A study on the effect of dilutions and buybacks on the pricing of equity and stock based claims using a finite difference mesh

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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy at the University of Cape Town. It has not been submitted before for any degree or examination in any other University.

October 8, 2022

Abstract

We study a model of the firm, with perpetual debt and a continuously payable coupon as well as the possibility to raise cash via equity issuance. Excess cash is paid back to shareholders either via dividends or via buybacks. The number of shares changes when equity is issued and when the firm buys back shares. Using this model we track the total number of shares in issue. Then we use finite difference methods to investigate the differences in pricing options on a fixed portion of equity and options linked to the share price, as well as implications for American options on equity.

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Chapter 1

Introduction

The value of options tied to a firm are affected by the underlying firm's capital structure. In Backwell *et al.* (2022) the effect of buybacks and dilutions on the value of a firm's stock options is studied, along with some interesting results about the value of American options on a firm's equity. Backwell *et al.* (2022) conclude that if one ignores dilutions and buybacks, one can misprice options. Our aim is to illustrate this phenomenon using finite difference methods.

In their seminal paper on capital structure, Modigliani and Miller (1958) studied the choices regarding leverage and the capital structure of the firm. They were able to show that, in an idealised setting, the capital structure of a firm does not affect the value of the firm. They went on to develop a qualitative examination of the conflict between the tax advantage of debt and the higher cost of debt for highly leveraged firms. This model was taken further by Brennan and Schwartz (1978) who developed a quantitative examination of optimal capital structure. A study into the pricing of corporate debt was carried out by Merton (1974) on a firm with an asset value described by a diffusion type stochastic differential equation. He modelled a firm with zero coupon debt which all matured at the same point in time. Merton (1974) also priced equity as a call option on the asset value of the firm, with strike price being the value of the debt and the maturity of the option being the same time as the expiry of the debt. On the pricing of equity options Geske (1979) developed a pricing formula using the Merton (1974) model by modelling options on equity as a compound option, i.e. an option where the underlying is itself an option. Leland (1994) developed the theory around endogenous bankruptcy, or bankruptcy declared to be optimal for shareholders rather than forced by a debt covenant. His model also avoids the issue of abrupt changes in capital structure at debt maturity, which is present in the Merton (1974) and Geske (1979) models. In the Leland (1994) model, debt requires a continuous coupon payment c rather than a lump sum payment at maturity. Toft and Prucyk (1997) use this capital structure model to price options on equity and develop a closed form solution for the option price.

Recently Backwell *et al.* (2022) extends the model from Toft and Prucyk (1997) to form a distinction between options on a firm's equity and options on a single stock. The Backwell *et al.* (2022) or "BMR" model allows for dilutions and buybacks of stock so the ratio of the value of a single stock to total equity is not necessarily constant. The firm's capital structure consists of equity and perpetual debt, with a constant continuous coupon payment. In the case that the firm is unable to generate sufficient cash to meet its debt obligations, it issues additional equity (provided the value of equity is positive) and current shareholders are diluted. In the case that the firm generates surplus cash, a proportion can be used to buyback stocks, in which case current stockholders concentrate their share of total equity or 'cash out' and sell shares back to the firm. In practice options are usually sold on stocks or on batches of stocks, rather than on equity. It makes sense to develop models which price stock options and account for the possibility that the number of stocks in issue can change rather than price options which are actually on equity. Despite this, until recently only equity options have been treated in the literature, e.g. Toft and Prucyk (1997).

To price stock options we use a partial differential equation (or PDE) for a stock based claim and do numerical calculations using a finite difference mesh. Using the results from the finite difference mesh we are able to replicate two key results from the Backwell *et al.* (2022) paper. Firstly that the value of a stock call option is sensitive to the buyback policy of the firm. Secondly we show that the value of an American call option is not dependant on the firm's payout policy, and that it may be optimal to exercise an American option prior to expiry in the absence of dividends.

In chapter 2 we will run through the theory behind pricing equity options on a leveraged firm, eventually constructing a PDE for a contingent claim on the firm's asset value. Initially we consider a European and then an American option on equity. Following this we discuss the model for the number of stocks in issue and then price options on a single share. In chapter 3 we use the PDEs developed in chapter 2 and construct a finite difference scheme so that we can arrive at numerical results for option prices. In chapter 4 we discuss some interesting results, and finally chapter 5 concludes the study and notes that buybacks and dilutions can have a material effect on the price of stock options. So when pricing options we should consider the payout policy of the firm and other parameters which might lead to a change in the number of shares in issue.

Chapter 2

Model to Price Equity and Stock Options on a Leveraged Firm

In his paper Merton (1974) develops a Black Scholes type pricing model for a firm with debt and total asset value V. The dynamics for the value of the firm, given by V, are assumed to follow an Ito process with stochastic differential equation:

$$dV_t = (r - q)V_t dt + \sigma V_t dW_t. \tag{2.1}$$

Here r is the risk free rate, q is the proportional payout rate and dW_t is the standard increment for a Brownian motion under the risk neutral measure. We denote the risk neutral measure as $\mathbb{Q}$. The value of the assets is assumed to be independent of the capital structure, in line with Modigliani and Miller (1958). In the BMR model debt is modelled consistently with Leland (1994), i.e. the firm has issued debt which pays out a continuous coupon of c per year. The assets produce cash at a rate of q per year. So $qV_t dt$ is the after tax cash available to debt and equity holders over time period dt. The amount paid to debt holders net of tax is $(1 - \rho)cdt$ where ρ is the corporate tax rate. The after tax cash flow rate available to equity holders is given by Backwell *et al.* (2022):

$$\text{CFR}_t = qV_t - (1 - \rho)c.$$

When CFR_t is positive cash is paid out to shareholders in the form of either dividends or stock buybacks. When CFR_t is negative, the firm can either raise the requisite cash from the equity holders, which manifests as a "synthetic negative dividend" to shareholders, or declare bankruptcy. As studied in Leland (1994), Backwell *et al.* (2022) consider the case that there are no debt covenants, and that bankruptcy can only be declared at the shareholders discretion. Bankruptcy will only be declared when the firm is not able to raise sufficient cash to meet debt obligations. Shareholders have limited liability and are not required to inject cash, but shareholders (or prospective shareholders) will inject cash if the value of the firm's equity is positive. Backwell *et al.* (2022) defines the value of equity as the present value of expected cash flows (both positive and negative) to shareholders until a stopping time τ when bankruptcy is declared:

$$\mathrm{EQ}_t = \sup_{\tau \geq t} \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{\tau} e^{-\int_t^u r_s d_s} \mathrm{CFR}_u du \right], \tag{2.2}$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ is the expectation taken under the risk neutral measure, $\mathbb{Q}$. Backwell *et al.* (2022) show that in the pricing of equity, negative cash flows need to be included as synthetic negative dividends. Define τ_b as the optimal stopping time associated with equation (2.2). Backwell *et al.* (2022) defines the value of debt as:

$$\mathrm{CB}_t = \mathbb{E}_t^{\mathbb{Q}} \left[\int_t^{\tau_b} e^{-\int_t^u r_s ds} c_u du + e^{\int_t^{\tau_b} r_s ds} (1-b) V_{\tau_b} \right], \tag{2.3}$$

where b is the bankruptcy cost, i.e. the proportion of the asset value lost to bankruptcy expenses. In (2.3) the first term is the expected coupon payments before bankruptcy and the second term is the expected cash recovered at bankruptcy. Given that the firm has not yet gone bankrupt, both equity and debt are Markov processes. So they depend on the current asset value only and not the asset value history. Both equation (2.2) and equation (2.3) are therefore functions of V_t

The analytical values of debt and equity of the firm derived in Leland (1994) are:

$$\mathrm{CB}_t = \left(c/r + ((1-b)V_b - c/r) \left(\frac{V_t}{V_b} \right)^{-x} \right), \tag{2.4}$$

$$\mathrm{EQ}_t = \left(V_t - A + B \left(\frac{V}{V_b} \right)^{-x} \right) \mathbb{I}_{V_t > V_b}, \tag{2.5}$$

where:

$$A = \frac{(1-\rho)c}{r}, \qquad B = A - V_b,$$

$$\mu = r - q - \frac{1}{2}\sigma^2, \qquad x = \frac{\mu + \sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2},$$

and V_b is the asset value at which the firm goes bankrupt, referred to as the bankruptcy boundary. Leland (1994) derives the value for V_b as:

$$V_b = \frac{(1-\rho)cx}{r(x+1)}.$$

Note that these expressions are derived using differential equations and not with equations (2.2) and (2.3). Although Backwell *et al.* (2022) did show that the two definitions for debt and equity agree.

2.1 Options on leveraged equity

We first look to calculate the value of a European call option on equity at time t. The payoff or terminal value of a European call option on equity is given by:

$$\text{EO}_T = \max((\text{EQ}(V_T) - K)\mathbb{I}_{T<\tau_b}, 0). \tag{2.6}$$

While the time t value of the option is given by:

$$\text{EO}_t = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\max((\text{EQ}(V_T) - K)\mathbb{I}_{T<\tau_b}, 0)\right]. \tag{2.7}$$

If $\tau_b < T$ then the firm is bankrupt at time T and the value of equity, and hence the call option, is zero. The indicator function $\mathbb{I}_{T<\tau_b}$ captures this. An analytical solution to equation (2.7) is given in Toft and Prucyk (1997), at time $t = 0$:

$$\begin{aligned}
\text{EO}_0 = Ve^{-qT}&\left[N(y^* + \sigma\sqrt{T}) - \left(\frac{V_b}{V}\right)^{2\mu/\sigma^2+2} N\left(y^* + \sigma\sqrt{T} + \left(\frac{2v}{\sigma\sqrt{T}}\right)\right)\right] \\
&+ B\left(\frac{V_b}{V}\right)^x \left[N(y^* - x\sigma\sqrt{T}) - \left(\frac{V_b}{V}\right)^{2\mu/\sigma^2-2x} N\left(y^* - x\sigma\sqrt{T} + \left(\frac{2v}{\sigma\sqrt{T}}\right)\right)\right] \\
&- (A+K)e^{-rT}\left[N(y^*) - \left(\frac{V_b}{V}\right)^{2\mu/\sigma^2} N\left(y^* + \left(\frac{2v}{\sigma\sqrt{T}}\right)\right)\right],
\end{aligned} \tag{2.8}$$

where $N(\cdot)$ is the cumulative normal distribution function, $v = \ln\frac{V_b}{V}$ f and y^* is the smallest solution to the equation in y:

$$\begin{aligned}
K =& V\exp\left[(r - q - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T}(-y)\right] - A \\
&+ B\left(\frac{V_b}{V}\right)^x \exp\left[-x(r - q - \frac{1}{2}\sigma^2)T - x\sigma\sqrt{T}(-y)\right].
\end{aligned}$$

In the case that $A > V_b$ there are two solutions to y^*, the larger solution is meaningless however and can only be reached by crossing the bankruptcy boundary. Under this condition the smaller value for y^* must be used. It is also worth noting that we could price an option on a fixed portion of equity rather than on all of equity by scaling appropriately.

We seek an alternative method to price this option using a partial differential equation for a (path independent) contingent claim on the firm's equity as in Black and Cox (1976). Note that the European call on equity is in reality not path independent, if the asset value falls below V_b then the firm goes bankrupt and the option becomes worthless. This is captured by the indicator function in equation (2.7). However we make the approximation to ignore the indicator. In other words we approximate that if a firm's asset value crosses the bankruptcy boundary then its equity value will not rise back above the strike price so that the option expires in the money. For a call option we believe this is a reasonable approximation to make. Once this approximation is made we can use a PDE to price the claim. Let $f(t, V_t)$ be the value of this claim at t, the Markov property of equity allows for the price of the claim to be given by the function f. Then we can develop the PDE (see Appendix A.1):

$$\frac{\partial f}{\partial t} + (r-q)V\frac{\partial f}{\partial V} + \frac{1}{2}V^2\sigma^2\frac{\partial^2 f}{\partial V^2} - rf = 0. \tag{2.9}$$

We can impose suitable boundary conditions and the terminal condition, equation (2.6). Then calculate the backwards evolution of the system.

2.2 Buybacks and dilutions

Negative cashflow to equity holders or "synthetic negative dividends" result from a shortfall of cash to meet the firm's debt obligations. The deficit needs to be met by raising equity capital, i.e. issuing new shares in the primary market. This action will dilute current shareholders. Let us introduce a variable n_t, the number of shares in circulation at time t. Then we have that the value of a stock S at time t is given by:

$$S_t = \frac{\text{EQ}_t}{n_t}.$$

Now in the case that capital needs to be raised, more shares are issued. The number of shares issued is the cash shortfall divided by the share price. Using the share price defined above we have the following dynamics (Backwell *et al.*, 2022):

$$\frac{dn_t}{n_t} = \frac{\text{CFR}_t^-}{\text{EQ}_t}dt, \tag{2.10}$$

where CFR_t^- is the magnitude of the negative part of CFR_t. When there is a cash shortfall the right hand side of (2.10) is positive so there will be an increase in the number of shares in issue, n_t. In the case that the firm generates a cash surplus and disburses positive cashflow to shareholders, we make the assumption that the firm will disburse a constant ratio of this cash B_p to shareholders via stock buybacks while the balance is disbursed via conventional dividends. Note that the value of equity is not dependent on the parameter B_p. Equity is calculated as the total expected present value of all future cashflows to equity holders. Where 'all future cashflows' includes both dividends and buybacks, we will refer to these collectively as 'synthetic positive dividends'. B_p does not affect the amount of synthetic positive dividends paid out to shareholders and so has no effect on the value of equity, only on the number of shares. We can then develop a differential equation describing changes in the number of shares (Backwell *et al.*, 2022):

$$\frac{dn_t}{n_t} = \left(\frac{\text{CFR}_t^- - B_p \text{CFR}_t^+}{\text{EQ}_t} \right) dt. \tag{2.11}$$

We are now able to model the total number of shares in issue, along with the total value of equity. This allows us to compute the value of each stock, and so we can price stock based claims rather than just equity based claims. The key difference being that a stock is worth a variable portion of equity, while equity based claims are linked to a fixed portion of equity.

2.3 Stock options versus equity options

Using equation (2.11) we can find a solution for the number of shares in issue at time t:

$$n_t = n_0 \exp \left(\int_0^t \frac{\text{CFR}_u^- - B_p \text{CFR}_u^+}{\text{EQ}_u} du \right). \tag{2.12}$$

We now have expressions for the value of equity, and the number of stocks in issue. Equity is strictly a function of current asset value. However the number of shares, and hence the share price, is a function of past dilutions and buybacks. Put differently, the share price is path dependant while the equity value is not. Additionally, shareholders are indifferent as to whether they receive excess cash through dividends or buybacks. Stock option holders on the other hand are sensitive to B_p, which affects the final ratio of number of shares to equity, and hence the final share price. To illustrate why there exists a difference in sensitivities, consider the following simplified case: a firm has an equity value of 10, including 1 unit of cash and 10 shares in issue. Currently the value of each share is 1. If the unit of cash is disbursed as a dividend the final equity value will be 9. So each share will be worth 0.9 and each shareholder will have 0.1 in cash. Now if the firm had instead decided to use the cash to buyback shares, the final equity value would still be 9, but there will only be 9 shares in issue so the value of each share will still be 1 (and the shareholder who sold their share to the company has cash equal to this amount). This illustrates the effect that B_p can have on the expected payout of an option on a share. The price of a stock call option is given in Backwell *et al.* (2022):

$$\text{SO}_t = \mathbb{E}_t^{\mathbb{Q}} \left[e^{-\int_t^T r_s ds} \left(\frac{\text{EQ}_T \mathbb{I}_{\tau_b > T}}{n_T} - K \right)^+ \right]. \tag{2.13}$$

The option payoff is clearly path dependant, both the bankruptcy indicator and the number of shares n_T depend on the path. This contrasts with the equity call option payoff, given in 2.6, where path dependence lies entirely in the indicator function, and can be approximated away.

For a contingent claim on equity, f, we can develop the PDE given in (2.9). Recall a claim on equity is dependant only on the firm's asset value, and so the PDE is in V and t. This partial differential equation can be used in a finite difference scheme to price an option, as will be done in chapter 3. In the European option case solutions can be compared to analytical values from equation (2.8). In order to price a stock based claim g, we need to develop a PDE in both asset value and number of shares (see Appendix A.2):

$$\frac{\partial g}{\partial t} + (r-q)V\frac{\partial g}{\partial V} + \frac{1}{2}V^2\sigma^2\frac{\partial^2 g}{\partial V^2} + n\left(\frac{\mathrm{CFR}_t^- - B_p\mathrm{CFR}_t^+}{\mathrm{EQ}_t}\right)\frac{\partial g}{\partial n} - rg = 0. \tag{2.14}$$

In chapter 3 we will calculate numerical solutions for option values using a finite difference scheme on (2.9) and (2.14).

Chapter 3

Solving for Option Price Using a Finite Difference Scheme

We can approximate the option pricing methods using equations (2.9) and (2.14) with a finite difference scheme and suitable boundary conditions. Let us initially focus on pricing a European call option on equity using equation (2.9).

3.1 European call option on equity

Since our "initial condition" on the PDE is in fact a terminal condition, i.e. the option payoff, we will time reverse the PDE to solve for the initial option price. The time reversed PDE describes the dynamics of the claim moving backwards in time. This is done by defining $\tau = T - t$ and replacing t with $T - \tau = t$. Then we move from $\tau = 0$ to $\tau = T$. In equation (2.9) the sole appearance of t is in the $\partial f/\partial t$ term; this is replaced with:

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial (T-\tau)} = -\frac{\partial f}{\partial \tau}.$$

So we can then write equation (2.9) as:

$$\frac{\partial f}{\partial \tau} - (r-q)V\frac{\partial f}{\partial V} - \frac{1}{2}V^2\sigma^2\frac{\partial^2 f}{\partial V^2} + rf = 0. \tag{3.1}$$

To price a European call option using (3.1) we first must construct a finite difference mesh. Create a τ axis by dividing the time $[0, T]$ into $M+1$ equally sized intervals. Choose $M-1$ regularly spaced points between 0 and T each of length $\delta_t = T/M$. These points will be denoted with a subscript m in our workings. We have that $m = 0$ corresponds to $\tau = 0$ and $m = M$ corresponds to $\tau = T$. To create a V axis choose a suitable minimum and maximum V value, such that approximations on the value of the option can be made at the boundaries. Again break the interval between V_{min} and V_{max} into $N+1$ equally sized intervals of length $\delta_V = (V_{\text{max}} - V_{\text{min}})/N$. We will denote the V grid points by i with $i = 0$ corresponding to V_{min} and $i = N$ corresponding to V_{max}. So f^i_m will denote the value of the option at $\tau = m\delta_t$ and asset value $V_{\text{min}} + i\delta_V$.

In order to avoid stability issues we will use an implicit scheme, which is unconditionally stable, see Crépey (2013, Chapter 8). This involves taking a backward difference derivative approximation on the τ derivative and a central difference approximation on both the V derivatives:

$$\frac{f^i_m - f^i_{m-1}}{\delta_t} - (r-q)V\frac{f^{i+1}_m - f^{i-1}_m}{2\delta_V} - \frac{1}{2}V^2\sigma^2\frac{f^{i+1}_m - 2f^i_m + f^{i-1}_m}{\delta_V^2} + rf^i_m = 0.$$

This implies that:

$$(1 + r\delta_t)f^i_m - \frac{(r-q)V\delta_t}{2\delta_V}\left(f^{i+1}_m - f^{i-1}_m\right) - \frac{V^2\sigma^2\delta_t}{2\delta_V^2}\left(f^{i+1}_M - 2f^i_m + f^{i-1}_m\right) = f^i_{m-1}.$$

If we let $f^{\cdot}_m$ denote a column vector containing all f^i_m values for $i = 1$ to $i = N - 1$. We can create a matrix equation for each time step m. :

$$(1 + r\delta_t)\mathbb{I}f^{\cdot}_m - D_1T_1f^{\cdot}_m - D_2T_2f^{\cdot}_m = f^{\cdot}_{m-1}, \tag{3.2}$$

with:

$$T_1 = \frac{1}{2}(r-q)\delta_t\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \ddots & \vdots \\ 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & -1 & 0 \end{pmatrix} \quad T_2 = \frac{1}{2}\sigma^2\delta_t\begin{pmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{pmatrix}$$

$$D_1 = \text{diag}[V_{\text{min}}/\delta_V + [1, 2, \ldots, N-1]] \qquad D_2 = D_1^2.$$

Here T_1, T_2, D_1 and D_2 are all $(N-1) \times (N-1)$ matrices. Then define matrix G by:

$$G = (1 + r\delta_t)\mathbb{I} - D_1T_1 - D_2T_2.$$

We can now write equation (3.2) as:

$$Gf^{\cdot}_m = f^{\cdot}_{m-1} \tag{3.3}$$

$$f^{\cdot}_m = G^{-1}f^{\cdot}_{m-1}. \tag{3.4}$$

As for any PDE we need boundary conditions in order to solve it. In our finite difference scheme we have $N - 1$ equations for all the interior points but $N + 1$ unknowns, including the two boundary points. For the boundary points we will use Neumann boundary conditions, or conditions on the slope at the boundary rather than conditions on the values. We impose that the derivative at the boundary be the same as the derivative between the two points adjacent to the boundary, for example at the $V = V_{\text{max}}$ boundary:

$$\frac{f_m^N - f_m^{N-1}}{\delta_V} = \frac{f_m^{N-1} - f_m^{N-2}}{\delta_V}, \tag{3.5}$$

$$\implies f_m^N = 2f_m^{N-1} - f_m^{N-2}. \tag{3.6}$$

Likewise for the V_{min} boundary we have:

$$f_m^0 = 2f_m^1 - f_m^2. \tag{3.7}$$

At the boundary the option is either deeply in or deeply out of the money, and we approximate that the option value remains linear in asset value when far away from the strike.

We can attempt to impose these boundary conditions directly onto matrix equation (3.4). Extend G to be $N+1 \times N+1$, first by recreating:

$$D_1 = \text{diag}[V_{\text{min}}/\delta_V + [0, 2, \ldots, N]].$$

Again take $D_2 = D_1^2$ and T_1, T_2 defined as before, for the larger dimension. To construct G^* take this enlarged G and replace the first and last rows with the vectors $[-1, 2, -1, 0, 0 \ldots 0]$ and $[0, 0, \ldots, 0, -1, 2, -1]$ respectively. Then create $f_{m-1}^{\cdot *}$ by concatenating onto $f_{m-1}^{\cdot}$ an initial and final entry of 0. The matrix equation:

$$G^* f_m^{\cdot} = f_{m-1}^{\cdot *} \tag{3.8}$$

$$\implies f_m^{\cdot} = (G^*)^{-1} f_{m-1}^{\cdot *} \tag{3.9}$$

directly imposes the boundary conditions (3.6) and (3.7). The initial condition of the time reversed PDE is simply the payoff function of the option. For a European option on equity we have that the initial condition is simply:

$$f_0^i = \max\left[\text{EQ}(V_{\text{min}} + i\delta_V) - K, 0\right]. \tag{3.10}$$

Given the initial condition we can cycle through m using equation (3.8) and solve for the price of the option. Note that the parameter B_p is not used in the calculation, and so the equity option value is not sensitive to the ratio of excess cash flow paid out as dividends.

In order to corroborate our results and method we can run a Monte Carlo simulation. This is done using the solution to (2.1):

$$V_T = V_0 \exp\left((r - q - \frac{1}{2}\sigma^2)T + \sigma W_T\right), \tag{3.11}$$

the expression for equity (equation (2.5)) and discounting the payoff for a call. We can compare both these solutions to the analytical expression for a call from (2.8). There is a high level of agreement as illustrated in figure 3.1a. The analytical solution is taken from Toft and Prucyk (1997). These results are for a firm with coupon $c = 4$, strike $K = 40$, bankruptcy cost $b = 0.25$ and tax $\rho = 0.35$.

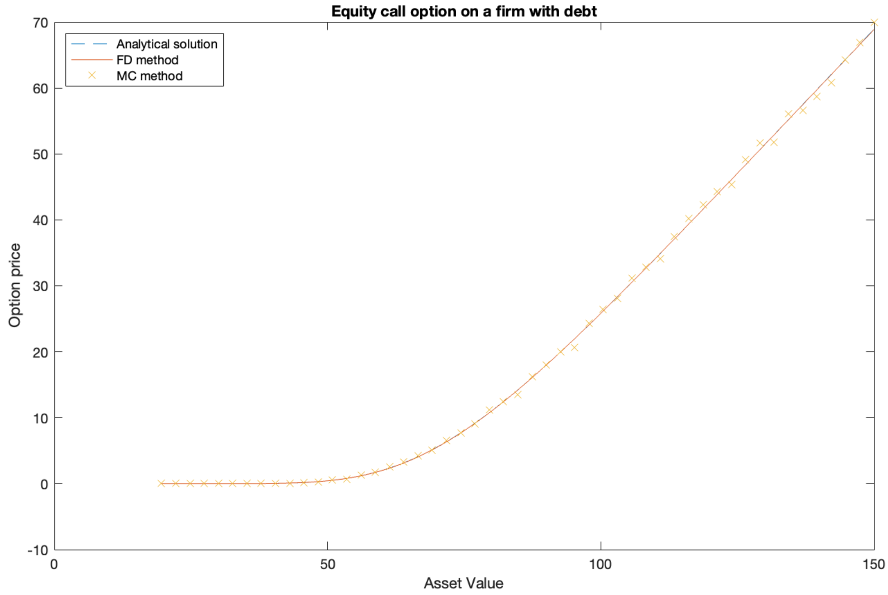
3.2 American call option on equity

We will also attempt to compute the value for an American call option on equity and show that there are some interesting properties in this model. To compute the value of the American option we will use the finite difference scheme from (3.8) along with a modified successive over relaxation (SOR) algorithm, see Wilmott (2006, Chapter 78) for more on the SOR algorithm for pricing American options.

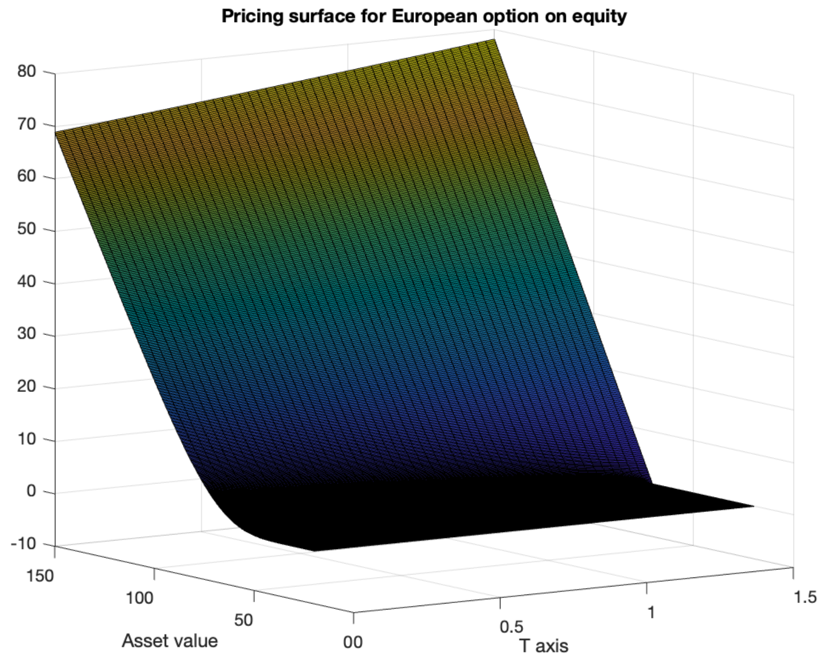
We apply the SOR algorithm to the system: $G^*\mathbf{x} = f_m^{\cdot *}$ and recursively find a solution for $\mathbf{x}$. First decompose $G^* = D + U + L$, where D is diagonal matrix, L is lower triangular and U is upper triangular. Use $\mathbf{x}^k$ as the k'th iteration of the solution for x. Then solve for $\mathbf{x}^{k+1}$ using:

$$\mathbf{x}^{k+1} = \max\left[(D+\omega L)^{-1}(\omega f_m^{\cdot *} + ((1-\omega)D - \omega U)\mathbf{x}^k), (\mathrm{EQ} - K)^+\right]. \tag{3.12}$$

Here EQ is a vector of equity values of the asset value vector from $V_{\min}$ to $V_{\max}$. We have $\omega > 1$ is the relaxation factor and the max function is taken elementwise over the vector. We use an initial estimate of $\mathbf{x}^0 = f_{m-1}^{\cdot}$, then at each time step m we repetitively compute $\mathbf{x}^{k+1}$ until the solution converges. At this point we can set $f_m^{\cdot} = \mathbf{x}^{k+1}$. Repeating from $m = 1$ to $m = M$ we can compute the value of the American option at inception. In figure 3.2a we compare the results of the SOR algorithm and the finite difference mesh with a least squares Monte Carlo simulation, see Crépey (2013, Chapter 6).

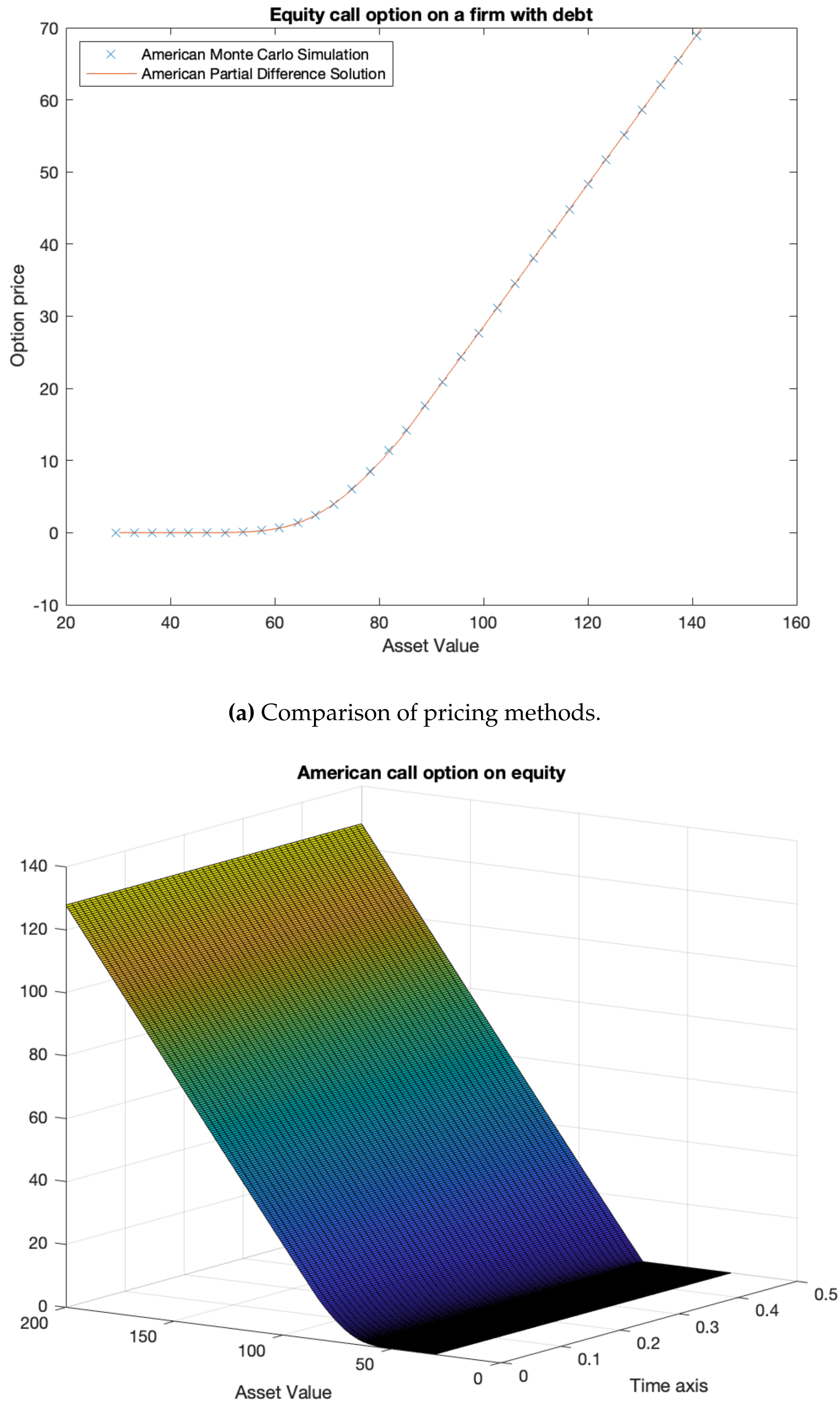


(a) Comparison of pricing methods.



(b) Pricing surface for option over time.

Fig. 3.1: Pricing a European call option on equity.



(a) Comparison of pricing methods.

(b) Pricing surface for option over time.

Fig. 3.2: Pricing an American call option on equity.

3.3 Call option on a single share

To price a European option on a single share we use the PDE in equation (2.14) with time reversal. We need to introduce a third dimension in n to our mesh. Again we must choose n_{min} and n_{max} appropriately. The interval is again broken into N spaces, each of length $\delta_n = (n_{\text{max}} - n_{\text{min}})/N$. So that the n and V vectors have the same number of elements. Then to make the computation easier we have chosen to take the n partial derivative at time step $m-1$ rather than at time step m:

$$
\begin{aligned}
0 =& \frac{g_m^{i,j} - g_{m-1}^{i,j}}{\delta_t} - (r-q)V\frac{g_m^{i+1,j} - g_m^{i-1,j}}{2\delta_V} - \frac{1}{2}V^2\sigma^2\frac{g_m^{i+1,j} - 2g_m^{i,j} + g_m^{i-1,j}}{\delta_V^2} \\
&- n^j\left(\frac{\text{CFR}^- - B_p\text{CFR}^+}{\text{EQ}}\right)^i \frac{g_{m-1}^{i,j+1} - g_{m-1}^{i,j-1}}{2\delta_n} + rg_m^{i,j} \\
\Longrightarrow& (1 + r\delta_t)g_m^{i,j} - \frac{(r-q)V\delta_t}{2\delta_V}\left(g_m^{i+1,j} - g_m^{i-1,j}\right) - \frac{V^2\sigma^2\delta_t}{2\delta_V^2}\left(g_m^{i+1,j} - 2g_m^{i,j} + g_m^{i-1,j}\right) \\
&= g_{m-1}^{i,j} + \frac{n^j\delta_t}{2\delta_n}\left(\frac{\text{CFR}^- - B_p\text{CFR}^+}{\text{EQ}}\right)^i\left(g_{m-1}^{i,j+1} - g_{m-1}^{i,j-1}\right) \\
\Longrightarrow& Gg_m^{\cdot,j} = g_{m-1}^{\cdot,j} + \frac{n^j\delta_t}{2\delta_n}\left(\frac{\text{CFR}^- - B_p\text{CFR}^+}{\text{EQ}}\right)\left(g_{m-1}^{\cdot,j+1} - g_{m-1}^{\cdot,j-1}\right)^{\cdot}.
\end{aligned}
$$

Note that in the above the expression EQ is a vector of the equity value calculated using equation (2.5) at each point on the V axis. Then $\left(\frac{\text{CFR}^- - B_p\text{CFR}^+}{\text{EQ}}\right)^{\cdot}$ is a vector of length $N-1$ over the V axis. This vector is constant through τ and n, let us call it the dilution vector as it specifies the dilution factor at each value of V. Impose the boundary conditions on V as before and we now have the equation:

$$
g_m^{\cdot,j} = (G^*)^{-1}\left(g_{m-1}^{*\cdot,j} + \frac{n^j\delta_t}{2\delta_n}\left(\frac{\text{CFR}^- - B_p\text{CFR}^+}{\text{EQ}}\right)^{\cdot}\left(g_{m-1}^{*\cdot,j+1} - g_{m-1}^{*\cdot,j-1}\right)\right). \quad (3.13)
$$

We can easily apply the above equation for $j = 2$ to $j = N-2$. For $j = 1$ and for $j = N-1$ we need to impose boundary conditions for the $j-1$ (corresponding to $j = 0$) and $j+1$ (corresponding to $j = N$) terms respectively. Again we impose Neumann boundary conditions that maintain the slope near the boundary into the boundary region:

$$
\begin{aligned}
g_{m-1}^{\cdot,0} &= 2g_{m-1}^{\cdot,1} - g_{m-1}^{\cdot,2} \\
g_{m-1}^{\cdot,N} &= 2g_{m-1}^{\cdot,N-1} - g_{m-1}^{\cdot,N-2}.
\end{aligned}
$$

So that we have at the boundaries:

$$g_m^{\cdot,1} = (G^*)^{-1}\left(g_{m-1}^{*\cdot,1} + \frac{n^1\delta_t}{2\delta_n}\left(\frac{\text{CFR}^- - B_p\text{CFR}^+}{\text{EQ}}\right)^{\cdot}\left(g_{m-1}^{*\cdot,2} - (2g_{m-1}^{*\cdot,1} - g_{m-1}^{*\cdot,2})\right)\right)$$

$$g_m^{\cdot,N-1} = (G^*)^{-1}\left(g_{m-1}^{*\cdot,N-1} + \frac{n^{N-1}\delta_t}{2\delta_n}\left(\frac{\text{CFR}^- - B_p\text{CFR}^+}{\text{EQ}}\right)^{\cdot}\left((2g_{m-1}^{*\cdot,N-1} - g_{m-1}^{*\cdot,N-2}) - g_{m-1}^{*\cdot,N-2}\right)\right).$$

We can again corroborate the results from our finite difference method using a Monte Carlo simulation (A closed form solution is not available for this calculation). As the value of the option is path dependant we need to simulate the path of the underlying instrument to value the option. We discretise equation (3.11) to calculate the asset and equity values. Then we use equation (2.11) to model the number of shares at each discrete time step and then the share price, so that we can value the stock option. Figure (3.3a) indicates that the Monte Carlo and finite difference method give matching results as expected.

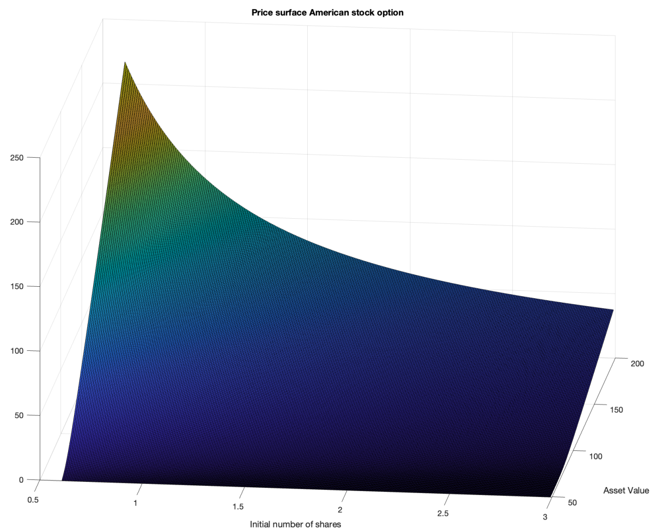
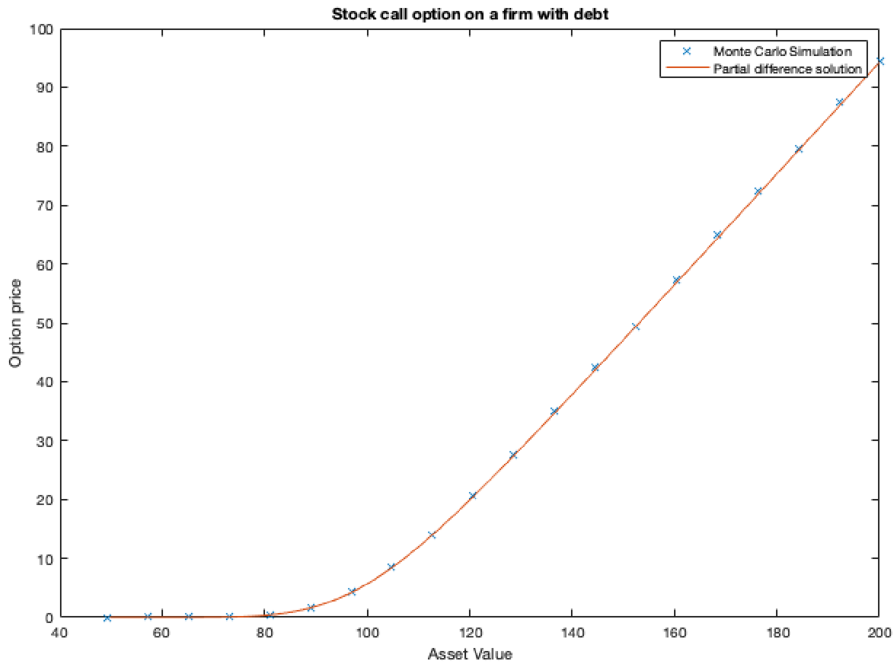
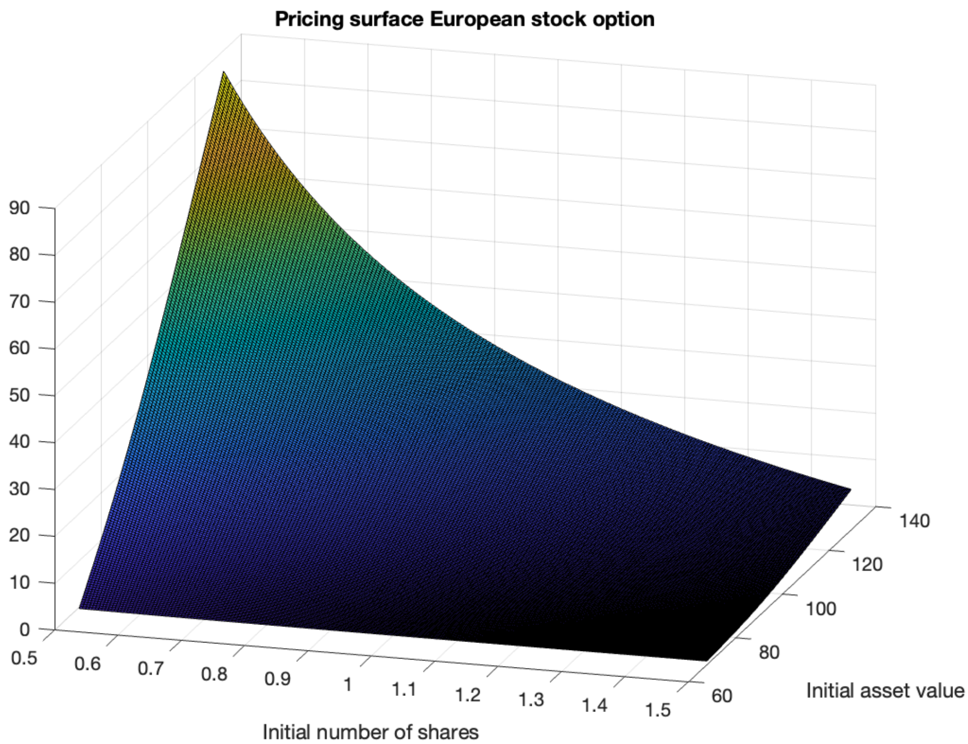


Fig. 3.4: Pricing surface at time 0 for an American call option on a stock.

We can now attempt to price an American option on a stock using the finite difference mesh, equation (3.13) and the SOR algorithm from equation (3.12). Figure 3.4 is a surface plot of the price for an American option at various initial asset values V, and initial shares in issue N_0. This option was priced with $\rho = 0.35$, $\sigma = 0.2$, $r = 0.08$, $q = 0.082$, $B_p = 0.4$, $b = 0.25$ and $T = 0.5$.



(a) Comparison of pricing methods.



(b) Pricing surface for option over time.

Fig. 3.3: Pricing an American call option on a stock.

Chapter 4

Results of Option Price Calculations

We analyse the results of our PDE computation described in chapter 3 based on the model introduced in chapter 2. First, we consider the American and European options on equity. We note firstly the counter-intuitive result that the value of an American option on equity is not dependent on the firm's payout policy and secondly that it might be optimal to exercise an American option early even in the absence of dividends. These properties, which seem to contradict standard option pricing principles are identified by Backwell *et al.* (2022) as a result of dilutions and Buybacks. Second, we compare the prices of a European option on equity and European options on stocks with different payout policies (i.e. the proportion of cash paid out via dividends versus buybacks) and note that the payout policy of the firm has a material effect on the price of the stock option.

4.1 Call options on equity for a highly leveraged firm

It is accepted in the literature that an option on an underlying should not be more valuable than the underlying itself, in the absence of holding costs. In the case that the firm has a high coupon, and hence a high chance of having to raise cash from equity holders, it is possible that an option on equity may be worth more than equity. In figure 4.1 we see that this is the case for a highly leveraged firm. The cost of servicing debt for equity holders is analogous to holding costs. Equity holders bear this cost while option holders forgo it, this is why in figure 4.1 the option value is greater than the equity value for low strikes.

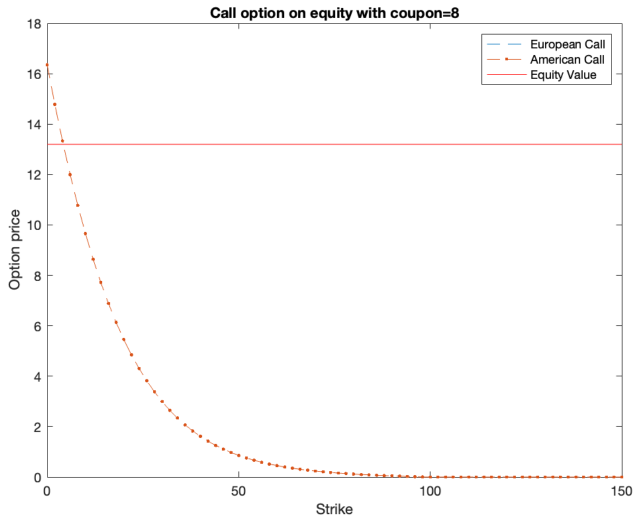


Fig. 4.1: Options on equity with $B_p = 1$ and $n_0 = 1$. When there is a large coupon an option on equity may be worth more than equity.

4.2 Early exercise premium for American call options on equity

It is a well known result in the literature that it is never optimal to exercise an American call option on a non-dividend paying asset before maturity. This results in an American and a European option on a non-dividend paying asset having equal value, as there is no early exercise premium. This is true in the case of an option on a single stock, however in the case of an equity option the value of the option is not dependant on the payout policy B_p as discussed in chapter 2. If the holder of an equity option were to exercise, they would control all (or a fixed portion of) the equity of the firm. So they would receive a fixed portion of dividends paid out, but they would also receive the proportion of payouts dedicated to stock buybacks. This phenomenon is illustrated in figure 4.2 where we can clearly see there is an early exercise premium for options deep in the money, despite the shares not paying out any dividends.

We also see that at strike $K = 0$, the European option is worth less than equity despite there being no dividends paid out. This again is due to the option being on a fixed portion of equity. If you were to hold equity you would still receive payouts in the form of buybacks, which the holder of an option would not receive. So despite there being no dividends paid out the value of a European option with strike $K = 0$ is less than the value of equity.

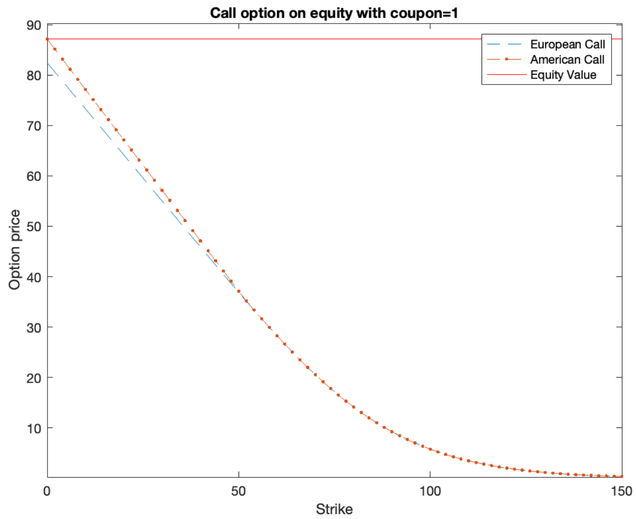


Fig. 4.2: Comparison of American and European options on equity with B_p=1, n_0=1. It may be optimal to exercise an American option before maturity despite there being no direct cash dividends.

4.3 European call options on stocks

It is important to note that the payout policy B_p has significant implications for the pricing of stock options. This is illustrated in figure 4.3, which graphs the value of options on equity and on a share against different coupon payments. When $n_0 = 1$ the price of the share is initially the same as the price of equity and so option values are comparable. Between the times 0 and T dilutions and buybacks will cause the two prices to differ. Of course the value of equity is not dependant on the payout policy, only on the asset value V, the coupon c and a few other parameters (see equation (2.5)). So the buyback policy B_p effects only the proportion of total equity that each share is worth. For a higher B_p, each share is expected to be worth a higher proportion of total equity by time T. This effect is exaggerated when the firm has a lower coupon, as a larger part of the cash generated by the firm is then distributed to shareholders. So the difference in price between stock options is most evident at lower coupon levels.

One can also see that the equity option has the same value as the $B_p = 0$ stock option for lower annual coupon values. This is because there are no buybacks and the low coupon value implies a low likelihood of dilution so the number of shares remains constant. Thus for low coupon levels and $B_p = 0$ the price of an option on equity and a stock option are virtually the same.

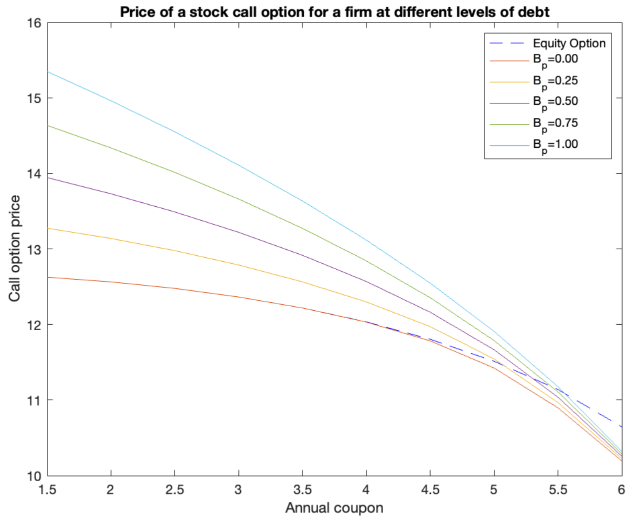


Fig. 4.3: Price for European call option for different payout policies B_p.

On the other hand, at high coupon levels it is more likely that shareholders will be diluted and so the proportion of total equity that a share is worth will decrease by time T. This phenomenon is observed on the right side of figure 4.3 where the option on equity is more valuable than the options on shares. It is also true that since less cash is paid out to shareholders at a higher coupon level, the payout policy has a smaller effect on the price of the option.

Chapter 5

Conclusion

In chapter 1 we revisited the literature on the capital structure of the firm and the effect on option pricing. In particular we noted that until recently the explicit effect of buybacks and dilutions has not been considered in the pricing of options.

In chapter 2 we continued to explore the theory behind pricing stock options on firms with debt. We adopted a model for the firm and expressions for debt and equity value from Leland (1994). We were able to develop a PDE with which we could price equity options. Then we worked through the BMR model theory around the number of shares in issue, buybacks and dilutions. This allowed us to develop a PDE to price stock options.

We implemented these PDE equations in chapter 3 using a finite difference mesh. We also priced American options on equity and on stocks using a SOR algorithm. Results were compared to analytical expressions for European options on equity, and Monte Carlo simulations for American options and options on stocks where analytical expressions are not available. We found that these instruments can be priced well using PDE methods and that there was agreement between the finite difference pricing and the other methods.

Using the algorithms we investigated some further results in chapter 4. Including that an option on equity may be more valuable than equity, it may be optimal to exercise an American call option on equity early even in the absence of dividends and the effect that the payout policy B_p has on the pricing of stock options. These results all indicate that the effect of buybacks and dilutions on the price of stock options can be significant. Further research on option pricing should consider differences between options on equity and options on stocks and the effect that possible dilutions and buybacks might have on price.

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Appendix A

Partial Differential Equation Derivations

A.1 Derivation of PDE for equity based claim

For an option on equity f, which is a function on asset value V we have from Ito's lemma:

$$\begin{aligned} df(V,t) &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial V}dV_t + \frac{1}{2}\frac{\partial^2 f}{\partial V^2}(dV_t)^2 \\ df(V,t) &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial V}(r-q)V + \frac{1}{2}\frac{\partial^2 f}{\partial V^2}\sigma^2 V^2\right)dt + \frac{\partial f}{\partial V}\sigma V dW_t. \end{aligned}$$

The second line is calculated using equation (2.1). We construct a portfolio Π to hedge one option f_t with $-\Delta$ units of the option's underlying asset, V (where V underlies equity). Of course in reality it may not be possible to trade V, but for the purposes of developing this theory let us imagine that one could trade V and then replicate equity or equity options and then value the instrument. So that the portfolio value is:

$$\Pi = f - \Delta V. \tag{A.1}$$

Then a differential in the portfolio value is given by:

$$\begin{aligned} d\Pi &= df - \Delta dV_t - \Delta qV dt \\ &= \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial V}(r-q)V + \frac{1}{2}\frac{\partial^2 f}{\partial V^2}\sigma^2 V^2 - \Delta(r-q)V - \Delta qV\right)dt \\ &+ \left(\frac{\partial f}{\partial V}\sigma V - \Delta\sigma V\right)dW_t. \end{aligned}$$

Recall that the asset V pays out cash at a rate q, which affects the change in portfolio value over time step dt. Choose $\Delta = \frac{\partial f}{\partial V}$ so that:

$$d\Pi = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial V^2}\sigma^2 V^2 - q\Delta V\right)dt.$$

Now Π is a riskless portfolio, and so it must grow by $r\Pi dt$ over time interval dt. So that:

$$\left(\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial V^2}\sigma^2 V^2 - q\Delta V\right) dt = r\Pi dt$$
$$\implies \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial V^2}\sigma^2 V^2 - q\Delta V = r\left(f - \Delta V\right)$$
$$\implies \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial V^2}\sigma^2 V^2 + (r-q)V\frac{\partial f}{\partial V} - rf = 0.$$

A.2 Derivation of PDE for stock based claim

We can likewise develop a PDE for a stock based claim g. Here the value of the claim is a function of both the asset value V and the number of stocks n. Once again starting with Ito's lemma:

$$dg(t,V,n) = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial V}dV_t + \frac{1}{2}\frac{\partial^2 g}{\partial g^2}(dV_t)^2 + \frac{\partial g}{\partial n}dn_t + \frac{1}{2}\frac{\partial^2 g}{\partial g^2}(dn_t)^2.$$

Then again use equation (2.1) as well as equation (2.11):

$$dg = \left(\frac{\partial g}{\partial t} + \frac{\partial g}{\partial V}(r-q)V + \frac{1}{2}\frac{\partial^2 g}{\partial V^2}\sigma^2 V^2 + \frac{\partial g}{\partial n}\left(\frac{\text{CFR}_t^- - B_p\text{CFR}_t^+}{\text{EQ}(V)}\right)n\right)dt$$
$$+ \frac{\partial g}{\partial V}\sigma V dW_t.$$

Again constructing a riskless portfolio $\Pi = g - \frac{\partial g}{\partial V}V$ we get that:

$$d\Pi = \left(\frac{\partial g}{\partial t} + \frac{1}{2}\frac{\partial^2 g}{\partial V^2}\sigma^2 V^2 - q\Delta V + \frac{\partial g}{\partial n}\left(\frac{\text{CFR}_t^- - B_p\text{CFR}_t^+}{\text{EQ}(V)}\right)n\right)dt$$

and so:

$$\frac{\partial g}{\partial t} + (r-q)\frac{\partial g}{\partial V}V + \frac{1}{2}\frac{\partial^2 g}{\partial V^2}\sigma^2 V^2 + \frac{\partial g}{\partial n}\left(\frac{\text{CFR}_t^- - B_p\text{CFR}_t^+}{\text{EQ}(V)}\right)n - rg = 0.$$