Quantitative Methods for Economics

Tutorial 5

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TUTORIAL 5

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ECO3021S

PART 1

1. A monopolist sets a price $p = q^{-1/\beta}$ where p is the price, q is the quantity of the good being produced, and β is the elasticity of demand for the good. We assume this elasticity of demand is constant. The monopolist faces a cost function $TC = c.q$.

   (a) What does the firm's cost function imply about the firm's marginal cost?

   (b) Find the equilibrium price and quantity that the monopolist should charge in order to maximize profits.

2. Find the extreme values of the function $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$. Classify the extrema as local (relative) or global (absolute) and as minima or maxima.

3. A South African car manufacturer has the following total revenue and total cost structures:

$$\begin{aligned} TR &= 8Q - Q^2 \\ TC &= \frac{1}{3}Q^3 - 10Q^2 + 80Q + 500 \end{aligned}$$

   How many cars should the firm produce if its goal is to maximize profits?

4. The production of Gautrain rolling stock (Q) requires two inputs, capital (K) and labour (L):

$$Q = f(K, L) = K^{1/4}L^{1/4}$$

   A unit of capital costs R8 and a unit of labour costs R4.

   (a) What is the cost minimising bundle of capital and labour to produce any *fixed* $Q = \overline{Q}$?

   (b) Confirm that your answer to (a) represents a constrained minimum.

   (c) Give an economic interpretation of the Lagrangian multiplier.

PART 2

5. Every individual faces a decision about how to allocate their time between leisure activities (R) and work activities (L). On any day, there are only 24 hours in which the individual can either be working or enjoying leisure activities i.e. $L + R = 24$. If the individual works, he earns an income (I) where $I = wL$. In other words, the income earned is equal to the wage paid (w), multiplied by the number of hours worked (L). Suppose the individual's utility function is given by:

$$U(I, R) = 4IR^2$$

The hourly wage rate is R4. How many hours should the individual devote to leisure, and how many should be devoted to work?

6. A perfectly competitive firm sells its output for $R12$ per unit. This firm has a total revenue function and total cost function given by:

$$\begin{aligned} TR &= P.Q \\ TC &= Q^3 - 4.5Q^2 + 18Q - 7 \end{aligned}$$

   (a) What level of output should this firm produce in order to maximize profits?

   (b) According to microeconomic theory, perfectly competitive firms will maximize profit by producing where $P = MC$, and where the slope of the MR curve is less than the slope of the MC curve. Show that theory holds in this example.

7. A monopolist can produce quantities x and y of two products X and Y respectively, at cost $4x^2 + xy + 2y^2$. The inverse demand functions are

$$\begin{aligned} p_X &= 150 - 5x + y \\ p_Y &= 30 + 2x - 2y \end{aligned}$$

where p_X and p_Y are the prices charged for X and Y.

   (a) Find the values of x, y, p_X and p_Y which maximise profit, and the maximal profit.

   (b) Confirm that your answer to (a) represents a maximum.

8. The utility a consumer derives from consuming two goods, A and B, is determined by the following utility function:

$$U = 40A^{0.25}B^{0.5}$$

If A costs $R4$ per unit, and B costs $R10$ per unit, and the consumer has $R600$ to spend, what combination of A and B will maximize the consumer's utility?

TUTORIAL 4 SOLUTIONS

ECO3021S

PART 1

1. A monopolist sets a price $p = q^{-1/\beta}$ where p is the price, q is the quantity of the good being produced, and β is the elasticity of demand for the good. We assume this elasticity of demand is constant. The monopolist faces a cost function $TC = c.q$.

   (a) What does the firm's cost function imply about the firm's marginal cost?

   $$MC(q) = \frac{d}{dq}\left[TC(q)\right] = c.$$

   This implies that marginal cost is constant for a monopolist.

   (b) Find the equilibrium price and quantity that the monopolist should charge in order to maximize profits.

   The monopolist will set its price where $MC = MR$. Here

   $$\begin{aligned} TC(q) &= cq \\ TR(q) &= pq = q(q^{-(1/\beta)}) = q^{1-(1/\beta)} = q^{(\beta-1)/\beta}. \end{aligned}$$

   So now

   $$\begin{aligned} MR &= MC \\ \Rightarrow \left(\frac{\beta-1}{\beta}\right) q^{-(1/\beta)} &= c \\ q^{-(1/\beta)} &= \left(\frac{\beta}{\beta-1}\right) \times c \\ q &= \left[\left(\frac{\beta}{\beta-1}\right) \times c\right]^{-\beta} \end{aligned}$$

   And

   $$\begin{aligned} p &= q^{-(1/\beta)} \\ &= \left(\left[\left(\frac{\beta}{\beta-1}\right) \times c\right]^{-\beta}\right)^{-(1/\beta)} \\ &= \left(\frac{\beta}{\beta-1}\right) \times c \end{aligned}$$

2. Find the extreme values of the function $z = 8x^3 + 2xy - 3x^2 + y^2 + 1$. Classify the extrema as local (relative) or global (absolute) and as minima or maxima.

$$\begin{aligned} \mathbf{D}z &= \begin{bmatrix} \partial z/\partial x \\ \partial z/\partial y \end{bmatrix} \\ &= \begin{bmatrix} 24x^2 + 2y - 6x \\ 2x + 2y \end{bmatrix} \end{aligned}$$

FOCS

$$24x^2 + 2y - 6x = 0 \quad (1)$$
$$2x + 2y = 0 \quad (2)$$

Solve simultaneously:

From (2)

$$y = -x \quad (3)$$

Substitute (3) into (1)

$$\begin{aligned} 24x^2 + 2(-x) - 6x &= 0 \\ \Rightarrow 24x^2 - 8x &= 0 \\ \Rightarrow x(24x - 8) &= 0 \\ \Rightarrow x &= 0, \frac{1}{3} \end{aligned}$$

Therefore, candidates for extrema are $x = 0, y = 0, z = 0$ and $x = \frac{1}{3}, y = -\frac{1}{3}, z = \frac{23}{27}$.

SOCS:

$$\begin{aligned} \mathbf{H} = \mathbf{D}^2 z &= \begin{bmatrix} \partial^2 z/\partial x^2 & \partial^2 z/\partial x \partial y \\ \partial^2 z/\partial y \partial x & \partial^2 z/\partial y^2 \end{bmatrix} \\ &= \begin{bmatrix} 48x - 6 & 2 \\ 2 & 2 \end{bmatrix} \end{aligned}$$

The definiteness of the Hessian matrix depends on the value of x, so it cannot be everywhere positive or negative definite.

We must determine the definiteness of the Hessian matrix at the candidate extrema:
At $x = 0, y = 0, z = 0:$

$$\begin{aligned} \mathbf{H} &= \begin{bmatrix} -6 & 2 \\ 2 & 2 \end{bmatrix} \\ |\mathbf{H}_1| &= |-6| = -6 < 0 \\ |\mathbf{H}_2| &= \begin{vmatrix} -6 & 2 \\ 2 & 2 \end{vmatrix} = -16 < 0 \end{aligned}$$

At this point, **H** is neither positive nor negative definite. So this point is neither a minimum nor a maximum.
At $x = \frac{1}{3}, y = -\frac{1}{3}, z = \frac{23}{27}:$

$$\begin{aligned} \mathbf{H} &= \begin{bmatrix} 10 & 2 \\ 2 & 2 \end{bmatrix} \\ |\mathbf{H}_1| &= |10| = 10 > 0 \\ |\mathbf{H}_2| &= \begin{vmatrix} 10 & 2 \\ 2 & 2 \end{vmatrix} = 16 > 0 \end{aligned}$$

At this point, **H** is positive definite. So this point is a *local minimum.*

3. A South African car manufacturer has the following total revenue and total cost structures:

$$\begin{aligned} TR &= 8Q - Q^2 \\ TC &= \frac{1}{3}Q^3 - 10Q^2 + 80Q + 500 \end{aligned}$$

How many cars should the firm produce if its goal is to maximize profits?

To maximize profits, set up the objective function and find the stationary points by setting the first derivative equal to zero and solving:

$$\begin{aligned} \pi(Q) &= TR - TC \\ &= 8Q - Q^2 - (\frac{1}{3}Q^3 - 10Q^2 + 80Q + 500) \\ &= -\frac{1}{3}Q^3 + 9Q^2 - 72Q - 500 \end{aligned}$$

$$\begin{aligned} \pi'(Q) &= 0 \\ \Rightarrow -Q^2 + 18Q - 720 &= 0 \\ \Rightarrow (Q - 12)(Q - 6) &= 0 \end{aligned}$$

Thus $Q = 12$ or 6.

Now we must check whether these stationary points are maxima or minima by evaluating the second derivative at the stationary points:

$$\pi''(Q) = -2Q + 18$$

So, we can see that there is a **relative maximum** where production is equal to 12 units. Therefore, the firm maximizes its profits where it produces 12 cars.

4. The production of Gautrain rolling stock (Q) requires two inputs, capital (K) and labour (L):

$$Q = f(K, L) = K^{1/4}L^{1/4}$$

A unit of capital costs R8 and a unit of labour costs R4.

(a) What is the cost minimising bundle of capital and labour to produce any *fixed* $Q = \overline{Q}$?

Problem:

$$\min_{K,L} 8K + 4L \text{ subject to } \overline{Q} = K^{1/4}L^{1/4}$$

Lagrangian:

$$= 8K + 4L + \lambda\left(\overline{Q} - K^{1/4}L^{1/4}\right)$$

FOCs:

$$\frac{\partial}{\partial K} = 8 - \lambda\frac{1}{4}K^{-3/4}L^{1/4} = 01 \quad (4)$$

$$\frac{\partial}{\partial L} = 4 - \lambda\frac{1}{4}K^{1/4}L^{-3/4} = 02 \quad (5)$$

$$\frac{\partial}{\partial \lambda} = \overline{Q} - K^{1/4}L^{1/4} = 03 \quad (6)$$

(4) ÷ (5)

$$2 = \frac{L}{K}$$

$$\Rightarrow L = 2K4 \quad (7)$$

Sub (7) into (6)

$$\overline{Q} - K^{1/4}(2K)^{1/4} = 0$$

$$K^* = \frac{\overline{Q}^2}{\sqrt{2}}$$

$$L^* = \sqrt{2}\overline{Q}^2$$

(b) Confirm that your answer to (a) represents a constrained minimum.

$$\overline{\mathbf{H}} = \begin{bmatrix} 0 & -\frac{1}{4}K^{-3/4}L^{1/4} & -\frac{1}{4}K^{1/4}L^{-3/4} \\ -\frac{1}{4}K^{-3/4}L^{1/4} & \frac{3}{16}\lambda K^{-7/4}L^{1/4} & -\frac{1}{16}\lambda K^{-3/4}L^{-3/4} \\ -\frac{1}{4}K^{1/4}L^{-3/4} & -\frac{1}{16}\lambda K^{-3/4}L^{-3/4} & \frac{3}{16}\lambda K^{1/4}L^{-7/4} \end{bmatrix}$$

$$\begin{aligned} \left|\overline{\mathbf{H}}\right| &= -\frac{1}{32}\lambda K^{-5/4}L^{-5/4} \\ &< 0 \\ &\Rightarrow \text{minimum} \end{aligned}$$

(c) Give an economic interpretation of the Lagrangian multiplier.

It is the marginal cost.

PART 2

5. Every individual faces a decision about how to allocate their time between leisure activities (R) and work activities (L). On any day, there are only 24 hours in which the individual can either be working or enjoying leisure activities i.e. $L + R = 24$. If the individual works, he earns an income (I) where $I = wL$. In other words, the income earned is equal to the wage paid (w), multiplied by the number of hours worked (L). Suppose the individual's utility function is given by:

$$U(I, R) = 4IR^2$$

The consumer has to maximize utility (which depends on either working and earning money or engaging in leisurely pursuits) subject to a time constraint:

$$\begin{aligned} \Rightarrow \max\ U(I, R) &= 4IR^2 \\ \text{subject to } L + R &= 24 \end{aligned}$$

Our objective function is in terms of I and R, so we have to re-write our constraint i.t.o. I and R as well.

We know $I = wL$ and that $w = 4$ (as given). Therefore, $L = \dfrac{I}{w} = \dfrac{I}{4}$ and so the constraint becomes

$$\frac{I}{4} + R = 24$$

. We can now set up our Lagrangian function:

$$\max_{I,R} = 4IR^2 + \lambda(24 - \frac{1}{4}I - R)$$

F.O.C.s:

$$\begin{aligned} \frac{\partial}{\partial I} &= 0 \Rightarrow 4R^2 - \frac{1}{4}\lambda = 01 & (8) \\ \frac{\partial}{\partial R} &= 0 \Rightarrow 8IR - \lambda = 02 & (9) \\ \frac{\partial}{\partial \lambda} &= 0 \Rightarrow 24 - \frac{1}{4}I - R = 03 & (10) \end{aligned}$$

Solve for the independent variables. To do this we can divide (8) by (9)

$$\begin{aligned} \Rightarrow \frac{4R^2}{8IR} &= \frac{\frac{1}{4}\lambda}{\lambda} \\ \Rightarrow \frac{R}{I} &= \frac{2}{4} \\ \Rightarrow R &= \frac{1}{2}I \end{aligned}$$

Substitute for R in equation (10) to get

$$\begin{aligned} 24 - \frac{1}{4}I - \frac{1}{2}I &= 0 \\ \Rightarrow \frac{3}{4}I &= 24 \\ \Rightarrow \bar{I} &= 32 \end{aligned}$$

$$\begin{aligned} \therefore \bar{R} &= \frac{1}{2}I = \frac{1}{2}(32) = 16 \\ \text{And } \bar{L} &= \frac{1}{w}I = \frac{1}{4}(32) = 8 \end{aligned}$$

The consumer therefore maximizes utility when he/she spends 16 hours on leisure activities and 8 hours on work (which add up to 24, as per the constraint). The value of the utility function for these values is equal to:

$$U^*(\bar{I}, \bar{R}) = 4(32)(16)^2 = 32768$$

Is this definitely a maximum? To answer this question we have to evaluate the S.O.C.s by constructing and interpreting the Bordered Hessian:

$$\begin{aligned}\overline{\mathbf{H}} &= \begin{bmatrix} 0 & g_x & g_y \\ g_x & xx & xy \\ g_y & yx & yy \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\frac{1}{4} & -1 \\ -\frac{1}{4} & 0 & 8R \\ -1 & 8R & 8I \end{bmatrix}\end{aligned}$$

We are now interested in the signs of the second and subsequent principal minors: (here there will be only a second principal minor).

$$\begin{aligned}|\overline{\mathbf{H}}_2| &= \frac{1}{4}\left[(-\frac{1}{4})(8I) - (-1)(8R)\right] - 1\left[(-\frac{1}{4})(8R) - 0\right] \\ &= \frac{1}{4}[-2I + 8R] + 2R \\ &= -\frac{1}{2}I + 4R\end{aligned}$$

Which is a maximum if $-\frac{1}{2}I + 4R > 0$.

6. A perfectly competitive firm sells its output for $R12$ per unit. This firm has a total revenue function and total cost function given by:

$$\begin{aligned}TR &= P.Q \\ TC &= Q^3 - 4.5Q^2 + 18Q - 7\end{aligned}$$

(a) What level of output should this firm produce in order to maximize profits?

$$\max_Q \pi = P.Q - Q^3 + 4.5Q^2 - 18Q + 7$$

You are told that $P = 12$, substitute this in and re-write the expression as:

$$\max_Q \pi = 12Q - Q^3 + 4.5Q^2 - 18Q + 7$$

F.O.C.:

$$\begin{aligned}\frac{\partial \pi}{\partial Q} = 12 - 3Q^2 + 9Q - 18 &= 0 \\ \Rightarrow -3Q^2 + 9Q - 6 &= 0 \\ \Rightarrow (3Q - 6)(Q - 1) &= 0\end{aligned}$$

So $Q = 1$ or 2.

Now, to be a maximum we need the second derivative to be negative:

$$\frac{\partial^2 \pi}{\partial Q^2} = -6Q + 9 \quad < \quad 0$$
$$\text{if } Q \quad > \quad \frac{9}{6}$$

Thus, $Q = 2$ gives us our **maximum**. (Alternatively, you could just plug in the two values into your S.O.C. and see which one is negative).

(b) According to microeconomic theory, perfectly competitive firms will maximize profit by producing where $P = MC$, and where the slope of the MR curve is less than the slope of the MC curve. Show that theory holds in this example.

Derive an expression for MR and MC

$$\begin{aligned} MR &= \frac{\partial TR}{\partial Q} = 12 \\ MC &= \frac{\partial TC}{\partial Q} = 3Q^2 - 9Q + 18 \end{aligned}$$

The firm maximizes its profits where

$$\begin{aligned} MR &= MC \\ \Rightarrow 12 &= 3Q^2 - 9Q + 18 \end{aligned}$$

and with a bit of re-arranging you will see that you have the exact same expression as you derived earlier for the first derivative. So, again, you'll solve for the values of $Q = 1$ and $Q = 2$.

Now you are asked to show that the firm only max. profits if the slope of MR curve is less than the slope of the MC curve. Essentially, you're being asked (in a roundabout way) to find the S.O.C.. To find the slope of the MR curve, take the second derivative of the TR:

$$\begin{aligned} TR''(Q) &= \frac{d}{dQ}(MR) = 0 \\ TC''(Q) &= \frac{d}{dQ}(MC) = 6Q - 9 \end{aligned}$$

If the slope of the MR curve is less than the slope of the MC curve, then this means that

$$\begin{aligned} 6Q - 9 &> 0 \\ \Rightarrow Q &> \frac{9}{6} \end{aligned}$$

which is again exactly what you showed earlier with your S.O.C.!!!

7. A monopolist can produce quantities x and y of two products X and Y respectively, at cost $4x^2 + xy + 2y^2$. The inverse demand functions are

$$\begin{aligned} p_X &= 150 - 5x + y \\ p_Y &= 30 + 2x - 2y \end{aligned}$$

where p_X and p_Y are the prices charged for X and Y.

(a) Find the values of x, y, p_X and p_Y which maximise profit, and the maximal profit.

$$\begin{aligned} \pi &= xp_X + yp_Y - \left(4x^2 + xy + 2y^2\right) \\ &= x\left(150 - 5x + y\right) + y\left(30 + 2x - 2y\right) - 4x^2 - xy - 2y^2 \\ &= 150x - 5x^2 + xy + 30y + 2xy - 2y^2 - 4x^2 - xy - 2y^2 \\ &= -9x^2 + 2xy + 150x - 4y^2 + 30y \end{aligned}$$

FOC:

$$\begin{aligned} \mathbf{D}\pi\left(x, y\right) = \begin{bmatrix} \partial\pi/\partial x \\ \partial\pi/\partial y \end{bmatrix} &= 0 \\ \Rightarrow \begin{bmatrix} -18x + 2y + 150 \\ 2x - 8y + 30 \end{bmatrix} &= 0 \end{aligned}$$

Solving simultaneously:

$$\begin{aligned} x^* &= 9 \\ y^* &= 6 \end{aligned}$$

Therefore:

$$\begin{aligned} p_X^* &= 150 - 5\left(9\right) + 6 = 111 \\ p_Y^* &= 30 + 2\left(9\right) - 2\left(6\right) = 36 \end{aligned}$$

And:

$$\begin{aligned} \pi_{\max} &= xp_X + yp_Y - \left(4x^2 + xy + 2y^2\right) \\ &= 9\left(111\right) + 6\left(36\right) - \left(4\left(9\right)^2 + 9\left(6\right) + 2\left(6\right)^2\right) \\ &= 765 \end{aligned}$$

(b) Confirm that your answer to (a) represents a maximum.

SOC:

$$\mathbf{H} = \mathbf{D}^2\pi(x,y) = \begin{bmatrix} \partial^2\pi/\partial x^2 & \partial\pi/\partial x\partial y \\ \partial\pi/\partial y\partial x & \partial^2\pi/\partial y^2 \end{bmatrix} = \begin{bmatrix} -18 & 2 \\ 2 & -8 \end{bmatrix}$$

$$|\mathbf{H}_1| = |-18| < 0$$
$$|\mathbf{H}_2| = \begin{vmatrix} -18 & 2 \\ 2 & -8 \end{vmatrix} = 140 > 0$$

Thus, **H** is negative definite and hence we have a (global) maximum.

8. The utility a consumer derives from consuming two goods, A and B, is determined by the following utility function:

$$U = 40A^{0.25}B^{0.5}$$

If A costs $R4$ per unit, and B costs $R10$ per unit, and the consumer has $R600$ to spend, what combination of A and B will maximize the consumer's utility?

$$U = 40A^{0.25}B^{0.5}$$
$$p_A = 4,\ p_B = 10,\ m = 600$$

Set up the Lagrangian function:

$$\max_{A,B} = 40A^{0.25}B^{0.5} + \lambda(600 - 4A - 10B)$$

F.O.C.s:

$$\frac{\partial}{\partial A} = 0 \implies 10A^{-0.75}B^{0.5} - 4\lambda = 01 \quad (11)$$
$$\frac{\partial}{\partial B} = 0 \implies 20A^{0.25}B^{-0.5} - 10\lambda = 02 \quad (12)$$
$$\frac{\partial}{\partial \lambda} = 0 \implies 600 - 4A - 10B = 03 \quad (13)$$

Now solve for the variables:

Divide (11) by (12)

$$\begin{aligned} \Rightarrow \frac{10A^{-0.75}B^{0.5}}{20A^{0.25}B^{-0.5}} &= \frac{4\lambda}{10\lambda} \\ \Rightarrow \frac{B}{2A} &= \frac{2}{5} \\ \Rightarrow B &= \frac{4A}{5} \end{aligned}$$

Now substitute B into equation (13) and solve for A

$$\begin{aligned} 600 - 4A - 10B &= 0 \\ \Rightarrow 600 - 4A - 10(\frac{4A}{5}) &= 0 \\ \Rightarrow 600 - 12A &= 0 \\ \Rightarrow 12A &= 600 \\ \Rightarrow \bar{A} &= 50 \end{aligned}$$

$$\therefore \bar{B} = \frac{4(50)}{5} = 40$$

The consumer therefore maximizes utility when he/she consumes 50 units of A and 40 units of B. However, we must now check if this is indeed a maximum. To do this we check the S.O.C.s by constructing the Bordered Hessian and evaluating it:

$$\begin{aligned} \overline{\mathbf{H}} &= \begin{bmatrix} 0 & g_x & g_y \\ g_x & xx & xy \\ g_y & yx & yy \end{bmatrix} \\ &= \begin{bmatrix} 0 & -4 & -10 \\ -4 & \left(-\frac{30}{4}A^{-1.75}B^{0.5}\right) & \left(5A^{-0.75}B^{-0.5}\right) \\ -10 & \left(5A^{-0.75}B^{-0.5}\right) & \left(-10A^{0.25}B^{-1.5}\right) \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \left|\overline{\mathbf{H}}_2\right| &= 4\left[(-\frac{30}{4}A^{-1.75}B^{0.5})(-10A^{0.25}B^{-1.5})\right] - 10\left[(5A^{-0.75}B^{-0.5})\right]^2 \\ &= 4\left[\frac{300}{4}A^{-1.5}B^{-1}\right] - 10\left[25A^{-1.5}B^{-1}\right] \\ &= 300A^{-1.5}B^{-1} - 250A^{-1.5}B^{-1} \\ &= 50A^{-1.5}B^{-1} > 0 \\ \therefore \left|\overline{\mathbf{H}}_2\right| &> 0 \end{aligned}$$

Because $\left|\overline{\mathbf{H}}_2\right| > 0 \ \forall A, B \in \mathbb{N}^+ \Rightarrow$ the Hessian is negative definite and the function has a **maximum**.