dK + \int_0^{\alpha} f''(K) (K - S)^+ dK \quad (8.20)$$

5.5 Discretization error replicating the fair variance with a log-contract

When the log-contract relation is used to find the risk-neutral expectation of the discretely (daily) sampled realized variance, the main source of error is of the third-order. The log-contract, together with a dynamic position in the underlying itself, replicates the continuously sampled variance over the contract period. By following [7] we show that the fair variance $\mathbb{E}^{\mathbb{Q}}(V_{0,T}^R) = V^S$ is underestimated, when the risk-neutral expectation of third-order (or cubed) returns are negative, and higher-order terms are ignored. For demonstration purposes, we assume the underlying asset is a future F that can be traded daily without market frictions, and a static position in a continuum of vanilla options is possible.

$$V_{0,T}^R = \frac{N}{n} \sum_{i=1}^n \left[\ln \left(\frac{F_i}{F_{i-1}} \right) \right]^2 \quad (9.1)$$

Where N is the number of trading days in a year. The daily return R_i for day i

$$R_i = \frac{F_i - F_{i-1}}{F_{i-1}} \quad i = 1, 2, 3, \dots, n \quad (9.2)$$

Using the Taylor series expansion, and let $O(R_i^p)$ be a function of g such that $g(x) = O(x^p)$ as $x \rightarrow 0$.

$$\begin{aligned} 2 \ln \frac{F_i}{F_{i-1}} &= 2(\ln F_i - \ln F_{i-1}) \\ &= 2 \left[\frac{\partial \ln F_{i-1}}{\partial F_{i-1}} (F_i - F_{i-1}) + \frac{1}{2!} \frac{\partial^2 \ln F_{i-1}}{\partial F_{i-1}^2} (F_i - F_{i-1})^2 + \frac{1}{3!} \frac{\partial^3 \ln F_{i-1}}{\partial F_{i-1}^3} (F_i - F_{i-1})^3 \right. \\ &\quad \left. + O(R_i^4) \right] \end{aligned}$$

$$2\ln \frac{F_i}{F_{i-1}} = 2 \left[\frac{2}{F_{i-1}} (F_i - F_{i-1}) + \frac{1}{2} \left(\frac{-1}{F_{i-1}^2} \right) (F_i - F_{i-1})^2 + \frac{1}{6} \left(\frac{2}{F_{i-1}^3} \right) (F_i - F_{i-1})^3 + O(R_i^4) \right]$$

$$\ln \frac{F_i}{F_{i-1}} = R_i - \frac{1}{2} R_i^2 + O(R_i^3) \quad (9.3)$$

Squaring both sides

$$\left(\ln \frac{F_i}{F_{i-1}} \right)^2 = R_i^2 - R_i^3 + O(R_i^4) \quad \left(\ln \frac{F_i}{F_{i-1}} \right)^2 + R_i^3 - O(R_i^4) = R_i^2 \quad (9.4)$$

Now substitute R_i^2 in the second equation

$$2\ln \frac{F_i}{F_{i-1}} = 2 \left[R_i - \frac{1}{2} R_i^2 + \frac{1}{3} R_i^3 + O(R_i^4) \right]$$

$$2\ln \frac{F_i}{F_{i-1}} = 2R_i - \left(\left(\ln \frac{F_i}{F_{i-1}} \right)^2 + R_i^3 - O(R_i^4) \right) + \frac{2}{3} R_i^3 + O(R_i^4)$$

$$2\ln \frac{F_i}{F_{i-1}} = 2R_i - \left(\ln \frac{F_i}{F_{i-1}} \right)^2 - \frac{1}{3} R_i^3 + O(R_i^4)$$

$$\left(\ln \frac{F_i}{F_{i-1}} \right)^2 = 2R_i - 2\ln \frac{F_i}{F_{i-1}} - \frac{1}{3} R_i^3 + O(R_i^4) \quad (9.5)$$

Apply summation to find the sum of the squared log-returns

$$\sum_{i=1}^n \left(\ln \frac{F_i}{F_{i-1}} \right)^2 = \sum_{i=1}^n 2R_i - 2 \sum_{i=1}^n (\ln F_i - \ln F_{i-1}) - \frac{1}{3} \sum_{i=1}^n R_i^3 + \sum_{i=1}^n O(R_i^4)$$

$$\sum_{i=1}^n \left(\ln \frac{F_i}{F_{i-1}} \right)^2 = \sum_{i=1}^n 2R_i - 2(\ln F_n - \ln F_0) - \frac{1}{3} \sum_{i=1}^n R_i^3 + \sum_{i=1}^n O(R_i^4) \quad (9.6)$$

Using a twice differentiable payoff result for $\ln F_n - \ln F_0$

$$\ln F_n = \ln F_0 + \frac{F_n - F_0}{F_0} + \int_0^{F_0} \frac{-1}{K^2} (K - F_n)^+ dK + \int_{F_0}^{\infty} \frac{-1}{K^2} (F_n - K)^+ dK \quad (9.7)$$

Substitute

$$\sum_{i=1}^n \left(\ln \frac{F_i}{F_{i-1}} \right)^2 = \sum_{i=1}^n 2R_i - 2 \frac{F_n - F_0}{F_0} + \int_0^{F_0} \frac{2}{K^2} (K - F_n)^+ dK + \int_{F_0}^{\infty} \frac{2}{K^2} (F_n - K)^+ dK - \frac{1}{3} \sum_{i=1}^n R_i^3$$

$$+ \sum_{i=1}^n O(R_i^4)$$

$$\begin{aligned} \sum_{i=1}^n \left(\ln \frac{F_i}{F_{i-1}} \right)^2 &= \sum_{i=1}^n 2 \frac{F_i - F_{i-1}}{F_{i-1}} - \frac{2}{F_0} \sum_{i=1}^n (F_i - F_{i-1}) + \int_0^{F_0} \frac{2}{K^2} (K - F_n)^+ dK \\ &+ \int_{F_0}^{\infty} \frac{2}{K^2} (F_n - K)^+ dK - \frac{1}{3} \sum_{i=1}^n R_i^3 + \sum_{i=1}^n O(R_i^4) \end{aligned}$$

Dividing by n to make it a variance and multiplying with N to annualize gives us the formula for realized variance

$$\begin{aligned} V_{0,T}^R &= \frac{N}{n} \sum_{i=1}^n \left[\ln \frac{F_i}{F_{i-1}} \right]^2 \tag{9.8} \\ V_{0,T}^R &= \frac{2N}{n} \sum_{i=1}^n \frac{F_i - F_{i-1}}{F_{i-1}} \left(\frac{1}{F_{i-1}} - \frac{1}{F_0} \right) + \int_0^{F_0} \frac{2N}{nK^2} (K - F_n)^+ dK + \int_{F_0}^{\infty} \frac{2N}{nK^2} (F_n - K)^+ dK \\ &- \frac{1}{3} \frac{N}{n} \sum_{i=1}^n R_i^3 + \sum_{i=1}^n O(R_i^4) \end{aligned}$$

For the first term one would hold $e^{-r(t_n-t_i)} \frac{2N}{n} \left(\frac{1}{F_{i-1}} - \frac{1}{F_0} \right)$ futures contract positions from day $i - 1$ to day i . One would hold and keep $\frac{2N}{nK^2} dK$ put options for every possible strike between zero and F_0 , and $\frac{2N}{nK^2} dK$ call options for all possible strikes between F_0 and infinity. The third term is the most prominent source of error when we want to perfectly replicate the discretely monitored realized variance $V_{0,T}^R$. When the cubed returns turns out to be a negative number, the third term is positive and results in the realized variance being bigger than the replicating strategy. If we want to calculate the risk-neutral expectation of realized variance $\mathbb{E}^{\mathbb{Q}}(V_{0,T}^R)$, and $\mathbb{E}^{\mathbb{Q}}(R_i^3) < 0$, then the standard approach of using the log-contract relation pricing method under-prices the variance swap, as the fixed variance swap rate will be higher than estimated by the log-contract relation valuation method.

6. Code and spreadsheets

Table 4 – Code to equation mapping

Matlab code name	Section	Implements equation	Calls function(s)
<i>funccHestonCall.m</i>	§2.1	(4)	<i>funccIntegral.m</i>
<i>funccIntegral.m</i>	§2.1	(19)	<i>funccIntegrand.m</i>
<i>funccIntegrand.m</i>	§2.1	(19)	<i>funccPhi.m</i>
<i>funccPhi.m</i>	§2.1	(19)	<i>funccC.m, funccD.m</i>
<i>funccC.m</i>	§2.1	(13)	
<i>funccD.m</i>	§2.1	(14)	
<i>BlackImplied.m</i>	§2.1, §2.2	(23)	<i>BlackOption.m, BlackOptionVega.m</i>
<i>BlackOption.m</i>	§2.1	(20)	
<i>BlackOptionVega.m</i>	§2.1	(22)	
<i>funccHestonCalib.m</i>	§2.2	(24)	<i>funccHestonSquares.m</i>
<i>funccHestonSquares.m</i>	§2.2	(24)	<i>funccHestonCall.m, BlackOption.m</i>
<i>simHestonProcess.m</i>	§2.3	(35)	
<i>fairAnnualVarianceHeston.m</i>	§3.1.1	(43)	
<i>simulaHestonVarianceSwapCap.m</i>	§3.1.2	(46)	
<i>logRelationVarianceSwap.m</i>	§3.1.3	(61), (65)	<i>funtionF.m, OptiononForward.m</i>
<i>simulaHestonVolatilitySwap.m</i>	§3.2.1	(71)	
<i>funFairVolIntegral.m</i>	§3.2.2	(73)	<i>funFairVolIntegrand.m</i>
<i>funFairVolIntegrand.m</i>	§3.2.2	(73), (74)	<i>funA.m, funB.m</i>
<i>funA.m</i>	§3.2.2	(75)	
<i>funB.m</i>	§3.2.2	(75)	
<i>simulaHestonVarianceOption.m</i>	§3.3.1	(76)	
<i>simHestonTimerr.m</i>	§3.4.1	(85)	

Table 5 - Figure to spreadsheet and/or code mapping

Figure and/or table	Calls code	Spreadsheet
Figure 1 Table 1	<i>VBA code in spreadsheet</i>	<i>Top40 returns distribution.xls</i> <i>ALSI returns distribution.xls</i>
Figure 2	<i>funcHestonCall.m</i> <i>BlackImplied.m</i>	<i>Heston Parameter effects.xls</i>
Figure 3	Static volatility surface input	<i>Heston Calibration.xls</i>
Figure 4	<i>BlackOption.m</i> <i>funcHestonCall.m</i> <i>BlackImplied.m</i> <i>Excel Solver</i>	<i>Heston Calibration.xls</i>
Figure 5	Static volatility surface input	<i>Heston Calibration.xls</i>
Figure 6 Table 2	<i>BlackOption.m</i> <i>funcHestonCall.m</i> <i>BlackImplied.m</i> <i>Excel Solver</i>	<i>Heston Calibration.xls</i>
Figure 7 Table 3	<i>simHestonProcess.m</i>	-
Figure 8	<i>fairAnnualVarianceHeston.m</i> <i>logRelationVarianceSwap.m</i> <i>simulaHestonVarianceSwapCap.m</i>	<i>Heston Variance Swap.xls</i>
Figure 9	<i>funFairVolIntegral.m</i> <i>simulaHestonVolatilitySwap.m</i>	<i>Heston Volatility Swap.xls</i>
Figure 10	<i>funFairVolIntegral.m</i> <i>fairAnnualVarianceHeston.m</i>	<i>Heston Volatility Swap.xls</i>
Figure 11	<i>simulaHestonVarianceOption.m</i>	<i>Heston Variance Option.xls</i>
Figure 12	<i>simHestonTimerr.m</i>	<i>Heston Timer Option.xls</i>

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