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Noncommutative Phenomena in Flat and Curved
Space-times:
A Master's Thesis

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Preface

This thesis aims to explore several facets of noncommutative geometry which arise in physics. In particular, our focus will be on string-inspired noncommutativity, and we will at all times try to justify the noncommutative models we study from a stringy perspective.

Chapter 1 comprises a review of noncommutative geometry and the reasons for the resurgence of interest in it in the 1990's. We begin with a brief historical introduction, followed by a discussion of the competing approaches to the subject in the literature. Next we present an in depth development of the Weyl-Moyal formalism, in which noncommutative geometry is induced by a noncommutative multiplication in the algebra of functions. In our discussion of noncommutative gauge theory, we show how this formalism may be used to generalize familiar physical systems relevant for our later work. The chapter concludes by reviewing the original discovery of noncommutative geometry in string theory. We consider both Matrix theory compactified on tori, and string theory in a background B -field, and finally illustrate the remarkable Seiberg-Witten Map.

Chapter 2 is largely a report on original work completed in collaboration with Jeff Murugan in 2003-4 and published in the Journal of High Energy Physics [1]. In it we study noncommutative solitons, in particular the vortices of the so-called critically coupled gauged linear sigma model with Fayet-Illiopolous D-terms (or semi-local model), and uncover a close relationship between these solutions and the lump solutions of the $\mathbb{C}P^N$ model. To make contact with chapter 1 we first present a few bridging results, including a brief general discussion of noncommutative solitons and their interpretation as D-branes in a background B -field. We then discuss the commutative semi-local model, and in particular derive its BPS equations whose solutions are the vortices in question. The model is then made noncommutative and we proceed to find solutions to the

corresponding noncommutative BPS equations. A study is made of their properties and in particular we show that in the limit of large gauge coupling, these solutions descend to the \mathbb{CP}^N instantons. We are able to interpret the noncommutative sigma model soliton as tilted D-strings stretched between an NS5-brane and a stack of D3-branes in type IIB superstring theory.

Chapter 3 is perhaps best thought of as the first installment in what is to be an in-depth study of whether cosmological singularities can be dealt with by including the effects of noncommutativity. In particular we focus on the problem of uniquely defining field evolution through the singularity in a big bang/big crunch space-time such as that discussed by Seiberg [66]. We first review the arguments for why a flat model which obeys the null energy condition cannot go from an expanding phase to a contracting phase in general relativity, and then discuss situations in which this seems to be possible if the two regions are separated by a singularity. We show that from the perspective of a higher dimensional gravity theory the big bang/big crunch geometry is regular, suggesting the possibility that a more fundamental theory may render field evolution through the singularity well defined. We then discuss quantum field theory on a big bang/big crunch space-time, making use of the analysis of Turok and Tolley [75] to justify our choice of vacua. Finally we introduce a simple spatial noncommutativity and study the effects this has on the quantum field theory. We calculate the 2- and 4-point correlation functions and tree-level scattering amplitudes for quadratic and quartic interactions. We discuss the dependence of the results on the noncommutativity parameter, and finally comment on possible future directions for this work.

Conventions: Unless otherwise stated, the metric has signature $(-, +, \dots, +)$. We will use i, j, \dots for spatial indices, a, b, \dots for world-sheet indices, and Greek letters for space-time indices.

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1

Introduction to Noncommutativity

1.1 A historical and conceptual introduction

Noncommutative space-times have been studied for some fifty years now, and yet it is only in the past five years that they have really come into the collective consciousness of the physics community. It began, as did so many other influential and prescient ideas, with Werner Heisenberg. This really isn't all that surprising; after all the step from the noncommutative phase space of Heisenberg's quantum mechanics to the co-ordinate algebra of noncommutative space-time, is not a large one. Heisenberg hoped that imposing some kind of space time uncertainty relation would help to fuzz out some of the divergences he had encountered (for instance the infinite self-energy of the electron) in his study of the quantized electromagnetic field. Heisenberg's idea was later picked up by a student of Oppenheimer, Hartland Snyder. His paper *Quantized Space Time*[16] was the first to take the idea that at a fundamental level, the space-time in which we live is 'fuzzy', and thus does not admit a description in terms of continuous functions with a commutative multiplication, seriously. His idea was to replace the co-ordinate functions x^i with Hermitean differential operators whose spectra were Lorentz invariant. We shall see that this approach is closely related to more sophisticated modern formulations. The next big step was the work of Moyal in 1949 [13], which placed the subject on a more respectable mathematical footing. He was able to find a noncommutative product for the algebra of functions over space-time which faithfully reproduced the canonical commutation relations for the co-ordinate functions. During the golden years of quantum field theory and the standard model, the subject of noncommutative space-times languished in the obscure backwaters of theoretical physics, much like string theory. There were bigger fish to fry. However, in the 80's and early 90's mathematicians took up the tune, and the subject of noncommutative ge-

ometry, as pioneered by Fields medalist Alain Connes [21], became a hot topic of current research. Perhaps fittingly, it was noncommutativity's partner in exile, string theory, that finally brought this beautiful subject back under the lime light; first with the advent of Matrix theory, and the work of Connes, Schwartz and Douglas in 1998 [3], and finally the seminal paper of Seiberg and Witten in 1999 [2], which established that in the $\alpha' \rightarrow 0$ limit, string theory in the presence of a constant background B -field is described by a minimally coupled super Yang-Mills theory on a noncommutative space. More than this, they were able to demonstrate a relation between ordinary and noncommutative gauge theory, now known as the Seiberg-Witten map.

While string theory has been the loving and good intentioned parent of present day approaches to noncommutativity, the child has unfortunately inherited some of the vices of its forebear. String theory is dogged by 'degeneracies' - which string theory, which vacuum, which compactification manifold? Due to the rather disjointed history of the subject, there have also arisen several different ways of approaching noncommutativity. While the study of the relations between the various alternatives in string theory received a jolt in 1995 with the discovery of dualities and D-branes, noncommutativity still awaits its 'revolution'. To be fair, the comparison is perhaps not quite a just one, as there is a definite conceptual unity to the various approaches to noncommutativity. The differences lie in the mathematical details. We will catch a brief glimpse of the main approaches to noncommutativity in current research in the next section.

Before moving on to noncommutativity proper, let us take a brief and rather heuristic look at the appearance of noncommutative space-time in conventional physics.

Firstly, let us study a very simple quantum system, first analyzed by Landau. Consider a particle of charge q constrained to move in the $x - y$ plane with a constant magnetic field \vec{B} pointing in the z direction. The Lagrangian for a charged particle moving in a magnetic field is

$$\begin{aligned} \mathcal{L} &= \frac{m}{2} v^2 - q\vec{v} \cdot \vec{A} \\ &= \frac{m}{2} \left[\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \right] - qBx \frac{dy}{dt}. \end{aligned} \quad (1.1)$$

Here \vec{v} is the particle's velocity and \vec{A} the vector potential, $\vec{B} = \nabla \times \vec{A}$. In the second line we have made the choice $A_y = Bx$ which clearly gives us the correct magnetic field. Upon quantization one finds that the energy spectrum of this system consists of infinitely degenerate *Landau levels* which are separated by an energy of $\frac{qB}{m}$. In the limit $m \rightarrow 0$ there is an infinite energy gap between the first energy level and the next, and thus

all the physics of the system is manifest in the lowest Landau level. In this case one may simplify the Lagrangian and consider only the last term in (1.2). The momentum canonically conjugate to y in this limit is

$$p_y = \frac{\partial \mathcal{L}}{\partial(dy/dt)} = -qBx. \quad (1.2)$$

The canonical commutation relation $[\hat{p}_y, \hat{y}] = -i$ ($\hbar = 1$) then implies that

$$[\hat{x}, \hat{y}] = \frac{i}{qB}. \quad (1.3)$$

We have a manifest noncommutativity in the spatial coordinate observables! Admittedly, this exposition is a bit heuristic, but it conveys the concept nicely. A less naive calculation (see [15]) shows that truncating the state space at *any* level* gives rise to a noncommutativity in the coordinates. So the notion of noncommutative space-times has a place even in conventional quantum mechanics. This effect may be seen as a simple analogue of what happens when we turn on the B field in string theory, but more on this later.

On a slightly more philosophical note, it is eminently necessary when a theory purports to describe nature, to ask whether nature is willing to admit such a description! Is noncommutativity part of the world we live in, and if it is what are the reasons for it? On a broad conceptual level it seems likely that quantum gravity, if and when it comes along, will entail some radical new view of space-time. In so much as we can consider string theory as a quantum theory of gravity, this has already been realized, at least in part, and we find that noncommutative geometry is a mandatory part of our description of nature. However even without invoking a particular candidate theory, conventional ideas of quantum mechanics and general relativity, when put together, lead naturally to some kind of 'fuzzy space-time' in the regime in which we cannot neglect either one of them. Thus if noncommutativity does indeed play a role in shaping our universe, its most important scene was at the beginning. As to what form this fuzzy space-time takes, the jury is still out. Noncommutative geometry, even if it proves not to be the correct description, is certainly a step towards it, and a fascinating subject in its own right.

1.2 Three roads to Fuzziness

In this section we shall take a brief look at the competing approaches to noncommutativity in the current literature.

* We have only considered the case $n = 1$, the lowest Landau level.

1.2.1 Co-ordinate algebra and the noncommutative product on \mathbb{R}^D

This is probably the most common approach to noncommutativity, as it is particularly well suited to considering noncommutative field theory. It will be the basis of what is to come in the following chapters, and thus deserves a section all to itself. Briefly, the idea is simply to consider a Heisenberg like co-ordinate algebra

$$[x^i, x^j] = i\theta^{ij}, \quad (1.4)$$

where the co-ordinates on \mathbb{R}^D are now represented as Hermitean operators over a Hilbert space.[†] The Hilbert space will be infinite dimensional as the operators are defined over \mathbb{R}^D . One can use the co-ordinate algebra to derive uncertainty relations of the form

$$\Delta x^i \Delta x^j \geq \frac{1}{2} |\theta^{ij}|. \quad (1.5)$$

In the case where all n th order commutators of co-ordinate operators vanish for some n , the algebra can be used to induce a noncommutative product on the algebra of functions over \mathbb{R}^D . Briefly, one can find an invertible map from the algebra of functions on \mathbb{R}^D to the ring of operators over the Hilbert space. One then uses the noncommutative product associated to operator composition to induce a noncommutative product on the algebra of functions. We shall have a great deal more to say about this in the section on Weyl-Moyal quantization.

Finally, one can in fact use this construction to define noncommutative compact spaces as well, by restricting attention to those operators that respect the defining equations or compactification conditions of the space, as the case may be. We shall have more to say on this in the case of the noncommutative torus when we come to M(atrrix) theory. For an example of this approach applied to the fuzzy disc see [23].

1.2.2 The fuzzy sphere - Matrix models

This approach, used to define a noncommutative version of embedded surfaces, was pioneered in [14]. The idea is to find a matrix representation for the co-ordinates which respects the symmetries of the embedded surface, and satisfies the equation defining it. Let's consider the case of the fuzzy sphere, \mathcal{S}^2 embedded in \mathbb{R}^3 .

The sphere can be defined through

$$g_{ab} x^a x^b = r^2, \quad (1.6)$$

[†] Note the similarity to Snyder's construction.

where the $g_{ab} = \delta_{ab}$ is the Euclidean metric, and $1 \leq a, b, \leq 3$. Let $\mathcal{C}(\mathcal{S}^2)$ denote the algebra of functions on \mathcal{S}^2 which can be written as polynomials in the co-ordinates:

$$f(x^a) = f_0 + f_a x^a + \frac{1}{2} f_{ab} x^a x^b + \dots \quad (1.7)$$

Note that $\mathcal{C}(\mathcal{S}^2)$ is dense in the algebra of smooth functions over \mathcal{S}^2 . We now define a succession of noncommutative algebras by truncating this series at the n th term, and replacing the co-ordinates with n dimensional irreducible matrix representations of $SU(2)$. For example, one may truncate the series at the quadratic term, and replace

$$x^a \rightarrow \tilde{x}^a = \kappa \sigma^a, \quad (1.8)$$

where the σ^a are the 2×2 Pauli spin matrices, and $\kappa = \frac{r}{\sqrt{3}}$ as required by (1.6). The co-ordinates now satisfy the Lie algebra of $SU(2)$:

$$[\tilde{x}^i, \tilde{x}^j] = 2i\kappa \epsilon_k^{ij} \tilde{x}^k. \quad (1.9)$$

This provides a noncommutative realization of the sphere, in which only the north and south poles may be distinguished. By considering larger and larger matrices and higher order truncations, the sphere becomes less and less fuzzy. In general, for an n dimensional representation of $SU(2)$, the sphere radius, κ and n are related by $r^2 = (n^2 - 1)\kappa^2$, from which we see that $\kappa \rightarrow 0$ as $n \rightarrow \infty$ and we regain commutativity. Note that the co-ordinate algebra used here does not fall into the Heisenberg-like class considered above, firstly since our matrices are operators on a finite dimensional Hilbert space (reflecting the finite volume of the sphere), and secondly because commutators of all orders for the generators of $SU(2)$ are non-vanishing.

This kind of noncommutativity arises naturally in string theory, in the context of D-branes in background fields. See for example [5]. While we have concentrated on the case of the fuzzy two-sphere, it is perfectly reasonable to believe that this construction can be extended to other embedded surfaces with different symmetry groups.

1.2.3 Noncommutative Differential Geometry

The study of noncommutative differential geometry was initiated by the mathematician Alain Connes, and its most complete exposition, which also contains a construction of a noncommutative standard model, may be found in his book [21], to which the reader is referred for the details of this subject. This approach differs from the previous two in that instead of regarding the algebra of functions over a commutative manifold as the

fundamental object one must deform in order to induce noncommutativity, one uses the algebra deformation to build noncommutativity straight into the differential structure of the manifold.

On a normal manifold, one expresses the fact that the derivation ∂_ν is dual to the differential dx^μ through the relation

$$dx^\mu(\partial_\nu) = \partial_\nu x^\mu = \delta_\nu^\mu . \quad (1.10)$$

In noncommutative differential geometry one defines a new derivation $e_\alpha = \text{ad}\lambda_\alpha$ such that

$$dx^\mu(\text{ad}\lambda_\alpha) = [\lambda_\alpha, x^\mu] . \quad (1.11)$$

λ^α may be thought of as a generalized Fourier transform whose particular form depends on the geometry and the particulars of the kind of noncommutativity one wishes to employ. For example, in flat space with a Heisenberg-like co-ordinate algebra with noncommutativity parameter $\theta^{\alpha\mu}$, $\lambda_\alpha = -i\theta_{\alpha\mu}^{-1}x^\mu$ is a simple linear transformation which gives rise to the standard expression, $dx^\mu e_\alpha = \delta_\alpha^\mu$. In general the co-ordinate algebra one employs is modified by the geometry. In fact, thinking of the curved noncommutative space as a perturbation of the flat noncommutative space one finds

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} (1 + O(\mu^2)) , \quad (1.12)$$

where μ^2 is a measure of the curvature perturbation.

This approach to noncommutativity has not received as much attention from physicists as the previous two, due to the fact that it is in a sense rather *ad hoc*. Unlike the matrix or Weyl-Moyal approach it is not motivated by problems which arise naturally in string theory or some other physical theory. That said, it is interesting and novel enough that it should certainly not be hastily dismissed. Some attempts have even been made to use this formulation of noncommutative geometry to resolve space-time singularities [22]. The third chapter of this thesis is devoted to precisely this question from the perspective of noncommutative field theory.

1.3 Weyl-Moyal Quantization

As we have seen, there are many approaches to the subject of noncommutative geometry. In choosing which one to use when answering a particular question about the effects of noncommutativity, one is motivated by physical considerations, from which it is often evident that one approach will provide a far more pertinent set of tools than another. Weyl-Moyal quantization is particularly suited to the study of field theory on noncommutative spaces, in part because of its close relationship to conventional

quantum mechanics, but also due to the fact that one may induce an explicit form for the noncommutative product of fields, which allows one to carry out perturbation theory in close analogy with conventional field theory. We shall be making heavy use of this machinery in the following chapters, thus making it in our interest to develop it in a thorough and rigorous (at least for a physicist) fashion.

Weyl quantization is a 'top-down' quantization procedure, in that one starts with a commutative manifold, and makes it noncommutative but by constructing a noncommutative product on the algebra of smooth functions over the manifold. That this method is successful relies on the fact that all the differential-geometric properties of a manifold can be extracted from this algebra (see [21]).

1.3.1 Weyl Symbols and the co-ordinate algebra

Suppose that we are given a Schwarzian[†] function $f(x) : \mathbb{R}^D \rightarrow \mathbb{C}$. We can find its Fourier transform via

$$\tilde{f}(k) = \int d^D x f(x) e^{-ik_i x^i} . \quad (1.13)$$

We induce a noncommutativity by replacing the co-ordinate functions x^i on \mathbb{R}^D with Hermitean operators \hat{x}^i over a Hilbert space \mathcal{H} which satisfy a Heisenberg-like co-ordinate algebra:

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij} , \quad (1.14)$$

where θ^{ij} is an anti-symmetric tensor which is in general a function of the co-ordinate operators. The function $f(x)$ is then mapped into $\hat{\mathcal{O}}(\mathcal{H})$, the ring of operators over \mathcal{H} , via an operator version of the inverse Fourier transform

$$\hat{W}[f] = \frac{1}{(2\pi)^D} \int d^D k \tilde{f}(k) e^{ik_i \hat{x}^i} . \quad (1.15)$$

This object is called the *Weyl symbol* of the function f . This mapping is the crux of the entire Weyl quantization procedure. Later we will prove that the map $W : f \rightarrow \hat{W}[f]$ is invertible.

The great insight of Weyl was that if one could indeed construct an invertible map such as W above, where $\hat{W}[f]$ is now a member of a set of objects which do not in general commute, then one could use the noncommutative product in this space to induce one in the algebra of functions over \mathbb{R}^D . This noncommutative \star -product is naturally defined through

[†] A Schwarzian function is one that satisfies a set of constraints such that its Fourier transform exists. See for example the discussion in [19]

$$f \star g(x) = W^{-1} \left(\hat{W}[f] \hat{W}[g] \right)$$

It is manifestly noncommutative, as $[\hat{W}[f], \hat{W}[g]] \neq 0$ in general. Since the map W is invertible, the pair $(\mathcal{C}^\infty(\mathbb{R}^D), \star)$ is isomorphic to the image of W with multiplication given by operator composition. Bearing this objective in mind we are now in a position to flesh out the details.

We can rewrite $\hat{W}[f]$ as

$$\hat{W}[f] = \int d^D x f(x) \hat{\Delta}(x), \quad (1.16)$$

where

$$\hat{\Delta}(x) = \frac{1}{(2\pi)^D} \int d^D k e^{ik_i \hat{x}^i} e^{-ik_i x^i}. \quad (1.17)$$

The object $\hat{\Delta}(x)$ interpolates between operators and fields. Looking at (1.16) we see that $f(x)$ can be viewed as the co-ordinate space representation of $\hat{W}[f]$ with the mixed Fourier basis function $\hat{\Delta}(x)$.

1.3.2 Derivatives, Integrals, and Non-locality

In order to consider field theory it is necessary to introduce derivative operators $\hat{\partial}_i$, and integrals. One of the most appealing features of the Weyl quantization procedure is the simple form these operations take in the operator formalism. Let's look at derivatives first:

We require derivative operators to satisfy the following properties, in analogy with the commutative case:

Linearity $\hat{\partial}_i(A\hat{f} + B\hat{g}) = A\hat{\partial}_i\hat{f} + B\hat{\partial}_i\hat{g}$

Leibniz Rule $\hat{\partial}_i(\hat{f}\hat{g}) = (\hat{\partial}_i\hat{f})\hat{g} + \hat{f}(\hat{\partial}_i\hat{g})$,

where A and B are commuting objects. Together these two conditions imply that the derivative of a commuting function is zero:

$$\begin{aligned} \hat{\partial}_i(C\hat{f}) &= C\hat{\partial}_i\hat{f} = (\hat{\partial}_i C)\hat{f} + C\hat{\partial}_i\hat{f} \\ \Rightarrow \hat{\partial}_i C &= 0. \end{aligned} \quad (1.18)$$

Note that from the Leibniz rule the following identity holds:

$$[\hat{\partial}_i, \hat{x}^j] = \hat{\partial}_i \hat{x}^j. \quad (1.19)$$

We also impose the following on the derivative operator:

Orthogonality $\hat{\partial}_i \hat{x}^j = [\hat{\partial}_i, \hat{x}^j] = \delta_i^j$

Mutual commutativity $[\hat{\partial}_i, \hat{\partial}_j] = 0$

The last condition is often generalized in physical applications, but it will serve our purposes. The operator $\hat{\partial}_i$ satisfying these conditions is anti-hermitean.

We wish now to find the operator which corresponds to the derivative of a function, $\partial_i f$. We will show that

$$\boxed{\hat{W}[\partial_i f] = [\hat{\partial}_i, \hat{W}[f]]}$$

Proof: By the Leibniz rule,

$$[\hat{\partial}_i, \hat{\Delta}(x)] = \hat{\partial}_i \hat{\Delta}(x) . \quad (1.20)$$

From (1.17) we see that

$$\begin{aligned} \hat{\partial}_i \hat{\Delta}(x) &= ik_i \hat{\Delta}(x) \\ &= -\partial_i \hat{\Delta}(x) , \end{aligned} \quad (1.21)$$

where we have used

$$\hat{\partial}_i e^{ik_i \hat{x}^i} = ik_i e^{ik_i \hat{x}^i} , \quad (1.22)$$

which follows immediately upon using the Taylor expansion definition of the exponential of an operator. It follows that

$$[\hat{\partial}_i, \hat{\Delta}(x)] = -\partial_i \hat{\Delta}(x) \quad (1.23)$$

Using this result we obtain

$$\begin{aligned} [\hat{\partial}_i, \hat{W}[f]] &= \left[\hat{\partial}_i, \int d^D x f(x) \hat{\Delta}(x) \right] \\ &= \int d^D x f(x) [\hat{\partial}_i, \hat{\Delta}(x)] \\ &= - \int f(x) \partial_i \hat{\Delta}(x) \end{aligned} \quad (1.24)$$

Integrating by parts and assuming that $f(x)$ falls off sufficiently quickly to make the boundary term vanish we find

$$[\hat{\partial}_i, \hat{W}[f]] = \int d^D x \partial_i f \hat{\Delta}(x) = \hat{W}[\partial_i f] . \quad (1.25)$$

■

Thus the operator corresponding to the derivative of a field is nothing but the commutator of the field operator and the derivative operator. Later we shall simplify this expression even further.

Having dealt with derivatives for now, let's see if we can find some analogue of the integral of a noncommuting function in the operator representation. We now prove that

$$\boxed{\int f(x) d^D x = \text{Tr} \hat{W}[f]}$$

where where we have chosen a convenient normalization, described below.

Proof: The normal partial derivatives are of course, the generators of translations:

$$e^{v^i \partial_i} f(x) = f(x + v) . \quad (1.26)$$

From (1.23) we can write the action of the translation generators on $\hat{\Delta}(x)$ as

$$\partial_i \hat{\Delta} = \hat{\Delta} \hat{\partial}_i - \hat{\partial}_i \hat{\Delta} . \quad (1.27)$$

From this it is clear that

$$e^{v^i \hat{\partial}_i} \hat{\Delta}(x) = \hat{\Delta}(x + v) = e^{v^i \hat{\partial}_i} \hat{\Delta}(x) e^{-v^i \hat{\partial}_i} . \quad (1.28)$$

If we now hypothesize a cyclic trace on $\hat{\mathcal{O}}(\mathcal{H})$ the above relation tells us that

$$\text{Tr} \left(e^{v^i \hat{\partial}_i} \hat{\Delta}(x) e^{-v^i \hat{\partial}_i} \right) = \text{Tr} \left(\hat{\Delta}(x) \right) = \text{Tr} \left(\hat{\Delta}(x + v) \right) , \quad (1.29)$$

or in words, $\text{Tr} \left(\hat{\Delta}(x) \right)$ is independent of x . We are thus free to set a normalization for this quantity. We will set

$$\text{Tr} \left(\hat{\Delta}(x) \right) = 1 . \quad (1.30)$$

From the definition (1.17) of $\hat{\Delta}(x)$ we see that this implies that

$$\text{Tr} \left(e^{i k_i \hat{x}^i} \right) = (2\pi)^D \prod_{i=0}^{D-1} \delta(k_i) . \quad (1.31)$$

Taking the trace of the definition (1.16) of $\hat{W}[f]$ we find, with the normalization (1.30), that

$$\int d^D x f(x) = \text{Tr} \left(\hat{W}[f] \right) . \quad (1.32)$$

■

A generic feature of noncommutative theories is an inherent non-locality. The following calculations make this explicit. Note that the result follows only from the co-ordinate algebra.

We wish to prove the following:

$$\boxed{e^{ik_j \hat{x}^j} f(x^i) e^{-ik_j \hat{x}^j} = f(x^i - \theta^{ij} k_j)}$$

Proof: First, it is easy to prove from the co-ordinate algebra that

$$[\hat{x}^i, (\hat{x}^j)^n] = in\theta^{ij}(\hat{x}^j)^{n-1} \quad (1.33)$$

where no summation is implied. From here it is trivial to show, upon expanding one of the exponentials, that

$$e^{ik_j \hat{x}^j} \hat{x}^i e^{-ik_j \hat{x}^j} = \hat{x}^i - \theta^{ij} k_j \quad (1.34)$$

The result then follows easily by applying this to a function $f(\hat{x}^i)$ defined through its Taylor expansion. ■

Physically, what this result tells us is that multiplication by plane waves leads to non-local correlations between separated space-time points.

We will now proceed to give a slightly simpler expression for the derivative operator in the case where θ^{ij} is a constant. First it is easy to see from (1.34) that

$$\theta^{ij} k_j e^{ik \cdot \hat{x}} = [\hat{x}^i, e^{ik \cdot \hat{x}}] \quad (1.35)$$

We can invert this equation to find

$$k_j e^{ik \cdot \hat{x}} = -(\theta^{-1})_{ji} [\hat{x}^i, e^{ik \cdot \hat{x}}] \quad (1.36)$$

provided of course that the matrix θ with components θ^{ij} is invertible. Since we must have $\theta = -\theta^T$, taking determinants of both sides of this equation leads us to the conclusion that θ is only invertible when the space-time dimension (or at least the number of noncommuting co-ordinates) is even. We will take this to be the case in all that follows.[§]

Recall from (1.25),(1.16) that

$$\hat{W}[\partial_i f] = [\hat{\partial}_i, \hat{W}[f]] \quad (1.37)$$

[§] Note that the construction is no less valid if D is odd, however one cannot in general find a closed form for the derivative in this case.

$$\begin{aligned}
&= \hat{\partial}_i \hat{W}[f] \\
&= \int d^D x f(x) \hat{\partial}_i \hat{\Delta}(x) \\
&= i \int d^D x f(x) \int \frac{d^D k}{(2\pi)^D} (k_i e^{ik \cdot \hat{x}}) e^{-ik \cdot x} \\
&= i \int d^D x f(x) \int \frac{d^D k}{(2\pi)^D} (-(\theta^{-1})_{ij} [\hat{x}^j, e^{ik \cdot \hat{x}}]) e^{-ik \cdot x} \\
&= -i(\theta^{-1})_{ij} \int d^D x f(x) [\hat{x}^j, \hat{\Delta}(x)] \\
&= -i(\theta^{-1})_{ij} [\hat{x}^j, \hat{W}[f]] .
\end{aligned}$$

We will use this result often in our discussion of noncommutative field theory.

1.3.3 The inverse Weyl map, the Baker-Campbell-Hausdorff formula, and the Moyal \star -product

Finding the inverse Weyl map $W^{-1} : \hat{\mathcal{O}}(\mathcal{H}) \rightarrow C^\infty(\mathbb{R}^D)$ such that $W^{-1}(\hat{W}[f]) = f$ is a crucial step in constructing a noncommutative \star -product on the algebra of functions, since we intend to define

$$f \star g = W^{-1}(\hat{W}[f] \hat{W}[g]) . \quad (1.38)$$

Looking at this we see that the noncommutativity is directly induced by the operator product $\hat{\Delta}_f(x) \hat{\Delta}_g(y)$ which in turn depends on the product $e^{ik \cdot x} e^{ik' \cdot x'}$. This term may be calculated via the Baker-Campbell-Hausdorff (BCH) formula which in full generality reads

$$\log(e^X e^Y) = X + \int_0^1 g(e^{\text{ad}_X} e^{t \text{ad}_Y})(Y) dt , \quad (1.39)$$

where X, Y are operators, $\text{ad}_X = [X, \cdot]$, and $g(z) = \left(\frac{1-z^{-1}}{\log z}\right)$. One can make a series expansion of this formula in which the n th term depends on the n th order commutator:

$$e^X e^Y = e^{X+Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X-Y, [X, Y]] + \dots} \quad (1.40)$$

In principal we could use the previous formula to calculate as many terms as we liked in this expansion. However, for practical purposes, in order to find a closed form for the \star -product we will require all commutators greater than a certain order to vanish. This is certainly the case in the

situation of most interest to us, in which θ^{ij} is a constant. We will treat this case in some detail in what follows, bearing in mind that the manipulations we make will be possible in the more general case of vanishing n th order commutator, but significantly more complex.

Before we go on to actually find the form of the \star -product, it is necessary to find an explicit form for the the map W^{-1} . In order to do this we will need an orthogonality relation which will allow us to invert the definition of $\hat{W}[f]$. We will find that for constant θ^{ij}

$$\boxed{\text{Tr} \hat{\Delta}(x) \hat{\Delta}(y) = \delta^D(x - y)}$$

Proof:

$$\begin{aligned} \hat{\Delta}(x) \hat{\Delta}(y) &= \int \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} e^{-i(k \cdot x + k' \cdot y)} e^{ik_i \hat{x}^i} e^{k'_j \hat{x}^j} \\ &= \int \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} e^{-i(k \cdot x + k' \cdot y)} e^{-\frac{i}{2} \theta^{ij} k_i k'_j} e^{i(k+k')_i \hat{x}^i}. \end{aligned} \quad (1.41)$$

Now it is easy to show that

$$e^{i(k+k')_i \hat{x}^i} = \int d^D z e^{i(k+k')_i z^i} \hat{\Delta}(z). \quad (1.42)$$

Substituting this into our expression for $\hat{\Delta}(x) \hat{\Delta}(y)$ we can now collect terms in the exponent that depend on k' and perform the k' integration to obtain

$$\begin{aligned} \hat{\Delta}(x) \hat{\Delta}(y) &= \int d^D z \hat{\Delta}(z) \int \frac{d^D k}{(2\pi)^D} e^{ik \cdot (z-x)} \prod_{j=0}^{D-1} \delta(z^j - y^j - \frac{1}{2} \theta^{ij} k_i) \\ &= \frac{1}{\pi^D |\det \theta|} \int d^D z \hat{\Delta}(z) e^{2i(\theta^{-1})_{ij} (z^i - x^i)(z^j - y^j)}, \end{aligned} \quad (1.43)$$

where we have used the well known fact

$$\delta(g(x)) = \sum_n \frac{1}{|g'(x_n)|} \delta(x - x_n) \quad \text{where } g(x_n) = 0. \quad (1.44)$$

If we now take the trace of this relation the $\hat{\Delta}(z)$ falls away and we can perform the integration using standard methods to find the desired result.

■

Using this theorem it is now trivial to show that

$$f(x) = \text{Tr} \left(\hat{W}[f] \hat{\Delta}(x) \right)$$

Proof: Just multiply the definition (1.16) of $\hat{W}[f]$ by $\hat{\Delta}(y)$ and take the trace of both sides. ■

This shows that the Weyl map is one-to-one, and hence an invertible map over its image.

We are now finally in a position to calculate the explicit form of the \star -product for constant θ^{ij} . We will find that it takes the following form:

$$f \star g(x) = e^{\frac{i}{2}\theta^{ij}\partial_i\partial'_j} f(x)g(x')|_{x'=x}$$

Proof: From (1.3.1) and the fact that W is invertible we must have $\hat{W}[f \star g] = \hat{W}[f]\hat{W}[g]$. Now on the one hand,

$$\hat{W}[f \star g] = \int \frac{d^D k}{(2\pi)^D} \widetilde{f \star g}(k) e^{ik \cdot \hat{x}}, \quad (1.45)$$

while on the other, from (1.42) and (1.13),

$$\begin{aligned} \hat{W}[f]\hat{W}[g] &= \int \int d^D x d^D y f(x)g(y) \hat{\Delta}(x) \hat{\Delta}(y) \\ &= \int \int d^D x d^D y f(x)g(y) \left[\int \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} e^{i(k+k') \cdot \hat{x}} e^{-\frac{i}{2}\theta^{ij} k_i k'_j} e^{-i(k \cdot x + k' \cdot y)} \right] \\ &= \int \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} \tilde{f}(k) \tilde{g}(k') e^{i(k+k') \cdot \hat{x}} e^{-\frac{i}{2}\theta^{ij} k_i k'_j}. \end{aligned} \quad (1.46)$$

Equating these two expressions allows one to solve for $\widetilde{f \star g}(k)$. The inverse Fourier transform then gives us

$$f \star g(x) = \int \int \frac{d^D k}{(2\pi)^D} \frac{d^D k'}{(2\pi)^D} \tilde{f}(k) \tilde{g}(k' - k) e^{-\frac{i}{2}\theta^{ij} k_i k'_j} e^{ik' \cdot x}, \quad (1.47)$$

where we have used $\tilde{g}(k' - k) = \tilde{g}(k') e^{ik \cdot x}$. The convolution theorem for Fourier integrals now gives the result. ■

Let's explore this remarkable result a little more. Firstly, it is easy to verify that

$$x^i \star f(x) - f(x) \star x^i = i\theta^{ij} \partial_j f(x). \quad (1.48)$$

This is the analogue of (1.38). Second, since the trace is cyclic, we have

$$\begin{aligned} \int f \star g d^D x &= \text{Tr} \left(\hat{W}[f \star g] \right) \\ &= \text{Tr} \left(\hat{W}[f] \hat{W}[g] \right) \\ &= \int g \star f d^D x . \end{aligned} \tag{1.49}$$

In fact, by using the exponential definition of the \star -product, and repeatedly integrating by parts, we can show

$$\int f \star g d^D x = \int f g d^D x . \tag{1.50}$$

This is a particularly significant property of the \star -product, as it tells us that at the level of free fields, noncommutative field theory is identical to commutative field theory. One must introduce cubic or higher order terms in the action in order to uncover the effects of the noncommutativity.

1.4 Noncommutative Gauge Theory

Noncommutative gauge theory arises in the study of D-branes in background fields [2]. D-branes are open-string degrees of freedom; open strings end on them. In the presence of a constant background Neveu-Schwartz B -field, the endpoints of the string - the co-ordinates of the D-brane world-volume - are noncommutative, and satisfy a canonical co-ordinate algebra with constant θ^{ij} which is related to the B field. In the $\alpha' \rightarrow 0$ limit the field theory which lives on the world-volume of a D-brane is a Yang-Mills theory, as may be seen by expanding the Dirac-Born-Infeld action which describes its low energy dynamics in powers of the gauge field-strength $F^{\mathfrak{A}}$. Hence we see that if we wish to study the world-volume theory in the presence of a background B field, we are going to need to know something about noncommutative gauge theory.

We will restrict our discussion to noncommutative Yang-Mills theory, with a view to working towards understanding noncommutative field theories that admit vortex-like solitonic solutions. In chapter 2 we will be looking at the noncommutative semi-local model, a generalisation of the Abelian-Higgs model which possesses a $U(1)$ gauge symmetry. Our discussion will be more general than what is needed to discuss this model, but the results are easily specialized, and generality often breeds greater understanding. Our discussion follows that of [19].

[¶] For a more sophisticated way of seeing this see [2]

Firstly, a word about which gauge groups it is possible to realize in the noncommutative theory. It is perhaps rather surprising that it is not possible to consider $SU(N)$ gauge theory. This is a simple consequence of the fact that for two matrix-valued objects A, B , $\det(A \star B) \neq \det(A) \star \det(B)$, and thus $SU(N)$ cannot close under the star product. The unitary groups, $U(N)$, are however still safe, owing to the fact that $(A \star B)^\dagger = B^\dagger A^\dagger$. This will be the only case of relevance for us.

In commutative $U(N)$ Yang-Mills theory, a Hermitean matrix-valued gauge field may be written as $A_i = A_i^a t_a$. Here t_a are the $N \times N$ orthogonal generators of $U(N)$:

$$[t_a, t_b] = i f_{ab}^c t_c, \quad (1.51)$$

$$\text{tr}_N(t_a t_b) = \delta_{ab}, \quad (1.52)$$

where tr_N is the matrix trace. The action is given by

$$S_{YM} = -\frac{1}{4g^2} \int \text{tr}_N (F^{ij} F_{ij}) d^D x, \quad (1.53)$$

where $F = dA + A \wedge A$ is the Yang-Mills field strength. In terms of components,

$$F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i, A_j]. \quad (1.54)$$

This action possesses a local gauge symmetry. Let $g \in U(N)$. Then under the gauge transformation $A_i \rightarrow g A_i g^\dagger - i g \partial_i g^\dagger$ the field-strength is transformed to $F_{ij} \rightarrow g F_{ij} g^\dagger$, and hence the action is invariant. We are interested in the noncommutative version of this model.

Obviously we are going to need a noncommutative gauge field. This is effortlessly provided by the Weyl map:

$$\hat{W}[A]_i = \int d^D x \hat{\Delta}(x) \otimes A_i(x), \quad (1.55)$$

where we have written a tensor product to remind us that we are dealing with the product of the noncommutative basis function with a matrix-valued object. The natural rule of thumb when going from a commutative to a noncommutative theory is to replace all conventional products with star products. Thus for instance the noncommutative field strength is

$$\begin{aligned} F_{ij}^\star &= \partial_i A_j - \partial_j A_i - i(A_i \star A_j - A_j \star A_i) \\ \hat{F}_{ij} &= [\hat{\partial}_i, \hat{W}[A]_j] - [\hat{\partial}_j, \hat{W}[A]_i] - i [\hat{W}[A]_i, \hat{W}[A]_j]. \end{aligned} \quad (1.56)$$

It is easy to show that for a unitary operator $\hat{W}[g]$, if the gauge field transforms as

$$\hat{W}[A]_i \rightarrow \hat{W}[g] \hat{W}[A]_i (\hat{W}[g])^\dagger - i \hat{W}[g] [\hat{\partial}_i, (\hat{W}[g])^\dagger] \quad (1.57)$$

then the field strength transforms like $\hat{F}_{ij} \rightarrow \hat{W}[g]\hat{F}_{ij}(\hat{O}[g])^\dagger$ and thus the action

$$S_{YM}^{nc} = -\frac{1}{g^2} \text{Tr} \left(\text{tr}_N \hat{F}_{ij} \hat{F}^{ij} \right)$$

is invariant. This action defines noncommutative $U(N)$ Yang-Mills theory.

Let's look a little closer at the unitary operator $\hat{W}[g]$. A moment's thought will reveal that we must have $\hat{W}[g] \in U(\mathcal{H}) \otimes U(N)$. In other words it must be unitary as a Hilbert space operator, and as a matrix. Thus in a certain sense, we have actually gauged the infinite dimensional symmetry group, $U(\mathcal{H})$. We may write $\hat{W}[g]$ using the Weyl map as

$$\hat{W}[g] = \int d^D x \hat{\Delta}(x) \otimes g(x) \quad (1.58)$$

where $g(x)$ is matrix valued. Since we require

$\hat{W}[g]^\dagger \hat{W}[g] = \hat{W}[g] \hat{W}[g]^\dagger = \hat{I} \otimes I_N$, it is easy to see that we must have

$$g(x)^\dagger \star g(x) = g(x) \star g(x)^\dagger = I_N . \quad (1.59)$$

We say that $g(x)$ is a star-unitary matrix. This provides a star-product realization of the gauge transformation rules for A_i^* and F_{ij}^* in the obvious manner.

We can now couple the Yang-Mills field to a (adjoint) scalar in the standard fashion by introducing the covariant derivative,

$$\begin{aligned} D_i \phi &= \partial_i \phi + i[A_i, \phi] \\ D_i^* \phi &= \partial_i \phi + i(A_i \star \phi - \phi \star A_i) \\ \hat{W}[D_i \phi] &= \left[\hat{\partial}_i, \hat{W}[\phi] \right] + i \left[\hat{W}[A]_i, \hat{W}[\phi] \right] \end{aligned} \quad (1.60)$$

The coupled action usually takes the form

$$S = \int dt \text{Tr} \left(\hat{W}[D_i \phi]^2 - \hat{F}_{ij} \hat{F}^{ij} - \hat{V}(\phi) \right) , \quad (1.61)$$

where $V(\phi)$ is a gauge invariant potential. The extension of this discussion to supersymmetric Yang-Mills theory is readily achieved by adding the relevant fermionic part to the above action, but this will not concern us here.

It turns out to be illuminating to simplify the covariant derivative operator by appealing to our expression (1.38) for the derivative in the operator formalism. If we define the covariant gauge variable

$$\hat{C}_i \equiv -i(\theta^{-1})_{ij} \hat{x}^j \otimes I_N + i\hat{W}[A]_i \quad (1.62)$$

then the (adjoint) covariant derivative is just

$$\hat{W}[D_i\phi] = [\hat{C}_i, \hat{W}[\phi]] . \quad (1.63)$$

Using this and the fact that $F_{ij} = i[D_i, D_j]$, one can then show that in terms of the \hat{C}_i , $F_{ij} = i[C_i, C_j] - (\theta^{-1})_{ij}$ and hence the pure Yang-Mills action takes the form,

$$S_{YM}^{nc} = -\frac{1}{4g^2} \text{Tr} \left(\text{tr}_N \left(i[\hat{C}_i, \hat{C}_j] - (\theta^{-1})_{ij} \right)^2 \right)$$

In the case $N = 1$ this is just an infinite dimensional matrix model. It has been noted that we could easily turn the argument just presented on its head by starting with a matrix model and working backwards to a noncommutative Yang-Mills theory, which becomes commutative in the limit $\theta \rightarrow 0$. Thus Yang-Mills theory may be arrived at from matrix theory. We will have more to say on this in the next section.

1.5 Noncommutativity in String/M(atrrix) Theory

The purpose of this section is to provide a physical context for the coming chapters. There are many situations in String/M-theory in which noncommutative geometry arises. We will only consider the two most famous and generic examples. The first shows that in a certain limit string theory in a constant background B -field describes a noncommutative field theory. This was first pointed out by Seiberg and Witten in 1999[2]. The second concerns M(atrrix) theory compactified on a torus, and was discovered by Connes, Schwartz, and Douglas in 1998[3]. Since then noncommutative phenomena have been unearthed in a wide variety of stringy scenarios, suggesting that it plays a fundamental and as yet only partially perceived role in whatever the ultimate formulation of the theory will be.

1.5.1 Matrix theory and compactification on tori

Matrix theory may be arrived at in a variety of ways. Historically, matrix theory was first seen as the strong coupling limit ($g_{string} \rightarrow \infty$) of type IIA string theory. It may also be viewed as the the $N \rightarrow \infty$ limit of the non-relativistic quantum mechanics of N D0-branes in the weakly coupled limit of the IIA theory, or as a regularized form of the 11 dimensional supermembrane theory. These are all sophisticated points of view which require significant justification (see for instance [9] and references therein). It will suit our purposes to view the subject from the point of view of dimensionally reduced Yang-Mills theory. That this has any relation to

string theory is a highly non-trivial and altogether amazing fact which we will unfortunately only have space to justify in part. Our presentation follows closely that of [8].

Consider the standard $U(N)$ Yang-Mills action functional,

$$S = -\frac{1}{4g_{YM}^2} \int d^D x \text{tr}_N (F_{ij} F^{ij}) , \quad (1.64)$$

where $F_{ij} = \partial_i A_j - \partial_j A_i - i[A_i, A_j]$. The trace appearing in the action above is actually a special case particular to the Lie algebra $u(N)$ of an invariant inner product $\langle \cdot, \cdot \rangle$ which satisfies $\langle [a, b], c \rangle = \langle a, [b, c] \rangle$ for all $a, b, c \in \mathfrak{g}$, where \mathfrak{g} is now a general Lie algebra. It is evident for instance that for $u(N)$, $\langle a, b \rangle = \text{Tr} a^* b$. One may define Yang-Mills theory over a general Lie algebra \mathfrak{g} with inner product $\langle \cdot, \cdot \rangle$ via

$$S = \frac{1}{4} \int \langle F_{ij}, F^{ij} \rangle , \quad (1.65)$$

where the field-strength is defined in the normal way, provided we interpret the commutator of the gauge fields as the Lie algebra commutator associated with \mathfrak{g} . We have also chosen to set $g_{YM} = 1$. This theory is invariant under the usual gauge transformations.

To reduce this general theory to a point we simply consider constant fields, and normalize the infinite space-time volume in the action integral to one. The resulting action has the form

$$S[X] = \frac{1}{4} \langle [X_i, X_j], [X^i, X^j] \rangle , \quad (1.66)$$

where we have relabelled $A_i \rightarrow X_i$. This action possesses an obvious residual symmetry of the full gauge symmetry, $X_i \rightarrow g X_i g^{-1}$ where $g \in G$, the Lie group associated to \mathfrak{g} .

In order to make contact with string theory, let's consider a special case of the above action in which our base space is a two dimensional symplectic manifold^{||} Σ with symplectic form $\omega = \omega(\tau, \sigma) d\tau \wedge d\sigma$ where σ and τ are local co-ordinates on Σ . We also take our Lie algebra \mathfrak{g} to be the set of smooth functions $X : \Sigma \rightarrow \mathbb{R}^D$. The natural commutator on \mathfrak{g} is the Poisson bracket,

$$[X, Y] \rightarrow \{X, Y\}_{PB} = \frac{i}{\omega(\tau, \sigma)} \left(\frac{\partial X}{\partial \tau} \frac{\partial Y}{\partial \sigma} - \frac{\partial X}{\partial \sigma} \frac{\partial Y}{\partial \tau} \right) . \quad (1.67)$$

^{||} A symplectic manifold is a pair (\mathcal{M}, ω) where \mathcal{M} is a smooth $2n$ dimensional manifold and ω a closed nondegenerate 2-form. ω^n is nowhere vanishing and hence defines a volume form. It can be shown that the algebra of functions over \mathcal{M} has a natural Lie algebra structure with bracket given by the Poisson bracket.

We can define an inner product of $X, Y \in \mathfrak{g}$ through

$$\langle X, Y \rangle = \int XY \omega(\tau, \sigma) d^2 \sigma . \quad (1.68)$$

Thus in this special case the reduced action takes the form

$$S[X, \omega] = \frac{1}{4} \int_{\Sigma} \{X^i, X^j\}_{PB}^2 \omega . \quad (1.69)$$

Now this action may be generalized slightly by adding a kind of cosmological constant term,

$$S[X, \omega] = \frac{\alpha}{4} \int_{\Sigma} \{X^i, X^j\}_{PB}^2 \omega + \frac{\beta}{2} \int_{\Sigma} \omega . \quad (1.70)$$

This extra term does not affect the gauge invariance. Note also that we have explicitly written the action as a functional of the symplectic form ω in order to emphasize its arbitrariness. In fact we can vary this action with respect to $\omega(\tau, \sigma)$ and find the equation of motion

$$-\frac{\alpha}{4[\omega(\tau, \sigma)]^2} (\epsilon^{ab} \partial_a X^i \partial_b X^j)^2 + \frac{\beta}{2} = 0 . \quad (1.71)$$

Using the readily verified fact that $(\epsilon^{ab} \partial_a X^i \partial_b X^j)^2 = 2 \det(\partial_a X^i \partial_b X^j)$ we can solve for $\omega(\tau, \sigma)$ and plug back into the action to find

$$S[X] = \sqrt{\alpha\beta} \int_{\Sigma} \sqrt{\det(\partial_a X^i \partial_b X^j)} d^2 \sigma . \quad (1.72)$$

This is of course nothing but the Nambu-Goto action for the bosonic string! Thus we see that our manifold Σ may be interpreted as the string world-sheet and the fields X^i as maps into D dimensional space-time.

This is the essence of this approach to Matrix theory. By a suitable choice of Lie algebra $(\mathfrak{g}, [\cdot, \cdot])$ and inner product $\langle \cdot, \cdot \rangle$ we may obtain string theory from a Yang-Mills gauge theory reduced to a point. So where do the matrices come in? This is answered in part by noticing that the action (1.70) can be interpreted as the $N \rightarrow \infty$ limit of

$$S[X] = \frac{\alpha}{4} \text{tr}_N ([X^i, X^j]^2) + \frac{\beta}{2} \text{tr}_N I , \quad (1.73)$$

where the X^i are now $N \times N$ Hermitean matrices. This can be justified by noting that in quantizing a theory on the symplectic manifold (Σ, ω) we associate to each function** f an operator \hat{f} on a Hilbert space of

** For instance via Weyl quantization

dimension N such that in the limit $\hbar \rightarrow 0$ we have $\{f, g\} \rightarrow \hbar^{-1}[\hat{f}, \hat{g}]$ and $\text{Tr} \hat{f} \rightarrow (2\pi\hbar)^{-1} \int f \omega d^2\sigma$. Thus for small but finite \hbar we obtain a finite dimensional Hilbert space since

$$N = \text{tr}_N I \approx \frac{\text{Vol}(\Sigma)}{2\pi\hbar}, \quad (1.74)$$

in which case the observable operators may be represented as $N \times N$ hermitean matrices. The classical limit $\hbar \rightarrow 0$ in which these correspondences become exact is equivalent to $N \rightarrow \infty$.

Now we notice something interesting. We have in fact encountered this before! Recall the expression we obtained for S_{YM}^{nc} in terms of the covariant gauge field \hat{C}_i , and specialize to the case $N = 1$. This action is then exactly equivalent, apart from irrelevant constant factors that can be absorbed into field definitions, to the $N \rightarrow \infty$ limit of the reduced Yang-Mills theory, in other words, the bosonic string! Thus we see that noncommutative geometry is intimately tied into string theory right from the start.

We have seen how we can arrive at bosonic string theory as the $N \rightarrow \infty$ limit of the action (1.66) with the X^i hermitean matrices. It therefore seems plausible at least that if we consider super Yang-Mills theory in ten dimensions, and go through similar manipulations, we should end up with one of the superstring theories. It turns out that this is indeed the case. It was shown in [10] that one could obtain IIB superstring theory in the Green-Schwarz formulation in the so-called Schild gauge using precisely this method. The resulting matrix theory, commonly known as the IKKT model, can be regarded as providing a nonperturbative definition of IIB string theory.

Suppose now we go back to the beginning and instead of reducing the Yang-Mills action (1.65) to a point, we reduce it to $(0+1)$ dimensions. We also take $\mathfrak{g} = u(N)$, and $D = 10$. The reduced action can then be shown to take the form

$$S = \int dt \text{Tr} \left(\frac{1}{2} D_t X^i D_t X_i - \frac{1}{4} [X_i, X_j]^2 \right), \quad (1.75)$$

where $D_t = \partial_t + iX_0$. This is the bosonic part of the low-energy matrix quantum mechanics action for a system of many D0-branes in the type IIA theory. Had we considered the reduction of 10 dimensional $U(N)$ Super Yang-Mills, we would have obtained extra fermionic terms. The resulting action defines the so-called BFSS model [11], which plays a central role in M-theory. This theory can also be obtained from the regularized supermembrane action [10]. The regularization one employs can be viewed as a geometric quantization of the algebra of functions on the membrane,

and is thus formally identical to Weyl quantization, except that the operators one uses act on a finite dimensional Hilbert space, and can thus be viewed as matrices. It is therefore reasonable to view the action (1.75) as that of a fuzzy membrane with matrix valued co-ordinates. Once again noncommutativity appears at the most fundamental level.

The resurgence of interest in noncommutativity in the late 90's was significantly contributed to by the consideration of the compactification of M(atrix) theory on tori. Let's briefly see how this is realized.

Suppose we wish to compactify one of the target-space dimensions, say X_1 , in either the IKKT or the BFSS matrix model on a circle of radius R . If X_1 were a conventional co-ordinate function this could be achieved by simply imposing the equivalence relation $X_1 \sim X_1 + 2\pi R$. However since X_1 is in fact a matrix gauge field in a gauge theory invariant under $X_i \rightarrow UX_iU^{-1}$ where $U \in U(N)$, the only way this equivalence makes sense is as a gauge equivalence. In other words to compactify the X^1 direction we must satisfy

$$UX_1U^{-1} = X_1 + 2\pi RI \quad (1.76)$$

for U a unitary matrix. It is clear, upon taking the trace of the above equation, that that no finite dimensional solutions exist.^{††} However solutions do exist in terms of operators on an infinite dimensional Hilbert space. Recall from our discussion on Weyl quantization that we derived the relation (1.34),

$$e^{ik \cdot \hat{x}} \hat{x}^i e^{-ik \cdot \hat{x}} = \hat{x}^i - \theta^{ij} k_j. \quad (1.77)$$

Lets suppose that we are in a two dimensional space, and set

$$\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad (1.78)$$

and $k_1 = 0$, $k_2 = 2\pi R$. Then this exactly reproduces (1.76) provided we make the identification $X_1 \rightarrow \hat{x}^1$ and $U \rightarrow e^{2\pi i \hat{x}^2 R}$. Note that for this choice the X_2 direction is still noncompact as $k_1 = 0 \Rightarrow UX_2U^{-1} = X_2$ as required if we identify $X_2 \rightarrow \hat{x}^2$. This solution is very suggestive, but a little contrived as we don't expect the X_1 co-ordinate to couple in any special way to one of the others, as is required to have an invertible θ matrix. It does however strongly suggest that we see what happens when we compactify another direction, say X_2 on a circle of radius R' . We would now need to satisfy $U'X_2U'^{-1} = X_2 + 2\pi R'I$ as well. This is achieved trivially by taking exactly the same solution as before, but

^{††} This can be seen by noting that the trace of the commutator of finite matrices vanishes. This implies that the above relation would only be satisfied if $\text{Tr}U = 0$. This is impossible for finite unitary matrices.

with $k_1 = -2\pi R'$ and $k_2 = 0$ this time. Then identifying $X_2 \rightarrow \hat{x}^2$ and $U' = e^{-2i\pi R' \hat{x}^1}$ we have the desired relation. The objects U and U' are said to generate the noncommutative torus, since in a sense they have the right 'periodicity'. One may in fact define the noncommutative torus in general through their commutation relation, which is obtained trivially from the BCH formula:

$$U^i U^j = e^{-2\pi i \theta^{ij}} U^j U^i \quad (1.79)$$

where we have set the radii to convenient values. Thus we see that Matrix theory compactified on a torus leads naturally to a theory defined over the noncommutative torus generated by U and U' .

One final word on Matrix theory. It has been shown that noncommutativity also arises naturally from Matrix theory on an arbitrary background geometry (see [9],[8] and references therein). From a string theory perspective this may be seen as the introduction of a non-trivial background field - the target-space metric perturbation. We shall see in the following section that switching on another background field also gives rise to noncommutative effects.

1.5.2 String Theory in a background B -field and the Seiberg-Witten map

The content of this section is based upon the remarkable paper [2]. Owing to the fact that this paper is 100 pages long, we can only hope to get a taste of the full potential of the results contained within it, and hence the reader is referred to the original article for a complete exposition of this subject.

We will be examining the effect that a constant background anti-symmetric tensor field (B) has on the open string embedding functions in the presence of Dp -branes. We shall show that the B -field induces a canonical commutation relation on the space-time co-ordinates that lie on the brane. We shall then pursue the line of reasoning that led Seiberg and Witten to conclude that there exists a correspondence between ordinary and noncommutative gauge fields. We state the equation defining the Seiberg-Witten map, and finally solve it.

The action^{††} for a string with world-sheet Σ in the presence of a Dp -brane, a background 2-form field with components B_{ij} , and with the space-time metric denoted g_{ij} is

$$S = \frac{1}{4\pi\alpha'} \int_{\Sigma} \left(g_{ij} \partial_a x^i \partial^a x^j - 2\pi i \alpha' B_{ij} \epsilon^{ab} \partial_a x^i \partial_b x^j \right) d^2\sigma \quad (1.80)$$

^{††} Note that we have used the symmetries of the string action to choose a gauge in which the world-sheet metric is that of Euclidean space.

$$= \frac{1}{4\pi\alpha'} \int_{\Sigma} (g_{ij} \partial_a x^i \partial^a x^j d^2\sigma) - \frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j ,$$

where ∂_t is a derivative tangential to $\partial\Sigma$ and we have used the fact that the second term is a total derivative for constant B . Since we are dealing with open strings the boundary term does not vanish. Moreover, since components of B that are not along the Dp -brane can be gauged away, we will take $0 \leq i, j \leq p$. We also assume that the target space-time is Lorentzian in which case we impose $B_{0i} = 0$, as a nonzero temporal component will later lead to nonzero θ_{0i} , which in turn leads to issues of the unitarity with which we do not wish to complicate things. The interested reader is referred to [20].

Varying the action leads to the standard 2-D wave equation for the x^i with the boundary conditions

$$g_{ij} \partial_n x^j + 2\pi\alpha' i B_{ij} \partial_t x^j |_{\partial\Sigma} = 0 . \quad (1.81)$$

We will be considering only the free propagation of an open string, in which the world sheet is a disc. This can be conformally mapped to the upper half-plane by defining $z = e^{\tau+i\sigma}$ and recalling that $0 \leq \sigma \leq \pi$. Hence the boundary of the world-sheet corresponds to $\text{Im}z = 0$. In these co-ordinates the boundary conditions take the form

$$g_{ij} (\partial - \bar{\partial}) x^j + 2\pi\alpha' B_{ij} (\partial + \bar{\partial}) x^j |_{z=\bar{z}} = 0 , \quad (1.82)$$

where $\partial = \frac{\partial}{\partial z}$, $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$ and $\text{Im}z \geq 0$. Now since the action is quadratic in the fields and their derivatives, all the content of the theory resides in the propagator. Since for open strings vertex operators are inserted on the boundary of the world-sheet, we must also evaluate the propagator at the boundary. The full propagator that satisfies the above boundary conditions is given on page 8 of [2]. It is then shown that on the boundary, i.e. for $\{z, z'\} = \{\tau, \tau'\} \in \mathbb{R}$ the propagator is

$$\langle 0|T \{x^i(\tau)x^j(\tau')\} |0\rangle = -\alpha' G^{ij} \log(\tau - \tau')^2 + \frac{i}{2} \theta^{ij} \epsilon(\tau - \tau') , \quad (1.83)$$

where $\epsilon(\tau)$ is 1 (-1) for positive (negative) τ and

$$\begin{aligned} G^{ij} &= \left(\frac{1}{g + 2\pi\alpha' B} \right)_S^{ij} , \\ \theta^{ij} &= 2\pi\alpha' \left(\frac{1}{g + 2\pi\alpha' B} \right)_A^{ij} , \end{aligned} \quad (1.84)$$

and S and A denote the symmetric and antisymmetric parts of the relevant matrix respectively. Let us now compute the equal time commutator

$[x^i(\tau), x^j(\tau)]$. Recalling that from the world-sheet perspective the fields x^i become operators in a conformal field theory, we may interpret time ordering as operator ordering, in which case

$$[x^i(\tau), x^j(\tau)] = \lim_{\epsilon \rightarrow 0} T(x^i(\tau)x^j(\tau - \epsilon) - x^i(\tau)x^j(\tau + \epsilon)) = i\theta^{ij} . \quad (1.85)$$

Hence the endpoints of the string - which furnish a system of co-ordinates for the Dp -brane - live on a noncommutative space. Now consider the product of two tachyon vertex operators. Using the propagator (1.83) it is easy to show that for $\tau' \leq \tau$

$$e^{ip \cdot x(\tau)} e^{iq \cdot x(\tau')} \sim (\tau - \tau')^{2\alpha' G^{ij} p_i q_j} e^{-\frac{i}{2} \theta^{ij} p_i q_j} e^{i(p+q) \cdot x(\tau')} + \dots , \quad (1.86)$$

where the extra terms are of order α' . This is very nearly the expression we obtained in Weyl quantization (cf. (1.40)), the only difference being the first term. In order to make contact with the Weyl-Moyal machinery we must take the $\alpha' \rightarrow 0$ limit (or equivalently the small momentum limit). In this case the analogy becomes exact, and we are free to replace operator multiplication by star products. We will take this limit to be implicit in all that follows.

The effect of this limit is to single out the low-energy behavior, but we also wish to decouple the closed string effects along the Dp -brane. This is realized by taking the limit $\alpha' \sim \sqrt{\epsilon} \rightarrow 0$ and $g_{ij} \sim \epsilon \rightarrow 0$ in the directions along the brane while keeping the open string parameters G^{ij} and θ^{ij} fixed. Looking at the boundary conditions (1.81) we see that provided that B is invertible this limit corresponds to setting $\partial_t x^j|_{\partial\Sigma} = 0$ for $j \leq p$. These are of course Dirichlet boundary conditions - each boundary of the string is attached to a point in the Dp -brane.

It is also easy to see that in this limit the action (on the brane) (1.81) is dominated by the term

$$-\frac{i}{2} \int_{\partial\Sigma} B_{ij} x^i \partial_t x^j , \quad (1.87)$$

and that the open string parameters on the brane become

$$\begin{aligned} G^{ij} &= -\frac{1}{(2\pi\alpha')^2} (B^{-1} g B^{-1})^{ij} , \\ \theta^{ij} &= (B^{-1})^{ij} . \end{aligned} \quad (1.88)$$

Now suppose we had included a background gauge field in the original action (1.81). This would couple to the string world-sheet through

$$-i \int d\tau A_i(x) \partial_\tau x^i . \quad (1.89)$$

Notice that if we set $A_i = -\frac{1}{2}B_{ij}x^j$ then the action (1.87) is reproduced exactly, with the field strength associated to A given by $F = B$. Thus we have shown that on the brane the string dynamics are described by a background gauge field coupled to the boundary of the string world-sheet in the $\alpha' \rightarrow 0$ limit. We have also shown that in this limit the fields live on a noncommutative space in which we may use the \star -product to multiply functions. It thus seems likely that the theory living on the brane must be some kind of noncommutative gauge theory. This is indeed the case as we shall now see.

The action (1.89) seems to be invariant under ordinary gauge transformations of the form $\delta A_i = \partial_i \lambda$, since it transforms by a total derivative. However in the quantum theory one must generically contend with short-distance singularities in operator multiplication and the theory must be regularized. There are many ways of doing this. The one we will use is called point-splitting regularization - the basic idea being never to evaluate two operators at the same point. Recall that in quantum field theory we require probability amplitudes to be gauge invariant, and that these are governed by a functional integral of the exponential of the action in the path integral formalism. Expanding the exponential of the action in powers of A we find that to first order the functional integral will transform by

$$- \int d\tau A_i(x) \partial_\tau x^i \cdot \int d\tau' \partial_{\tau'} \lambda(x) \quad (1.90)$$

under the normal gauge transformation. This product of operators will be divergent at $\tau = \tau'$. We therefore split the τ' integration range such that $|\tau - \tau'| \geq \delta$ and take the limit $\delta \rightarrow 0$ in the end. Since the τ' integral is a total derivative, it contributes surface terms at $\tau' = \tau \pm \delta$. We must also normal order the operators to remove any infinite vacuum energies. Using this regularization this term transforms by

$$\begin{aligned} \lim_{\delta \rightarrow 0} \int d\tau : A_i(x(\tau)) \partial_\tau x^i(\tau) :: (\lambda(x(\tau - \delta)) - \lambda(x(\tau + \delta))) : & \quad (1.91) \\ = - \int d\tau : (A_i(x) \star \lambda - \lambda \star A_i(x)) \partial_\tau x^i, & \end{aligned}$$

where we have once again interpreted time ordering as operator ordering, made use of the correspondence between operator products and \star -products, and made use of the fact that there is no term in the propagator that depends on $\partial_\tau x$ in the $\alpha' \rightarrow 0$ limit. In order to cancel this term we must change the transformation rule for A to

$$\delta A_i = \partial_i \lambda + i\lambda \star A_i - iA_i \star \lambda. \quad (1.92)$$

This is exactly the transformation rule for a noncommutative Yang-Mills field! It turns out that this transformation rule also cancels all the con-

tributions from terms of higher order in A and is thus an exact gauge symmetry of the regularized quantum theory.

Seiberg and Witten go on to show that one may generate the elements of the S-matrix for this theory by considering a noncommutative pure Yang-Mills effective action with field strength exactly as in (1.57), thus completing the identification of string theory in this limit as a noncommutative theory.

This remarkable conclusion may not sit well with the reader. It has been argued on very general grounds that string theory in background fields may in fact be regularized to yield a commutative field theory. This may be achieved in the case in question by appealing to Pauli-Villars regularization, in which one introduces fictitious fields of mass M , and after computing the relevant quantities, decouples them by sending $M \rightarrow \infty$ (see [2] for details). Thus it seems that whether we end up with a commutative or a noncommutative theory depends only on our choice of regularization. Now regularization is nothing but a mathematical tool; the physics should be independent of which such tool we use. It was this observation, coupled with the desire to reconcile the two approaches to the problem above, that led Seiberg and Witten to conjecture that there must be a map between commutative and noncommutative gauge fields. Let us have a quick look at their reasoning.

It is important to note that we cannot hope to find an isomorphism between commutative and noncommutative gauge theory. This much is evident from the fact that an ordinary rank one gauge field A transforms by $\delta A_i = \partial_i \lambda$ which is Abelian, whereas a noncommutative rank one gauge field must transform by $\delta A_i = \partial_i \lambda + i\lambda \star A_i - iA_i \star \lambda$, which is nonabelian. Clearly a nonabelian group cannot be isomorphic to an abelian group. What Seiberg and Witten realized is that one does not require an isomorphism to show that the two descriptions are equivalent. All one needs is a map which preserves gauge equivalence.

Let all functions with hats on them be functions over a noncommutative space with multiplication given by the \star -product. We seek to describe the noncommutative gauge field \hat{A} and infinitesimal gauge parameter $\hat{\lambda}$ in terms of their commutative counterparts, A and λ . $\hat{\lambda}$ cannot depend on λ alone since we would then have an isomorphism, so we relax this and take $\hat{A} = \hat{A}(A)$ and $\hat{\lambda} = \hat{\lambda}(A, \lambda)$. With this choice there is no mapping between the gauge groups, but one can construct a map between the two notions of gauge equivalence. Physically this is all that is required, as all we need to know to build a gauge theory is when two fields are gauge equivalent. Having come to this realization, it is easy to write down such a map. We must have

$$\hat{A}(A) + \hat{\delta}_{\hat{\lambda}(A,\lambda)} \hat{A}(A) = \hat{A}(A + \delta_\lambda A)$$

where $\delta_\lambda A_i = \partial_i \lambda + i[\lambda, A_i]$ and $\hat{\delta}_{\hat{\lambda}} \hat{A}_i = \partial_i \hat{\lambda} + i\hat{\lambda} \star \hat{A}_i - i\hat{A}_i \star \hat{\lambda}$. In words, this map will make ordinary gauge transformations of A by λ equivalent to noncommutative gauge transformations of \hat{A} by $\hat{\lambda}$. Writing $\hat{A} = A + A'(A)$ and $\hat{\lambda} = \lambda + \lambda'(A, \lambda)$ we may use the exponential form of the star product to expand this equation in powers of θ . To first order it becomes

$$A'_i(A + \delta_\lambda A) - A'_i(A) - \partial_i \lambda' - i[\lambda', A'_i] - i[\lambda, A'_i] = -\frac{1}{2} \theta^{kl} (\partial_k \lambda \partial_l A_i + \partial_l A_i \partial_k \lambda) + \mathcal{O}(\theta^2), \quad (1.93)$$

where all products are now ordinary matrix products. This equation is solved to first order in θ by putting

$$A'_i(A) = -\frac{1}{4} \theta^{kl} \{A_k, \partial_l A_i + F_{li}\} + \mathcal{O}(\theta^2), \quad (1.94)$$

$$\lambda'(A, \lambda) = \frac{1}{4} \theta^{ij} \{\partial_i \lambda, A_j\} + \mathcal{O}(\theta^2),$$

where $\{, \}$ is an anti-commutator. From this one may proceed to write down the formula for \hat{F}_{ij} to first order.

Now if one thinks about it carefully, it is not hard to convince oneself that the problem we have just solved is in practice identical to a slightly different one. Consider attempting to map a noncommutative gauge theory with noncommutativity parameter θ to a closely related noncommutative gauge theory with noncommutativity $\theta + \delta\theta$. We wish to do this in such a way that the physics in the two theories is the same. To first order in θ , finding a map that gives us $\hat{A}(\theta + \delta\theta)$ in terms of $\hat{A}(\theta)$ is exactly the problem solved above. We can now use the solution found above to write down a differential equation that tells us how $\hat{A}(\theta)$ and $\hat{\lambda}(\theta)$ must change in order to leave the physics unchanged (See [2] for the full equations). Since these are differential equations, they give us the desired change of variables to *all* orders in θ ! These equations define the Seiberg-Witten map [2].

In general the Seiberg-Witten equations are very complex and difficult to solve, however one simple and highly relevant case readily admits a solution. Recall that in the context of the $\alpha' \rightarrow 0$ limit with constant B , the entire string dynamics was described by the background gauge field with $F = B$. Thus the case of constant \hat{F} is of significant physical importance. In this case the Seiberg-Witten equations become simply

$$\delta \hat{F} = -\hat{F} \delta \theta \hat{F}, \quad (1.95)$$

which for the boundary condition $\hat{F}(\theta = 0) = F$ has solution

$$\hat{F} = \frac{1}{1 + F\theta} F. \quad (1.96)$$

This can be inverted in which case

$$F = \hat{F} \frac{1}{1 - \theta \hat{F}} . \quad (1.97)$$

We may also write these equations in terms of B using the fact that in this limit $\theta = B^{-1}$.

These solutions illustrate a very important point about the Seiberg-Witten map. One could naively argue that the Seiberg-Witten map renders noncommutative gauge theory superfluous, as each noncommutative theory can be transformed to a commutative one. However these solutions illustrate that this is not so in general. In particular, for $\hat{F} = \theta^{-1}$ the commutative description fails as F is singular. Similarly the noncommutative theory fails to faithfully reproduce the commutative one for $F = -\theta^{-1}$. Furthermore, there is as yet no proof that the Seiberg-Witten equations admit solutions for arbitrarily complex \hat{F} .

We now have almost all the tools we will require to make sense of what is to come. Moreover, we have provided a sound physical basis for the study of noncommutative field theory in the context of string/M theory. In the coming chapters we will be examining certain noncommutative systems which may at first glance appear significantly more complex than those dealt with in this chapter. However all the techniques used will be natural extensions or combinations of these results. We shall refer the reader to the literature for details we exclude. The few extra background details we will need will be developed as they arise.

2

Transmogrifying Fuzzy Vortices

2.1 Some precursors

In this chapter we will be studying soliton solutions of noncommutative field theories in (1+2) dimensions. The plane wave basis we used in our discussion of Weyl quantization in the previous chapter is very well suited to a discussion of perturbation theory, and will be the basis of our work in chapter 3, however solitons are of course nonperturbative objects, and it turns out that a translation of the results to a harmonic oscillator basis will aid our present purpose. At the end of this chapter we will also be exploiting the interpretation of noncommutative solitons as D-branes in the presence of a constant background B -field. In order to facilitate understanding of the rather complex scenario used in this discussion, we introduce the reader to this interpretation with a simpler example from bosonic string theory.

2.1.1 The harmonic oscillator basis

We will be working in (1+2) dimensions, where the time co-ordinate is commutative. It is convenient to define $\hat{z} = \frac{\hat{x}^1 + i\hat{x}^2}{\sqrt{2}}$, and $\hat{\bar{z}} = \frac{\hat{x}^1 - i\hat{x}^2}{\sqrt{2}}$, in which case the canonical commutation relation $[\hat{x}^1, \hat{x}^2] = i\theta$ becomes

$$[\hat{z}, \hat{\bar{z}}] = \theta . \quad (2.1)$$

If we now define $\hat{z} = \sqrt{\theta}\hat{a}$ and $\hat{\bar{z}} = \sqrt{\theta}\hat{a}^\dagger$ then we find that

$$[\hat{a}, \hat{a}^\dagger] = 1 . \quad (2.2)$$

This is exactly the algebra of creation and annihilation operators for a quantum harmonic oscillator. Thus, using the Weyl machinery, functions on the noncommutative plane may now be associated to operators on the

one particle Hilbert space $\mathcal{H} = \bigoplus_n \mathbb{C}|n\rangle$ built out of harmonic oscillator eigenstates. The action of the coordinate operators on the basis states is given by the standard relations

$$\begin{aligned}\widehat{a}|n\rangle &= \sqrt{n}|n-1\rangle \\ \widehat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle,\end{aligned}\quad (2.3)$$

with the vacuum $|0\rangle$ defined by $\widehat{a}|0\rangle = 0$. A general operator in this basis may be written

$$\widehat{O} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_{mn}|m\rangle\langle n| \quad (2.4)$$

with $f_{mn} \in \mathbb{C}$, and the regularized operator trace (introduced in [60]) is given by

$$\text{Tr}\widehat{O} = \lim_{N \rightarrow \infty} \sum_{n=0}^N \langle n|\widehat{O}|n\rangle. \quad (2.5)$$

Further, under the Weyl map the expression (1.38) for the derivative and the corresponding result for the integral become

$$\partial_z \rightarrow -\frac{1}{\sqrt{\theta}}[\widehat{a}^\dagger, \cdot] \quad (2.6)$$

$$\int_{\mathbb{C}_\theta} d^2z f(z, \bar{z}) \rightarrow 2\pi\theta \text{Tr}_{\mathcal{H}} \widehat{O}_f(\widehat{z}, \widehat{\bar{z}}) \quad (2.7)$$

with an analogous expression for $\partial_{\bar{z}}$.

We will also have cause to make use of the explicit form of the inverse Weyl map for cylindrically symmetric operators. First notice that the number operator is

$$\widehat{a}\widehat{a}^\dagger = \frac{1}{2\theta} [(\widehat{x}^1)^2 + (\widehat{x}^2)^2] + \frac{1}{2}. \quad (2.8)$$

If we interpret $(\widehat{x}^1)^2 + (\widehat{x}^2)^2$ as a noncommutative 'radius', we see that without loss of generality we may consider cylindrically symmetric operators to be of the form $\widehat{O}_{cyl} = \sum_{n=0}^{\infty} f_n|n\rangle\langle n+q|$ for some $q \in \mathbb{N}$. One may now appeal to the formula for the inverse Weyl transform to find the function $f_{n(n+q)}(r, \phi)$ which corresponds to $|n\rangle\langle n+q|$. The calculation is of no particular interest, so we simply state the result:

$$|n\rangle\langle n+q| \rightarrow f_{n(n+q)}(r, \phi) = 2e^{-r^2} \sqrt{\frac{n!}{(n+q)!}} (-1)^n (2r^q) e^{iq\phi} L_n^q(2r^2) \quad (2.9)$$

where $L_n^q(x)$ is a Laguerre polynomial, and (r, ϕ) are polar co-ordinates on the plane. The interested reader is referred to [18] for details.

2.1.2 Noncommutative solitons...

So far we have discussed the formulation of noncommutative field theory in some depth, but we have as yet said nothing about the kinds of solutions these theories admit. In fact, one of the most attractive aspects of noncommutative models is their rich spectrum of solutions, which may often be seen to generalize the commutative case in a natural way. Of the most interesting and certainly the most important such solutions are the noncommutative solitons - fuzzy analogues of the localized solutions familiar from classical field theory. In particular, our focus will be on the so-called Bogomolnyi-Prasad-Sommerfeld (BPS) solitons [24]- topologically stable solitons which generically saturate an energy bound. The reader unfamiliar with these ubiquitous and very important solutions is referred to any text on soliton theory, for example [25]. Appendix 1 also contains an introduction to the $\mathbb{C}\mathbb{P}^n$ model as a specific (and relevant) example.

The study of noncommutative solitons originated with the work of Gopakumar, Minwalla, and Strominger [31], in which it was shown that even extremely simple field theories which are trivial in the commutative case, can exhibit noncommutative solitons. Their reasoning is worth repeating.

Consider the action

$$S = \int dt d^2x V(\phi) \rightarrow 2\pi\theta \int dt \text{Tr}_{\mathcal{H}} V(\phi) \quad (2.10)$$

where

$$V(\phi) = \frac{m^2}{2} \phi \star \phi + c_1 \phi \star \phi \star \phi + \dots \quad (2.11)$$

is a polynomial potential where the fields are multiplied using the star-product. The equation of motion of this system is simply,

$$\frac{\partial V}{\partial \phi} = c\phi(\phi - \lambda_1) \dots (\phi - \lambda_n) = 0 \quad (2.12)$$

where the λ_i are the critical points of V . In the commutative case this has the decidedly uninteresting solution $\phi = \lambda_i$. However in the noncommutative case, the guess

$$\phi = \lambda_i P \quad (2.13)$$

where P is a projection operator on the Hilbert space of states, may easily be seen to solve (2.12). Thus even in this simplest of field theories we have a non-trivial solution! If we take the simplest possible projector $P = |0\rangle\langle 0|$ to get a feel for what these solutions look like, and perform an inverse Weyl transform using (2.9) we see that this solution corresponds

in function space to $f(q, p) = e^{-(q^2+p^2)/\theta}$ [18]. Thus we obtain a localized solution even in a field theory without any derivative terms!

Note that the action (2.10) is invariant under transformations

$$\phi \rightarrow U\phi U^\dagger, \quad U^\dagger U = U U^\dagger = 1. \quad (2.14)$$

This transformation also preserves the equations of motion since under it

$$\frac{\partial V}{\partial \phi} \rightarrow U \frac{\partial V}{\partial \phi} U^\dagger. \quad (2.15)$$

We may use this symmetry to generate new solutions from old. Take a trivial solution ($\phi_{triv} = \lambda_i I$ will do in this case). Then

$$\phi_{new} = U \phi_{triv} U^\dagger \quad (2.16)$$

will also be a solution. An important point to note is that only the property $U^\dagger U = 1$ is used in showing (2.15), thus leaving open the possibility of U being a non-unitary isometry, since $U^\dagger U = 1 \Rightarrow U U^\dagger = 1$ only in finite dimensional Hilbert spaces. The standard example of such an operator is the shift operator,

$$S = \sum_{n=0}^{\infty} |n+1\rangle \langle n|. \quad (2.17)$$

In general S^n will be a non-unitary isometry. Thus a solution to the theory defined in (2.10) is

$$\phi = \lambda_i S^n (S^\dagger)^n. \quad (2.18)$$

This very simple technique can be used to generate interesting solutions in far more complex situations.

2.1.3 ...as D -branes

In the light of our discussion of the previous chapter, it seems natural to try to interpret these noncommutative solitons in a stringy context. It is well known that some D -branes may be viewed as BPS solutions of the string equations. This suggests that, via the Seiberg-Witten machinery, it is possible to interpret noncommutative solitons as D -branes in a background B -field. This is indeed the case, as we shall see in the following simple example*. We shall use this correspondence again at the end of this chapter.

* This section follows the discussion in [18].

To illustrate the concept, we'll consider a simple bosonic model. Of course, all D -branes are unstable in bosonic string theory so the results obtained should be taken with a grain of salt. The effective world-volume action for a space filling $D25$ -brane in bosonic string theory with constant tachyon and gauge field strength F may be written ([32],[33]) as

$$S_{BI} = T_{25} \int d^{26}x \left[-V(\phi - 1) \sqrt{-\det(g + 2\pi\alpha' F)} + \frac{1}{2} \sqrt{g} f(\phi - 1) \partial^\mu \phi \partial_\mu \phi + \dots \right] \quad (2.19)$$

where T_{25} is the brane tension, and \dots indicates the presence of higher derivative terms. The potential $V(\phi - 1)$ is chosen so that $\phi = 0$ corresponds to a local maximum of V with $V(-1) = 1$, which represents the unstable $D25$ -brane, and $\phi = 1$ corresponds a local minimum of V , with $V(0) = 0$ representing the vacuum.

Let us now turn on a B -field in a plane in space-time. Lets pick $B_{24,25} = b < 0$. The results of Seiberg and Witten tell us how to write the open string parameters so that we can work in the framework of non-commutative field theory:

$$\begin{aligned} G_{ij} &= \text{diag}(1, -1, \dots, -1, -(2\pi\alpha'b)^2, -(2\pi\alpha'b)^2) \quad (2.20) \\ G_s &= g_s(2\pi\alpha'|b|) \\ \theta^{24,25} &\equiv \theta = \frac{1}{|b|} \end{aligned}$$

where in the second line, we have used the fact that in the $\alpha' \rightarrow 0$ limit,

$$G_s = g_s \sqrt{\det(2\pi\alpha' B g^{-1})}. \quad (2.21)$$

This follows by comparing the constant term in the Dirac-Born-Infeld effective action in the commutative picture (with a B -field) with the same term in the noncommutative picture (in which the B dependence is absorbed into the \star -product). See page 18 of [2] for the calculation. Under these transformations the action becomes,

$$S_{BI(nc)} = 2\pi\theta \int d^{24}x \text{Tr}_{\mathcal{H}} \left[-V(\phi - 1) \sqrt{\det(G_{\mu\nu} 2\pi\alpha' (F - B)_{\mu\nu})} + \frac{1}{2} \sqrt{G} f(\phi - 1) D^\mu \phi D_\mu \phi + \dots \right]. \quad (2.22)$$

Now reparametrize the gauge field strength operator through $F_{24,25} - B_{24,25} = F_{24,25} - 1/\theta = [K^\dagger, K]/\theta^\dagger$. The idea now is to use the solution generating technique to find solutions to this theory from the trivial

[†] The operator K is the covariant gauge field of (1.62). More on this later

vacuum configuration given by

$$\phi = 1, \quad K = a, \quad A_\rho = 0 \quad \rho = 0 \dots 23. \quad (2.23)$$

ϕ, K and A_μ all transform in the adjoint representation of $U(\mathcal{H})$, thus $U\phi U^\dagger, UKU^\dagger$ and $UA_\mu U^\dagger$ is a solution for U a non-unitary isometry. This is true since, although the form of the higher order terms in (2.22) is unknown to us, we do know that they transform in the same way as the fields. Picking $U = S^n$ as in (2.17) gives

$$\phi = S^n(S^\dagger)^n, \quad K = S^n a (S^\dagger)^n, \quad A_\rho = 0. \quad (2.24)$$

As illustrated previously this solution is localised in the 24-25 plane and as such should represent a D23-brane!

To 'prove' this assertion we should show that the tension and spectrum of fluctuations corresponding to this solution exactly match those we would expect from a D23-brane. We will just demonstrate the match for the tension by substituting our solution back into the action (2.22).

First note that the fact that $D\phi = 0$ in the vacuum and $D\phi$ also transforms in the adjoint representation of $U(\mathcal{H})$ means that both the second term in (2.22), and all terms in $D\phi$, vanish. This argument may also be applied to the derivatives of F . Now it is easy to show that $S^n(S^\dagger)^n = 1 - P_n$ where

$$P_n = \sum_{k=0}^{n-1} |k\rangle\langle k| \quad (2.25)$$

is a projection operator. This may be used to dispatch all terms that go like $(F - B)^2$ since

$$V(\phi - 1)[K^\dagger, K]^2 = V(-P_n)(P_n - 1) = V(-1)P_n(P_n - 1) = 0. \quad (2.26)$$

Thus using

$$\text{Tr}V(\phi - 1) = \text{Tr}P_n = n \quad (2.27)$$

and

$$\frac{\sqrt{G}\theta}{G_s} = \frac{2\pi\alpha'}{g_s}, \quad (2.28)$$

which follows from (2.21), we find

$$S_{BI(nc)} = (2\pi)^2 \alpha' n T_{25} \int d^{24}x. \quad (2.29)$$

We are thus able to deduce the tension of our soliton solution,

$$T_{soliton} = (2\pi)^2 \alpha' n T_{25} = n T_{23}. \quad (2.30)$$

This is consistent with known results for the tension of a bosonic D23-brane (see for example [6],[5]). We have thus shown that our noncommutative soliton is in fact a D23-brane in the bosonic theory with constant background B -field. We should now go on to compute the fluctuations of the fields around this solution and obtain their masses to make sure that they match those of a D23-brane, but for our purposes this analysis will be sufficient. The interested reader is referred to [38].

2.2 Introduction to the problem

A little more than a decade ago, the study of electroweak strings in a modified Abelian-Higgs theory initiated in [58] revealed a curious new vortex solution. As the story goes, vortices are indeed enigmatic objects [57] and the *semilocal vortices* found in [58] are no exception. Firstly, standard lore holds that a non-simply-connected vacuum manifold is a necessary condition for the existence of stable, finite energy cosmic string solutions. If this is anything to go by, the very existence of these semilocal vortices should be called into question since the vacuum manifold of the modified Abelian-Higgs theory is S^3 . Yet exist they do. Consequently, a more consistent condition was offered in [42]. Semilocal vortices (actually, this holds for other defects as well) form in theories exhibiting spontaneous symmetry breaking and whose vacuum manifold is fibred by the action of the gauge group in some non-trivial way. In this same work it was realised also that the low momentum dynamics of these vortices bear a striking resemblance to the 2-dimensional lump solutions of the $\mathbb{C}\mathbb{P}^N$ nonlinear sigma model. Since then, this similarity between the modified Abelian-Higgs theory (a.k.a gauged linear sigma model) and the $\mathbb{C}\mathbb{P}^N$ (or, more generally, Grassmannian) sigma model has found itself the subject of much attention [36, 55, 63]. Nevertheless, much of what is known about the semilocal vortex is only asymptotic. Even its descent to the lump in the infinite coupling limit is only exact at spatial infinity and suffers Skyrme term corrections at smaller radial distances. This is the allure and frustration of vortices; as simple as their defining equations seem, they are also remarkably unyielding.

Until a short time ago, the only avenue toward tractable vortex equations was a curvature deformation of the background space in which the vortices live [56, 61]. These are, of course, not without their own puzzles. Recently however, as we have seen, we have acquired a new weapon in our arsenal - noncommutative geometry. Fuzzy deformations of the background space have, in only a few years, not only yielded a wealth of new solitonic solutions but also several new insights into old solutions to a host of field theories (see [17, 18, 19] for excellent reviews). The *noncommu-*

tative Abelian-Higgs model, for example, exhibits exact vortex solutions [26, 27, 48, 60] whose moduli space metric can be computed explicitly in the large noncommutativity limit [57].

In this work we extend this idea to the $(2 + 1)$ -dimensional, critically coupled, gauged linear sigma model with an $N + 1$ component Higgs field. The BPS spectrum of the resulting fuzzy theory is studied and, like its commutative counterpart, shown to have quite rich structure. In particular, we use an extension of the computational technique of [48] to explicitly construct a family of exact semilocal vortices. As expected, our family contains the Abelian-Higgs vortices of [26, 48, 60] as well as the fluxons of [39] as special cases. As suggested by the title, the metamorphosis of the semilocal vortex is of central importance in this paper. By turning up the gauge coupling, we demonstrate conclusively, at the level of the solutions, the descent of the semilocal vortex into the instanton solution of the fuzzy $\mathbb{C}\mathbb{P}^N$ model of the same degree. Interestingly, unlike the commutative case, this “transmogrification” of the vortex is exact at a certain point in the parameter space of the theory. Finally, we turn our attention toward the physical[†] interpretation of the k -lump solution of the noncommutative $\mathbb{C}\mathbb{P}^N$ model of [47]. Without much additional work, the brane configuration in type II-B string theory that realises the fuzzy lump may be read off from the construction of [36] as tilted D -strings suspended between an $NS5$ - and $D3$ -brane.

2.3 The Gauged Linear Sigma Model

2.3.1 Definitions

Among the many extensions to the Abelian-Higgs model, one of the most natural is the gauged linear sigma model with Fayet-Illiopolous D-terms [55, 63]. This is certainly true if the aim is the construction of a model that supports solitonic excitations saturating BPS-like bounds. With its \mathbb{C}^{N+1} -valued scalar fields and $U(1)^{N+1}$ gauge symmetry, the linear sigma model is a natural springboard for our discussion of the relation between noncommutative semilocal vortices, fuzzy sigma model lumps and the braney systems they are associated with. To this end then it will prove useful to briefly review some of the ideas and notation used to extract the vortex excitations from the solution spectrum of the semilocal model. Following [55] we write the linear sigma model action as

$$S_{SL} = - \int_{\mathbb{R}^{(1,2)}} d^3x \left[(D_\mu \Phi)(D^\mu \Phi)^\dagger + \sum_{a=1}^{N+1} \frac{1}{4e_a^2} (F^a_{\mu\nu})^2 \right]$$

[†] By which we mean ‘stringy’.

$$+ \sum_{a=1}^N \frac{e_a^2}{2} (R_a - \Phi \tau_a \Phi^\dagger)^2 \quad (2.31)$$

The dynamical degrees of freedom in this model are encoded in a \mathbb{C}^{N+1} -valued spacetime scalar $\Phi = (\phi_1, \dots, \phi_{N+1})$ and the $N+1$ $U(1)$ -valued gauge 1-forms $A^a = A^a_\mu dx^\mu$ with associated curvature 2-forms $F^a = dA^a$. The τ_a are the $N+1$ generators of $U(1)^{N+1}$. The gauge covariant derivative we will take as

$$D := d - i \sum_{a=1}^{N+1} \tau_a A^a \quad (2.32)$$

There are two sets of parameters in the theory; the $N+1$ coupling constants e_a of dimension $(mass)^{1/2}$ and $N+1$ Fayet-Illiopolous (FI) parameters R_a - effectively the vacuum expectation values of the components of Φ . Without loss of generality (and because we can always re-scale the fields to absorb them anyway) we set the latter to unity. The coupling constants we retain because they control the energy scales of the model.

In the temporal gauge (in which we pick $A_0^a = 0$), the static energy corresponding to the action (2.31) is

$$E = \int_{\mathbb{R}^2} d^2x \left[(D_i \Phi)(D_i \Phi)^\dagger + \sum_{a=1}^{N+1} \frac{1}{4e_a^2} (F^a_{ij})^2 + \sum_{a=1}^{N+1} \frac{e_a^2}{2} (\Phi \tau_a \Phi^\dagger - 1)^2 \right] \quad (2.33)$$

For instance, in the case $N=1$, following [55] the energy functional becomes (in exhaustive detail)

$$E = \int_{\mathbb{R}^2} d^2x \left[(D_1 \Phi)(D_1 \Phi)^\dagger + (D_2 \Phi)(D_2 \Phi)^\dagger + \frac{1}{4e_1^2} (F_{ij})^2 + \frac{1}{4e_2^2} (G_{ij})^2 + \frac{e_1^2}{2} (\Phi \tau_1 \Phi^\dagger - 1)^2 + \frac{e_2^2}{2} (\Phi \tau_2 \Phi^\dagger - 1)^2 \right] \quad (2.34)$$

where the $GL(2, \mathbb{R})$ -valued connections $A = \tau_1 A^1$ and $B = \tau_2 A^2$ are associated to the curvature forms $F = dA$ and $G = dB$ respectively. For our purposes, it will suffice to turn off B and e_2 and take $\tau_1 = \mathbf{1}_2$ giving

$$E = \int_{\mathbb{R}^2} d^2x \left[(D_1 \Phi)(D_1 \Phi)^\dagger + (D_2 \Phi)(D_2 \Phi)^\dagger + \frac{1}{2e_1^2} (F_{ij})^2 + \frac{e_1^2}{4} (\Phi \Phi^\dagger - 1)^2 \right] \quad (2.35)$$

Even under such restricted circumstances, the resulting linear sigma model is still remarkably rich, exhibiting a wealth of solitonic structure and enjoying intimate relations with nonlinear sigma models on toric varieties

[55, 63].

In what follows, it will prove convenient to rewrite the energy in terms of the complex coordinates $z := (x^1 + ix^2)/\sqrt{2}$ and $\bar{z} := (x^1 - ix^2)/\sqrt{2}$, so that we can later make contact with our discussion of noncommutativity in the harmonic oscillator basis. This particular normalization means that

$$\partial_z := \frac{\partial}{\partial z} = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2) \quad \partial_{\bar{z}} := \frac{\partial}{\partial \bar{z}} = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2) \quad (2.36)$$

This in turn induces a complexification of the gauge covariant derivative so that

$$D_z := \frac{1}{\sqrt{2}}(D_1 - iD_2) \quad D_{\bar{z}} := \frac{1}{\sqrt{2}}(D_1 + iD_2). \quad (2.37)$$

when $A_z := (A_1 - iA_2)/\sqrt{2}$ and $A_{\bar{z}} := (A_1 + iA_2)/\sqrt{2}$. These are of course now $GL(2, \mathbb{C})$ -valued objects. With these definitions,

$$E = \int_{\mathbb{C}} d^2z \left[(D_z\Phi)(D_z\Phi)^\dagger + (D_{\bar{z}}\Phi)(D_{\bar{z}}\Phi)^\dagger + \frac{1}{2e^2}(F_{12})^2 + \frac{e^2}{2}(\Phi\Phi^\dagger - 1)^2 \right] \quad (2.38)$$

2.3.2 Solitons on the Plane

To see the emergence of the semilocal vortex in the spectrum of the gauged linear sigma model, the usual method of “completing the square” in the energy functional may be followed. After some straightforward manipulations, (2.38) may be put into the form

$$E = \int_{\mathbb{C}} d^2z \left[2(D_{\bar{z}}\Phi)(D_{\bar{z}}\Phi)^\dagger + \frac{1}{2e^2} \left| F_{12} + e^2(\Phi\Phi^\dagger - 1) \right|^2 \right] + \int_{\mathbb{C}} d^2z T + \underbrace{\int_{\mathbb{C}} d^2z F_{12}}_{2\pi k} \quad (2.39)$$

where $T = \partial_{\bar{z}}(\Phi D_z \Phi^\dagger) - \partial_z(\Phi D_{\bar{z}} \Phi^\dagger)$. As such, the second to last term is a total derivative whose integral vanishes. Consequently, a nonvanishing lower bound of $E \geq 2\pi k$ is established on finite energy field configurations. As usual, the bound saturates when the first order system

$$\begin{aligned} D_{\bar{z}}\Phi &= 0 \\ F_{12} &= e^2(\Phi\Phi^\dagger - 1) \\ \int_{\mathbb{C}} d^2z F_{12} &= 2\pi k \end{aligned} \quad (2.40)$$

is satisfied. The first of these is, of course, really two equations, one for each component of the \mathbb{C}^2 -valued field Φ . The equations in (2.40) form a closed system whose solutions are precisely the semilocal vortices of [41, 55, 58].

Although such solitonic solutions are vortex-like in many respects, a little analysis soon reveals that their asymptotic behavior is very different from the exponential falloff of Abelian-Higgs vortices [41, 42]. In fact the fields of the semilocal model exhibit a distinctive power-law behavior at spatial infinity, a symptom of the fact that the width of the flux tube is an arbitrary parameter of the theory. This should be contrasted with Abelian-Higgs vortices where the width is controlled by the Compton wavelength of the gauge boson. In this sense, these vortex solutions are rather reminiscent of $\mathbb{C}\mathbb{P}^N$ instantons. This is no mere coincidence. In fact, the correspondence can be made precise in the large coupling limit in which the semilocal vortices of the $U(1)^{N+1}$ -gauged linear sigma model descend to the instanton solutions of a $\mathbb{C}\mathbb{P}^N$ nonlinear sigma model [55, 63]. While this is quite clear at the levels of the action and equations of motion, its realization at the level of the solutions is obscured by the fact that only the asymptotic forms of the vortex solutions are known to exist. This is not unlike the situation with the conventional Nielsen-Olesen vortex. However this particular hurdle was recently surmounted in [26, 27, 48, 60] where a noncommutative deformation of the two-dimensional configuration space of the Abelian-Higgs model allows for the construction of *exact vortex solutions*. As we have already mentioned, the fact that the noncommutative version of the theory seems so much richer than its commutative counterpart is by now not surprising [31, 34]. It would seem then, that a noncommutative deformation of the base space of the two-dimensional gauged linear sigma model might offer an interesting avenue to explore the construction of exact semilocal vortices.

2.3.3 The Noncommutative Semilocal Model

Using the conventions of the previous section, and our discussion on the harmonic oscillator basis, the noncommutative semilocal energy functional (2.38) can be written as

$$E_\theta = 2\pi\theta\text{Tr}_{\mathcal{H}} \left[(\widehat{D}_z\widehat{\Phi})(\widehat{D}_z\widehat{\Phi})^\dagger + (\widehat{D}_{\bar{z}}\widehat{\Phi})(\widehat{D}_{\bar{z}}\widehat{\Phi})^\dagger + \frac{1}{2e^2}\widehat{F}_{12}^2 + \frac{e^2}{2}(\widehat{\Phi}\widehat{\Phi}^\dagger - 1)^2 \right] \quad (2.41)$$

where, now $\widehat{D}_z \widehat{\Phi} = (\widehat{\Phi} \widehat{a}^\dagger + \widehat{C}^\dagger \widehat{\Phi})/\sqrt{\theta}$, $\widehat{D}_z \widehat{\Phi} = -(\widehat{\Phi} \widehat{a} + \widehat{C} \widehat{\Phi})/\sqrt{\theta}$, the gauge field is parameterized as $\widehat{A}_z = (i/\sqrt{\theta})(\widehat{a}^\dagger + \widehat{C}^\dagger)$, and $\widehat{F}_{12} = (1 + [\widehat{C}^\dagger, \widehat{C}])/\theta$ [§]. As in the commutative case, this can be massaged into a Bogomol'nyi form which is saturated when the BPS equations

$$\begin{aligned} \widehat{\Phi} \widehat{a} + \widehat{C} \widehat{\Phi} &= 0 \\ 1 + [\widehat{C}^\dagger, \widehat{C}] &= \theta e^2 (\widehat{\Phi} \widehat{\Phi}^\dagger - 1) \\ \text{Tr}_{\mathcal{H}}(1 + [\widehat{C}^\dagger, \widehat{C}]) &= -k \end{aligned} \quad (2.42)$$

are satisfied. As in the commutative case, this is a system of three first order equations, subject to the flux constraint. Solutions of this system will be the noncommutative generalizations of the semilocal vortex of [42]. In the spirit of [27, 48], we begin with an *ansatz* for the Higgs doublet and the gauge field. As we have noted, the most general vortex-like solution of the BPS equations which maintain the cylindrical symmetry is of the form[¶]

$$\widehat{\Phi} = \widehat{\phi}_1 \otimes \langle \text{I} | + \widehat{\phi}_2 \otimes \langle \text{II} | \quad (2.43)$$

where $\langle \text{I} | = (1, 0)$, $\langle \text{II} | = (0, 1)$ and

$$\widehat{\phi}_i = \sum_{m=0}^{\infty} f_m^{(i)} |m\rangle \langle m + q^{(i)}| \quad (2.44)$$

where $\{q^{(1)}, q^{(2)}\}$ is a set of integers related to the topological charge and angular momentum quantum number of the vortex respectively, as we show below. For the $U(1)$ gauge field we take the cylindrically symmetric ansatz

$$\widehat{C} = \sum_{m=0}^{\infty} g_m |m\rangle \langle m + 1|. \quad (2.45)$$

Without loss of generality all coefficients are taken to be real. The construction of exact vortex solutions to the semilocal model now hinges on determining the various coefficients in the above *ansatze* that satisfy the appropriate boundary conditions. In terms of the coefficients $f_n \equiv f_n^{(1)}$ and $h_n \equiv f_n^{(2)}$, the first of eqs.(2.42) become

$$\begin{aligned} f_m \sqrt{m + q + 1} + g_m f_{m+1} &= 0 \\ h_m \sqrt{m + 1} + g_m h_{m+1} &= 0 \end{aligned} \quad (2.46)$$

[§] Note that \widehat{C} is the analogue of (1.62). This choice justifies the form we took for F in the discussion of noncommutative solitons as bosonic D -branes

[¶] The generalization to an $N + 1$ component Higgs field is quite straightforward so we persist in restricting our attention to the $N = 1$ case for the moment.

with $q^{(1)} = q$ and $q^{(2)} = 0$. An explanation for this choice will follow below. For the moment though, notice that eqs.(2.46) mean that the coefficients of each of the components of the Higgs doublet are not independent. Indeed

$$h_{m+1} = \sqrt{\frac{m+1}{m+q+1}} \left(\frac{f_{m+1}}{f_m} \right) h_m. \quad (2.47)$$

This is a simple recurrence relation which is easily solved for h_m to give

$$h_m = \sqrt{\frac{m!q!}{(m+q)!}} \kappa f_m \quad (2.48)$$

with $\kappa = h_0/f_0$ determining the relationship between the initial conditions of each coefficient sequence. With the convenient definitions of $Q_n \equiv f_n^2$ and $P_n \equiv h_n^2$, this may be combined with the second of the BPS equations to give

$$\begin{aligned} Q_1 &= \frac{(q+1)Q_0}{1+\gamma-\gamma(1+\kappa^2)Q_0} \\ Q_{m+1} &= \frac{(m+q+1)Q_m^2}{Q_m + (m+q)Q_{m-1} - \gamma Q_m \left[\left(1 + \frac{m!q!}{(m+q)!} \kappa^2 \right) Q_m - 1 \right]} \quad m > 0 \end{aligned} \quad (2.49)$$

where, following [57], the dimensionless combination θe^2 is denoted γ . In principle then, the noncommutative vortex solution of the critically coupled linear sigma model may be determined by solving the recurrence relation (2.50) and consequently (2.48) subject to the ‘‘boundary conditions’’ $f_n \rightarrow 1$, $h_n \rightarrow 0$ as $n \rightarrow \infty$. Well, almost. The attentive reader would of course have noticed that there is still the matter of the arbitrary integer q . Fortunately, there is also the third of the BPS equations, the flux constraint. Using (2.46) and the *ansatz* for the gauge field it may be shown that

$$\begin{aligned} \text{Tr}_{\mathcal{H}} \left(1 + [\widehat{C}^\dagger, \widehat{C}] \right) &= \text{Tr}_{\mathcal{H}} \left(\sum_{m=0}^{\infty} \left(1 + g_{m-1}^2 - g_m^2 \right) |m\rangle \langle m| \right) \\ &= \sum_{l,m=0}^{\infty} \left(1 + g_{m-1}^2 - g_m^2 \right) \langle l|m\rangle \langle m|l\rangle \\ &= \lim_{M \rightarrow \infty} \left[M + 1 - (M + q + 1) \frac{Q_M}{Q_{M+1}} \right]. \end{aligned} \quad (2.50)$$

In the last step, the cutoff of (2.5) was employed to regulate the trace. In the large M limit, the convergence of the coefficient sequence means

that the ratio of successive Q 's approaches unity. Consequently, the flux constraint equation implies that $q = k$. Indeed, a quick comparison with the analogous commutative result confirms that this is the only physically meaningful conclusion; the index of ϕ_1 is equal to the topological number of the vortex. Interestingly enough, choosing $q^{(2)} \neq 0$ does not affect this conclusion. Again, this is not altogether unexpected since $q^{(2)}$ is just the angular momentum quantum number of the vortex [41]. Convergence of the coefficient sequence for ϕ_2 bounds the angular momentum quantum number to the range $0 \leq q^{(2)} < k$. However, since none of the arguments presented here depends essentially on $q^{(2)}$ we can, without any loss of generality, set $q^{(2)} = 0$. It is also worth noting that when $q = 0$, $h_m = \kappa f_m$ and both boundary conditions can only be simultaneously satisfied if $h_m \equiv 0$ which reduces to the $k = 0$ vortex of the noncommutative Abelian-Higgs model [26, 27, 48, 57, 60]. Instead of solving eqs.(2.50) in full generality, it is perhaps more illuminating to focus on a few examples.

2.3.4 Examples

1. To begin with we consider the case $h_m = 0$ for all m . In this case the Higgs doublet $\Phi = (\phi_1, 0)$ satisfies exactly the equations of motion of the noncommutative Abelian-Higgs model and it is quite easy to check that the solutions of (2.50) reduce to the degree- k vortices found in [48] for which

$$\begin{aligned}
 Q_1 &= \frac{(q+1)Q_0}{1+\gamma(1-Q_0)} & (2.51) \\
 Q_{m+1} &= \frac{(m+q+1)Q_m^2}{Q_m + (m+q)Q_{m-1} - \gamma Q_m(Q_m - 1)} & m > 0
 \end{aligned}$$

This set of equations has been studied extensively and numerically shown to exhibit regular vortex solutions with $+k$ units of magnetic flux for a large γ range. In particular, for small θ (and consequently γ) the regular commutative Nielsen-Olesen vortex solutions of [28, 53] are obtained. In addition, an obvious solution to (2.52) that satisfies the boundary conditions of the semilocal model is $Q_m \equiv 1$. As noted in [48], these are exactly the fluxon solutions of [39].

2. Moving on now to the more interesting case of non-vanishing h_m , it will suffice to restrict our attention to $k = 1$ for which the BPS recurrence relations become

$$P_m = \frac{\kappa^2}{m+1} Q_m \quad (2.52)$$

$$Q_1 = \frac{2Q_0}{1 + \gamma - \gamma(1 + \kappa^2)Q_0}$$

$$Q_{m+1} = \frac{(m+2)Q_m^2}{Q_m + (m+1)Q_{m-1} - \gamma Q_m \left[\left(1 + \frac{\kappa^2}{m+1}\right) Q_m - 1 \right]} \quad m > 0.$$

The vortex solutions of the noncommutative semilocal model are constructed by solving eqs.(2.53) subject to the convergence constraint $(P_m, Q_m) \rightarrow (0, 1)$ as $n \rightarrow \infty$. From the first of these it is clear that when the Q_m sequence converges and κ is of order unity, $P_m \sim 1/m$ for large m . Again, this remains true for any fixed value of the angular momentum quantum number. We solve the above system numerically using a double precision, split-step shooting algorithm. At first glance, the shooting-parameter space looks to be two-dimensional (corresponding to the different values of the pair (P_0, Q_0)) but a prescient choice of $\kappa^2 = 1/\theta$ fixes one of these parameters in terms of the other and reduces the dimension to one. With the initial value Q_0 as the shooting parameter, we solve (2.53) for various values of γ and tabulate our results below.

θ	e^2	γ	Q_0
0.2	1	0.2	0.099732894
0.2	4	0.8	0.140471163
0.2	16	3.2	0.158732886334
0.2	36	7.2	0.16297094403243935
0.5	1	0.5	0.215729007
0.5	4	2	0.2895665841653
0.5	16	8	0.32043540606185
0.5	36	18	0.32737242959721649

Each of these initial values for the Q_m results in a coefficient sequence that converges (with varying degrees of accuracy) to one. Once determined, the P_m and Q_m may then be used to compute other characteristic quantities associated with the semilocal vortex. For example, the magnetic field of the semilocal vortex may easily be computed as

$$\hat{B} = \frac{\gamma}{\theta} \sum_{n=0}^{\infty} \left[1 - \left(\frac{n+1+\theta^{-1}}{n+1} \right) Q_n \right] |n\rangle \langle n|. \quad (2.53)$$

Substituting this, together with the covariant derivative

$$\hat{D}_z \hat{\Phi} = \sum_{m=0}^{\infty} \frac{1}{\sqrt{\theta}} \left(f_m \sqrt{m+1} + g_{m-1} f_{m-1} \right) |m\rangle \langle m| \otimes |I\rangle \quad (2.54)$$

$$+ \sum_{m=0}^{\infty} \frac{1}{\sqrt{\theta}} \left(h_{m+1} \sqrt{m+1} + g_m f_m \right) |m+1\rangle \langle m| \otimes \langle \mathbb{I} |$$

into eq.(2.41) allows for the energy density of the vortex to be computed quite straightforwardly as

$$\begin{aligned} \hat{\mathcal{E}} &= \frac{1}{\theta} \sum_{m=0}^{\infty} \left[\frac{m+1}{Q_m} (Q_m - Q_{m-1})^2 + \frac{m}{P_m} (P_m - P_{m-1})^2 \right. \\ &\quad \left. + \gamma \left(1 + \left(\frac{m+1+\theta^{-1}}{m+1} \right) Q_m \right)^2 \right] |m\rangle \langle m|. \end{aligned} \quad (2.55)$$

It may be verified numerically that up to the first few hundred terms the above expression for the energy density sums to $1/(2\pi\theta)$ to within a few percent as is expected for the 1-vortex solution. To make contact with the primary aim of this chapter, it will be convenient to visualize the profile of the vortex, especially as γ is turned up. However, both eqs.(2.53) and (2.55) are Fock space representations. Fortunately, these can be turned into (noncommutative) coordinate space representations by appealing to (2.9). In fig.1 we plot the magnetic field as a function of r for various values of the dimensionless parameter γ . Fig.2. contains a series of snapshots of the energy profile of the vortex as gamma increases from 0.2 to 28.8.

2.4 The large coupling limit

Having presented a general algorithm for the construction of degree- k semilocal vortex solutions of the gauged noncommutative linear sigma model and explicitly constructed the 1-vortex solution we proceed now to study one of the more interesting limits of the semilocal model: its large coupling limit. At the level of the action (2.41), the $e^2 \rightarrow \infty$ limit decouples the gauge field dynamics and any finite energy static solution has

$$E = 2\pi\theta \text{Tr}_{\mathcal{H}} \left[(\hat{D}_z \hat{\Phi})(\hat{D}_z \hat{\Phi})^\dagger + (\hat{D}_{\bar{z}} \hat{\Phi})(\hat{D}_{\bar{z}} \hat{\Phi})^\dagger \right] \quad (2.56)$$

subject to the constraint $\hat{\Phi}\hat{\Phi}^\dagger = 1$. In this limit, the gauge field is relegated to an auxiliary field, completely determined by $\hat{\Phi}$. Recalling that $\hat{\Phi}$ is an $(n+1)$ -component complex vector leads to the conclusion that this is, of course, nothing but the noncommutative version of the $\mathbb{C}\mathbb{P}^N$ sigma model (reviewed in Appendix 1). At the level of the action this observation is certainly not new; in the commutative case^{||}, this relation

^{||} Indeed, even in the noncommutative case it has not gone entirely unnoticed. In [57] a formal $2k$ -parameter solution to the vortex equations of the noncommutative

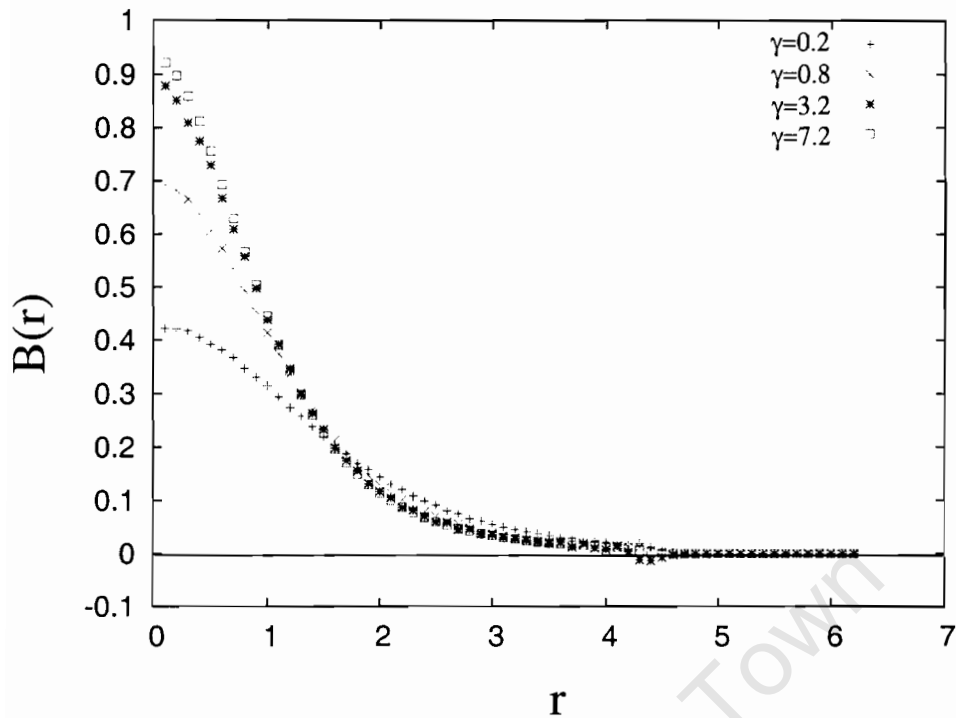


Fig. 2.1. The magnetic field trapped in the vortex core for varying γ

has been commented on by several authors in many different contexts [41, 42, 55, 63]. However, it remains to be seen whether this correspondence persists at the level of the solutions. If it does we will have produced an explicit descent from the vortices of the fuzzy linear sigma model to the instantons of the noncommutative $\mathbb{C}\mathbb{P}^N$ model. In the interests of self-containment, we review now the derivation of the lump solutions of the noncommutative sigma model.

With (2.56) as a starting point, we relabel and reparameterize the $(N + 1)$ -component Higgs field as $\hat{\Phi} \rightarrow \hat{U} = (1/\sqrt{\widehat{W}\widehat{W}^\dagger})\widehat{W}$. A subsequent definition of the Hermitian projector $P \equiv \widehat{W}^\dagger(\widehat{W}\widehat{W}^\dagger)^{-1}\widehat{W}$ allows for the

Abelian-Higgs model was found to all orders in γ^{-1} and, in particular, the metric on the moduli space of vortices explicitly computed in the limit $\gamma \rightarrow \infty$. There it was also noted that while this limit is usually taken to mean $\theta \rightarrow \infty$, it could equally well correspond to the large coupling limit. It is this latter view that we advocate.

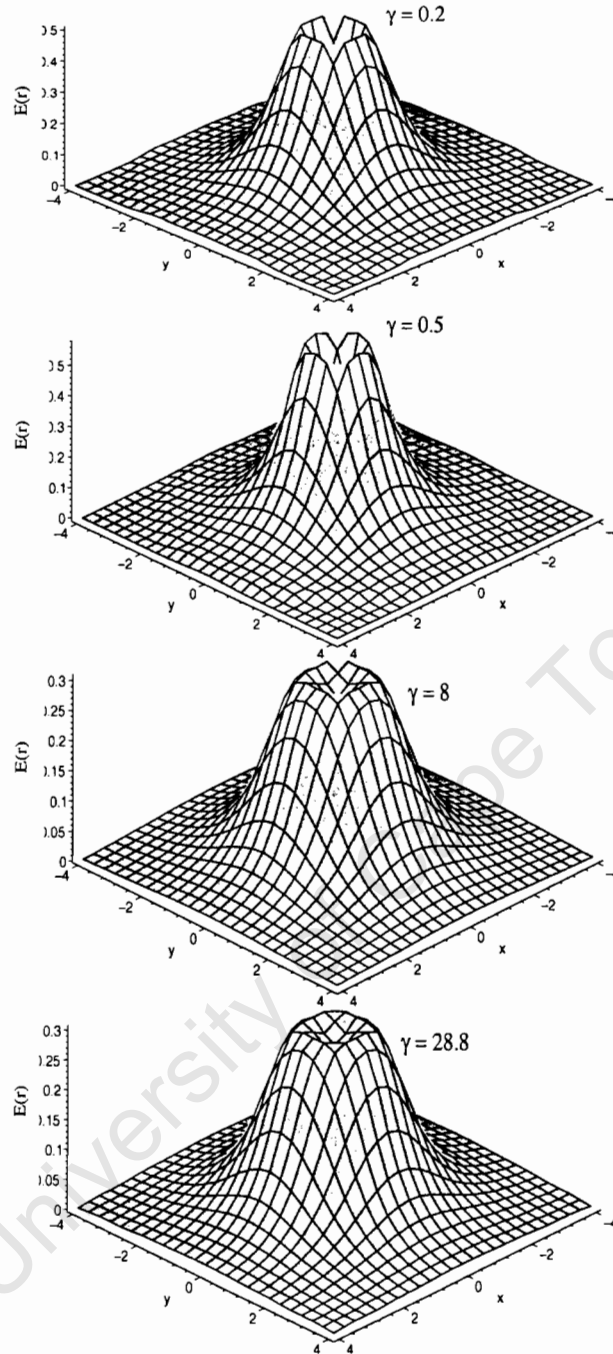


Fig. 2.2. The metamorphosis of the semilocal vortex into the $\mathbb{C}\mathbb{P}^N$ lump

static energy (or two-dimensional action) to be written as

$$E = 2\pi \operatorname{Tr}_{\mathcal{H}} \operatorname{tr} \left([P, \widehat{a}^\dagger] [\widehat{a}, P] \right). \quad (2.57)$$

In this form, the $\mathbb{C}\mathbb{P}^N$ energy is remarkably similar to the kinetic term of the static energy of a $(2+1)$ -dimensional noncommutative scalar field (see equation (2.2) of [34]) with the crucial difference of the additional matrix trace in (2.57). Indeed it was shown in [45, 46, 50] that the quantity $\operatorname{Tr}_{\mathcal{H}} \operatorname{tr} [\widehat{a}, \widehat{a}^\dagger P]$ contributes a nonvanishing boundary term to the energy and some care needs to be exercised in the derivation of the noncommutative Bogomol'nyi bound. With this in mind, the energy may correctly be written as

$$E = 2\pi \operatorname{Tr}_{\mathcal{H}} \operatorname{tr} \left(2F_+(P)^\dagger F_+(P) \right) + 2\pi Q_+ \geq 2\pi Q_+ \quad (2.58)$$

with the topological charge $Q_+ \equiv \operatorname{Tr}_{\mathcal{H}} \operatorname{tr} (P - [\widehat{a}, \widehat{a}^\dagger P])$ and $F_+(P) \equiv (1 - P)\widehat{a}P$. A similar expression holds for the anti-BPS states. Focusing on the BPS states though, saturation of the bound on the energy is obtained when $F_+(P) = 0$. As first shown in [47], solutions are not difficult to find; any Hermitian projector constructed from an $(n+1)$ -vector \widehat{W} whose components are holomorphic polynomials in \widehat{z} will satisfy the above BPS equation. These are precisely the noncommutative extension of the instanton solutions of the conventional $\mathbb{C}\mathbb{P}^N$ sigma model. For example, the static, 1- and 2-lump solutions of the noncommutative $\mathbb{C}\mathbb{P}^1$ model are given by

$$\widehat{W}_1 = (\widehat{z} - a_1, b_1) \quad \widehat{W}_2 = (\widehat{z}^2 - a_2, 2b_2\widehat{z} + c_2) \quad (2.59)$$

where the soliton parameters $a_1, \dots, d_2 \in \mathbb{C}$ are chosen to coincide with the standard way of writing the solutions in the commutative theory [59]. These are the complex moduli of the $\mathbb{C}\mathbb{P}^1$ instanton. To facilitate comparison with the vortices, these may be written in the harmonic oscillator basis so that, for example, the 1-lump solution becomes

$$\widehat{U}_1 = \sum_{n=0}^{\infty} \sqrt{\frac{\theta(n+1)}{\theta(n+1)+1}} |n\rangle \langle n+1| \otimes \langle \mathbb{I} | + \sum_{n=0}^{\infty} \sqrt{\frac{1}{\theta(n+1)+1}} |n\rangle \langle n| \otimes \langle \mathbb{I} | \quad (2.60)$$

Returning to the degree- k semilocal vortex of the last section, notice that (2.50) may be recast as

$$\left(1 + \frac{n!k!}{(n+k)!} \kappa^2 \right) Q_n - 1 + \frac{1}{\gamma} \left((n+k+1) \frac{Q_n}{Q_{n+1}} - (n+k) \frac{Q_{n-1}}{Q_n} - 1 \right) = 0 \quad (2.61)$$

In the infinite coupling limit $e^2 \rightarrow \infty$ (or equivalently $\gamma \rightarrow \infty$), the above recurrence relation may be solved exactly to give

$$Q_n = \left(1 + \frac{n!k!}{(n+k)!}\kappa^2\right)^{-1}. \quad (2.62)$$

In particular, for $k = 1$ we find

$$Q_n = \frac{n+1}{n+1+\kappa^2} \quad P_n = \frac{\kappa^2}{n+1+\kappa^2}. \quad (2.63)$$

Finally, matching coefficients to all orders in (2.60) and (2.63) means that the descent from noncommutative vortex to fuzzy lump only occurs when $\kappa^2 = 1/\theta$. Indeed, this is exactly the choice we made in our numerical computations to reduce the dimension of the shooting-parameter space. As a check, we expect that for a fixed value of θ , $Q_0 \rightarrow \theta/(\theta+1)$ as $\gamma \rightarrow \infty$. A quick glance at the table of our numerical results verifies that this is indeed the case for $\theta = 0.2$ and 0.5 . Figure (2.3) illustrates the convergence for $\theta = 0.5$. Moreover, hindsight reveals that the set of

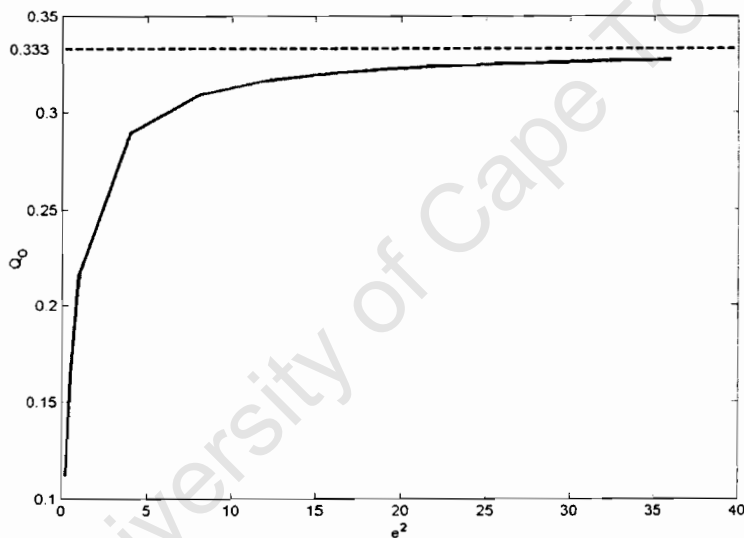


Fig. 2.3. The approach of Q_0 to the expected \mathbb{CP}^1 1-lump value as $e^2 \rightarrow \infty$.

energy densities in figure (2.2) is in fact a series of snapshots of the $k = 1$ vortex of the noncommutative semilocal model morphing into a fuzzy \mathbb{CP}^1 1-lump. The case $k = 2$ is no less straightforward. With its center of mass localised at the origin, the \mathbb{CP}^1 2-lump in eq.(2.59) can be written

as

$$\begin{aligned} \widehat{U}_2 &= \sum_{n=0}^{\infty} \sqrt{\frac{\theta^2(n+1)(n+2)}{\theta^2(n+1)(n+2)+1}} |n\rangle\langle n+2| \otimes \langle \mathbf{I}| \\ &+ \sum_{n=0}^{\infty} \sqrt{\frac{1}{\theta^2(n+1)(n+2)+1}} |n\rangle\langle n| \otimes \langle \mathbf{II}| \end{aligned} \quad (2.64)$$

when b_2 , the frozen out modulus [29] is set to zero. A comparison with the general expression for the infinite coupling coefficients (2.62) reveals a matching at all levels only if $\kappa^2 = 1/2\theta$. Generalisation to larger k follows in much the same way so no further attention is paid to it here.

At this juncture, a few comments are in order. The BPS equations of the commutative gauged linear sigma model admit a one parameter family of vortex solutions [42]. This single complex parameter w is to the commutative theory what the ratio of initial coefficients κ is to our noncommutative model with $w = 0$ corresponding to the conventional Nielsen-Olesen string. One of the distinguishing characteristics of the $w \neq 0$ semilocal vortices is the power law behavior exhibited by the scalar and gauge fields as they relax to their respective vacuum values. Consequently, the magnetic field** $B \sim 2|w|^2/\xi^4$ and the width of the flux tube trapped in the vortex core is an arbitrary parameter instead of the Compton wavelength of the vector boson as in the Nielsen-Olesen vortex. In the noncommutative model we once again find a one parameter family of vortices only now the parameter, κ , is not at all arbitrary. Indeed, we find that there exists a point in the κ parameter space dependent on the degree of the vortex and the deformation parameter θ at which the semilocal vortex *exactly* descends to the corresponding noncommutative \mathbb{CP}^N lump. Correspondingly, the width of the magnetic flux tube associated with the semilocal vortex is set by the scale of noncommutativity. This observed exact metamorphosis of the vortex into the lump should be compared to the results of section 3. of [42]. There an expansion of the 1-instanton solution of the commutative \mathbb{CP}^N model in powers of $|w|/|z|$ was used to establish that the vortex-instanton matching was exact at spatial infinity with differences emerging at $\mathcal{O}(|w|^4/|z|^4)$ in this expansion.

2.5 Brane Realisations

Quite apart from their intrinsic field theoretic value [58, 41, 42], the vortices of gauged linear sigma models also have a remarkably rich stringy structure. Beginning with the ground-breaking work of [37] in which the

** Following [42] ξ is a dimensionless radial variable on the plane.

(2 + 1)–dimensional, $\mathcal{N} = 4$ $U(N)$ Yang-Mills-Higgs theory was recognised as the worldvolume theory on a stack of N $D3$ –branes suspended between two parallel $NS5$ –branes, an intricate tapestry of ideas can be woven, leading inexorably to a realisation of the noncommutative semilocal vortex as a D –brane configuration in type IIB string theory [36]. In this section, we review some of these ideas and cast them into a form that better facilitates comparison with our results.

As in [36] the description of the system begins with a (2+1)–dimensional, $\mathcal{N} = 4$, $U(N)$ Yang-Mills-Higgs theory. The field content of the theory consists of a $U(N)$ vector multiplet made up of a gauge field A_μ and a triplet of adjoint scalars ϕ^r together with their fermionic super partners. Coupled to these are N fundamental hypermultiplets each of which contain a doublet of complex scalars q and \tilde{q} and their super partners. The Lagrangian for the theory is endowed with a global $SU(N + M)$ flavour symmetry as well as a local $U(N)$ gauge symmetry. Consequently, under these two groups and with $N_f \equiv N + M$ denoting the number of flavours, q and \tilde{q} transform as $(\mathbf{N}, \overline{\mathbf{N}}_f)$ and $(\overline{\mathbf{N}}, \mathbf{N}_f)$ respectively; the fundamental scalars are represented by $N \times (N + M)$ matrices. The dynamical content of the bosonic sector of the theory is contained in the Lagrangian

$$\begin{aligned} \mathcal{L} = & -\text{Tr} \left[\frac{1}{4e^2} F^2 + \frac{1}{2e^2} (D\phi^r)^2 + (Dq)^2 + (D\tilde{q})^2 + e^2 |q\tilde{q}|^2 \right. \\ & \left. + \frac{1}{2e^2} [\phi^r, \phi^s]^2 + (q^2 + \tilde{q}^2) \phi^r \phi^r + \frac{e^2}{2} (q^2 - \tilde{q}^2 - \zeta \mathbf{1}_N) \right]. \end{aligned} \quad (2.65)$$

where the Fayet-Iliopolous (FI) parameter, ζ , in the final D-term in (2.65) is chosen to be positive. This theory exhibits a Higgs branch of vacua which possess BPS vortices only if \tilde{q} and ϕ^r both vanish. This constraint defines a so-called reduced Higgs branch, $\mathcal{N}_{N,M} = Gr(N, N + M)$, the Grassmannian manifold of N –dimensional hyperplanes in \mathbb{C}^{N+M} . A particular vacuum choice^{††} is made by picking

$$q_{\text{vac}} = \begin{cases} \sqrt{\zeta} \delta_i^a & a, i = 1, \dots, N \\ 0 & i = N + 1, \dots, N + M \end{cases} \quad (2.66)$$

In our abelian case, for example, $N = 1$, the reduced Higgs branch $\mathcal{N}_{1,M} = Gr(1, 1 + M) \simeq \mathbb{C}\mathbb{P}^M$ and $q_{\text{vac}} = (\sqrt{\zeta}, 0)$. Relabelling $q \rightarrow \Phi$, setting the FI parameter $\zeta = 1$ and restricting to time-independent solutions trivially establishes the equivalence of the action in this branch with the static energy (2.35). As discussed earlier, the spectrum of solutions

^{††} Since the Grassmannian is, after all, a symmetric space, no generality is lost in this choice.

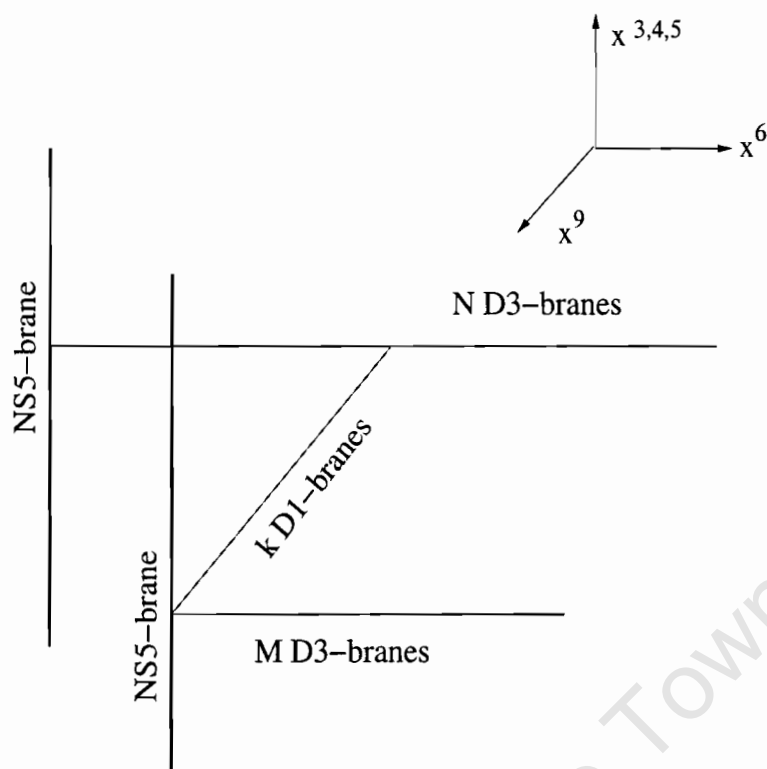


Fig. 2.4. The degree- k BPS vortex as k stretched D-strings.

of this theory is rich with BPS vortices. The brane realization of these vortices is built up from the $U(N)$ Yang-Mills-Higgs described in (2.65). It consists of N $D3$ -branes suspended between two parallel $NS5$ -branes and a further $N + M$ $D3$'s attached to the right hand $NS5$ -brane to add flavour (see figure 2.4).

In the Higgs branch, one of the $NS5$ -branes is separated from the others. This separation is proportional to the FI parameter ζ . The degree- k BPS vortices manifest as k D -strings stretched between the $D3$ -branes and the separated $NS5$ -brane - an identification made on the basis of the fact that the stretched $D1$ -branes are the *only* BPS states of the brane configuration with the correct mass. More than just a pretty picture, the geometry of the D -brane configuration in figure 2.4 encodes vital information about the FI parameter, ζ as well as the gauge coupling e as

$$\frac{1}{e^2} = \frac{\Delta x^6}{2\pi g_s}; \quad \zeta = \frac{\Delta x^9}{4\pi^2 g_s l_s^2} \quad (2.67)$$

where l_s and g_s are the string length and coupling respectively and Δx^6 and Δx^9 are the separation distances between the $NS5$ -branes defined as in figure 2.4. It is now clear that the sigma model limit ($e^2 \rightarrow \infty$) of the vortex occurs precisely when the separation of the $NS5$ -branes in the 6-direction vanishes. The configuration that realises the k -lump solution of the (commutative) $\mathbb{C}\mathbb{P}^M$ nonlinear sigma model then is as above only with $N = 1$. As we have seen, in string theory the transition from commutative to noncommutative worldvolume theories is achieved by turning on an NS-NS B -field in the appropriate direction [2]. In the present context, the transition from the semilocal action (2.35) to its noncommutative counterpart (2.41) translates into turning on a constant NS-NS B -field $B_{12} = \theta dx^1 \wedge dx^2$ in the (1, 2)-directions in a background of two $NS5$ -branes with a $D3$ -brane stretched between them and a further $M + 1$ $D3$'s attached to the right hand $NS5$ -brane. What of the vortices? The effect of the B -field on the D -strings stretched between the $NS5$ -brane and the $D3$ is quite remarkable. The basic physics is analogous to the situation of a D -string suspended between two $D3$ -branes studied in [40] and was first described for the vortex case in [36]. The NS-NS 2-form manifests on the $D3$ -worldvolume as a constant magnetic flux \mathcal{F}_{12} while the D -string endpoint appears as a magnetic source. Since on the 4-dimensional worldvolume of the $D3$ -brane $\mathcal{F}_{12} = \star\mathcal{F}_{06}$, the magnetic endpoint of the $D1$ -brane feels the same force as an electric charge in a constant electric field in the 6-direction. However, as the other end of the D -string remains married to the $NS5$ -brane, the D -string responds to this force by tilting as in figure 2.5. The effect of the tilting was investigated in [40] by studying the D -string Born-Infeld action at weak string coupling

$$S = \frac{1}{2\pi l_s^2} \int_0^{\Delta x^9} dx^9 \left(\frac{1}{g_s} \sqrt{1 + \left(\frac{dx^6}{dx^9}\right)^2} + A_{06} \frac{dx^6}{dx^9} \right) \quad (2.68)$$

where the RR 2-form A_{06} that couples to the D -string worldvolume is induced by \mathcal{B}_{06} . The result of that investigation translated into the language of the vortex theory [36] is that the displacement of the $D1$ -brane endpoint is given by^{††} $\delta = (\theta \Delta x^9)/(2\pi l_s^2)$. With this and some straightforward algebra, the distance between the D -string endpoint and the left $NS5$ -brane can be computed. With the choice of $\zeta = 1$ for the FI parameter, the result is

$$r = 2\pi g_s \left(\frac{1}{e^2} + \theta \right). \quad (2.69)$$

^{††} Note the sign difference from [36] and the difference it has on the tilt of the D -strings.

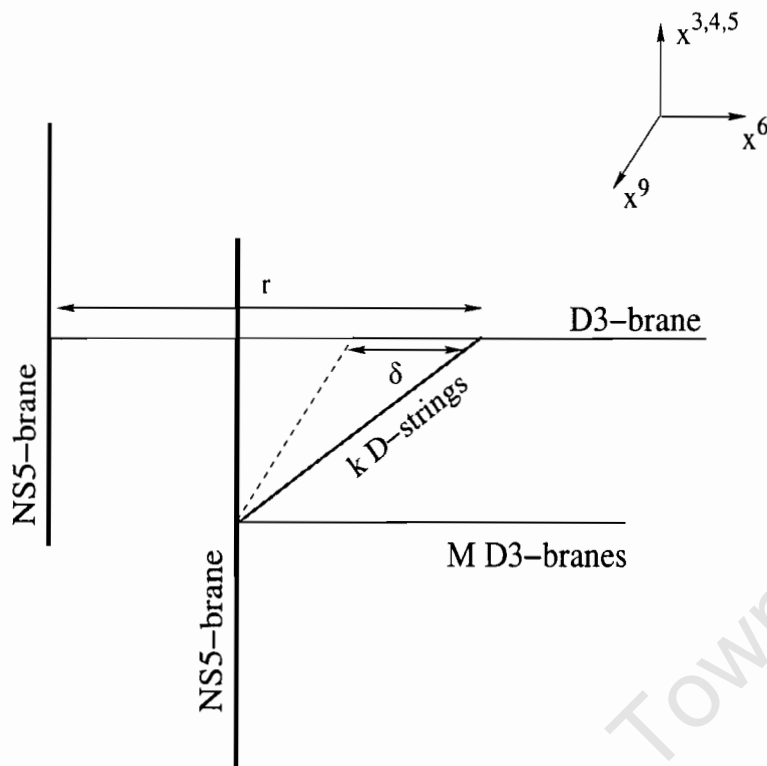


Fig. 2.5. The magnetic field induced on the $D3$ -worldvolume causes the D -strings to tilt.

This distance is, in fact, the FI parameter of the theory living on the $D1$ -branes (see [36] for a lucid discussion of this aspect). Having fixed Δx^9 with the choice $\zeta = 1$ the magnitude of r is completely determined by the size of the gauge coupling as determined by the $NS5$ -brane separation in the 6-direction and the noncommutativity. Since the latter is also fixed, the transition from vortex to lump can be studied by changing the distance between the $NS5$ -branes. As Δx^6 is decreased to zero, the separation between the D -string endpoint and the left $NS5$ -brane decreases to $r_* = 2\pi g_s \theta$. It is this configuration of the k tilted D -strings stretched between the (formerly right hand) $NS5$ -brane and the $D3$ -brane that realises the degree- k instanton of the $\mathbb{C}\mathbb{P}^M$ sigma model. This concludes our treatment of D -brane realisation of the noncommutative $\mathbb{C}\mathbb{P}^M$ lump.

More than just an academic exercise, this identification of the semilocal vortex and $\mathbb{C}\mathbb{P}^M$ instanton has proven invaluable in the understanding of the low energy dynamics of both the vortex and instanton as encoded

in the geometry of their respective moduli spaces [57]. We refer the interested reader to [36] for a nice discussion of the structure of the moduli spaces and content ourselves with merely summarising some of their most pertinent results. The moduli space of degree- k semilocal vortices $\hat{\mathcal{V}}_{k,(1,M)}$ is a $2k(1+M)$ -dimensional space with a natural Kähler metric defined by the overlap of zero modes. However, this metric is afflicted with some non-normalisable zero modes that, classically, correspond to the moduli with infinite moments of inertia and that make the quantum mechanical treatment of these solitonic objects quite subtle. Fortunately these subtleties may be circumvented with a little help from the branes. A study of the theory on the $D1$ -brane predicts that the Higgs branch, $\hat{\mathcal{M}}_{k,(1,M)}$, constructed by a $U(k)$ Kähler quotient of $\mathbb{C}^{k(1+M+k)}$ is isomorphic to the moduli space $\hat{\mathcal{V}}_{k,(1,M)}$. While the metric on $\hat{\mathcal{M}}_{k,(1,M)}$ retains all the symmetries of the Kähler metric on the vortex moduli space, it is finite and suffers from none of the non-normalisability problems of the latter. Consequently, the study of the quantum theory of semilocal vortices may be simplified somewhat by replacing the natural metric on the vortex moduli space with the metric on the Higgs branch of the D -string theory inherited from the Kähler quotient construction of [36].

2.6 Conclusions and Discussion

The primary concern of this chapter has been the construction and study of a noncommutative extension of $(2+1)$ -dimensional critically coupled, gauged linear sigma model. Like its commutative counterpart this theory possesses a rich spectrum of BPS solutions. By extending the systematic construction of [48] we have explicitly constructed a family of vortex solutions to the BPS equations (2.42) for arbitrary positive values of the noncommutativity parameter θ . As expected, these fuzzy vortices reduce to the exact Nielsen-Olesen strings of the noncommutative Abelian-Higgs model [26, 27, 48, 60] on the $\kappa = 0$ surface of the parameter space. Despite retaining many of the properties of their commutative analogues [42, 58], the introduction of a new length scale set by the noncommutativity parameter θ induces several remarkable differences. Among these we find that the width of the magnetic flux tube trapped in the vortex core no longer exhibits the characteristic arbitrariness of the commutative semilocal vortex. In the noncommutative model, this width is set by the scale of the noncommutativity.

The detailed investigation of the large coupling ($e^2 \rightarrow \infty$) regime of the θ -deformed gauged linear sigma model confirms, both numerically and analytically, the commutative intuition of the vortex metamorphosing into

a lump of the (fuzzy) $\mathbb{C}\mathbb{P}^M$ sigma model. Additionally, while the agreement between vortex and lump in the $\theta = 0$ case is precise only asymptotically [42], we find an *exact* matching at all levels of the harmonic oscillator expansion at finite θ . Indeed, insisting that this agreement holds selects a preferred set of values for κ , dependent on the scale of noncommutativity and the degree of the vortex. This effectively reduces the dimension of the parameter space by one. While we have explicitly constructed solutions for the 1- and 2-vortex cases, the construction of higher degree solutions follows in much the same way and we do not expect any further surprises. Finally, we reviewed the elegant constructions of [36] that lead to a realisation of the noncommutative $\mathbb{C}\mathbb{P}^M$ k -lump as k tilted D -strings stretched between an isolated $NS5$ -brane (on which a stack of M semi-infinite $D3$ -branes end) and a semi-infinite $D3$ whose one endpoint ends on a second $NS5$ (see figure 2.5). This identification is built on the foundation of a study of the $\mathcal{N} = 4$ $U(N)$ Yang-Mills-Higgs $D3$ -worldvolume theory hinges on the metamorphosis of vortices into lumps. Of course, to be sure that this configuration really does correspond to the lump solution requires more work than just a comparison of the masses of both configurations; the spectrum of fluctuations around each object needs to be computed and compared. This is a more difficult endeavor which, together with a more thorough investigation of the spectrum of BPS objects of the noncommutative gauged linear sigma model is left to future work [51]. Curiously, this realisation of fuzzy $\mathbb{C}\mathbb{P}^M$ lumps is not unique, at least for $M = 1$. Drawing on the tree level equivalence between $\mathcal{N} = 2$ open string theory and self-dual Yang-Mills theory in $(2+2)$ -dimensions [54], it was argued in [45, 46] that the effective field theory induced on the worldvolume of N $D2$ -branes by $\mathcal{N} = 2$ open strings in a Kähler B -field background is a noncommutative $U(N)$ sigma model. Using a modified “method of dressing” soliton solutions of the latter were constructed and their various scattering properties investigated. In this context, the k -lump solution of the $\mathbb{C}\mathbb{P}^1$ sigma model may be interpreted as k $D0$ -branes in the worldvolume of a stack of $D2$ -branes [46, 43]. Again, while this assertion needs to be tested beyond the level of a mass comparison, the possibility of a duality between $\mathcal{N} = 2$ open string theory and the type II-B superstring is, to say the least, intriguing and certainly deserves more attention.

3

Noncommutative Big Bang/Big Crunch Space-times

3.1 Introduction

The idea that our universe undergoes a kind of reincarnation, rises like a Phoenix from its own ashes, is an old and beautiful one. In the scientific literature alone it has been considered as a viable cosmological model almost from the very beginning, and doubtless its roots may be traced back into the mythologies of time immemorial. As a cosmological scenario, its popularity has waxed and waned; often left to flounder in obscurity only to resurface in a new and unexpected guise.

In the era of general relativity and the modern approach to cosmology, it was perhaps Lemaitre who in 1933 first turned to this enchanting possibility: *'The solutions where the universe successively expands and contracts...have an incontestable poetic charm and bring to mind the Phoenix of the legend.'* [68] The following year Tolman studied the thermodynamic implications of a bouncing universe [69]. Since then the scenario has emerged in variety of guises. Perhaps most notably, Smolin used it in [71] as the basis for his evolving universe theory in an attempt to explain the values of the physical constants of particle physics and cosmology. Closely related is the idea of universe generation from within black holes [72]. For a synopsis of some of the older literature see [70].

Most of the literature mentioned above focuses on the cosmological implications of a Big Bang/Big Crunch space time. However, in the light of the Hawking-Penrose singularity theorems, the scenario plays a different role. With the inevitability of a space-time singularity in mind (at least from the perspective of classical general relativity), the need for a theory which either resolves this cosmological singularity, or yields a well defined evolution through it becomes apparent. It is in this context that the scenario is of interest to us. As a bonus, this possibility also provides an elegant solution to the problem of the beginning of time - in this model

there need not be one. Time may be continued through the singularity, perhaps infinitely far.

Up until recently it has been difficult to make any concrete statements about cosmological singularity resolution from the perspective of a particular candidate quantum theory of gravity. Even string theory has been noticeably tight-lipped owing to the difficulty of formulating the theory on time-dependent backgrounds. Yet if string theory is to fulfill the promise of a true quantum theory of gravity, it must rise to the challenge. There have been several attempts to describe the big bang singularity in a stringy context. Amongst the most notable are the so called Ekpyrotic [73] and Cyclic [74] universe scenarios, which model the Big Bang as a brane collision which is everywhere well defined from the higher dimensional perspective, except at the collision point. The only manifestation of the lower dimensional singularity is that the fifth dimension shrinks to zero size at the collision event. However, in order for such scenarios to be taken seriously, we need a better understanding of how one may uniquely pass through the singularity corresponding to the vanishing of the extra dimension.*

This issue has been approached from the perspective of quantum fields on a Big Bang/Big Crunch space-time [75], with some success. However, as the authors admit, the results obtained have limited utility owing to the neglect of gravitational backreaction in this approximation. It is hoped that a full stringy calculation will verify their results. In this chapter we will be interested in trying to take the analysis of [75] one step further towards this ultimate goal by making space-time noncommutative. This can be motivated in two ways: First it may be seen as an *ad hoc* attempt to model the quantum geometry we expect to prevail at early times, and second as an attempt to include the effects of the stringy background B -field we expect to be present in the quantum gravity era.

3.2 The Big-Bang/Big-Crunch scenario

Note: In this chapter capital Latin indices take values in the field target-space.

* The reader should note that we have neglected to mention any approaches to the problem which stem from other candidate theories of quantum gravity. In particular, results in Loop Quantum Cosmology suggest that the universe may in fact be fundamentally nonsingular in the quantum gravity era. Aside from being wholly outside the scope of this discussion, these results come with their own barrage of problems, and we will simply refer the interested reader to the literature - [87], [88].

3.2.1 A no-go theorem

In conventional general relativity a reversal from a contracting to an expanding Friedmann-Robertson-Walker (FRW) model is forbidden for flat models if we assume the null energy condition. To see this, consider minimally coupled scalar fields ϕ^K in a 4d space-time. Using Weyl rescaling we may write the action as

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2\kappa} \mathcal{R} - \frac{1}{2} g^{\mu\nu} G_{IJ}(\phi^K) \nabla_\mu \phi^I \nabla_\nu \phi^J - V(\phi^K) \right). \quad (3.1)$$

Here κ is a constant, $g_{\mu\nu}$ is the space-time metric, and G_{IJ} the metric on the field target space. For the theory to be unitary we require $G_{IJ} \geq 0$. The energy momentum tensor associated to the variation of the matter action with respect to $g^{\mu\nu}$ is

$$T_{\mu\nu} = G_{IJ} \partial_\mu \phi^I \partial_\nu \phi^J - g_{\mu\nu} \left[\frac{1}{2} g^{\alpha\beta} G_{IJ} \partial_\alpha \phi^I \partial_\beta \phi^J + V \right]. \quad (3.2)$$

For homogenous isotropic models $g_{\mu\nu}$ is a FRW metric and the scalar field may be thought of as a perfect fluid with energy density ρ and pressure p given by

$$\begin{aligned} \rho &= T_{00} = \frac{1}{2} G_{IJ} \dot{\phi}^I \dot{\phi}^J + V, \\ p &= -\frac{1}{3} g^{ij} T_{ij} = \frac{1}{2} G_{IJ} \dot{\phi}^I \dot{\phi}^J - V, \end{aligned} \quad (3.3)$$

where a dot signifies differentiation with respect to the cosmic time t . Note that homogeneity requires $\phi = \phi(t)$. From these relations it is clear that the scalar fields satisfy the null energy condition,

$$\rho + p = G_{IJ} \dot{\phi}^I \dot{\phi}^J \geq 0. \quad (3.4)$$

For flat models ($k = 0$) the metric may be written as

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1}^3 (dx^i)^2. \quad (3.5)$$

The Friedmann equation for $k = 0$ is

$$3H^2 = \kappa\rho + \Lambda, \quad (3.6)$$

where Λ is the cosmological constant and $H = \frac{\dot{a}}{a}$. Using this along with the Raychaudhuri equation which may be written as

$$3\dot{H} + 3H^2 + \frac{\kappa}{2}(\rho + 3p) - \Lambda = 0, \quad (3.7)$$

one finds

$$\dot{H} = -\frac{\kappa}{2}(\rho + p) \leq 0. \quad (3.8)$$

Thus we see that reversal from contraction to expansion is impossible in this model. Another way of looking at this is to note that the only way for a contracting phase and expanding phase to coexist in a flat, unitary model of this type is for them to be separated by a singularity.

There have been a number of suggestions as to how to circumvent this problem. They are mostly based on the realization that the action (3.1) cannot hold if our space-time passes through a singularity, since there it must be replaced by some more fundamental theory. In particular, notice that the action (3.1) is only of second order in the field derivatives. It is widely believed that this is modified by higher derivative terms when the universe enters the quantum gravity regime. These may be provided by a stringy effective action which contains higher order curvature couplings, as well as the possibility of a extra background fields. See the discussion in [64] and references therein.

Another more controversial remedy is to introduce some kind of exotic matter which violates the null energy condition. This however has consequences for causality, and is rather *ad hoc*.

Instead of adding anything extra at this stage we will investigate the possibility that the evolution through a singularity is better behaved from the perspective of a higher dimensional gravity theory. We shall see that here there is a set of variables which parameterize the evolution which remain finite at the singularity.

3.2.2 Solutions and perspectives

We consider a simplified model with action

$$S = \int d^d x \sqrt{-g} \left(\mathcal{R} - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right). \quad (3.9)$$

Note that we are now working in d dimensions and we have chosen units in which $2\kappa = 1$. We have also set $V = 0$. This model may be obtained as a low energy approximation to IIA or IIB string theory for $d = 10$. Furthermore, we restrict ourselves to the spatially flat, homogenous and isotropic space-times with metric

$$ds^2 = a^2(t) \left[-N^2(t) dt^2 + \sum_{i=1}^{d-1} (dx^i)^2 \right]. \quad (3.10)$$

We now define the following variables,

$$a_0 = a^{\frac{d-2}{2}} \left(e^{-\gamma\phi} + e^{\gamma\phi} \right) \quad (3.11)$$

$$a_1 = a^{\frac{d-2}{2}} \left(e^{-\gamma\phi} - e^{\gamma\phi} \right)$$

where $\gamma = \sqrt{(d-2)/8(d-1)}$. Note that $a_0 \geq |a_1|$. The old variables may be written in terms of the new via

$$\begin{aligned} \phi &= -\frac{1}{2\gamma} \ln \left(\frac{a_0 + a_1}{a_0 - a_1} \right) \\ a &= \left(\frac{a_0^2 - a_1^2}{4} \right)^{\frac{1}{d-2}}. \end{aligned} \quad (3.12)$$

In terms of the new variables the Lagrangian corresponding to (3.9) takes the form

$$\mathcal{L} = \frac{d-1}{(d-2)N(t)} [-\dot{a}_0^2 + \dot{a}_1^2]. \quad (3.13)$$

We can choose a gauge in which $N = 1$ and impose the equation of motion for N as a constraint. This is

$$\dot{a}_0^2 = \dot{a}_1^2. \quad (3.14)$$

The equations of motion for a_0 and a_1 are simply $\ddot{a}_0 = 0 = \ddot{a}_1$. These are easily solved subject to the above constraint. We may then write these solutions in terms of the old variables to find

$$\begin{aligned} a(t) &= a(1)|t|^{\frac{1}{d-2}}, \\ \phi(t) &= \phi(1) \pm \frac{1}{2\gamma} \ln |t|. \end{aligned} \quad (3.15)$$

The \pm comes from the quadratic nature of the constraint. Note that as $t \rightarrow 0$, $a \rightarrow 0$ and $\phi \rightarrow \mp\infty$. We can interpret the singular behaviour of the scalar field in terms of its embedding in string theory. In the IIA theory the string coupling is $g_s = e^\phi$, in which case the two solutions correspond to weak and strong coupling respectively as $t \rightarrow 0$. We will consider the solutions with the + sign - weak coupling.

For this choice there are in fact two solutions, one for positive t , the other for negative t . Note that the new variables are finite at the singularity, and that we have a bounce at $t = 0$ - the one solution is contracting for $t \leq 0$ and the other expanding for $t \geq 0$. It is an obvious conjecture that these two solutions can be connected at the singularity. Clearly this doesn't violate the no-go theorem as we pass through a singularity. We must now attempt to justify this conjecture from the perspective of some more fundamental theory which encodes microscopic information about the evolution through the singularity.

Lets look at the model we've been using from a different perspective. Firstly, we make the change of variables,

$$\tilde{g}_{\mu\nu} = e^{-\frac{4}{d-2}\gamma\phi} g_{\mu\nu} . \quad (3.16)$$

Since this is a conformal transformation, it is easy to calculate how the new Ricci scalar $\tilde{\mathcal{R}}$ is related to the old one, \mathcal{R}^\dagger . We find

$$\tilde{\mathcal{R}} = e^{\frac{4}{d-2}\gamma\phi} \left(\mathcal{R} - \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 4\frac{d-1}{d-2}\gamma g^{\lambda\sigma}\nabla_\lambda\nabla_\sigma\phi \right) . \quad (3.17)$$

Notice that the first two terms inside the bracket are precisely those used in the action (3.9). In fact it is easy to show that, defining $\psi = e^{\gamma\phi}$, the action (3.9) can be written in terms of the new variables as

$$S = \int d^d x \sqrt{-\tilde{g}} \psi^2 \tilde{\mathcal{R}} . \quad (3.18)$$

The unwanted third term in the above expression for \tilde{R} is a total derivative and thus falls away. The classical solution (3.16) can also be written in terms of these new variables (up to rescaling of the co-ordinates) as

$$\tilde{g}_{\mu\nu} = \eta_{\mu\nu} \quad \psi = \psi(1)\sqrt{|t|} . \quad (3.19)$$

Note that the metric $\tilde{g}_{\mu\nu}$ does not see the singularity, and is smooth at $t = 0$. ψ however is singular at $t = 0$ and the Planck scale goes to zero there.

As a first attempt at viewing this solution from the perspective of a more fundamental microscopic theory, we will show how the model described above can be thought of as a compactification of a $d + 1$ dimensional theory - *a la* Kaluza-Klein.

Consider a metric on a $d + 1$ dimensional space of the form

$$ds^2 = g_{\alpha\beta}^{d+1} dx^\alpha dx^\beta = \psi^4 dw^2 + \tilde{g}_{\mu\nu} dx^\mu dx^\nu \quad (3.20)$$

where we have deliberately set the (w, μ) components to zero in order to avoid an unwanted gauge field once we've done the reduction. We also take ψ and $\tilde{g}_{\mu\nu}$ to be independent of w . Consider a pure gravity action on this space:

$$S = \int d^d x dw \sqrt{-g^{(d+1)}} \mathcal{R}(g^{(d+1)}) . \quad (3.21)$$

If we now compactify the w direction on a circle or interval such that $w \sim w + 1$ then this is equivalent to (3.18). The classical solution (3.19) from this $d + 1$ dimensional perspective is then

$$ds^2 = (-d\tau^2 + A^2\tau^2 dw^2) + \delta_{ij} dx^i dx^j , \quad (3.22)$$

[†] The properties of the the Ricci scalar under conformal transformations can be found in any good text on general relativity. See for example Appendix D of [67]

where A is a constant and $\tau = x^0$. The part in brackets is the metric on \mathcal{M}_C^2 - the (1+1) dimensional Milne universe compactified in the spatial direction. This space is locally flat, as may be seen by embedding it in Minkowski space. Defining light-cone co-ordinates $x^\pm = \pm\tau e^{\pm A w}$, the metric on \mathcal{M}_C^2 is $ds^2 = dx^+ dx^-$ and the compactification relation becomes $x^\pm \sim e^{\pm A} x^\pm$. Thus \mathcal{M}_C^2 may be embedded in Minkowski space quotiented out by a boost. This space has been studied in detail in [76].

We have seen that our model of a scalar field in a spatially flat, homogeneous and isotropic space-time that exhibits reversal from contraction to expansion may be viewed as the solution $\mathcal{M}_C^2 \times \mathbb{R}^{d-1}$ of $d + 1$ dimensional Einstein gravity. Our conjecture that the expanding region can be connected to the contracting region in (3.16) amounts to saying that we must consider the whole double cone of \mathcal{M}_C^2 , i.e. $-\infty \leq \tau \leq \infty$, and the two regions are joined at $\tau = 0$. Although we've been able to represent the Big Bang/Crunch solution in terms of the 'more fundamental' higher dimensional gravity, this can at best be thought of as a heuristic (and suggestive) first step towards justifying our conjecture. What we really want to do is consider string theory in such a background, in the hope that the string's singularity resolving abilities may be brought to bare on the problem. At this stage however, it is not clear that \mathcal{M}_C^2 can be embedded naturally in string theory. See [66] for some suggestions.

As a first approximation, Tolley and Turok [75] have studied quantum field theory on $\mathcal{M}_C^2 \times \mathbb{R}^{d-1}$. The limitation of this approach is that we do not expect the approximation of quantum field theory on curved space time to be valid as $\tau \rightarrow 0$. This is because the field energy diverges at the singularity [77] and we therefore expect a large back-reaction on the space-time geometry. The formalism employed is only valid for static geometries and thus is an invalid approximation at precisely the point of interest.

As a further step towards the ultimate goal of doing a full string theory calculation, we will introduce spatial noncommutativity which is decoupled from gravity[†]. There are a number of ways of justifying this. Firstly, one may think of it as a modification of the original action (3.1) to include higher derivative terms. These come from replacing normal multiplication with the star product which depends on derivatives of all orders of the fields, as is clear from its exponential definition which we uncovered in chapter 1. One may justify this particular modification by appealing to our intuition about quantum gravity effects. As mentioned in the introduction we expect space-time geometry in the quantum gravity regime to

[†] That we consider the geometry as inert with respect to the noncommutativity may be justified by noting that (at least perturbatively) closed strings are unaffected by the presence of a B -field.

be replaced by some kind of quantum geometry. String theory suggests that at least in certain limits this may be described by the noncommutative geometry we have been studying. Thus, at least as a suggestive toy model, it seems that the study of noncommutative quantum fields in big bang/big crunch space-times may illuminate the problem to some degree.

3.3 Quantum fields on a conformally flat Big Bang/Big Crunch space-time

To investigate the effects of noncommutativity on field evolution through the singularity, we will study fields on the background given by the classical solution (3.16) to the action (3.9). As we will assume that the noncommutativity decouples from gravity, the solution for the conformal factor $a(t)$ remains valid[§] and we will consider quantum fields on a space-time with metric

$$ds^2 = A|t|^{\frac{2}{d-2}} \left(-dt^2 + \sum_{i=1}^{d-1} (dx^i)^2 \right) \quad (3.23)$$

where A is a constant. It will prove fruitful to review the study of commutative quantum fields on such space-times. For general reviews of quantum field theory on curved space-time see [78], [79].

Firstly, note that the kinetic term for a scalar field in such a space-time is conformally invariant. A mass term however will in general carry a positive power of the conformal factor since it goes like $\sqrt{-g}\phi^2$. Since our particular choice for $a(t)$ vanishes as $t \rightarrow 0$, the mass term is dominated by the kinetic term near the singularity. For this reason, and because we wish to make contact with our discussion following (3.9), we consider only massless fields, at least initially.

The action is

$$S = -\frac{1}{2} \int d^d x \sqrt{-g} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi, \quad (3.24)$$

from which we obtain the massless Klein-Gordon equation by varying with respect to ϕ ,

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0. \quad (3.25)$$

A calculation of the Christoffel symbols for a conformally flat space-time with metric $ds^2 = a^2(t)\eta_{\mu\nu}$ gives

$$\Gamma_{\beta\gamma}^\alpha = H(\delta_0^\alpha \eta_{\beta\gamma} + \delta_\gamma^0 \delta_\beta^\alpha + \delta_\beta^0 \delta_\gamma^\alpha) \quad (3.26)$$

[§] Provided that $\theta^{0\mu} = 0$.

where $H = \frac{\dot{a}}{a}$. The Klein-Gordon equation for conformally flat space-times may be written as

$$\frac{1}{a^2} \eta^{\mu\nu} (\partial_\mu \partial_\nu \phi - \Gamma_{\mu\nu}^\alpha \partial_\alpha \phi) = 0. \quad (3.27)$$

Using (3.26) this is easily shown to reduce to

$$\ddot{\phi} - \nabla^2 \phi + (d-2)H\dot{\phi} = 0 \quad (3.28)$$

where ∇^2 is the spatial Laplacian. Picking $a(t) = A|t|^{\frac{1}{d-2}}$ this becomes,

$$\ddot{\phi} - \nabla^2 \phi + \frac{1}{t} \dot{\phi} = 0. \quad (3.29)$$

This is the equation of motion for a massless scalar field in the background (3.23).

Since we wish to consider quantum fields, which in the canonical approach are represented by integrals over orthogonal mode functions, it is important to find solutions to this equation of the form $\phi = \phi(t)e^{i\vec{k}\cdot\vec{x}}$. Substituting this *ansatz* into (3.29) we find that the time dependent amplitude must be a solution of

$$\ddot{\phi} + \frac{1}{t} \dot{\phi} + k^2 \phi = 0 \quad (3.30)$$

where $k = |\vec{k}|$. This is Bessel's equation of zeroth order, and its solutions for $k \neq 0$ may be represented in terms of the Hankel functions, which have the following integral representation:

$$\begin{aligned} H_0^{(1)}(z) &= \frac{1}{i\pi} \int_{-\infty}^{\infty} du \exp(iz \cosh u), \\ H_0^{(2)}(z) &= -\frac{1}{i\pi} \int_{-\infty}^{\infty} du \exp(-iz \cosh u) \end{aligned} \quad (3.31)$$

for $z \in \mathbb{C}$, from which it is clear that $H_0^{(1)}(-z) = -H_0^{(2)}(z)$. Also, note that $H_0^{(1)}$ is analytic in the upper half plane, while $H_0^{(2)}$ is analytic in the lower half plane. The properly normalized[¶] positive and negative frequency modes are respectively (for $t > 0$):

$$\psi^+(k, t) = \sqrt{\frac{\pi}{4}} H_0^{(2)}(kt), \quad \psi^-(k, t) = \sqrt{\frac{\pi}{4}} H_0^{(1)}(kt). \quad (3.32)$$

The $k = 0$ mode satisfies (3.30) without the term in ϕ . The solution is easily found to be

$$\phi(t) = \alpha + \beta \ln |t| \quad (k = 0) \quad (3.33)$$

[¶] With respect to the Klein-Gordon inner product. See for example [78]

which blows up as $t \rightarrow 0$. Indeed if one analyzes the short distance behaviour of the Hankel functions one finds a similar logarithmic divergence as $t \rightarrow 0$. This is an artefact of the singular geometry at $t = 0$, and will lead to divergent particle production effects in the quantum theory. We thus need to regularize the theory in some way.

Perhaps the simplest way of regularizing the theory is by analytically continuation of the mode functions in t . Using the analyticity properties of the Hankel functions and their transformation under temporal reflections it is possible to naturally extend the definition of the mode functions to negative values of t . The fact that we are able to do this consistently means that essentially we have defined the vacua of the two space-time regions separated by the singularity at $t = 0$ equal to one another: $|in\rangle = |out\rangle$ (see figure 3.1). Trivially then there is no particle production as we pass through the singularity. This choice of vacua may seem a little contrived from this perspective, and indeed it is. What we would really like is to be able to justify this choice in the context of a regularization and renormalization scheme. This is precisely what is achieved in [75]. Here a form of dimensional regularization is used to introduce a local counter-term into the action at $t = 0$ which removes the field divergences and preserves unitarity. Once one stipulates that the propagator be of Hadamard form (see [79]), and imposes time reflection symmetry this counter-term is found to be unique. Also, the regularized fields are precisely (up to an irrelevant minus sign) those one obtains from the analytic continuation procedure. This justifies our choice of vacua. We may thus now write down the mode function solutions for all time:

$$\psi^+(k, t) = \sqrt{\frac{\pi}{4}} H_0^{(2)}(|kt|), \quad \psi^-(k, t) = \sqrt{\frac{\pi}{4}} H_0^{(1)}(|kt|), \quad (3.34)$$

and the quantum field is expressed as

$$\phi = \int \frac{d^d \vec{k}}{(2\pi)^d} \left[a_{\vec{k}} \psi^+(k, t) e^{i\vec{k} \cdot \vec{x}} + a_{\vec{k}}^\dagger \psi^-(k, t) e^{-i\vec{k} \cdot \vec{x}} \right], \quad (3.35)$$

where a^\dagger and a are the creation and annihilation operators which satisfy

$$\left[a_{\vec{k}}, a_{\vec{k}'}^\dagger \right] = (2\pi)^{d-1} \delta^{d-1}(\vec{k} - \vec{k}'). \quad (3.36)$$

3.4 Can noncommutativity cope?

We would now like to introduce a spatial noncommutativity and see what effects this has on correlation functions and scattering amplitudes in our model. Before we do this however, perhaps a few words of justification for the approach we will take are in order.

In thinking about how to consistently introduce noncommutativity on a curved space-time with physical applications in mind, we explored several different approaches which have been used in the literature. We shall give a brief discussion of these below.

In a paper entitled '*Cosmological perturbations and short distance physics from noncommutative geometry*' [80] the authors attempt to construct a manifestly covariant formulation of noncommutative field theory on curved space-time by replacing the partial derivatives which occur in the exponential definition of the star product with covariant derivatives. This prescription suffers from the rather alarming drawback that the multiplication is no longer associative once one performs this replacement. The authors are then forced to perform an expansion in powers of the noncommutativity parameter θ , and find that if one considers only terms cubic in θ or lower, associativity is restored. They thus use an artificially truncated expansion to define their multiplication. This method, while intuitive, seems a little *ad hoc* and it is not immediately obvious whether it can be naturally justified from a more fundamental stringy perspective.

As mentioned previously, studies have also been made of noncommutative curved space-time from the perspective of noncommutative differential geometry [22], [21]. These however do not readily lend themselves to physical applications, and in particular there is no indication of how to formulate quantum field theory in such a context.

One promising possibility is that of using matrix models to study the effects of noncommutativity in curved space-time. For example, [75] makes use of a duality between the small neighbourhood around the singularity in the Milne double-cone space-time and the bulk of de Sitter space to define unitary field evolution through the singularity. It is conceivable that one could use methods similar to those in our discussion of matrix models of noncommutativity in chapter 1 to construct a version of de Sitter space in which the co-ordinates are matrix valued, and exploit the duality to see what effect this will have on field theory near the Milne singularity. While this is an intriguing idea, we must unfortunately leave a detailed exploration of it to a later work.

The approach we will take in what follows is perhaps the simplest one which maintains a close relationship with the formalisms of quantum field theory on curved space-times and noncommutative field theory, and is hence closest to a string inspired modification. It may be helpful to have a picture of what we are attempting to do in mind. See figure 3.1. We will only make one very slightly new definition. We define the star product of functions at two different space-time points via,

$$f(x_1) \star g(x_2) = e^{\frac{i}{2}\theta^{ij}\partial_i\partial'_j} f(x)g(x')|_{x=x_1, x'=x_2} . \quad (3.37)$$

This definition is also used in [81],[82], and will enable us to compute

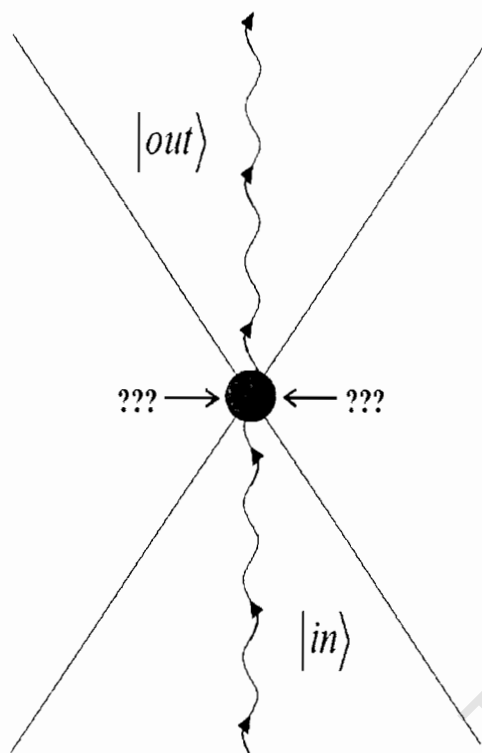


Fig. 3.1. Field evolution in our conformally flat Big Bang/Big Crunch space-time. We wish to investigate the effect of a spatial noncommutativity at early times on the quantum theory.

correlation functions and scattering amplitudes. An obvious criticism of a formulation of noncommutative field theory on curved space-time based on this definition is that it is not covariant. However, if we allow θ^{ij} to be time (or temperature) dependent we can envisage a situation in which a commutative and covariant theory emerges at late times if θ^{ij} drops off to zero. In fact this is precisely the behaviour we will assume, and partially justify later on.

3.4.1 Correlation functions

Let us begin with a computation of some representative correlation functions. We define the n -point correlation function I_n through

$$I_n(\mathbf{x}_1, \dots, \mathbf{x}_n) = \langle 0_{out} | \phi(\mathbf{x}_1, t) \star \dots \star \phi(\mathbf{x}_n, t) | 0_{in} \rangle . \quad (3.38)$$

However, as we have mentioned, we define $|0_{out}\rangle = |0_{in}\rangle$, which simplifies this definition dramatically. Essentially what this means is that there is no particle production due to gravitational effects in the transition through the singularity, or from the point of view of the correlation functions, that they are independent of Bogoliubov coefficients. Thus we will need to introduce interaction terms which break the conformal symmetry if we wish to study the effect of our mode definitions on particle production rates. We will do this shortly, and explore their dependence on the noncommutativity.

For now however, let us content ourselves with a calculation of some representative correlation functions. Firstly, notice that if we allow θ^{ij} to depend only on temporal and not spatial co-ordinates, and since the conformal factor multiplying the flat metric is also only time dependent, we may make use of all the Weyl-Moyal machinery we developed in chapter 1, all formulae remaining unchanged up to the possible introduction of a time-dependent factor^{||}. In particular (1.50) generalizes to

$$\int \int dt d^{d-1}x \sqrt{-g} f(x) \star h(y) = \int \int dt d^{d-1}x \sqrt{-g} f(x) h(y). \quad (3.39)$$

This means that we can expect noncommutative effects to appear only at cubic order or higher. However, it is important to note that even at the level of free fields the noncommutativity alters the quantum theory through the correlation functions. Certainly, the 2-point function will remain unchanged and is easily calculated. Using (3.36) and (3.34) we find

$$I_2(\mathbf{x}_1, \mathbf{x}_2) = I_2(\mathbf{x}_1 - \mathbf{x}_2) = \int_{\mathbf{k}} \frac{\pi}{4} \eta(|kt|) e^{i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_2)}, \quad (3.40)$$

where we have defined $\int_{\mathbf{k}} \equiv \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}}$ and $\eta(z) = H_0^{(2)}(z)H_0^{(1)}(z)$. This is manifestly independent of θ^{ij} . However if we compute the 4-point function (since clearly $I_{2n+1} = 0$), a lengthy but straightforward calculation shows that

$$\begin{aligned} I_4(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) &= I_2(\mathbf{x}_1 - \mathbf{x}_2)I_2(\mathbf{x}_3 - \mathbf{x}_4) + I_2(\mathbf{x}_1 - \mathbf{x}_4)I_2(\mathbf{x}_2 - \mathbf{x}_3) \\ &+ \int_{\mathbf{k}} \int_{\mathbf{k}'} \frac{\pi^2}{16} \eta(|kt|) e^{i\mathbf{k} \cdot (\mathbf{x}_1 - \mathbf{x}_3)} \eta(|k't|) e^{i\mathbf{k}' \cdot (\mathbf{x}_2 - \mathbf{x}_4)} e^{i\theta^{ij} k_i k'_j}, \end{aligned} \quad (3.41)$$

where use has been made of the easily verified fact that

$$e^{i\mathbf{k}' \cdot \mathbf{x}} e^{-i\mathbf{k} \cdot \mathbf{y}} = e^{i\theta^{ij} k_i k'_j} e^{-i\mathbf{k} \cdot \mathbf{y}} e^{i\mathbf{k}' \cdot \mathbf{x}}, \quad (3.42)$$

^{||} With the caveat that as usual the number of noncommutative co-ordinates must be even.

where multiplication is performed with the star product implicitly from now on. So evidently, unlike commutative free field theory, it is not possible to factorize n th order correlators into products of 2-point functions. It has been noted in [82] that this leads to a non-gaussian spectrum of density perturbations in the context of an inflationary background.

3.4.2 Scattering amplitudes

We will now attempt to calculate some tree level scattering amplitudes. We will work in the so-called interaction picture in which quantum state evolution is given by the unitary operator U defined through

$$|\Phi[\Sigma]\rangle = U[\Sigma, \Sigma_0]|\Phi[\Sigma_0]\rangle, \quad (3.43)$$

where $\Sigma(x)$ is a space-like Cauchy hypersurface through x . The states satisfy the Schrodinger equation,

$$\mathcal{H}_I(x)|\Phi[\Sigma]\rangle = i \frac{\delta|\Phi[\Sigma]\rangle}{\delta\Sigma(x)} \quad (3.44)$$

where \mathcal{H}_I is the interaction Hamiltonian. This implies an evolution equation for U , the Tomonaga-Schwinger equation. Supplemented by the initial condition $U[\Sigma_0, \Sigma_0] = 1$ this equation has solution

$$U[\Sigma, \Sigma_0] = \mathcal{P} \exp \left(-i \int_{\Sigma_0}^{\Sigma} \mathcal{H}_I(x) d^d x \right), \quad (3.45)$$

where \mathcal{P} is the path-ordering symbol. The S matrix, whose elements are the scattering amplitudes, is defined by taking Σ in the infinite future and Σ_0 in the infinite past in this expression. Performing an expansion in powers of the coupling constant implicitly contained in \mathcal{H}_I we see that the first nontrivial contribution to the S matrix is a term of the form

$$\begin{aligned} S^{(1)} &= -i \int d^d x \mathcal{H}_I(x) \\ &= i \int d^d x \mathcal{L}_I(x), \end{aligned} \quad (3.46)$$

where \mathcal{L}_I is the interaction Lagrangian density and use has been made of the fact that $\mathcal{H}_I = -\mathcal{L}_I$ for nonderivative coupling.

As a warm-up exercise, we will consider a mass term interaction, $\mathcal{L}_I = -\frac{m^2}{2}\phi^2\sqrt{-g}$. Since this is quadratic, we do not expect the S -matrix to depend on the noncommutativity. However we will illustrate that, while our choice of vacua implies no particle production for free fields, there are processes which can occur in the interacting theory which are forbidden in

Minkowski space-time. This is a consequence of the fact that in a general curved space-time Poincaré symmetry is broken. We consider a process in which two particles of momenta k and k' are created from the vacuum. Our asymptotic states are

$$|in\rangle = |0\rangle, \quad |out\rangle = a_{\mathbf{k}}^\dagger a_{\mathbf{k}'}^\dagger |0\rangle. \quad (3.47)$$

The relevant tree-level scattering amplitude is thus

$$\Gamma \equiv \langle out|S^{(1)}|in\rangle = -i\frac{m^2}{2} \int d^d x \sqrt{-g} \langle 0|a_{\mathbf{k}} a_{\mathbf{k}'} \phi^2|0\rangle, \quad (3.48)$$

where ϕ is given by (3.35). The only term which contributes is of the form $\langle 0|a_{\mathbf{k}} a_{\mathbf{k}'} a_1^\dagger a_1^\dagger|0\rangle$ which may be easily evaluated using (3.36):

$$\langle 0|a_{\mathbf{k}} a_{\mathbf{k}'} a_1^\dagger a_1^\dagger|0\rangle = (2\pi)^d [\delta^d(\mathbf{k}-1)\delta^d(\mathbf{k}'-1') + \delta^d(\mathbf{k}-1')\delta^d(\mathbf{k}'-1)]. \quad (3.49)$$

Hence,

$$\begin{aligned} \Gamma &= -im^2 \int d^d x \sqrt{-g} \left(\psi^-(k, t) \psi^-(k', t) e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} \right) \\ &= i\frac{m^2}{4\pi V} (2\pi)^{d-1} \delta^{d-1}(\mathbf{k}+\mathbf{k}') \int \int \int dt du du' |t|^{\frac{d}{d-2}} e^{it(k \cosh u + k' \cosh u') - \epsilon|t|} \end{aligned} \quad (3.50)$$

where we have used (3.32), $V = (2\pi)^{d-1} \delta^{d-1}(0)$ is a volume factor that normalizes the final states, and we have added in a convergence factor $e^{-\epsilon|t|}$. The t integral may be performed easily in certain cases using a trick. For the $[0, \infty)$ part we have

$$\begin{aligned} \int_0^\infty t^{\frac{d}{d-2}} e^{-tv} dt &= (-1)^{\frac{d}{d-2}} \left(\frac{d}{dv} \right)^{\frac{d}{d-2}} \int_0^\infty e^{-tv} dt \\ &= \frac{6}{v^4} \quad (d=3), \end{aligned} \quad (3.51)$$

where $v = -i(k \cosh u + k' \cosh u') + \epsilon$ and we have chosen $d=3$ in the last line, since this is the only positive odd value of d that makes $\frac{d}{d-2}$ a positive integer. It is this case that we will focus on in this computation - (1+2) dimensional conformally flat space-time with the spatial co-ordinates noncommuting. The $(-\infty, 0]$ part contributes equally to the integral. Taking the limit $\epsilon \rightarrow 0$ we find

$$\Gamma_{d=3} = i\frac{3m^2}{\pi V} (2\pi)^2 \delta^2(\mathbf{k}+\mathbf{k}') \int \int du du' \frac{1}{(k \cosh u + k' \cosh u')^4}. \quad (3.52)$$

The remaining double integral is difficult to compute explicitly, however it certainly converges, as may be seen by noting that $\cosh X \geq 1 + X^2$

and computing the integral with this upper bound. We will simply write

$$\int \int du du' \frac{1}{(k \cosh u + k' \cosh u')^4} \equiv \frac{1}{k^4} F\left(\frac{k'}{k}\right) \quad (3.53)$$

for some undetermined function F . Hence the transition probability per unit volume for a process that creates two particles with momenta \mathbf{k} , $-\mathbf{k}$ from the vacuum, and \mathbf{k} in $d^2\mathbf{k}$ is

$$|\Gamma_{d=3}|^2 V = \frac{9m^4}{\pi^2} \frac{1}{k^8} F^2(1) \frac{d^2\mathbf{k}}{(2\pi)^2}. \quad (3.54)$$

Using the bound mentioned previously, one can show that $0 < F(1) \leq \frac{\pi}{48}$.

There are several important points to note about this result. Firstly the integration over \mathbf{k} is infrared divergent. See [75] for a discussion of how this might be remedied in string theory in a related context. Despite this, the fact that for a given \mathbf{k} the theory yields a finite probability for this process to occur suggests that in some sense, the evolution through the singularity is indeed well-defined, at least from the perspective of this mass term interaction.

The previous calculation says nothing of the effect of noncommutativity on scattering amplitudes. In order to investigate this we must either consider loop graphs, or interactions of cubic order or higher. We will not be considering loop graphs in this work, and instead focus on a quartic interaction,

$$\mathcal{L}_I = \frac{\lambda}{4!} \phi^4 \sqrt{-g}, \quad (3.55)$$

as an illustrative example. Since we are interested in following the field evolution through the singularity, we'll define the asymptotic states to be

$$|in\rangle = a_{\mathbf{k}}^\dagger |0\rangle, \quad |out\rangle = a_{\mathbf{k}'}^\dagger |0\rangle. \quad (3.56)$$

Intuitively, the S-matrix element corresponding to these states encodes the quantum analogue of the classical notion of geodesic completeness. If the scattering amplitude is well defined, in some sense at least the particle that was created in the 'in' region has managed to traverse the singularity at $t = 0$, and get itself annihilated in the 'out' region. Of course this is grossly over-stating the case. What we really want is for the S-matrix to be well-defined to all orders, and for all reasonable interactions, but this is asking too much at the moment and we will content ourselves with investigating the effects of the noncommutativity in this single case.

It is easy to convince oneself that the noncommutativity will only affect the phase of the integrand in the scattering amplitude. This is obvious from the fact that the field depends on the spatial co-ordinates only

through the exponential basis functions. It turns out that the precise form of this phase factor depends on the topology of the particular Feynman graph being considered [83, 84]. We will find it instructive to perform the calculation explicitly.

Let the internal momenta be p, q, r, s , and recall that now the order does matter, we cannot simply swap them around. There are $\frac{4!}{2 \times 2!} - 1 = 5$ terms that will contribute to the first order S -matrix element:

$$\begin{aligned} A_1 &= \langle 0 | a_{k'} a_p a_q^\dagger a_r^\dagger a_s a_k^\dagger | 0 \rangle \\ A_2 &= \langle 0 | a_{k'} a_p^\dagger a_q a_r a_s^\dagger a_k^\dagger | 0 \rangle \\ A_3 &= \langle 0 | a_{k'} a_p a_q a_r^\dagger a_s^\dagger a_k^\dagger | 0 \rangle \\ A_4 &= \langle 0 | a_{k'} a_p^\dagger a_q a_r^\dagger a_s a_k^\dagger | 0 \rangle \\ A_5 &= \langle 0 | a_{k'} a_p a_q^\dagger a_r a_s^\dagger a_k^\dagger | 0 \rangle . \end{aligned} \quad (3.57)$$

In the full amplitude, which we will refer to as Δ , to each internal creation operator is associated a factor of the form $\psi_l^-(t)e^{-il \cdot x}$, and to each annihilation operator a factor of the form $\psi_l^+(t)e^{il \cdot x}$, where l is the internal momentum label. We will not go through the laborious business of evaluating the contribution from each one of these terms, as the procedure is much the same for each. Let us just do one, A_2 for example.

It is easy to show, using (3.36), that

$$A_2 = (2\pi)^{3(d-1)} \delta(k' - p) [\delta(r - s)\delta(q - k) + \delta(r - k)\delta(q - s)] \quad (3.58)$$

Putting in the relevant factors that accompany the combination of operators involved, we may reduce this term's contribution to Δ to

$$\begin{aligned} \Delta_{A_2} &= i \frac{\lambda}{4!} \int d^d x \sqrt{-g} \int \frac{d^{d-1} s}{(2\pi)^{d-1}} \psi_{k'}^- \psi_k^+ \psi_s^+ \psi_s^- \left(e^{-ik' \cdot x} e^{ik \cdot x} + e^{-ik' \cdot x} e^{is \cdot x} e^{ik \cdot x} e^{-is \cdot x} \right) \\ &= i \frac{\lambda}{4!} \int d^d x \sqrt{-g} \int \frac{d^{d-1} s}{(2\pi)^{d-1}} \psi_{k'}^- \psi_k^+ \psi_s^+ \psi_s^- e^{-ik' \cdot x} e^{ik \cdot x} \left(1 + e^{-i\theta^{ij} k_i s_j} \right) \end{aligned} \quad (3.59)$$

where in the last line we have used (3.42). Now using (1.31) we have

$$\begin{aligned} \int d^{d-1} x e^{-ik' \cdot x} e^{ik \cdot x} &= \text{Tr} \left(e^{i(k-k') \cdot x} \right) e^{\frac{i}{2} \theta^{ij} k'_i k_j} \\ &= (2\pi)^{d-1} \delta(k - k') e^{\frac{i}{2} \theta^{ij} k'_i k_j} . \end{aligned} \quad (3.60)$$

So we can write generally that

$$\Delta_{A_2} = i \frac{\lambda}{4!} \int dt \sqrt{-g} \psi_{k'}^- \psi_k^+ e^{\frac{i}{2} \theta^{ij} k'_i k_j} \int d^{d-1} s \psi_s^+ \psi_s^- \left(1 + e^{-i\theta^{ij} k_i s_j} \right) . \quad (3.61)$$

Notice that the noncommutativity mixes not only k and k' , but the internal momentum s as well. In fact, if θ^{ij} is time independent, then the phase factor that depends on k and k' falls out of the transition probability, since it is the square modulus of Δ ; but the noncommutativity still modifies the probability through the internal factor. In general, the contributions from the other $A_{(i)}$ will be much the same, the only difference being the labels on the mode functions, and whether k or k' mixes with the residual internal momentum.

For our particular space-time, we can go a little further, by inserting the explicit form of the mode functions and performing the s integration, at least for the first term in the brackets in (3.61). The calculation is a laborious and messy one, and we will spare the reader the details. The result is

$$\Delta_{A_2} = i \frac{\lambda}{4!} \delta^{d-1}(k - k') [\kappa(d)T(k, k'; d) + \Delta_\theta] \quad (3.62)$$

where

$$\begin{aligned} \kappa(d) &= i^{d-1} \frac{(2\pi)^{\frac{d-1}{2}} (d-2)!}{(4\pi)^2 \Gamma(\frac{d}{2} - \frac{1}{2})} \int \int du du' \frac{1}{(\cosh u - \cosh u')^{d-1}} \\ T(k, k'; d) &= \int dv dv' \int dt |t|^{\frac{d}{d-2} - \frac{1}{d-1}} e^{i|t|(k' \cosh v - k \cosh v')} e^{\frac{i}{2} \theta^{ij} k_i k_j} \\ \Delta_\theta &= \int d^{d-1} s \int dt |t|^{\frac{d}{d-2}} \psi_{k'}^- \psi_k^+ \psi_s^+ \psi_s^- e^{i\theta^{ij} (\frac{1}{2} k'_i k_j - k_i s_j)}. \end{aligned} \quad (3.63)$$

At this point things unfortunately become a little unwieldy, particularly if we allow θ^{ij} to depend on time. Perhaps now is a good time to consider whether it is reasonable to believe θ^{ij} to be time-dependent. From the perspective of string theory, the answer is yes. As we have learned in chapter 1, the noncommutativity is related to the value of the string background field B^{ij} . This field is generically coupled to the metric via a set of field equations which arise from requiring that the conformal β -functions of the theory vanish. Thus, seeing as our metric is most certainly time dependent, we must expect B^{ij} , and hence θ^{ij} to be time dependent as well. What form this time dependence takes is a difficult question to answer, and we will not attempt to do so here.

Given the complicated expression for Δ_{A_2} above, can we expect the transition probability to be finite, and what rôle does the noncommutativity play in answering this question? Certainly, from our experience with the quadratic interaction, we expect $\kappa(d)$ to be a finite constant. The form of $T(k, k'; d)$ and Δ_θ depends strongly on the exact form of θ^{ij} . It is perhaps illuminating to consider their behaviour for $\theta^{ij} = 0$. In this case $T(k, k'; d)$ reduces to an integral very similar to the one encountered in our study of the mass term interaction, although we cannot use the

same trick to evaluate it, as the exponent of $|t|$ can never be an integer. Nonetheless we still expect it to converge (at least once we insert an appropriate regulating factor), as the exponent is smaller by a factor $\frac{1}{d-1}$ than that of the convergent integral we calculated. For $\theta^{ij} = 0$, Δ_θ gives nothing new, it being an exact copy of the previous term. Thus, even without noncommutativity, we expect the transition probability to be well defined. So how do things change when we include the noncommutativity? For one thing, the contribution of Δ_θ is no longer trivial - this is a purely noncommutative phenomenon. More importantly, θ^{ij} introduces an additional set of time dependent functions into the amplitude, over and above the mode functions. The time dependence of the mode functions is a direct consequence of the space-time geometry, since they are solutions of an equation of motion whose form is directly determined by the metric. But the θ^{ij} are, as far as this analysis is concerned, independent functions of time. This is very promising, as it is the singular nature of the *geometry*, as encoded by the mode functions, which generically causes problems in defining particle evolution through the singularity. It so happens that due to our choice of vacua, the scattering amplitudes are well defined anyway, but we have been considering a very special example - the fact that our space-time is conformally flat allows us to calculate most quantities analytically and unambiguously, and also permits the particular choice of vacuum we have made. In a more general situation one would not expect to have these luxuries, yet we have shown that for *any* background geometry, the noncommutativity modifies the scattering amplitude by time dependent phase factors in the integrand, which mix the external momenta amongst themselves, but also with the internal momenta. This additional time dependence provides the hope that in a more general situation, in which it is no longer possible to make very special choices of vacua consistently, the noncommutativity will come to the rescue, and play a soothing, sobering rôle in the field evolution through the singularity.

3.5 The end of the beginning...

As mentioned in the preface, this chapter is best viewed as a first installment, a feasibility study. What is the prognosis?

We managed to show that for a simple conformally flat big bang/big crunch space time, a quantum field theory may be defined on it which produces sensible answers when we ask it questions about transition probabilities for particles propagating through the singularity. This seems to give credence to the arguments presented in the first section of this chapter, in which it was conjectured that evolution through the singularity in such a space-time should be well defined from the perspective of some

theory which is more fundamental than general relativity. Of course, this conclusion is a naive one. We have only considered two simple cases, and have said nothing about the rôle of gravitational back-reaction, or general S -matrix elements at arbitrary order. However, the question of back-reaction, in a related context, has been addressed in [85], and the outcome seems to be favourable.

We also studied a hybrid theory, a minimal combination of quantum field theory on curved space-time with noncommutative field theory, and explored the effects of a purely spatial noncommutativity on the correlation functions and scattering amplitudes. We found that while quadratic quantities were unaffected by the noncommutativity, higher order objects exhibited interesting dependence on it. For the correlation functions, even for free field theory, we found that the 4-point function no longer factorises over the 2-point functions, and thus we generically expect some non-gaussianity in the spectrum of density perturbations. This may have interesting observational consequences, as noted in [82]. However this was not our main concern. More importantly for us, we showed that the introduction of a time-dependent noncommutativity parameter gives rise to a new time dependence in the quartic (and higher) scattering amplitude, which may be regarded as distinct from the geometry. This will generically affect the transition probabilities, and provide a new set of functions with which it may be possible to ameliorate the bad behaviour which arises through the geometrically determined mode functions. The simplicity of the specific model we studied allowed us to explicitly calculate scattering amplitudes for two different processes, one quadratic and one quartic, and to see exactly how the noncommutativity affects these quantities. We found that for the quartic amplitude, the noncommutativity mixes both the external and internal momenta, and even when constant, makes a non-trivial contribution to the scattering amplitude. This is a phenomenon one expects to continue to hold true at higher orders in perturbation theory.

There are many directions which beg exploration from this point. Firstly, can we make the claim that noncommutativity provides additional scope for control of singular behaviour concrete by providing a specific example? This a technically difficult problem, but intuitively it is clear what we would require. We would like to find a singular space-time, and a particular time dependence for θ^{ij} which smooths out the singularity from the perspective of the transition probabilities. One would expect that θ^{ij} would have to have a strong peak at the time of the singularity, and then fall off quickly away from it. Heuristically, the fuzziness, and inherent non-locality of the field theory near the singularity, would hopefully smooth out the amplitudes.

What would be truly remarkable is if string theory in some way con-

spired to make this the case. As we mentioned, in string theory the metric and noncommutativity are coupled through a set of field equations. If it were the case that one could plug a metric into these equations, solve for θ^{ij} , put this back into the amplitudes, and miraculously find that everything works out, this would indeed be a triumph for the string. At least in this form, this notion smacks of wishful thinking. In fact, even for our simple big bang/big crunch space-time, it is easy to see that this line of thought is unlikely to succeed. For the bosonic string, the relevant string equations would involve the Ricci tensor $R_{\mu\nu}$, which diverges quadratically as $t \rightarrow 0$, thus making it very unlikely indeed that any solution to the equations will remain valid at $t = 0$. Of course, even if we could solve the equations, this is no guarantee that the particular form we get for θ^{ij} will do the job for us.

This is not to say that string theory may not be capable of dealing with the problem. Indeed it offers several promising alternatives. For one thing, it would be very interesting to see whether a noncommutative matrix model would deal with the singularity in a different way. As mentioned in the text, it has been shown that the region near the singularity in a double cone Milne space-time can be related to de Sitter space [75], which may be described as an embedded surface in a higher dimensional space-time. It is possible that one could find a matrix representation for the co-ordinates on this surface, and use this to study field evolution through the singularity. This would be an interesting direction for future work.

Perhaps most notably, while this work was being completed, a membrane approach to the problem was put forward [86]. Here the geometrical singularity is manifest as the point of collision of two tensionless orbifold planes moving towards each other at constant non-relativistic velocity. It was shown that winding states, which include the graviton, can be smoothly propagated across the singularity, and that the gravitational back-reaction remains under control. This approach says nothing about the bulk states though. From our perspective, it would be interesting to see what happens if we couple the world-volume of the membranes to a B -field in the action. Will this introduce noncommutative effects, and how will they affect the transition through the singularity?

Clearly, there is much still to be done. In the immortal words of Winston Churchill, after the battle of El Alemein: 'This is not the end. Nor is it the beginning of the end. It is however, the end of the beginning.'

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4

Appendix 1: The $\mathbb{C}\mathbb{P}^n$ model

Consider an $(n + 1)$ component row vector $Z = (Z_1, \dots, Z_{n+1})$ that satisfies the constraint,

$$ZZ^\dagger = 1 \quad (4.1)$$

Now consider the Lagrangian

$$\mathcal{L} = \partial_i Z \partial^i Z^\dagger + [(\partial_i Z) Z^\dagger]^2 \quad (4.2)$$

This expression is invariant under $U(1)$ gauge transformations, $Z \rightarrow Z' = Ze^{i\Lambda(x)}$, as is easily verified. Now n -dimensional complex projective space ($\mathbb{C}\mathbb{P}^n$) is the set of all complex $(n+1)$ vectors w satisfying

$$ww^\dagger = 1 \quad (4.3)$$

$$w \sim \Lambda w \quad (\Lambda \in \mathbb{C}) \quad (4.4)$$

The field Z in our model clearly satisfies these conditions. We thus refer to it as a $\mathbb{C}\mathbb{P}^n$ model. We may cast the Lagrangian into a more familiar form by defining an 'artificial' real gauge field A_i in terms of the Z fields. Consider the following,

$$\mathcal{L} = (D_i Z)(D^i Z)^\dagger \quad (4.5)$$

where $D_i Z = \partial_i Z - iA_i Z$. Treating A_i as an independent field one may find its equation of motion,

$$A_i = -i(\partial_i Z) Z^\dagger \quad (4.6)$$

Thus A_i has no independent dynamics. Substituting back into (4.5) one finds that it reduces to (4.2). Thus one may consider the $\mathbb{C}\mathbb{P}^n$ models as free gauge theories (with non-dynamical gauge field) subject to the

constraint (4.1). We can make this explicit at the level of the action by writing

$$S = \int d^2x \left[D_i Z D^i Z^\dagger + \lambda (ZZ^\dagger - 1) \right] \quad (4.7)$$

where λ is a Lagrange multiplier. Varying with respect to Z and using the property (4.1) then allows on to solve for λ and find the equations of motion:

$$D^j D_j Z - (Z^\dagger D^i D_i Z) Z = 0 \quad (4.8)$$

From (4.5) we may write the action for static fields as

$$\begin{aligned} S &= \int d^2x \left[D_i Z (D^i Z)^\dagger \right] \\ &= \frac{1}{2} \int d^2x (D_j Z \pm i\epsilon_{jk} D_k Z) (D^j Z \pm i\epsilon^{jk} D^k Z)^\dagger \pm i \int d^2x \epsilon_{jk} D^j Z (D^k Z)^\dagger \end{aligned} \quad (4.9)$$

The second integral defines the topological charge for this model, as we shall now demonstrate. Define

$$\lim_{|x| \rightarrow \infty} Z_i = \rho_i e^{i\phi_i} \quad (4.10)$$

Requiring finite energy (static) solutions means $\partial_i Z - iA_i Z \rightarrow 0$. Equating real and imaginary parts we see $\partial_j \rho_i = 0$ and $\partial_i \phi = A_i$ on $\partial\mathbb{R}^2$. The simplest guess for the behavior of Z at the boundary that satisfies these conditions is

$$Z = Z_0 e^{i\phi(\theta)} \quad (4.11)$$

where Z_0 is a constant vector, and θ is the polar angle. The index of the field will then be given by

$$Q = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\phi}{d\theta} d\theta \quad (4.12)$$

Now in polar coordinates, the θ component of A_i will be

$$A_\theta = \frac{i}{r} (\partial_\theta Z) Z^\dagger \rightarrow -\frac{1}{r} \frac{d\phi}{d\theta} \text{ as } r \rightarrow \infty \quad (4.13)$$

So

$$Q = -\frac{1}{2\pi} \int_0^{2\pi} (A_\theta r)_{r \rightarrow \infty} d\theta = -\frac{1}{2\pi} \oint_{\partial\mathbb{R}^2} \vec{A} \cdot d\vec{l} = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \epsilon_{jk} \partial^j A^k d^2x \quad (4.14)$$

where we have used Stokes' theorem. It follows from (4.6) that $\partial_j A_k = -i((\partial_j \partial_k Z) Z^\dagger + \partial_k Z \partial_j Z^\dagger)$. Using the fact that ϵ_{jk} is anti symmetric we may write

$$Q = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \epsilon_{jk} (\partial_j A_k - \partial_k A_j) d^2x = -\frac{i}{2\pi} \int_{\mathbb{R}^2} \epsilon_{jk} \partial_j Z \partial_k Z^\dagger d^2x \quad (4.15)$$

This is equivalent to

$$Q = -\frac{i}{2\pi} \int_{\mathbb{R}^2} \epsilon_{jk} D^j Z (D^k Z)^\dagger d^2x \quad (4.16)$$

as is easily shown by using (4.6). Hence on comparison with (4.10) we see that,

$$S \geq 2\pi|Q| \quad (4.17)$$

and the bound is saturated for

$$D_j Z \pm i\epsilon_{jk} D_k Z = 0 \quad (4.18)$$

These are the BPS equations of the $\mathbb{C}\mathbb{P}^n$ model. We now show that these reduce to the Cauchy-Riemann equations.

Pick some region $\mathcal{R}_1 \in \mathbb{R}^2$ in which $Z_{n+1} \neq 0$. In \mathcal{R}_1 define a new variable W through $Z = WZ_{n+1}$. Noting that, for two arbitrary functions f and g ,

$$\begin{aligned} D_j(fg) &= \partial_j(fg) - iA_j(fg) \\ &= (D_j f)g + f\partial_j g \end{aligned} \quad (4.19)$$

we see that on setting $f = Z_{n+1}$ and $g = W$ equation (4.18) becomes

$$[W D_j Z_{n+1} + Z_{n+1} \partial_j W] \pm i\epsilon_{jk} [W D_k Z_{n+1} + Z_{n+1} \partial_k W] = 0 \quad (4.20)$$

The first terms of each of the brackets cancel off due to the $(n+1)$ th component of (4.18) and we are left with

$$\partial_j W \pm i\epsilon_{jk} \partial_k W = 0 \quad (4.21)$$

These are of course the Cauchy-Riemann equations. Thus if the components of W are holomorphic functions they will solve the BPS equations in \mathcal{R}_1 . By the constraint (4.1) there will always be at least one non-zero component of Z so we can extend this analysis to the whole plane by defining $Z = WZ_i$ on the region \mathcal{R}_i in which $Z_i \neq 0$, and we are assured $\bigcup \mathcal{R}_i = \mathbb{R}^2$.

Of course, to satisfy the complete set of BPS equations we must also make sure that our solution gives the correct (integer) topological charge (4.16). The functions that achieve this are the so-called $\mathbb{C}\mathbb{P}^n$ -lump solutions given by

$$W = \frac{1}{P_{n+1}(z)} (P_1(z), \dots, P_n(z)) \quad (4.22)$$

where the P_i are polynomial functions of the holomorphic variable $z = x + iy$ with degree equal to the topological charge Q . These are the self-dual solutions. One may obtain the anti-self-dual solutions by replacing

$z \rightarrow \bar{z}$. Obviously, this solution is a meromorphic rather than holomorphic function, as it has poles at the zeros of P_{n+1} . However, W blows up at poles, so the (anti)self-duality equations are still satisfied by this solution.

The \mathbb{CP}^n models arise in a host of physical scenarios. Perhaps most interestingly it can also be shown that the nonlinear sigma model with a \mathbb{CP}^n target-space is a \mathbb{CP}^n model. See [50] and references therein.

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