Lie Analysis for Partial Differential Equations in Finance
by

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Declaration of Authorship

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"All truths are easy to understand once they are discovered, the point is to discover them. "

(Galileo Galilei)

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Abstract

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Weather derivatives are financial tools used to manage the risks related to changes in the weather and are priced considering weather variables such as rainfall, temperature, humidity and wind as the underlying asset. Some recent researches suggest to model the amount of rainfall by considering the mean reverting processes. As an example, the Ornstein Uhlenbeck process was proposed by Allen [3] to model yearly rainfall and by Unami et al. [52] to model the irregularity of rainfall intensity as well as duration of dry spells. By using the Feynman-Kac theorem and the rainfall indexes we derive the partial differential equations (PDEs) that governs the price of an European option. We apply the Lie analysis theory to solve the PDEs, we provide the group classification and use it to find the invariant analytical solutions, particularly the ones compatible with the terminal conditions.

Keywords - Lie symmetry analysis, Ornstein-Uhlenbeck process, Partial differential equations, Rainfall index, Weather derivatives.

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May GOD Bless You All !!!

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With love to my parents, sisters, brothers, nieces, nephews and my future children . . .

Chapter 1

Introduction

In general, companies activities face risks related to changes in the weather. The agriculture and energetic industry are examples of sectors that can be influenced by weather variables such as rainfall, snowfall, temperature, humidity and wind. For example, the electricity supply and the crop yield can be influenced negatively by the rain, wind. To face these risks, companies must choose according to the nature of theirs activities, which kind of weather protections they must use. In agriculture for example, the crop yield can depend on quantities of rainfall. There is a strong correlation between the amount of rainfall and wheat yield [41] and [50]. A strong correlation between the amount of rainfall and maize yield was also mentioned in [21]. This suggest that the crop yield is dependent of an expected amount of rainfall. If the observed amount of rain is less or more than the amount required, the farmers can lose or get few crop yield than expected. If they can protect themselves against adverse weather patterns the risk will substantially decrease.

The financial instruments used to manage weather risks are called financial Weather contract since they are related to weather variables and can take the form of a weather derivative contract (WD) with the main purpose to cover non-catastrophic weather events or the form of a weather insurance contract (WI) with the main purpose to cover catastrophic weather events. Depending on the circumstances, the flexibility and efficiency of WD make them more attractive than the WI.

As stated in [1] and [22], the WD is formulated by specifying some parameters and among them the underlying meteorological index which can be, for example temperature degree days (DD) and their variants "heat degree days (HHD) or cool degree days (CDD)", rainfall. Generally for pricing WD one can use methods based on arbitrage-free pricing principles [47], [48], equilibrium models [12] and actuarial approach [8], since the WD underlying is not tradable asset and the Black-Scholes methodology can not be implemented directly. The methods based on PDEs were suggested to model temperature derivatives by Pirrong and Jermakyan [47],[48] and posteriori adopted by other researchers, such as Alaton, Djehiche, and Stillberger [1], Balter and Pelsser [4] and Li [36].

I found few studies on pricing rainfall derivatives considering methods based on PDEs and mostly are treated in the context of discrete models. This fact can be justified by the nature of the rainfall that is observed in discrete time. But there are some attempt to use the stochastic differential models to price also rain derivatives such as [19]. They used the mean reverting process as a base model for rainfall and then applied numerical integration schemes to price rainfall options derivatives underlying to cumulative daily average amount of rain. In [52] and [3] was also suggested a mean reverting process (Ornstein-Uhlenbeck process) as a base model to describe the dynamics of the rainfall.

Motivated by these studies and after see how the model can fit the rainfall data considering the public rainfall data from the German weather station (www.dwd.de), we use the Girsanov's theorem to derive the PDE of the Ornstein-Uhlenbeck process that govern the price of any option underlying to the rainfall. The rainfall index is constructed by a similar principle to the degree-days indexes used to model temperature derivatives and it was shown that it gives higher hedging effectiveness [41]. We provide the analytical solutions of associated equations by applying the Lie group analysis.

The application of the Lie group analysis, can allow to synthesize symmetries of differential equations and construct their analytical invariant solutions. It was introduced by Sophus Lie, with his work *"On integration of a class of linear partial differential equations by means of definite integrals "* [37]. He identified a set of equations which could be integrated or reduced to lower-order equations by group theoretic algorithms and proposed the group classification of the linear seconder-order partial differential equation with two independent variables. On his classification he pointed that all parabolic equations admitting the symmetry group of the highest order could be reduced to the heat conduction equation. The collections of his and others results on group analysis of differential equation can be found in [31].

In financial mathematics the Lie analysis was applied firstly by Gazizov and Ibragimov in [20]. They started by analyzing the complete symmetry of one dimensional Black-Scholes model and showed that this equation is included in Sophus Lie's classification of linear second-order partial differential equation with two independents variables, and it can be reduced to the heat equation. For Jacobs-Jones models, they carried out the classification according to their symmetry groups, providing a theoretic background for constructing exact (invariant) solutions for this equation, since it does not admit the symmetry group of the highest order.

More research on the same direction can be found, for example in [46], provided a Lie group classification of the Lie point symmetries for the Black-Scholes-Merton Model for European options with stochastic volatility. The volatility was defined following stochastic differential equation with an Ornstein-Uhlenbeck term. In this model, the value of the option is given by a linear $(1+2)$ dimensional evolution partial differential equation. They found that for arbitrary functional form of the volatility, the evolution PDE always admits two Lie point symmetries in addition to the automatic linear symmetry and the infinite number of symmetries solution. However, for a particular value of the functional volatility and the price of the option depending on the second Brownian motion in which the volatility is defined, the evolutionary PDE is not reduced to the Black-Scholes-Merton equation. The model admits five Lie point symmetries in addition to the linear symmetry and the infinite number of symmetries solution. By applying the zero-order invariants of the Lie symmetries they reduced the $(1+2)$ dimensional evolutionary PDE to a linear second-order ordinary differential equation. They also studied the Heston model and the Stein-Stein model. Lo and Hui [38], applying Wei–Norman theorem derived the analytical closed-form for pricing weather derivatives by exploiting the dynamical symmetry of the (1+1) dimensional pricing PDE describing financial derivatives with time-dependent parameters.

Our aim in this study is to make a Lie group classification for (1+2) dimensional evolution PDE that govern the price of weather derivative underlyed to the weather variable modeled by Ornstein-Uhlenbeck process with constant volatility and deterministic mean. By making a group classification one can realize that the PDE can be reduced to the heat equation and if not, can use the subalgebras to find their invariant solutions.

Some PDEs can be reduced to the heat equation by change of variables if they have a symmetry of highest order [37]. As we know the heat equation has a known fundamental solutions. If we are interested to find the solution for a PDE, we can reduce it to heat equation, but some times, as illustrated in [14], if it is a boundary value problem, the solutions produced may not be necessary a probability density. In order to show this, they considered a problem which vanishes in the boundary, but by changing the variables they found that some fundamental solutions were not defined in the boundaries. Also by making further change of variables they produced a problem which was more complicated than the original, showing the importance of the techniques which allows to solve PDE avoiding change of variables. One way to do so, is to consider all the symmetries admitted by the PDEs and find the ones mapping the boundary and final conditions to the values of the original problem. This will be the technique that we will firstly implement in order to solve the (1+2) dimensional PDE derived from the Ornstein-Uhlenbeck process. But, since the application of the symmetry compatible with the terminal conditions in (1+2) dimensional PDE, produced an (1+1) dimensional parabolic PDE with complicated coefficients that made impossible to compute its symmetries directly, we use the results from [39], that allow to reduce the (1+1) dimensional parabolic PDE to heat equation in order to use its fundamental solution to find the candidate to fundamental solution for the (1+1) dimensional parabolic PDE. This fundamental solution can only be used to produce the solution compatible with terminal condition for (1+1) dimensional PDE, if provided that its limit when $t \to T$ is Dirac function [49].

This thesis is divided in six chapters. Chapter one is the introduction. The second Chapter has two sections. We present some of the basic notions and terminologies of stochastic calculus and financial derivatives. Chapter three is subdivided in two section focus on the methodology of pricing weather derivatives and in the rainfall derivative model. We show how the Ornstein-Uhlenbeck process fits the model and we derive the corresponding PDE. The Chapter four is also subdivided in two sections where we present a brief background of theory of one-parameter transformation of the Lie group analysis and we provide a summary of concepts, main theorems and the methodology of Lie symmetry theory for PDEs. Chapter five is subdivided in four sections, we apply the methodology of Lie symmetries to the PDE of the Ornstein-Uhlenbeck process. We present some basic transformations of the PDE, the result of the infinitesimal operators and the Lie group classification of the equation with the coefficients satisfying the restrictions $\sigma \neq 0, \;\; k(k^2 + \pi^2) \neq 0$, showing that the dimension of the symmetry group depends on the values of the parameters σ and k. We also present the extension of the principal Lie algebra, the constructions of invariant solutions including the ones compatible with the terminal conditions and the determination of the one dimensional optimal system. We finalize the thesis in Chapter 6 where we present the conclusions and the future work.

Chapter 2

Background on financial Mathematics

Weather derivatives are financial tools used to manage financial risks related to the weather and are priced under consideration of some weather indexes, which among others could be the temperature, rainfall and wind.The prices and the weather variables evolve as stochastic processes, and in continuous time context can be modeled by stochastic differential equations. This chapter focus on some of the basic notions and terminologies of stochastic calculus and financial derivatives which will be used freely throughout this study, sometime without further references.

2.1 stochastic calculus

The observations of some processes such as natural phenomenons, the prices of stocks, commodities are depending on time t. For each fixed time t_i the values of the process are random i.e., we only known the set for which the value of the process belongs. These phenomenons are usually called stochastic processes in continuous or discrete time. The evolution of the stochastic processes can be modeled and the probability that their values belong to any chosen set can be calculated. The stochastic calculus gives the tools to operate with stochastic phenomenons. In this section we give a brief review of the basic notions and terminologies of stochastic processes in continuous time context. The principal results of this section is the Feynman-Kac formula, which will be used to derive the PDEs that governs the prices of the rainfall derivatives. In order to explain these formulas we will also present some of the keys concepts such as martingales, Markov properties, Brownian motion, Itô's formula and the Girsanov's theorem. Further details can be found in [32], [33], [42] and [35].

2.1.1 Stochastic Processes

Some processes depending on the time can take random values for an fixed time, so are random functions of time. We could be interested in modeling this uncertainty or to model the flow of the information of the process. The stochastic process is a mathematical model used to represent the occurrence of a random phenomenon at each moment after the initial time. The important concept to define the stochastic process is the concept of probability space, which represent the notion of probability measure.

Definition 2.1.1 (of σ-algebra, measurable space, probability measure, probability space)
If Ω is a given set, a σ-field or σ-algebra $\mathcal{F}$ in Ω is a collection or family of subsets of Ω satisfying the following conditions:

- non empty: $\emptyset \in \mathcal{F}$ and $\Omega \in \mathcal{F}$,
- if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, where $A^c = \Omega - A$ is the complement of A in Ω;
- if $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ and $\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space. A probability measure P in a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ such that

- $P(\emptyset) = 0, \;\; P(\Omega) = 1;$
- if $A_1, A_2, \ldots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ disjoint then $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$.

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

With a notion of measurable space and the correspondent σ-field $\mathcal{F}$ we can define the stochastic process.

Definition 2.1.2 (of stochastic process)
Consider the probability space $(\Omega, \mathcal{F}, P)$. A stochastic process is a parametrized collection of random variables $X = \{X_t\}_{0 \le t < \infty}$ on $(\Omega, \mathcal{F}, P)$, taking values in $\mathbb{R}^n$.

From the above definition, each realization also called sample path of the stochastic process is a function of time t. The stochastic processes can be classified depending on the time or according to its values, both cases being discrete or continuous. The stochastic process in discrete time is considered when the values of the stochastic process change at certain fixed points in time. In other hand when the changes of the stochastic process occur at any time we will have continuous time stochastic process. In term of the values that the stochastic process can take, it can be classified as discrete if it takes only discrete values and continuous if it takes any value in any continuous subset. If we conjugate the two classifications we found in total four categories

- Discrete time-discrete variable,
- Discrete time-continuous variable,
- Continuous time-discrete variable,
- Continuous time- continuous variable.

Example 1
Consider a sample point $\omega \in \Omega$ representing the state of the weather variable such as a rainfall, the sample path or trajectory of the process X associated with this weather condition ω is a function $t \mapsto X_t(\omega)$, $t \geq 0$ which represent the quantity of rainfall per unit of time (for example, day, week, month,year). In this case $X_t(\omega) \in \mathbb{R}_0^+$. If we consider for example the day as a unit of time, it can be categorized on Discrete time-continuous variable, since the variable take values in $\mathbb{R}_0^+$. In other hand if we are interested to measure the quantity of rainfall during an interval, for example 3 months, we can categorize it on continuous time-continuous variable.

Our focus will be in stochastic calculus in continuous time since the weather variables and the weather derivatives are manly modeled by stochastic differential equations. In order to describe the distribution and the probabilities of the uncertain future, we require that the stochastic process has a finite-dimensional distributions, see for example [35]. We assume that the observed processes are càdlàg (regular right-continuous) functions. In this context the stochastic processes are different only in sets of null measures and are called versions one of another therefore there is no distinction between them. The next example illustrates two processes which are versions one of another.

Example 2 (Klebaner [35],Pag.48)
Let $X_t = 0 \, \forall t$, $t \in [0,1]$, and τ be a uniformly distributed random variable in $[0,1]$. Let $Y_t = 0$ for $t \neq \tau$ and $Y_t = 1$ if $t = \tau$. Then for any fixed t, $P(Y_t \neq 0) = P(\tau = t) = 0$, hence $P(Y_t = 0) = 1$, so that all one-dimensional distributions of X_t and Y_t are the same. Similarly all finite-dimensional distributions of X and Y are the same. However the functions X_t, $t \in [0,1]$ (the sample paths of the process X) are continuous in t, whereas every sample path Y_t, $t \in [0,1]$ has a jump at the point τ. Additionally $P(X_t = Y_t) = 1, \forall t \in [0,1]$.

Definition 2.1.3 (of the versions one of another processes)
Two stochastic process X_t and Y_t are called versions (modifications) one of another

if

$$P(X_t = Y_t) = 1, \ \forall t \in [0, T].$$

Additionally if happens that $P(\bigcup_{t\in[0,T]} X_t \neq Y_t) = 0$, then the process are called indistinguishable and the set $N = \bigcup_{t\in[0,T]} X_t \neq Y_t$ is called *evanescent set*.

In the Example 2, $P(X_t \neq Y_t) = P(\tau = t) = 0$ for any $t \in [0, 1]$. But $P(\bigcup_{t\in[0,1]} X_t \neq Y_t) = P(\tau = t \text{ for some } t \text{ in } [0, 1]) = 1$, since the union of single sets $X_t \neq Y_t$ contains uncountable many null sets and in this case the probability of the union set is one, the two processes X_t and Y_t are not called indistinguishable.

More generally, if two processes X_t and Y_t are versions one of another and they are also càdlàg functions, then they are indistinguishable.

Theorem 2.1.1 (conditions of existence of càdlàg versions of a stochastic process)
Let $X_t, \ t \in [0, T]$ be an stochastic process with values in $\mathbb{R}$, if

1. there existe $\alpha > 0$ and $\epsilon > 0$, such that for any $0 \leq u \leq t \leq T$,

$$E|X_t - X_u|^\alpha \leq C(t - u)^{(1+\epsilon)}, \tag{2.1}$$

for some constant C, then there exists a version of X with continuous sample paths, which are Hölder continuous of order $h < \frac{\epsilon}{\alpha}$,

2. there exist $C > 0$, $\alpha_1 > 0$, $\alpha_2 > 0$ and $\epsilon > 0$, such that for any $0 \leq u \leq v \leq t \leq T$,

$$E(|X_v - X_u|^{\alpha_1}|X_t - X_v|^{\alpha_2}) \leq C(t - u)^(1 + \epsilon), \tag{2.2}$$

then, there exists a version of X_t with paths that may have discontinuities at any interior point, both right and left limits exist, and one-sided limits exist at the boundaries.

Definition 2.1.4 (of the Markov process)
Let $\mathcal{F}_t$ denote the σ-field generated by the process up to time t. X is Markov process if for any t and $s > 0$, the conditional distribution of X_{t+s} given $\mathcal{F}_t$ is the same as the conditional distribution of X_{t+s} given X_t, that is

$$P(X_{t+s} \leq y|\mathcal{F}_t) = P(X_{t+s} \leq y|X_t), a.s. \tag{2.3}$$

From the Markov property in **Definition** 2.1.4, we can understand that the future behavior of the Markov process depends only on the present states of the process and not on the past.
Definition 2.1.5 (of filtration)

The *filtration* is defined as non-decreasing family $\{\mathcal{F}_t\}_{t\geq 0}$ of $\sigma-$field $\mathcal{F}_t \subseteq \mathcal{F}$ such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for $0 \leq s < t < \infty$.

Definition 2.1.6 (of stochastic process adapted to Filtration)
The process X is called adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if the random variable X_t is $(\mathcal{F}_t, \mathcal{F})$ measurable function for each t.

Definition 2.1.7 (of supermartingale,submartingale and martingale process)
A stochastic process $\{X_t, t \leq 0\}$ adapted to a filtration $\mathcal{F}$ is a supermatingale (submartingale), if for any t it is integrable, $E[|X_t|] < \infty$, and for any $s < t$

$$E(X_t|\mathcal{F}_s) \leq X_s, \quad (E(X_t|\mathcal{F}_s) \geq X_s). \tag{2.4}$$

If $E[X_t|\mathcal{F}_s] = X_s$, then the process X_t is called martingale.

The condition $E[X_t|\mathcal{F}_s] = X_s$ means that the expectation of the stochastic variable is conditioned in the past values of X_t up to s. $\mathcal{F}_s$ is a *filtration* associated with the measure space $(\Omega, \mathcal{F})$ and is defined in **Definition** 2.1.5.

2.1.2 Brownian motion

The Brownian motion or Wiener process is a particular type of the Markov process with mean zero and variance t.

Definition 2.1.8 (of Brownian motion)
A stochastic process $B = (B_t, t \in \mathbb{R}^+)$ is called Brownian motion or Wiener process if the following conditions are satisfied:

- it start at zero: $B_0 = 0$,
- the function $t \to B_t$ is almost surely continuous and non differentiable,
- it has independents increments with distribution $B_t - B_s \sim N(0, t-s)$ i.e., the random variable $B_t - B_s$ is independent of the random variable $B_u - B_v$ when $t > s \geq u > v \geq 0$.

Brownian motion is a process whose movements are similar to the movement described firstly by Robert Brown in 1828. Robert Brown described a motion of the a pollen particle suspended in fluid that was moving describing irregular, random continuous paths. In the figures below we can see examples of this movements.

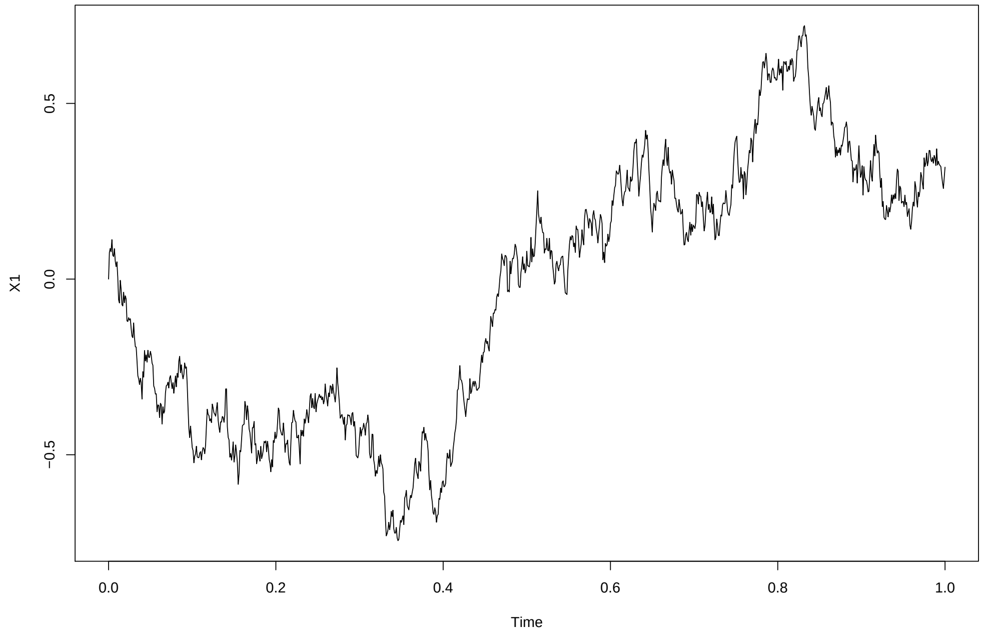

FIGURE 2.1: One path of realization of Brownian motion

In finance the same term was used by Lours Bachelier in 1900 in his PhD thesis to describe the movements of the stock prices, but the mathematical foundation for Brownian motion as a stochastic process was developed in 1931 by Norbert Wiener and, therefor the process is also called Wiener process. Generally it is denoted by B_t or W_t, for Brown or Wiener respectively.

Theorem 2.1.2 (Martingale property for Brownian motion)
Let B_t be a Brownian motion, then

1. B_t is a martingale,
2. $B_t^2 - t$ is a martingale,
3. For any u, $e^{uB_t - \frac{u^2}{2}t}$ is a martingale.

The proof of this theorem is made firstly by considering that B_t is Brownian motion in $\mathbb{R}^n$, so that $E[|B_t|]^2 \leq E[|B_t|^2] = |B|^2 + nt$ and if $s \leq t$ then $E[B_t|\mathcal{F}_s] = E[B_t - B_s + B_s|\mathcal{F}_s] = E[B_t - B_s|\mathcal{F}_s] + E[B_s|\mathcal{F}_s] = E[B_t - B_s] + B_s = B_s$, since $E[B_t - B_s] = 0$ ($B_t - B_s$ is independent of $\mathcal{F}_s$ and $E[B_s|\mathcal{F}_t] = B_s$, since B_s is $\mathcal{F}_s$ measurable.

Theorem 2.1.3 (Markov property for Brownian motion)
Brownian motion B_t possesses Markov property.

The proof of this theorem can be made by using the moment generating function to find the conditional distribution of B_{t+s} given $\mathcal{F}_t$ and by using the property that $e^{u(B_{s+t}-B_t)}$ is independent of $\mathcal{F}_t$ and the Brownian increments are normal distributed with mean 0 and variance s i.e., $N(0,s)$ so that

$$E[e^{uB_t+s}|\mathcal{F}_t] = e^{uB_t}E[e^{u(B_{t+s}-B_t)}|\mathcal{F}_t] = e^{uB_t}E[e^{uB_{t+s}-B_t}] = e^{uB_t}e^{\frac{u^2s}{2}} =$$

$$= e^{uB_t}E[e^{uB_{t+s}-B_t}|B_t] = E[e^{uB_{t+s}}|B_t].$$

2.1.3 stochastic differential equations

Stochastic process have been proposed in many applications and the categories of "continuous time-continuous variable" have been proved to be more commonly used. In the definition of the Brownian motion we can see that B_t has zero mean (drift rate) and variance t i.e., $E(B_t) = 0$ and $Var(B_t) = t$. The definition of the Brownian motion can be extended in order to retrieve a stochastic process with any drift rate b and variance in terms of the Brownian independent increments dB_t, and this is called generalized Wiener process or a Brownian motion and can be defined as:

$$dX_t = bdt + \sigma dW_t. \tag{2.5}$$

In (2.5) the parameters b, σ are constants and X_t has drift rate equal to b and variance σ^2 per unit time. If we consider that the expected proportional change in a short period of time remains constant, whereas the expected absolute change in short period vary in time and that the uncertainty regarding the magnitude of future changes in variable is proportional to the variable, the model (2.5) take the form

$$dX_t = bX_tdt + \sigma X_tdW_t. \tag{2.6}$$

The model (2.6) is called geometric Brownian motion and was mostly used to model the stock price changes. In Geometric Brownian motion both, the drift rate and variance rate are functions of X_t that change over the time, and the process is called Itô process. The Itô process is the generalization of the Wiener process or Brownian motion, b and σ being functions of the underlying variable X and time t, written in general form as

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t)dW_t \tag{2.7}$$

where W_t is m - dimensional Brownian motion.

As we can see, all these equations are differential equations with the second term being a random term (the white noise), therefore they are called stochastic differential equations. In general the stochastic differential equations are obtained by allowing randomness coefficients $\sigma(t, X_t)W_t$ in the differential equation

$$\frac{dX_t}{dt} = b(t, X_t) \tag{2.8}$$

and being

$$\frac{dX_t}{dt} = b(t, X_t) + \sigma(t, X_t)W_t \tag{2.9}$$

where $X_t \in \mathbb{R}^n$, $b(t, X_t) \in \mathbb{R}^n$ is the drift rate of the variable X per unit time. $W_t \in \mathbb{R}^m$ denotes "white noise" and $\sigma(t,x) \in \mathbb{R}^{n\times m}$ the variance rate of X or the intensity of the noise at X per unit time. The equations (2.9) implies that the variations in the variable in short period of time are explained by an know rate (the deterministic term) plus an unknown rate (the "white noise").

There are various interpretations of equation (2.9), but here we will focus on the Itô interpretation, which is given by (2.7). The Itô equation is found by replacing W_t in (2.9) by $W_t = \frac{dB_t}{dt}$ and multiplying the resultant equation by dt. In the stochastic differential equation (2.7), the deterministic coefficient b is called the drift coefficient and the stochastic coefficient σ is called the diffusion coefficient.

If the solution of (2.7) is thought as representation of the mathematical trajectory of the motion of a small particle in a moving fluid, such stochastic process is called Itô diffusions [42]. The Itô diffusions have the property of being time-homogeneous, in the sense of the Markov property (the future behaviour of the Itô diffusion is not influenced by the past).

2.1.4 Itô's formula for Itô processes

The Itô formula is a main tool of the stochastic calculus and can be thought as an Itô integral version of the chain rule. We will use the Itô formula in the demonstration of the Feynman-Kac theorem.

First we present the notion of the integration of the stochastic differential equation. The integrator of the Itô integral is a Brownian motion B_t with an associated filtration $\mathcal{F}_t$ and the integrand process X possess the following properties:

- X is adapted to the Brownian motion in $[0, T]$,
- The integral $\int_0^T E[X_s^2]ds < \infty$.

Definition 2.1.9 (of the Itô integrable stochastic process)
A stochastic process X_s is called Itô integrable in the interval $[0, t]$ if X_s is adapted for $s \in [0, t]$ and $\int_0^t [X_s^2] ds < \infty$.

The Itô integral is defined as the random variable

$$I_t(X) = \int_0^t X_s dB_s = \lim_{n \to \infty} \sum_{i=1}^{n-1} X(s_i)(B(s_{i+1}) - B(s_i)). \tag{2.10}$$

The notion of Itô process presented above in terms of a differential form can also be given rigorously in term of integral form.

Definition 2.1.10 (of Itô processes or Itô integral)
Let B_t be Brownian motion in $(\Omega, \mathcal{F}, P)$. A Itô process (or Itô stochastic integral) is a stochastic process X_t in $(\Omega, \mathcal{F}, P)$ of the form

$$X_t = X_0 + \int_0^t b(s, w) ds + \int_0^t \sigma(s, w) dB_s \tag{2.11}$$

for $\sigma \in \mathcal{W}_{\mathcal{F}} = \bigcap_{T \geq 0} \mathcal{W}_{\mathcal{F}}(0, T)$, $\mathcal{W}_{\mathcal{F}}(0, T)$ denotes the class of process $\sigma(t, w) \in \mathbb{R}$ satisfying:

- $(t, w) \rightarrow \sigma(t, w)$ is $\mathcal{B} \times \mathcal{F}$- measurable ($\mathcal{B}$ denotes the Borel σ-algebra on $[0, \infty)$);
- There existe an increasing family of σ-algebra $\mathcal{F},\ t \geq 0$ such that B_t is a martingale with respect to $\mathcal{F}_t$ and σ is $\mathcal{F}_t$-adapted;
- $$P\left[\int_0^t \sigma^2(s, w) ds < \infty \text{ for all } t \geq 0\right] = 1, \tag{2.12}$$

$b(s, w)$ is $\mathcal{F}_t-$adapted and

$$P\left[\int_0^t |b(s, w)| ds < \infty \text{ for all } t \geq 0\right] = 1. \tag{2.13}$$

The itô integral is characterized by the following properties:

1. (Filtration adapted) for each $t \in [0, T]$, I_t is $\mathcal{F}_t-$mensurable,
2. (Linearity) Given two processes X^1, X^2 and constants c_1, c_2 for $t \in [0, T]$, we have

$$\int_0^t [c_1 X_s^1 + c_2 X_s^2] dB_s = c_1 \int_0^t X_s^1 dB_s + c_2 \int_0^t X_s^2 dB_s,$$

3. $I_t(X)$ is martingale,
4. $I_t(X)$ has expectations zero i.e., $E[\int_0^t X_s dB_s] = 0$,
5. (Continuity) $I_t(X)$ is a continuous function of the upper limit of the integration t,
6. (Isometry) $E[\int_0^t X_s dB_s]^2 = \int_0^t E[X_s^2] ds$.

Theorem 2.1.4 (the Itô formula)
Let X_t be an Itô process given by

$$dX_t = b(t, x_t)dt + \sigma(t, X_t)dB_t. \tag{2.14}$$

If $f(t, x)$ is twice continuously differentiable on $[0, \infty) \times \mathbb{R}$, then $Y_t = f(t, X_t)$ is again an Itô process, and the stochastic differential equation of the process exists and is given by

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)(dX_t)^2 \tag{2.15}$$

where $(dX_t)^2 = (dX_t)(dX_t)$ is computed according to the following rules

$$dtdt = dtdB_t = dB_t dt = 0, \;\; dB_t dB_t = dt. \tag{2.16}$$

Following the definition of an Itô process, one can prove that the process $f(t, X_t)$ admits the following representation

$$\begin{aligned} f(t, X_t) \;\; &= \;\; f(0, X_0) + \int_0^t \sigma_s \frac{\partial f}{\partial x}(s, X_s)dB_s + \\ &+ \int_0^t \left(\frac{\partial f}{\partial s}(s, X_s) + b_s \frac{\partial f}{\partial x}(s, x_s) + \frac{\sigma_s^2}{2}\frac{\partial^2 f}{\partial x^2}(s, X_s) \right) ds. \end{aligned} \tag{2.17}$$

The proof can be made by substitution (2.14) in (2.15) and with application of the rules (2.16) we get an Itô process in the sense of **Definition** 2.11 i.e., (2.17), where $b_s = b(s, w), \;\; \sigma_s = \sigma(s, w)$. The idea is to assume that the functions $f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ are bounded. The assunption is suported by the fact that generally f can be approximated by functions $f_n \in C^2$, such that $f_n, \frac{\partial f_n}{\partial t}, \frac{\partial f_n}{\partial x}, \frac{\partial^2 f_n}{\partial x^2}$ are bounded for each n and converge uniformly to $f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ respectively in a compact subsets of $\mathbb{R}_0^+ \times \mathbb{R}$. In addition, the functions $b(t, X_t)$ and $\sigma(t, X_t)$ are elementary functions so that, can be approximated respectively by functions b_n and σ_n. The interval $[0, t]$ is divided by n equals sub-intervals and, the increment $f(t, X_t) - f(0, X_0)$ is represented as $\sum_0^{n-1}(f(t_{j+1}, X_{t_{j+1}}) - f(t_j, X_{t_j}))$ so that, the function $f(t, X_t)$ is approximated by the Taylor expansion up to the second order derivatives terms and is evaluated at points (t_j, X_{t_j}). Applying the assumption that b and σ are elementary functions and after substituting $(\Delta X_j)^2$ by $(b_j\Delta t_j + \sigma_j \Delta B_j)^2$, the limit is taken when $\Delta t_j \to 0$. More details can be found in [42].

The Itô formula can be used to calculate the values of the stochastic integrals as we illustrate in the following example.

Example 3
Suppose that we choose $X_t = B_t$ and $f(t,x) = x^2$ where B_t is the Brownian motion. The first and second derivatives of $f(t,x)$ are respectively $f'_t(t,x) = 0,\ f'_x(t,x) = 2x$ and $f''_{xx}(t,x) = 2$. So, $Y_t = f(t, B_t) = B_t^2$. The Itô formula will be

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial t}(t, X_t)dt + \frac{\partial f}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)\cdot(dX_t)^2 \\ &= 0dt + 2B_t dB_t + (dB_t)^2 = 2B_t dB_t + dt \end{aligned}$$

hence

$$d(B_t^2) = 2B_t dB_t + dt.$$

In the integral form, for $s \in [0, t]$ we will have

$$B_t^2 = 2\int_0^t B_s dB_s + t,$$

from the last representation we can deduce that

$$\int_0^t B_s dB_s = \frac{B_t^2}{2} - \frac{t}{2}$$

By the Itô formula we can see that there is a link between the stochastic process and a second order partial differential equation. This link usually is defined in terms of the generators defined as bellow.

Definition 2.1.11 (of generator of Itô diffusion)
Let X_t be a (time-homogeneous) Itô diffusion in $\mathbb{R}^n$. The (infinitesimal) generator L of X_t is defined by

$$Lf(x) = \lim_{t \downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t};\ \ x \in \mathbb{R}^n. \tag{2.18}$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exists at x is denoted by $\mathcal{D}_L(x)$, while $\mathcal{D}_L$ denotes the set of functions for which the limit exists for all $x \in \mathcal{R}^n$.

The deduction of the generator formula L for an Itô diffusion process can be found detailed in [42], but the principal idea is to consider that the expectation $E^x[f(X_t)]$

in (2.18) can be calculated by

$$E^x[f(X_t) = f(x) + \sum_{ik} E^x[\sigma_{ik}\frac{\partial f}{\partial x_i}(X)dB_k] + \\ +E^x\left[\int_0^t \left(\sum_i b_i \frac{\partial f}{\partial x_i}(X) + \frac{1}{2}\sum_{ij}(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}(X)\right) ds\right] \quad (2.19)$$

in addition we assume that f is bounded Borel function, so that the limit of the second term in (2.19) can be vanished and any Itô diffusion process X_t in $\mathbb{R}^n$ will alwalys be associeted with the generator L given by

$$Lf = \sum_i b_i \frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{ij}(\sigma\sigma^T)_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j}; \; f = f(x) \in\in C_0^2(\mathbb{R}^n). \quad (2.20)$$

2.1.5 Girsanv's theorem for Brwnian motion

The Girsanov's theorem plays an important role in the stochastic analysis. Named after Igor Vladimirovich Girsanov, tells how the dynamics of the stochastic process changes when the original measure is changed to an equivalent measure. When pricing derivatives, the underlying asset measure is usually converted from the real measure to the risk-neutral measure.

Two measures P and Q in a probabilistic space $(\Omega, \mathcal{F}, P)$ are equivalent if they have the same null sets. If it happens, there exists a random variable M, called Radon-Nikodym derivative

$$M = \frac{dQ}{dP}$$

such that the probabilities under Q are given by

$$Q(A) = \int_A M dP, \quad \forall A \in \mathcal{F}.$$

By the Girsanov's theorem we find the form of M.

Theorem 2.1.5 (Girsanov'e theorem)
Let $B_t,\ t \in [0, T]$, be a Brownian motion under probability measure P. Consider $W_t = B_t + \lambda t$. Define the measure Q by the stochastic process

$$M_t = \frac{dQ}{dP}(B_t) = e^{(\lambda B_t - \frac{1}{2}\lambda^2 t)}, \; t \in [0, T]. \quad (2.21)$$

Q is equivalent to P, and W_t is a $Q-$ Brownian motion.

$$\frac{dP}{dQ}(W_t) = e^{(\lambda W_t - \frac{1}{2}\lambda^2 t)}, \; t \in [0, T]. \quad (2.22)$$

The proof is made using the Levy's characterization of the Brownian motion, as a continuous martingale with quadratic variation process t. Quadratic variation is the same under P and Q, by convergence in probability (on the result of the general Bayes formula). Since λt is a smooth function it has no contribution to the quadratic variation, therefore

$$[W_t, W_t] = [B_t + \lambda t, B_t + \lambda t] = [B_t, B_t] = t$$

To prove that W_t is $Q-$martingale we consider that $M_t = E^P(M|\mathcal{F}_t)$ and one can show that $M_t W_t$ is P - martingale by direct calculations i.e.,

$$E^P(W_t M_t|\mathcal{F}_t) = E^P((B_t + \lambda t)e^{(-\lambda B_t - \frac{1}{2}\lambda^2 t)}|\mathcal{F}_t) = W_t M_t.$$

2.1.6 Feynman-Kac formula

In many applications a stochastic process X_t can be associated to a second order partial differential operator L in the sense that L is the generator of the process X_t. The Feynman-Kac theorem allow the probabilistic representation of the solutions of the PDEs whose their infinitesimal operators are associated with the stochastic process. The connection between the Markov property of X_t and the PDE is made by applying the Itô's formula on the martingale term.

Let X_t be a diffusion process satisfying the following SDE

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dB_t \text{ and } X_s = x. \tag{2.23}$$

Theorem 2.1.6 (Feynman-Kac Formula)
For given bounded functions $r(x, t)$ and $g(x)$ let

$$V(x, t) = E\left(e^{-\int_t^T r(X_u, u)du} g(X_T)|X_t = x\right) \tag{2.24}$$

assume that there is a solution to a Cauchy problem

$$\frac{\partial f}{\partial t}(x, t) + L_t f(x, t) = r(x, t) f(x, t), \text{ with } f(x, T) = g(x) \tag{2.25}$$

then the solution is unique and equal to $V(x, t)$.

The principal item in the Feynman-Kac theorem is the Itô formula associated to solutions of the linear SDE. By applying the Itô formula in the solution of (2.25) we get

$$df(X_t,t) = \left(\frac{\partial f}{\partial t}(X_t,t) + L_t f(X_t,t)\right) dt + \frac{\partial f}{\partial x}(X_t,t)\sigma(X_t,t)dB_t. \tag{2.26}$$

The last term is a martingale term, so it can be write as dM_t. Substituting the first equation of (2.25) into (2.26) we obtain the linear SDE

$$df(X_t,t) = r(X_t,t)f(X_t,t)dt + dM_t \tag{2.27}$$

Integrating this SDE between t and T, and using $T \geq t$ as a time variable and t as the origin we get

$$f(X_T,T) = f(X_t,t)e^{\int_t^T r(X_u,u)du} + e^{\int_t^T r(X_u,u)du} \int_t^T e^{\int_t^T r(X_u,u)du} dM_s. \tag{2.28}$$

By substituting the terminal conditions $f(X_T,T) = g(X_T)$ of the Cauchy problem (2.25) into (2.28) and rearranging, we get

$$g(X_T)e^{-\int_t^T r(X_u,u)du} = f(X_t,t) + \int_t^T e^{\int_t^T r(X_u,u)du} dM_s. \tag{2.29}$$

Now we take the expectation of (2.29) given $X_t = x$.

$$\begin{aligned} &E\left(g(X_T)e^{-\int_t^T r(X_u,u)du}|X_t = x\right) = \\ &= E\left[\left(f(X_t,t) + \int_t^T e^{\int_t^T r(X_u,u)du} dM_s\right)|X_t = x\right], \end{aligned} \tag{2.30}$$

applying the linearity property of expectation in (2.30) we get

$$\begin{aligned} &E\left(g(X_T)e^{-\int_t^T r(X_u,u)du}|X_t = x\right) = \\ &= E\left(f(X_t,t)|X_t = x\right) + E\left(\int_t^T e^{\int_t^T r(X_u,u)du} dM_s|X_t = x\right). \end{aligned} \tag{2.31}$$

Note that the last term is an integral of a bounded function with respect to martingale, it is itself a martingale with zero mean, then

$$E\left(g(X_T)e^{-\int_t^T r(X_u,u)du}|X_t = x\right) = E\left(f(X_t,t)|X_t = x\right) \tag{2.32}$$

So, that $V(x,t) = f(x,t)$.

Note also that if r is constant, the quantity $e^{-\int_t^T r du} E\left(g(X_T)|X_t = x\right)$ is well known and represent in finance the discounted expected payoff.

2.2 Financial derivatives

This section provide an overview of financial tools used to manage financial risks. We give the concepts used for evaluation of financial derivatives which will be used during this study. Our principal focus will be in weather derivatives. More details about financial derivatives can be found in [2], [25], [24], [28], [40], [7], [13] and [15].

2.2.1 Products and fundamentals

For the companies survive and prosper there will always be a risk to handle. These risks can be equity risks such as interest rates, exchange rates, commodities prices or in less traditional markets can be weather risk, energy price risk, and insurance risks. In general the companies have no expertise to predict such variables and as a solution they prefer to hedge the risks associated with their activities. The fundamental idea is that by hedging they can avoid unpleasant surprises and concentrate only on their production. There is also the idea of the existence of the trade-off between the risk and return, since the higher expected returns can only be achieved by taking higher risks. The risk management is a principal tool to understand the portfolio of risks currently taken and the risks planed to take in the future. In the financial market there are many financial tools that can be used to manage the financial risk and they are called contingent claims.

Definition 2.2.1 (of contingent claims)
Contingent claim-T or T-claim is a contract which pays to the holder a stochastic amount X at time T. The random variable X is $\mathcal{F}_T$-measurable and T is called exercise time of the contingent claim or maturity of the contract.

The common characteristic of the contingent claims is that they a defined in terms of the underlying asset and according to the nature of the financial products to be exchanged, and can define specifics financial markets such as:

- Stock markets: familiar notion of stock exchange markets such as New York, London, Tokyo, Milan;
- Bond markets: products with fixed return, usually issued by the central banks;
- Currency markets or foreign exchange: currencies and their prices are determined;
- Commodities markets: commodities prices such as oil, gold, are fixed;
- Future, forward, swaps, options markets: derivatives products based on one or more other underlying products typically of the previous markets.

Also into futures and options markets one can consider sub-markets according to the specific underlying asset such as, power derivative, oil derivatives, weather derivatives. The underlying asset are respectively the power, oil and weather. Derivatives are contracts applied to financial products and can be in form of standard products or plain vanilla products, structured or exotics products. The last group is designed to meet particular needs of the corporate treasure. The most traded plain vanilla products are options, future, swaps and forward contracts.

- **Forward contract**
  Is an agreement made between two parties to buy an asset in the future at a certain price. One of the parties assumes a long position (i.e., agrees to buy the underlying asset at a certain specified future date for a certain specified price) and the other party assumes a short position (i.e., agrees to sell the asset in the same date for the same price). These contracts are traded usually over-the-counter market (OTC).

  The OTC markets is not a organized market. The term of the contract do not have to be those specified by an exchange and the market participants are free to negotiate any mutually attractive deal. In contrast of over-the-counter market, we have the exchange-traded market, which is a organized market where the contracts and the trading among the participants are respectively defined and organized by the exchange so that the traders can be sure that the trades they agree to, will be honored.

- **Futures Contracts**
  Like forwards contracts the futures contracts are agreement to buy an asset at a future time. But unlike forward contracts, futures are traded on an exchange. This means that the contracts are standardized. The exchange defines the amount of the underlying asset of the contract, when delivery can be made, what can be delivered, and so on. This contract can be closed before the delivery month is reached.

- **Swaps**
  Is an agreement between two companies to exchange cash flows in the future. The agreement defines the dates when the cash flows are to be paid and the way in which they are to be calculated. The forward contracts can be viewed as swap in the sense that they lead to exchange of cash flows on just one future date whereas the swaps lead to cash flow exchanges taking place on several future dates.

- **Options**
  Are financial contracts that are traded on both exchanges and OTC markets, in two types, a *call option* and a *put option*. A *call option* gives to the holder the rights to buy the underlying asset at a certain date for certain price. A

put option gives the holder the rights to sell the underlying asset at a certain date for a certain price. The price in the contract is called *exercise price* or *strike price*, the date is called *expiration date* or *maturity*. In the point of view of the exercise date the options can be splited into two types, *American option* and *European options*. *American options* can be exercised at any time up to the expiration date, whereas the *European options* can be exercised only on the expiration date. In contrast to the forward or futures contracts, the option gives the holder the rights, with no obligation to sell or buy the underlying asset and there is a cost to take position on these contracts. On the exercise date the option can be in one of the three different situations, *at-the-money*, *out-of-the money* and *in-the-money*. An *at-the-money option* is an option where the strike price is close to the price of the underlying asset. An *out-of-the-money option* is *call option* where the strike price is above of the price of the underlying asset or a *put option* where the strike price is below this price. An *in-the money option* is a *call option* where the strike price is below the price of the underlying asset or a *put option* where the strike price is above to this price.

Futures and forwards contracts provide a hedger an exposure at one particular time. The use of future contracts sub-intend that the holder intend to close the contract prior to maturity and as a result the hedge performance is reduced somewhat because there is uncertainty about the difference between the futures price and the spot price on the close-out date, called basis risk. The swaps contracts can provide a hedge for cash flows that will occur in a regular basis over a period of time. But option are different type of hedging instrument from forwards, futures, and swap. Whereas this last consider the prices for future sales or purchases of an asset, an option provides an insurance.

With this, contingent claims are expected that to catch up the price in the market, but it will be worth depending on the time t and on the price $S(t)$ of the underlying asset. The requirement $X \in \mathcal{F}_T$ in the **Definition** 2.2.1 means that, at maturity T it will be possible to determine the value to be paid. The associated payoff function $f(\cdot)$ is usually calculated at point S_T (the final value of the asset price at maturity). Bellow we present a payoff function of the European option, $f(x) = (x - K, 0)^+$ for a call, and $f(x) = (K - x, 0)^+$ for a put. Here K represent the strike price.

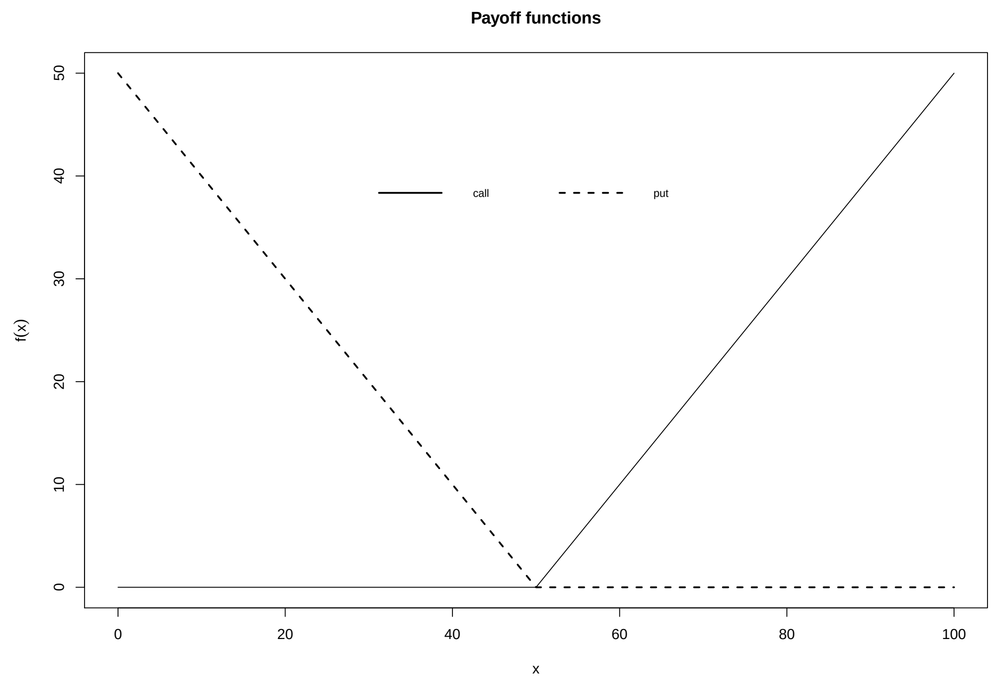

FIGURE 2.2: The payoff functions of call and put options.

From the **Figure** 2.2, we can see that the holder of a call option will make a profit if the spot price at maturity S_T is higher than the strike price $K = 50m.u$ so that, if $S_T = 55m.u$ we will have $(S_T - K, 0)^+ = (55m.u - 50m.u, 0)^+ = 5m.u$. Otherwise if $S_T - K \leq 0$ we will have $(S_T - K, 0)^+ = 0$. In other hand the holder of a put option will make a profit if the spot price at maturity is lower than the strike price K so that if $S_T = 45m.u$ we have $(K - S_T, 0)^+ = (50m.u - 45m.u, 0)^+ = 5m.u$. Otherwise $K - S_T \leq 0$, we will have $(K - S_T, 0)^+ = 0$.

The main problem is to determine a fair price for the contingent claim corresponding to the price process S_t. Under some assumptions, the Black-Scholes formula gives the unique price of the option. Although these assumptions are not generally true in the real world, they determine the start point to investigate the practical problems. The principal assumption is that the market is efficient in the sense that it is free of arbitrage possibilities. The arbitrage concept and among others which we define bellow, are some of the keys concepts on the theory of financial investments. More details can be found in [7].

Definition 2.2.2 (of self financing portfolio)
Consider price process $\{S(t),\ t \geq 0\}$.

- A portfolio strategy is any $\mathcal{F}_t-$ adapted $N-$dimensional process

$$\{h(t),\ t \geq 0\}$$

where $h_i(t)$ is the number of shares of the type i holded during the period $[t, t + \Delta t]$;

- The value process V^h corresponding to the portfolio h is given by

$$V^h(t) = \sum_{i=1}^{N} h_i(t) S_i(t), \tag{2.33}$$

where $S_i(t)$is a stock price of the underlying asset to the share i at time t;

- A consumption process is any $\mathcal{F}_t-$adapted one- dimensional process;
- A portfolio-consumption pair (h, c) is called self-financing if the value V^h satisfies the condition

$$dV^h(t) = h(t)dS(t) - c(t)dt. \tag{2.34}$$

Definition 2.2.3 (of risk free asset)
The price process S is a risk free asset if it has the dynamics

$$dS(t) = r(t)S(t)dt \tag{2.35}$$

where r is any adapted process.

The definition of the risk free asset suggest that the dynamics of the asset is not driven by stochastic term. It can correspond to a bank deposit with the short interest rate r, or for deterministic constant rate r, a price of a bond.

Definition 2.2.4 (of arbitrage)
An arbitrage possibility on a financial market is a self-financing portfolio h such that

$$V^h(0) = 0, \;\; P(V^h(T) \geq 0) = 1, \;\; P(V^h(T) > 0) > 0. \tag{2.36}$$

A market is arbitrage free if there are no arbitrage possibilities (absence of arbitrage).

The arbitrage possibilities can be interpreted as a possibilities of making a positive values of money without taking any risk. This is a serious problem in the financial market and generally is assumed that the market is efficient if there is no arbitrage possibilities, although is not generally true in the real world.

Theorem 2.2.1 (condition for no arbitrage)
Suppose that there exist a self-financing portfolio h such that the value process V^h has the dynamics

$$dV^h(t) = k(t)V^h(t)dt, \tag{2.37}$$

where k is an adapted process. Then $k(t) = r(t)$ for all t, or there exists arbitrage possibilities.

The proof of this theorem can be found in [7], but the principal idea is to consider that for $k > r$, we can borrow money from the bank at the rate r and immediately invest in the portfolio strategy h and it will grow at the rate k so that, from zero initial investment we get profit at any time $t > 0$ (it mean that we have the possibilities of arbitrage). If in other hand we consider $r > k$, we sell the portfolio h (short) and we immediately invest the money in the bank, and again there are arbitrage possibilities.

2.2.2 Weather derivatives

Many companies activities face risks related to changes in the weather and it can adversely influence their performance. The weather risks are managed by using the financial instruments, and they are called financial Weather contract. Following Dischel and Barrieu in [17], we can define a financial weather contract as a weather contingent contract whose payoff is determined by a future weather events and the settlement value of these weather events is determined from a weather index expressed as values of a weather variable measured at a stated location. The commonly underlying weather indexes are

- Temperature,
- Rainfall,
- Humidity,
- Wind,
- Snowfall.

The financial weather contract can take the form of a WD or of a WI contract. The significant difference between the Weather derivatives and weather insurance contracts as appointed in [21] and [50], following on regulatory and legal point of views are:

- the insurance contracts cover only high risks, with low probability of occurrence, whereas weather derivatives also cover low risks, with high probability of occurrence;
- the WI usually are more expensive and require a demonstration of loses whereas the WD are cheaper and do not depend on loses, it only depend on the observation of the weather indexes;
- the payoff on weather derivatives must be proportional to the magnitude of

the phenomena whereas on weather insurance contracts, it can only depend on the amount of losses.

Depending on the circumstances, the flexibility and efficiency of WD make them more attractive than the WI. WD belong to the group of less traditional markets in early stage of development, where usually are considered the risks such as weather risk, energy price risk, and insurance risks.

The main purpose of using WD is to cover non-catastrophic weather events. As example the rainfall derivatives can be used by the companies to protect themselves against fluctuation on their revenues caused by frequent rains or dry periods. In agriculture for example, the crop yield can depend on quantities of rainfall. As shown, for example in [41] and [50] there is a strong correlation between the amount of rainfall and wheat yield. Also in [21] is showed that there is a strong correlation between the amount of rainfall and maize yield. Then the crop yield will depend on an expected(normal) amount of rainfall. If the observed amount of rain is less or more than the amount required, the farmers can lose or get few crop yield than expected. If the farmers can protect themselves against adverse rainfall patterns during the critical stages of growth, the crop yield risk will substantially decrease. Geyser proposed in [21] some possible rainfall options strategies for maize yield. She suggested a options risk protection strategy called *long strangle*, where a long call and a long put are combined. This combination provides to the farmer a hedge traditionally associated in the financial markets with high volatility of the underlying risk exposure.

The first weather derivative was executed in the United State of America in 1997, between two energy companies (Koch Industries and Enron), using a swap on temperature indexes to hedge against warm days in winter. Two years later, the expansion of the climatic contracts gave birth to an organized electronic platform launched by the Chicago Mercantile Exchange (CME) [34]. The first contracts traded were essentially degree days in temperature contracts. In 2003, the CME opened at two subsidiary respectively in Europe and in Japan. In Africa, few countries started to offer weather derivatives contracts and in very small volume. Morocco and South Africa have launched a few OTC contracts. Other initiatives by the World Bank associated with private companies to reduce natural extreme weather risks in developing countries have shown to be very important demanded by small farm holders notably for example in Ethiopia. The proportion of all type of climatic contracts negotiated was appointed in [34] to be more significantly on CME market in 2005, with 95% of contracts against 5% of OTC contracts.

2.2.3 Complete and Incomplete markets

The concept of completeness of a system of markets is related to existence of enough commodities in the markets. A market is complete when any asset can be replicated into a portfolio. But in the real world such markets does not exist, since the time and the uncertainty are included in the definition of the complete market. Can happen that some commodities are not tradable and the market is said to be incomplete. The notion of completeness is very important because it can allow to assess the inefficiency of a particular market so that we can use specifics mechanism in order to make such markets less incomplete. In the incomplete markets the no-arbitrage theory of valuation based in the principle of self-financing replicating portfolio is not applicable since the martingale measure is not unique as in complete markets [7]. Details about completeness can be found in [15] and [7].

The identification of completeness of markets can be made via the Meta-theorem as follow:

Theorem 2.2.2 (Meta-theorem)
Let M denote the number of underlying assets in the model excluding the risk free asset, and R denote the number of random sources. Generically the following relations are valid

1. The model is arbitrage free if and only if $M \leq R$,
2. The model is complete if and only if $M \geq R$,
3. The model is complete and arbitrage free if and only if $M = R$.

In the meta-theorem the concepts of completeness and absence of arbitrage (arbitrage free) works in opposite directions. The idea is, if we fix the number of the random sources and we increase the number of underlying assets we will create an arbitrage portfolio, therefore to avoid an arbitrage free market the number of underlying asset must be small than the number of random sources. On the other hand, by adding new underlying asset to the model, gives us new possibilities of replicating a given contingent claim. The completeness requires the number of the underlying assets to be greater than the number of the random sources. This theorem is also used in the problems of untradable assets. As an example, in the Black-Scholes model we have one underlying asset with one risk asset ($M = 1$). The model is driven by one Wiener process ($R = 1$), so that we have the number of the underlying asset equal the number of random sources. By the meta-theorem we can conclude that the Black-Scholes model is in a arbitrage free as well as complete market.

Theorem 2.2.3 (First fundamental theorem in mathematical finance)
A necessary and sufficient condition for the absence of arbitrage opportunities is the existence of the martingale measure of the underlying asset process.

The proof of this theorem is made in two parts namely, the necessity and the sufficiency. In the necessity is proved that the absence of arbitrage implies existence of equivalent martingale measure. The demonstration is made by assuming that all asset price process are bounded in order to guarantee the integrability so that, the arbitarge is viewed in the sence of "bounded arbitrage". By the Girsanov theorem, we have to prove the existence of a Radon-Nikodyn derivative L in $\mathcal{F}_T$ which will transform the $P-$measure into a Q martingale measure. This is made by considering that $L \in L^1 = L^1(\Omega, \mathcal{F}_T, P)$ and the existence of the duality between the spaces L^1 and $L^m = L^\infty(\Omega, \mathcal{F}_T, P)$ although, this assumption is not generally true (the detailed discussion can be found in [7]). Furthermore we define the sets

$$\begin{aligned} \mathcal{K} &= \mathcal{K}_0 \cap L^\infty, \\ L^\infty_+ &= \text{the set of non-negative random variables in } L^\infty, \\ C &= \mathcal{K} - L^\infty_+, \end{aligned} \tag{2.38}$$

where $\mathcal{K}_0$ is the space of all claims which can be reached by a self financing portfolio at zero initial cost, $\mathcal{K}$ consist of all bounded claims which are reachable by a self financing portfolio at zero initial cost, and C consist of claims in $\mathcal{K}$ that can be reached by self financing portfolio with zero initial cost if the investor also allow himself to throw away the money. By absence of arbitrage assumption we deduce that $C \cap L^\infty_+ = \{0\}$ and both C and L^∞_+ are convex sets in L^m with only one common point. The nonzero random variable $L \in L^1$ such that

$$\begin{aligned} (a): &\quad E^P[LX] \geq 0, \ \ \forall X \in L^\infty_+ \ \text{ and} \\ (b): &\quad E^P[LX] \leq 0, \ \ \forall X \in C \end{aligned} \tag{2.39}$$

is guaranted by the convex separtion. From (2.39(a)) one can deduce that in fact $L \geq 0$ and by scaling one can choose L such that $E^P[L] = 1$ so that, one can use L as a Radon-Nikodym derivative to define a new measure Q by $dQ = LdP$ in $\mathcal{F}_T$ and Q is a natural candidate as a martingale measure.

The sufficiency is proved from the fact that the existence of a martingale measure implies absence of arbitrage, for that we assume the existence of martingale measure Q and then, we apply the Girsanov theorem so that, all price process can be expressed with zero drift under Q. Furthermore, the no arbitrage possibility is proved by assume the existence some self financing process h which is also uniformly bounded and satisfy

$$P(V(T,h) \geq 0) = 1 \ \wedge \ P(V(T,h) > 0) > 0, \tag{2.40}$$

where $V(T,h)$ is value of the process h at maturity T. Since, the condition (2.40) can suggest h as potential arbitrage portfolio, we have to show that $V(0,h) > 0$ in order to guarantee the absence of arbitrage. This is made by considering that from $Q \sim P$ we also get the condition (2.40) under Q. In the other hand, since h is self financing and bounded it is shown that $V(t,h)$ is $Q-$martingale so that, $V(0,h) = E^Q[V(T,h)]$ and $V(0,h) > 0$ since (2.40) under Q imply that $E^Q[V(T,h)] > 0$.

Theorem 2.2.4 (Second fundamental theorem in mathematical finance)
Assume the absence of arbitrage opportunities. Then a necessary and sufficient condition for the completeness of the market is the uniqueness of the martingale measure.

This theorem is also proved in two steps. In the sufficiency we assume that the martingale measure Q is unique, then $M = \{Q\}$ so Q is trivially an extremal point of M thus, every $Q-$ martingale M can be represented by the stochastic integral of the form

$$M(t) = x + \sum_{i=1}^{N} \int_0^t h_i(s) dZ_i(s). \tag{2.41}$$

This imply that M can be hedged in the $S-$ economy therefore the model is complete. The necessity is obtained by considering that if the market is complete then, every claim X can be replicated so that, for hedging portfolio h

$$V(t,h) = E^Q[e^{-\int_t^T r(s)ds} X|\mathcal{F}_t].$$

More details about the demonstration of the first and second fundamental theorems in Mathematical Finance can be found in [7]. From these two theorems, one can determine the conditions of arbitrage free and completeness of the financial market in the point of view of martingale approach. So if we consider a complete market and the assumption of arbitrage free principle, the unique price $\pi(S)$ of a risk free contigent claim S is determined by

$$\pi(S) = E_Q[e^{-rT} S] \tag{2.42}$$

where Q is the unique martingale measure and r is the interest rate of the risk free asset. In the case where the market satisfies the no-arbitrage assumption but does not satisfy the completeness assumption, the price $\pi(S)$ is supposed to belong to the interval

$$\pi(S) \in \left[inf_{Q \in \mathcal{M}} E_Q[e^{-rT}S], sup_{Q \in \mathcal{M}} E_Q[e^{-rT}S] \right], \tag{2.43}$$

where $\mathcal{M}$ is the set of all equivalent martingale measures.

Chapter 3

Weather derivative pricing

As appointed in [2], the market is still relatively illiquid while practitioners and risk management companies keep WD data private and do not publish their models making difficulty to develop pricing models for weather risks. There is no general framework for pricing weather derivatives. The companies mostly use historical analysis pricing methodology [18] because of the simplicity in terms of its implementation. Some researchers, for example in [9] argued that this methodology gives inaccurate results. In recent years the continuous model have been proposed. The evolution of the weather variables are described by the dynamics models and these are used to price weather derivatives.

3.1 Overview on pricing weather derivatives

The determination of the price is made in two principal steps. First the development of the model for dynamics of the weather variable and, in the second step the dynamics of the weather variable are used to price WD. More generally the WD can be formulated as in [54] specifying the following parameters:

- Contract type, if it is European or American option, Future, Swap, among others;
- Contract period, usually 1 month or 6 months (Six months corresponds to hedging either the winter or summer season);
- The referential point, from which the meteorological data is obtained;
- The underlying index of the contract (can be, Temperature DD and their variants "HHD or CDD", rainfall, etc) for each commodity;
- Pre-negotiated threshold or strike level for weather index (S);
- Tick or constant payment for a linear or binary payment scheme "τ" (translate the payoff into monetary terms), and
- The premium.

In the WD market the underlying is not tradable asset, then weather models do not follow Geometric Brownian motion [1], therefore Black-Scholes methodology can not be implemented directly. Alternative methods based on arbitrage-free pricing methods [47], [48], equilibrium models [12] and actuarial approach [8] was proposed. The methods based on PDEs were suggested to model temperature derivatives in [47], [48] and posteriori adopted by other researchers in [1], [4] and [36]. Since the underlying is non-tradable asset the price is determined under the theory of incomplete markets, where the risk neutral equivalent probability is not necessarily unique as in theory of the Black Scholes framework (the underlying is tradable asset). The claims can not be hedged by the principle of self-financing portfolio. They have suggested to calculate the arbitrage-free price of weather option by using the market price of risk extracted from the quotations of the weather futures and the price of the weather option determined as discounted conditional expectation

$$V = \mathbb{E}\left[e^{-\int_t^T r(u)du} V(T, X_T, Y_T) | X_t = x, Y_t = y\right]. \tag{3.1}$$

The Feynman-Kac theorem allow to derive the PDE that governs the price of the WD under a risk neutral probability Q. If $V(t, X_t, Y_t)$ is a solution of the PDE, such market price of risk λ extracted from the quotations minimize the following objective function [22]

$$Min_\lambda \sum_{t=1}^{M} (V(t, X_t, Y_t) - V_t^\theta)^2. \tag{3.2}$$

The market price of risk is the difference between the expected rate of return of the underlying and the risk-less interest rate reported to the quantity of risk measured by the volatility [22]. Since the risk has to be extracted from quotations of the weather futures, the arbitrage-free pricing method is applicable only when quotations are available for the weather contracts in order to extract a risk-neutral distribution or to infer market prices.

3.2 Rainfall derivative model

The rainfall derivative are not commonly investigated by researchers, the reason can come from the fact that they are not frequently traded in the market and their complexity of the treatment from the fact that the rainfall is a local weather event. It can happen that in two closed location the rainfall is not correlated [19]. Traditionally the rainfall derivatives are treated in a context of discrete models, maybe by the nature of the rainfall that tend to be observed in discrete time. But there are some attempt to use the stochastic differential models to price rainfall derivatives so that the continuous models for rainfall must be considered. The mean reverting process model was proposed to model rainfall in [19]. They were based on the fact that as well as the other Weather variables the rainfall exhibits seasonal patterns and usually reverts to the mean. Additionally the value of the mean is dependent on the time of the year and does not grow or fall indefinitely. The Ornstein-Uhlenbeck process was applied in [52]. The simplest realization of the mean reversion property as a model to assessing drought and flood risks. The same model is also referred in [3] as a model for annual rainfall at a certain location over a period of decades.

3.2.1 The Ornstein-Uhlenbeck process on modelling Rainfall dynamics

In [19] the empirical studies showed that the precipitation dynamics can be characterized by being

- stochastic,
- high in volatility,
- in fluctuation about a mean,
- seasonal in their effects.

Four mean reverting process were tested namely, mean reversion with constant mean, mean reversion with deterministic mean, both driven by standardized Brownian motion. The other two processes were obtained refining the previous two, replacing the Brownian motion by the fractional Brownian motion to allow a long term relationship. The precipitation dynamics exhibits long-range temporal dependencies in the sense that the present weather condition is influenced significantly by the previous weather conditions. Moving averages do not strongly start from the mean, sometimes the medium term average is above or below the long term mean. The Ornstein-Uhlenbeck process was proposed by many authors to model weather variables possessing the same charateristics and mostly for temperature variable. But in more recent studies it has been used to model drought and flood risks. For example in [52] the Ornstein-Uhlenbeck process was proposed to model the point rainfall evolving with cumulative rainfall depth. The variable becomes smaller or larger during a drought or a flood but reverts to an average when such an event ends. They suggested the mean reverting Ornstein-Uhlenbeck process as the simplest model having this property of mean reversion. The same model was referred before in [3] as a model for the annual rainfall at certain locations over a period of decades.

Its clear that more studies still need to be made on the evolution of rainfall process by mean reverting Ornstein-Uhlenbeck process. Motivated by the previous studies we consider the most general Ornstein-Uhlenbeck process with deterministic mean as a model for monthly rainfall over a period of one year.

If X_t represent the total rainfall at time t, the possible changes in the total rainfall over a very small interval of time dt, can be assumed to be represented by the deterministic mean reverting Ornstein-Uhlenbeck process given by

$$dX_t = [k(\theta(t) - X_t) + \theta'(t)]dt + \sigma_t dW_t \tag{3.3}$$

where k is the rate of the mean reversion, $\theta(t)$ is the long term mean of the process, as in [19] it is given by

$$\theta(t) = m + \Sigma_{i=0}^{n} \alpha_i \sin\left(\frac{(2i+1)2\pi(t-\nu)}{12}\right). \tag{3.4}$$

Here m is the mean of the sine curve, α determines the oscillation and ν represent the shift of the $X - axis$ (to scale up to months we divide by 12). σ_t is the volatility of the rainfall and dW_t represent the Brownian increment under the real probability.

The number of sine terms can be found individually by analyzing the data from the weather station which will be used to estimate the parameters. The appropriate choice is $n = 3$ [19]. For simplicity we will consider the case when $n = 0$ (one sine term). Then the deterministic function $\theta(t)$ is defined as

$$\theta(t) = m + \alpha \sin\left(\frac{\pi(t-\nu)}{6}\right). \tag{3.5}$$

The derivative of $\theta(t)$ is

$$\theta'(t) = \frac{\pi}{6}\alpha \cos\left(\frac{\pi(t-\nu)}{6}\right). \tag{3.6}$$

In order to test this model, the public rain data was obtained from www.dwd.de. The data refer to the weather in a German weather service (Deutscher Wetterdienst, DWD). We consider the Schleswig weather station with the complete data between 1947 and 2017. Below the figures show the dynamics of rain considering month as units of time and period of one year. In total we have 71 periods.

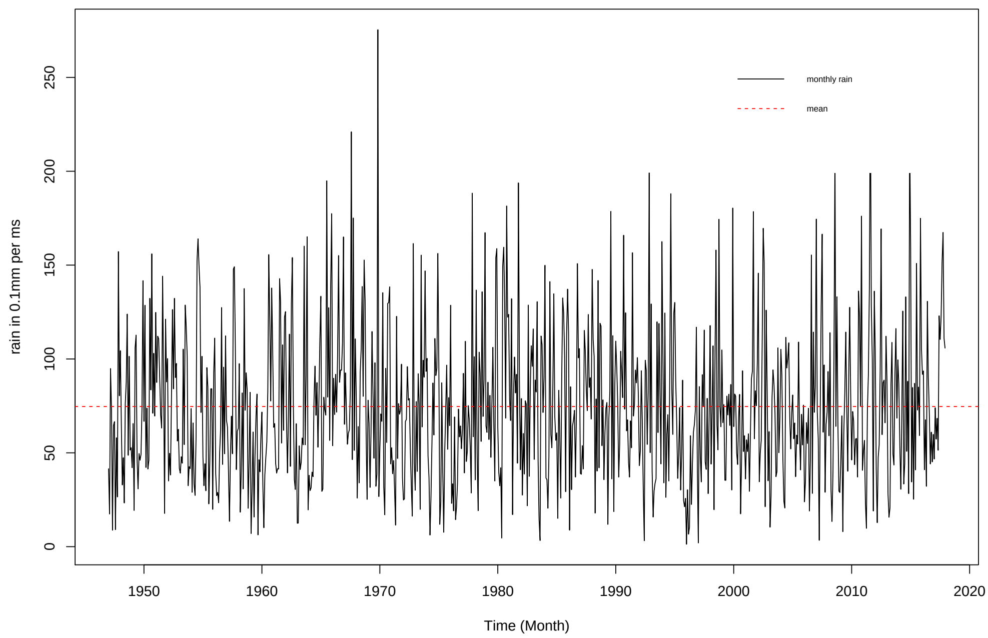

FIGURE 3.1: Monthly rain at Schleswig, 1947-2017.

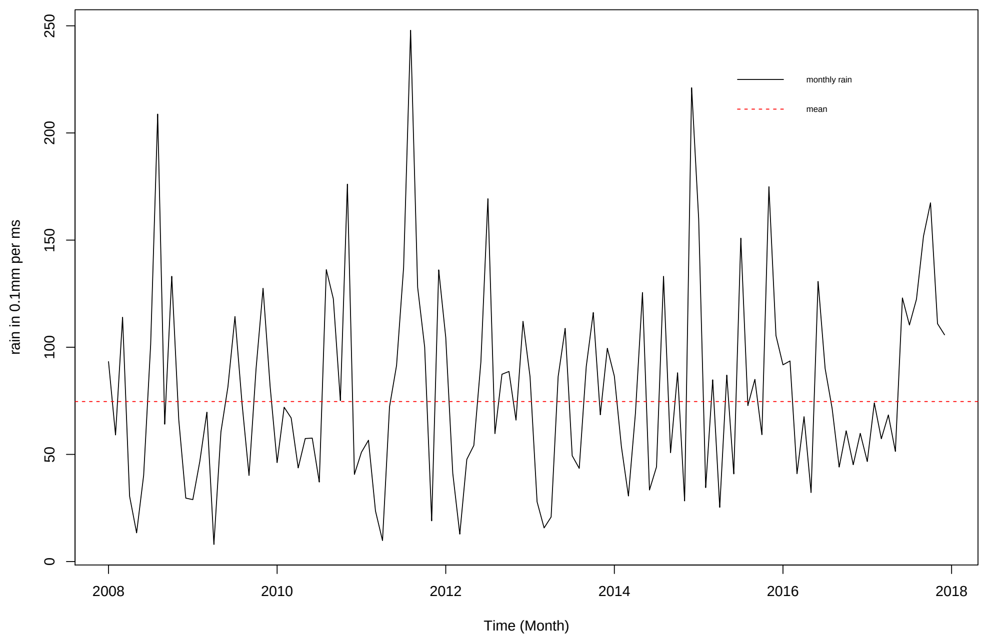

FIGURE 3.2: Monthly rain at Schleswig, 2008-2017.

From the visual inspection on the above figure we can see the suggested characteristics described for rainfall variable. There are strong suspicious that it can be modeled by the Ornstein-Uhlenbeck process (3.3). As proposed in [1], the choice of the wienner process for noise also can be justified by the good fit to the normal distribution of daily rainfall differences, as shown in the figure below.

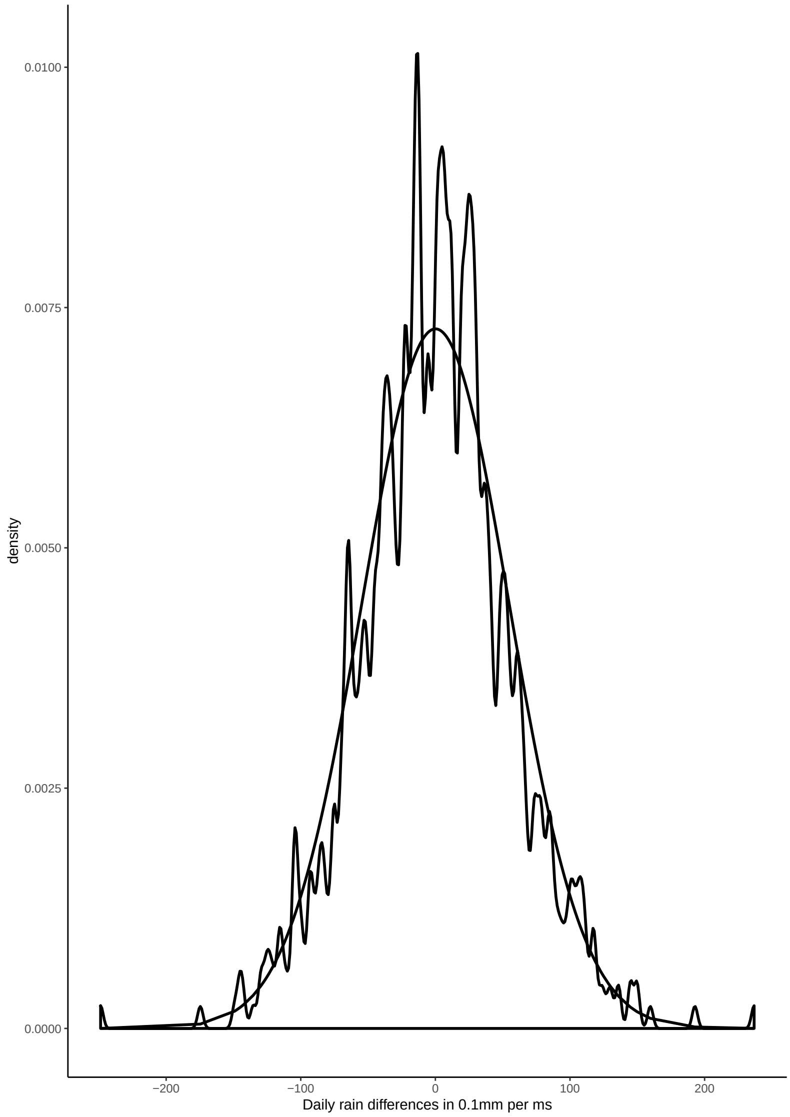

FIGURE 3.3: The density of the monthly rain differences

3.2.2 Parameter estimation and simulated path

The problem of parameter estimation for diffusion processes based on discrete observed data has been studied by several authors and can be solved into different schemes, depending on the properties of the observed data. The principal properties of the observations can be summarized as in [29]:

- Large sample scheme: the most natural, the time lag between consecutive observations is fixed and the number of observations increases as well as the path(window) of observations $[0, n * lag = T]$. The assumptions of stationarity and/or ergodicity are required on the underlying continuous model. The ergodicity or positive recurrent means that the process is recurrent and admits a stationary distribution i.e., the expected time that the process hit a ball around any point $y \in \mathbb{R}^n$ after leaving from start point $x \in \mathbb{R}^n$ is finite.

- High-frequency scheme: the path(window) of observation is fixed and the lag goes to zero as n increases.
- Rapidly increasing design: the lag of observations goes to zero as n increases, but the path(window) of observations also increase as n grows. In addition, the condition of stationarity or ergodicity is required. The rapidly convergence rate for lag is defined by $n * lag_n^k \to 0,\ k \geq 2$. Otherwise, for high values of k is considered the slowly convergence of the lag between observations.

There are many schemes that can be applied such as least square approach, maximum likelihood estimation, adapative Bayes estimation. But in general each model will require an adaptation to the specific situation of at least one of the elements: drifts parameters, diffusion parameters, high dimension of parameter space, non-ergodic case, non stationary case, etc. For high dimension of the parameter space, the combinations estimating functions is applied. Here we apply the combinations of estimating schemes proposed in [1] to estimate the parameter of the temperature model as governed by Ornstein-Uhlenbeck process. Three schemes were used to estimate the parameters of the model, namely: method of least square to estimate parameter of the mean function, quadratic variation to estimating the volatility and martingale estimation functions to estimate the mean reverting parameter. On the last method the estimator is obtained as an adaptation of the estimator obtained by replacing the Lebesgue and Itô integral on the continuous time likelihood functions by Riemann-Itô sums, for the cases when the time between observations is bounded away from zero. Since the estimators based only on Riemann-Itô approach, works well when the observation time are closely spaced as stated in [6]. They also proposed and proved the efficiency of the adapted estimators that are improved by constructing a martingale estimating function from the Riemann-Itô approximation of the likelihood function. The estimation were performed with software *R*.

Drift parameters

The drift parameters m, α and ν in the mean function $\theta(t)$ (3.8) were estimated by the least-square method, applying numerical schemes the Gauss-Newton algorithm. For referred data and considering one sine terms we got the values of parameter $\hat{m} = 74.6736mm$, $\hat{\alpha} = 19.6698$ and $\hat{\nu} = 6.4262$ by minimising

$$\min_{\cdot,m,\alpha,\nu} \|\Theta(\cdot, m, \alpha, \nu) - D(\cdot)\| \tag{3.7}$$

with

$$\Theta(\cdot, m, \alpha, \nu) = m + \alpha \sin\left(\frac{\pi(t-\nu)}{6}\right) \tag{3.8}$$

and

$$D(j) = \frac{1}{N}\sum_{i=1}^{N} R(j,i), \tag{3.9}$$

$N = 71$ is a number of periods (years) to be considered and is given by $N = T - t_0 + 1 = 2017 - 1947 + 1$. The quantity $R(j,i)$ means quantity of rainfall in month j and in year i when $j = 1,\ldots,12$.

Thus if the corresponding estimated value for θ is found to be $\hat{\theta} = 74.67mm$, it means that there was an average precipitation of $74.67mm$ per m^2 at the weather station Schleswig during the years 1947-2017.

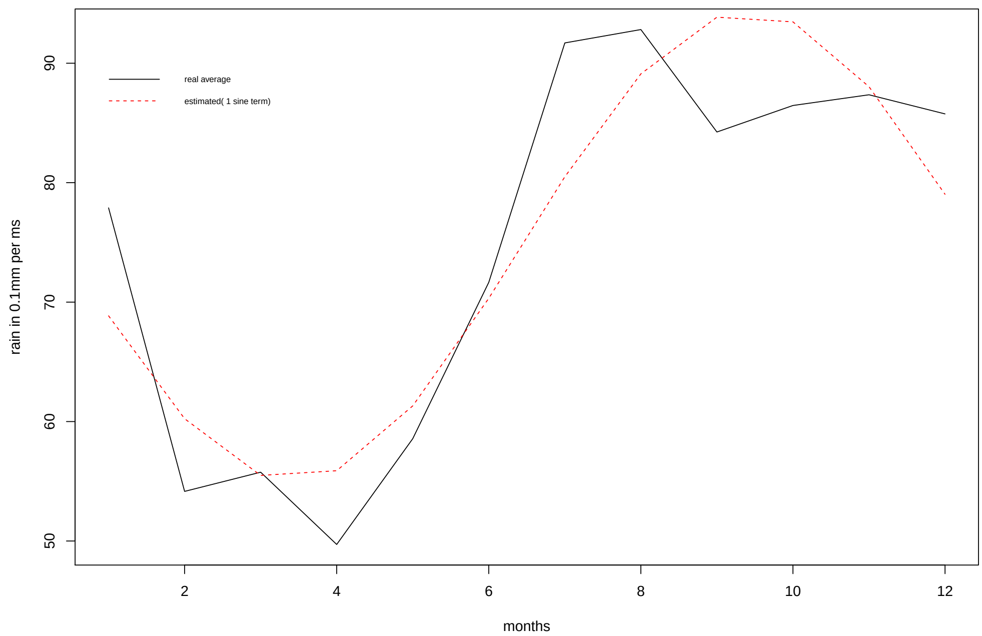

FIGURE 3.4: Monthly average vs $\theta(t)$: Schleswig, data series (1947-2017)

The mean function with one sine terms does not cover all monthly average curve, the approximation become more closely to the real average if we consider more than one sine terms, **Figure** 3.5.

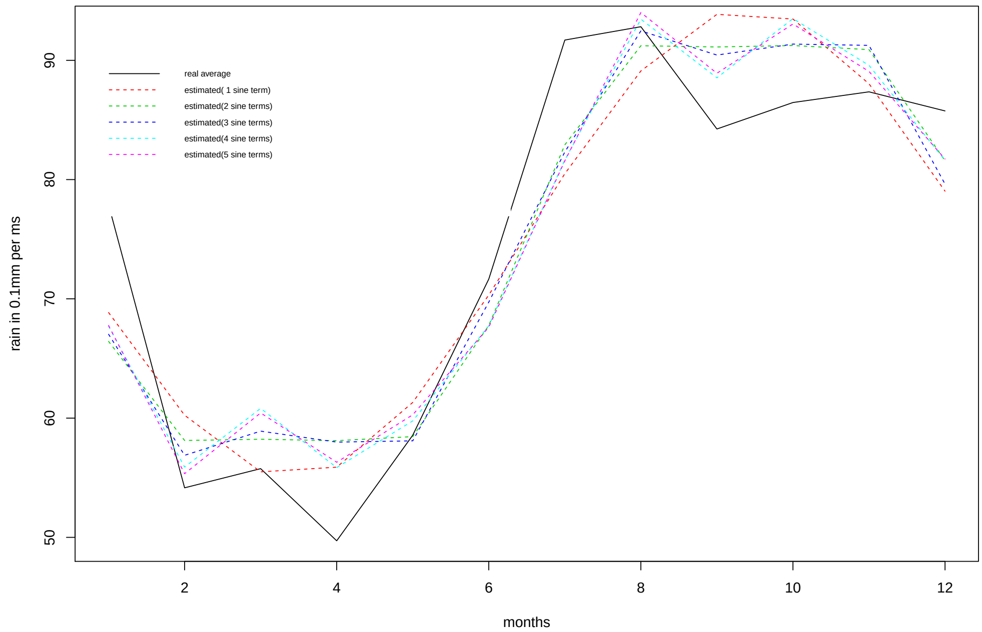

FIGURE 3.5: Monthly average vs $\theta(t)$: Schleswig, data series (1947-2017)-Aproximation with 1,2,3,4 and 5 sine terms

Volatility

To estimate the volatility σ we follow [1]. They analysed two estimators, namely the quadratic variation proposed in [5] and the second derived by discretizing the process proposed in [10], where they found that the use of second estimator can induce an error in the price of the derivative because it under-estimate the value of the mean-reversion parameter k. The estimator based on quadratic variation of X_t is given by

$$\hat{\sigma}_\mu^2 = \frac{1}{N_\mu} \sum_{j=0}^{N_\mu - 1} (X_{j+1} - X_j)^2. \tag{3.10}$$

The volatility σ_t is considered constant over the year. The output result of the volatility based in the quadratic variation is 54.8156.

Mean-reversion Parameter

A martingale estimation functions method was applied in [1] to estimate k. Justified by the fact that the lag between observation is not close to zero (one day). Based in collated data over n days, the efficient estimator $\hat{k}_n$ for k is obtained as a root of the equation

$$G_n(\hat{k}_n) = 0 \tag{3.11}$$

where

$$G_n(\hat{k}_n) = \sum_{i=1}^{n} \frac{\dot{b}(X_{i-1},k)}{\sigma_{i-1}^2}\{X_i - E[X_i|X_{i-1}]\} \tag{3.12}$$

and $\dot{b}(X_i,k)$ is the derivative of the drift term of the process (3.3), written on k and given by

$$\dot{b}(X_i,k) = \theta(t) - X_t. \tag{3.13}$$

Note that $\theta(t)$ is the mean function given by

$$\theta = m + \alpha \sin\left(\frac{\pi(t-\nu)}{6}\right).$$

By solving the equation, the unique estimator (unique zero of equation) is given by

$$\hat{k}_n = -log\left[\frac{\sum_{i=1}^{n} Y_{i-1}(X_i - \theta(i))}{\sum_{i=1}^{n} Y_{i-1}(X_{i-1} - \theta(i-1))}\right] \tag{3.14}$$

with

$$Y_{i-1} \equiv \frac{\theta(i-1) - X_{i-1}}{\sigma_{i-1}^2},\ i = 1,2,\ldots,n. \tag{3.15}$$

Based on the discretized score function

$$\dot{\tilde{l}}_n(k) = \sum_{i=1}^{n} \frac{\dot{b}(X_{i-1},k)}{\sigma_{i-1}^2}(X_i - X_{i-1}) - \sum_{i=1}^{n} \frac{b(X_{i-1};k')\dot{b}(X_{i-1};k)}{\sigma_{i-1}^2}, \tag{3.16}$$

from where we have the equation $\dot{\tilde{l}}_n(k) = 0$ given by

$$\sum_{i=1}^{n} Y_{i-1}(X_i - X_{i-1}) - \sum_{i=1}^{n} Y_{i-1}(\theta'(i-1) + k'(\theta(i-1)) - X_{i-1}) = 0. \tag{3.17}$$

The estimator of k (which is unique solution of (3.17)), is

$$\hat{k}_n = \frac{\sum_{i=1}^{n} Y_{i-1}[X_i - X_{i-1} - \frac{\pi}{6}\alpha\cos(\frac{\pi(t-\nu)}{6})]}{\sum_{i=1}^{n} Y_{i-1}[\theta(i-1) - X_{i-1}]} \tag{3.18}$$

where Y_{i-1} is defined in (3.15). Next we calculate the value of the mean reverting parameter applying the quadratic variation approach, which is found to be $\hat{k} = 3.3759$.

Simulated tragectory of Ornstein-Uhlenbeck process

The **figure** 3.6 represent one simulated trajectory of Ornstein-Uhlenbeck process (3.3). We use the public rainfall data from German weather service, www.dwd.de, Schleswig weather station, between the years 1947 and 2017.

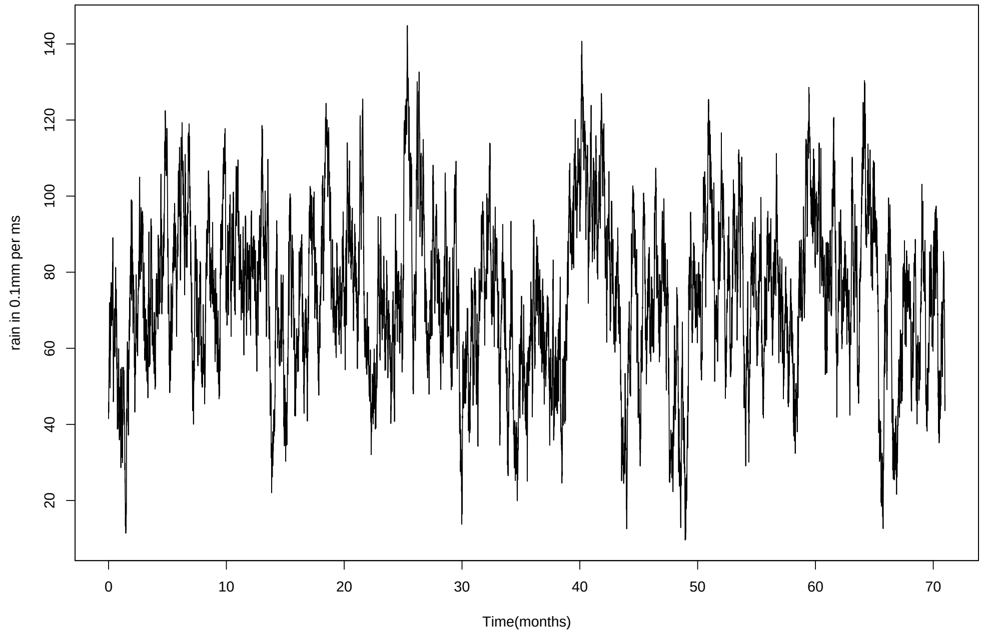

FIGURE 3.6: One trajectory of the Ornstein-Uhlenbeck process that will be used to model the monthly rain

The visual inspection suggest similar appearance to the original monthly rainfall data presented in **Figure** 3.1. Although the performance of the parameters can be improved by incorporating on the mean function (3.8) more than one sine terms as proposed in [19].

3.2.3 The PDE of the Ornstein-Uhlenbeck process

First we apply the Girsanov's theorem to change the measure. Under the risk neutral measure Q characterized by the market price of risk λ, we define

$$W_t^Q = W_t + \lambda t, \tag{3.19}$$

we have that $dW_t^Q = dW_t + \lambda dt$, so that $dW_t = dW_t^Q - \lambda dt$ and we replace it into the equation (3.3) and we get

$$dX_t = [k(\theta(t) - X_t) + \theta'(t) - \lambda\sigma_t]dt + \sigma_t dW_t^Q. \tag{3.20}$$

where dW_t^Q is the Brownian increment under risk neutral measure Q, see for example [47], [48], [36]. By the Girsanov's theorem W_t^Q is a standard Brownian motion under the equivalent probability measure Q and the solution of (3.20) is martingale under Q.

Weather derivative incomes depend on the evolution of an underlying meteorological index [1], [22]. Mostly the rainfall is modeled through an cumulative of daily average amount of rainfall [19]. But, was shown for example in [41] that the indexes constructed by the principle similar to that of degree-days indexes used to modeling temperature derivatives, gives higher hedging effectiveness. Hence, we present two ways of modeling rainfall derivatives contracts similar to the ones used to model temperature derivatives by degree-days indexes. Bellow we present an adaptation from [1] of the two alternatives.

Definition 3.2.1
We define *rainfall defice day and denote RDD* as the number of millimeters by which the daily average rainfall X_t is below the base rainfall X_{ref} i.e:

$$f(X_t, t) = (X_{ref} - X_t)^+. \tag{3.21}$$

Definition 3.2.2
We define *rainfall excess day and denote RED* as the number of millimeters by which the daily average rainfall X_t is above the base rainfall X_{ref} i.e:

$$f(X_t, t) = (X_t - X_{ref})^+. \tag{3.22}$$

The "RDD" can be thought in terms of necessity of water in non raining periods whereas, "RED" can be thought in terms of existence of more water than required in raining periods. Hence, an investor wants to protect himself against higher levels of rain, he can take position on RED contracts and the payment has a payoff defined according to the equation (3.22). On the other hand if the protection is against lower levels of rain he can take position on RDD contracts and contract will pays according to the equation (3.21). For the strategy proposed in [21] with the expected amount of rainfall per year between $200mm$ and $800mm$, the investors must take position on both RDD and RED contracts, considering the annual base raining of $200mm$ for RDD contracts and the annual base raining of $800mm$ for RED contracts.

The cumulative index Y_t, of the underlying weather variable can be described by the following equation (see for example [36])

$$dY_t = f(X_t, t)dt. \tag{3.23}$$

In this case the indexes Y_t represents the amount of rainfall over all period $t \in [0, T]$. The index is a quantity of RDD or RED over all considered period. Additionally we consider $t \geq 0$, $k \geq 0$ and the initial condition $X_0 = x_0$.

Using financial theory from the rainfall model, one can measure the amount of the rainfall in certain weather station and period, depending on the contract (RDD or RED) one can use equation (3.21) or (3.22) as the payoff functions.

Under risk neutral probability measure, a contingent claim for example an option price $V(X_t, Y_t, t)$ can be given by (3.1), the discounting conditionally expected payoff at maturity. By the Feynman-Kac theorem the value of the weather option $V(X_t, Y_t, t)$ is also a unique solution of the following bivariate PDE,

$$\frac{\partial V}{\partial t} = rV - f(x,t)\frac{\partial V}{\partial y} - \gamma(x,t)\frac{\partial V}{\partial x} - \frac{1}{2}\sigma_t^2\frac{\partial^2 V}{\partial x^2} \tag{3.24}$$

where $\gamma(x,t) = k(\theta(t) - X_t) + \theta'(t) - \lambda\sigma_t$, and the terminal conditions depends on derivatives to be analyzed. The payoff is defined as $f(x,t) = f(x)$.

When the underlying variable follow Ornstein-Uhlenbeck process, if one consider some weather indexes, the prices for weather derivatives can be governed by a convection-diffusion equation (3.24), that belongs to the wider class of Kolmogorov backward equations. The diffusion effects are much smaller than the convection effects. Pirrong and Jermakyan in [47] suggested a method based on PDEs to price weather derivatives. They obtained the arbitrage-free prices of weather options by inducing the market prices of risk from the quotations of the weather futures, considering the liquidity of the weather options market.

In practise the weather option contract does not have a negotiable underlying index, and the model is still far from the reality. For practical uses, improvements of weather derivatives pricing by PDEs can be found for example in [11]. Assuming mean-self financing portfolio and partial hedging he derived a PDE introducing a hedging instrument H that is imperfectly correlated with the underlying index. Another improvement of (3.24), can be made if one consider a stochastic volatility, which will allow to compute the market prices of risk instead of extracting them from quotations. But as mentioned in [22], it is still necessary to have available quotations of the weather contracts in order to extract a risk-neutral distribution. Both risk-neutral distribution and market prices of risk requires the liquidity of the quoted weather contracts.

The rain risk can be managed buying RDD or RED (American, Asian or European) options, taking short or long positions. The limit on the financial gains or losses are defined by the following terminal condition:

- for an RDD European put

$$V(x,y,t) = tick \times (S - y_T)^+, \tag{3.25}$$

- for an RDD European call

$$V(x, y, t) = tick \times (y_T - S)^+, \tag{3.26}$$

where y_T is the value of RDD or RED index at maturity, S is strike level (that is defined at time t) and "*tick*" is used to convert the quantity $(y_T - K)^+$ into monetary terms, see for example [47], [48], [11], [51].

Chapter 4

Lie symmetry analysis

Generally to solve differential equation we apply some special methods such as, separation of variables, homogeneous or exact equations. These methods are special cases of the general integration procedure based on the invariance of the differential equation under a continuous group of symmetries developed by Sophus Lie (1842-1899) and now universally known as Lie Groups. In his paper, "*On integration of a class of linear partial differential equations by means of definite integrals* " [37], Lie identified a set of equations that could be integrated or reduced to a lower-order equations by group theoretic algorithms and proposed the group classification of the linear seconder-order partial differential equation with two independent variables. Besides the PDEs, the Lie symmetry group theory can be applied in other fields such as algebraic topology, differential geometry, invariant theory, bifurcation theory, special functions, numerical analysis, control theory, classical mechanics, quantum mechanics, relativity, continuum mechanics [44].

The symmetry group of a system of differential equations is a group which transforms solutions of the system to other solutions and consist of geometric transformations on the space of independent and dependent variables for the system, and act on solutions by transforming their images. As examples of continuous symmetry groups we can consider the group of translations, dilatation (scaling symmetries) and rotations. Also exist discrete symmetry groups such as, reflections. But the continuous symmetries can take more advantage in the point of view of the computation.

By application of the Lie symmetry analysis the non linear characteristics of invariance of the system under the continuous local group transformation can be replaced by other equivalent and more simpler linear conditions in form of infinitesimal invariance of the system under the group generators [44]. In most cases the infinitesimal symmetries conditions can be solved explicitly in closed form and the symmetry group determined explicitly then, the computer programs play a principal role. Once we have constructed the symmetry group, we can use it to determine new solutions to the system from known ones or to use it to make a group classification i.e., to affect a classification of families of differential equations depending on arbitrary parameters of functions.

On the case of the ordinary differential equations, the invariance under a one-particlar symmetry group allow to reduce the order of the equation by one, and the solution of the original equation can be found from those of the reduced equations by single quadrature. If the equations is first order differential equation, the reduction gives a general solution. But in general case i.e., on multi-parameter symmetries groups sucessives reductions must be made and, in additional the conditions of solvability have to be satiesfied in order to be able to reduce the original equation by quadrature. In other hand, for partial differential equation, we can not determine the general solution, but we can use the general group symmetry to find special types of solutions which are invariant under some subgroup of the symmetry group of the system of PDEs.

We provide a summary of concepts and main theorems of Lie symmetry theory for PDEs. We gives the general description of the Lie symmetry method for computation of the infinitesimal symmetries and the invariant solutions of the PDEs. Furthermore, one can refer for more details to [45], [30], [27], [44] and [43].

4.1 Lie groups

Lie groups are continuous transformations groups and represents a subject in which the algebraic groups and topological structures are combined. The differential equations are regarded as a surface in the space of independent and dependent variables together with the derivatives involved, so that its Lie group consist with the geometric transformations which transform their solutions. Generally the differential equations are given in Banach spaces and the group actions are those formulated in the point of view of local Lie groups. In this section we give an overview of concepts of Lie groups and some examples, most of them from [44].

Definition 4.1.1 (of group)
A *group* is a set G together with a group operation, usually called multiplication, such that for any three elements g, h and k from G, the following axioms are satisfied

1. *Closure:* the product $g \cdot h$ is again an element of G
2. *Associativity:* $g \cdot (h \cdot k) = (g \cdot h) \cdot k$,
3. *Identity element:* there is a distinguished element $\mathbf{e} \in G$, called the identity element, which has the property that $\mathbf{e} \cdot g = g = g \cdot \mathbf{e}$, $\forall g \in G$,

4. *Inverses:* for each $g \in G$ there is an inverse, denoted g^{-1} with the property $g \cdot g^{-1} = \mathbf{e} = g^{-1} \cdot g$

If in addition the condition $g \cdot h = h \cdot g$, the group is called *abelian*.

Example 4
The usual sets $\mathbb{Z}$ and $\mathbb{R}$ of respectively integers and real numbers. In both sets the group operation being addition. The addition in integers or in real numbers is closed and associative, in both sets the identity is 0 and the inverse of an integer or real number x is $-x$. Both groups are abelian, since the addition of real numbers is commutative.

Example 5
Consider $G = GL(n, \mathbb{Q})$ called general linear set of $n \times n$ invertible matrices with rational numbers on entries or $G = GL(n, \mathbb{R})$ called general linear set of $n \times n$ invertible matrices with real numbers on entries. In both sets the group operation is multiplication. The multiplication of matrices in $\mathbb{Q}$ or $\mathbb{R}$ is closed and associative. In both cases the identity element is identity matrix I and the inverse of a matrix A, denoted A^{-1} is the ordinary matrix inverse, which has respectively the rational or real number on entries.

In the second case on both examples, the elements of the group can change continuously as structure of an smooth manifold. This groups are those called Lie group and formally defined bellow.

Definition 4.1.2 (r-parameter Lie group)
An r-parameter Lie group is a group G which also carries the structure of an r-dimensional smooth manifold in such a way that both

- the group operation: $m : G \times G \to G,\ m(g,h) = g \cdot h,\ \ \forall g,h \in G,$
- and the inversion: $i : G \to G,\ i(g) = g^{-1},\ g \in G,$

are smooth maps between manifolds.

Example 6
On the second case of Example 4 we have, $G = \mathbb{R}$ and the group operation is addition $(x,y) \mapsto x+y$. The inverse of x, $-x$. Both operation are smooth, so $\mathbb{R}$ is one-parameter abelian Lie group. Generally all set of the form $G = \mathbb{R}^r$, where $\mathbb{R}$ is a set of real numbers can be used as example of $r-$ parameter abelian Lie group.

Example 7
A set $G = SO(2)$ of rotations in the plane

$$\Gamma_a : \bar{x} = x\cos a - u\sin a,\ \bar{u} = u\cos a + x\sin a,\ -\pi < a < \pi$$

a denoting the rotation angle, is a one-parameter group and can be represented as unit circle

$$S^1 = \{(\cos a, \sin a) : 0 \leq a \leq 2\pi\},$$

which is a subset of $\mathbb{R}^2$ and the parameter is a. The group $SO(2)$ is a special group of 2×2 orthogonal matrix A, called special orthogonal group $O(2)$, with additional property that $det A = 1$. Besides the rotations, the group $O(2)$ contains also the reflections and it has the manifold structure of two disconnected copies of S^1. Generally the special orthogonal groups for $n \times n$ matrices A is represented by

$$SO(n) = \{A \in O(n) : det A = 1\}, \tag{4.1}$$

where $O(n)$ is a group of orthogonal $n \times n$ matrices A, defined by

$$O(n) = \{A \in GL(n) : A^T A = I\}. \tag{4.2}$$

The general linear set $GL(n)$ is a subset of the set of all $n \times n$ dimensional matrices $M_{n\times n}$, isomorphic to $\mathbb{R}^{n^2}$, and is also n^2-dimensional manifold. The set $SO(n)$ and $O(n)$ are $\frac{1}{2}n(n-1)$-parameter Lie group, since it is defined by n^2 equations $A^T A - I = 0$ involving the elements of A, where precisely $\frac{1}{2}n(n+1)$ of these equations, corresponding to the matrix elements on or above the diagonal are independent and satisfy the condition of maximal rank [43].

Definition 4.1.3 (Lie group homomorphism)
A Lie group homomorphism is a smoth map $\phi : G \to H$ between two Lie groups, G and H, which respects the group operations:

$$\phi(g \cdot \tilde{g}) = \phi(g) \cdot \phi(\tilde{g}),\ g, \tilde{g} \in G. \tag{4.3}$$

Note that if ϕ is smooth inverse, the Lie group homomorphism determines an isomorphism between the Lie groups G and H.

Example 8
Consider that $G = \mathbb{R}$ and $H = \mathbb{R}^+$. The set H with the group operation being the usual multiplication is a Lie group and is isomorphic to the Lie group G with group operation being the usual addition. The Lie group homomorphism which is also an

isomorphism between $\mathbb{R}$ and $\mathbb{R}^+$ is the exponential function $\phi : \mathbb{R} \to \mathbb{R}^+,\ \phi(t) = e^t$.

Generally in the theory of Lie groups we works with groups that are subgroup of others Lies groups. As an example, the special orthogonal groups, $SO(n)$, are subgroups of orthogonal groups $O(n)$ and these subgroups of general linear groups $GL(n)$. Since our interest is to study the Lie groups, we will consider only the subgroups of Lie groups in the sense of **Definition** 4.1.3, as defined below.

Definition 4.1.4 (of Lie subgroup)
A Lie subgroup H of a Lie group G is given by a submanifold $\phi : \tilde{H} \to G$, where $\tilde{H}$ itself is a Lie group, $H = \phi(\tilde{H})$ is the image of ϕ which is a Lie group homomorphism.

Example 9
For $w \in \mathbb{R}$, the submanifold

$$H_2 = \{(t, 2t) mod 2\pi : t \in \mathbb{R}\},$$

is a one-parameter Lie subgroup of the torus T^2 since is isomorphic to the special orthogonal group $SO(2)$. The elements of H_2 are classes of equivalence which are circles so that, two elements, we say $(t, 2t)$ and $(t', 2t')$, belong to the same class if the remainder of the quotients $\frac{t}{2\pi}$ and $\frac{t'}{2\pi}$ is the same.

4.1.1 Local Lie Groups

As we can see from the above examples, the Lie groups can be connected or not connected. The set of orthogonal $n \times n$ matrices $O(n)$ is example of an not connected Lie group since it contains the discrete transformation such as reflections. In the other hand the special orthogonal groups $SO(n)$ are connected Lie groups. These Lie groups only contains continuous transformations which are rotations. Both the connected and the disconnected groups are important on the study of differential equations but the theory of local Lie group based on infinitesimal transformations is only relevant for connected groups.

Definition 4.1.5 (of r-parameter local Lie group)
An r-parameter local Lie group consists of connected open subsets $V_0 \subset V \subset \mathbb{R}$ containing the origin 0, and smooth maps

$$m : V \times V \to \mathbb{R}^r$$

defining the group operation, and

$$i : V_0 \to V$$

defining the group inversion, with the following properties

- Associativity: If $x, y, z \in V$, and also $m(x, y)$ and $m(y, z) \in V$, them $m(x, m(y, z)) = m(m(x, y), z)$
- Identity element: For all $x \in V$, $m(0, x) = x = m(x, 0)$
- Inverses: For all $x \in V_0$, $m(x, i(x)) = m(i(x), x)$.

In contrast with the usual group axioms, the rules in **Definition** 4.1.5, are not necessarily defined everywhere, only make sense for x and y sufficiently near to 0, since the identity element of the group is the origin 0. The definition is in term of local coordinates and only the group elements close to the identity element are considered.

Example 10
As an example of local one-parameter group, we consider the subset of $\mathbb{R}$, $V = \{x : |x| < 1\}$ with group multiplication,

$$m(x, y) = \frac{2xy - x - y}{xy - 1}, \quad x, y \in V.$$

If $m(y, z) = \frac{2yz - y - z}{yz - 1}$, then

- Associativity: $m(x, m(y, z)) = m(m(x, y), z)$,

$$\begin{aligned} m(x, m(y, z)) &= \frac{2x \cdot m(y, z) - x - m(y, z)}{x \cdot m(y, z) - 1} = \frac{2x \frac{2yz - y - z}{yz - 1} - x - \frac{2yz - y - z}{yz - 1}}{x \frac{2yz - y - z}{yz - 1} - 1} \\ &= \frac{3xyz - 2xy - 2xz + x - 2yz + y + z}{2xyz - xy - xz - yz + 1}, \\ m(m(x, y), z) &= \frac{2m(x, y) \cdot z - m(x, y) - z}{m(x, y) \cdot z - 1} = \frac{2 \frac{2xy - x - y}{xy - 1} z - \frac{2xy - x - y}{xy - 1} - z}{\frac{2xy - x - y}{xy - 1} z - 1} \\ &= \frac{3zxy - 2zx - 2zy - 2xy + x + y + z}{2zxy - zx - zy - xy + 1} \end{aligned}$$

  As we can see, $m(x, m(y, z)) = m(m(x, y), z)$.

- Identity: 0 is identity element

$$m(x, 0) = m(0, x) = \frac{-x}{-1} = x$$

- Inverse: $i(x) = \frac{x}{2x-1}$, $x \in V_0 = \{x : |x| < \frac{1}{2}\}$ is inverse element, so that,

$$m(x, i(x)) = m(i(x), x) = \frac{x\frac{x}{2x-1} - x - \frac{x}{2x-1}}{x\frac{x}{2x-1} - 1} = \frac{2x^2 - 2x^2 + x - x}{x^2 - 2x + 1} = 0.$$

Most Lie group G arise as group of transformations of some manifold M, if for each group element $g \in G$ there is associative map from M to itself. As example, the special orthogonal group $SO(2)$ arises as the group of rotations in the plane $M = \mathbb{R}^2$. The general lineal group $GL(n)$ appears as the the group of invertible linear transformation on $\mathbb{R}^n$.

Definition 4.1.6 (of transformation of the set)
Let M be a set. A one to one mapping of the set M onto M, $\tau(M)$, is called a transformation of the set M.

As we will see in the examples of the next definition, the totality of such transformations $\tau(M)$, of the set M can form a group, where the group operation is the composition of mappings. The identity transformation is denoted by $\mathbf{e}_M$. In that case, the group τ is called a transformation group of the set M.

In order to present a concept of one-parameter local Lie group of transformations, first we recall the representation of an smooth curve on manifold by their parametric form. Suppose that $\mathcal{C}$ is a smooth curve on a manifold M, parametrized by $\phi : I \subset \mathbb{R} \to M$. In local coordinates $x = (x^1, \ldots, x^m)$, the curve $\mathcal{C}$ is given by m smooth functions

$$\phi(a) = (\phi^1(a), \ldots, \phi^m(a))$$

where $a \in I$ is the parameter.

Definition 4.1.7 (of one-parameter local Lie group of transformations)
A set G of transformations

$$\Gamma_a : \bar{x}^i = \phi^i(x, a), \; \alpha = 1, \ldots, m, \tag{4.4}$$

where a is a real parameter which continuously ranges in values from a neighbourhood $\mathbb{D} \subset \mathbb{R}$ of $a = 0$ and ϕ^i are differentiable functions, is continuous one-paremeter local Lie group of transformations in $\mathbb{R}^m$ if it satisfy the properties of group, namely

- Closure: If Γ_a, $\Gamma_b \in G$ and $a, b \in \mathbb{D}' \subset \mathbb{D}$, then $\Gamma_b\Gamma_a = \Gamma_c \in G$, $c = \varphi(a, b) \in \mathbb{D}$,
- Associativity: $(\Gamma_a\Gamma_b)\Gamma_c = \Gamma_a(\Gamma_b\Gamma_c)$,

- Identity: There exists element $\Gamma_0 \in G$ such that $\Gamma_0\Gamma_a = \Gamma_a\Gamma_0 = \Gamma_a$ for any $a \in \mathbb{D}' \subset \mathbb{D}$ and $\Gamma \in G$,
- Inverse: For any $\Gamma_a \in G$, $a \in \mathbb{D}' \subset \mathbb{D}$, there exists $\Gamma_a^{-1} = \Gamma_{a^{-1}} \in G$, $a^{-1} \in \mathbb{D}$ such that $\Gamma_a\Gamma_{a^{-1}} = \Gamma_0 = \Gamma_{a^{-1}}\Gamma_a$.

The concept of the one-parameter local Lie group of transformation will be the principal key on the symmetries of PDEs.

Example 11
A set G of transformations

$$\Gamma_a : \bar{x} = x + a,\ \bar{u} = e^{ka}u,\ k = constante,\ a \in \mathbb{R}$$

forms a local group, since satisfy the property of group namely:

- Closure: $\Gamma_b\Gamma_a$:

  $$\bar{\bar{x}} = \bar{x} + b = x + a + b = x + c,\ \bar{\bar{u}} = \bar{u}e^{kb} = ue^{k(a+b)} = ue^{kc},$$

  since $c = a + b \in \mathbb{R}$.
- Associativity: $(\Gamma_b\Gamma_a)\Gamma_c$:

  $$\begin{aligned} \bar{\bar{\bar{x}}} &= (x + a + b) + c = b + (x + a + c), \\ \bar{\bar{\bar{u}}} &= (ue^{k(a+b)})e^{kc} = ue^{k(a+b)+kc} = ue^{kb+k(a+c)} = e^{kb}(ue^{k(a+c)}) \end{aligned}$$

  the last members correspond to $\Gamma_b(\Gamma_a\Gamma_c)$.
- Identity: $\Gamma_0\Gamma_a = \Gamma_a\Gamma_0 = \Gamma_a$:

  $$\bar{\bar{x}} = \bar{x} + a = x + a + 0 = x + a,\ \ \bar{\bar{u}} = \bar{u}e^{ka} = ue^{k(a+0)} = ue^{ka}$$

- Inverse: $\Gamma_{a^{-1}} = \Gamma_{-a}$, so that, $\Gamma_{-a}\Gamma_a = \Gamma_a\Gamma_{-a} = \Gamma_0$:

  $$\bar{\bar{x}} = \bar{x} - a = x + a - a = x\ \bar{\bar{u}} = \bar{u}e^{-ka} = ue^{k(a-a)} = ue^{k0} \text{ that is } \Gamma_0$$

Example 12
A set G of transformations

$$\Gamma_a : \bar{x} = x + a,\ \bar{u} = u + a^2,\ a \in \mathbb{R}$$

Does not form a local group, since the property of closure is not satisfied:

$$\Gamma_b\Gamma_a : \bar{\bar{x}} = \bar{x} + b = x + a + b = x + c,\ \bar{\bar{u}} = \bar{u} + b^2 = u + a^2 + b^2 = u + c^2$$

to satisfy the closure property $c = a + b = \sqrt{a^2 + b^2}$, which is not necessary true for all $a, b \in \mathbb{R}$.

For an given manifold M, our goal is to construct a group of local Lie group of transformations i.e., the transformation that map M to itself. The first step is to construct the one-parameter local Lie group of transformation and, if the given manifold is $r-$ dimensional, the one-parameter local Lie group is used to find the $r-$ prolonged local Lie group of transformation. Since we are looking for local Lie group of transformation, the transformation are approximated by the infinitesimal transformation using Taylor's series for integral curves generated by the vector field on each point x in manifold M. As said at preview sections, only the local Lie groups homomorphism are considered.

If we consider the curve in a parametrized form, the tangent vector of the smooth curve $\mathcal{C}$ given in local coordinates $x = (x^1, \ldots, x^m)$ at each point $x = \phi(a)$ is given by the derivatives

$$v|_x = \phi'(a) = \phi_a^1(a)\frac{\partial}{\partial x^1} + \phi_a^2(a)\frac{\partial}{\partial x^2} + \cdots + \phi_a^m(a)\frac{\partial}{\partial x^m} \tag{4.5}$$

Example 13
Consider in $\mathbb{R}^3$ a curve parametrized by $\phi(a) = (\sin a, a^2, a)$. At each point $(x, y, z) = \phi(a)$, the tangent vector will be

$$v|_x = \cos a\frac{\partial}{\partial x} + 2a\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

If $x = \phi(a) = (\phi^1(a), \ldots, \phi^m(a))$ is smooth curve expressed in term of local coordinates $x = (x^1, \ldots, x^m)$ and $y = \Psi(\phi(a))$ is correspondent curve in term of local coordinate y, then the tangent vector $V|_x$ in $y-$ coordinates is given by

$$v|_{y=\Psi(x)} = \sum_{j=1}^{m}\frac{d}{da}\Psi^j(\phi(a))\frac{\partial}{\partial y^j} = \sum_{j=1}^{m}\sum_{k=1}^{m}\frac{\partial\Psi^j}{\partial x^k}(\phi(a))\frac{d\phi^k}{da}\frac{\partial}{\partial y^j}. \tag{4.6}$$

Consider the tangent space $TM|_x$ to M at x, which is defined as the collection of all tangent vectors to all possible curves passing through a given point $x \in M$. Each smooth tangent vector $v|_x \in TM|_x$ is assigned to each point $x \in M$ by the vector field v in M. In local coordinates $(x^1, \dots, x^m)$, a vector field has the form

$$v|_x = \left(\xi^1(x)\frac{\partial}{\partial x^1} + \xi^2(x)\frac{\partial}{\partial x^2} + \cdots + \xi^m(x)\frac{\partial}{\partial x^m}\right)|_x, \tag{4.7}$$

where each $\xi^i(x),\ i = 1, \dots, m$ are smooth function of x.

Now consider the integral curve of a vector field v, the smooth parametric curve $x = \phi(a)$(a is a parameter) whose tangent vector at any point coincides with the values of v at the same point $\phi'(a) = v|_{\phi(a)}$ for all a. In local coordinates, the integral curve $x = \phi(a)$ is a solution of linear system of ordinary differential equations

$$\frac{dx^i}{da}\xi^i(a),\ \ i = 1, \dots, m \tag{4.8}$$

where $\xi^i(a)$ are the coefficients of vector field v at x. If we consider the initial values

$$\phi(0) = x_0, \tag{4.9}$$

the problem of existence and uniqueness of solution for system (4.8-4.9) is guarantied by the standard theorems. The parametrized maximal integral curve passing through $x \in M$, a curve not containing any longer other integral curve $\Psi(a, x)$ is the same as a one-parameter local Lie group of transformations [44]. The vector v is called the infinitesimal generator of the action. The basic idea is that, in local coordinates the parametrized maximal integral curve can be approximated by the Taylor's series. The expressions obtained by the approximation are called infinitesimal transformation and formally defined bellow.

Definition 4.1.8 (of infinitesimal transformation)
The infinitesimal transformation is called the approximation by the Taylor series expansion in a of the transformation (4.4) about $a = 0$ taking into account the initial conditions $\phi^i|_{a=0} = x^i$,

$$\bar{x}^i \approx x^i + a\xi^i(x) \tag{4.10}$$

where,

$$\xi^i(x) = \frac{\partial \phi^i(x, a)}{\partial a}\ |_{a=0} \tag{4.11}$$

components of the vector field $v = (\frac{\partial \phi^1(x,a)}{\partial a}, \dots, \frac{\partial \phi^m(x,a)}{\partial a})$ at local coordinates $(x^1, \dots, x^m)$.

There is one to one correspondence between local one parameter groups of transformations and their infinitesimals generators [44]. The computation of this one-parameter group is often referred to, as exponentiation of the vector field and is denoted by

$$e^{(av)}x \equiv \Psi(a,x) \tag{4.12}$$

With vectors fields we can realize some specific operations and the most important is the Lie bracket or commutator.

Definition 4.1.9 (of Lie brackets)
If v and w are vector fields on manifold M, then their Lie bracket , denoted $[v,w]$, is the unique vector field satisfying

$$[v,w](f) = v(w(f)) - w(v(f)) \tag{4.13}$$

for all smooth functions $f : M \to \mathbb{R}$.

Easily we can verify that $[v,w]$ is in reality a vector field since for

$$v = \sum_{i=1}^{m} \xi^i(x)\frac{\partial}{\partial x^i} \text{ and } w = \sum_{i=1}^{m} \eta^i(x)\frac{\partial}{\partial x^i} \tag{4.14}$$

in local coordinates we have that

$$\begin{aligned}
[v,w] &= vw - wv = \\
&= v\left(\sum_{i=1}^{m} \eta^i(x)\frac{\partial}{\partial x^i}\right) - w\left(\sum_{i=1}^{m} \xi^i(x)\frac{\partial}{\partial x^i}\right) = \\
&= \sum_{i=1}^{m} \left(v(\eta^i) - w(\xi^i)\right)\frac{\partial}{\partial x^i} = \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{m} \xi^j(x)\frac{\partial}{\partial x^j}(\eta^i) - \sum_{j=1}^{m} \eta^j(x)\frac{\partial}{\partial x^j}(\xi^i)\right)\frac{\partial}{\partial x^i} = \\
&= \sum_{i=1}^{m}\sum_{j=1}^{m} \left(\xi^j(x)\frac{\partial \eta^i}{\partial x^j} - \eta^j(x)\frac{\partial \xi^i}{\partial x^j}\right)\frac{\partial}{\partial x^i}.
\end{aligned} \tag{4.15}$$

Note that on Lie bracket form, the terms involving the higher derivatives of a function f cancels.The next theorem gives the properties of the Lie bracket.

Theorem 4.1.1
The Lie bracket is characterized by the following properties:

1. Bilinearity

$$\begin{aligned}
[cv + c'v', w] &= c[v, w] + c'[v', w], \\
[v, cw + c'w'] &= c[v, w] + c'[v, w']
\end{aligned}$$

where c, c' are constants.

2. Skew-Symmetry

$$[v, w] = -[w, v]$$

3. Jacobi Identity

$$[u, [v, w]] + [w, [u, v]] + [v, [w, u]] = 0$$

To proof this theorem one can apply the definition of the Lie bracket (4.13) and take in attention the fact that the vector multiplication is not commutative. c, c' are constants so that, we have for the bilinearity property:

$$\begin{aligned}
[cv + c'v', w] &= (cv + c'v')w - w(cv + c'v') = cvw + c'v'w - wcv - wc'v' \\
&= c(vw - wv) + c'(v'w - wv') = c[v, w] + c'[v', w],
\end{aligned}$$

for skew-symmetry:

$$[v, w] = vw - wv = -(wv - vw) = -[w, v],$$

and for Jacob identity:

$$\begin{aligned}
[u, [v, w]] + [w, [u, v]] + [v, [w, u]] &= u[v, w] - [v, w]u + w[u, v] + \\
&\quad -[u, v]w + v[w, u] - [w, u]v = \\
&= u(vw - wv) - (vw - wv)u + \\
&\quad w(uv - vu) - (uv - vu)w + \\
&\quad v(wu - uw) - (wu - uw)v = \\
&= uvw - uvw - uwv + uwv - vwu + \\
&\quad vwu + wvu - wvu + wuv - wuv + \\
&\quad vuw - vuw = 0
\end{aligned}$$

4.1.2 Lie Algebras

Some vector fields on Lie group G can be characterized by their invariance under the group multiplication. A set of all these invariant vector fields (infinitesimal generator) form a finite-dimensional vector space, called the Lie algebra of G. The important realization of the Lie group theory is that almost all information in the group G is contained in its Lie algebra and, the applications to differential equation is linked to the idea that the nonlinear conditions of invariance under a group action can be replaced by relatively simpler linear infinitesimal conditions.

Definition 4.1.10 (of Lie algebra)
The Lie algebra of a Lie group G, usually denoted by the lowercase German letter g, is the vector space of all right-invariant vector fields on G i.e., satisfies the condition

$$dR_g(v|_h) = v|_{R_g(h)} = v|_{hg} \tag{4.16}$$

for all g and h in G. Where R_g is a diffeomorphism, right multiplication map $R_g : G \to G$, for any group element $g \in G$ defined by

$$R_g(h) = h \cdot g \tag{4.17}$$

with inverse $R_{g^{-1}} = (R_g)^{-1}$.

Since $R_g(\mathbf{e}) = g$, the right-invariant vector field is uniquely determined by its value at the identity because

$$V|_g = dR_g(v|_{\mathbf{e}}) \tag{4.18}$$

and the Lie algebra g of G can be identified by the tangent space to G at the identity element

$$g \simeq TG|_{\mathbf{e}}. \tag{4.19}$$

In other hand any tangent vector to G at $\mathbf{e}$ uniquely determines a right-invariant vector field on G by (4.18) as following

$$dR_g(v|_h) = dR_g(R_h(V|_{\mathbf{e}})) = d(R_g \circ R_h)(v|_{\mathbf{e}}) = dR_{gh}(v|_{\mathbf{e}}) = v|_{hg}.$$

The vector space structure of the Lie algebra, also satisfy the properties of the Lie bracket namely, bilinearity, skew-symmetric and Jacob identity. If the vectors fields v and w are right-invariant in G, their Lie bracket $[v, w]$ is also right invariant in G. Since by definition of the Lie brackets of differential map dF, induced by an smooth function $F : M \to N$ from the tangent spaces $TM|_x$ to the $TN|_{F(x)}$

$$dR_g[v, w] = [dR_g(v), dR_g(w)] = [v, w]$$

the Lie algebra can be also defined in term of Lie brackets as following.

Definition 4.1.11 (of the Lie algebra)
A lie algebra is a vector space g together with a bilinear operation

$$[\cdot,\cdot] : g \times g \to g$$

called the Lie bracket for g, satisfying the bilinearity, skew-symmetry and Jocobi Identity conditions.

Example 14
Consider the set of infinitesimals generators

$$\{v_1, v_2, v_3\} = \{\frac{\partial}{\partial x}, \frac{\partial}{\partial u}, e^u \frac{\partial}{\partial x}\}.$$

we have

- $[v_1, v_1] = v_1 v_2 - v_2 v_1 = \frac{\partial}{\partial x}(\frac{\partial}{\partial u}) - \frac{\partial}{\partial u}(\frac{\partial}{\partial x}) = 0 - 0 = 0,$
- $[v_1, v_3] = v_1 v_3 - v_3 v_1 = \frac{\partial}{\partial x}(e^u \frac{\partial}{\partial x}) - e^u \frac{\partial}{\partial x}(\frac{\partial}{\partial x}) = 0 - 0 = 0,$
- $[v_2, v_3] = v_2 v_3 - v_3 v_2 = \frac{\partial}{\partial u}(e^u \frac{\partial}{\partial x}) - e^u \frac{\partial}{\partial x}(\frac{\partial}{\partial u}) = e^u \frac{\partial}{\partial x} - 0 = e^u \frac{\partial}{\partial x} = v_3.$

These vectors span a three dimensional Lie algebra g also denoted L_3.

The following theorems shows the relationship between some subsets of the Lie algebra and the subsets of its corresponding Lie group G. They are important because tells how the Lie sub-algebras are related to the correspond Lie groups through the subgroups.

Theorem 4.1.2 (relation between one-dimensional subspace of Lie group and connected one-parameter subgroup of its Lie group)
Let $v \neq 0$ be a right-invariant vector field on a Lie group G. Then the infinitesimals generators of v

$$g_a = e^{av}\mathbf{e} \equiv e^{av} \tag{4.20}$$

is defined for all $a \in \mathbb{R}$ and forms a one-parameter subgroup of G, with

$$g_{a+b} = g_a \cdot g_b, \;\; g_0 = \mathbf{e} \;\; g_{\mathbf{e}}^{-1} = g_{-a} \tag{4.21}$$

isomorphic to either $\mathbb{R}$ itself or the circle group $SO(2)$. Conversely, any connected one-dimensional subgroup of G is generated by such a right-invariant vector field in the above manner.

The proof of this theorem is made in two steps. To proof the direct assumption, one can consider the definition of F-related vectors fields i.e., two vectors v from M and w from N are F-related if $dF(v|_x) = w|_{F(x)}$ for all $x \in M$. Therefore if v and $w = dF(v)$ are F-related then F maps its integral curves with

$$F(e^{\mathbf{e}}x) = e^{adF(v)}F(x).$$

In addition of the definition of the right-invariance of vector field we will have:

$$\begin{aligned} g_b \cdot g_a &= R_{g_a}(g_b) = R_{g_a}[e^{bv}\mathbf{e}] = e^{b \cdot dR_{g_a}(v)} \cdot R_{g_a}(\mathbf{e}) = e^{bv}g_a = e^{bv}e^{av}\mathbf{e} = \\ &= e^{(b+a)v}\mathbf{e} = g_{b+a} \end{aligned}$$

to show that g_a is at least a local one-parameter subgroup and satisfy (4.21) since that maximal integral curves are globally defined and form a subgroup. The interval of definition of g_a is at least $-\frac{1}{2}a_0 \leq a \leq \frac{1}{2}a_0$, for some $a_0 > 0$ so that for $g_a = g_b$ for some $a \neq b$, g_a has period a_0 ($g_{a+a_0} = g_a$), that imply the isomorphism with $SO2$ (by taking $\theta = 2\frac{\pi a}{a_0}$). Otherwise $g_a \neq g_b$ for all $a \neq b$, g_a is isomorphic to $\mathbb{R}$. The inverse assumption can be demonstrated by considering that the Lie algebra g of G can be identified with the tangent space to G at the identity element g isomorphic to $TG|_{\mathbf{e}}$ and one extended v to a right invariant vector field in all G, where v is nonzero tangent vector to $H \subset G$ at the identity. Since H is a subgroup it follows that $v|_h$ is a tangent to H at any $h \in H$ and therefore H is the integral curve of v passing through the identity $\mathbf{e}$. More details can be found in [44]

Theorem 4.1.3 (relation between Lie subgroup and Lie sub-algebra)
Let G be a Lie group with Lie algebra g. If $H \subset G$ is a Lie subgroup, its Lie algebra is a sub-algebra of g. Conversely, if g is any s-dimensional sub-algebra of g, there is a unique connected s-parameter Lie subgroup H of G with Lie algebra g.

The proof of this theorem can be made first by defining the system of vector fields on G generated by basis of sub-algebra g' for each element of G. Since g' is a sub-algebra, then it is closed under the Lie bracket operation so that g' is completely integrable (also called involutive system of vector fields in G). Its mean that, one can consider a maximal connected submanifold H, passing through identity $\mathbf{e}$, which corresponds to a sub-algebra g'. To show that H is subgroup, we can consider the Lie group hommomorphism ϕ and an element of $\phi(t)$, since the set of subspaces associated with each base of the system of vector fields in G is invariant under the *"left"* translations $(H, \ell_{a^{-1}} \circ \phi)$ is also a maximal submanifold of the set of subspaces associated with each base trough the identity $\mathbf{e}$, and $\ell_{\sigma^{-1}} \circ \phi(H) \subset \phi(H)$, and if $\sigma, \tau \in \phi(H)$ implies that $\sigma^{-1}\tau \in \phi(H)$. We can now conclude that $\phi(H)$ is an abstract subgroup of G. The group structure can be induced in H by the homomorphism $\phi : H \to G$, of abstract groups. By showing the existence of an smooth map $\alpha : H \times H \to H$, one can conclude that (H, ϕ) is a Lie subgroup of G and g' is the Lie algebra of H, and is isomorphic to the sub-algebra g. The uniqueness can be proved by considering another connected Lie group of G, which is also maximal submanifold at identity $\mathbf{e}$, by maximality of H is proved that there is uniquely determined map which is an injective and subjective Lie group homomorphism. This mean that the Lie group homomorphism is also Lie group isomorphism, so that the two connected subgroups are equivalents. More details of this demonstraction can be found in [53] and [44].

An r dimensional Lie algebra g, has a vector space whose elements are infinitesimals generators represented by their bases

$$\{v_1, \ldots, v_r\}. \tag{4.22}$$

The Lie bracket of any two basis vectors must be again in Lie algebra g. Thus there are certain constants $c_{ij}^k,\ i, j, k = 1, \ldots, r$ such that

$$[v_i, v_j] = \sum_{k=1}^{r} c_{ij}^k v_k,\ i, j = 1, \ldots, r. \tag{4.23}$$

The elements $c_{ij}^k,\ i, j, k = 1, \ldots, r$ satisfying the condition (4.23) are called struture of constants of the Lie algebra g.

Since the infinitesimals generators v_i's form a basis of the Lie Algebra g, it can be recovered by their structure of constants(4.23) and the bilinearity of the Lie bracket. The other two conditions of the Lie bracket gives two additional constraints on the structure of constants as bellow

- Skew-symmetry
$$c_{ij}^k = -c_{ij}^k, \tag{4.24}$$

- Jacobi identity
$$\sum_{k=1}^{r} (c_{ij}^k c_{kl}^m + c_{li}^k c_{kj}^m + c_{jl}^k c_{ki}^m) = 0. \tag{4.25}$$

In the other hand, any set of constants c_{ij}^k which satisfy (4.24) and (4.25) are the structure of constants for some Lie algebra g since we can represent it by an different basis with form $v_i' = \sum_j a_{ij} v_j$. The structure of constants will take the form

$$c_{ij}' = \sum_{l,m,n} a_{il} a_{jm} b_{nk} c_{lm}^n \tag{4.26}$$

where (b_{ij}) is the inverse matrix of (a_{ij}). The (4.26) is a condition that two different sets of structure of constants determine the same Lie algebra. By the existence and uniqueness of connected, simply-connected Lie group of some Lie algebra there is a one-to-one correspondence between equivalente classes of structure constants c_{ij}^k satisfying (4.24), (4.25) and connected, simply-connected Lie groups G whose Lie algebras have the given structure constants. As we will see in the next chapter, the entire theory of Lie groups can be reduced to the study of the structure of constants relative to correspondent Lie algebra.

In practise, the convenient way to display the structure of constants of a given Lie algebra is trough an tabular form, called *commutator table*. If the Lie algebra is r-dimensional and $v_1, \dots, v_r$ are their basis then, the commutator table for the Lie algebra will be an $r \times r$ table whose (i, j)-th entry expresses the Lie bracket $[v_i, v_j]$. Since the structure of constants are skew-symmetric, the commutator table will be skew-symmetric and in particular the diagonal entries are all zero. On the commutator table, we can extract easily the structure of constant, since each element c_{ij}^k is the coefficient of infinitesimal generator v_k in the position (i, j).

4.2 Symmetries of Partial Differential Equations

The first step on application of Lie symmetry analysis for differential equation is a construction of the symmetries of differential equation, since we do not known a prior the local one-parameter Lie group of transformation of the differential equation.

Consider a partial differential equation

$$E^\sigma(x, u, u_{(1)}, u_{(2)}, \cdots, u_{(k)}) = 0, \ \ \sigma = 1, \cdots, \tilde{m} \tag{4.27}$$

$u = (u^1, \cdots, u^m)$ is a function of the independent variable $x = (x^1, \cdots, x^n)$. $u^1, \cdots, u^m$ are the sets of all first, second up to kth-order partial derivatives:

$$\begin{aligned}
u_{(1)} &= \{u_i^\alpha\} = \{u^1_{x^1}, \cdots, u^1_{x^n}, \cdots, u^\alpha_{x^1}, \cdots, u^\alpha_{x^n}\},\\
u_{(2)} &= \{u_{ij}^\alpha\} = \{u^1_{x^1x^1}, u^1_{x^1x^2}, \cdots, u^1_{x^1x^n}, \cdots, u^\alpha_{x^1x^1}, u^\alpha_{x^1x^2}, \cdots, u^\alpha_{x^1x^n}\},\\
&\vdots\\
u_{(k)} &= \{u_{i_k}^\alpha\} = \{u^1_{\underbrace{x^1\cdots x^1}_{\times k}}, u^1_{x^1x^2\dots x^k}, u^1_{x^1\underbrace{x^2\cdots x^2}_{\times(k-1)}} \cdots, u^1_{\underbrace{x^n\cdots x^n}_{\times k}}, \cdots\\
&\quad \cdots, u^\alpha_{\underbrace{x^1\cdots x^1}_{\times k}}, u^\alpha_{x^1x^2\dots x^k}, u^\alpha_{x^1\underbrace{x^2\cdots x^2}_{\times(k-1)}} \cdots, u^\alpha_{\underbrace{x^n\cdots x^n}_{\times k}}\}.
\end{aligned}$$

$\alpha = 1, \cdots, m$ and $i, j, i_1, \cdots, i_k = 1, \cdots, n$. Since we assume that $u_{ij}^\alpha = u_{ji}^\alpha$, $u_{(2)}$

contains only the terms u^{α}_{ij} for which $i \leq j$, $u_{(3)}$ contains only the terms for which $i \leq j \leq k$, and so for $u_{(4)}$, $u_{(5)}$, $\cdots$. There is a natural ordering in $u_{(k)}$ and the number of elements is $m \binom{k}{n+k-1}$.

We say that the system (4.27) admits the invertible transformation of the variables x and u i.e.,

$$\bar{x}^i = f^i(x,u), \quad \bar{u}^{\alpha} = \phi^{\alpha}(x,u), \quad i = 1, \cdots, n; \alpha = 1, \cdots, m \tag{4.28}$$

if it is form-invariant in the new variables $\bar{x}$ and $\bar{u}$, i.e.,

$$E^{\sigma}(\bar{x}, \bar{u}, \bar{u}_{(1)}, \bar{u}_{(2)}, \cdots, \bar{u}_k) = 0, \quad \sigma = 1, \cdots, \tilde{m}, \tag{4.29}$$

whenever (4.27) holds. The invertible transformations are said to be a symmetry transformation of the system (4.27) and, the set of all these transformation are defined by

$$\Gamma_a : \bar{x}^i = f^i(x,u,a), \quad \bar{u}^{\alpha} = \phi^{\alpha}(x,u,a), \quad i = 1, \cdots, n; \alpha = 1, \cdots, m \tag{4.30}$$

where a is real continuous parameter from a neighborhood of $a = 0$ and f^i, ϕ^{α} are differentiable functions that forms a *local continuous one-parameter Lie group of transformation* G in $\mathbb{R}$, since Γ_a satisfy the properties of the group, Definitions 4.1.3 and 4.1.7.

If the transformation (4.30) of a group G are symmetry transformations of (4.27), then G is called a symmetry group of (4.27) and (4.27) is said to admit G as a group. The symmetries transformations can be used to construct the solutions of differential equations and, as said previously it form a one-parameter local Lie group.

According to *Lie's theory*, the construction of a one-parameter group G is equivalent to the determination of the corresponding infinitesimal transformations obtained by the Taylor series expansion in a of the equation (4.30) about $a = 0$, taking into account the initial conditions, i.e:

$$f^i \mid_{a=0} = x^i, \phi^{\alpha} \mid_{a=0} = u^{\alpha}. \tag{4.31}$$

The infinitesimal transformations are:

$$\bar{x}^i \approx x^i + a\xi^i(x,u), \quad \bar{u}^{\alpha} \approx u^{\alpha} + a\eta^{\alpha}(x,u) \tag{4.32}$$

where

$$\xi^i(x,u) = \frac{\partial f^i(x,u,a)}{\partial a} \mid_{a=0}, \quad \eta^{\alpha}(x,u) = \frac{\partial \phi^{\alpha}(x,u,a)}{\partial a} \mid_{a=0}. \tag{4.33}$$

By introducing the operator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \tag{4.34}$$

the infinitesimal transformation (4.32) can be written as:

$$\bar{x}^i \approx (1 + aX)x^i, \;\; \bar{u}^\alpha \approx (1 + aX)u^\alpha. \tag{4.35}$$

The operator (4.34) is known as the infinitesimal operator or generator of the group G. Therefore, if G is admitted by (4.27), X is also admitted operator of (4.27).

The one-parameter groups are obtained trough their generators as said on the *Lie's theorem* bellow.

Theorem 4.2.1 (Lie's theorem)
Given the infinitesimal transformations (4.32) or X, the corresponding one-parameter group G is obtained by solution of the Lie equations,

$$\frac{d\bar{x}^i}{da} = \xi^i(\bar{x},\bar{u}), \;\; \frac{d\bar{u}^\alpha}{da} = \eta^\alpha(\bar{x},\bar{u}), \tag{4.36}$$

subject to the initial conditions (4.31),

$$\bar{x}^i \mid_{a=0} = x^i, \;\; \bar{u}^\alpha \mid_{a=0} = u^\alpha. \tag{4.37}$$

The base idea of proof this theorem, is that, if we consider the Lie equations, which are obviously associated to the vector field X, is required to find out whether there is a one parameter group G, for which the given X is its tangent vector field. The proof is made by consider the existence of unique solutions of the Cauchy problem (4.36 - 4.37) and that this map form a one parameter family of local transformations. It is shown that this set satisfy the conditions of definition of local Lie group, using the principal idea that if two solutions are solutions of the Cauchy problem they are coincident. More details can be found in [45] (Pag.14).

In the space $(x, u, u_{(1)}, u_{(2)}, u_{(3)}, \cdots, u_{(k)})$, the infinitesimal transformation are obtained by constructing the prolonged group $G^{[k]}$ of G, where k is the highest order of the derivatives in the system (4.27). For example if one consider the space $(x, u, u_{(1)})$, the prolonged group will be $G^{[1]}$, and if the space considered is $(x, u, u_{(1)}, u_{(2)})$ then the prolonged group will be $G^{[2]}$.

Since the transformation (4.30) is a symmetry group G of the system (4.27), the function $\bar{u} = \bar{u}(\bar{x})$ satisfies (4.29), whenever the function $u = u(x)$ satisfies (4.27). The transformation of the derivatives $\bar{u}_{(1)}, \bar{u}_{(2)}, \bar{u}_{(3)}, \cdots, \bar{u}_{(k)}$, are found from (4.30) by using the formulae of change of variables in the derivatives with respect to each of the parameter x^i,

$$D_i = D_i(f^j)\bar{D}_j \tag{4.38}$$

where

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots \tag{4.39}$$

is the *total derivative operator* with respect to x^i and $\bar{D}_j$ is likewise given in terms of the transformed variables. If we consider

$$\bar{u}_i^\alpha = \bar{D}_i(\bar{u}^\alpha), \quad \bar{u}_{ij}^\alpha = \bar{D}_i(\bar{u}_i^\alpha) = \bar{D}_i(\bar{u}_j^\alpha), \cdots$$

and apply (4.38) in $\bar{u}^\alpha$, (4.30) can be written as

$$D_i(\phi^\alpha) = D_i(f^j)\bar{D}_j(\bar{u}^\alpha) = D_i(f^j)\bar{u}_j^\alpha. \tag{4.40}$$

Taking the first and the last terms of (4.40), we get

$$D_i(f^j)\bar{u}_j^\alpha = D_i(\phi^\alpha) \Leftrightarrow (\frac{\partial f^j}{\partial x^i} + u_i^\beta \frac{\partial f^j}{\partial u^\beta})\bar{u}_j^\alpha = \frac{\partial \phi^\alpha}{\partial x^i} + u_i^\beta \frac{\partial \phi^\alpha}{\partial u^\beta}. \tag{4.41}$$

By solving (4.41) with respect to u_i^α, since it is locally invertible equation, we found the transformation of the derivatives u_i^α in u_i^α, given by

$$\bar{u}_i^\alpha = \psi_i^\alpha(x, u, u_{(1)}, a), \quad \psi_i^\alpha \mid_{a=0} = u_i^\alpha. \tag{4.42}$$

The transformation of $u_{(1)}$ on $\bar{u}_{(1)}$ together with the transformations of (x, u) on $(\bar{x}, \bar{u})$ will form the first prolongation of the one-parameter group $G^{[1]}$ acting in the space $(x, u, u_{(1)})$. From the first prolongation by using the total derivatives, we obtain the second prolongation one-parameter group $G^{[2]}$ acting in the space $(x, u, u_{(1)}, u_{(2)})$ and successively we obtain the kth prolongation one-parameter group $G^{[k]}$ acting in the space $(x, u, u_{(1)}, \cdots, u_{(k)})$.

Since we have the prolonged groups $G^{[1]}$ to $G^{[k]}$, we have to find their infinitesimal transformation. Remember that the infinitesimal transformation for (x, u) is given by (4.32). For (4.42), if we apply the Taylor's series expansion in the neighborhood of $a = 0$ and taking into account the initial conditions $\psi_i^\alpha \mid_{a=0} = u_i^\alpha$, we will get:

$$\begin{aligned}
\bar{u}_i^\alpha &\approx u_i^\alpha + a\zeta_i^\alpha(x, u, u_{(1)}), \\
\bar{u}_{ij}^\alpha &\approx u_{ij}^\alpha + a\zeta_{ij}^\alpha(x, u, u_{(1)}, u_{(2)}), \\
&\vdots
\end{aligned} \tag{4.43}$$

$$\bar{u}^{\alpha}_{i_1\cdots i_2} \approx u^{\alpha}_{i_1\cdots i_k} + a\zeta^{\alpha}_{i_1\cdots i_2}(x, u, u_{(1)}, \cdots, u_{(k)}).$$

Now we have to find the functions ζ^{α}_i, ζ^{α}_{ij} and $\zeta^{\alpha}_{i_1\cdots i_k}$. Taking equation (4.41), considering only the expressions of the equation in the right side, taking into account the values of f^j and ϕ^{α} from (4.32), and of $\bar{u}^{\alpha}_j$ from (4.43), considering that

$$\delta^j_i = \begin{cases} 0, & i \neq j; \\ 1, & i = j. \end{cases}$$

Also we considered the fact that this transformation is local, i.e., a is very small and a^2 will be smaller therefore, negligible, we obtain:

$$\begin{aligned} D_i(x^j + a\xi^j)(u^{\alpha}_j + a\zeta^{\alpha}_j) &= D_i(u^{\alpha} + a\eta^{\alpha}) \\ (\delta^j_i + aD_i\xi^j)(u^{\alpha}_j + a\zeta^{\alpha}_j) &= u^{\alpha}_i + aD_i\eta^{\alpha} \\ u^{\alpha}_i + a\zeta^{\alpha}_i + au^{\alpha}_j D_i\xi^j &= u^{\alpha}_i + aD_i\eta^{\alpha}. \end{aligned} \tag{4.44}$$

By simplification we find ζ^{α}_i, which is $\bar{u}^{\alpha}_i$, in the first prolongation formula (4.43). Higher prolongation formulas are obtained by introducing the Lie characteristic function $W^{\alpha} = \eta^{\alpha} - \xi^j u^{\alpha}_j$, giving

$$\begin{aligned} \zeta^{\alpha}_i &= D_i(W^{\alpha}) + \xi^j u^{\alpha}_{ji} \\ \zeta^{\alpha}_{ij} &= D_iD_j(W^{\alpha}) + \xi^k u^{\alpha}_{kij} \\ &\vdots \\ \zeta^{\alpha}_{i_1\cdots i_k} &= D_{i_1}\cdots D_{i_k}(W^{\alpha}) + \xi^j u^{\alpha}_{ji_1\cdots i_1}. \end{aligned} \tag{4.45}$$

From (4.45) and (4.46), we can conclude that the functions ζ^{α}_i, ζ^{α}_{ij} and $\zeta^{\alpha}_{i_1\cdots i_k}$ in the equation (4.43) are given recursively by the prolongation formulas:

$$\begin{aligned} \zeta^{\alpha}_i &= D_i(\eta^{\alpha}) - u^{\alpha}_j D_j(\xi^j) \\ \zeta^{\alpha}_{ij} &= D_j(\zeta^{\alpha}_i) - u^{\alpha}_{il} D_j(\xi^l) \\ &\vdots \\ \zeta_{i_1\cdots i_k} &= D_{i_k}(\zeta_{i_1\cdots i_{k-1}}) - u^{\alpha}_{i_1\cdots i_k l} D_j(\xi^l). \end{aligned} \tag{4.46}$$

The generators of the prolonged groups are determined in the same way using equation (4.34) to obtain the generator of the group G, which are also referred to as the k-th prolonged generators

$$X^{[1]} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^{\alpha}(x,u)\frac{\partial}{\partial u^{\alpha}} + \zeta^{\alpha}_i(x,u,u_1)\frac{\partial}{\partial u^{\alpha}_i}$$

$$\vdots \tag{4.47}$$

$$X^{[k]} = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} + \zeta_i^\alpha(x,u,u_1)\frac{\partial}{\partial u_i^\alpha} + \\ +\zeta^\alpha_{i_1\cdots i_k}(x,u,\cdots,u_k)\frac{\partial}{\partial u^\alpha_{i_1\cdots i_k}}.$$

4.2.1 Exact Solutions

By considering the symmetry transformation of (4.27), one can use some solutions of the system to obtain general solutions of the system *(group transformation of known solutions)*. If the solutions are not known, one can look for the solutions that are invariant of the group generated by the particular X *(invariant solutions)*. Let us consider first some definitions and theorems about invariant points and invariant functions:

Definition 4.2.1 (of invariant point)
A point $(x,u) \in \mathbb{R}^{n+m}$ is an invariant point if it remains unaltered by all transformation of the group G, i.e, $(\bar{x},\bar{u}) = (x,u)$ $\forall a \in D' \subset D$.

Theorem 4.2.2 (necessary and sufficient condition for a point to be invariant of a group)
A point $(x,u) \in R^{n+m}$ is an invariant point of a group G with generator given by (4.34) if and only if $\xi^i(x,u) = \eta^\alpha = 0$.

To give the idea on the proof of the theorem, first we call *"orbit of the group through point (x,y)"*, the set of points to which (x,y) can be mapped by a suitable choice of parameter a. This set is smooth curve and is invariant under the action of a Lie group (i.e., the action of a Lie group maps every point from an orbit to a point into the same orbit), [27]. If X is a tangent vector to the orbit at the point $(\bar{x},\bar{u})$, then X is smooth curve. If an orbit crosses any curve C transversely at a point (x,u), then there are Lie symmetries that map (x,u) to points that are not in C. Any curve is invariant if and only if no orbit cross it, in other words, C is an invariant curve if and only if the tangent to C at point (x,u) is parallel to the tangent vector X i.e., we have the inner product

$$\eta(x,u) - u'(x)\xi(x,u) = 0,$$

$(u'(x),1)$ being the tangent vector of the curve ate $(x,u(x))$.

Definition 4.2.2 (of invariant group)
A function $F(x,u)$ is an invariant of a group G if and only if $F(\bar{x},\bar{u}) = F(x,u)$, $\forall$ x,

u and the parameter $a \in D' \subset D$.

Theorem 4.2.3 (necessary and sufficient condition for invariance of function)
A function $F(x,u)$ is an invariant of a group G with generator given in (4.34) if and only if

$$X(F) = \xi^i(x,u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial F}{\partial u^\alpha} = 0. \tag{4.48}$$

Following the idea of demonstration of the **Theorem** 4.2.2, we can say that the surface F is invariant provides that the tangent field to F at (x^i, u^α) is parallel to (ξ^i, η^α), i.e.,

$$\xi^i(x,u)\frac{\partial F}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial F}{\partial u^\alpha} = 0$$

In other hand if (4.48) is true then, the tangent field to F at (x^i, u^α) is parallel to (ξ^i, η^α) so that, no orbit cross transversely the surface i.e., the action of the Lie group maps every point on the surface to a another point into the same surface therefore F is invariant.

The equation (4.48) is a linear PDE and can be solved using method of characteristics that will give the invariant curves which are tangent to the vector (ξ^i, η^α), for $i = 1, \cdots, n$ and $\alpha = 1, \cdots, m$. The local invariant surface $u(x)$, which is a solution of a PDE will be a union of these invarian curves in the neighborhood of the point $a = 0$. The characteristics equations are:

$$\frac{dx^1}{\xi^1(x,u)} = \cdots = \frac{dx^n}{\xi^n(x,u)} = \frac{du^1}{\eta^1(x,u)} = \cdots = \frac{du^m}{\eta^m(x,u)}. \tag{4.49}$$

Equation (4.49) holds for exactly $m + n - 1$ functionally independent first integrals invariant of a one-parameter group G, called *basis of invariant*

$$I_1(x,u) = c_1, \cdots, I_{m+n-1}(x,u) = c_{m+n-1},$$

where $i = 1, \cdots m + n - 1$ and c_i are constants. But there are other invariant functions which are given by the general solution

$$F = \Lambda(I_1(x,u), \cdots, I_{m+n-1}(x,u)) \quad \text{of} \quad X(F) = 0$$

for an arbitrary function Λ.

Let us now define and present theorems for the invariant points and invariant functions for prolonged groups.

Definition 4.2.3 (of differential invariant of a group)
A differential function $F(x, u, \cdots, u_p)$, $p \geq 0$, is a pth-order differential invariant

of a group G if

$$F(x, u, \cdots, u_p) = F(\bar{x}, \bar{u}, \cdots, \bar{u}_p)$$

i.e., if F is an invariant under the prolonged group $G^{[p]}$, for $p = 0,\ u_0 \equiv u$ and $G^{[0]} \equiv G$.

Theorem 4.2.4 (sufficient condition for invariance of differential equation)
A differential function $F(x, u, \cdots, u_{(p)}),\ \ p \geq 0$, is a pth-order differential invariant of a group G if,

$$X^{[p]}F = 0,$$

$X^{[p]}$ is the pth prolongation of X and for $p = 0,\ \ X^{[0]} = X$.

The idea for proof of this theorem is similar to the **Theorem** 4.2.3 therefore we can regard the derivatives on differential function F also as another p additional variables and $X^{[p]}$ being the infinitesimal generator correspondent to the function F with the variables (x, u) plus p new variables, so that the invariance criterion (4.48) become $X^{[p]}F = 0$.

Similarly, we will write down the characteristic system corresponding to the linear PDE in the **Theorem 4.2.3** and solve it for the differential invariants. To compute the symmetries of the system (4.27), we start by applying (4.43) into (4.29) i.e.,

$$E^{\sigma}(\bar{x}, \bar{u}, \bar{u}_1, \cdots, \bar{u}_k) \approx E^{\sigma}(x, u, u_1, \cdots, u_k) + a(X^{[k]}E^{\sigma}),\ \ \sigma = 1, \cdots, \tilde{m}. \qquad (4.50)$$

For the invariance of (4.27), as can be seen in (4.50) we require that

$$X^{[k]}E^{\sigma}(x, u, u_1, \cdots, u_k) = 0,\ \ \sigma = 1, \cdots, \tilde{m} \qquad (4.51)$$

whenever (4.27) is satisfied. The converse also applies.

Theorem 4.2.5 (about how to define all infinitesimal symmetries of system of PDEs)
Equation (4.51) define all infinitesimal symmetries of the system (4.27).

The demonstration of **Theorem** 4.2.5, is immediate from the **Theorems** 4.2.3 and 4.2.4.

Equations (4.51) are called the *determining equations* of (4.27). They are compactly written as

$$X^{[k]}E^{\sigma}(x, u, u_1, \cdots, u_k)\ |_{(4.27)} = 0,\ \ \sigma = 1, \cdots, \tilde{m}, \qquad (4.52)$$

where $|_{(4.27)}$ means that the equation is evaluated on the surface (4.27). Generally the transformations Γ_a generated by the Lie equations (4.27) are a result of a composition of r one-parameter groups, as we can see in the following theorem.

Theorem 4.2.6 (necessary and sufficient conditions for a product of "r" one parameter groups to be r-parameter group)
Let $\mathcal{L}_r$ be an r-dimensional vector space of operators

$$X_l = \xi_l^i(x,u)\frac{\partial}{\partial x^i} + \eta_l^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \ \ l = 1,\cdots,r.$$

The product $\Gamma_a = \Gamma_{a_r}\cdots\Gamma_{a_1}$ of r one-parameter groups of transformations Γ_{a_l} generated by each X_l via the Lie equations

$$\frac{d\bar{x}^i}{da_l} = \xi_l^i(\bar{x},\bar{u}), \ \ \frac{d\bar{u}^\alpha}{da_l} = \eta_l^\alpha(\bar{x},\bar{u}) \tag{4.53}$$

subject to the initial conditions

$$\bar{x}^i \mid_{a=0} = x^i, \ \ \bar{u}^\alpha \mid_{a=0} = u^\alpha$$

is a local $r-$parameter group G_r if and only if $\mathcal{L}_r$ is a Lie algebra.

If we consider that a multi-parameter group is a composition of a various one-parameter groups, its invariants can be defined as Definition 4.2.2. Thus, a function $F(x,u)$ is an invariant of an $r-$parameter group G_r with generators

$$X_l = \xi_l^i(x,u)\frac{\partial}{\partial x^i} + \eta_l^\alpha(x,u)\frac{\partial}{\partial u^\alpha}, \ \ l = 1,\cdots,r$$

if and only if

$$X_l(F) = \xi_l^i(x,u)\frac{\partial F}{\partial x^i} + \eta_l^\alpha(x,u)\frac{\partial F}{\partial u^\alpha} = 0, \ \ l = 1,\cdots,r.$$

4.2.2 Group Classification

The treatment given to the equations without arbitrary elements is different from the one given to the equations with arbitrary elements. If the equation or system does not contain arbitrary elements, the Lie group analysis will be merely to calculate its full group (the full Lie algebra of operators) and, if the equation or the system contains parameters, we have to realize a group classification relative to these parameters.

Consider the class of generalized (1+2) dimensional equations of form (3.24). The principal idea of group classification is, once we have calculated the determining equation, we realize that some of them depend on arbitrary elements. By solving it for arbitrary elements, we find the *principal Lie algebra* of the equation. The *principal Lie group* of the PDE, is the group admitted by all equation of the these form. But some elements of the group can admit extension of the principal or full Lie group. We find this extension by considering particular values of the arbitrary elements and we extend the kernel of the full Lie algebra. Then we solve this equations with respect to the arbitrary elements and we find the additional condition for infinitesimal transformation. By substitution of this conditions into the determining equation we generate the general structure of the classification equation which is responsible for the group classification of the equation or system. The values of the arbitrary elements that can extend the full Lie algebra must be the solution of the general structure [45]. For further explanation about Lie group classification of PDE we refer for example to Lie [37], [31] and [30].

4.2.3 Optimal Algebras

In many situations some subalgebras are similar i.e., they are connected each other by a transformation from the symmetry group with Lie algebra $\mathcal{L}_r$. Their corresponding invariant solutions are also connected by the same transformation. By putting into one class all subalgebra of a given dimension n, the problem of finding invariant solutions of the Lie algebra $\mathcal{L}_r$ can be reduced to the problem of finding an optimal system of invariant solutions. The optimal system of order n can be a set of all invariant solutions of selected representative from each class of subalgebras of dimension n. Unfortunately, there is no an efficient method which can be used to find optimal system. Some algorithms are presented in [23], [43] and [45].

The principal idea is, suppose we have to construct two non similar sub-groups of group G, say H and K. The invariant solution S under the subgroup H will satisfy the condition $S = hS$ for all $h \in H$. Since H and K are non similar sub-groups, the solutions S will be transformed to $\bar{S}$ under the sub-group K i.e., $\bar{S} = k\bar{S}$, for all $k \in K$. To find k, we look for $g \in G$ such that $\bar{S} = gS$. Since S is invariant under H, we can make the transformation $\bar{S} = g(hS) = ghS = ghg^{-1}gS = ghg^{-1}\bar{S}$ and $k = ghg^{-1}$. K is called the adjoint subgroup of H under the symmetry group G.

Chapter 5

Application to the PDE of the Ornstein-Uhlenbeck process

In this section we apply the Lie symmetry method described in **Section** 4.2, to the PDE of Ornstein-Uhlenbeck process. With help of the computer subprogram *Wolfram Mathematica*, *SYMLie* we make a group classification of this PDE and we use its infinitesimal to find some invariant solutios and construct its one dimensional optimal system.

5.1 The Basic Equation

We consider a PDE (3.24), the equation derived from the prices of rainfall derivatives when the rainfall follow the Ornstein-Uhlenbeck Process. We consider the function

$$\theta(t) = m + \alpha \sin \frac{\pi(t-\nu)}{6}$$

and

$$f(x,t) = f(x) = (X_{ref} - X_t)^+ \quad \text{or} \quad f(x,t) = (X_t - X_{ref})^+,$$

where X_{ref} is the reference level (the base from which the excess or deficit of rainfall is determined). Differentiating $\theta(t)$ we will get

$$\theta'(t) = \frac{\pi}{6}\alpha \cos \frac{\pi(t-\nu)}{6}.$$

In order to reduce the complexity of the computation, we consider the following basic mathematic transformations

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \quad \wedge \quad \sin(a \pm b) = \sin a \cos b \pm \sin b \cos a$$

for $\theta(t)$ and $\theta'(t)$, the equation (3.24) can take the following form

$$\frac{\partial V}{\partial t} = rV - f(x)\frac{\partial V}{\partial y} - \left(w_1 \cos\frac{\pi t}{6} + w_2 \sin\frac{\pi t}{6} + w_3 + kx\right)\frac{\partial V}{\partial x} - \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} \quad (5.1)$$

t, x, y are independent variables, V is dependent variable, r, x_{ref}, w_1, w_2, w_3, k, σ are parameters and w_1, w_2, w_3 are defined by the following system of equation,

$$\begin{aligned} w_1 &= \frac{\alpha\pi}{6}\cos\frac{\pi\nu}{6} - k\alpha\sin\frac{\pi\nu}{6}, \\ w_2 &= \frac{\alpha\pi}{6}\sin\frac{\pi\nu}{6} + k\alpha\cos\frac{\pi\nu}{6}, \\ w_3 &= km - \lambda\sigma. \end{aligned}$$

5.2 Infinitesimal Operators and Group Classification

We construct now the prolonged generator,

$$\begin{aligned} X^{[2]} &= X + \zeta_1\frac{\partial}{\partial V_t} + \zeta_2\frac{\partial}{\partial V_x} + \zeta_3\frac{\partial}{\partial V_y} + \zeta_{12}\frac{\partial}{\partial V_{tx}} + \\ &+ \zeta_{13}\frac{\partial}{\partial V_{ty}} + \zeta_{23}\frac{\partial}{\partial V_{xy}} + \zeta_{11}\frac{\partial}{\partial V_{tt}} + \zeta_{22}\frac{\partial}{\partial V_{xx}} + \zeta_{33}\frac{\partial}{\partial V_{yy}} \end{aligned} \quad (5.2)$$

for

$$X = \xi^1(t,x,y,V)\frac{\partial}{\partial t} + \xi^2(t,x,y,V)\frac{\partial}{\partial x} + \xi^3(t,x,y,V)\frac{\partial}{\partial y} + \eta(t,x,y,V)\frac{\partial}{\partial V} \quad (5.3)$$

and we find solutions of the determined equation *(the infinitesimals)*. This solution can be explicitly determined by hand calculation, but it requires a lot of computations that can be avoided if one use *Wolfram Mathematica*, *SYMLie* [16]. For more details in the hand calculation, please see [45].

Generally the system of over-determined equations can contain many equations. In our case we obtained more than hundred equations. Since the determining equation are linear homogeneous PDEs of order two for the unknown functions ξ^i and η, we generate an over-determined system of algebraic equation with $n + m = 3 + 1 = 4$ unknowns functions. By solving the over-determined system we obtain the unknowns function ξ^i, η and consequently ζ_i and ζ_{ij}. The values of functions ξ^i and η depends on different values of the parameter k and σ. For $a \neq 0 \wedge k(36k^2 + \pi^2) \neq 0$ we will have:

$$\begin{aligned} \xi^1 &= \frac{\pi}{36k^2+\pi^2}\left[(\pi w_1 + 6kw_2)\cos\frac{\pi t}{6} + (-6kw_1 + \pi w_2)\sin\frac{\pi t}{6}\right]\mathbf{c}_1 + \\ &+ \frac{e^{-kt}\left(-e^{2kt}\mathbf{c}_2 + \mathbf{c}_3\right)}{k} - \mathbf{c}_5, \end{aligned}$$

$$
\begin{aligned}
\xi^2 &= \frac{6}{36k^2+\pi^2}\left[(-6kw_1+\pi w_2)\cos\frac{\pi t}{6}-(\pi w_1+6kw_2)\sin\frac{\pi t}{6}+\right]\mathbf{c}_1+ \\
&\quad +\frac{e^{kt}\mathbf{c}_2+e^{-kt}\mathbf{c}_3}{k^2}+\mathbf{c}_4+t\mathbf{c}_5, \\
\xi^3 &= \mathbf{c}_1, \\
\eta &= \frac{V}{\sigma^2 k}\left[2e^{kt}(w_3-kx)\mathbf{c}_2+k\left(k(tw_3-x)\mathbf{c}_5+k^2(-st+y)\mathbf{c}_5+a^2\mathbf{c}_6\right)\right]+ \\
&\quad \frac{-6V}{\sigma^2\pi\left(36k^2+\pi^2\right)}\left[2e^{kt}\pi(-6kw_1+\pi w_2)\mathbf{c}_2+k\left(36k^2+\pi^2\right)w_2\mathbf{c}_5\right]\cdot \\
&\quad \cdot\cos\frac{\pi t}{6}+\frac{6V}{\sigma^2\pi\left(36k^2+\pi^2\right)}\cdot \\
&\quad \cdot\left[2e^{kt}\pi(\pi w_1+6kw_2)\mathbf{c}_2+k\left(36k^2+\pi^2\right)w_1\mathbf{c}_5\right]\sin\frac{\pi t}{6}+\omega(x,y,t),
\end{aligned}
$$

where $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ are constants and $\omega(x,y,t)$ satisfies (5.1). The infinitesimal generators are obtained by representing the general solutions of the determining equations as linear combinations of independent solutions defined by the constants i.e., the number of independents solutions will depend on the number of the constants in the general solution.

The principal Lie algebra $\mathcal{L}_p$ i.e., the Lie algebra of operators admitted by the linear PDE (5.1) containing arbitrary elements is spanned by the generators

$$
X_1=\frac{\partial}{\partial y},\ \ X_2=V\frac{\partial}{\partial V},\ \ X_\omega=w(t,x,y)\frac{\partial}{\partial V}. \tag{5.4}
$$

where the function $\omega(t,x,y)$ satisfies Equation (5.1).

5.3 Extension of Principal Lie Algebra

For particular choices of the arbitrary elements, we found the possible extensions for non-degenerate PDE (5.1), satisfying the following conditions:

1. $\sigma \neq 0$ and $(36k^2+\pi^2)=0$ (in two cases, $k=\pm i\frac{\pi}{6}$): the principal Lie algebra $\mathcal{L}_p$ admits extension only by one operator.
2. $\sigma \neq 0$ and $k(36k^2+\pi^2)\neq 0$: the principal Lie algebra $\mathcal{L}_p$ admits extension by four operators.

$$
\begin{aligned}
X_3 &= -\frac{e^{kt}\partial_x}{k}+\frac{e^{kt}\partial_y}{k^2}, \\
X_4 &= \frac{k\left(-kx_{ref}\pi t-\pi w_3 t+\pi x+k\pi y+6w_2\cos\frac{\pi t}{6}\right)}{\sigma^2\pi}V\partial_V+ \\
&\quad -\frac{6kw_1}{\sigma^2\pi}\sin\frac{\pi t}{6}V\partial_V-\partial_x+t\partial_y,
\end{aligned}
$$

$$
\begin{aligned}
X_5 &= \partial_t + \pi\left[\frac{(\pi w_1 - 6kw_2)}{36k^2+\pi^2}\cos\frac{\pi t}{6} + \frac{(\pi w_2 + 6kw_1)}{36k^2+\pi^2}\sin\frac{\pi t}{6}\right]\partial_x + \\
&+ \left[\frac{(6\pi w_2 + 36kw_1)}{36k^2+\pi^2}\cos\frac{\pi t}{6} + \frac{(36kw_2 - 6\pi w_1)}{36k^2+\pi^2}\sin\frac{\pi t}{6}\right]\partial_y, \\
X_6 &= \frac{72ke^{-kt}}{\sigma^2(36k^2+\pi^2)}\left(-w_3 - \frac{\pi^2 w_3}{36k^2} - kx - \frac{\pi^2 x}{36k} - w_1\cos\frac{\pi t}{6}\right)V\partial_V \\
&+ \frac{72ke^{-kt}}{\sigma^2(36k^2+\pi^2)}\left[\frac{-\pi w_2}{6k}\cos\frac{\pi t}{6} + (-\frac{\pi w_1}{6k} + w_2)\sin\frac{\pi t}{6}\right]V\partial_V \\
&+ \frac{e^{-kt}}{k}\partial_x + \frac{e^{-kt}}{k^2}\partial_y.
\end{aligned}
$$

3. $\sigma \neq 0$ and $k = 0$: the principal Lie algebra $\mathcal{L}_p$ admits extension by 6 operators.

$$
\begin{aligned}
X_3 &= -\partial_x + t\partial_y, \\
X_4 &= \partial_t + \left(w_1\cos\frac{\pi t}{6} + w_2\sin\frac{\pi t}{6}\right)\partial_x + 6\frac{w_2\cos\frac{\pi t}{6} - w_1\sin\frac{\pi t}{6}}{\pi}\partial_y, \\
X_5 &= \frac{2\left(\pi w_3 t - \pi x - 6w_2\cos\frac{\pi t}{6} + 6w_1\sin\frac{\pi t}{6}\right)}{\sigma^2\pi}V\partial_V - 2t\partial_x + t^2\partial_y, \\
X_6 &= \frac{3}{\sigma^2}\left[2x_{ref}t + w_3t^2 - 2tx - 2y + \frac{12(6w_1 - w_2 t)}{\pi}\cos\frac{\pi t}{6}\right]V\partial_V \\
&+ \frac{36}{\sigma^2}(\pi w_1 t + 6w_2)\sin\frac{\pi t}{6}V\partial_V - 3t^2\partial_x + t^3\partial_y, \\
X_7 &= 2t\partial_t + \left[2rt + \frac{6w_3\left(\frac{\pi w_3 t - \pi x}{6} - w_2\cos\frac{\pi t}{6} + w_1\sin\frac{\pi t}{6}\right)}{\sigma^2\pi}\right]V\partial_V \\
&+ \left[x - x_{ref} + \frac{2\pi t w_1 + 6w_2}{\pi}\cos\frac{\pi t}{6} - \frac{6w_1 - 2\pi t w_2}{\pi}\sin\frac{\pi t}{6}\right]\partial_x \\
&+ 12\left[\frac{y}{4} - \frac{9w_1 - \pi w_2 t}{\pi^2}\cos\frac{\pi t}{6} - \frac{9w_2 + \pi t w_1}{\pi^2}\sin\frac{\pi t}{6}\right]\partial_y, \\
X_8 &= 2t^2\partial_t + 4\left(+\frac{rt^2}{2} + \frac{x_{ref}w_3 t}{2\sigma^2} + \frac{w_3^2 t^2}{4\sigma^2} - \frac{2x_{ref}x}{\sigma^2} - \frac{w_3 x t}{2\sigma^2}\right)V\partial_V \\
&- \left[4t + \frac{48w_3}{\sigma^2}\left(\frac{x_{ref}w_2}{\pi w_3} + \frac{9w_1}{2\pi^2} + \frac{w_2 t}{4\pi} - \frac{w_2 x}{\pi w_3}\right)\cos\frac{\pi t}{6}\right]V\partial_V \\
&+ \frac{24}{\sigma^2}\left(\frac{2x_{ref}w_1}{\pi} + \frac{w_1 w_3 t}{2\pi} - \frac{9w_2 w_3}{\pi^2} - \frac{2w_1 x}{\pi}\right)\sin\frac{\pi t}{6}V\partial_V + \\
&+ \left[\frac{4x^2 + 6w_3 y}{\sigma^2} - 72\frac{(w_1^2 - w_2^2)\cos\frac{\pi t}{3} + w_1 w_2\sin\frac{\pi t}{3}}{\pi^2\sigma^2}\right]V\partial_V \\
&+ \left[-2x_{ref}t + \left(\frac{216w_1 + 2\pi^2 t^2 w_1}{\pi^2} + \frac{tw_2}{\pi}\right)\cos\frac{\pi t}{6}\right]\partial_x + \\
&+ \left[2tx - 6y + \left(\frac{-12tw_1}{\pi} + \frac{216w_2}{\pi^2} + 2t^2 w_2\right)\sin\frac{\pi t}{6}\right]\partial_x + \\
&+ 6t\left(y - \frac{36w_1 + 2\pi t w_2}{\pi^2}\cos\frac{\pi t}{6} - \frac{36w_2 + 2\pi t w_1}{\pi^2}\sin\frac{\pi t}{6}\right)\partial_y,
\end{aligned}
$$

We can see that the equation (5.1) admits a maximum extension by six operators, therefore equation (5.1) cannot be transformed, for any choice of its coefficients into heat equation,

$$u_t = u_{xx} + u_{yy} \tag{5.5}$$

since we know that the heat equation can only be extended by seven additional operators, see [20] and [30].

5.4 Invariant Solutions

As a result of the group classification we realized that one can not reduce the PDE (5.1) to the heat equation. But we can use the Lie analysis to find the invariant (exact) solution of the equation (5.1). The invariant solutions are those solutions that transform into themselves under a particular group of symmetries. If equation (4.27) admits a Lie algebra $\mathcal{L}_r$ of dimension $r > 1$ we could consider invariants solutions based on many (infinite) number of subalgebra of $\mathcal{L}_r$. We consider only the symmetries generated in the case where

$$\sigma \neq 0 \ \wedge k(k^2 + \pi^2) \neq 0.$$

i.e., the volatility and the mean reverting factor are non zero. The first six symmetries generating the finite dimensional Lie algebra $\mathcal{L}$, allow us to construct a commutator **Table.** 5.1, where the commutator of two symmetries X_i and X_j is given by,

$$[X_i, X_j] = X_iX_j - X_jX_i; \ i, j = 1, 2, \cdots, 6 \tag{5.6}$$

TABLE 5.1: The Commutator Table of Subalgebras

$[,]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	$\frac{k^2}{\sigma^2}X_2$	0	0
X_2	0	0	0	0	0	0
X_3	0	0	0	0	$-kX_3$	$\frac{2}{k\sigma^2}X_2$
X_4	$-\frac{k^2}{\sigma^2}X_2$	0	0	0	$-X_1 + \frac{k^2x_{ref}+kw_3}{\sigma^2}X_2$	0
X_5	0	0	kX_3	$X_1 - \frac{k^2x_{ref}+kw_3}{\sigma^2}X_2$	0	$-kX_6$
X_6	0	0	$-\frac{2}{k\sigma^2}X_2$	0	kX_6	0

From the commutator **Table.** 5.1 we see that the set of operators (or Lie point symmetry generators) $X_1, X_2, X_3, X_4, X_5, X_6$ is anti-symmetric and closed under the product $[\cdot, \cdot]$. Since the commutator is bilinear and satisfy the Jacobi identity we can confirm that the set of these symmetries form the infinite dimensional Lie algebra of the PDE (5.1). The infinite dimension of the Lie algebra explain the fact that the PDE has infinitely many linearly independents solutions. The PDE (5.1) belong to a wide class of (1+2) evolutionary equation. To construct their invariant solution, we have to use two dimensional Lie subalgebra in order to use the two linearly independent invariants to reduce to an ordinary differential equation. We will apply the algorithm presented in [20], following the steps:

1. First we choose a set of n operators that are admitted by the equation which form a subalgebra (for example X_1 and X_2). This algebra has n functionally independents invariants (for the case $n = 2$, I_1 and I_2) and their rank is n.

2. We determine the invariants by the system of n differential equations

$$X_1 I = 0, \quad X_2 I = 0, \quad \cdots, \quad X_n I = 0.$$

   The invariants solution exists if $rank(\partial_V I_1, \cdots, \partial_V I_n) = 1$.

3. We find the invariant solution in the form

$$I_n = \phi(I_1, \cdots, I_{(n-1)}). \tag{5.7}$$

By substituting the solution (5.7) into (5.1), we get a differential equation for the function ϕ. Considering the case $k \neq 0$ and $\sigma \neq 0$, and the subalgebra spanned by X_2, X_3 i.e., $< X_2, X_3 >$, the independent invariant solutions of these subalgebra are $I_1 = t$ and $I_2 = kxe^{kt} - \ln V$. The invariant solution of (5.1) take the form,

$$kxe^{kt} - \ln V = \phi(t) \Longleftrightarrow \ln V = kxe^{kt} - \phi(t) \Longleftrightarrow V = e^{kxe^{kt} - \phi(t)}, \tag{5.8}$$

where $\phi(t)$ is determined by the following ordinary differential equation

$$-\phi'(t) = r + k(-w_3 - w_1 \cos \frac{\pi t}{6} - w_2 \sin \frac{\pi t}{6})e^{kt} - \frac{k^2}{2}\sigma^2 e^{2kt}. \tag{5.9}$$

This equation can be integrated by the standard methods and ϕ is given by

$$\begin{aligned} \phi(t) &= -rt - [-w_3 - w_1 \frac{36k^2}{36k^2 + \pi^2}(\cos \frac{\pi t}{6} + \frac{\pi}{6k} \sin \frac{\pi t}{6}) + \\ &\quad -w_3 \frac{36k^2}{36k^2 - \pi^2}(\sin \frac{\pi t}{6} - \frac{\pi}{6k} \cos \frac{\pi t}{6})]e^{kt} + \frac{\sigma^2}{4} e^{2kt} + C. \end{aligned} \tag{5.10}$$

For the subalgebra spanned by X_1 and X_2, the invariants are

$$I_1 = t, \;\; I_2 = x, \;\; I_3 = Ve^{-y}$$

and the relation among the invariants is given by

$$I_3 = \phi(I_1, I_2).$$

Then the invariant solution of the PDE is $V = e^y \phi(t,x)$, where the function $\phi(t,x)$ is solution of the equation

$$\begin{aligned}\phi_t &= (r - x_r ef + x)\,\phi - \left(kx + w_3 + w_1 \cos\frac{\pi t}{6} + w_2 \sin\frac{\pi t}{6}\right)\phi_x + \\ &\quad -\frac{1}{2}\sigma^2 \phi_{xx}.\end{aligned} \tag{5.11}$$

Again we realized a group classification of equation (5.11) since we found that the determining equations depends on arbitrary elements k and σ. In the case $k(36k^2 + \pi^2) \neq 0 \wedge \sigma \neq 0$ of its group classification we have the symmetry $\Gamma_2 = \frac{e^{kt}}{k}\partial_\phi + e^{kt}\partial_x$ and the invariants are given by,

$$I_1 = t, \;\; I_2 = \phi e^{-\frac{x}{k}}.$$

The relation between the invariants is $I_2 = \psi(I_1)$ and $\phi = e^{\frac{x}{k}}\psi(t)$. The function $\psi(t)$ is a solution of the first order ordinary differential equation

$$\psi_t = \left[r - x_r ef - \left(w_3 + w_1 \cos\frac{\pi t}{6} + w_2 \sin\frac{\pi t}{6}\right)\frac{1}{k} - \frac{1}{2k^2}\sigma^2\right]\psi. \tag{5.12}$$

For equation (5.12) we can apply the standard methods of resolution and we get the following solution

$$\psi(t) = e^{\left(r - x_r ef - \frac{w_3}{k} - \frac{\sigma^2}{k^2}\right)t - \frac{6}{k\pi}\left(w_1 \sin(\frac{\pi t}{6}) - w_2 \cos(\frac{\pi t}{6})\right) + C} \tag{5.13}$$

the final solution of the PDE under the subalgebra $\langle X_1, X_2\rangle$ given by $V = e^y \phi(t,x) = e^{(y + \frac{x}{k})}\psi(t)$ will take the form

$$V = e^{y + \frac{x}{k} + \left(r - x_r ef - \frac{w_3}{k} - \frac{\sigma^2}{k^2}\right)t - \frac{6}{k\pi}\left(w_1 \sin(\frac{\pi t}{6}) - w_2 \cos(\frac{\pi t}{6})\right) + C} \tag{5.14}$$

5.4.1 Invariant Solutions Compatible with the Terminal Conditions

In financial applications, the only relevant invariant solutions are those that are compatible with the terminal conditions. Among all symmetries, we will seek for those which satisfies the system of equation and terminal conditions and we use them to find the invariant solutions. By substituting the invariant solutions into the equation (5.1), we find the solution consistent with the terminal condition by solving the resulting ordinary differential equation. We recall the terminal condition (3.25), and we transform it in a double conditions:

$$t = T \wedge V = tick \times (S - y_T)^+. \tag{5.15}$$

The general Lie point symmetry for maximal finite Lie algebra of the equation (5.1) (case: $\sigma \neq 0$ and $k(k^2+\pi^2) \neq 0$) will be given by

$$X = \sum_{i=1}^{6} a_i X_i \tag{5.16}$$

where the constants $a_i,\ i = 1, \cdots, 6$ must be determined. The application of X for each terminal condition (5.15) gives:

$$\begin{aligned}
t = T \quad &: \quad a_5 = 0, \\
V = \tau \cdot (S - y_T) \quad &: \quad \frac{kT\left(w_3 - kx_{ref} + \frac{ky_T - x}{T} - 6\frac{w_2\cos\frac{\pi T}{6} - w_1\sin\frac{\pi T}{6}}{\pi T}\right)}{\sigma^2} V a_4 \\
&+ \frac{72ke^{kT}\left(w_3 + \frac{\pi^2 w_3}{36k^2} - kx - \frac{\pi^2}{36k}x + w_1\cos\frac{\pi t}{6}\right)}{\sigma^2(36k^2+\pi^2)} V a_6 + \\
&V a_2 + \frac{12\pi e^{kT}\left[\left(w_1 + \frac{6kw_2}{\pi}\right)\sin\frac{\pi T}{6} - w_2\cos\frac{\pi t}{6}\right]}{\sigma^2(36k^2+\pi^2)} V a_6 \\
&= a_1 + \frac{e^{kT}}{k^2}a_3 + Ta_4 + \frac{e^{-kT}}{k^2}a_6.
\end{aligned}$$

Solving this equations, and equating the coefficients of the same powers of the variables x and y_T we get

$$\begin{aligned}
a_5 &= 0,\ a_4 = \frac{2}{k}e^{-kT}a_6,\ a_6 = 0 \\
a_1 &= -\frac{e^{kT}}{k^2}a_3
\end{aligned} \tag{5.17}$$

As a solution, we have a one parameter symmetry a_3 which is compatible with the terminal conditions (3.25). Then we can write the symmetry as one-parameter point symmetries:

$$\Lambda = -\frac{e^{kt}}{k}\partial_x + \left(-\frac{e^{kT}}{k^2} + \frac{e^{kt}}{k^2}\right)\partial_y, \tag{5.18}$$

From the invariance under Λ we get

$$I_1 = t,\ I_2 = ke^{kt}y + (e^{kt} - e^{kT})x, \ \text{ and } V = I_3$$

The relation among the invariants is given by $I_3 = \phi(I_1, I_2)$. Then the invariant solution of the PDE compatible with the terminal condition is $V = \phi(t, z)$, where

$z = I_2 = ke^{kt}y + (e^{kt} - e^{kT})x$. The function $\phi(t, z)$ is solution of the equation

$$\begin{aligned}\phi_t &= r\phi - \left[kz + kx_{ref}e^{kt} + (w_3 + w_1 \cos\frac{\pi t}{6} + w_2 \sin\frac{\pi t}{6})(e^{kt} - e^{kT})\right]\phi_z + \\ &\quad -\frac{1}{2}\sigma^2(e^{kt} - e^{kT})^2\phi_{zz}. \end{aligned} \tag{5.19}$$

With the symmetry Λ, the PDE was reduced by one independent variable. We can find the symmetries of the reduced equation (5.19) and we use it to find their invariants solutions, which obviously will give the solution of the equation (5.1). To make the treatment of the equation (5.19) we perform some simpliplications on the Lie equations, so that the invariants become

$$I_1 = t,\ I_2 = y + \frac{(1 - e^{k(T-t)})}{k}x,\ \text{ and } V = I_3,$$

and $\phi(t, z)$ solution of the equation

$$\begin{aligned}\phi_t &= r\phi - \left[x_{ref} + (w_3 + w_1 \cos\frac{\pi t}{6} + w_2 \sin\frac{\pi t}{6})\frac{(1 - e^{k(T-t)}}{k})\right]\phi_z + \\ &\quad -\frac{1}{2k^2}\sigma^2(1 - e^{k(T-t)})^2\phi_{zz}. \end{aligned} \tag{5.20}$$

For the PDE (5.20), the terminal conditions (3.25) and (3.26) become respectively

- for an RDD European put

$$\phi(T, z) = tick \times (S - z)^+ \tag{5.21}$$

- and, for an RDD European call

$$\phi(T, z) = tick \times (z - S)^+ \tag{5.22}$$

The equation (5.20) belong to the classe of $(1 + 1)$ parabolic PDE with general form

$$u_t = a(t, z)u_z z + b(t, z)u_z + c(t, z)u. \tag{5.23}$$

a, b and c are continuous functions in t and z. Lie gave the complete group classification of the PDE (5.23) providing all the canonical forms for which the PDE admits nontrivial point symmetry. In [39] was presented the complete characterization of the PDE (5.23) in terms of the invariant and its reduction to the four Lie canonical forms. The general conditions for the PDE (5.23) to be reduced to the heat equation and the transformation to be done were summarised in the following theorem.

Theorem 5.4.1 (theorem 3 in [39])
The following statements are equivalent:

1. the scalar linear $(1+1)$ parabolic PDE (5.23) has six nontrivial point symmetries in addition to the infinite number of superposition symmetries;

2. the coefficients of the parabolic equation (5.23) satisfies the invariant equation

$$2L_z + 2M_z - N_z = 0, \tag{5.24}$$

where

$$L = a^{\frac{1}{2}}[a^{\frac{1}{2}}J_z]_z,\ M = a^{\frac{1}{2}}[a^{\frac{1}{2}}\partial_t(\frac{b}{2a})]_z,\ N = a^{\frac{1}{2}}\partial_t^2(\frac{1}{a^{\frac{1}{2}}}) \tag{5.25}$$

and J is given by

$$J = c - \frac{b_z}{2} + \frac{ba_z}{2a} + \frac{a_{zz}}{4} - \frac{3a_z^2}{16a} - \frac{a_t}{2a} - \frac{b^2}{4a}; \tag{5.26}$$

3. the linear parabolic equation (5.23) is reducible to the classical heat PDE $\bar{u}_{\bar{t}} = \bar{u}_{\bar{z}\bar{z}}$ via the transformation

$$\begin{aligned}
\bar{t} &= \varphi(t),\\
\bar{z} &= \pm\int[\dot{\varphi}a(t,z)^{-1}]^{\frac{1}{2}}dz + \beta(t),\\
\bar{u} &= \nu(t)|a(t,z)|^{\frac{-1}{4}}u\times \\
&\times Exp\left\{\int\frac{b(t,z)}{2a(t,z)}dz - \frac{\ddot{\varphi}}{8\dot{\varphi}}\left[\int\frac{dz}{a(t,z)^{\frac{1}{2}}}\right]^2\right\}\times\\
&\times Exp\left\{-\frac{1}{2}\int\frac{1}{a(t,z)^{\frac{1}{2}}}\partial_t\left[\int\frac{dz}{a(t,z)^{\frac{1}{2}}}\right]dz \mp \frac{\dot{\beta}}{2\dot{\varphi}^{\frac{1}{2}}}\int\frac{dz}{a(t,z)^{\frac{1}{2}}}\right\}
\end{aligned} \tag{5.27}$$

for which φ, β and ν are constructed from

$$\begin{aligned}
f(t) &= \frac{\ddot{\varphi}^2}{16\dot{\varphi}^2} - \frac{1}{8}\left(\frac{\ddot{\varphi}}{\dot{\varphi}}\right)_t,\\
g(t) &= \pm\frac{\ddot{\varphi}\dot{\beta}}{4\dot{\varphi}\dot{\varphi}^{\frac{1}{2}}} \mp \frac{1}{2}\left(\frac{\dot{\beta}}{\dot{\varphi}^{\frac{1}{2}}}\right)_t,\\
h(t) &= \frac{\ddot{\varphi}}{4\dot{\varphi}} + \frac{\dot{\beta}^2}{4\dot{\varphi}} + \frac{\dot{\nu}}{\nu}
\end{aligned} \tag{5.28}$$

with functions f, g and h constrained by the relation

$$\begin{aligned}
&J + \partial_t\int\frac{b}{2a}dz - \frac{1}{2}\int\frac{1}{a^{\frac{1}{2}}}\partial_t^2\left(\int\frac{1}{a^{\frac{1}{2}}}dz\right)dz + f(t)\left(\int\frac{1}{a^{\frac{1}{2}}}dz\right)^2 +\\
&+g(t)\int\frac{1}{a^{\frac{1}{2}}}dz + h(t) = 0.
\end{aligned} \tag{5.29}$$

Now we apply the invariant criteria provided with the **Theorem** 5.4.1 to reduce the equation (5.20) into first Lie canonical form. Comparing the PDE (5.20) with the (1+1) general parabolic PDE we can writte the following coefficients:

$$
\begin{aligned}
a(t,z) &= -\frac{1}{2k^2}\sigma^2\left(1-e^{k(T-t)}\right)^2, \\
b(t,z) &= -x_{ref} - \left(w_3 + w_1\cos\frac{\pi t}{6} + w_2\sin\frac{\pi t}{6}\right)\frac{1-e^{k(T-t)}}{k}, \\
c(t,z) &= r.
\end{aligned} \tag{5.30}
$$

To evaluate the invariant condition (5.24), we need the values of J, L, M and N given also in the **Theorem** 5.4.1 so that,

$$
\begin{aligned}
J &= r - \frac{ke^{k(T-t)}}{1-e^{k(T-t)}} - \frac{\left(\frac{kx_{ref}}{\left(1-e^{-kt+kT}\right)^2} + \left(w_3 + w_1\cos\frac{\pi t}{6} + w_2\sin\frac{\pi t}{6}\right)\right)^2}{2\sigma^2}, \\
L &= 0, \\
M &= 0, \\
N &= \frac{e^{kt}(e^{kt}+e^{kT})k^2}{(e^{kt}-e^{kT})^2}.
\end{aligned} \tag{5.31}
$$

Note that the functions L, M and N are independent of z, so that we get $L_z = M_z = N_z = 0$. With the substitutions of these values to the invariant condition of **Theorem** 5.4.1, we can see that the invariant condition is satisfied. In the **Theorem** 5.4.1 all the statements are equivalent, then the PDE (5.20) can be reduced to the heat equation. We obtain the transformations (5.27) defined in **Theorem** 5.4.1. The functions φ, β and ν are obtained by solving the ODEs in (5.28). First we need to obtain the functions $f(t)$, $g(t)$ and $h(t)$ by equating the coefficients of the same power of z in (5.29), we have the following result:

$$
\begin{aligned}
f(t) &= \frac{e^{kT}\left(e^{kt}+e^{kT}\right)k^2}{4\left(e^{kt}-e^{kT}\right)^2}, \\
g(t) &= \frac{-6k\left[-e^{2kT}w_3 + e^{k(t+T)}(w_3 + 2kx_{ref})\right]}{6\sqrt{2}e^{2kt}\sqrt{-\left(e^{kt}-e^{kT}\right)^4\sigma^2}} + \\
&\quad -\frac{\left(e^{kt}-e^{kT}\right)\left[-e^{kt}\pi w_2 + e^{kT}(6kw_1 + \pi w_2)\right]\cos\left[\frac{\pi t}{6}\right]}{6\sqrt{2}e^{2kt}\sqrt{-\left(e^{kt}-e^{kT}\right)^4\sigma^2}} + \\
&\quad -\frac{\left(e^{kt}-e^{kT}\right)\left[e^{kt}\pi w_1 + e^{kT}(-\pi w_1 + 6kw_2)\right]\sin\left[\frac{\pi t}{6}\right]}{6\sqrt{2}e^{2kt}\sqrt{-\left(e^{kt}-e^{kT}\right)^4\sigma^2}}, \\
h(t) &= \frac{e^{kT}k}{e^{kt}-e^{kT}} - r +
\end{aligned} \tag{5.32}
$$

$$-\frac{\left[kx_{ref}-\left(-1+e^{k(-t+T)}\right)\left(w_3+w_1\cos\left[\frac{\pi t}{6}\right]+w_2\sin\left[\frac{\pi t}{6}\right]\right)\right]^2}{2\left(\sigma-e^{k(-t+T)}\sigma\right)^2}.$$

By solving the ODEs in (5.28) we obtain the functions φ, β and ν

$$\begin{aligned}
\varphi(t) &= -\frac{e^{2kt}}{2k\left[-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1)\right]},\\
\beta(t) &= \pm\int[\int\frac{e^{kt}}{3(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))\sqrt{-2\left(e^{kt}-e^{kT}\right)^2\sigma^2}}\times\\
&\times(-6k(-e^{2kT}w_3+e^{k(t+T)}(w_3+2kx_{ref}))+\\
&-\left(e^{kt}-e^{kT}\right)\left(-e^{kt}\pi w_2+e^{kT}(6kw_1+\pi w_2)\right)\cos\left[\frac{\pi t}{6}\right]+\\
&-\left(e^{kt}-e^{kT}\right)\left(e^{kt}\pi w_1+e^{kT}(-\pi w_1+6kw_2)\right)\sin\left[\frac{\pi t}{6}\right])dt]dt,\\
\nu(t) &= \frac{e^{-\frac{kt}{2}}\left(e^{kt}-e^{kT}\right)^{-\frac{1}{2}}\left(e^{kt}-e^{kT}\right)}{e^{rt+kt}\left(e^{2kT}-4e^{k(t+T)}-2e^{2kt}(kt+c_1)\right)^{-\frac{1}{2}}}\times\\
&\times Exp\{\int[-\frac{(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))^2}{(e^{2kt}-e^{k(t+T)})^2}\times\\
&\times[\int\frac{e^{kt}}{(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))\sqrt{2}6\sqrt{-\left(e^{kt}-e^{kT}\right)^2\sigma^2}}\times\\
&\times(-6k(-e^{2kT}w_3+e^{k(t+T)}(w_3+2kx_{ref}))+\\
&-\left(e^{kt}-e^{kT}\right)\left(-e^{kt}\pi w_2+e^{kT}(6kw_1+\pi w_2)\right)\cos\left[\frac{\pi t}{6}\right]+\\
&-\left(e^{kt}-e^{kT}\right)\left(e^{kt}\pi w_1+e^{kT}(-\pi w_1+6kw_2)\right)\sin\left[\frac{\pi t}{6}\right])dt]^2]dt+\\
&-\frac{\left(kx_{ref}-\left(-1+e^{k(-t+T)}\right)\left(w_3+w_1\cos\left[\frac{\pi t}{6}\right]+w_2\sin\left[\frac{\pi t}{6}\right]\right)\right)^2}{2\left(\sigma-e^{k(-t+T)}\sigma\right)^2}\}
\end{aligned}\tag{5.33}$$

where c_1 is a integration constant. The transformations which reduce the PDE (5.20) into the heat equation are

$$\begin{aligned}
\bar{t} &= -\frac{e^{2kt}}{2k\left(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1)\right)},\\
\bar{z} &= \pm\frac{k\sqrt{2}e^{2kt}z}{\sqrt{-\sigma^2}\left(e^{2kT}-4e^{k(t+T)}-2e^{2kt}(kt+c_1)\right)}+\\
&\pm 2\int[\int\frac{e^{kt}}{6(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))\sqrt{-2\left(e^{kt}-e^{kT}\right)^2\sigma^2}}\times
\end{aligned}$$

$$
\begin{aligned}
&\times(-6k(-e^{2kT}w_3+e^{k(t+T)}(w_3+2kx_{ref}))+\\
&-\left(e^{kt}-e^{kT}\right)\left(-e^{kt}\pi w_2+e^{kT}(6kw_1+\pi w_2)\right)\cos\left[\frac{\pi t}{6}\right]+\\
&-\left(e^{kt}-e^{kT}\right)\left(e^{kt}\pi w_1+e^{kT}(-\pi w_1+6kw_2)\right)\sin\left[\frac{\pi t}{6}\right])dt]dt,\\
\bar{\phi} \;=\;& \frac{2^{\frac{1}{4}}\sqrt{k\left(e^{2kT}-4e^{k(t+T)}-2e^{2kt}(kt+c_1)\right)}}{e^{2kt+rt}\sqrt{\sigma}}\phi\times\\
&\times Exp\{\frac{k^2(x_{ref}+(w_3+w_1\cos\frac{\pi t}{6}+w_2\sin\frac{\pi t}{6})\frac{(1-e^{k(T-t)})}{k})z}{\sigma^2(1-e^{k(T-t)})^2}+\\
&\frac{k^3\left(2e^{3kt}-e^{3kT}+2e^{k(t+2T)}-2e^{k(2t+T)}(3+c_1)-2e^{k(2t+T)}kt\right)z^2}{2\sigma^2e^{kt}\left(1-e^{k(T-t)}\right)^3\left(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}c_1+2e^{2kt}kt\right)}+\\
&+\frac{k^3e^{kt}z^2}{2\sigma^2(1-e^{k(T-t)})^3}\mp\frac{\sqrt{2}(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))}{\sqrt{-\sigma^2}e^{kt}(e^{kt}-e^{kT})^2}\times\\
&\times[\int\frac{e^{kt}}{6(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))\sqrt{-2\left(e^{kt}-e^{kT}\right)^2\sigma^2}}\times\\
&\times(-6k(-e^{2kT}w_3+e^{k(t+T)}(w_3+2kx_{ref}))+\\
&-\left(e^{kt}-e^{kT}\right)\left(-e^{kt}\pi w_2+e^{kT}(6kw_1+\pi w_2)\right)\cos\left[\frac{\pi t}{6}\right]+\\
&-\left(e^{kt}-e^{kT}\right)\left(e^{kt}\pi w_1+e^{kT}(-\pi w_1+6kw_2)\right)\sin\left[\frac{\pi t}{6}\right])dt]z+\\
&+\int[-\frac{(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))^2}{(e^{2kt}-e^{k(t+T)})^2}\times\\
&\times[\int\frac{e^{kt}}{(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))\sqrt{2}6\sqrt{-\left(e^{kt}-e^{kT}\right)^2\sigma^2}}\times\\
&\times(-6k(-e^{2kT}w_3+e^{k(t+T)}(w_3+2kx_{ref}))+\\
&-\left(e^{kt}-e^{kT}\right)\left(-e^{kt}\pi w_2+e^{kT}(6kw_1+\pi w_2)\right)\cos\left[\frac{\pi t}{6}\right]+\\
&-\left(e^{kt}-e^{kT}\right)\left(e^{kt}\pi w_1+e^{kT}(-\pi w_1+6kw_2)\right)\sin\left[\frac{\pi t}{6}\right])dt]^2]+\\
&-\frac{\left(kx_{ref}-\left(-1+e^{k(-t+T)}\right)\left(w_3+w_1\cos\left[\frac{\pi t}{6}\right]+w_2\sin\left[\frac{\pi t}{6}\right]\right)\right)^2}{2\left(\sigma-e^{k(-t+T)}\sigma\right)^2}dt\}
\end{aligned}
\tag{5.34}
$$

By using the transformations (5.34) we reduce the equation (5.20) to the heat equation and we can use the well known fundamental solution of the heat equation. In [49] the fundamental solution of heat equation was presented in barred coordinates,

$$\bar{\phi}=\frac{1}{2\sqrt{\bar{t}\pi}}e^{-\frac{\bar{z}^2}{4\bar{t}}}. \tag{5.35}$$

We solve the transformation (5.34) in term of ϕ, and the solution of the PDE (5.20) is given by:

$$\begin{aligned}
\phi &= \frac{e^{2kt+rt}\sqrt{\sigma}}{2^{\frac{1}{4}}\sqrt{k\left(e^{2kT}-4e^{k(t+T)}-2e^{2kt}(kt+c_1)\right)}}\bar{\phi}\times \\
&\times Exp\{-[\frac{k^2(x_{ref}+(w_3+w_1\cos\frac{\pi t}{6}+w_2\sin\frac{\pi t}{6})\frac{(1-e^{k(T-t)})}{k})z}{\sigma^2(1-e^{k(T-t)})^2}+ \\
&\frac{k^3\left(2e^{3kt}-e^{3kT}+2e^{k(t+2T)}-2e^{k(2t+T)}(3+c_1)-2e^{k(2t+T)}kt\right)z^2}{2\sigma^2 e^{kt}\left(1-e^{k(T-t)}\right)^3\left(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}c_1+2e^{2kt}kt\right)}+ \\
&+\frac{k^3e^{kt}z^2}{2\sigma^2(1-e^{k(T-t)})^3}\mp\frac{\sqrt{2}(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))}{\sqrt{-\sigma^2}e^{kt}(e^{kt}-e^{kT})^2}\times \\
&\times[\int\frac{e^{kt}}{6(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))\sqrt{-2\left(e^{kt}-e^{kT}\right)^2\sigma^2}}\times \qquad (5.36)\\
&\times(-6k(-e^{2kT}w_3+e^{k(t+T)}(w_3+2kx_{ref}))+ \\
&-\left(e^{kt}-e^{kT}\right)\left(-e^{kt}\pi w_2+e^{kT}(6kw_1+\pi w_2)\right)\cos\left[\frac{\pi t}{6}\right]+ \\
&-\left(e^{kt}-e^{kT}\right)\left(e^{kt}\pi w_1+e^{kT}(-\pi w_1+6kw_2)\right)\sin\left[\frac{\pi t}{6}\right])dt]z+ \\
&+\int[-\frac{(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))^2}{(e^{2kt}-e^{k(t+T)})^2}\times \\
&\times[\int\frac{e^{kt}}{(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))\sqrt{2}6\sqrt{-\left(e^{kt}-e^{kT}\right)^2\sigma^2}}\times \\
&\times(-6k(-e^{2kT}w_3+e^{k(t+T)}(w_3+2kx_{ref}))+ \\
&-\left(e^{kt}-e^{kT}\right)\left(-e^{kt}\pi w_2+e^{kT}(6kw_1+\pi w_2)\right)\cos\left[\frac{\pi t}{6}\right]+ \\
&-\left(e^{kt}-e^{kT}\right)\left(e^{kt}\pi w_1+e^{kT}(-\pi w_1+6kw_2)\right)\sin\left[\frac{\pi t}{6}\right])dt]^2]+ \\
&-\frac{\left(kx_{ref}-\left(-1+e^{k(-t+T)}\right)\left(w_3+w_1\cos\left[\frac{\pi t}{6}\right]+w_2\sin\left[\frac{\pi t}{6}\right]\right)\right)^2}{2\left(\sigma-e^{k(-t+T)}\sigma\right)^2}]dt\}
\end{aligned}$$

we substitute the solution of heat equation (5.35), $\bar{z}$ and $\bar{t}$ in the solution (5.36), the solution of the PDE (5.20) become

$$\begin{aligned}
\phi &= \frac{\sqrt{\sigma}e^{rt+kt}}{2^{\frac{3}{4}}\sqrt{-\pi\left(1-e^{k(T-t)}\right)^{\frac{x_{ref}(x_{ref}+2w_3)}{k\sigma^2}}}}\times \\
&\times Exp\{-[\frac{k^2(x_{ref}+(w_3+w_1\cos\frac{\pi t}{6}+w_2\sin\frac{\pi t}{6})\frac{(1-e^{k(T-t)})}{k})z}{\sigma^2(1-e^{k(T-t)})^2}+ \\
&\frac{k^3\left(2e^{3kt}-e^{3kT}+2e^{k(t+2T)}-2e^{k(2t+T)}(3+c_1)-2e^{k(2t+T)}kt\right)z^2}{2\sigma^2 e^{kt}\left(1-e^{k(T-t)}\right)^3\left(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}c_1+2e^{2kt}kt\right)}+
\end{aligned}$$

$$
\begin{aligned}
&+\frac{k^3 e^{kt} z^2}{2\sigma^2(1-e^{k(T-t)})^3} \mp \frac{\sqrt{2}(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))}{\sqrt{-\sigma^2}e^{kt}(e^{kt}-e^{kT})^2} \times \\
&\times [\int \frac{e^{kt}}{6(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))\sqrt{-2\left(e^{kt}-e^{kT}\right)^2\sigma^2}} \times \\
&\times (-6k(-e^{2kT}w_3+e^{k(t+T)}(w_3+2kx_{ref}))+ \\
&-\left(e^{kt}-e^{kT}\right)\left(-e^{kt}\pi w_2+e^{kT}(6kw_1+\pi w_2)\right)\cos\left[\frac{\pi t}{6}\right]+ \\
&-\left(e^{kt}-e^{kT}\right)\left(e^{kt}\pi w_1+e^{kT}(-\pi w_1+6kw_2)\right)\sin\left[\frac{\pi t}{6}\right])dt]z+ \\
&+\int[-\frac{(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))^2}{(e^{2kt}-e^{k(t+T)})^2} \times \\
&\times [\int \frac{e^{kt}}{(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))\sqrt{2}6\sqrt{-\left(e^{kt}-e^{kT}\right)^2\sigma^2}} \times \\
&\times (-6k(-e^{2kT}w_3+e^{k(t+T)}(w_3+2kx_{ref}))+ \\
&-\left(e^{kt}-e^{kT}\right)\left(-e^{kt}\pi w_2+e^{kT}(6kw_1+\pi w_2)\right)\cos\left[\frac{\pi t}{6}\right]+ \qquad (5.37)\\
&-\left(e^{kt}-e^{kT}\right)\left(e^{kt}\pi w_1+e^{kT}(-\pi w_1+6kw_2)\right)\sin\left[\frac{\pi t}{6}\right])dt]^2]+ \\
&-\frac{1}{2\left(\sigma-e^{k(-t+T)}\sigma\right)^2}[kx_{ref}^2 e^{k(-t+T)}+w_3^2\left(-1+e^{k(-t+T)}\right)^2+ \\
&+x_{ref}(x_{ref}+w_3)(k-e^{k(T-t)})-2\left(kx_{ref}-w_3(-1+e^{k(T-t)})\right)\times \\
&\times\left(-1+e^{k(-t+T)}\right)\left(w_3+w_1\cos\left[\frac{\pi t}{6}\right]+w_2\sin\left[\frac{\pi t}{6}\right]\right)+ \\
&+\left(\left(-1+e^{k(-t+T)}\right)\left(w_3+w_1\cos\left[\frac{\pi t}{6}\right]+w_2\sin\left[\frac{\pi t}{6}\right]\right)\right)^2]]dt+ \\
&+\frac{2k\left(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1)\right)}{4e^{kt}}\times \\
&\times[\pm\frac{k\sqrt{2}e^{2kt}z}{\sqrt{-\sigma^2}\left(e^{2kT}-4e^{k(t+T)}-2e^{2kt}(kt+c_1)\right)}+ \\
&\pm 2\int[\int\frac{e^{kt}}{6(-e^{2kT}+4e^{k(t+T)}+2e^{2kt}(kt+c_1))\sqrt{-2\left(e^{kt}-e^{kT}\right)^2\sigma^2}}\times \\
&\times (-6k(-e^{2kT}w_3+e^{k(t+T)}(w_3+2kx_{ref}))+ \\
&-\left(e^{kt}-e^{kT}\right)\left(-e^{kt}\pi w_2+e^{kT}(6kw_1+\pi w_2)\right)\cos\left[\frac{\pi t}{6}\right]+ \\
&-\left(e^{kt}-e^{kT}\right)\left(e^{kt}\pi w_1+e^{kT}(-\pi w_1+6kw_2)\right)\sin\left[\frac{\pi t}{6}\right])dt]dt]^2\}.
\end{aligned}
$$

In [49] the following results was applied. Consider the evolution of the Cauchy problem

$$
u_t + a(t,z)u_{zz} + b(t,z)u_z + c(t,z)u = f(t,z), \qquad (5.38)
$$

$$u(0,z) \quad = \quad u_0(x), \tag{5.39}$$

where $a(t,z)$, $b(t,z)$, $c(t,z)$, $u_0(x)$ and $f(t,z)$ are sufficiently smooth functions. In addition, we assume the existence and uniqueness for the Cauchy problem (5.38) its solution is given by

$$u(t,z) = \int_{\mathbb{R}} S(t,z,\varsigma,0)u_0(\varsigma)d\varsigma + \int_{\mathbb{R}\times[t,0]} S(t,z,\varsigma,\tau)f(\tau,\varsigma)d\varsigma d\tau, \tag{5.40}$$

with the function $S(t,z,\varsigma,\tau)$ being the fundamental solution of the Cauchy problem (5.38).

The solution (5.37) will be the fundamental solution for the Cauchy problem (5.20)-(5.22) [49], if is provide that

$$\lim_{t\to t_0} \phi(t,z) = \delta(z-z_0), \tag{5.41}$$

where $\delta(z-z_0)$ is the Dirac function and is defined by the well-known limit

$$\lim_{p\to 0} \frac{1}{\sqrt{p\pi}} e^{\frac{(z-z_0)^2}{4p}} = \delta(z-z_0). \tag{5.42}$$

The big challenge is to verify if the solution (5.37) satisfies the condition (5.41) for $t_0 = T$ and use it in (5.40) to determine the solution of the Cauchy problem (5.20)-(5.22). If condition (5.41) is satisfied, then the solution of the Cauchy problem (5.20)-(5.22) will be given by

$$\phi(t,z) = \int_{-\infty}^{\infty} \phi(t,z-\varsigma) tick\cdot(\varsigma - S)^+ d\varsigma \tag{5.43}$$

where $\phi(t,z-\varsigma)$ is fundamental solution (5.37). The solution (5.43) must have the following property

$$\lim_{t\to T} \phi(t,z) = tick\cdot(z-S)^+. \tag{5.44}$$

5.5 One Dimensional Optimal System of the PDE

We construct an optimal system of one dimensional subalgebra of equation (5.1) and apply the direct algorithm of one-dimensional optimal system presented in [23]. The general operator takes the form

$$X = a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 + a_6X_6 \tag{5.45}$$

and corresponding to a vector of their coefficients

$$a = (a_1, a_2, a_3, a_4, a_5, a_6).$$

The goal is to simplify as many as possible the elements a_i being zero by application of the adjoint transformation in X. We applied the direct algorithm by following the steps:

1. From the commutator **Table.**5.1 we take the matrix of the structure of constants $C(j),\ j = 1, \cdots, 6$, i.e., each column determine one matrix;
2. We determine all the vectors rows

$$(a_1, a_2, a_3, a_4, a_5, a_6)C(j),\ j = 1, \cdots, 6;$$

3. The $R - rank(K(a))$ functionally independents invariants $I(a)$ are found by solving the linear system $aC(j)\nabla I(a) = 0$, $K(a)$ is the $R \times R$ matrix whose j th row is $aC(j), j = 1, \cdots, R$ (also called Killing matrix in [45] from which the invariant is determined as a trace of $K^2(a)$).
4. We determine the adjoint matrix

$$A(j, \varepsilon) = \exp^{(\varepsilon C(j))} = \Sigma_0^\infty C(j)^n \frac{\varepsilon^n}{n!},\ j = 1, \cdots 6 \tag{5.46}$$

from the adjoint representation **Table.** 5.2. Each row corresponding to one matrix $A(j, \varepsilon)$ by using each position to fill its rows;

5. We construct the general adjoint transformation matrix

$$A = \prod_{j=1}^{6} A(j, \varepsilon_j) \tag{5.47}$$

and we use to simplify a, we look for a_i associated with the invariants and consider all subcases. We select the simplest representative $\tilde{X}$ by solving the adjoint transformation equation

$$\tilde{a} = aA \text{ or } a = \tilde{a}A, \tag{5.48}$$

so that X must be equivalent to $\tilde{X}$ under the adjoint action, if the system has solution.

The following adjoint representation table was constructed by using the *SYM package* of the software *Wolfram Mathematica* [16].

TABLE 5.2: The Adjoint Table of Subalgebras

Ad	X_1	X_2	X_3	X_4	X_5	X_6
X_1	X_1	X_2	X_3	$-\frac{k^2\varepsilon}{\sigma^2}X_2 + X_4$	X_5	X_6
X_2	X_1	X_2	X_3	X_4	X_5	X_6
X_3	X_1	X_2	X_3	X_4	$k\varepsilon X_3 + X_5$	$-\frac{2\varepsilon}{k\sigma^2}X_2 + X_6$
X_4	$X_1 + \frac{k^2\varepsilon}{\sigma^2}X_2$	X_2	X_3	X_4	$\varepsilon X_1 + \frac{-k^2x_{ref}+kw_3}{\sigma^2}\varepsilon X_2 + \frac{k^2\varepsilon^2}{2\sigma^2}X_2 + X_5$	X_6
X_5	X_1	X_2	$e^{-k\varepsilon}X_3$	X_4	X_5	$-e^{k\varepsilon}X_6$
X_6	X_1	X_2	$\frac{2\varepsilon}{k\sigma^2}X_2 + X_3$	$-\varepsilon X_1 + \frac{k^2x_{ref}-kw_3}{\sigma^2}\varepsilon X_2 + X_4$	$X_5 - k\varepsilon X_6$	X_6

The coefficients of the linear system $aC(j)\nabla I(a) = 0$ can be extracted directly from the commutator **Table.**5.1, and the system become,

$$\begin{cases} \frac{k^2a_4}{\sigma^2}\frac{\partial I}{\partial a_2} = 0 \\ \frac{2a_6}{k\sigma^2}\frac{\partial I}{\partial a_2} - ka_5\frac{\partial I}{\partial a_3} = 0 \\ -a_5\frac{\partial I}{\partial a_1} + \left[-\frac{k^2a_1}{\sigma^2} + \frac{(k^2x_{ref}+kw_3)a_5}{\sigma^2}\right]\frac{\partial I}{\partial a_2} = 0 \\ a_4\frac{\partial I}{\partial a_1} - \frac{(k^4x_{ref}+kw_4)a_4}{\sigma^2}\frac{\partial I}{\partial a_2} + ka_3\frac{\partial I}{\partial a_3} - ka_6\frac{\partial I}{\partial a_6} = 0 \\ -\frac{2a_3}{k\sigma^2}\frac{\partial I}{\partial a_2} + ka_5\frac{\partial I}{\partial a_6} = 0 \end{cases} \tag{5.49}$$

By Solving the system (5.49), we can found the invariants $I_1 = a_5$ and $I_2 = a_4$.

According to [23], the first step in constructing the one-dimensional optimal system of the finite dimensional Lie algebra is to scale the invariants as in the following situations:

- **A.** If the degree of the invariant is odd, we need to consider two cases: $I = 0$ and $I \neq 0$ (for simplicity we scale it to 1 or -1);
- **B.** If the degree of the invariant is even (excluding zero), three cases are considered: $I = 0$, $I > 0$ and $I < 0$ (if we scale we get respectively the correspondents cases $I = 0$, $I = 1$ and $I = -1$);
- **C.** Once one of the invariants is scaled (not zero), the other invariants (if any) can not be adjusted.

The invariants of the equation (5.1) are $I_1 = a_5$ and $I_2 = a_4$, both with degree one, then we can consider the cases

$$\{I_1 \neq 0, I_2 \neq 0\}, \ \{I_1 = 0, (I \neq 0 \vee I_2 = 0)\} \tag{5.50}$$

and the subcases given by the new invariant

$$I = a_3 a_6 \tag{5.51}$$

found by substitute $a_5 = 0$ in the system $aC(j)\nabla I(a) = 0$.

Once we have the invariants of the equation (5.1), we need to determine the general adjoint transformation matrix A, which is given in (5.47). The components $A(j,\varepsilon)$ extracted from ajoint **Table.**5.2 are:

$$A(1,\varepsilon_1) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{k^2\varepsilon_1}{\sigma^2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad A(2,\varepsilon_2) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A(3,\varepsilon_3) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k\varepsilon_3 & 0 & 1 & 0 \\ 0 & -\frac{2\varepsilon_3}{k\sigma^2} & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A(4,\varepsilon_4) = \begin{pmatrix} 1 & \frac{k^2\varepsilon_4}{\sigma^2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \varepsilon_4 & -\frac{k(w_3+kx_ref)\varepsilon_4}{\sigma^2} + \frac{k^2\varepsilon_4^2}{2\sigma^2} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A(5,\varepsilon_5) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{-k\varepsilon_5} & 0 & 0 & 0 \\ -\varepsilon_5 & \frac{k(w_3+kx_ref)\varepsilon_5}{\sigma^2} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{k\varepsilon_5} \end{pmatrix},$$

$$A(6,\varepsilon_6) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{2\varepsilon_6}{k\sigma^2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -k\varepsilon_6 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

then,

$$A = \begin{pmatrix} 1 & \frac{k^2\varepsilon_4}{\sigma^2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{2e^{-k\varepsilon_5}\varepsilon_6}{k\sigma^2} & e^{-k\varepsilon_5} & 0 & 0 & 0 \\ -\varepsilon_5 & -\frac{k^2\varepsilon_1}{\sigma^2} + \frac{k(w_3+kx_ref)\varepsilon_5}{\sigma^2} & 0 & 1 & 0 & 0 \\ \varepsilon_4 & -\frac{k(w_3+kx_ref)\varepsilon_4}{\sigma^2} + \frac{k^2\varepsilon_4^2}{2\sigma^2} + \frac{2e^{-k\varepsilon_5}\varepsilon_3\varepsilon_6}{\sigma^2} & e^{-k\varepsilon_5}k\varepsilon_3 & 0 & 1 & -k\varepsilon_6 \\ 0 & -\frac{2\varepsilon_3}{k\sigma^2} & 0 & 0 & 0 & e^{k\varepsilon_5} \end{pmatrix}. \quad (5.52)$$

The adjoint transformation equation (5.48) is

$$\begin{cases} \tilde{a}_1 = a_1 + a_5\varepsilon_4 - a_4\varepsilon_5 \\ \tilde{a}_2 = a_2 + a_4\left(-\frac{k^2\varepsilon_1}{\sigma^2} + \frac{k(w_3+kx_ref)\varepsilon_5}{\sigma^2}\right) + \\ \qquad +a_5\left(-\frac{k(w_3+kx_ref)\varepsilon_4}{\sigma^2} + \frac{k^2\varepsilon_4^2}{2\sigma^2} + \frac{2e^{-k\varepsilon_5}\varepsilon_3\varepsilon_6}{\sigma^2}\right) + \\ \qquad -\frac{2a_6\varepsilon_3}{k\sigma^2} + \frac{a_1k^2\varepsilon_4}{\sigma^2} + \frac{2a_3e^{-k\varepsilon_5}\varepsilon_6}{k\sigma^2} \\ \tilde{a}_3 = a_3e^{-k\varepsilon_5} + a_5e^{-k\varepsilon_5}k\varepsilon_3 \\ \tilde{a}_4 = a_4 \\ \tilde{a}_5 = a_5 \\ \tilde{a}_6 = a_6e^{k\varepsilon_5} - a_5k\varepsilon_6 \end{cases} \quad (5.53)$$

The goal is to find for each case (5.50)-(5.51) the representative $\tilde{a}$ for which the system has solution. Some algebraic details for computation and simplifications in resolution of the system of linear equations $\tilde{a} = aA$ will be omitted.

Case 1: $I_1 = a_5 = 1,\ I_2 = a_4 = c,\ c \in \mathbb{R}$
We can simplify (5.53) by vanishing a_1, a_2, a_3 and a_6, choosing

$$\begin{cases} \varepsilon_1 \in \mathbb{R}, \\ \varepsilon_4 = \frac{-a_1 + a_4\varepsilon_5}{a_5}, \\ \varepsilon_3 = \frac{-a_3}{ka_5}, \\ \varepsilon_5 = \frac{-\sigma^2 a_2}{k(w_3+kx_ref)} + a_4 \frac{k\varepsilon_1}{(w_3+kx_ref)} - a_5\left(-\varepsilon_4 + \frac{k\varepsilon_4^2}{2(w_3+kx_ref)} + \frac{2e^{-k\varepsilon_5}\varepsilon_3\varepsilon_6}{k(w_3+kx_ref)}\right) + \\ \quad + \frac{2a_6\varepsilon_3}{k^2(w_3+kx_ref)} - \frac{a_1k\varepsilon_4}{(w_3+kx_ref)} - \frac{2a_3e^{-k\varepsilon_5}\varepsilon_6}{k^2(w_3+kx_ref)}, \\ \varepsilon_6 = \frac{a_6e^{k\varepsilon_5}}{ka_5} \end{cases}$$

The corresponding representative adjoint equivalent vector for all vectors of the form $a = (a_1, a_2, a_3, a_4, 1, a_6)$ is $\tilde{a} = (0, 0, 0, a_4, 1, 0)$.

Case 2: $I_1 = a_5 = 0$,

- $I_2 = a_4 = 1$: we can vanish the elements a_1 and a_2 by choosing ε_3, ε_4, $\varepsilon_6 \in \mathbb{R}$

$$\begin{aligned} \varepsilon_5 &= \frac{a_1}{a_4}, \\ \varepsilon_1 &= \frac{\sigma^2 a_2}{k^2} + a_4\frac{(w_3 + kx_ref)\varepsilon_5}{k} - \frac{2a_6\varepsilon_3}{k^3} + a_1\varepsilon_4 + \frac{2a_3e^{-k\varepsilon_5}\varepsilon_6}{k^3}. \end{aligned}$$

  The representative of all the vectors of the form $a = (a_1, a_2, a_3, 1, 0, a_6)$ is $\tilde{a} = (0, 0, a_3, 1, 0, a_6)$

- $I_2 = a_4 = 0$: we can vanish a_2 by choosing

$$\varepsilon_4 = \frac{-k\sigma^2 a_2 + 2a_6\varepsilon_3 - 2a_3\varepsilon_6}{k^3 a_1}, \quad a_1 \neq 0 \quad \wedge \varepsilon_3, \varepsilon_6 \in \mathbb{R}$$

  If we scale a_1 by $a_1 = 1$ we have the representative vector for all vectors of the form $a = (1, a_2, a_3, 0, 0, a_6)$ given by $\tilde{a} = (1, 0, a_3, 0, 0, a_6)$.

- $a_3a_6 = \pm 1$: we have to consider two cases, namely for $a_4 \neq 0$ and $a_4 = 0$ and we substitute a_3 in (5.53) by $a_3 = \pm\frac{1}{a_6}$. For $a_4 \neq 0$ we can vanish a_1 and a_2, the solution is included in the case $I_1 = a_5 = 0,\ \wedge I_2 = a_4 = 1$. For $a_4 = 0$ we can vanish a_2 and the solution is included in the case $I_1 = a_5 = 0,\ \wedge I_2 = a_4 = 0$

- $a_3a_6 = 0$: we have to consider the following three possibilities.

  - $a_3 \neq 0$ and $a_6 = 0$: for $a_4 \neq 0$ we can vanish a_1 and a_2. If we scale a_3 by $a_3 = 1$, the representative for all vectors of the form $a = (a_1, a_2, 1, a_4, 0, 0)$ is $\tilde{a} = (0, 0, 1, a_4, 0, 0)$. For $a_4 = 0$, we can vanish a_2 and after scaling a_3, the representative of all the vectors of the form $a = (a_1, a_2, 1, 0, 0, 0)$ is $\tilde{a} = (a_1, 0, 1, 0, 0, 0)$.

  - $a_3 = 0$ and $a_6 \neq 0$: we can also consider two possibilities. For $a_4 \neq 0$, we can vanish a_1 and a_2. If we scale a_6, all vectors of the form $a = (a_1, a_2, 0, a_4, 0, 1)$ has the representative $\tilde{a} = (0, 0, 0, a_4, 0, 1)$. For $a_4 = 0$, we can vanish a_2 and, after scaling a_6, the representative of all vectors of the form $a = (a_1, a_2, 0, 0, 0, 1)$ is $a_1, 0, \tilde{0}, 0, 0, 1$.

– $a_3 = a_6 = 0$: We also have two possibilities. For $a_4 \neq 0$, we can vanish a_1 and a_2, so that, after scaling a_4, all the vectors of the form $a = (a_1, a_2, 0, 1, 0, 0)$ are represented by $\tilde{a} = (0, 0, 0, 1, 0, 0)$. For $a_4 = 0$, we can vanish only a_2 for $a_1 \neq 0$, and after scaling a_1, the representative of all vectors of the form $a = (1, a_2, 0, 0, 0, 0)$ is $\tilde{a} = (1, 0, 0, 0, 0, 0)$.

– The Last possibility is to consider $a_1 = a_3 = a_4 = a_5 = a_6 = 0$, and after scaling a_2, the representative is $\tilde{a} = (0, 1, 0, 0, 0, 0)$.

The one dimensional optimal subalgebra for the equation (5.1) is,

$$X_1, X_2, X_4, X_3 + a_1X_1, X_3 + a_4X_4, X_5 + a_4X_4, X_6 + a_1X_1, X_6 + a_4X_4,$$
$$X_1 + a_3X_3 + a_6X_6, X_4 + a_3X_3 + a_6X_6.$$

Using the subalgebra $X_3 + aX_1$, we get the following invariants for the generator,

$$I_1 = t,\ I_2 = z = \frac{e^{kt}}{k}y + (\frac{e^{kt}}{k^2} + a)x,\ I_3 = V$$

and the invariant solution for the equation (5.1) is $V = \phi(t, z)$. The substitution of V into (5.1) yields to the reduced equation by one independent variable

$$\begin{aligned}\phi_t &= r\phi - \left[S + kz + (w_3 + w_1 \cos\frac{\pi t}{6} + w_2 \sin\frac{\pi t}{6})(\frac{e^{kt}}{k^2} + a)\right]\phi_z + \\ &\quad -\frac{1}{2}\sigma^2(\frac{e^{kt}}{k^2} + a)^2\phi_{zz} \qquad (5.54)\end{aligned}$$

then, $\phi(t, z)$ is a solution of the reduced equation (5.54).

Chapter 6

Conclusions and Future work

We applied Lie analysis of the partial differential equations with three independent variables, equation (3.24). The PDE was derived by applying the Fayman-Kac theorem for pricing weather derivatives when the rainfall process follow the Ornstein-Uhlenbeck process with constant volatility.

By the group classification we have shown that the Lie algebra of the PDE (3.24) depends on the parameters k and σ. The principal Lie algebra admits the symmetries ∂_y, $u\partial_u$ and $w(x,y,t)\partial_u$, where $w(x,y,t)$ is an solution of the equation (3.24). The PDE admits the maximal extension by 6 symmetries for $\sigma \neq 0 \wedge k = 0$, extension by 4 symmetries for $\sigma \neq 0, \wedge k(k^2+\pi^2) \neq 0$ and by 1 operators for $\sigma \neq 0, \wedge (k^2+\pi^2) = 0$.

We realized that the PDE can not be reduced to heat equations for any values of parameters, since it admits the extension by seven symmetries. We have used some subalgebra to find some solutions of the equation (3.24), although the solution is not compatible with our boundary conditions. By determining the symmetries which are compatible with our boundary conditions, we found a subalgebra with only one symmetry. We used it to reduce the equation (3.24) by one independent variable. We was not able to compute the symmetries of the reduced (1+1) parabolic PDE (5.20) by the computer programs. As alternative we applied the results from [39] that allowed the reduction of the PDE (5.20) to heat equation. From the fundamental solution of the heat equation we derived the candidate of the fundamental solution for parabolic PDE (5.20). As stated in [49] the produced solution is fundamental solution if provide that its limit is Dirac function.

Furthermore, we determined the one dimensional optimal system of the algebra admitted by the PDE (3.24) through algorithm suggested by Yu, Li and Chen in [23]. The optimal system allows to dived the set of all invariant solutions of the PDE into equivalent classes. The solutions which can be mapped to the other solution by a point symmetry of the PDE, are equivalent and belong to the same class. Once we have constructed a optimal system, we need only to find one invariant solution from each class, and the whole class can be constructed by applying the symmetries. We found ten classes of equivalence and we used one which reduced the PDE by one variable.

As future work We expect to verify that the solution (5.37) satisfy the conditions (5.41) for $t_0 = T$ so that if it is provide to be we will use it as a fundamental solution for the PDE (3.24) in order to derive the solutions compatible with terminal conditions and illustrate simulations of the derivative prices. Also we expect to investigate the reduction by others classes of equivalence and find all solutions.

Other research can be made to analytically investigate if this PDE can admit a complete number of symmetries, because can happen that the result of the group classification was influenced by the limitation of the computer programs in finding all symmetries of the equations. In addition for future works we also expect to investigate the discrete symmetries of this PDE since are also important and can be used as alternatives to simplify the numerical computations of the solution of the PDEs, to create new exact solutions from know solutions and to study the stability and integrability of the PDEs. More details about discrete symmetries for PDEs we refer for exemple to [26]

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