General Relativity and Penrose process

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Abstract

Using the concept of parallel transport of vectors in curved manifolds, the Riemann curvature tensor in terms of Christoffel symbols is obtained. Making use of the Riemann curvature tensor's symmetry properties, the Ricci curvature tensor and Einstein's tensor are derived. Through Einstein's tensor and the Poisson equation for Newtonian gravity, the Einstein field equations are introduced. Upon using Kerr metric (Kerr, 1963) as a solution for Einstein's field equations, extraction of energy from a rotating black hole is proved. This is called the Penrose process (Penrose and Floyd, 1971).

Declaration

I, the undersigned, hereby declare that the work contained in this research project is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.

Signed by candidate

Derhham Abdelfattah Ibrahim, 19 May 2016

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1. Introduction

1.1 Objective and Aims

We provide in detail, an introduction to the general theory of relativity. We then derive Einstein's field equations, which are a set of non-linear partial differential equations with the metric tensor as a solution. The metric tensor of any space is the quantity that describes its geometry. In the context of general relativity, gravity is defined as a curvature of spacetime due to the presence of matter and radiation. In this essay, we are interested in studying spacetime geometry in the vacuum around black holes which may be defined using a special form of the metric tensor. Black hole, by definition, is a region from which no energy or matter can escape. We introduce the Kerr metric or Kerr solution (Kerr, 1963) which is considered as one of the most important solutions to Einstein's field equations. Kerr metric depicts the spacetime in the vacuum around a rotating mass distribution, which fits for rotating black holes. Unlike the Schwarzschild solution(Dunsby, 2012), which describes spacetime around spherically symmetric mass distribution which is non-rotating and uncharged. Kerr solution is the generalisation of the Schwarzschild solution. At zero angular momentum limit, the Kerr metric becomes the Schwarzschild metric. Finally, we use the Kerr metric to discuss the main problem involved here which is the Penrose process (Penrose and Floyd, 1971). Following the Penrose process, we can extract energy from rotating black holes. Analysing the Kerr metric for rotating black hole results in an ergoregion around this black hole. This ergoregion is a region of spacetime influenced by black hole rotation in a way that drags spacetime in the vicinity of the black hole. Consequently, spacetime in that region acquires rotational kinetic energy. It is expected that once the wave enters that region, the recoiling wave becomes more energetic, and this is the so-called Penrose process.

1.2 Outlook

Starting from the discovery of the Kerr solution for Einstein's field equation for rotating black holes in 1963, up to 1973 when Stephen Hawking proved that black holes radiate particles with black body spectrum(Page, 2005), this period is called the golden age of black hole physics. We may mention briefly black hole accomplishments during that period. In 1969, Roger Penrose discussed the Penrose process for the extraction of the spin energy from a Kerr black hole, and this is the problem included in that essay. In 1972, Jacob Bekenstein (Bekenstein, 1973) suggested that black holes have an entropy proportional to their surface area due to information loss effects. In 1972, Stephen Hawking proved that the area of a classical black hole's event horizon is a non-decreasing function of time. In 1973, James Bardeen, Brandon Carter, and Stephen Hawking propose four laws of black hole mechanics in analogy with the laws of thermodynamics (Bardeen et al., 1973). Firstlly, the horizon area is a non decreasing function of time or $\frac{dA}{dt} \geq 0$. Secondly, for a stationary black hole the horizon has constant surface gravity. Thirdly, the change of energy for any black hole is given in terms of change of area, angular momentum, and electric charge as follows: $dE = \frac{\kappa}{8\pi} dA + \Omega dJ + \Phi dQ$ with κ: surface gravity, Ω: angular velocity, J: angular momentum, Φ: electro-static potential, Q:electric charge. Finally, For any black hole surface gravity cannot vanish. $\kappa \neq 0$.

1.3 Conventions

In the context of general relativity we use the metric of signature (3,1) which has the following matrix representation:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.3.1}$$

Also, we contract Riemann tensor to Ricci tensor as follows:

$$R_{\alpha\beta} = R^{\mu}_{\alpha\mu\beta}. \tag{1.3.2}$$

2. Mathematical preliminaries

According to the mentioned definition of gravity as a curvature of spacetime, we then need to describe gravity in tensorial language. This is simply because in differential geometry, the most generalised quantity that describes curvature in pseudo-Riemanian manifold is the Riemann curvature tensor. We define pseudo-Riemanian manifold and Riemann curvature tensors in this chapter. As a main feature of tensors being frame independent quantities, we introduce the concept of covariant derivatives which are the tensors derivative in general coordinate frames. In fact, this covariant derivative is a very useful tool to prove all the laws in general coordinate frames. These laws may then be applied in any different inertial frame, since tensors are frame independent quantities. Frame independent quantities are those quantities that take the same mathematical form in all coordinate systems and under any coordinate transformation. This concept is called general covariance or general invariance. This is also a basic feature of general relativity as a general covariant theory.

2.1 One-forms and Tensors

In order to define tensors properly we first define one-forms. A one-form is a quantity that maps vector into real number, for example when we dot product a vector $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$ with the Del operator $\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$, we end up with real number. So the Del operator with the dot product define one-form. As a very neat generalisation to vectors and one-forms, we define (M/N) tensor as multi linear mapping of M one-forms and N-vectors into real numbers. In the same way, we define (0/N) tensor as a linear mapping of N-vectors into real numbers, as well as (M/0) tensor as a linear mapping of M one-forms to the real numbers, where (M/N) is the tensor rank

2.2 Manifolds

A manifold is defined as a continuous space on which coordinates are assigned to each point in that space. A smooth manifold is a manifold which has a defined metric. In this case, we define a scalar function ϕ such that all orders of the derivatives of that function exist. A Riemannian manifold is a smooth manifold equipped with a symmetric metric tensor at every point that satisfies $g(\vec{V}, \vec{V}) > 0$. Where $\vec{V}$ is a general vector defined in the manifold. If the metric of a Riemannian manifold has an indefinite sign, it is said to be a pseudo-Riemannian manifold.

Flat manifold

It is a manifold that maintains the global definition of parallelism. If two straight lines are parallel they will continue to be parallel through the manifold. Flat manifolds are characterized by zero Riemann curvature tensor.

2.3 Covariant derivatives and Christoffel symbols

In Minkowski space which is the four dimensional spacetime, any vector can be represented as:

$$\vec{V} = V^\alpha \vec{e_\alpha}. \tag{2.3.1}$$

$\vec{e_\alpha}$: Basis vectors, the subscript alpha runs from zero to three defining four dimensions. We define the derivative of that vector as:

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} \vec{e_\alpha}. \tag{2.3.2}$$

In the above case, the basis vectors are considered constant because they have constant magnitude and direction. In general those basis vectors are not constant, we then need to take into consideration that change when we are calculating the derivative. For a general basis, we obtain:

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} \vec{e_\alpha} + V^\alpha \frac{\partial \vec{e_\alpha}}{\partial x^\beta}. \tag{2.3.3}$$

We express the quantity $\frac{\partial \vec{e}_\alpha}{\partial x^\beta}$ as $\Gamma^\mu_{\alpha\beta} \vec{e}_\mu$, where $\Gamma^\mu_{\alpha\beta}$ is the Christoffel symbol, and it geometrically describes the change in the basis vectors as we move from one point to another in the general coordinate system. This change is zero in Minkowski spacetime and Euclidean space and that is why this symbol did not appear. In curved manifolds, we expect non-zero Christoffel symbols because basis vectors will change direction from one point to another along curved manifold. The above derivative now reads:

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} \vec{e}_\alpha + \Gamma^\mu_{\alpha\beta} V^\alpha \vec{e}_\mu. \tag{2.3.4}$$

Changing dummy indices (μ and α) yields:

$$\frac{\partial \vec{V}}{\partial x^\beta} = \frac{\partial V^\alpha}{\partial x^\beta} \vec{e}_\alpha + \Gamma^\alpha_{\mu\beta} V^\mu \vec{e}_\alpha, \tag{2.3.5}$$

$$\frac{\partial \vec{V}}{\partial x^\beta} = \left[\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\beta} V^\mu \right] \vec{e}_\alpha. \tag{2.3.6}$$

The above derivative for the vector is called the covariant derivative, the quantity inside the bracket is called the component of the derivative and denoted by a semi-colon with a Greek subscript index as follows: $V^\alpha_{;\beta} = \left[\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\beta} V^\mu \right]$. Therefore the above equation may be written as:

$$\frac{\partial \vec{V}}{\partial x^\beta} = \left[\frac{\partial V^\alpha}{\partial x^\beta} + \Gamma^\alpha_{\mu\beta} V^\mu \right] \vec{e}_\alpha = V^\alpha_{;\beta} \vec{e_\alpha}. \tag{2.3.7}$$

The most important property about covariant derivatives is that they transform in the same way that tensors transform under general coordinate transformation, which means that covariant derivative of any quantity in any frame is the same in any other frame. We use that property intensively to prove many results firstly in local frames, where calculations are easier, and then promote those results to general frames as a result of invariance under tensor transformation. Covariant derivatives transform according to the following equation:

$$V^{\alpha'}_{;\beta'} = \Lambda^{\alpha'}_\nu \Lambda^\mu_{\beta'} V^\nu_{;\mu}, \tag{2.3.8}$$

where $\Lambda^{\alpha'}_\nu = \frac{\partial x^{\alpha'}}{\partial x^\nu}$, and the same for $\Lambda^\mu_{\beta'} = \frac{\partial x^\mu}{\partial x^{\beta'}}$, where this means that covariant derivatives are tensors.

2.4 Christoffel symbols in terms of metric tensor

We aim to express Christoffel symbols in terms of metric tensor, which is the most convenient way to calculate the Christoffel symbols. To find the Christoffel symbol in terms of the metric tensor, we first prove that the covariant derivative of metric tensor is equal to zero. Then we use that result to find that expression. Consider the covariant derivative for the components of an one-form as follows:

$$V_{\alpha;\beta} = g_{\alpha\mu} V^{\mu}_{;\beta} = (g_{\alpha\mu} V^{\mu})_{;\beta} = g_{\alpha\mu;\beta} V^{\mu} + g_{\alpha\mu} V^{\mu}_{;\beta}. \tag{2.4.1}$$

Consequently we have $g_{\alpha\mu;\beta} V^{\mu} = 0$, Since V^{μ} is not zero, therefore $g_{\alpha\mu;\beta} = 0$. Also since the metric tensor is a (0/2) tensor, therefore its covariant derivative is given as

$$g_{\alpha\beta;\mu} = g_{\alpha\beta,\mu} - g_{\nu\beta} \Gamma^{\nu}_{\alpha\mu} - g_{\alpha\nu} \Gamma^{\nu}_{\beta\mu}, \tag{2.4.2}$$

equating $g_{\alpha\beta;\mu}$ with zero gives:

$$g_{\alpha\beta,\mu} = g_{\nu\beta} \Gamma^{\nu}_{\alpha\mu} + g_{\alpha\nu} \Gamma^{\nu}_{\beta\mu}. \tag{2.4.3}$$

Swapping the indices β and μ in the above equation and rewriting

$$g_{\alpha\mu,\beta} = g_{\nu\mu} \Gamma^{\nu}_{\alpha\beta} + g_{\alpha\nu} \Gamma^{\nu}_{\mu\beta}. \tag{2.4.4}$$

Swapping α and β in the above equation

$$g_{\beta\mu,\alpha} = g_{\nu\mu} \Gamma^{\nu}_{\beta\alpha} + g_{\beta\nu} \Gamma^{\nu}_{\mu\alpha}. \tag{2.4.5}$$

Adding the first two of the above three equations and subtracting the third one results in

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = (g_{\nu\beta} \Gamma^{\nu}_{\alpha\mu} + g_{\alpha\nu} \Gamma^{\nu}_{\beta\mu}) + (g_{\nu\mu} \Gamma^{\nu}_{\alpha\beta} + g_{\alpha\nu} \Gamma^{\nu}_{\mu\beta}) - (g_{\nu\mu} \Gamma^{\nu}_{\beta\alpha} + g_{\beta\nu} \Gamma^{\nu}_{\mu\alpha}) = 2\Gamma^{\nu}_{\beta\mu} g_{\alpha\nu}. \tag{2.4.6}$$

To obtain the above result we used the symmetry property over the downstairs indices $\Gamma^{\nu}_{\alpha\mu} = \Gamma^{\nu}_{\mu\alpha}$. Multiplying by the inverse metric $g^{\alpha\gamma}$ gives

$$\Gamma^{\gamma}_{\beta\mu} = \frac{1}{2} g^{\alpha\gamma} [g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}]. \tag{2.4.7}$$

There are some important remarks as a result of defining the Christoffel symbol in terms of the inverse metric tensor and partial derivatives of the metric tensor. Firstly, Christoffel symbols are not tensors because they do not transform the same way tensors do under general coordinate transformation. Secondly, in any local inertial frame Christoffel symbols vanish. In the context of manifolds, we assign constant metric tensor for local coordinates therefore partial derivatives for the metric are zero. Consequently, the Christoffel symbol vanishes. However, in non-inertial frames Christoffel symbols never vanish even in the case of zero curvature.

2.5 Geodesics and parallel transportation

According to the general theory of relativity, in the absence of all forms of force, and with the gravity no longer force, particles follow a geodesic path. Geodesic is defined as the curve representing shortest distance between two points in a curved manifold. Geodesic is the generalization of the straight line. If the manifold is the Euclidean space, the geodesic is the straight line while for spherical space geodesics are great circles. We now aim to find the equation that represents the paths of particles in spacetime. To do that, we introduce the concept of parallel transportation. Consider a curve with tangent $\vec{u} = \frac{dx}{d\lambda}$, a vector $\vec{V}$ is parallel transported along that curve if and only if

$$\frac{dV^\alpha}{d\lambda} = 0, \tag{2.5.1}$$

here λ is the curve parameter which in our case is the proper time τ. If the curve parameter is the proper time τ, the curve is called a time-like curve. If we have a vector V^α parallel transported along a curve which has a curve parameter λ, we write

$$\frac{dV^\alpha}{d\lambda} = \frac{dV^\alpha}{dx^\beta}\frac{dx^\beta}{d\lambda} = V^\alpha_{,\beta} u^\beta = 0. \tag{2.5.2}$$

In a general coordinate frame, we write the same equation but with the partial derivatives being replaced with the covariant derivatives as follows

$$u^\beta V^\alpha_{;\beta} = 0. \tag{2.5.3}$$

If we define geodesic as the curve which parallel transports the tangent vector along itself, we replace V^α by u^α and write

$$u^\alpha_{,\beta} u^\beta + \Gamma^\alpha_{\gamma\beta} u^\beta u^\gamma = 0, \tag{2.5.4}$$

this can be expressed as

$$\frac{du^\alpha}{dx^\beta}\frac{dx^\beta}{d\tau} + \Gamma^\alpha_{\gamma\beta}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0. \tag{2.5.5}$$

Simplifying the above equation gives:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\gamma\beta}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0. \tag{2.5.6}$$

This is called geodesic equation and it represents the path followed by any particle in curved spacetime in the absence of all forces. We need only one proof to derive Einstein's field equations. This is the proof of the Riemann curvature tensor in terms of the Christoffel symbols. We obtain Riemann curvature tensor by considering the parallel transport of an arbitrary vector along a loop in a curved manifold.

2.6 Riemann curvature tensor

Consider the following closed loop in a curved manifold Fig. 2.1. We want to transport a vector along that loop keeping it parallel to itself as it moves along the loop. Finally, when the vector returns to its original point, we observe that the vector is rotated by 90 degrees relative to its original orientation. This is a result of curvature. In flat manifolds we cannot observe that effect after parallel transport of vectors. Since parallel transportation of vectors keeps its covariant derivative equals to zero. Therefore, for a vector V^μ we obtain

$$\frac{\partial V^\mu}{\partial x^\beta} = -\Gamma^\mu_{\nu\beta} V^\nu. \tag{2.6.1}$$

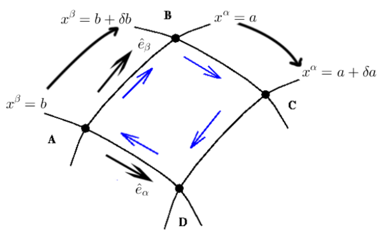

Figure 2.1: parallel transport of vector in a curved manifold.

For the line $x^\alpha = a$ where a is a constant, multiplying by dx^β and integrate from A to B as follows:

$$\int_A^B \frac{\partial V^\mu}{\partial x^\beta} dx^\beta = -\int_A^B \Gamma^\mu_{\nu\beta} V^\nu dx^\beta. \tag{2.6.2}$$

The whole integral can be written as

$$V^\mu(B) = V^\mu(A) - \int_{x^\alpha = a} \Gamma^\mu_{\nu\beta} V^\nu dx^\beta. \tag{2.6.3}$$

Along the line $x^\beta = b + \delta b$, by equating the covariant derivative for the vector V^μ by zero and multiplying by dx^α then integrating we find

$$V^\mu(C) = V^\mu(B) - \int_{x^\beta = b+\delta b} \Gamma^\mu_{\nu\alpha} V^\nu dx^\alpha. \tag{2.6.4}$$

Following the same procedure along the line $x^\alpha = a + \delta a$ we find

$$V^\mu(D) = V^\mu(C) + \int_{x^\alpha = a+\delta a} \Gamma^\mu_{\nu\beta} V^\nu dx^\beta. \tag{2.6.5}$$

Similarly for the line $x^\beta = b$

$$V^\mu(A_{final}) = V^\mu(D) + \int_{x^\beta = b} \Gamma^\mu_{\nu\alpha} V^\nu dx^\alpha, \tag{2.6.6}$$

From the above results, we have two values for vector V^μ one at the beginning $V^\mu(A)$ and the other one after moving around the loop $V^\mu(A_{final})$. The strategy followed is to subtract the initial value from the final one and see the result, since that change is due to curvature, therefore the Riemann curvature tensor should manifest itself in the result

$$\begin{aligned} V^\mu(A_{final}) - V^\mu(A) = \Big[\int_{x^\alpha = a+\delta a} \Gamma^\mu_{\nu\beta} V^\nu dx^\beta - \int_{x^\alpha = a} \Gamma^\mu_{\nu\beta} V^\nu dx^\beta \Big] \\ - \Big[\int_{x^\beta = b+\delta b} \Gamma^\mu_{\nu\alpha} V^\nu dx^\alpha - \int_{x^\beta = b} \Gamma^\mu_{\nu\alpha} V^\nu dx^\alpha \Big]. \end{aligned} \tag{2.6.7}$$

Consider the quantity $\Gamma^\mu_{\nu\beta} V^\nu$ as a function and consider its partial derivatives along the lines x^α and x^β respectively, also express $V^\mu(A_{final}) - V^\mu(A)$ as δV^μ. We end up with

$$\delta V^\mu = \int_b^{b+\delta b} \delta a \frac{\partial}{\partial x^\alpha}(\Gamma^\mu_{\nu\beta} V^\nu) dx^\beta - \int_a^{a+\delta a} \delta b \frac{\partial}{\partial x^\beta}(\Gamma^\mu_{\nu\alpha} V^\nu) dx^\alpha. \tag{2.6.8}$$

For a very small loop in a curved manifold, the change in the Christoffel symbol is almost constant, so the above equation will be given as:

$$\delta V^{\mu} = \delta a \frac{\partial}{\partial x^{\alpha}}(\Gamma^{\mu}_{\beta\nu} V^{\nu}) \int_{b}^{b+\delta b} dx^{\beta} - \delta b \frac{\partial}{\partial x^{\beta}}(\Gamma^{\mu}_{\alpha\nu} V^{\nu}) \int_{a}^{a+\delta a} dx^{\alpha} = \delta a \delta b \Big[\frac{\partial}{\partial x^{\alpha}}(\Gamma^{\mu}_{\beta\nu} V^{\nu}) - \frac{\partial}{\partial x^{\beta}}(\Gamma^{\mu}_{\alpha\nu} V^{\nu})\Big]. \tag{2.6.9}$$

Deriving the quantities inside the brackets,

$$\delta V^{\mu} = \delta a \delta b \Big[\Gamma^{\mu}_{\beta\nu,\alpha} V^{\nu} + \Gamma^{\mu}_{\beta\nu} \frac{\partial V^{\nu}}{\partial x^{\alpha}} - \Gamma^{\mu}_{\alpha\nu,\beta} V^{\nu} - \Gamma^{\mu}_{\alpha\nu} \frac{\partial V^{\nu}}{\partial x^{\beta}}\Big], \tag{2.6.10}$$

from the definitions of covariant derivatives, and changing dummy index ν to be λ in first and third term we find

$$\delta V^{\mu} = \delta a \delta b \Big[\Gamma^{\mu}_{\beta\lambda,\alpha} - \Gamma^{\mu}_{\beta\nu}\Gamma^{\nu}_{\alpha\lambda} - \Gamma^{\mu}_{\alpha\lambda,\beta} + \Gamma^{\mu}_{\alpha\nu}\Gamma^{\nu}_{\beta\lambda}\Big] V^{\lambda}, \tag{2.6.11}$$

from the above equation, the quantity inside the square bracket is the Riemann curvature tensor. It is clear that the Riemann curvature tensor should be written as (1/3) tensor according to the indices of the quantity inside the bracket.

$$R^{\mu}_{\beta\alpha\lambda} = \Gamma^{\mu}_{\beta\lambda,\alpha} - \Gamma^{\mu}_{\beta\nu}\Gamma^{\nu}_{\lambda\alpha} - \Gamma^{\mu}_{\beta\alpha,\lambda} + \Gamma^{\mu}_{\nu\lambda}\Gamma^{\nu}_{\beta\alpha}. \tag{2.6.12}$$

In a local coordinate system we have zero Christoffel symbol, consequently in local coordinate system Riemann curvature tensor takes the following form

$$\tilde{R}^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu}. \tag{2.6.13}$$

We express Riemann curvature tensor for local coordinate system in terms of metric tensor, this by using the definition of the Christoffel symbol in terms of metric tensor(2.4.7), and then by taking the first derivative with respect to coordinates (x^{μ}) knowing the fact that on local coordinates the derivative of the inverse metric tensor is zero we find

$$\Gamma^{\alpha}_{\beta\nu,\mu} = \frac{1}{2} g^{\alpha\lambda}(g_{\lambda\beta,\nu\mu} + g_{\lambda\nu,\beta\mu} - g_{\beta\nu,\lambda\mu}). \tag{2.6.14}$$

Similarly for $\Gamma^{\alpha}_{\beta\mu,\nu}$, and by subtracting $\Gamma^{\alpha}_{\beta\mu,\nu}$ from $\Gamma^{\alpha}_{\beta\nu,\mu}$ and using the fact that partial derivatives commute we obtain

$$\tilde{R}^{\alpha}_{\beta\mu\nu} = \frac{1}{2} g^{\alpha\lambda}\Big[g_{\lambda\nu,\beta\mu} - g_{\beta\nu,\lambda\mu} - g_{\lambda\mu,\beta\nu} + g_{\beta\mu,\lambda\nu}\Big]. \tag{2.6.15}$$

This is the expression for Riemann curvature tensor in a local inertial frame, where tilde denotes Riemann tensor in a local inertial frame. Although this equation is in a local frame it applicable in a general coordinate frame because it is a tensorial equation. Additionally, according to the principle of equivalence free falling frame is an inertial frame. So this equation is applicable in a free falling frame. By using the above definition for Riemann curvature tensor we derive very important results for Riemann tensor that are helpful for proving Einstein's field equations. We also define a (0/4) version for Riemann curvature tensor upon multiplying it by the metric tensor. The metric tensor lowers the indices as follows:

$$\tilde{R}_{\sigma\beta\mu\nu} = \frac{1}{2}\Big[g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu}\Big]. \tag{2.6.16}$$

Using the above definition of Riemann curvature tensor in terms of metric tensor. Three symmetry properties for Riemann curvature tensor can be proved. Anti-symmetric over the first two indices

$$\tilde{R}_{\alpha\beta\mu\nu} = -\tilde{R}_{\beta\alpha\mu\nu}.$$

Anti-symmetric over the second two indices

$$\tilde{R}_{\alpha\beta\mu\nu} = -\tilde{R}_{\alpha\beta\nu\mu}.$$

This symmetry relation could also be expressed for the (1/3) version as follows:

$$\tilde{R}^{\alpha}_{\beta\mu\nu} = -\tilde{R}^{\alpha}_{\beta\nu\mu}.$$

Symmetric over exchanging first and second pair of indices as follows:

$$\tilde{R}_{\alpha\beta\mu\nu} = \tilde{R}_{\mu\nu\alpha\beta}.$$

One of the most important relations that Riemann tensor satisfies is

$$\tilde{R}_{\alpha\beta\mu\nu,\lambda} + \tilde{R}_{\alpha\beta\lambda\mu,\nu} + \tilde{R}_{\alpha\beta\nu\lambda,\mu} = 0. \tag{2.6.17}$$

This is the so-called Bianchi identity which is used to prove Einstein's field equations. It reveals that addition over the cyclic permutation of the last three indices is zero. This identity can easily be proved by defining first derivative of Riemann tensor as follows:

$$\tilde{R}_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}\Big[g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda}\Big]. \tag{2.6.18}$$

Upon defining $\tilde{R}_{\alpha\beta\mu\nu,\lambda}$ and $\tilde{R}_{\alpha\beta\mu\nu,\lambda}$ in the same manner, and then adding up we obtain Equation (2.6.17), but we have to use the commutative property of partial derivatives and the symmetric property of metric tensor to prove that result.

2.7 Ricci tensor and Ricci scalar

The Ricci tensor is defined as the contraction of the Riemann tensor by inverse metric tensor as follows:

$$R_{\alpha\beta} = g^{\mu\nu} R_{\alpha\beta\mu\nu} = R^{\mu}_{\alpha\mu\beta} = R^{\nu}_{\alpha\nu\beta}. \tag{2.7.1}$$

The Ricci scalar is defined as the contraction of the Ricci tensor as follows:

$$R = g^{\alpha\beta} R_{\alpha\beta} = R^{\alpha}_{\alpha} = R^{\beta}_{\beta}. \tag{2.7.2}$$

2.8 Einstein's tensor

Given Bianchi identity (2.6.17)

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0, \tag{2.8.1}$$

multiplying by the inverse metric tensor $g^{\alpha\mu}$ we find

$$R^{\mu}_{\beta\mu\nu;\lambda} + R^{\mu}_{\beta\lambda\mu;\nu} + R^{\mu}_{\beta\nu\lambda;\mu} = 0. \tag{2.8.2}$$

Using the anti-symmetric property of the Riemann tensor $R^{\mu}_{\beta\lambda\mu;\nu} = -R^{\mu}_{\beta\mu\lambda;\nu}$ back into the above equation.

$$R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^{\mu}_{\beta\nu\lambda;\mu} = 0. \tag{2.8.3}$$

Multiplying by the inverse metric $g^{\beta\nu}$ to contract again, then using the anti-symmetry property of the Riemann tensor for the last term of the above equation, and suppressing the dummy indices we obtain

$$R_{;\lambda} - R^{\nu}_{\lambda;\nu} - R^{\mu}_{\lambda;\mu} = 0, \tag{2.8.4}$$

ν and μ are dummy indices, so adding the last two terms in the above equation we find

$$R_{;\lambda} - 2R^{\mu}_{\lambda;\mu} = 0, \tag{2.8.5}$$

expressing $R_{;\lambda}$ as $g^{\mu}_{\lambda}R_{;\mu}$

$$2R^{\mu}_{\lambda;\mu} - g^{\mu}_{\lambda}R_{;\mu} = 0. \tag{2.8.6}$$

Taking the derivative out as a common factor and raising the indices by the inverse metric $g^{\lambda\nu}$

$$\Big[2R^{\mu\nu} - g^{\mu\nu}R\Big]_{;\mu} = 0, \tag{2.8.7}$$

this can be written as

$$\Big[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\Big]_{;\mu} = G^{\mu\nu}_{;\mu} = 0. \tag{2.8.8}$$

The quantity inside the bracket is defined as the Einstein's tensor $G^{\mu\nu}$ which fits into Poisson equation instead of the second derivative of the potential. This is because it has zero divergence as the Energy-momentum tensor. Additionally it has in its components the second derivative of the metric tensor. Since Poisson equation contains second derivative of the potential and the metric tensor is interpreted as the generalised potential, Field equations in higher dimensions have to include second derivative of the metric tensor.

2.9 Geodesic deviation equation

Consider two points on curved spacetime A and B. A is a local inertial frame which implies that $\Gamma_A = 0$, consequently at point A geodesic equation is given as

$$\frac{dx^{\beta}_{A}}{d\tau^2} = 0. \tag{2.9.1}$$

At B $\Gamma_B \neq 0$, to find the Christoffel symbol we use taylor's expansion as follows:

$$\Gamma^{\alpha}_{\mu\nu}|_B = \Gamma^{\alpha}_{\mu\nu}|_A + \Gamma^{\alpha}_{\mu\nu,\beta}\zeta^{\beta} = \Gamma^{\alpha}_{\mu\nu,\beta}\zeta^{\beta}, \tag{2.9.2}$$

ζ^β is defined as $x^\beta_B - x^\beta_A$, this means geodesic equation at B will be given as

$$\frac{d^2x^\alpha_B}{d\tau^2} = -\Gamma^\alpha_{\mu\nu,\beta}\zeta^\beta \dot{x^\mu}\dot{x^\nu}, \tag{2.9.3}$$

since

$$\frac{d^2\zeta^\beta}{d\tau^2} = \frac{d^2x^\beta_B}{d\tau^2} - \frac{d^2x^\beta_A}{d\tau^2}$$

with $\frac{d^2x^\beta_A}{d\tau^2} = 0$, Equation (2.9.3) will be given as

$$\frac{d^2\zeta^\alpha}{d\tau^2} = -\Gamma^\alpha_{\mu\nu,\beta}\zeta^\beta \dot{x^\mu}\dot{x^\nu}. \tag{2.9.4}$$

The above equation is a frame dependent equation, because it is the equation of motion for the separation vector ζ^β as viewed from A, but we are not interested in frame dependent equations, we take the covariant derivative for the separation vector twice instead. As we have showed covariant derivatives transform the same way tensors do, this means that the equation will be frame independent. At B and taking the covariant derivative twice

$$\Big[D_{\vec{v}}D_{\vec{v}}\Big]\zeta^\alpha = D_{\vec{v}}\Big[\zeta^\alpha_{;\beta}V^\beta\Big] = \Big[\zeta^\alpha_{,\beta}V^\beta + \Gamma^\alpha_{\beta\nu}\zeta^\nu V^\beta\Big]_{;\beta}V^\beta. \tag{2.9.5}$$

Since

$$V^\beta = \frac{\partial x^\beta}{\partial\tau}$$

V^β is the tangent to the geodesic. Considering a local inertial frame

$$;\beta = \frac{\partial}{\partial x^\beta}$$

$$\Big[D_{\vec{v}}D_{\vec{v}}\Big]\zeta^\alpha = \frac{d}{d\tau}\Big[\zeta^\alpha_{,\beta}V^\beta + \Gamma^\alpha_{\mu\beta}\zeta^\mu V^\beta\Big] = \frac{d^2\zeta^\alpha}{d\tau^2} + \frac{d}{d\tau}(\Gamma^\alpha_{\mu\beta}\zeta^\mu V^\beta), \tag{2.9.6}$$

$$\Big[D_{\vec{v}}D_{\vec{v}}\Big]\zeta^\alpha = \frac{d^2\zeta^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\beta}\frac{d}{d\tau}(\zeta^\mu V^\beta) + \zeta^\mu V^\beta V^\nu \Gamma^\alpha_{\mu\beta,\nu}, \tag{2.9.7}$$

upon using the fact that at A the Christoffel symbol $\Gamma^\alpha_{\mu\beta}$ vanishes we find

$$\Big[D_{\vec{v}}D_{\vec{v}}\Big]\zeta^\alpha = (\Gamma^\alpha_{\mu\beta,\nu} - \Gamma^\alpha_{\mu\nu,\beta})\zeta^\mu V^\beta V^\nu, \tag{2.9.8}$$

$$\Big[D_{\vec{v}}D_{\vec{v}}\Big]\zeta^\alpha = R^\alpha_{\mu\nu\beta}\zeta^\mu V^\beta V^\nu. \tag{2.9.9}$$

This is the so called geodesic deviation equation. We see that for the flat manifold, Riemann curvature tensor is zero which means we have zero deviation or in other word geodesics are straight lines which keep parallel.

2.10 Riemann curvature tensor for 2-sphere

We introduce the Riemann curvature tensor for the 2-sphere as an illustrative example. This may help in understanding how we calculate Riemann curvature tensor for manifolds. In order to find Riemann curvature tensor for any manifold we need the metric tensor representing that manifold, through the metric tensor we can find Christoffel symbols. Implementing that in the definition given for Riemann tensor we can easily find the components of the tensor. Firstly, the line element for a 2-sphere is given as

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2. \tag{2.10.1}$$

Consequently, we represent the metric as follows:

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}, \tag{2.10.2}$$

and the inverse metric as

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2\theta} \end{pmatrix}. \tag{2.10.3}$$

We have to find components of the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$. Since we have only two coordinates (θ, ϕ) and three indices α, β, and γ, therefore we expect to have six components for Christoffel symbols as follow $\Gamma^{\theta}_{\theta\theta}$, $\Gamma^{\theta}_{\theta\phi} = \Gamma^{\theta}_{\phi\theta}$, $\Gamma^{\phi}_{\theta\theta}$, $\Gamma^{\theta}_{\phi\phi}$, $\Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta}$ and $\Gamma^{\phi}_{\phi\phi}$. Using the definition given for the Christoffel symbol in terms of metric tensor (2.4.7) we show that all these components are equal zero apart from $\Gamma^{\theta}_{\phi\phi}$ and $\Gamma^{\phi}_{\theta\phi}$ which are given as follows:

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2} g^{\theta\lambda} \left[g_{\lambda\phi,\phi} + g_{\lambda\phi,\phi} - g_{\phi\phi,\lambda} \right]. \tag{2.10.4}$$

Summing over λ and substituting by the required metric components we obtain

$$\Gamma^{\theta}_{\phi\phi} = -\frac{1}{2} g^{\theta\theta} g_{\phi\phi,\theta} = -\sin\theta\cos\theta. \tag{2.10.5}$$

Following the same procedure we find $\Gamma^{\phi}_{\phi\theta}$ to be

$$\Gamma^{\phi}_{\phi\theta} = \cot\theta. \tag{2.10.6}$$

From the above two equations and using the definition of Riemann tensor in terms of the Christoffel symbols. We verify that all components of Riemann tensor vanish apart from $R^{\theta}_{\phi\theta\phi}$ which is calculated as follows:

$$R^{\theta}_{\phi\theta\phi} = \Gamma^{\theta}_{\phi\phi,\theta} - \Gamma^{\theta}_{\theta\phi,\phi} + \Gamma^{\theta}_{\alpha\theta}\Gamma^{\alpha}_{\phi\phi} - \Gamma^{\theta}_{\alpha\phi}\Gamma^{\alpha}_{\phi\theta}. \tag{2.10.7}$$

Since all the quantities are functions of θ, therefore the second term in the above equation will vanish. Since $\Gamma^{\theta}_{\theta\alpha} = 0$ for $\alpha = \theta, \phi$, consequently Riemann curvature tensor will be given as

$$R^{\theta}_{\phi\theta\phi} = \Gamma^{\theta}_{\phi\phi,\theta} - \Gamma^{\theta}_{\alpha\phi}\Gamma^{\alpha}_{\phi\theta}. \tag{2.10.8}$$

Summing over α

$$R^{\theta}_{\phi\theta\phi} = \Gamma^{\theta}_{\phi\phi,\theta} - \Gamma^{\theta}_{\theta\phi}\Gamma^{\theta}_{\phi\theta} - \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta}, \tag{2.10.9}$$

the second term of the above equation will be zero because of $\Gamma^{\theta}_{\phi\theta}$. Substituting by the values we have for $\Gamma^{\phi}_{\phi\theta}$ and $\Gamma^{\theta}_{\phi\phi}$ in the above equation we find

$$R^{\theta}_{\phi\theta\phi} = \sin^2\theta. \tag{2.10.10}$$

3. Einstein's field equations

3.1 What is wrong with Newtonian gravity?

After the release of special relativity as the consistent theory describing motion at very high speeds (close to that of light), some contradiction appeared between special relativity which was believed to be correct at that time and Newtonian gravity. According to the special relativity light is the ultimate speed in the universe. Taking a glance at Newton's law of universal gravitation $\vec{F} = -\frac{Gm_1 m_2}{r^2}\hat{r}$, we see the forementioned contradiction. Simply, this law considers that gravity travels faster than light. Imagining two masses which are separated by very large distance that light cannot cover instantaneously. According to the universal law of gravitation any two masses in the universe are attracting each other with the gravitational force defined above. This implies that in order for those two masses to attract each other regardless of their separation distance, this gravitational force has to travel faster than light to make that law always verified. That contradiction was the main motivation behind constructing new theory describing gravity, and also takes special relativity into account. Most of physicists at that time tried to extend Poisson equation $\nabla^2\phi = 4\pi\rho G$ which is the field equation for Newtonian gravity to higher dimensions. They approached that using tensors instead of scalars used (potential and density) but restricting those tensors to flat spacetime. This approach did not seem to be working until Einstein introduced the idea of curved spacetime as a result of matter and energy. After he had developed special theory of relativity which does not involve sophisticated mathematics, he started to think about that contradiction but he was unequipped with the needed mathematical tools to describe his new idea for gravity in terms of curvature. That is why Einstein started his work in general theory of relativity by studying mass in a fundamental bases, and then introducing the principle of equivalence.

3.2 Inertial mass, passive and active gravitational masses

In general and according to the concepts of inertia and gravitation introduced in Newton's laws we can define three masses to any body inertial mass, passive and active gravitational masses.

Inertial mass

From the way Newton introduced his second law $F = m^I a$, inertial mass is the mass defined in that equation, and is a measure of body resistance to the change in its state of motion which we call inertia, and this mass has nothing to do with gravity.

Passive gravitational mass

Passive gravitational mass is a measure of a body reaction to the gravitational field or to what extent a body can be affected by a gravitational field. This mass can de defined in terms of Newton's second law as follows

$$F = -m^p \nabla\phi,$$

where ϕ: gravitational potential. Consider two masses have been thrown from a building in the Earth gravitational field, both masses will experience the same gravitational acceleration "g". The two masses will be subjected to those two forces as follows

$$F_1 = m_1^I a_1 = -m_1^p \nabla\phi, \tag{3.2.1}$$
$$F_2 = m_2^I a_2 = -m_2^p \nabla\phi. \tag{3.2.2}$$

For free falling $a_1 = a_2 = g$, and ϕ is the same for two bodies falling in the same field, consequently we can write

$$\frac{m_1^I}{m_2^I} = \frac{m_1^P}{m_2^P}. \tag{3.2.3}$$

We can set the above equality to be equal to some universal constant. Then by choosing some system of units such that this universal constant is equal one. We can equate inertial mass and passive gravitational mass.

$$m^p = m^I.$$

Active gravitational mass

Active gravitational mass is a measure of the strength of gravitational field produced by the mass. If we considered the gravitational forces between two masses and by using Newtons third law for the mutual interaction between any two bodies. We can show that active and passive gravitational masses are equal. From the result for inertial mass and passive gravitational mass we can say

$$m^I = m^P = m^A.$$

The above result is one of the pillars of general theory of relativity, and disrupting this result means explicitly breaking down of general relativity.

3.3 The principle of equivalence

The principle of equivalence meant to show the equivalence between gravitation and acceleration as well as gravitational force and inertial force. Einstein did two experiments clarified that locally gravitation and acceleration are equivalent, as well as gravitational force and inertial force.

Experiment one

Imagining an observer inside a space rocket who does not know anything about the outside world where this space rocket is moving at uniform acceleration "g". If the observer throws a ball, the observer will see the ball moving at acceleration 'g' downward. If the same observer is in an elevator which is at rest on the Earth surface, then dropped the ball, similarly the ball will be moving at acceleration 'g' downward Fig. 3.1. This explicitly means that we cannot locally differentiate between acceleration and gravitation, or acceleration and gravitation are equivalent. This leads us to the first version of the principle of equivalence. "No local experiment can distinguish between a non rotating free fall motion

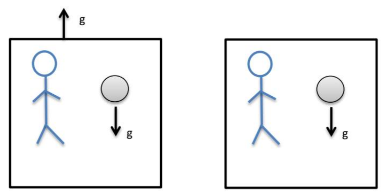

Figure 3.1: Experiment one

in a gravitational field (elevator at earth surface) and uniform motion in absence of gravitational field (moving space rocket)". The word local here means that the observer is restricting himself to a very small region of spacetime, but what if the observer took a large scale picture from outside both the space rocket and the elevator. The ball at the surface of the earth will be moving in path which is converging toward earth center, it is extremely difficult to observe that deviation from straight line motion but it exists. On the other hand the ball inside that space rocket will maintain straight line motion. This also means that non-locally we can differentiate between acceleration and gravitation, or non-locally acceleration and gravitation are not equivalent.

Experiment two

Consider the same observer who does not know anything about the outside world inside a space rocket which has switched off its engine, this means that the rocket is moving at constant speed. If the observer drops a ball it will stay at rest relative to the observer. If the observer is inside an elevator which is freely falling in gravitational field and drops the ball, the ball will stay at rest because the elevator, observer and the ball are freely falling together Fig. 3.2. This also explicitly means that, we cannot locally distinguish between free falling in a gravitational field and uniform motion in absence of a gravitational field, which leads us also to the second version of the principle of equivalence. "No local experiment can distinguish between a free falling in a gravitational field and uniform motion in absence of gravitational field". Again non-locally if we have two balls we can see that they will approach each other as a result of their paths toward centre of earth Fig. 3.3. the path for the ball falling in gravitational field will deviate from straight line motion a bit as a result of gravitation. This means that non-locally this statement for the principle does not hold.

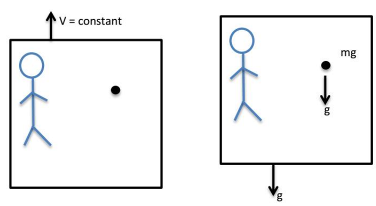

Figure 3.2: Experiment two

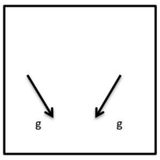

Figure 3.3: Experiment two

3.4 Gravity as geometry

The idea of depicting gravity as curvature of spacetime by matter and energy up till now seems to be the best approach and description for gravity. Firstly, in the context of the theory speed of light cannot be exceeded. This implicitly means that general relativity is the unification of Newtonian gravity and special relativity. Secondly, the theory added many corrections to Newtonian gravity results. This includes orbits of the planets around the sun and bending of light beams by gravitational fields. Additionally it has its own pure predictions such as gravitational waves, black holes and worm holes. Finally, one could say that we look at the large scale universe in the context of general relativity, currently all the cosmological models are built in general relativity as the most acceptable theory describing gravity.

3.5 Vacuum field equation

Starting from geodesic deviation (2.9.9) equation we promote Poisson equation to Einstein's vacuum field equations. The strategy is to start from Newton's second law of motion for two particles and find the equation of motion for the separation vector between their paths. Comparing that to geodesic deviation equation we will be able to write the vacuum field equations.

Consider two particles having the following equations of motion

$$\frac{d^2x^i}{dt^2} = -\frac{\partial\phi(\vec{x})}{\partial x^i}, \tag{3.5.1}$$

$$\frac{d^2(x^i+\zeta^i)}{dt^2} = -\frac{\partial\phi(\vec{x}+\vec{\zeta})}{\partial x^i}. \tag{3.5.2}$$

Subtracting the above two equations we find,

$$\frac{d^2(x^i+\zeta^i)}{dt^2} - \frac{d^2x^i}{dt^2} = -\frac{\partial\phi(\vec{x}+\vec{\zeta})}{\partial x^i} + \frac{\partial\phi(\vec{x})}{\partial x^i}, \tag{3.5.3}$$

which can be expressed as

$$\frac{d^2\zeta^i}{dt^2} = \frac{\partial}{\partial x^i}\Big[\phi(\vec{x}) - \phi(\vec{x}+\vec{\zeta})\Big], \tag{3.5.4}$$

using Taylor's expansion for $\phi(\vec{x}+\vec{\zeta})$ as $\phi(\vec{x}+\vec{\zeta}) = \phi(x) + \zeta^j\frac{\partial\phi}{\partial x^j}$ we find

$$\frac{d^2\zeta^i}{dt^2} = -\zeta^j\frac{\partial^2\phi}{\partial x^j\partial x^i}. \tag{3.5.5}$$

This is obviously equation of motion for the separation vector ζ^i, by comparing that to geodesic deviation equation (2.9.9), we obtain

$$-\zeta^\alpha \nabla^2 \phi = R^\alpha_{\mu\nu\beta} V^\mu V^\nu \zeta^\beta, \tag{3.5.6}$$

where V^μ and V^ν are tangent vectors. Since for vacuum $\nabla^2\phi = 0$, therefore Riemann tensor should be zero, multiplying by g^ν_α we find

$$R_{\mu\beta} = 0, \tag{3.5.7}$$

these are Einstein's vacuum field equations which we use to find the full non-vacuum field equations. We see that it looks very simple, in fact Ricci tensor is defined in terms of second derivative of metric tensor and it is a symmetric tensor which means we have ten equations.

3.6 Non-vacuum field equations

Talking about non-vacuum means that energy momentum tensor $T^{\mu\nu}$ has non-zero value. As we know all conservation laws for energy and momentum could be expressed in terms of energy momentum tensor as follows

$$T^{\mu\nu}_{;\nu} = 0. \tag{3.6.1}$$

Usually if we find any quantity with zero divergence, most likely it represents conservation law. If we are to implement energy momentum tensor in field equations as the source of curvature we have to take into our considerations that it has zero divergence. This means we cannot equate energy momentum tensor to Ricci tensor which has non-zero divergence. Accordingly we have to implement tensor has zero divergence and function of the metric as well. Using Bianchi identity for Riemann tensor we have shown that the tensor $G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ satisfies these conditions. Also as we can see Ricci tensor contains inside second derivative of metric tensor. This is in analogue to Poisson equation which contains second derivative of potential. This means when we promote Poisson equation to Einstein's field equations it should take the form

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = \kappa T^{\mu\nu}, \tag{3.6.2}$$

in general we add constant Λ so that field equations are given as

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu}, \tag{3.6.3}$$

The constant Λ is called cosmological constant, it has been put by Einstein because universe is static. Later on in 1929 when Hubble discovered that universe is expanding the constant removed, but upon discovering of dark energy it has introduced again as source of curvature even if we have zero Ricci curvature. Remembering that energy and mass are equivalent so if matter is source of curvature, energy will be. Cosmological constant is energy density of vacuum. We find Λ by taking the trace of the above equation when $T^{\mu\nu} = 0$ as follows:

$$g_{\mu\nu}\Big[R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu}\Big] = 0, \tag{3.6.4}$$

$$R - 2R + 4\Lambda = 0, \tag{3.6.5}$$

$$\Lambda = \frac{R}{4}. \tag{3.6.6}$$

The constant κ is analogues to the constant $4\pi\rho G$ in Poisson equation and we have to find its value. To accomplish this we will try the so-called weak field approximation, which is the case of weak gravitational field and slow moving systems. Through these two considerations we can compare Einstein's field equation to Poisson equation and get κ

3.7 Weak field approximation

Weak field approximation is the classical limit of Einstein's field equations on which it should reduce to Poisson equation, and geodesic equation to Newton's second law. The main characteristics of weak field approximation are

- Speeds are much lower than that of light $\frac{v}{c} << 1$, consequently we neglect time derivatives of different quantities, as a result of moving at low speed, change of all quantities relative to time is negligible.
- Gravitational field is very weak $\frac{\phi}{c^2} << 1$ and $\frac{p}{c^2} \to 0$

Our main goal is to find the constant κ. Starting from the geodesic equation

$$\frac{d^2x^\gamma}{d\tau^2} = -\Gamma^\gamma_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}. \tag{3.7.1}$$

Since $v << c$, therefore coordinate time replaces proper time in geodesic equation as follows

$$\frac{d^2x^\gamma}{dt^2} = -\Gamma^\gamma_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt}, \tag{3.7.2}$$

denoting the coordinates by "i" instead of Greek letters and substituting for "i" by different coordinates.

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} - 2\Gamma^i_{0j}\frac{dx^0}{dt}\frac{dx^j}{dt} - \Gamma^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt}, \tag{3.7.3}$$

we see that in the above equation, the first term is of the order of O(c^2), second term is of order O(vc) while the last term is of order O(v^2). Obviously we neglect second and third term relative to the first one obtaining

$$\frac{d^2x^i}{dt^2} = -\Gamma^i_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} = -c^2\Gamma^i_{00}. \tag{3.7.4}$$

We need to find Γ^i_{00}, then comparing with Newton's second law to find an expression for the potential ϕ. To do that we use the definition of the Christoffel symbol in terms of metric tensor (2.4.7) as follows

$$\Gamma^i_{00} = \frac{1}{2}g^{i\gamma}\Big[g_{\gamma 0,0} + g_{\gamma 0,0} - g_{00,\gamma}\Big], \tag{3.7.5}$$

ignoring time derivatives, the above equation will be

$$\Gamma^i_{00} = -\frac{1}{2}g^{i\gamma}g_{00,\gamma}. \tag{3.7.6}$$

For the case of weak gravitational fields the metric tensor should be a very little deviation from Minkowski metric $\eta_{\alpha\beta}$. We consider the following form for the metric tensor

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \epsilon h_{\alpha\beta}, \tag{3.7.7}$$

where $\epsilon << 1$ such that ϵ^2 can be neglected. While $h_{\alpha\beta}$ is a small perturbation to Minkowski metric, consequently the inverse metric is given as

$$g^{\alpha\beta} = \eta^{\alpha\beta} - \epsilon h^{\alpha\beta}. \tag{3.7.8}$$

Substituting back in (3.7.6)

$$\Gamma^i_{00} = -\frac{1}{2}\Big[\eta^{i\gamma} - \epsilon h^{i\gamma}\Big]\Big[\eta_{00,\gamma} + \epsilon h_{00,\gamma}\Big] = -\frac{1}{2}\Big[\eta^{i\gamma}\epsilon h_{00,\gamma} - \epsilon^2 h^{i\gamma} h_{00,\gamma}\Big], \tag{3.7.9}$$

where $\eta_{00,\gamma} = 0$, neglecting terms of order ϵ^2

$$\Gamma^i_{00} = -\frac{1}{2}\epsilon h^{,i}_{00}. \tag{3.7.10}$$

Substituting back in equation (3.7.4)

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}c^2\epsilon h^{,i}_{00} = \frac{\partial}{\partial x^i}(\frac{c^2\epsilon h_{00}}{2}), \tag{3.7.11}$$

comparing with $\frac{d^2x^i}{dt^2} = -\frac{\partial\phi}{\partial x}$, we find

$$\phi = -\frac{c^2\epsilon h_{00}}{2}.$$

Considering energy-momentum tensor for dust $T_{\alpha\beta} = \rho U_\alpha U_\beta$ with ρ as density while U_α and U_β as four velocities, and substituting in field equations $R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \kappa T_{\alpha\beta}$ we find

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \kappa\rho U_\alpha U_\beta, \tag{3.7.12}$$

multiplying by the inverse metric $g^{\alpha\beta}$, we obtain $R - 2R = -k\rho c^2$, therefore Ricci scalar is given as $R = \kappa\rho c^2$. If we substitute Ricci scalar back in field equations we obtain

$$R_{\alpha\beta} - \frac{1}{2}\kappa\rho c^2 g_{\alpha\beta} = \kappa\rho c^2. \tag{3.7.13}$$

Taking the "00" component for the above equation,

$$R_{00} + \frac{1}{2}\kappa\rho c^2 = \kappa\rho c^2, \tag{3.7.14}$$

$$R_{00} = \frac{1}{2}\kappa\rho c^2, \tag{3.7.15}$$

to find Ricci tensor we use its definition in terms the Christoffel symbols as follows:

$$R_{\alpha\beta} = R^\mu_{\alpha\mu\beta} = \Gamma^\mu_{\alpha\beta,\mu} - \Gamma^\mu_{\alpha\mu,\beta}. \tag{3.7.16}$$

In the above equation we have neglected the terms which contain products of the Christoffel symbol, because they are of order ϵ^2, consequently R_{00} will be given as

$$R_{00} = \Gamma^i_{00,i} - \Gamma^i_{0i,0}. \tag{3.7.17}$$

Neglect the second term which is time derivative

$$R_{00} = \Gamma^i_{00,i} = (-\frac{1}{2}\epsilon h^{,i}_{00})_{,i}. \tag{3.7.18}$$

We used the value given for $\Gamma^i_{00,i}$ in (3.7.10). Using $\phi = \frac{-c^2 \epsilon h_{00}}{2}$, we express R_{00} as

$$R_{00} = \frac{\nabla^2 \phi}{c^2} = \frac{1}{2}\kappa \rho c^2, \tag{3.7.19}$$

$$\kappa = \frac{8\pi G}{c^4}.$$

Consequently we write the full non-vacuum equations as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{3.7.20}$$

The above are full non-vacuum Einstein's field equations. It reduces to vacuum field equations $R_{\mu\nu} = 0$ for zero energy momentum tensor $T_{\alpha\beta} = 0$. Einstein's field equations relate matter and radiation represented in energy momentum tensor to curvature represented in Ricci curvature tensor.

4. Kerr metric and Penrose process

Kerr metric is a solution to Einstein's field equation. Kerr metric describes the geometry of spacetime in the vacuum $(T_{\alpha\beta} = 0)$ around a rotating uncharged black hole. According to Kerr metric any rotating black hole has a region around called ergoregion. This region is rotating as a result of black hole rotation, consequently characterized by existence of rotational kinetic energy. If any wave for example electromagnetic wave or gravitational wave enters inside that region, it gains energy. This is called the Penrose process. The size of that region depends on the mass and angular momentum of the black hole, this is determined by studying the kerr metric singularities. Kerr metric is given as

$$g_{\mu\nu} = \begin{pmatrix} -(1-\frac{2Mr}{\Sigma}) & 0 & 0 & -\frac{2Mra\sin^2\theta}{\Sigma} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ \frac{-2Mra\sin^2\theta}{\Sigma} & 0 & 0 & (r^2+a^2+\frac{2Mra^2\sin^2\theta}{\Sigma})\sin^2\theta \end{pmatrix}, \tag{4.0.1}$$

where M is mass of the black hole, a is the angular momentum per unit mass,

$\Delta = r^2 - 2Mr + a^2 = 0,$

$\Sigma = r^2 + a^2\cos^2\theta = 0.$

Kerr metric has two singularities one at $\Delta = 0$ which corresponds to

$$r_- = M + \sqrt{M^2 - a^2}, \tag{4.0.2}$$

$$r_+ = M - \sqrt{M^2 - a^2}. \tag{4.0.3}$$

r_- and r_+ represent inner and outer horizons respectively, while the second one at $\Sigma = 0$, which corresponds to

$$r_{s_-} = M - \sqrt{M^2 - a^2\cos^2\theta}, \tag{4.0.4}$$

$$r_{s_+} = M + \sqrt{M^2 - a^2\cos^2\theta}, \tag{4.0.5}$$

r_{s_-} and r_{s_+}, represent inner and outer ergospheres respectively. These radii are arranged as follow $r_{s_-} \leq r_- \leq r_+ \leq r_{s_+}$. The outer ergosphere and outer event horizon are given as

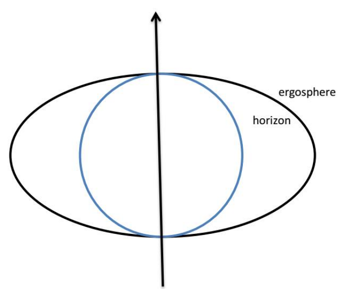

Figure 4.1: Outer horizon and outer ergosphere

The region between outer event horizon and outer ergosphere is called the ergoregion which is characterized by the existence of rotational kinetic energy. This is due to the effects of black hole rotation. If any wave enters that region it starts to acquire rotational kinetic energy. We are interested in calculating that gain in energy. To do that we solve the wave equation for arbitrary wave function Φ at different regions, first far away from the black hole which represents ingoing at outgoing waves, then at the boundaries of the ergoregion ($r = r_+$). When we find the form of Φ inside that ergoregion, we will use it to calculate the change in energy and verify that phenomenon. We start with writing the wave equation for the field Φ, to do that we use this identity

$$\Box\Phi = g^{-\frac{1}{2}}\Big[g^{\frac{1}{2}}g^{\alpha\beta}\Phi_{,\beta}\Big]_{,\alpha}, \tag{4.0.6}$$

where $\Box$ is the D'Alembert operator $\Box\Phi = g^{\alpha\beta}\partial_\alpha\partial_\beta = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$, Δ is the Laplace operator in 3-dimensional space. Φ is a wave function, g:determinant of the metric, $g^{\alpha\beta}$ is the inverse metric tensor.

To find the metric determinant we use the matrix representation for the metric given above (4.0.1).

$$g = det(g_{\mu\nu}) = -\frac{\Sigma^2}{\Delta}\Big[(1-\frac{2Mr}{\Sigma})(r^2+a^2)\sin^2\theta + (1-\frac{2Mr}{\Sigma})(\frac{2Mra^2\sin^4\theta}{\Sigma}) + \frac{4M^2r^2a^2\sin^4\theta}{\Sigma^2}\Big], \tag{4.0.7}$$

$$-g = \frac{\Sigma^2\sin^2\theta}{\Delta}\Big[(1-\frac{2Mr}{\Sigma})(r^2+a^2) + \frac{2Mra^2\sin^2\theta}{\Sigma}\Big], \tag{4.0.8}$$

the above bracket is equal to $\Delta = r^2 - 2Mr + a^2$, this means the determinant of the metric will be given as

$$-g = \Sigma^2\sin^2\theta, \tag{4.0.9}$$

$$(-g)^{\frac{1}{2}} = \Sigma\sin\theta. \tag{4.0.10}$$

For the above metric (4.0.1), the inverse metric is given as

$$g^{\mu\nu} = \begin{pmatrix} -\frac{r^2+a^2+\frac{2Mra^2\sin^2\theta}{\Sigma}}{\Delta} & 0 & 0 & \frac{-2Mra}{\Sigma\Delta} \\ 0 & \frac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & .0 \\ \frac{-2Mra}{\Sigma\Delta} & 0 & 0 & \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta} \end{pmatrix}, \tag{4.0.11}$$

undergoing Einstein's summation over both α and β from zero to three in equation (4.0.6)

$$\begin{aligned}&(-g)^{\frac{1}{2}}g^{00}\Phi_{,00} + (-g)^{\frac{1}{2}}g^{30}\Phi_{,30} + (-g)^{\frac{1}{2}}_{,1}g^{11}\Phi_1 + (-g)^{\frac{1}{2}}g^{11}_{,1}\Phi_{,1} + (-g)^{\frac{1}{2}}g^{11}\Phi_{,11} \\ &+ (-g)^{\frac{1}{2}}_{,2}g^{22}\Phi_{,2} + (-g)^{\frac{1}{2}}g^{22}_{,2}\Phi_{,2} + (-g)^{\frac{1}{2}}g^{22}\Phi_{,22} + (-g)^{\frac{1}{2}}g^{03}\Phi_{03} + (-g)^{\frac{1}{2}}g^{33}\Phi_{,33} = 0,\end{aligned} \tag{4.0.12}$$

using equation (4.0.11) for the inverse metric and equation (4.0.10) for the determinant. The above equation is given as

$$\begin{aligned}&\frac{-\Sigma\sin\theta}{\Delta}\Big[(r^2+a^2) + \frac{2Mra^2}{\Sigma}\sin^2\theta\Big]\frac{\partial^2\Phi}{\partial t^2} - \frac{4Mra\sin\theta}{\Delta} + \frac{2r\sin\theta\Delta}{\Sigma}\frac{\partial\Phi}{\partial r} + \sin\theta\frac{\Sigma(2r-2m)}{\Sigma}\frac{\partial\Phi}{\partial r} + \\ &\frac{\Delta\sin\theta}{\Sigma}\frac{\partial^2\Phi}{\partial r^2} + \frac{\Sigma\cos\theta - 2a^2\cos\theta\sin^2\theta}{\Sigma}\frac{\partial\Phi}{\partial\theta} + \frac{2a^2\cos\theta\sin^2\theta}{\Sigma}\frac{\partial\Phi}{\partial\theta} + \sin\theta + \sin\theta\frac{\partial^2\Phi}{\partial\phi^2} = 0,\end{aligned} \tag{4.0.13}$$

this can be put into the form

$$\Big[\frac{-(r^2+a^2)^2}{\Delta}+a^2\sin^2\theta\Big]\frac{\partial^2\Phi}{\partial t^2}-\frac{4Mra}{\Delta}\frac{\partial^2\Phi}{\partial\phi\partial t}+\frac{\partial}{\partial r}(\Delta\frac{\partial\Phi}{\partial r})+\cot\theta\frac{\partial\Phi}{\partial\theta}+\frac{\partial^2\Phi}{\partial\theta^2}+(\frac{1}{\sin^2\theta}-\frac{a^2}{\Delta})\frac{\partial^2\Phi}{\partial\phi^2}=0. \tag{4.0.14}$$

For the above equation choosing $\Phi = e^{-i\omega t}e^{im\phi}R(r)S(\theta)$, ω is the angular velocity of the black hole while m is quantum number that represents the projection of angular momentum. Calculating all the derivatives in (4.0.14) using the above expression for Φ. Upon substituting back into the equation and dividing by Φ we find

$$a^2\omega^2\sin^2\theta+\frac{m^2}{\sin^2\theta}-\frac{1}{S(\theta)\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dS}{d\theta})=\frac{\omega^2(r^2+a^2)^2}{\Delta}+\frac{1}{R}\frac{d}{dr}(\Delta\frac{dR}{dr})+\frac{a^2m^2}{\Delta}. \tag{4.0.15}$$

For the above equation, one side is function of θ and the other side is function of r. This means every side is equal to constant. This will be written in terms of two ordinary differential equations as follows

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\Big(\sin\theta\frac{dS}{d\theta}\Big)-\Big(a^2\omega^2\sin^2\theta+\frac{m^2}{\sin^2\theta}-C\Big)S=0, \tag{4.0.16}$$

$$\frac{d}{dr}\Big(\Delta\frac{dR}{dr}\Big)+\Big(\frac{\omega^2(r^2+a^2)^2-4Marm\omega+a^2m^2}{\Delta}-C\Big)R=0. \tag{4.0.17}$$

Introduce the so-called tortoise coordinates to our problem through the following transformation $\frac{dr^*}{dr}=\frac{r^2+a^2}{\Delta}$, implementing that transformation into equation (4.0.17) we find

$$\frac{d^2R}{dr^{*2}}+\frac{2r\Delta}{(r^2+a^2)^2}\frac{dR}{dr^*}+\Big(\omega^2+\frac{a^2m^2-4Marm\omega-\Delta C}{(r^2+a^2)^2}\Big)R=0. \tag{4.0.18}$$

For the wave that is fired toward a black hole which is far away from the Earth. This corresponds to r^* goes to infinity. Upon considering $r^*\to\infty$, equation (4.0.18) will be

$$\frac{d^2R}{dr^{*2}}+\frac{2}{r}\frac{dR}{dr^*}+\omega^2R=0, \tag{4.0.19}$$

the above equation has the following solution

$$R=\frac{c_1e^{i\omega r^*}+c_2e^{-i\omega r^*}}{r}. \tag{4.0.20}$$

The second case for equation (4.0.18) is when Δ goes to zero. In fact this is the most interesting case because as Δ goes to zero it corresponds to two values for r given as $r_+ = M+\sqrt{M^2-a^2}$ and $r_- = M-\sqrt{M^2-a^2}$ which are horizon radii. As mentioned earlier the ergoregion represents a region around the black hole which starts at $r = r_+$ and is rotating with the black hole thus has rotational kinetic energy. This is where we expect the gain in energy for the field Φ is happening. As Δ goes to zero we consider the case $r = r_+$ which is the inner boundary of the ergoregion, and equation (4.0.18) takes the following form

$$\frac{d^2R}{dr^{*2}}+\Big[\omega^2-\frac{2ma\omega}{2Mr_+}+\frac{a^2m^2}{(2Mr_+)^2}\Big]R=0, \tag{4.0.21}$$

if we considered the following substitution $\omega_+ = \frac{a}{2Mr_+}$, where ω_+ represents the angular velocity of the ergoregion at the horizon $(r = r_+)$, the above equation will be given as

$$\frac{d^2R}{dr^{*2}} + [\omega - m\omega_+]^2 R = 0, \tag{4.0.22}$$

which has the following solution

$$R = c_1 e^{i(\omega - m\omega_+)r^*} + c_2 e^{-i(\omega - m\omega_+)r^*}, \tag{4.0.23}$$

in general Φ inside the ergoregion is given as

$$\Phi(t, r, \theta, \phi) = e^{-i\omega t} e^{im\phi} e^{i(\omega - m\omega_+)r^*} S(\theta). \tag{4.0.24}$$

To show that energy can be extracted from a rotating black hole. We use the solution obtained for the scalar field Φ in the ergoregion, then we calculate the energy loss by the region. In order to calculate change of energy we have to deal with energy momentum tensor, as we know by definition for energy momentum tensor the components T^{tr} represent energy flux across surface of constant "r". In our case the surface is the horizon with the constant radius $r = r_+$. The flux for any physical quantity is the quantity per unit area per unit time. Therefore rate of change of energy is given as

$$\frac{dE}{dt} = \int T^r_t (g)^{\frac{1}{2}} d\theta d\phi. \tag{4.0.25}$$

We need T^r_t to calculate the above integral. Energy momentum tensor for a scalar field is given as

$$T_{\alpha\beta} = \frac{1}{4\pi}\Big[\Phi_{,\alpha}\Phi_{,\beta} - \frac{1}{2}g_{\alpha\beta}|\Phi_{,\gamma}\Phi^{,\gamma}|\Big]. \tag{4.0.26}$$

Using the above formula for $\alpha = r$ and $\beta = t$, we see that second term in the above equation will be zero because $g_{rt} = g_{tr} = 0$, therefore T_{rt} is given as

$$T_{rt} = \frac{1}{4\pi}\Phi_{,t}\Phi^*_{,r}, \tag{4.0.27}$$

here we considered Φ^* because Φ has a complex representation. Calculating the above formula for Φ we obtain

$$T_{rt} = \frac{1}{4\pi}\frac{\partial\Phi}{\partial t}\frac{\partial\Phi^*}{\partial r} = \frac{1}{4\pi}(-i\omega\Phi)\frac{\partial\Phi^*}{\partial r}, \tag{4.0.28}$$

since $\frac{\partial\Phi^*}{\partial r} = i(\omega - m\omega_+)(\frac{r^2+a^2}{\Delta})\Phi^*$, therefore by substituting that back at T_{rt} we obtain

$$T_{rt} = \frac{1}{4\pi}\omega(\omega - m\omega_+)\frac{r^2 + a^2}{\Delta}S^2(\theta). \tag{4.0.29}$$

To find T^r_t we have to raise the index "r" in T_{rt}, to do that we use the g^{rr} component of inverse metric tensor as follows $T^r_t = g^{rr}T_{tr} = \frac{\Delta}{\Sigma}T_{tr}$, substituting by the value of T_{tr}

$$T^r_t = \frac{1}{4\pi\Sigma}\Big[\omega(\omega - m\omega_+)(r^2 + a^2)S^2(\theta)\Big]. \tag{4.0.30}$$

Since at $r = r_+$ we have $\Delta = 0$, therefore $r^2 + a^2 = 2Mr_+$ and T^r_t is given as

$$T^r_t = \frac{2Mr_+}{4\pi\Sigma}\Big[\omega(\omega - m\omega_+)S^2(\theta)\Big]. \tag{4.0.31}$$

Given that $g^{\frac{1}{2}} = \Sigma \sin\theta$, we have the rate of change of energy given as

$$\frac{dE}{dt} = \omega(\omega - m\omega_+)\frac{Mr_+}{2\pi}\int S^2(\theta)\sin\theta d\theta d\phi. \tag{4.0.32}$$

According to the above equation $\frac{dE}{dt}$ is smaller than zero only if $\omega - m\omega_+ < 0$, which is satisfied for the solution chosen. This means that energy flows out of the surface $r = r_+$, consequently energy is extracted from a rotating black hole.

5. Conclusion

The aim of the current work is to discuss the Penrose mechanism. In chapter 2, we started with describing our mathematical framework, which is the Riemannian Geometry, as an introduction to General Relativity. We then presented a detailed description of general relativity. We next introduced the Kerr metric and discussed the Penrose process. We summarize the whole general theory of relativity in only two statements as follow:

- Mass and energy interact with spacetime in a way that curves spacetime
- Particles follow geodesics as their path in curved spacetime

Einstein's field equations are the mathematical implementation to the first statement, where field equations are relating mass and energy represented in energy-momentum tensor to curvature represented in Ricci tensor which is contained in Einstein's tensor. The cosmological constant Λ is basically defined as energy density of vacuum. Cosmological constant has been added because Einstein thought that the universe is static. When Hubble discovered that universe is expanding, Einstein removed the cosmological constant. Cosmological constant added Again after discovering of dark energy which as any form of energy curves spacetime. With the cosmological constant we can find non-zero curvature even if energy momentum tensor is zero. On the other hand geodesic equation is proved using the parallel transport of vectors along a curve.

Schwarzschild metric is the first solution for Einstein's field equations and it describes the vacuum around a spherically symmetric mass distribution. Applying Birkhoff's theorem to Schwarzschild metric implies that spacetime around any spherically symmetric bodies like stars, planets even stones is static and asymptotically flat. Kerr metric is another solution to Einstein's field equation. Kerr metric describes the geometry of vacuum ($T_{\alpha\beta} = 0$) around a rotating uncharged mass distribution. Kerr solution is the generalization to Schwarzschild solution. kerr metric reduces to Schwarzschild metric if we substitute for angular momentum per unit mass "a" by zero. According to Kerr metric any rotating black hole has a region around called ergoregion. This region is rotating as aresult of black hole rotation, thus characterized by existence of rotational kinetic energy. If any scalar wave like electromagnetic wave or gravitational wave enters that region, it gains energy in what is called Penrose process. Definitely we can not expect such energy in the case of Schwarzschild black hole (non rotating black hole), because we have only one event horizon corresponding to the singularity at $r = 2M$.

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