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**On Doitchinov's quietness for arbitrary
quasi-uniform spaces**

by

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A thesis prepared under the supervision of Professor Hans-Peter Albert
Künzi in the fulfilment of the requirement for the degree of Doctor of
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Abstract

The notion of a quiet quasi-uniform space was introduced by Doitchinov in 1988 when he developed an interesting completion theory for this class of quasi-uniform spaces. At the same time Doitchinov developed a similar completion theory for a class of T_0 -balanced quasi-pseudometric spaces.

In my Masters thesis I showed that the Doitchinov completion theory for balanced quasi-pseudometric spaces can be extended to arbitrary T_0 -quasi-pseudometric spaces. That completion was called *the B-completion*.

The principal aim of this thesis is to investigate whether the Doitchinov completion theory for quiet quasi-uniform space can be extended to arbitrary T_0 -quasi-uniform spaces. The main result in this thesis is negative and leads us to conclude that Doitchinov's completion theory for quiet quasi-uniform spaces cannot be fully extended to arbitrary quasi-uniform spaces, because investigations due to Deák indicate that no suitable concept of a quiet Cauchy filter pair exists which could replace the quasi-pseudometric concept of a balanced Cauchy filter pair in the quasi-uniform setting. Under these circumstances we therefore suggest that in an arbitrary quasi-uniform space, we should work with a nonempty subbasic family of quasi-pseudometrics and an appropriate concept of balancedness of Cauchy filter pairs with respect to that family. In this way we obtain a general theory of the *B-completion* for a subbasic T_0 -family of quasi-pseudometrics that can be applied to the study of any quasi-uniform space.

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To my wife Fyfy Kinsamba Makitu and our children Exaurdi and Daniel Newton Makitu, thank you for all your support and love through out my life.

Dedication

I dedicate this thesis to the memory of my late father, Francois Makitu Samba, who passed away two months before its completion, to my wife Fyfy Kinsamba and my two boys Exaurdi Makitu Kassata and Daniel Newton Makitu Mbala for their patience and understanding.

To all members of the Youth Department of the Good Shepherd Christian community, who are willing to carry out any scientific work.

University of Cape Town

Declaration

I, CHARLY MAKITU KIVUVU

hereby declare that this thesis is my own unaided work which is being submitted for the degree of Doctor of Philosophy at the University of Cape Town. It has not been submitted for any degree or examination in any other university.

SIGNED:

DATE:

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Chapter 0

Introduction

In 1988 Doitchinov [11] developed an interesting completion theory for balanced T_0 -quasi-metric spaces. At the same time he developed a similar completion theory for T_0 -quiet quasi-uniform spaces [12, 13].

In our recent work [28] we have successfully extended the theory of balanced quasi-pseudometric spaces to arbitrary T_0 -quasi-pseudometric spaces. The resulting completion was called *the B-completion*. The B -completion was built as an extension of the bicompletion of the original space. We introduced and investigated a new class of maps called *balanced* maps. It was proved that in the case of a balanced T_0 -quasi-metric space the B -completion yields up to isometry Doitchinov's completion.

The main purpose of this thesis is first to establish some new results from the theory of the B -completion and secondly to extend the Doitchinov completion theory of quiet T_0 -quasi-uniform spaces to arbitrary quasi-uniform spaces. The main result in this part of the thesis is negative and we have to conclude that Doitchinov's completion theory of quiet quasi-uniform spaces cannot be extended to arbitrary quasi-uniform spaces, because investigations due to Deák indicate that no suitable concept of a quiet Cauchy filter pair exists which could replace the quasi-pseudometric concept of a balanced Cauchy filter pair in the quasi-uniform setting. In order to obtain reasonable results, we therefore suggest that in an arbitrary quasi-uniform space we instead work with a chosen nonempty subbasic family of quasi-pseudometrics and a concept of balancedness with respect to that family. In this way we obtain a general theory of the B -completion for a subbasic T_0 -family of quasi-

pseudometrics that can be applied to the study of any quasi-uniform space.

We now give a more detailed discussion of the contents of each chapter.

In Chapter 1 we give a brief overview of certain well-known definitions from the theory of quasi-pseudometric spaces. We summarize the construction of the B -completion of a T_0 -quasi-pseudometric space that we have developed in [28]. This leads us to establish some new results about the theory of the B -completion that are given in Chapter 2.

Chapter 2 presents an example which shows that B -completeness is a property of quasi-pseudometric spaces which need not be preserved under quasi-uniform isomorphisms. We exhibit some examples of balanced maps in the theory of the B -completion of a T_0 -quasi-pseudometric space. We also prove a result showing that the B -completion commutes in the expected way with an appropriate form of a countable product of quasi-pseudometric spaces. Similar results for balanced quasi-metrics were obtained by Doitchinov. We finally prove that the B -completion of a totally bounded T_0 -quasi-pseudometric space is totally bounded. Note that some of the results in this chapter were published in [29].

Chapter 3 deals with some results on quasi-uniform spaces and properties of filter pairs. We explain in a detailed way the notions of Cauchy, fully Cauchy, locally quiet and filter symmetric quasi-uniformities introduced by Deák in [6, 8, 9]. We also give detailed proofs of some results due to Deák in order to establish connections between these different classes of quasi-uniformities.

Chapter 4 is the main chapter of the thesis which deals with Doitchinov's quietness for quasi-uniform spaces. We try to extend the Doitchinov completion theory of quiet quasi-uniform spaces to general T_0 -quasi-uniform spaces. In order to understand the notion of quietness of a quasi-uniform space, we begin first by studying the notion of uniformly concentrated Cauchy filter pairs on a quasi-uniform space. We describe an example of a quasi-pseudometrizable quasi-uniform space due to Deák that has a family of weakly concentrated filter pairs that are minimal Cauchy, but the family is not uniformly weakly concentrated.

We then introduce and construct the B -completion of a family of quasi-pseudometrics \mathcal{D} on a set X . We give a new definition of a balanced (resp.

B -isometric) map for a family of quasi-pseudometrics and investigate the extension theorem of a balanced map on B -complete spaces. This leads us to characterize the B -completion of a family of quasi-pseudometrics. We also present an example of the B -completion of a family of quasi-pseudometrics to illustrate our investigations.

Chapter 5 presents the conclusion of the thesis and mentions some open problems which can constitute the topics for further research. The study of B -completeness of quasi-pseudometric spaces and quietness of quasi-uniform spaces leads to many open problems. For instance one can investigate whether the theory of the B -completion can be applied to paratopological groups. The construction of the B -completion of a T_0 -quasi-pseudometric induced by an absolute quasi-valued function on a paratopological group may be an interesting application of this theory in paratopological group theory.

The notion of a balanced fuzzy quasi-metric space introduced by V. Gregori, J.A. Mascarell and A. Sapena [21] leads them to develop an interesting completion theory for a class of balanced fuzzy quasi-metric spaces. They show that each balanced fuzzy-quasi-metric space has a completion in Doitchinov's sense. Similarly one can also investigate the possibility of applying the theory of the B -completion to fuzzy quasi-metrics and the possibility of constructing the B -completion of a fuzzy quasi-metric space.

Chapter 1

Preliminaries

In [28] we have successfully extended Doitchinov's completion theory for balanced quasi-pseudometric spaces, which he studied in [11, 12]. The resulting completion was called *the B-completion*. We have shown that each T_0 -quasi-pseudometric space admits a B -completion which is larger than the bicompletion of the original space.

In this chapter we present the summary of the construction of the B -completion of a T_0 -quasi-pseudometric space which we developed in [28]. We first recall some basic concepts from the theory of quasi-pseudometric spaces and then establish some interesting examples (Example 1.0.2, Example 1.0.3) and propositions (Proposition 1.0.2, Proposition 1.0.3) that help us better understand the theory of the B -completion. We also give an example (Example 1.0.5) which shows that for a general quasi-pseudometric d on a set X a balanced Cauchy filter pair need not be \mathcal{U}_d -equivalent to the Cauchy filter pair generated by a balanced Cauchy pair of sequences on X .

We start this section by recalling some basic concepts from the theory of quasi-pseudometric spaces.

Definition 1.0.1. Let X be a set and let $d : X \times X \rightarrow [0, \infty)$ be a function mapping into the set $[0, \infty)$ of the non-negative reals. Then d is called a quasi-pseudometric on X if

- (a) $d(x, x) = 0$ whenever $x \in X$,
- (b) $d(x, z) \leq d(x, y) + d(y, z)$ whenever $x, y, z \in X$.

We say that d is a T_0 -quasi-pseudometric if d also satisfies the following condition: For each $x, y \in X$, $d(x, y) = 0 = d(y, x)$ implies that $x = y$. A quasi-pseudometric d is called quasi-metric if for each $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$.

Note that a quasi-pseudometric d is called *pseudometric* when it satisfies the symmetry condition.

Let d be a quasi-pseudometric on X , then $d^{-1} : X \times X \rightarrow [0, \infty)$ defined by $d^{-1}(x, y) = d(y, x)$ whenever $x, y \in X$ is also a quasi-pseudometric, called the *conjugate quasi-pseudometric* of d . As usual, a quasi-pseudometric d on X such that $d = d^{-1}$ is called a *pseudometric*. For any quasi-pseudometric d , $d^s = \max\{d, d^{-1}\}$ is a pseudometric.

Let (X, d) be a quasi-pseudometric space. For each $\epsilon > 0$ set $U_\epsilon = \{(x, y) \in X \times X : d(x, y) < \epsilon\}$. (Instead of U_ϵ we shall also write $U_{d, \epsilon}$ in cases where the quasi-pseudometric d may not be obvious from the context.) In the following \mathcal{U}_d denotes the quasi-uniformity on X generated by the base $\{U_{d, \epsilon} : \epsilon > 0\}$ on $X \times X$. It is called *the quasi-uniformity induced by d on X* . The topology $\tau(\mathcal{U}_d)$ is called *the topology induced by d on X* and is often denoted by $\tau(d)$. For each $x \in X$, $\mathcal{U}_d(x)$ denotes the $\tau(\mathcal{U}_d)$ -neighbourhood filter at x .

The concept of a quasi-uniform space will be presented with more details in the third chapter of the thesis.

A map $f : (X, d) \rightarrow (Y, e)$ between two quasi-pseudometric spaces (X, d) and (Y, e) is called an *isometry* provided that $e(f(x), f(y)) = d(x, y)$ whenever $x, y \in X$. Two quasi-pseudometric spaces (X, d) and (Y, e) will be called *isometric* provided that there exists a bijective isometry $f : (X, d) \rightarrow (Y, e)$. A map $f : (X, d) \rightarrow (Y, e)$ between two quasi-pseudometric spaces (X, d) and

(Y, e) will be called *uniformly continuous* provided that for each $\epsilon > 0$ there is $\delta > 0$ such that for all $x, y \in X$, $d(x, y) < \delta$ implies that $e(f(x), f(y)) < \epsilon$.

We next repeat a well-known lemma which deals with the T_0 -quotient of a quasi-pseudometric space.

Lemma 1.0.1. [28, Lemma 1] *Let (X, d) be a quasi-pseudometric space. Define an equivalence relation \sim on X by setting $x \sim y$ if $d(x, y) = 0 = d(y, x)$. Let \widehat{X} be the set of equivalence classes $q_X(x)$, where $x \in X$, with respect to \sim . Then \widehat{d} on \widehat{X} defined by $\widehat{d}(q_X(x), q_X(y)) = d(x, y)$ whenever $x, y \in X$ determines a T_0 -quasi-pseudometric \widehat{d} on \widehat{X} . (In the following $q_X : X \rightarrow \widehat{X}$ will denote the isometric quotient map defined by $x \mapsto q_X(x)$ whenever $x \in X$.)*

Let $f : (X, d) \rightarrow (Y, e)$ be a uniformly continuous map between quasi-pseudometric spaces (X, d) and (Y, e) . Then $\widehat{f} : (\widehat{X}, \widehat{d}) \rightarrow (\widehat{Y}, \widehat{e})$ defined by $\widehat{f}(q_X(x)) := (q_Y \circ f)(x)$ whenever $x \in X$ is a well-defined uniformly continuous map between the T_0 -quasi-pseudometric quotient spaces $(\widehat{X}, \widehat{d})$ and $(\widehat{Y}, \widehat{e})$. It is an isometry provided that f is an isometry.

Let (X, d) be a quasi-pseudometric space and let A, B be nonempty subsets of X . We define the *2-diameter from A to B* by $\Phi_d(A, B) = \sup\{d(a, b) : a \in A, b \in B\}$. Of course ∞ is a possible value of a 2-diameter. For singleton $\{x\}$ we write $\Phi_d(x, A)$ and $\Phi_d(B, x)$ instead of $\Phi_d(\{x\}, A)$ and $\Phi_d(B, \{x\})$, respectively.

Let X be a set. For each $x \in X$, by \underline{x} we shall denote the filter on X generated by the filter base $\{\{x\}\}$ on X .

Definition 1.0.2. [28, Definition 2] *Let (X, d) be a quasi-pseudometric space. We shall say that a pair $\langle \mathcal{F}, \mathcal{G} \rangle$ of filters \mathcal{F} and \mathcal{G} on X is a Cauchy filter pair on (X, d) if $\inf_{F \in \mathcal{F}, G \in \mathcal{G}} \Phi_d(F, G) = 0$.*

Definition 1.0.3. [28, Definition 3] *Let (X, d) be a quasi-pseudometric space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be two Cauchy filter pairs on X . Then the following formula defines the distance*

$$d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) := \inf_{F \in \mathcal{F}, G' \in \mathcal{G}'} \Phi_d(F, G') = \inf_{F \in \mathcal{F}, G' \in \mathcal{G}'} \sup_{f \in F, g' \in G'} d(f, g')$$

from $\langle \mathcal{F}, \mathcal{G} \rangle$ to $\langle \mathcal{F}', \mathcal{G}' \rangle$.

According to [28, Lemma 2] the distance d^+ only attains values in $[0, \infty[$. Of course, for any Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-pseudometric space (X, d) we have $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) = 0$.

The next concept of a balanced Cauchy filter pair is useful in the study of the B -completion and should be compared with the notion of a weakly concentrated Cauchy filter pair on a quasi-uniform space that we shall study in the third chapter.

Definition 1.0.4. [28, Definition 4] Let (X, d) be a quasi-pseudometric space. A Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) is said to be *balanced* on (X, d) if for each $x, y \in X$ we have

$$d(x, y) \leq \inf_{G \in \mathcal{G}} \Phi_d(x, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, y).$$

The following notion of C -balancedness for Cauchy filter pairs will be useful in our investigation in Chapter 4.

Definition 1.0.5. Let C be a chosen real constant larger than or equal to 1. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on a quasi-pseudometric space (X, d) . We shall call $\langle \mathcal{F}, \mathcal{G} \rangle$ *C -balanced*, provided for each $x, y \in X$ we have

$$d(x, y) \leq C(\inf_{G \in \mathcal{G}} \Phi_d(x, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, y))$$

(Note that this condition becomes weaker when C gets larger.) We shall call a quasi-pseudometric space (X, d) *C -balanced* provided that each Cauchy filter pair on (X, d) is C -balanced. Of course, 1-balanced is balanced.

Let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be two filter pairs on a set X . Then $\langle \mathcal{F}, \mathcal{G} \rangle$ is called *coarser* than $\langle \mathcal{F}', \mathcal{G}' \rangle$ provided that both $\mathcal{F} \subseteq \mathcal{F}'$ and $\mathcal{G} \subseteq \mathcal{G}'$. In this case we say that $\langle \mathcal{F}', \mathcal{G}' \rangle$ is *finer* than $\langle \mathcal{F}, \mathcal{G} \rangle$.

Let (X, d) be a quasi-pseudometric space. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be two Cauchy filter pairs on (X, d) such that $\langle \mathcal{F}, \mathcal{G} \rangle$ is coarser than $\langle \mathcal{F}', \mathcal{G}' \rangle$. Then $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced if $\langle \mathcal{F}', \mathcal{G}' \rangle$ is balanced.

We next give an example of a balanced Cauchy filter pair.

Example 1.0.1. [28, Example 2] Let (X, d) be a quasi-pseudometric space. Then for each $x \in X$, $\alpha_X(x) := \langle \underline{x}, \underline{x} \rangle$ as well as $\langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle$ are balanced Cauchy filter pairs on (X, d) . In fact, for each $x \in X$, $\langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle$ is a minimal (balanced) Cauchy filter pair on (X, d) .

Our next example shows that in Definition 1.0.3, d^+ is defined as a limit superior. We cannot change the order of the operations \inf and \sup in the definition of d^+ without possibly changing the value of d^+ , even for balanced Cauchy filter pairs.

Example 1.0.2. Let $X = \{a_n, b_n, x_n, y_n : n \in \mathbb{N}\} \cup \{0^-, 0^+\}$ be a collection of four sequences $(a_n)_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$, $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ (such that all these terms are distinct) and two additional special points $0^-, 0^+$. We define a T_0 -quasi-pseudometric d on X as follows:

For each $x \in X$ set $d(x, x) = 0$. Furthermore for each $n, k \in \mathbb{N}$ set $d(a_n, x_k) = d(a_n, y_k) = d(b_n, x_k) = d(b_n, y_k) = \frac{1}{2n} + \frac{1}{2k}$. Moreover, for each $n \in \mathbb{N}$ set $d(b_n, 0^-) = d(0^+, y_n) = \frac{1}{2n}$. Finally, set $d(x, y) = 1$ otherwise. Note that $d \leq 1$. Since it is impossible that $x, y, z \in X$, $x \neq y$, $y \neq z$, $d(x, y) < 1$ and $d(y, z) < 1$, we deduce that d satisfies the triangle inequality.

For each $n \in \mathbb{N}$ set $F_n = \{a_k, b_k : k \in \mathbb{N}, k \geq n\}$ and $G_n = \{x_k, y_k : k \in \mathbb{N}, k \geq n\}$ and let \mathcal{F} be the filter on X generated by the base $\{F_n : n \in \mathbb{N}\}$ and let \mathcal{G} be the filter on X generated by the base $\{G_n : n \in \mathbb{N}\}$. Evidently $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair on (X, d) . Let us show that $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced on (X, d) .

Suppose that $x, y \in X$ and $n, k \in \mathbb{N}$ such that $\Phi_d(x, G_n) < 1$ and $\Phi_d(F_k, y) < 1$. Then there is $s \in \mathbb{N}$ such that $x = a_s$ or $x = b_s$, and there is $t \in \mathbb{N}$ such that $y = x_t$ or $y = y_t$. We conclude that $d(x, y) = \frac{1}{2s} + \frac{1}{2t} = \inf_{r \in \mathbb{N}} \Phi_d(x, G_r) + \inf_{p \in \mathbb{N}} \Phi_d(F_p, y)$. It follows that $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced.

We note that $d(0^+, 0^-) = 1$, but

$$\sup_{n \in \mathbb{N}} d(0^+, G_n) + \sup_{n \in \mathbb{N}} d(F_n, 0^-) = 0,$$

where as usual in a quasi-pseudometric space (X, d) , for nonempty $A, B \subseteq X$ we make use of the convention that $d(A, B) := \inf_{a \in A, b \in B} d(a, b)$.

This shows that we cannot change the order of the operations \inf and \sup in the definition of balancedness. Similarly, we observe that for the balanced Cauchy filter pairs $\alpha_X(0^+)$ and $\langle \mathcal{F}, \mathcal{G} \rangle$ the value

$$d^+(\alpha_X(0^+), \langle \mathcal{F}, \mathcal{G} \rangle) = \inf_{n \in \mathbb{N}} \Phi_d(0^+, G_n) = \inf_{n \in \mathbb{N}} \sup_{g \in G_n} d(0^+, g) = 1$$

is strictly larger than $\sup_{n \in \mathbb{N}} \inf_{g \in G_n} d(0^+, g) = 0$.

Furthermore, for each $n \in \mathbb{N}$ let $G'_n = \{y_k : k \in \mathbb{N}, k \geq n\}$. Moreover, let \mathcal{G}' be the filter on X generated by the base $\{G'_n : n \in \mathbb{N}\}$. Then $\langle \mathcal{F}, \mathcal{G}' \rangle$ is a Cauchy filter pair on (X, d) finer than $\langle \mathcal{F}, \mathcal{G} \rangle$, which is not balanced, since $1 = d(0^+, x_1) \not\leq \inf_{G' \in \mathcal{G}'} \Phi_d(0^+, G') + \inf_{F \in \mathcal{F}} \Phi_d(F, x_1) = 0 + \frac{1}{2}$. In particular $d^+(\alpha_X(0^+), \langle \mathcal{F}, \mathcal{G}' \rangle) = 0$. Hence refining a Cauchy filter pair may destroy the property of balancedness.

We shall now explain the construction of the B -completion of a T_0 -quasi-pseudometric space (X, d) and its relation to the bicompletion of (X, d) .

Proposition 1.0.1. *[28, Theorem 1] Let (X, d) be a quasi-pseudometric space and let X^+ be the set of all balanced Cauchy filter pairs on (X, d) . Then (X^+, d^+) is a quasi-pseudometric space, where for convenience in the following d^+ will also denote the restriction of d^+ to $X^+ \times X^+$.*

Definition 1.0.6. *([28, Definition 6]) Let (X, d) be a quasi-pseudometric space. An arbitrary Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on X is said to converge to $x \in X$ provided that $\inf_{G \in \mathcal{G}} \Phi_d(x, G) = 0$ and $\inf_{F \in \mathcal{F}} \Phi_d(F, x) = 0$. A quasi-pseudometric space (X, d) is called B -complete provided that each balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ converges in X .*

Note that for any quasi-pseudometric space (X, d) , the space (X, d^{-1}) is B -complete if and only if (X, d) is B -complete [28, Remark 8]. We recall that a quasi-pseudometric space (X, d) is bicomplete provided that each d^s -Cauchy filter on X converges in (X, d^s) . Furthermore a filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a set X is called *linked* provided that $F \cap G \neq \emptyset$ whenever $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

Lemma 1.0.2. (a) *Each linked Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-pseudometric space (X, d) is balanced.*

(b) [28, Lemma 3] *Let \mathcal{F} be a d^s -Cauchy filter on a quasi-pseudometric space (X, d) . Then $\langle \mathcal{F}, \mathcal{F} \rangle$ is a balanced Cauchy filter pair on (X, d) .*

Proof. (a) The assertion follows for instance from a slight modification of the proof of [28, Lemma 3] by replacing $\langle \mathcal{F}, \mathcal{F} \rangle$ by $\langle \mathcal{F}, \mathcal{G} \rangle$.

(b) The statement immediately follows from (a). □

We conclude that each quasi-pseudometric space (X, d) that is B -complete is bicomplete [28, Proposition 2].

Proposition 1.0.2. *Given two Cauchy filter pairs $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ on a quasi-pseudometric space (X, d) we have that*

$$\inf_{G' \in \mathcal{G}'} \Phi_{d^+}(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(G')) \leq d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle)$$

with equality if $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced; similarly we have that

$$\inf_{F \in \mathcal{F}} \Phi_{d^+}(\alpha_X(F), \langle \mathcal{F}', \mathcal{G}' \rangle) \leq d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle)$$

with equality if $\langle \mathcal{F}', \mathcal{G}' \rangle$ is balanced.

Proof. The statements follow from Lemmas 9,10 and 11 of [28]. □

The last stated result indeed means that for balanced Cauchy filter pairs we can compute d^+ as a limit superior componentwise in either order:

Corollary 1.0.1. [28, Corollary 8] *Let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be two balanced Cauchy filter pairs on a quasi-pseudometric space (X, d) . Then*

$$d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) = \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} \inf_{F \in \mathcal{F}} \sup_{f \in F} d(f, g') = \inf_{F \in \mathcal{F}} \sup_{f \in F} \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} d(f, g').$$

Note that by [28, Lemma 4] any isometry $g : (X, d) \longrightarrow (Y, e)$ from a T_0 -quasi-pseudometric space into a quasi-pseudometric space (Y, e) is injective. It follows from [28, Remark 11] that if (X, d) is a T_0 -quasi-pseudometric space, then the map $\alpha_X : (X, d) \longrightarrow (X^+, d^+)$ is an isometric embedding of (X, d) into (X^+, d^+) .

We next introduce a helpful terminology for sequences. Of course the underlying construction was already employed in Example 1.0.2.

Let $(x_n)_{n \in \mathbb{N}}$ and $(y_n)_{n \in \mathbb{N}}$ be two sequences in a set X . Let $\mathcal{F}(x_n)$ be the filter generated by the filter base $\{\{x_k : k \geq n, k \in \mathbb{N}\} : n \in \mathbb{N}\}$ and let $\mathcal{F}(y_n)$ be the filter generated by the filter base $\{\{y_k : k \geq n, k \in \mathbb{N}\} : n \in \mathbb{N}\}$ on X . Then we shall say that $\langle \mathcal{F}(x_n)_{n \in \mathbb{N}}, \mathcal{F}(y_n)_{n \in \mathbb{N}} \rangle$ is the filter pair generated by the pair $\langle (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \rangle$ of sequences in X . Let (X, d) be a quasi-pseudometric space. A pair of sequences $\langle (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \rangle$ in (X, d) will be called a (balanced) Cauchy pair of sequences provided that the filter pair $\langle \mathcal{F}(x_n)_{n \in \mathbb{N}}, \mathcal{F}(y_n)_{n \in \mathbb{N}} \rangle$ generated by $\langle (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \rangle$ is a (balanced) Cauchy filter pair on (X, d) . Our next example shows that equality need not hold in Proposition 1.0.2 for arbitrary Cauchy filter pairs.

Example 1.0.3. Let $X = \{-\frac{1}{n}, \frac{1}{n} : n \in \mathbb{N}\} \cup \{0^-, 0^+\}$, where 0^- and 0^+ are two distinct special points. We define a T_0 -quasi-pseudometric d on X as follows: For each $x \in X$, set $d(x, x) = 0$. For each $m \in \mathbb{N}$ set $d(0^+, \frac{1}{m}) = \frac{1}{m}$. For each $p, m \in \mathbb{N}$ such that $p \neq m$, let $d(-\frac{1}{m}, \frac{1}{p}) = \frac{1}{p}$. Furthermore set $d(-\frac{1}{m}, 0^-) = \frac{1}{m}$ whenever $m \in \mathbb{N}$. Finally set $d(x, y) = 1$ otherwise. One readily checks that d is a T_0 -quasi-metric: Indeed since $0 \neq d(x, y) < 1$ implies that $(x, y) = (0^+, \frac{1}{p}), (\frac{1}{n}, 0^-)$, or $(\frac{1}{n}, \frac{1}{p})$, with $p > 0$ and $n < 0$, we see that d satisfies the triangle inequality. Also note that $\langle 0^+, (\frac{1}{p})_{p \in \mathbb{N}} \rangle$ and $\langle (-\frac{1}{m})_{m \in \mathbb{N}}, 0^- \rangle$ are Cauchy pairs of sequences, where here 0^+ resp. 0^- denote constant sequences.

Furthermore in self-explanatory notation that suppresses the operation \mathcal{F} , we get that

$$d^+(\langle (-\frac{1}{m})_{m \in \mathbb{N}}, 0^- \rangle, \langle 0^+, (\frac{1}{p})_{p \in \mathbb{N}} \rangle) = \inf_{s, t \in \mathbb{N}} \Phi_d(\{-\frac{1}{m} : m \in \mathbb{N}, m \geq s\}, \{\frac{1}{p} : p \in \mathbb{N}, p \geq t\}) = 1,$$

since $d(-\frac{1}{m}, \frac{1}{m}) = 1$ whenever $m \in \mathbb{N}$. Moreover

$$\inf_{t \in \mathbb{N}} \Phi_{d^+}(\langle (-\frac{1}{m})_{m \in \mathbb{N}}, 0^- \rangle, \alpha_X(\{\frac{1}{p} : p \in \mathbb{N}, p \geq t\})) =$$

$$\inf_{t \in \mathbb{N}} \sup_{p \in \mathbb{N}, p \geq t} d^+(\langle (-\frac{1}{m})_{m \in \mathbb{N}}, 0^- \rangle, \alpha_X(\frac{1}{p})) = \inf_{t \in \mathbb{N}} \sup_{p \in \mathbb{N}, p \geq t} \frac{1}{p} = 0.$$

In fact $\langle (-\frac{1}{m})_{m \in \mathbb{N}}, 0^- \rangle$ is not balanced, since, for instance, we have that $1 = d(-\frac{1}{4}, \frac{1}{4}) \not\leq \Phi_d(-\frac{1}{4}, 0^-) + \inf_{s \in \mathbb{N}} \Phi_d(\{-\frac{1}{m} : m \in \mathbb{N}, m \geq s\}, \frac{1}{4}) = \frac{1}{4} + \frac{1}{4}$.

We now turn our attention to the minimal Cauchy filter pairs on a given quasi-pseudometric space (X, d) . Minimal Cauchy filter pairs that are balanced will be used to construct the B -completion of the space (X, d) .

Lemma 1.0.3. *If $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ are Cauchy filter pairs on a quasi-pseudometric space (X, d) and $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) = 0 = d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}, \mathcal{G} \rangle)$, then $\langle \mathcal{F} \cap \mathcal{F}', \mathcal{G} \cap \mathcal{G}' \rangle$ is a Cauchy filter pair on (X, d) .*

Proof. Let $U \in \mathcal{U}_d$. Since $\langle \mathcal{F}, \mathcal{G} \rangle$, $\langle \mathcal{F}', \mathcal{G}' \rangle$, $\langle \mathcal{F}', \mathcal{G} \rangle$ and $\langle \mathcal{F}, \mathcal{G}' \rangle$ are Cauchy filter pairs on (X, d) , there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$, and similarly $F' \in \mathcal{F}'$ and $G' \in \mathcal{G}'$ such that $F \times G \subseteq U$, $F \times G' \subseteq U$, $F' \times G \subseteq U$ and $F' \times G' \subseteq U$. Therefore $(F \cup F') \times (G \cup G') \subseteq U$ and we deduce that $\langle \mathcal{F} \cap \mathcal{F}', \mathcal{G} \cap \mathcal{G}' \rangle$ is a Cauchy filter pair on (X, d) . \square

According to [28, Lemma 5] we obtain an equivalence relation \cong on the quasi-pseudometric space (X^+, d^+) if we define two balanced Cauchy filter pairs $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ to be *equivalent* provided that $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) = 0$ and $d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) = 0$. We can identify (\tilde{X}, \tilde{d}) with the subspace of all balanced Cauchy filter pairs on (X, d) that are minimal in the space (X^+, d^+) (see below).

Observe also that this is exactly the equivalence relation associated with the T_0 -reflection of (X^+, d^+) (see e.g. [28]). Since balancedness is preserved by coarsening a filter pair and comparable balanced Cauchy filter pairs on a quasi-pseudometric space (X, d) are \mathcal{U}_d -equivalent, we can conclude from Lemma 1.0.3 that each \mathcal{U}_d -equivalence class of a balanced Cauchy filter pair

$\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) that contains a minimal Cauchy filter pair, possesses a smallest element. Indeed it follows from the next lemma that each such class contains a coarsest element, which is a minimal Cauchy filter pair (compare [28, Lemma 5]).

The following explicit construction of that filter pair is related to a quasi-uniform result due to Deák [6, p. 412], (compare for instance Lemma 3.1.7).

Lemma 1.0.4. *Let (X, d) be a quasi-pseudometric space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on (X, d) . For each $n \in \mathbb{N}$ set $F_n = \{x \in X : U_{2^{-n}}(x) \in \mathcal{G}\}$ and $G_n = \{x \in X : U_{2^{-n}}^{-1}(x) \in \mathcal{F}\}$. Let \mathcal{F}_{m_1} be the filter on X generated by the base $\{F_n : n \in \mathbb{N}\}$ and let \mathcal{G}_{m_2} be the filter on X generated by the base $\{G_n : n \in \mathbb{N}\}$. Then $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ is the (unique minimal and) coarsest (balanced) Cauchy filter pair coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$. It is the coarsest element of the \mathcal{U}_d -equivalence class of $\langle \mathcal{F}, \mathcal{G} \rangle$.*

Proof. Fix $n \in \mathbb{N}$. Then $F_n \times G_n \subseteq U_{2^{-(n-2)}}$, because $U_{2^{-n}}(x) \in \mathcal{G}$ and $U_{2^{-n}}^{-1}(y) \in \mathcal{F}$ imply that $\Phi_d(x, U_{2^{-n}}(x)) \leq 2^{-n}$ and $\Phi_d(U_{2^{-n}}^{-1}(y), y) \leq 2^{-n}$, and hence $(x, y) \in U_{2^{-(n-2)}}$ by balancedness of $\langle \mathcal{F}, \mathcal{G} \rangle$. Thus $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair on (X, d) . Suppose that $\langle \mathcal{F}', \mathcal{G}' \rangle$ is any Cauchy filter pair on (X, d) such that $\langle \mathcal{F}', \mathcal{G}' \rangle$ is coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$. Let $n \in \mathbb{N}$. There are $F' \in \mathcal{F}'$ and $G' \in \mathcal{G}'$ such that $F' \times G' \subseteq U_{2^{-n}}$. By definition of G_n we see that $G' \subseteq G_n$, since $y \in G'$ implies that $U_{2^{-n}}^{-1}(y) \in \mathcal{F}' \subseteq \mathcal{F}$. Similarly we obtain $F' \subseteq F_n$. Consequently, $G_n \in \mathcal{G}'$ and $F_n \in \mathcal{F}'$ and thus $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ is coarser than $\langle \mathcal{F}', \mathcal{G}' \rangle$. It follows that $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ is coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$ and it is the (unique minimal and) coarsest Cauchy filter pair coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$. In particular $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ is balanced, since $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced. By Lemma 1.0.3 it is obvious that it is the coarsest element of the equivalence class of $\langle \mathcal{F}, \mathcal{G} \rangle$, because if $\langle \mathcal{F}'', \mathcal{G}'' \rangle$ is a Cauchy filter pair on X that is \mathcal{U}_d -equivalent to $\langle \mathcal{F}, \mathcal{G} \rangle$, then $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ must be coarser than the Cauchy filter pair $\langle \mathcal{F} \cap \mathcal{F}'', \mathcal{G} \cap \mathcal{G}'' \rangle$. \square

Remark 1.0.1. (a) *As noted in [28, p. 257], two balanced Cauchy filter pairs of the form $\langle \mathcal{F}, \mathcal{G}_1 \rangle$ and $\langle \mathcal{F}, \mathcal{G}_2 \rangle$ on a quasi-pseudometric space (X, d) are \mathcal{U}_d -equivalent. Hence our notation \mathcal{F}_{m_1} does not lead to confusion.*

(b) *(compare e.g. [3, Lemma 12.4]) Let \mathcal{F} be a d^s -Cauchy filter on a quasi-pseudometric space (X, d) . Then \mathcal{F}_{m_1} is the filter $\mathcal{U}_d^{-1}(\mathcal{F})$ having the base*

$\{U^{-1}(F) : U \in \mathcal{U}_d, F \in \mathcal{F}\}$ and \mathcal{F}_{m_2} is the filter $\mathcal{U}_d(\mathcal{F})$ having the base $\{U(F) : U \in \mathcal{U}_d, F \in \mathcal{F}\}$.

Proof. The Cauchy filter pair $\langle \mathcal{U}_d^{-1}(\mathcal{F}), \mathcal{U}_d(\mathcal{F}) \rangle$ is finer than $\langle \mathcal{F}_{m_1}, \mathcal{F}_{m_2} \rangle$ by Lemma 1.0.4. Let $F' \in \mathcal{F}$ and $n \in \mathbb{N}$. Furthermore let $U_{2^{-n}}(x) \in \mathcal{F}$ for some $x \in X$. Thus $x \in U_{2^{-n}}^{-1}(F')$. It follows that $\{x \in X : U_{2^{-n}}(x) \in \mathcal{F}\} \subseteq U_{2^{-n}}^{-1}(F')$. Therefore $\mathcal{U}_d^{-1}(\mathcal{F}) \subseteq \mathcal{F}_{m_1}$. We conclude that $\mathcal{F}_{m_1} = \mathcal{U}_d^{-1}(\mathcal{F})$ and similarly $\mathcal{F}_{m_2} = \mathcal{U}_d(\mathcal{F})$. \square

Theorem 1.0.1. [28, Theorem 2] Let (X, d) be a T_0 -quasi-pseudometric space and let \tilde{X} be the set of all balanced Cauchy filter pairs on (X, d) which are minimal Cauchy filter pairs. Define $\tilde{d} : \tilde{X} \times \tilde{X} \rightarrow [0, \infty[$ as the restriction of d^+ to $\tilde{X} \times \tilde{X}$. Then (\tilde{X}, \tilde{d}) is a B -complete T_0 -quasi-pseudometric space.

[28, Definition 7] Let (X, d) be a quasi-pseudometric space. Then the T_0 -quasi-pseudometric space (\tilde{X}, \tilde{d}) defined as above will be called the (*standard*) B -completion of (X, d) . We set $\beta_X = q_X^+ \circ \alpha_X$ where $q_X^+ : (X^+, d^+) \rightarrow (\tilde{X}, \tilde{d})$ is the T_0 -quotient map according to Lemma 1.0.1. For each $x \in X$, we set $\beta_X(x) = \langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle$. Then $\beta_X(x) : X \rightarrow \tilde{X}$ is an isometric embedding. [28, Corollary 5] Let (X, d) be a T_0 -quasi-pseudometric space. Then its B -completion (\tilde{X}, \tilde{d}) is a bicomplete T_0 -space. In particular it contains the bicompletion of (X, d) as an extension of X .

In the following we shall make use of a characterization of the B -completion of a T_0 -quasi-pseudometric space that is based on the concept of a balanced embedding. Therefore we recall the definition of a balanced map.

Definition 1.0.7. [28, Definition 8] A uniformly continuous map $f : (X, d) \rightarrow (Y, e)$ between quasi-pseudometric spaces (X, d) and (Y, e) is called balanced provided that for each balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) , the Cauchy filter pair $\langle f\mathcal{F}, f\mathcal{G} \rangle$ is balanced on (Y, e) . (Note first that $\langle f\mathcal{F}, f\mathcal{G} \rangle$ is a Cauchy filter pair on (Y, e) , because f is uniformly continuous).

Lemma 1.0.5. [28, Lemma 9] Let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be Cauchy filter pairs on a quasi-pseudometric space (X, d) .

Then

$$\inf_{G' \in \mathcal{G}'} \Phi_{d^+}(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(G')) = \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} \inf_{F \in \mathcal{F}} \sup_{f \in F} d(f, g').$$

and

$$\inf_{F \in \mathcal{F}} \Phi_{d^+}(\alpha_X(F), \langle \mathcal{F}', \mathcal{G}' \rangle) = \inf_{F \in \mathcal{F}} \sup_{f \in F} \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} d(f, g').$$

Lemma 1.0.6. [28, Lemma 10] Let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be Cauchy filter pairs on a quasi-pseudometric space (X, d) . Then

$$\inf_{F \in \mathcal{F}} \sup_{f \in F} \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} d(f, g') \leq d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle)$$

and

$$\inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} \inf_{F \in \mathcal{F}} \sup_{f \in F} d(f, g') \leq d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle).$$

Corollary 1.0.2. [28, Corollary 9, 10] Let (X, d) be a quasi-pseudometric space. Then

a) The map $\alpha_X : (X, d) \rightarrow (X^+, d^+)$ is balanced.

b) The map $\beta_X : (X, d) \rightarrow (\tilde{X}, \tilde{d})$ is balanced.

Theorem 1.0.2. [28, Theorem 3] Let $f : (X, d) \rightarrow (Y, e)$ be a balanced map between T_0 -quasi-pseudometric spaces (X, d) and (Y, e) . Then there is a unique balanced map $\tilde{f} : (\tilde{X}, \tilde{d}) \rightarrow (\tilde{Y}, \tilde{e})$ such that $\tilde{f} \circ \beta_X = \beta_Y \circ f$. If f is also an isometry, then \tilde{f} is an isometry.

The next result establishes a characterization of the B -completion of a T_0 -quasi-pseudometric space.

Theorem 1.0.3. [28, Theorem 4] Let (X, d) be a subspace of the B -complete T_0 -quasi-pseudometric space (Y, e) . Suppose that the embedding $i : (X, d) \rightarrow (Y, e)$ is balanced and that for each $y \in Y$ there is a balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) such that the filter pair $\langle i\mathcal{F}, i\mathcal{G} \rangle$ converges to y . Then the B -completion (\tilde{X}, \tilde{d}) of (X, d) is isometric to (Y, e) under the isometric balanced extension \tilde{i} of i to \tilde{X} .

The next proposition is an application of the preceding theorem, which illustrates some of the introduced concepts.

Proposition 1.0.3. *Let (X, d) be a T_0 -quasi-pseudometric space and let A be a subset of \tilde{X} such that $X \subseteq A \subseteq \tilde{X}$. Then the extension \tilde{j} of the balanced isometric embedding $j : (A, \tilde{d}|_A) \rightarrow (\tilde{X}, \tilde{d})$ to \tilde{A} yields a bijective balanced isometry between $(\tilde{A}, \widetilde{\tilde{d}|_A})$ and (\tilde{X}, \tilde{d}) . (Here $\tilde{d}|_A$ denotes the restriction of \tilde{d} to $A \times A$.)*

Proof. As the statement of the proposition indicates, we shall consider X and A as subspaces of (\tilde{X}, \tilde{d}) . We first want to show that the inclusion map $j : (A, \tilde{d}|_A) \rightarrow (\tilde{X}, \tilde{d})$ is balanced. To this end, consider a balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on $(A, \tilde{d}|_A)$. In particular, we have that $d(x, y) \leq \inf_{G \in \mathcal{G}} \Phi_{\tilde{d}|_A}(x, G) + \inf_{F \in \mathcal{F}} \Phi_{\tilde{d}|_A}(F, y)$ whenever $x, y \in X$. Consider arbitrary $\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle \in \tilde{X}$. Let $F' \in \mathcal{F}'$ and $G'' \in \mathcal{G}''$. Furthermore, let $f' \in F'$ and $g'' \in G''$. Then $d(f', g'') \leq \Phi_{\tilde{d}|_A}(F', G) + \Phi_{\tilde{d}|_A}(F, G'')$ whenever $G \in \mathcal{G}$ and $F \in \mathcal{F}$. Hence $\Phi_d(F', G'') \leq \Phi_{\tilde{d}|_A}(F', G) + \Phi_{\tilde{d}|_A}(F, G'')$ whenever $G \in \mathcal{G}$ and $F \in \mathcal{F}$.

Consequently,

$$\inf_{F' \in \mathcal{F}', G'' \in \mathcal{G}''} \Phi_d(F', G'') \leq \inf_{G \in \mathcal{G}} \inf_{F' \in \mathcal{F}'} \Phi_{\tilde{d}|_A}(F', G) + \inf_{F \in \mathcal{F}} \inf_{G'' \in \mathcal{G}''} \Phi_{\tilde{d}|_A}(F, G'').$$

The Cauchy filter pair generated by $\langle \mathcal{F}', \mathcal{G}' \rangle$ (resp. $\langle \mathcal{F}'', \mathcal{G}'' \rangle$) on \tilde{X} is balanced, since the map $\beta_X : (X, d) \rightarrow (\tilde{X}, \tilde{d})$ is balanced. We next use that obviously the restrictions $\langle \mathcal{F}'_A, \mathcal{G}'_A \rangle$ (resp. $\langle \mathcal{F}''_A, \mathcal{G}''_A \rangle$) of those filter pairs to A are balanced Cauchy filter pairs on A . Therefore by employing the inequality just established, we get that

$$\begin{aligned} & \tilde{d}(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) \leq \\ & \inf_{G \in \mathcal{G}} \inf_{F' \in \mathcal{F}'} \Phi_{\tilde{d}|_A}(F', G) + \inf_{F \in \mathcal{F}} \inf_{G'' \in \mathcal{G}''} \Phi_{\tilde{d}|_A}(F, G'') = \\ & \inf_{G \in \mathcal{G}} \Phi_{(\tilde{d}|_A)^+}(\langle \mathcal{F}'_A, \mathcal{G}'_A \rangle, \alpha_A(G)) + \inf_{F \in \mathcal{F}} \Phi_{(\tilde{d}|_A)^+}(\alpha_A(F), \langle \mathcal{F}''_A, \mathcal{G}''_A \rangle) = \\ & \inf_{G \in \mathcal{G}} \sup_{g \in G} \inf_{F' \in \mathcal{F}'} \sup_{f' \in F'} (\tilde{d}|_A)(f', g) + \inf_{F \in \mathcal{F}} \sup_{f \in F} \inf_{G'' \in \mathcal{G}''} \sup_{g'' \in G''} (\tilde{d}|_A)(f, g'') = \end{aligned}$$

$$\inf_{G \in \mathcal{G}} \Phi_{\tilde{d}}(\langle \mathcal{F}', \mathcal{G}' \rangle, j(G)) + \inf_{F \in \mathcal{F}} \Phi_{\tilde{d}}(j(F), \langle \mathcal{F}'', \mathcal{G}'' \rangle),$$

where in these computations we have also used that by Proposition 1.0.2

$$(\tilde{d}|A)^+(\langle \mathcal{F}'_A, \mathcal{G}'_A \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) =$$

$$\inf_{G \in \mathcal{G}} \inf_{F' \in \mathcal{F}'} \Phi_{\tilde{d}|A}(F', G) = \inf_{G \in \mathcal{G}} \Phi_{(\tilde{d}|A)^+}(\langle \mathcal{F}'_A, \mathcal{G}'_A \rangle, \alpha_A(G))$$

and

$$(\tilde{d}|A)^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}''_A, \mathcal{G}''_A \rangle) =$$

$$\inf_{F \in \mathcal{F}} \inf_{G'' \in \mathcal{G}''} \Phi_{\tilde{d}|A}(F, G'') = \inf_{F \in \mathcal{F}} \Phi_{(\tilde{d}|A)^+}(\alpha_A(F), \langle \mathcal{F}''_A, \mathcal{G}''_A \rangle).$$

Hence we have shown that $\langle j(\mathcal{F}), j(\mathcal{G}) \rangle$ is balanced on (\tilde{X}, \tilde{d}) and so the map $j : (A, \tilde{d}|A) \rightarrow (\tilde{X}, \tilde{d})$ is balanced.

Let $y \in \tilde{X}$. Similarly, as above we argue that according to the density condition of X in \tilde{X} mentioned in Theorem 1.0.3 there is a balanced Cauchy filter pair $\langle \mathcal{H}, \mathcal{K} \rangle$ on (X, d) such that the balanced Cauchy filter pair $\langle \beta_X(\mathcal{H}), \beta_X(\mathcal{K}) \rangle$ converges to y in (\tilde{X}, \tilde{d}) . This implies that the restriction $\langle \mathcal{H}_A, \mathcal{K}_A \rangle$ of $\langle \beta_X(\mathcal{H}), \beta_X(\mathcal{K}) \rangle$ to A is a balanced Cauchy filter pair on $(A, \tilde{d}|A)$ such that $\langle j(\mathcal{H}_A), j(\mathcal{K}_A) \rangle$ converges to y in (\tilde{X}, \tilde{d}) . The statement now follows from Theorem 1.0.3. \square

Corollary 1.0.3. *The B -completion of the bicompletion of a T_0 -quasi-pseudometric space (X, d) can be identified with the B -completion of (X, d) .*

Proof. The statement follows from the preceding result by setting A equal to the ground set of the bicompletion of (X, d) . \square

We next explain the connections between the B -completion and the Doitchinov completion theory developed in [11]. We have shown in [28] that for a balanced quasi-pseudometric space, our completion is equivalent to the one of Doitchinov.

Definition 1.0.8. ([11, Definition 11]) *A quasi-pseudometric space (X, d) is called balanced if whenever $(x'_n)_{n \in \mathbb{N}}$ and $(x''_m)_{m \in \mathbb{N}}$ are two sequences in (X, d) and $x', x'' \in X$, then from $d(x', x'_n) \leq r'$ for each $n \in \mathbb{N}$, and $d(x''_m, x'') \leq r''$ for each $m \in \mathbb{N}$ and $\lim_{m, n \rightarrow \infty} d(x''_m, x'_n) = 0$, it follows that $d(x', x'') \leq r' + r''$.*

Proposition 1.0.4. ([28, Proposition 3]) *The following conditions are equivalent for a quasi-pseudometric space (X, d) :*

(a) *(X, d) is balanced.*

(b) *Every Cauchy pair of sequences in (X, d) is balanced.*

(c) *Every Cauchy filter pair on (X, d) is balanced.*

Although balanced quasi-pseudometrics are known to satisfy an interesting condition of separate continuity (see e.g. [28, Remark 7]), our next example (compare Example 1.0.2) shows that even in a balanced T_0 -quasi-pseudometric space (balanced) Cauchy filter pairs $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ may exist such that

$$\sup_{F \in \mathcal{F}, G' \in \mathcal{G}'} d(F, G') < d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle).$$

Hence in such a space the limit superior $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle)$ need not be a limit as stated in [11, Proposition 9], since the corresponding limit may not exist.

Example 1.0.4. *Let $X = \{-\frac{1}{n}, \frac{1}{n} : n \in \mathbb{N}\} \cup \{0^-, 0^+\}$. Set $d(x, x) = 0$ whenever $x \in X$. For each $n \in \mathbb{N}$ let $d(-\frac{1}{n}, 0^-) = \frac{1}{n}$, $d(0^+, \frac{1}{n}) = \frac{1}{n}$ and $d(-\frac{1}{n}, \frac{1}{n}) = 1$. Furthermore set $d(x, y) = 2$ otherwise. We note that d is a T_0 -quasi-metric. Indeed for $x, y, z \in X$ with $x \neq y$ and $y \neq z$ we have that $d(x, y) = 2$ or $d(y, z) = 2$. Hence d satisfies the triangle inequality. We next want to show that d is balanced.*

Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on (X, d) . There are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $\Phi_d(F, G) < 1$. Considering the cases that F or G can be chosen as a singleton or not, we see that only three cases are possible: (1) $\langle \mathcal{F}, \mathcal{G} \rangle = \langle \underline{x}, \underline{x} \rangle$ for some $x \in X$, (2) $\langle \mathcal{F}, \mathcal{G} \rangle = \langle \underline{0^+}, \mathcal{G} \rangle$ or (3) $\langle \mathcal{F}, \mathcal{G} \rangle = \langle \mathcal{F}, \underline{0^-} \rangle$, where \mathcal{F} and \mathcal{G} are appropriate filters on X which are not of the form \underline{y} for some $y \in X$.

It therefore remains to be checked that filter pairs of the second and third kind are balanced. So let us consider a Cauchy filter pair of the form $\langle \mathcal{F}, \underline{0^-} \rangle$ where \mathcal{F} is an appropriate filter on X which is not of the form \underline{y} with $y \in X$.

Let $a, b \in X$. If $b = 0^-$, then $d(a, b) = d^+(\alpha_X(a), \langle \mathcal{F}, \underline{0^-} \rangle)$. If $b \neq 0^-$, then obviously $\inf_{F \in \mathcal{F}} \Phi_d(F, b) = 2$, since $\langle \mathcal{F}, \underline{0^-} \rangle$ is a Cauchy filter pair. Hence in either case we have that $d(a, b) \leq d^+(\alpha_X(a), \langle \mathcal{F}, \underline{0^-} \rangle) + d^+(\langle \mathcal{F}, \underline{0^-} \rangle, \alpha_X(b))$. So $\langle \mathcal{F}, \underline{0^-} \rangle$ is a balanced Cauchy filter pair on (X, d) . Analogously, it can be shown that all Cauchy filter pairs of the second kind are balanced. We conclude that d is indeed a balanced T_0 -quasi-pseudometric.

One finally computes that $d^+(\langle (-\frac{1}{n})_{n \in \mathbb{N}}, 0^- \rangle, \langle 0^+, (\frac{1}{p})_{p \in \mathbb{N}} \rangle) = 2$, but

$$\sup_{k, \ell \in \mathbb{N}} d(\{-\frac{1}{n} : n \in \mathbb{N}, n \geq k\}, \{\frac{1}{p} : p \in \mathbb{N}, p \geq \ell\}) = 1.$$

Remark 1.0.2. ([28, Remark 12]) Let (X, d) be a balanced quasi-pseudometric space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on (X, d) . Then $\langle \mathcal{F}, \mathcal{G} \rangle$ is equivalent to a Cauchy filter pair generated by a balanced Cauchy pair of sequences. Therefore the B -completion of a balanced T_0 -quasi-pseudometric space can be built with the help of (balanced) Cauchy pairs of sequences only, since such sequences can represent all equivalence classes of X^+ .

Lemma 1.0.7. ([28, Lemma 12]) Let (X, d) be a balanced quasi-pseudometric space and let $\langle (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \rangle$ be a Cauchy pair of sequences in (X, d) . If for some $x \in X$, $\lim_{n \in \mathbb{N}} d(x, y_n) = 0$, then $\lim_{n \in \mathbb{N}} d(x_n, x) = 0$ (compare [11, Lemma 5]).

We next try to motivate why we work with filters instead of (equivalence classes of) sequences.

Remark 1.0.3. [28, Remark 12] Let (X, d) be a balanced quasi-pseudometric space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on (X, d) . Then $\langle \mathcal{F}, \mathcal{G} \rangle$ is \mathcal{U}_d -equivalent to a Cauchy filter pair generated by a balanced Cauchy pair of sequences. (Indeed the B -completion of a balanced T_0 -quasi-pseudometric space was built by Doitchinov with the help of (balanced) Cauchy pairs of sequences only.)

On the other hand, the following example shows that for a general quasi-pseudometric d on a set X a balanced Cauchy filter pair (with both filters of the pair possessing a countable base) need not be \mathcal{U}_d -equivalent to the Cauchy filter pair generated by a balanced Cauchy pair of sequences on X .

Example 1.0.5. By Y we shall denote a fixed uncountable set. Let $X = (Y \times \{0^-\}) \cup (Y \times \{0^+\}) \cup (Y \times \mathbb{N} \times \{0^-\}) \cup (Y \times \mathbb{N} \times \{0^+\})$.

Set $d(x, x) = 0$ whenever $x \in X$. Let $d((a, 0^-), (y, n, 0^-)) = \frac{1}{2n}$, if $a, y \in Y, n \in \mathbb{N}$ and $a \neq y$; set $d((x, m, 0^+), (b, 0^+)) = \frac{1}{2m}$, if $x, b \in Y, m \in \mathbb{N}$ and $x \neq b$; and let $d((x, m, 0^+), (y, n, 0^-)) = \frac{1}{2m} + \frac{1}{2n}$, where $x, y \in Y$ and $m, n \in \mathbb{N}$. Furthermore, set $d(x, y) = 1$ otherwise. We want to check that d is a T_0 -quasi-pseudometric on X .

If $x, y, z \in X$ with $x \neq y$ and $y \neq z$, then $d(x, z) \leq d(x, y) + d(y, z)$, since $d(x, y)$ or $d(y, z)$ is equal to 1. Therefore d satisfies the triangle inequality. We conclude that (X, d) is a T_0 -quasi-metric space. For each $n \in \mathbb{N}$ set $G_n = \{(y, k, 0^-) : y \in Y, k \geq n, k \in \mathbb{N}\}$ and $F_n = \{(x, m, 0^+) : x \in Y, m \geq n, m \in \mathbb{N}\}$. Let \mathcal{F} be the filter on X generated by the base $\{F_n : n \in \mathbb{N}\}$ and \mathcal{G} be the filter on X generated by the base $\{G_n : n \in \mathbb{N}\}$. By definition of d it is obvious that $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair on (X, d) .

We want to show that for any $s, t \in X$ we have that $d(s, t) \leq \inf_{k \in \mathbb{N}} \Phi_d(s, G_k) + \inf_{p \in \mathbb{N}} \Phi_d(F_p, t)$. Since $d \leq 1$, it suffices to consider the case that both the two summands on the right hand side are smaller than 1. Since for all $n \in \mathbb{N}$ and $y \in Y$ we have $d((y, 0^-), (y, n, 0^-)) = 1$, and since for all $n \in \mathbb{N}$ and $x \in Y$ we have $d((x, n, 0^+), (x, 0^+)) = 1$, we conclude that s is of the form $(x, m, 0^+)$, and t is of the form $(y, n, 0^-)$, where $x, y \in Y$ and $m, n \in \mathbb{N}$. Therefore we see that the inequality $\frac{1}{2m} + \frac{1}{2n} \leq d(s, t) \leq \inf_{k \in \mathbb{N}} \Phi_d(s, G_k) + \inf_{p \in \mathbb{N}} \Phi_d(F_p, t) = \frac{1}{2m} + \frac{1}{2n}$ holds. Hence $\langle \mathcal{F}, \mathcal{G} \rangle$ is a balanced Cauchy filter pair on (X, d) .

Suppose that there exists a Cauchy pair $\langle (s_k)_{k \in \mathbb{N}}, (t_p)_{p \in \mathbb{N}} \rangle$ of sequences in (X, d) which is \mathcal{U}_d -equivalent to $\langle \mathcal{F}, \mathcal{G} \rangle$. We are going to prove that the Cauchy filter pair $\langle \mathcal{F}(s_k)_{k \in \mathbb{N}}, \mathcal{F}(t_p)_{p \in \mathbb{N}} \rangle$ is not balanced:

Since by our assumption $\langle \mathcal{F}(s_k)_{k \in \mathbb{N}}, \mathcal{G} \rangle, \langle \mathcal{F}, \mathcal{F}(t_p)_{p \in \mathbb{N}} \rangle$ are Cauchy filter pairs, we conclude that for each $\ell \in \mathbb{N}$ there is $m \in \mathbb{N}$ such that $m \geq \ell$, $\Phi_d(\{s_k : k \in \mathbb{N}, k \geq m\} \times G_m) < \frac{1}{2\ell}$ and $\Phi_d(F_m \times \{t_p : p \in \mathbb{N}, p \geq m\}) < \frac{1}{2\ell}$. It follows that $\{s_k : k \geq m, k \in \mathbb{N}\} \subseteq F_\ell$ and $\{t_p : p \geq m, p \in \mathbb{N}\} \subseteq G_\ell$ by definition of d . We have shown that $\mathcal{F} \subseteq \mathcal{F}(s_k)_{k \in \mathbb{N}}$ and $\mathcal{G} \subseteq \mathcal{F}(t_p)_{p \in \mathbb{N}}$.

We now find $a, b \in Y \setminus (\{\text{first coordinate of } s_k : k \in \mathbb{N}\} \cup \{\text{first coordinate of } t_p : p \in \mathbb{N}\})$. Then $1 = d((a, 0^-), (b, 0^+))$, but

$$\inf_{p \in \mathbb{N}} \Phi_d((a, 0^-), \{t_r : r \in \mathbb{N}, r \geq p\}) + \inf_{k \in \mathbb{N}} \Phi_d(\{s_n : n \in \mathbb{N}, n \geq k\}, (b, 0^+)) = 0 + 0.$$

We conclude that $\langle (s_k)_{k \in \mathbb{N}}, (t_p)_{p \in \mathbb{N}} \rangle$ is not balanced on (X, d) . It follows that the \mathcal{U}_d -equivalence class of $\langle \mathcal{F}, \mathcal{G} \rangle$ does not contain a filter pair generated by a balanced Cauchy pair of sequences.

Proposition 1.0.5. ([28, Proposition 4]) *A balanced quasi-pseudometric space (X, d) is B -complete if and only if each Cauchy pair $\langle (x_n)_{n \in \mathbb{N}}, (y_n)_{n \in \mathbb{N}} \rangle$ of sequences converges (that is, there is $x \in X$ such that the sequences $(d(x, y_n))_{n \in \mathbb{N}}$ and $(d(x_n, x))_{n \in \mathbb{N}}$ both converge to 0) (compare [13, Theorem 9]).*

Theorem 1.0.4. ([28, Theorem 4]) *Let (X^λ, d^λ) be the Doitchinov completion of the balanced T_0 -quasi-pseudometric space (X, d) with isometric imbedding $i : (X, d) \rightarrow (X^\lambda, d^\lambda)$. Then the B -completion (\tilde{X}, \tilde{d}) of (X, d) is isometric to (X^λ, d^λ) under the balanced extension $\tilde{i} : (\tilde{X}, \tilde{d}) \rightarrow (X^\lambda, d^\lambda)$ of the balanced map i .*

Chapter 2

On B -completeness of a T_0 -quasi-pseudometric space

In the previous chapter we have shown that each T_0 -quasi-pseudometric space has a B -completion which contains the bicompletion of the original space. We also gave a summary of the construction of the B -completion of a quasi-pseudometric space. In this chapter we establish some new results about the B -completion of a T_0 -quasi-pseudometric space. In the first section we show that B -completeness is a property of quasi-pseudometric spaces but not a quasi-uniform property. We give an example which shows that two distinct quasi-pseudometrics d and d' on a set X can induce the same quasi-uniformity $\mathcal{U}_d = \mathcal{U}_{d'}$ although d' is B -complete, while d is not.

In the second section, we present some examples of balanced maps on the B -completion of a T_0 -quasi-pseudometric space. We prove a result showing that the B -completion commutes in the expected way with an appropriate form of countable product of quasi-pseudometric spaces. Similar results for balanced quasi-metrics were obtained by Doitchinov.

In the last section we prove a result that the B -completion of a totally bounded T_0 -quasi-pseudometric space is totally bounded and mention that even for totally bounded T_0 -quasi-pseudometric spaces the B -completion can be strictly larger than the bicompletion.

2.1 B-completeness as a quasi-pseudometric completeness property

In this section we prove that B -completeness is a property of quasi-pseudometric spaces which need not be preserved under quasi-uniform isomorphisms.

Let us recall that a quasi-pseudometric space (X, d) is called B -complete provided that each balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ converges in X .

We have shown in the previous chapter that every T_0 -quasi-pseudometric space has a B -completion which contains the bicompletion. Our next example shows that two distinct quasi-pseudometrics d and d' on a set X can induce the same quasi-uniformity $\mathcal{U}_d = \mathcal{U}_{d'}$ although d' is B -complete, while d is not.

Example 2.1.1. Let $X = \{-\frac{1}{n}, \frac{1}{n} : n \in \mathbb{N}\}$. For any $x, y \in X$ define $d : X \times X \rightarrow [0, \infty[$ as follows: Set $d(x, y) = 0$ if $x = y$; furthermore set $d(x, y) = y - x$ if $x < 0 < y$; and set $d(x, y) = 2$ otherwise.

Let A be any subset of $\mathbb{N} \setminus \{1\}$. We shall also consider $d_A : X \times X \rightarrow [0, \infty[$ defined as follows: Given $n \in A$, set $d_A(-\frac{1}{n}, \frac{1}{n}) = \frac{3}{n}$ and $d_A(x, y) = d(x, y)$ otherwise. In particular we have $d_\emptyset = d$. Furthermore for instance $d_{\{2\}}(x, y) = d(x, y)$, except that $d_{\{2\}}(-\frac{1}{2}, \frac{1}{2}) = \frac{3}{2}$.

We first prove that each d_A satisfies the triangle inequality: We have to show that $d_A(x, z) \leq d_A(x, y) + d_A(y, z)$ whenever $x, y, z \in X$. Since $d_A \leq 2$, similarly as above it clearly suffices to consider the case that $x \neq y$, $y \neq z$ and neither $d_A(x, y)$ nor $d_A(y, z)$ is equal to 2. But this is impossible, since for instance $0 \neq d_A(x, y) < 2$ implies that $x < 0 < y$. Hence d_A is a T_0 -quasi-metric on X . We also note that $d \leq d_A \leq \frac{3}{2}d$. Therefore d and d_A induce the same quasi-uniformity $\mathcal{U}_d = \mathcal{U}_{d_A}$ on X .

Let \mathcal{F} be the filter on X generated by $\{]-\epsilon, 0[\cap X : \epsilon > 0\}$, and let \mathcal{G} be the filter on X generated by $\{]0, \epsilon[\cap X : \epsilon > 0\}$. Then $\langle \mathcal{F}, \mathcal{G} \rangle$ is a nonconvergent Cauchy filter pair on (X, d_A) where A is an arbitrary subset of $\mathbb{N} \setminus \{1\}$. Next we want to show that (X, d_A) is a balanced T_0 -quasi-metric space if and only if A is empty.

So let A be any subset of $\mathbb{N} \setminus \{1\}$. Consider an arbitrary Cauchy filter pair $\langle \mathcal{F}', \mathcal{G}' \rangle$ on (X, d_A) . According to Proposition 1.0.4 we have to investigate whether $\langle \mathcal{F}', \mathcal{G}' \rangle$ is balanced. Similarly as in Example 1.0.4 we see by the definition of d_A that there is $x \in X$ such that $\langle \mathcal{F}', \mathcal{G}' \rangle = \alpha_X(x)$, or $\langle \mathcal{F}, \mathcal{G} \rangle$ is coarser than $\langle \mathcal{F}', \mathcal{G}' \rangle$. Since $\alpha_X(x)$ is balanced, in the following we need only study the second case further.

Assume now that A is empty, and let $a, b \in X$. Because $d \leq 2$, in order to prove balancedness of $\langle \mathcal{F}', \mathcal{G}' \rangle$, it obviously suffices to consider the case that both $\inf_{G' \in \mathcal{G}'} \Phi_d(a, G') < 2$ and $\inf_{F' \in \mathcal{F}'} \Phi_d(F', b) < 2$. Then $a < 0 < b$. Thus $d(a, b) = -a + b = \inf_{G' \in \mathcal{G}'} \Phi_d(a, G') + \inf_{F' \in \mathcal{F}'} \Phi_d(F', b)$. Hence $\langle \mathcal{F}', \mathcal{G}' \rangle$ is balanced on (X, d) . We conclude that (X, d) is balanced. In particular we also deduce that (X, d) is not B -complete.

On the other hand let A be nonempty. We find $n \in \mathbb{N}$ such that $n \in A$. Then $\frac{3}{n} = d_A(-\frac{1}{n}, \frac{1}{n}) \not\leq \inf_{\epsilon > 0} \Phi_{d_A}(-\frac{1}{n},]0, \epsilon[\cap X) + \inf_{\epsilon > 0} \Phi_{d_A}(] - \epsilon, 0[\cap X, \frac{1}{n}) = \frac{2}{n}$. Hence for nonempty A the Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d_A) is not balanced. We then conclude that also the finer Cauchy filter pair $\langle \mathcal{F}', \mathcal{G}' \rangle$ is not balanced. Consequently each balanced Cauchy filter pair on (X, d_A) is equal to some $\alpha_X(x)$ (with $x \in X$) provided that A is nonempty. Hence we have proved that (X, d_A) is B -complete if and only if A is nonempty. Note that our proof shows that given a nonempty subset A of $\mathbb{N} \setminus \{1\}$, the identity map $id : (X, d_A) \rightarrow (X, d)$ is balanced, but the inverse map is only uniformly continuous and not balanced.

Remark 2.1.1. If $\mathcal{U}_{d_1} = \mathcal{U}_{d_2}$ for quasi-pseudometrics d_1 and d_2 on a set X , then two Cauchy filter pairs $\langle \mathcal{F}_1, \mathcal{G}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{G}_2 \rangle$ on (X, d_1) (equivalently, (X, d_2)) are \mathcal{U}_{d_1} -equivalent if and only if they are \mathcal{U}_{d_2} -equivalent, since $\mathcal{U}_{d_1} = \mathcal{U}_{d_2}$ implies that (X, d_1) and (X, d_2) have the same Cauchy filter pairs.

In order to investigate the phenomenon discussed in Example 2.1.1 further we introduce the following concept.

Definition 2.1.1. Two quasi-pseudometrics d_1 and d_2 on a set X are called B -equivalent provided that the following conditions are all satisfied:

(1) $\mathcal{U}_{d_1} = \mathcal{U}_{d_2}$.

(2) each balanced Cauchy filter pair on (X, d_1) is \mathcal{U}_{d_1} -equivalent to a balanced Cauchy filter pair on (X, d_2) .

(3) Each balanced Cauchy filter pair on (X, d_2) is \mathcal{U}_{d_2} -equivalent to a balanced Cauchy filter pair on (X, d_1) .

Lemma 2.1.1. *Let d_1 and d_2 be two quasi-pseudometrics on a set X such that $\mathcal{U}_{d_1} = \mathcal{U}_{d_2}$. Then d_1 and d_2 are B -equivalent if and only if (X, d_1) and (X, d_2) have the same minimal balanced Cauchy filter pairs.*

Proof. Assume that (X, d_1) and (X, d_2) have the same minimal balanced Cauchy filter pairs. Since each balanced Cauchy filter pair on a quasi-pseudometric space (X, d) is \mathcal{U}_d -equivalent to a minimal balanced Cauchy filter pair on (X, d) (compare Lemma 1.0.4), we immediately deduce that d_1 and d_2 are B -equivalent by definition.

In order to establish the converse, suppose that d_1 and d_2 are B -equivalent quasi-pseudometrics on a set X . Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a minimal balanced Cauchy filter pair on (X, d_1) . By B -equivalence of d_1 and d_2 there is a balanced Cauchy filter pair $\langle \mathcal{F}', \mathcal{G}' \rangle$ on (X, d_2) which is \mathcal{U}_{d_1} -equivalent to $\langle \mathcal{F}, \mathcal{G} \rangle$. Of course, without loss of generality we can suppose that $\langle \mathcal{F}', \mathcal{G}' \rangle$ is a minimal balanced Cauchy filter pair on (X, d_2) (see Lemma 1.0.4). Thus by Lemma 1.0.3 $\langle \mathcal{F} \cap \mathcal{F}', \mathcal{G} \cap \mathcal{G}' \rangle$ is a balanced Cauchy filter pair both on (X, d_1) and (X, d_2) . Therefore by minimality of $\langle \mathcal{F}, \mathcal{G} \rangle$ resp. $\langle \mathcal{F}', \mathcal{G}' \rangle$ we get that $\langle \mathcal{F}, \mathcal{G} \rangle = \langle \mathcal{F} \cap \mathcal{F}', \mathcal{G} \cap \mathcal{G}' \rangle = \langle \mathcal{F}', \mathcal{G}' \rangle$.

By the symmetry of the situation we conclude that (X, d_1) and (X, d_2) have the same minimal balanced Cauchy filter pairs. \square

Remark 2.1.2. *If d_1 and d_2 are two B -equivalent quasi-pseudometrics on a set X and d_1 is B -complete, then d_2 is B -complete, too.*

Proof. A quasi-pseudometric space (X, d) is B -complete if and only if the collection $\{\langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle : x \in X\}$ is equal to the set of all minimal balanced Cauchy filter pairs on (X, d) (compare e.g. [28, Example 3]). Since by assumption $\mathcal{U}_{d_1} = \mathcal{U}_{d_2}$, the result follows from Lemma 2.1.1. \square

Proposition 2.1.1. *Let d_1 and d_2 be T_0 -quasi-pseudometrics on a set X that are B -equivalent. Then we have $\mathcal{U}_{\tilde{d}_1} = \mathcal{U}_{\tilde{d}_2}$ for the B -completions (\tilde{X}, \tilde{d}_1) of (X, d_1) and (\tilde{X}, \tilde{d}_2) of (X, d_2) .*

Proof. As noted in Lemma 2.1.1, the set \tilde{X} of the minimal balanced Cauchy filter pairs on (X, d_1) is equal to the set of the minimal balanced Cauchy filter pairs on (X, d_2) . Therefore our notation is appropriate. Let $\epsilon > 0$. Then there is $\delta > 0$ such that $U_{d_1, \delta} \subseteq U_{d_2, \frac{\epsilon}{2}}$, since $\mathcal{U}_{d_1} = \mathcal{U}_{d_2}$. Consider $\langle \mathcal{F}_1, \mathcal{G}_1 \rangle, \langle \mathcal{F}_2, \mathcal{G}_2 \rangle \in \tilde{X}$ such that $\tilde{d}_1(\langle \mathcal{F}_1, \mathcal{G}_1 \rangle, \langle \mathcal{F}_2, \mathcal{G}_2 \rangle) < \delta$. By definition of \tilde{d}_1 , there are $F_1 \in \mathcal{F}_1$ and $G_2 \in \mathcal{G}_2$ such that $\Phi_{d_1}(F_1, G_2) < \delta$. Consequently $\Phi_{d_2}(F_1, G_2) \leq \frac{\epsilon}{2}$ and hence $\tilde{d}_2(\langle \mathcal{F}_1, \mathcal{G}_1 \rangle, \langle \mathcal{F}_2, \mathcal{G}_2 \rangle) < \epsilon$. We conclude that $\mathcal{U}_{\tilde{d}_2} \subseteq \mathcal{U}_{\tilde{d}_1}$. Similarly one shows that $\mathcal{U}_{\tilde{d}_1} \subseteq \mathcal{U}_{\tilde{d}_2}$. It follows that $\mathcal{U}_{\tilde{d}_1} = \mathcal{U}_{\tilde{d}_2}$. \square

Remark 2.1.3. *Two balanced quasi-pseudometrics d_1 and d_2 on a set X that induce the same quasi-uniformity are B -equivalent: Indeed since $\mathcal{U}_{d_1} = \mathcal{U}_{d_2}$, (X, d_1) and (X, d_2) have the same Cauchy filter pairs. Because both d_1 and d_2 are balanced, all these Cauchy filter pairs are balanced on (X, d_1) and (X, d_2) by Proposition 1.0.4. The statement now immediately follows from the definition of B -equivalence.*

2.2 Some examples of balanced maps

In this section we consider how balancedness of filter pairs is preserved under some well-known constructions applied to quasi-pseudometrics. In particular we shall prove a result showing that the B -completion commutes in the expected way with an appropriate form of a countable product of quasi-pseudometric spaces. We shall give some examples of balanced maps. Similar results for balanced quasi-metrics were obtained by Doitchinov in [11]. We start with an auxiliary result.

Lemma 2.2.1. *Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on a quasi-pseudometric space (X, d) and let $x, y \in X$. Then $\Phi_d(x, G) < \infty$ for some $G \in \mathcal{G}$ and $\Phi_d(F, y) < \infty$ for some $F \in \mathcal{F}$.*

Proof. The assertion follows from the fact that $d^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{G} \rangle) < \infty$ and $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(y)) < \infty$, see [28, Lemma 2]. \square

We now present several examples of balanced maps.

Remark 2.2.1. Let (X, d) be a quasi-pseudometric space and let c be a positive real constant. Then the identity map $id : (X, d) \rightarrow (X, c \cdot d)$ and its inverse $(id)^{-1}$ are both balanced. (Note first that $c \cdot d$ is a quasi-pseudometric on X .) Indeed $c \cdot d$ and d have the same (balanced) Cauchy filter pairs.

Lemma 2.2.2. Let (X, d) be a quasi-pseudometric space and let t be a positive real constant. Set $d_t = \min\{t, d\}$. Then d_t is a quasi-pseudometric on X and the identity map $id : (X, d) \rightarrow (X, d_t)$ is balanced. The inverse map $(id)^{-1} : (X, d_t) \rightarrow (X, d)$ need not be balanced.

Proof. It is well known (compare e.g. [14, Theorem 4.1.3]) that d_t is a quasi-pseudometric on X such that $\mathcal{U}_d = \mathcal{U}_{d_t}$. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on (X, d) . Clearly $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair on (X, d_t) , because the identity map $id : (X, d) \rightarrow (X, d_t)$ is uniformly continuous. Let $x, y \in X$. Suppose first that $\inf_{G \in \mathcal{G}} \Phi_{d_t}(x, G) + \inf_{F \in \mathcal{F}} \Phi_{d_t}(F, y) \geq t$. Then by definition of d_t , $d_t(x, y) \leq t \leq \inf_{G \in \mathcal{G}} \Phi_{d_t}(x, G) + \inf_{F \in \mathcal{F}} \Phi_{d_t}(F, y)$. So we can suppose that $\inf_{G \in \mathcal{G}} \Phi_{d_t}(x, G) + \inf_{F \in \mathcal{F}} \Phi_{d_t}(F, y) < t$. In particular $\inf_{G \in \mathcal{G}} \Phi_{d_t}(x, G) = \inf_{G \in \mathcal{G}} \Phi_d(x, G)$ and $\inf_{F \in \mathcal{F}} \Phi_{d_t}(F, y) = \inf_{F \in \mathcal{F}} \Phi_d(F, y)$. It follows from balancedness of $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) and the definition of d_t that $d_t(x, y) \leq d(x, y) \leq \inf_{G \in \mathcal{G}} \Phi_d(x, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, y) = \inf_{G \in \mathcal{G}} \Phi_{d_t}(x, G) + \inf_{F \in \mathcal{F}} \Phi_{d_t}(F, y)$. Hence $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced on (X, d_t) . We use Example 2.1.1 to prove the statement about the inverse map: By the proof given there, $(X, d_{\{2\}})$ is not balanced. But $(X, \min\{\frac{1}{2}, d_{\{2\}}\})$ is balanced, since $\min\{\frac{1}{2}, d_{\{2\}}\} = \min\{\frac{1}{2}, d\}$ is balanced, because an arbitrary Cauchy filter pair on $(X, \min\{\frac{1}{2}, d\})$ is balanced on the balanced space (X, d) and thus on $(X, \min\{\frac{1}{2}, d\})$ according to the statement established in the first part of this proof. \square

Lemma 2.2.3. Let (X, d) be a quasi-pseudometric space. Set $d_b = \frac{d}{1+d}$. Then d_b is a quasi-pseudometric on X and $id : (X, d) \rightarrow (X, d_b)$ is balanced. The inverse map $(id)^{-1} : (X, d_b) \rightarrow (X, d)$ need not be balanced.

Proof. It is well known (compare e.g. [14, Exercise 4.1.B(a)]) that d_b is a quasi-pseudometric on X such that $\mathcal{U}_d = \mathcal{U}_{d_b}$. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on (X, d) . Then for each $x, y \in X$ we have $d(x, y) \leq \inf_{G \in \mathcal{G}} \Phi_d(x, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, y)$. Of course, $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair on (X, d_b) , since the identity map $id : (X, d) \rightarrow (X, d_b)$ is uniformly continuous. Fix $x, y \in X$. By Lemma 2.2.1 there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $\Phi_d(F, y) < \infty$ and $\Phi_d(x, G) < \infty$. Consider any $F' \in \mathcal{F}$ and $G' \in \mathcal{G}$ such that $F' \subseteq F$ and $G' \subseteq G$ and let $\epsilon > 0$. There are $f' \in F'$ and $g' \in$

G' such that $\Phi_d(x, G') \leq d(x, g') + \frac{\epsilon}{2}$ and $\Phi_d(F', y) \leq d(f', y) + \frac{\epsilon}{2}$. Then $d(x, y) \leq d(x, g') + \frac{\epsilon}{2} + d(f', y) + \frac{\epsilon}{2}$. Since the real-valued function $f(t) = \frac{t}{1+t}$ is monotonically increasing where $t \in [0, \infty[$, we get that

$$d_b(x, y) = \frac{d(x, y)}{1 + d(x, y)} \leq \frac{d(x, g') + \frac{\epsilon}{2} + d(f', y) + \frac{\epsilon}{2}}{1 + d(x, g') + \frac{\epsilon}{2} + d(f', y) + \frac{\epsilon}{2}}.$$

Thus $d_b(x, y) \leq \frac{d(x, g') + \frac{\epsilon}{2}}{1 + d(x, g') + \frac{\epsilon}{2}} + \frac{d(f', y) + \frac{\epsilon}{2}}{1 + d(f', y) + \frac{\epsilon}{2}} \leq \frac{d(x, g')}{1 + d(x, g')} + \frac{\epsilon}{2} + \frac{d(f', y)}{1 + d(f', y)} + \frac{\epsilon}{2} \leq \Phi_{d_b}(x, G') + \frac{\epsilon}{2} + \Phi_{d_b}(F', y) + \frac{\epsilon}{2}$. Because ϵ was arbitrary, it follows that $d_b(x, y) \leq \inf_{G \in \mathcal{G}} \Phi_{d_b}(x, G) + \inf_{F \in \mathcal{F}} \Phi_{d_b}(F, y)$. Hence $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced on (X, d_b) and so $\text{id} : (X, d) \rightarrow (X, d_b)$ is balanced.

Again we use Example 2.1.1 to prove the statement about the inverse map. There we noted that $(X, d_{\{2\}})$ is not balanced, because the inequality for balancedness does not hold for the pair of points $(-\frac{1}{2}, \frac{1}{2})$ and all Cauchy filter pairs on X finer than the filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ defined in Example 2.1.1. Indeed, since d is balanced, and since $d = d_{\{2\}}$ except at $(-\frac{1}{2}, \frac{1}{2})$, one readily checks that these are the only instances where $d_{\{2\}}$ does not satisfy the inequality of balancedness.

Hence the argument presented in the first part of this proof establishes that the condition of balancedness holds for all pairs of points in $X \times X$ and all Cauchy filter pairs on $(X, (d_{\{2\}})_b)$, except maybe for the pair $(-\frac{1}{2}, \frac{1}{2})$ of points and all Cauchy filter pairs $\langle \mathcal{F}', \mathcal{G}' \rangle$ on X finer than $\langle \mathcal{F}, \mathcal{G} \rangle$.

However the space $(X, (d_{\{2\}})_b)$ is balanced, since indeed $(d_{\{2\}})_b(-\frac{1}{2}, \frac{1}{2}) = \frac{\frac{3}{2}}{1 + \frac{3}{2}} < \frac{2}{3} = \frac{\frac{1}{2}}{1 + \frac{1}{2}} \cdot 2 = \inf_{G' \in \mathcal{G}'} \Phi_{(d_{\{2\}})_b}(-\frac{1}{2}, G') + \inf_{F' \in \mathcal{F}'} \Phi_{(d_{\{2\}})_b}(F', \frac{1}{2})$ whenever $\langle \mathcal{F}', \mathcal{G}' \rangle$ is a Cauchy filter pair finer than $\langle \mathcal{F}, \mathcal{G} \rangle$ on X . \square

In the following we shall consider products of countably many quasi-pseudometric spaces. Various methods are known to equip products of finitely or countably many quasi-pseudometric spaces with a product quasi-pseudometric that induces the product quasi-uniformity. In our context it seems best to endow such products with the supremum quasi-pseudometric (see [31, p. 3 and p. 233]), of course, appropriately scaled in the case of countably many factor spaces. Therefore, for instance, given for each $i = 1, \dots, n$ a quasi-pseudometric space (X_i, d_i) , we put the maximum quasi-pseudometric

$$d((x_i)_{i=1}^n, (y_i)_{i=1}^n) := \max\{d_i(x_i, y_i) : i = 1, \dots, n\}$$

on the product $\prod_{i=1}^n X_i$, where $x_i, y_i \in X_i$ ($i = 1, \dots, n$). In the following we shall leave the case of finitely many factor spaces to the reader. We only deal with the case of countably infinitely many factor spaces. To simplify the notation we shall use $a \vee b$ to denote the maximum of two real numbers a and b in the next proof. Let us still introduce some notation first.

For each $i \in \mathbb{N}$ let $f_i : X_i \rightarrow Y_i$ be a map. The map $(\prod_{i \in \mathbb{N}} f_i) : \prod_{i \in \mathbb{N}} X_i \rightarrow \prod_{i \in \mathbb{N}} Y_i$ is defined as follows: $(\prod_{i \in \mathbb{N}} f_i)((x_i)_{i \in \mathbb{N}}) = (f_i(x_i))_{i \in \mathbb{N}}$ whenever $(x_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} X_i$. For each $i \in \mathbb{N}$ let \mathcal{F}_i be a filter on the set X_i . Then $\prod_{i \in \mathbb{N}} \mathcal{F}_i$ will denote the filter on $\prod_{i \in \mathbb{N}} X_i$ that is generated by the filter base consisting of the sets $\prod_{i \in \mathbb{N}} F_i$ where $F_i \in \mathcal{F}_i$ whenever $i \in \mathbb{N}$ and $F_i = X_i$ for all but finitely many $i \in \mathbb{N}$.

Lemma 2.2.4. *Let a sequence $((X_j, d_j))_{j \in \mathbb{N}}$ of quasi-pseudometric spaces be given, where for each $j \in \mathbb{N}$ we have that $d_j \leq 1$. For any $(x_j)_{j \in \mathbb{N}}, (y_j)_{j \in \mathbb{N}} \in \prod_{j \in \mathbb{N}} X_j$, we put*

$$d((x_j)_{j \in \mathbb{N}}, (y_j)_{j \in \mathbb{N}}) := \sup_{j \in \mathbb{N}} \frac{1}{2^j} d_j(x_j, y_j).$$

Then d is a quasi-pseudometric on $\prod_{j \in \mathbb{N}} X_j$. Furthermore for each $i \in \mathbb{N}$, the projection $\pi_i : (\prod_{j \in \mathbb{N}} X_j, d) \rightarrow (X_i, d_i)$ is balanced.

Proof. It is well known and easy to see that d is a quasi-pseudometric on $\prod_{j \in \mathbb{N}} X_j$, which induces the product quasi-uniformity $\prod_{j \in \mathbb{N}} \mathcal{U}_{d_j}$ on $\prod_{j \in \mathbb{N}} X_j$ (compare e.g. [32, Theorem 20.5] or [14, p. 439]).

Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on $(\prod_{j \in \mathbb{N}} X_j, d)$. Given $i \in \mathbb{N}$, we have to show that $\langle \pi_i \mathcal{F}, \pi_i \mathcal{G} \rangle$ is balanced on (X_i, d_i) . (It is a Cauchy filter pair on (X_i, d_i) , because the projection map π_i is uniformly continuous.) In order to reach a contradiction, suppose that there are $a, b \in X_i$, $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $d_i(a, b) > \Phi_{d_i}(a, \pi_i G) + \Phi_{d_i}(\pi_i F, b)$. Set $\delta = d_i(a, b) - \Phi_{d_i}(a, \pi_i G) - \Phi_{d_i}(\pi_i F, b)$. Since $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair on $(\prod_{j \in \mathbb{N}} X_j, d)$, we can choose $F' \in \mathcal{F}$, $G' \in \mathcal{G}$, $F' \subseteq F$ and $G' \subseteq G$ such that $\Phi_d(F', G') < \frac{\delta}{5 \cdot 2^i}$. Furthermore find $x = (x_j)_{j \in \mathbb{N}} \in F'$ and $y = (y_j)_{j \in \mathbb{N}} \in G'$. Define $s = (s_j)_{j \in \mathbb{N}}$ and $t = (t_j)_{j \in \mathbb{N}}$ in $\prod_{j \in \mathbb{N}} X_j$ as follows: Set $s_j = x_j$ if $j \in \mathbb{N}$ and $j \neq i$; furthermore let $s_i = a$. Set $t_j = y_j$ if $j \in \mathbb{N}$, and $j \neq i$; furthermore let $t_i = b$. By Lemma 2.2.1 there are $F'' \in \mathcal{F}$ and $G'' \in \mathcal{G}$ such that $F'' \subseteq F'$,

$G'' \subseteq G'$, $\Phi_d(F'', t) < \infty$ and $\Phi_d(s, G'') < \infty$.

Since $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced on $(\prod_{j \in \mathbb{N}} X_j, d)$, we have that $d(s, t) \leq \Phi_d(s, G'') + \Phi_d(F'', t)$. Therefore for some $g = (g_j)_{j \in \mathbb{N}} \in G''$ and $f = (f_j)_{j \in \mathbb{N}} \in F''$, we see that $\Phi_d(s, G'') \leq d(s, g) + \frac{\delta}{5 \cdot 2^i}$ and $\Phi_d(F'', t) \leq d(f, t) + \frac{\delta}{5 \cdot 2^i}$.

Then

$$\begin{aligned} \frac{1}{2^i} d_i(a, b) &= \frac{1}{2^i} d_i(s_i, t_i) \leq d(s, t) \leq \Phi_d(s, G'') + \Phi_d(F'', t) \\ &\leq d(s, g) + d(f, t) + \frac{2\delta}{5 \cdot 2^i} \leq \\ &\quad \sup_{j \in \mathbb{N}, j \neq i} \left\{ \frac{1}{2^j} d_j(s_j, g_j) \right\} \vee \frac{1}{2^i} d_i(a, g_i) + \\ &\quad \sup_{j \in \mathbb{N}, j \neq i} \left\{ \frac{1}{2^j} d_j(f_j, t_j) \right\} \vee \frac{1}{2^i} d_i(f_i, b) + \frac{2\delta}{5 \cdot 2^i} \leq \\ &\quad \Phi_d(x, G'') \vee \frac{1}{2^i} d_i(a, g_i) + \Phi_d(F'', y) \vee \frac{1}{2^i} d_i(f_i, b) + \frac{2\delta}{5 \cdot 2^i} \leq \\ &\quad \frac{\delta}{5 \cdot 2^i} + \frac{1}{2^i} d_i(a, g_i) + \frac{\delta}{5 \cdot 2^i} + \frac{1}{2^i} d_i(f_i, b) + \frac{2\delta}{5 \cdot 2^i}. \end{aligned}$$

So $d_i(a, b) \leq d_i(a, g_i) + d_i(f_i, b) + \frac{4\delta}{5}$. Therefore $d_i(a, b) \leq \Phi_{d_i}(a, \pi_i G'') + \Phi_{d_i}(\pi_i F'', b) + \frac{4\delta}{5} \leq \Phi_{d_i}(a, \pi_i G) + \Phi_{d_i}(\pi_i F, b) + \frac{4\delta}{5}$. Then $\delta = d_i(a, b) - \Phi_{d_i}(a, \pi_i G) - \Phi_{d_i}(\pi_i F, b) \leq \frac{4\delta}{5}$ —a contradiction. Thus $\langle \pi_i \mathcal{F}, \pi_i \mathcal{G} \rangle$ is balanced on (X_i, d_i) . We have shown that π_i is balanced. \square

Lemma 2.2.5. *For each $i \in \mathbb{N}$ let (X_i, d_i) be a (nonempty) quasipseudometric space such that $d_i \leq 1$. As above, on the product $\prod_{i \in \mathbb{N}} X_i$ put*

$$d((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) := \sup\{2^{-i} d_i(x_i, y_i) : i \in \mathbb{N}\}$$

where for each $i \in \mathbb{N}$, $x_i, y_i \in X_i$.

Then a filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ is a balanced Cauchy filter pair on $(\prod_{i \in \mathbb{N}} X_i, d)$ if and only if $\langle \pi_i \mathcal{F}, \pi_i \mathcal{G} \rangle$ is a balanced Cauchy filter pair on (X_i, d_i) whenever $i \in \mathbb{N}$, where $\pi_i : (\prod_{i \in \mathbb{N}} X_i, d) \rightarrow (X_i, d_i)$ denotes the projection map.

Proof. It is well known and easy to see that $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair on $(\prod_{i \in \mathbb{N}} X_i, d)$ if and only if $\langle \pi_i \mathcal{F}, \pi_i \mathcal{G} \rangle$ is a Cauchy filter pair on (X_i, d_i) whenever $i \in \mathbb{N}$ (compare for instance with [24, Lemma 9]). Suppose now that $\langle \pi_i \mathcal{F}, \pi_i \mathcal{G} \rangle$ is a balanced Cauchy filter pair on (X_i, d_i) whenever $i \in \mathbb{N}$. We are going to show that $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced on $(\prod_{i \in \mathbb{N}} X_i, d)$. For each $i \in \mathbb{N}$ consider $x_i, y_i \in X_i$. By our assumption and the definition of d for each $i \in \mathbb{N}$ we have $2^{-i} d_i(x_i, y_i) \leq 2^{-i} \Phi_{d_i}(x_i, \pi_i G) + 2^{-i} \Phi_{d_i}(\pi_i F, y_i) \leq \Phi_d((x_i)_{i \in \mathbb{N}}, G) + \Phi_d(F, (y_i)_{i \in \mathbb{N}})$ whenever $G \in \mathcal{G}$ and $F \in \mathcal{F}$. Therefore by the definition of d , $d((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) \leq \Phi_d((x_i)_{i \in \mathbb{N}}, G) + \Phi_d(F, (y_i)_{i \in \mathbb{N}})$ whenever $G \in \mathcal{G}$ and $F \in \mathcal{F}$. We conclude that $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced on $(\prod_{i \in \mathbb{N}} X_i, d)$. Furthermore if $\langle \mathcal{F}, \mathcal{G} \rangle$ is a balanced Cauchy filter pair on $(\prod_{i \in \mathbb{N}} X_i, d)$, then for each $i \in \mathbb{N}$, $\langle \pi_i \mathcal{F}, \pi_i \mathcal{G} \rangle$ is a balanced Cauchy filter pair on X_i , since the projection map π_i is balanced by Lemma 2.2.4. \square

Corollary 2.2.1. *Let $((X_i, d_i))_{i \in \mathbb{N}}$ be a family of quasi-pseudometric spaces such that $d_i \leq 1$ whenever $i \in \mathbb{N}$. Suppose that $\prod_{i \in \mathbb{N}} X_i \neq \emptyset$. Then $(\prod_{i \in \mathbb{N}} X_i, \sup_{i \in \mathbb{N}} \frac{d_i}{2^i})$ is B -complete if and only if (X_i, d_i) is B -complete whenever $i \in \mathbb{N}$.*

Proof. As before, let us set $d := \sup_{i \in \mathbb{N}} \frac{d_i}{2^i}$. Suppose that for each $i \in \mathbb{N}$, (X_i, d_i) is B -complete. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on $(\prod_{i \in \mathbb{N}} X_i, d)$. Then by Lemma 2.2.4 $\langle \pi_i \mathcal{F}, \pi_i \mathcal{G} \rangle$ is a balanced Cauchy filter pair on (X_i, d_i) whenever $i \in \mathbb{N}$. Hence for each $i \in \mathbb{N}$, $\langle \pi_i \mathcal{F}, \pi_i \mathcal{G} \rangle$ converges to some x_i in X_i by B -completeness of (X_i, d_i) . We conclude that $\langle \mathcal{F}, \mathcal{G} \rangle$ converges to $(x_i)_{i \in \mathbb{N}}$ on $(\prod_{i \in \mathbb{N}} X_i, d)$. Therefore $(\prod_{i \in \mathbb{N}} X_i, d)$ is B -complete. For the converse suppose that $\prod_{i \in \mathbb{N}} X_i \neq \emptyset$ and $(\prod_{i \in \mathbb{N}} X_i, d)$ is B -complete. Given fixed $j \in \mathbb{N}$, let $\langle \mathcal{F}_j, \mathcal{G}_j \rangle$ be a balanced Cauchy filter pair on (X_j, d_j) . For each $i \in \mathbb{N}$ such that $i \neq j$ choose $x_i \in X_i$ and set $\mathcal{F}'_i = \mathcal{U}_{d_i}^{-1}(x_i)$; furthermore let $\mathcal{F}'_j = \mathcal{F}_j$. Similarly for each $i \in \mathbb{N}$ such that $i \neq j$ set $\mathcal{G}'_i = \mathcal{U}_{d_i}(x_i)$, and let $\mathcal{G}'_j = \mathcal{G}_j$. Then by Lemma 2.2.5 $\langle \prod_{i \in \mathbb{N}} \mathcal{F}'_i, \prod_{i \in \mathbb{N}} \mathcal{G}'_i \rangle$ is a Cauchy filter pair on $(\prod_{i \in \mathbb{N}} X_i, d)$ which is balanced. By B -completeness of $(\prod_{i \in \mathbb{N}} X_i, d)$, it converges to some point $(y_i)_{i \in \mathbb{N}}$ in $(\prod_{i \in \mathbb{N}} X_i, d)$. Then $\langle \mathcal{F}_j, \mathcal{G}_j \rangle$ converges to y_j . Thus (X_j, d_j) is B -complete. \square

Proposition 2.2.1. *For each $i \in \mathbb{N}$ let $f_i : (X_i, d_i) \rightarrow (Y_i, e_i)$ be a family of maps between quasi-pseudometric spaces (X_i, d_i) and (Y_i, e_i) , where $d_i \leq 1$ and $e_i \leq 1$ whenever $i \in \mathbb{N}$. Suppose that $\prod_{i \in \mathbb{N}} X_i \neq \emptyset$. Then the map $\prod_{i \in \mathbb{N}} f_i : (\prod_{i \in \mathbb{N}} X_i, \sup_{i \in \mathbb{N}} \frac{d_i}{2^i}) \rightarrow (\prod_{i \in \mathbb{N}} Y_i, \sup_{i \in \mathbb{N}} \frac{e_i}{2^i})$ is balanced if and only if each f_i ($i \in \mathbb{N}$) is balanced.*

Proof. As above for convenience we set $d := \sup_{i \in \mathbb{N}} \frac{d_i}{2^i}$ and $e := \sup_{i \in \mathbb{N}} \frac{e_i}{2^i}$. It is well known and easy to see that $\Pi_{i \in \mathbb{N}} f_i$ is uniformly continuous if and only if each f_i ($i \in \mathbb{N}$) is uniformly continuous (compare for instance [14, Theorem 8.2.1]). Suppose now that the map $\Pi_{i \in \mathbb{N}} f_i$ is balanced.

Fix $j \in \mathbb{N}$. Choose $a_i \in X_i$ whenever $i \in \mathbb{N}$ and $i \neq j$. Let $t_j : X_j \rightarrow \Pi_{i \in \mathbb{N}} X_i$ be defined by $t_j(x) = (b_i)_{i \in \mathbb{N}}$ where $b_j = x \in X_j$ and $b_i = a_i$ whenever $i \in \mathbb{N}$ and $i \neq j$. Then $t_j : (X_j, d_j) \rightarrow (\Pi_{i \in \mathbb{N}} X_i, d)$ is a balanced map, as we show next: Indeed let $\epsilon > 0$ and $x, y \in X_j$ be such that $d_j(x, y) < \epsilon$. Then $d(t_j(x), t_j(y)) = \frac{1}{2^j} d_j(x, y) < \epsilon$. Hence t_j is uniformly continuous. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on (X_j, d_j) and let $(x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}} \in \Pi_{i \in \mathbb{N}} X_i$. Note that for each $i \in \mathbb{N}$ with $i \neq j$, we have by the triangle inequality and the definition of d that $\frac{1}{2^i} d_i(x_i, y_i) \leq \frac{1}{2^i} d_i(x_i, a_i) + \frac{1}{2^i} d_i(a_i, y_i) \leq \Phi_d((x_i)_{i \in \mathbb{N}}, t_j(G)) + \Phi_d(t_j(F), (y_i)_{i \in \mathbb{N}})$ whenever $F \in \mathcal{F}$ and $G \in \mathcal{G}$.

Furthermore by balancedness of $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X_j, d_j) we have

$$\frac{1}{2^j} d_j(x_j, y_j) \leq \frac{1}{2^j} d_j(x_j, G) + \frac{1}{2^j} d_j(F, y_j) \leq \Phi_d((x_i)_{i \in \mathbb{N}}, t_j(G)) + \Phi_d(t_j(F), (y_i)_{i \in \mathbb{N}})$$

whenever $F \in \mathcal{F}$ and $G \in \mathcal{G}$. We conclude that

$$d((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) \leq \Phi_d((x_i)_{i \in \mathbb{N}}, t_j(G)) + \Phi_d(t_j(F), (y_i)_{i \in \mathbb{N}})$$

whenever $F \in \mathcal{F}$ and $G \in \mathcal{G}$. We have shown that $t_j : (X_j, d_j) \rightarrow (\Pi_{i \in \mathbb{N}} X_i, d)$ is balanced.

Evidently $f_j = \pi_j \circ (\Pi_{i \in \mathbb{N}} f_i) \circ t_j$ where $\pi_j : \Pi_{i \in \mathbb{N}} Y_i \rightarrow Y_j$ is the projection map. The composition on the right hand side is balanced as the composition of three balanced maps. We conclude that $f_j : (X_j, d_j) \rightarrow (Y_j, e_j)$ is balanced.

For the converse suppose that each f_i ($i \in \mathbb{N}$) is balanced. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on $(\Pi_{i \in \mathbb{N}} X_i, d)$.

The filter pair $\langle (\Pi_{i \in \mathbb{N}} f_i) \mathcal{F}, (\Pi_{i \in \mathbb{N}} f_i) \mathcal{G} \rangle$ clearly is a Cauchy filter pair on $(\Pi_{i \in \mathbb{N}} Y_i, e)$, since $\Pi_{i \in \mathbb{N}} f_i$ is uniformly continuous. We see that $\langle (\Pi_{i \in \mathbb{N}} f_i) \mathcal{F}, (\Pi_{i \in \mathbb{N}} f_i) \mathcal{G} \rangle$ is balanced as follows:

Given for each $i \in \mathbb{N}$ some $x_i, y_i \in Y_i$, we deduce by balancedness of $\pi'_i : \Pi_{i \in \mathbb{N}} X_i \rightarrow X_i$ and f_i , as well as the definition of e , that

$$\frac{1}{2^i} e_i(x_i, y_i) \leq \frac{1}{2^i} \Phi_{e_i}(x_i, f_i(\pi'_i G)) + \frac{1}{2^i} \Phi_{e_i}(f_i(\pi'_i F), y_i) \leq \\ \Phi_e((x_i)_{i \in \mathbb{N}}, (\Pi_{i \in \mathbb{N}} f_i)(G)) + \Phi_e((\Pi_{i \in \mathbb{N}} f_i)(F), (y_i)_{i \in \mathbb{N}})$$

whenever $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Consequently

$$e((x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}) \leq \Phi_e((x_i)_{i \in \mathbb{N}}, (\Pi_{i \in \mathbb{N}} f_i)(G)) + \Phi_e((\Pi_{i \in \mathbb{N}} f_i)(F), (y_i)_{i \in \mathbb{N}})$$

whenever $F \in \mathcal{F}$ and $G \in \mathcal{G}$. We conclude that $\langle (\Pi_{i \in \mathbb{N}} f_i) \mathcal{F}, (\Pi_{i \in \mathbb{N}} f_i) \mathcal{G} \rangle$ is a balanced Cauchy filter pair on the space $(\Pi_{i \in \mathbb{N}} Y_i, e)$. Hence the map $\Pi_{i \in \mathbb{N}} f_i$ is balanced. \square

Proposition 2.2.2. *Let $((X_i, d_i))_{i \in \mathbb{N}}$ be a family of T_0 -quasi-pseudometric spaces with B -completions $((\widetilde{X}_i, \widetilde{d}_i))_{i \in \mathbb{N}}$ such that for each $i \in \mathbb{N}$, $d_i \leq 1$.*

Then $(\Pi_{i \in \mathbb{N}} \widetilde{X}_i, \sup_{i \in \mathbb{N}} \frac{\widetilde{d}_i}{2^i})$ yields the B -completion of the T_0 -quasi-metric space $(\Pi_{i \in \mathbb{N}} X_i, \sup_{i \in \mathbb{N}} \frac{d_i}{2^i})$.

Proof. Note first that for each $i \in \mathbb{N}$, $\widetilde{d}_i \leq 1$ by definition of $(d_i)^+$. For each $j \in \mathbb{N}$ the natural isometric embedding $\beta_j : (X_j, d_j) \rightarrow (\widetilde{X}_j, \widetilde{d}_j)$ is balanced (see [28, Corollary 10]).

It follows from Proposition 2.2.1 that the isometric embedding

$$\beta := \Pi_{j \in \mathbb{N}} \beta_j : (\Pi_{i \in \mathbb{N}} X_i, \sup_{i \in \mathbb{N}} \frac{d_i}{2^i}) \rightarrow (\Pi_{i \in \mathbb{N}} \widetilde{X}_i, \sup_{i \in \mathbb{N}} \frac{\widetilde{d}_i}{2^i})$$

is balanced. Furthermore $(\Pi_{i \in \mathbb{N}} \widetilde{X}_i, \sup_{i \in \mathbb{N}} \frac{\widetilde{d}_i}{2^i})$ is B -complete by Corollary 2.2.1. Let $\langle \mathcal{F}_j, \mathcal{G}_j \rangle \in \widetilde{X}_j$ whenever $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ we have that $\langle \mathcal{F}_j, \mathcal{G}_j \rangle$ is a minimal balanced Cauchy filter pair on (X_j, d_j) such that $\langle \beta_j(\mathcal{F}_j), \beta_j(\mathcal{G}_j) \rangle$ converges to $\langle \mathcal{F}_j, \mathcal{G}_j \rangle$ in $(\widetilde{X}_j, \widetilde{d}_j)$ (see remarks following Theorem 1.0.1). Thus $\langle \beta(\Pi_{j \in \mathbb{N}} \mathcal{F}_j), \beta(\Pi_{j \in \mathbb{N}} \mathcal{G}_j) \rangle$ converges to $\langle (\mathcal{F}_j, \mathcal{G}_j)_{j \in \mathbb{N}} \rangle$ and $\langle \Pi_{j \in \mathbb{N}} \mathcal{F}_j, \Pi_{j \in \mathbb{N}} \mathcal{G}_j \rangle$ is a balanced Cauchy filter pair on $(\Pi_{j \in \mathbb{N}} X_j, d)$ by Lemma 2.2.5. The result follows from the characterization of the B -completion given in Theorem 1.0.3. \square

2.3 B -completion versus bicompletion

In this section we shall characterize those T_0 -quasi-pseudometric spaces for which the B -completion coincides with the bicompletion.

We show that the B -completion coincides with the bicompletion if the T_0 -quasi-pseudometric space is B -filter-symmetric.

Definition 2.3.1. *A Cauchy filter pair of the form $\langle \mathcal{F}, \mathcal{F} \rangle$ on a quasi-pseudometric space (X, d) will be called constant.*

Proposition 2.3.1. *Let (X, d) be a T_0 -quasi-pseudometric space. Then the following conditions are equivalent: (a) The bicompletion of (X, d) is equal to the B -completion of (X, d) . (b) Each minimal balanced Cauchy filter pair on (X, d) is linked. (c) Each balanced Cauchy filter pair is \mathcal{U}_d -equivalent to a constant Cauchy filter pair on (X, d) .*

Proof. (a) \Rightarrow (b) : Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a minimal balanced Cauchy filter pair on (X, d) . Evidently by hypothesis $\langle \mathcal{F}, \mathcal{G} \rangle$ is \mathcal{U}_d -equivalent to a constant (balanced) Cauchy filter pair $\langle \mathcal{H}, \mathcal{H} \rangle$ on (X, d) (see remarks after Theorem 1.0.1). Then by [28, Lemma 5] $\langle \mathcal{F}, \mathcal{G} \rangle$ is coarser than $\langle \mathcal{H}, \mathcal{H} \rangle$ and thus $\langle \mathcal{F}, \mathcal{G} \rangle$ is linked.

(b) \Rightarrow (c) : Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on (X, d) . Furthermore let $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ be the minimal Cauchy filter pair coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) (see Lemma 1.0.4). Then by our assumption the filter $\mathcal{F}_{m_1} \vee \mathcal{G}_{m_2}$ exists. It obviously is a d^s -Cauchy filter. Furthermore $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ and the constant Cauchy filter pair $\langle \mathcal{F}_{m_1} \vee \mathcal{G}_{m_2}, \mathcal{F}_{m_1} \vee \mathcal{G}_{m_2} \rangle$ are \mathcal{U}_d -equivalent. Hence $\langle \mathcal{F}, \mathcal{G} \rangle$ is \mathcal{U}_d -equivalent to that constant Cauchy filter pair on (X, d) .

(c) \Rightarrow (a) : We know that for any T_0 -quasi-pseudometric space the B -completion is an extension of the bicompletion [28, Corollary 5]. By assumption each balanced Cauchy filter pair on (X, d) is \mathcal{U}_d -equivalent to a constant Cauchy filter pair. Hence the two constructions coincide, because all points of the B -completion are represented in the bicompletion. \square

Corollary 2.3.1. *Let d_1 and d_2 be two quasi-pseudometrics on a set X such that $\mathcal{U}_{d_1} = \mathcal{U}_{d_2}$. Furthermore suppose that each minimal balanced Cauchy filter pair on (X, d_1) is linked and that each minimal balanced Cauchy filter pair on (X, d_2) is linked. Then d_1 and d_2 are B -equivalent.*

Proof. Since every linked Cauchy filter pair on a quasi-pseudometric space is balanced, in the light of Lemma 1.0.4 the hypothesis of the corollary implies that each balanced Cauchy filter pair on (X, d_1) is \mathcal{U}_{d_1} -equivalent to a balanced Cauchy filter pair on (X, d_2) , and similarly that each balanced Cauchy filter pair on (X, d_2) is \mathcal{U}_{d_2} -equivalent to a balanced Cauchy filter pair on (X, d_1) . Hence the conditions stated in the definition of B -equivalence are satisfied. \square

We next define a concept of symmetric Cauchy filter pair on a quasi-pseudometric space and the notion of B -filter-symmetry of a T_0 -quasi-pseudometric space. (The similar notion of filter symmetric will be defined on a quasi-uniform space in the next chapter).

A Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) will be called *symmetric* provided that $\langle \mathcal{G}, \mathcal{F} \rangle$ is a Cauchy filter pair on (X, d) .

A quasi-pseudometric space (X, d) will be called *B -filter-symmetric* provided that for each balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) , $\langle \mathcal{G}, \mathcal{F} \rangle$ is a Cauchy filter pair on (X, d) . (Note that $\langle \mathcal{G}, \mathcal{F} \rangle$ is then also balanced, since otherwise there are $x, y \in X$, $\delta > 0$ and $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $d(x, y) > \Phi_d(x, F) + \Phi_d(G, y) + \delta$ and $\Phi_d(F, G) < \delta$; therefore for any $f \in F$ and any $g \in G$ we get that $d(x, y) \leq d(x, f) + d(f, g) + d(g, y) \leq \Phi_d(x, F) + \delta + \Phi_d(G, y) < d(x, y)$ —a contradiction.) We also observe that each pseudometric space is (B -)filter-symmetric.

Corollary 2.3.2. *For any B -filter-symmetric T_0 -quasi-pseudometric space (X, d) the B -completion coincides with the bicompletion.*

Proof. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a balanced Cauchy filter pair on (X, d) . By B -filter-symmetry $\langle \mathcal{G}, \mathcal{F} \rangle$ is a Cauchy filter pair. By the triangle inequality, it follows that $\langle \mathcal{F}, \mathcal{F} \rangle$ and $\langle \mathcal{G}, \mathcal{G} \rangle$ are Cauchy filter pairs on (X, d) . Altogether it follows that $\langle \mathcal{F} \cap \mathcal{G}, \mathcal{F} \cap \mathcal{G} \rangle$ is a balanced Cauchy filter pair on (X, d) (see Lemma 1.0.3). We conclude that each minimal balanced Cauchy filter pair on (X, d) is constant and so the assertion follows from Proposition 2.3.1. \square

Example 2.3.1. *Let \mathbb{R} be the set of the reals equipped with its Sorgenfrey T_0 -quasi-metric s . Then (\mathbb{R}, s) is B -complete [11, p. 132] and so each minimal balanced Cauchy filter pair on this space is of the form*

$\langle \mathcal{U}_s^{-1}(x), \mathcal{U}_s(x) \rangle$ for some $x \in \mathbb{R}$. Hence it is linked. But the minimal balanced Cauchy filter pairs on (\mathbb{R}, s) are clearly not constant. Indeed the latter condition would imply that $\mathcal{U}_s(x) = \mathcal{U}_s^{-1}(x)$ whenever $x \in \mathbb{R}$.

2.4 Totally bounded quasi-pseudometrics

In this last section of Chapter 2 we show that total boundedness is preserved by the B -completion and that even for totally bounded T_0 -quasi-pseudometric spaces the B -completion can be strictly larger than the bicompletion.

As usual, we call a quasi-pseudometric d *totally bounded* provided that d^s is a totally bounded pseudometric. Totally bounded balanced quasi-pseudometrics are known to induce uniformities (see [15, 25, 9]). The following proposition makes use of techniques of Deák (compare [4, Proposition 6.5] and [6, Proposition 2.3]).

Proposition 2.4.1. *Let (X, d) be a totally bounded T_0 -quasi-pseudometric space and let (\tilde{X}, \tilde{d}) be its B -completion. Then (\tilde{X}, \tilde{d}) is totally bounded (and bicomplete). (Hence $\tau((\tilde{d})^s)$ is a compact Hausdorff topology.)*

Proof. As mentioned before, it is known that (\tilde{X}, \tilde{d}) is bicomplete [28, Corollary 5]. We shall identify the points of X with their images in \tilde{X} under the map β_X . Let $\epsilon > 0$. Since \mathcal{U}_d is totally bounded, there is a finite cover $\{A_i : i \in F\}$ of X such that $A_i \times A_i \subseteq U_{d, \frac{\epsilon}{4}}$ whenever $i \in F$. For convenience let us set $V := U_{d, \frac{\epsilon}{4}}$. Since X is dense in \tilde{X} , we conclude that $\bigcup_{i \in F} \text{cl}_{\tau(\mathcal{U}_{\tilde{d}})} A_i = \tilde{X}$. Consider the finite partition \mathcal{P} of \tilde{X} that is determined by the cover $\{\text{cl}_{\tau(\mathcal{U}_{\tilde{d}})} A_i : i \in F\}$ of \tilde{X} . Let $P \in \mathcal{P}$. Hence for some $J \subseteq F$, $P \in \mathcal{P}$ is equal to $\bigcap_{i \in J} \text{cl}_{\tau(\mathcal{U}_{\tilde{d}})} A_i \setminus (\bigcup_{i \in F \setminus J} \text{cl}_{\tau(\mathcal{U}_{\tilde{d}})} A_i)$. In order to establish that (\tilde{X}, \tilde{d}) is totally bounded, it will suffice to show that $P \times P \subseteq U_{\tilde{d}, \epsilon}$. Let $(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) \in P \times P$. Then there is $W \in \mathcal{U}_{\tilde{d}}$ such that $W(\langle \mathcal{F}', \mathcal{G}' \rangle) \cap \bigcup_{i \in F \setminus J} A_i = \emptyset$, so $W(\langle \mathcal{F}', \mathcal{G}' \rangle) \cap X \subseteq \bigcup_{i \in J} A_i$.

Because $\beta_X(\mathcal{G}')$ converges to $\langle \mathcal{F}', \mathcal{G}' \rangle$ with respect to the topology $\tau(\mathcal{U}_{\tilde{d}})$, we conclude from the latter inequality that $\bigcup_{i \in J} A_i \in \mathcal{G}'$. Fix $i \in J$. Since

$A_i \times A_i \subseteq V$, we have $V^{-1}(A_i) \times A_i \subseteq V^2$. Because $\langle \mathcal{F}, \mathcal{G} \rangle \in \text{cl}_{\tau(\mathcal{U}_{\tilde{d}})} A_i$ and hence $U_{\tilde{d}, \frac{\epsilon}{4}}(\langle \mathcal{F}, \mathcal{G} \rangle) \cap A_i \neq \emptyset$, there are $F_i \in \mathcal{F}$ and $a_i \in A_i$ such that $F_i \times \{a_i\} \subseteq V$. Thus $F_i \subseteq V^{-1}(a_i) \subseteq V^{-1}(A_i)$. Hence $\bigcap_{i \in J} F_i \subseteq \bigcap_{i \in J} V^{-1}(A_i)$.

It follows that $\bigcap_{i \in J} V^{-1}(A_i) \times \bigcup_{i \in J} A_i \subseteq V^2 \subseteq U_{d, \frac{\epsilon}{2}}$. By definition of \tilde{d} , we conclude that $(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) \in U_{\tilde{d}, \epsilon}$ and thus $P \times P \subseteq U_{\tilde{d}, \epsilon}$. Hence we have verified that $\mathcal{U}_{\tilde{d}}$ is totally bounded. \square

Our final example in this chapter shows that even for a totally bounded T_0 -quasi-pseudometric space (X, d) the B -completion can be strictly larger than the bicompletion of (X, d) . Similar constructions have been used by Deák (see [2, Section 3]).

Example 2.4.1. (see also [28, Example 4]) Let $Z_- = \{-\frac{1}{n+1} : n \in \mathbb{N}\}$ and $Z_+ = \{\frac{1}{n+1} : n \in \mathbb{N}\}$. Furthermore set $Z = Z_- \cup Z_+$ and $X = Z \times \{1, 2\}$. By $|x-y|$ we shall denote the usual distance between two real numbers x and y .

Define $d : X \times X \rightarrow [0, \infty[$ as follows. Set $d(x, x) = 0$ whenever $x \in X$. For distinct points (z_1, i_1) and (z_2, i_2) in X with $z_1, z_2 \in Z$ and $i_1, i_2 \in \{1, 2\}$ set $d((z_1, i_1), (z_2, i_2)) = |z_1 - z_2|$ if (1) $z_1 < 0 < z_2$ and i_1, i_2 are arbitrary or if (2) $z_1 \cdot z_2 > 0$ and $i_1 = i_2$. Otherwise set $d(x, y) = 2$. Note that $d \leq 2$. We next verify that d satisfies the triangle inequality. So let $x_1 = (z_1, i_1)$, $x_2 = (z_2, i_2)$, $x_3 = (z_3, i_3)$ in X with $z_1, z_2, z_3 \in Z$ and $i_1, i_2, i_3 \in \{1, 2\}$. Since $d \leq 2$, it suffices to consider the case that $d((z_1, i_1), (z_2, i_2)) < 2$ and $d((z_2, i_2), (z_3, i_3)) < 2$. Hence $d(x_1, x_2) = |z_1 - z_2|$ and $d(x_2, x_3) = |z_2 - z_3|$. By definition of d , therefore $z_1 \in Z_-$ and $z_2 \in Z_+$, or $z_1, z_2 \in Z_-$, or $z_1, z_2 \in Z_+$. Moreover $z_2 \in Z_-$ and $z_3 \in Z_+$, or $z_2, z_3 \in Z_-$, or $z_2, z_3 \in Z_+$. Combining these possibilities, we can distinguish the following four cases.

Case 1: $z_1 \in Z_-$ and $z_2, z_3 \in Z_+$.

Case 2: $z_1, z_2 \in Z_-$ and $z_3 \in Z_+$.

Case 3: $z_1, z_2, z_3 \in Z_-$.

Case 4: $z_1, z_2, z_3 \in Z_+$.

By definition of d , cases 1 and 2 imply that $d(x_1, x_3) = |z_1 - z_3|$. On the other hand, in cases 3 and 4, $d(x_1, x_2) = |z_1 - z_2|$ and $d(x_2, x_3) = |z_2 - z_3|$ yield $i_1 = i_2$ and $i_2 = i_3$. Hence $i_1 = i_3$ and thus $d(x_1, x_3) = |z_1 - z_3|$ in either case. It follows that in all four cases under consideration the triangle inequality holds. Hence d is a T_0 -quasi-metric on X . Since the four subspaces $Z_- \times \{1\}$, $Z_- \times \{2\}$, $Z_+ \times \{1\}$, and $Z_+ \times \{2\}$ of (X, d) are clearly all totally bounded (and metric), the space (X, d) is totally bounded.

Let \mathcal{G} be the filter generated by $\{(\downarrow 0, \epsilon[\times\{1, 2\}) \cap X : \epsilon > 0\}$. Similarly let \mathcal{F} be the filter generated by $\{(\downarrow -\epsilon, 0[\times\{1, 2\}) \cap X : \epsilon > 0\}$. Clearly $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair on (X, d) . Next we are going to prove that $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced on (X, d) . Since $d \leq 2$, it suffices to consider the case that

$$\inf_{\epsilon > 0} \Phi_d((a, i), (\downarrow 0, \epsilon[\times\{1, 2\}) \cap X) < 2$$

and

$$\inf_{\epsilon > 0} \Phi_d((\downarrow -\epsilon, 0[\times\{1, 2\}) \cap X, (b, j)) < 2$$

with $a, b \in Z$ and $i, j \in \{1, 2\}$.

Observe next that for $(z_1, i), (z_2, j) \in X$ with $z_1, z_2 \in Z$ and $i, j \in \{1, 2\}$ we have that $d((z_1, i), (z_2, j)) = 2$ if $i \neq j$ and $z_1 \cdot z_2 > 0$ (*).

Hence our assumptions imply that $a < 0 < b$ and thus $d((a, i), (b, j)) = b - a = \inf_{\epsilon > 0} \Phi_d((a, i), (\downarrow 0, \epsilon[\times\{1, 2\}) \cap X) + \inf_{\epsilon > 0} \Phi_d((\downarrow -\epsilon, 0[\times\{1, 2\}) \cap X, (b, j))$. We conclude that $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced on (X, d) . We shall now show that the B -completion is larger than the bicompletion of (X, d) . In order to reach a contradiction assume that $\langle \mathcal{F}, \mathcal{G} \rangle$ is \mathcal{U}_d -equivalent to $\langle \mathcal{H}, \mathcal{H} \rangle$ where \mathcal{H} is a d^s -Cauchy filter on X (compare with Proposition 2.3.1).

Consequently $\langle \mathcal{F}, \mathcal{H} \rangle$ and $\langle \mathcal{H}, \mathcal{G} \rangle$ are Cauchy filter pairs on (X, d) . Note that a nonempty set $H \subseteq X$ satisfying $\Phi_d(H, H) < 2$ can only hit one class of the partition $\mathcal{P} = \{Z_- \times \{1\}, Z_- \times \{2\}, Z_+ \times \{1\}, Z_+ \times \{2\}\}$ of X , since $d^s(x, y) = 2$ whenever x and y are points of X belonging to distinct classes of this partition, as it is straightforward to check. Since there is such an H that belongs to the filter \mathcal{H} , that filter \mathcal{H} must contain exactly one member of the partition \mathcal{P} . On the other hand, since $\langle \mathcal{F}, \mathcal{H} \rangle$ is a Cauchy filter pair on (X, d) , by (*) and the definition of \mathcal{F} , \mathcal{H} cannot contain $Z_- \times \{1\}$ and

similarly it cannot contain $Z_- \times \{2\}$. Analogously, because $\langle \mathcal{H}, \mathcal{G} \rangle$ is a Cauchy filter pair on (X, d) , both $Z_+ \times \{1\}$ and $Z_+ \times \{2\}$ cannot belong to \mathcal{H} by (*) and the definition of \mathcal{G} . Consequently \mathcal{H} does not contain any set of the partition. Hence we have reached a contradiction and conclude that $\langle \mathcal{F}, \mathcal{G} \rangle$ is not \mathcal{U}_d -equivalent to any constant Cauchy filter pair on (X, d) . Hence the B -completion of (X, d) is strictly larger than the bicompletion of (X, d) although (X, d) is totally bounded.

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Chapter 3

Some results on quasi-uniform spaces and properties of filter pairs on quasi-uniform spaces

In this chapter we shall discuss some interesting results related to the notion of quasi-uniform spaces and investigate interesting properties of filter pairs on quasi-uniform spaces.

In the first section we recall some basic notions in quasi-uniform spaces and those of filter pairs on a quasi-uniform space. We finally discuss the concept of a weakly concentrated Cauchy filter pair on a quasi-uniform space.

In the second section we explain the notions of Cauchy, fully Cauchy, locally quiet and filter symmetric quasi-uniformities introduced by Deák in [6, 8, 9]. We present results showing that a filter symmetric quasi-uniform space is Cauchy and each Cauchy filter pair on that quasi-uniform space is stable and costable. Furthermore a filter symmetric quasi-uniform space is locally quiet and doubly costable.

In the last section we shall explain the notion of D -completeness and C -completeness developed by Fletcher and Hunsaker and their relationship in a quasi-uniform space. We point out that in quiet quasi-uniform spaces, the two notions of completeness coincide.

3.1 Quasi-uniformities and basic results

We start this section by recalling some elementary concepts and facts from the theory of quasi-uniform spaces and some properties of filter pairs on quasi-uniform spaces. Indeed quasi-pseudometrics that we presented in the previous chapters induce quasi-uniformities.

Definition 3.1.1. [15] Let X be a set and \mathcal{U} be a filter on $X \times X$ such that

(a) for each $U \in \mathcal{U}$, U contains the diagonal $\{(x, x) : x \in X\}$ of X ,

(b) for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V \circ V = \{(x, z) \in X \times X : (x, y) \in V, (y, z) \in V\} \subseteq U$. Then we say that \mathcal{U} is a quasi-uniformity on X . The tuple (X, \mathcal{U}) is called a quasi-uniform space.

Note that for any quasi-uniformity \mathcal{U} on X , the filter of inverse relation \mathcal{U}^{-1} is also a quasi-uniformity on X called *the conjugate* of \mathcal{U} .

We denote by \mathcal{U}^s the uniformity $\mathcal{U} \vee \mathcal{U}^{-1}$ on a set X . (The definition of a quasi-uniformity and those of an uniformity on a set X are given in [15, Definition 1.1]).

A quasi-uniformity \mathcal{U} on a set X is called T_0 -quasi-uniformity provided that $\bigcap \mathcal{U} \cap \bigcap \mathcal{U}^{-1}$ is equal to the diagonal $\{(x, x) : x \in X\}$ of X and $\bigcap \mathcal{U}$ is a partial order.

A map $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ between two quasi-uniform spaces (X, \mathcal{U}) and (Y, \mathcal{V}) is called *quasi-uniformly continuous* (see [15, Definition 1.11]) if for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V$ whenever $(x, y) \in U$. A bijection $f : (X, \mathcal{U}) \longrightarrow (Y, \mathcal{V})$ is called quasi-isomorphism if f and f^{-1} are quasi-uniformly continuous. In this case we say that (X, \mathcal{U}) and (Y, \mathcal{V}) are quasi-isomorphic.

In a quasi-uniform space (X, \mathcal{U}) we shall say that a filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a set X is called a *Cauchy filter pair* (compare Definition 1.0.2) provided that for each $U \in \mathcal{U}$ there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$. A Cauchy filter pair on a quasi-uniform space (X, \mathcal{U}) will be called *constant* provided that $\mathcal{F} = \mathcal{G}$. (This was defined early for quasi-pseudometric spaces).

The following concept of bicompleteness can be generalized to arbitrary quasi-uniform spaces.

A quasi-uniform space (X, \mathcal{U}) is called *bicomplete* if each \mathcal{U}^s -Cauchy filter converges with respect to the topology $\tau(\mathcal{U}^s)$ and the bicompletion of a T_0 -quasi-uniform space (X, \mathcal{U}) is a bicomplete T_0 -quasi-uniform space (Y, \mathcal{V}) that has a $\tau(\mathcal{U}^s)$ -dense subspace quasi-uniform isomorphic to (X, \mathcal{U}) .

Definition 3.1.2. [2] A Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-uniform space (X, \mathcal{U}) is called a *round Cauchy filter pair* provided that for $G \in \mathcal{G}, F \in \mathcal{F}$ there are $U \in \mathcal{U}$ and $T_1 \in \mathcal{F}, T_2 \in \mathcal{G}$ such that $U^{-1}[T_1] \subseteq F$ and $U[T_2] \subseteq G$.

We next recall the definition of a weakly concentrated Cauchy filter pair on a quasi-uniform space.

Definition 3.1.3. [4, Lemma 7.7] A Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-uniform space (X, \mathcal{U}) is called *weakly concentrated* provided that for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$ imply that $(x, y) \in U$ whenever $x, y \in X$.

Let us recall that a family Ψ of Cauchy filter pairs is called *uniformly weakly concentrated* [3, Lemma 7.15] provided that each member of Ψ satisfies the condition of weak concentration with V not depending on the considered filter pair but only on the family Ψ . A quasi-uniform space (X, \mathcal{U}) is called *quiet* if the family of all Cauchy filter pairs on (X, \mathcal{U}) is uniformly weakly concentrated. In this case, we say that V is *quiet* for U .

Lemma 3.1.1. [3, Lemma 7.7 and Definition 7.6] Let (X, \mathcal{U}) be a quasi-uniform space. A Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-uniform space (X, \mathcal{U}) is weakly concentrated if and only if the following condition (*) is satisfied: if for each $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $F_1, F_2 \in \mathcal{F}, G_1, G_2 \in \mathcal{G}, F_1 \times G_1 \subseteq V$ and $F_2 \times G_2 \subseteq V$ imply that $F_1 \times G_2 \subseteq U$.

Proof. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair such that for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$ imply that $(x, y) \in U$. Suppose

that $F_1 \times G_1 \subseteq V$ and $F_2 \times G_2 \subseteq V$ with $F_1, F_2 \in \mathcal{F}$ and $G_1, G_2 \in \mathcal{G}$. Let $(f_1, g_2) \in F_1 \times G_2$. Then $V(f_1) \in \mathcal{G}$ and $V^{-1}(g_2) \in \mathcal{F}$. Thus $(f_1, g_2) \in U$ and $F_1 \times G_2 \subseteq U$. So the given condition (*) is satisfied.

To prove the converse, suppose that the condition (*) is satisfied and that $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$. Find $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \subseteq V^{-1}(y)$, $G \subseteq V(x)$ and $F \times G \subseteq V$. Then $(F \cup \{x\}) \times G \subseteq V$ and $F \times (G \cup \{y\}) \subseteq V$. By condition (*) it follows that $(F \cup \{x\}) \times (G \cup \{y\}) \subseteq U$. Thus $(x, y) \in U$ and $\langle \mathcal{F}, \mathcal{G} \rangle$ is weakly concentrated. \square

Note that a Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-uniform space (X, \mathcal{U}) is said to converge to x provided that the filter $\langle \mathcal{U}^{-1}(x), \mathcal{U}(x) \rangle$ is coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$.

Lemma 3.1.2. *(Compare [28, Proposition 1]) Let (X, \mathcal{U}) be a T_0 -quasi-uniform space. Then the limit of a weakly concentrated Cauchy filter pair on (X, \mathcal{U}) is unique if it exists.*

Proof. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a weakly concentrated Cauchy filter pair on (X, \mathcal{U}) such that $\langle \mathcal{F}, \mathcal{G} \rangle$ converges to x as well as to y in X . Then for each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that for each $a, b \in X$ with $V(a) \in \mathcal{G}$ and $V^{-1}(b) \in \mathcal{F}$ we have that $(a, b) \in U$. By our assumption on convergence we conclude that $(x, y) \in U$ as well as $(y, x) \in U$ whenever $U \in \mathcal{U}$. Thus $x = y$, since (X, \mathcal{U}) is a T_0 -quasi-uniform space. \square

Lemma 3.1.3. *[3, Lemma 7.9a] Let (X, \mathcal{U}) be a quasi-uniform space. If $\langle \mathcal{F}, \mathcal{G} \rangle$ is a linked filter pair on (X, \mathcal{U}) , then $\langle \mathcal{F}, \mathcal{G} \rangle$ is weakly concentrated.*

Proof. Let $U \in \mathcal{U}$. There is $V \in \mathcal{U}$ such that $V^2 \subseteq U$. Let $x, y \in X$. Suppose that $V(x) \in \mathcal{F}$ and $V^{-1}(y) \in \mathcal{G}$ such that $V^{-1}(y) \cap V(x) \neq \emptyset$. Then $(x, y) \in V^2 \subseteq U$. So we conclude that $\langle \mathcal{F}, \mathcal{G} \rangle$ is weakly concentrated. \square

Our next result is related to Remark 1.0.1 part (b).

Lemma 3.1.4. *[3, Lemma 7.9b] Let (X, \mathcal{U}) be a quasi-uniform space. Then the envelope $\langle \mathcal{U}^{-1}(\mathcal{F}), \mathcal{U}(\mathcal{G}) \rangle$ of a linked Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-uniform space (X, \mathcal{U}) is a minimal Cauchy filter pair that is weakly concentrated.*

Proof. Let $\langle \mathcal{F}', \mathcal{G}' \rangle$ be a Cauchy filter pair on X such that $\mathcal{F}' \subseteq \mathcal{U}^{-1}(\mathcal{F})$ and $\mathcal{G}' \subseteq \mathcal{U}(\mathcal{G})$. Let $U \in \mathcal{U}$ and $G \in \mathcal{G}$. There exist $F' \in \mathcal{F}'$ and $G' \in \mathcal{G}'$ such that $F' \times G' \subseteq U$, since $\langle \mathcal{F}', \mathcal{G}' \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . By our assumption there is $x \in G \cap G' \cap F'$. Then $G' \subseteq U(x) \subseteq U(G) \in \mathcal{U}(\mathcal{G})$. Therefore $\mathcal{U}(\mathcal{G}) \subseteq \mathcal{G}'$. Similarly we can show that $\mathcal{U}^{-1}(\mathcal{F}) \subseteq \mathcal{F}'$. Hence we get that $\langle \mathcal{U}^{-1}(\mathcal{F}), \mathcal{U}(\mathcal{G}) \rangle$ is a minimal Cauchy filter pair on (X, \mathcal{U}) , which is weakly concentrated by the preceding lemma. \square

Note that a filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ is called *concentrated* if it is weakly concentrated and minimal Cauchy.

Definition 3.1.4. A filter \mathcal{G} on a quasi-uniform space (X, \mathcal{U}) is \mathcal{U} -stable provided that $\bigcap_{G \in \mathcal{G}} U(G) \in \mathcal{G}$ whenever $U \in \mathcal{U}$. Note that a \mathcal{U}^s -Cauchy filter is called doubly stable if it is both \mathcal{U}^{-1} -stable and \mathcal{U} -stable.

A filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-uniform space (X, \mathcal{U}) is called stable provided that \mathcal{G} is \mathcal{U} -stable and \mathcal{F} is \mathcal{U}^{-1} -stable.

A Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-uniform space (X, \mathcal{U}) is called costable if \mathcal{F} is \mathcal{U} -stable and \mathcal{G} is \mathcal{U}^{-1} -stable.

Lemma 3.1.5. [3, Lemma 7.17] Let (X, \mathcal{U}) be a quasi-uniform space. If $\langle \mathcal{F}, \mathcal{G} \rangle$ is a linked Cauchy filter pair on (X, \mathcal{U}) , then $\langle \mathcal{F}, \mathcal{G} \rangle$ is stable.

Proof. Let $U \in \mathcal{U}$. There are $F_U \in \mathcal{F}$ and $G_U \in \mathcal{G}$ such that $F_U \times G_U \subseteq U$. Consider any $G \in \mathcal{G}$. Then there is $x_G \in F_U \cap G$. Thus $G_U \subseteq U(x_G) \subseteq U(G)$. Hence \mathcal{G} is \mathcal{U} -stable. Similarly we show that \mathcal{F} is \mathcal{U}^{-1} -stable. \square

Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a weakly concentrated Cauchy filter pair. There exists a unique weakly concentrated Cauchy filter pair coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$ which can be described as the coarsest one among the Cauchy filter pairs coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$. In the literature, Deák [3, 4] used several ways to describe a minimal weakly concentrated Cauchy filter. In the following we discuss some of the constructions of a weakly concentrated minimal Cauchy filter pair ([4, p.351],[3, Lemma 7.11]) that are useful in our present study. So these studies will generalize the previous Lemma 1.0.4.

Lemma 3.1.6. [6] Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a weakly concentrated Cauchy filter pair on a quasi-uniform space (X, \mathcal{U}) . For each $U \in \mathcal{U}$, set $M_{-1}(U) = \bigcup \{S \in \mathcal{F} : \text{there is } K \in \mathcal{G} \text{ such that } S \times K \subseteq U\}$ and $M_2(U) = \bigcup \{K \in \mathcal{G} : \text{there is } S \in \mathcal{F} \text{ such that } S \times K \subseteq U\}$. Let $\mathcal{F}_{\mathcal{U}^{-1}}$ be the filter generated by the filter base $\{M_{-1}(U) : U \in \mathcal{U}\}$ and let $\mathcal{G}_{\mathcal{U}}$ be the filter generated by the filter base $\{M_2(U) : U \in \mathcal{U}\}$. Then $\langle \mathcal{F}_{\mathcal{U}^{-1}}, \mathcal{G}_{\mathcal{U}} \rangle$ is the unique minimal weakly concentrated Cauchy filter pair coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$.

Proof. Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ be such that $V^3 \subseteq U$. There are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$. Let $x \in M_{-1}(U)$ and $y \in M_2(U)$. Then there is $G \in \mathcal{G}$ such that $\{x\} \times G \subseteq V$ and there is $F \in \mathcal{F}$ such that $F \times \{y\} \subseteq V$. Therefore $\{x\} \times \{y\} \subseteq V^3 \subseteq U$. Thus $M_{-1}(U) \times M_2(U) \subseteq U$. So we have that $\langle \mathcal{F}_{\mathcal{U}^{-1}}, \mathcal{G}_{\mathcal{U}} \rangle$ is a Cauchy filter pair.

Let $\langle \mathcal{F}', \mathcal{G}' \rangle$ be a Cauchy filter pair on X coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$. For $U \in \mathcal{U}$ there are $F' \in \mathcal{F}'$ and $G' \in \mathcal{G}'$ such that $F' \times G' \subseteq U$. Let $S \in \mathcal{F}_{\mathcal{U}^{-1}}$ be such that $M_{-1}(U) \subseteq S$. Then we have that $F' \in \mathcal{F}'$ and $F' \subseteq M_{-1}(U) \subseteq S$. So $\mathcal{F}' \subseteq \mathcal{F}_{\mathcal{U}^{-1}}$. Since $\mathcal{F}_{\mathcal{U}^{-1}}$ is a minimal filter, then $\mathcal{F}' = \mathcal{F}_{\mathcal{U}^{-1}}$. Similarly, we prove that $\mathcal{G}' = \mathcal{G}_{\mathcal{U}}$. Hence we conclude that $\langle \mathcal{F}_{\mathcal{U}^{-1}}, \mathcal{G}_{\mathcal{U}} \rangle$ is the coarsest Cauchy filter pair coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$. \square

Lemma 3.1.7. (Compare Lemma 1.0.4) Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a weakly concentrated Cauchy filter pair on a quasi-uniform space (X, \mathcal{U}) . For each $U \in \mathcal{U}$, set $F_U = \{x \in X : U(x) \in \mathcal{G}\}$ and $G_{U^{-1}} = \{x \in X : U^{-1}(x) \in \mathcal{F}\}$. Let \mathcal{F}_{m_1} be the filter generated by the filter base $\{F_U : U \in \mathcal{U}\}$ and let \mathcal{G}_{m_2} be the filter generated by the filter base $\{G_{U^{-1}} : U \in \mathcal{U}\}$. Then $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ is the unique minimal weakly Cauchy filter pair coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$.

Proof. The proof is similarly as the one in Lemma 1.0.4. \square

Corollary 3.1.1. Let (X, \mathcal{U}) be a quasi-uniform space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a weakly concentrated Cauchy filter pair on (X, \mathcal{U}) . Then $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ and $\langle \mathcal{F}_{\mathcal{U}^{-1}}, \mathcal{G}_{\mathcal{U}} \rangle$ are equal filter pairs; that is, the two pairs of bases used to generate them are equivalent.

Proof. The proof is straightforward. \square

Proposition 3.1.1. *Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on a quasi-uniform space (X, \mathcal{U}) .*

- (a) *Then $\mathcal{F}_{m_1} \subseteq \mathcal{F}$ and $\mathcal{G}_{m_2} \subseteq \mathcal{G}$.*
- (b) *If $\langle \mathcal{F}', \mathcal{G} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) , then $\mathcal{F}_{m_1} \subseteq \mathcal{F}'$. Similarly if $\langle \mathcal{F}, \mathcal{G}' \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) , then $\mathcal{F}_{m_2} \subseteq \mathcal{G}'$.*
- (c) *The filter pair $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ is Cauchy on (X, \mathcal{U}) if and only if $\langle \mathcal{F}, \mathcal{G} \rangle$ is weakly concentrated on (X, \mathcal{U}) .*
- (d) *[6, Lemma 5.2] If \mathcal{G} is \mathcal{U} -stable, then $\langle \mathcal{F}_{m_1}, \mathcal{G} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Similarly if \mathcal{F} is \mathcal{U}^{-1} -stable, then $\langle \mathcal{F}, \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) .*
- (e) *[6, Lemma 5.6] Each stable, minimal Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-uniform space (X, \mathcal{U}) is (weakly) concentrated.*

Proof. (a) Since $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) , we have that $\mathcal{F}_{m_1} \subseteq \mathcal{F}$; indeed $U \in \mathcal{U}, F \in \mathcal{F}, G \in \mathcal{G}$ and $F \times G \subseteq U$ imply that $\mathcal{F}_{m_1} \subseteq F_U$. Analogously $\mathcal{G}_{m_2} \subseteq \mathcal{G}$.

(b) The proof is similar to the one in part (a): Let $U \in \mathcal{U}, F' \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F' \times G \subseteq U$. Then $F' \subseteq F_U$. Thus $\mathcal{F}_{m_1} \subseteq \mathcal{F}'$. The second statement is proved analogously.

(c) The proof is straightforward. It follows from the definition of weak concentration: For each $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ such that $G_V \times F_V \subseteq U$. The latter part of the statement exactly means that $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$ with $x, y \in X$ imply that $(x, y) \in U$.

(d) Let $U \in \mathcal{U}$. Since \mathcal{G} is \mathcal{U} -stable, we know that $\bigcap_{G \in \mathcal{G}} U(G) \in \mathcal{G}$. If $(x, y) \in U(G) \times \bigcap_{G \in \mathcal{G}} U(G)$, then $y \in U^2(x)$, since $U(x) \in \mathcal{G}$, and thus $(x, y) \in U^2$. Therefore $\langle \mathcal{F}_{m_1}, \mathcal{G} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Similarly, \mathcal{U}^{-1} -stability of \mathcal{F} implies that $\langle \mathcal{F}, \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) .

(e) Since $\langle \mathcal{F}, \mathcal{G} \rangle$ is a minimal Cauchy filter pair, by (b) and (d) we have

$\mathcal{F}_{m_1} = \mathcal{F}$ and $\mathcal{G}_{m_2} = \mathcal{G}$. Hence we conclude that $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair. So $\langle \mathcal{F}, \mathcal{G} \rangle$ is weakly concentrated by (c). \square

The next lemma is related to the notion of quasi-proximity and quasi-proximity induced by a quasi-uniformity which is defined in [15, Definition 1.22, Proposition 1.28].

Lemma 3.1.8. *Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on (X, \mathcal{U}) . Then for any $F \in \mathcal{F}, G \in \mathcal{G}$ we have that $F\delta G$, where δ denotes the quasi-proximity induced by \mathcal{U} on X .*

Proof. For any $U \in \mathcal{U}$ there are $F_U \in \mathcal{F}, G_U \in \mathcal{G}$ such that $F_U \times G_U \subseteq U$. Consider any $F \in \mathcal{F}, G \in \mathcal{G}$. Therefore $(F \cap F_U) \times (G \cap G_U) \subseteq U$ where $F \cap F_U \neq \emptyset \neq G \cap G_U$ and $(F \times G) \cap U \neq \emptyset$. Thus $F\delta G$. \square

3.2 Cauchy and fully Cauchy quasi-uniformities

In this section we present the notions of Cauchy, fully Cauchy and locally quiet quasi-uniformities introduced by Deák in [9]. We improve some results related to these classes of quasi-uniformities and write down with more details some of the proofs found in [9]. We shall establish the following two new results: The first one shows that a filter symmetric quasi-uniform space (X, \mathcal{U}) is Cauchy and each Cauchy filter pair on (X, \mathcal{U}) is stable and costable. The second result states that a filter symmetric quasi-uniform space (X, \mathcal{U}) is locally quiet and doubly costable. These results are closely related to Deák's result that a quasi-uniform space is filter symmetric if and only if it is quiet and doubly costable [9, Proposition 5.1]. He had also shown that a locally quiet quasi-uniform space is Cauchy and fully Cauchy.

We start by recalling first the definitions of a Cauchy and a fully Cauchy quasi-uniformity.

Definition 3.2.1. [9, p.318] *A quasi-uniform space (X, \mathcal{U}) is called Cauchy provided that whenever the filter pairs $\langle \mathcal{F}_1, \mathcal{G}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{G}_2 \rangle$ are Cauchy and $\mathcal{F}_1 \vee \mathcal{F}_2$ and $\mathcal{G}_1 \vee \mathcal{G}_2$ are well-defined filters on (X, \mathcal{U}) , then $\langle \mathcal{F}_1 \cap \mathcal{F}_2, \mathcal{G}_1 \cap \mathcal{G}_2 \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) .*

A quasi-uniform space (X, \mathcal{U}) is called fully Cauchy if for any Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, \mathcal{U}) there is a coarsest one among the Cauchy filter pairs coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$.

Lemma 3.2.1. *Each stable Cauchy filter pair on a Cauchy quasi-uniform space is weakly concentrated.*

Proof. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a stable Cauchy filter pair on a Cauchy quasi-uniform space (X, \mathcal{U}) . Then we have that $\langle \mathcal{F}_{m_1}, \mathcal{G} \rangle$ and $\langle \mathcal{F}, \mathcal{G}_{m_2} \rangle$ are Cauchy filter pairs coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$. Hence by Cauchyness of \mathcal{U} , we have that $\langle \mathcal{F}_{m_1} \cap \mathcal{F}, \mathcal{G} \cap \mathcal{G}_{m_2} \rangle = \langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Then it follows from Proposition 3.1.1 that $\langle \mathcal{F}, \mathcal{G} \rangle$ is weakly concentrated. \square

Definition 3.2.2. [9] *A quasi-uniform space (X, \mathcal{U}) is called locally quiet provided that each Cauchy filter pair is weakly concentrated.*

A quasi-uniform space (X, \mathcal{U}) is called costable provided that whenever $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair, then \mathcal{F} is stable.

A quasi-uniform space (X, \mathcal{U}) is called double costable if for each Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$, the filter \mathcal{F} is \mathcal{U} -stable and \mathcal{G} is \mathcal{U}^{-1} -stable, that is, $\langle \mathcal{F}, \mathcal{G} \rangle$ is costable.

Lemma 3.2.2. [9, Proposition 2.1a] *a) Each locally quiet quasi-uniform space (X, \mathcal{U}) is fully Cauchy.*

b) [9, Proposition 2.1b] Each fully Cauchy quasi-uniform space (X, \mathcal{U}) is Cauchy.

Proof. a) Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on a quasi-uniform space (X, \mathcal{U}) . Since (X, \mathcal{U}) is locally quiet, then $\langle \mathcal{F}, \mathcal{G} \rangle$ is weakly concentrated. So it contains a coarsest Cauchy filter pair coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$. Hence (X, \mathcal{U}) is fully Cauchy.

b) Let $\langle \mathcal{F}_1, \mathcal{G}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{G}_2 \rangle$ be Cauchy filter pairs on (X, \mathcal{U}) such that $\mathcal{F}_1 \vee \mathcal{G}_1$ and $\mathcal{F}_2 \vee \mathcal{G}_2$ are well-defined filters. Since (X, \mathcal{U}) is fully Cauchy, $\langle \mathcal{F}_1 \vee \mathcal{G}_1, \mathcal{F}_2 \vee \mathcal{G}_2 \rangle$ contains a coarsest Cauchy filter pair $\langle \mathcal{H}_1, \mathcal{H}_2 \rangle$. Then we have that $\langle \mathcal{H}_1, \mathcal{H}_2 \rangle$ is coarser than the filter pair $\langle \mathcal{F}_1 \cap \mathcal{G}_1, \mathcal{F}_2 \cap \mathcal{G}_2 \rangle$. Thus $\langle \mathcal{F}_1 \cap \mathcal{G}_1, \mathcal{F}_2 \cap \mathcal{G}_2 \rangle$ is a Cauchy filter pair. Hence we conclude that (X, \mathcal{U}) is Cauchy. \square

Corollary 3.2.1. [9, Proposition 2.1] *Each locally quiet quasi-uniform space (X, \mathcal{U}) is Cauchy.*

Proof. The proof follows from the previous lemma. \square

We next recall the notion of a filter symmetric quasi-uniform space (X, \mathcal{U}) .

Definition 3.2.3. *Let (X, \mathcal{U}) be a quasi-uniform space. A Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, \mathcal{U}) will be called symmetric provided that $\langle \mathcal{G}, \mathcal{F} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) , too. A quasi-uniform space (X, \mathcal{U}) is called filter symmetric [9, Definition 5.1] if each Cauchy filter pair on (X, \mathcal{U}) is symmetric.*

Lemma 3.2.3. *Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on a quasi-uniform space (X, \mathcal{U}) that is weakly concentrated such that \mathcal{F} is a \mathcal{U} -stable ultrafilter. Then \mathcal{F} is a \mathcal{U}^s -Cauchy filter.*

Proof. Let $U \in \mathcal{U}$. By weak concentration of $\langle \mathcal{F}, \mathcal{G} \rangle$, there is $V \in \mathcal{U}$ such that $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$ imply that $(x, y) \in U$. Find $F_0 \in \mathcal{F}$ and $G_0 \in \mathcal{G}$ such that $F_0 \times G_0 \subseteq V$. Since \mathcal{F} is \mathcal{U} -stable, we have that $F_0 \cap \bigcap_{F \in \mathcal{F}} V(F) \in \mathcal{F}$. Let $a, b \in F_0 \cap \bigcap_{F \in \mathcal{F}} V(F) \in \mathcal{F}$. Then $V(a) \in \mathcal{G}$ and $V^{-1}(b) \cap F \neq \emptyset$ whenever $F \in \mathcal{F}$. Therefore $V^{-1}(b) \in \mathcal{F}$, since \mathcal{F} is an ultrafilter on X . Then $(a, b) \in U$. Hence we conclude that \mathcal{F} is a \mathcal{U}^s -Cauchy filter on X . \square

Note that the following two propositions are closely related to Deák's results in [9, Proposition 5.1].

Proposition 3.2.1. *A quasi-uniform space (X, \mathcal{U}) is filter symmetric if and only if it is Cauchy and each Cauchy filter pair is stable and costable.*

Proof. Let (X, \mathcal{U}) be a filter symmetric quasi-uniform space. Suppose that $\langle \mathcal{F}, \mathcal{G} \rangle$ is Cauchy filter pair on (X, \mathcal{U}) . Then $\langle \mathcal{G}, \mathcal{F} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Consequently \mathcal{F} and \mathcal{G} are \mathcal{U}^s -Cauchy filters on X . (It follows that $\mathcal{F} \cap \mathcal{G}$ is a \mathcal{U}^s -Cauchy filter .) In particular, the Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ is stable and costable.

Let $\langle \mathcal{F}_1, \mathcal{G}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{G}_2 \rangle$ be Cauchy filter pairs on (X, \mathcal{U}) such that $\mathcal{F}_1 \vee \mathcal{F}_2$ and $\mathcal{G}_1 \vee \mathcal{G}_2$ are well-defined filters. Then $\langle \mathcal{F}_1 \vee \mathcal{F}_2, \mathcal{G}_1 \vee \mathcal{G}_2 \rangle$ is a Cauchy

filter pair on (X, \mathcal{U}) . By filter symmetry, $\langle \mathcal{G}_1 \vee \mathcal{G}_2, \mathcal{F}_1 \vee \mathcal{F}_2 \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . By Lemma 3.1.8, $H\delta E$ whenever $H \in \mathcal{G}_1 \vee \mathcal{G}_2$ and $E \in \mathcal{F}_1 \vee \mathcal{F}_2$.

Let $U \in \mathcal{U}$ and let $V \in \mathcal{U}$ be such that $V^3 \subseteq U$. Find $F_1 \in \mathcal{F}_1$ and $H \in \mathcal{G}_1 \vee \mathcal{G}_2$ such that $F_1 \times H \subseteq V$. Similarly, find $E \in \mathcal{F}_1 \vee \mathcal{F}_2$ and $G_2 \in \mathcal{G}_2$ such that $E \times G_2 \subseteq V$. Since $H\delta E$, we conclude that $F_1 \times G_2 \subseteq V^3 \subseteq U$.

It follows that $\langle \mathcal{F}_1, \mathcal{G}_2 \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Hence $\langle \mathcal{F}_1 \cap \mathcal{F}_2, \mathcal{G}_1 \cap \mathcal{G}_2 \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Therefore (X, \mathcal{U}) is a Cauchy quasi-uniform space. We conclude that filter symmetry implies Cauchyness in a quasi-uniform space.

For the converse of the assertion in the proposition we argue as follows : Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on (X, \mathcal{U}) . Since $\langle \mathcal{F}, \mathcal{G} \rangle$ is stable by assumption, $\langle \mathcal{F}, \mathcal{G} \rangle$ is weakly concentrated according to Lemma 3.2.1, because (X, \mathcal{U}) is Cauchy. Hence it contains the coarsest Cauchy filter pair $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ according to Lemma 1.0.4 and Lemma 3.1.7. Suppose that \mathcal{H} is an ultrafilter finer than \mathcal{F}_{m_1} on X . Then by assumption the Cauchy filter pair $\langle \mathcal{H}, \mathcal{G}_{m_2} \rangle$ is stable and costable. So \mathcal{H} is doubly stable and since it is an ultrafilter, \mathcal{H} is a \mathcal{U}^s -Cauchy filter on X .

Therefore $\langle \mathcal{H}, \mathcal{H} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Since $\langle \mathcal{H}, \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) , trivially $\langle \mathcal{H}, \mathcal{H} \cap \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Then $\langle \mathcal{H}, \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair on X that contains the Cauchy filter pairs $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ and $\langle \mathcal{H}, \mathcal{H} \cap \mathcal{G}_{m_2} \rangle$. By Cauchyness of \mathcal{U} , $\langle \mathcal{F}_{m_1}, \mathcal{H} \cap \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair. Since $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ is minimal, we conclude that $\mathcal{F}_{m_1} \subseteq \mathcal{H} \cap \mathcal{G}_{m_2}$ and therefore $\mathcal{G}_{m_2} \subseteq \mathcal{H}$. Since \mathcal{F}_{m_1} , as any filter can be written as the intersection of ultrafilters \mathcal{H} on X , we obtain that $\mathcal{F}_{m_1} \subseteq \mathcal{G}_{m_2}$. Similarly $\mathcal{G}_{m_2} \subseteq \mathcal{F}_{m_1}$, by applying the same argument to \mathcal{U}^{-1} . Therefore we get that $\mathcal{F}_{m_1} = \mathcal{G}_{m_2}$. Then $\langle \mathcal{G}, \mathcal{F} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) , since it is finer than the constant Cauchy filter pair $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$. Hence the quasi-uniformity \mathcal{U} is filter symmetric. \square

Proposition 3.2.2. *A quasi-uniform space (X, \mathcal{U}) is filter symmetric if and only if it is locally quiet and doubly costable.*

Proof. Let (X, \mathcal{U}) be a filter symmetric quasi-uniform space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair. Then $\langle \mathcal{G}, \mathcal{F} \rangle$ is also a Cauchy filter pair on (X, \mathcal{U}) . We have that \mathcal{F} and \mathcal{G} are \mathcal{U}^s -Cauchy filters. Let $U \in \mathcal{U}$. There is $V \in \mathcal{U}$ such that $V^3 \in U$. There are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$.

Consider $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$ whenever $x, y \in X$. There are $f \in F$ and $g \in G$ such that $f \in V^{-1}(y), g \in V(x)$, which implies that $(x, y) \in V^3 \subseteq U$. Then the Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ is weakly concentrated. Thus we get that (X, \mathcal{U}) is locally quiet. Since the filters \mathcal{F}, \mathcal{G} are \mathcal{U}^s -Cauchy, they are \mathcal{U}^s -stable, that is, \mathcal{F} is \mathcal{U} -stable and \mathcal{G} is \mathcal{U}^{-1} -stable. Hence we get that the quasi-uniform space (X, \mathcal{U}) is doubly costable.

To prove the converse, suppose that (X, \mathcal{U}) is a locally quiet and doubly costable quasi-uniform space. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on (X, \mathcal{U}) . Since (X, \mathcal{U}) is locally quiet, then the Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ is weakly concentrated. Therefore it contains the unique minimal Cauchy filter pair $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$. Let \mathcal{H} be an ultrafilter finer than \mathcal{F}_{m_1} on X . By Lemma 3.2.3 \mathcal{H} is doubly stable. Since it is an ultrafilter, for each $U \in \mathcal{U}$, $\bigcap_{H \in \mathcal{H}} U(H) \cap \bigcap_{H \in \mathcal{H}} U^{-1}(H) \in \mathcal{H}$. For any $x \in \bigcap_{H \in \mathcal{H}} U(H) \cap \bigcap_{H \in \mathcal{H}} U^{-1}(H)$ we have that $U(x) \cap H \neq \emptyset$. So $U(x) \in \mathcal{H}$. Similarly we prove that $U^{-1}(x) \in \mathcal{H}$. Then \mathcal{H} is a \mathcal{U}^s -Cauchy filter on X .

Then $\langle \mathcal{H}, \mathcal{H} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Since $\langle \mathcal{H}, \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) , trivially $\langle \mathcal{H}, \mathcal{H} \cap \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Then $\langle \mathcal{H}, \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair on X that contains the Cauchy filter pairs $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$ and $\langle \mathcal{H}, \mathcal{H} \cap \mathcal{G}_{m_2} \rangle$. By local quietness of \mathcal{U} , $\langle \mathcal{F}_{m_1}, \mathcal{H} \cap \mathcal{G}_{m_2} \rangle$ is a Cauchy filter pair. It follows from the last part of the proof in the preceding proposition that $\langle \mathcal{G}, \mathcal{F} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) , since it is finer than the constant Cauchy filter pair $\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle$. Hence we conclude that (X, \mathcal{U}) is filter symmetric quasi-uniform space. \square

Definition 3.2.4. *A quasi-uniform space (X, \mathcal{U}) is called proximally symmetric or Smyth symmetric (compare [9, p.325]) provided that the quasi-proximity δ induced by \mathcal{U} on X is a proximity.*

Proposition 3.2.3. *[9, Proposition 5.2] Each proximally symmetric quasi-uniform space (X, \mathcal{U}) is filter symmetric.*

Proof. According to Proposition 3.2.1 it will suffice to show that \mathcal{U} is Cauchy and that each Cauchy filter pair on (X, \mathcal{U}) is stable and costable. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be any Cauchy filter pair on (X, \mathcal{U}) . Let $F \in \mathcal{F}$ and $G \in \mathcal{G}$. Then $F\delta G$ by Lemma 3.1.8. Since \mathcal{U} is proximally symmetric, it follows that $G\delta F$.

We next note that if $\langle \mathcal{F}', \mathcal{G}' \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) and \mathcal{F}' is an ultrafilter on X , then \mathcal{F}' is a \mathcal{U}^s -Cauchy filter on X : Indeed by the end of the preceding paragraph $\mathcal{U}(\mathcal{G}') \vee \mathcal{F}'$ is a well-defined filter. It obviously is an \mathcal{U}^s -Cauchy filter on X , which is finer than \mathcal{F}' . Since \mathcal{F}' is an ultrafilter, $\mathcal{U}(\mathcal{G}') \vee \mathcal{F}'$ agrees with \mathcal{F}' , and our claim is established.

We still need to show that proximal symmetry implies Cauchyness. This is similar to the result that symmetry implies Cauchyness: Let $\langle \mathcal{F}_1, \mathcal{G}_1 \rangle$ and $\langle \mathcal{F}_2, \mathcal{G}_2 \rangle$ be Cauchy filter pairs on (X, \mathcal{U}) such that $\mathcal{F}_1 \vee \mathcal{F}_2$ and $\mathcal{G}_1 \vee \mathcal{G}_2$ are well-defined filters. Then $\langle \mathcal{F}_1 \vee \mathcal{F}_2, \mathcal{G}_1 \vee \mathcal{G}_2 \rangle$ is a Cauchy filter pair. Thus by Lemma 3.1.8, if $F \in \mathcal{F}_1 \vee \mathcal{F}_2$ and $G \in \mathcal{G}_1 \vee \mathcal{G}_2$, then $F \delta G$. Let $U \in \mathcal{U}$ and let $V \in \mathcal{U}$ be such that $V^3 \subseteq U$. Find $F_1 \in \mathcal{F}_1$ and $G \in \mathcal{G}_1 \vee \mathcal{G}_2$ such that $F_1 \times G \subseteq V$. Similarly, find $F \in \mathcal{F}_1 \vee \mathcal{F}_2$ and $G_2 \in \mathcal{G}_2$ such that $F \times G_2 \subseteq V$. Since $G \delta F$ by proximal symmetry, we conclude that $F_1 \times G_2 \subseteq V^3 \subseteq U$. It follows that $\langle \mathcal{F}_1, \mathcal{G}_2 \rangle$ is a Cauchy filter pair. Analogously, $\langle \mathcal{F}_2, \mathcal{G}_1 \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Hence $\langle \mathcal{F}_1 \cap \mathcal{F}_2, \mathcal{G}_1 \cap \mathcal{G}_2 \rangle$ is a Cauchy filter pair. Thus (X, \mathcal{U}) is a Cauchy quasi-uniform space. \square

Proposition 3.2.4. [9, Theorem 1.3] *Each totally bounded Cauchy quasi-uniform space (X, \mathcal{U}) is symmetric, that is, \mathcal{U} is uniformity.*

Proof. Let \mathcal{U} be a totally bounded Cauchy quasi-uniformity on a set X and let δ be the quasi-proximity induced by \mathcal{U} on X . Since trivially any totally bounded proximally symmetric quasi-uniformity is a uniformity, we only have to show that δ is a proximity. We suppose that δ is not a proximity and choose $A, B \subseteq X$ such that $A \delta B$, but $B \bar{\delta} A$. For the proof we recall that \mathcal{U} is generated by the subbasic entourages $(X \times X) \setminus (E \times F)$ where $E, F \subseteq X$ and $E \bar{\delta} F$ [15, Theorem 1.33].

In the following we use some ideas developed in [28]. Since $A \delta B$, we have that $A \neq \emptyset$ and $B \neq \emptyset$. Let $\mathcal{M} = \{ \langle \mathcal{F}_1, \mathcal{F}_2 \rangle : \mathcal{F}_1, \mathcal{F}_2 \text{ are filters on } X \text{ such that } A \in \mathcal{F}_1, B \in \mathcal{F}_2 \text{ and such that } C \in \mathcal{F}_1, D \in \mathcal{F}_2 \text{ imply } C \bar{\delta} D \}$ partially ordered by the usual coarser relation between filter pairs, that is, $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle \leq \langle \mathcal{G}_1, \mathcal{G}_2 \rangle$ if $\mathcal{F}_1 \subseteq \mathcal{G}_1$ and $\mathcal{F}_2 \subseteq \mathcal{G}_2$. Since the union of a chain of filters is a filter, we see that we can apply Zorn's Lemma. We conclude that \mathcal{M} has a maximal element $\langle \mathcal{H}_1, \mathcal{H}_2 \rangle$. We next note that \mathcal{H}_1 and \mathcal{H}_2 are ultrafilters on X : Suppose that \mathcal{H}_1 is not an ultrafilter on X . Then there is $E \subseteq X$ such that $E \notin \mathcal{H}_1$ and $X \setminus E \notin \mathcal{H}_1$. Let \mathcal{K}_1 be the filter generated by $\mathcal{H}_1 \cup \{E\}$ on X and let \mathcal{K}_2 be the filter generated by $\mathcal{H}_2 \cup \{X \setminus E\}$ on X . Since $\langle \mathcal{H}_1, \mathcal{H}_2 \rangle$ is maximal

in (\mathcal{M}, \leq) and $\mathcal{K}_1, \mathcal{K}_2$ are strictly finer than \mathcal{H}_1 , there are $H_1, H'_2 \in \mathcal{H}_1$ and $H_2, H'_2 \in \mathcal{H}_2$ such that $H_1 \cap E\bar{\delta}H_2$ and $H'_1 \cap (X \setminus E)\bar{\delta}H'_2$. It follows that $H_1 \cap H'_1 \cap E\bar{\delta}H_2 \cap H'_2$ and $H_1 \cap H'_1 \cap (X \setminus E)\bar{\delta}H_2 \cap H'_2$. Thus $H_1 \cap H'_1 \bar{\delta} H_2 \cap H'_2$ —a contradiction. Hence \mathcal{H}_1 is an ultrafilter on X . Similarly, one proves that \mathcal{H}_2 is an ultrafilter on X .

Next we show that $\langle \mathcal{H}_1, \mathcal{H}_2 \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Assume the contrary. Then it is obvious that by total boundedness of \mathcal{U} there are $C, D \subseteq X$ such that $C\bar{\delta}D$, but $(H_1 \times H_2) \cap (C \times D) \neq \emptyset$ whenever $H_1 \in \mathcal{H}_1$ and $H_2 \in \mathcal{H}_2$. Hence $C \in \mathcal{H}_1$ and $D \in \mathcal{H}_2$, because \mathcal{H}_1 and \mathcal{H}_2 are ultrafilters—contradicting the fact that $\langle \mathcal{H}_1, \mathcal{H}_2 \rangle \in \mathcal{M}$. Thus $\langle \mathcal{H}_1, \mathcal{H}_2 \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) .

Then $\langle \mathcal{H}_1, \mathcal{H}_1 \rangle$ and $\langle \mathcal{H}_2, \mathcal{H}_2 \rangle$ are Cauchy filter pairs, since by total boundedness each ultrafilter is \mathcal{U}^s -Cauchy [15, Proposition 3.14]. Consequently obviously $\langle \mathcal{H}_1, \mathcal{H}_1 \cap \mathcal{H}_2 \rangle$ and $\langle \mathcal{H}_1 \cap \mathcal{H}_2, \mathcal{H}_2 \rangle$ are Cauchy filter pairs on (X, \mathcal{U}) . Therefore by Cauchyness of \mathcal{U} , $\langle \mathcal{H}_1 \cap \mathcal{H}_2, \mathcal{H}_1 \cap \mathcal{H}_2 \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Hence $\langle \mathcal{H}_2, \mathcal{H}_1 \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) , which implies by Lemma 3.1.8 that $B\bar{\delta}A$, since $B \in \mathcal{H}_2$ and $A \in \mathcal{H}_1$. We have reached a contradiction and we conclude that δ is a proximity. \square

3.3 D -completeness and C -completeness in a quasi-uniform space

In this last section of this chapter we investigate the notion of D -completeness and C -completeness. In [17], P. Fletcher and W. Hunsaker have considered three notions of completeness: D -completeness, strong D -completeness and pair completeness. They showed that in a uniformly regular quasi-uniform space D -completeness implies pair completeness. Note that the notion of pair completeness coincides with bicompleteness in a quasi-uniform space. We shall show that any D -complete quiet quasi-uniform space is C -complete.

A filter \mathcal{G} on a quasi-uniform space (X, \mathcal{U}) is said to be a D -Cauchy filter if there is a filter \mathcal{F} on X such that $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair. We call \mathcal{F} a cofilter of \mathcal{G} .

Definition 3.3.1. A quasi-uniform space (X, \mathcal{U}) is called *C-complete* provided that each Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ converges.

A quasi-uniform space (X, \mathcal{U}) is *D-complete* (compare [17, p. 150]) if each *D-Cauchy filter* converges, that is, each second filter of the Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ converges with respect to $\tau(\mathcal{U})$.

Definition 3.3.2. ([4, 17]) A quasi-uniform space (X, \mathcal{U}) is *uniformly regular* if for any $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $cl_{\tau(\mathcal{U})}V(x) \subseteq U(x)$ whenever $x \in X$.

The next lemma and proposition are due to P. Fletcher and W. Hunsaker in [17]. They may be used to prove some of the results in this section.

Lemma 3.3.1. ([17, Lemma 2.1]) Let (X, \mathcal{U}) be a quasi-uniform space. If (X, \mathcal{U}) is uniformly regular and *D-complete*, then (X, \mathcal{U}^{-1}) is *D-complete*.

Proposition 3.3.1. ([17, Proposition 2.2]) Let (X, \mathcal{U}) be a quasi-uniform space. If (X, \mathcal{U}) is uniformly regular and *D-complete*, then it is *pair complete*.

Note that any quiet quasi-uniform space is uniformly regular (see [17, Proposition 1.2]).

Proposition 3.3.2. Let (X, \mathcal{U}) be a quiet quasi-uniform space. If (X, \mathcal{U}) is *D-complete* then (X, \mathcal{U}) is *C-complete*.

Proof. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on (X, \mathcal{U}) . Since (X, \mathcal{U}) is quiet and *D-complete*, there is $x \in X$ such that \mathcal{G} converges to x with respect to $\tau(\mathcal{U})$. Let $U \in \mathcal{U}$. There is $V \in \mathcal{U}$ such that for each $y \in X$ $(cl_{\tau(\mathcal{U})}V(y)) \subseteq U(y)$ because (X, \mathcal{U}) is uniformly regular.

There are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq V$. Let $y \in F$. Then $\overline{G} \subseteq \overline{V(y)} \subseteq U(y)$ and $\overline{G} \subseteq U(y)$. Since \mathcal{G} converges to x with respect to $\tau(\mathcal{U})$, then $x \in \overline{G}$ and $F \times \{x\} \subseteq U$. Then $F \subseteq U^{-1}(x)$. We get that \mathcal{F} converges to x with respect to $\tau(\mathcal{U}^{-1})$. Hence we conclude that (X, \mathcal{U}) is *C-complete*. \square

A quasi-uniform space is said to be *strongly D-complete* [17, p. 150] provided that for each Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$, the filter \mathcal{F} has a cluster point.

Note that every uniformly regular strongly D -complete quasi-uniform space is quiet [17, Proposition 3.6].

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Chapter 4

Doitchinov's quietness for quasi-uniform spaces

In [12] Doitchinov introduced the notion of quiet quasi-uniform spaces and developed an interesting completion theory. In this chapter we try to extend the Doitchinov completion theory of quiet quasi-uniform spaces to general T_0 -quasi-uniform spaces. We have been motivated by our successful approach to generalize Doitchinov's completion theory of balanced T_0 -quasi-pseudometric spaces that we have presented in [28]. We point out that our results for quasi-uniform spaces are negative and conclude that Doitchinov's completion theory for quiet T_0 -quasi-uniform spaces cannot be fully generalized to arbitrary T_0 -quasi-uniform spaces.

We first investigate the notion of a set of uniformly weakly concentrated Cauchy filter pairs on a quasi-uniform space. We then describe an example of a quasi-pseudometrizable quasi-uniform space due to Deák, having a family of weakly concentrated filter pairs that are minimal Cauchy, but is not uniformly weakly concentrated (see [3, Example 7.15]).

We secondly define and construct the B -completion of a T_0 -family of quasi-pseudometrics \mathcal{D} on a set X . In this way we obtain a completion theory for a family of quasi-pseudometrics that can be applied to the study of T_0 -quasi-uniform spaces. We recall that for singleton family \mathcal{D} the B -completion depends on the quasi-pseudometric that was chosen to generate the underlying quasi-uniformity. We also introduce a new definition of a balanced (resp. B -isometric) map for a family of quasi-pseudometrics and investigate the ex-

tension theorem of balanced maps onto B -complete spaces. This leads us to characterize the B -completion of a family of quasi-pseudometrics. We finally present an example of the B -completion of a family of quasi-pseudometrics to illustrate our investigations.

4.1 Quietness in quasi-uniform spaces

In this section we present the notion of a quiet quasi-uniform space introduced by Doitchinov in [12]. We investigate the possibility of extending Doitchinov's theory of quiet quasi-uniform spaces to general quasi-uniform spaces. We also study the notion of a family of uniformly weakly concentrated Cauchy filter pairs on a quasi-uniform space (X, \mathcal{U}) . We investigate the connections between the notion of a family of uniformly weakly concentrated Cauchy filter pairs and the notion of C -balanced Cauchy filter pairs (see for instance Proposition 1.0.5).

Let \tilde{X} be the set of Cauchy filter pairs of a quasi-uniform space (X, \mathcal{U}) . We use the following technique to construct a quasi-uniformity on the set \tilde{X} . For any $U \in \mathcal{U}$ we let $\tilde{U} = \{(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) \in \tilde{X} \times \tilde{X} : \text{There are } F \in \mathcal{F} \text{ and } G' \in \mathcal{G}' \text{ such that } F \times G' \subseteq U\}$. We have that each \tilde{U} contains the diagonal of \tilde{X} , since \tilde{X} consists of Cauchy filter pairs on (X, \mathcal{U}) . However in general the filter $\tilde{\mathcal{U}}$ on $\tilde{X} \times \tilde{X}$ generated by the base $\{\tilde{U} : U \in \mathcal{U}\}$ will not be a quasi-uniformity.

Unfortunately, in general the entourage V used in the definition of a weakly concentrated Cauchy filter pair (see Definition 3.1.3) depends on the Cauchy filter pair so that we cannot prove that $\tilde{\mathcal{U}}$ is a quasi-uniformity on the collection of all weakly concentrated Cauchy filter pairs (see Proposition 4.1.1).

Note that if Ψ is a uniformly weakly concentrated family of Cauchy filter pairs on a quasi-uniform space (X, \mathcal{U}) , then $\Psi_m = \{\langle \mathcal{F}_{m_1}, \mathcal{G}_{m_2} \rangle : \langle \mathcal{F}, \mathcal{G} \rangle \in \Psi\}$ is uniformly weakly concentrated, too. We also observe that on a given quasi-uniform space (X, \mathcal{U}) , the union of finitely many collections of uniformly weakly concentrated Cauchy filter pairs yields another uniformly weakly concentrated collection of Cauchy filter pairs on (X, \mathcal{U}) . It immediately follows from the definitions that for any quasi-uniform space (X, \mathcal{U}) , the collection

Ψ_L of all linked Cauchy filter pairs is uniformly weakly concentrated Cauchy filter pair (see [3, Lemma 7.16]). In particular, for any quasi-uniform space (X, \mathcal{U}) the collection $\{\langle \mathcal{U}^{-1}(x), \mathcal{U}(x) \rangle : x \in X\}$ is uniformly weakly concentrated.

Example 4.1.1. For a quasi-uniform space (X, \mathcal{U}) possessing a base \mathcal{B} consisting of transitive entourages, we may want to work with those Cauchy filter pairs on X that for any $U \in \mathcal{B}$ satisfy the condition of weak concentration with respect to $V = U$ in order to obtain a collection of Cauchy filter pairs that is indeed uniformly weakly concentrated on (X, \mathcal{U}) . Note that this collection of Cauchy filter pairs contains all linked Cauchy filter pairs on (X, \mathcal{U}) .

Example 4.1.2. Given a base \mathcal{B} of a quasi-uniformity \mathcal{U} on a set X with a function $M : \mathcal{B} \rightarrow \mathcal{B}$ such that $M(B)^2 \subseteq B$ whenever $B \in \mathcal{B}$, we can consider all Cauchy filter pairs on (X, \mathcal{U}) such that on these filter pairs $M(B)$ is quiet for B . Evidently in this way we obtain a uniformly weakly concentrated family of Cauchy filter pairs on (X, \mathcal{U}) which contains all linked Cauchy filter pairs on (X, \mathcal{U}) . Note that the family depends on \mathcal{B} and the chosen M .

Proposition 4.1.1. (Compare [3, Theorem 8.13]) Let Ψ be a collection of Cauchy filter pairs of a quasi-uniform space (X, \mathcal{U}) containing all neighborhood filter pairs $\langle \mathcal{U}^{-1}(x), \mathcal{U}(x) \rangle$ where $x \in X$.

(a) Then Ψ is uniformly weakly concentrated if and only if $\tilde{\mathcal{U}}|(\Psi \times \Psi)$ is a quasi-uniformity of Ψ . (In the following, we shall often write $\tilde{\mathcal{U}}$ instead of $\tilde{\mathcal{U}}|(\Psi \times \Psi)$ to denote the restriction of $\tilde{\mathcal{U}}$ to $(\Psi \times \Psi)$.)

(b) If Ψ is a uniformly weakly concentrated family, then the map $\beta_X : (X, \mathcal{U}) \rightarrow (\Psi, \tilde{\mathcal{U}})$ defined by $x \mapsto \langle \mathcal{U}^{-1}(x), \mathcal{U}(x) \rangle$ yields a quasi-uniform embedding for the T_0 -quasi-uniform space (X, \mathcal{U}) .

Proof. (a) Suppose that Ψ is a collection of Cauchy filter pairs on (X, \mathcal{U}) such that $\tilde{\mathcal{U}}|(\Psi \times \Psi)$ is a quasi-uniformity. Let $U \in \mathcal{U}$. There is $V \in \mathcal{U}$ such that $\tilde{V}^2 \subseteq \tilde{U}$. Choose $W \in \mathcal{U}$ such that $W^2 \subseteq V$. Consider $\langle \mathcal{F}, \mathcal{G} \rangle \in \Psi$. Let $x, y \in X$ be such that $W(x) \in \mathcal{G}$ and $W^{-1}(y) \in \mathcal{F}$. Then there is $G \in \mathcal{G}$ such that $\{x\} \times G \subseteq W$. Thus $W^{-1}(x) \times G \subseteq V$ and therefore

$(\langle \mathcal{U}^{-1}(x), \mathcal{U}(x) \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) \in \tilde{V}$. Similarly $(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{U}^{-1}(y), \mathcal{U}(y) \rangle) \in \tilde{V}$. Then by assumption we get $(\langle \mathcal{U}^{-1}(x), \mathcal{U}(x) \rangle, \langle \mathcal{U}^{-1}(y), \mathcal{U}(y) \rangle) \in \tilde{U}$ and therefore we have that $(x, y) \in U$. We conclude that Ψ is uniformly weakly concentrated.

We note that for any uniformly weakly concentrated collection Ψ of Cauchy filter pairs of a quasi-uniform space (X, \mathcal{U}) the set of all relations \tilde{U} with U defines a base of a quasi-uniformity $\tilde{\mathcal{U}} \mid (\Psi \times \Psi)$: Let $U \in \mathcal{U}$. There is $V \in \mathcal{U}$ such that for any $\langle \mathcal{F}, \mathcal{G} \rangle \in \Psi$, $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$ with $x, y \in X$ we have $(x, y) \in U$. We show that $(\tilde{V})^2 \subseteq \tilde{U}$: Let $(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) \in \tilde{V}$ and $(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) \in \tilde{V}$. There are $F \in \mathcal{F}, G' \in \mathcal{G}'$ such that $F \times G' \subseteq V$ and there are $F' \in \mathcal{F}', G'' \in \mathcal{G}''$ such that $F' \times G'' \subseteq V$. Thus $f \in F$ and $g'' \in G''$ imply that $V(f) \in \mathcal{G}'$ and $V^{-1}(g'') \in \mathcal{F}'$. Therefore $F \times G'' \subseteq U$ by assumption and we conclude that $(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) \in \tilde{U}$. Hence $\tilde{\mathcal{U}} \mid (\Psi \times \Psi)$ is a quasi-uniformity.

(b) Let $U \in \mathcal{U}$ be $\tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})$ -open. Then $(x, y) \in U$ if and only if

$$(\langle \mathcal{U}^{-1}(x), \mathcal{U}(x) \rangle, \langle \mathcal{U}^{-1}(y), \mathcal{U}(y) \rangle) \in \tilde{U}.$$

Since $\{U \in \mathcal{U} : U \text{ is a } \tau(\mathcal{U}^{-1}) \times \tau(\mathcal{U})\text{-open}\}$ is a base for a quasi-uniformity \mathcal{U} (see [15, Corollary 1.17]). Hence we conclude that β_X is a quasi-uniform embedding map. \square

Proposition 4.1.2. [10, Proposition 5.1] *Each filter symmetric quasi-uniform space (X, \mathcal{U}) is quiet.*

Proof. Let $U \in \mathcal{U}$ and $V \in \mathcal{U}$ be such that $V^3 \subseteq U$. Suppose that $\langle \mathcal{F}, \mathcal{G} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) and $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$ with $x, y \in X$. By filter symmetry $\langle \mathcal{G}, \mathcal{F} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . According to Lemma 3.1.8, $G\delta F$ whenever $F \in \mathcal{F}$ and $G \in \mathcal{G}$, where δ is the quasi-proximity induced by \mathcal{U} on X . It follows that $V(x)\delta V^{-1}(y)$ and thus $(x, y) \in V^3 \subseteq U$. Hence the set of all Cauchy filter pairs on (X, \mathcal{U}) is uniformly weakly concentrated. \square

In order to single out those Cauchy filter pairs on a quasi-uniform space (X, \mathcal{U}) that are suitable for our construction of the completion, some fixed connection between entourages V and U that appear in the definition of

weakly concentrated is required. As we stated in Chapter 2, we next consider the approach which is motivated by Doitchinov's work dealing with balanced T_0 -quasi-pseudometrics.

As it was defined in the second chapter, we recall that a Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ is C -balanced on a quasi-pseudometric space (X, d) provided that

$$d(x, y) \leq C(\inf_{G \in \mathcal{G}} \Phi_d(x, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, y))$$

whenever $x, y \in X$. A quasi-pseudometric space (X, d) is C -balanced provided that each Cauchy filter pair on (X, d) is C -balanced.

A quasi-pseudometric space (X, d) is called B_C -complete provided that each C -balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) converges.

Observe that if a Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ is C -balanced on a quasi-pseudometric space (X, d) , then any coarser Cauchy filter pair will be C -balanced, too.

Suppose that C_1 and C_2 are two real constants such that $C_1 \geq C_2 \geq 1$. Note that for any quasi-pseudometric space, B_{C_1} -completeness implies B_{C_2} -completeness. Observe that 1-balancedness (see Definition 1.0.4) is exactly the property of balancedness and B_1 -completeness is the notion of B -completeness, as it was presented in Chapters 1 and 2.

The next result leads us to understand the uniqueness of the B -completion of a T_0 -quasi-pseudometric space, which was discussed in [28, Theorem 4] and makes use of the case $C = 1$.

Proposition 4.1.3. *Let (X, d) be a quasi-pseudometric space and let $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ be C -balanced Cauchy filter pairs converging to x and to y . Then*

$$d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) \leq d(x, y) \leq C^2 d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle).$$

Proof. Let $F \in \mathcal{F}$ and $G' \in \mathcal{G}'$. Then we have that $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) \leq \Phi_d(F, x) + d(x, y) + \Phi_d(y, G')$, and hence $d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) \leq d(x, y)$, because $\langle \mathcal{F}, \mathcal{G} \rangle$ converges to x and $\langle \mathcal{F}', \mathcal{G}' \rangle$ to y .

Furthermore by C -balancedness of $\langle \mathcal{F}', \mathcal{G}' \rangle$ we obtain

$$\begin{aligned}
d(x, y) &\leq C \left(\inf_{G' \in \mathcal{G}'} \Phi_d(x, G') + \inf_{F' \in \mathcal{F}'} \Phi_d(F', y) \right) = \\
&C \inf_{G' \in \mathcal{G}'} \Phi_d(x, G') \leq C \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} d(x, g') \leq \\
&C \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} (C \inf_{G \in \mathcal{G}} \Phi_d(x, G) + C \inf_{F \in \mathcal{F}} \Phi_d(F, g')) \leq \\
C^2 \inf_{G' \in \mathcal{G}'} \sup_{g' \in G'} \inf_{F \in \mathcal{F}} \sup_{f \in F} d(f, g') &\leq C^2 \inf_{G' \in \mathcal{G}'} \inf_{F \in \mathcal{F}} \sup_{g' \in G'} \sup_{f \in F} d(f, g') = \\
&C^2 d^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle).
\end{aligned}$$

□

In the following, we consider a family of quasi-pseudometrics on a set X .

If \mathcal{D} is a nonempty family of quasi-pseudometrics on a set X , then we set $\mathcal{U}_{\mathcal{D}}$ equal to the supremum quasi-uniformity of the family of quasi-pseudometric quasi-uniformities $(\mathcal{U}_d)_{d \in \mathcal{D}}$ on X and call \mathcal{D} a *subbasic family of quasi-pseudometrics* for the generated quasi-uniformity $\mathcal{U}_{\mathcal{D}}$. We shall say that a Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on $(X, \mathcal{U}_{\mathcal{D}})$ is \mathcal{D} -balanced if $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced in (X, d) whenever $d \in \mathcal{D}$. For each nonempty finite subset F of \mathcal{D} set $d_F = \max_{d \in F} d$ and $\mathcal{D}_F = \{d_F : \emptyset \neq F \subseteq \mathcal{D} \text{ finite}\}$. Then \mathcal{D}_F is a family of quasi-pseudometrics on X such that $\mathcal{U}_{\mathcal{D}} = \mathcal{U}_{\mathcal{D}_F}$. It is obvious that a filter pair on X is Cauchy and balanced with respect to \mathcal{D} if and only if it is Cauchy and balanced with respect to \mathcal{D}_F .

Proposition 4.1.4. *A collection Ψ of Cauchy filter pairs on a quasi-uniform space (X, \mathcal{U}) is uniformly weakly concentrated if and only if there is a nonempty subbasic family \mathcal{D} of quasi-pseudometrics for (X, \mathcal{U}) such that each Cauchy filter pair of Ψ is 2-balanced in any quasi-pseudometric space (X, d) with $d \in \mathcal{D}$.*

Proof. Suppose first that there is a nonempty family \mathcal{D} of quasi-pseudometrics on X such that $\mathcal{U} = \bigvee_{d \in \mathcal{D}} \mathcal{U}_d$ and each Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ of Ψ is C -balanced (with $C \geq 1$) in each (X, d) with $d \in \mathcal{D}$. (In particular, we could set $C = 2$). For each $d \in \mathcal{D}$ and $\delta > 0$ we let $U_{d, \delta} = \{(x, y) \in X \times X : d(x, y) < \delta\}$.

Consider any $\varepsilon > 0$. Set $\delta = \frac{\varepsilon}{4C}$. Let $\langle \mathcal{F}, \mathcal{G} \rangle \in \Psi$ and let $x, y \in X$ be such that $U_{d,\delta}(x) \in \mathcal{G}$ and $U_{d,\delta}^{-1}(y) \in \mathcal{F}$. Then $d(x, y) \leq C(\Phi_d(x, U_{d,\delta}(x)) + \Phi_d(U_{d,\delta}^{-1}(y), y)) \leq C \cdot 2\delta \leq \varepsilon$. We conclude that Ψ is uniformly weakly concentrated on (X, \mathcal{U}) . In order to establish the converse, suppose that Ψ is a uniformly weakly concentrated family of Cauchy filter pairs on (X, \mathcal{U}) . Let $U \in \mathcal{U}$. Since Ψ is uniformly weakly concentrated, inductively we can construct a sequence $(V_n)_{n \in \omega}$ of entourages of \mathcal{U} such that $V_0 = X \times X$, $V_1 \subseteq U$, $V_{n+1} \subseteq V_n$ whenever $n \in \omega$, and $\langle \mathcal{F}, \mathcal{G} \rangle \in \Psi$, $V_{n+1}(x) \in \mathcal{G}$ and $V_{n+1}^{-1}(y) \in \mathcal{F}$ with $x, y \in X$ imply that $(x, y) \in V_n$. For each $x \in X$, let $\alpha_X(x)$ be the Cauchy filter pair $\langle \underline{x}, \underline{x} \rangle$ on X where \underline{x} is the filter on (X, \mathcal{U}) generated by the filter base $\{\{x\}\}$ (compare for instance [28, Example 1]).

Then by the Quasi-Pseudometrization Lemma [15, Lemma 1.5] there is a quasi-pseudometric $d_U : X \times X \rightarrow [0, 1]$ such that $\{(x, y) \in X \times X : d_U(x, y) < 2^{-(n+1)}\} \subseteq V_{n+1} \subseteq \{(x, y) \in X \times X : d_U(x, y) < 2^{-n}\}$ whenever $n \in \omega$. Consider an arbitrary $\langle \mathcal{F}, \mathcal{G} \rangle \in \Psi$. Suppose that $x, y \in X$, $(d_U)^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{G} \rangle) = \delta$ and $(d_U)^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(y)) = \varepsilon$. We next want to show that

$$d_U(x, y) \leq 8[(d_U)^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{G} \rangle) + (d_U)^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(y))].$$

If either $\delta \geq \frac{1}{4}$ or $\varepsilon \geq \frac{1}{4}$, then we obtain $8(\delta + \varepsilon) \geq 2 \geq d_U(x, y)$, since $d_U \leq 1$. So the inequality is satisfied. Then it suffices to consider the case where we find $n \in \omega$ such that $\max\{\delta, \varepsilon\} < 2^{-(n+2)}$. Since $\max\{\delta, \varepsilon\} < 2^{-(n+2)}$, there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $\{x\} \times G \subseteq V_{n+2}$ and $F \times \{y\} \subseteq V_{n+2}$. Therefore we have that $V_{n+2}(x) \in \mathcal{G}$ and $V_{n+2}^{-1}(y) \in \mathcal{F}$. We deduce that $(x, y) \in V_{n+1}$ by our construction.

Suppose that $\varepsilon = \delta = 0$. From the preceding argument it follows that $(x, y) \in V_n$ whenever $n \in \omega$ and so $d_U(x, y) = 0$. Therefore trivially $d_U(x, y) \leq 8[(d_U)^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{G} \rangle) + (d_U)^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(y))]$ in this case.

Otherwise (that is, $\max\{\delta, \varepsilon\} \neq 0$) we can suppose that $n \in \omega$ is maximal such that $\max\{\delta, \varepsilon\} < 2^{-(n+2)}$. Since then $(x, y) \in V_{n+1}$, as noted above, by the construction of d_U , it follows that $d_U(x, y) < 2^{-(n+3)}$. Since $2^{-(n+3)} \leq \max\{\delta, \varepsilon\}$, then we have

$$d_U(x, y) < 8 \cdot 2^{-(n+3)} \leq 8 \cdot \max\{\delta, \varepsilon\} \leq 8(\delta + \varepsilon) =$$

$$8[(d_U)^+(\alpha_X(x), \mathcal{F}, \mathcal{G})) + (d_U)^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(y))].$$

Then we have shown that each Cauchy filter pair of Ψ is 8-balanced with respect to d_U .

Now set $e_U = (d_U)^{\frac{1}{3}}$. Making use of an obvious inequality we see that e_U is a quasi-pseudometric on X , because $(d_U(x, z))^{\frac{1}{3}} \leq (d_U(x, y) + d_U(y, z))^{\frac{1}{3}} \leq (d_U(x, y))^{\frac{1}{3}} + (d_U(y, z))^{\frac{1}{3}}$ whenever $x, y, z \in X$.

Since $d_U \leq 1$, we have that $d_U \leq e_U$. For each positive ρ we get that for any $x, y \in X$, $d_U(x, y) < \rho$ implies that $e_U(x, y) < \rho^{\frac{1}{3}}$. We conclude that $\mathcal{U}_{d_U} = \mathcal{U}_{e_U}$. Furthermore, for any $x, y \in X$ and $\langle \mathcal{F}, \mathcal{G} \rangle \in \Psi$ we obtain

$$\begin{aligned} e_U(x, y) &= d_U(x, y)^{\frac{1}{3}} \leq 8^{\frac{1}{3}}[(d_U)^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{G} \rangle) + (d_U)^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(y))]^{\frac{1}{3}} \leq \\ &2[(e_U)^+(\alpha_X(x), \langle \mathcal{F}, \mathcal{G} \rangle) + (e_U)^+(\langle \mathcal{F}, \mathcal{G} \rangle, \alpha_X(y))]. \end{aligned}$$

It follows that $\langle \mathcal{F}, \mathcal{G} \rangle$ is 2-balanced with respect to the quasi-pseudometric e_U (with $U \in \mathcal{U}$) on X .

Let us now set $\mathcal{D} = \{e_U : U \in \mathcal{U}\}$. Then \mathcal{D} is a family of quasi-pseudometrics on X such that $\bigvee_{d \in \mathcal{D}} \mathcal{U}_d = \bigvee_{U \in \mathcal{U}} \mathcal{U}_{e_U} = \mathcal{U}$. Thus any Cauchy filter pair of Ψ is 2-balanced with respect to any $d \in \mathcal{D}$. \square

Corollary 4.1.1. *Let \mathcal{D} be a family of quasi-pseudometrics on a set X . For any $C \geq 1$, each family of Cauchy filter pairs on $(X, \mathcal{U}_{\mathcal{D}})$ that is C -balanced on (X, d) whenever $d \in \mathcal{D}$, is uniformly weakly concentrated on $(X, \mathcal{U}_{\mathcal{D}})$.*

Proof. The proof is related to the first part of the preceding proof. \square

Lemma 4.1.1. *Each symmetric Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on a quasi-pseudometric space (X, d) is balanced.*

Proof. Let $a, b \in X$. In order to reach a contradiction, suppose that there are $F \in \mathcal{F}, G \in \mathcal{G}$ and $\varepsilon > 0$ such that $d(a, b) > \Phi_d(a, G) + \Phi_d(F, b) + \varepsilon$. Since by our assumption $\langle \mathcal{G}, \mathcal{F} \rangle$ is a Cauchy filter pair on (X, d) , there are $F' \in \mathcal{F}$ and $G' \in \mathcal{G}$ such that $F' \subseteq F, G' \subseteq G$, and $\Phi_d(G', F) < \varepsilon$. Let $f' \in F'$ and $g' \in G'$. Then $d(a, b) \leq d(a, g') + d(g', f') + d(f', b) \leq \Phi_d(a, G') + \varepsilon + \Phi_d(F', b) \leq \Phi_d(a, G) + \varepsilon + \Phi_d(F, b) < d(a, b)$. We have reached

a contradiction and conclude that $d(a, b) < \inf_{G \in \mathcal{G}} \Phi_d(a, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, b)$. Hence $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced. \square

Corollary 4.1.2. *Let (X, d) be a quasi-pseudometric space such that the quasi-pseudometric quasi-uniformity \mathcal{U}_d is filter symmetric. Then (X, d) is balanced.*

Proof. Since \mathcal{U}_d is filter symmetric, all Cauchy filter pairs on (X, d) are symmetric. Then each Cauchy filter pair is balanced on (X, d) by the preceding lemma. Hence (X, d) is balanced according to [28, Proposition 3].

Remark 4.1.1. *Let (X, \mathcal{U}) be a filter symmetric quasi-uniformity and \mathcal{D} be a nonempty family of quasi-pseudometrics on a set X such that $\mathcal{U} = \mathcal{U}_{\mathcal{D}}$. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be an arbitrary \mathcal{U} -Cauchy filter pair on X . Then $\langle \mathcal{G}, \mathcal{F} \rangle$ is a Cauchy filter pair by filter symmetry of \mathcal{U} . So $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{G}, \mathcal{F} \rangle$ are Cauchy filter pairs on (X, \mathcal{U}_d) whenever $d \in \mathcal{D}$. By Lemma 4.1.1, for each $d \in \mathcal{D}$, $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced on (X, d) . So $\langle \mathcal{F}, \mathcal{G} \rangle$ is \mathcal{D} -balanced. It follows that the set of all Cauchy filter pairs on (X, \mathcal{U}) is uniformly weakly concentrated. Hence in particular, (X, \mathcal{U}) is quiet (see Proposition 4.1.2).*

Proposition 4.1.5. *Let \mathcal{D} be a nonempty family of quasi-pseudometrics on a set X such that $(X, \mathcal{U}_{\mathcal{D}})$ is B -symmetric (that is, for each \mathcal{D} -balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{G}, \mathcal{F} \rangle$ is Cauchy filter pair). Then each \mathcal{D} -balanced minimal Cauchy filter pair is constant.*

Proof. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a \mathcal{D} -balanced minimal Cauchy filter pair on $(X, \mathcal{U}_{\mathcal{D}})$. Then $\langle \mathcal{G}, \mathcal{F} \rangle$ is a Cauchy filter pair on $(X, \mathcal{U}_{\mathcal{D}})$ by the assumption of B -symmetry. It follows that \mathcal{F} and \mathcal{G} are $\mathcal{U}_{\mathcal{D}}^s$ -Cauchy filters so that clearly altogether $\langle \mathcal{F} \cap \mathcal{G}, \mathcal{F} \cap \mathcal{G} \rangle$ is a Cauchy filter pair on $(X, \mathcal{U}_{\mathcal{D}})$. Hence $\mathcal{F} = \mathcal{G}$ by minimality. \square

Corollary 4.1.3. *Let \mathcal{D} be a family of quasi-pseudometrics on a set X such that $(X, \mathcal{U}_{\mathcal{D}})$ is B -symmetric. Then $\tau(\mathcal{U}_{\mathcal{D}})$ is completely regular.*

Proof. For each $x \in X$, $\langle \mathcal{U}_{\mathcal{D}}^{-1}(x), \mathcal{U}_{\mathcal{D}}(x) \rangle$ is a minimal Cauchy filter pair that is obviously \mathcal{D} -balanced on $(X, \mathcal{U}_{\mathcal{D}})$ (compare [28, Example 2]). Hence

by the preceding result $\mathcal{U}_{\mathcal{D}}(x) = \mathcal{U}_{\mathcal{D}}^{-1}(x)$ whenever $x \in X$. Since for any quasi-uniformity \mathcal{U} on X , $\mathcal{U}^s(x)$ is the neighborhood filter at $x \in X$ of a completely regular topology on X , the assertion follows. \square

Since any collection of uniformly weakly concentrated Cauchy filter pairs on a quasi-uniform space remains uniformly weakly concentrated after adding a weakly concentrated Cauchy filter pair to it, only those quasi-uniform spaces in which the set of all weakly concentrated filter pairs that are minimal Cauchy is uniformly weakly concentrated possess a canonical maximal ground set for our completion. We next describe an example of a quasi-pseudometrizable quasi-uniform space due to Deák for which that condition is not satisfied. We shall conclude from this example that Doitchinov's completion theory cannot be extended from a quiet quasi-uniform space to arbitrary T_0 -quasi-uniform spaces.

Example 4.1.3. (see [3, Example 7.15]) Let $X = (\mathbb{R} \setminus \{0\}) \times \mathbb{N}$ and consider \mathcal{U}_d , where the T_0 -quasi-metric d on X is defined as follows:
 $d((s, n), (t, k)) = \min\{1, (t - s)^n\}$ if $n = k$ and $s < 0 < t$. Furthermore set d equal to 0 on the diagonal of X and $d = 1$ otherwise. One readily verifies that d is a T_0 -quasi-metric on X .

For each $n \in \omega$, let \mathcal{F}_n be the filter generated on X by the base $\{]-\varepsilon, 0[\times \{n\} : \varepsilon > 0\}$ and let \mathcal{G}_n be the filter on X generated by the base $\{]0, \varepsilon[\times \{n\} : \varepsilon > 0\}$. Set $\Psi = \{\langle \mathcal{F}_n, \mathcal{G}_n \rangle : n \in \omega\}$. One checks that for each $n \in \omega$, $\langle \mathcal{F}_n, \mathcal{G}_n \rangle$ is a minimal Cauchy filter pair on (X, \mathcal{U}_d) .

Furthermore for each $n \in \omega$, let $\langle \mathcal{F}_n, \mathcal{G}_n \rangle$ be 2^n -balanced: Since $d \leq 1$, it suffices to consider the case that $x, y \in X$ such that $\inf_{G \in \mathcal{G}_n} \Phi_d(x, G) \leq 1$ and $\inf_{F \in \mathcal{F}_n} \Phi_d(F, y) \leq 1$. Then there are $u, v \in \mathbb{R} \setminus \{0\}$ such that $x = (u, n)$ and $y = (v, n)$ with $u \leq 0 < v$. Consequently, $(v - u)^n \leq \sum_{k=0}^n \binom{n}{k} (-u)^k v^{n-k} \leq 2^n (\max\{v, -u\})^n \leq 2^n ((-u)^n + v^n) = 2^n (\inf_{G \in \mathcal{G}_n} \Phi_d(x, G) + \inf_{F \in \mathcal{F}_n} \Phi_d(F, y))$. So $\langle \mathcal{F}_n, \mathcal{G}_n \rangle$ is indeed 2^n -balanced on (X, d) . Let $\varepsilon = \frac{1}{2}$. Consider any $\delta > 0$. Then for any $n \in \mathbb{N}$ with $2^{n-1} \delta > 1$, we have that $V_{d, \delta}(-(\frac{\delta}{2})^{\frac{1}{n}}, n) \in \mathcal{G}_n$ and $V_{d, \delta}^{-1}((\frac{\delta}{2})^{\frac{1}{n}}, n) \in \mathcal{F}_n$, but $d(-(\frac{\delta}{2})^{\frac{1}{n}}, n), ((\frac{\delta}{2})^{\frac{1}{n}}, n) = 1 > \frac{1}{2}$. Hence Ψ is not uniformly weakly concentrated.

Therefore $\tilde{\mathcal{U}}_d$ is not a quasi-uniformity on Ψ according to Proposition 4.1.1,

but for any $n \in \mathbb{N}$ it is certainly a quasi-uniformity restricted to $\Psi_n \times \Psi_n$ where $\Psi_n = \{\langle \mathcal{F}_k, \mathcal{G}_k \rangle : k \in \mathbb{N} \text{ and } k \leq n\}$, because Ψ_n is a (finite) collection of Cauchy filter pairs, which are all 2^n -balanced. So there does not exist a largest ground set for our quasi-uniform completion of the quasi-uniform space (X, \mathcal{U}_d) .

4.2 B -completeness of a T_0 -family of quasi-pseudometrics

In this section we generalize the theory of the B -completion of a quasi-pseudometric space to families of quasi-pseudometrics. We extend the study of the B -completion from one quasi-pseudometric space to a T_0 -family of quasi-pseudometrics by constructing the B -completion of a family of quasi-pseudometrics. Our B -completion depends on the chosen family \mathcal{D} of quasi-pseudometrics. We introduce a new concept of balanced maps for a family of quasi-pseudometrics, and investigate the extension theorem of balanced maps to B -complete spaces and show that each T_0 -family of quasi-pseudometrics has a unique B -completion up to isometry.

In the following we consider the quasi-uniformity $\mathcal{U}_{\mathcal{D}} = \bigvee_{d \in \mathcal{D}} \mathcal{U}_d$ on X . We say that a nonempty family \mathcal{D} of a quasi-pseudometrics on a set X is a T_0 -family provided that $\mathcal{U}_{\mathcal{D}}$ is a T_0 -quasi-uniformity. We first give the definition of B -completeness for a family of quasi-pseudometrics.

Definition 4.2.1. *Let \mathcal{D} be a nonempty family of quasi-pseudometrics on a set X . We say that the space (X, \mathcal{D}) is B -complete provided that each $\mathcal{U}_{\mathcal{D}}$ -Cauchy filter pair that is \mathcal{D} -balanced converges in the quasi-uniform space $(X, \mathcal{U}_{\mathcal{D}})$.*

For better understanding, we introduce the following notations.

For each $d \in \mathcal{D}$, let X_d^+ be the set of all balanced Cauchy filter pairs on (X, d) and let (\widetilde{X}_d^+, d^+) be the B -completion of (X, d) . For each $x \in X$, let $j_d(x) : (X, d) \longrightarrow (\widetilde{X}_d^+, d^+)$ be the map defined by $j_d(x) = \langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle$. Then j_d is an isometry.

Furthermore let $X_{\mathcal{D}}^+ = \bigcap_{d \in \mathcal{D}} X_d^+$ and let $\tilde{X} \subseteq X_{\mathcal{D}}^+$ be the set of all minimal \mathcal{D} -balanced Cauchy filter pairs on X . Let $\mathcal{D}^+ = \{d^+ : d \in \mathcal{D}\}$ be the family of quasi-pseudometrics on $X_{\mathcal{D}}^+$ (resp. \tilde{X}) where d^+ denotes the restriction of d to $X_{\mathcal{D}}^+ \times X_{\mathcal{D}}^+$ (resp. $\tilde{X} \times \tilde{X}$) whenever $d \in \mathcal{D}$. We define the map $j_{\mathcal{D}} : (X, \mathcal{D}) \longrightarrow (\tilde{X}, \mathcal{D}^+)$ by $j_{\mathcal{D}}(x) = \langle \mathcal{U}_{\mathcal{D}}^{-1}(x), \mathcal{U}_{\mathcal{D}}(x) \rangle$ whenever $x \in X$. We then note that for each $d \in \mathcal{D}$ and $x, y \in X$, $d(x, y) = d^+(j_{\mathcal{D}}(x), j_{\mathcal{D}}(y))$.

Lemma 4.2.1. *Let (X, \mathcal{U}) be a quasi-uniform space with a chosen sub-basic family \mathcal{D} of quasi-pseudometrics on X . Let \tilde{X} be the set of all \mathcal{D} -balanced minimal $\mathcal{U}_{\mathcal{D}}$ -Cauchy filter pairs on X . Then $(\tilde{X}, \mathcal{U}_{\mathcal{D}^+})$ is a T_0 -quasi-uniform space, where $\mathcal{U}_{\mathcal{D}^+}$ denotes the restriction of the quasi-uniformity $\mathcal{U}_{\mathcal{D}^+}$ from $X_{\mathcal{D}}^+ \times X_{\mathcal{D}}^+$ to $\tilde{X} \times \tilde{X}$.*

Proof. Let $U \in \mathcal{U}_{\mathcal{D}}$ and let $\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle$ be minimal (\mathcal{D} -balanced) Cauchy filter pairs on $(X, \mathcal{U}_{\mathcal{D}})$ such that $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$ belong to $\bigcap \mathcal{U}_{\mathcal{D}^+} \cap \bigcap (\mathcal{U}_{\mathcal{D}^+})^{-1}$. There are $F \in \mathcal{F}, G \in \mathcal{G}, F' \in \mathcal{F}', G' \in \mathcal{G}'$ such that $F \times G \subseteq U, F' \times G' \subseteq U, F \times G' \subseteq U, F' \times G \subseteq U$. Thus $(F \cup F') \times (G' \cup G) \subseteq U$. Therefore $\langle \mathcal{F} \cap \mathcal{F}', \mathcal{G} \cap \mathcal{G}' \rangle$ is a minimal Cauchy filter pair coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$ and $\langle \mathcal{F}', \mathcal{G}' \rangle$. We have that $\langle \mathcal{F}, \mathcal{G} \rangle = \langle \mathcal{F} \cap \mathcal{F}', \mathcal{G} \cap \mathcal{G}' \rangle = \langle \mathcal{F}', \mathcal{G}' \rangle$. Thus $(\tilde{X}, \mathcal{U}_{\mathcal{D}^+})$ is a T_0 -quasi-uniform space. \square

Our results in the following lemma and corollary will be useful in the proof of B -completeness of the family \mathcal{D}^+ on $X_{\mathcal{D}}^+$.

Lemma 4.2.2. *Let \mathcal{D} be a nonempty T_0 -family of quasi-pseudometrics on a set X . Let $x \in X, \varepsilon > 0, d \in \mathcal{D}$ and $\langle \mathcal{G}, \mathcal{H} \rangle$ be a Cauchy filter pair on a quasi-uniform space $(X, \mathcal{U}_{\mathcal{D}})$ such that $d^+(j_{\mathcal{D}}(x), \langle \mathcal{G}, \mathcal{H} \rangle) > \varepsilon$. Then for any $W \in \mathcal{U}_{\mathcal{D}}$ there exists $h \in X$ such that $(\langle \mathcal{G}, \mathcal{H} \rangle, j_{\mathcal{D}}(h)) \in \tilde{W}$ and $d(x, h) > \varepsilon$.*

Proof. Let $P \in \mathcal{U}_{\mathcal{D}}$ such that $P^2 \subseteq W$. There are $G \in \mathcal{G}$ and $H \in \mathcal{H}$ such that $G \times H \subseteq P$. Thus $G \times P(H) \subseteq W$. Let $r > 0$ be such that $r + \varepsilon < d^+(j_{\mathcal{D}}(x), \langle \mathcal{G}, \mathcal{H} \rangle)$. If $\{x\} \times H \subseteq V_{d, \varepsilon}$, then $V_{d, \varepsilon}^{-1}(x) \times H \subseteq V_{d, r + \varepsilon}$ by the triangle inequality—a contradiction to our assumption on d^+ . So $\{x\} \times H \not\subseteq V_{d, \varepsilon}$. Thus $d(x, h) > \varepsilon$ for some $h \in H$. Furthermore $(\langle \mathcal{G}, \mathcal{H} \rangle, j_{\mathcal{D}}(h)) \in \tilde{W}$, since $P(h) \in \mathcal{U}_{\mathcal{D}}(h)$. \square

Corollary 4.2.1. *Let \mathcal{D} be a nonempty family of quasi-pseudometrics on a set X . Let $y \in X, \varepsilon > 0, d \in \mathcal{D}$ and $\langle \mathcal{G}, \mathcal{H} \rangle$ be a Cauchy filter pair on a*

quasi-uniform space $(X, \mathcal{U}_{\mathcal{D}})$ such that $d^+(\langle \mathcal{G}, \mathcal{H} \rangle, j_{\mathcal{D}}(y)) > \varepsilon$. Then for any $W \in \mathcal{U}_{\mathcal{D}}$ there exists $g \in X$ such that $(j_{\mathcal{D}}(g), \langle \mathcal{G}, \mathcal{H} \rangle) \in \widetilde{W}$ and $d(g, y) > \varepsilon$.

Proof. The proof is conjugate to the preceding one. \square

We next prove that the family of quasi-pseudometrics \mathcal{D}^+ is B -complete on $X_{\mathcal{D}}^+$.

Theorem 4.2.1. *Let (X, \mathcal{U}) be a quasi-uniform space equipped with a subbasic family \mathcal{D} of quasi-pseudometrics on X . Then \mathcal{D}^+ is B -complete on $X_{\mathcal{D}}^+$.*

Proof. It is readily verified that $X_{\mathcal{D}_f}^+ = X_{\mathcal{D}}^+$, that is, a filter pair on X is Cauchy and balanced with respect to \mathcal{D} if and only if it is Cauchy and balanced with respect to \mathcal{D}_f . We shall assume in the following that $\mathcal{D} = \mathcal{D}_f$. For any $V \in \mathcal{U}_{\mathcal{D}}$, there are $d \in \mathcal{D}$ and $\varepsilon > 0$ such that $V_{d, \varepsilon} \subseteq V$. Hence if we choose an arbitrary basic $V \in \mathcal{U}$, we can assume without loss of generality that there are $d \in \mathcal{D}$ and $\varepsilon > 0$ such that $V = V_{d, \varepsilon}$. Let us recall that for any $d \in \mathcal{D}$ and any $\varepsilon > 0$ we have that $\widetilde{V}_{d, \varepsilon} \subseteq V_{d^+, 2\varepsilon} \subseteq \widetilde{V}_{d, 3\varepsilon}$ on $X_{\mathcal{D}}^+ \times X_{\mathcal{D}}^+$.

Let $\langle \Xi, \Upsilon \rangle$ be a \mathcal{D}^+ -balanced Cauchy filter pair on $X_{\mathcal{D}}^+$. Then $\langle \Xi, \Upsilon \rangle$ is a weakly concentrated Cauchy filter pair on $(X_{\mathcal{D}}^+, \mathcal{U}_{\mathcal{D}^+})$.

Note that for each $\langle \mathcal{F}, \mathcal{G} \rangle \in X_{\mathcal{D}}^+$, the Cauchy filter pair $\langle j_{\mathcal{D}}(\mathcal{F}), j_{\mathcal{D}}(\mathcal{D}) \rangle$ on $X_{\mathcal{D}}^+$ converges to the point $\langle \mathcal{F}, \mathcal{G} \rangle$ in $(X_{\mathcal{D}}^+, \mathcal{U}_{\mathcal{D}^+})$: Indeed given $U, V \in \mathcal{U}$ such that $U^2 \subseteq V$, there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$ and thus $U^{-1}(F) \times G \subseteq V$. Therefore $j_{\mathcal{D}}(F) \times \{\langle \mathcal{F}, \mathcal{G} \rangle\} \subseteq \widetilde{V}$. So $j_{\mathcal{D}}(\mathcal{F})$ $\tau(\mathcal{U}_{\mathcal{D}^+}^{-1})$ -converges to $\langle \mathcal{F}, \mathcal{G} \rangle$. Similarly we prove that $j_{\mathcal{D}}(\mathcal{G})$ $\tau(\mathcal{U}_{\mathcal{D}^+})$ -converges to $\langle \mathcal{F}, \mathcal{G} \rangle$.

For each $U \in \mathcal{U}$, set $F'_U = \{x \in X : \widetilde{U}(j_{\mathcal{D}}(x)) \in \Upsilon\}$ and $G'_U = \{x \in X : \widetilde{U}^{-1}(j_{\mathcal{D}}(x)) \in \Xi\}$. By the density argument above, it follows that for each $U \in \mathcal{U}$, $F'_U \neq \emptyset \neq G'_U$. Indeed for $W \in \mathcal{U}$ such that $W^4 \subseteq U$ there are $X_W \in \Xi$ and $Y_W \in \Upsilon$ such that $X_W \times Y_W \subseteq \widetilde{W}$. Then $\widetilde{W}^{-1}(X_W) \times \widetilde{W}(Y_W) \subseteq \widetilde{W}^3 \subseteq \widetilde{U}$. Thus $\{x \in X : j_{\mathcal{D}}(x) \in \widetilde{W}^{-1}(X_W)\} \subseteq F'_U$ and $\{x \in X : j_{\mathcal{D}}(x) \in \widetilde{W}(Y_W)\} \subseteq G'_U$. So we have that $F'_U \neq \emptyset \neq G'_U$.

Let \mathcal{F}_{M_1} be the filter on X generated by the filter base $\{F'_U : U \in \mathcal{U}\}$. Furthermore, let \mathcal{G}_{M_2} be the filter on X generated by the filter base $\{G'_U : U \in \mathcal{U}\}$. We next show that $\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle$ is a \mathcal{D} -balanced minimal Cauchy filter pair on (X, \mathcal{U}) . Since $\langle \Xi, \Upsilon \rangle$ is weakly concentrated on $(X_{\mathcal{D}}^+, \mathcal{U}_{\mathcal{D}^+})$, we immediately see that $\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle$ is a Cauchy filter pair on (X, \mathcal{U}) . Let $d \in \mathcal{D}$. We next show that $\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle$ is balanced on (X, d) .

We want to show that for any $x, y \in X$,

$$\inf_{T \in \Upsilon} \Phi_{d^+}(j_{\mathcal{D}}(x), T) \leq \inf_{G \in \mathcal{G}} \Phi_d(x, G)$$

and

$$\inf_{S \in \Upsilon} \Phi_{d^+}(S, j_{\mathcal{D}}(y)) \leq \inf_{F \in \mathcal{F}} \Phi_d(F, y).$$

Then because $d(x, y) = d^+(j_{\mathcal{D}}(x), j_{\mathcal{D}}(y))$ whenever $x, y \in X$, the latter two inequalities imply that $\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle$ is d -balanced, since $\langle \Xi, \Upsilon \rangle$ is d^+ -balanced. In order to reach a contradiction, suppose that there are $\varepsilon > 0$ and $x \in X$ with $\inf_{T \in \Upsilon} \Phi_{d^+}(j_{\mathcal{D}}(x), T) > \varepsilon$, but $\inf_{G \in \mathcal{G}} \Phi_d(x, G) < \varepsilon$.

Let $V := V_{d, \varepsilon}$. Consequently, we have that $\tilde{V}(j_{\mathcal{D}}(x)) \notin \Upsilon$, but $V(x) \in \mathcal{G}$. Then there is $H \in \mathcal{U}$ such that $H \subseteq V$ and $G'_H \subseteq V(x)$. Find $W \in \mathcal{U}$ such that $W^2 \subseteq H$. Choose $X_W \in \Xi$ and $Y_W \in \Upsilon$ such that $X_W \times Y_W \subseteq \tilde{W}$. Since $\Phi_{d^+}(j_{\mathcal{D}}(x), Y_W) > \varepsilon$, we can choose $\eta \in Y_W$ such that $d^+(j_{\mathcal{D}}(x), \eta) > \varepsilon$.

By Lemma 4.2.2 we find $h \in X$ such that $(x, h) \notin V$ and $(\eta, j_{\mathcal{D}}(h)) \in \tilde{W}$. Since we have $\eta \in Y_W$, thus $X_W \subseteq \tilde{W}^{-1}(\eta)$ and so $\tilde{W}^{-1}(\eta) \in \Xi$. Consequently, $\tilde{W}^{-1}(\eta) \subseteq \tilde{W}^2(j_{\mathcal{D}}(h)) \subseteq \tilde{H}^{-1}(j_{\mathcal{D}}(h)) \in \Xi$. Then by definition $h \in G'_H$, but $h \notin V(x)$. We have reached a contradiction and therefore the inequality above for x must hold. Similarly one can show that the inequality for $y \in X$ holds.

We next show that $\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle$ is minimal. Since $\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle$ is balanced on (X, \mathcal{U}) , let $\langle (\mathcal{F}_{M_1})_{m_1}, (\mathcal{G}_{M_1})_{m_1} \rangle$ be a \mathcal{D} -Cauchy filter pair coarser than $\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle$.

Since $(\mathcal{F}_{M_1})_{m_1} \subseteq \mathcal{F}_{M_1}$, it suffices to show that $\mathcal{F}_{M_1} \subseteq (\mathcal{F}_{M_1})_{m_1}$. Let $U = V_{d, \varepsilon}$ for some $d \in \mathcal{D}$ and $\varepsilon > 0$ and let $W = V_{d, \frac{\varepsilon}{3}}(x)$. Suppose that $x \in X$ such that $V_{d, \frac{\varepsilon}{3}}(x) \in \mathcal{G}_{M_2}$. By the inequality just proved, it follows that

$V_{d^+, \frac{2\varepsilon}{3}}(j_{\mathcal{D}}(x)) \in \Upsilon$ and thus $\tilde{U}(j_{\mathcal{D}})(x) \in \Upsilon$. So $x \in F'_U$. We conclude that $\mathcal{F}_{M_1} \subseteq (\mathcal{F}_{M_1})_{m_1}$ as asserted. Hence $\mathcal{F}_{M_1} = (\mathcal{F}_{M_1})_{m_1}$. Similarly we show that $\mathcal{G}_{M_1} = (\mathcal{G}_{M_1})_{m_1}$.

It remains to show that $\langle \Xi, \Upsilon \rangle$ converges to $\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle$ in $X_{\mathcal{D}}^+$.

Let $P \in \mathcal{U}$. Choose $W \in \mathcal{U}$ such that $W^2 \subseteq P$. Since $\langle \Xi, \Upsilon \rangle$ is weakly concentrated on $(X^+, \mathcal{U}_{\mathcal{D}^+})$, there is $U \in \mathcal{U}$ such that $U^2 \subseteq W$ and $\tilde{U}(\xi) \in \Upsilon$ and $\tilde{U}^{-1}(\eta) \in \Xi$ with $\xi, \eta \in X_{\mathcal{D}}^+$ implies that $(\xi, \eta) \in \tilde{W}$.

Choose $X_U \in \Xi$ and $Y_U \in \Upsilon$ such that $X_U \times Y_U \subseteq \tilde{U}$. Observe next that $\tilde{U}(\mathcal{A}, \mathcal{B}) \in \Upsilon$, because $X_U \times Y_U \subseteq \tilde{U}$. Furthermore, let $Z \in \mathcal{U}$ such that $Z^2 \subseteq U$. There are $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A \times B \subseteq Z$. Let $a \in A$ and $y \in G'_U$.

Then $Z^{-1}(A) \times B \subseteq Z^2 \subseteq U$ and therefore $(j_{\mathcal{D}}(a), \langle \mathcal{A}, \mathcal{G} \rangle) \in \tilde{U}$. By definition of G'_U , $\tilde{U}^{-1}(j_{\mathcal{D}}(y)) \in \Xi$. Thus by our assumption on weak concentration of $\langle \Xi, \Upsilon \rangle$, we have $(\langle \mathcal{A}, \mathcal{B} \rangle, j_{\mathcal{D}}(y)) \in \tilde{W}$. Hence $(j_{\mathcal{D}}(a), j_{\mathcal{D}}(y)) \in \tilde{W}^2 \subseteq \tilde{P}$ and $A \times G'_U \subseteq P$. Therefore $(\langle \mathcal{A}, \mathcal{B} \rangle, \mathcal{F}, \mathcal{G}) \in \tilde{P}$, thus $X_U \subseteq \tilde{P}^{-1}(\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle)$.

We conclude that Ξ converges to $\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle$ with respect to $\tau(\mathcal{U}_{\mathcal{D}^+}^{-1})$. Similarly, one can show that Υ converges to $\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle$ with respect to $\tau(\mathcal{U}_{\mathcal{D}^+})$. Hence $(X_{\mathcal{D}}^+, \mathcal{D}^+)$ is B -complete. \square

Theorem 4.2.2. *Let \mathcal{D}^+ be a family of quasi-pseudometrics restricted to $\tilde{X} \times \tilde{X}$. Then the family \mathcal{D}^+ of restrictions is a subbasic family for the restricted quasi-uniformity $\mathcal{U}_{\mathcal{D}^+}$ to $\tilde{X} \times \tilde{X}$, which is B -complete. (Then we shall say that $(\tilde{X}, \mathcal{D}^+)$ is the B -completion of (X, \mathcal{D})).*

Proof. The proof is similar to the one in Theorem 4.2.1, since \mathcal{D}^+ restricted to $X^+ \times X^+$ is B -complete. \square

4.3 Extension of mappings and example

In this section we investigate the extension of a balanced map to B -complete spaces. We first introduce a new concept of a balanced map for a family of quasi-pseudometrics and define the (balanced) isometry map between

quasi-pseudometric spaces which we call *B-isometry* map. At the end of the section, we present an example of the *B-completion* of a family of quasi-pseudometrics which illustrates our investigations.

Definition 4.3.1. *A map $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is called balanced provided that for each $e \in \mathcal{E}$, there is $d \in \mathcal{D}$ such that the map $f : (X, d) \longrightarrow (Y, e)$ is balanced in the sense of [28] that is $f : (X, \mathcal{U}_d) \longrightarrow (Y, \mathcal{U}_e)$ is quasi-uniformly continuous and for any balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, d) , the filter pair $\langle f\mathcal{F}, f\mathcal{G} \rangle$ is balanced on (Y, e) .*

Remark 4.3.1. *Note that for each balanced map $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ the map $f : (X, \mathcal{U}_{\mathcal{D}}) \longrightarrow (Y, \mathcal{U}_{\mathcal{E}})$ is quasi-uniformly continuous; furthermore for each \mathcal{D} -balanced Cauchy filter pair on $(X, \mathcal{U}_{\mathcal{D}})$, $\langle f\mathcal{F}, f\mathcal{G} \rangle$ is \mathcal{E} -balanced Cauchy filter pair on $(Y, \mathcal{U}_{\mathcal{E}})$.*

We next define a *B-isometry* map.

Definition 4.3.2. *Let \mathcal{D} be a nonempty family of quasi-pseudometrics on a set X and let \mathcal{E} be a nonempty family of quasi-pseudometrics on a set Y . Then a map $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ will be called an isometry provided that $\mathcal{D} = \{d_e : e \in \mathcal{E}\}$ where for each $e \in \mathcal{E}$ and $x, y \in X$ we have that $d_e(x, y) = e(f(x), f(y))$. (Note that for singletons \mathcal{D} and \mathcal{E} we obtain the standard definition of an isometry between quasi-pseudometric spaces). We shall say that f is a *B-isometry* provided that moreover for each $e \in \mathcal{E}$, $f : (X, d_e) \longrightarrow (Y, e)$ is balanced. In particular each *B-isometry* is balanced.*

Lemma 4.3.1. *Let \mathcal{D} be a nonempty family of quasi-pseudometrics on a set X and let \mathcal{E} be a nonempty family of quasi-pseudometrics on a set Y . Furthermore let $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ be a balanced map (resp. a *B-isometry*). Suppose that $g : X \longrightarrow Y$ is a map such that for each $e \in \mathcal{E}$ and $x \in X$ we have that $e(f(x), g(x)) = 0 = e(g(x), f(x))$. Then $g : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is balanced (resp. a *B-isometry*), too.*

Proof. Let $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ be balanced and let $e \in \mathcal{E}$. There is $d \in \mathcal{D}$ such that $f : (X, d) \longrightarrow (Y, e)$ is balanced. In the case of f being a *B-isometry*, we choose $d = d_e$ and note that $d_e(x, y) = e(f(x), f(y)) = e(g(x), g(y))$ whenever $x, y \in X$. By our assumption and the triangle inequality, in particular we conclude that $g : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is an isometry

whenever $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is a B -isometry.

Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be a Cauchy filter pair on (X, d) . Consider $x, y \in Y$. By our assumption, we see that

$$e(x, y) \leq \inf_{G \in \mathcal{G}} \Phi_e(x, f(G)) + \inf_{F \in \mathcal{F}} \Phi_e(f(F), y).$$

Thus for any $x, y \in Y$, we have that

$$e(x, y) \leq \inf_{G \in \mathcal{G}} \Phi_e(x, g(G)) + \inf_{F \in \mathcal{F}} \Phi_e(g(F), y),$$

since $e(g(z), f(z)) = 0 = e(f(z), g(z))$ whenever $z \in X$, and therefore by the triangle inequality, $e(f(a), b) = e(g(a), b)$ and $e(b, g(a)) = e(b, f(a))$ whenever $a \in X$ and $b \in Y$. We conclude that $g : (X, d) \longrightarrow (Y, e)$ is balanced thus, $g : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is balanced (resp. a B -isometry). \square

Proposition 4.3.1. (Compare [28, Corollary 9]) *Let \mathcal{D} be a nonempty family of quasi-pseudometrics on a set X . Then the map $i_{\mathcal{D}} : (X, \mathcal{D}) \longrightarrow (\tilde{X}, \mathcal{D}^+)$ defined by $i_{\mathcal{D}}(x) = \langle \mathcal{U}_{\mathcal{D}}^{-1}(x), \mathcal{U}_{\mathcal{D}}(x) \rangle$ whenever $x \in X$ is a B -isometry. It is injective if and only if $\mathcal{U}_{\mathcal{D}}$ is a T_0 -quasi-uniformity.*

Proof. It follows from the preceding lemma that $i_{\mathcal{D}}$ is an isometry, since for any $d \in \mathcal{D}$ we have $d_{d^+} = d$. Let $d \in \mathcal{D}$ and let $\langle \mathcal{F}, \mathcal{G} \rangle \in X_d^+$. Then $d(x, y) \leq \inf_{G \in \mathcal{G}} \Phi_d(x, G) + \inf_{F \in \mathcal{F}} \Phi_d(F, y)$ whenever $x, y \in X$. We know that $j_d : (X, d) \longrightarrow (X_d^+, d^+)$ defined by $x \longmapsto \langle \mathcal{U}_d^{-1}(x), \mathcal{U}_d(x) \rangle$ whenever $x \in X$ is balanced. Then

$$d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) \leq \inf_{G \in \mathcal{G}} \Phi_{d^+}(\langle \mathcal{F}', \mathcal{G}' \rangle, j_d(G)) + \inf_{F \in \mathcal{F}} \Phi_{d^+}(j_d(F), \langle \mathcal{F}'', \mathcal{G}'' \rangle)$$

whenever $\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle \in \tilde{X} \subseteq X_d^+$. Therefore

$$\begin{aligned} d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) &\leq \inf_{G \in \mathcal{G}} \Phi_{d^+}(\langle \mathcal{F}', \mathcal{G}' \rangle, j_d(G)) + \inf_{F \in \mathcal{F}} \Phi_{F \in \mathcal{F}} \Phi_{d^+}(j_d(F), \langle \mathcal{F}'', \mathcal{G}'' \rangle) \\ &= \inf_{G \in \mathcal{G}} \Phi_{d^+}(\langle \mathcal{F}', \mathcal{G}' \rangle, i_{\mathcal{D}}(G)) + \inf_{F \in \mathcal{F}} \Phi_{F \in \mathcal{F}} \Phi_{d^+}(i_{\mathcal{D}}(F), \langle \mathcal{F}'', \mathcal{G}'' \rangle), \end{aligned}$$

since for each $x \in X$,

$$d^+(j_d(x), i_{\mathcal{D}}(x)) = 0 = d^+(i_{\mathcal{D}}(x), j_d(x))$$

so that we have

$$d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, j_d(x)) = d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, i_{\mathcal{D}}(x))$$

and

$$d^+(j_d(x), \langle \mathcal{F}'', \mathcal{G}'' \rangle) = d^+(\langle \mathcal{F}'', \mathcal{G}'' \rangle, i_{\mathcal{D}}(x)).$$

Therefore $i_{\mathcal{D}} : (X, d) \longrightarrow (\tilde{X}, d^+)$ is balanced and, consequently, $i_{\mathcal{D}} : (X, \mathcal{D}) \longrightarrow (\tilde{X}, \mathcal{D}^+)$ is a B -isometry. \square

Lemma 4.3.2. *Let \mathcal{D} be a nonempty family of quasi-pseudometrics on a set X . Then for each $d \in \mathcal{D}$ the map $i : (\tilde{X}, d^+ | (\tilde{X} \times \tilde{X})) \longrightarrow (X_d^+, d^+)$ defined by $i(\langle \mathcal{F}, \mathcal{G} \rangle) = \langle \mathcal{F}, \mathcal{G} \rangle$ whenever $\langle \mathcal{F}, \mathcal{G} \rangle \in \tilde{X}$ is a balanced isometry.*

Proof. For simplicity set $q = d^+ | (\tilde{X} \times \tilde{X})$. Note first that i is an isometry. We now consider a map $i_{\mathcal{D}} : (X, d) \longrightarrow (\tilde{X}, q)$ defined by $i_{\mathcal{D}}(x) = \langle \mathcal{U}_{\mathcal{D}}^{-1}(x), \mathcal{U}_{\mathcal{D}}(x) \rangle$ whenever $x \in X$. In the proof above we proved that the map $i_{\mathcal{D}} : (X, d) \longrightarrow (\tilde{X}, q)$ is balanced.

Let $\langle \mathcal{F}', \mathcal{G}' \rangle$ and $\langle \mathcal{F}'', \mathcal{G}'' \rangle$ be balanced Cauchy filter pairs on (X, d) . Then $\langle \mathcal{F}'_{\mathcal{D}}, \mathcal{G}'_{\mathcal{D}} \rangle = \langle i_{\mathcal{D}}(\mathcal{F}'), i_{\mathcal{D}}(\mathcal{G}') \rangle$ and $\langle \mathcal{F}''_{\mathcal{D}}, \mathcal{G}''_{\mathcal{D}} \rangle = \langle i_{\mathcal{D}}(\mathcal{F}''), i_{\mathcal{D}}(\mathcal{G}'') \rangle$ are balanced Cauchy filter pairs on (\tilde{X}, q) .

Applying our construction to the quasi-pseudometric space (\tilde{X}, q) by equipping the set $(\tilde{X})_q^+$ with the quasi-pseudometric q^+ , we also introduce the map $j_q : (\tilde{X}, q) \longrightarrow ((\tilde{X})_q^+, q^+)$ defined by $j_q(x) = \langle \mathcal{U}_q^{-1}(x), \mathcal{U}_q(x) \rangle$ whenever $x \in \tilde{X}$. Since $i_{\mathcal{D}}$ is an isometry, $d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) = q^+(\langle \mathcal{F}'_{\mathcal{D}}, \mathcal{G}'_{\mathcal{D}} \rangle, \langle \mathcal{F}''_{\mathcal{D}}, \mathcal{G}''_{\mathcal{D}} \rangle)$. Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be any balanced Cauchy filter pair on (\tilde{X}, q) . Then by the triangle inequality,

$$q^+(\langle \mathcal{F}'_{\mathcal{D}}, \mathcal{G}'_{\mathcal{D}} \rangle, \langle \mathcal{F}''_{\mathcal{D}}, \mathcal{G}''_{\mathcal{D}} \rangle) \leq q^+(\langle \mathcal{F}'_{\mathcal{D}}, \mathcal{G}'_{\mathcal{D}} \rangle, \langle \mathcal{F}, \mathcal{G} \rangle) + q^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}''_{\mathcal{D}}, \mathcal{G}''_{\mathcal{D}} \rangle)$$

With the help of [29, Proposition 2] we see that the latter sum is equal to $\inf_{G \in \mathcal{G}} \Phi_{q^+}(\langle \mathcal{F}'_{\mathcal{D}}, \mathcal{G}'_{\mathcal{D}} \rangle, j_q(G)) + \inf_{F \in \mathcal{F}} \Phi_{q^+}(j_q(F), \langle \mathcal{F}''_{\mathcal{D}}, \mathcal{G}''_{\mathcal{D}} \rangle)$, which is equal to $\inf_{G \in \mathcal{G}} \Phi_{d^+}(\langle \mathcal{F}', \mathcal{G}' \rangle, i(G)) + \inf_{F \in \mathcal{F}} \Phi_{d^+}(i(F), \langle \mathcal{F}'', \mathcal{G}'' \rangle)$, as one sees as follows:

We have that

$$\inf_{G \in \mathcal{G}} \Phi_{q^+}(\langle \mathcal{F}'_{\mathcal{D}}, \mathcal{G}'_{\mathcal{D}} \rangle, j_q(G)) =$$

$$\begin{aligned}
& \inf_{G \in \mathcal{G}} \sup_{g \in \mathcal{G}} q^+(\langle \mathcal{F}'_{\mathcal{D}}, \mathcal{G}'_{\mathcal{D}} \rangle, \langle \mathcal{U}_q^{-1}(g), \mathcal{U}_q(g) \rangle) = \\
& \inf_{G \in \mathcal{G}} \sup_{g \in \mathcal{G}} \inf_{F' \in \mathcal{F}'} \inf_{U \in \mathcal{U}_q} \Phi_q(i_{\mathcal{D}}(F'), U(g)) = \\
& \inf_{G \in \mathcal{G}} \sup_{g \in \mathcal{G}} \inf_{F' \in \mathcal{F}'} \Phi_q(i_{\mathcal{D}}(F'), g) = \\
& \inf_{G \in \mathcal{G}} \sup_{g \in \mathcal{G}} \inf_{F' \in \mathcal{F}'} \sup_{f' \in F'} d^+(\langle \mathcal{U}_{\mathcal{D}}^{-1}, \mathcal{U}_{\mathcal{D}}(f') \rangle, g) = \inf_{G \in \mathcal{G}} \sup_{g \in \mathcal{G}} \sup_{f' \in F'} d^+(\alpha_d(f'), g) = \\
& \inf_{G \in \mathcal{G}} \sup_{g \in \mathcal{G}} \inf_{F' \in \mathcal{F}'} \Phi_{d^+}(\alpha_d(F'), g) = \inf_{G \in \mathcal{G}} \sup_{g \in \mathcal{G}} d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, g) = \\
& \inf_{G \in \mathcal{G}} \sup_{g \in \mathcal{G}} d^+(\langle \mathcal{F}', \mathcal{G}' \rangle, i(g)) = \inf_{G \in \mathcal{G}} \Phi_{d^+}(\langle \mathcal{F}', \mathcal{G}' \rangle, i(G)).
\end{aligned}$$

The second equality $\inf_{F \in \mathcal{F}} \Phi_{q^+}(j_q(F), \langle \mathcal{F}''_{\mathcal{D}}, \mathcal{G}''_{\mathcal{D}} \rangle) = \inf_{F \in \mathcal{F}} \Phi_{d^+}(i(F), \langle \mathcal{F}''_{\mathcal{D}}, \mathcal{G}''_{\mathcal{D}} \rangle)$ is established similarly. Then we have that $\langle i(\mathcal{F}), i(\mathcal{G}) \rangle$ is a balanced Cauchy filter pair on (X_d^+, d^+) . \square

Theorem 4.3.1. *Let \mathcal{D} be a nonempty T_0 -family of quasi-pseudometrics on a set X and let \mathcal{E} be a nonempty T_0 -family of quasi-pseudometrics on a set Y . Furthermore let $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ be a balanced map (resp. a B -isometry), and let $i_{\mathcal{D}} : (X, \mathcal{D}) \longrightarrow (\tilde{X}, \mathcal{D}^+)$ and $i_{\mathcal{E}} : (Y, \mathcal{E}) \longrightarrow (\tilde{Y}, \mathcal{E}^+)$ be the embeddings of the corresponding B -completions.*

Define $f_{\mathcal{D}\mathcal{E}}^\dagger : (\tilde{X}, \mathcal{D}^+) \longrightarrow (\tilde{Y}, \mathcal{E}^+)$ as follows: For each $\langle \mathcal{F}, \mathcal{G} \rangle \in \tilde{X}$ set $f_{\mathcal{D}\mathcal{E}}^\dagger(\langle \mathcal{F}, \mathcal{G} \rangle) = (\langle (f\mathcal{F})_{M_1}, (f\mathcal{G})_{M_2} \rangle)$ where $\langle (f\mathcal{F})_{M_1}, (f\mathcal{G})_{M_2} \rangle$ denotes the coarsest Cauchy filter pair on $(Y, \mathcal{U}_{\mathcal{E}})$ coarser than $\langle f\mathcal{F}, f\mathcal{G} \rangle$. Then the map $f_{\mathcal{D}\mathcal{E}}^\dagger : (\tilde{X}, \mathcal{D}^+) \longrightarrow (\tilde{Y}, \mathcal{E}^+)$ is balanced (resp. a B -isometry). It is the unique balanced map having the property that $i_{\mathcal{E}} \circ f = f_{\mathcal{D}\mathcal{E}}^\dagger \circ i_{\mathcal{D}}$.

Proof. Observe that $f_{\mathcal{D}\mathcal{E}}^\dagger$ is well defined by Remark 4.3.1 and Corollary 4.1.1. We next note that $i_{\mathcal{E}} \circ f = f_{\mathcal{D}\mathcal{E}}^\dagger \circ i_{\mathcal{D}}$, since for each $x \in X$ we have that

$$\langle (f(\mathcal{U}_{\mathcal{D}}^{-1}(x)))_{M_1}, f(\mathcal{U}_{\mathcal{D}}(x))_{M_2} \rangle = \langle \mathcal{U}_{\mathcal{E}}^{-1}(f(x)), \mathcal{U}_{\mathcal{E}}(f(x)) \rangle$$

by uniform continuity of f .

Note next that if $f : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is an isometry, then $f_{\mathcal{D}\mathcal{E}}^\dagger : (\tilde{X}, \mathcal{D}^+) \longrightarrow (\tilde{Y}, \mathcal{E}^+)$ is an isometry, too: Indeed if f is an isometry, then $\mathcal{D} = \{d_e : e \in \mathcal{E}\}$.

Therefore $\mathcal{D}^+ = \{(d_e)^+ : e \in \mathcal{E}\} = \{d_{e^+} : e^+ \in \mathcal{E}^+\}$, in the sense that for any $\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle \in \tilde{X}$ we have that

$$d_{e^+}(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle) = e^+(f_{\mathcal{D}\mathcal{E}}^\dagger(\langle \mathcal{F}, \mathcal{G} \rangle), f_{\mathcal{D}\mathcal{E}}^\dagger(\langle \mathcal{F}', \mathcal{G}' \rangle)) = \\ \inf_{F \in \mathcal{F}} \inf_{G' \in \mathcal{G}'} \Phi_e(f(F), f(G')) = \inf_{F \in \mathcal{F}} \inf_{G' \in \mathcal{G}'} \Phi_{d_e}(F, G') = (d_e)^+(\langle \mathcal{F}, \mathcal{G} \rangle, \langle \mathcal{F}', \mathcal{G}' \rangle).$$

Hence we conclude that $f_{\mathcal{D}\mathcal{E}}^\dagger$ is an isometry.

Suppose that f is balanced (resp. a B -isometry). Let $e \in \mathcal{E}$. There is $d \in \mathcal{D}$ such that $f : (X, d) \rightarrow (Y, e)$ is balanced where we choose $d = d_e$ in case that f is a B -isometry. According to the proof of [28, Theorem 3] we obtain a balanced map $f_{d_e}^+ : (X_d^+, d^+) \rightarrow (Y_e^+, e^+)$, where for each $\langle \mathcal{F}, \mathcal{G} \rangle \in X_d^+$ we set $f_{d_e}^+(\langle \mathcal{F}, \mathcal{G} \rangle) = \langle f\mathcal{F}, f\mathcal{G} \rangle$.

Let $\langle \mathcal{H}, \mathcal{K} \rangle$ be a balanced Cauchy filter pair on (\tilde{X}, d^+) . Since the map $i : (\tilde{X}, d^+) \rightarrow (X_d^+, d^+)$ is a balanced isometry, we see that $f_{d_e}^+ \circ i : (\tilde{X}, d^+) \rightarrow (Y_e^+, e^+)$ is balanced (resp. a balanced isometry).

So $\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle \in Y_e^+$ implies that

$$e^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) \leq \\ \inf_{K \in \mathcal{K}} \Phi_{e^+}(\langle \mathcal{F}', \mathcal{G}' \rangle, (f_{d_e}^+ \circ i)(K)) + \inf_{H \in \mathcal{H}} \Phi_{e^+}((f_{d_e}^+ \circ i)(H), \langle \mathcal{F}'', \mathcal{G}'' \rangle).$$

Let now $\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle \in \tilde{Y} \subseteq Y_e^+$. We see by a similar argument as in the proof of Lemma 4.3.1 that therefore

$$e^+(\langle \mathcal{F}', \mathcal{G}' \rangle, \langle \mathcal{F}'', \mathcal{G}'' \rangle) \leq \\ \inf_{K \in \mathcal{K}} \Phi_{e^+}(\langle \mathcal{F}', \mathcal{G}' \rangle, (f_{\mathcal{D}\mathcal{E}}^\dagger \circ i)(K)) + \inf_{H \in \mathcal{H}} \Phi_{e^+}(f_{\mathcal{D}\mathcal{E}}^\dagger \circ i)(H), \langle \mathcal{F}'', \mathcal{G}'' \rangle),$$

since $e^+(f_{\mathcal{D}\mathcal{E}}^\dagger(x), (f_{d_e}^+ \circ i)(x)) = 0 = e^+((f_{d_e}^+ \circ i)(x), f_{\mathcal{D}\mathcal{E}}^\dagger(x))$ whenever $x = \langle \mathcal{F}, \mathcal{G} \rangle \in \tilde{X}$, because the Cauchy filter pair $\langle (f\mathcal{F})_{M_1}, (f\mathcal{G})_{M_2} \rangle$ is coarser than the Cauchy filter pair $\langle f\mathcal{F}, f\mathcal{G} \rangle$.

We conclude that $f_{\mathcal{D}\mathcal{E}}^\dagger : (\tilde{X}, d^+) \rightarrow (\tilde{Y}, e^+)$ is balanced (resp. a balanced isometry) and, thus $f_{\mathcal{D}\mathcal{E}}^\dagger : (\tilde{X}, \mathcal{D}^+) \rightarrow (\tilde{Y}, \mathcal{E}^+)$ is balanced (resp. a B -isometry).

It remains to show that $f_{\mathcal{D}\mathcal{E}}^\dagger$ is unique. Suppose that $g : (\tilde{X}, \mathcal{D}^+) \longrightarrow (\tilde{Y}, \mathcal{E}^+)$ is balanced such that $f_{\mathcal{D}\mathcal{E}}^\dagger \circ i_{\mathcal{D}} = g \circ i_{\mathcal{D}}$. Let $\langle \mathcal{F}, \mathcal{G} \rangle \in \tilde{X}$. It follows from the construction of the B -completion that the Cauchy filter pair $\langle i_{\mathcal{D}}(\mathcal{F}), i_{\mathcal{D}}(\mathcal{G}) \rangle$ converges to $\langle \mathcal{F}, \mathcal{G} \rangle$ in $(\tilde{X}, \mathcal{U}_{\mathcal{D}}^+)$. Thus by uniform continuity of g , $\langle (g \circ i_{\mathcal{D}})(\mathcal{F}), (g \circ i_{\mathcal{D}})(\mathcal{G}) \rangle$ converges to $g(\langle \mathcal{F}, \mathcal{G} \rangle)$ and similarly $\langle (f_{\mathcal{D}\mathcal{E}}^\dagger \circ i_{\mathcal{D}})(\mathcal{F}), (f_{\mathcal{D}\mathcal{E}}^\dagger \circ i_{\mathcal{D}})(\mathcal{G}) \rangle$ converges to $f_{\mathcal{D}\mathcal{E}}^\dagger(\langle \mathcal{F}, \mathcal{G} \rangle)$. But by the aforementioned property of g , $\langle (g \circ i_{\mathcal{D}})(\mathcal{F}), (g \circ i_{\mathcal{D}})(\mathcal{G}) \rangle = \langle (f_{\mathcal{D}\mathcal{E}}^\dagger \circ i_{\mathcal{D}})(\mathcal{F}), (f_{\mathcal{D}\mathcal{E}}^\dagger \circ i_{\mathcal{D}})(\mathcal{G}) \rangle$. Because all maps are balanced, the latter Cauchy filter pair is \mathcal{E}^+ -balanced so that $f_{\mathcal{D}\mathcal{E}}^\dagger(\langle \mathcal{F}, \mathcal{G} \rangle) = g(\langle \mathcal{F}, \mathcal{G} \rangle)$, since the limit of a weakly concentrated Cauchy filter pair is unique in the T_0 -quasi-uniform space $(\tilde{Y}, \mathcal{U}_{\mathcal{E}^+})$. \square

Theorem 4.3.2. *Let \mathcal{E} be a nonempty T_0 -family of quasi-pseudometrics on a set Y such that (Y, \mathcal{E}) is B -complete and $\mathcal{U}_{\mathcal{E}}$ is a T_0 -quasi-uniformity, and let X be a subset of Y . For each $e \in \mathcal{E}$ let d_e be the restriction of e to $X \times X$. Furthermore let $\mathcal{D} = \{d_e : e \in \mathcal{E}\}$. Suppose that the inclusion map $j : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ is a B -isometry and that for each $y \in Y$ there is a \mathcal{D} -balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on (X, \mathcal{D}) such that $\langle j(\mathcal{F}), j(\mathcal{G}) \rangle$ converges to y in $(Y, \mathcal{U}_{\mathcal{E}})$. Then $(Y, \mathcal{U}_{\mathcal{E}})$ can be identified with the B -completion of (X, \mathcal{D}) .*

Proof. Note first that since (Y, \mathcal{E}) is B -complete and $\mathcal{U}_{\mathcal{E}}$ is a T_0 -quasi-uniformity, (Y, \mathcal{E}) can be identified with $(\tilde{Y}, \mathcal{E}^+)$ via the bijective B -isometry $i_{\mathcal{E}} : (Y, \mathcal{E}) \longrightarrow (\tilde{Y}, \mathcal{E}^+)$, because $e(x, y) = e^+(i_{\mathcal{E}}(x), i_{\mathcal{E}}(y))$ whenever $x, y \in Y$ and $e \in \mathcal{E}$.

By Theorem 4.3.1 the map $j : (X, \mathcal{D}) \longrightarrow (Y, \mathcal{E})$ can be extended to a B -isometry $\tilde{j} : (\tilde{X}, \mathcal{D}^+) \longrightarrow (\tilde{Y}, \mathcal{E}^+)$. Similarly as in [28, Theorem 4] one sees that \tilde{j} is injective and surjective: The map \tilde{j} is injective, since $(\tilde{X}, \mathcal{U}_{\mathcal{D}}^+)$ is a T_0 -quasi-uniform space and \tilde{j} is an isometry by the second paragraph of the preceding proof. Given $y \in Y$ there is a \mathcal{D} -balanced Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on $(X, \mathcal{U}_{\mathcal{D}})$ such that the \mathcal{E} -balanced Cauchy filter pair $\langle j\mathcal{F}, j\mathcal{G} \rangle$ converges to y . Thus the minimal Cauchy filter pair $\langle \mathcal{U}_{\mathcal{E}}^{-1}(y), \mathcal{U}_{\mathcal{E}}(y) \rangle$ is coarser than the Cauchy filter pair $\langle j\mathcal{F}, j\mathcal{G} \rangle$. Since the minimal Cauchy filter pair $\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle$ is coarser than $\langle \mathcal{F}, \mathcal{G} \rangle$, we have that the Cauchy filter pair $\langle j(\mathcal{F}_{M_1}), j(\mathcal{G}_{M_2}) \rangle$ is coarser than $\langle j(\mathcal{F}), j(\mathcal{G}) \rangle$, too. Then we have that $\tilde{j}(\langle \mathcal{F}_{M_1}, \mathcal{G}_{M_2} \rangle) = \langle \mathcal{U}_{\mathcal{E}}^{-1}(y), \mathcal{U}_{\mathcal{E}}(y) \rangle$, where the latter filter pair is identified with y according to the first line of this proof. \square

We next give an example of the B -completion of a family of quasi-pseudometrics.

Example 4.3.1. Let I be a nonempty set. Let \mathbb{Q} be the set of the rationals and let \mathbb{R} be the set of the reals. Set $X = \mathbb{Q}^I = \prod_{i \in I} \mathbb{Q}_i$ and $Y = \mathbb{R}^I = \prod_{i \in I} \mathbb{R}_i$. Furthermore, let $s(x, y) = y - x$ if $y \geq x$ and 1 otherwise. That is, s is the usual Sorgenfrey quasi-metric on \mathbb{R} . For each $j \in I$, let $e_j((x_i)_{i \in I}, (y_i)_{i \in I}) = s(x_j, y_j)$ whenever $(x_i)_{i \in I}, (y_i)_{i \in I} \in X$. Let $\mathcal{E} = \{e_j : j \in I\}$ and for $j \in I$, let d_j be the restriction of e_j to $\mathbb{Q}^I \subseteq \mathbb{R}^I$ and let $\mathcal{D} = \{d_j : j \in I\}$. Then (Y, \mathcal{E}) is the B -completion of (X, \mathcal{D}) .

Proof. It is well known that (\mathbb{R}, s) is the B -completion of $(\mathbb{Q}, s/(\mathbb{Q} \times \mathbb{Q}))$. We have that $\mathcal{U}_{\mathcal{E}}$ is a T_0 -quasi-uniformity. For each $i \in I$, $j_i : (\mathbb{Q}_i, s_i/(\mathbb{Q}_i \times \mathbb{Q}_i)) \rightarrow (\mathbb{R}_i, s_i)$ is a balanced embedding for the B -completion of i^{th} copy of the rationals. Let us define the inclusion map $j : \mathbb{Q}^I \rightarrow \mathbb{R}^I$ by $j((q_i)_{i \in I}) = (j_i(q_i)_{i \in I})$ for any $(q_i)_{i \in I} \in \mathbb{Q}^I$. It is obvious that for each $i \in I$, $j : (\mathbb{Q}^I, d_i) \rightarrow (\mathbb{R}^I, e_i)$ is a balanced isometry. We conclude that $j : (\mathbb{Q}^I, \mathcal{D}) \rightarrow (\mathbb{R}^I, \mathcal{E})$ is a B -isometry.

Let $\langle \mathcal{F}, \mathcal{G} \rangle$ be an \mathcal{E} -balanced Cauchy filter pair on \mathbb{R}^I and let $pr_i : \mathbb{R}^I \rightarrow \mathbb{R}_i$ denote the projection map onto the i^{th} -factor of \mathbb{R}^I . Then for each $i \in I$, the Cauchy filter pair $\langle pr_i \mathcal{F}, pr_i \mathcal{G} \rangle$ is balanced in (\mathbb{R}_i, s_i) and since (\mathbb{R}_i, s_i) is B -complete, that latter filter pair converges to some $x_i \in \mathbb{R}_i$. Thus $\langle \mathcal{F}, \mathcal{G} \rangle$ converges to $(x_i)_{i \in I} \in I$ on $(\mathbb{R}^I, \mathcal{U}_{\mathcal{E}})$. Hence $(\mathbb{R}, \mathcal{E})$ is B -complete.

Let $(y_i)_{i \in I} \in \mathbb{R}^I$. Then for each $i \in I$ there is a balanced Cauchy filter pair $\langle \mathcal{F}_i, \mathcal{G}_i \rangle$ on \mathbb{Q}_i that converges to y_i . For each $i \in I$ let $\pi_i : \mathbb{Q}^I \rightarrow \mathbb{Q}_i$ denote the projection map onto the i^{th} -factor of \mathbb{Q}^I . Then one checks that the Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on \mathbb{Q}^I , where \mathcal{F} is generated by the subbase $\cup_{i \in I} \pi_i^{-1} \mathcal{F}_i$ and \mathcal{G} is generated by the subbase $\cup_{i \in I} \pi_i^{-1} \mathcal{G}_i$, is a \mathcal{D} -Cauchy filter pair on $(\mathbb{Q}^I, \mathcal{U}_{\mathcal{D}})$ such that $\langle j(\mathcal{F}), j(\mathcal{G}) \rangle$ converges to $(y_i)_{i \in I}$ on $(\mathbb{R}^I, \mathcal{U}_{\mathcal{E}})$. By Theorem 4.3.2 we therefore conclude that we can identify $(\mathbb{R}^I, \mathcal{E})$ with the B -completion of $(\mathbb{Q}^I, \mathcal{D})$. \square

Chapter 5

Conclusion and open problems

In this last chapter of this thesis, we draw the conclusions of our investigations and underline some open problems found throughout the work that can constitute the topics of further research.

The thesis achieved the task of first establishing some new results from the theory of the B -completion that we developed in [28], and partially extended Doitchinov's completion theory for quiet quasi-uniform spaces. For this second part of the work, our result is negative and so, we conclude that Doitchinov's completion theory for quiet quasi-uniform spaces cannot be fully extended to arbitrary T_0 -quasi-uniform spaces because investigations due to Deák show that the concept of a weakly concentrated Cauchy filter pair seems to be too weak in general quasi-uniform spaces to yield a satisfactory completion. In arbitrary quasi-uniform spaces we worked with a nonempty subbasic family of quasi-pseudometrics and with an appropriate concept of balancedness of Cauchy filter pairs with respect to that family.

Below we give a summary of the work which we studied in each chapter of the thesis and then suggest two important areas of future research which are related to the B -completion theory.

5.1 Summary of the achieved work

In Chapter 1, we presented some preliminaries and overviewed certain well-known definitions from the theory of quasi-pseudometric spaces. We also summarized the construction of the B -completion of a T_0 -quasi-pseudometric space that we had developed in [28] and gave some examples related to the B -completion theory.

In Chapter 2, we presented some new results about the B -completion of a T_0 -quasi-pseudometric space. We showed that B -completeness is a property of quasi-pseudometric spaces but not a quasi-uniform space property. We found an example which shows that two distinct quasi-pseudometrics d and d' on a set X can induce the same quasi-uniformity $\mathcal{U}_d = \mathcal{U}_{d'}$ although d' is B -complete, while d is not. Some examples of balanced maps were presented. We showed in the last section of the chapter that the B -completion of a totally bounded quasi-pseudometric space is totally bounded and, possibly, larger than the bicompletion.

In Chapter 3, we investigated some properties of Cauchy filter pairs on a quasi-uniform space. We showed that a quasi-uniform space is filter symmetric if and only if it is Cauchy and each Cauchy filter pair is stable and costable.

Chapter 4 was the main chapter of the thesis where we showed that Doitchinov's completion theory for quiet quasi-uniform spaces cannot be fully extended to arbitrary quasi-uniform spaces. We constructed a B -completion for a family of quasi-pseudometrics on a set X . We introduced a new definition of a balanced (resp. B -isometric) map for a family of quasi-pseudometrics and investigated the extension theorem of balanced maps to B -complete spaces. This led us to the characterization and the uniqueness of the B -completion of a T_0 -family.

The following problems are related to our investigations in Chapter 4.

Problem 5.1.1. *Is there a weakly concentrated Cauchy filter pair on a quasi-pseudometrizable quasi-uniform space (X, \mathcal{U}) that is not balanced with respect to any quasi-pseudometric d on X such that $\mathcal{U}_d = \mathcal{U}$?*

Problem 5.1.2. *Suppose that the collection Ψ of all weakly concentrated minimal Cauchy filter pairs is uniformly weakly concentrated on a quasi-uniform space (X, \mathcal{U}) . Does there exist a subbasic family \mathcal{D} of quasi-pseudometrics for \mathcal{U} such that all members of Ψ are balanced in each (X, d) with $d \in \mathcal{D}$?*

The theory of the B -completion may have some interesting applications in other structures of mathematics. In the following we point out two areas where the theory of the B -completion can lead to reasonable applications, namely in paratopological groups and fuzzy quasi-pseudometric spaces.

5.2 Two possible areas for future work

5.2.1 B -completeness in paratopological groups

The completion theory of a paratopological group was studied by several authors: In [33], J. Marín and others studied the bicompletion for the left quasi-uniformity of a paratopological group. The results have been used to characterize those paratopological groups for which the bicompletion of the left quasi-uniformity is a group.

In [25], H. Künzi, S. Romaguera and O. Sipacheva showed that the two-sided quasi-uniformity of a regular paratopological group is quiet and for an Abelian group the Doitchinov completion of a regular paratopological group yields a paratopological group.

The theory of B -completeness may have some applications in paratopological groups, too. However, we point out that the necessary completion of a paratopological group may be different from the B -completion since in general the product of two balanced Cauchy filter pairs is not necessarily balanced (see [25, Example 3]). In order to obtain a reasonable completion theory in paratopological groups, we below suggest to introduce a new concept of a Cauchy filter pair called G -balanced Cauchy filter pair which is closely related to our notion of a balanced Cauchy filter pair in a quasi-pseudometric space.

We can try to construct a completion for a T_0 -paratopological group with the same approach as for the B -completion. The completion of a T_0 -paratopological group would be called the G -completion. We point out that the G -completion of a T_0 -paratopological group contains the bicompletion of the induced quasi-pseudometric space. We shall also observe that the B -completion of a T_0 -paratopological group might be larger than its G -completion .

Let us recall the definition of a paratopological group.

A paratopological group [33, p.104] is a pair (X, τ) where X is a group and τ is a topology on X such that the map $\phi : (X \times X, \tau \times \tau) \longrightarrow (X, \tau)$ defined by $\phi(x, y) = xy$ is continuous. If (X, τ) is a paratopological group, then so is (X, τ^{-1}) where $\tau^{-1} = \{A \subseteq X : A^{-1} \in \tau\}$ is called the conjugate of τ .

Let (X, τ) be a paratopological group. For each $U \in \eta(e)$, let $U_L = \{(x, y) : x^{-1}y \in U\}$ and $U_R = \{(x, y) : yx^{-1} \in U\}$. Then $\{U_L : U \in \eta(e)\}$ and $\{U_R : U \in \eta(e)\}$ are bases for the left quasi-uniformity \mathcal{U}_L and the right quasi-uniformity \mathcal{U}_R on X such that $\tau(\mathcal{U}_L) = \tau(\mathcal{U}_R) = \tau$ and $\tau(\mathcal{U}_L^{-1}) = \tau(\mathcal{U}_R^{-1}) = \tau^{-1}$. Note that the quasi-uniformity $\mathcal{U}_B = \mathcal{U}_L \vee \mathcal{U}_R$ is called *the two-sided* quasi-uniformity for (X, τ) .

Let φ be an absolute quasi-valued function on X (see [34, Definition 2]). Then the function d_L defined on $X \times X$ by $d_L(x, y) = \varphi(x^{-1}y)$ (or by $d_R(x, y) = \varphi(yx^{-1})$) is a quasi-pseudometric on X such that (X, τ_d) is a paratopological group (see [34, Proposition 3]). It induces the left(or the right) quasi-uniformity \mathcal{U}_L (or \mathcal{U}_R) on X .

Note that in an abelian group, $d_L(x, y) = d_R(x, y)$ whenever $x, y \in X$.

As an example for the suggested theory, we shall define the notion of a G -balanced Cauchy filter pair on a paratopological group equipped with an absolute quasi-valued function on X .

Let (X, τ) be a paratopological group and let φ be an absolute quasi-valued function on X inducing a quasi-pseudometric d_L . A Cauchy filter pair $\langle \mathcal{F}, \mathcal{G} \rangle$ on X is called G -balanced provided that

$$\varphi(x^{-1}y) \leq \inf_{F \in \mathcal{F}, G \in \mathcal{G}} \sup_{f \in F, g \in G} \varphi(x^{-1}gf^{-1}y) \text{ whenever } x, y \in X.$$

The following results have been established. Here we omit the proof. Given a

paratopological group (X, τ) , if $\langle \mathcal{F}, \mathcal{G} \rangle$ is a G -balanced Cauchy filter pair on (X, τ) , then $\langle \mathcal{F}, \mathcal{G} \rangle$ is balanced (in our sense) with respect to d_L . Let (X, τ) be an abelian paratopological group. Then the product of two G -balanced Cauchy filter pairs is G -balanced, too. So we have the following question:

Problem 5.2.1. *Let (X, τ) be a paratopological group and let φ be an absolute quasi-valued function on X . Let us denote by \tilde{X} the set of all G -balanced Cauchy filter pairs on X . Can we prove that (\tilde{X}, τ) is a paratopological group?*

5.2.2 A B -completion of fuzzy quasi-pseudometric spaces

The notion of a fuzzy set has been introduced by Zadeh in 1965 and the notion of a fuzzy metric space has been discussed by several authors. In [20] V. Gregori and S. Romaguera introduced a concept of fuzzy quasi-metric spaces. Later in 2005, V. Gregori, J.A. Mascarell and A. Sapena [21] developed an interesting completion theory for a class of balanced fuzzy quasi-metric spaces.

However it is shown in [21, Example 2] that there exists a fuzzy metric space which does not admit any fuzzy metric completion. The question of obtaining necessary and sufficient conditions for a fuzzy quasi-metric space to be completable was studied in [22, Theorem 1]. Indeed, every standard fuzzy metric space has an unique fuzzy metric completion.

The study of the B -completion of a fuzzy quasi-metric space and the investigation of those fuzzy quasi-metrics that are B -completable may lead to an interesting theory.

In this discussion an interesting open question is to define a reasonable notion of a balanced Cauchy filter pair on fuzzy quasi-metric spaces that will help to construct a B -completion. A fuzzy quasi-metric space will then be called B -complete if each balanced Cauchy filter pair converges, and it will be called B -completable if it admits a fuzzy quasi-metric B -completion. Then we have the following question:

Problem 5.2.2. *Given a fuzzy quasi-metric space. Find necessary and sufficient conditions for a fuzzy quasi-metric space to be B -completable and construct a B -completion of these fuzzy quasi-metric spaces.*

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