Volatility Transformation in a Multi-Curve Setting Applied to Caps and Swaptions

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy at the University of the Cape Town. It has not been submitted before for any degree or examination to any other University.

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Abstract

The effects of the 2007-08 financial crisis have resulted in a sharp change in the way interest rate markets are viewed as well as modelled. As a result of the crisis, the general market framework has transitioned from a single curve framework to what is commonly known as the ‘multiple-curve’ framework. In addition to this, there is debate as to which curve to use for discounting.

This dissertation will initially aim to give a succinct, yet thorough overview of the changes affecting interest rate modelling as a result of the financial crisis. In particular pricing methods that are consistent with the multi-curve framework are presented. Adaptations of the popular Libor Market Model (LMM) and Stochastic Alpha-Beta-Rho (SABR) consistent with the new market framework are also presented.

The second aim of the dissertation is to outline and implement methods of transforming volatilities within this new market framework. The market quotes available for caps/floors and swaptions often assume a particular payment tenor, for example swaption volatilities are typically quoted assuming payment legs of six months. As such, if one wanted to price an identical swaption based on payment legs of three months, or even monthly payments, some form of transformation is needed. The methods presented and implemented are largely based on the work of Kienitz (2013). The methods described are implemented to transform six month cap and swaption volatility surfaces to three month surfaces.

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Chapter 1

Introduction

The aim of this section is to provide a succinct overview of the changes experienced during and as a result of the credit crisis of 2007. The dissertation will aim to draw the readers attention to the main features of the current interest rate market framework, including the pricing of linear instruments such as Forward Rate Agreements (FRAs) and Interest Rate Swaps (IRSs), as well as the popular vanilla interest rate derivatives, namely caps, floors and swaptions. This is followed by a brief overview of the recent attempts towards adapting interest rate models from the single curve framework to the modern multi-curve framework, particular emphasis is given to the Libor Market Model (LMM), which has become widely used. Finally, an extension to the Stochastic Alpha-Beta-Rho (SABR) model is presented.

Chapter 2

The Crisis

The financial crisis which started in 2007 has had several consequences and has resulted in a so called 'market evolution' (Bianchetti and Carlicchi, 2011). Many of these changes, or rather realisations, are based around credit and liquidity risk in markets and their effects on market instruments. There is a vast amount of recent literature presenting and analysing the major market anomalies that occurred during the crisis, many of which have persisted. This has resulted in practitioners accepting that the establishment of a new market framework is required. Below is a brief outline of some of these market anomalies.

- **Divergence in Euribor and OIS rates**
  Bianchetti and Carlicchi (2011) present a detailed analysis of the divergence in Euribor (6M) and the Eonia/OIS (6M) rates which we summarise here. Before the crisis, historical data showed the series of 6M Euribor and 6M Eonia/OIS rates were 'strictly overlapping' and the difference between the rates was never more than about six basis points (Bianchetti and Carlicchi, 2011). This basis spread was often seen as negligible.
  When the crisis occurred, there was a simultaneous increase in Euribor rates and a decrease in OIS rates, and as such the basis spread between the two 'exploded', peaking at 222 basis points and stabilizing over time at between 30 to 40 basis points (Bianchetti and Carlicchi, 2011). The high peak of the basis spread during the crisis could be explained by the fact that at the time the solvency of the entire market became a major worry as well as the general market being in a state of panic. However, the persistence of the basis spread post-crisis is generally viewed as a result of the different liquidity and credit risk levels implicit in these rates. The OIS rate, being the reference rate for overnight Over-The-Counter (OTC) transactions, has significantly lower credit risk than the Euribor rates which are the reference rates for longer deposits in

the inter-bank market (Bianchetti and Carlicchi, 2011).

- **Divergence of Euribor FRA and forward rates**
Mercurio (2009) analyses the divergence of Euribor FRAs (e.g., 3x6 FRA) and the forward rates implied by corresponding Eonia/OIS 3M and 6M deposits. These rates followed each other very closely and the spread between them was seen as negligible. However, during and since the crisis, there has been a significant increase in this spread. Once again, the reason for this permanent divergence is perceived credit and liquidity risks. In the pre-crisis framework a divergence between FRA and forward rates would suggest an arbitrage opportunity, however Mercurio (2009) shows that the divergence can be explained with the use of a simple credit model. Making use of similar notation, this is shown below:

Let τ_t be the default time of an interbank counterparty. Assume independence between the default time and interest rates. Let R denote the fixed recovery amount.
The value of a defaultable deposit, $D(t,T)$, at time t with maturity T is given by:

$$\begin{aligned} D(t,T) &= E^Q\left[e^{\int_t^T -r(u)du}\left(R + (1-R)1_{\{\tau_t > T\}}\right)|\mathcal{F}_t\right] \\ &= RP(t,T) + (1-R)P(t,T)E^Q\left[1_{\{\tau_t>T\}}|\mathcal{F}_t\right], \end{aligned}$$

where $P(t,T)$ is the price of a default-free zero coupon bond, and the measure Q is the risk-neutral measure. Now let $\mathcal{Q}(t,T) = E^Q\left[1_{\{\tau_t>T\}}|\mathcal{F}_t\right]$, then one can solve for the simple Libor rate, $(L(T_1,T_2))$ implied by the defaultable deposit:

$$L(T_1,T_2) = \frac{1}{\tau_{1,2}}\left[\frac{1}{D(T_1,T_2)} - 1\right] = \frac{1}{\tau_{1,2}}\left[\frac{1}{P(T_1,T_2)}\frac{1}{R+(1-R)\mathcal{Q}(T_1,T_2)} - 1\right],$$

where $\tau_{1,2}$ is the year fraction between T_1 and T_2. If one assumes the FRA, F_X, has no counterparty risk, then its value at $t=0$ is:

$$\begin{aligned} 0 &= E^Q\left[\exp^{\int_0^{T_1} -r(u)du}\frac{\tau_{1,2}(L(T_1,T_2) - F_X)}{1+\tau_{1,2}L(T_1,T_2)}\right] \\ &= P(0,T_1) - (1+\tau_{1,2}F_X)P(0,T_2)(R+(1-R)E^Q[\mathcal{Q}(T_1,T_2)]). \end{aligned}$$

The final line is achieved by substituting for $L(T_1,T_2)$ and simplifying. Solving for the FRA rate F_X yields:

$$F_X = \frac{1}{\tau_{1,2}}\left[\frac{P(0,T_1)}{P(0,T_2)}\frac{1}{(R+(1-R)E^Q[\mathcal{Q}(T_1,T_2)])} - 1\right].$$

Mercurio (2009) then shows that since $0 \leq R \leq 1$ and $0 < \mathcal{Q}(T_1, T_2) < 1$, we know that $0 < R + (1 - R)E^Q[\mathcal{Q}(T_1, T_2)] < 1$. Hence, one can conclude that

$$F_X > \frac{1}{\tau_{1,2}} \left[\frac{P(0, T_1)}{P(0, T_2)} - 1 \right].$$

Thus the FRA rate is greater than the forward rate implied by two default free bonds.

- **Divergence in Basis Swap spreads**.

  Basis swaps are swaps where both parties receive floating payments, however the frequency/tenor of each leg is different. Subsequently, this can be seen as the difference between two swaps with different floating leg payments, but the same fixed legs that both pay annually and hence cancel each other out. Using the same logic, the quoted basis swap rates are the differences in the swap rates of the two underlying swaps. For example the Euribor 3M vs Euribor 6M basis swap (with fixed maturity) has swap rate equal to the difference between the fair swap rate for the Euribor 3M vs annual fixed, and the fair swap rate for Euribor 6M vs annual fixed.

  Prior to the crisis, the existence of these spreads was acknowledged, but again they were assumed negligible. These basis spreads 'exploded' during the crisis and also have yet to reach pre-crisis levels. The spreads provide evidence of the markets perceived difference in liquidity and credit risk in rates corresponding to different underlying tenors/payment legs (Bianchetti and Carlicchi, 2011).

For an overview of the rates that make up the Euro and London Interbank markets, the risk levels corresponding to each rate, the number of banks which set the rates and other information, see Bianchetti and Carlicchi (2011).

Chapter 3

The Market Response

Whilst it is generally accepted that the divergence in the above mentioned market rates as well as many others are due to credit and liquidity risk levels, the development of a sound credit and liquidity risk theory is a very difficult task, and a market accepted risk approach has yet to be introduced. The market instead has taken the approach of modelling multiple rates, each with its own credit and liquidity risk level.

Below we outline the two main shifts in market dynamics after the crisis: market rate segmentation and the discounting curve.

3.1 Market Segmentation

The markets reaction to the crisis and its effects was to segment the interest rate market into multiple sub-markets. According to Bianchetti and Carlicchi (2011) these sub-markets are “characterized, in principle, by different internal dynamics, liquidity and credit risk premia, reflecting the different views and interests of the market players”. The division of the sub-markets is based on the underlying rate tenor, with the common divisions being the 1M, 3M, 6M and 12M sub-markets. The idea being that instruments with different rate tenors have different credit and liquidity levels. As such, the pre-crisis single curve pricing methodology is no longer consistent with this market framework. In accordance with this new framework, multiple curves are constructed, one for each sub-market. These are bootstrapped using market quotes for instruments with the same rate tenor. It is assumed all such instruments are homogeneous in credit and liquidity risk levels. The multiple curves (i.e 1M, 3M, 6M and 12M) are known as the “Forwarding Curves”, and are used to generate forward rates. Their exact role in the pricing of instruments is detailed in the section 4.5. Another common sub-market is the OIS sub-market, and its main use comes when dealing with discounting cashflows. This is detailed below.

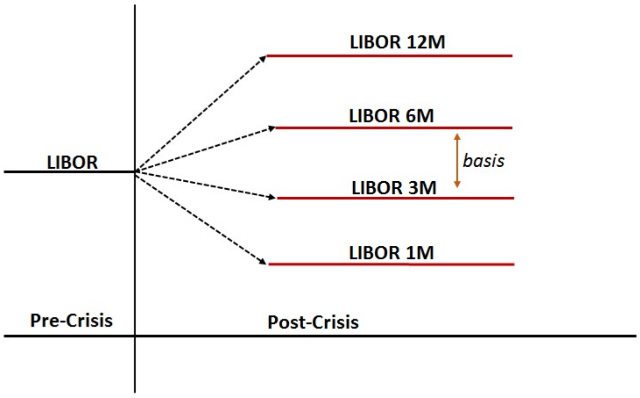

Fig. 3.1: Graphical illustration of single to multi-curve transition

3.2 Discounting Debate

The concept of the 'risk-free' rate is a vitally important concept in finance, particularly in the pricing of derivatives, where it is used to discount cashflows. Prior to the crisis, the rate mostly used to proxy the risk-free rate was simply the LIBOR rate. In other words, using our terminology from the previous section, the risk-free curve coincided with the single forwarding curve. As the rate used to compute the cash flows was the same rate used to discount the cash flows, it resulted in very straightforward valuation formulae/methodologies, contributing to the popularity of this rate being used to proxy the risk-free rate (Hull and White, 2013).

In the post-crisis multi-curve framework, the OIS (Overnight Indexed Swap) rate has become the main choice for discounting, particularly for collateralized contracts. An OIS is a swap whereby the fixed payment corresponding to a usual tenor (e.g 3M, 6M, 1yr) is exchanged for a floating payment. However the floating payment is an average of the overnight rates during each payment leg. In other words the floating payment corresponds to the return earned by rolling over an overnight deposit each day for the duration of the payment leg. As such the OIS rate is seen as having the credit and liquidity risk levels of the inter-bank overnight market. The justification for the use of the OIS rate as a proxy for the risk-free rate is closely related to collateralisation.

The number of collateralized contracts increased greatly in the aftermath of the crisis. This had several effects, one of which was the view that "the CSA margination rate and the discount rate of future cashflows must match" (Bianchetti and Carlicchi, 2011). The idea behind this view is that collateralised derivatives are essentially funded by the collateral posted, and as such this should be very similar to the risk-free rate because collateral should not earn excess return. The CSA margination rate is the interest earned on collateral, and is typically an overnight rate. Thus the need for a discounting curve that carries similar 'funding levels [as] in an overnight collateralized inter-bank market' (Bianchetti and Carlicchi, 2011) arose, and hence the natural choice of the OIS rate.

Conversely, for non-collateralized transactions, the common view is that the Libor rate should be used for discounting as it is a 'better estimate of the dealers cost of funding [rather] than the OIS rate' Hull and White (2013).

Hull and White (2013) suggest there are more factors to be considered when considering a discounting curve beyond simply collateralized versus uncollateralized, as these arguments are based mainly on the manner in which a contract is funded. Hull and White (2013) conclude that the OIS curve is the better proxy for the risk-free rate in both collateralized and non-collateralized dealings and gives a better estimate of the no-default value of a transaction. Also Hull warns that Libor discounting used together with CVA and DVA credit adjustments can "lead to double counting for credit risk" Hull and White (2013).

Thus, there is some debate, but Bianchetti and Carlicchi (2011) summarise the general market approach to selecting a discount curve as follows:

- Contract under CSA with daily margination and overnight rate: **use OIS based Curve**
- Contract does not have CSA: **use funding curve**
- Contract under a non-standard CSA (different margination rate or frequency): **an appropriate curve would need to be constructed**

Similarly to the forwarding curves, an OIS curve can be bootstrapped using market quoted OIS rates. As such, the OIS is typically considered the fifth common sub-market. The sub-markets are now, in order of perceived credit risk level; {OIS, 1M, 3M, 6M and 12M}.

3.3 Difficulties as a Result of New Market Framework

This new market framework has added several layers of complexity to the single curve framework. Bianchetti (2008) suggests that the current multi-curve framework is more demanding than the single curve framework in at least the following ways:

- Selecting a discounting curve:
  The common approaches to selecting a discounting curve were outlined in the previous section.

- Constructing multiple curves:
  The construction of yield curves was already a non-trivial task, and in the mutli-curve framework this becomes an even more taxing task. Constructing multiple curves requires more input instruments, as each curve requires its own set of liquidly traded instruments out to the required maturity. This then results in the need for complex interpolation schemes and bootstrapping methodologies to deal with poor liquidity levels and possibility of multiple market quotes per instrument.

- Hedging:
  More bootstrapping instruments implies more hedging instruments, this coupled with liquidity issues means that hedging becomes more complicated.

Chapter 4

Modern Pricing of Interest Rate Instruments

4.1 Mathematical Representation of The New Market Framework

The definitions, assumptions as well as notation presented here are similar to that found in the literature, particularly Bianchetti (2008) and Mercurio (2009).

Assume, within a single-currency market, there exist multiple interest rate sub-markets, $x = \{1, ..., N\}$, distinguishable by the underlying rate tenor, $\{\delta_1, ..., \delta_N\}$. Corresponding to each sub-market is a yield curve C_x in the form of a term structure of discount factors (or zero-coupon bond prices) $P_x(t, T_i)$ for some $T_i \geq t$.
Assume the existence of a discounting curve D, and corresponding discount factors $P_D(t, T_i)$ for the discounting of cash flows.

Note that each sub-market has a time structure consistent with the tenor defining that particular sub-market, hence we have time structures $\{T_i^x\}$ for $i = 0, 1, 2, ...$

Simple forward rates at time t corresponding to a time interval $[T, S]$ are defined for each sub-market as follows:

$$F_x(t, T, S) = \frac{1}{\tau_x(T, S)} \left[\frac{P_x(t, T)}{P_x(t, S)} - 1 \right],$$

where $x \in \{1, ..., N, D\}$ and $\tau_x(T, S)$ is the particular year fraction.
As a short-hand we define the following:

$$F_x^k(t) = F_x(t, T_{k-1}^x, T_k^x).$$

A common assumption is that some arbitrage principles that were believed to hold in the single curve framework now hold within each sub-market. The assumption made here is that the martingale property of forward rates holds within each sub-market:

$$F_x(t,T,S) = E^{Q_x^S}[F_x(T,T,S)|\mathcal{F}_t] = E^{Q_x^S}[R_x(T,S)|\mathcal{F}_t],$$

where $R_x(T,S)$ is the time T spot Libor rate for maturity S. Q_x^S is the sub-market x S-forward measure, with numeraire $P_x(t,S)$. Hence there is an added layer of complexity when considering the pricing measure. Consistent with the literature and the single-curve setting, the pricing measures are those associated with the discounting curve.

4.2 Measures of Importance

Select probability measures from the discounting curve are presented here as in Mercurio (2009):

- **The T-forward Measure** Q_D^T
  The numeraire asset is $P_D(.,T)$

- **The Spot Libor Measure** $Q_D^{\mathcal{T}}$
  The numeraire asset is the discretely balanced bank account $B_D^{\mathcal{T}}$, rebalanced at times $\mathcal{T} = \{T_0^x, T_1^x, ..., T_M^x\}$ dictated by the cash flows being discounted. $B_D^{\mathcal{T}}$ is computed as

$$B_D^{\mathcal{T}}(t) = \frac{P_D(t,T_m^x)}{\prod_{j=0}^{m} P_D(T_{j-1}^x, T_j^x)},$$

  where $T_{m-1}^x < t \leq T_m^x$ for $m = 1, .., M$.

- **The Forward Swap Measure** $Q_D^{c,d}$
  The numeraire is the annuity factor $C_D^{c,d}$, determined by swap fixed legs $\{T_c^s, T_{c+1}^s, ..., T_{d-1}^s, T_d^s\}$. $C_D^{c,d}$ is computed as

$$C_D^{c,d}(t) = \sum_{j=c+1}^{d} \tau_j^s P_D(t, T_j^s),$$

  where $\tau_j^s = \tau_D(T_{j-1}^s, T_j^s)$

4.3 Valuation of Linear Interest Rate Instrument

Following the steps taken by Mercurio (2009), we value an interest rate swap. Assume a set of times $\{T_a^x, ..., T_b^x\}$ consistent with sub-market x. At each time T_k^x the floating leg pays the Libor rate set at time T_{k-1}^x, for $k = a+1, ..., b$. Consider the floating cash flow payoff at time T_k^x

$$FL(T_k^x; T_{k-1}^x, T_k^x) = \tau_k^x F_k^x(T_{k-1}^x) = \tau_k^x R_x(T_{k-1}^x, T_k^x).$$

The value of this payoff at time t can be computed by taking the discounted expectation of the payoff

$$FL(t; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) E^{Q_D^{T_k^x}}[F_k^x(T_{k-1}^x)|\mathcal{F}_t].$$

Note the significant difference in the multiple curve setting compared to the single curve setting. If the discounting curve and the forwarding curve are one and the same (as was the case in the single curve framework), the martingale principle for forward rates could be applied at this point. Hence the expected value of the forward rate would simply be the current Libor spot rate. However, in this case the forwarding curve and discount curve are distinct, and hence the martingale property will not generally hold.

The time t FRA rate is the fixed rate that gives this payment leg zero value, it is clearly defined as

$$L_k^x(t) = FRA(t; T_{k-1}^x, T_k^x) = E^{Q_D^{T_k^x}}[F_k^x(T_{k-1}^x)|\mathcal{F}_t].$$

The value of the floating payment can be written as

$$FL(t; T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) L_k^x(t).$$

Summing all of the floating payments gives the value of the floating leg of the swap

$$FL(t; T_a^x, ..., T_b^x) = \sum_{k=a+1}^{b} FL(t; T_{k-1}^x, T_k^x) = \sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) L_k^x(t).$$

Let K denote the swaps fixed rate and assume the fixed payment dates are $\{T_c^s, ..., T_d^s\}$. The value of the fixed leg is given by:

$$K \sum_{j=c+1}^{d} \tau_j^s P_D(t, T_j^s).$$

Thus, the value of an interest rate swap is given by

$$V_{IRS}(t, K; T_a^x, .., T_b^x; T_c^s, ..., T_d^s) = \alpha \left(K \sum_{j=c+1}^{d} \tau_j^s P_D(t, T_j^s) - \sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) L_k^x(t) \right),$$

where $\alpha = 1$ if receiving fixed and $\alpha = -1$ if paying fixed.

The fair forward swap rate is the rate that gives the swap zero value and is computed as

$$S_{a,b,c,d}^x(t) = \frac{\sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) L_k^x(t)}{\sum_{j=c+1}^{d} \tau_j^s P_D(t, T_j^s)}.$$

In the case of a spot starting swap (i.e., $t = 0$) the spot swap rate is

$$S^x_{0,b,0,d}(0) = \frac{\sum_{k=1}^b \tau^x_k P_D(0, T^x_k) L^x_k(0)}{\sum_{j=1}^d \tau^s_j P_D(0, T^s_j)}.$$

Mercurio (2010) puts forward shifted log-normal dynamics for the FRA rate and discount curve forward rate, and hence presents formulae for the FRA and interest rate swap with 'convexity adjustments'. These results are also quoted by Bianchetti and Carlicchi (2011).
A second approach is by Bianchetti (2008), who applies quanto techniques to fixed income modelling in the multi curve framework. Using the developed theory as well as stating dynamics of the forward rate, he models the Radon-Nikodym derivative $dQ_D^{T_k^x}/dQ_x^{T_k^x}$ and shows how the pricing of linear instruments as well as vanilla derivatives can be tackled with so-called 'Quanto-Adjustments'.

4.4 Valuation of Caps/Floors and Swaptions

Once again, the approaches presented here follow the approach of Mercurio (2009).

4.4.1 Caplets/Floorlets

The caplet price at some time t is given by the discounted expected payoff

$$Cplt(t, K; T^x_{k-1}, T^x_k) = \tau^x_k P_D(t, T^x_k) E^{Q_D^{T_k^x}}[(F^x_k(T^x_{k-1}) - K)^+ | \mathcal{F}_t].$$

As was stated, the main difference between this computation under the single and multiple curve frameworks is that the forward rate is no longer a martingale under the pricing measure.

The approach taken is to replace the forward rate with the FRA rate as this is easier to handle under this pricing measure. The idea behind this is outlined below.
The FRA rate is the conditional expectation of the forward rate

$$L^x_k(t) = E^{Q_D^{T_k^x}}[F^x_k(T^x_{k-1}) | \mathcal{F}_t].$$

At time T^x_{k-1} the Libor rate resets, and, as such, the FRA rate and the forward rate coincide

$$L^x_k(T^x_{k-1}) = F^x_k(T^x_{k-1}) = R_x(T^x_{k-1}, T^x_k).$$

Thus, we can replace the forward rate $F^x_k(T^x_{k-1})$ in the caplet payoff with the FRA rate $L^x_k(T^x_{k-1})$, and the time t price of the caplet is now

$$Cplt(t, K; T^x_{k-1}, T^x_k) = \tau^x_k P_D(t, T^x_k) E^{Q_D^{T_k^x}}[(L^x_k(T^x_{k-1}) - K)^+ | \mathcal{F}_t.$$

Now the logic behind the approach becomes clear. As the forward rate was a martingale under the pricing measure in the single curve framework, the FRA rate is a martingale, by definition, under the current pricing measure $Q_D^{T_k^x}$.

Thus, as in the single curve setting, a smart choice of dynamics for the underlying can result in closed-form solutions.

Define the Black formula as

$$Bl(K, F, v) = F\mathcal{N}\left(\frac{\ln(\frac{F}{K}) + \frac{v^2}{2}}{v}\right) - K\mathcal{N}\left(\frac{\ln(\frac{F}{K}) - \frac{v^2}{2}}{v}\right),$$

where K is the strike, F is the rate of the underlying and v is its volatility. Select the following dynamics for the FRA rate under measure $Q_D^{T_k^x}$

$$dL_k^x(t) = v_k L_k^x(t) dW_k(t), \quad \text{for } t \leq T_k^x,$$

where $W_k(t)$ is a $Q_D^{T_k^x}$ Brownian motion. This results in the closed form solution

$$Cplt(t, T_{k-1}^x, T_k^x) = \tau_k^x P_D(t, T_k^x) Bl\left(K, L_k^x, v_k\sqrt{T_{k-1}^x - t}\right).$$

Generally, log-normal dynamics can lead to Black formulae similar to the single curve setting, but with slightly different inputs (FRA rate as opposed to forward rate and discount factor from the discounting curve).

Floorlets can be priced in the same manner. And to price a cap (or floor), one simply sums up the individual caplets (or floorlets).

4.4.2 Swaptions

Consider a payer swaption, which gives the right, but not the obligation, to enter into an interest rate swap at time $T_a^x = T_c^s$, with floating legs given by $T_{a+1}^x, ..., T_b^x$ and fixed legs given by $T_{c+1}^s, ..., T_d^s$, where $T_b^x = T_d^s$.

The payoff of the payer swaption at time $T_a^x = T_c^s$ is given by

$$\left(S_{a,b,c,d}^x(T_a^x) - K\right)^+ \sum_{j=c+1}^{d} \tau_j^s P_D(T_c^s, T_j^s).$$

Earlier it was shown that

$$S_{a,b,c,d}^x(t) = \frac{\sum_{k=a+1}^b \tau_k^x P_D(t, T_k^x) L_k^x(t)}{\sum_{j=c+1}^d \tau_j^s P_D(t, T_j^s)}.$$

To price using the forward swap measure, an annuity factor must be selected. Define this as follows

$$C_D^{c,d}(t) = \sum_{j=c+1}^{d} \tau_j^s P_D(t, T_j^s).$$

The price of the payer swaption, PS, is then the discounted expected value of the payoff under the forward swap measure, and is computed as follows

$$\begin{aligned}
&PS(t, K; T_{a+1}^x, ..., T_b^x; T_{c+1}^s, ..., T_d^s) \\
&= \sum_{j=c+1}^{d} \tau_j^s P_D(T_c^s, T_j^s) E^{Q_D^{c,d}} \left[\frac{\left(S_{a,b,c,d}^x(T_a^x) - K\right)^+ \sum_{j=c+1}^{d} \tau_j^s P_D(T_c^s, T_j^s)}{C_D^{c,d}(T_c^s)} | \mathcal{F}_t \right] \\
&= \sum_{j=c+1}^{d} \tau_j^s P_D(T_c^s, T_j^s) E^{Q_D^{c,d}} \left[\left(S_{a,b,c,d}^x(T_a^x) - K\right)^+ | \mathcal{F}_t \right].
\end{aligned}$$

Hence, pricing a Swaption is analogous to pricing an option on the underlying swap rate. The forward swap rate can be viewed as a tradable asset, being the floating leg of a swap, denominated by the numeraire asset

$$S_{a,b,c,d}^x(t) = \frac{\sum_{k=a+1}^{b} \tau_k^x P_D(t, T_k^x) L_k^x(t)}{\sum_{j=c+1}^{d} \tau_j^s P_D(t, T_j^s)} = \frac{Fl(t; T_a^x, ..., T_b^x)}{C_D^{c,d}(t)}.$$

As such, the forward swap rate is a martingale under the measure $Q_D^{c,d}$. Thus, as in the caplet case, the choice of dynamics for the forward swap rate can lead to Black like solutions. Assume $S_{a,b,c,d}^x(t)$ has the following dynamics under $Q_D^{c,d}$

$$dS_{a,b,c,d}^x(t) = v_{a,b,c,d} S_{a,b,c,d}^x(t) dW_{a,b,c,d}(t), \quad \text{for } t \leq T_a^x,$$

where $W_{a,b,c,d}(t)$ is a $Q_D^{c,d}$ Brownian motion. These dynamics result in the following closed form price

$$PS(t, K; T_{a+1}^x, ..., T_b^x; T_{c+1}^s, ..., T_d^s) = Bl\left(K, S_{a,b,c,d}^x(t), v_{a,b,c,d}\sqrt{T_a^x - t}\right) \sum_{j=c+1}^{d} \tau_j^s P_D(T_c^s, T_j^s).$$

4.5 The General Market Pricing Practice

Given the new market framework as defined above, at this point one can outline a general pricing algorithm. This is taken from the algorithms presented by Bianchetti (2008) and Bianchetti and Carlicchi (2011)

1. Construct a single **discounting** curve, C_d, using vanilla interest rate instruments and an appropriate bootstrapping scheme.

2. Construct multiple **forwarding** curves, C_x where $x = \{1M, 3M, 6M, 12M\}$. Once again, using an appropriate bootstrapping scheme, each curve is constructed using vanilla interest rate instruments homogeneous in rate tenor. Assume we are valuing interest rate cash flows $k \in \{1, ..., m\}$.

3. For each cash flow compute an estimate of the relevant FRA rate, $L_k^x(t)$, from the relevant forwarding curve, C_x. For example, using the following:

$$L_k^x(t) = F_x(t, T_{k-1}, T_k) = \frac{1}{\tau_{x,k}} \left[\frac{P_x(t, T_{k-1})}{P_x(t, T_k)} - 1 \right],$$

where $t \leq T_{k-1} \leq T_k$, $\tau_{x,k}$ is the year fraction between T_{k-1} and T_k and $P_x(t, .)$ is the zero coupon bond price given by curve C_x.

4. Compute the expected cash flows cf_k as the time t expectation of the interest rate related payoff π_k under the T_k forward measure, $Q_D^{T_k}$, associated with the discounting curve (the numeraire is $P_D(t, T_k)$)

$$cf_k(t, T_k, \pi_k) = E^{Q_D^{T_k}}[\pi_k | \mathcal{F}_t].$$

5. Compute the relevant discount factor from the discounting curve $P_D(t, T_k)$.

6. Compute the derivative value as the sum of the discounted expected cashflows

$$V(t) = \sum_{k=1}^{m} P_D(t, T_k) cf_k(t, T_k, \pi_k) = \sum_{k=1}^{m} P_D(t, T_k) E^{Q_D^{T_k}}[\pi_k | \mathcal{F}_t].$$

Bianchetti and Carlicchi (2011) tested this methodology as well as new market pricing formula against the pre-crisis pricing methodology. The market quotes for instruments clearly showed that post crisis the pricing methodology had shifted, and appeared consistent with the above algorithm. Thus, the single curve approach to pricing, and, by extension, hedging may lead to incorrect results.

Chapter 5

Modelling of Rates in the New Multi-Curve Framework

Several authors (e.g., Mercurio (2009) and Bianchetti (2008)) have acknowledged that the most consistent and complete model of the current market framework would be one that jointly models the interest rates with credit and liquidity risks. However, this is an extremely difficult task. Morini (2009) explores this approach further, looking into the market quotes, the types of liquidity risk faced by market players and several other facets of the current market framework.

A more popular approach to interest rate modelling, that is in line with the market approach of market segmentation, is the modelling of the different curves directly, hence implicitly dealing with credit and liquidity risks. Below, we outline some of the work done in this vein.

5.1 Development of Short Rate Models and HJM Extensions Consistent with Multiple Curves

Kenyon (2010) presents pseudo-analytic swaption pricing in a short rate setting. The approach taken here is to state short rate dynamics to drive the discounting curve, as well as short rate dynamics to drive the 'fixings (forwarding) curve'. This is done generally first, followed by a specific example using the one-factor Hull-White models.
There have also been some extensions to the popular Heath-Jarrow-Morton model. For examples see Pallavicini and Tarenghi (2010) and Moreni and Pallavicini (2010).

5.2 Extensions of the LMM

For a thorough introduction to the now ‘classic’ Libor Market Model (LMM) see Filipovic (2009). The classic LMM modelled the joint evolution of a set of consecutive forward rates, dictated by a certain time structure (Mercurio, 2010). Presented here are some ideas present in the literature investigating extensions of the LMM such that it is consistent with the current framework.

There are two major complications that a model must address, namely the multiple yield curves as well as the distinction between forward rates used to compute cashflows and those computed from the discounting curve (Mercurio, 2010).

The FRA rate has a few features that make it an ideal extension to the forward rate in the single curve setting. The FRA rate coincides with the Libor rate at reset times, as did the forward rate in the single curve setting. The FRA rate is also a martingale under the pricing measure, as was the forward rate in the single curve setting. Also, the forward swap rate is a linear combination of FRA rates, weighted by factors computed from the discount curve (Mercurio, 2010). In fact, one sees that modelling the FRA rate is a more general approach allowing for distinct forwarding and discounting curves, and in the case when these curves coincide this approach reduces to the single curve approach.

Modelling of the FRA rates from the forwarding curve does not complete the extension. These FRA rates must be modelled jointly with the forward rates from the respective discounting curve. Mercurio (2010) cites two reasons for this: firstly, as we are pricing under measures associated with the discounting curve, a full instantaneous covariation structure between the FRA rates and the discount curve forward rates is needed to allow for dynamics of the underlyings to be computed under different measures. Secondly, to simulate swap rates requires the joint simulation of FRA rates as well as rates from the discount curve. From this point, assume the discount curve is the OIS curve.

Carrying out the joint modelling of the FRA rate and OIS forward rate can be done directly, or via the definition of the spread. Defining the OIS forward rate as follows:

$$F_k^{D,x}(t) = F_D(t, T_{k-1}^x, T_k^x) = \frac{1}{\tau_k^x}\left[\frac{P_D(t, T_{k-1}^x)}{P_D(t, T_k^x)} - 1\right],$$

then the spread $S_k^x(t)$ can be defined as

$$S_k^x(t) = L_k^x(t) - F_k^{D,x}(t).$$

Note as the FRA rate and the OIS forward rate are martingales under the pricing measure $Q_D^{T_k^x}$, the spread must also be a martingale under this measure.

This decomposition allows for some variation in modelling approach:

1. Jointly model $L_k^x(t)$ and $F_k^{D,x}(t)$; or
2. Jointly model $S_k^x(t)$ and $L_k^x(t)$; or
3. Jointly model $S_k^x(t)$ and $F_k^{D,x}(t)$,

   for $k = 1, ..., M$.

One then specifies LMM dynamics for each set of rates.

Mercurio (2010) looks at the benefits and pit falls of each approach. Mercurio (2009) models the FRA rate and the OIS forward rate (choice one above) under an extended LMM, and using these dynamics derives formulae for caplets and swaptions. Mercurio (2010) models the OIS rates and the spread (choice three above) under an extended LMM, and also derives generalised formulae for caplets and swaptions. He then carries out a more specific computation using dynamics for the OIS forward rates with SABR style volatility and a driftless Brownian motion for the spreads.

A brief presentation of the LMM extension used by Mercurio (2009) is given below.

Consider the time structure $\mathcal{T} = \{T_0^x, T_1^x, ..., T_M^x\}$ consistent with curve x. Each FRA rate is modelled under its respective forward measure $Q_D^{T_k^x}$ as a driftless geometric Brownian motion:

$$dL_k^x(t) = \sigma_k(t) L_k^x(t) dW_k(t), \quad t \leq T_{k-1}^x.$$

The $W_k(t)$ is the k-th element of an M-dimensional $Q_D^{T_k^x}$ Brownian motion W, with correlation matrix $(\rho_{k,j})_{k,j=1,\ldots,M}$ such that $dW_k(t)dW_j(t) = \rho_{k,j}dt$ (Mercurio, 2009).

The existence of a distinct discounting curve requires the modelling of the evolution of $F_h^{D,x}$. Once again, we model these rates under their respective forward measure $Q_D^{T_h^x}$ as follows:

$$dF_h^{D,x}(t) = \sigma_h^D(t) F_h^{D,x}(t) dW_h^D(t), \quad t \leq T_{h-1}^x.$$

Again, $W_h^{D,x}(t)$ is the h-th element of an M-dimensional $Q_D^{T_h^x}$ Brownian motion with correlation structure

$$dW_k^D(t)dW_h^D(t) = \rho_{k,h}^{D,D}dt$$
$$dW_k(t)dW_h^D(t) = \rho_{k,h}^{x,D}dt.$$

Mercurio (2009) proceeds to state the bounds the correlation structure must satisfy, as well as to derive general dynamics for FRA rates and forward rates under any of the forward and spot Libor measures.

Chapter 6

Presentation of SABR Model Consistent with Multi-Curve Framework

To conclude this review of multi-curve modelling, below is a presentation of the SABR model in the classic setting and an extension in the multi-curve setting. For a more in depth introduction to the SABR model see Hagan and Lesniewski (2008) and for detailed implementation approaches see Kienitz and Wetterau (2012).

6.1 Classic SABR

The SABR model was introduced by Hagan and Lesniewski (2008). It is a generalization of Blacks model, allowing for stochastic volatility and resulting in closed form Black-like formulae for caps/floors and swaptions (Bianchetti and Carlicchi, 2011). Its relative simplicity as well as the fact it allows calibration to market data has led to this model becoming a "market standard for pricing and hedging financial instruments" (Bianchetti and Carlicchi, 2011).

6.1.1 Model Dynamics

The classic SABR dynamics for the forward rate, as given by Bianchetti and Carlicchi (2011), under the terminal measure Q^{T_k} are

$$
\begin{aligned}
\frac{dF_k(t)}{F_k^{\beta}(t)} &= V(t)dW_1^{Q^{T_k}}(t), \quad F_k(0) = f, \\
\frac{dV(t)}{V(t)} &= vdW_2^{Q^{T_k}}(t), \quad V(0) = \alpha, \\
dW_1^{Q^{T_k}}(t)dW_2^{Q^{T_k}}(t) &= \rho dt,
\end{aligned}
$$

for k = 1,...,M.

6.1.2 SABR volatility

Hagan and Lesniewski (2008) compute, using singular perturbation techniques, an approximation formula for the implied volatility curve observed in market data. The formula for the volatility smile is

$$\sigma(K,f) = \frac{\alpha}{(fK)^{0.5(1-\beta)}\left(1+\frac{(1-\beta)^2}{24}\log(\frac{f}{K})^2+\frac{(1-\beta)^4}{1920}\log(\frac{f}{K})^4\right)}.\frac{z}{x(z)}.$$
$$\left(1+\left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{\alpha\beta\rho\nu}{4(fK)^{0.5(1-\beta)}}+\frac{2-3\rho^2}{24}\nu^2\right]T\right),$$

where

$$z = \frac{\nu}{\alpha}(fK)^{0.5(1-\beta)}\log\left(\frac{f}{K}\right)$$

and

$$x(z) = \log\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right).$$

6.2 Multi-Curve SABR

This extension of SABR is as presented by Bianchetti and Carlicchi (2011), and is consistent with the multi-curve framework.

6.2.1 Model Dynamics

The dynamics for the FRA rates under the discount curve terminal measure $Q_D^{T_k^x}$, as presented by Bianchetti and Carlicchi (2011), are

$$\frac{dL_k^x(t)}{(L_k^x(t))^\beta} = V^x(t)dW_{x,1}^{Q_D^{T_k^x}}(t), \quad L_k^x(0)=f,$$
$$\frac{dV^x(t)}{V^x(t)} = vdW_{x,2}^{Q_D^{T_k^x}}(t), \quad V^x(0)=\alpha,$$
$$dW_{x,1}^{Q^{T_k^x}}(t)dW_{x,2}^{Q^{T_k^x}}(t) = \rho dt,$$

for k = 1,...,M.

6.2.2 SABR volatility

The smile formula remains the same as in the single curve case.

6.3 Overview of the Parameters

Note the SABR model deals with one forward rate/FRA rate at a time, and hence each FRA rate (and subsequently volatility smile) has its own set of parameters. The four parameters, their bounds and effects on the implied smile shape are detailed below (Gauthier and Rivaille, 2009):

- α, where $\alpha > 0$.
  α is the spot level of volatility and effects the level of the curve. As alpha increases, the level of the curve increases.

- β, where $0 \leq \beta \leq 1$.
  β is the Constant Elasticity of Variance (CEV) parameter (note the SABR model is an extension of the well-known CEV model, allowing for stochastic variance). It influences the slope of the curve, and as β decreases, the curve steepens.

- ρ, where $-1 \leq \rho \leq 1$.
  ρ is the correlation between the Brownian motions driving the forward/FRA rate and the stochastic volatility. It has the same effect as β, hence as ρ decreases the curve steepens.

- v, where $v > 0$.
  v is the volatility of the volatility. It effects the curvature of the curve, or the 'degree of skew'. As v increases the amount of curvature also increases.

Part II

Tranformation of Cap and Swaption Volatility

Chapter 7

Linking Volatilities of Caps and Floors

The need for linking volatilities is a result of the nature of the volatility quotes in the market. There is typically an assumption about the frequency of the floating rate payments on the instrument, the most common of which is to quote cap, floor and swaption volatilities assuming an underlying rate tenor of 6 months (Kienitz, 2013). This market convention creates the need for multiple complete volatility surfaces. For example, if one wanted to price a 6 year swaption on a 5 year Swap, one would need to somehow adjust the market quoted volatility if the underlying swaption had quarterly payments as opposed to semi-annual.

Not a great deal of work has been done in terms of volatility modelling in the multi-curve framework. Bianchetti and Carlicchi (2011) present a multi-curve generalisation of the SABR model which has gained "acceptance as [a] standard valuation and risk management model" (Hagan and Lesniewski, 2008). Mercurio (2009) presents an overview of general stochastic volatility dynamics in the multi-curve framework and focuses mainly on how to apply the relevant changes of measure.
There has been even less literature investigating how to link volatilities between rates with different tenors. Jackel and Rebonato (2003) were one of the first to put forward the idea of linking volatilities, the context was, however, linking volatilities between caplets and swaptions. Kienitz (2013) develops this much further, showing how to link volatilities of, and ultimately pricing, caps, floors and swaptions with different rate tenors.

The investigation below will aim to contribute to the idea of linking volatilities in the multi-curve framework. The method outlined by Kienitz (2013) forms the basis of the approaches implemented.

In order to show the complexity added by multiple yield curves as well as OIS discounting, below is a brief overview of a common approach used to link volatilities of forward rates with different tenors in the now outdated single curve framework. See Kienitz (2013) for more detail on this approach.

7.1 Single Curve Approach

Using the usual definition of the forward rate as

$$F_{i,j}(t) = \frac{1}{\tau_{i,j}} \left(\frac{P(t,T_i)}{P(t,T_j)} - 1 \right),$$

where $\tau_{i,j} = T_j - T_i$ and $t \leq T_i \leq T_j$, an approach used to link volatilities is by making use of the following no-arbitrage condition

$$F_{1,3}(t) = \tau_{1,2} F_{1,2}(t)(1 + \tau_{2,3} F_{2,3}) + \tau_{2,3} F_{2,3}(t). \tag{7.1}$$

For this purpose, $F_{1,2}(t)$ and $F_{2,3}(t)$ are adjacent forward rates of equal tenor. These are the short tenor forward rates. The forward rate $F_{1,3}(t)$ is the long tenor forward rate. Assuming dynamics for the above forward rates allows us to use (7.1) to relate the volatilities. Assume the following dynamics for the short tenor rates:

$$\begin{aligned} dF_{1,2}(t) &= ...dt + \sigma_{1,2} F_{1,2}(t) dW_1(t) \\ dF_{2,3}(t) &= ...dt + \sigma_{2,3} F_{2,3}(t) dW_2(t) \\ d\langle W_1(t), W_2(t) \rangle &= \rho dt. \end{aligned}$$

Assume the following dynamics for the long tenor rate:

$$dF_{1,3}(t) = ...dt + \sigma_{1,3} F_{1,3}(t) dW_3(t).$$

We do not specify the dt terms as they are not needed. Applying Ito's Lemma on (7.1) yields the following dynamics for the long tenor forward rate:

$$\begin{aligned} dF_{1,3}(t) =& ...dt + \frac{\sigma_{1,2}}{\tau_{1,3}} \left(\tau_{1,2} F_{1,2}(t) + \tau_{1,2}\tau_{2,3} F_{1,2}(t) F_{2,3}(t) \right) dW_1(t) \\ &+ \frac{\sigma_{2,3}}{\tau_{1,3}} \left(\tau_{2,3} F_{2,3}(t) + \tau_{1,2}\tau_{2,3} F_{1,2}(t) F_{2,3}(t) \right) dW_2(t). \end{aligned}$$

Taking quadratic variation of both SDEs for $F_{1,3}(t)$ and comparing terms yields

$$\sigma_{1,3}^2 = V_1(t)^2 \sigma_{1,2}^2 + V_2(t)^2 \sigma_{2,3}^2 + 2\rho V_1(t) V_2(t) \sigma_{1,2} \sigma_{2,3},$$

where

$$V_i = \frac{\tau_{i,i+1} F_{i,i+1}(t) + \tau_{1,2}\tau_{2,3} F_{1,2}(t) F_{2,3}(t)}{\tau_{1,3} F_{1,3}(t)}.$$

Using a 'freezing of the drifts' argument allows us to approximate this using current market data, thus one can approximate $\sigma_{1,3}$ as

$$\sigma_{1,3}^2 \approx V_1(0)^2\sigma_{1,2}^2 + V_2(0)^2\sigma_{2,3}^2 + 2\rho V_1(0)V_2(0)\sigma_{1,2}\sigma_{2,3}.$$

Note the above example was assuming the short term tenor divided the long term tenor in two, but this could easily be extended beyond this case. Moving from the shorter tenor volatilities to a longer tenor volatility requires multiple input volatilities to compute a single volatility. Moving from the longer tenor to a shorter tenor volatility leads to a situation whereby one must compute multiple volatilities from a single input volatility, and as such an assumption on the structure of the short tenor volatilities must be made. A simple assumption that yields plausible results is the constant volatility assumption, but one can also fit a functional form for the volatilities (Kienitz, 2013).

In the multiple curve framework, a general no arbitrage equation, such as (7.1) between forward rates of different tenors cannot formulated, hence this approach becomes invalid. Kienitz (2013) presents an expansion of this methodology that is consistent with the multiple curve framework and takes the basis spreads between FRA rates of different tenors into account by making use of displaced diffusion models. This is detailed below.

7.2 Multi Curve Extension

This method is based on the use of displaced diffusion models. Before outlining the method, below we introduce the concept and uses of displaced diffusion models.

Displaced diffusion models were introduced by Rubinstein (1983) and have since gained popularity as a means of modelling the volatility skews inherent in certain market data. This is achieved by modelling the dynamics of a stochastic variable plus a constant, known as the 'displacement' or 'displacement coefficient' (Lee and Wang, 2012). Along with the ability to allow for skewed implied volatility, it also allows pricing of vanilla instruments using Black-Scholes formulae with adjusted inputs, making it 'exceptionally easy and efficient in practical applications' (Jäckel, 2009). Jäckel (2009) demonstrates in the case of constant coefficients how one can use a transformation of the displaced diffusion model dynamics to recover Black-Scholes pricing formulae as well as various other results. A particular result presented allows one to compute the Black-Scholes implied volatility, σ_{ATM}^{BS}, for an at-the-money vanilla option in terms of a displaced diffusion volatility σ^{DD} and parameters σ and

β (these are simple transformation of the displacement parameter b and σ^{DD}). Below is an outline of the exact displaced diffusion model in this multi-curve context.

As was mentioned earlier, in the multi curve framework the common approach is to model the FRA rate. Using the notation of Kienitz (2013), for some given T_1 and T_2, we define the forward rate on the OIS curve as

$$F_{1,2}^{OIS}(t) = F^{OIS}(t, T_1, T_2),$$

and the FRA rate on the nM curve (where $nM = \{1M; 3M; 6M; 12M\}$) as

$$F_{1,2}^{nM}(t) = F^{nM}(t, T_1, T_2).$$

We can then define the basis spread, b^{nM} with respect to these two rates

$$b_{1,2}^{nM} = b^{nM}(T_1, T_2) = F_{1,2}^{nM}(t) - F_{1,2}^{OIS}(t),$$

more appropriately written as

$$F_{1,2}^{nM}(t) = F_{1,2}^{OIS}(t) + b_{1,2}^{nM}.$$

This basis spread is a direct result of the difference in credit and liquidity risk levels between the OIS and nM 'sub-markets'. Now, we assume log-normal dynamics for the FRA rate

$$dF_{1,2}^{nM}(t) = ...dt + \sigma_{1,2} F_{1,2}^{nM}(t) dW(t).$$

If we assume the basis spread is deterministic, we can rewrite these dynamics in displaced diffusion form as

$$dF_{1,2}^{nM}(t) = ...dt + \sigma_{1,2}(F_{1,2}^{OIS}(t) + b_{1,2}^{nM}),$$

where $\sigma_{1,2}$ is now a displaced diffusion volatility σ^{DD} and $b_{1,2}^{nM}$ is the displacement coefficient b. At this point we make use of the aforementioned result derived by Jäckel (2009). As shown by Kienitz (2013), we define β and ξ in terms of σ^{DD} and b:

$$\sigma_{1,2} = \sigma^{DD} = \beta\xi \tag{7.2}$$

$$b_{1,2}^{nM} = b = \frac{1-\beta}{\beta} F_{1,2}^{OIS}(0). \tag{7.3}$$

Let σ_{ATM}^{BS} denote the at-the-money (ATM) implied Black volatility. We can use the following formulae to compute σ_{ATM}^{BS} given σ^{DD} and $b_{1,2}^{nM}$:

$$\sigma_{ATM}^{BS} = \frac{2}{\sqrt{T}} N^{-1}\left(\frac{1}{\beta} N\left(\frac{\xi\beta\sqrt{T}}{2}\right) - \frac{1-\beta}{2\beta}\right) \tag{7.4}$$

$$\xi = \frac{2}{\beta\sqrt{T}} N^{-1}\left(\beta N\left(\frac{\sigma_{ATM}^{BS}\sqrt{T}}{2}\right) + \frac{1-\beta}{2}\right). \tag{7.5}$$

Note Kienitz (2013) shows alternate formulas that can be used instead of the above analytic formula. At this point, we are able to convert ATM implied Black volatilities to displaced diffusion volatilities, and vice versa, where the displacement coefficient is dependent on basis spreads.

The next point to realise is that (7.1) holds within the OIS sub-market. So we can write

$$F_{1,3}^{OIS}(t) = \tau_{1,2}F_{1,2}^{OIS}(t)(1 + \tau_{2,3}F_{2,3}^{OIS}) + \tau_{2,3}F_{2,3}^{OIS}(t). \tag{7.6}$$

Hence, the single curve approach detailed above would work sufficiently as long as we are linking volatilities of rates only on the OIS curve. This leads us to the idea put forward by Kienitz (2013). Using the displaced diffusion dynamics for the FRA rates, we can transform these displaced diffusion volatilities to Black volatilities. These Black volatilties would be the volatilties of the forward rates along the OIS curve, hence this corresponds to a move from the nM curve to the OIS curve. Now that we have volatilities for forward rates on the OIS curve, we can exploit (7.6) and carry out the approach detailed in the single curve framework, moving either from long tenor to short tenor or short to long. Once this is complete, we would have transformed volatilities within the OIS sub-market from one tenor to another. At this point, we convert these OIS volatilities, which are Black volatilities, to displaced diffusion volatilities using the basis spread specific to the new tenor. This corresponds to a move from the OIS curve to the mM curve. The result is a set of transferred volatilities from sub-market nM to sub-market mM.

The above approach allows for transfer of only the ATM caplet volatilities. Transforming an entire volatility surface requires more effort and some assumptions. The method proposed by Kienitz (2013) is to make use of the aforementioned SABR model. Since we have no preconceived idea as to the shape of the transformed volatility surface, we make an assumption on shape. We assume that the over-all shape of the volatility surface will remain the same as that of the input volatility surface. The approach is to calibrate the SABR model to the input volatility surface, then carry out the transformation on the set of ATM caplet volatilities. Once these are transformed into the new sub-market, we assume that three of the four parameters of the SABR model remain constant, all except the α parameter. As will be shown, the α parameter can be computed using the ATM caplet volatility and the other three parameters. So, we can compute the new set of α parameters implied by the transformed volatilities, and hence using the SABR model, one has an entire volatility surface. The approach taken is detailed in the section 9.

Chapter 8

Linking Volatilities of Swaptions

In this section we link swaption volatilities with different underlying tenors. Once again we introduce a common approach in the single curve framework. Then we present one of the multi-curve methods suggested by Kienitz (2013), again making use of displaced diffusion models.

8.1 Single Curve Approach

This approach is presented in Kienitz (2013). In the single curve framework we can specify the dynamics of swap rates with different underlying tenors. Assume we have swap rates with tenor nM and tenor mM, and hence swap rates $S^{nM}_{T_a,T_b}(t)$ and $S^{mM}_{T_a,T_b}(t)$. These swap rates correspond to swaps commencing at time T_a and reaching maturity at time T_b. Assume the following dynamics:

$$\begin{aligned} dS^{nM}_{T_a,T_b}(t) &= (...)dt + \sigma^{nM}_{T_a,T_b} S^{nM}_{T_a,T_b} dW_1(t) \\ dS^{mM}_{T_a,T_b}(t) &= (...)dt + \sigma^{mM}_{T_a,T_b} S^{mM}_{T_a,T_b} dW_2(t) \\ d\langle W_1(t), W_2(t)\rangle &= \rho dt. \end{aligned}$$

In the single curve setting a common assumption was that $S^{nM}_{T_a,T_b}(t) - S^{mM}_{T_a,T_b}(t)$ was constant (and often even negligible). This would imply that

$$dS^{nM}_{T_a,T_b}(t) - dS^{mM}_{T_a,T_b}(t) = 0.$$

This in turn implies that the coefficient of the dt, the $dW_1(t)$ and the $dW_2(t)$ terms are each zero. Hence the following must hold

$$0 = \sigma^{nM}_{T_a,T_b} S^{nM}_{T_a,T_b}(t) - \sigma^{mM}_{T_a,T_b} S^{mM}_{T_a,T_b}(t).$$

So, given the Swap rates and a single volatility, one can compute the other

$$\sigma^{mM}_{T_a,T_b} = \sigma^{nM}_{T_a,T_b} \frac{S^{nM}_{T_a,T_b}(t)}{S^{mM}_{T_a,T_b}(t)}.$$

Thus in a single curve setting, a volatility surface and the corresponding set of swap rates would allow one to compute a new volatility surface corresponding to swaption volatilities with exact same specifications except a different underlying tenor.

8.2 Multi Curve Extension

The approach suggested by Kienitz (2013) is based on similar ideas as to that presented for transforming caplet volatilities. Assuming we have swap rates $S^{nM}_{T_a,T_b}(t)$ and $S^{mM}_{T_a,T_b}(t)$ with dynamics as given by (8.1) and (8.2) above. Also assume we have the swap rate $S^{OIS}_{T_a,T_b}(t)$. We can then define the 'basis spreads' b^{nM} and b^{mM} in the same manner as they were computed in terms of forward and FRA rates in the section 7.2.

Note, the main difference between this swaption linking method and that presented above for caplets is that we do not have an arbitrage relationship (such as (7.1)) to exploit, hence we do not need to carry out the transformation along the OIS submarket. We can move directly from one displaced diffusion volatility to another.

Let us assume we have $\sigma^{mM}_{T_a,T_b}(t)$ and we wish to find $\sigma^{nM}_{T_a,T_b}(t)$. Also, we have computed b^{nM} and b^{mM}. Again, we define the variables β^i and ξ^i using

$$b^i = \frac{1-\beta^i}{\beta^i} S^i_{T_a,T_b} \quad \text{and} \quad \sigma^i_{T_a,T_b} = \xi^i \beta^i,$$

for $i = \{nM, mM\}$.

Once again this is done on the ATM volatilities, in other words for swap rates equal to the ATM swaption strike. Let σ^{OIS} denote the OIS volatility, which we compute, but do not specifically transform. Kienitz (2013) then shows that we can move from the input $\sigma^{mM}_{T_a,T_b}$ to σ^{OIS} and finally to $\sigma^{nM}_{T_a,T_b}$ using

$$\sigma^{OIS} = \frac{2}{\sqrt{T}} N^{-1}\left(\frac{1}{\beta^{mM}} N\left(\frac{\xi^{mM}\beta^{mM}\sqrt{T}}{2}\right) - \frac{1-\beta^{mM}}{2\beta^{mM}}\right), \tag{8.1}$$

$$\xi^{nM} = \frac{2}{\beta^{nM}\sqrt{T}} N^{-1}\left(\beta^{nM} N\left(\frac{\sigma^{OIS}\sqrt{T}}{2}\right) - \frac{1-\beta^{nM}}{2}\right), \tag{8.2}$$

$$\sigma^{nM}_{T_a,T_b} = \xi^{nM}\beta^{nM}. \tag{8.3}$$

Hence, the result is the converted ATM swaption volatility. As with the cap transformation case, Kienitz (2013) shows alternate transformation formulae that can be used instead of equations 8.1-8.4. To take the entire surface into account, one can apply the same approach as was taken on the caplet volatilities with the use of the SABR model.

Chapter 9

Implementation 1 - Transforming Cap Volatility Surface

We aim to take as input a 6M cap volatility surface and compute a 3M cap volatility surface. Although we carry out our transformation on caplet volatilities, only caps are traded in the market, and hence we have market data for cap volatilities. And even then the market quotes for a specific strike range and set of maturities. Before one can carry out the implementation of the above methodologies, one must back out the caplet volatility surface from the market quoted cap volatility surface. This process is known as the 'Stripping' of caplet volatilities and is addressed below.

9.1 Stripping Caplet Volatilities using the SABR model

Before we introduce the stripping approach followed, we must first outline the difference between flat volatilities, and spot volatilities.

- Flat Volatility: This is the single volatility value inserted in Black's formula to give the correct cap price
- Spot Volatility: This is the volatility specific to a particular caplet which prices that caplet correctly. The cap price can be computed by summing the series of caplets, each priced with its individual spot volatility.

Market quotes are in the from of flat volatilities, and our aim is to find the series of spot volatilities implied from the flat volatilities.

Caplet stripping is typically an iterative procedure, making use of the following equation:

$$\sum_{i=1}^{n} V_{caplet}(F_i^{nM}, \sigma_{flat}(n), T_i) = \sum_{i=1}^{n-1} V_{caplet}(F_i^{nM}, \sigma_{flat}(n-1), T_i) + V_{caplet}(F_n^{nM}, \sigma_{spot}(n), T_n). \tag{9.1}$$

Using this equation, plus the fact that the first quoted cap only consists of a single caplet (hence the spot and the flat volatilities are equal), allows us to iterate cap by cap and compute the price of each additional caplet. Using Black's formula and the caplet price, we can compute the implied caplet volatility. Note this requires cap volatilities at every 6 month tenor, and as 'full' surfaces are often not quoted, interpolation is required.

Market quotes are often in the form of volatilities across a certain strike range, and these are quoted for a set of option tenors. Also available are quotes for the ATM strikes for caps at each tenor. Taking these ATM quotes into account requires a more careful approach. If we did not take into account the ATM strikes, one could simply carry out the iterative approach described above along each strike in the strike range. This way we are comparing 'like for like' in terms of strike. The ATM strike is typically different for every tenor. Hence each volatility smile (the volatility smile is the set of volatilities against strikes for a particular tenor) we have quotes for the standard strike range as well as a unique ATM volatility. The approach here is to find a way of interpolating along these smiles for volatilities corresponding to strikes not in the standard strike range. One such method is 'SABR interpolation'.

The SABR interpolation entails parametrising SABR parameters for each volatility smile including, where relevant, the ATM volatilities. An overview of the approach is given below:

- The first cap only has one caplet, hence we have the caplet volatilities. We calibrate the SABR parameters to this smile (including the ATM volatility)

- At each subsequent 6 month tenor, we check if there is an ATM cap quote for that tenor.

  ▶ If there is an ATM quote: For the standard strike range, strip the latest caplet volatility using the method described above and (9.1). To compute the caplet volatility for the ATM strike, we use the SABR model fitted to the previous tenor(s) to compute the caplet volatilities at the ATM strike for all previous tenors. We then value the previous caplets using these volatilities.

Then using the following formula:

$$\sum_{i=1}^{n} V_{caplet}(F_i^{nM}, \sigma_{flat}(n), T_i) = \sum_{i=1}^{n-1} V_{caplet}(F_i^{nM}, \sigma_{spot}(i), T_i) + V_{caplet}(F_n^{nM}, \sigma_{spot}(n), T_n),$$

we can compute the price of the latest caplet at the ATM strike, and invert Black's formula to find the spot volatility for this ATM caplet. We then fit the SABR parameters to this smile (including the ATM caplet volatility).

▶ If there is no ATM quote: We then strip the latest caplet volatilities for the standard strike range using the method described above and (9.1). We then fit SABR parameters to this smile.

The result of this method is a set of caplet volatilities for each tenor for the standard strike range, and more importantly it results in a set of SABR parameters for each tenor, calibrated to volatility smiles which incorporate the ATM quotes where applicable. These parameters will be used to compute the ATM caplet volatilities for the transformation based on the multi curve extension described above. We will also use these parameters to facilitate the transformation of the entire volatility surface using the method suggested by Kienitz (2013).

9.2 SABR Model Calibration

In the previous section, we have glanced over how the calibration of the SABR model to the volatility smiles is carried out. This is detailed here.

The SABR model, as introduced in Part 1, has four parameters, α, β, ρ and ν. Note, it is possible to fit functional forms for each parameter across the tenor range, however the approach taken here is a parametric approach. We solve for the parameters of the volatility smile at each tenor independently. One of the most useful features of the SABR model is the functional form it describes for the volatility smile given these four parameters. This functional form is:

$$\sigma(K, f) = \frac{\alpha}{(fK)^{0.5(1-\beta)}\left(1 + \frac{(1-\beta)^2}{24}\log(\frac{f}{K})^2 + \frac{(1-\beta)^4}{1920}\log(\frac{f}{K})^4\right)} . \frac{z}{x(z)} .$$

$$\left(1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}} + \frac{\alpha\beta\rho\nu}{4(fK)^{0.5(1-\beta)}} + \frac{2-3\rho^2}{24}\nu^2\right] T\right),$$

where

$$z = \frac{\nu}{\alpha}(fK)^{0.5(1-\beta)} \log\left(\frac{f}{K}\right)$$

and

$$x(z) = \log\left(\frac{\sqrt{1 - 2\rho z + z^2} + z - \rho}{1 - \rho}\right).$$

A common approach is simply to set the β parameter to a fixed value, typically 0.5. A similar approach is taken here, except after optimising for the remaining three parameters, we compute another optimisation holding the three parameters constant and optimising with respect to the β parameter. The purpose here was twofold, a large change in the β parameter resulting in a better model fit may indicate that simply setting $\beta = 0.5$ may not be wise, secondly it could be used to 'fine-tune' the calibration. Note in the implementation, β was found to always remain significantly close to 0.5.

The approach taken at each tenor is as follows:

- set $\beta = 0.5$;
- calibrate α, ρ and ν to the observed volatility smile:

$$argmin_{\alpha,\rho,\nu} = \sum_i (\hat{\sigma}(K_i, F, \alpha, \rho, \nu) - \sigma_{K_i}^{mkt})^2,$$

  thus minimizing the error between the model implied volatility and the market observed volatility with respect to the three parameters;

- We then fix α, ρ, ν and repeat the optimisation with respect to β

$$argmin_{\beta} = \sum_i (\hat{\sigma}(K_i, F, \beta; \alpha, \rho, \nu) - \sigma_{K_i}^{mkt})^2,$$

Hence, we have a set of four parameters for every tenor.

An alternate method is the vega-weighted SABR model. In this version, the error term for each strike is multiplied by the vega of each option at that strike. The idea behind this is that more weighting should be given to the volatilities in and around the ATM strike. This method was implemented, but it was found that improved fitting to the ATM options was minimal. This, coupled with poor fit in the far out-the-money strikes has resulted in our choice of the method descried above and not the vega-weighted method.

In the multi curve extension approach, we transfer the ATM volatilities. The SABR functional from simplifies in the case when computing the ATM volatility, $\hat{\sigma}(F, F)$. We have:

$$\hat{\sigma}(f, f) = \frac{\alpha}{f^{1-\beta}} \left(1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}} + \frac{\alpha\beta\rho\nu}{4f^{1-\beta}} + \frac{2-3\rho^2}{24}\nu^2\right] T\right).$$

From this we can construct a cubic equation in α.

$$\left(\frac{(1-\beta)^2 T}{24 f^{2-2\beta}}\right)\alpha^3 + \left(\frac{\beta\rho\nu T}{4 f^{1-\beta}}\right)\alpha^2 + \left(1 + \frac{2-3\rho^2}{24}\nu^2 T\right)\alpha - \sigma_{ATM} f^{1-\beta} = 0 \quad (9.2)$$

Hence given σ_{ATM}, β, ρ and ν we can solve this cubic expression for α. Typically, there will be one real root, but if not we take the smallest real root as our α.

Below we outline the algorithms followed to tranform a 6M Cap volatility surface to a 3M volatility surface using the methods outlined above. Henceforth, to avoid conflicts in notation, the SABR parameters are superscripted by 'SABR'.

Assume as input from the market we have the 6M cap volatility surface, we have the OIS curve, 6M curve and 3M curves.

9.3 Algorithm for the Multi Curve Extension Method

- Interpolate the volatilities across the tenors, resulting in a 'complete' cap volatility surface
- Implement the caplet stripping algorithm detailed above. Resulting in caplet volatilities with 6 month tenors as well as SABR smile parameters.
- Compute forward curves for the OIS, 6M and 3M curves. From these we estimate the basis spreads $b^{3M}_{T_i,T_{i+1}}$ and $b^{6M}_{T_i,T_{i+1}}$.

  For every tenor we carry out the following to convert the ATM volatility:

- Find K^{6M}_{ATM} from the 6M forward curve
- Use the fitted SABR parameters to find $\sigma^{6M} = \hat{\sigma}_{SABR}(f, f)$
- Apply the displaced diffusion dynamics and convert this volatility to an OIS volatility using (8.1)
- Transform the 6 month OIS volatility into two 3 month OIS volatilities using the single curve methodology and the constant volatility assumption
- Apply the displaced diffusion dynamics and convert this 3 month OIS volatility to a 3M volatility, corresponding to strike K^{3M}_{ATM}. Hence we have $\sigma^{3M}_{K^{3M}_{ATM}} = \sigma^{3M}$.
- Transform the entire surface by assuming the $\beta^{SABR}, \rho^{SABR}$ and ν^{SABR} parameters are the same as for the 6M smile corresponding to K^{6M}_{ATM}, and use these and σ^{3M} to solve for a new α^{SABR} as in (9.2).

- Using the new set of SABR parameters to compute the caplet volatility surface
- Find the 3M cap volatility surface implied by this caplet surface by solving iteratively for the 3M flat volatilities. Note, the first cap from the 6M volatility surface will give us two 3-month caplets, with tenors 0.5 to 0.75 years and 0.75 to 1 year. Thus a crucial input into the 3M cap volatility surface is the 0.25 to 0.5 year caplet, for which the 6M cap surface holds no information. The approach taken here is to calibrate the 1 year cap on the 3M surface to that of the Bloomberg market quoted 1 year 3M cap. Thus, we solve for the 0.25 to 0.5 year caplet such that model implied 1 year 3M cap is as close as possible to that quoted by Bloomberg.

Chapter 10

Implementation 2 - Transforming Swaption Volatility Surface

The transformation of swaption volatilities is less computationally intensive than the transformation of cap volatilities, mainly because there is no stripping involved and hence we can transform on the market quoted tenors without having to interpolate to create a 'complete' surface. The SABR calibration carried out here is identical to the way it was carried out on the caplet volatilities, however now we consider Swaption volatilities and the rate F is no longer a forward rate but a forward Swap rate.

Assume as input we have a 6M swaption volatility surface with ATM strike rates and volatilities, 3M ATM strikes as well as OIS ATM strikes computed from the 3M and OIS yield curves respectively. The swap rate is computed using the multi-curve formula presented in Part 1, with the FRA rates being estimated using forward rates.

10.1 Algorithm for the Multi Curve Extension Method

- Calibrate the SABR model for each swaption volatility smile. Include in the ATM volatility.
- Compute the basis spreads b^{3M} and b^{6M} as the spreads between the ATM swap rates for each tenor.
- Compute the terms β^i and ξ^i for each tenor. Where $i = \{3M, 6M\}$.
- Convert the ATM volatility at each tenor from 6M to 3M using (8.1), (8.2) and (8.3), or the alternate formulae presented in Kienitz (2013).

- Assume β^{SABR}, ρ^{SABR} and ν^{SABR} are identical to that of the 6M surface for each tenor. Use (9.2) and the ATM strikes on the 3M curve to compute the α^{SABR} parameter for each tenor.

- Using the new SABR parameters, compute the 3M swaption surface.

Chapter 11

Results 1 - Cap Volatility Surface

11.1 Multi Curve Extension

11.1.1 Market Inputs

As inputs we take the Bloomberg 6M cap volatility surface, quoted on 02-12-2014. A surface plot of this is shown in Figure 11.1 below. Also quoted are the cap ATM strikes and volatilities, these are shown in table 1 in Appendix 1. Note these are Black volatilities, assume a rate tenor of 6 months as well as OIS discounting. The yield curve data taken from Bloomberg is available in Appendix 4.

In order to carry out the caplet stripping we need to interpolate for every 6 month tenor. Due to fewer data points as maturity increases, we truncate the data at the 20 year point. Carrying out the interpolation we obtain the 'complete' cap volatility surface shown in Figure 11.2.

11.1.2 Caplet Stripping

The caplet surface produced as a result of the stripping procedure is shown in Figure 11.3.

11.1.3 SABR Calibration

The SABR calibrations, both the normal and the vega-weighted calibration, were carried out for each caplet volatility smile. The plots of the Volatility surfaces implied by each set of SABR parameters are shown in Figures 11.4 and 11.5, along with the absolute error plots with respect to the input caplet surface. The normal SABR model had a lower total absolute error and was selected as the model of choice.

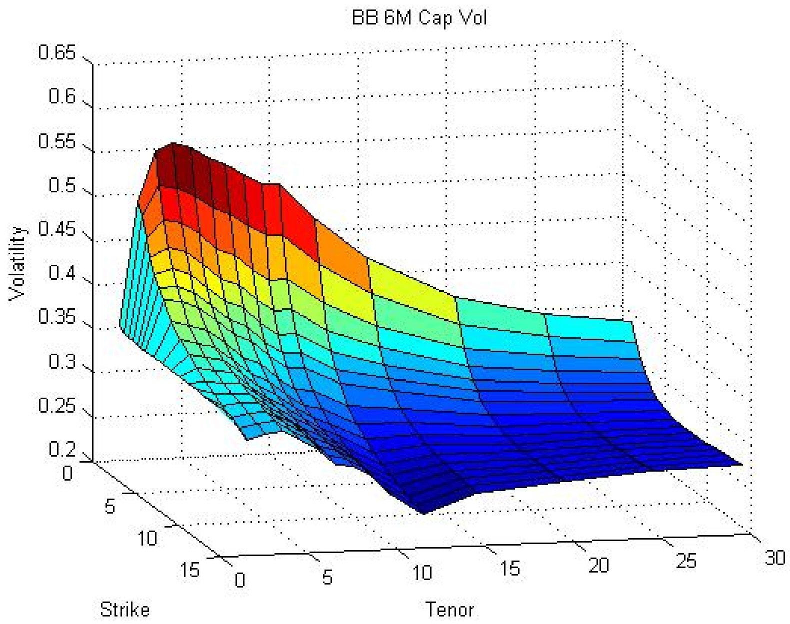

Fig. 11.1: (Input 6M Volatility Surface. Source- Bloomberg)

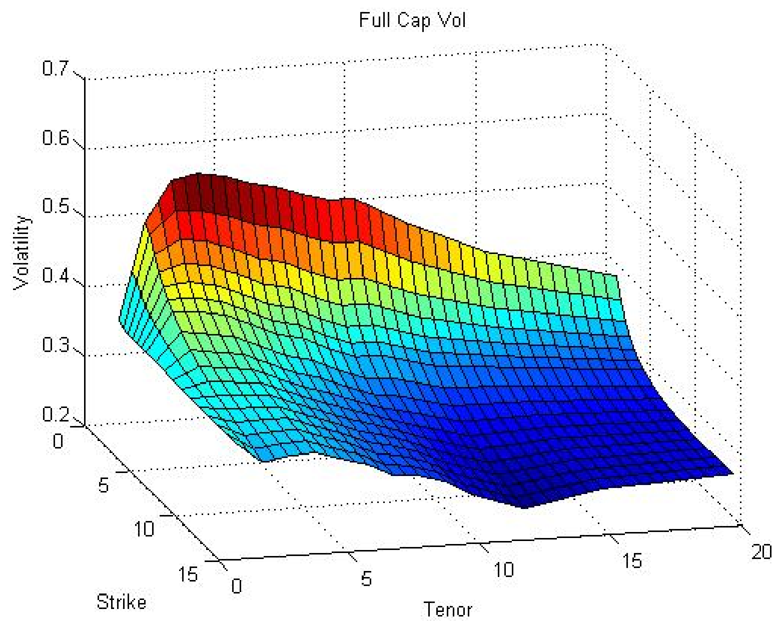

Fig. 11.2: The Interpolated 'Full' Volatility Surface

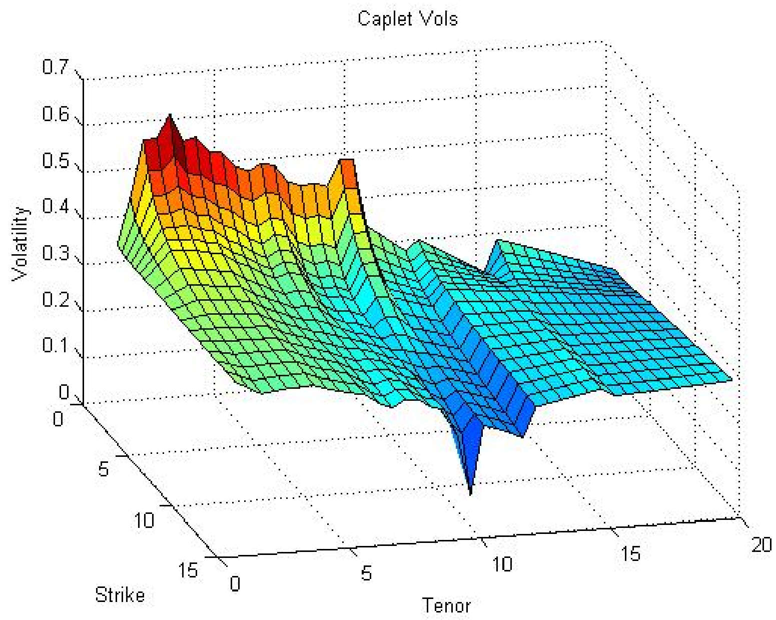

Fig. 11.3: Caplet Volatility Surface

11.1.4 Transformation

The basis spreads computed from the forward rates on each curve are shown in Appendix 2. Note that as the FRA rates are not model inputs, they are approximated using the forward rates from the respective sub-market curves.

Figure 11.6 shows the transformed 3M caplet surface. These are used to compute the 3M cap volatility surface. This is shown in Figure 11.7 for the tenor range quoted in the market.

11.1.5 Comparison with Market 3M Surface

Available on Bloomberg is a 3M cap volatility surface, constructed from a combination of market data where liquidity is adequate and interpolation methods where data is scarce. This surface, as quoted 02-12-2014, is shown in Figure 11.8. The percentage difference between the output 3M surface and the Bloomberg 3M surface is computed and shown in Figure 11.9. This figure shows that, besides the second volatility smile, all the model output volatilities are within 5% of the market quoted volatilities.

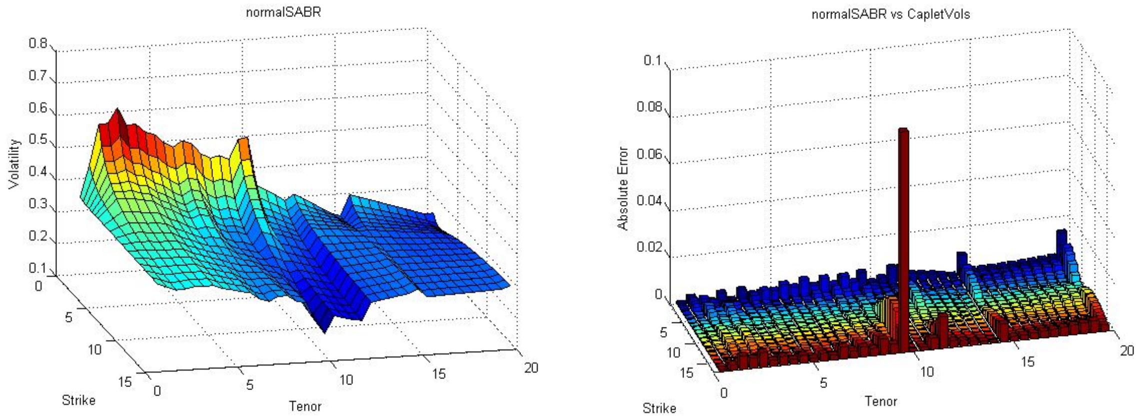

Fig. 11.4: Normal SABR calibration surface (left) as well as absolute error plot (right)

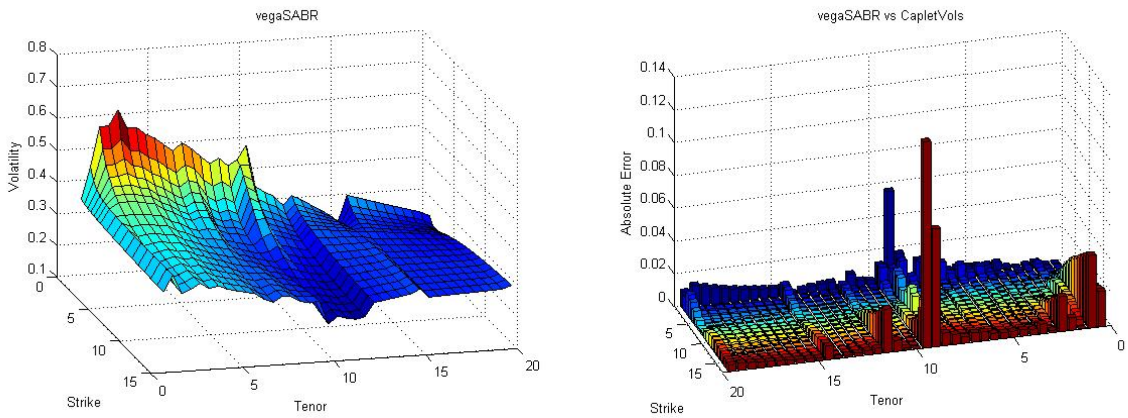

Fig. 11.5: Vega SABR calibration surface (left) as well as absolute error plot (right)

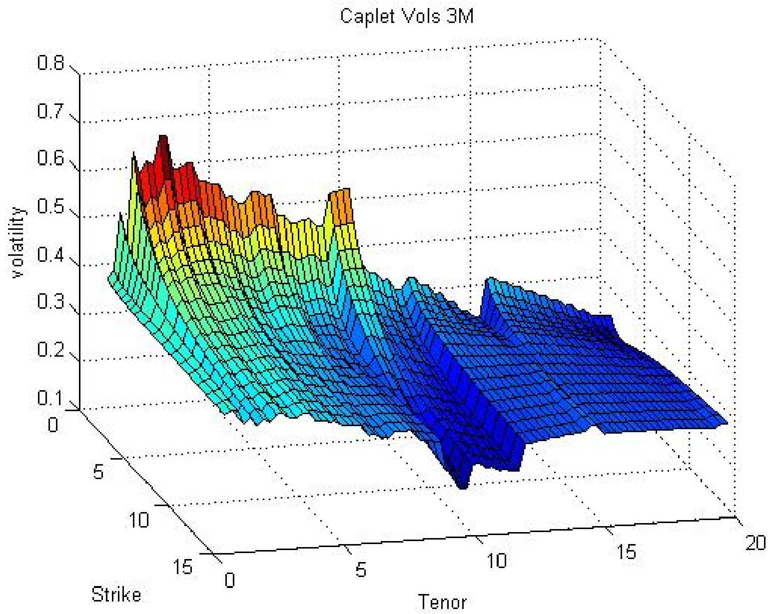

Fig. 11.6: Transformed 3M Caplet Volatilities

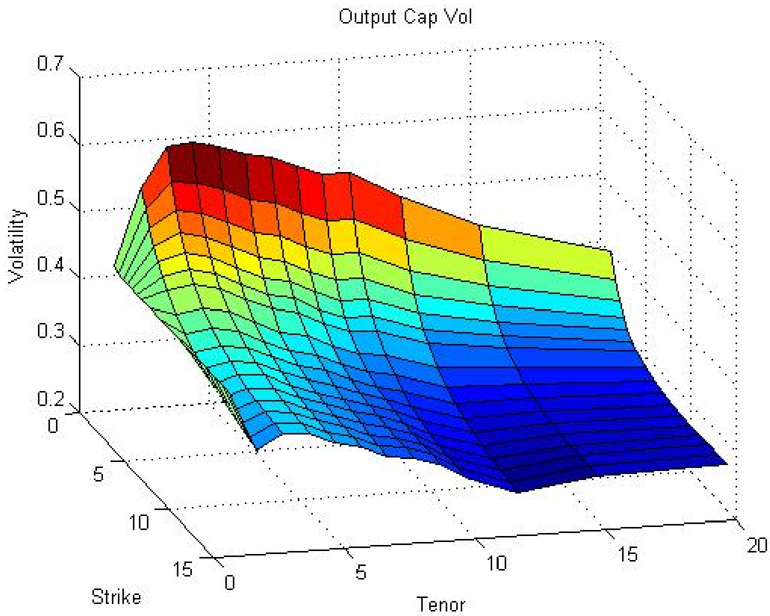

Fig. 11.7: Output 3M Cap volatility surface

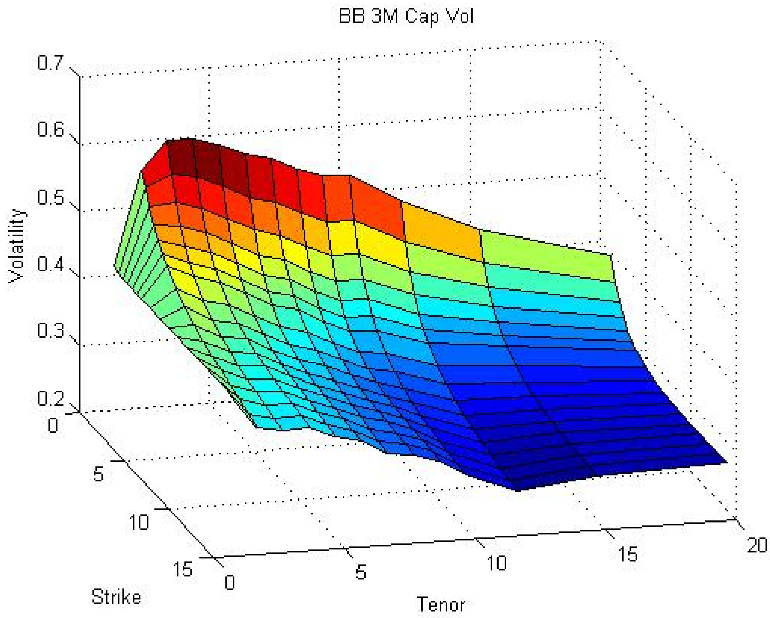

Fig. 11.8: Bloomberg 3M Cap volatility surface

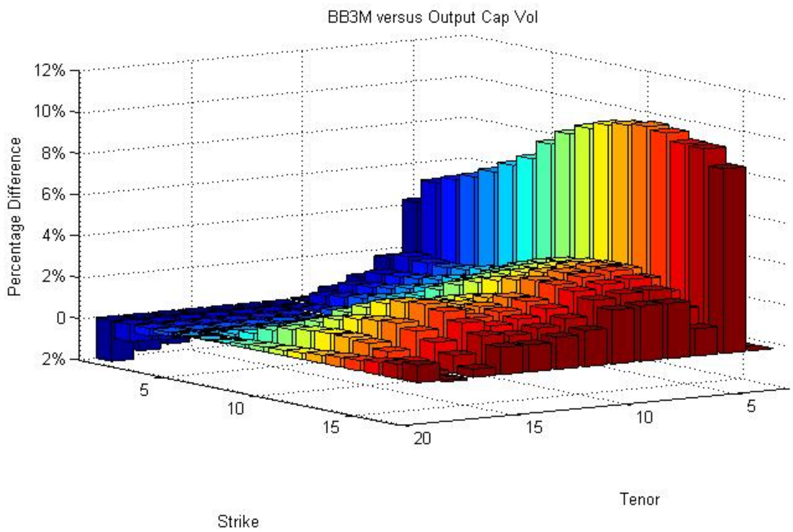

Fig. 11.9: Percentage difference between the Bloomberg 3M cap volatility surface and the output 3M Cap surface

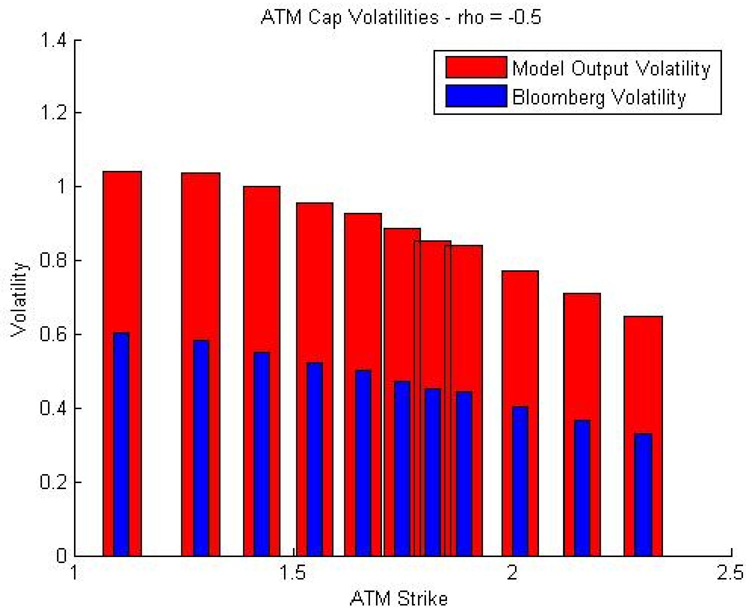

Fig. 11.10: Plot of ATM volatilites for ρ^{OIS} = -0.5

11.1.6 Altering ρ^{OIS} Parameter

The ρ^{OIS} parameter was defined as the correlation between the forward rates along the OIS curve. Whilst its theoretical interpretation is of importance, it is also an input parameter in this methodology and as such can be used to 'tune' the results. Here we demonstrate the effect on the output volatilities by varying the ρ^{OIS} parameter.

Appendix 1 shows the ATM strikes and volatilities quoted on Bloomberg for 3M caps. Note as the first two quotes are not in our strike range, and the last two quotes not in our tenor range, these are not considered for comparison. Figures 11.10 through to 11.14 show the effects of various ρ^{OIS} parameters.

The flexibility the parameter allows is very clear. Note the final output cap volatility surface was computed with $\rho^{OIS} = 0.95$.

11.1.7 Plausibility Check: 3M to 6M

As a plausibility check for the entire approach, we check whether the methodology is consistent. We do this by using our computed 3M cap volatility surface to compute a 6M cap volatility surface, which ideally should be very similar to the 6M cap surface we used as in input. Holding the assumptions ($\rho^{OIS} = 0.95$) and approach the same,

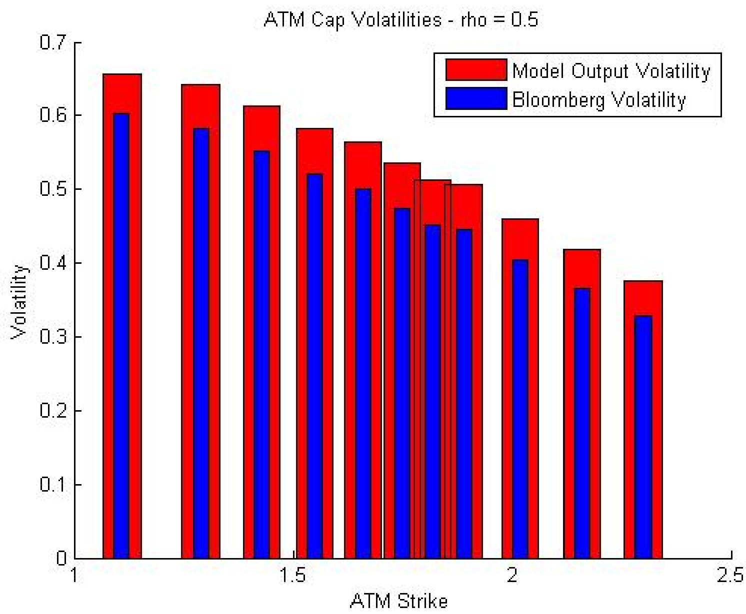

Fig. 11.11: Plot of ATM volatilites for $\rho^{OIS} = 0.5$

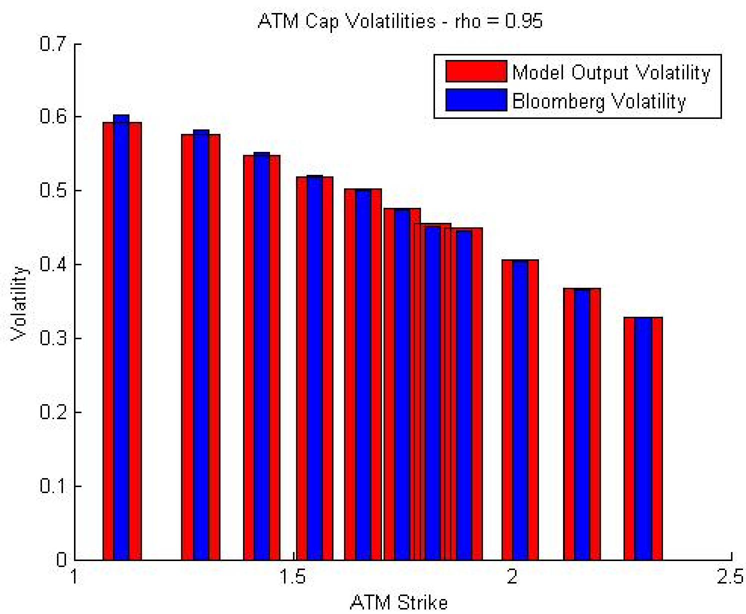

Fig. 11.12: Plot of ATM volatilites for $\rho^{OIS} = 0.95$

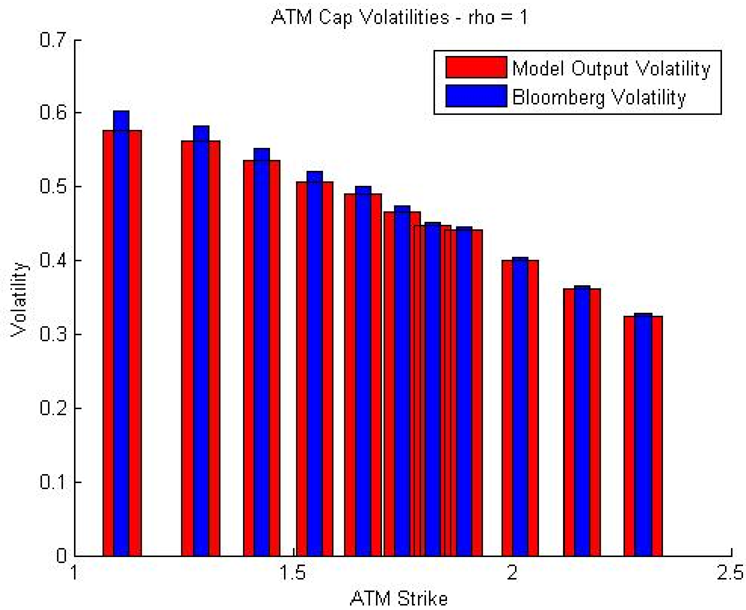

Fig. 11.13: Plot of ATM volatilities for $\rho^{OIS} = 1$

the output 6M cap surface can be seen in Figure 11.14, along with the percentage difference to the input 6M surface in Figure 11.15. As can be seen in Figure 11.14, along the first smile (at tenor 0.5) there appears to be a kink in the surface at the far out the money strikes, this is mimicked by the spike in figure 11.15. Besides this, the output volatility surface is always within 4% of the input surface, and most often within under 2%. This is a very positive result and it shows the methodology is within itself consistent.

11.1.8 Arbitrage Considerations

Having already shown the algorithm to self-consistent, a final consideration into possible arbitrage is presented here. Conditions for the avoidance of arbitrage opportunities in the equity and FX market volatility surfaces is well documented, however this is not as easy in the interest rate derivatives case (Johnson and Nonas, 2009). This is due in part to the complexity of the products, as well as the liquidity issues surrounding market quotes.

Two arbitrage checks were carried out on the caplet and cap prices, as presented by Carr and Madan (2005), shown in Figure 11.16 and 11.17. The first tests for butterfly spread arbitrage. Let $Cplt_{i,j}$ be the price of a 3 month caplet with strike

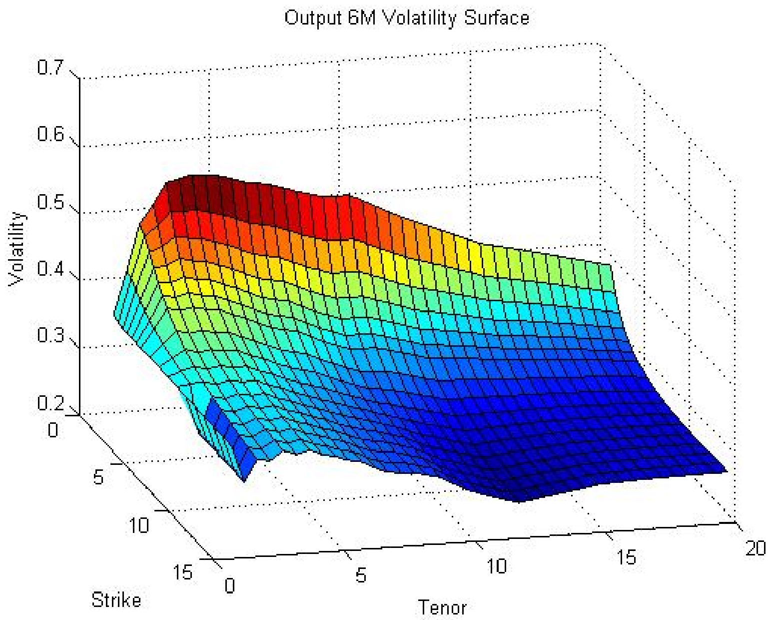

Fig. 11.14: Surface plot of the plausibility check output 6M cap surface

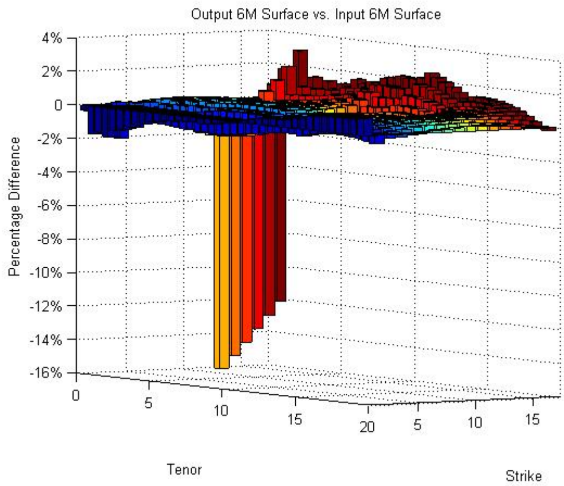

Fig. 11.15: Percentage difference plot between model input 6M cap surface and plausibility check output 6M Cap surface

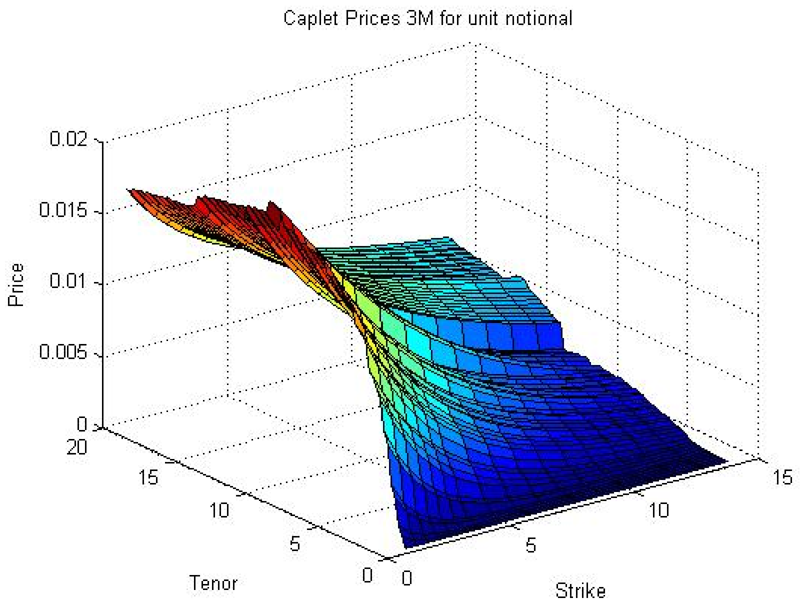

Fig. 11.16: Prices of the three month caplets

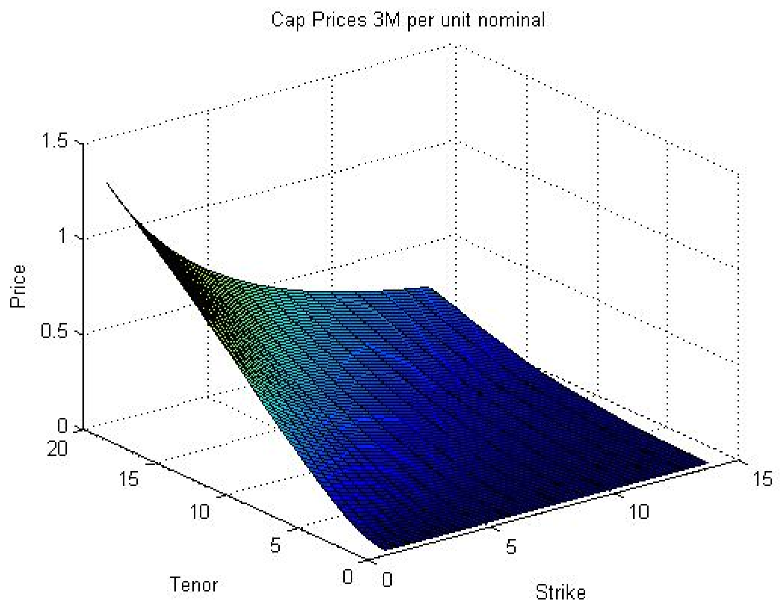

Fig. 11.17: Prices of the three month caps

K_i and maturity j, then we compute

$$BS_{i,j} = Cplt_{i-1,j} - \frac{K_{i+1} - k_{i-1}}{K_{i+1} - K_i} Cplt_{i,j} + \frac{K_i - k_{i-1}}{K_{i+1} - K_i} Cplt_{i+1,j}.$$

We then require $BS_{i,j} \geq 0$. A surface plot of BS is shown in Appendix 3. We find that this inequality is violated in only three instances.

The final check is for increasing Cap prices, hence we check that $Cap_{i,j+1} - Cap_{i,j} \geq 0$ where $Cap_{i,j}$ is the price of a three month cap today with strike K_i and maturity j. This condition was never violated and a plot of this is shown in Appendix 3. Note none of the volatilities implied negative caplet prices.

Johnson and Nonas (2009) puts forward the idea of triangle or 'in plane' arbitrage which should also be tested along with the butterly and call-spread arbitrage. However, this approach requires a great deal of market data as one would need caplet volatilities based on an increasing set of underlying rate tenors (6m, 9m, 12m etc).

Chapter 12

Results 2 - Swaption Volatility Surface

12.1 Multi Curve Extension

12.1.1 Market Inputs

Below is a surface plot of the input 6M swaption volatility surface quoted from Bloomberg 2-12-2014. The underlying swap tenor is 5 years, with rate tenor 6 months and OIS discounting. Also as input we have the 6M, 3M and OIS yield curves, and the 6M and 3M fair swap rates.

12.1.2 SABR calibration

Using the normal SABR calibration approach, Figure 12.2 shows the swaption volatility surface implied by the SABR parameters, as well as the absolute error plot.

12.1.3 Transformation

The final Output 3M swaption volatility surface is shown in Figure 12.3.

12.1.4 Comparison with Market 3M Surface

Bloomberg quotes a 3M swaption surface. Once again the it is populated by a combination of market quotes and interpolated volatilities. Figure 12.3 shows the Bloomberg 3M surface (Swap tenor of 5 years) and Figure 12.4 shows the percentage difference between the model output 3M surface and the market quoted surface. Once again the output volatility surface is largely within 5% of the market quoted volatilities.

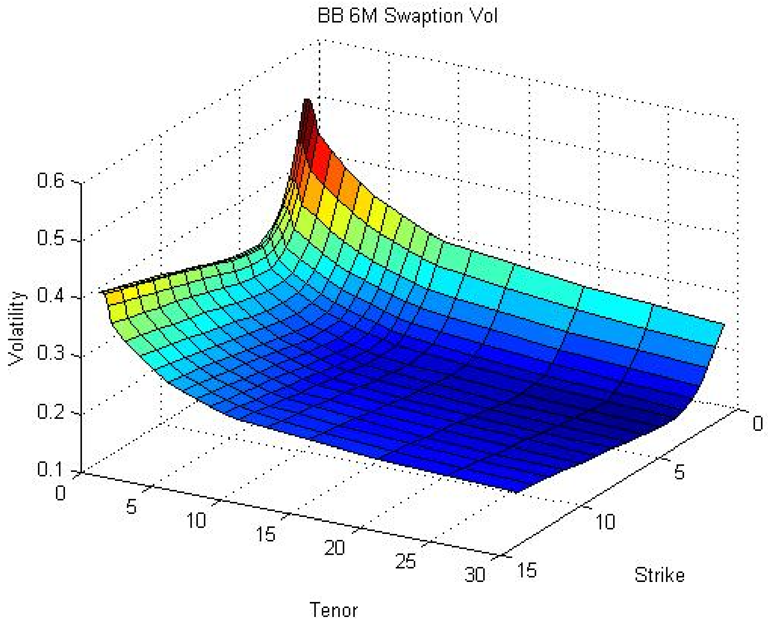

Fig. 12.1: Input 6M swaption volatility surface. Source- Bloomberg

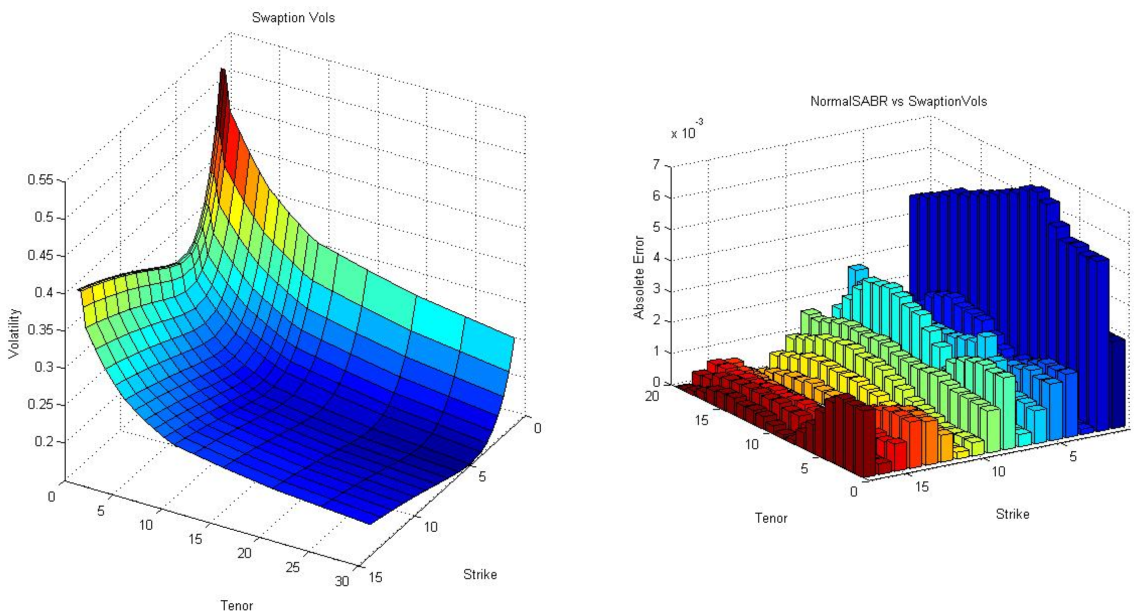

Fig. 12.2: Normal SABR calibration(left) as well as absolute error plot(right)

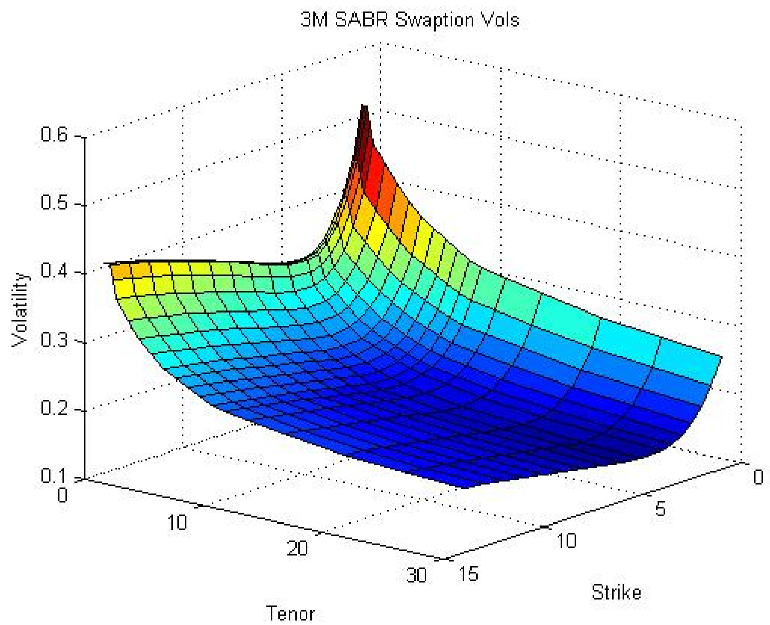

Fig. 12.3: The output 3M swaption volatility surface

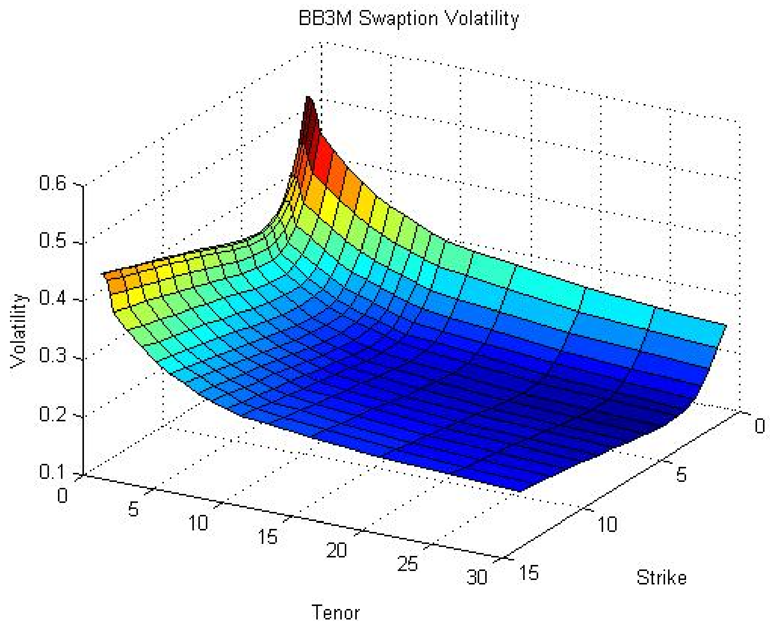

Fig. 12.4: Bloomberg 3M swaption volatility surface

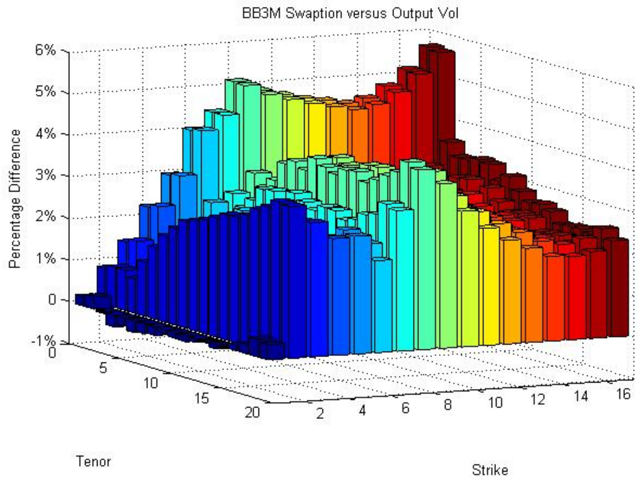

Fig. 12.5: Percentage difference between Bloomberg 3M swaption volatility surface and the model output 3M volatility surface

12.1.5 Plausibility Check: 3M to 6M

Once again, to assess the consistency within the model, we use the output 3M surface as input to the model and carrying out the same algorithm as detailed in Section 10.1 we compute a 6M swaption volatility surface and compare this with the original input 6M surface. Figure 12.6 shows the resultant 6M surface and Figure 12.7 shows the percentage error plot. Here we see that the output 6M swaption volatilities are always within 5% of the input 6M swaption volatilities.

12.1.6 Arbitrage Considerations

As with the caplets, two tests for arbitrage were computed on the swaption prices (shown in Figure 12.8), namely the butterfly spread condition as well as the call spread condition. These were computed similarly to the caplet case, using the swaption prices instead of caplet and cap prices. No violation of the constraints were found in either case.

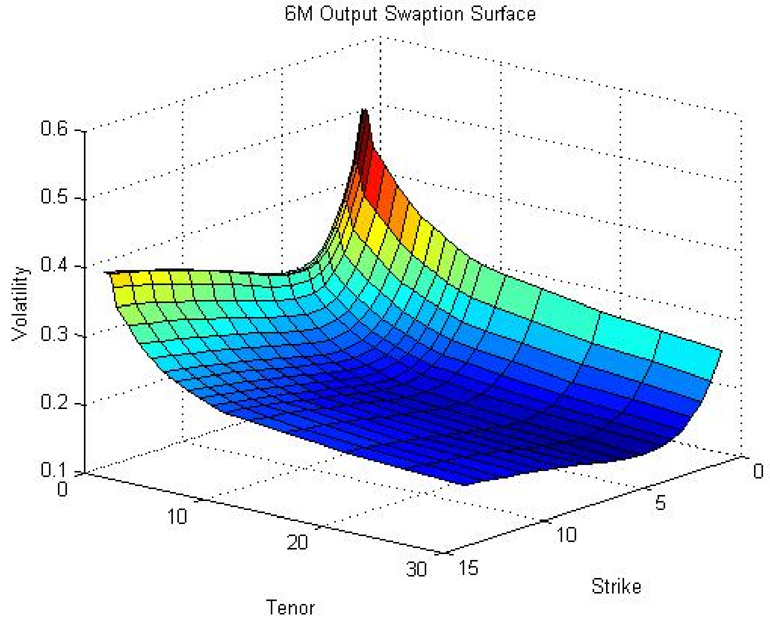

Fig. 12.6: Surface plot of the plausibility check output 6M swaption surface

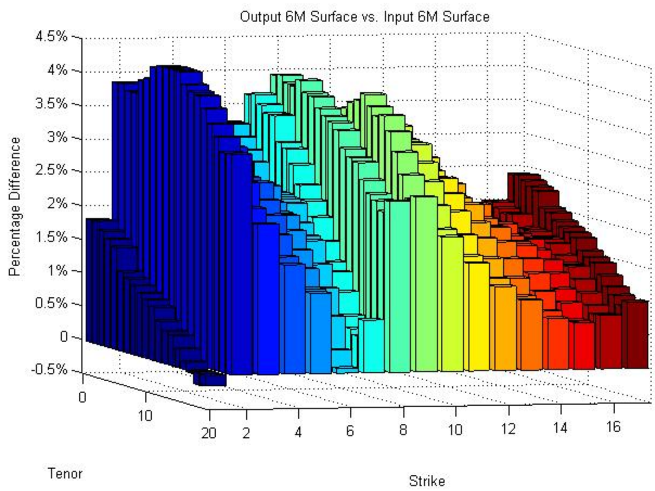

Fig. 12.7: Percentage difference plot between model input 6M swaption surface and plausibility check output 6M swaption surface

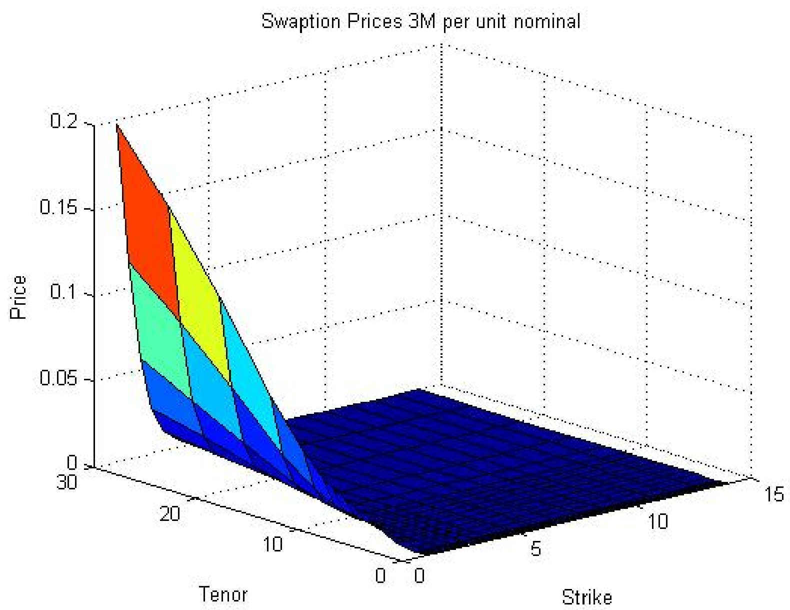

Fig. 12.8: Prices of the 3 month swaptions

Chapter 13

Discussion

The effects of the crisis of 2007-08 on interest rate markets has been outlined in Part 1 of this dissertation. It has resulted in a new pricing methodology as well as a new approach to modelling interest rates. This has resulted in numerous difficulties market practitioner must now face that were not perceived to be an issue in the single curve framework, such as bootstrapping multiple curves within a single economy.

Along with the change in pricing of interest rate derivatives, many applications and methodologies surrounding interest rate derivatives must also be addressed. One such methodology is that of linking volatilities. We have implemented a solution to this put forward by Kienitz (2013). We have seen that this method takes into account the basis spreads between forward/FRA rates, which is necessary as these spreads carry the different liquidity and credit risk levels within the different sub-markets. The model is able to reproduce the market quoted ATM volatilities as well as the entire surface to a reasonable degree. The flexibility allowed by the ρ parameter is also an attractive feature, as one can 'tune' the surface to reprice the market quoted ATM caps, floors and swaptions. Also it is self-consistent in that the output 3M surface obtained from an input 6M surface will, when carrying out the same methodology just reversing the direction, give a 6M surface very similar to the input. Finally some arbitrage conditions were investigated and the output volatility surfaces appeared to perform adequately.

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Appendix 1

The Table below shows the ATM strikes and volatilities from the input **6M** Bloomberg Cap Surface.

ATM Strike	ATM Vol
0,79	36,52
1,03	49,95
1,26	53,86
1,45	52,23
1,6	49,74
1,73	47,06
1,83	45,39
1,93	43,01
2,01	41,11
2,08	40,6
2,2	37,15
2,33	34,09
2,45	30,98
2,48	29,86
2,5	29,24

The Table below shows the ATM strikes and volatilities from the input **3M** Bloomberg Cap Surface.

ATM Strike	ATM Vol
0,67	43,85
0,89	57,09
1,11	60,31
1,29	58,16
1,43	55,13
1,55	52,02
1,66	50,06
1,75	47,35
1,82	45,16
1,89	44,48
2,02	40,37
2,16	36,59
2,3	32,8
2,35	31,37
2,38	30,52

Appendix 2

Figure 13.1 shows the basis spreads between the 6M FRA rates and the OIS forward rates. Figure 13.2 shows the basis spreads between the 3M FRA rates and the OIS forward rates.

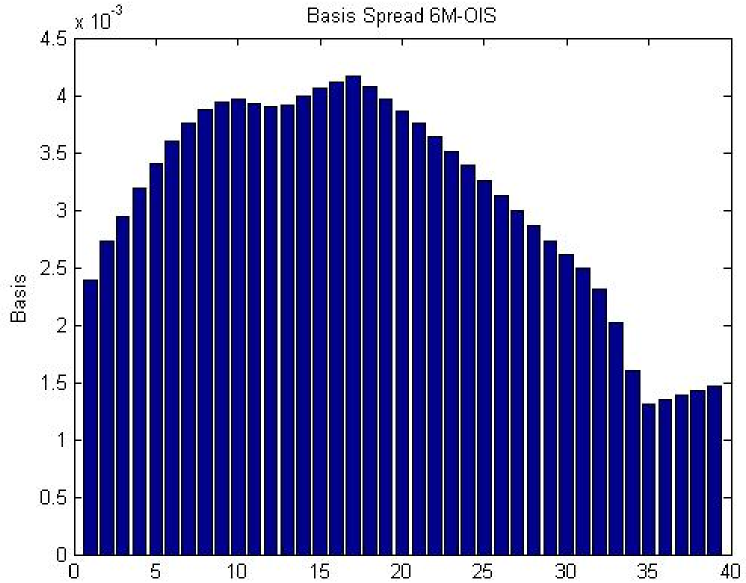

Fig. 13.1: Plot of the 6M-OIS basis spreads

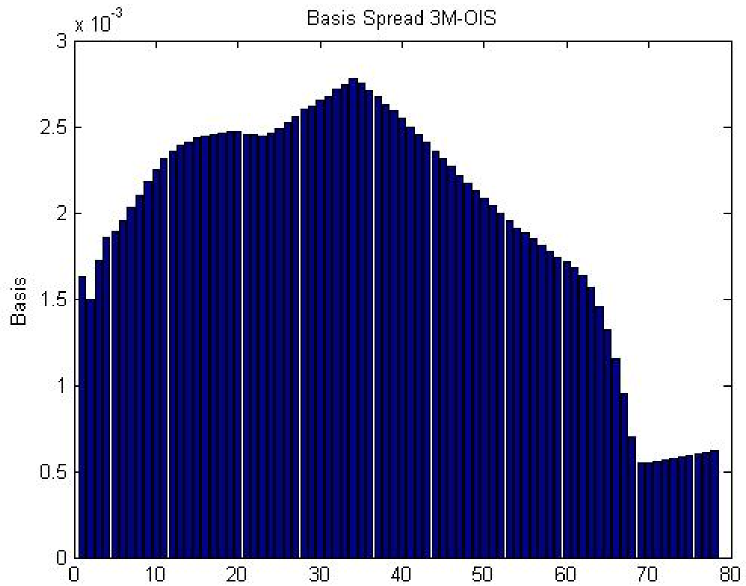

Fig. 13.2: Plot of the 3M-OIS basis spreads

Appendix 3

Figure 13.3 shows the butterfly spread condition for the three month caplets. Figure 13.4 shows the call spread condition applied to three month caps. Figure 13.5 shows the butterfly spread condition for the three month swaptions, and figure 13.6 shows the call spread condition applied to the three month swaptions.

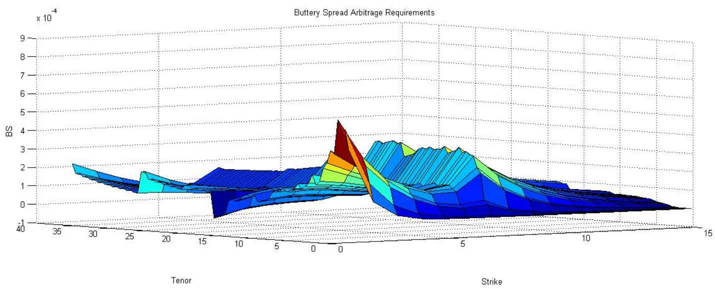

Fig. 13.3: Butterly Spread requirement for three month caplets

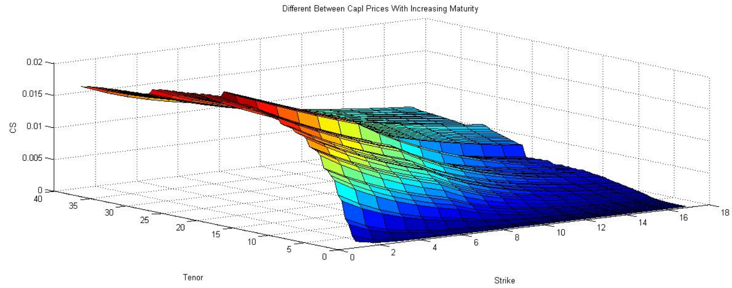

Fig. 13.4: Call spread requirement for three month caps.

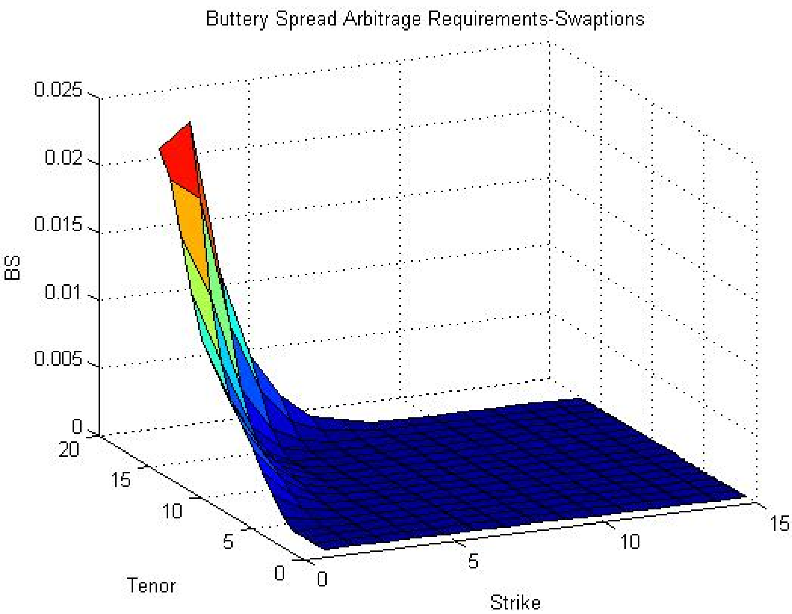

Fig. 13.5: Butterly Spread requirement for three month caplets

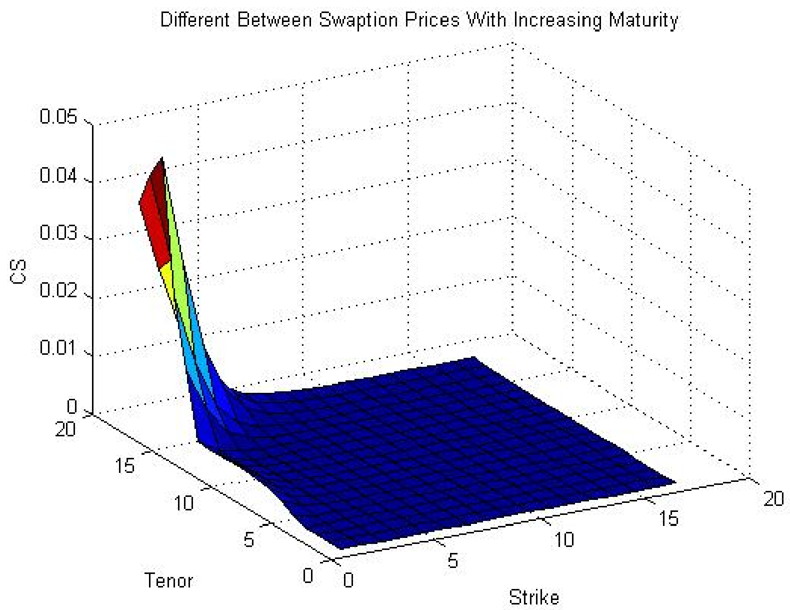

Fig. 13.6: Call spread requirement for three month caps.

Appendix 4

Below is the yield curve data used in the above computations.

Time(yrs)	**OIS**	**Time(yrs)**	**3M**	**Time (yrs)**	**6M**
0,0000	1,0000	0,0000	1,0000	0,0000	1,0000
0,0027	1,0000	0,2466	0,9986	0,4986	0,9966
0,0192	0,9999	0,2904	0,9984	0,5808	0,9960
0,0384	0,9998	0,3671	0,9979	0,6685	0,9954
0,0849	0,9996	0,4630	0,9973	0,7507	0,9947
0,1699	0,9993	0,5397	0,9968	0,8329	0,9941
0,2466	0,9990	0,7890	0,9952	0,9178	0,9934
0,3315	0,9986	1,0384	0,9933	1,0000	0,9927
0,4219	0,9982	1,2877	0,9911	1,2493	0,9904
0,4986	0,9978	1,5370	0,9886	1,5014	0,9877
0,5808	0,9974	1,8055	0,9855	2,0027	0,9814
0,6685	0,9970	2,0548	0,9824	3,0082	0,9657
0,7507	0,9965	3,0082	0,9686	4,0055	0,9473
0,8329	0,9961	4,0055	0,9515	5,0027	0,9273
0,9178	0,9956	5,0027	0,9326	6,0055	0,9061
1,0000	0,9951	6,0055	0,9128	7,0055	0,8843
1,5014	0,9913	7,0055	0,8924	8,0055	0,8619
2,0027	0,9865	8,0055	0,8715	9,0110	0,8393
3,0082	0,9740	9,0110	0,8501	10,0082	0,8168
4,0055	0,9590	10,0082	0,8287	12,0082	0,7717
5,0027	0,9426	12,0082	0,7849	15,0137	0,7074
6,0055	0,9252	15,0137	0,7213	20,0192	0,6126
7,0055	0,9071	20,0192	0,6270	25,0164	0,5359
8,0055	0,8882	25,0164	0,5498	30,0219	0,4709
9,0110	0,8686	30,0219	0,4840	40,0274	0,3713
10,0082	0,8489	40,0274	0,3834	50,0356	0,2925
12,0082	0,8077				
20,0192	0,6535				
25,0164	0,5769				
30,0219	0,5107				
40,0274	0,4082				
50,0356	0,3255				