A study of the computations done by Grade 9 learners in a Western Cape high school when simplifying algebraic expressions involving the negative symbol

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BRNHES004

Dissertation submitted in partial fulfilment of the requirements for the award of the degree of Master of Education specialising in Mathematics Education

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June 2025

Abstract

The computations that learners do when simplifying algebraic expressions are multiple and diverse, with some determined by reasoning aligned with mathematics whilst others rely on idiosyncratic constructs like mnemonics or solution templates. Research in mathematics education highlights symbol sense and negative number concepts as persistent difficulties in learning algebra and categorises learners' errors, but it is wanting in explanations of learners' computations and how they might relate to the way learners think. This study identifies, describes and offers possible explanations for some of the computations that learners did when attempting to simplify algebraic expressions involving the negative symbol. Grade 9 learners from one class in a Western Cape high school were given a set of algebraic expressions to simplify after which interviews were conducted with some learners to discuss their solutions.

In the computational analysis of the data, cognitive science and universal algebra were used as lenses for a deeper understanding of learners' mathematical (and non-mathematical) thinking. The data indicates learners' tendency to read algebraic expressions as strings of characters constituting different types of objects, classified in this study as *operators*, *signs*, *numerals*, *letters*, and *superscripts*. Many learners resorted to replacing standard mathematical operations with various *operation-like* manipulations taking different types of objects as arguments. As suggested by the literature, the negative symbol presented learners with additional challenges, given its polysemic nature in mathematics. Many learners were prompted by the negative symbol to remove certain objects entirely in their attempts to simplify expressions.

Plausible reasons for learners' type-sensitivity and idiosyncratic computations offered by this study include: humans' innate capacity for recognising and categorising different objects and symbols; the biases produced from language; and the reliance on existing mental structures for the assimilation of new knowledge. In considering learners' computations at a fundamental level, this study contributes to a more complete view of what learners do computationally and, importantly, why.

Acknowledgements

Fundamental to this study is the contribution of its participants, so first thanks must go to the learners for sharing with me their time and computations, and to the teachers and principal for welcoming me into their school and generously helping to organise my data collection.

I was very fortunate to receive the Master's Research Scholarship Award from UCT for both 2023 and 2024, which aided me in pursuing my studies, and therefore this research, in mathematics education.

The many hours spent writing a dissertation can become rather solitary, so I am extremely grateful to those who stood behind me throughout this process for lessening that solitude. My greatest thanks go to my wonderful supervisors, Shaheeda Jaffer and Zain Davis, for bringing me into the fascinating world of mathematics education and instilling confidence in me from the very beginning of my Master's. Your academic support and reliability have been as invaluable to me as your enjoyable company in our meetings and lectures. I am looking forward to continuing my studies under your supervision.

I am indebted to my parents (and editors), Tobea and Chris, for their perennial encouragement to study what interests me and for taking curiosity in whatever I choose. I am sure the kitchen chalk board was not intended for mathematical diagrams, but I am glad I could repurpose it! Thank you, Ma, for being my sounding board and for lightening my load in challenging times. And Pa, thank you for inspiring and supporting me to pursue a career in academics (and of course for that illustrious "maths brain").

A special mention must go to Ross, for listening to my ideas, to do lists and findings, and for his unwavering moral support. Thank you for dependably lifting my spirits in times of stress. I am also thankful to my close friends, particularly Bella, whose regular check-ins and motivation provided me much comfort. And finally, to the family, friends and colleagues who took interest in my studies, whose good company energised me during study breaks and who cheered me on from start to finish, thank you! I did it!

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Abbreviations

Abbreviation	**Meaning**
BODMAS	Brackets, order, division, multiplication, addition, subtraction (see Glossary)
DBE	Department of Basic Education
DFMS	Detachment from the minus sign
DSB	Different, subtract, bigger (see Glossary)
FOIL	First, outer, inner, last (see Glossary)
ICM	Integrated Causal Model
SSAK	Same sign, add, keep (see Glossary)
$(\mathbb{R}, +, \times)$	The field of real numbers

Glossary

There are several terms and abbreviations used in this dissertation which may not be familiar to all. There are also terms which may have multiple or general meanings, but which are used in this dissertation unambiguously. These terms are defined here for the reader's clarity. Where one term in the list appears in the explanation of another, that term is followed by an asterisk (*) in the explanation.

Term	**Meaning**
Additive inverse	The additive inverse of a real number x is the number $-x$, i.e. the number that when added to x yields the additive identity of the real numbers: $x+(-x)=x-x=0$. For example, the numbers 2 and -2 are additive inverses. Additive inverses in an expression may be simplified to 0 (see cancellation theorem*).
Algebraic expression (or simply "expressions")	A string of symbols representing the composition of operations defined over a domain. This study takes the set of real numbers ($\mathbb{R}$) as the domain intended for computations required by school mathematics. Typographically, algebraic expressions are alphanumeric strings comprised of letters and/or numerals along with symbols denoting operations. An algebraic expression may be comprised of a single alphanumeric character (a singlet) or a string of alphanumeric characters representing the composition of operations. For example, x, 2 and $x-3xy+2$ all constitute algebraic expressions.
Argument	An input of a transformation*.
Arithmetic	Elementary school arithmetic, i.e. arithmetic involving only numbers (i.e. no letters) and basic operations* (addition, subtraction, multiplication and division).
Binomial	A polynomial* with only two terms*.

BODMAS	An acronym mnemonic that many learners (and teachers) use to recall the hierarchy of number operations: **b**rackets, **o**rder (i.e. exponents), **d**ivision, **m**ultiplication, **a**ddition, **s**ubtraction. Although “brackets” does not refer to an operation, it is used to indicate that the contents of the brackets should always be simplified first. There are variants of this mnemonic such as PEMDAS or BIDMAS, but BODMAS is typical of South African schooling. Note that the “O” is alternatively also taken as standing for “of,” indicating multiplication such as, e.g. “half of 2” is $\frac{1}{2} \times 2$, but when dealing with exponents it is commonly taken as “order” (see for example, Maboya et al. (2020, p.5418; Fonseca, 2011, p.40).
Cancellation theorem (addition of real numbers)	For any real numbers x, y, z, if $x + z = y + z$ then $x = y$. When simplifying an expression*, if two terms in the expression add up to 0 (i.e. are additive inverses*), they can be “cancelled out.” For example, $2 + 4x - 2 = 4x$ because $2 - 2 = 0$ so we can “cancel the 2’s.” Learners may try to use “cancellation” when it is not appropriate, for example, simplifying $3x - x$ to 3 by “cancelling the x’s,” as discussed in this dissertation. (See, for example, the VOID mapping in Section 5.6).
Codomain (of a transformation)	The set of possible outputs of a transformation*.
Coefficient (of a variable)	A number or any symbol representing a constant value that is multiplied by the variable. For example, in the term $-2x$ the coefficient is -2.
Compossible	Compatible or possible in coexistence with something else. For example, the basic operation* of subtraction is compossible with the real numbers, but not with alphanumeric characters.

Concatenate	To link in a series or chain. For example, the concatenation of the strings $/a/$ and $/b/$ is the string $/ab/$. (See the definition of the CON mapping in Section 5.6)
Disjoint union (⊔)	The disjoint union of two sets A and B, denoted $A \sqcup B$, is the union of those sets with the condition that the information regarding which set each element originally belonged to is retained. For example, the disjoint union of $\{a\}$ and $\{b\}$ is $\{a\} \sqcup \{b\} = \{a, b\}$.
Domain (of a transformation)	The set of possible inputs of a transformation*.
DSB method	The abbreviation DSB stands for "different, subtract, bigger" and is used as mnemonic for a method for simplifying an expression of the form $\pm a \mp b$, where a and b are unsigned real numbers. When the signs are *different*, subtract the bigger number from the smaller number (where the numbers are detached from the signs) and keep the sign attached to the *bigger* number. For example, $3 - 5 = -(5 - 3) = -2$ (but the middle step is left implicit).
FOIL method	FOIL is an acronym mnemonic used in schooling to recall the series of operations used to multiply two binomials $(a + b)(c + d)$. The acronym provides the order in which two terms—one from inside each set of brackets—should be multiplied together to expand the expression: **f**irst, **o**uter, **i**nner, **l**ast. The four products are then added together for a final solution: $(a + b)(c + d) = ac + ad + bc + bd$. This method constitutes the FOIL method.
Invisible 1	The "invisible 1" is a pedagogic recontextualisation of the multiplicative identity of the real numbers that is used to teach learners that a variable term x with no written coefficient is equivalent to the term $1x$ (as well as $x = \frac{x}{1}$ and $x = x^1$).

Like terms	Two terms which have the same variable raised to the same power. For example, $2x^2$ and $-4x^2$ are like terms. (Whereas $2x^2$ and $-4x$ or $-4x^3$ are *unlike* terms.)
Matric	The final year of high school in South Africa and the matric exams are the school exit exams.
Monomial	A polynomial* consisting of only one term*.
Operation	A foundational mathematical process (function) by which one or more inputs (arguments*) are mapped to a well-defined output (value). The basic operations that take real numbers as arguments are the binary operations of addition ($+$), subtraction ($-$), multiplication ($\times$), and division ($\div$), each of which take two real numbers as their arguments and return a single real number as an output. For example, $2+3=5$ and $3-1=2$. An operation is a type of transformation*.
Operator	A symbol indicating an operation*.
Parse (an expression)	To analyse in terms of or partition into particular components. For example, one might parse the expression* $3x-2x-5$ into the components $3x, -2x, -5$ by recognising like* terms, or the components $3, x, -,2, x, -,5$ by recognising individual characters. The parsing of an expression may be thought of as generating subcollections containing the objects constituting the expression. Those objects can be selected for further computation (see SEL mapping in Section 5.6).
Polynomial	A type of algebraic expression which consists of the sum of products of numbers and one or more variables raised to nonnegative powers. The numbers are usually referred to as *coefficients* of the variables. For example, the polynomial $-2x+3y$ consists of the sum of the products $-2x$ and $3y$, where -2 and 3 are the respective coefficients of the variables x and y. The general form of a polynomial is

	$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$, where, in this study, a_i is an integer ($a_i \in \mathbb{Z}$), x is a real number ($x \in \mathbb{R}$), and n is a natural number ($n \in \mathbb{N}$). Alternatively, the general form may be taken as $a_nx^n - a_{n-1}x^{n-1} - \cdots - a_2x^2 - a_1x - a_0$, given the focus on subtraction problems in this study.
Relative complement	The relative complement of a set B contained in the set A, is the set of elements in A that are *not* in B, it is denoted $A \setminus B$. For example, $\{a, b, c\} \setminus \{a\} = \{b, c\}$.
SSAK method	The abbreviation SSAK stands for "same sign, add, keep" and is used as a mnemonic for a method for simplifying an expression of the form $\pm a \pm b$, where a and b are unsigned real numbers. When the signs are the *same*, *add* the numbers (detached from the signs) and *keep* the common sign. For example, $-3 - 5 = -(3 + 5) = -8$ (but the middle step is left implicit).
Sunder	To split apart. With regards to strings, if a string (i.e. algebraic expression*) is sundered into smaller strings, the order of the characters constituting those strings is not changed, such that concatenating* the smaller strings reproduces the whole string. For example, the string $/3x - 2x - 5/$ might be sundered into the parts $/3x - 2x//-5/$ or $/3x//-2x - 5/$, but in either case the smaller strings can be concatenated to reproduce $/3x - 2x - 5/$. (See the definition of the SUN mapping in Section 5.6)
Term	Any singlet (i.e. number, variable, or product) comprising a subexpression. For example, the terms of $-2x + 3y$ are the subexpression $-2x$ and $3y$, and the terms of $6xy - x - 1$ are $6xy$, $-x$ and -1 (or x and 1 if the operation is taken to be subtraction). In addition, the expression $(3x - 5) - (3x - 5)$ consists of the terms $(3x - 5)$ and $-(3x - 5)$, if the operation is taken as addition, whereas the subexpression $3x - 5$ consists of the terms $3x$ and -5.

	(In other words, the notion of a *term* is fluid, depending on the focus of computational attention—an expression or a subexpression.)
Transformation	A relation that associates inputs (members of its domain*) to outputs (members of its codomain*). Also referred to as a mapping. For example, the basic operations* are transformations over the domain of real numbers, e.g. $-: 5 - 3 \mapsto 2$. A transformation need not take numbers as inputs, it may take other objects like characters, for example $/3/ \mapsto 3$ is a transformation that maps the string $/3/$ (a character) to the real number 3. The inputs and outputs of a transformation need not belong to the same set (unlike those of operations).

1 Introduction

Algebra is internationally recognised as a topic that learners struggle with, yet a topic that is "foundational and fundamental to success in STEM subjects" (Stewart & Reeder, 2016, p.4). In South Africa, many learners entering high school still struggle with the essential mathematical concepts required for later learning in algebra. The Trends in International Mathematics and Science Study (TIMMS) conducted in 2019 showed that "59% of [Grade 9] learners had not acquired basic mathematical knowledge" (Reddy et al., 2020, p.5). In 2024, 47.9% of matric learners achieved 40% or above for mathematics in the National Senior Certificate (NSC) final examinations (Department of Basic Education (DBE), 2024, p.217). Of course, there are many factors at play for poor performance, particularly in South Africa, but these results highlight the need for attention to the general lack of mathematical understanding.

Research on algebra problem solving (e.g., Booth et al., 2014; Küchemann, 1978, 1981; Vlassis, 2004, 2008) shows that learners often struggle most with variables, negative numbers, and subtraction. Literature focusing specifically on negative numbers and subtraction, however, is far from plentiful and attention to *why* learners struggle with certain concepts appears to be lacking in discussions too. The natural numbers are certain; we can count with them and imagine them as quantities of physical objects that we are familiar with, but letters are far more ambiguous and intangible. Similarly, negative numbers too are more difficult, since they are not as easy to work with as 1, 2 or 3. What, then, are learners thinking when they work with these concepts?

We tend to think of mathematics problems as being very rigid and having only one correct answer or one set of correct answers, but the paths that learners use to arrive at those answers can vary greatly and deviate from what they were taught at school. There are many "tricks" that learners use to solve different mathematical problems; whether they be shortcuts based on mathematical reasoning or idiosyncrasies that rely on language or gestures or drawings. It is easy to identify when a learner has made a mistake, but the interesting part comes from trying to classify what that mistake really is; *why* is it a mistake? Why *that* mistake? What do learners' methods tell us about their mathematical understanding and the "tricks" they may have used?

Literature on algebraic problem solving looks at the kinds of mistakes learners make and the concepts or topics they struggle with, often categorising errors, but it is wanting in explanations of where these mistakes come from and how they might relate to the way learners think. Research in cognitive science offers useful insight into how humans think and learn and on their understanding of number concepts (e.g., Feigenson et al., 2004; Gelman, 2000, 2015; Spelke, 2022; Stahl & Feigenson, 2018). Although there is mathematics education literature that draws on cognitive science and considers the computations involved in mathematics problem solving (e.g., Davis, 2012, 2013; Davis et al., 2022; Jaffer, 2018, 2023), this intersection is comparatively small. Limited is the use of universal algebra and cognitive science as lenses in mathematics education literature. Scholars applying these lenses (e.g., Davis, 2012, 2013; Jaffer 2018) contribute to a comprehensive view of the relations between different structures and computations used in algebraic problem solving.

In identifying, describing and offering possible explanations for learners' computations using the lenses mentioned above, this study seeks to contribute to a more complete view of those computations and their relations to mathematical structures. Explaining these computations may allow teachers and researchers to be more aware of how they can help learners to grasp mathematics concepts and answer mathematics problems correctly.

This dissertation is structured as follows. Chapter 2 situates this study with respect to the mathematics education literature by reviewing relevant existing research. Chapter 3 outlines the concepts, theories and propositions that underpin the methodology and guide the analysis and discussion of the data. The methodology in Chapter 4 sets out the research problem and questions as well as the research design by which the data was generated and analysed. The framework for the analysis of the data is given in Chapter 5, in which the various concepts and objects serving as analytical tools are explained. Chapter 6 is concerned with the representation and analysis of the data, offering insight into learners' computations, which are then further described and explained in Chapter 7 in relation to the context and situatedness of the study.

2 Literature Review

2.1 Introduction

The introduction of algebraic concepts and relations in mathematics can be difficult to grasp, requiring learners to work with concepts which may seem to contradict their previous understanding of number properties, particularly those related to natural numbers, which are learnt from a young age. Young and Booth (2020) assert that fundamental to success in mathematics is an understanding of the negative symbol and negative number concepts. The mathematics education literature relevant to this study can largely be divided into two focus areas; one concerned with algebra-problem solving (e.g., Adnan et al., 2021; Booth et al., 2014; Booth & Koedinger, 2008; Küchemann, 1978, 1981; Pournara et al., 2016; Stewart & Reeder, 2016; Susac et al., 2014; Tirosh et al., 1998) and the other with symbol sense and dealing with negative numbers (e.g., Bofferding, 2010; Lamb et al., 2012; Vlassis, 2004, 2008; Young & Booth, 2020). Of course, these two categories are not mutually exclusive, with scholars like Vlassis and Demonty (2022), McGowen (2016) and McGowen and Tall (2010) explicitly focusing on both issues. However, this intersection is limited, highlighting the need for further research. Also lacking in the literature is the focus on why learners choose to do certain computations over others. Küchemann (1981), for example, classifies different ways in which learners interpret letters in mathematics, but offers no explanation for where these interpretations may stem from. Similarly, Bofferding (2010) explores how children solve problems with negative numbers, focusing largely on the effects of their learning experiences, without discussing *why*, fundamentally, children may choose certain approaches.

Cognitive science offers useful insight into how learners think and learn and why certain concepts may be more challenging for them to grasp than others. Therefore, drawing on findings from this field can help researchers to explain how and why learners may arrive at specific solutions, rather than simplify classifying their mistakes. The work by scholars like Davis (2012, 2013) and Jaffer (2018) demonstrates that using mathematics as an analytical tool allows for comprehensive descriptions and analyses of learners' computations. This study adopts a similar analytical framework, using cognitive science and universal algebra as lenses to offer greater insight into what learners do computationally and, importantly, why. To identify and evaluate the foundations and context of this study, the literature review focuses on texts investigating algebraic problem solving and learners' understanding of negative numbers and the negative symbol.

2.2 Algebraic problem solving

Mathematical reasoning, misconceptions, and the errors learners make seem to be the predominant foci of the literature on algebraic problem solving. Booth et al. (2014) consider the algebraic misconceptions that Algebra 1 students (around age 13) had at different points in the school year when solving algebraic problems and identify those that are most common and most pernicious, such as misinterpreting symbols and misunderstanding variables that represent more than one value. These scholars, along with others like Stewart and Reeder (2016) and Susac et al. (2014), suggest that awareness of such common misconceptions can help to better organise interventions aimed at tackling them, including dedicating instructional time to focus on remedying learners' presuppositions, misunderstandings, and errors. This highlights the importance of the present study in that being able to identify the computations that learners frequently do when simplifying algebraic expressions could allow teachers to organise their instructional time so that extra attention is paid to leading learners away from incorrect or detrimental strategies and towards efficient and effective ones.

Giving name to a notion that emerges from the literature but often remains implicit, McGowen and Tall (2010, p.169) make use of the term *met-before* to refer to "a mental structure that we have now as a result of experiences we have met-before." They highlight the effects that learners' met-befores can have on their learning in mathematics, both beneficial and detrimental, regarding the negative symbol in particular. Pournara et al. (2016, p.2) assert that learners' existing knowledge of arithmetic can negatively affect their learning of algebra, adding that this is "typically manifest in errors relating to 'lack of closure' where learners do not accept expressions such as $a + 2$ as final answers." Before the introduction of variables, the solutions to school mathematics problems are explicit real numbers that learners can quantify. The shift to multiterm expressions in algebra requires them to overcome their notion, i.e. met-before, that a solution always takes the form of a single object. Several other studies (e.g., Booth et al., 2014; Küchemann, 1981; Stewart & Reeder,

2016) also highlight learners' difficulties with accepting solutions consisting of more than one term i.e. those presenting a lack of closure.

Analysing the difficulties learners have with negative numbers when reducing polynomials and their readings of the minus sign, Vlassis (2004) explains that learners have an existing conceptual framework which they try to use to assimilate new information, but if the existing framework is incompatible with the new content this can easily lead to misinterpretations and errors. Küchemann (1978, 1981) discusses some of the misinterpretations that Grade 9 (14-year-old) learners have of algebraic variables, but he focuses predominantly on identifying and describing their misconceptions rather than providing possible reasons for why they may occur. Although Küchemann (1981, p.118) asserts that "children frequently tackle mathematics problems with methods that have little or nothing to do with what has been taught," he does not acknowledge that correct answers may also be the result of non-standard or erroneous procedures or interpretations. There is little literature which considers the computations that learners may use to arrive at the *correct* solutions which *differ* from those computations that they are taught to use. Vlassis and Demonty (2022) acknowledge this in their study on the effects that learners' algebraic thinking capacity has on their performance operating with negative numbers. They state that "the possibility cannot be ruled out … that even the correct answers did not indicate an understanding of negative numbers, but rather derived from superficial approaches" (Vlassis & Demonty, 2022, p.1246). Learners may arrive at the correct answers without fully understanding how to work with negative numbers, or without using the computations they are taught at school. Learners' computations can largely be deduced from their written work, and many studies rely on the data from paper-and-pencil tests for their findings (e.g., Bofferding, 2010; Booth & Koedinger, 2008; Küchemann, 1978, 1981; Vlassis, 2004, 2008; Vlassis & Demonty, 2022; Young & Booth, 2020), however, as Vlassis & Demonty (2022) highlight, using *only* paper-and-pencil tests has the limitation that learners' written answers may not accurately reflect the computations they used to solve a problem. They suggest that learner interviews would be beneficial in providing greater insight in this regard. The present research study aims to remedy the oversight of certain computations by conducting post-task interviews to gain insight into those computations which may not be deducible from or wholly represented in written solutions alone.

The literature offers various taxonomies of interpretations and errors which are useful for analysing learners' solutions to algebra problems. Booth et al. (2014, p.11) distinguish six categories of concepts that are commonly misunderstood: variables, negative signs, equality/inequality, operations, fractions, and mathematical properties. Similarly, Adnan et al. (2021, p.4) refer to five different types of errors that learners make when solving problems with algebraic expressions, namely: reading, comprehension, transformation, process skill, and encoding. These classifications overlap with the six categories suggested by Ying et al. (2020, p.5408) of difficulties that learners face. Küchemann (1978, p.23) rather focuses on six interpretations of letters in mathematical problems and aligns these to Piaget's stages of cognitive development. Despite drawing on Piaget's stages, Küchemann (1978) does not refer to propositions from cognitive science and offers little reason for the computations that learners do. Discussing the interpretation *letter as variable*, Küchemann (1981, p.111-112) refers to the test item "Which is larger, $2n$ or $n + 2$?" and claims that learners with "sufficient 'processing capacity' to be able to cope with complex [second-order] relationships" would be apt to think about how n affects which of $2n$ and $n + 2$ are larger, whereas learners who lack this capacity would choose a simple answer, for example based on one possible value of n. He does not explain how "processing capacity" is measured beyond being able to solve this type of problem nor does he define what it means "to cope" with second order relationships. Notwithstanding his suggestion that learners' cognitive levels (rather than their experiences of algebra) determine their performance in solving algebraic problems, Küchemann (1981) does not provide sufficient evidence for the link between cognitive level and algebra level and how learners' cognitive level can be determined. Susac et al. (2014, p.5), however, who tested learners' cognitive abilities using Raven's Progressive Matrices in their study focusing on the development of abstract reasoning in 13- to 17-year-old learners, provide evidence that learners "with higher cognitive abilities were generally more efficient in equation solving." They do not, however, attempt to draw on cognitive science to explain learners' algebraic problem-solving computations. Nonetheless, Susac et al. (2014) do provide a more integrated discussion of the relationship between cognitive levels and mathematical problem-solving capacities by drawing on Küchemann (1981), Piaget's theory of cognitive development, and on learners' results in the cognitive ability tests. Notwithstanding the contributions of scholars like Susac et al. (2014) and McGowen and Tall (2010), in general, the literature shows little evidence of taking propositions from cognitive science into account when considering the ways in which learners approach algebraic problem solving and why. The cognitive science lens of this study will contribute to addressing this gap in the literature.

Research in cognitive science (e.g., Feigenson et al., 2004; Gelman, 2015; Spelke, 2022; Stahl & Feigenson, 2018) asserts that humans are born with a core knowledge system for number. Natural number concepts are learnt from a young age and the literature (e.g., McGowen & Tall, 2010; McGowen, 2016) suggests that many of the computations that learners use in incorrect solutions to algebraic problems are a result of them trying to apply natural number concepts in cases where they are not appropriate. For example, the notion that adding two numbers together results in a larger number is not appropriate if one or both numbers are negative. Another met-before highlighted by McGowen and Tall (2010, p.172) as problematic is the notion that the negative symbol necessarily indicates a negative value; when it comes to algebraic terms like $-x$, where x need not be a positive number, learners struggle to recognise that $-x$ could in fact be positive. Young and Booth (2020, p.393) maintain that if "students hold a flawed or incomplete concept of negative numbers, they may make improper generalizations which then affect their algebra performance." Despite the importance that understanding negative number concepts holds in algebraic problem solving—a notion also supported by Vlassis (2004) and Vlassis and Demonty (2022)—Booth et al. (2014, p.19) point out that a "surprisingly limited amount of research has been conducted on students' understanding of negative numbers" and even less on the relation between negative numbers and solving algebraic problems. The following section discusses some of the literature from this limited field of research.

2.3 Symbol sense, subtraction, and negative numbers

Symbol sense and dealing with negative numbers are highlighted as difficulties across the literature focusing on algebraic problem solving (e.g., Booth et al., 2014; Booth & Koedinger, 2008; Küchemann, 1978, 1981; Stewart & Reeder, 2016). Vlassis and Demonty (2022) provide valuable insight into how developing learners' relational thinking may affect their capacity to do operations with negative numbers. Relational thinking allows learners to recognise the structure of an expression and how the terms in the expression may be related, and to use that knowledge to choose certain procedures or computations which allow them to solve a given problem. The lack of research on this topic and on negative numbers in general is highlighted by Vlassis and Demonty (2022), which is significant given the recency of the study. The authors suggest that developing learners' algebraic thinking will address their difficulties with negative numbers, and the study serves as a foundation for further research to be done on the relationship between algebra and negative number concepts.

In their study, Vlassis and Demonty (2022) suggest that most learners who performed well when operating with negative numbers were able to understand subtraction as a transformation where the minus sign is seen as attached to the number following it and where that signed number serves as an operation, rather than the minus sign serving as a binary operator. For example, in the equation $7 - 5 = 2$, learners would recognise "-5" as a transformation being applied to 7, rather than seeing the minus sign as a binary operator and 7 and 5 as unsigned numbers. Understanding the three major functions of the negative symbol is highlighted in the literature (e.g., Lamb et al., 2012; Vlassis, 2004, 2008; Vlassis & Demonty, 2022) as fundamental to learners' capacity for algebraic problem solving. Vlassis (2004, 2008) refers to the three functions as *unary*, *binary*, and *symmetric* and classifies the negative symbol as either a *structural signifier* or an *operational signifier*[1], according to its function. The term 'negativity' is used by Vlassis (2004, 2008) to refer to these functions of the minus sign generally. The unary function of the negative symbol "corresponds to the role of the minus sign as a sign attached to the number to form the negative numbers" (Vlassis, 2008, p.561) and determines the negative symbol as a structural signifier. This description lacks the specificity that the number needs to be an unsigned—i.e. positive—real number for the attachment of the negative symbol to produce a negative number. Vlassis' (2004, 2008) use of "function" in this case is misleading because the unary function does not correspond to a function from the point of view of mathematics, rather "function" in this case is used as a synonym for "role" or "use," whereas the binary and symmetric functions (i.e. roles) *can* be thought of as representing functions which take inputs and produce outputs. Since the unary function simply refers to the role of the negative symbol as an indicator of negative numbers, it seems more appropriate to classify the negative symbol in this role as an *existential signifier*. According to Vlassis (2004, 2008), as an *operational signifier* the function of the negative symbol is either binary or symmetric. In the first case the negative symbol denotes a binary

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[1] Structural signifiers represent mathematical objects, and operational signifiers represent mathematical operations.

operation which takes two input values, the most common understanding of which is subtraction or "taking away." For example, $5-3$ represents "5 take away 3" or "5 minus 3." The symmetric function of the negative symbol is to "take the opposite" of the single input (Vlassis, 2004, p.472). Although Vlassis (2004, p.472) mentions "inversion," she does not refer to negation which ultimately is what the symmetric function represents. For example, $-(-3)=3$ represents the negation of -3, or "take the opposite of -3." Note that the negative symbol in -3 could also be considered a unary operator indicating the negation of the positive number 3. Referring to negation as a symmetric function may be misleading and so the classification of the negative symbol as a unary operator by Lamb et al. (2012) in this case (and as a binary operator in the preceding case) is more suitable. The classification of the functions of the negative symbol is useful because a learner's reading of the negative symbol in a given problem informs how they choose to solve it and therefore their solutions can be analysed to determine whether they correctly read the negative symbol or not. Vlassis (2004, p.482) asserts that "students have a template-driven use of the symbols used in the polynomials" and the template[2] they choose depends on their reading of the minus sign or the context of the minus sign.

A common misconception amongst learners is that the negative symbol is detachable. Vlassis and Demonty (2022) refer to the common error as *detachment from the minus sign* (DFMS) and explain that learners operate on the term following a minus sign as if it were separate from the minus sign. For example, given $x-2+5$ learners may simplify the expression to $x-7$ by detaching the 2 from the negative symbol and calculating $2+5$ to obtain 7, then reattaching the negative symbol as a binary operator for the answer $x-7$. Vlassis and Demonty (2022, p.1254) identify DFMS as a "significant and persistent obstacle in the learning of both algebraic and numerical operations" and explain that learners exhibit DFMS even in numerical expressions and this practice easily transitions into dealing with algebraic expressions. McGowen and Tall (2010) also consider this phenomenon and suggest that the met-before developed from left-to-right reading[3] as a possible explanation. To tackle the persistent challenges that learners face in grasping negative number concepts, Vlassis and

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[2] Templates are discussed in more detail in Section 5.4.

[3] Naturally this only applies to learners whose language does in fact require left-to-right reading, although similar met-befores may develop from any one-directional type of reading.

Demonty (2022) and Lamb et al. (2012), amongst others, suggest ways in which teachers can develop learners’ algebraic thinking and their understanding of negative numbers.

2.4 Summary

The existing literature (e.g., Küchemann, 1978, 1981; Vlassis, 2004, 2008) provides a general taxonomy of learners’ readings of mathematical symbols which the present study can draw on in its analysis of learners’ solutions to algebra problems. An effective general research design for investigating learners’ solutions also emerges from the literature (e.g., Bofferding, 2010; Küchemann, 1978, 1981; Vlassis & Demonty, 2022), which the present study will follow. Paper-and-pencil tests provide a useful initial insight into learners’ understanding of mathematical concepts and the computations they use to solve algebraic problems. Following the analysis of learners’ test scripts, the literature suggests that interviewing a sample of the learners provides the necessary further insight into their mathematical reasoning and the procedures they follow to arrive at different solutions. Using a similar research design as those outlined above allows this study to draw on related findings from the literature.

Setting this study apart from those done previously, which have largely overlooked the possible explanations of learners’ computational choices, is its theoretical framework, set out in the following chapter, which draws on theories from different disciplines like cognitive science and linguistics, allowing for the contribution of a new classification of learners’ understandings and (mis)conceptions.

3 Theoretical Framework

3.1 Introduction

The previous chapter highlights that the mathematics education literature is wanting in studies that explore why learners make certain errors. Why are certain misconceptions so dominant? And what fundamentally explains the common errors that learners make? Beyond simply identifying what learners do computationally, this study seeks to explain learners' computations and consider where they stem from. In accordance with a rationalist epistemology (see Chomsky & Ronat, 1979), this study assumes that knowledge is based on reason and that it can be constructed independently of sensory experience; in other words, understanding can be deduced using an *a priori* knowledge base. According to an Integrated Causal Model (ICM) (see Tooby & Cosmides, 1992), which serves as the paradigm for this study, the social sciences and the natural sciences are inherently linked, and those links must be considered when studying the human mind. An ICM approach to research entails drawing on propositions and theories from different fields of study, without hard boundaries, where those propositions and theories are mutually consistent and relevant to the research objects. In the case of this study, those fields of study include cognitive science, semiotics and mathematics, allowing for the construction of holistic depictions and explanations of the computational phenomena found in the data. The propositions and concepts outlined in this chapter construct the theoretical framework for this study, serving as a basis for the research design set out in Chapter 4 and guiding the interpretation, analysis and discussion of the data in Chapter 6 and Chapter 7. This theoretical lens allows for a more integrated analysis which not only describes what learners do computationally but attempts to explain *why* as well.

3.2 Underpinning theories and propositions

An ICM recognises the mind as "a set of evolved information-processing mechanisms instantiated in the human nervous system" (Tooby & Cosmides, 1992, p.24). This notion is also supported by Chomsky, who asserts that, like systems of the physical body, the mind is a system of "mental organs" which are "organized according to a genetic program that determines their function, their structure, [and] the process of their development" (Chomsky

& Ronat, 1979, p.83). There appear to be few studies in mathematics education that draw on research in the natural sciences to explain observed phenomena. Many of the studies mentioned Chapter 2 neglect to consider where learners' (mis)conceptions come from or why learners think the way they do.

Drawing on the field of cognitive science (e.g., Feigenson et al., 2004; Gelman, 2000, 2015; Spelke, 2017, 2022), this study holds that human knowledge belongs to *core domains* and *non-core domains*, where *domain* refers to a body of knowledge whose contents is determined by a particular set of principles. The acquisition of knowledge in a particular domain depends on those "information-processing mechanisms" that constitute the mind. Knowledge belonging to a core domain develops naturally from infancy, without particular attention to learning. Core domains are "represented in an innately given skeletal form" (Gelman, 2015, p.185), enabling spontaneous learning, whereas non-core domains require new mental structures to be developed through instruction and dedication to learning. Humans' innate capacities to learn their native language, to distinguish between animate and inanimate objects, and to represent numerical magnitudes are all examples of core domain knowledge (see, for example Gelman, 2015; Feigenson et al., 2004). School mathematics, however, and algebra specifically, constitutes a non-core domain for which learners "must acquire the structure and the domain relevant content pretty much from scratch" (Gelman, 2000, p.855), thus requiring dedicated attention to learning.

Fundamental to this study is the proposition that the mind is a computational system (see, for example Gallistel & King, 2010; Neumann, 2017) and that mental processes like problem solving are, consequently, computations. Neumann (2017, p.25) describes a process of computation as one in which "some output is produced from some input through certain operations." By nature, computational systems satisfy the Markov property, i.e. that the next state of the system is only dependent on the current state, not on any past states[4]. Chomsky et al. (2023, p.17) explain that at "each step in a derivation, a computational system operates on a certain representation; it does not have the capacity to access earlier steps of computation." When learners are given an algebraic expression to simplify, there are several processes they need to go through to arrive at a solution. First, they read the expression, i.e.

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[4] See Arbib (1969, p.328) for a formal definition.

they process visual data to produce an internal representation of the expression. Next, they identify the transformations required to simplify the expression. These transformations generate new internal representations which are externalised when the learner writes down their working out, until finally they arrive at their solution. Learners' computations are what constitute these working-out processes. Note that by the Markovian nature of computational systems, each simpler expression is derived only from the one preceding it. Although not wholly representative of their thinking, learners' written working out gives great insight into the computations that they likely used.

3.3 Principles of rationalist enquiry

Conditioned by a rationalist approach to research, this study is driven by a need to contribute explanations, rather than mere descriptions, of observed phenomena. To ensure the reliability and validity of both the descriptions and explanations of learners' computations, this dissertation is governed by three principles of rationalist enquiry, namely: observational adequacy, descriptive adequacy, and explanatory adequacy[5]. Observational adequacy is concerned with accurately capturing the primary data. Regarding this study, it requires recognising what learners are doing computationally, i.e. the sundry transformations they use, and going beyond simple classifications. For descriptive adequacy to be achieved, the variations and complexities of the observed data need to be accounted for and represented accurately. This entails detailed and faithful descriptions of the various transformations that learners use, which the use of interviews helped to achieve. Finally, explanatory adequacy requires consideration of the plausible reasons for why the data is the way it is, i.e. why learners choose certain computations and where those computations stem from.

3.4 Definitions and descriptions

To maintain descriptive and explanatory adequacy, this section offers definitions and descriptions of certain concepts and terminology such that they may be used unambiguously in the discussion of learners' computations.

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[5] See Chomsky (1964) for a discussion of these principles with regards to linguistics.

3.4.1 Conceptual and procedural knowledge

In describing and explaining what learners do computationally, references are made to their conceptual knowledge and procedural knowledge. Conceptual knowledge is taken here to refer to the knowledge associated with the meanings of and relations between things. For example, the meanings that learners associate with the objects of their computations. Procedural knowledge is assumed to refer to knowledge related to the processes required for completing tasks, like simplifying algebraic expressions. Jaffer (2018, pp.25-27) discusses, and contests, a dichotomy established in the mathematics education literature between conceptual knowledge[6] and procedural knowledge. She highlights that the latter is often considered lesser than the former and that many scholars' descriptions of the two forms of knowledge "imply that connections between knowledge only occur with respect to conceptual knowledge and that procedural knowledge is devoid of such connections" (Jaffer, 2018, p.25). In this dissertation, the two forms of knowledge are not considered in opposition, rather it is recognised that "procedures cannot be divorced from concepts because concepts are central to cognition" (Jaffer, 2018, p.26). Where a learner makes use of procedures considered non-standard from the point of view of mathematics, it is not because they lack conceptual knowledge, rather it is because their conceptual knowledge *differs* from what is considered standard or necessary by mathematics. As McGowen and Tall (2010, p.172) state: "One cannot have a misconception without first having a conception." In this case, it is learners' conceptions of the denotation of objects, based on their existing knowledge, which may generate misconceptions regarding how to deal with those objects.

3.4.2 Computational syntax categories

By Chomsky's (2007) argument that language does not have a reference function; what a learner is referring to is conveyed by *how* they use language (including written productions) more so than *what* language they use. Learners' computations involving the components of the expression associated with mathematics may not always be those considered standard

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[6] Crooks and Alibali (2014) investigate the various definitions of conceptual knowledge found in research on mathematical thinking.

from a mathematics perspective but may nonetheless produce the required outcome[7] (Jaffer, 2018).

In this study, learners' computational activity is inferred by examining their written work, gestures, and verbal explanations. The transformations that learners use reveal, for the most part, the referents that they assign to the objects of their computations[8]. Learners' computations are classified in three ways:

i. Typographic: referring to the typographic composition of expressions rather than to the denoted mathematical composition.
ii. Indexical: an indication to perform a transformation.
iii. Mathematical: keeping with the axioms for the field $(\mathbb{R},+,\times)$.

For example, a learner might simplify $5x - 2x$ to $3x$ by first recognising that both terms contain x (typographic), then "lifting out" the numbers for subtraction (indexical), but not recognising that the distributive property of $(\mathbb{R},+,\times)$ is enabling of such a transformation. Their computation is typographic and indexical but mathematically restricted to subtraction over $\mathbb{N}$.

3.4.3 Merge and the computational workspace

An operation which is adopted in this dissertation for the analysis and discussion of learners' computations is the operation known as *Merge*, introduced by Chomsky (1994, see also 2004, 2013) in the field of linguistics. Chomsky (2015, p.16) explains that Merge "takes objects X and Y already constructed and forms a new object Z." For the purposes of this study, Merge is taken more generally as an operation which takes two inputs and produces an output in which those inputs are contained. For example, if the algebraic expression $3 + x$ is (incorrectly) simplified to $3x$, this exhibits the conjoining of "3" and "x" which can be denoted by $\mathrm{Merge}(3, x) = 3x$. Chomsky (2015, p.16) posits Merge as "the simplest computational operation, embedded in some manner in every relevant computational procedure." In their close examination of Merge as a computational device, Chomsky et al. (2023, p.2) introduce the notion of a *workspace*, which they gloss as "the set consisting of the material available for

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[7] See Section 4.1.

[8] See the three inter-related levels comprising computational activity proposed by Davis (2013; see also, Jaffer, 2018).

computation at a given derivational stage.” When the objects in a workspace are taken as the arguments of a computation—e.g. Merge—then the output of the computation constitutes a new object, thus modifying the workspace. The objects in this modified workspace may then be taken as the arguments of new computations. Note that workspaces are therefore Markovian by nature.

3.5 Conclusion

Driven by a rationalist epistemology and adopting an ICM approach to research, this study accepts that humans are born with certain innate mental structures which allow for the acquisition of core domain knowledge (Gelman, 2000, 2015; Spelke, 2017, 2022) as well as the later development of new mental structures supporting non-core domain knowledge, such as school mathematics, and algebra in particular. In accepting the mind as a computational system (Gallistel & King, 2010), learners’ attempts at simplifying expressions are taken as comprised of various computations which this study seeks to describe and offer possible explanations for. The propositions and theories set out in this chapter underpin the methodology described in the following chapter and serve to guide the production, analysis and discussion of the data as presented in Chapter 6 and Chapter 7.

4 Methodology

4.1 Introduction

With the context and situatedness of this study having been established in the previous three chapters, this chapter goes into detail about the research problem and questions, and the research design that was followed in order produce data which could be analysed and discussed to address those questions.

4.2 Research problem and questions

When simplifying algebraic expressions, learners rely on various computations to arrive at their solution. Often there is more than one way in which a solution can be reached, but these pathways ultimately represent the same transformations just in different compositions. Learners may use tricks or shortcuts which take them from the problem to its solution along a path that differs to the one they were taught to take. In some instances, a learner may arrive at the correct solution through transformations that *cannot* be mapped to those considered correct from the point of view of mathematics.

For example, asked to simplify the expression $(3-x)-(3-x)$, a learner might immediately recognise that $(3-x)$ and $-(3-x)$ are additive inverses and give an answer of 0, or they might choose to multiply out the brackets:

$$(3-x)-(3-x)=3-x-3+x=3-3-x+x=0.$$

These two methods are both correct, the first one is simply a contraction of the second. Alternatively, the learner might incorrectly simplify $3-x$ to $-3x$ by concatenation (i.e. merge), but still come to the correct solution as follows:

$$(3-x)-(3-x)\mapsto(-3x)-(-3x)=-3x+3x=0.$$

This method disobeys mathematical axioms but nonetheless shows some seemingly logical reasoning and produces the correct final solution. Looking only at learners' solutions disregards the possibility that they may have been correct by chance rather than by mathematical reasoning. This study seeks to identify and describe the paths (computations) that learners choose when simplifying algebraic expressions and to explain *why* they may choose certain paths above others.

Both the theoretical framework (Chapter 3) and the analytical framework (Chapter 5) of this study appear to be relatively novel to the focus area. There is comparatively little literature which i) uses universal algebra as a tool for describing and analysing learners' computations, or ii) draws on cognitive science to attempt to explain those computations. To remedy this gap in the literature, this study addresses the question: *What do Grade 9 learners do computationally when simplifying algebraic expressions involving the negative symbol?* The following sub-questions are also considered: *What transformations do learners use when simplifying algebraic expressions? What computations are elicited by the negative symbol?*

4.3 Research approach

As can be deduced from the literature (e.g., Bofferding, 2010; Booth et al., 2014; Küchemann, 1981; Susac et al., 2014), the data sources most suitable for analysing learners' algebra-problem-solving behaviours and methods, and which were therefore used in this study, are algebra test scripts and post-test interviews. Given that tests can induce stress or anxiety in some learners and since the learners' solutions were kept anonymous and had no impact on their school marks, the set of algebra problems was instead referred to as a "task." An initial interpretation of learners' computations could be developed based on their written solutions; however, these did not always capture the internal representations that they generated when problem solving. Therefore, post-task interviews were necessary to gain more insight into the transformations that learners used. Before each round of interviews, a preliminary analysis of the available data was done to guide the interview schedules for each learner. This triangulation process allowed for a more holistic understanding of learners' computations as well as greater validity and reliability of the data.

This research study is qualitative and explanatory in nature, with the interviews serving as the most important source of information. The literature (e.g., Booth et al., 2014; Küchemann, 1978, 1981; Tirosh et al., 1998) served to guide the formulation of the task and interview schedules for primary data collection. There are several studies (e.g., Lamb et al., 2012; Vlassis, 2004, 2008; Vlassis & Demonty, 2022) which classify the ways in which learners understand (or misunderstand) the functions of the negative symbol. The algebra problems and corresponding understandings of the negative symbol outlined in those studies served as a general guide to which types of expressions would elicit certain computations or misinterpretations. The task in this research study, generating primary data, was formulated using that general guide whilst keeping in line with the senior phase mathematics curriculum (DBE, 2011).

4.4 Research design

This research study, a case study of one school, comprised three stages. The first stage consisted of the curation and administration of the task, followed by a preliminary analysis of learners' scripts. The task was designed to elicit a range of computations, particularly those highlighted in the literature, to gain insight into the transformations that learners use when simplifying expressions. Learners were given the task to complete individually and without intervention, after which their scripts were preliminarily analysed for common approaches, errors and misinterpretations. The second stage comprised the curation of a general interview schedule along with specific solutions chosen for each learner, first-round interviews, a preliminary analysis of the first-round interviews, and follow-up interviews. The interview schedules were designed according to the preliminary script analysis. Learners were questioned on their responses to the task and asked to explain their working out. Certain learners were selected for follow-up interviews to get a better understanding of their mathematical thinking and to see their responses to new expressions. The analysis of all scripts, interview notes and interview video-recordings constituted the final stage. These were analysed in full using cognitive science and universal algebra as lenses to identify, describe and explain the computations learners did when simplifying algebraic expressions.

4.4.1 Data collection

4.4.1.1 Participants

The participants of this study were a group of 22 learners who belonged to one Grade 9 mathematics class in a suburban public high school in the Western Cape. The class had 30 learners but eight of them did not consent to their data being used and are therefore excluded from the study. The school has a Quintile 5 ranking, meaning that it is classified as being situated in a community with a low unemployment rate and a high literacy rate (unlike lower ranked quintile schools)[9]. Over the last five years the school's matric pass rates have all been above 95%, with more than 70% of learners achieving Bachelor's Passes, positioning the school as high performing. The school has one mathematics class for the top-performing Grade 9 learners, whilst the rest of the learners are in mixed-ability classes, the participants of this study included. The 22 participants each attempted the algebraic subtraction task and interviews were conducted with 13 of them (those who agreed to this part of the study) to gain further insight into their computations. The remaining nine learners did not agree to being interviewed.

The algebraic subtraction task was administered in the first term, before learners had begun learning new algebraic concepts as part of the Grade 9 curriculum. Therefore, the learners were expected to have a grasp on the conventions of algebraic language and manipulating algebraic expressions only up to Grade 8 level. Interestingly, after disruptions to teaching resulting from school closures and emergency remote learning in response to the COVID-19 pandemic, the DBE removed the introduction of algebraic concepts from the Grade 7 Mathematics Annual Teaching Plan (ATP) for 2023/2024 (DBE, 2023a). Learners in Grade 7 were therefore not eligible to participate in this study at all and, with the data collection taking place in the first term, neither were learners in Grade 8 as they would not be introduced to algebraic language and algebraic expressions until late into Term 2 (DBE, 2023b). Grade 9 learners were therefore the only eligible cohort within the senior phase and focusing on Grade 9 learners in the first term allowed for an in-depth analysis of the computations that form the base of algebraic problem solving, which could be drawn on for research involving learners in higher grades. This cohort is also the most focused on in the literature (e.g., Booth et al., 2014; Küchemann, 1978, 1981; Susac et al., 2014; Tirosh et al., 1998; Vlassis, 2004, 2008), which allowed for existing data and analyses to be drawn on for guidance and for comparison.

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[9] It must be noted that quintile allocational designations are problematic, particularly in the Western Cape, and should not be taken as a strict marker of socio-economic class. See Dieltiens and Motala (2014) and White and Van Dyk (2019).

4.4.1.2 Research Instruments and Procedures

The first stage of the study began with the curation of the algebraic subtraction task. A review of the literature dealing with algebraic problem solving and negative symbol sense served to highlight which kinds of algebraic problems, with a focus on subtraction, elicit which kinds of computations and mappings. To ensure that the task was at the appropriate level for the participants, Grade 8 and Grade 9 mathematics textbooks and algebra worksheets were examined for sample algebraic expressions. A Universe of Possibilities mind map (Appendix 1) was constructed to set out the various forms that the subtraction problems could take when containing combinations of integers and variables. For example, an expression containing integers and a single variable x can be formulated such that the only arguments available for subtraction are:

i. integers as constants
   e.g. $3x - 4 - 5 = 3x - 9$
ii. integers as coefficients
   e.g. $-8x - 4x - 3 = -12x - 3$
iii. integers as constants and integers as coefficients
   e.g. $8 - 3x - 2x - 1 = 7 - 5x$

The Universe of Possibilities presents 15 different possibilities for expressions containing integers and one or more variables; these possibilities can be considered the different subtraction *types*.

The subtraction operation $a - b$, where a and b are integers, can vary according to whether a and b are positive or negative. A table was created to outline the different categories, each with two variants, under which $a - b$ could be classified (Figure 1).

	$a > 0$	$a < 0$
$b > 0$	**F1a**: $\|a\| > \|b\|$ e.g. $8 - 5 = 3$	**F2a**: $\|a\| < \|b\|$ e.g. $-5 - 8 = -13$
	F1b: $\|a\| < \|b\|$ e.g. $5 - 8 = -3$	**F2b**: $\|a\| > \|b\|$ e.g. $-8 - 5 = -13$
$b < 0$	**F3a**: $\|a\| > \|b\|$ e.g. $8 - (-5) = 13$	**F4a**: $\|a\| < \|b\|$ e.g. $-5 - (-8) = 3$
	F3b: $\|a\| < \|b\|$ e.g. $5 - (-8) = 13$	**F4b**: $\|a\| > \|b\|$ e.g. $-8 - (-5) = 3$

Figure 1: Categories of subtraction

The variations (e.g. F1a, F1b) can be considered the different subtraction *sub-forms*. A set of subtraction problems was devised so that each subtraction *type* was represented at least once and so that there were at least five questions employing each subtraction *form* (e.g. F1, F2). The idea was to cover the various functions of the negative symbol as much as possible and to elicit an array of computations. The task (Appendix 2) consisted of 19 algebraic expressions which learners were asked to simplify. A memorandum (Appendix 3) was created to serve as the control script against which to compare the learners' calculations and solutions.

The task was administered to learners during a one-hour mathematics lesson towards the end of Term 1. Learners worked individually and without intervention, attempting one question at a time on separate sheets of paper. The scripts were collected as learners worked through the task to prevent them from returning to questions and changing their solutions. This was done to capture learners' initial computations, giving insight into the mappings that they resorted to most naturally, so that these could be analysed and explored further during the interviews. Once they had all been collected, the scripts were organised by learner. In preparation for the interviews, the solutions by learners being interviewed were preliminarily analysed and a mix of correct and incorrect solutions was chosen for each learner according to the computations they exhibited.

Beginning the second stage, a general interview schedule (Appendix 4) was drawn up to probe learners for explanations of how they understood the different expressions and what computations they did to simplify them. Some interview questions were prepared based on certain questions in the task and the solutions they had elicited, but those questions were open and could prompt further questioning depending on learners' responses. The objective of the interview questions was to elicit explanations that clearly (to the extent that is possible) show the transformations and computational resources that learners used in their approach to simplifying the algebraic expressions. Interviews were roughly 20-25 minutes in length and took place during the learners' mathematics period in an unoccupied classroom. Learners were interviewed individually, and all the interviews were video recorded (with the learners' and parents' consent), although the camera was not directed at their faces. Before the interview, it was explained to learners that they would be given their scripts one at a time and asked to explain what they did to try to simplify the given expression. A blank workbook was put in front of each learner, this being the focal point of the camera, for writing calculations or notes if they wanted to. In cases where learners expressed that they would change their answer, they were asked to write down their calculations and revised solution in the workbook. Where learners exhibited interesting computations, they were sometimes asked to simplify new but similar expressions written in the workbook and explain their working out. Probing questions were asked throughout the interview to gain further insight into the computational resources that learners employed and how they understood the numbers, symbols and operations. Once the selected questions had been worked through, learners were asked about their remaining solutions in the same manner, depending on how much time was left and on their previous responses and explanations.

After a brief reflection on the first nine interviews, it seemed certain responses had been skimmed over and required further investigation to explore the computational resources being used, so follow-up interviews were organised with those nine learners. The remaining four learners were interviewed only once with more probing questions. The number of solutions on which the first nine learners were interviewed varied between 9-14 and was dependent on the extent to which solutions were discussed and how much time was available. The last four learners were asked to explain between 5-8 of their solutions and were given extra questions to further explore their computations.

In preparation for the follow-up interviews, the video-recordings were briefly analysed to identify the computations that could be explored further. It emerged from the task solutions that there were certain kinds of expressions that learners struggled with, namely: expressions containing brackets, expressions containing superscripts, expressions with multiple variables, and expressions which triggered cancellation-like manipulations (e.g. $ax \pm x$ or $ax \pm a$). Where a learner's solution exhibited transformations involving one (or more) of these kinds of expressions, a code was used to categorise that solution, with the codes corresponding to the kinds of expressions above: *brackets, superscripts, multiple variables,* and *cancellation* (see Figure 2).

The questions which elicited marked transformations were recorded and a set of 16 extra expressions was generated (Appendix 5). These extra expressions were designed to be like the recorded original expressions so that learners' computations could be compared across simple changes. For each learner, six extra expressions were chosen depending on the codes that had been logged from their interview recordings. The reason for drawing from the same set of expressions in the follow-up interviews was, again, to be able to compare learners' solutions for the same questions (wherever there was overlap) and consider whether certain computations were more common than others. Learners were also given other expressions during their follow-up interviews specific to the responses they had given.

Code	**Common errors**	**Example (Learner)**
Brackets	Misinterpretation of brackets as implying multiplication (e.g. Q5)	$(3x - 5) - (3x - 5)$ $= -9x + 25$ (L4, L10, L11)
Cancellation	Removal of variables or of numbers (typically when the letters or numbers, respectively, are the same) (e.g. Q2, Q9)	$3x - 3 = x$ (L1) $-x^3 + 3x^3 = 3$ (L24)
Exponents	Addition or subtraction of exponents (e.g Q7, Q17)	$-5x^2 - 12x^2 - 4x^2$ $= -7x^4 + 4x^2$ $= 11x^6$ (L24)
Multiple variables	Treating ab-terms and ba-terms as unlike (e.g Q15)	$13ab - 4ba - 13 - 2ba - (-9) - 1$ $= 13ab - 6ba - 5$ (L1, L3, L6, L10, L13, L20)

Figure 2: Coding system for task solutions

The follow-up interviews lasted between 15-25 minutes each and were conducted in the same way as in the first round, but with the extra questions as the focus. For each selected question, the expression was written in the blank workbook, and learners were asked to write their solution and explain their working out. In some cases, learners were given other expressions to simplify to elicit similar computations and asked to explain what they were thinking. If the six chosen questions had been worked through, learners were asked to simplify the remaining expressions, time depending. All video-recordings (both first round and follow-up interviews), scans of the workbook pages (Appendix 8 and Appendix 10) and transcriptions of the interviews (Appendix 9 and Appendix 11) were saved to a secure folder.

These sources along with learners' task solutions constitute the information archive from which the data of this study was generated.

4.4.2 Data analysis

The final stage of the study included the in-depth analysis of the solution scripts (Appendix 7), interview notes (Appendix 8 and Appendix 10), and all video-recordings. The information archive was preliminarily analysed, for each learner and across learners, generating the data set for in-depth analysis of learners' computations as presented in Chapter 6 and Chapter 7. Three tables were created to capture i) learners' overall performance on the task (Appendix 6), ii) the frequency of different solutions to each task question, and iii) a general profile for each learner. Each learner profile consisted of notes regarding the transformations that emerged from their written work and verbal explanations (if they had been interviewed), the consistency of their methods, and any misconceptions they had. Given the limited scope of the dissertation, focus areas were identified for analysis and discussion by identifying non-standard transformations, errors and misconceptions from the learner profiles as explanations of these are wanting in the literature. Extracts from learners' written work and interview transcripts were selected to best demonstrate and give insight into the computations of relevance to each focus area. The focus areas correspond to the subheadings in Chapter 6.

Comparing the transformations that learners used across their written work and drawing on their own descriptions and explanations of their working out made for greater observational adequacy in the analysis and discussion of learners' computations. The analytical framework, presented in Chapter 5, was established to ensure descriptive adequacy in the written, schematic, and diagrammatic representations of learners' computations in Chapter 6. This framework draws on universal algebra and sets out definitions and explications of the objects, transformations and concepts used in the analysis. Finally, explanatory adequacy was attained by drawing on propositions and research from cognitive science, driven by the theoretical framework (see Chapter 3), and the relevant mathematics education literature (see Chapter 2), to explain what learners did computationally and, importantly, offer explanations as to *why*.

4.5 Ethical considerations

Ethics approval for this study was granted by the School of Education ethics committee at the University of Cape Town (reference: EDNREC20231104) and permission to conduct the research in a public school was granted by the Western Cape Education Department (reference: 1655D9DDF000007-20231124). Permission was also granted by the school and the teacher of the mathematics class, who was consulted before data collection to determine which days would be suitable for the administration of the task and for the interviews. Given that the participants were under 18 years of age, both they and their parents/guardians were asked for written consent to participate in the study, the objectives of which were clearly presented. To protect the identities of the participants and their schools, no actual learner names were used in the study and only my supervisors and I have access to the video recordings of the interviews. No personal information was collected from participants and, to further protect their identity, the video recordings did not focus on the participants' faces only on the workbook put in front of them. The task scripts were also stored securely, and all electronic data were stored in a secure folder that only my supervisors and I can access.

To minimise the chances of learners feeling stressed or anxious when completing the subtraction task, it was made clear to them that their task scripts and results would not be seen by their teacher or parents, only by me and my supervisors. They were also reassured that it did not matter whether their answers were correct or incorrect as the focus of the study was their mathematical thought. Beyond this the study posed no harm to its participants. Several learners in the class chose not to participate in the study at all and their autonomy was respected, as was the case with learners who opted not to be interviewed.

The next chapter sets out the analytic framework used to generate and analyse the data.

5 Analytical Framework

5.1 Introduction

Having discussed the context (with regards to the literature), epistemological situatedness and research methodology of this study in the first four chapters, the remaining chapters of this dissertation are concerned with the presentation, analysis and discussion of the data. To ensure that the principles of rationalist enquiry outlined in Chapter 3 are satisfied in the remaining chapters, this chapter sets out an analytical framework which allows for learners' computations do be identified, described and explained in a consistent and reliable manner. The chapter begins by describing the notion of a *computational workspace*[10] after which certain objects and mappings relevant from mathematics as well as those generated pedagogically or by learners are described and explained for their use in Chapter 6 and Chapter 7. The transformations generated by learners (or sometimes their teachers) in school mathematics are based on those from the field of mathematics, but often take non-mathematical arguments, like symbols rather than real numbers.

5.2 The computational workspace

It can be said then that when confronted with a mathematical task, the *reading* of a mathematical task necessarily generates a *workspace* populated by the objects available for completing that task. For example, given an algebraic expression to simplify, a learner generates a workspace containing objects derived from the expression. Those objects are taken as the arguments of their computations which are driven by the need to reduce the expression. It is important here to understand that the term *reading* may refer to the material expression of a text, as in speech or writing, or to the interpretation of a text, i.e. the signification attributed to it[11]. Naturally, different learners can be expected to generate different workspaces according to the computations employed and these workspaces may or may not align with the mathematical denotation.

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[10] Introduced in Chapter 3.

[11] See Dowling's (1998, p.127) use of Barthes' (1981) distinction between *text-as-work* and *text-as-text*.

Consider one of the task items used in this study: “Simplify $11x - 8x - 3$.” Generated by the reading of this task, an initial workspace, W, might be

$$W = (11x,_{-}\, 8x,_{-}\, 3).$$

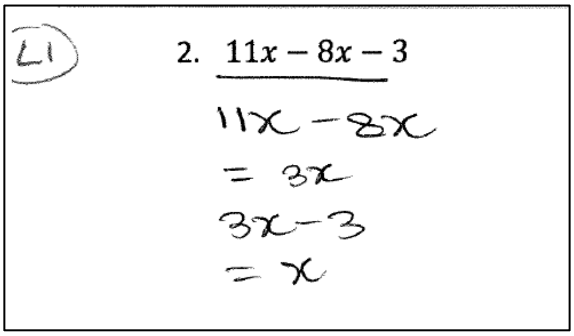
The negative symbols are subscripted here to indicate that the objects $8x$ and 3 are recognised in the expression as arguments of the operation of subtraction (these negative symbols are not objects in the workspace). This workspace shows an acceptable parsing of the expression to elicit the computation $11x - 8x$, generating a new workspace

$$W' = ([11x - 8x],_{-}\, 3).$$

Hard brackets are used here to indicate a computation that will produce a new object (notice that the operator symbol is written inline when it appears between hard brackets). The computation $11x - 8x$ produces the output $3x$, constituting a new object and generating a new workspace

$$W'' = (3x,_{-}\, 3).$$

The expression $3x - 3$ can then be extracted as a solution from W''. The transformations $W \to W' \to W''$ are framed by the instruction to simplify $11x - 8x - 3$. Learners might arrive at this, or another, solution without necessarily thinking of the objects in the workspaces as real numbers. For example, consider a solution offered by one of the participants, Learner 1 (L1), to the same task (Figure 3).



L1

2. $11x - 8x - 3$

$11x - 8x$

$= 3x$

$3x - 3$

$= x$

Figure 3: L1's solution to Q2

Here we can deduce that W, W', and W'' are the same as those above since L1 arrived at the expression $3x - 3$. However, we see that L1 did not stop there, reducing the expression $3x - 3$ to x. It appears then that L1’s reading of the expression $3x - 3$ prompted them to treat $3x$ as representing two independent objects available for computation, thus generating a new workspace

$$W''' = (3_1, x,_{-} 3_2),$$

in which the 3_1 can be taken as an argument independently of x. Subscripted numbers are used here indicate different copies of the same object. This indicates an existential shift of the object 3 from number to character. From their solution x we can deduce that L1 did the computation $3 - 3 = 0$ and then mapped 0 to *nothing*, i.e. "cancelled the 3's," represented by the absence of a symbol in the position preceding x, since it is unconventional to write $0x$[12]. Let us denote this mapping of 0 to *nothing* by $\mathrm{VOID}: 0 \mapsto \Phi$, where Φ indicates the absence of a symbol in a particular position within an expression. From W''', L1's computations therefore seem to have generated the following transformation of workspaces:

$$W''' \to W'''' = ([3_1 - 3_2], x) \to W''''' = (0, x) \to W'''''' = (\Phi, x).$$

Since *nothing* concatenated with x is simply x, L1's solution can be extracted from W''''''. This series of computations indicates that i) algebraic objects of the form ab are not treated as a product, $a \times b$, but rather as a simple concatenation of a and *b*, and that ii) numerals and letters appear to be treated as referring to different types of objects. We can infer that L1's computations involved the standard mathematical objects and operations along with operation-*like* manipulations requiring arguments that do not reference the real numbers. It follows that observational and descriptive adequacy cannot be achieved by treating the computational inscriptions of learners as referring to only the range of objects and operations consistent with the axioms for the field of real numbers.

5.3 Learner-generated computational objects

Since an algebraic expression represents a composition of operations, the point of simplifying the expression is to reduce the number of operations that are being composed without altering the expected output. Davis (2013, p.4) highlights that it is "not unusual to find alternate operations, or even *operation-like manipulations* [emphasis added], replacing the operations indicated by mathematical statements in the pedagogic situations of schooling."

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[12] The term $0x$ is equal to 0 so it does not affect the value of an expression when added or subtracted, therefore it is typically omitted, unless it is the only term in the expression in which case it is simply written as 0.

What emerges from learners' written solutions and verbal responses to the items in this study is the use of transformations that do not necessarily share a domain (set of inputs) with the operations that they are attempting to replace, and, more importantly, a strong type-sensitivity regarding the arguments of those transformations.

Worth highlighting here is that transformations and their arguments are compossible, meaning that transformations imply the presence of objects that are consistent with those transformations, and vice versa. Naturally, algebraic expressions indicate the presence of operations related to real numbers, regardless of whether those real numbers be represented by letters or by numerals. However, the operation-like manipulations emerging from the data are often *not* compossible with real numbers, implying the presence of other objects that serve as the arguments of learners' computations. Those objects, it appears, are not restricted to the set of real numbers.

Five types of objects were revealed through learners' productions of algebraic expressions: *sign*, *numeral*, *letter*, *superscript* (*sup*), and *operator (opr)*. The objects constituting an expression each belong to a particular set corresponding to their type, as defined in the following. Note that these definitions are specific to this study, but that they could be extended to include more objects as required.

- *Sign* objects are elements of the set $\{\Phi, +, -\}$, where Φ here denotes the textual absence of an object, i.e. an empty string //. For example, the expression $-2x$ contains a sign object $/-/$ whilst the expression $3y$, being unsigned, can be read as containing the sign object Φ. *Sign* objects indicate whether the *numeral* or *letter* objects they precede are positive or negative.

- *Numeral* objects belong to the set $\mathbb{Z}^{+} \cup \{\Phi\} \cup C$, where $\mathbb{Z}^{+}$ denotes the set of non-negative integers and C denotes the set of numeral characters representing such integers. Note that $/0/$ is typically represented by Φ. The inclusion of C here is important because sometimes the arguments of learners' transformations are not numbers but numerals representing numbers. The transformation $3 + x \mapsto 3x$, for example, takes the numeral $/3/$ as an argument, not the number 3. Note that numeral objects may be referred to as the arguments of number operations (e.g. subtraction) in an explanation, however in such a case the arguments are in fact the numbers being represented, not the character symbols[13].

- $*$ *Letter* objects are elements of the set $X \cup \{\Phi\}$, where X denotes the set of one- or two-letter strings, e.g. $/x/$ or $/xy/$ (with two-letter strings consisting of distinct letters). In the field of mathematics, letter objects are used to represent unknown real numbers, i.e. variables, which adhere to the axioms of $(\mathbb{R}, +, \times)$. In school mathematics however, letter objects seem often to be read simply as alphabetic characters which elicit various operation-like transformations from learner to learner, as discussed in this dissertation.

- $*$ *Superscript* objects belong to the set $\mathbb{Z} \cup \{\Phi\} \cup S$, where S denotes the set of character symbols representing integers. In the field of mathematics, superscript objects are used to represent exponentiation i.e. raising one object to the power of another, which is an operation that conforms to certain axioms. For example, x^2 represents "x raised to the power of 2" or "x squared." In school mathematics, learners' computations do not always conform to the axioms of mathematics regarding exponentiation nor align with the mathematical denotation of exponents. Consequently, the term *superscript* is used rather than *exponent*.

- $*$ *Operator* objects are elements of the set $\{+, -, \times\}$, although in this study $/\times/$ is only used in one or two cases where learners wrote $/\times/$ in their solutions. The objects $/+/$, $/-/$, and $/\times/$ represent the binary operations of addition, subtraction, and multiplication, respectively, which are mappings $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that take the terms in algebraic expressions as their arguments, and which conform to the axioms of $(\mathbb{R}, +, \times)$. As some learner solutions suggest, however, the mathematical properties of these operations are not always respected. Note that although the object $/\times/$ is absent from the items, the operation of multiplication is certainly ubiquitous implicitly[14]. It is also important to note that an operator object is written in the same way whether learners do a standard mathematical operation or an operation-like

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[13] See NUM mapping on p.39.

[14] See Figure 7 (p.41)

manipulation, but in the latter case the arguments of the transformation are strings rather than real numbers. For example, in “$3 + 2 = 5$” the operator object $/+/$ represents addition of real numbers 2 and 3, whereas in “$3 + x = 3x$” the operator symbol represents some transformation that takes the strings $/3/$ and $/x/$ as arguments and produces a new string $/3x/$.

5.3.1 Workspace notation

To keep the representation of workspaces concise, the notation will not involve the use of forward slashes given that the definition of an algebraic expression implies that such objects are always-already strings of symbols. Objects in the workspace are written in the order in which they appear in an expression and are separated by commas. Different copies of the same object are indicated by subscripted numbers, e.g. (x_1, x_2). Operator objects ($/-/$ or $/+/$) are subscripted preceding an object to indicate that object as the argument of the relevant operation or operation-like manipulation i.e. the subtrahend/subtrahend-like object or addend/addend-like object, but the subscripted operator objects are not considered objects available as arguments in the workspace. Sign objects are written in-line and they might be taken as arguments for a transformation. An operator object ($/-/$ or $/+/$) may be changed to a sign object and vice versa depending on the computational focus. Where an expression is parsed by types of objects, but adjacent objects are recognised as belonging to a single component, those objects will be grouped together in brackets. Hard brackets indicate a transformation that will produce a new object. Note that learners’ choice of transformation dictates the nature of the objects (e.g. numbers or characters) in an expression which in governs the workspace. The following examples serve as a guide for the notation.

- $W = (4a,{}_{+}\,3a,{}_{-}\,1)$ generated by $4a + 3a - 1$
  e.g. $3a$ ($\in \mathbb{R}$) is an argument of addition, or $/3a/$ is an argument of a transformation represented by the operator object $/+/$ (like concatenation or some form of merge).
- $W = \big((-5, x,{}^{2}),{}_{-}\,(12, x,{}^{2}),{}_{-}\,(4, x,{}^{2})\big)$ generated by $-5x^2 - 12x^2 - 4x^2$
  e.g. $/-5/,/x/$ and $/{}^{2}/$ are considered together but can be taken as arguments independently.
- $W = \big((-, x), (-,15), (-,11)\big)$ generated by $-x - 15 - 11$

e.g. The sign objects /—/ are considered as arguments.

5.4 Character distribution matrices as computational templates

Learners' written solutions and verbal responses suggest that they work with the terms in an algebraic expression as being comprised of the different types of objects in a particular arrangement, as shown in Figure 4.

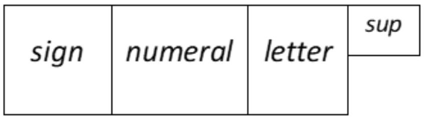
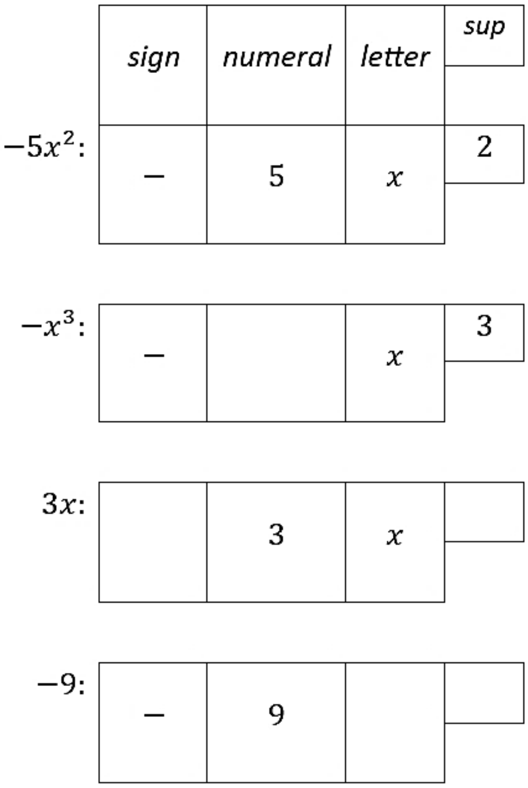


Figure 4: Arrangement of objects to constitute an algebraic term

Each fixed position takes one type of object. Learners seem to think of algebraic terms as a concatenation (merging) of different object types, a typographical rather than mathematical computation. This arrangement is an example of what Johnson and Davis (2010) call a *character distribution matrix,* i.e. a pedagogic mechanism which serves to regulate the transformations used in mathematical activities that require particular spatial distributions of symbols. Johnson and Davis (2010. P.143) posit that for "the spatial transformation of symbols not to violate their symbolic value the symbols have to be organised in very specific orders, enabling their symbolic value to be maintained implicitly when, or even because, the problem solver pays no explicit attention to it." Learners parse (i.e. unmerge) the arrangement in Figure 4, taking individual objects as the arguments of their transformations such that, if they follow the appropriate procedures, the symbolic values of those arguments are implicitly preserved and their solutions will be of the same arrangement. The objects in Figure 4 serve as the inscriptions which exist in a learner's workspace at a particular derivational stage, whether they be generated from a given algebraic expression (e.g. the items in the task) or by some computation in the process of simplifying said given expression. Chomsky et al. (2023, p.12) posit that the inscriptions in the workspace are "irreducible lexical features of 'form' (i.e., sound/sign) and meaning that are combined to comprise the atoms of syntax." In this study, the different types of objects defined above serve as the "irreducible lexical features" required for working with algebraic expressions (and their inscriptions more specifically). Figure 5 provides examples of algebraic terms represented according to the arrangement provided in Figure 4, and subsequently of the kinds of objects that may be entered into a learner’s workspace for computation.



	sign	*numeral*	*letter*	*sup*
$-5x^2$:	$-$	5	x	2
$-x^3$:	$-$		x	3
$3x$:		3	x	
-9:	$-$	9		

Figure 5: Algebraic terms represented by types of objects

When a term is unsigned, has no written variable component (i.e. is a constant), no written coefficient (i.e. its coefficient is equal to ± 1), or no written exponent (i.e. its exponent is equal to 1), then the corresponding position or “box” is left empty, in which case the object is Φ. One could replace Φ by the relevant object for an equivalent representation, as shown in Figure 6, but this may not coincide with learners’ reading of the expression. In cases where there is no superscript in an expression, the superscript “box” may be omitted from the representation for simplicity, however when there is no letter object it is important to note that the superscript object in that case is 0.

When terms containing the negative symbol are in isolation, take $-x^3$ for example, the negative symbol is a sign object, as shown in Figure 6. In an expression like $2x^3 - x^3$,

	sign	*numeral*	*letter*	*sup*
$-x^3$:	$-$	1	x	3
$3x$:	$+$	3	x	1
-9:	$-$	9	x	0

Figure 6: Algebraic terms represented by types of (non-Φ) objects

however, the negative symbol may be read by the learner as an operator object, indicating that the (unsigned) term x^3 is term being subtracted from the term $2x^3$. In this case, drawing on the three different uses of the negative symbol proposed in the literature (e.g., Lamb et al., 2012; Vlassis, 2004, 2008), the operator object /—/ functions as a binary operator.

5.5 The uses of the negative symbol

Although it is agreed in the literature that the negative symbol has three distinct uses, the descriptions of those uses are not unanimous. To avoid ambiguity in the computational analysis, the three uses are explicitly defined below within the classification of the negative symbol as either an *existential signifier*[15] or an *operational signifier*.

- *Existential signifier*: A negative symbol in front of a number indicates an element of the set of negative real numbers $\mathbb{R}^-$. For example, $-3 \in \mathbb{R}^-$.
- *Operational signifier*:
  - *Binary operator*: A negative symbol separating two (like) terms indicates that those terms be taken as the arguments of the binary operation of subtraction. For example, $8x - 3x = -(8x, 3x) = 5x$ (with prefix notation being used here to make the functional nature of operations more explicit).

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[15] *Existential signifier* is used in place of *structural signifier* and *unary function* following the critique of this terminology in Chapter 2.

- *Unary operator*: A negative symbol in front of a term indicates the operation on that term as a single input described as “taking the opposite,” i.e., negation. For example, $-x$ represents the negation of x, so if $x > 0$ then $-x$ is a negative number and if $x < 0$ then $-x$ is a positive number, like with $-(-8) = 8$ where the “opposite” of -8 is 8.

The computational focus determines in which way the negative symbol is used. When learners use the word “minus” it can be difficult to determine which use they are referring to because “minus” is polysemic. For example, “minus five” can mean “negative five,” “subtract five,” or “take the opposite of five.” To determine which of these three roles a learner is referring to, one relies on their explanation and the context given. Recognising which conception of the negative symbol is required by a problem is very important for choosing appropriate computations. For example, to simplify “minus five minus three” $[-5 - 3]$, one needs to recognise the first “minus” as indicating that 5 is negative whilst the second “minus” indicates that 3 is being subtracted, therefore the computational focus should be the operation of subtraction with -5 and 3 as arguments: $-(-5{,}3)$. Note that one could, alternatively, read both negative symbols as indicating negative numbers if the operation is taken to be addition, $+(-5, -3)$, in which case the reading might instead be “minus five *and* minus three” with *and* indicating addition.

Given a term like $-(-3a)$, the first negative symbol can be taken as an operator object acting as a unary operator, and the second negative symbol as a sign object indicating that the coefficient of a is negative. Negating a negative number gives a positive number, so the output of this unary operation is the term $3a$, i.e. $-(-3a) = 3a$. An alternative reading of $-(-3a)$ is the product $-1 \times -3a$. The mnemonic “minus and minus makes plus” (and many renditions of it) is taught in many classrooms as a tool for recalling that the product of two negative numbers is a positive number, or that subtracting a negative number is equivalent to adding its opposite. One result of this mnemonic is a mapping, let us denote it MMP (defined below), which allows learners to replace the term $-(-3a)$ with the term $+3a$, without thinking of the mathematical transformations taking place.

5.6 Learner-generated operation-like mappings

The following are definitions of a variety of mappings that will be used to describe the operations and operation-like manipulations used by learners, and to analyse their computations in detail. The first four of these mappings, namely: STR, SUN, NUM, and CON, come from Davis (2013, p.11) and their definitions are adapted in certain details for relevance to the items of this study.

- $*$ $\mathrm{STR}(\lambda)$: This mapping takes an algebraic expression λ as its input and returns the alphanumeric string $/\lambda/$. For example, $\mathrm{STR}(4x^2-2x)$ returns the string $/4x^2-2x/$ which is a sequence of the single-character strings $/4/$, $/x/$, $/^2/$, $/-/$, $/2/$, and $/x/$. Notice that the superscripted character remains at a different spatial level. Alternatively, we may have $\mathrm{STR}(4x^2-2x)=/\Phi 4x^2-2x^{\Phi}/$, where the absence of a sign object and a superscript object, respectively, represented by Φ, are part of the computational focus. In written format, the input and output of this mapping look the same, however an existential shift takes place as the algebraic expression, a representation of a composition of operations on real numbers, changes to a string of characters. Since algebraic expression are always-already strings of symbols, STR will only be used in cases where it is necessary to explicitly highlight a transformation $\lambda \mapsto /\lambda/$ within a series of computations.

- $*$ $\mathrm{NUM}(/\lambda/)$: For any string $/\lambda/$ representing an integer[16] (i.e. $/\lambda/$ can consist only of alphanumeric characters which represent an integer when concatenated), NUM maps the string $/\lambda/$ to the integer $\lambda \in \mathbb{Z}$ itself. For example, $\mathrm{NUM}(/-4/)=-4$. An existential shift takes places under this mapping too with a string of characters changing to a representation of a composition of operations. This mapping is implicitly used when numeral objects are referred to as the arguments of number operations.

- $*$ $\mathrm{SUN}(/\lambda/)$: A string $/\lambda/$ is sundered by this mapping into a list of two more strings $(/\lambda_1/, \ldots, /\lambda_n/)$, where $n \in \mathbb{N}$ and $n \geq 2$. Where a string $/\lambda/$ consists of more than two characters, the output of this mapping can differ according to the computation being

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[16] According to the items of this study.

performed. For example, the input $/3x - 4/$ may result in any combination of strings comprised of the characters $/3/$, $/x/$, $/-/$, and $/4/$ (and Φ where appropriate) as an output, such as $(/3x/, /-4/)$ or $(/3x/, /-/, /4/)$, or $(\Phi,/3/,/x/,/-/,/4/)$, etc. The strings into which an expression is sundered may be considered independently for further computation. In diagrams a single SUN mapping may be represented by separate SUN arrows pointing to the different strings which together constitute the whole expression (see Figure 28).

- $\mathrm{CON}(/\lambda_1/, \ldots ,/\lambda_n/)$: This mapping takes a list of strings $(/\lambda_1/, \ldots ,/\lambda_n/)$ and produces their concatenation $(/\lambda_1 \ldots \lambda_n/)$, i.e. links them in a series. Again, the arrangement of types constituting a term plays a role in that the concatenation of strings conforms to the ordering of objects in Figure 4. For example, $\mathrm{CON}(/-/, /3/,/x/) = -3x$. Note that concatenation is not commutative, i.e. $\mathrm{CON}(\lambda_1, \lambda_2) \neq \mathrm{CON}(\lambda_2, \lambda_1)$.

- $\mathrm{SEL}_\tau(/\lambda/)$: Taking τ to be a type-marker, this mapping selects and returns a list of the objects of type τ (output) in the order they appear in the string of alphanumeric characters λ (input), such as those constituting an algebraic expression. This mapping is used in the parsing of an expression. The domain and codomain of $\mathrm{SEL}_\tau(/\lambda/)$ are the class of alphanumeric strings. According to the object types defined earlier, we therefore have five distinct $\mathrm{SEL}_\tau(/\lambda/)$ mappings, denoted by choosing τ from the set $\{\sigma, n, l, s, o\}$ to represent *signs*, *numerals*, *letters*, *superscripts*, and *operators*, respectively, and defined as follows:
    - $\mathrm{SEL}_\sigma(/\lambda/)$: Selects one or more singlet strings consisting of a sign object, typically a minus sign associated with a numeral object in the data set under consideration. For example, given $-2x^2 - 3x$, if the negative symbol preceding $3x$ is taken as an operator object, then $\mathrm{SEL}_\sigma(/-2x^2 - 3x/)$ returns the sign object $/-/$ of $-2x^2$.
    - $\mathrm{SEL}_n(/\lambda/)$: Selects one or more singlet strings consisting of a numeral object. For example, $\mathrm{SEL}_n(/-6x - 8/) = (/6/,/8/)$.
    - $\mathrm{SEL}_l(/\lambda/)$: Selects one or more singlet strings consisting of a letter object. For example, $\mathrm{SEL}_l(/8ab - 3ba - 1/) = (/ab/,/ba/)$.

- $\mathrm{SEL}_s(/\lambda/)$: Selects one or more singlet strings consisting of a superscript object. For example, $\mathrm{SEL}_s(/-5x^4-3x^2/)=(/^4/,/^2/)$.
- $\mathrm{SEL}_o(/\lambda/)$: Selects one or more singlet strings consisting of an operator object. For example, $\mathrm{SEL}_o(/5x-2x+1/)=(/-/,/+/)$.

The number of outputs returned by $\mathrm{SEL}_\sigma, \mathrm{SEL}_l$ and SEL_s for an input containing copies of the same object depends on the computational focus. For example, given $-5x^2-2x^2$, if both negative symbols are taken as sign objects, and the focus is on the terms having the *same* sign object, then $\mathrm{SEL}_\sigma(-5x^2-2x^2/)=/-/$, whereas if the focus is on the *number* of sign objects in the expression, then $\mathrm{SEL}_\sigma(-5x^2-\ 2x^2/)=\ (/-/,/-/)$. Similarly, SEL_l might return only the common letter object $/x/$, or both copies of the letter object, i.e. $(/x/,/x/)$, and SEL_s might return $/\,^2/$, or $(/\,^2/,/\,^2/)$. In cases where two (usually adjacent) types of objects are selected together, the notation $\mathrm{SEL}_{\tau_1\tau_2}(/\lambda/)$ is used. For example, SEL_{nl} selects pairs of numeral and letter objects, e.g. $\mathrm{SEL}_{nl}(/4x-2x/)=(/4x/,/2x/)$.

* $\mathrm{MMP}(/-(-/)$: This mapping is used for application of the "minus minus makes plus" mnemonic to simplify an algebraic expression. Let Y be the set of algebraic terms whose sign object is Φ. Given an algebraic expression containing the term $-(-\lambda)$, where $\lambda \in Y$, MMP takes as input the string of characters $/-(-/$ and selects one of two outputs according to whether $-(-\lambda)$ is preceded by another term in the expression:

$$\mathrm{MMP}(/-(-/)=\begin{cases}\ \ \Phi, & \text{if} -(-\lambda) \text{ has no preceding term}\\ /+/, & \text{if} -(-\lambda) \text{ has a preceding term}\end{cases}$$

Note that when the MMP mapping is applied to $/-(-/$, the closing bracket is also removed from $/-(-\lambda)/$ for the output $/+\lambda/$ or $/\lambda/$.For example, given the expression $2-(-5x^2)$, the term $-(-5x^2)$ is preceded by the term 2, therefore $\mathrm{MMP}(/-(-)/\ )=/+/$ and the expression becomes $2+\ 5x^2$. For the expression $-(-4a)+3b$, however, $\mathrm{MMP}(/-(-/\ )=\Phi$, since the term $-(-4a)$ is not preceded by another term. In the items of this study, most terms $-(-\lambda)$ had a preceding term.

* $\mathrm{VOID}(/\lambda_1/,\ldots/\lambda_n/)$: Taking any list of strings $(/\lambda_1/,\ldots/\lambda_n/)$ as input, this mapping simply sends that list to the object Φ (with the type of object that Φ belongs to being

determined by the computational focus). For example, $\mathrm{VOID}(/x - x/) = \Phi$, and the transformation $/x + 0/ \mapsto /x/$ can be described using $\mathrm{VOID}(/0/)$ since $/0/$ is replaced by Φ.

5.7 Computational pathways

The mappings outlined above are helpful in representing learners' computations that do not conform to the properties of real numbers. Another mapping that is used in the representations of learners' computations, but one which does come from mathematics, is an inclusion mapping. Denoted $i_x\colon a \to b$, an inclusion mapping maps an object a found in the workspace to some expression b containing a. For example, given the expression $-2x + 3y$, we may identify the following naturally occurring inclusion mappings:

$$i_x\colon -2x \to -2x + 3y \text{ and } i_x\colon +3y \to -2x + 3y.$$

The term $-2x$ is included in the expression $-2x + 3y$ by i_x. The use of the inclusion mapping is always implied when expressions are composed in the process of computing and is, therefore, usually left out of descriptions of computations. However, where the outputs of previous computations are used as inputs to subsequent computations, explicit inscription of the inclusion mapping is used. For example, the composition of operations in $-2x + 3y$ can be represented diagrammatically as in Figure 7.

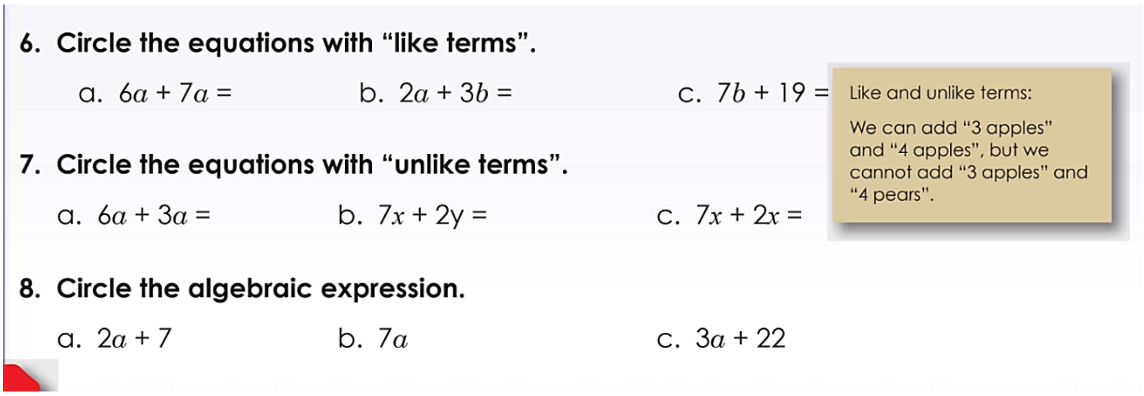
$$\begin{tikzcd}
-2, x \arrow[r, "\times"] & -2x \arrow[dr, "i_x"] & & \\
& & -2x, 3y \arrow[r, "+"] & -2x + 3y \\
3, y \arrow[r, "\times"'] & 3y \arrow[ur, "i_x"'] & &
\end{tikzcd}$$

Figure 7: Diagrammatic representation of the composition $-2x + 3y$

The expression $-2x + 3y$ cannot be simplified further because it consists of two *unlike terms*. The notion of *like* and *unlike* terms is specific to mathematics in pedagogical contexts; it is introduced in the teaching of algebra to help learners recognise when an expression can be simplified and when not. When two terms have the same variable raised to the same power they are considered like terms, and they can be reduced to a single term by adding their coefficients. For example, in the expression $5x^2 + 12x^2 + 4x^2$, the terms all have a variable x raised to the power of 2 , they are all like terms. The coefficients of the terms are 5, 12 and 4 and their sum is 21, therefore the expression can be reduced to the singlet $21x^2$. Relying on the distributive property of the real numbers, the calculation behind this reduction is as follows:

$$5x^2 + 12x^2 + 4x^2 = (5 + 12 + 4)x^2 = 21x^2$$

The concept of *like* terms is introduced to learners in the second term of Grade 8 (see DBE, 2023b) when they first start learning about algebraic expressions, but factorising algebraic expressions is only taught in the second term of Grade 9. Figure 8[17] and Figure 9 (p.43) are extracts from a Grade 8 workbook and textbook, respectively.



6. Circle the equations with "like terms".

a. $6a + 7a =$ b. $2a + 3b =$ c. $7b + 19 =$

7. Circle the equations with "unlike terms".

a. $6a + 3a =$ b. $7x + 2y =$ c. $7x + 2x =$

Like and unlike terms:

We can add "3 apples" and "4 apples", but we cannot add "3 apples" and "4 pears".

8. Circle the algebraic expression.

a. $2a + 7$ b. $7a$ c. $3a + 22$

Figure 8: Extract on like terms from Grade 8 mathematics workbook
(DBE, 2024, p.62)

Associating like terms with objects like apples, as in Figure 8, makes it very easy for learners to think of coefficients as cardinal-like values. For example, simplifying the expression $5x^2 + 12x^2 + 4x^2$, the term $5x^2$ may be thought of as "five x^2's," with the variable component x^2 becoming an arbitrary object, an "apple," and the coefficient 5 the "amount" of that object. In Figure 9, like terms are "combined" rather than "added"; using ambiguous language like "combine" can very easily prompt learners to do transformations that do not

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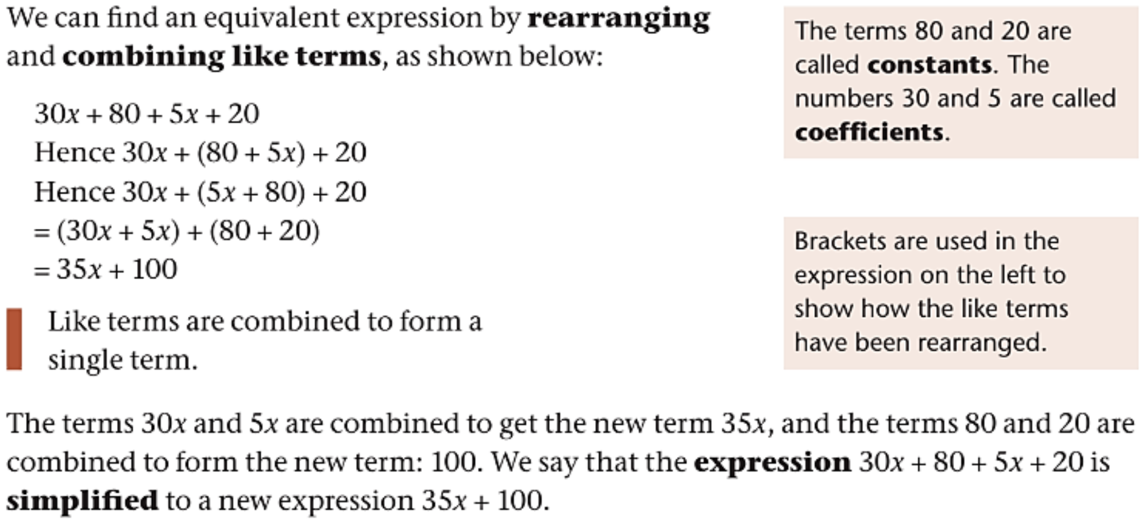
[17] It is curious in this extract that expressions like "$6a + 7a =$" are referred to as "equations" although they are not expressing equivalence. See Kieran (1981) for a discussion of the equal sign as an operator symbol.



We can find an equivalent expression by **rearranging** and **combining like terms**, as shown below:

$30x + 80 + 5x + 20$
Hence $30x + (80 + 5x) + 20$
Hence $30x + (5x + 80) + 20$
$= (30x + 5x) + (80 + 20)$
$= 35x + 100$

Like terms are combined to form a single term.

The terms 80 and 20 are called **constants**. The numbers 30 and 5 are called **coefficients**.

Brackets are used in the expression on the left to show how the like terms have been rearranged.

The terms $30x$ and $5x$ are combined to get the new term $35x$, and the terms 80 and 20 are combined to form the new term: 100. We say that the **expression** $30x + 80 + 5x + 20$ is **simplified** to a new expression $35x + 100$.

Figure 9: Extract on like terms from Grade 8 mathematics textbook
(DBE, 2017, p.71)

coincide with those that the language intends to depict (in this case, addition). Neither of the extracts make explicit use of the distributive or associative properties of the real numbers. Rather, the coefficients seem to be “lifted out” of the expression and added before their sum is rejoined with the variable. This process is similar enough to factorisation to be successful, but it requires transformations that are not mathematical operations, as demonstrated by Figure 10 and Figure 11 (p.44) (note that SUM simply adds all inputs together).

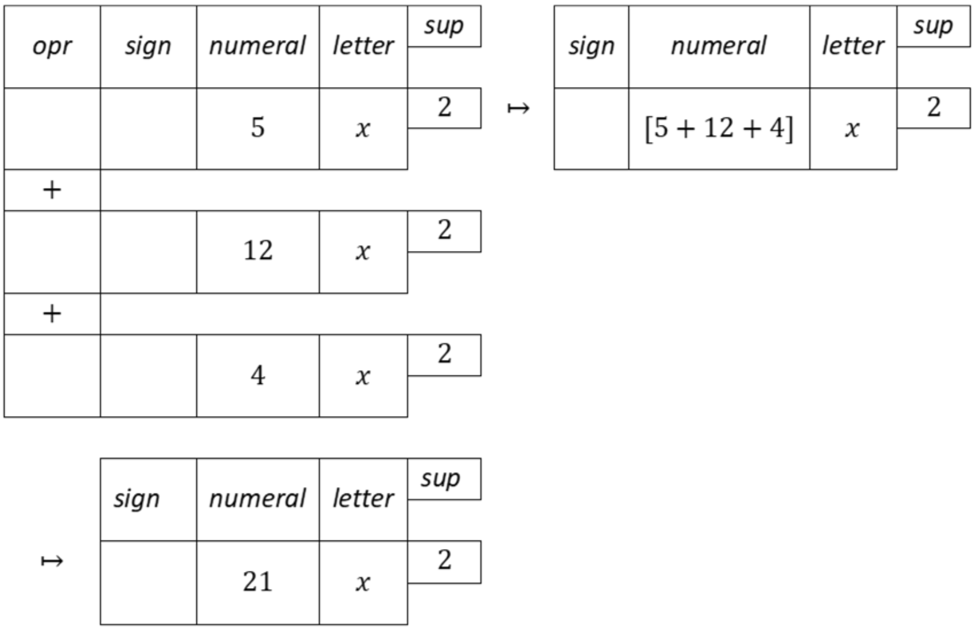


Figure 10: Schematic representation of type-specific computations for
$5x^2 + 12x^2 + 4x^2$

$$\begin{tikzcd}
 & /5x^2+12x^2+4x^2/ \arrow[dl, "\mathrm{SEL}_n"'] \arrow[dr, "\mathrm{SEL}_{ls}"] & \\
(/5/,/12/,/4/) \arrow[d, "\mathrm{NUM}"'] & & /x^2/ \arrow[ddd, "i_x"] \\
(5,12,4) \arrow[d, "\mathrm{SUM}"'] & & \\
21 \arrow[d, "\mathrm{STR}"'] & & \\
/21/ \arrow[rr, "i_x"'] & & (/21/,/x^2/) \arrow[d, "\mathrm{CON}"] \\
 & & /21x^2/
\end{tikzcd}$$

Figure 11: Diagrammatic representation of transformations for $5x^2 + 12x^2 + 4x^2$

Unlike terms are described as terms with different variables or the same variables but raised to different powers. Two *unlike* terms cannot be reduced to a single term. For example, the expression $x - 26$ cannot be reduced any further because the two terms x and -26 do not have a common variable, they are unlike. In Figure 8 unlike terms are compared to different kinds of objects like apples and pears: “we cannot add ‘3 apples’ and ‘4 pears.’” It is not true that we cannot “add” apples and pears; if we use the collective noun *fruit*, then it is very easy to “add” (i.e. combine) the apples and pears to constitute a collection of 7 fruits. This may be where learners struggle to keep unlike terms distinct.

5.8 Procedure for data analysis

To deduce what learners did computationally in their solutions[18], using the analytical tools set out above, the following general method was followed. For a selected written solution, the working out was analysed by checking whether each expression was equivalent to the preceding one. Where this was not the case, the differences (typographic and mathematical) between the two expressions were considered. A plausible workspace generated by the first expression was determined as well as one or more plausible transformations that may have

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[18] Appendix 7, Appendix 8 and Appendix 10

been used to produce the second expression. Where possible, the video recording or transcript[19] was consulted to see if the learner's verbal explanation coincided with the observed use of transformations. If not, the observed transformations (operation/operation-like manipulations together with compossible objects) were changed to match (to the extent possible) those revealed by the interview. Diagrams, schemas and descriptions were produced using the mappings and objects outlined above to represent and demonstrate learners' computations. These representations can be found in the next chapter, in which the computations that emerge from the data are dissected, described and analysed to provide insight into what learners may be doing and thinking in their attempts to simplify algebraic expressions.

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[19] Appendix 9 and Appendix 11

6 Analysis

6.1 Introduction

A plethora of computations are revealed in the data, but given the limited scope of this dissertation only a handful of them can be explored in detail. The computations represented and analysed in this, and the next chapter were selected for their pertinence to the negative symbol and for their exhibition of type-specific computations, according to object types identified in Section 5.3. In attempting to simplify expressions, many learners went beyond reducing like terms to reducing *unlike* terms by taking particular types of objects as the arguments of their computations. Following an overview of learners' performance in the task and interviews, those type-specific computations are analysed in detail according to the analytical framework set out in Chapter 5, demonstrating learners' idiosyncratic conceptions of the objects constituting algebraic expressions. For brevity, the task items and the learners are referred to by number, such as Q1 for Question 1 and L1 for Learner 1[20].

6.2 Overview

The performance on the task overall was quite poor, with many learners[21] answering less than half of the questions correctly (see Appendix 6). Three general performance levels were classified according to the number of correct solutions given in the task, namely high, medium, or low. The task scores required for each level as well as the number of learners who performed at that level are presented in Figure 12. The chart in Figure 13 shows how learners fared overall on each question.

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[20] L9, L12, L15, L16, L21, L23, L25, and L30 are excluded from the study.

[21] 14 out of the total 22.

Performance Level **(No. of correct answers)**	**Number of learners**
High **(13-19)**	5
Medium **(6-12)**	9
Low **(0-5)**	8

Figure 12: Task performance overview

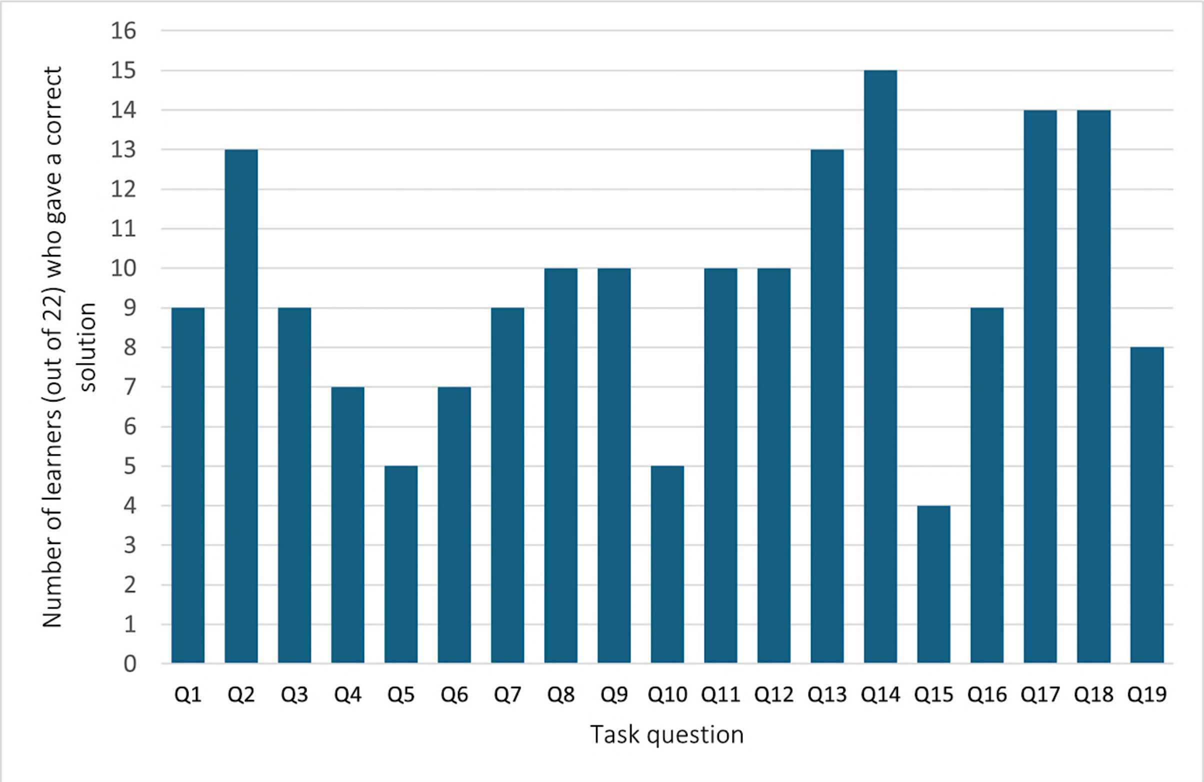

Figure 13: Number of correct solutions for each task question

Notice that despite Q14 being the most correctly answered question, there were still seven learners (roughly a third of the cohort) who answered it incorrectly, indicating that the task challenged learners overall. The performance level classification in Figure 12 is of course very reductionist and is intended only to give an overview. There were many cases where incorrect solutions nonetheless exhibited rational reasoning and knowledge of algebraic concepts beyond a “low performing” level. Many of the learners corrected their solutions during interviews, suggesting that it may have been a lack of practice which contributed to their poor results.

The data is rich in sundry transformations, some conforming to mathematical axioms and others portraying idiosyncratic (and non-mathematical) readings of mathematical objects and symbols. The task questions were designed to present learners with combinations of different types of subtraction but, ultimately, the same methods could be used for most of the questions. Interestingly, however, there was a high rate of inconsistency in learners’ methods across the task; this seems to have been prompted by a strong sensitivity to the types of objects constituting the expressions. In addition to type sensitivity, the negative symbol played a significant role in learners’ choice of transformations, eliciting, for example: sign changes, the removal of objects, and operations on numeral objects of unlike terms. During interviews many of the learners struggled to read and explain their own solutions and seemed generally uncertain of which methods to follow, often changing tactics as the interview progressed. Reflecting on their solutions by verbally explaining them or simply seeing the task items anew sometimes made learners aware of their mistakes and helped them to recall how to simplify the expressions correctly (or at least partly correctly). More often, however, learners did not recognise where their procedures had gone awry because their conceptual understanding of the objects with which they were working was not aligned with the mathematical denotation of those objects.

The sections that follow consider learners’ computations in more detail and offer various representations of the transformations that emerged from their solutions across the task and interviews, with Section 6.3, specifically, exploring learners’ attempts to simplify like and *unlike* terms.

6.3 Like and unlike terms

To simplify the kinds of algebraic expressions that constitute the items of this study, learners are typically taught some general procedure as comprised of the following[22]:

1. Expand the brackets (i.e. multiply out the brackets).
2. Identify like terms.
3. Reduce like terms to a single term by addition or subtraction.

Most learners were quick to identify like terms in the task items, often using different markings for different like terms, as Figure 14[23] exhibits.

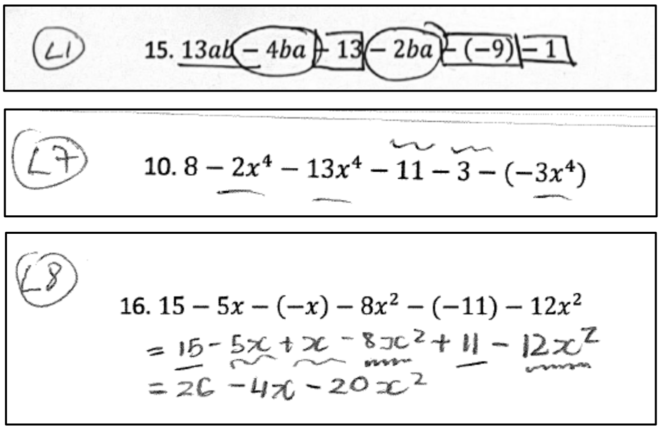

Figure 14: Solutions exhibiting markings for like terms

Simplifying an expression like $2x - 8x$, many learners will explain that, since there are two like terms with different signs, they subtract the "smaller" number from the "bigger" number and "keep the sign of the bigger number." In their interview (see Extract 1, p.50), L8 offered a name for this, and another method, namely: DSB and SSAK.

The SSAK method is used instead when two (or more) like terms have the same sign, in which case learners add together the numeral objects and keep the common sign and variable. The DSB and SSAK methods are used for subtraction of real numbers (or more typically integers), which allow them to be applied in algebraic expressions to the coefficients of like terms or to constants. Although no other learners referred to these two methods by name, their habitual

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[22] See, for example, DBE (2017, pp.67-71).

[23] Notice in Figure 14 that L1 used different markings for $13ab$ and $-4ba$, despite the products ab and ba being equivalent. See Section 6.5 for further mention.

L8: So, in primary school we did something that was called DSB. So, we would like, DSB, so the difference, whichever one is bigger, then you subtract them and then you bring down the bigger sign.

INT: OK, let me write that down, DSB. So, different, subtract, the bigger one?

L8: Yes Miss, and then same sign, add, keep.

⋮

INT: Oh, SSAK [writes SSAK].

Extract 1: L8 (first interview)

use emerged from interviews, with learners saying things like "take the sign of the bigger number."

The expression $2x - 8x$ contains two numbers, 2 (with an implicit $/+/$ in front) and -8, and one letter, x. Since $-8 + 10 = 2$ we know that 2 is a bigger number than -8. However, the "bigger" number that learners refer to in this expression is the whole number 8. In other words, learners look exclusively at the numbers 2 and 8 (represented by the numeral objects) and determine which of them is bigger so that they can settle on an order of subtraction, that being $8 - 2$ (bigger minus smaller). Because there is a negative sign in front of the "bigger" number, they know that their final answer should start with the same sign, and since both terms have a letter object $/x/$, so will the answer. Then, having simplified $8 - 2 = 6$, they can concatenate $/-/$, $/6/$ and $/x/$ for a final answer $-6x$, as depicted in Figure 15[24]. Note that square brackets are used here to indicate a transformation taking place (as in workspace notation).

sign	*numeral*	*letter*
	2	x
$-$	8	x

↦

sign	*numeral*	*letter*
$-$	$[8-2]$	x

↦

sign	*numeral*	*letter*
$-$	6	x

Figure 15: Schematic representation of type-specific computations for $2x - 8x$

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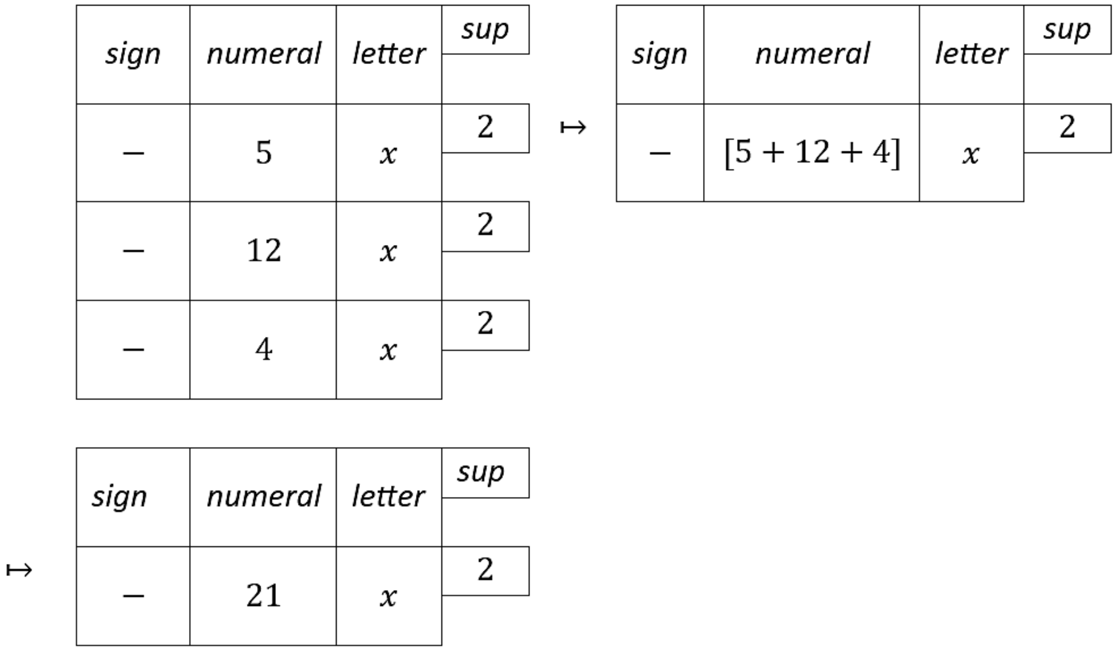
[24] See Davis (2013) for an in-depth computational analysis of the DSB method.

The expression $-5x^2 - 12x^2 - 4x^2$ (Q7) contains three like terms each with a negative coefficient, therefore the SSAK method can be used, i.e. keep the common sign and add the (whole) numbers. Again, this method requires parsing the expression according to types of objects, as shown in Figure 16 and Figure 18 (p.52).



sign	*numeral*	*letter*	*sup*
$-$	5	x	2
$-$	12	x	2
$-$	4	x	2

↦

sign	*numeral*	*letter*	*sup*
$-$	$[5 + 12 + 4]$	x	2

↦

sign	*numeral*	*letter*	*sup*
$-$	21	x	2

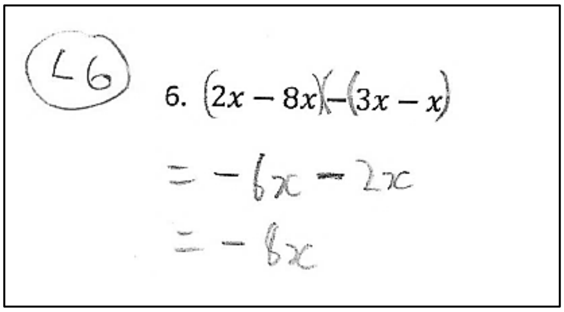
Figure 16: Schematic representation of type-specific computations for Q7

When working with like terms, the letter and superscript objects are of least importance because they can automatically be used in the final answer, as is the case for the sign object when terms have the same sign. As L4 said in their first interview: "you basically don't have to worry about the x squared, you must just put it in the next number." Therefore, the only objects being taken as arguments of transformations are the numeral objects, which are added then concatenated with the other objects for a final answer—in this case being $-21x^2$. Of course, the act of doing these computations is far more effortless than the act of describing them, as the complexity of Figure 18 may exhibit.

$$\begin{tikzcd}
 & /-5x^2-12x^2-4x^2/ \arrow[ddl, "\mathrm{SEL}_\sigma"'] \arrow[d, "\mathrm{SEL}_n"'] \arrow[ddr, "\mathrm{SEL}_{ls}"] & \\
 & (/5/,/12/,/4/) \arrow[d, "\mathrm{NUM}"'] & \\
/-/ \arrow[dddr, "i_x"'] & 5,12,4 \arrow[d, "\mathrm{SUM}"'] & /x^2/ \arrow[dddl, "i_x"] \\
 & 21 \arrow[d, "\mathrm{STR}"'] & \\
 & /21/ \arrow[d, "i_x"'] & \\
 & (/-/,/21/,/x^2/) \arrow[d, "\mathrm{CON}"] & \\
 & /-21x^2/ &
\end{tikzcd}$$

Figure 18: Diagrammatic representation of transformations used for Q7 (SSAK)

The DSB and SSAK methods only work for expressions constituted by two like terms. This means that any expression with more than two like terms needs to be sundered into digestible two-term subexpressions first, which many learners' solutions suggest being particularly challenging. For example, the expression $2x - 8 - 3x - x$ (Q6) can be sundered into two two-term subexpressions, with the obvious choice being $/2x - 8x\,//\,{-3x} - x/$, and then the DSB and SSAK methods can be applied to each subexpression, respectively. As demonstrated by L6's solution in Figure 17, how to sunder the expression was not necessarily obvious to learners.


L6

6. $(2x - 8x)(-(3x - x)$

$= -6x - 2x$

$= -8x$

Figure 17: Solution to Q6 exhibiting incorrect sundering by L6

The two sets of brackets drawn by L6 outline two subexpressions, $2x - 8x$ and $3x - x$, with the negative symbol in front of $3x$ existing as a binary operator (i.e. an operational signifier) between the two. In their first interview, L6 confirmed the use of brackets as a tactic for organising like terms and establishing an order of computations (Extract 2).

INT: In this one [Q6] you've also done brackets – is that to organise which terms you're going to subtract?

L6: Yes.

INT: … does it matter it in which order you do the subtraction?

L6: No.

INT: But would you use a specific order? Would you normally do the first two if they are like terms?

L6: Yes, I normally do the first two.

INT: And then the other two. So, you did these two first [pointing to $2x$ and $-8x$] and then those two second [pointing to $3x$ and $-x$], and then you've worked with those two numbers to bring it back down.

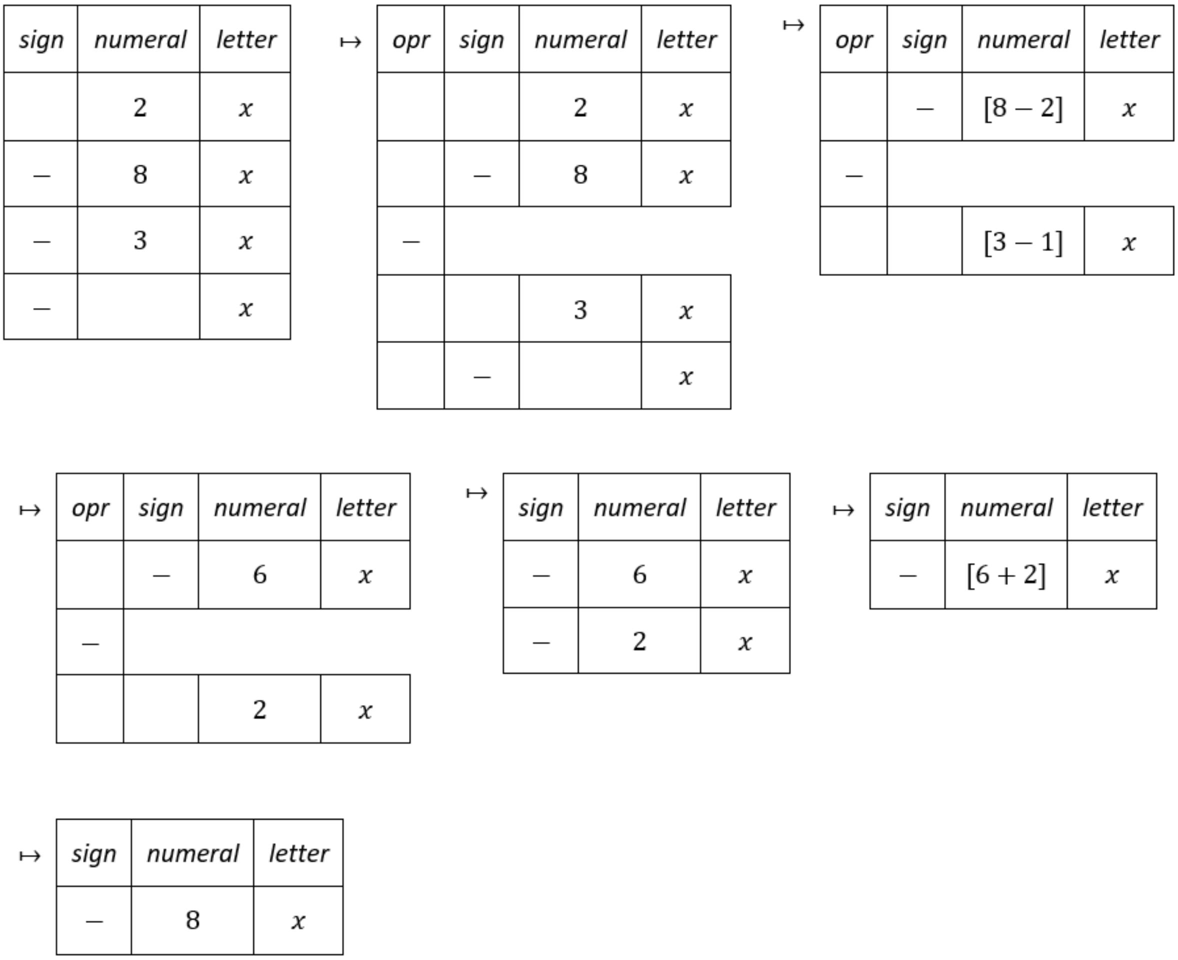
L6: Yes, just to make it more simple for me so I don't have to add all of it at once.

INT: OK, by putting these brackets here you're saying you'll do two x minus eight x first to get minus six x, and then you're subtracting, and then you're doing three x minus x gives you two x, and you've just brought down the minus.

L6: Yes.

Extract 2: L6 (first interview)

Likely using the DSB method, L6 simplified the subexpressions concurrently to $-6x$ and $2x$, with the operator object maintaining its position between the two subexpressions. Then, situated in a new two-term expression, $-6x - 2x$, the negative symbol in front of $2x$ was treated as a sign object attached to $2x$ long enough for L6 to recognise that they should apply the SSAK method. The resulting objects $/-/$, $/8/$, and $/x/$ could then be concatenated for the final answer $-8x$. Figure 19 depicts L6's type-specific computations.



sign	*numeral*	*letter*
	2	x
−	8	x
−	3	x
−		x

↦

opr	*sign*	*numeral*	*letter*
		2	x
	−	8	x
−			
		3	x
	−		x

↦

opr	*sign*	*numeral*	*letter*
	−	$[8-2]$	x
−			
		$[3-1]$	x

↦

opr	*sign*	*numeral*	*letter*
	−	6	x
−			
		2	x

↦

sign	*numeral*	*letter*
−	6	x
−	2	x

↦

sign	*numeral*	*letter*
−	$[6+2]$	x

↦

sign	*numeral*	*letter*
−	8	x

Figure 19: Schematic representation of type-specific computations by L6 for Q6

The incorrect sundering exemplified by L6's solution appeared as a common error in learners' solutions to various questions, as shown in Figure 20 (p.55). The terms outlined in red are those constituting a subexpression of the form $\mu - \nu$, where μ and ν consist of a numeral object and/or letter object, and where the negative symbol in front of μ has been detached and kept as an operator object preceding the subexpression. The terms outlined in blue are the reductions $(\mu - \nu)$.

When *adding* two subexpressions together, the plus sign separating them can be taken as a partition and the second subexpression need not undergo any sign changes after being detached from the plus. Understandably then, learners may think they can similarly take the negative symbol preceding the second subexpression as a partition, without needing to change the signs[25] of the second subexpression. This, however, changes the value of the

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[25] Dividing out a negative requires a sign change to adhere to the distributive property: $-a - b = -(a + b)$.

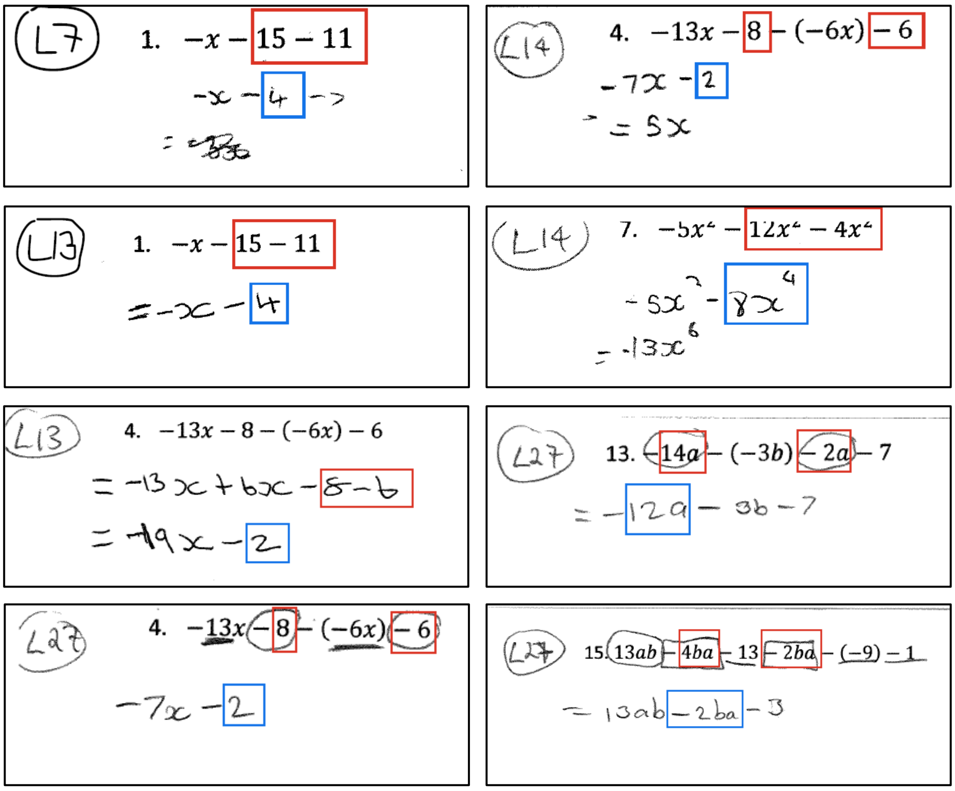


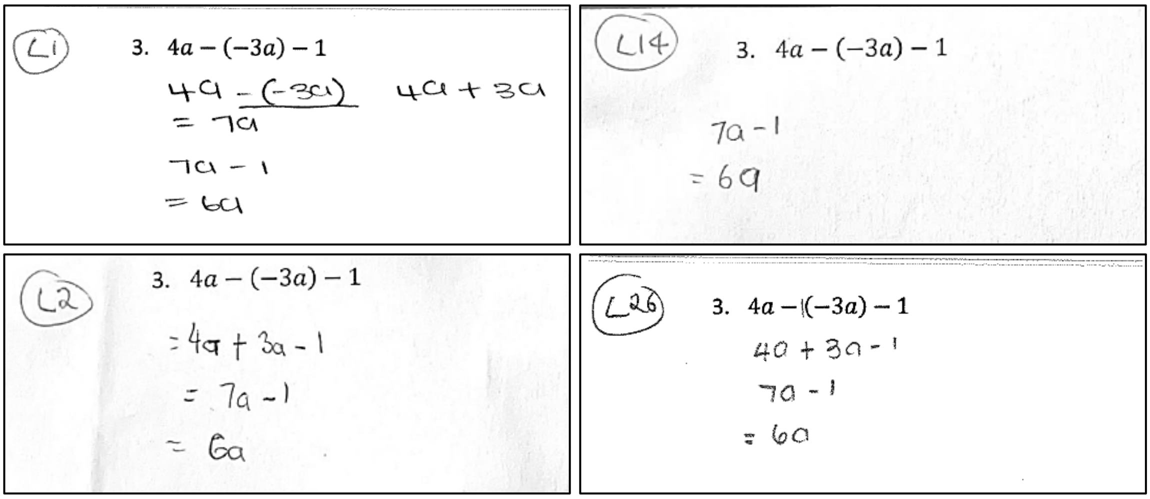
Figure 20: Examples of the negative symbol being taken as a partition

expression. For example, $2x - 8x - 3x - x = (2x - 8x) - (3x + x)$ and is not equal to $(2x - 8x) - (3x - x) = 2x - 8x - 3x + x$ (as L6 thought) because dividing a factor of -1 out of the terms $3x$ and $-x$ changes the sign object of $-x$ to /+/. Where learners simply draw (or imagine) brackets without effecting the necessary sign changes (when dividing out a negative), as in Figure 20, they demonstrate detachment from the minus sign (DFMS).

Unlike in L6's solution to Q6, the brackets around like terms preceded by a minus in these cases are imaginary. Whether by drawing or imagination, introducing brackets into an expression can quickly change the value of that expression. Certainly, this is the case with the brackets that allow for DFMS. For example, in L13's solution to Q1 above, imaginary brackets have been drawn around $15 - 11$ such that the expression is transformed to $-x - (15 - 11)$, which no longer has the same value as $-x - 15 - 11$. It is likely that L13's reading of the expression (and L7's) was "minus x minus, fifteen minus eleven," thus detaching the negative symbol in front of $/15/$ and sundering the expression $/-x -//15 - 11/$. By the calculation $15 - 11 = 4$ their final workspace would then be $(-x, -, 4)$, generating their final answer $-x - 4$ by concatenation (i.e. merge). This solution is not unreasonable if one considers the verbal externalisation of the expression. In fact, many of the incorrect solutions that learners gave nonetheless demonstrated rational reasoning, albeit not conforming to the axioms of mathematics. The section that follows explores computations that learners did involving numeral objects specifically, which did not always adhere to the concepts of *likeness* and *unlikeness*.

6.4 Numeral objects

Since the letter object of a constant is Φ, constants and terms with variables are unlike terms. Interestingly, however, there were many cases where learners disregarded this unlikeness in their computations by simplifying numeral objects as if they did belong to like terms. For example, of the 13 learners[26] who correctly simplified the expression $4a-(-3a)-1$ (Q3) to $7a-1$, only nine of them provided a correct final answer. The other four went one step too far, subtracting the constant -1 from the coefficient 7 and offering the solution $6a$, as shown in Figure 21.



L1

3. $4a-(-3a)-1$

$\underline{4a-(-3a)}$ $4a+3a$

$=7a$

$7a-1$

$=6a$

L14

3. $4a-(-3a)-1$

$7a-1$

$=6a$

L2

3. $4a-(-3a)-1$

$=4a+3a-1$

$=7a-1$

$=6a$

L26

3. $4a-|(-3a)-1$

$4a+3a-1$

$7a-1$

$=6a$

Figure 21: Learners' incorrect solutions of $6a$ to Q3

Under the instruction "simplify," these learners seemingly wanted to simplify *each type of object* in the expression. Since /7/ and /1/ are both numeral objects, they transformed them into a single numeral object, as represented by Figure 22.

---

[26] L1, L2, L3, L4, L5, L8, L10, L11, L13, L14, L17, L19, L26

sign	*numeral*	*letter*
	7	a
$-$	1	

↦

sign	*numeral*	*letter*
	$[7-1]$	a

↦

sign	*numeral*	*letter*
	6	a

Figure 22: Schematic representation of type-specific computations for Q3

When there were no more like terms to simplify, a transition seemingly took place to what we may think of as a new *level of simplification*, that being a level concerning *like objects*, i.e. objects of the same type. Let Level 1 be where like terms are simplified, and Level 2 where like objects are simplified. Figure 23[27] presents the workspaces likely generated by the computations for Q3 shown above.

Level 2

$$(7, a,_{-} 1) \to ([7-1], a) \to (6, a)$$

Level 1

$$(4a,_{-} - 3a,_{-} 1\ ) \to (4a,_{+}\ 3a,_{-} 1\ ) \to ([4a+3a],_{-} 1) \to (7a,_{-} 1)$$

Figure 23: Possible workspaces generating $6a$ as a solution to Q3

Reading the expression $7a - 1$ aloud, "seven a minus one," it is reasonable to generate a workspace $(7, a,_{-} 1)$ containing of one letter object, two numeral objects and one operator object (or one sign object, depending on the reading). Reducing the two numeral objects to just one by subtraction (prompted by the operator object /−/) may be an attempt to simplify the workspace to contain *at most* one object of each type. To decipher these computations, L14 was asked to explain their solution to Q4 which exhibited a similar reduction (Figure 24).

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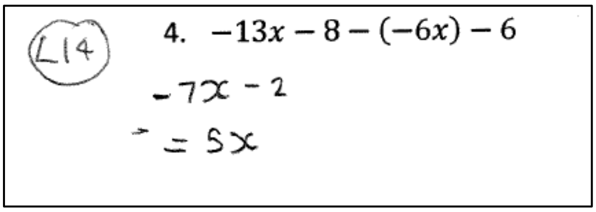
[27] Notice $(7a,_{-} 1)$ at Level 1 becomes $(7, a, -,1)$ at Level 2. At Level 2 the objects are considered anew because there are no more like terms, i.e. the expression is parsed by types instead. This demonstrates the Markovian nature of computational systems (see Chapter 3).



L14

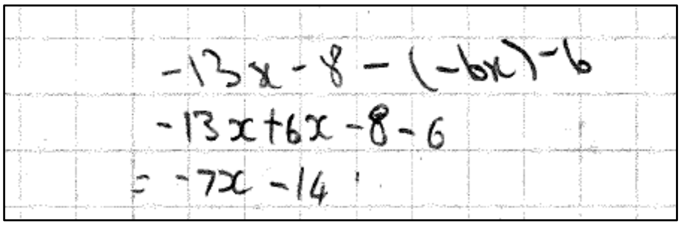
4. $-13x - 8 - (-6x) - 6$

$-7x - 2$

$= 5x$

Figure 24: L14's solution to Q4 (task)

Identifying their solution as incorrect, L14 retried the question in the workbook and came to the correct solution (Figure 25).



$-13x - 8 - (-6x) - 6$

$-13x + 6x - 8 - 6$

$= -7x - 14$

Figure 25: L14's solution to Q4 (first interview)

With the unlike terms having this time been kept distinct, L14 was prompted to explain why they changed approach from Q3 to Q4. Their responses (see Extract 3, p.59) imply that they considered simplifying the expression to a singlet to be an optional extra step. Although there were several other learners who used the same tactic (Level 2 simplification) for various items of the task[28], L14 was by far the most consistent. Their solution to Q15 in Figure 26[29] (p.59) below provides a clear (albeit peculiar) case of the same computations. From the transformation $13ab - 2ba \mapsto 11ab$ in Figure 26 one would assume that L14 recognised $13ab$ and $-2ba$ as like terms, yet in the interview they expressed that the terms were different, they were simply combining them[30]. Level 2 simplification was therefore done prematurely on the first two numeral objects rather than completing simplification of the like terms (Level 1) first. Figure 27 (p.60) and Figure 28 (p.60) offer two different representations of L14's computations for Q15.

---

[28] See for example: L1 (Q2, Q4, Q5, Q6); L2 (Q2); L13 (Q5); L14 (Q12); L24 (Q4, Q6); L26 (Q2); L29 (Q2, Q9).

[29] Note the sign change $-2ba \mapsto +2ba$ in this figure. Many learners explained that when one negative term is "taken over" another negative term the sign changes to a plus—see Section 6.6.

[30] Combining letter objects is discussed in Section 6.5.

INT: You wouldn’t want to simplify it further?

⋮

So here you had seven a minus one and then you said it was six a, but now you've got minus seven x minus fourteen…

L14: You *can* simplify it further.

INT: *Would* you simplify further?

L14: Negative… no… ya, negative twenty-one x.

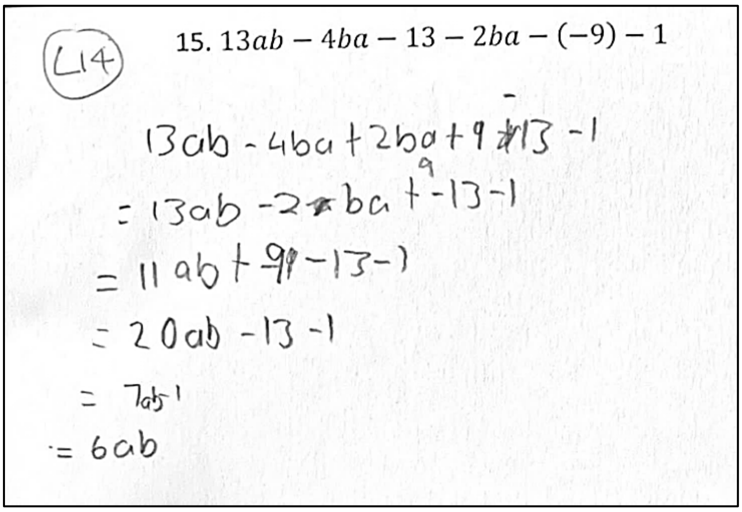
INT: OK, but you say you're starting with like terms, so the x’s and x’s and the numbers, but then … once you've simplified it as far as you can with the x’s and with the numbers, then you would just keep simplifying it? Is that what you're saying?

L14: Hmm, ya, you can. You can simplify it, or if you want to.

INT: But you don't have to, is that what you're saying?

L14: I mean if you, you know, if you wanna get an extra mark then you can.

Extract 3: L14 (first interview)



L14

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$13ab - 4ba + 2ba + 9 \overset{-}{\cancel{+}}13 - 1$

$= 13ab - 2\cancel{a}ba \overset{9}{+} -13 - 1$

$= 11ab + 9\cancel{1} - 13 - 1$

$= 20ab - 13 - 1$

$= 7ab\,1$

$= 6ab$

Figure 26: L14's solution to Q15

sign	*numeral*	*letter*
	11	ab
$+$	9	
$-$	13	
$-$	1	

↦

sign	*numeral*	*letter*
	$[11+9]$	ab
$-$	13	
$-$	1	

↦

sign	*numeral*	*letter*
	$[20-13]$	ab
$-$	1	

↦

sign	*numeral*	*letter*
	$[7-1]$	ab

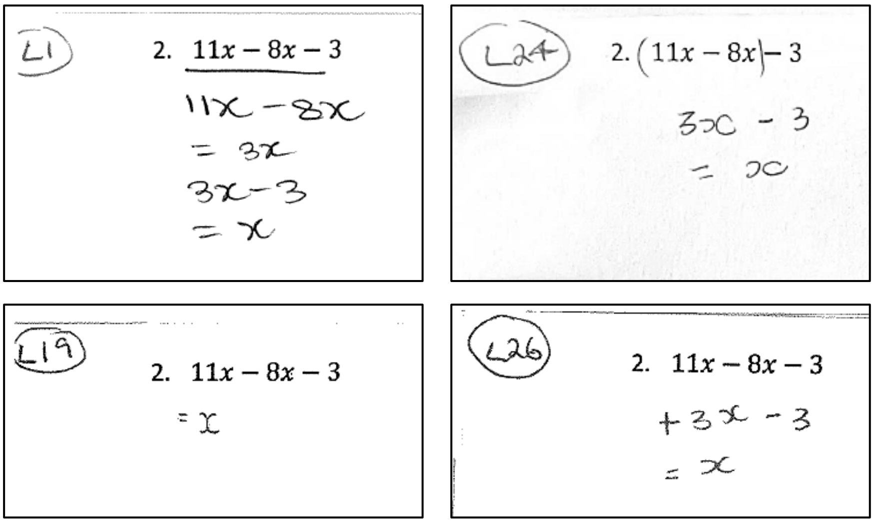
sign	*numeral*	*letter*
	6	ab

Figure 27: Schematic representation of type-specific computations for Q15

$$\begin{tikzcd}
 & & /11ab+9-13-1/ \arrow[dl, "\mathrm{SUN}"'] \arrow[dr, "\mathrm{SUN}"] & & \\
(/11/,/9/) \arrow[d, "\mathrm{NUM}"'] & /11ab+9/ \arrow[l, "\mathrm{SEL}_n"'] \arrow[dd, "\mathrm{SEL}_l"] & & /-13-1/ \arrow[r, "\mathrm{SUN}"] \arrow[dd, "\mathrm{SUN}"'] & /-1/ \arrow[ddd, "i_x"] \\
11,9 \arrow[d, "+"'] & & & & \\
20 \arrow[d, "\mathrm{STR}"'] & /ab/ \arrow[ddl, "i_x"] & & /-13/ \arrow[dddl, "i_x"] & \\
/20/ \arrow[d, "i_x"'] & & & & (/7ab/,/-1/) \arrow[d, "\mathrm{CON}"] \\
(/20/,/ab/) \arrow[d, "\mathrm{CON}"'] & & & & /7ab-1/ \arrow[dddl, "\mathrm{SEL}_n"'] \arrow[ddd, "\mathrm{SEL}_l"] \\
/20ab/ \arrow[rr, "i_x"'] & & (/20ab/,/-13/) \arrow[dll, "\mathrm{SEL}_n"] \arrow[ddl, "\mathrm{SEL}_l"] & & \\
/20/,/13/ \arrow[d, "\mathrm{NUM}"'] & & & & \\
20,13 \arrow[d, "-"'] & /ab/ \arrow[dd, "i_x"] & & (/7/,/1/) \arrow[d, "\mathrm{NUM}"'] & /ab/ \arrow[ddd, "i_x"] \\
7 \arrow[d, "\mathrm{STR}"'] & & & 7,1 \arrow[d, "-"'] & \\
/7/ \arrow[r, "i_x"'] & (/7/,/ab/) \arrow[r, "\mathrm{CON}"'] & /7ab/ \arrow[uuuuuurr, "i_x"] & 6 \arrow[d, "\mathrm{STR}"'] & \\
 & & & /6/ \arrow[r, "i_x"'] & (/6/,/ab/) \arrow[d, "\mathrm{CON}"] \\
 & & & & /6ab/
\end{tikzcd}$$

Figure 28: Diagrammatic representation of transformations likely used by L14 for Q15

The solutions discussed above exhibit a reluctance to accept expressions like $7a - 1$ and $7ab - 1$ as final solutions because of the lack of closure they present in containing operator symbols. The same reluctance was demonstrated in some learners' solutions to Q2, as shown in Figure 29.



L1

2. $11x - 8x - 3$

$11x - 8x$

$= 3x$

$3x - 3$

$= x$

L24

2. $(11x - 8x) - 3$

$3x - 3$

$= x$

L19

2. $11x - 8x - 3$

$= x$

L26

2. $11x - 8x - 3$

$+ 3x - 3$

$= x$

Figure 29: Learners' incorrect solution of x to Q2

The like terms $11x$ and $8x$ were simplified to $3x$ first, producing the expression $3x - 3$, which learners then reduced by taking the numeral objects as arguments of subtraction. Interestingly, because these numeral objects have a difference of 0, and because it is unconventional to write $0x$, learners instead opted for a solution of just x, replacing the numeral object $/0/$ with Φ by VOID. This corroborates the argument that learners' computations are type-specific since only the numeral objects were taken as the arguments of transformation and the letter object was left in position. This implies that some learners did not consider that one type of object may affect another, i.e. that a coefficient of 0 could effect the transformation $0x \mapsto \Phi$ in a polynomial, or $0x \mapsto 0$ for a monomial.

Considering the verbal reproduction of the expression $3x - 3$, "three x minus three," learners may have read the expression as "three, x, take away 3," generating the following series of workspaces:

$$(3_1, x,_{-}\, 3_2) \to ([3_1 - 3_2]\,, x) \to (0, x) \to (\Phi, x).$$

The related computations are depicted in Figure 30.

$$\begin{tikzcd}
& /3x-3/ \arrow[dl, "\mathrm{SEL}_n"'] \arrow[dr, "\mathrm{SEL}_l"] & \\
(/3/,/3/) \arrow[d, "\mathrm{NUM}"'] & & /x/ \arrow[dd, "i_x"] \\
3,3 \arrow[d, "-"'] & & \\
0 \arrow[r, "\mathrm{VOID}"'] & \Phi \arrow[r, "i_x"'] & \Phi,/x/ \arrow[d, "\mathrm{CON}"] \\
& & /x/
\end{tikzcd}$$

Figure 30: Diagrammatic representation of transformations for $3x - 3$

The solutions in Figure 29 (p.61) demonstrate a non-referential way of thinking about the objects in the expression in that learners simply kept $/x/$ in place regardless of the numeral object being 0 or *nothing* (Φ); they did not seem to think of the referents beyond numerals and letters. Interestingly, L28 gave an answer of 0 to Q2. One might assume that their approach was also to subtract the numeral objects (like above), but then to extract the solution 0 from $(0,x)$ instead of Φx, since $0x = 0$. Inspecting L28's other solutions, however, it emerges that their answer was 0 because they "subtracted" *both* the numeral objects *and* the *letter* objects, as depicted in Figure 31 (p.63).

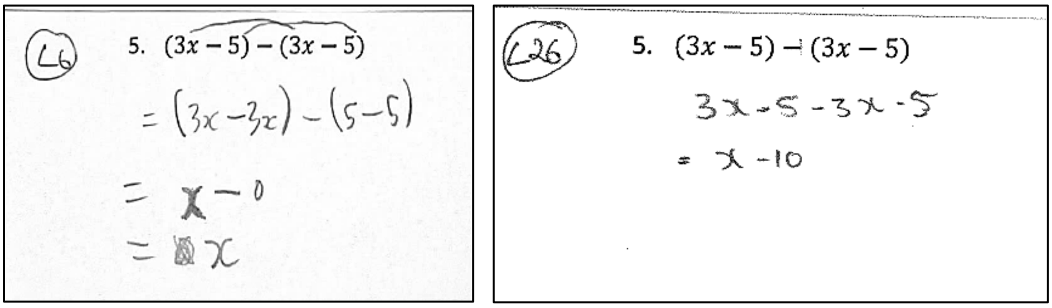
A similar transformation used by some learners was $ax - ax \mapsto x$, where a is some unsigned real number, by which the numeral objects were subtracted to get 0, which was then mapped to Φ by VOID such that the variable x no longer had a written coefficient. Rather than recognising the terms as additive inverses, learners did so only for the numeral objects and therefore kept the letter object $/x/$ in place, or "brought it down" to the next line in their working out, again demonstrating type-specific computations. The workspaces generated by the computations driving this reduction may be depicted as follows:

$$\big((a_1,x),_{-}(a_2,x)\big) \to ([a_1-a_2],x) \to (0,x) \to (\Phi,x)$$

$$\begin{tikzcd}
 & & & /11x-8x-3/ \arrow[dll, "\mathrm{SUN}"'] \arrow[dr, "\mathrm{SUN}"] & \\
 & /11x-8x/ \arrow[dl, "\mathrm{SEL}_n"'] \arrow[dr, "\mathrm{SEL}_l"] & & & /-3/ \arrow[d, "\mathrm{SEL}_n"] \\
(/11/,/8/) \arrow[d, "\mathrm{NUM}"'] & & (/x/,/x/) \arrow[d, "\mathrm{VOID}"] & & /3/ \arrow[d, "\mathrm{NUM}"] \\
(11,8) \arrow[d, "-"'] & & \Phi \arrow[dd, "i_x"] & & 3 \arrow[dd, "i_x"] \\
3 \arrow[d, "\mathrm{STR}"'] & & & & \\
/3/ \arrow[rr, "i_x"'] & & /3/ \arrow[r, "\mathrm{NUM}"'] & 3 \arrow[r, "i_x"'] & (3,3) \arrow[d, "-"'] \\
 & & & & 0
\end{tikzcd}$$

Figure 31: Diagrammatic representation of transformations likely used by L28 for Q2

Further examples of this approach are exhibited in solutions to Q5 (Figure 32), where learners simplified $3x - 3x$ to x, and in solutions to Q14 (Figure 33), where learners simplified $3y - 3y$ to y.



L6

5. $(3x - 5) - (3x - 5)$

$= (3x - 3x) - (5 - 5)$

$= x - 0$

$= x$

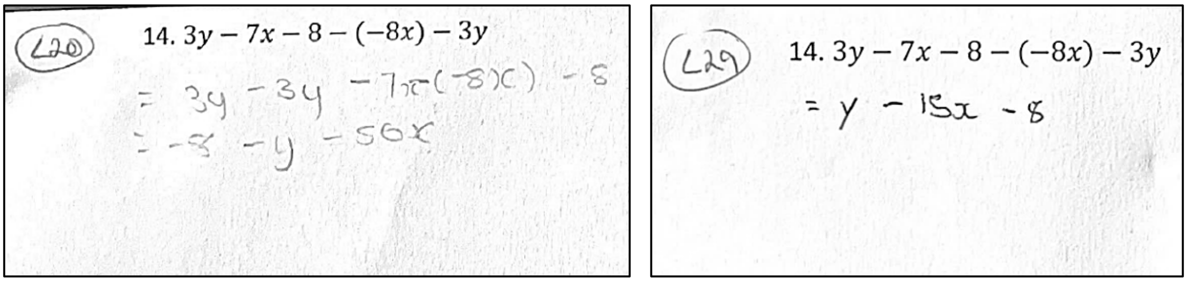
L26

5. $(3x - 5) - (3x - 5)$

$3x - 5 - 3x - 5$

$= x - 10$

Figure 32: Solutions to Q5 by L6 and L26



L20

14. $3y - 7x - 8 - (-8x) - 3y$

$= 3y - 3y - 7x(-8x) - 8$

$= -8 - y - 56x$

L29

14. $3y - 7x - 8 - (-8x) - 3y$

$= y - 15x - 8$

Figure 33: Solutions to Q14 by L20 and L29

In both sets of solutions, subtraction was used to simplify numeral objects (coefficients) by $3-3=0$ and then the VOID mapping was applied: $0 \mapsto \Phi$. The solution to Q5 (Figure 32) was explained by L6 as follows (Extract 4).

L6: I put that three over there [gesturing from second $3x$ to position of first 5] so it will automatically become a negative and then five will go over [gesturing from first 5 to position of second $3x$], that will become a positive so it will be positive five minus five equals now zero and then three x minus three x equals zero. So, it will be x.

INT: OK so you kept them in brackets to show that these are the like terms and then you said five minus five is zero and then with the three x and the three x? How did you get to x?

L6: Uh, I minus the three.

INT: OK, if you've got three and three then you're cancelling them out?

L6: Ya.

Extract 4: L6 (first interview)

Note that the sign change of $3x$ is another example of changing signs when one term is "taken over" a negative, discussed in Section 6.6. Evidently L6 (and L20, L26 and L29 as seen above)[31] recognised whole number additive inverses, but they did not infer from their calculation $3-\ 3=0$ that a coefficient of 0 would result in product $0x$ being equal to 0. The same can be assumed of the other learners simplifying $ax-ax$ expressions to x. The term x has a coefficient of 1, but from the solutions discussed here that is clearly not the coefficient that learners had in mind when they gave the term x as a solution to expressions of the form $ax-ax$. Interestingly, on two occasions where an expression contained a variable with no written coefficient, L6 mentioned that there was an "invisible 1." That L6 was aware of the "invisible 1" in other expressions yet still gave an answer of x to Q5 corroborates the argument that learners parse expressions into types of objects and do type-specific transformations in that L6 believed that there was still a letter object x despite the coefficients, i.e. numeral objects,

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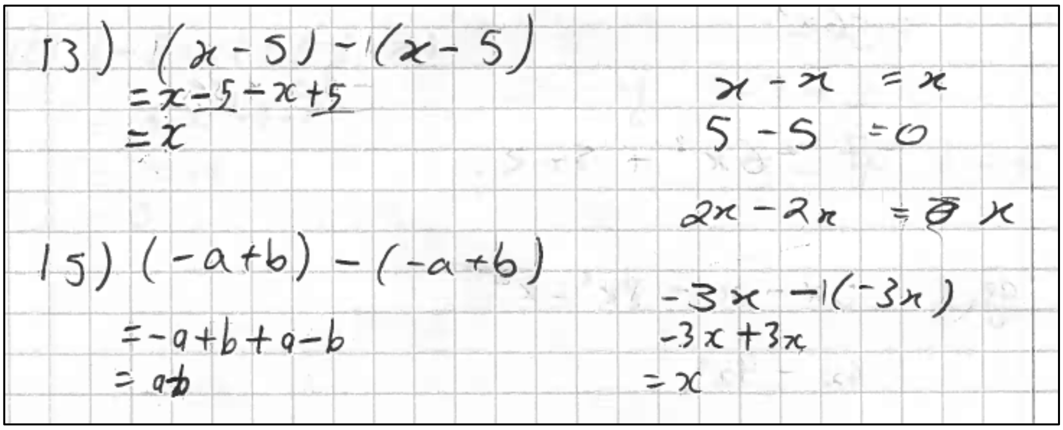
[31] See also L14's written work in their follow-up interview.

cancelling each other out. Figure 34 shows that that L6 was consistent in their approach even for expressions of the form $x - x$, where the numeral objects were both Φ.



13) $(x-5)-(x-5)$

$=x\underline{-5}-x\underline{+5}$

$=x$

$x - x = x$

$5 - 5 = 0$

$2x - 2x = 0x$

15) $(-a+b)-(-a+b)$

$=-a+b+a-b$

$=ab$

$-3x - (-3x)$

$-3x+3x$

$=x$

Figure 34: L6's solutions exhibiting additive inverses (follow-up)

It appears that L6 read the expression $x - x$ as demonstrating no transformation on x and a trivial operation of subtracting *nothing* from *nothing* regarding numeral objects. Verbally, one could think of this as “nothing x minus nothing x is still nothing x,” where for L6 *nothing* x is represented symbolically by $/x/$. Figure 35 presents the corresponding transformations.

$$\begin{tikzcd}
 & /x-x/ \arrow[dl, "\mathrm{SEL}_n"'] \arrow[dr, "\mathrm{SEL}_l"] & \\
(\Phi,\Phi) \arrow[d, "\mathrm{NUM}"'] & & /x/ \arrow[ddd, "i_x"] \\
(0,0) \arrow[d, "-"'] & & \\
0 \arrow[d, "\mathrm{VOID}"'] & & \\
\Phi \arrow[rr, "i_x"'] & & (\Phi,x) \arrow[d, "\mathrm{CON}"] \\
 & & /x/
\end{tikzcd}$$

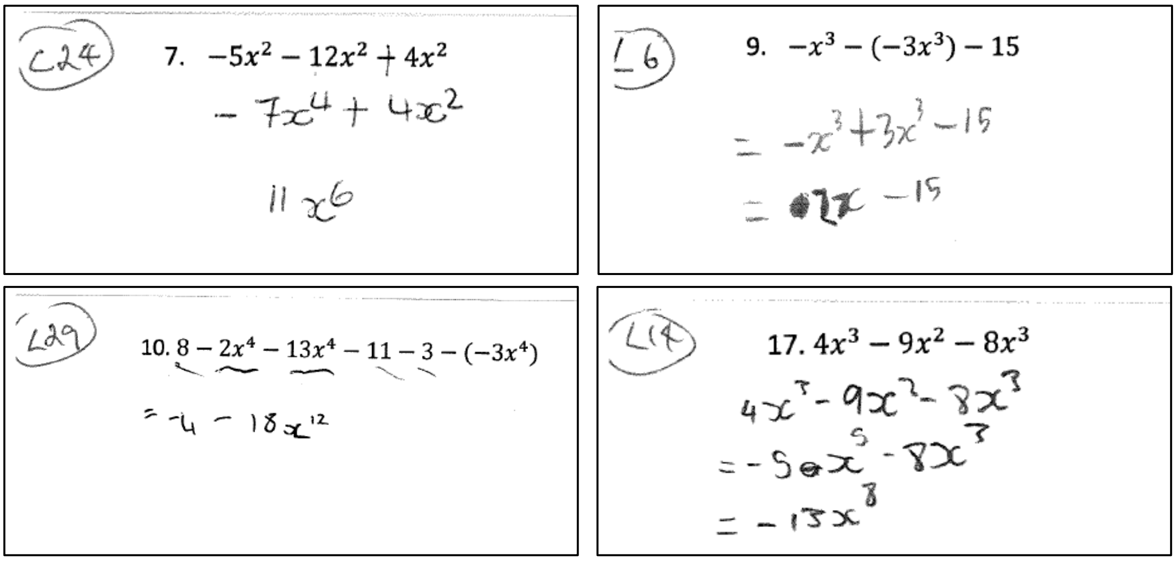
Figure 35: Diagrammatic representation of transformations likely used by L6 for $x - x$

In the solutions in Figure 34 it seems the negative symbol was taken as applying to the numeral objects independently of the letter objects, such that with no written coefficients, i.e. no numeral objects, the negative symbol was taken as applying trivially to (Φ, Φ) rather than L6 recognising that x could be subtracted from x and give 0. The letter object therefore remained fixed whilst the numeral objects were "subtracted."

The presence of the negative symbol and/or letter objects in expressions seems to play a key role in eliciting transformations that are non-standard from a mathematics perspective, suggesting that learners did have a strong sense of the mathematical denotation of expressions, and especially not of letter objects. The section that follows considers the transformations that learners did on letter objects, particularly those that do not respect the properties of real numbers.

6.5 Letter and superscript objects

Important to algebraic problem solving is the understanding that the letters in algebraic expressions are representations of real numbers and should behave as such, despite those numbers being unknown. Several learners' solutions suggest that they were thinking of the letters as arbitrary symbols or objects that they could manipulate rather than real numbers. Consequently, the transformations that these learners used to simplify expressions were often not driven by mathematical axioms, but rather by their own idiosyncratic notions. Many solutions exhibited the removal of certain letter objects, prompted by the negative symbol, corroborating the argument here that learners parse expressions into types of objects for type-specific transformations. Further evidence of this is found in the solutions of four learners who resorted to taking superscript objects as arguments for addition (and in L6's case subtraction) independent of their bases (see Figure 36, p.67). Detailed analysis of the computations involving superscript objects falls beyond the limited scope of this dissertation, but the curious reader may consult Appendix 12 for further insight.



L24 7. $-5x^2 - 12x^2 + 4x^2$

$-7x^4 + 4x^2$

$11x^6$

L6 9. $-x^3 - (-3x^3) - 15$

$= -x^3 + 3x^3 - 15$

$= 2x - 15$

L29 10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$= -4 - 18x^{12}$

L14 17. $4x^3 - 9x^2 - 8x^3$

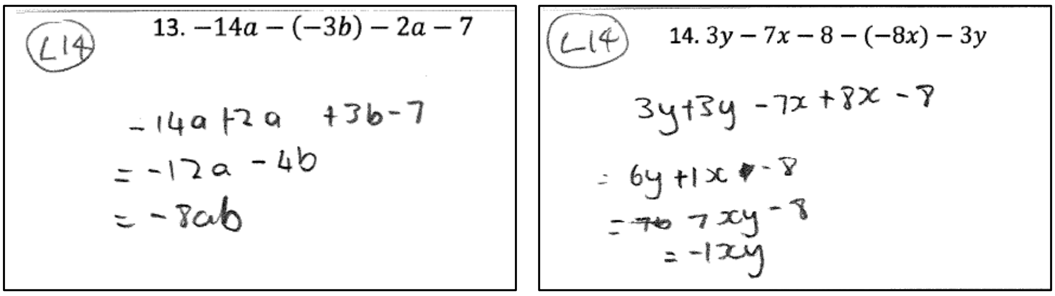
$4x^3 - 9x^2 - 8x^3$

$= -5x^5 - 8x^3$

$= -13x^8$

Figure 36: Examples of addition and subtraction of superscript objects

An interesting method that arose under the instruction “simplify,” was L14’s approach of simplifying nearly all the task items to singlets, often combining terms that were unlike (despite referring to like terms in their interviews), as seen in Figure 37.



L14 13. $-14a - (-3b) - 2a - 7$

$-14a + 2a \quad +3b - 7$

$= -12a - 4b$

$= -8ab$

L14 14. $3y - 7x - 8 - (-8x) - 3y$

$3y + 3y - 7x + 8x - 8$

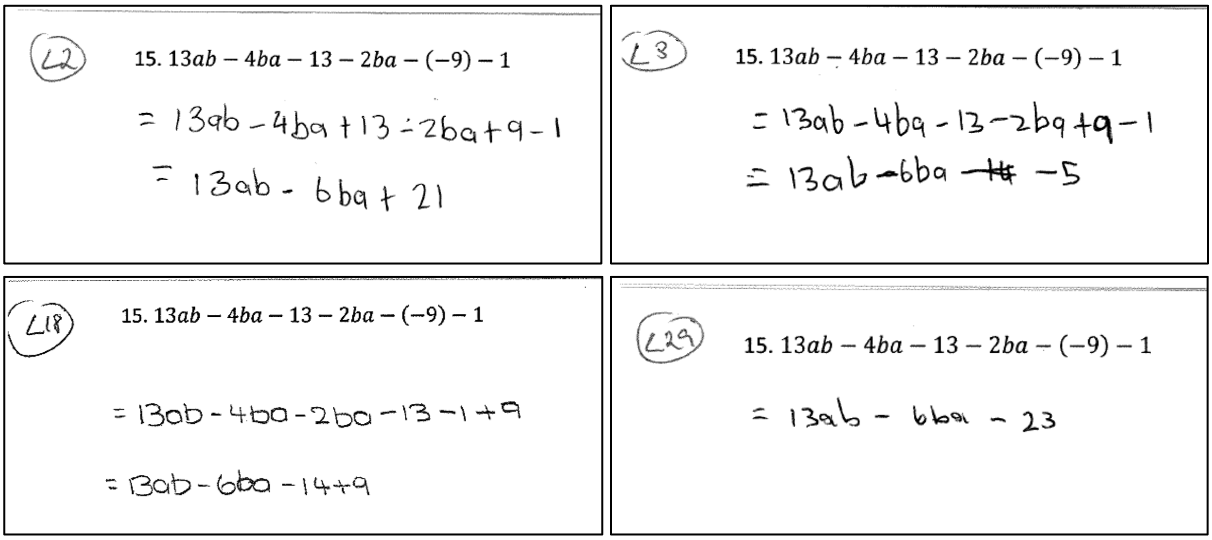
$= 6y + 1x - 8$

$= 7xy - 8$

$= -1xy$

Figure 37: L14's solutions exhibiting conjoining of unlike terms

Regardless of the operation, L14 conjoined unlike letter objects to arrive at a singlet as their solution, with the operator object being used for simplifying numeral objects. There were no other learners who conjoined unlike letter objects, recognising that they differentiated terms. Interestingly, however, when it came to multi-variable terms, this sense of differentiation meant that half of the learners considered ab and ba to be *unlike*, simplifying the ba-terms but leaving the ab-term as it was (see Figure 38, p.68). Evidently, these learners did not associate /ab/ and /ba/ with the equivalent products ab and ba (see Extract 5, p.68). This indicates that their computations were typographic but not entirely indexical—they did not recognise that the transformation $ab \mapsto ba$ was required—nor mathematical.



L2

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= 13ab - 4ba + 13 - 2ba + 9 - 1$

$= 13ab - 6ba + 21$

L3

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= 13ab - 4ba - 13 - 2ba + 9 - 1$

$= 13ab - 6ba$ ~~-14~~ -5

L18

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= 13ab - 4ba - 2ba - 13 - 1 + 9$

$= 13ab - 6ba - 14 + 9$

L29

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= 13ab - 6ba - 23$

Figure 38: Solutions with distinct ab- and ba-terms

INT: OK. So, you're saying now that the ba is the same as ab.

L22: Oh no that's wrong.

INT: Would you say they're different?

L22: Yes, because they're two different numbers.

INT: Why do you say so?

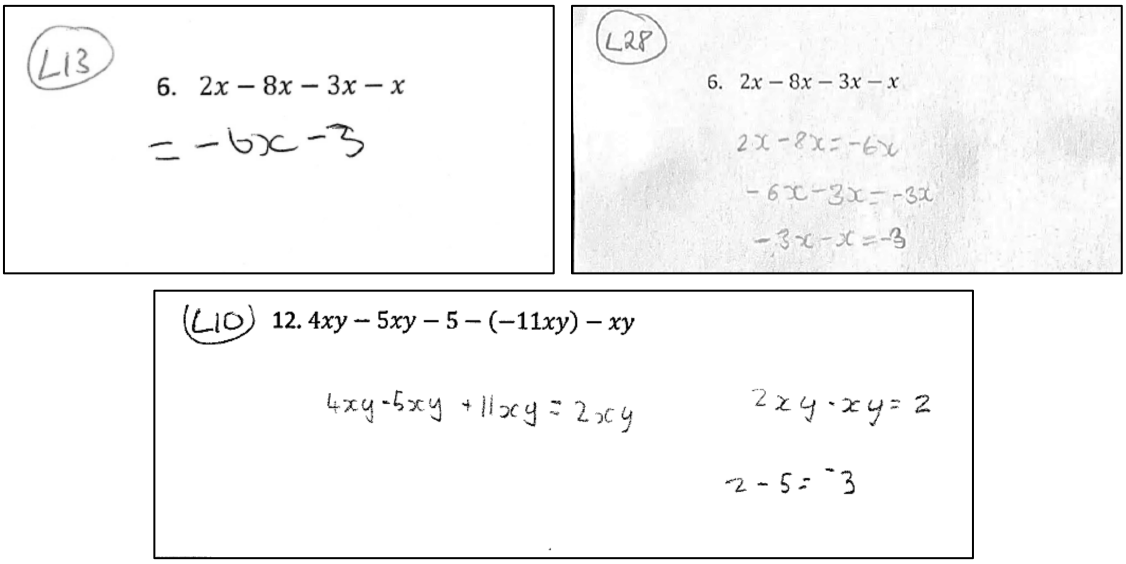
L22: Because a isn't equal to b and b isn't equal to a [pointing to a and b].

INT: OK, so you're saying that if you have a a next to b it's not the same as b next to a?

L22: No.

Extract 5: L22 (first interview)

Another indication that some learners focused on the typographic features of expressions more so than their mathematical denotation, was the use of the transformation $ax - x \mapsto a$, whereby the letter objects were "subtracted" or "cancelled" and the result mapped to Φ. As seen in Figure 39 (p.69), for example, L28 simplified $-3x - x$ to -3, despite having dealt with like terms in the preceding lines of their solution, similarly to L6, and L10 "subtracted" $/xy/$ from $/2xy/$. During their first interview, L10 "corrected" their solution to Q12, changing their reduction of the numeral objects from $/2/$ to $/10/$, but still removing the letter object $/xy/$ as before (see Figure 40, p.69).



L13

6. $2x - 8x - 3x - x$

$= -6x - 3$

L28

6. $2x - 8x - 3x - x$

$2x - 8x = -6x$

$-6x - 3x = -3x$

$-3x - x = -3$

L10

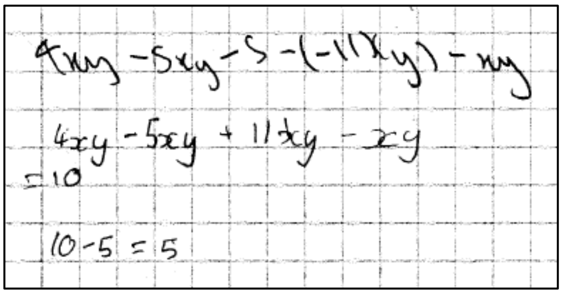
12. $4xy - 5xy - 5 - (-11xy) - xy$

$4xy - 5xy + 11xy = 2xy$

$2xy \cdot xy = 2$

$-2 - 5 = {}^{-}3$

Figure 39: Solutions exhibiting the transformation $ax \mapsto a$



$4xy - 5xy - 5 - (-11xy) - xy$

$4xy - 5xy + 11xy - xy$

$= 10$

$10 - 5 = 5$

Figure 40: L10's solutions exhibiting the transformation $ax \mapsto a$

It is interesting that their "correction" was of the written coefficients rather than of the transformation $ax - x \mapsto a$. This suggests that L10 was confident in their use of the transformation as a simplifying technique. Reaching an expression of the form $ax - x$, the negative symbol was read as an operational signifier indicating subtraction and where the learners were unable to subtract numbers—having only one written coefficient—they resorted to "subtracting" the letter objects, of which there were two, instead. It was only in the follow-up interview that L10 appeared to recognise that the term $-x$ has a coefficient of -1 which can be used for subtraction. What makes the transformation $ax - x \mapsto a$ particularly interesting is that learners used it despite being aware that $ax + x = (a + 1)x$, suggesting a nescience regarding the reversibility of certain computations. This very interesting notion falls beyond the scope of this dissertation, but the curious reader may turn to Appendix 13 for brief consideration of why learners may struggle to recognise reversible computations.

Three other transformations also demonstrating the "cancellation" of letter objects, but emerging in only a few solutions were:

i. $-x + ax \mapsto ax$, used by L13
ii. $ax - bx \mapsto (a - b)$, used by L28 and (on one occasion) L10
iii. $-x - x \mapsto \Phi$, used by L13 and L4

(where a and b are unsigned real numbers). With several learners removing letter objects in their solutions, it seems as though the negative symbol played a significant role in learners' identification of the transformations compossible with each expression, and that often those transformations were related to the general notion of "taking something away" more so than the operation of subtracting real numbers. The section that follows explores learners' readings and applications of the negative symbol in looking at how they dealt with sign and operator objects.

6.6 Sign and operator objects

Standing out from the data is learners' tendency to treat the negative symbol as detachable[32]. The common error of DFMS, which may be described by the transformation $-\mu \pm \nu \mapsto -(\mu \pm \nu)$, appeared numerous times across learners' solutions—as already demonstrated in Figure 20 (p.55). Figure 41 provides further examples of $\mu \pm \nu$ being separated from the minus preceding μ.

The negative symbol that precedes μ is kept fixed in position whilst $\mu \pm \nu$ is calculated in the space that follows. Notice that the numeral object of μ is greater than that of ν in each of the $\mu \pm \nu$ expressions outlined in Figure 41 (p.71), allowing learners to resort to the "bigger minus smaller" understanding of subtraction that they are so accustomed to. Looking at L5's solution to Q4, for example, they clearly read the subexpression $-13x + 6x$ as "minus, thirteen x plus six x" such that the first minus became the sign object for the term representing the sum of

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[32] E.g.: L7 (Q1); L13 (Q1, Q4, extra questions); L14 (Q4, Q7, Q13; L24: extra questions); L27 (Q4, Q13, Q15); L28 (Q6, Q7).

$13x$ and $6x$, i.e. $19x$. Figure 42 depicts the transformations L5 seemly used to simplify the subexpression.

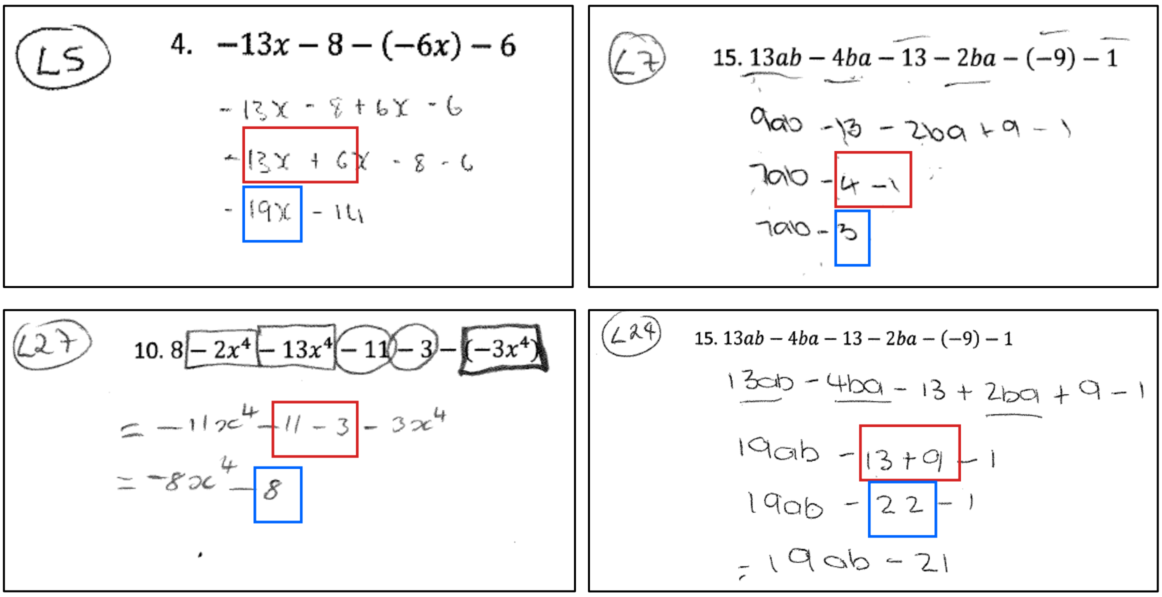

Figure 41: Examples of DFMS

$$\begin{tikzcd}
 & & /-13x+6x/ \arrow[ddll, "\text{SUN}"'] \arrow[dr, "\text{SUN}"] & & \\
 & & & /13x+6x/ \arrow[dl, "\text{SEL}_n"'] \arrow[dr, "\text{SEL}_l"] & \\
/-/ \arrow[ddddrr, "i_x"] & & (/13/,/6/) \arrow[d, "\text{NUM}"'] & & /x/ \arrow[ddddll, "i_x"'] \\
 & & (13,6) \arrow[d, "+"'] & & \\
 & & 19 \arrow[d, "\text{STR}"'] & & \\
 & & /19/ \arrow[d, "i_x"'] & & \\
 & & /-19x/ & &
\end{tikzcd}$$

Figure 42: Diagrammatic representation of transformations likely used by L5 for $-13x+6x$

Another commonly used transformation was $-(-\mu) \mapsto +\mu$, although learners were not necessarily concerned with the mathematical properties on which it relies. Likely, they were simply using the MMP mapping $/-(-/\mapsto/+/$ based on the mnemonic "minus and minus makes plus" (or some rendition of it), which many referred to in their interviews to justify their computations. Learners mostly seemed to be referring to the multiplication of two negatives rather than to subtraction or negation, often indicated by an arc drawn across the opening bracket between two negative symbols, as seen in Figure 43.

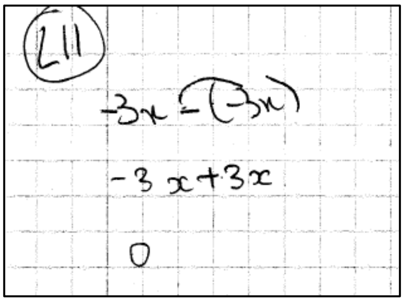

Figure 43: L11 solution to $-3x - (-3x)$

Describing their solution in, L11 said: "So BODMAS[33], so I'd first do the brackets, so I'd times the negative." Similarly, asked how they would simplify the expression $3 - (-3)$ in their first interview, L1 responded: "So negative times negative is equal to positive. So, then I'll give three [writes 3 + 3] then equal to six." Most learners began their solutions by rewriting the expressions, replacing any terms in brackets $-(-\mu)$ with $+\mu$, because, as L4 said in their interview, when "there's a bracket you must always work out the bracket first."

Interestingly, asked what they thought brackets implied in their first interview, L24 responded: "when there's brackets, doesn't it automatically mean multiplication?" In fact, the overwhelming response from learners regarding brackets was that they imply multiplication. This was particularly evident with regards to Q5, which was included in the task to see if learners would recognise additive inverses. Of the 22 learners there were 13[34] whose solutions demonstrated attempted use of the well-known FOIL method[35] to simplify the expression, as exemplified in Figure 44 (p.73).

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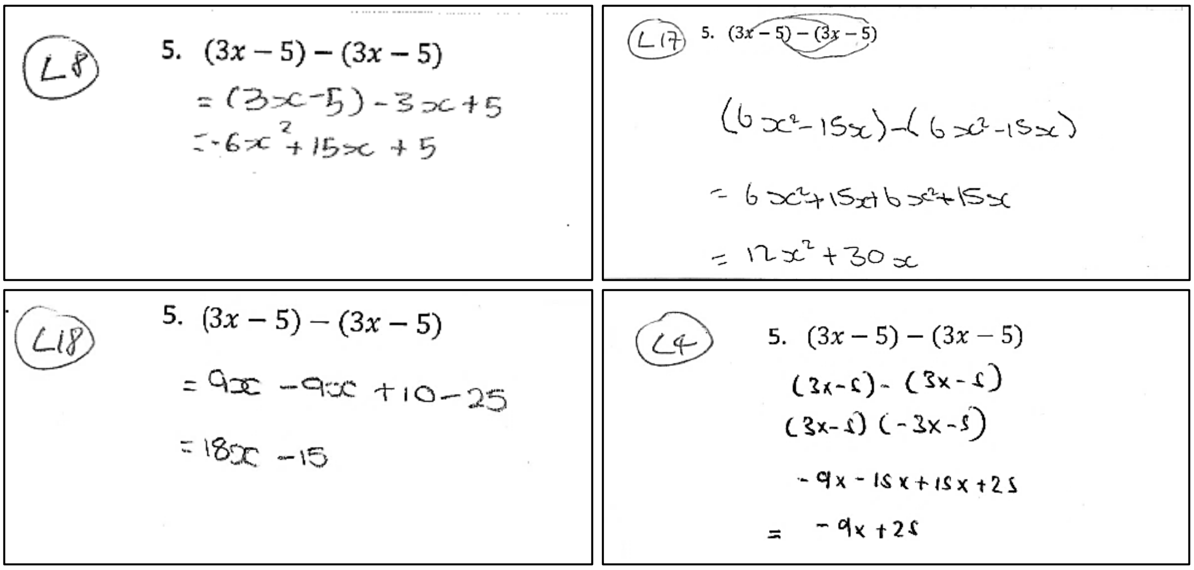
[33] See Glossary (p.viix).

[34] L1, L4, L7, L8, L10, L11, L17, L18, L20, L22, L24, L28, L29.

[35] See Glossary (p.vii).



L8

5. $(3x-5)-(3x-5)$

$= (3x-5) - 3x + 5$

$= -6x^2 + 15x + 5$

L17

5. $(3x-5)-(3x-5)$

$(6x^2 - 15x) - (6x^2 - 15x)$

$= 6x^2 + 15x + 6x^2 + 15x$

$= 12x^2 + 30x$

L18

5. $(3x-5)-(3x-5)$

$= 9x - 9x + 10 - 25$

$= 18x - 15$

L4

5. $(3x-5)-(3x-5)$

$(3x-5) - (3x-5)$

$(3x-5)(-3x-5)$

$-9x - 15x + 15x + 25$

$= -9x + 25$

Figure 44: Examples of learners attempting to use the FOIL method (Q5)

Clearly, the presence of two binomials in brackets immediately drove learners to taking the FOIL method as the required approach despite the negative symbol separating the brackets. This demonstrates a sensitivity to the form of expressions. Learners' sense of hierarchy regarding operations, or at least regarding brackets being dealt with first, meant that most of them wanted to remove the brackets from the expression in Q5 first to then cancel individual terms, despite the contents of the brackets clearly being the same. The tendency to think of brackets as requiring multiplication seems therefore to have outweighed the association of the negative symbol with subtraction, with most learners not immediately recognising additive inverses when brackets were involved.

Several learners also resorted to multiplication when presented with expressions containing only two like terms, one or both in brackets, as shown in Figure 45 (p.74). In these cases, learners used the MMP mapping to change the second term $-(-v)$ to (v) instead of $+v$, thus changing the operation between the first two terms to multiplication instead of addition. For example, rather than using the transformation $4a-(-3a) \mapsto 4a+3a$, L7 used the transformation $4a-(-3a) \mapsto 4a(3a)$. Although there were several examples of this type of transformation, for the most part learners did correctly apply the MMP mapping when expressions contained a term of the form $-(-\mu)$. The MMP mapping was typically the first thing learners did, prompted by BODMAS, or "brackets first." Although learners may have been concentrating on the negative symbols as strings rather than on the operations that the negative symbols represented, the "minus and minus makes plus" mnemonic nonetheless enabled them to simplify terms of the form $-(-\mu)$ to $+\mu$ correctly.

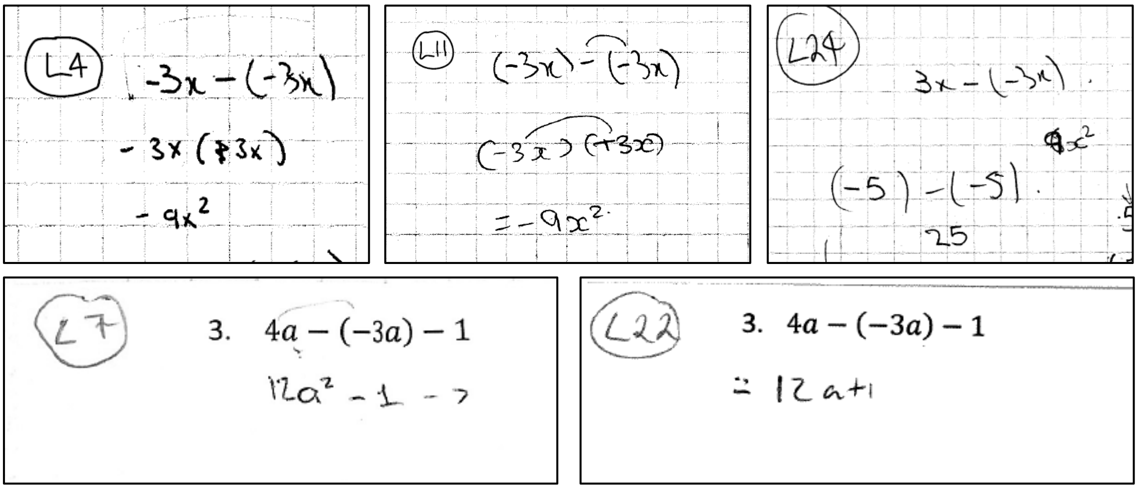


Figure 45: Examples of multiplication elicited by brackets

An error that arose from the misapplication of "minus and minus makes plus" is exhibited by the transformation $-\mu - \nu \mapsto (\mu + \nu)$. Examples of this transformation from L5's solutions are given in Figure 46, where the subexpressions $-\mu - \nu$ are outlined in red and the sums $(\mu + \nu)$ in blue.

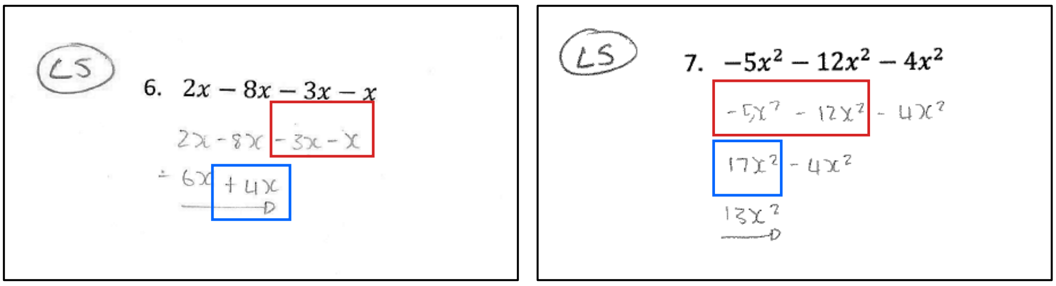


Figure 46: Solutions by L5 exhibiting the transformation $-\mu - \nu \mapsto (\mu + \nu)$

Here, it is the "minus and minus" of consecutive terms which "make plus" by fixing the sign object of the outcome to $/+/$, or to Φ if there is no preceding term. For example, in L5's solution to Q6, it is likely that they read the subexpression $-3x - x$ as "minus $3x$ and minus x," giving them a "minus and minus" which they transformed to "plus, $3x$ and x" and then simplified to $+4x$.

Although $-\mu - \nu \neq (\mu + \nu)$, there is certainly rationality in learners' reasoning if one thinks only of the mnemonic; "minus and minus makes plus" gives no indication of where the two minuses should be in relation to one another. Working only with the symbols in an expression without being cognisant of the mathematical objects being represented and then resorting to a rather vague mnemonic can misguide learners' computations without them realising where or why they have gone wrong.

The same mnemonic, or likely its rendition "minus minus *then* plus," misled L2 and L24 who seem to have taken it as indicating a particular sequence of sign objects. Seemingly, for L2, the mnemonic indicated that every second minus should be changed to plus, i.e. $-\mu - \nu \mapsto -\mu + \nu$, whilst L24 understood that every third minus should be changed to a plus, as seen in Figure 47.

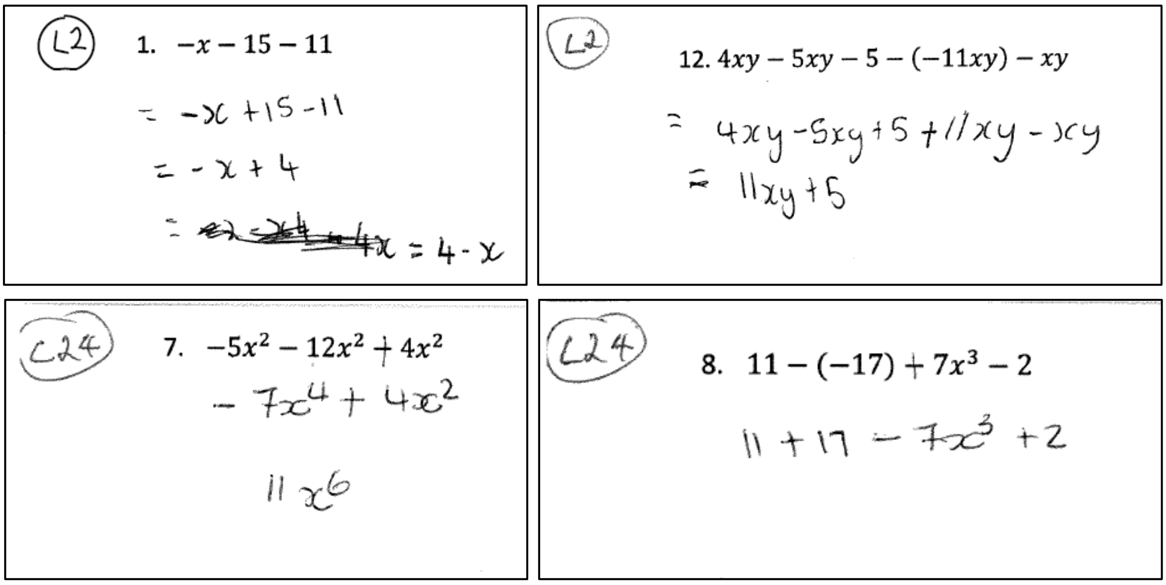
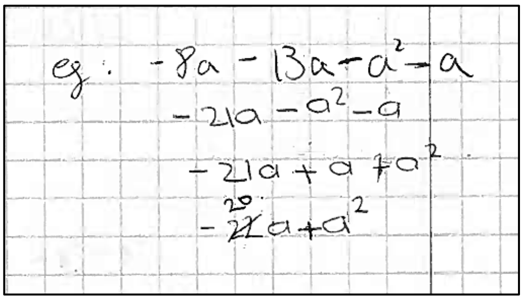


Figure 47: Examples of L2 and L24 changing sign objects

These sign changes were done first, except when expressions contained a term $-(-\mu)$, in which case L2 concurrently used the transformation $-(-\mu) \mapsto +\mu$, thus "resetting" the sequence of minuses, as seen in their solution to Q12 with the term $-(-11xy)$. In both cases, the learners' readings of the expressions seemingly drove them to do typographic computations based on the sequence of sign objects, without considering the composition of operations that the plusses and minuses indicate, nor how changing the signs can affect those operations.

Sign changes were also justified in some cases by one term being "taken over" another (particularly one with a negative sign object). For example, Figure 48 (p.76) shows that L24 rewrote $-21a - a^2 - a$ as $-21a + a + a^2$, because "I took this [pointing to $-a^2 - a$] and I, like, I made it swap places, so I changed the sign."



eg: $-8a-13a-a^2-a$

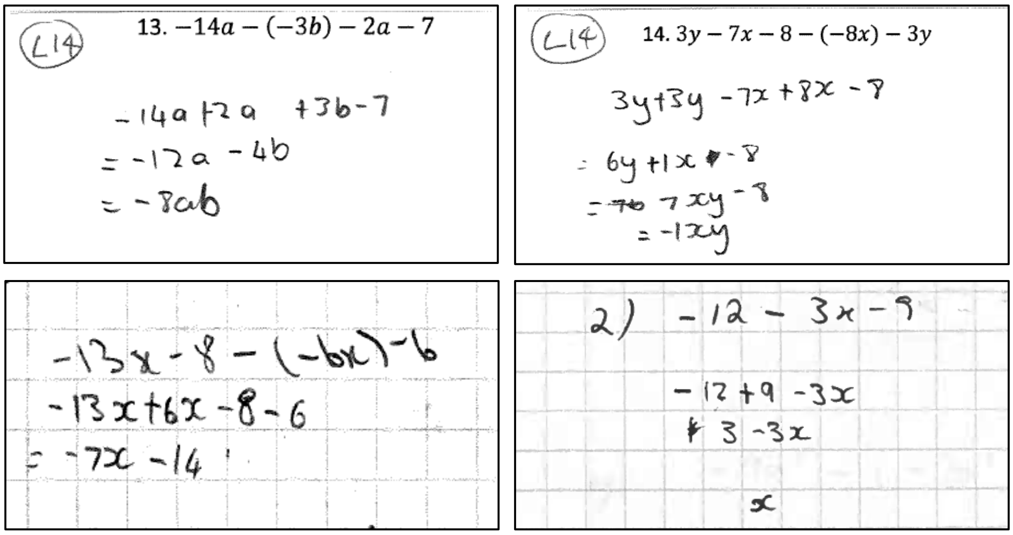
$-21a-a^2-a$

$-21a+a+a^2$

$-\overset{20}{\cancel{22}}a+a^2$

Figure 48: Example of L24 changing signs when rearranging terms

This notion that the sign object needs to change when a term is “taken over” stems from the way in which learners are taught to solve equations using the addition and subtraction of terms: “take over and change the sign.” In Figure 48, L24 “moved” the terms $-a^2$ and $-a$, prompting them to change their sign objects like they would when “taking terms over” an equal sign. In other solutions L24 simplified like terms that were not adjacent without rearranging the expression, so it is interesting that they chose to rearrange the expression above and felt that the sign objects needed to change as a result. As shown in Figure 49, L14 changed signs by the same reasoning. For example, in their second attempt at Q4 (bottom left in Figure 49), L14 explained that “you bring this over [gesturing arc from $-6x$ to $13x$] then it becomes a positive,” with no reference to the MMP mapping or the “minus and minus makes plus” mnemonic.



L14

13. $-14a-(-3b)-2a-7$

$-14a+2a+3b-7$

$=-12a-4b$

$=-8ab$

L14

14. $3y-7x-8-(-8x)-3y$

$3y+3y-7x+8x-8$

$=6y+1x\cancel{+}-8$

$=\cancel{76}\ 7xy-8$

$=-1xy$

$-13x-8-(-6x)-6$

$-13x+6x-8-6$

$=-7x-14$

2) $-12-3x-9$

$-12+9-3x$

$\cancel{+}3-3x$

x

Figure 49: Examples of L14 changing signs when rearranging terms

The examples discussed in this section demonstrate learners' manipulations of sign objects (and/or operators objects) as though they were independent objects within the expression. Mnemonics like "minus and minus makes plus" and "take over and change the sign" are certainly useful for solving algebraic problems when the mathematical axioms or properties that they are based on are understood, but when learners do not have a sense of the mathematical denotation of algebraic expressions or how certain transformations can change them, these mnemonics can easily lead them astray. Evidently, even when learners gave the correct solutions, the computations that they used to reach them were not necessarily those considered standard from the point of view of mathematics.

6.7 Summary

In general, learners did not demonstrate a strong grasp of algebraic concepts, with only 43% of all the solutions to the task being correct, yet still they described and justified many of their computations by rational reasoning and references to mathematical properties and operations. Despite learners' apparent procedural knowledge concerning the steps for simplifying algebraic expressions, some of them went too far in their computations, attempting to reduce expressions to a singlet despite terms being unlike. Learners' reluctance to accept lack of closure highlight that their conceptual knowledge regarding the objects constituting an expression (and the transformations compossible with them) did not align with the mathematical denotation of those objects. The argument here that learners are sensitive to the types of objects that constitute expressions and parse expressions accordingly is corroborated by their solutions demonstrating, for example: the removal of numeral and letter objects by transformations like $ax - x \mapsto a$, $ax - a \mapsto x$ and $ax - ax \mapsto x$, the transformations on superscript objects independently of their bases, and the changing of sign objects based on the mnemonic "minus and minus makes plus."

One can easily say that incorrect solutions indicate that learners do not know what to do or do not know what objects they are working with, which may be true from the perspective of mathematics, but this disregards that learners have their own conceptions of the objects constituting algebraic expressions and the transformations with which they are compossible. The next chapter seeks to explain and uncover where learners' choices of transformations may come from.

7 Discussion

7.1 Introduction

As Hart (1981, p.213) expressed over 40 years ago and as seems still to be the case, we "tend often to assess the progress of a child by stating what he does that is correct and what he does that is incorrect rather than asking ourselves why he is correct or why he is wrong." Chapter 6 demonstrated and discussed learners' type sensitivity, their reluctance to accept lack of closure and their reliance on the typographic features of expressions, amongst other findings. Drawing on the theories and propositions set out in Chapter 3 and those relevant from the literature considered in Chapter 2, this chapter seeks to describe what learners are doing fundamentally, across computations, and offer possible explanations for why they chose certain computations above others.

7.2 Type sensitivity

Presented with an operation, learners are forced to identify the objects that can be taken as its arguments. In school mathematics these objects are assumed to belong to the domain of real numbers, however, many learners resorted to transformations that are *not* compossible with the real numbers, taking different objects as arguments instead. Learners were prone to doing type-specific computations, such as operating on numeral objects regardless of whether they belonged to like terms or removing like letter objects as a form of "subtraction." This suggests that learners parse expressions into types of objects in their internal representations and determine which computations to do according to the objects that they identify. For example, the transformations $ax - a \mapsto x$ and $ax - ax \mapsto x$ demonstrate the parsing of expressions to operate only on numeral objects, keeping the letter object fixed in position. For the curious reader, Appendix 14 offers a brief discussion on how this parsing by type can be seen mathematically as constructing equivalence classes.

Letter objects and the negative symbol, as a sign object or an operator object, posed a challenge to learners because they were either i) less sure of which transformations to use to simplify an expression, or ii) misled by an idiosyncratic understanding regarding the denotation of those objects. Gelman (2015, p.198) highlights that the "mastery of mathematics topics like algebra and calculus require conceptual changes as to what counts as numbers.” The various non-standard computations that learners did involving letter objects suggest that they did not think of letter objects as representing real numbers nor did they consider the relations between objects in expressions, treating numeral objects as independent of letter objects and each type as compossible with their own set of transformations.

Spelke (2022, p.204) suggests that “children harness the ancient core place and form systems, shared with other animals, to learn symbol systems that are unique to humans,” such as alphabetic characters and Arabic numerals. Part of acquiring a symbol system is the ability to distinguish it from *other* symbol systems, like one may distinguish between unfamiliar forms or objects of different kinds—both of which humans have an innate capacity for (Spelke, 2022). Radford (2006, p.14) highlights that humans’ “ability to notice differences in things is one of our basic cognitive components.” Children are taught early on what symbols for numbers and letters look like, but letters are not associated with numbers until the introduction of algebra in Grade 7 (and even then, letters are often thought of as objects like “apples” rather than numbers (see McGowen & Tall, 2010)). It is no surprise then that learners in this study distinguished letter objects from numeral objects and wanted to take them as independent arguments. Learners’ differentiation between object types within expressions was clear in interviews from their use of gestures to specific objects and between objects of the same type (e.g. pointing to numeral objects or letter objects of different terms). Marghetis et al. (2016, p.1) contend that practising mathematics is “undeniably perceptual” and indeed learners seemed reliant on visual prompts like written representations, markings and gestures both to understand the denotation of expressions and to explain their computations. Spelke (2022, p.202) states that “vision is the primary modality by which we identify and classify objects.” This may explain learners’ tendency to rewrite solutions in that they wanted to have visual access to the objects available for computation through reading. Presmeg et al. (2016, p.26) posit that the “emerging meanings [ascribed to algebraic expressions by learners] are deeply rooted in significations that come from natural language and perception.” A learners’ reading of their own working out determines what they will do next. Naturally, then, language plays a key role in learners’ computations.

7.3 The effects of language

Children are taught to read long before they are introduced to algebraic language. Although language and reading are necessary for learning algebra, the biases and met-befores developed from learning a language can be problematic in several ways. For example, because of the met-before produced by left-to-right reading, the “knowledge of how grouping symbols change the order of operations may fail to make a strong impression on many learners” (McGowen & Tall, 2010, p.175). This is evidenced by the number of solutions exhibiting DFMS, such as three learners’ reduction of $-x - 15 - 11$ (Q1) to $-x - 4$. Reading the expression $-x - 15 - 11$, these learners likely generated the workspace $\big((-,x),_{-}15,_{-}11\big)$, containing two objects of the same type (i.e. numeral) which could be simplified. However, reading from left to right and having reached the end of the expression, the learners had to return to some point in the expression from which they could reread and determine how to simplify the like terms. The same way that when reading a passage, if something is unclear one goes back to the nearest point from which rereading will clarify whatever was unclear. Going back in the expression to the numeral object $/15/$ provided them with enough like objects to do a reduction, i.e., $15 - 11 = 4$. This modified the expression to $-x - 4$. Rereading the expression, they recognised that there were no like terms to simplify further and therefore accepted the modified expression as their answer.

Consider the term $3x$, which is read as “three x.” That this expression indicates a product of two numbers is implicit since the syntax offers no reference to multiplication (whether the term is produced verbally or in writing). Marghetis et al. (2016, p.2) state that algebraic notation “expresses relations that are both abstract and hierarchical, but the notation itself relies heavily on visuospatial features to represent those relations.” Understanding how the locations of different symbols affect computations is fundamental to mathematical thinking. Mathematically, $/3/$ right next to $/x/$ represents the product $3 \times x$, but verbally expressing “three” and then immediately “x” need not have the same referent in everyday language. It is up to the learner to decipher “three x” as “three times x” when working in a mathematical context. Sfard (1991, p.17) argues that algebraic symbols “do not speak for themselves” and that the way they are interpreted depends on what the problem requires as well as “on what one is able to perceive and prepared to notice.” Told to simplify an expression, learners naturally want to reduce what they see before them. If what they see is a collection of distinct objects generated by the reading of the expression, then it is not irrational to reduce the collection by simplifying the subcollection containing like objects, as learners did when using transformations like $ax - a \mapsto x$ and $ax - x \mapsto a$.

The transition that several learners made from Level 1 simplification to Level 2 simplification suggests an existential shift in the objects of their computations. It seems that initially, learners looked for like terms to simplify, thus cognisant of a relation between numeral objects and their adjacent letter objects (even if that relation was simply "is adjacent to"). At Level 1, a term like $3x$ may have been read as "three x's" or as "three and x" with the condition that the two objects be kept together. Once no like terms remained however, the reluctance to accept lack of closure elicited a shift to Level 2 where $3x$ was likely read as "three and x" or "three, x," with the objects changing to become independent. Learners do not seem to have thought of letter objects as representing numbers at either level.

As mentioned in Section 5.7, the use of everyday objects to represent variables in the teaching of algebra can quickly dissociate variables from real numbers. The idea of $3a$ representing "three apples" immediately implies that $a = 1$ because an apple is a single unit, limiting learners' understanding of variables in an expression indeed *being variable* i.e. being able to take on any real number value. McGowen and Tall (2010, p.172) highlight that even replacing variables of *like* terms with objects can pose a challenge: "'3 apples take away 5 apples' makes no sense, so that – as a met-before – this can cause a later obstacle when handling negatives." Drawing on Küchemann's (1981, pp.104-107) taxonomy of children's uses of letters, it appears that when learners operate on like terms the letters are either *letters not used* or *letters used as objects*. This puts the focus on the coefficients of like terms which encourages the parsing of expressions according to types of objects. According to Radford (2006, p.3): "Using letters does not amount to doing algebra." If learners ignore letters in their computations, they are doing arithmetic more so than algebra.

The tendency to read letters as representing objects or as words also seemed to arise when it came to multi-variable terms like ab and ba, which half of the learners took to be distinct in their computations for Q15. Spelke (2022, p.383) posits that "speech fosters young infants' categorization of objects." From a linguistic perspective, it is entirely reasonable to classify ab and ba as different because, read out loud, they do not sound the same nor do they conjure up the same object. Why should “a-b” and “b-a” (or “ab” and “ba” if read together) be any more alike than “apples” and “pears”[36]? If ab and ba are thought of as objects “a-b” and “b-a,” then $13ab$ represents “thirteen a-b’s” and $4ba$ represents “four b-a’s” and certainly we cannot subtract “b-a’s” from “a-b’s.” Identifying ab- and ba-terms as like terms requires, firstly, a typographic computation, to recognise that they contain the same letters, and secondly, an indexical computation, to prompt the transformation $ab \mapsto ba$ (or $ba \mapsto ab$) such that the terms have the same letter object. To perceive the letters as representing unknown real numbers, a mathematical computation is also necessary. Where learners left ab- and ba-terms distinct, their computation was likely only typographic in that they recognised ab and ba as containing the same letters but not as constituting the same letter objects. Overall, it seems that typographic and indexical computations were more natural to learners than mathematical computations. This was particularly apparent during interviews when learners often failed to recognise that their solutions were incorrect until they were asked to explain their computations.

Explaining their solutions forced learners to redo their computations, often resulting in them noticing a mistake or incongruity. As Hart (1981, p.213) claims: “children were learning simply by voicing their thoughts.” Research in cognitive science suggests that this may be explained by learners’ tendency to focus their attention on learning about things that behave unexpectedly, i.e. that violate their perception (Stahl & Feigenson, 2018). When learners’ written work did not correspond with their internal computations while explaining their solutions, they wanted to correct themselves. Notwithstanding, their amendments were not always correct. Of course, there were also several errors of which learners remained unaware when their written work corresponded to their internal representations. McGowen and Tall (2010, p.169) assert that “supportive met-befores give confidence in handling any context in which they work,” yet they do not mention the *false*-confidence that *problematic* met-befores can give when learners think they know what to do but apply their pre-existing knowledge inappropriately.

In their interviews, learners often referred to tricks and mnemonics as explanation or justification of their computations. Although mnemonics can be useful, they can also be

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[36] Or “23” and “32”, see Extract 5 (p.68).

pernicious when learners do not recall their mathematical referents correctly. Hart (1981, p.212) asserts that algorithms are often taught too soon while learners' understanding of the content is still developing, and consequently "they are not remembered or sometimes remembered in a form that was never taught." The rote-learned mnemonic "minus and minus makes plus" (and its renditions) for example, offers very little information and can thus be interpreted in sundry ways if learners do not recall to what kinds of expressions it is relevant. What does "minus and minus" really mean? Do the two minuses need to be adjacent or simply close to one another? This mnemonic was adapted by some learners to produce transformations that did not adhere to mathematical axioms but rather to typographical constructs involving the *presence* of two negative symbols, such as:

i. $-\mu - \nu \mapsto (\mu + \nu)$
   The sum of two negatives becomes a positive.
ii. $-\mu - \nu \mapsto -\mu + \nu$
   Every second minus changes to a plus.
iii. $-\mu - \nu - \pi \mapsto -\mu - \nu + \pi$
   Every third minus changes to a plus.

Vlassis (2004) offers examples of the same transformations, referring to the mnemonic as the "signs rule." Some learners went so far as to refer to the same mnemonic for different transformations on the same expression, such as the MMP mapping and transformation (ii) above. Another typographically driven transformation was the changing of signs when one negative term was "taken over" another (i.e. written earlier or later in the expression), as shown in Section 6.6. As Pournara et al. (2016) highlight, learners using transformations of this type (unintentionally) treat the expression as if there were an equal sign between the two positions of the term being "moved." Note that learners' explanations gave no indication that they were thinking of the equal sign, only that they were concerned with the terms changing positions (which stems from solving equations). Relying on the typography of expressions to determine which transformations to use to simplify them indicates that learners made use of character distribution matrices in their solutions, although not always correctly.

7.4 Character distribution matrices

Often corresponding to mnemonics, character distribution matrices also appeared to play a significant role in learners' computations, particularly with the parsing of expressions according to types of objects. Johnson and Davis (2010, p.144) highlight that the use of character distribution matrices as regulative texts can result in solutions that are "not concerned with symbolic meaning of mathematical expressions (i.e. mathematics) but rather how symbols are distributed (spatially) to format solution procedures". Learners seemed to expect their solutions to have certain typographic features such that if their solution "looked right" it was assumed to be correct (demonstrating typographic computations). That learners were often unaware of whether their solutions were suitable to a given question may be explained by the Markovian nature of computational systems. With respect to linguistics, Chomsky et al. (2023, p.17) explain that "Merge has access to the WS [workspace] only at a fixed derivational moment." Adapting this to the present study, at any point, learners have access only to the objects of the most recent workspace and thus think only of the transformations compossible with those objects. Once the original workspace has been modified, the transformations learners choose may not be compatible with the original expression. The final solution may be suitable following on from the most recent workspace, but unsuitable to the original expression.

A few character distribution matrices emerged from learners' solutions, prompting methods like DSB, SSAK, FOIL, and the use of the MMP mapping. Although the DSB and SSAK methods escape the bounds of mathematics into a more general realm of symbol manipulation[37], they are taught in schools because they *work*. Where learners rely on character distribution matrices, the mathematics they are doing "generates reliable mathematical outcomes but via quasi-mathematical pathways" (Johnson & Davis, 2010, p.140). These two methods do not encourage mathematical thinking beyond the level of basic arithmetic because learners need only focus on working with the natural numbers and can easily become accustomed to parsing algebraic terms.

Terms of the form $-(-\mu)$ prompted learners to use mapping $\mathrm{SUN}:/-(-\mu)/\mapsto\ /-(-//\ \mu)/$ followed by $\mathrm{MMP}:/-(-/\mapsto/+/$ without necessarily giving any thought to the role of the

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[37] See Davis' (2013) discussion of the transformation from numbers to symbols.

negative symbol as a unary operator on the term $-\mu$. The character distribution matrix $-(-\mu)$ thus elicits a transformation that requires no mathematical computation. Learners' sensitivity to the form of an expression (rather than its mathematical denotation) was made particularly clear by how many of them read expressions like $(-3x-5)-(-3x-5)$ (Q15) as a character distribution matrix for the FOIL method, i.e. binomial multiplication, despite the negative symbol between the brackets indicating a *difference* of binomials.

Mathematics teaching in South African schools often focuses on developing learners' procedural knowledge without ensuring that their conceptual knowledge related to those procedures aligns with mathematical axioms (see, for example, Engelbrecht et al., 2005; Johnson & Davis, 2010; Sorto & Sapire, 2011). McGowen (2016, p.20) highlights the same issue in the US: "the ongoing instructional emphasis is predominantly to show students how to use a rule to get the "right" answer." The problem is that if learners blindly follow rules and set procedures without understanding the mathematical axioms that they are based on, then when something goes wrong, they have no way of knowing how to correct themselves. Notwithstanding learners' improving recollection of algebraic concepts throughout the task and during interviews, the number of incorrect solutions being produced indicated a general nescience regarding the mathematical denotation of the expressions they were working with.

7.5 Dealing with lack of closure

Instructed to simplify, learners naturally generate a final workspace containing as few objects as possible. What learners considered "possible" in such instances, however, did not always align with the axioms of mathematics, as the number of solutions exhibiting Level 2 simplification indicate. Arithmetic expressions can always be reduced to a single number, but for expressions containing variables this is often not the case because unlike terms cannot be simplified. In keeping with the literature (e.g., Booth et al., 2014; Küchemann, 1981; Pournara et al., 2016), several learners struggled to accept the lack of closure presented by expressions containing an operator object, especially early on in the task. Learners' met-befores regarding number problems and their indexical computations involving the operator object drove them to reduce expressions to singlets notwithstanding the unlike terms. Drouhard and Teppo (2004, p.235) posit that "students with poor capabilities to recognize this aspect of the meaning of an expression often make endless calculations because they do not know in what direction to go and when to stop.” Learners continued to reduce expressions until there were no visible operator symbols remaining and hence nothing left to simplify, as they would with arithmetic problems. Old habits die hard. Pournara et al. (2016, p.5) discuss the same phenomenon, reporting that “learners ‘simplify’ the binomial by attending first to the numbers, then appending the letter.” They add that when learners “conjoin unlike terms, they ignore the letter and then append it as one writes units in a measurement problem” (Pournara et al., 2016, p.5). Learners seemed largely to overcome the met-before that solutions should be singlets as they worked through the task, but during interviews the tendency to resort to Level 2 simplification recurred in some of them.

Told to simplify an expression, say $7a - 1$, learners will recognise the operator object indicating subtraction. In algebra, learners need to be aware of whether the arguments of the operation are like terms, since subtraction can only take like terms as arguments if the difference is to be represented as a singlet. Working at the level of terms (Level 1), the given expression evidently has no like terms and cannot be reduced. If learners parse the expression into types of objects, they may generate the workspace $\big((7, a),\_\ 1\big)$. Being accustomed to single number solutions, it seems that when there are no like terms the next best thing for learners to deal with the lack of closure is to simplify like objects (Level 2). Taking the like objects /7/ and /1/ as the arguments of subtraction allows learners to generate the workspace $(6, a)$ which contains no operator object and no more than one of each other type of object. Having nothing left to do to the objects (since there is no operator object) learners can extract the solution $6a$ by concatenating (i.e. merging) the final objects (note then that $6a$ does not necessarily mean $6 \times a$), offering closure.

The conjoining of unlike terms would not have been explicitly taught to learners. The literature (e.g., Bofferding, 2010; Booth et al., 2014; Booth & Koedinger, 2008; Küchemann, 1981; McGowen & Tall, 2010) demonstrates that regardless of where learners are, they produce the same errors and use the same procedures that their teachers (presumedly) do *not* teach them. It must be then that these procedures stem from how learners’ minds work, from their already established mental structures, and not from their surroundings. The set theoretic correlate of collecting, conjoining or concatenating objects is taking the disjoint union of sets in which those objects are considered elements. For example, concatenating /6/ and /a/ can be thought of as the disjoint union $\{/6/\} \sqcup \{/a/\}$. The core domain correlate of disjoint union is Merge. With Merge considered fundamental to computation (see Chomsky et al., 2023) it is not surprising that learners may resort to the concatenation or conjoining of unlike terms. If one thinks of concrete objects, it is always possible to “add” them together in the sense that one can consider them as constituting a single collection. For example, one might “add” apples and pears to a bowl, constituting a single collection of fruit. In this case, the collection can be referred to as containing one type of object because there is a collective noun “fruit,” encompassing both apples and pears. In everyday language, *any* collection of objects can typically be described by a collective noun, albeit as vague as “things” or “stuff.” If learners think of algebraic expressions as collections of objects, then it is not surprising that they may try to come up with some algebraic equivalent of a collective noun by conjoining or concatenating objects.

Another approach that learners used to deal with lack of closure was the “cancellation,” i.e. removal, of like numeral or letter objects by transformations like those earlier mentioned:

iv. $ax - x \mapsto a$
Remove like letter objects.

v. $ax - a \mapsto x$
Remove like numeral objects.

vi. $ax - ax \mapsto x.$
Remove like numeral objects.

Notice that expressions of the form $ax - x$, or $ax - \ ax$ can each be simplified at Level 1, yet learners using the transformations above chose to work at Level 2 instead. Each of these transformations highlights learners’ tendency to do type-specific operations to reduce the workspace and their nescience regarding the mathematical denotation of the expressions.

The use of transformation (iv) by learners who also recognised that $ax - bx = (a - b)x$ was particularly interesting, highlighting their reluctance to accept lack of closure as well their parsing of expressions by type. With $ax - bx$, the use of the negative symbol can be “exhausted” by subtracting the numeral objects and keeping the letter object fixed since the terms are like. The term $(a - b)x$ offers closure[38] so learners need not (and cannot) simplify it further. However, with subtraction being binary and there only being one (visible) numeral

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[38] $(a - b)$ represents a single numeral object.

object in $ax - x$, learners seemingly thought that the only way to "exhaust" the use of the negative symbol (and achieve closure) was to take the like letter objects as arguments of "subtraction" and remove $/x/$ entirely. The removal of $/x/$ can be described by the VOID mapping, as shown in Figure 50, and highlights that learners did not consider x as a number.

$$\begin{tikzcd}
 & & /ax - x/ \arrow[ddll, "\mathrm{SEL}_n"'] \arrow[dr, "\mathrm{SEL}_l"] & & \\
 & & & (/x/, /x/) \arrow[dr, "\mathrm{VOID}"] & \\
/a/ \arrow[drr, "i_x"'] & & & & \Phi \arrow[dll, "i_x"] \\
 & & /a/ & &
\end{tikzcd}$$

Figure 50: Diagrammatic representation of the transformation $ax - x \mapsto a$

Regarding the transformations (v) and (vi), the use of the negative symbol could be "exhausted" on the numeral objects, producing the numeral object $/0/$ mapped to Φ by VOID, leaving only the letter object $/x/$, as demonstrated in Figure 51[39] (p.89).

The VOID mapping appeared in numerous solutions, suggesting that the notion of additive inverses and the cancellation theorems are not given explicit attention in school mathematics and when they are applied implicitly learners misinterpret the operations as requiring the manipulation of symbols (e.g. letter objects) rather than the use of transformations conforming to mathematical axioms (Davis, 2012). The negative symbol elicited learners' use of the VOID mapping, particularly at Level 2, to "take things away" so as to simplify the expression to a singlet. The use of the transformations (iv), (v), and (vi) suggest that number operations took precedence, and where numerals were the same the VOID mapping was used. Where an expression contained only one numeral object and a negative symbol as an

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[39] Note that the same diagrammatic representation can be used for transformation (v) simply replacing $/ax - ax/$ with $/ax - a/$.

$$\begin{tikzcd}
 & & /ax - ax/ \arrow[dl, "\mathrm{SEL}_n"'] \arrow[ddrr, "\mathrm{SEL}_l"] & & \\
 & (/a/, /a/) \arrow[dl, "\mathrm{NUM}"'] & & & \\
(a, a) \arrow[d, "-"'] & & & & /x/ \arrow[ddd, "i_x"] \\
0 \arrow[d, "\mathrm{STR}"'] & & & & \\
/0/ \arrow[d, "\mathrm{VOID}"'] & & & & \\
\Phi \arrow[rrrr, "i_x"'] & & & & /x/
\end{tikzcd}$$

Figure 51: Diagrammatic representation of the transformation
$ax - ax \mapsto x$

operator object, the negative symbol was considered as available for use on the letter objects to reduce the expression, which prompted learners to use the VOID mapping to remove like letter objects entirely (as in Figure 50, p.88).

7.6 The negative symbol

The negative symbol posed a challenge to many of the learners when requiring computations that were incompatible with the principles of the core domain of number. Subtraction of natural numbers is a concept easily grasped thanks to humans' innate number systems (Dehaene, 2011; Gelman, 2015; Gelman & Gallistel, 1986), but only when that subtraction is restricted to the natural numbers. Concrete objects are often used to present children with subtraction problems, but with concrete objects the *least* amount that one could end up with is zero. If a child has four toys, it makes no sense to tell them five of their toys are being taken away—where is the fifth toy? This establishes the notion of subtraction as "bigger minus smaller," or at the most, "some things minus the same things[40]." *Taking the difference* is specifically defined as subtracting the smaller from the larger. McGowen and Tall (2010,

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[40] The mathematical correlate of which is additive inverses: $a - a = 0$.

p.172) explain that “the word ‘difference’ is often used non-directionally, so that the difference between 2 and 5 is the same as the difference between 5 and 2, which is 3.” *Taking the difference* is therefore a commutative operation, which poses a challenge to learners’ developing understanding of subtraction, which is non-commutative. In the DSB method, learners take the difference between two numeral objects, allowing them to continue with the met-before of “bigger minus smaller” regardless of which numeral object belongs to the bigger number, and promoting the idea that the order of the inputs does not matter (which is true of addition but not of subtraction). Similarly, by the SSAK method learners could add the numeral objects in any order and simply attach the negative symbol afterwards. During interviews, many of the learners referred to “bigger” and “smaller” numbers when explaining their working out.

This met-before is likely also a reason behind the common error of DFMS (see Section 6.3), which Vlassis (2004, p.476) glosses as “a widespread type of reasoning which puts imaginary brackets around like terms when preceded by the minus sign.” Note that learners were less likely to detach the minus in expressions $-\mu - \nu$ where ν had a bigger numeral object than μ, because $\mu - \nu$ was not of the form “bigger minus smaller.” Vlassis (2004, p.481) explains that the “minus sign is considered as a barrier that divides the polynomial,” a clear example of which appears in Figure 17 (p.52) where L6 drew brackets around subexpressions on either side of the minus. The negative symbol was treated as detachable by learners using the DSB and SSAK methods as well as the MMP mapping in many cases. Terms of the form $-\mu$ appear to have been treated as a composition of two strings[41] $/-//\mu/$, prompting learners to treat $/\mu/$ as an independent argument. The positive symbol can be detached from the argument $+\mu$ without making a difference to the outcome of the transformation since $+\mu = \mu$ i.e. the positive symbol is implicit. As a met-before, this possibly misled learners into thinking that the negative symbol could be detached too—further reinforced by learning methods like DSB and SSAK. Regarding the MMP mapping, when given a term $-(-\mu)$, learners seemed to consider the entire string $/-(-/$ as detachable because many of them referred only to the two negative symbols when explaining where the plus came from, rather than referring to the whole term. Some learners used the transformation $\mu - \ (-\nu) \mapsto \mu(\nu)$ thus replacing the string $/-(-/$ with the string $/(/$ and consequently choosing to multiply μ and ν because of

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[41] Or more if μ consists of, say, a numeral object and a letter object and is parsed according to types.

the brackets. With brackets being strongly associated to multiplication, the negative symbols on either side of the bracket may have confused learners as to what the arguments of multiplication should be. Some learners therefore did "multiplication" of the two negative symbols, but did not remove the brackets thus producing $\mu(\nu)$ instead of $\mu + \nu$. With there still being brackets they then took μ and ν as the arguments of multiplication.

Learners' tendency to simplify terms in brackets before doing any other computations demonstrated their awareness of the hierarchy of mathematical operations i.e. BODMAS. As Chomsky et al. (2023, p.8) highlight: "sensorimotor mechanisms cannot see structure and are sensitive to linear order." The mnemonic "BODMAS" is used as a tool for remembering the mathematical order of operations as superior to the order from left to right, i.e. the order in which operations are read. What emerged from some learners' solutions, however, is a hierarchy of *types* with respect to which types of objects should be dealt with first when doing type-specific computations. With many learners effecting sign changes prompted by the presence of one or more negative symbols, as mentioned in Section 7.4, it seems that sign objects are dealt with first where learners deemed necessary. As mentioned in Section 7.5, transformations of numeral objects appear to have taken precedence over transformations of letter objects. If learners do not think of letter objects as real numbers, then it is understandable that operating on numeral object would take preference as these relate easily to the principles of the core domain of number and learners have far more experience dealing mathematically with numbers than with letters. Where learners operated on superscript objects, these transformations appeared to take place at a different spatial level concurrent to operations (or operation-like manipulations) being done at the base level (see Appendix 12).

Parsing expressions by type of object meant that learners considered one type of object at a time for their computations and in cases where there were no longer numeral objects, some learners resorted to operation-like manipulations on letter objects instead, often by non-standard methods. As discussed above, many learners applied the negative symbol to letter objects by "cancelling" them entirely, i.e. by taking them as arguments of the VOID mapping. In terms of language, there exists a single word which universally captures addition (or more generally disjoint union and merge): the word "and." There is no single word which captures subtraction as universally as the word "and" captures addition, so it is unsurprising that subtraction poses more of a challenge to learners simply from a linguistic perspective. The negative symbol is often read as "minus" regardless of whether it is an existential signifier, binary operator or unary operator, which may explain learners' tendency to want to take things away even when there are no like terms.

Although learners resorted to non-standard transformations, often those transformations were fundamentally not very different from the standard transformations they strayed from. For example, consider the expression $3x - x$ and the transformation (iv) $ax - x \mapsto a$ above. By subtraction the expression can be simplified to $2x$. If one thinks of $3x$ as the set $\{x, x, x\}$, where $x \in \mathbb{R}$, then subtracting x is the same as taking the relative complement of $\{x\}$:

$$\{x, x, x\} \setminus \{x\} = \{x, x\}.$$

Figure 52 demonstrates how the two readings of $3x - x$ are related by the transformation f.

$$\boxed{f : ax \to \{ \underbrace{x, \ldots, x}_{a \text{ copies of } x} \}}$$

$$\begin{tikzcd}
3x \arrow[r, "-x"] \arrow[d, "f"'] & 2x \arrow[d, "f"] \\
\{x, x, x\} \arrow[r, "\setminus\{x\}"'] & \{x, x\}
\end{tikzcd}$$

Figure 52: Diagrammatic representation of $3x - x = 2x$

Suppose, alternatively, that $3x$ is thought of as the set $\{/3/, /x/\}$, containing a numeral object and a letter object. This time, "subtracting" x is the same as taking the relative complement of $\{/x/\}$:

$$\{/3/, /x/\} \setminus \{/x/\} = \{/3/\}.$$

Figure 53 presents how the two readings are related by the transformation g. Note that VOID_l indicates VOID on letter objects and set simply maps $/a/$ and $/x/$ to the set containing them as elements.

$$\boxed{g : ax \xrightarrow{\text{STR}} /ax/ \xrightarrow{\text{SUN}} (/a/,/x/) \xrightarrow{\text{set}} \{/a/,/x/\}}$$

$$\begin{tikzcd}
/3x/ \arrow[r, "\text{VOID}_l"] \arrow[d, "g"'] & /3/ \arrow[d, "g"] \\
\{/3/,/x/\} \arrow[r, "\setminus\{/x/\}"'] & \{/3/\}
\end{tikzcd}$$

Figure 53: Diagrammatic representation of $3x - x \mapsto 3$

As we can see these two diagrams are in fact very similar to one another. Both readings above ultimately represent $3x - x$ as the relative complement $X \setminus Y$ where X represents $3x$, Y represents x and $Y \subset X$. To consider learners' computations in this way is useful because it allows us to consider their fundamental differences and similarities and to compare them with those computations considered standard from the point of view of mathematics.

7.7 Conclusion

Across the data learners' computations often strayed from those considered standard from a mathematical perspective. Although it is useful to identify the errors that learners make and the misconceptions they hold, it is important to go a step further and consider what it is that fundamentally drives them to do computations which do not align with what they have been taught.

Many of the computations that learners did constituted procedures that their teachers would certainly not have taught them, but that nonetheless often had rational reasoning behind them from a non-mathematics perspective. From learners' written work and explanations in interviews, it seems that their computations were largely driven by the same mental structures, met-befores and cognitive biases. The natural capacity to learn and distinguish between different symbol systems and to categorise different objects provides reason for the type-sensitivity displayed in learners' work and their tendency to treat different objects independently in their computations. Many of the computations discussed in Chapter 6 can be explained by biases and met-befores developed from language learning and arithmetic as well as a general nescience regarding the mathematical denotation of expressions. With algebraic concepts still being relatively new to the participants of this study, where they did not have a strong sense of the denotation of expressions, they relied on their procedural knowledge, idiosyncratic concepts and whichever mental structures seemed most relevant for dealing with the information with which they were presented. Often, however, these structures were incompatible with the axioms of real numbers. Evident from the plethora of non-standard transformations exhibited in the data is the need to ensure that learners' conceptual skills related to simplifying algebraic expressions are aligned with those intended by mathematics. What this chapter has shown is that often various transformations can in fact be explained by the same mental structures or met-befores, such that helping learners to overcome those biases would be far more fruitful than attempting to correct each of their idiosyncratic methods. With algebra constituting a non-core domain of knowledge, it is important to focus on helping learners to develop new mental structures which can support their acquisition of new concepts and their understanding of the mathematical denotation of the objects with which they are working.

8 Conclusion

8.1 Summary

Notwithstanding the value that taxonomies of learners' interpretations and common errors (e.g., Adnan et al., 2021; Booth et al., 2014; Küchemann, 1978, 1981) hold as analytical tools, or the usefulness of discussions considering learners' symbol sense and algebraic reasoning (e.g., Booth & Koedinger, 2008; Pournara et al., 2016; Vlassis, 2004, 2008; Vlassis & Demonty, 2022), the mathematics education literature is wanting in in-depth descriptions and, more importantly, explanations of what learners do computationally when solving algebraic problems and why. The work presented in this dissertation contributes to the limited body of literature (e.g., Davis, 2012, 2013; Jaffer, 2018) which addresses that gap.

The principal proposition of this study, corroborated by the analysis of learners' written work and verbal explanations, holds that learners are sensitive to the types of objects constituting algebraic expressions that they take as the arguments of their computations. Five such types of objects are identified, namely: *signs*, *numerals*, *letters*, *superscripts* and *operators*. A result of this type-sensitivity is learners' preference for type-specific computations, elicited under instruction to simplify a given expression. Reluctance to accept lack of closure—highlighted as a common occurrence by the literature (e.g., Booth et al., 2014; Pournara et al., 2016; Stewart & Reeder, 2016)—prompted many learners to resort to simplifying like *objects* (Level 2 simplification) when expressions no longer contained like terms. At Level 2 it seems learners wanted to modify their computational workspaces such that they contained no operator objects and at-most one of each other type of object. Transformations used for Level 2 simplification often resulted in the removal of objects entirely, highlighting that learners' conceptual understanding of the objects constituting algebraic expressions often did not coincide with the mathematical denotation of those objects. This was further instantiated in learners' reliance on character distribution matrices and mnemonics as prompts for which procedures to follow when simplifying expressions. Emerging clearly from the data analysis was the key role of the negative symbol in prompting learners to resort to transformations considered non-standard from the point of view of mathematics.

Plausible reasons for learners' type-sensitivity and idiosyncratic computations offered here include: humans' innate capacity for recognising and categorising different objects and symbols; the biases produced from language, such as left-to-right reading; the effects of met-befores related to arithmetic; and the reliance on existing mental structures for the assimilation of new knowledge. Drawing on these explanations, this study shows that learners' reasoning may often be deemed as rational, conceptual or seemingly logical despite not coinciding with the axioms of mathematics. And, considering learners' computations at this fundamental level reveals that often those considered non-standard are structurally not that different from the standard ones that they replace.

8.2 Limitations of the study

Given that this study had only 22 participants, it is limited in that, relative to the sample size, transformations that emerged as "common" from the data do not necessarily constitute common transformations universally. Nonetheless, it certainly provides a snapshot of some of the computations that learners do which are likely (as corroborated by the literature) apparent amongst learners across the country (if not the world). Further, given that not all participants were interviewed and that those who were did not explain every one of their solutions, or may not have described their computations accurately, it can be argued that inferences regarding certain computations may have been based on incomplete representations. However, the analysis focused on computations which emerged across different learners' solutions, or which were explained by one or more learners in their interviews to ensure, to the extent that was possible, that the transformations used to describe learners' computations satisfied the principles of observational and descriptive adequacy.

Since an investigation into what learners had been taught in mathematics prior to the task was not included in this study, it could only be assumed that learners had not been taught the non-standard practices[42] exhibited in their solutions. However, these assumptions were based on the Senior Phase Curriculum, Assessment and Policy Statement (DBE, 2011) and on annual teaching plans for Grade 7 and Grade 8 (DBE, 2023a, 2023b), ensuring their reliability

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[42] Note that this excludes, for example, the DSB and SSAK methods which are considered standard practices in school mathematics although they may diverge from standard mathematical operations.

to the extent possible without specific investigation. In addition, these assumptions were corroborated by the variance in leaners' computations and their performance on the task.

Given that this study's analytical framework is still novel to the focus area and that even a limited insight into the possible computations that learners use in problem solving can be helpful to understanding how they learn and how they can be helped to grasp certain concepts, the limitations discussed above do not prevent this study from making a useful contribution to the field and serving as a base from which further research may be conducted.

8.3 Recommendations and implications for practice

Having mentioned the sample size of this study as a limitation, conducting the same or a similar study with more participants would allow for further conclusions to be drawn regarding which computations are privileged by learners. Investigating the teaching of algebra preceding the data collection would be useful for revealing the effects that teachers' use of language and their delivery of pedagogic texts, for example, have on learners' conceptual understanding of mathematical objects. This study could also be adapted for investigating the computations of learners in higher grades.

This study highlights the need for dedicated attention to the uses of the negative symbol and the denotation of mathematical objects and expressions generally in mathematics teaching. Developing learners' conceptual understanding to align with mathematical denotation would help in tackling learners' reluctance to accept lack of closure, their misconceptions concerning the cancellation of objects and their reliance on mnemonics and character distribution matrices as prompts for which procedures to follow.

Given the novelty that the lenses used in this study[43] present to the mathematics education literature, further research using these lenses would certainly be beneficial for developing a greater body of literature that draws on cognitive science and mathematics for the descriptions and explanations of learners' computations. Identifying and explaining learners' computations, as has been done here, may bring awareness to teachers and other researchers regarding the idiosyncratic conceptions that several learners appear to have of the objects

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[43] Like those of, for example, Davis (2012, 2013) and Jaffer (2018).

constituting algebraic expressions as well as the transformations compossible with those objects. This would allow teachers to understand why, fundamentally, their learners might be struggling with certain concepts and allow them to develop learners’ foundational understanding.

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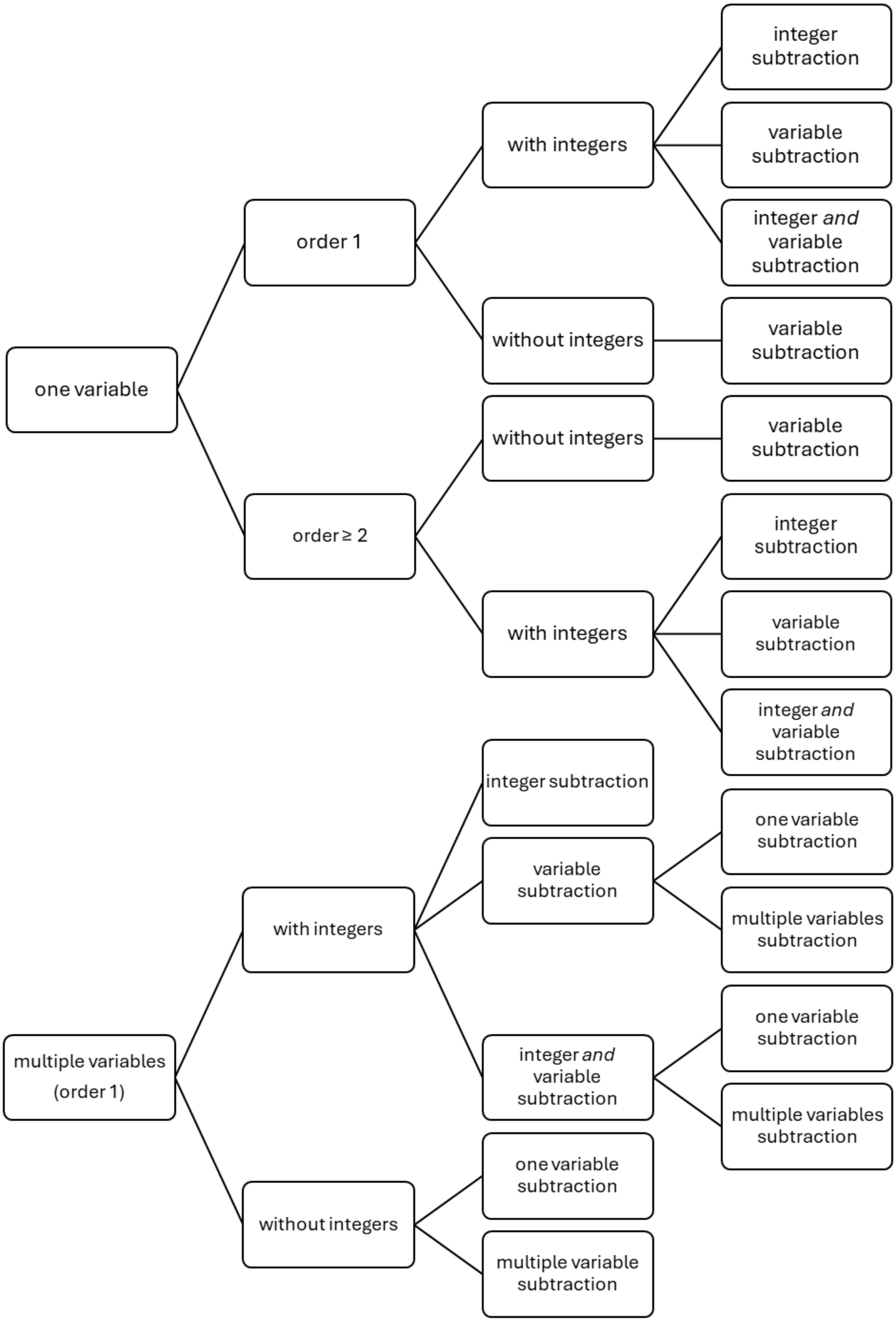
Appendices

Appendix 1 Universe of Possibilities



Appendix 2 Task

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write your <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

2. $11x - 8x - 3$

3. $4a - (-3a) - 1$

4. $-13x - 8 - (-6x) - 6$

5. $(3x - 5) - (3x - 5)$

6. $2x - 8x - 3x - x$

7. $-5x^2 - 12x^2 - 4x^2$

8. $11 - (-17) - 7x^3 - 2$

9. $-x^3 - (-3x^3) - 15$

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

11. $1 - 16x - (-13) - y - 14$

12. $4xy - 5xy - 5 - (-11xy) - xy$

13. $-14a - (-3b) - 2a - 7$

14. $3y - 7x - 8 - (-8x) - 3y$

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

17. $4x^3 - 9x^2 - 8x^3$

18. $17a - (-4a) - 7b - 6a$

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

Appendix 3 Task memorandum

1. $-x-15-11$

   $=-x-26$

2. $11x-8x-3$

   $=3x-3$

3. $4a-(-3a)-1$

   $=4a+3a-1$

   $=7a-1$

4. $-13x-8-(-6x)-6$

   $=-13x-8+6x-6$

   $=-7x-14$

5. $(3x-5)-(3x-5)$

   $=3x-5-3x+5$

   $=0$

6. $2x-8x-3x-x$

   $=-6x-4x$

   $=-10x$

7. $-5x^2-12x^2-4x^2$

   $=-21x^2$

8. $11-(-17)-7x^3-2$

   $=11+17-7x^3-2$

   $=26-7x^3$

9. $-x^3-(-3x^3)-15$

   $=-x^3+3x^3-15$

$= 2x^3 - 15$

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$= 8 - 2x^4 - 13x^4 - 11 - 3 + 3x^4$

$= -12x^4 - 6$

11. $1 - 16x - (-13) - y - 14$

$= 1 - 16x + 13 - y - 14$

$= -16x - y$

12. $4xy - 5xy - 5 - (-11xy) - xy$

$= 4xy - 5xy - 5 + 11xy - xy$

$= 9xy - 5$

13. $-14a - (-3b) - 2a - 7$

$= -14a + 3b - 2a - 7$

$= -16a + 3b - 7$

14. $3y - 7x - 8 - (-8x) - 3y$

$= 3y - 7x - 8 + 8x - 3y$

$= x - 8$

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= 13ab - 4ab - 13 - 2ab + 9 - 1$

$= 7ab - 5$

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$= 15 - 5x + x - 8x^2 + 11 - 12x^2$

$= -20x^2 - 4x - 26$

17. $4x^3 - 9x^2 - 8x^3$

$= -4x^3 - 9x^2$

18. $17a - (-4a) - 7b - 6a$

$= 17a + 4a - 7b - 6a$

$= 15a - 7b$

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$= -13a^2 - 5a^4 + 6a^4 - 14a^2 - 5a^2$

$= -32a^2 + a^4$

Appendix 4 Interview schedule (first round)

Proceedings

1. Set up: Position the blank workbook on the desk in front of the participant's chair and install the camera so that it is focused on the workbook.
2. Upon the student's arrival, ask them to sit in the designated chair and explain to them the proceedings and purpose of the interview.
   - Proceedings: *I have chosen some of the solutions you wrote to the task I gave you; I am going to give them to you one by one and I would like you to explain your working out. If you want to change your answer or make any notes you are welcome to do so in the workbook in front of you. I might also give you some extra expressions to simplify in the workbook.*
   - Purpose*: The purpose of this interview is for me to gain insight into your mathematical thinking by going through some of your solutions to the task.*
3. Begin video recording.
4. For each chosen question, put the student's solution script in front of them and ask them to explain their working out and how they came to their final answer.
5. Use the students' solutions as prompts for general questions as well as for questions specific to the computations they exhibit.
6. In cases where they indicate that they would change their solution, ask them to write their new solution in the workbook and treat that answer in the same way as the original answers. If they seem unsure of their computations or if it would be beneficial to explore their computations in more depth, write down a new (similar) expression in the workbook and ask them to simplify it and explain their working out.
7. Once all the selected solutions have been explored, other solutions may be worked through to gain further insight into the students' computations if there is still time remaining (interviews should last no more than 25 minutes).
8. End interview and video recording when sufficient solutions have been explored (or 25 minutes have passed).

Questions

The interview questions should be used to prompt students to give clear explanations of their mathematical thinking and probe their use of computational resources surrounding subtraction, negative numbers and the negative symbol. Given that the interviews are semi-structured, the wording need not follow the questions below exactly and students' responses will direct the consequent wording of intervening and supplementary questions.

Ask every student:

- How did you come to this answer? Can you please explain your working out?
- How did you know that this term [choose a term in their solution] should be positive/negative?

The following questions should be used depending on students' responses and written solutions but need not be posed to every student:

- How do you know if terms are like or unlike?
- When you do subtraction, do you look at the numbers and the symbols to get to your final answer or do you imagine a number line?
- Does it matter in which order we subtract? (Task Q6)
- Do you work with certain terms first when you simplify an expression?
- What do the brackets tell us? Can brackets mean different things in different situations? (e.g. Task Q3, Q4, Q5)
- Are these brackets necessary? Can we remove them? (Task Q5)
- What do you think the minus in between the brackets is telling you to do? (Task Q5)
- Does a minus in between the brackets change the way you simplify the expression? (Task Q5)
- How would you simplify the expression $3x - 3x$?
- How would you simplify the expression $-3x - (-3x)$?
- If there are brackets around the first term, like in $(-3x) - (-3x)$, does that make a difference to the way you would simplify the expression?
- Do you think ab and ba are the same or different? Why? (Task Q15)
- What do the exponents mean to you? (e.g. Task Q7, Q8, Q9)

In cases where a student's working out is minimal or difficult to interpret, the following questions can be posed:

- What were you thinking when you wrote this answer?
- How would you simplify this expression now?
- What would you do to simplify an expression like this? [write similar expression in workbook]?

If students struggle to explain their thinking or the steps that they used, it may be helpful to prompt them with the computational resources that they may have used and ask them whether they think their steps were different or similar.

Task solutions selected pre-interviews

The following solutions were selected for each student to explain. Further solutions can be explored once these have been worked through, time permitting.

	Learner												
Question	L1	L4	L6	L7	L8	L10	L11	L13	L14	L17	L22	L24	L26
1	X	X	X	X	X	X	X	X		X	X	X	X
2									X				
3									X				
4									X				
5	X	X	X	X	X	X	X	X	X	X	X	X	X
6		X		X	X		X	X		X			X
7												X	
8							X						
9												X	
10	X	X	X	X	X	X	X	X		X	X	X	X
11							X						
12				X	X	X	X			X		X	X
13							X						
14													
15	X	X	X	X		X	X	X	X	X		X	
16												X	
17													
18	X	X	X	X		X		X			X	X	
19													
Total	5	6	5	7	5	6	9	6	5	6	4	9	5

Appendix 5 Interview schedule (follow-up)

Proceedings

9. Set up workbook and camera as in first-round interviews.
10. Upon the student's arrival, ask them to sit in the designated chair and explain to them the proceedings and purpose of the interview.
    - Proceedings: *In this interview I am going to write down expressions similar to the ones you attempted in the task, and I would like you to simplify them and explain your working out.*
    - Purpose*: The purpose of this interview is for me to get a better understanding of your mathematical thinking by working through new expressions.*
11. Begin video recording.
12. For each chosen expression, write down the expression and ask the student to simplify it, showing their working out.
13. Ask the student to explain their solution, probing them with questions to highlight the computations that they did.
14. In cases where they exhibit an interesting computation, another similar expression can be given to them to explore their thinking further.
15. Once all the selected expressions have been explored, other expressions may be worked through to gain further insight into the students' computations if there is still time remaining (interviews should last no more than 25 minutes).
16. End interview and video recording when sufficient expressions have been worked through (or 25 minutes have passed).

Extra Expressions

1) $-12 - 3x - 9$
2) $7a - 3 - 4$
3) $-x^2 - 4x^3 - 2x^2$
4) $4x^2 - x^2 - 6x^2 - x^2$
5) $-4x^4 - (-6x^4) - 6$
6) $8xy - 8 - 5x - 5$
7) $-6xy - x - y - 5yx$
8) $-5x^3 - 3 - 8x^3 - (-x^3) - 12 - 1$

9) $2ab - 8 - (-ab) - 5ab - 1$

10) $-7 - 5x^3 - x^3 - (-5) - (-2)$

11) $-12ab - 8ab - 3 - (-2ba) - (-6)$

12) $9ab - (-3b) - (-10a) - b - 3$

13) $(x - 5) - (x - 5)$

14) $-(x - 1) - (-x + 1)$

15) $(-a + b) - (-a + b)$

16) $(-8 - x) - (-8 - x)$

Extra expressions selected pre-interview

	Learner								
Expression	L1	L4	L6	L10	L11	L13	L14	L22	L24
1							X		
2	X	X	X	X	X	X	X	X	X
3							X		X
4		X	X	X		X	X	X	X
5					X				X
6	X							X	
7	X			X	X	X	X	X	
8									
9									
10	X	X	X		X	X			
11		X	X	X				X	
12									
13			X		X		X	X	X
14		X							
15	X			X	X	X			
16	X	X	X	X		X			
Total	6	6	6	6	6	6	6	6	5

Questions

The following questions may be asked again (or anew) to determine whether students' computations when simplifying new expressions are consistent with the computations they did in the task:

- Does it matter in which order we subtract?
- When you do subtraction, do you look at the numbers and the symbols to get to your final answer or do you imagine a number line?
- How do you know if terms are like or unlike?

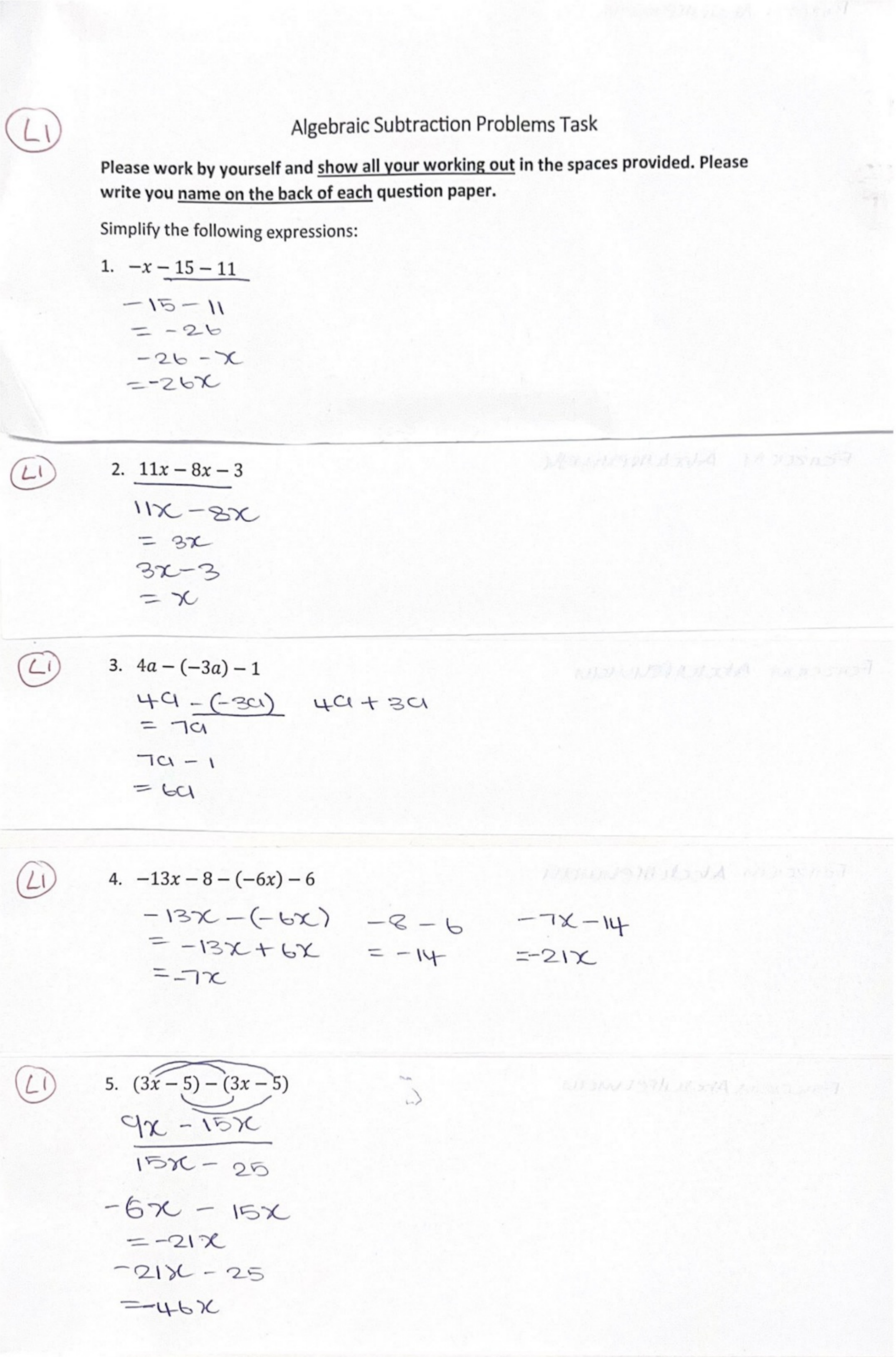
- What do brackets mean to you?
- What does the negative symbol between the brackets mean to you? (e.g. E13-16)

Extra expressions not listed above can be given to students to simplify according to the computations they exhibit and the explanations they offer. Students' responses will prompt the questions they are asked, which will aim to elicit clear explanations of the computations they have done.

Appendix 6 Results table

Learner	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Q13	Q14	Q15	Q16	Q17	Q18	Q19	Total incorrect	Performance
L1	0	0	0	0	0	1	1	0	1	0	1	0	1	0	0	1	1	1	1	**10**	Medium
L2	0	0	0	0	1	0	0	0	1	0	1	0	0	0	0	1	0	0	0	**15**	Low
L3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	1	1	1	**2**	High
L4	0	1	1	1	0	1	0	1	1	1	1	0	1	1	0	1	1	1	1	**5**	High
L5	1	0	1	0	1	0	0	1	1	0	1	1	1	1	1	0	1	1	0	**7**	Medium
L6	1	1	0	0	0	0	0	1	0	0	1	1	1	1	0	1	0	1	0	**10**	Medium
L7	0	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	1	1	0	**13**	Medium
L8	1	1	1	1	0	1	0	1	1	0	1	1	1	1	1	1	1	1	0	**4**	High
L10	0	1	1	0	0	0	1	1	0	0	0	0	1	1	0	0	1	1	1	**10**	Medium
L11	0	1	1	1	0	1	1	0	1	1	0	1	1	1	1	1	1	1	1	**4**	High
L13	0	1	1	0	0	0	1	1	0	0	0	0	1	1	0	1	1	1	1	**9**	Medium
L14	1	1	0	0	0	0	0	B	B	B	B	0	0	0	0	B	0	B	B	**17**	Low
L17	1	1	1	1	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	**3**	High
L18	1	0	0	0	0	0	1	0	0	1	1	0	1	1	0	0	1	1	0	**11**	Medium
L19	0	0	1	1	1	0	1	1	1	0	0	1	1	1	0	0	0	1	1	**8**	Medium
L20	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**18**	Low
L22	1	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	1	0	B	**14**	Low
L24	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	**15**	Low
L26	0	0	0	1	0	1	1	0	1	0	1	1	1	1	0	1	0	0	0	**10**	Medium
L27	0	1	0	0	1	0	0	0	0	0	0	1	0	1	0	0	1	0	0	**14**	Low
L28	B	0	0	0	0	0	0	0	0	0	0	0	0	0	B	B	B	B	B	**19**	Low
L29	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	**19**	Low
Total incorrect	**13**	**9**	**13**	**15**	**17**	**15**	**13**	**12**	**12**	**17**	**12**	**12**	**9**	**7**	**18**	**13**	**8**	**8**	**14**		
Total correct	**9**	**13**	**9**	**7**	**5**	**7**	**9**	**10**	**10**	**5**	**10**	**10**	**13**	**15**	**4**	**9**	**14**	**14**	**8**		

Appendix 7 Task solutions



(L1)

Algebraic Subtraction Problems Task

Please work by yourself and show all your working out in the spaces provided. Please write you name on the back of each question paper.

Simplify the following expressions:

1. $-x - \underline{15 - 11}$

$-15 - 11$
$= -26$
$-26 - x$
$= -26x$

(L1)

2. $\underline{11x - 8x} - 3$

$11x - 8x$
$= 3x$
$3x - 3$
$= x$

(L1)

3. $4a - (-3a) - 1$

$\underline{4a - (-3a)}$ $4a + 3a$
$= 7a$
$7a - 1$
$= 6a$

(L1)

4. $-13x - 8 - (-6x) - 6$

$-13x - (-6x)$
$= -13x + 6x$
$= -7x$

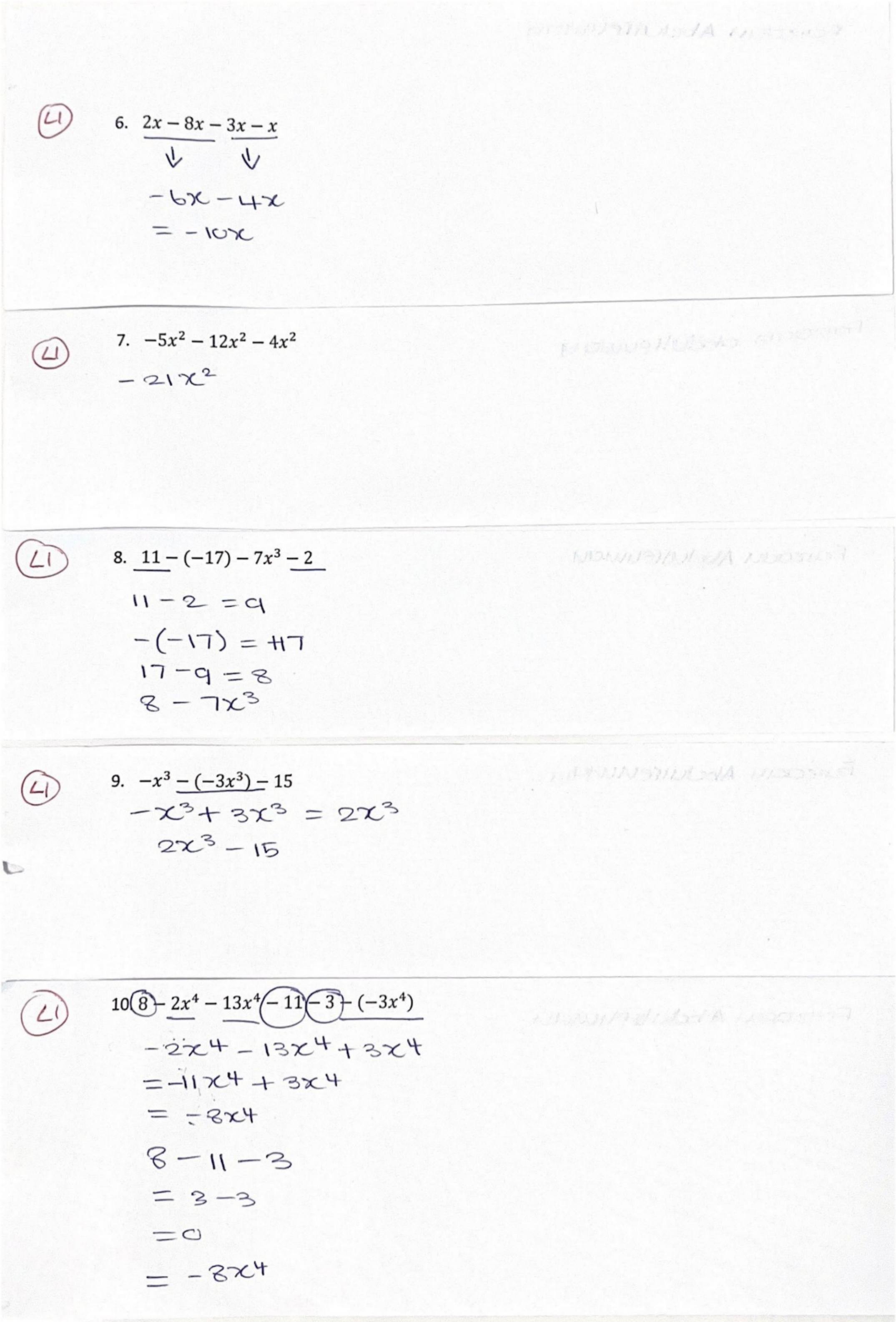
$-8 - 6$
$= -14$

$-7x - 14$
$= -21x$

(L1)

5. $(3x - 5) - (3x - 5)$

$\underline{9x - 15x}$
$15x - 25$
$-6x - 15x$
$= -21x$
$-21x - 25$
$= -46x$



L1

6. $\underline{2x - 8x} - \underline{3x - x}$

$\Downarrow \quad \Downarrow$

$-6x - 4x$

$= -10x$

L1

7. $-5x^2 - 12x^2 - 4x^2$

$-21x^2$

L1

8. $\underline{11} - (-17) - 7x^3 \underline{- 2}$

$11 - 2 = 9$

$-(-17) = +17$

$17 - 9 = 8$

$8 - 7x^3$

L1

9. $-x^3 \underline{- (-3x^3)} - 15$

$-x^3 + 3x^3 = 2x^3$

$2x^3 - 15$

L1

10. $8 - \underline{2x^4} - \underline{13x^4} - 11 - 3 \underline{- (-3x^4)}$

$-2x^4 - 13x^4 + 3x^4$

$= -11x^4 + 3x^4$

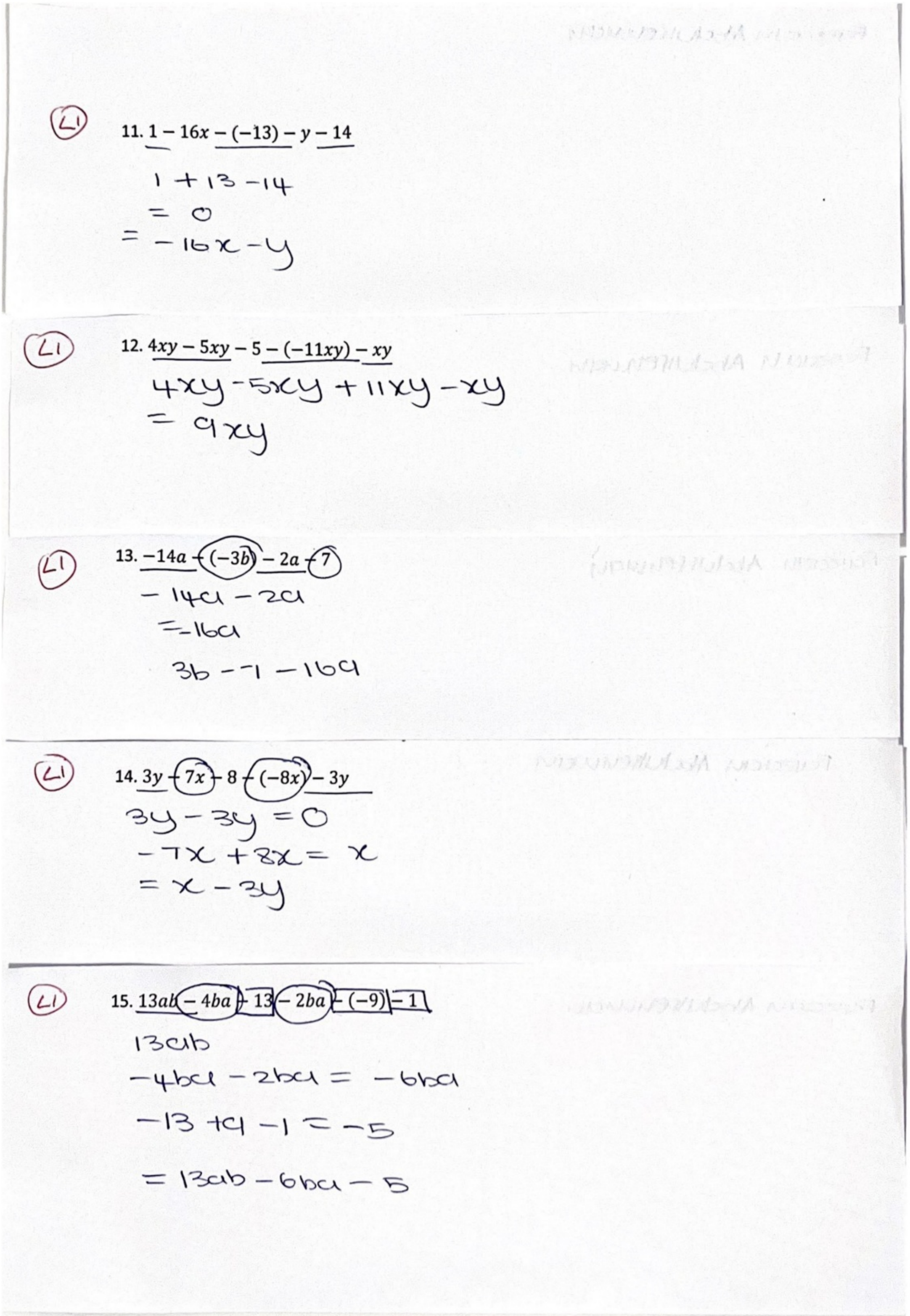
$= -8x^4$

$8 - 11 - 3$

$= 3 - 3$

$= 0$

$= -8x^4$



L1

11. $1 - 16x - (-13) - y - 14$

$1 + 13 - 14$

$= 0$

$= -16x - y$

L1

12. $4xy - 5xy - 5 - (-11xy) - xy$

$4xy - 5xy + 11xy - xy$

$= 9xy$

L1

13. $-14a - (-3b) - 2a - 7$

$-14a - 2a$

$= -16a$

$3b - 7 - 16a$

L1

14. $3y - 7x - 8 - (-8x) - 3y$

$3y - 3y = 0$

$-7x + 8x = x$

$= x - 3y$

L1

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$13ab$

$-4ba - 2ba = -6ba$

$-13 + 9 - 1 = -5$

$= 13ab - 6ba - 5$

L1

16. $\boxed{15} \underline{- 5x - (-x)} - 8x^2 \boxed{- (-11)} - 12x^2$

$15 + 11 = 26$

$-5x + x = -4x$

$-8x^2 - 12x^2 = 20x^2$

$26 - 4x - 20x^2$

L1

17. $\underline{4x^3} - 9x^2 \underline{- 8x^3}$

$4x^3 - 8x^3 = -4x^3$

$-4x^3 - 9x^2$

18. $\underline{17a} \underline{- (-4a)} - 7b \underline{- 6a}$

L1

$17a + 4a - 6a$

$= 15a - 7b$

L1

19. $\underline{-13a^2} - 5a^4 - (-6a^4) \underline{- 14a^2} \underline{- 5a^2}$

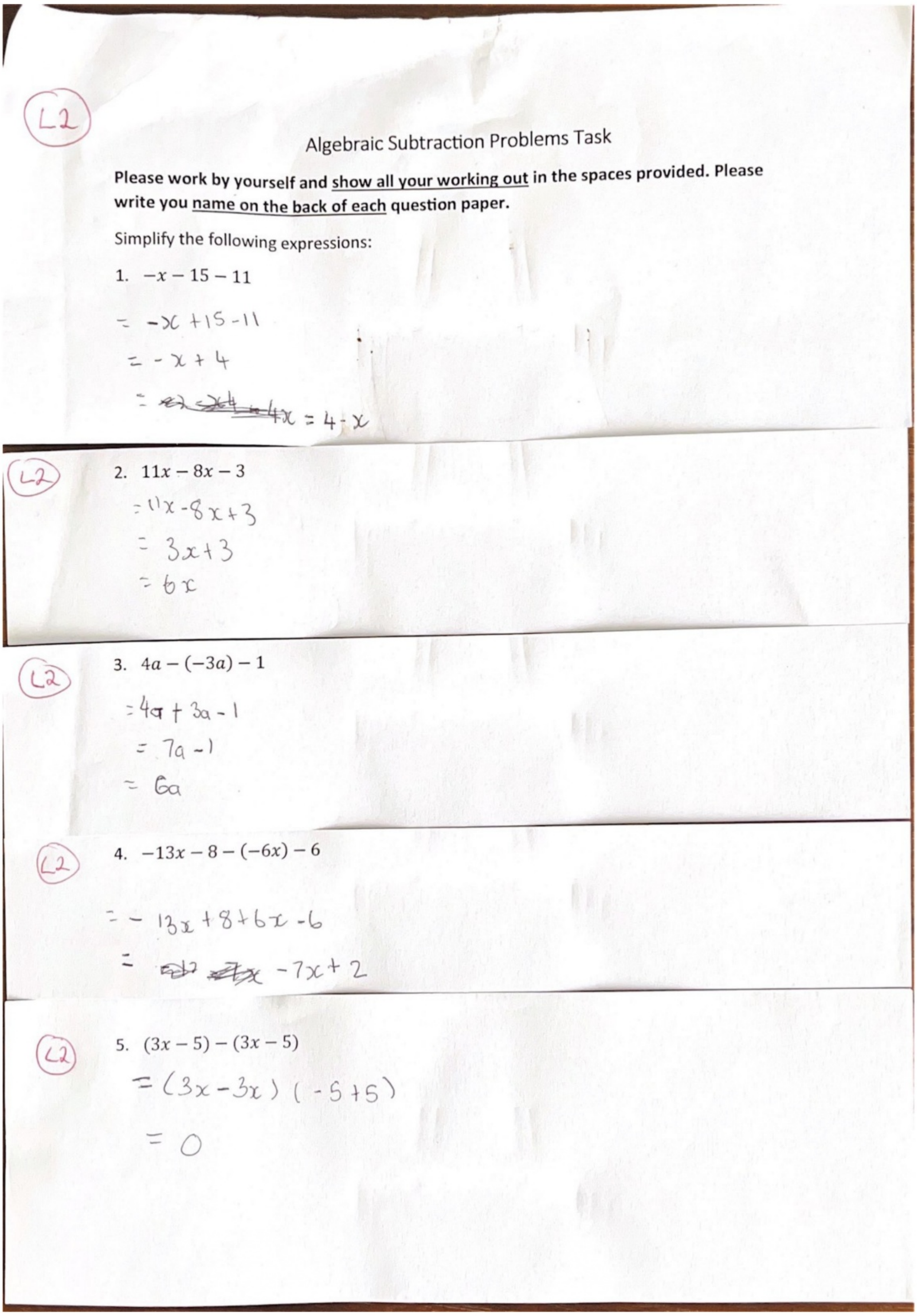
$-13a^2 - 14a^2 - 5a^2$

$= -32a^2$

$-5a^4 + 6a^4$

$= a^4$

$= -32a^2 + a^4$



L2

Algebraic Subtraction Problems Task

Please work by yourself and show all your working out in the spaces provided. Please write you name on the back of each question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$= -x + 15 - 11$

$= -x + 4$

$= 4x = 4 - x$

L2

2. $11x - 8x - 3$

$= 11x - 8x + 3$

$= 3x + 3$

$= 6x$

L2

3. $4a - (-3a) - 1$

$= 4a + 3a - 1$

$= 7a - 1$

$= 6a$

L2

4. $-13x - 8 - (-6x) - 6$

$= -13x + 8 + 6x - 6$

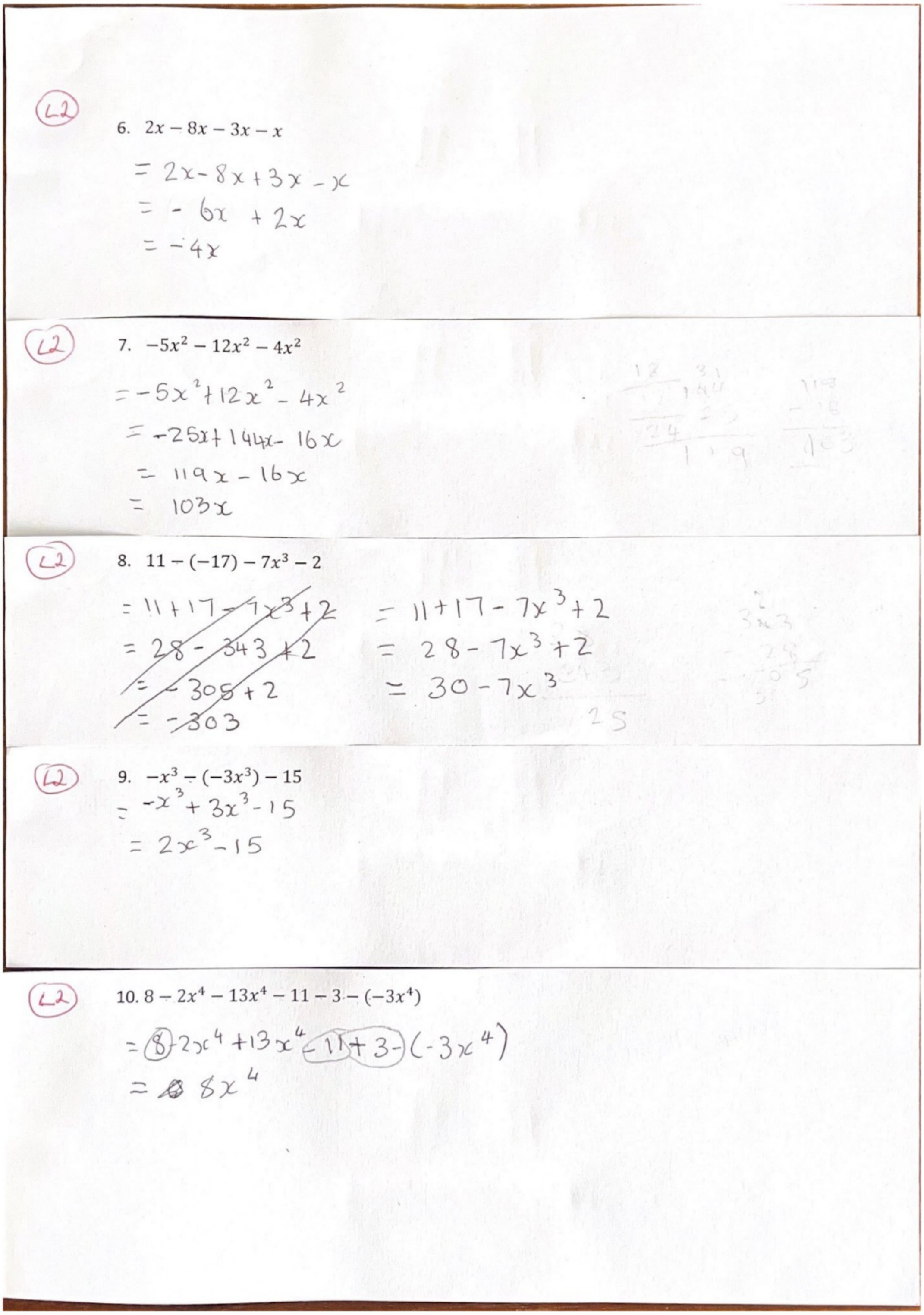
$= -7x + 2$

L2

5. $(3x - 5) - (3x - 5)$

$= (3x - 3x)(-5 + 5)$

$= 0$



L2

6. $2x - 8x - 3x - x$

$= 2x - 8x + 3x - x$

$= -6x + 2x$

$= -4x$

L2

7. $-5x^2 - 12x^2 - 4x^2$

$= -5x^2 + 12x^2 - 4x^2$

$= -25x + 144x - 16x$

$= 119x - 16x$

$= 103x$

L2

8. $11 - (-17) - 7x^3 - 2$

~~$= 11 + 17 - 7x^3 + 2$~~

~~$= 28 - 343 + 2$~~

~~$= -305 + 2$~~

~~$= -303$~~

$= 11 + 17 - 7x^3 + 2$

$= 28 - 7x^3 + 2$

$= 30 - 7x^3$

L2

9. $-x^3 - (-3x^3) - 15$

$= -x^3 + 3x^3 - 15$

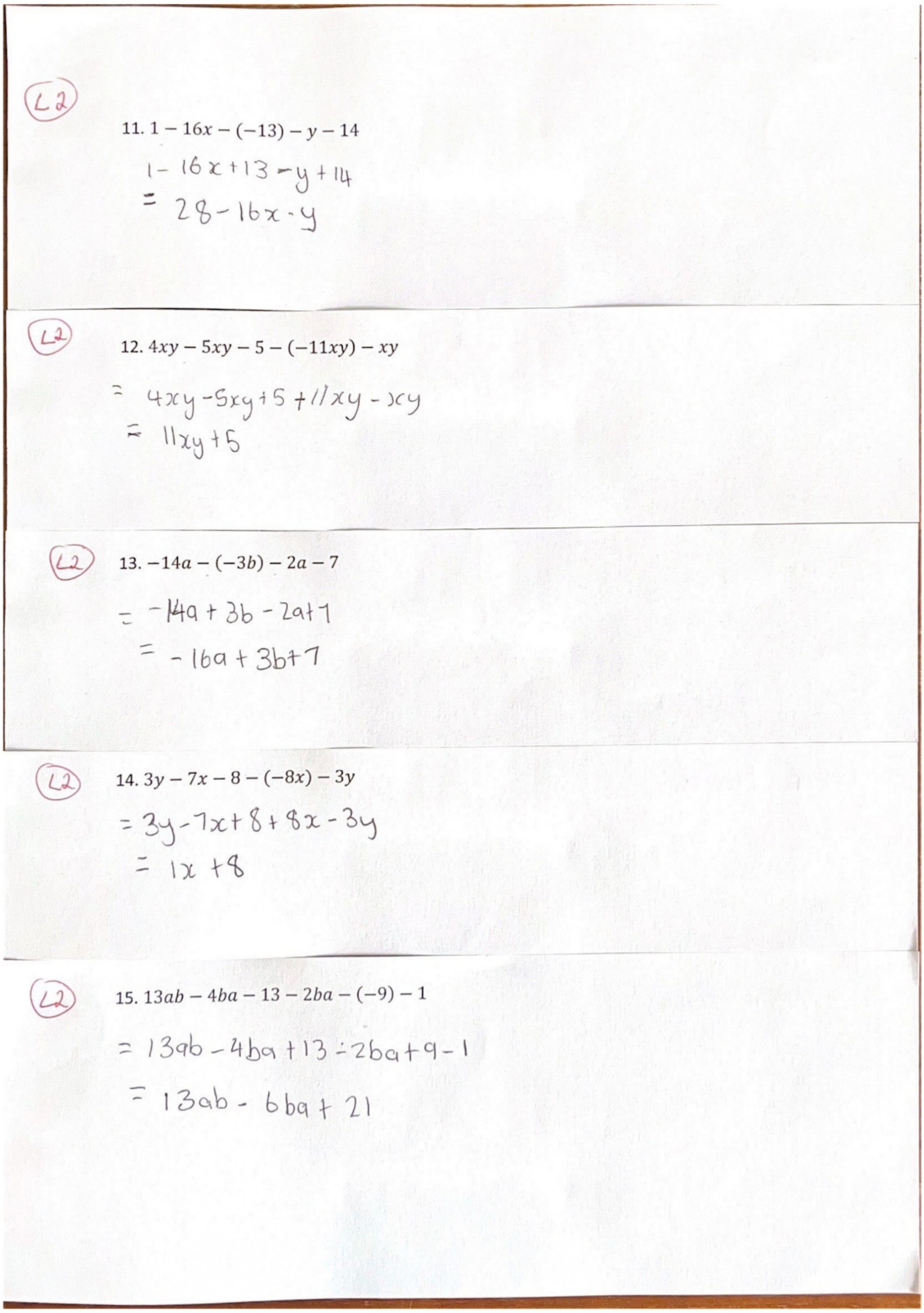
$= 2x^3 - 15$

L2

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$= 8 - 2x^4 + 13x^4 - 11 + 3 - (-3x^4)$

$= \cancel{8}\ 8x^4$



L2

11. $1 - 16x - (-13) - y - 14$

$1 - 16x + 13 - y + 14$

$= 28 - 16x - y$

L2

12. $4xy - 5xy - 5 - (-11xy) - xy$

$= 4xy - 5xy + 5 + 11xy - xy$

$= 11xy + 5$

L2

13. $-14a - (-3b) - 2a - 7$

$= -14a + 3b - 2a + 7$

$= -16a + 3b + 7$

L2

14. $3y - 7x - 8 - (-8x) - 3y$

$= 3y - 7x + 8 + 8x - 3y$

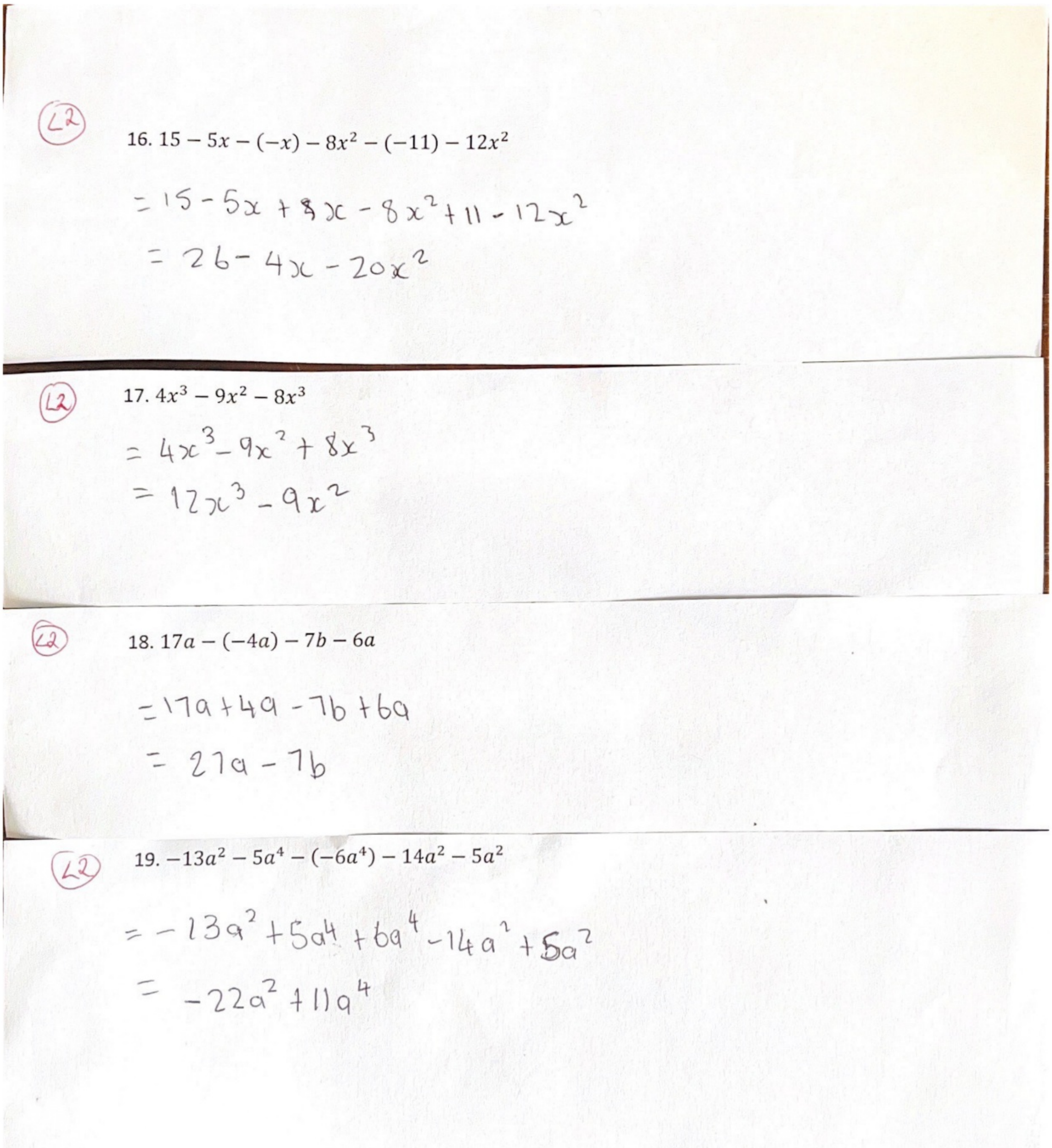
$= 1x + 8$

L2

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= 13ab - 4ba + 13 - 2ba + 9 - 1$

$= 13ab - 6ba + 21$



L2

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$= 15 - 5x + 8x - 8x^2 + 11 - 12x^2$

$= 26 - 4x - 20x^2$

L2

17. $4x^3 - 9x^2 - 8x^3$

$= 4x^3 - 9x^2 + 8x^3$

$= 12x^3 - 9x^2$

L2

18. $17a - (-4a) - 7b - 6a$

$= 17a + 4a - 7b + 6a$

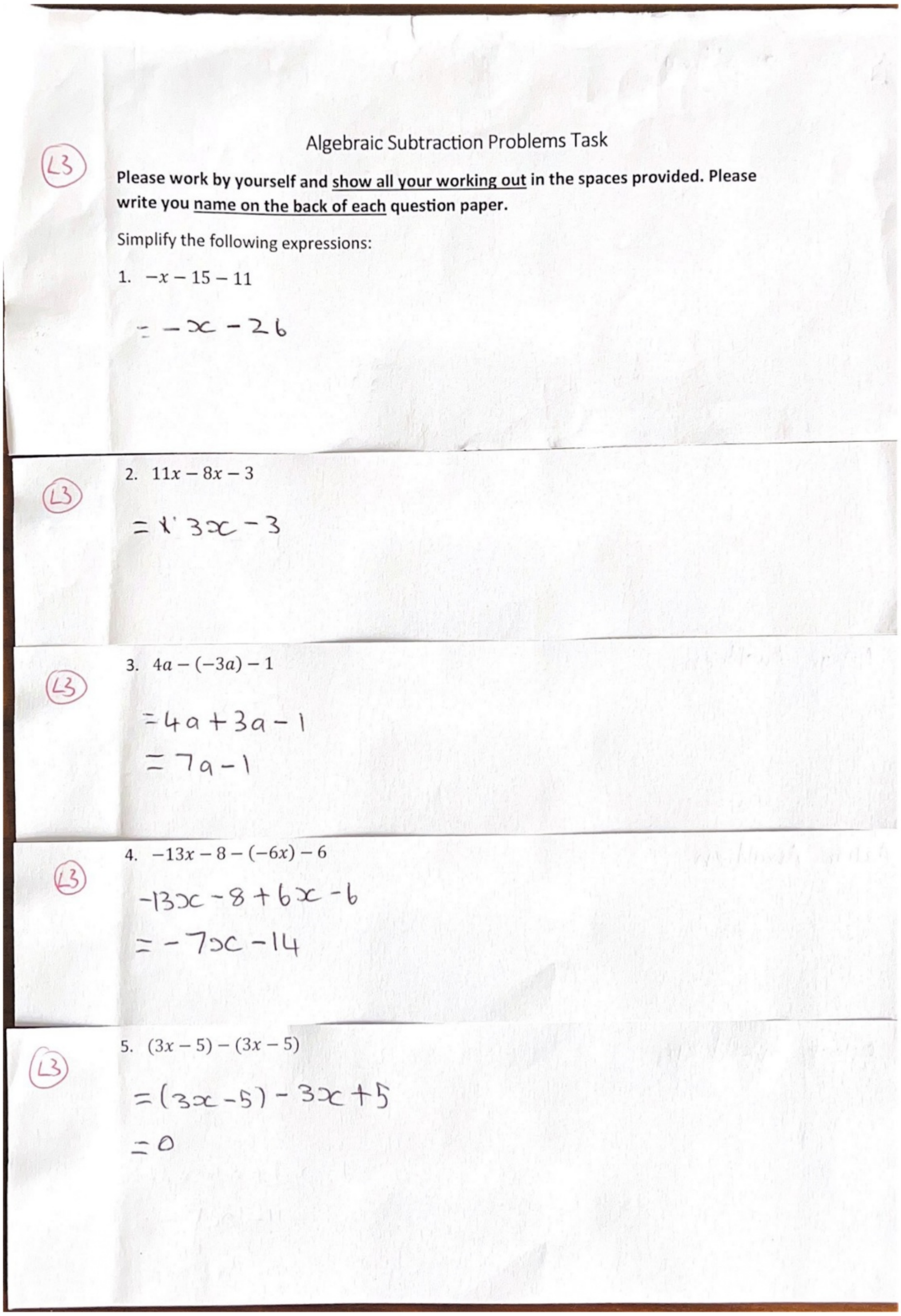
$= 21a - 7b$

L2

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$= -13a^2 + 5a^4 + 6a^4 - 14a^2 + 5a^2$

$= -22a^2 + 11a^4$



Ⓛ3

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$= -x - 26$

L3

2. $11x - 8x - 3$

$= 3x - 3$

L3

3. $4a - (-3a) - 1$

$= 4a + 3a - 1$

$= 7a - 1$

L3

4. $-13x - 8 - (-6x) - 6$

$-13x - 8 + 6x - 6$

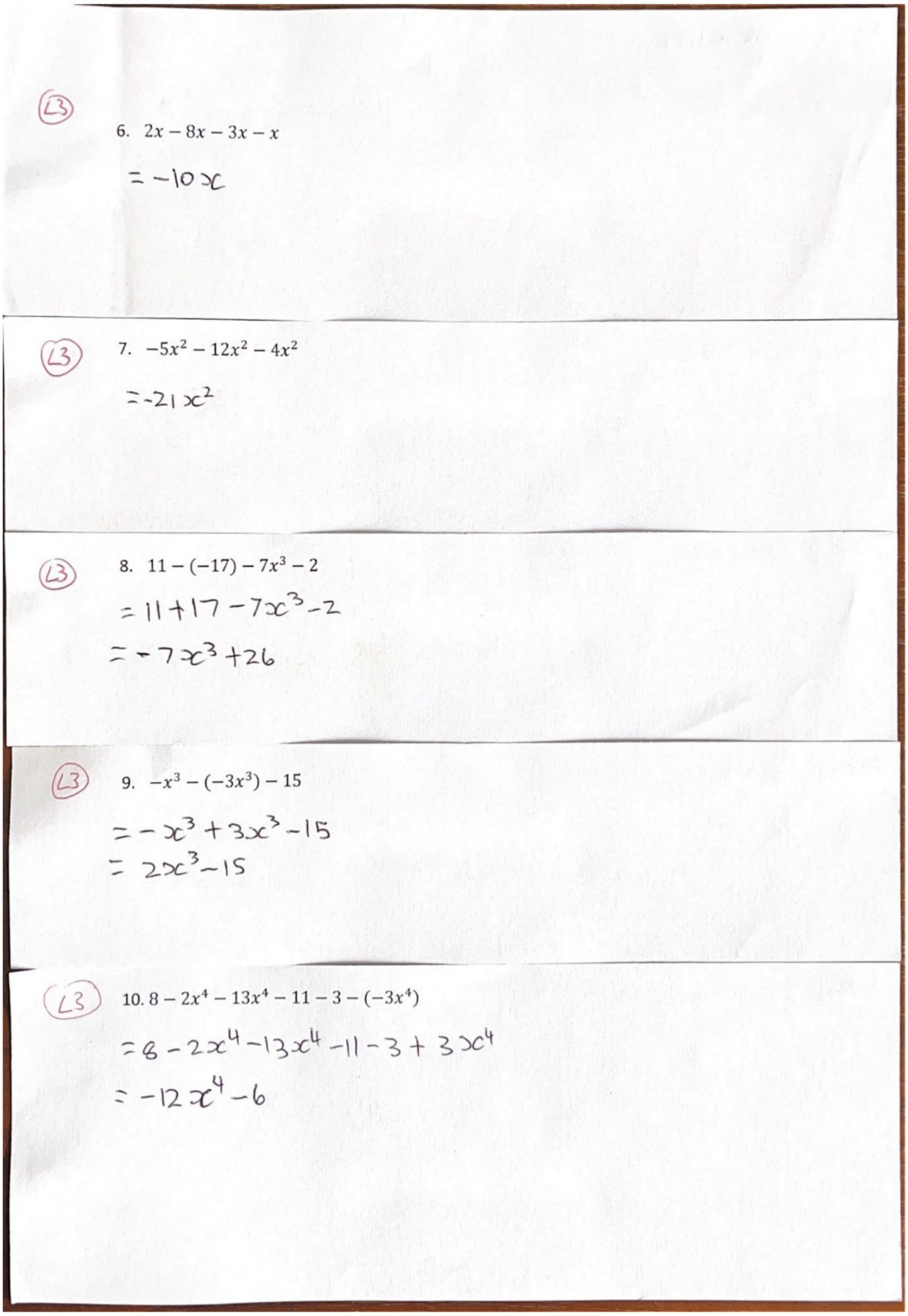
$= -7x - 14$

L3

5. $(3x - 5) - (3x - 5)$

$= (3x - 5) - 3x + 5$

$= 0$



L3

6. $2x - 8x - 3x - x$

$= -10x$

L3

7. $-5x^2 - 12x^2 - 4x^2$

$= -21x^2$

L3

8. $11 - (-17) - 7x^3 - 2$

$= 11 + 17 - 7x^3 - 2$

$= -7x^3 + 26$

L3

9. $-x^3 - (-3x^3) - 15$

$= -x^3 + 3x^3 - 15$

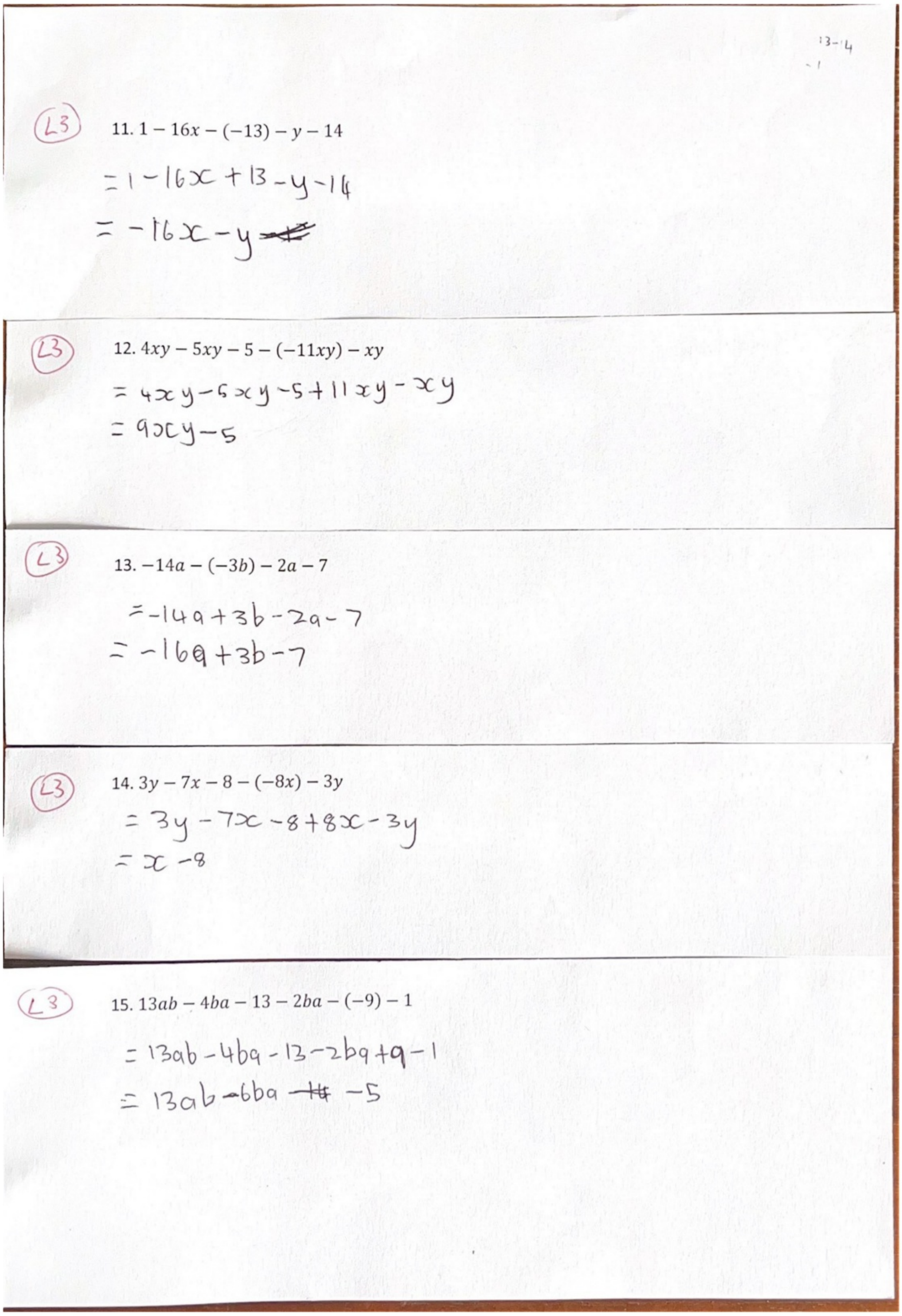
$= 2x^3 - 15$

L3

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$= 8 - 2x^4 - 13x^4 - 11 - 3 + 3x^4$

$= -12x^4 - 6$



$13-14$
-1

L3 11. $1-16x-(-13)-y-14$

$=1-16x+13-y-14$

$=-16x-y$ ~~−14~~

L3 12. $4xy-5xy-5-(-11xy)-xy$

$=4xy-5xy-5+11xy-xy$

$=9xy-5$

L3 13. $-14a-(-3b)-2a-7$

$=-14a+3b-2a-7$

$=-16a+3b-7$

L3 14. $3y-7x-8-(-8x)-3y$

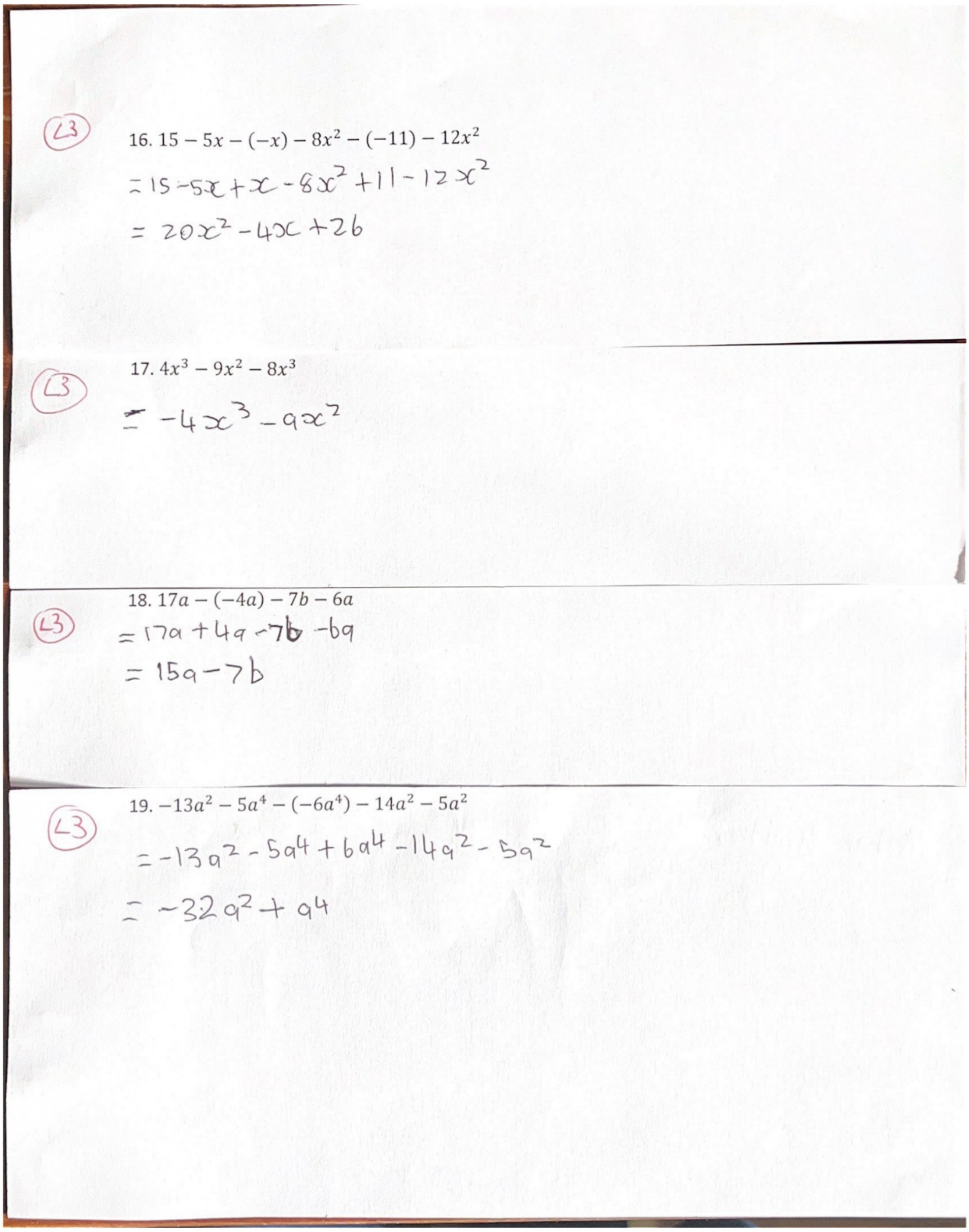
$=3y-7x-8+8x-3y$

$=x-8$

L3 15. $13ab-4ba-13-2ba-(-9)-1$

$=13ab-4ba-13-2ba+9-1$

$=13ab-6ba$ ~~−14~~ -5



L3

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$= 15 - 5x + x - 8x^2 + 11 - 12x^2$

$= 20x^2 - 4x + 26$

L3

17. $4x^3 - 9x^2 - 8x^3$

$= -4x^3 - 9x^2$

L3

18. $17a - (-4a) - 7b - 6a$

$= 17a + 4a - 7b - 6a$

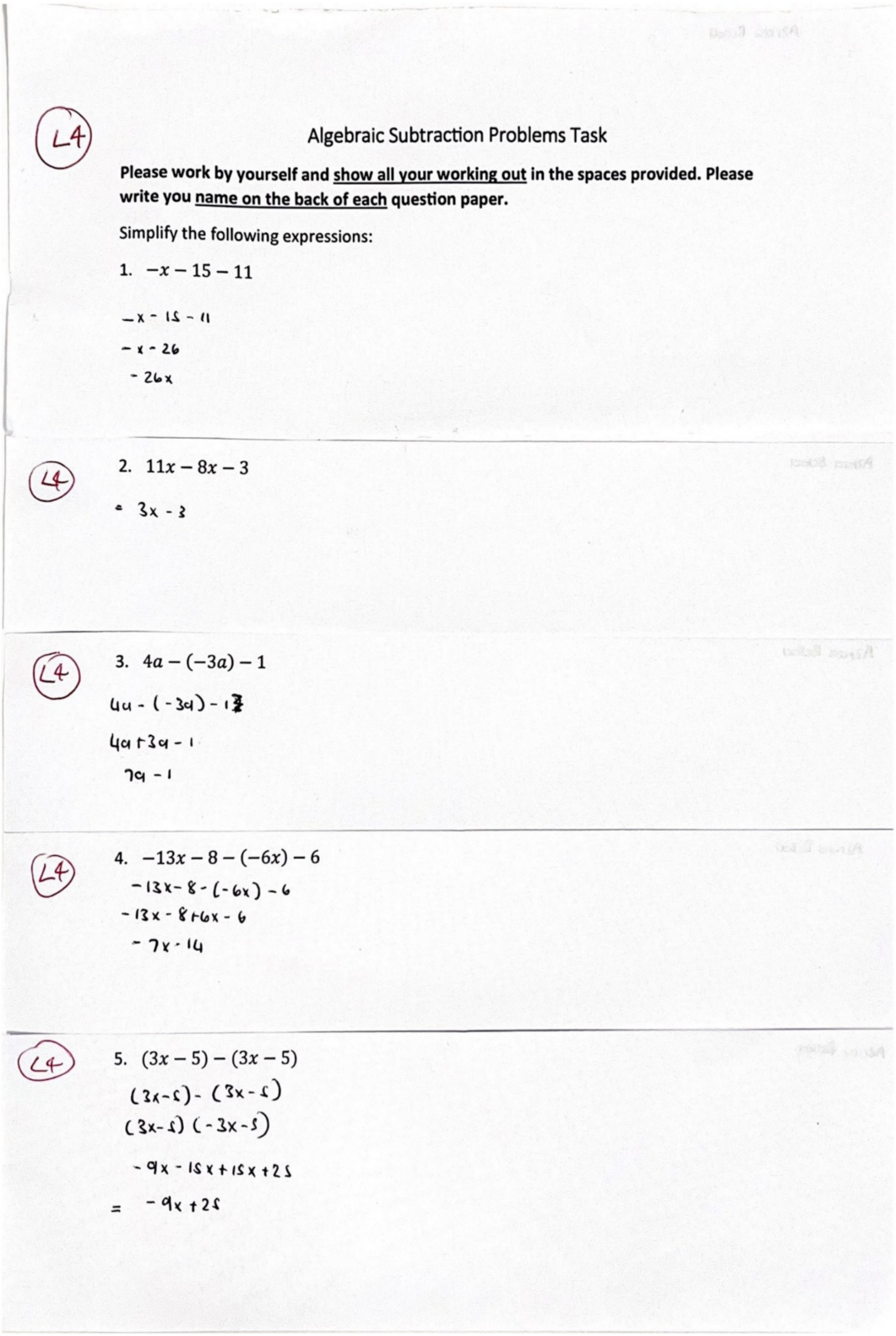
$= 15a - 7b$

L3

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$= -13a^2 - 5a^4 + 6a^4 - 14a^2 - 5a^2$

$= -32a^2 + a^4$



L4

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$-x - 15 - 11$

$-x - 26$

$-26x$

L4

2. $11x - 8x - 3$

$= 3x - 3$

L4

3. $4a - (-3a) - 1$

$4a - (-3a) - 1$

$4a + 3a - 1$

$7a - 1$

L4

4. $-13x - 8 - (-6x) - 6$

$-13x - 8 - (-6x) - 6$

$-13x - 8 + 6x - 6$

$-7x - 14$

L4

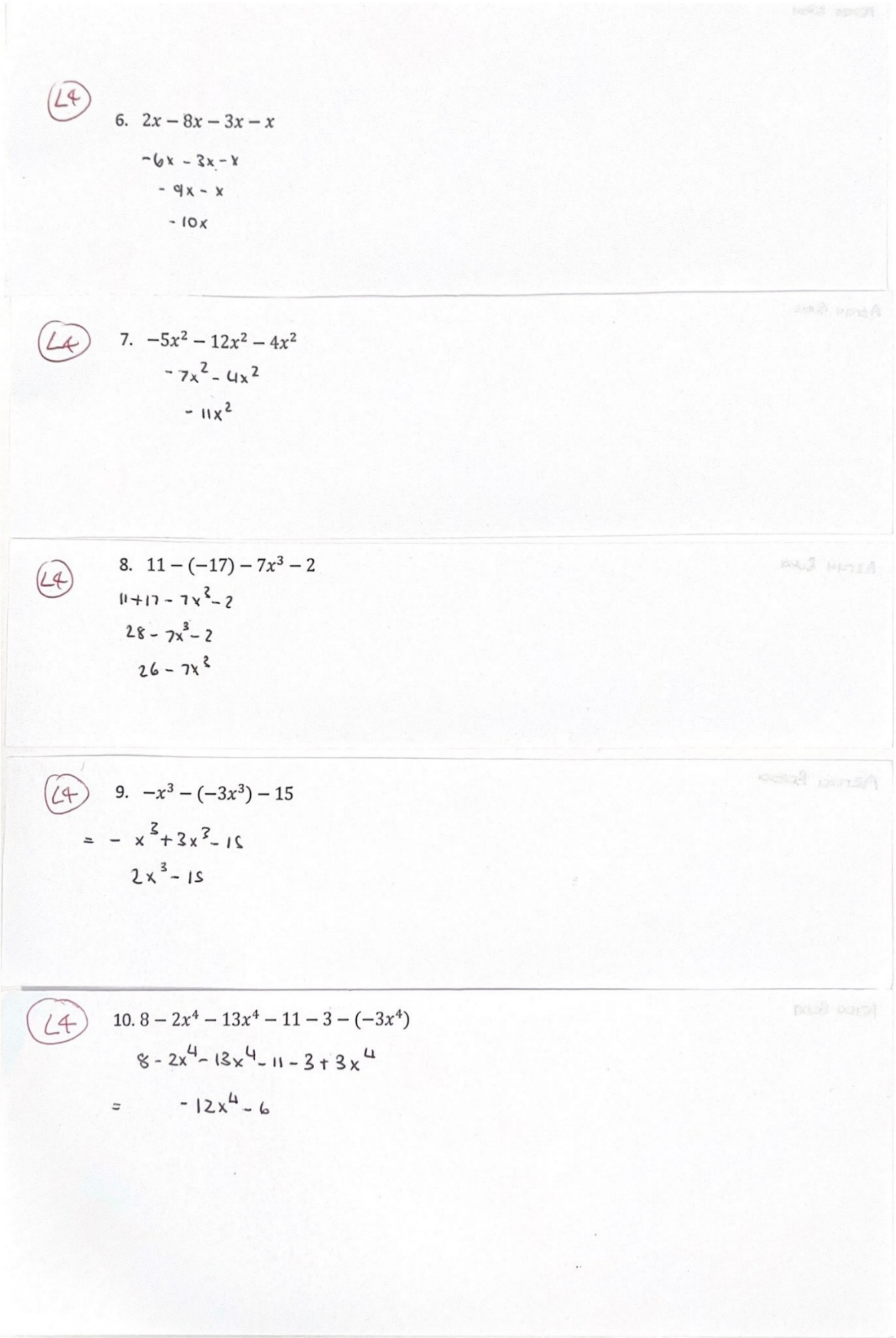
5. $(3x - 5) - (3x - 5)$

$(3x - 5) - (3x - 5)$

$(3x - 5)(-3x - 5)$

$-9x - 15x + 15x + 25$

$= -9x + 25$



L4

6. $2x - 8x - 3x - x$

$-6x - 3x - x$

$-9x - x$

$-10x$

L4

7. $-5x^2 - 12x^2 - 4x^2$

$-7x^2 - 4x^2$

$-11x^2$

L4

8. $11 - (-17) - 7x^3 - 2$

$11 + 17 - 7x^3 - 2$

$28 - 7x^3 - 2$

$26 - 7x^3$

L4

9. $-x^3 - (-3x^3) - 15$

$= -x^3 + 3x^3 - 15$

$2x^3 - 15$

L4

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$8 - 2x^4 - 13x^4 - 11 - 3 + 3x^4$

$= -12x^4 - 6$

L4

11. $1 - 16x - (-13) - y - 14$

$1 - 16x + 13 - y - 14$

$= -16x - y$

L4

12. $4xy - 5xy - 5 - (-11xy) - xy$

$4xy - 5xy - 5 + 11xy - xy$

$-1xy - 5 + 11xy - xy$

$= -5 + 11xy$

L4

13. $-14a - (-3b) - 2a - 7$

$-14a + 3b - 2a - 7$

$-16a + 3b - 7$

14. $3y - 7x - 8 - (-8x) - 3y$

L4

$3y - 7x - 8 + 8x - 3y$

$= 1x - 8$

L4

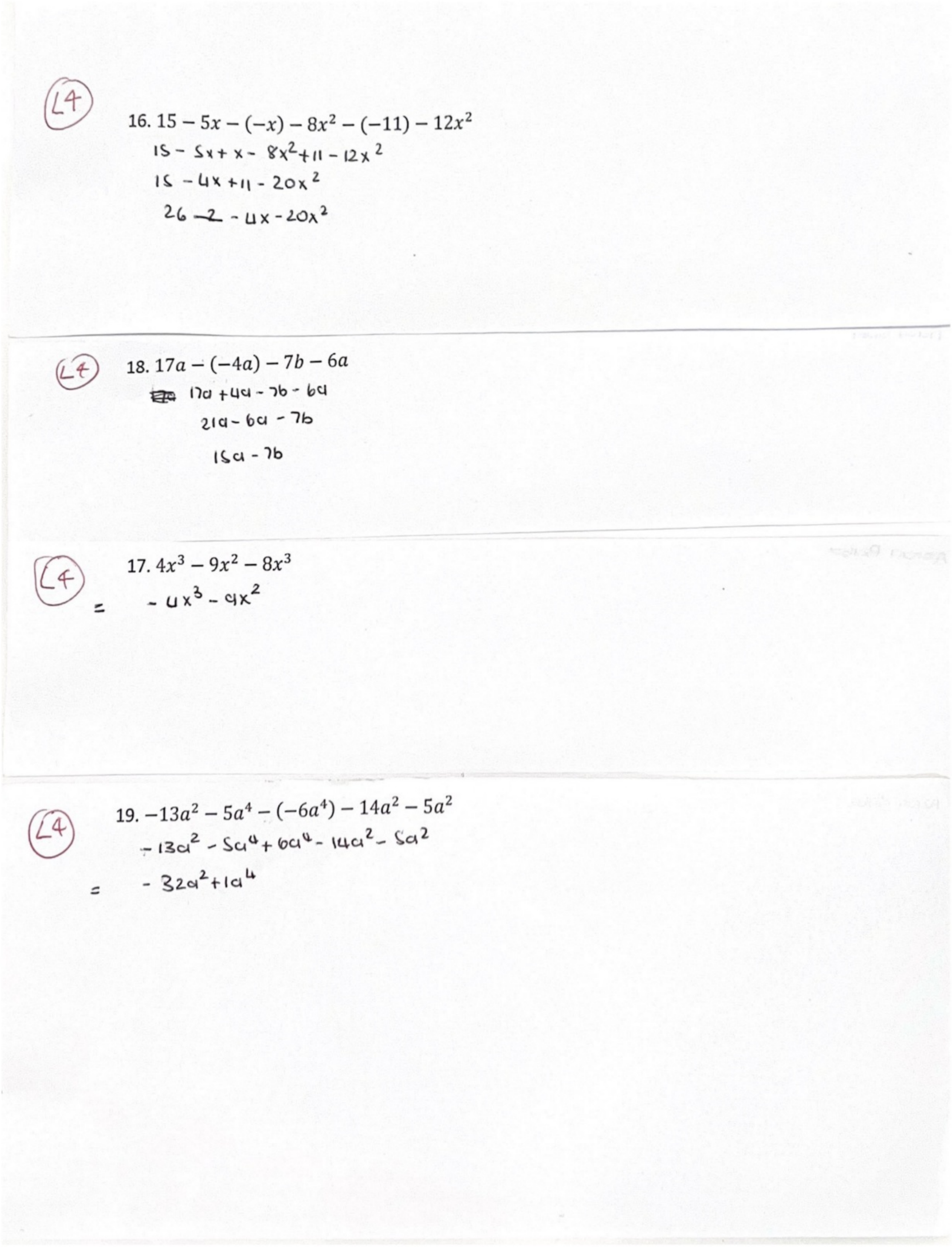
15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$13ab - 4ba - 13 - 2ba + 9 - 1$

~~13ab~~

$11ab - 2ba - 13 + 9 - 1$

$9ab + 3$



(L4)

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$15 - 5x + x - 8x^2 + 11 - 12x^2$

$15 - 4x + 11 - 20x^2$

$26 \underline{2} - 4x - 20x^2$

(L4)

18. $17a - (-4a) - 7b - 6a$

$17a + 4a - 7b - 6a$

$21a - 6a - 7b$

$15a - 7b$

(L4)

17. $4x^3 - 9x^2 - 8x^3$

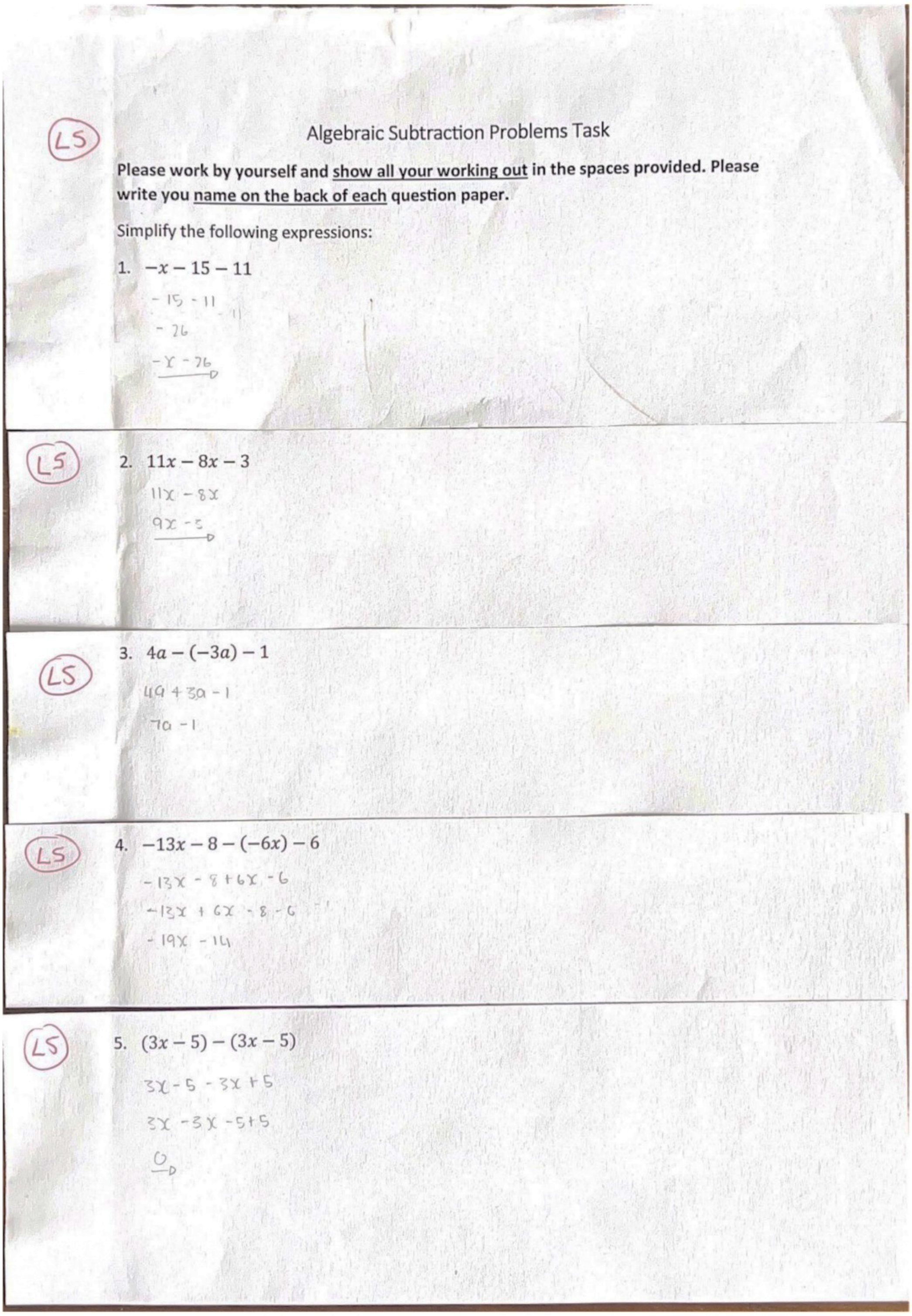
$= -4x^3 - 9x^2$

(L4)

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$-13a^2 - 5a^4 + 6a^4 - 14a^2 - 5a^2$

$= -32a^2 + 1a^4$



LS

Algebraic Subtraction Problems Task

Please work by yourself and show all your working out in the spaces provided. Please write you name on the back of each question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$-15 - 11$ -11

-26

$\underrightarrow{-x - 26}$

LS

2. $11x - 8x - 3$

$11x - 8x$

$\underrightarrow{9x - 3}$

LS

3. $4a - (-3a) - 1$

$4a + 3a - 1$

$7a - 1$

LS

4. $-13x - 8 - (-6x) - 6$

$-13x - 8 + 6x - 6$

$-13x + 6x - 8 - 6$

$-19x - 14$

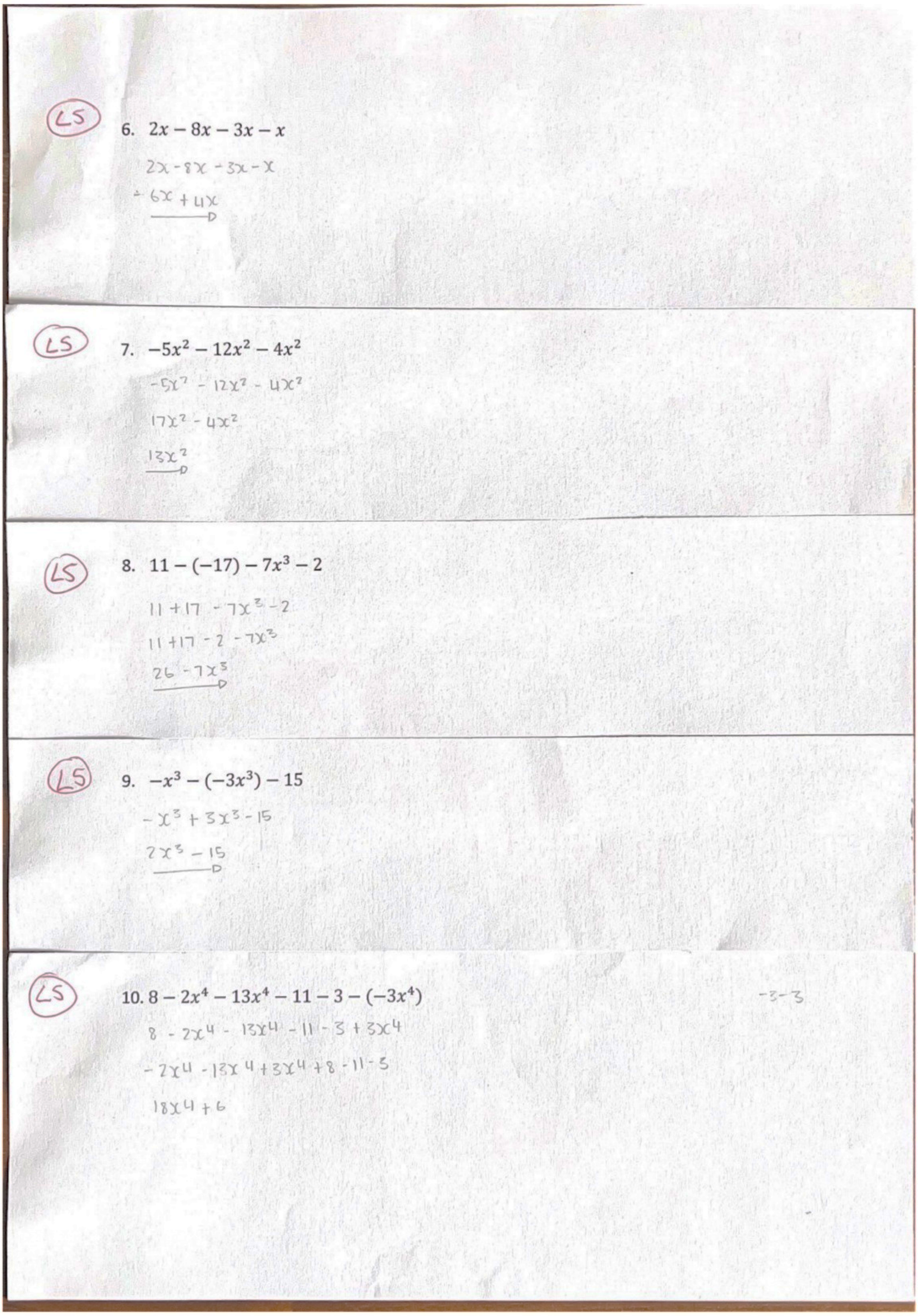
LS

5. $(3x - 5) - (3x - 5)$

$3x - 5 - 3x + 5$

$3x - 3x - 5 + 5$

$\underrightarrow{0}$



LS

6. $2x - 8x - 3x - x$

$2x - 8x - 3x - x$

$\underrightarrow{-6x + 4x}$

LS

7. $-5x^2 - 12x^2 - 4x^2$

$-5x^7 - 12x^2 - 4x^2$

$17x^2 - 4x^2$

$\underrightarrow{13x^2}$

LS

8. $11 - (-17) - 7x^3 - 2$

$11 + 17 - 7x^3 - 2$

$11 + 17 - 2 - 7x^3$

$\underrightarrow{26 - 7x^3}$

LS

9. $-x^3 - (-3x^3) - 15$

$-x^3 + 3x^3 - 15$

$\underrightarrow{2x^3 - 15}$

LS

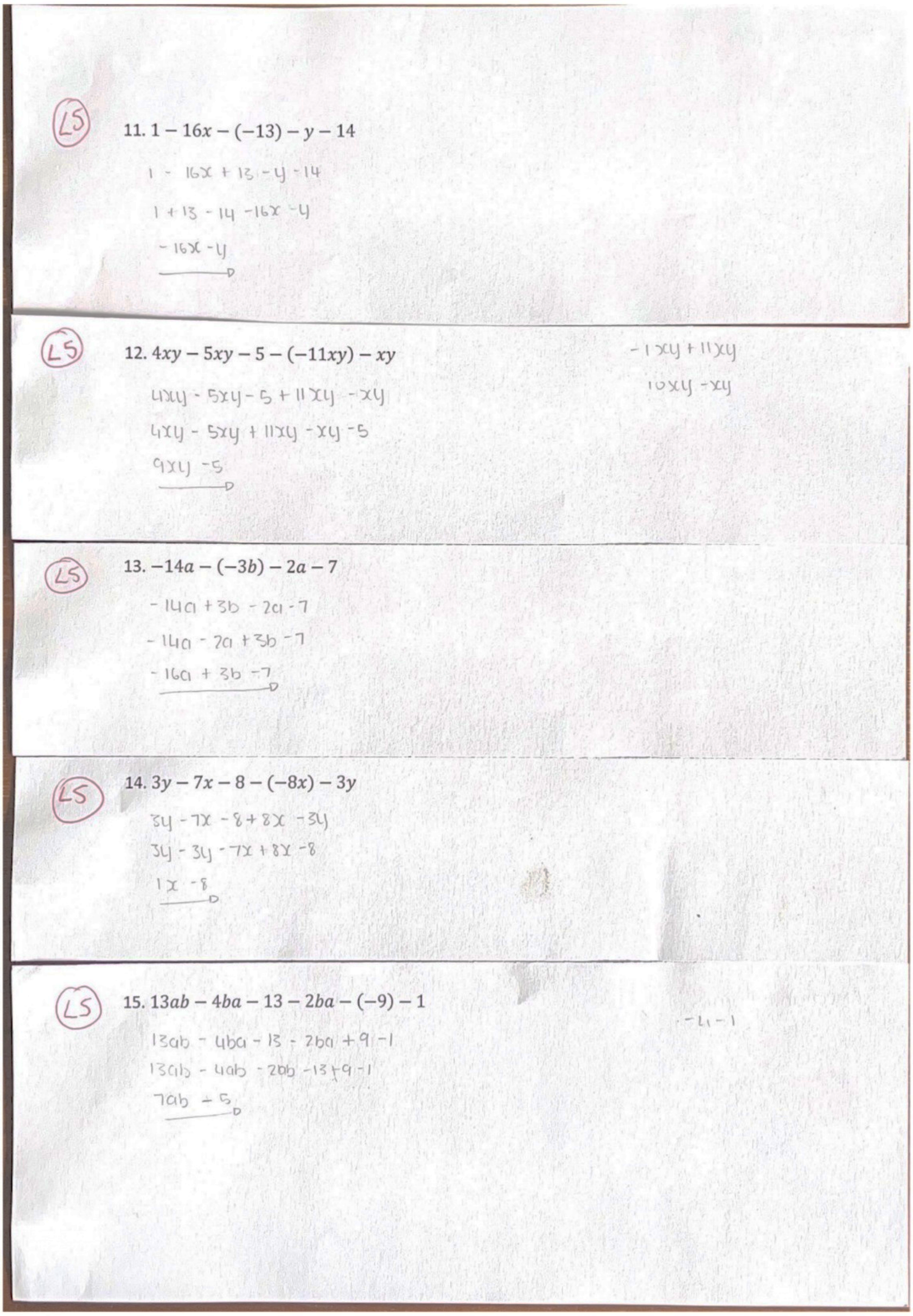
10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$-3 - 3$

$8 - 2x^4 - 13x^4 - 11 - 3 + 3x^4$

$-2x^4 - 13x^4 + 3x^4 + 8 - 11 - 3$

$18x^4 + 6$



LS

11. $1 - 16x - (-13) - y - 14$

$1 - 16x + 13 - y - 14$

$1 + 13 - 14 - 16x - y$

$\underrightarrow{-16x - y}$

LS

12. $4xy - 5xy - 5 - (-11xy) - xy$

$-1xy + 11xy$

$10xy - xy$

$4xy - 5xy - 5 + 11xy - xy$

$4xy - 5xy + 11xy - xy - 5$

$\underrightarrow{9xy - 5}$

LS

13. $-14a - (-3b) - 2a - 7$

$-14a + 3b - 2a - 7$

$-14a - 2a + 3b - 7$

$\underrightarrow{-16a + 3b - 7}$

LS

14. $3y - 7x - 8 - (-8x) - 3y$

$3y - 7x - 8 + 8x - 3y$

$3y - 3y - 7x + 8x - 8$

$\underrightarrow{1x - 8}$

LS

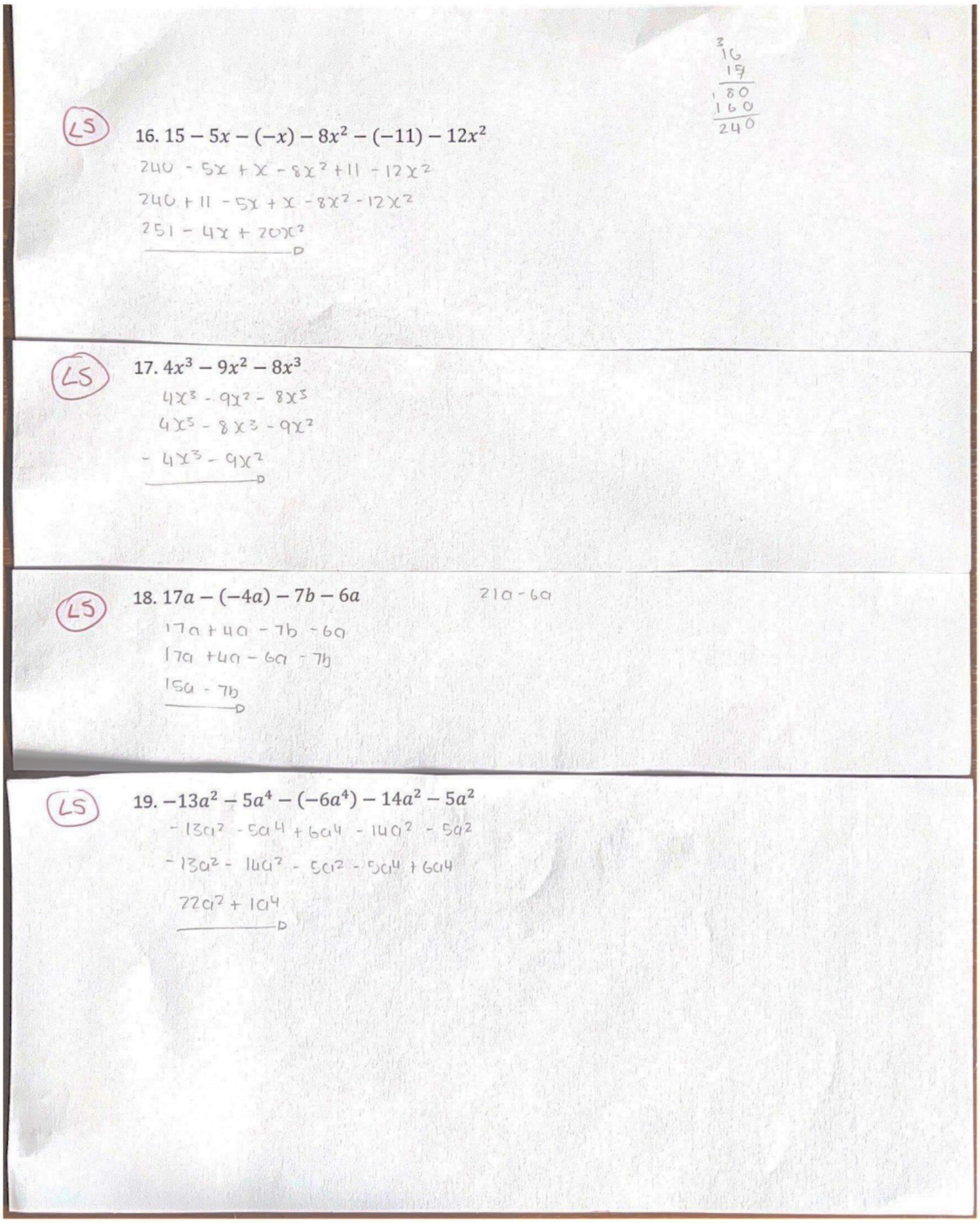
15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$-4 - 1$

$13ab - 4ba - 13 - 2ba + 9 - 1$

$13ab - 4ab - 2ab - 13 + 9 - 1$

$\underrightarrow{7ab + 5}$



$^{3}16$
15
80
160
240

LS

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$240 - 5x + x - 8x^2 + 11 - 12x^2$

$240 + 11 - 5x + x - 8x^2 - 12x^2$

$251 - 4x + 20x^2$

$\longrightarrow$

LS

17. $4x^3 - 9x^2 - 8x^3$

$4x^3 - 9x^2 - 8x^3$

$4x^3 - 8x^3 - 9x^2$

$-4x^3 - 9x^2$

$\longrightarrow$

LS

18. $17a - (-4a) - 7b - 6a$

$21a - 6a$

$17a + 4a - 7b - 6a$

$17a + 4a - 6a - 7b$

$15a - 7b$

$\longrightarrow$

LS

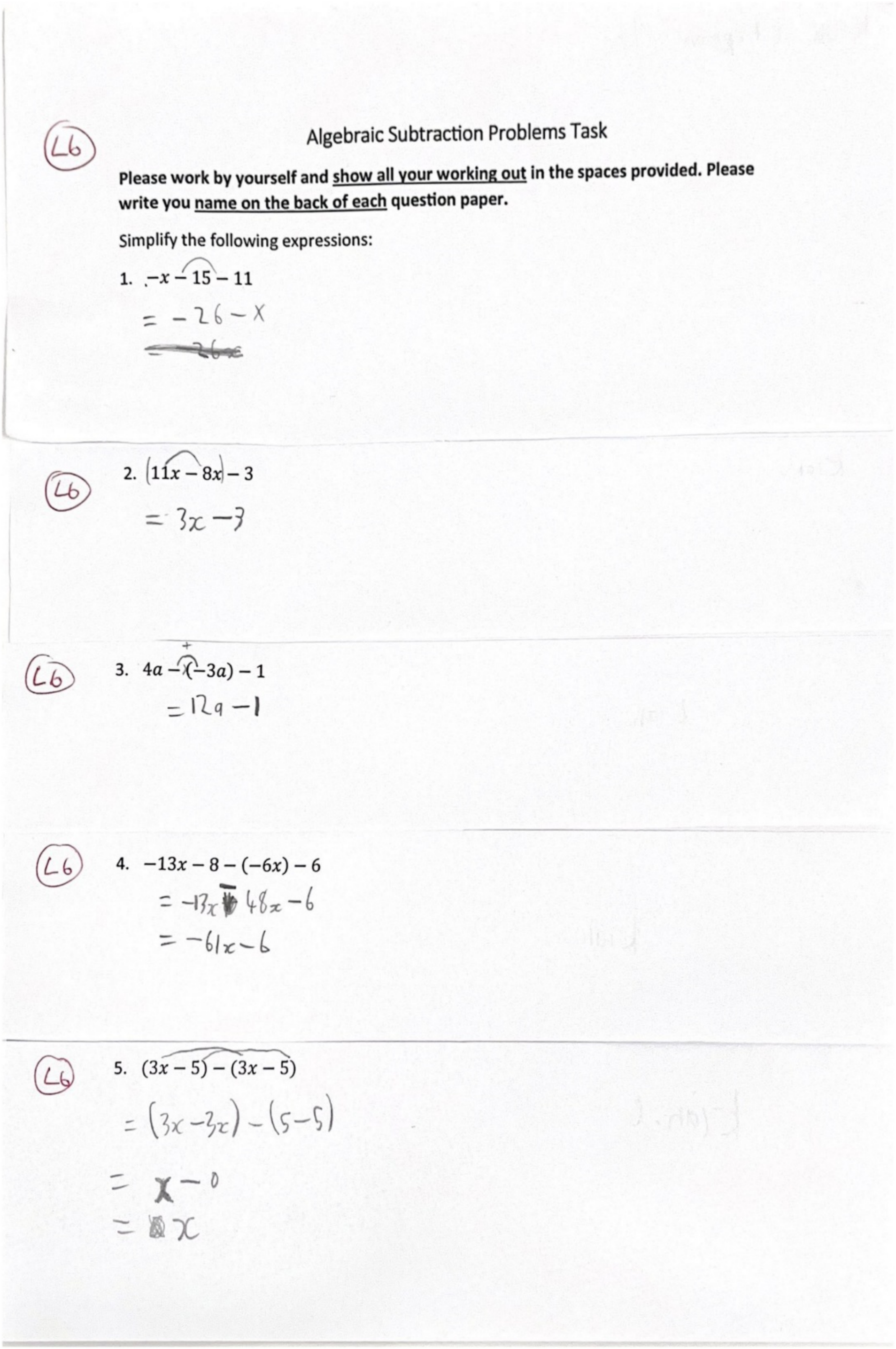
19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$-13a^2 - 5a^4 + 6a^4 - 14a^2 - 5a^2$

$-13a^2 - 14a^2 - 5a^2 - 5a^4 + 6a^4$

$72a^2 + 1a^4$

$\longrightarrow$



L6

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$= -26 - x$

~~$= 26x$~~

L6

2. $(11x - 8x) - 3$

$= 3x - 3$

L6

3. $4a - \overset{+}{(}-3a) - 1$

$= 12a - 1$

L6

4. $-13x - 8 - (-6x) - 6$

$= -13x \; 48x - 6$

$= -61x - 6$

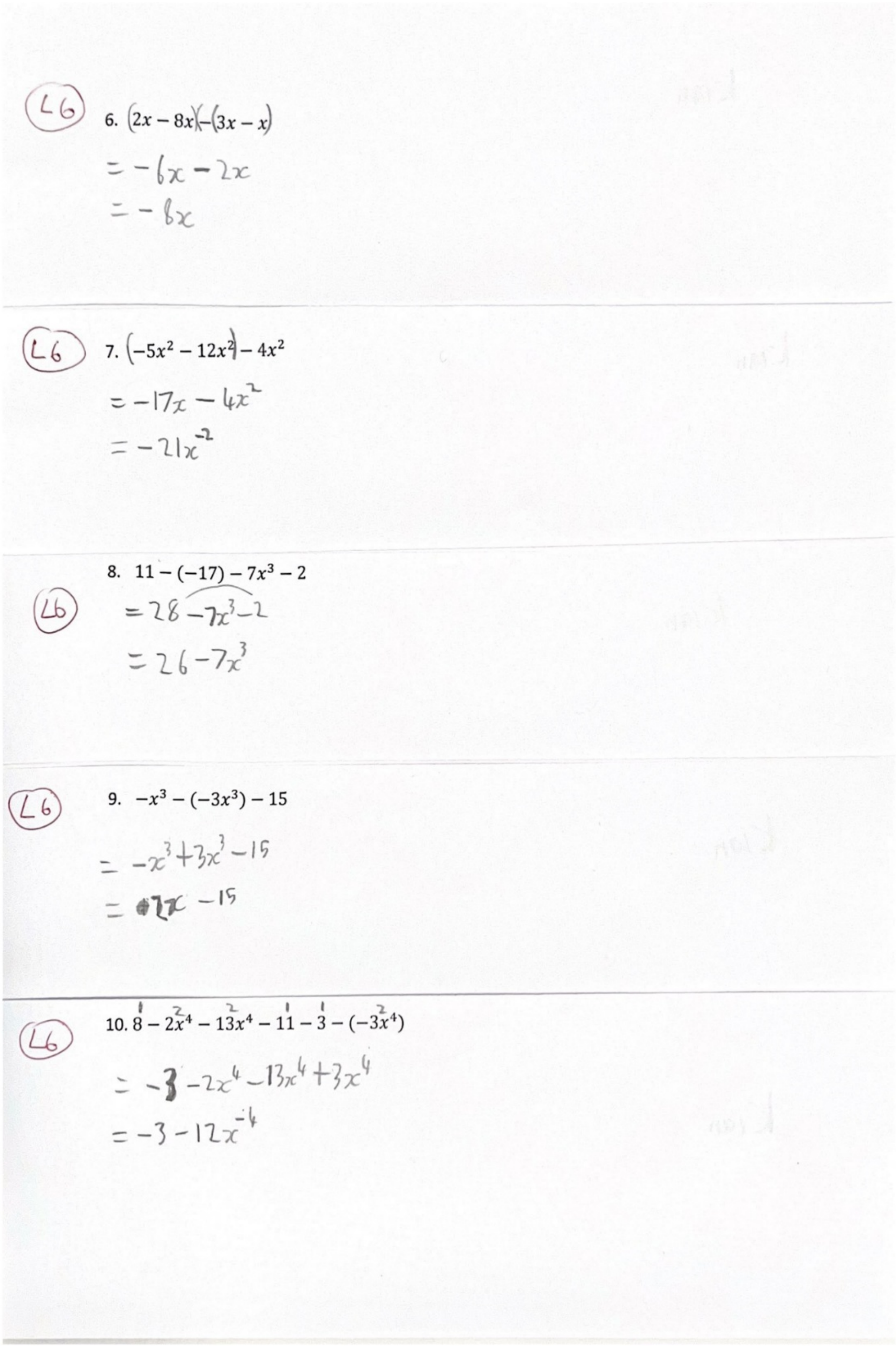
L6

5. $(3x - 5) - (3x - 5)$

$= (3x - 3x) - (5 - 5)$

$= x - 0$

$= x$



L6

6. $(2x - 8x) - (3x - x)$

$= -6x - 2x$

$= -8x$

L6

7. $(-5x^2 - 12x^2) - 4x^2$

$= -17x - 4x^2$

$= -21x^{-2}$

8. $11 - (-17) - 7x^3 - 2$

L6

$= 28 \overset{\frown}{-7x^3 - 2}$

$= 26 - 7x^3$

L6

9. $-x^3 - (-3x^3) - 15$

$= -x^3 + 3x^3 - 15$

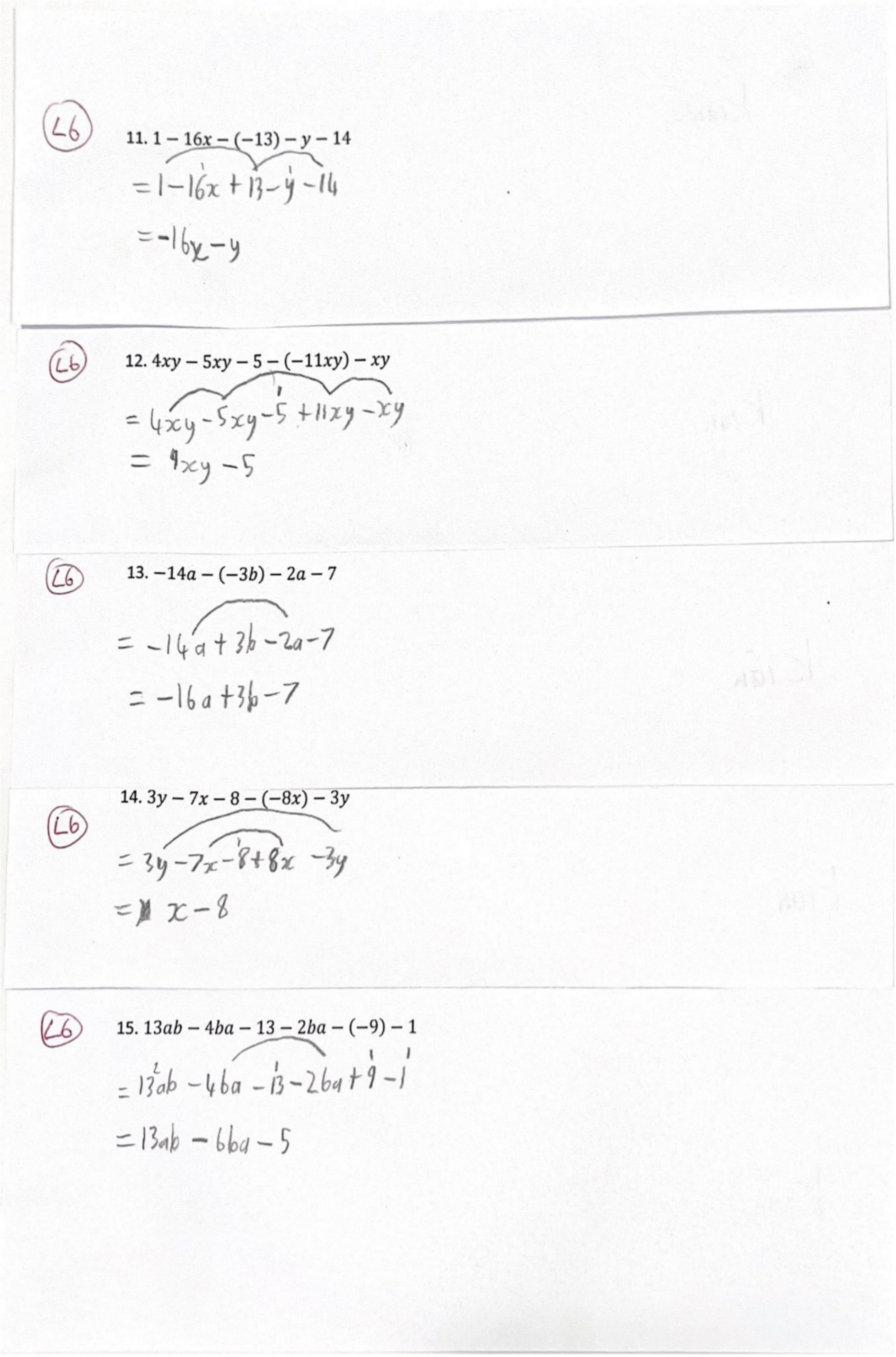
$= 2x - 15$

L6

10. $\overset{1}{8} - \overset{2}{2x^4} - \overset{2}{13x^4} - \overset{1}{11} - \overset{1}{3} - (\overset{2}{-3x^4})$

$= -3 - 2x^4 - 13x^4 + 3x^4$

$= -3 - 12x^{-4}$



L6

11. $1 - 16x - (-13) - y - 14$

$= 1 - 16x + 13 - y - 14$

$= -16x - y$

L6

12. $4xy - 5xy - 5 - (-11xy) - xy$

$= 4xy - 5xy - 5 + 11xy - xy$

$= 9xy - 5$

L6

13. $-14a - (-3b) - 2a - 7$

$= -14a + 3b - 2a - 7$

$= -16a + 3b - 7$

14. $3y - 7x - 8 - (-8x) - 3y$

L6

$= 3y - 7x - 8 + 8x - 3y$

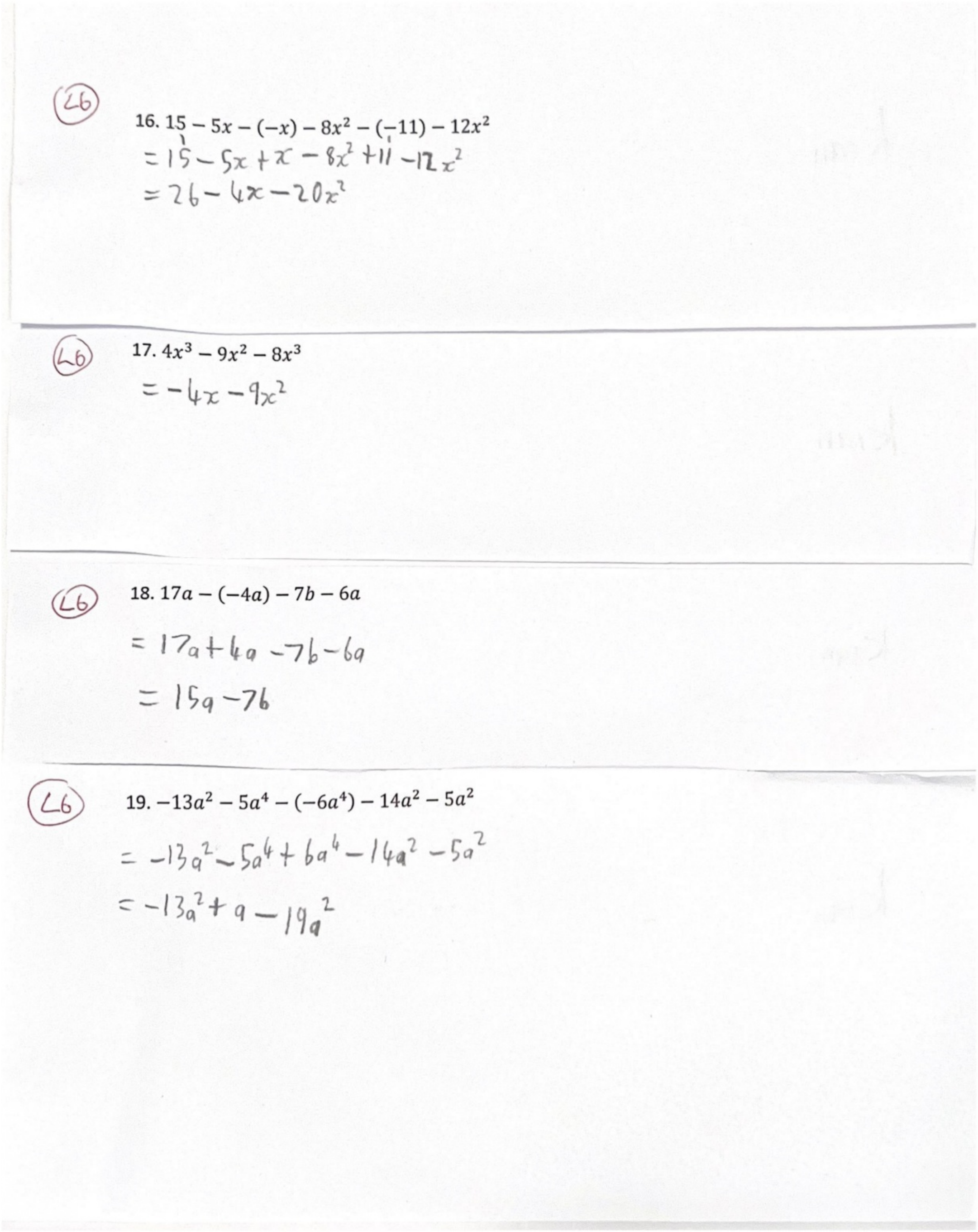
$= x - 8$

L6

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= 13ab - 4ba - 13 - 2ba + 9 - 1$

$= 13ab - 6ba - 5$



L6

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$= 15 - 5x + x - 8x^2 + 11 - 12x^2$

$= 26 - 4x - 20x^2$

L6

17. $4x^3 - 9x^2 - 8x^3$

$= -4x - 9x^2$

L6

18. $17a - (-4a) - 7b - 6a$

$= 17a + 4a - 7b - 6a$

$= 15a - 7b$

L6

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$= -13a^2 - 5a^4 + 6a^4 - 14a^2 - 5a^2$

$= -13a^2 + a - 19a^2$

X

L7

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$-x - 4$ ->

= [illegible]

L7

2. $\underline{11}x - \underline{8}x - \underline{3}$

✓

$3x - 3$ ->

L7

3. $4a - (-3a) - 1$

X

$12a^2 - 1$ ->

L7

5. $(3x - 5) - (3x - 5)$

X

[illegible]

$(3x - 5)^2$

$-8x - 3x + 2x - 1x$

X

L7

6. $2x - 8x - 3x - 1x$

$= -5x + 2x - 1x$

$= 7x - 1x$

$= 6x$

L7

7. $-5x^2 - 12x^2 - 4x^2$

X

$= -12x^2 - 5x^2 - 4x^2$

$= 7x^2 - 4x^2$

$= 3x^2$

L7

8. $11 - (-17) - 7x^3 - 2$

✓

$= 11 + 17 - 7x^3 - 2$

$= 28 - 7x^3 - 2$

$= 26 - 7x^3$

L7

9. $-x^3 - (-3x^3) - 15$

X

$= -x^3 + 3x^3 - 15$

$= -2x^3 - 15$

L7

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

X

$= -2x^4 - 13x^4 - 11 - 3 + 3x^4$

$= -15x^4 - 11 - 3 + 3x^4$

$= -12x^4 - 11 - 3$

$= -12x^4 - 14$

✗

L7

11. $1 - 16x - (-13) - y - 14$

$-16x + 13 - y - 14$

$-16 - 1 - y$

✓

L7

12. $\underline{4xy} - \underline{5xy} - \overline{5} - (-1\underline{1xy}) - x\underline{y}$

$4xy - 5xy - 5 + 11xy \; 9xy$

$= -1xy - 5 + 10xy$

$= 9xy - 5$

✗

L7

13. $-\underline{14a} - \overline{(-3b)} - \underline{2a} - \underline{7}$

$-14a + 3b - 2a - 7$

$= 12a + 3b - 7$

✓

L7

14. $\cancel{3y} - \overset{\sim}{7}x - \overline{8} - (\overset{\sim}{-8}x) - \cancel{3y}$

$= -7x - 8 + 8x$

$= 1x - 8$

✗

L7

15. $\underline{13ab} - \underline{4ba} - \overline{13} - \underline{2ba} - (\overline{-9}) - \overline{1}$

$9ab - 13 - 2ba + 9 - 1$

$7ab - 4 - 1$

$7ab - 3$

✗

L7

16. $\underline{15} - \overline{5x} - (\overline{-x}) - \underset{\sim}{8x^2} - \underline{(-11)} - \underset{\sim}{12x^2}$

$15 - 5x + x - 8x^2 + 11 - 12x^2$

$-4x - 8x^2 + 11 - 12x^2$

$-4x + 11 - 4x^2$

L7

17. $4x^3 - 9x^2 - 8x^3$

✓

$-4x^3 - 9x^2$

L7

18. $17a - (-4a) - 7b - 6a$

✓

$17a + 4a - 7b - 6a$

$21a + 7b - 6a$

$15a - 7b$

L7

19. $-\underline{13a^2} - \overset{\frown}{5a^4} - (\overset{\frown}{-6a^4}) - \underline{14a^2} - \underline{5a^2}$

✗

$-13a^2 - 5a^4 + -6a^4 - 14a^2 - 5a^2$

$-1a^2 - 1a^4 - 5a^2$

$= 4a^2 - 1a^4$

L8

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$= -x - 26$

=

L8

2. $11x - 8x - 3$

$= 3x - 3$

L8

3. $4a - (-3a) - 1$

$= 4a + 3a - 1$

$= 7a - 1$

4. $-13x - 8 - (-6x) - 6$

L8

$= \underline{-13x} - 8 + \underline{6x} - 6$

$= -7x - 14$

L8

5. $(3x - 5) - (3x - 5)$

$= (3x - 5) - 3x + 5$

$= -6x^2 + 15x + 5$

L8

6. $2x - 8x - 3x - x$

$= 2x - 11x - x$

$= 2x - 12x$

$= +10x$

L8

7. $\underline{-5}x^2 - 1\underline{2x}^2 - \underline{4x}^2$

$= -27x$

L8

8. $11 - (-17) - 7x^3 - 2$

$= \underline{11} + \underline{17} - \utilde{7x^3} - \underline{2}$

$= 26 - 7x^3$

L8

9. $-x^3 - (-3x^3) - 15$

$= -x^3 + 3x^3 - 15$

$= 2x^3 - 15$

L8

10. $\underline{8} - 2x^4 - 13x^4 - \underline{11} - \underline{3} - (-3x^4)$

$= \underline{8} - \utilde{2x^4} - \utilde{13x^4} - \underline{11} - \underline{3} + \utilde{3x^4}$

$= -4 - 12x^4$

✓

L8

11. $1 - 16x - (-13) - y - 14$

$= \underline{1} \utilde{-16x} + \underline{13} - \underline{\underline{y}} - \underline{14}$

$= -16x - y$

✓

L8

12. $4xy - 5xy - 5 - (-11xy) - xy$

$= \underline{4xy} - \underline{5xy} - \utilde{5} + \underline{11xy} - \underline{xy}$ 6

$= \utilde{15xy} - \utilde{6xy} - \underline{5}$

$= 9xy - 5$

✓

L8

13. $-14a - (-3b) - 2a - 7$

$= \underline{-14a} + \utilde{3b} - \underline{2a} - \utilde{7}$

$= -16a + 3b - 7$

✓

L8

14. $3y - 7x - 8 - (-8x) - 3y$

$= \cancel{3y} - \utilde{7x} - \underline{8} + \utilde{8x} - \cancel{3y}$

$= 1x - 8$

✓

L8

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= \utilde{13ab} - \utilde{4ba} - \underline{13} - \utilde{2ba} + \underline{9} - \underline{1}$

$= 7ab - 5$

✓

L8

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$= 15 - 5x + x - 8x^2 + 11 - 12x^2$

$= 26 - 4x - 20x^2$

L8

17. $4x^3 - 9x^2 - 8x^3$ ✓

$= -4x^3 - 9x^2$

L8

18. $17a - (-4a) - 7b - 6a$ ✓

$= 17a + 4a - 7b - 6a$

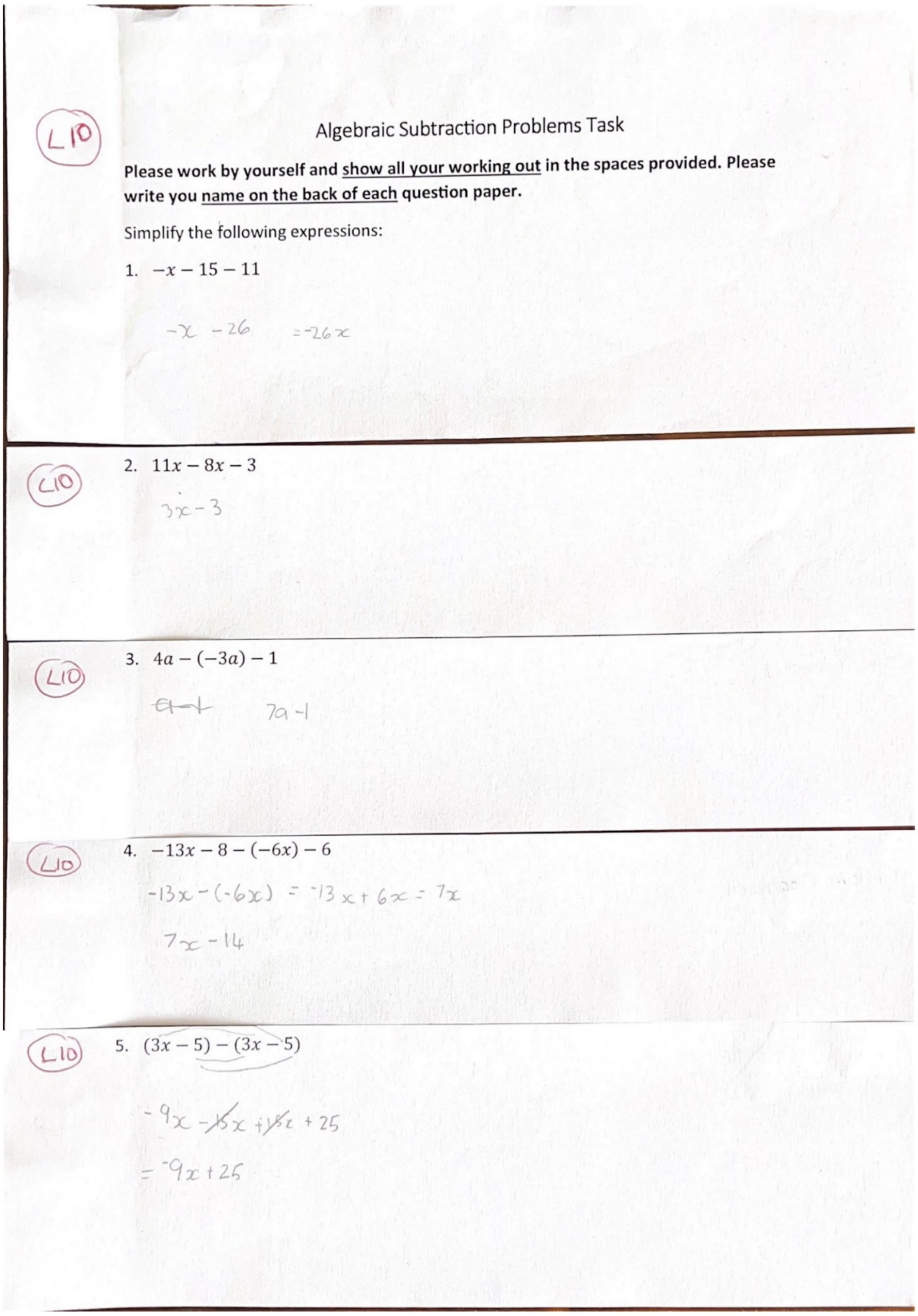
$= 15a - 7b$

L8

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$ ×

$= -13a^2 - 5a^4 + 6a^4 - 14a^2 - 5a^2$

$= -32a^2 - 11a^4$



L10

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$-x \ -26 \quad = -26x$

L10

2. $11x - 8x - 3$

$3x - 3$

L10

3. $4a - (-3a) - 1$

~~a-1~~ $\quad 7a - 1$

L10

4. $-13x - 8 - (-6x) - 6$

$-13x - (-6x) = {}^{-}13x + 6x = 7x$

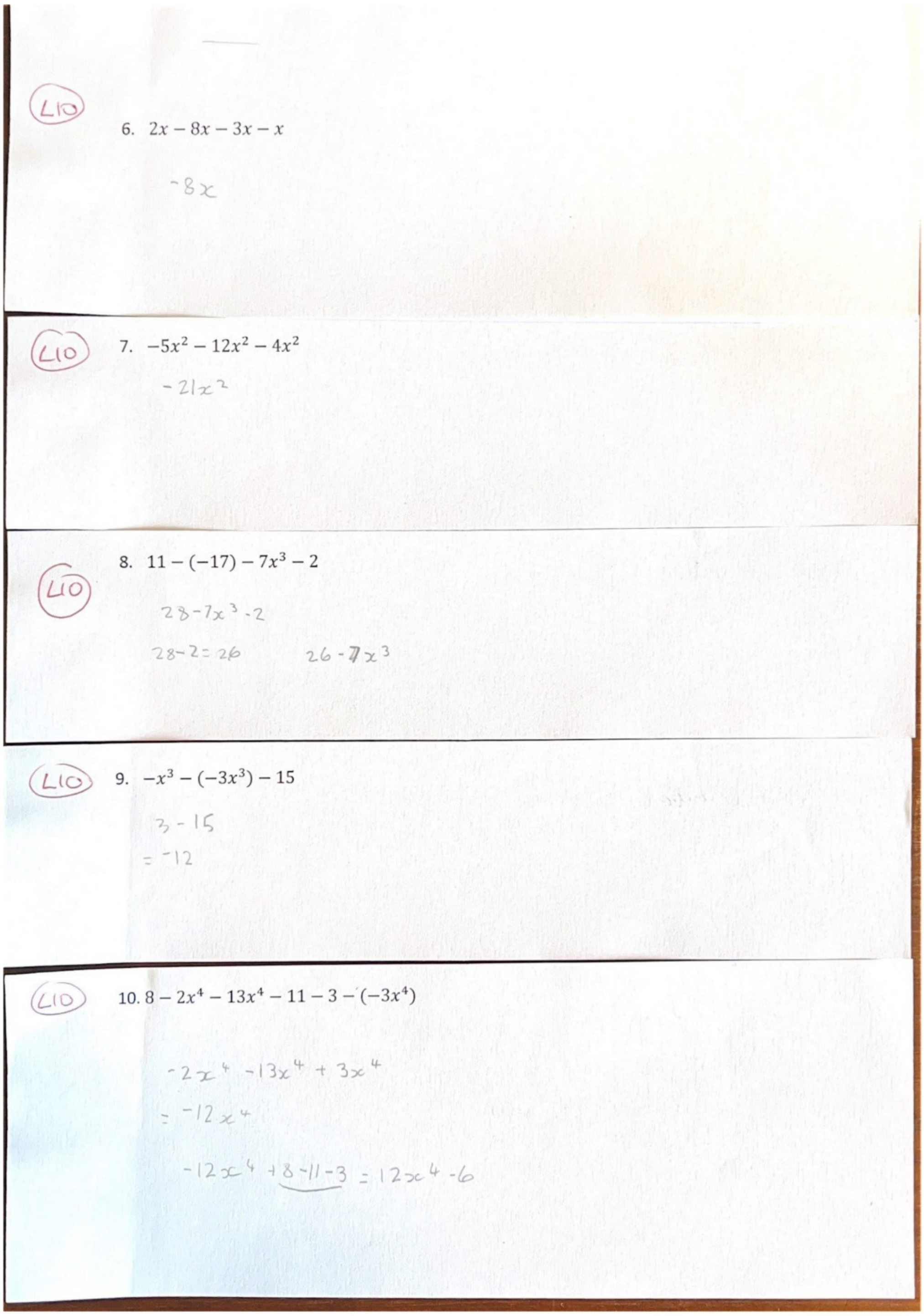
$7x - 14$

L10

5. $(3x - 5) - (3x - 5)$

$-9x - \cancel{15}x + \cancel{15}x + 25$

$= {}^{-}9x + 25$



L10

6. $2x - 8x - 3x - x$

$^{-}8x$

L10

7. $-5x^2 - 12x^2 - 4x^2$

$-21x^2$

L10

8. $11 - (-17) - 7x^3 - 2$

$28 - 7x^3 - 2$

$28 - 2 = 26$ $26 - 7x^3$

L10

9. $-x^3 - (-3x^3) - 15$

$3 - 15$

$= {}^{-}12$

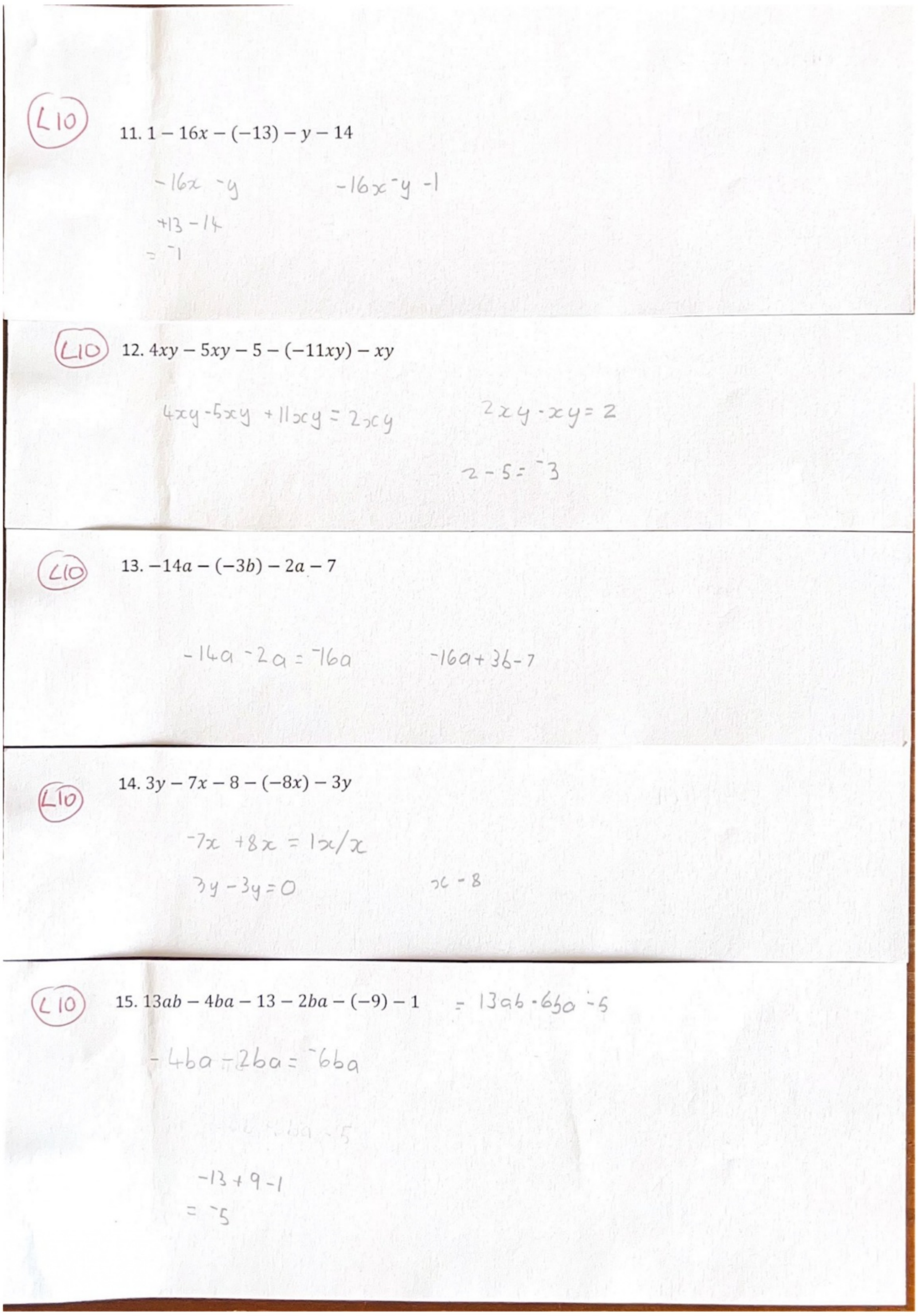
L10

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$-2x^4 - 13x^4 + 3x^4$

$= {}^{-}12x^4$

$-12x^4 + \underbrace{8 - 11 - 3} = 12x^4 - 6$



L10

11. $1 - 16x - (-13) - y - 14$

$-16x\ {}^{-}y$ $\quad$ $-16x\,{}^{-}y\ -1$

$+13-14$

$= {}^{-}1$

L10

12. $4xy - 5xy - 5 - (-11xy) - xy$

$4xy-5xy\ +11xy = 2xy$ $\quad$ $2xy \cdot xy = 2$

$2-5 = {}^{-}3$

L10

13. $-14a - (-3b) - 2a - 7$

$-14a\ {}^{-}2a = {}^{-}16a$ $\quad$ ${}^{-}16a + 3b - 7$

L10

14. $3y - 7x - 8 - (-8x) - 3y$

${}^{-}7x\ +8x = 1x/x$

$3y - 3y = 0$ $\quad$ $x - 8$

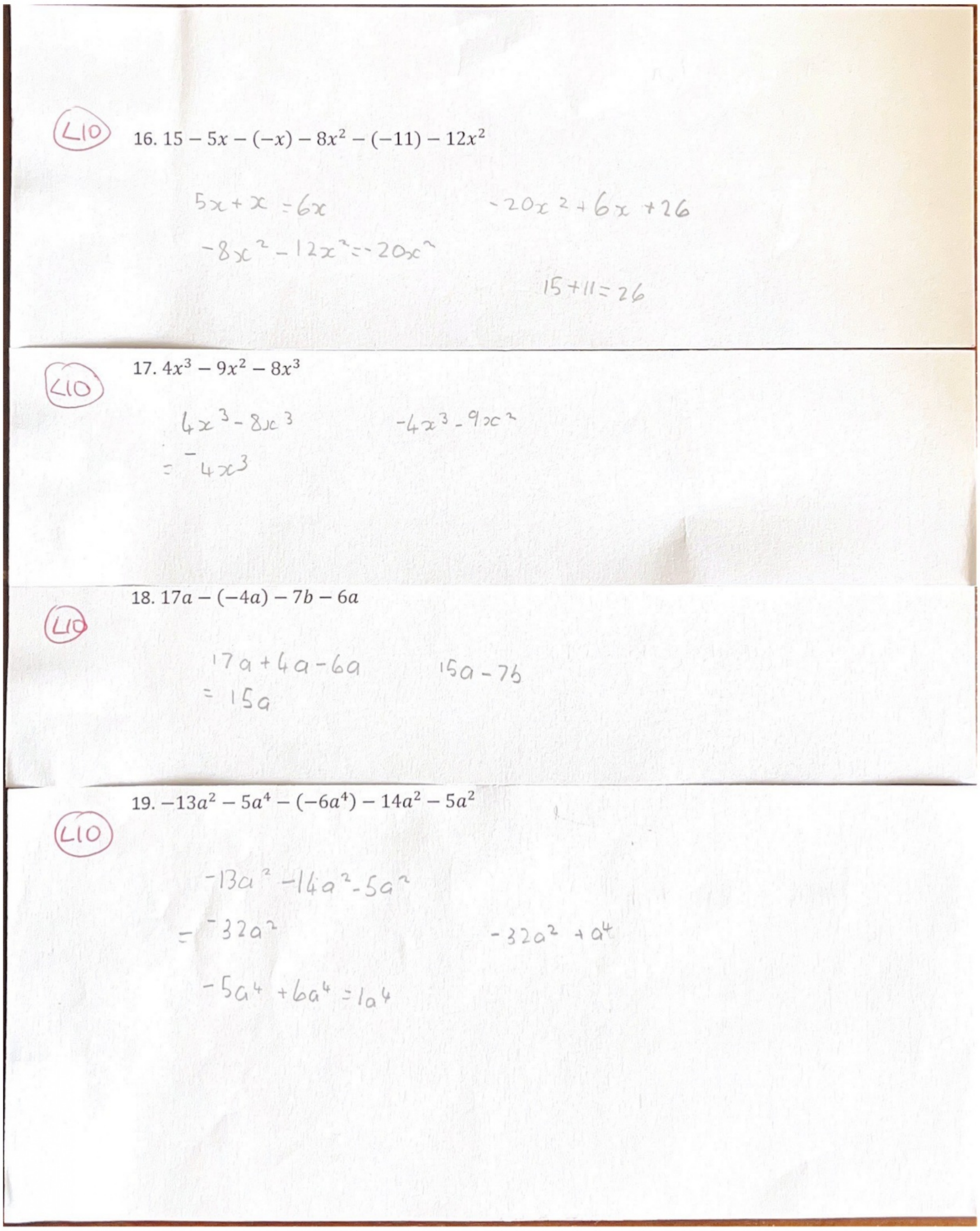
L10

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$ $\quad$ $= 13ab \cdot 6ba\ {}^{-}5$

$-4ba - 2ba = {}^{-}6ba$

$-13 + 9 - 1$

$= {}^{-}5$



L10

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$5x + x = 6x$

$-8x^2 - 12x^2 = -20x^2$

$-20x^2 + 6x + 26$

$15 + 11 = 26$

L10

17. $4x^3 - 9x^2 - 8x^3$

$4x^3 - 8x^3$

$= {}^{-}4x^3$

$-4x^3 - 9x^2$

L10

18. $17a - (-4a) - 7b - 6a$

$17a + 4a - 6a$

$= 15a$

$15a - 7b$

L10

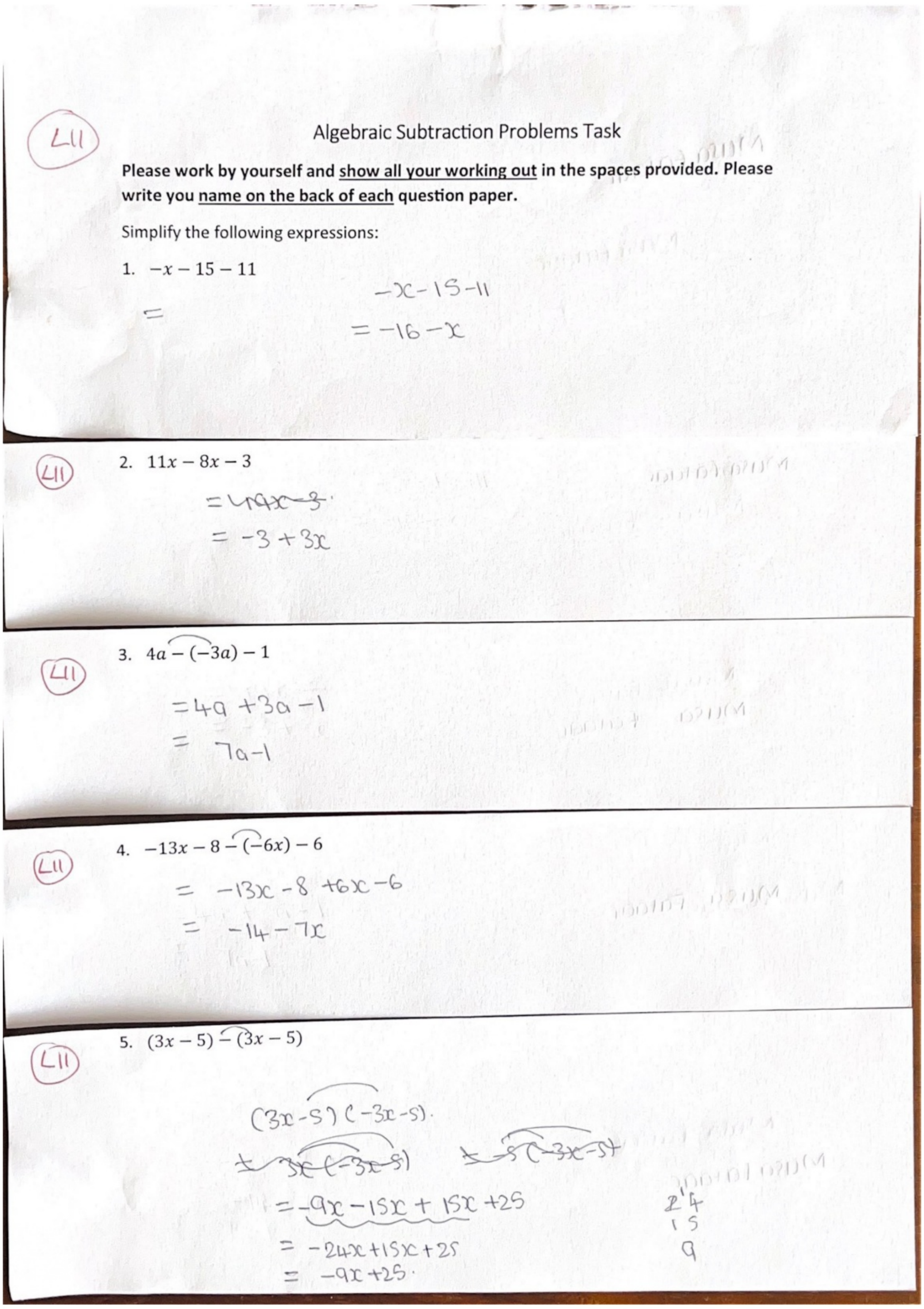
19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$-13a^2 - 14a^2 - 5a^2$

$= {}^{-}32a^2$

$-32a^2 + a^4$

$-5a^4 + 6a^4 = 1a^4$



L11

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

=

$-x-15-11$

$= -16 - x$

L11

2. $11x - 8x - 3$

$=$ ~~[illegible]x-3~~

$= -3 + 3x$

L11

3. $4a - (-3a) - 1$

$= 4a + 3a - 1$

$= 7a - 1$

L11

4. $-13x - 8 - (-6x) - 6$

$= -13x - 8 + 6x - 6$

$= -14 - 7x$

L11

5. $(3x - 5) - (3x - 5)$

$(3x-5)(-3x-5).$

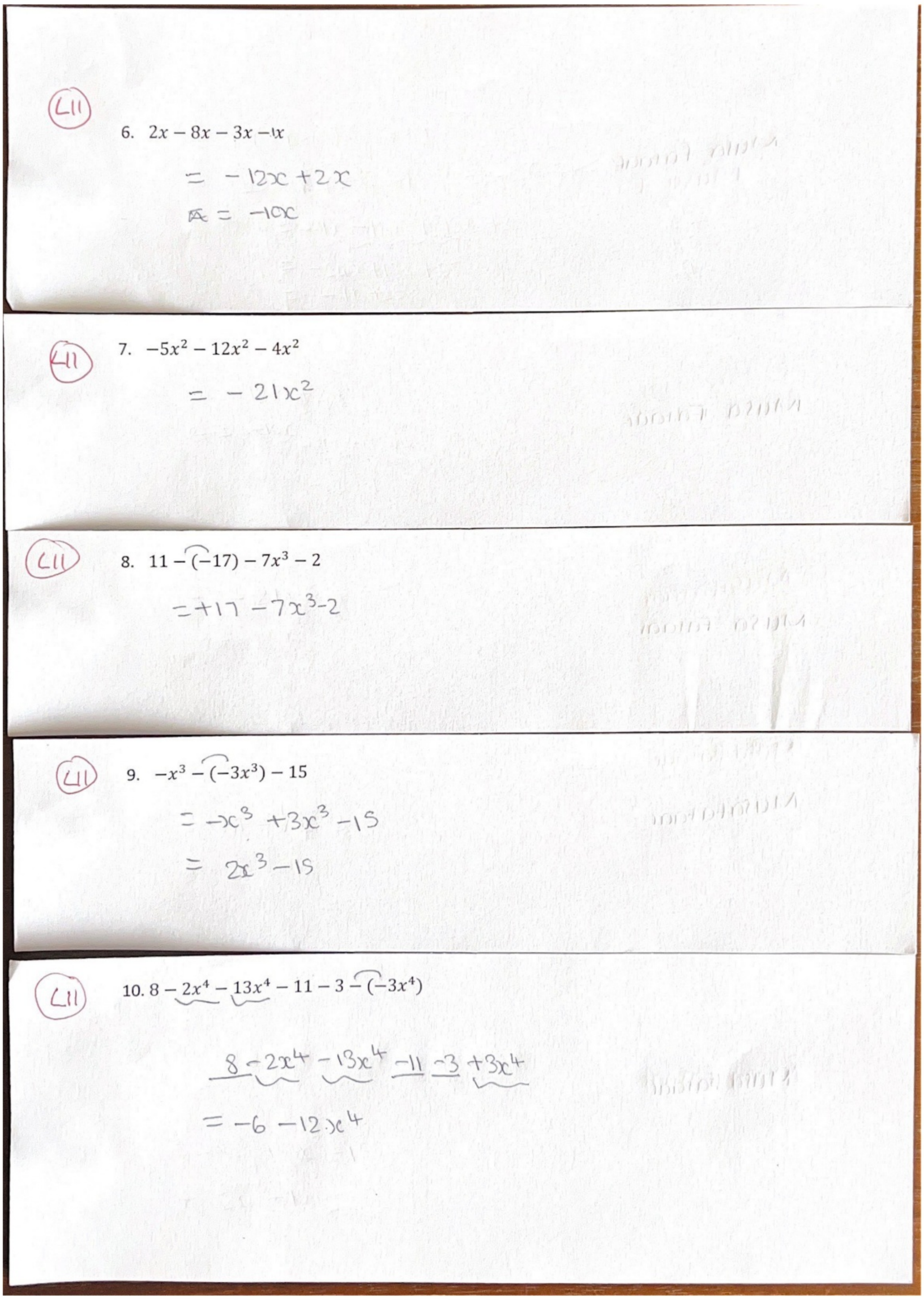
~~= 3x(-3x-5)~~ ~~= -5(-3x-5)~~

$= \underbrace{-9x - 15x} + 15x + 25$

$= -24x + 15x + 25$

$= -9x + 25.$

24
15
9



L11

6. $2x - 8x - 3x - 1x$

$= -12x + 2x$

~~A~~ $= -10x$

L11

7. $-5x^2 - 12x^2 - 4x^2$

$= -21x^2$

L11

8. $11 - \overset{\frown}{(-17)} - 7x^3 - 2$

$= +17 - 7x^3 - 2$

L11

9. $-x^3 - \overset{\frown}{(-3x^3)} - 15$

$= -x^3 + 3x^3 - 15$

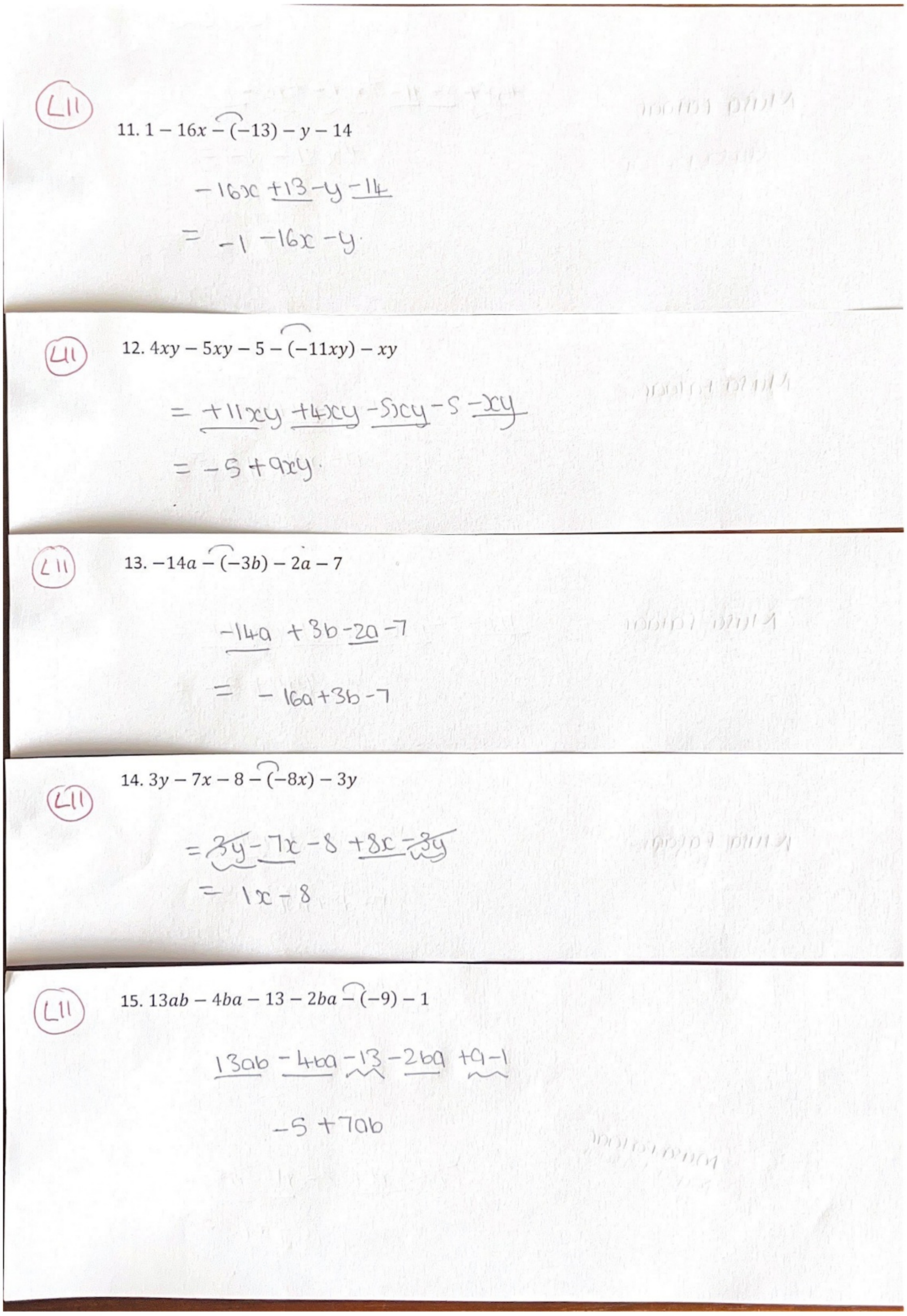
$= 2x^3 - 15$

L11

10. $8 - \underbrace{2x^4} - \underbrace{13x^4} - 11 - 3 - \overset{\frown}{(-3x^4)}$

$\underline{8} \underbrace{-2x^4} \underbrace{-13x^4} \underline{-11} \underline{-3} \underbrace{+3x^4}$

$= -6 - 12x^4$



L11

11. $1 - 16x - (-13) - y - 14$

$-16x + 13 - y - 14$

$= -1 - 16x - y$

L11

12. $4xy - 5xy - 5 - (-11xy) - xy$

$= +11xy + 4xy - 5xy - 5 - xy$

$= -5 + 9xy$

L11

13. $-14a - (-3b) - 2a - 7$

$-14a + 3b - 2a - 7$

$= -16a + 3b - 7$

L11

14. $3y - 7x - 8 - (-8x) - 3y$

$= 3y - 7x - 8 + 8x - 3y$

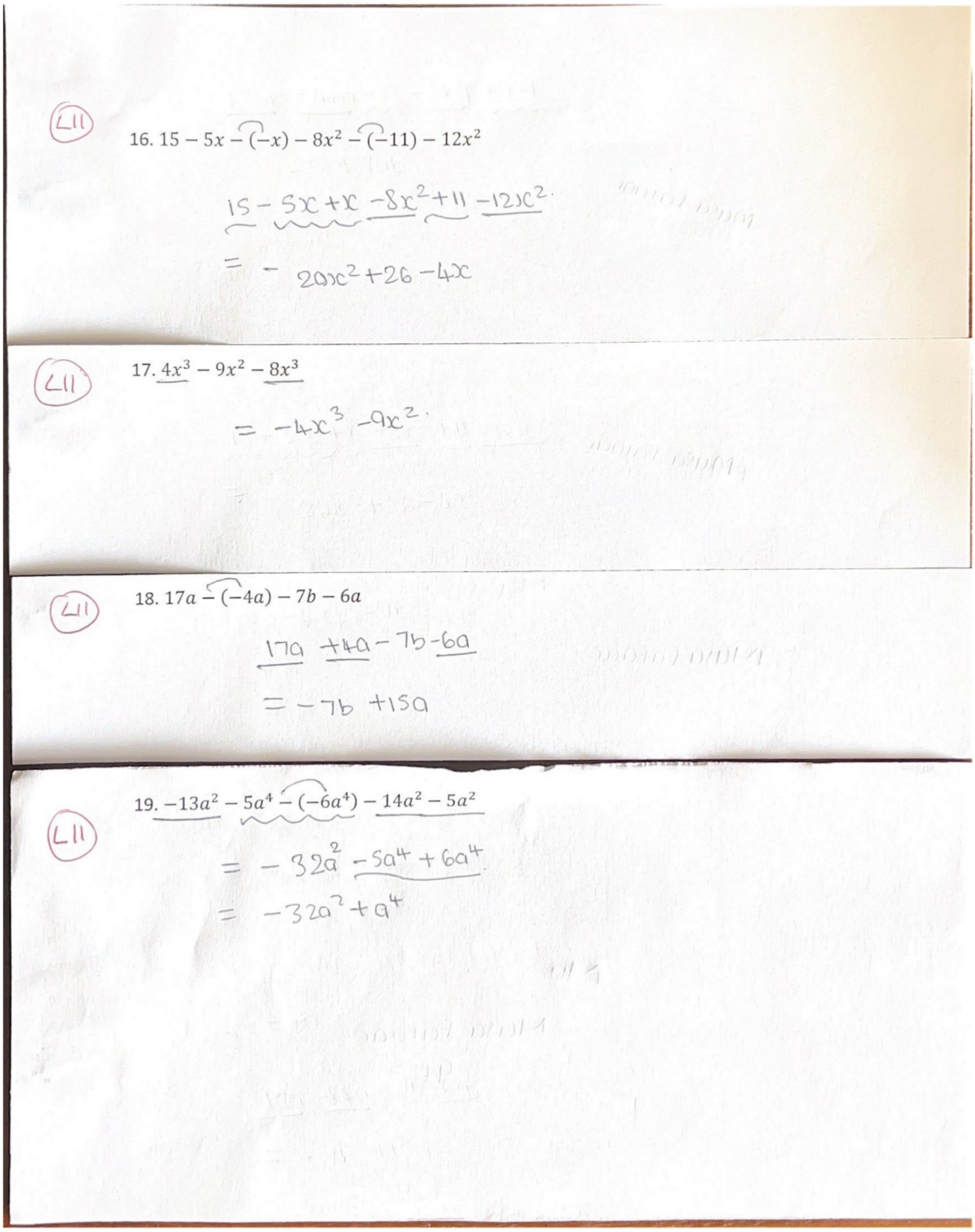
$= 1x - 8$

L11

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$13ab - 4ba - 13 - 2ba + 9 - 1$

$-5 + 7ab$



L11

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$15 - 5x + x - 8x^2 + 11 - 12x^2$

$= -\ 20x^2 + 26 - 4x$

L11

17. $4x^3 - 9x^2 - 8x^3$

$= -4x^3 - 9x^2$

L11

18. $17a - (-4a) - 7b - 6a$

$17a + 4a - 7b - 6a$

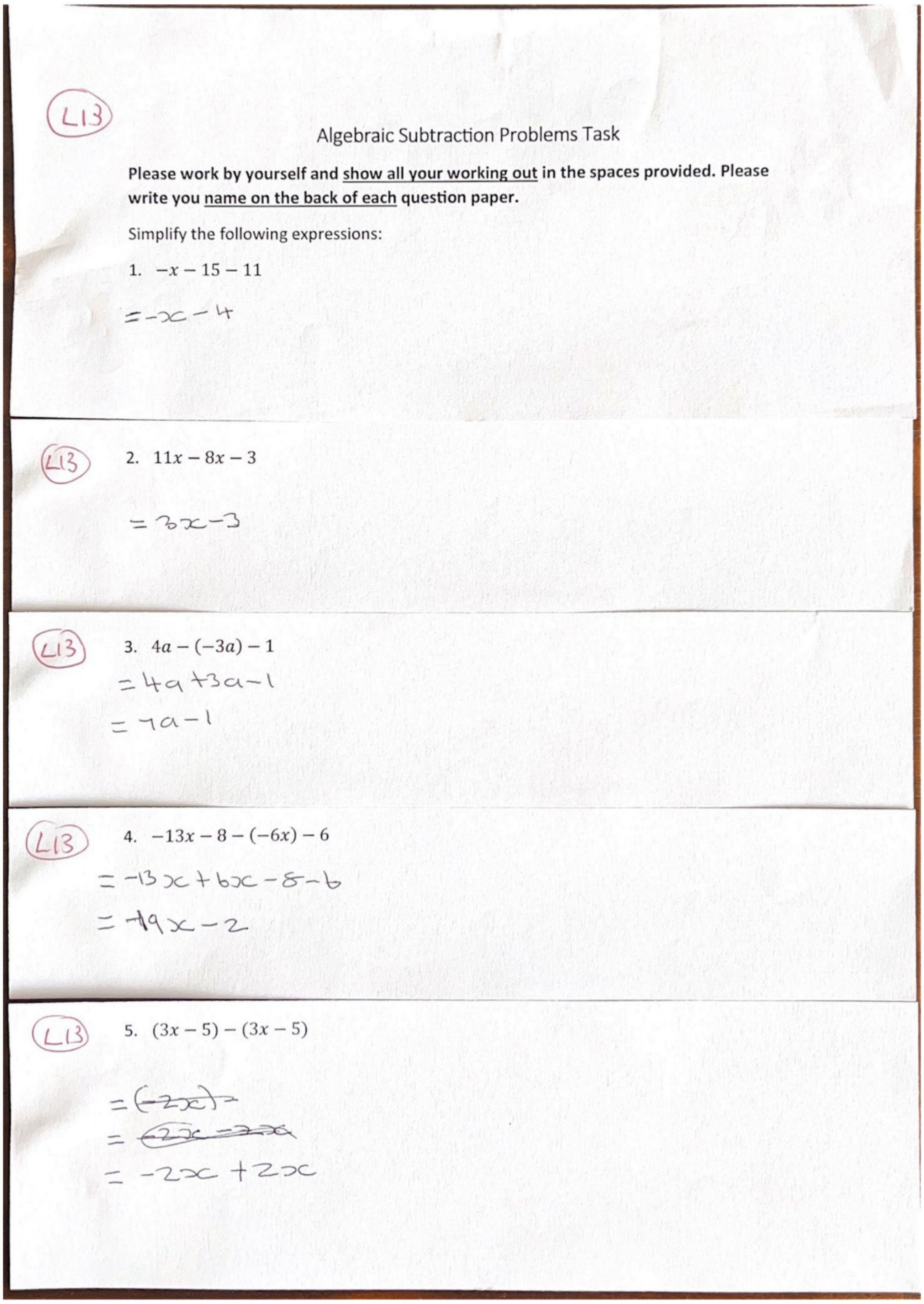
$= -7b + 15a$

L11

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$= -32a^2 - 5a^4 + 6a^4$

$= -32a^2 + a^4$



L13

Algebraic Subtraction Problems Task

Please work by yourself and show all your working out in the spaces provided. Please write you name on the back of each question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$= -x - 4$

L13

2. $11x - 8x - 3$

$= 3x - 3$

L13

3. $4a - (-3a) - 1$

$= 4a + 3a - 1$

$= 7a - 1$

L13

4. $-13x - 8 - (-6x) - 6$

$= -13x + 6x - 8 - 6$

$= -19x - 2$

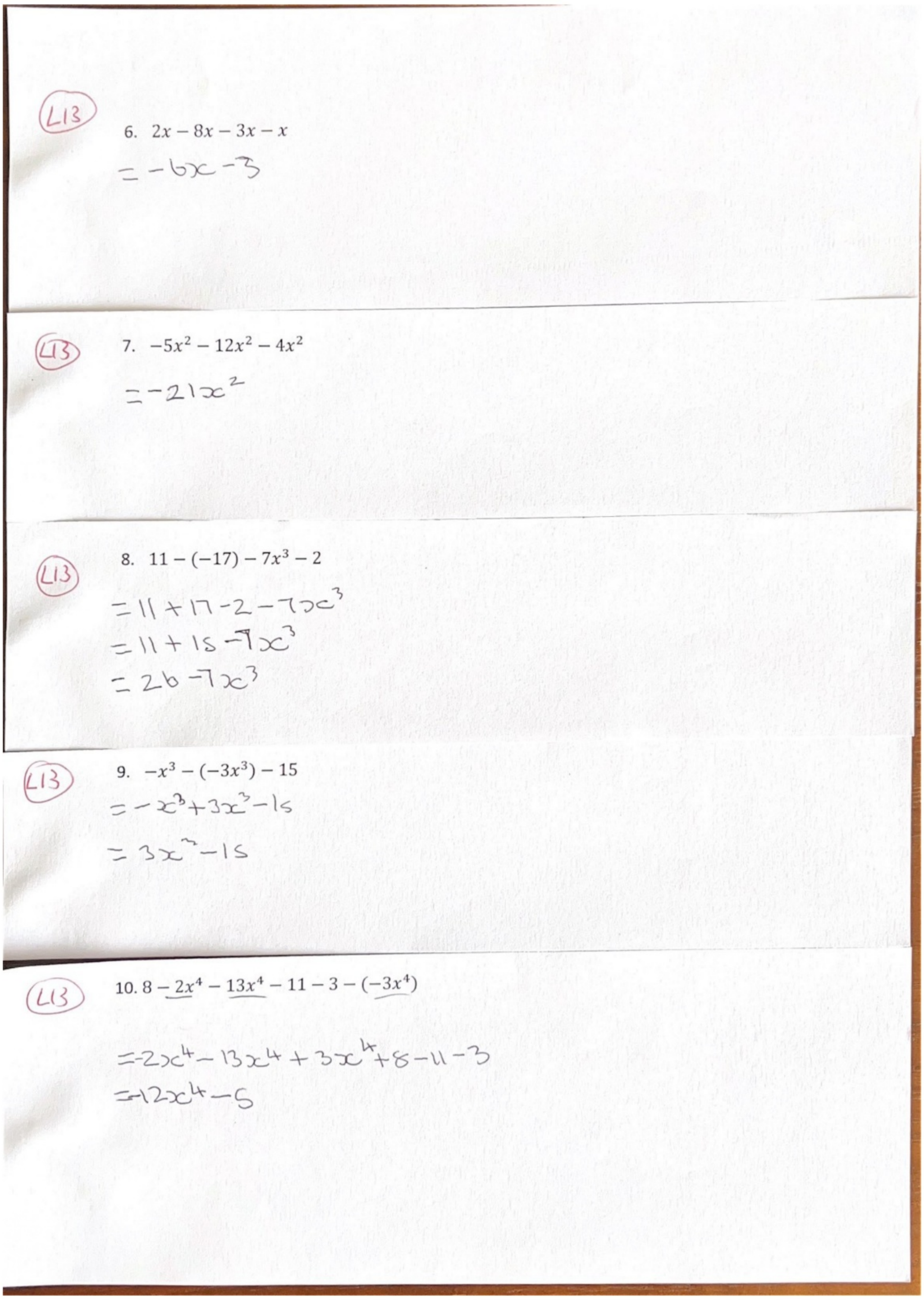
L13

5. $(3x - 5) - (3x - 5)$

= ~~$(-2x) =$~~

= ~~$-2x - 2x$~~

$= -2x + 2x$



L13

6. $2x - 8x - 3x - x$

$= -6x - 3$

L13

7. $-5x^2 - 12x^2 - 4x^2$

$= -21x^2$

L13

8. $11 - (-17) - 7x^3 - 2$

$= 11 + 17 - 2 - 7x^3$

$= 11 + 15 - 7x^3$

$= 26 - 7x^3$

L13

9. $-x^3 - (-3x^3) - 15$

$= -x^3 + 3x^3 - 15$

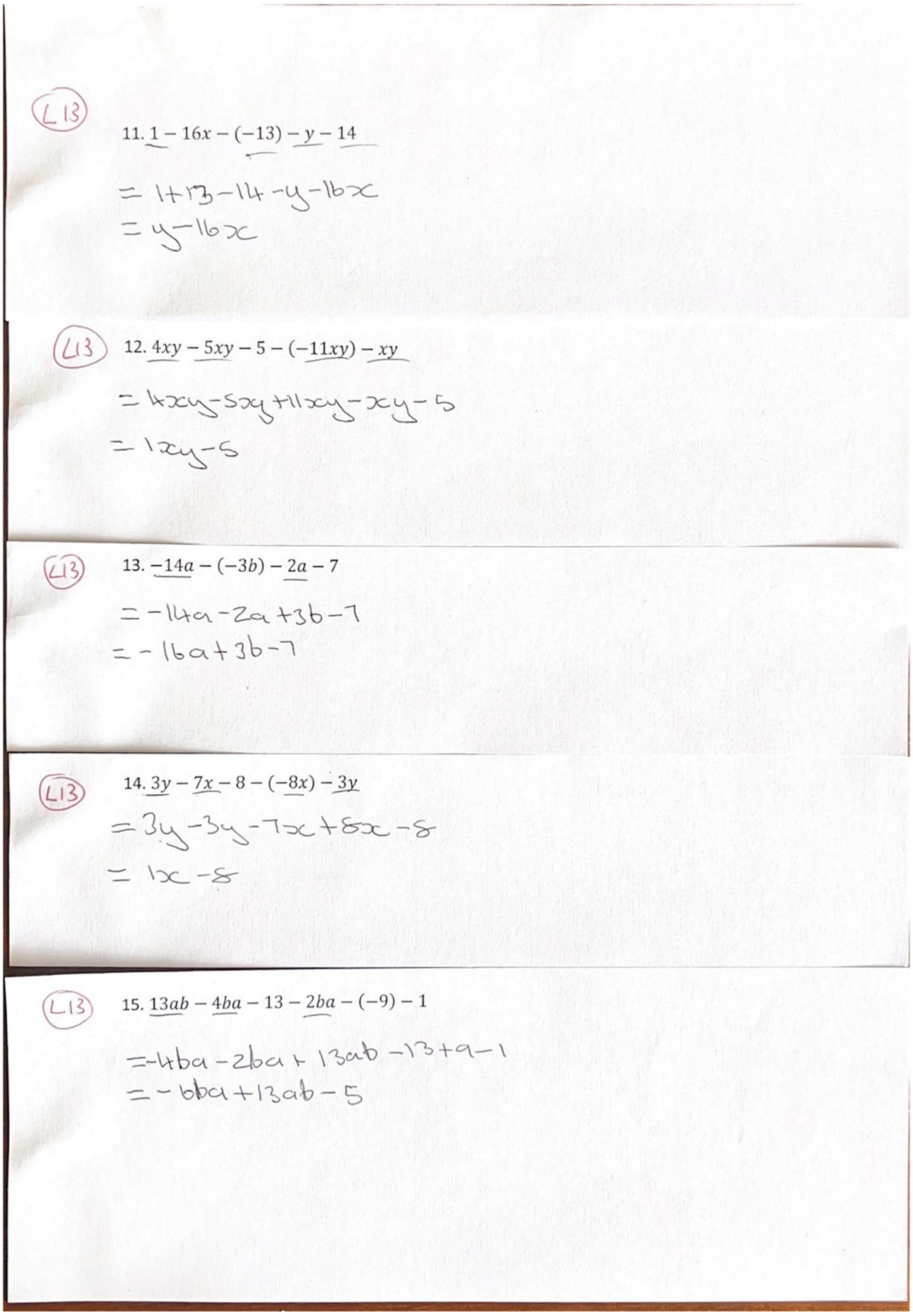
$= 3x^3 - 15$

L13

10. $8 \underline{- 2x^4} - \underline{13x^4} - 11 - 3 - (\underline{-3x^4})$

$= -2x^4 - 13x^4 + 3x^4 + 8 - 11 - 3$

$= -12x^4 - 5$



L13

11. $\underline{1} - 16x - (\underline{-13}) \underline{-y} \underline{-14}$

$= 1 + 13 - 14 - y - 16x$

$= y - 16x$

L13

12. $\underline{4xy} - \underline{5xy} - 5 - (\underline{-11xy}) \underline{-xy}$

$= 4xy - 5xy + 11xy - xy - 5$

$= 1xy - 5$

L13

13. $\underline{-14a} - (-3b) - \underline{2a} - 7$

$= -14a - 2a + 3b - 7$

$= -16a + 3b - 7$

L13

14. $\underline{3y} - \underline{7x} - 8 - (\underline{-8x}) - \underline{3y}$

$= 3y - 3y - 7x + 8x - 8$

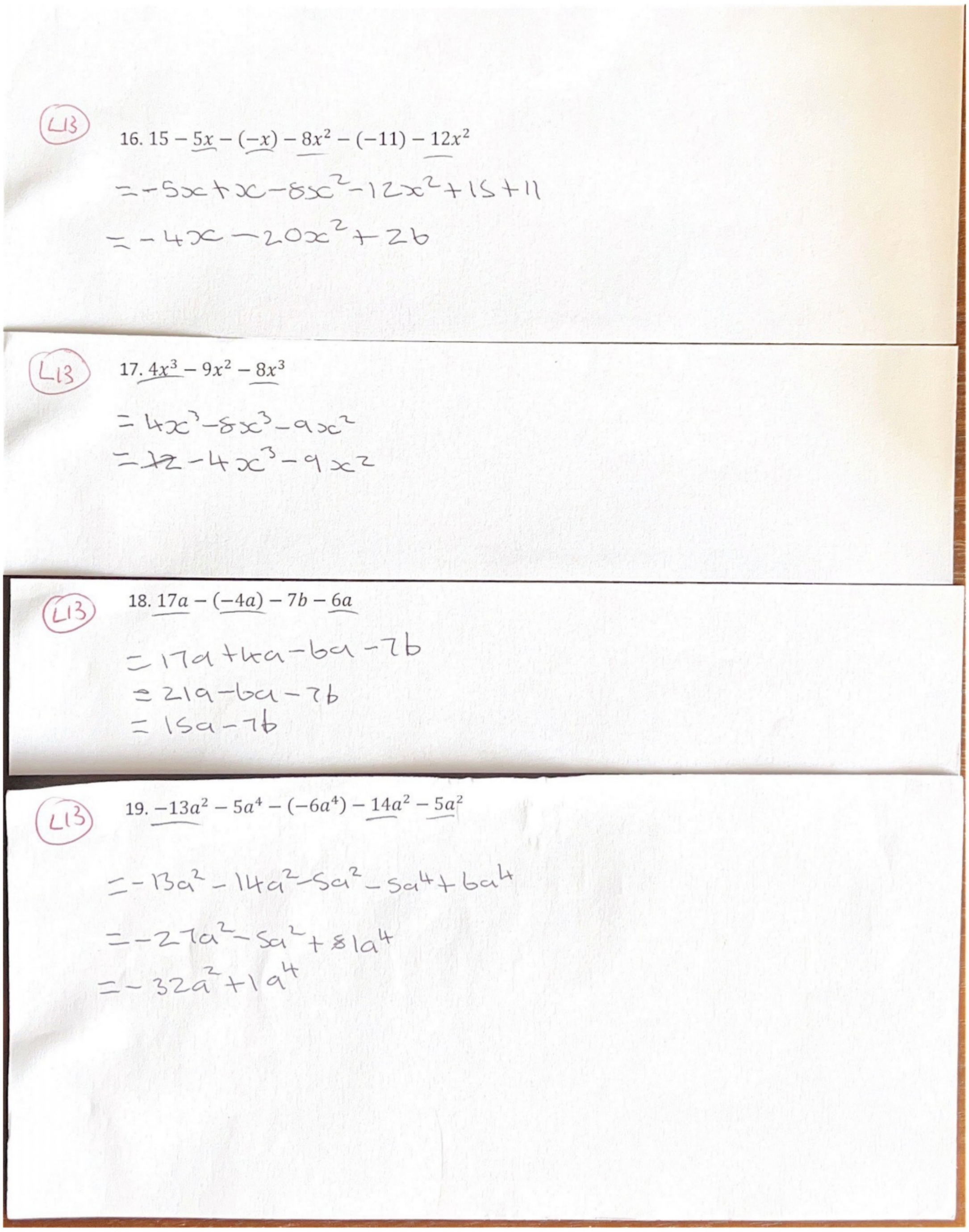
$= 1x - 8$

L13

15. $\underline{13ab} - \underline{4ba} - 13 - \underline{2ba} - (-9) - 1$

$= -4ba - 2ba + 13ab - 13 + 9 - 1$

$= -6ba + 13ab - 5$



L13

16. $15 - \underline{5x} - \underline{(-x)} - \underline{8x^2} - (-11) - \underline{12x^2}$

$= -5x + x - 8x^2 - 12x^2 + 15 + 11$

$= -4x - 20x^2 + 26$

L13

17. $\underline{4x^3} - 9x^2 - \underline{8x^3}$

$= 4x^3 - 8x^3 - 9x^2$

= ~~12~~ $-4x^3 - 9x^2$

L13

18. $\underline{17a} - \underline{(-4a)} - 7b - \underline{6a}$

$= 17a + 4a - 6a - 7b$

$= 21a - 6a - 7b$

$= 15a - 7b$

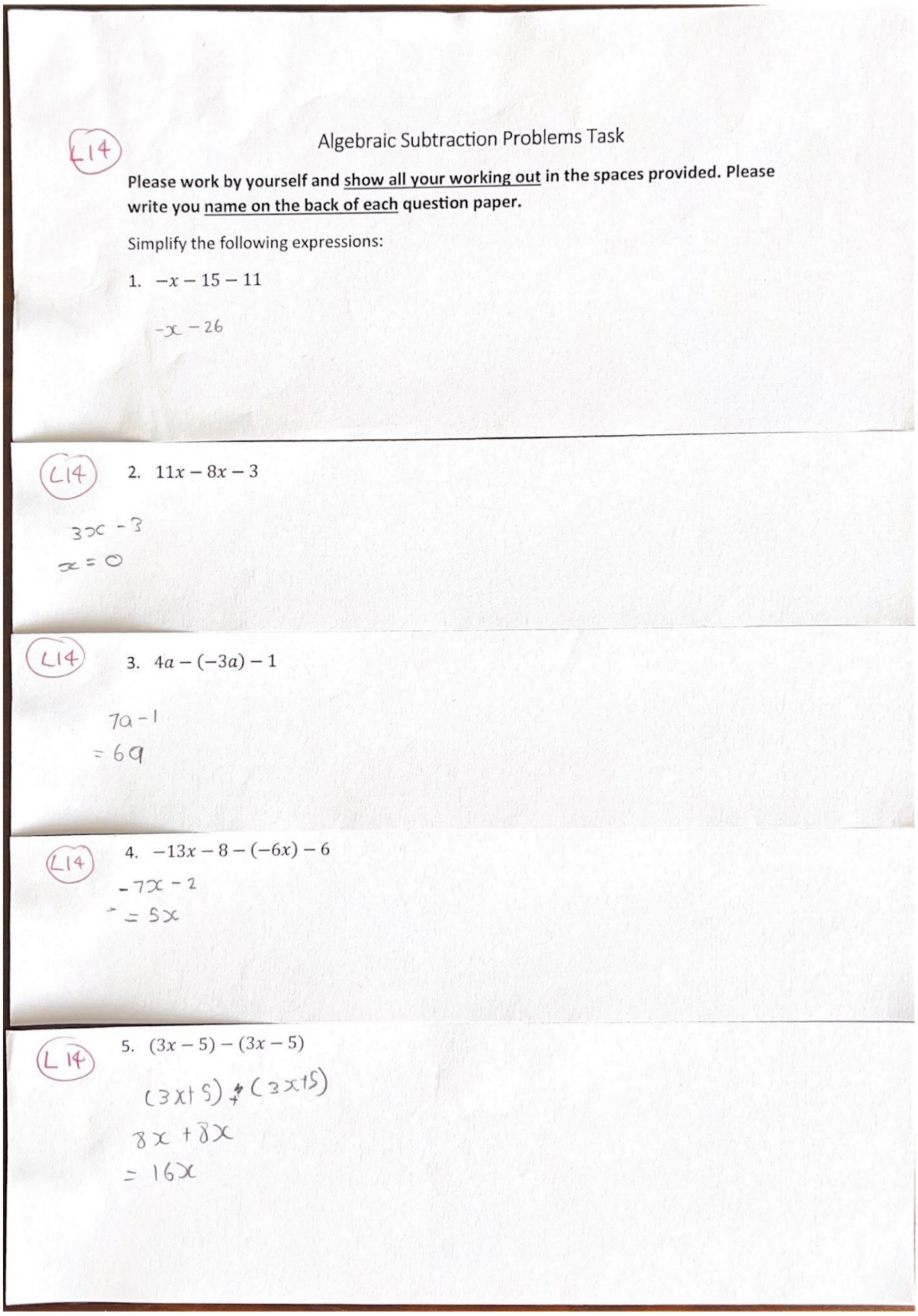
L13

19. $\underline{-13a^2} - 5a^4 - (-6a^4) - \underline{14a^2} - \underline{5a^2}$

$= -13a^2 - 14a^2 - 5a^2 - 5a^4 + 6a^4$

$= -27a^2 - 5a^2 + 81a^4$

$= -32a^2 + 1a^4$



L14

Algebraic Subtraction Problems Task

Please work by yourself and show all your working out in the spaces provided. Please write you name on the back of each question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$-x - 26$

L14

2. $11x - 8x - 3$

$3x - 3$

$x = 0$

L14

3. $4a - (-3a) - 1$

$7a - 1$

$= 69$

L14

4. $-13x - 8 - (-6x) - 6$

$-7x - 2$

$= 5x$

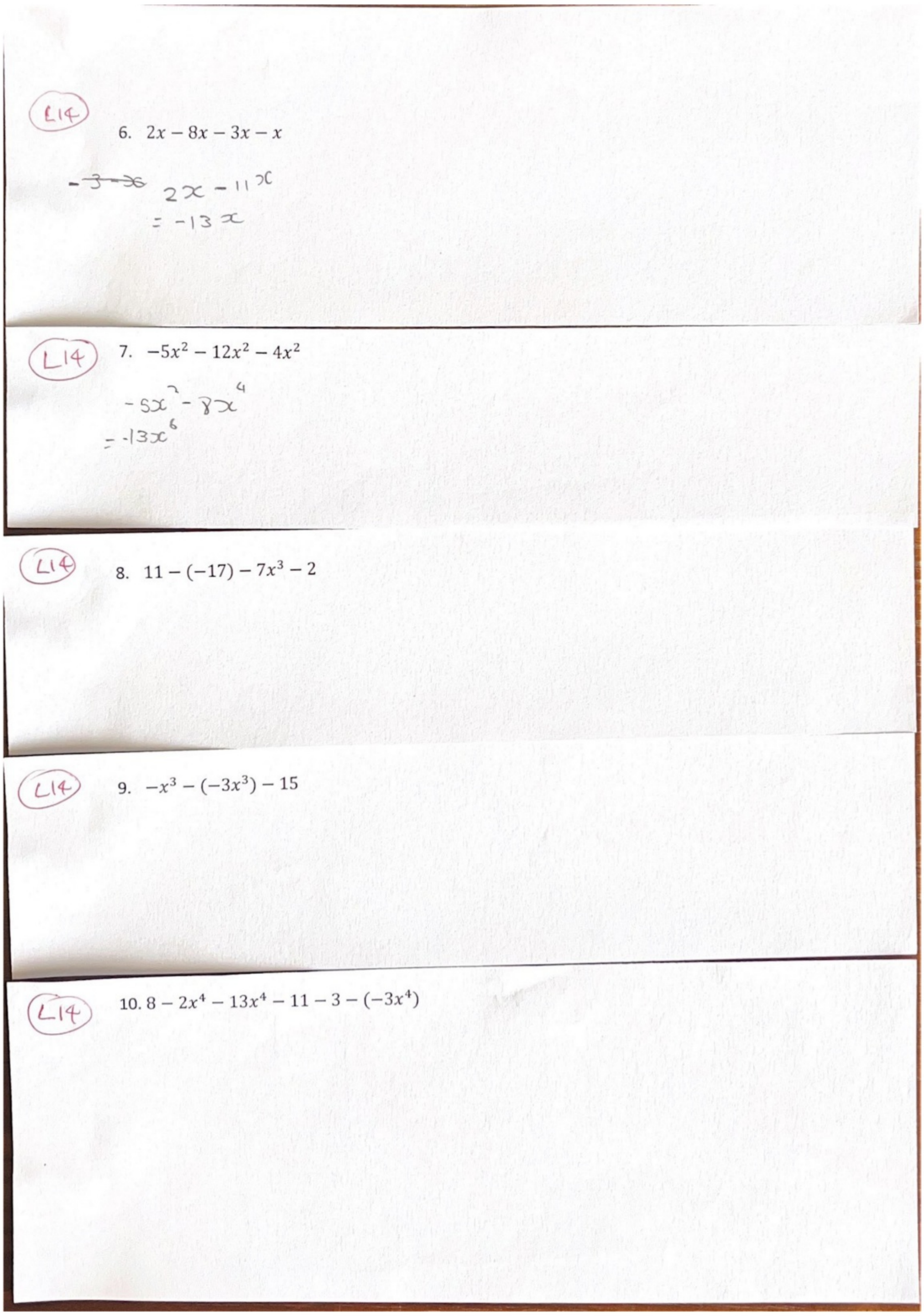
L14

5. $(3x - 5) - (3x - 5)$

$(3x+5) + (3x+5)$

$8x + 8x$

$= 16x$



L14

6. $2x - 8x - 3x - x$

~~$-3-x$~~ $2x - 11x$

$= -13x$

L14

7. $-5x^2 - 12x^2 - 4x^2$

$-5x^2 - 8x^4$

$= -13x^6$

L14

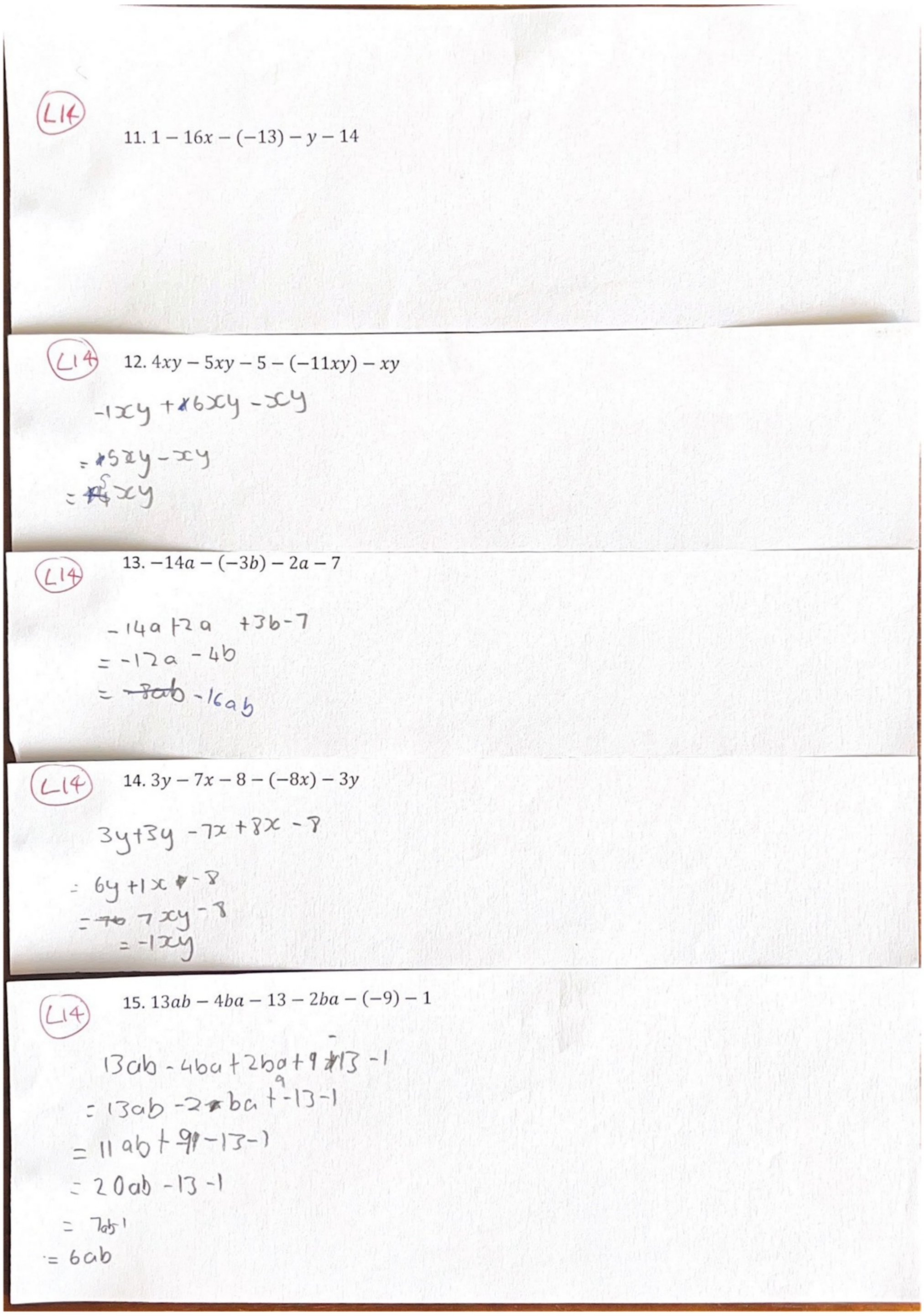
8. $11 - (-17) - 7x^3 - 2$

L14

9. $-x^3 - (-3x^3) - 15$

L14

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$



L14

11. $1 - 16x - (-13) - y - 14$

L14

12. $4xy - 5xy - 5 - (-11xy) - xy$

$-1xy + \cancel{1}6xy - xy$

$= \cancel{1}5xy - xy$

$= \overset{5}{\cancel{4}}xy$

L14

13. $-14a - (-3b) - 2a - 7$

$-14a + 2a + 3b - 7$

$= -12a - 4b$

$= \cancel{-8ab} - 16ab$

L14

14. $3y - 7x - 8 - (-8x) - 3y$

$3y + 3y - 7x + 8x - 8$

$= 6y + 1x - 8$

$= \cancel{-7} 7xy - 8$

$= -1xy$

L14

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$13ab - 4ba + 2ba + 9 \overset{-}{\cancel{+}}13 - 1$

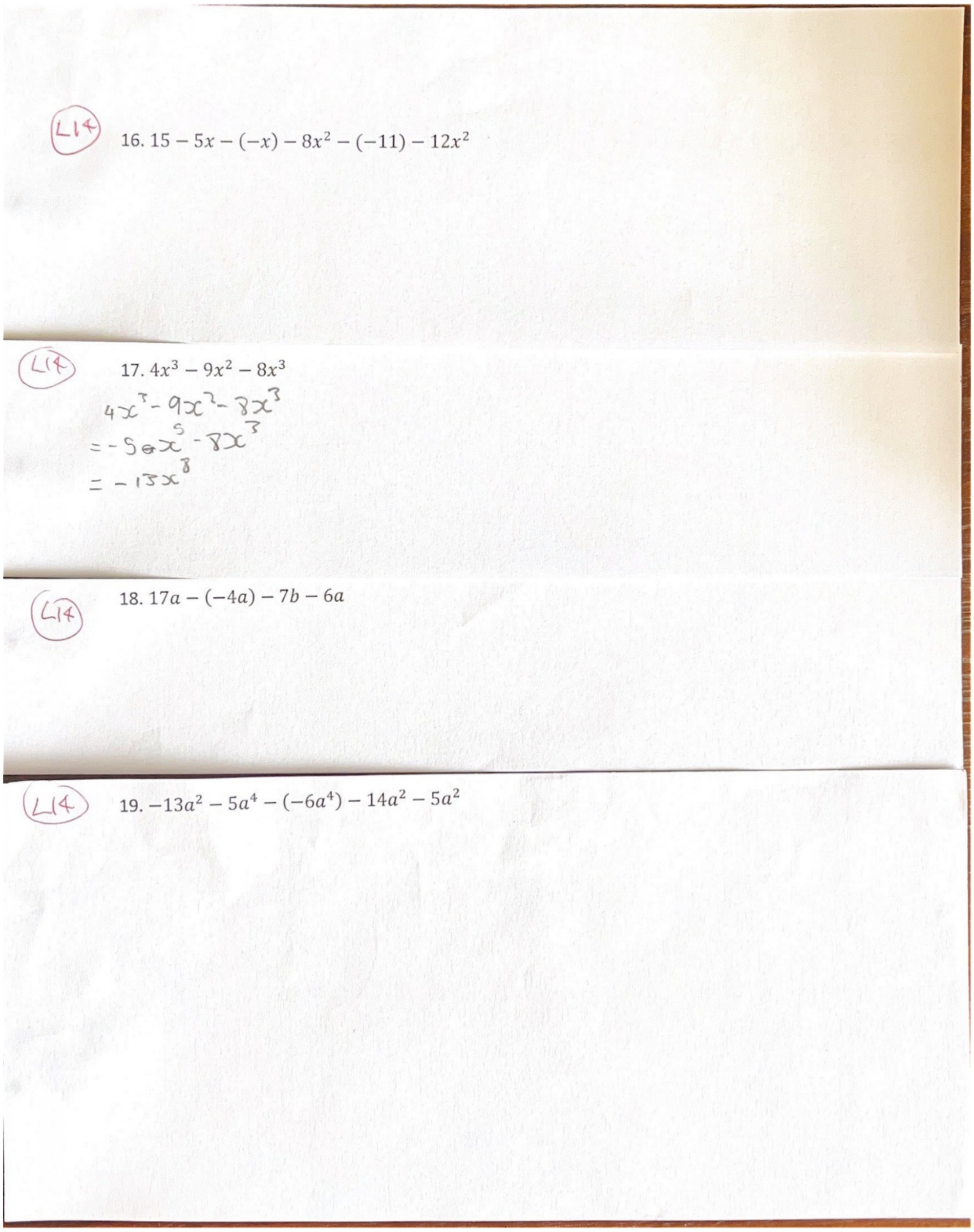
$= 13ab - 2ba \overset{9}{+} - 13 - 1$

$= 11ab + \cancel{9} - 13 - 1$

$= 20ab - 13 - 1$

$= 7ab\ 1$

$= 6ab$



L14

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

L14

17. $4x^3 - 9x^2 - 8x^3$

$4x^3 - 9x^2 - 8x^3$

$= -5x^5 - 8x^3$

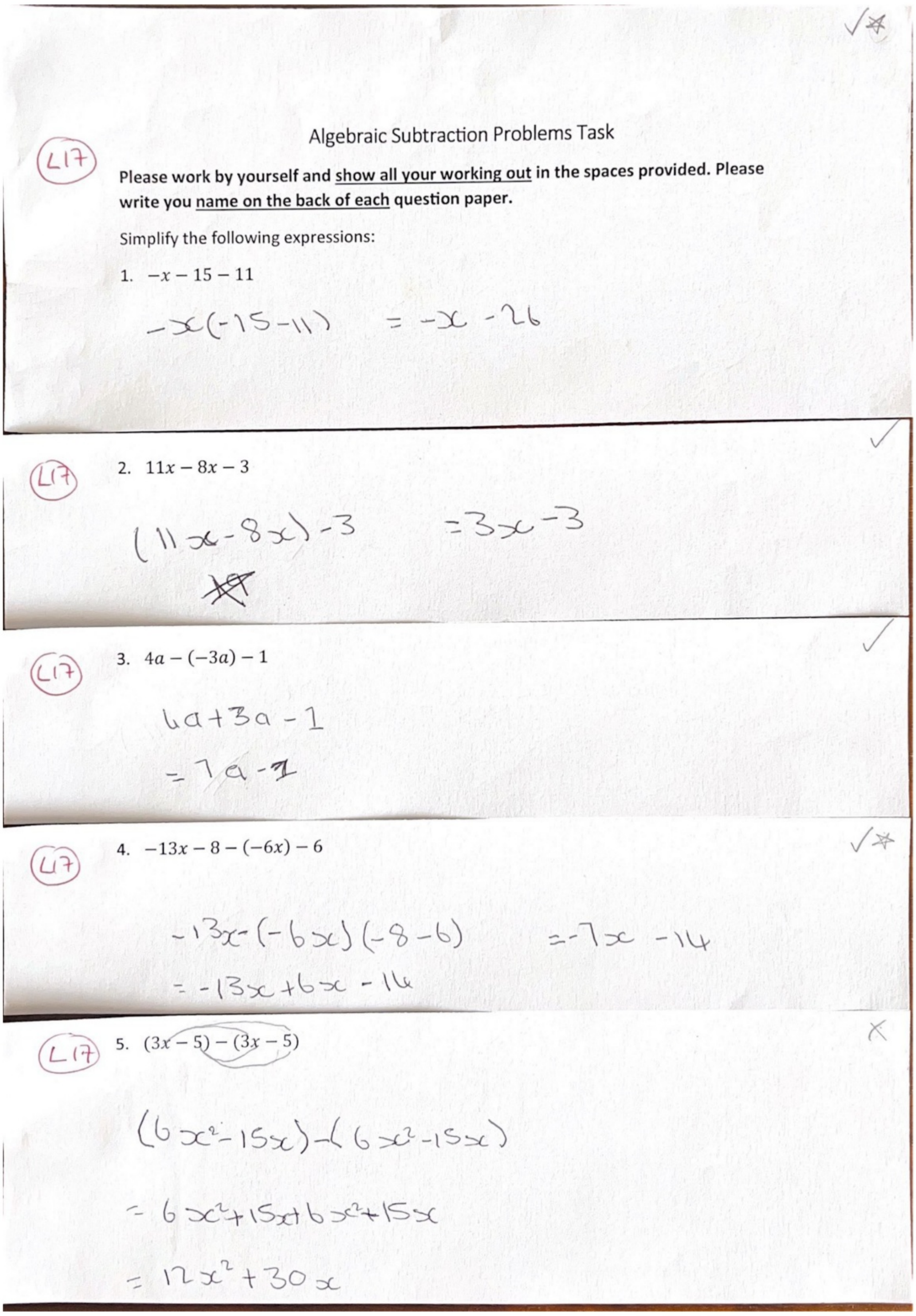
$= -13x^8$

L14

18. $17a - (-4a) - 7b - 6a$

L14

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$



✓☆

L17

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$-x(-15-11) \quad = -x - 26$

L17 ✓

2. $11x - 8x - 3$

$(11x - 8x) - 3 \quad = 3x - 3$

L17 ✓

3. $4a - (-3a) - 1$

$4a + 3a - 1$

$= 7a - 1$

L17 ✓☆

4. $-13x - 8 - (-6x) - 6$

$-13x - (-6x)(-8-6) \quad = -7x - 14$

$= -13x + 6x - 14$

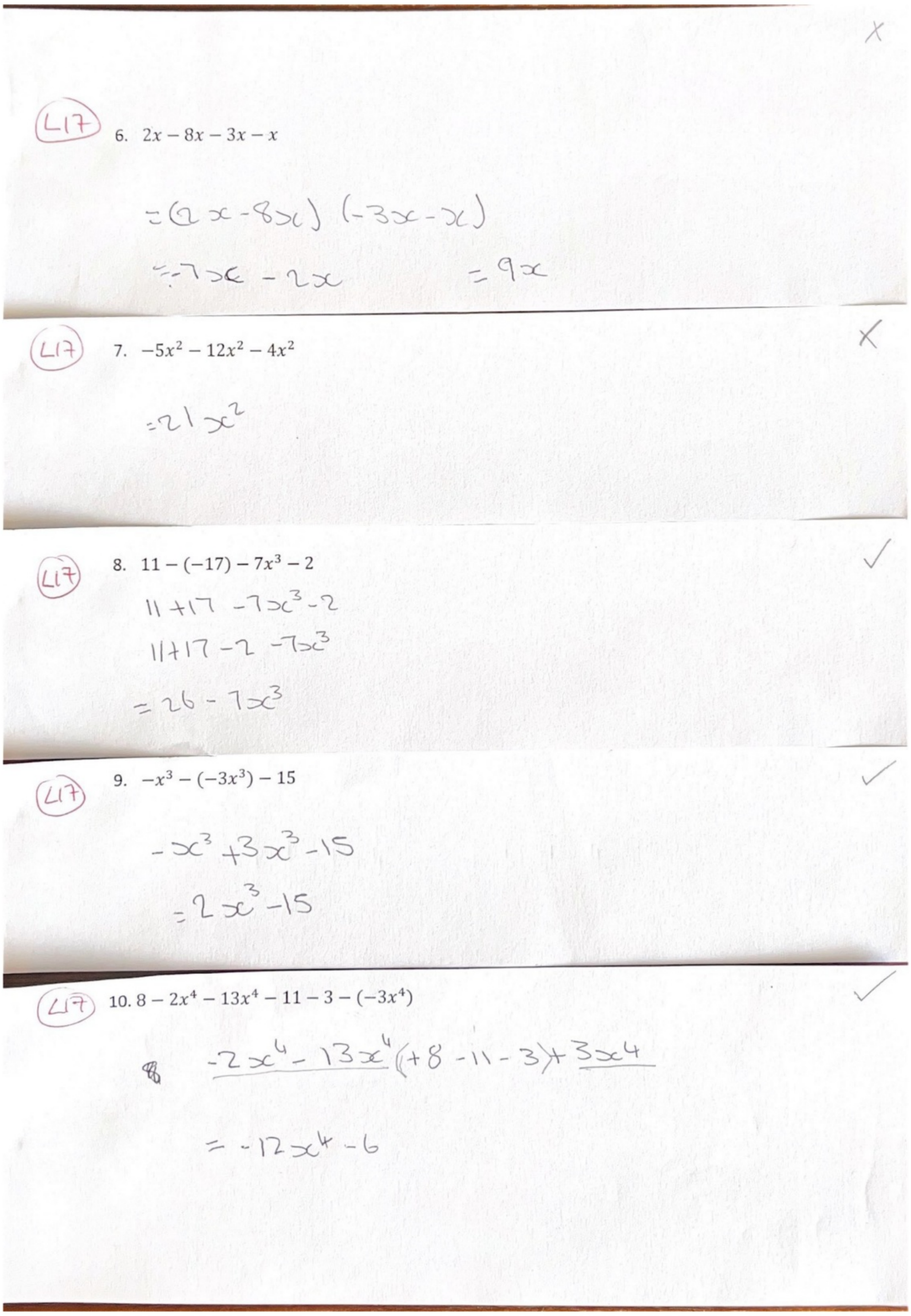
L17 ✗

5. $(3x - 5) - (3x - 5)$

$(6x^2 - 15x) - (6x^2 - 15x)$

$= 6x^2 + 15x + 6x^2 + 15x$

$= 12x^2 + 30x$



L17 ✗

6. $2x - 8x - 3x - x$

$= (2x - 8x)\ (-3x - x)$

$= -7x - 2x \qquad = 9x$

L17 ✗

7. $-5x^2 - 12x^2 - 4x^2$

$= 21x^2$

L17 ✓

8. $11 - (-17) - 7x^3 - 2$

$11 + 17 - 7x^3 - 2$

$11 + 17 - 2 - 7x^3$

$= 26 - 7x^3$

L17 ✓

9. $-x^3 - (-3x^3) - 15$

$-x^3 + 3x^3 - 15$

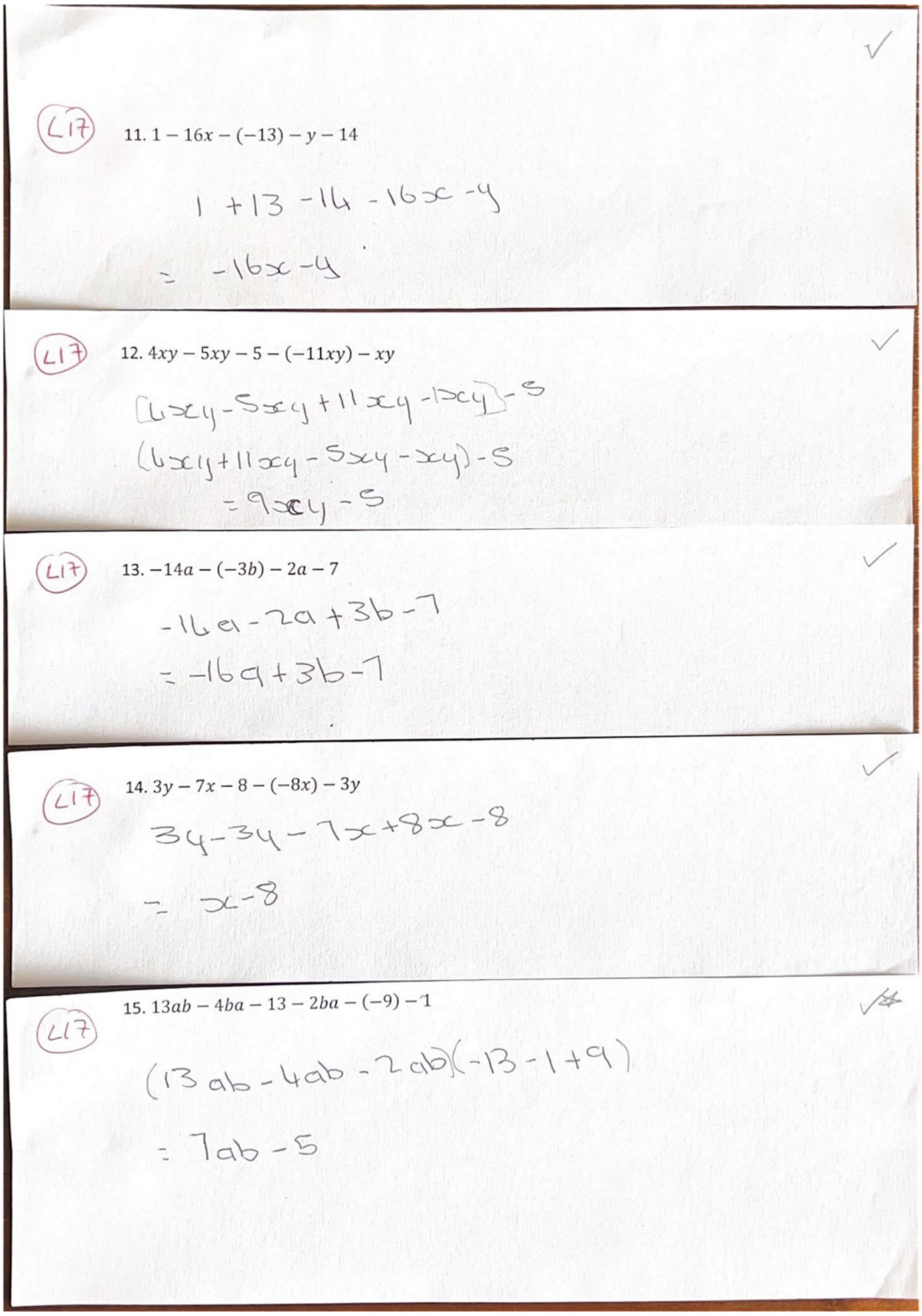
$= 2x^3 - 15$

L17 ✓

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$\underline{-2x^4 - 13x^4}\ (+8 - 11 - 3) + \underline{3x4}$

$= -12x^4 - 6$



L17

11. $1 - 16x - (-13) - y - 14$

$1 + 13 - 14 - 16x - y$

$= -16x - y$

✓

L17

12. $4xy - 5xy - 5 - (-11xy) - xy$

$[4xy - 5xy + 11xy - 1xy] - 5$

$(4xy + 11xy - 5xy - xy) - 5$

$= 9xy - 5$

✓

L17

13. $-14a - (-3b) - 2a - 7$

$-14a - 2a + 3b - 7$

$= -16a + 3b - 7$

✓

L17

14. $3y - 7x - 8 - (-8x) - 3y$

$3y - 3y - 7x + 8x - 8$

$= x - 8$

✓

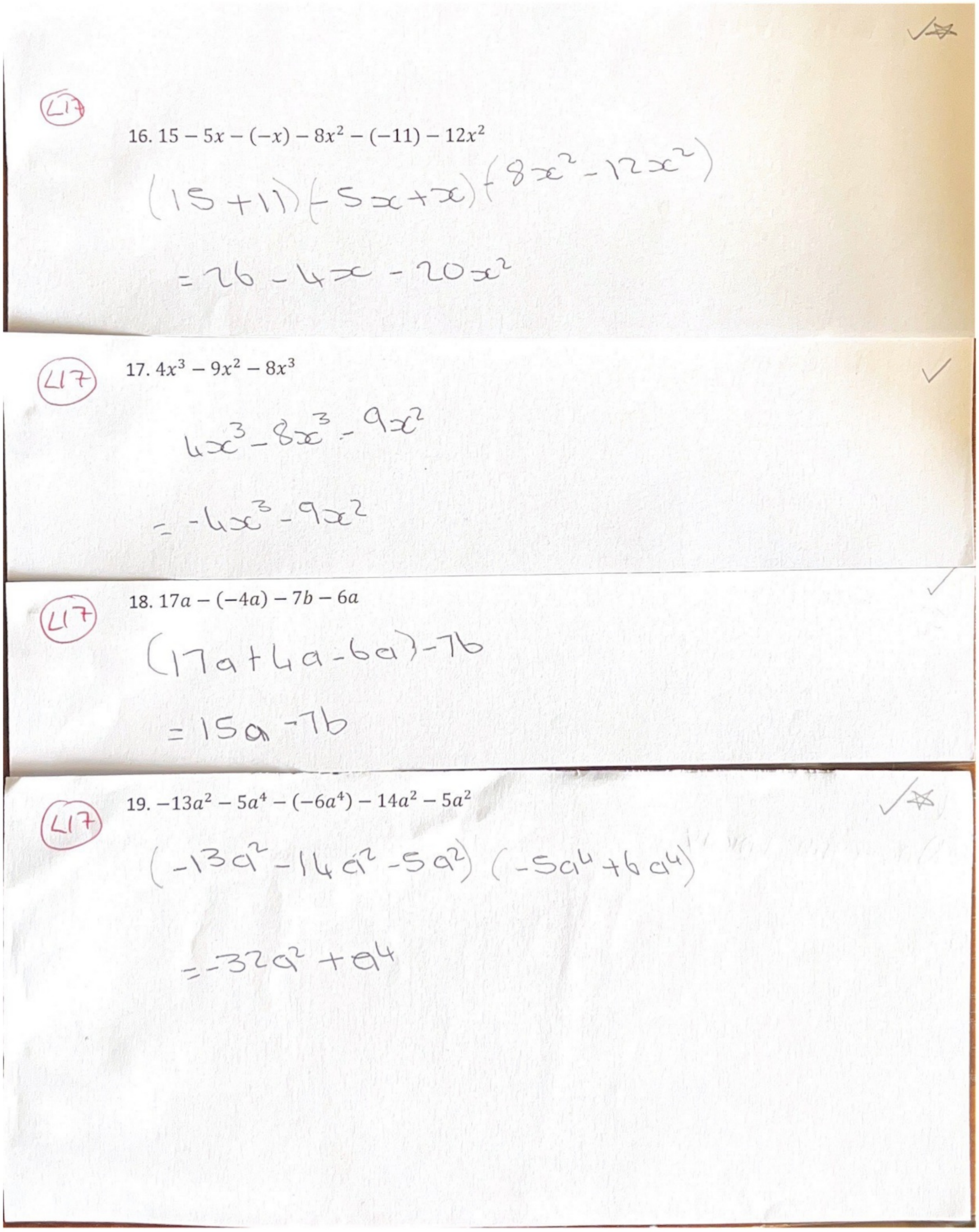
L17

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$(13ab - 4ab - 2ab)(-13 - 1 + 9)$

$= 7ab - 5$

✓



✓☆

L17

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$(15+11)(-5x+x)(-8x^2-12x^2)$

$= 26 - 4x - 20x^2$

L17

17. $4x^3 - 9x^2 - 8x^3$

✓

$4x^3 - 8x^3 - 9x^2$

$= -4x^3 - 9x^2$

L17

18. $17a - (-4a) - 7b - 6a$

✓

$(17a + 4a - 6a) - 7b$

$= 15a - 7b$

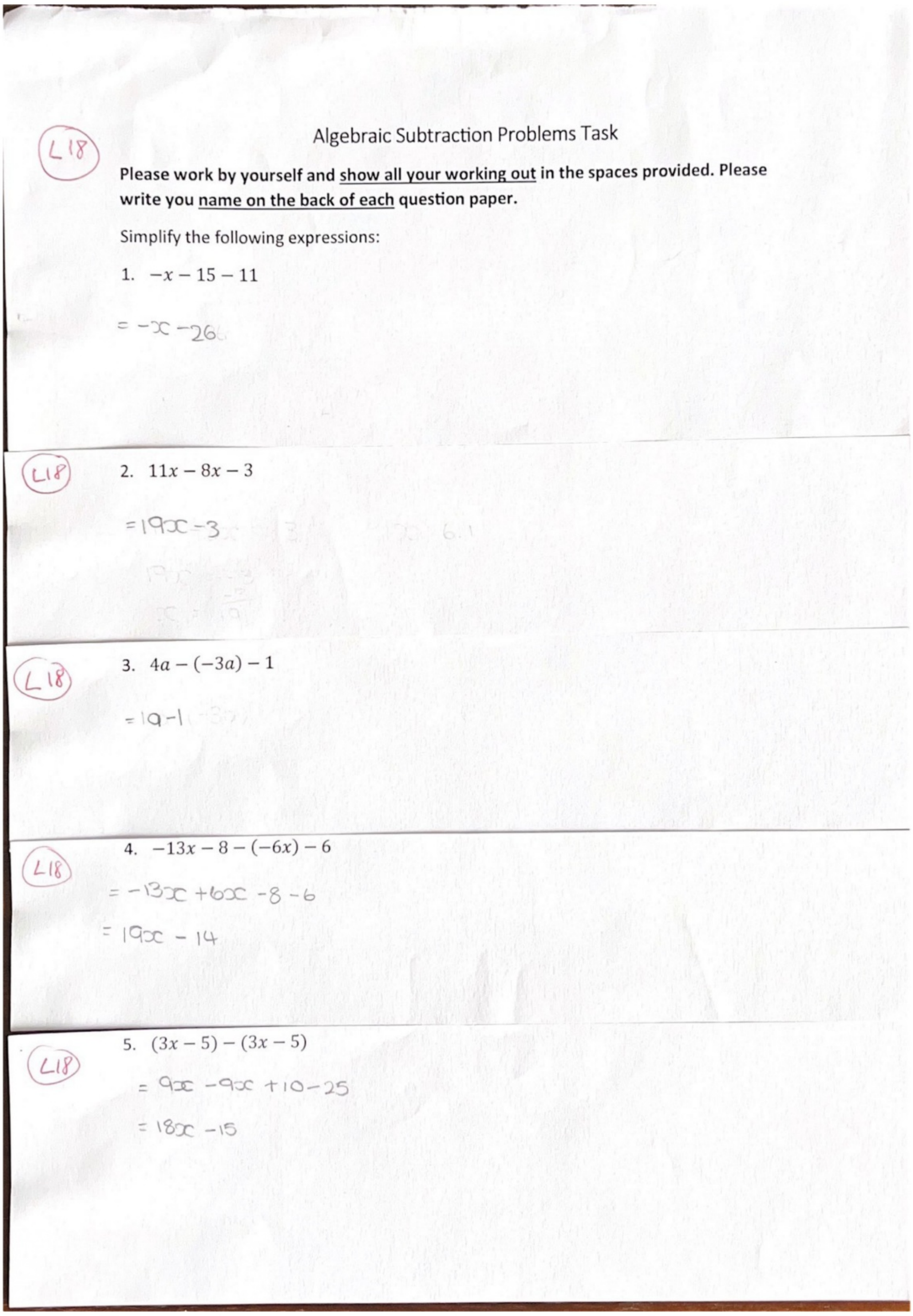
L17

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

✓☆

$(-13a^2 - 14a^2 - 5a^2)(-5a^4 + 6a^4)$

$= -32a^2 + a^4$



L18

Algebraic Subtraction Problems Task

Please work by yourself and show all your working out in the spaces provided. Please write you name on the back of each question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$= -x - 26$

L18

2. $11x - 8x - 3$

$= 19x - 3$

L18

3. $4a - (-3a) - 1$

$= 1a - 1$

L18

4. $-13x - 8 - (-6x) - 6$

$= -13x + 6x - 8 - 6$

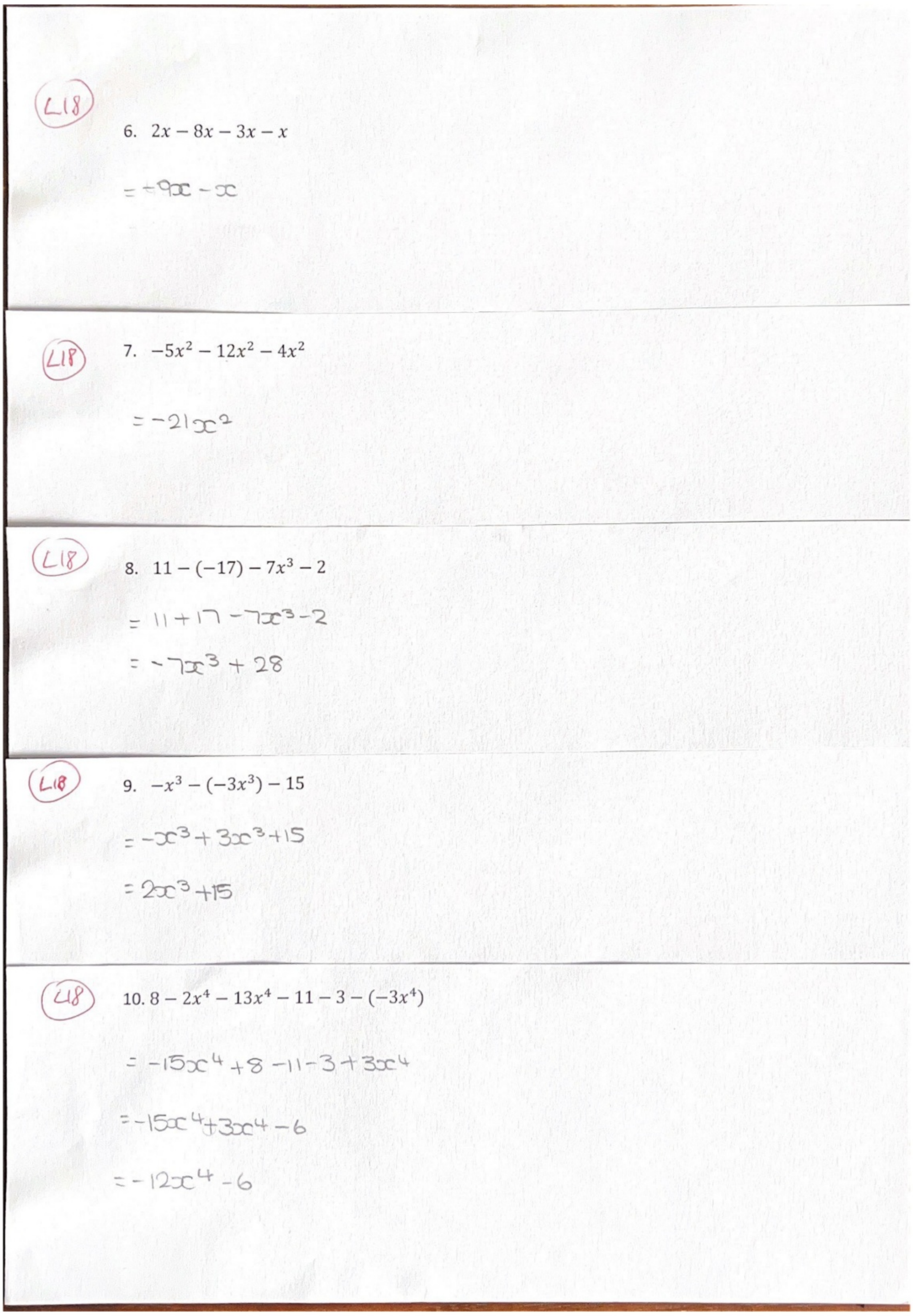
$= 19x - 14$

L18

5. $(3x - 5) - (3x - 5)$

$= 9x - 9x + 10 - 25$

$= 18x - 15$



L18

6. $2x - 8x - 3x - x$

$= +9x - x$

L18

7. $-5x^2 - 12x^2 - 4x^2$

$= -21x^2$

L18

8. $11 - (-17) - 7x^3 - 2$

$= 11 + 17 - 7x^3 - 2$

$= -7x^3 + 28$

L18

9. $-x^3 - (-3x^3) - 15$

$= -x^3 + 3x^3 + 15$

$= 2x^3 + 15$

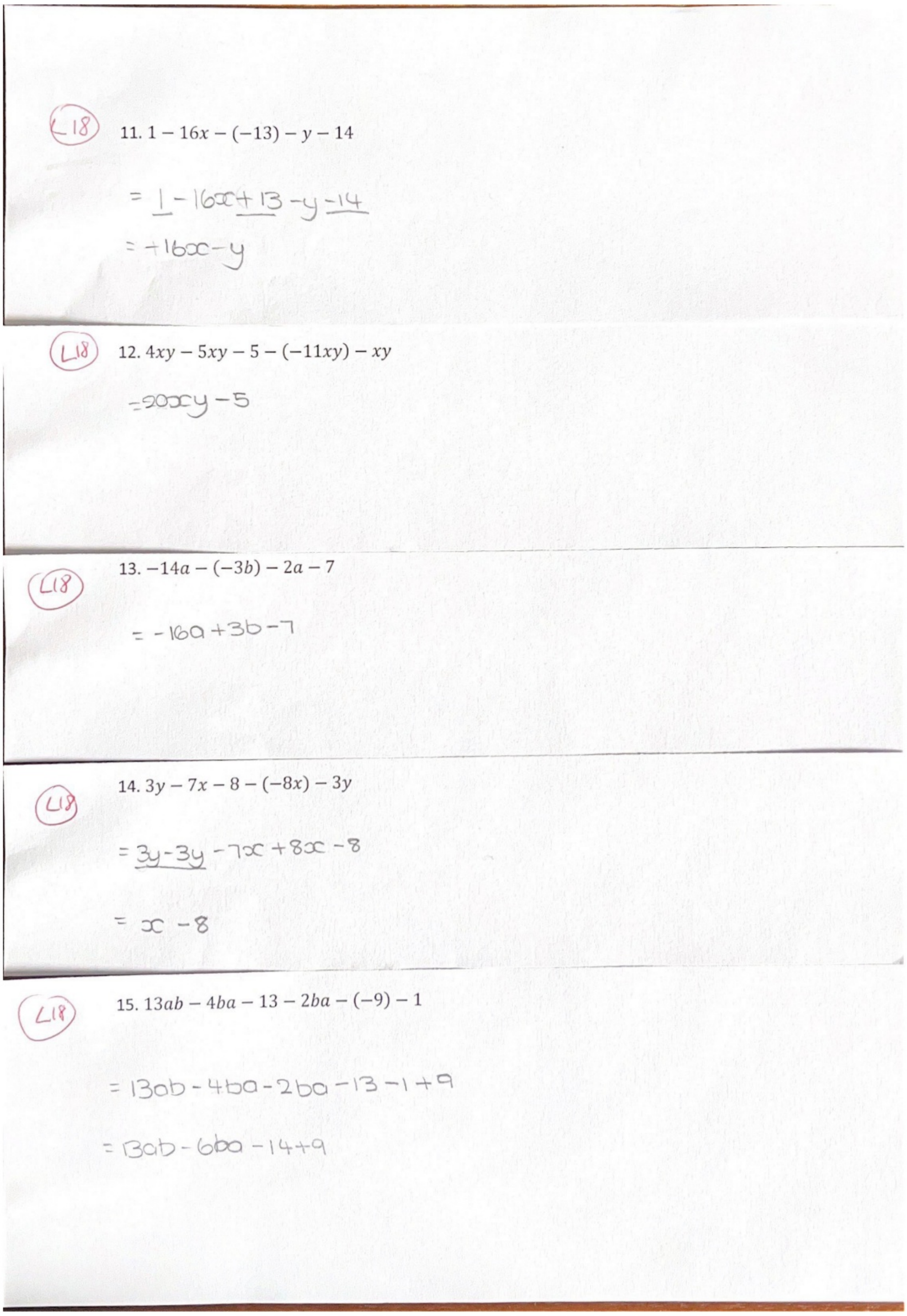
L18

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$= -15x^4 + 8 - 11 - 3 + 3x^4$

$= -15x^4 + 3x^4 - 6$

$= -12x^4 - 6$



L18

11. $1 - 16x - (-13) - y - 14$

$= \underline{1} - 16x \underline{+ 13} - y \underline{- 14}$

$= +16x - y$

L18

12. $4xy - 5xy - 5 - (-11xy) - xy$

$= 20xy - 5$

L18

13. $-14a - (-3b) - 2a - 7$

$= -16a + 3b - 7$

L18

14. $3y - 7x - 8 - (-8x) - 3y$

$= \underline{3y - 3y} - 7x + 8x - 8$

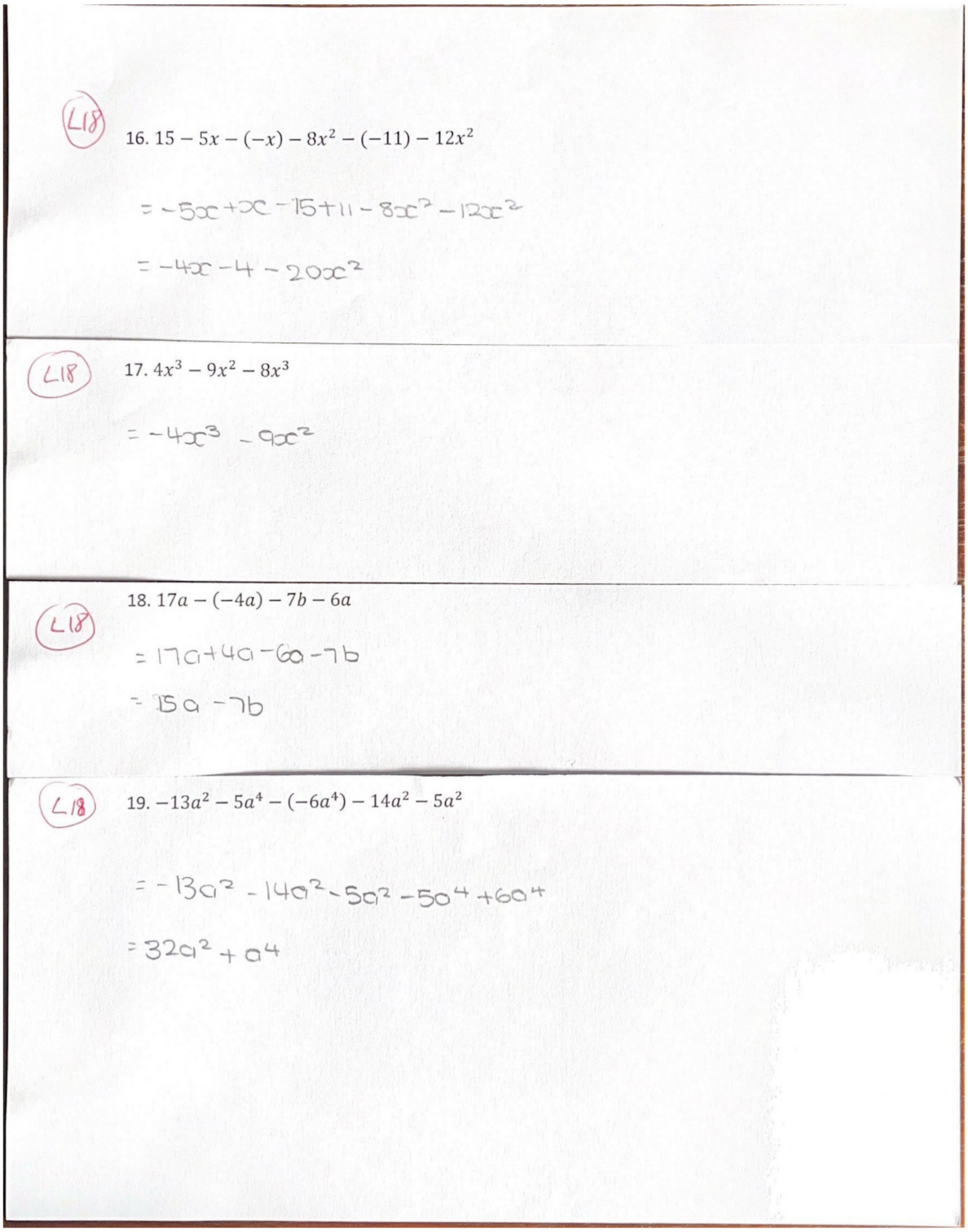
$= x - 8$

L18

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= 13ab - 4ba - 2ba - 13 - 1 + 9$

$= 13ab - 6ba - 14 + 9$



L18

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$= -5x + x - 15 + 11 - 8x^2 - 12x^2$

$= -4x - 4 - 20x^2$

L18

17. $4x^3 - 9x^2 - 8x^3$

$= -4x^3 - 9x^2$

L18

18. $17a - (-4a) - 7b - 6a$

$= 17a + 4a - 6a - 7b$

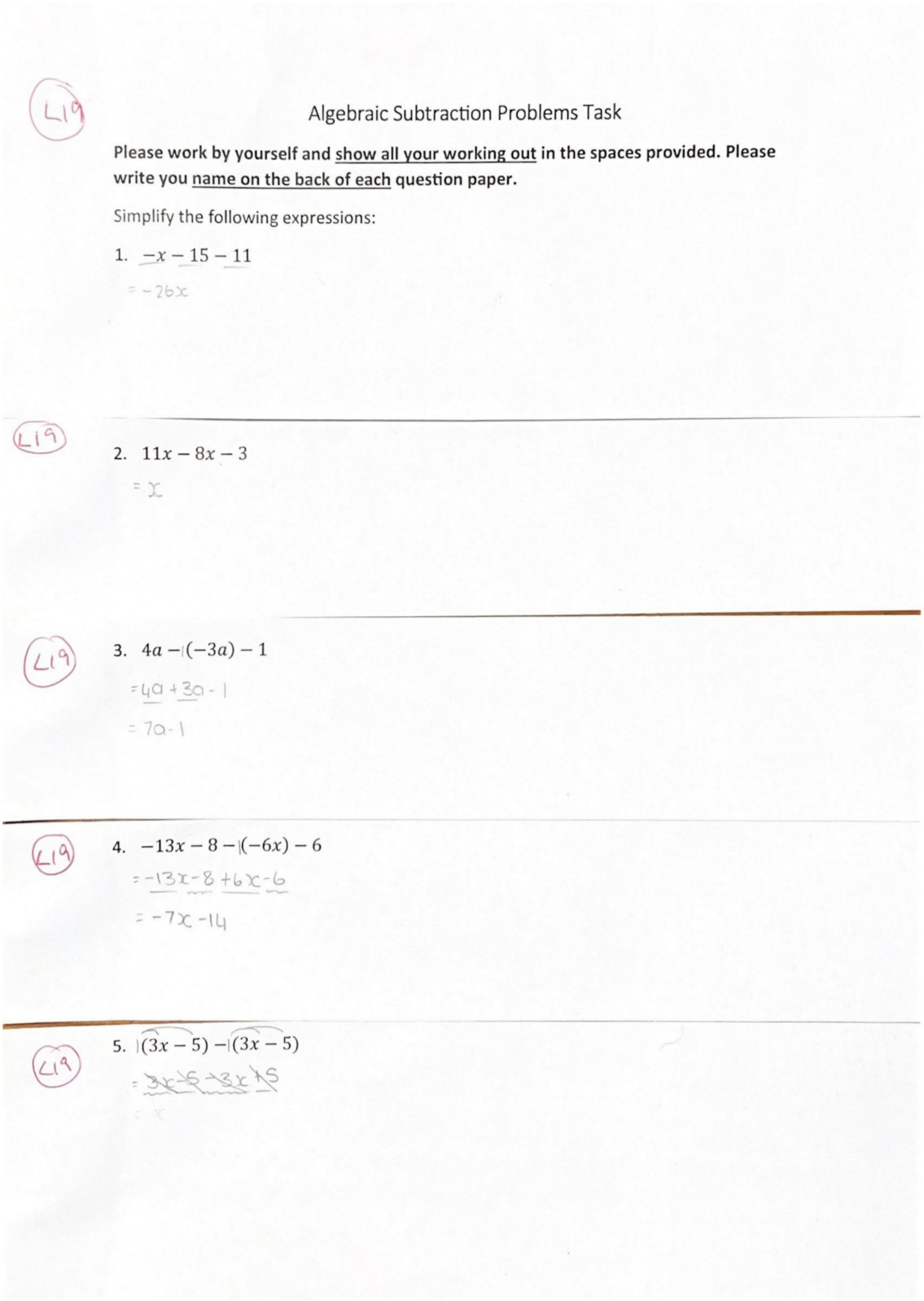
$= 15a - 7b$

L18

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$= -13a^2 - 14a^2 - 5a^2 - 5a^4 + 6a^4$

$= 32a^2 + a^4$



L19

Algebraic Subtraction Problems Task

Please work by yourself and show all your working out in the spaces provided. Please write you name on the back of each question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$= -26x$

L19

2. $11x - 8x - 3$

$= x$

L19

3. $4a - (-3a) - 1$

$= 4a + 3a - 1$

$= 7a - 1$

L19

4. $-13x - 8 - (-6x) - 6$

$= -13x - 8 + 6x - 6$

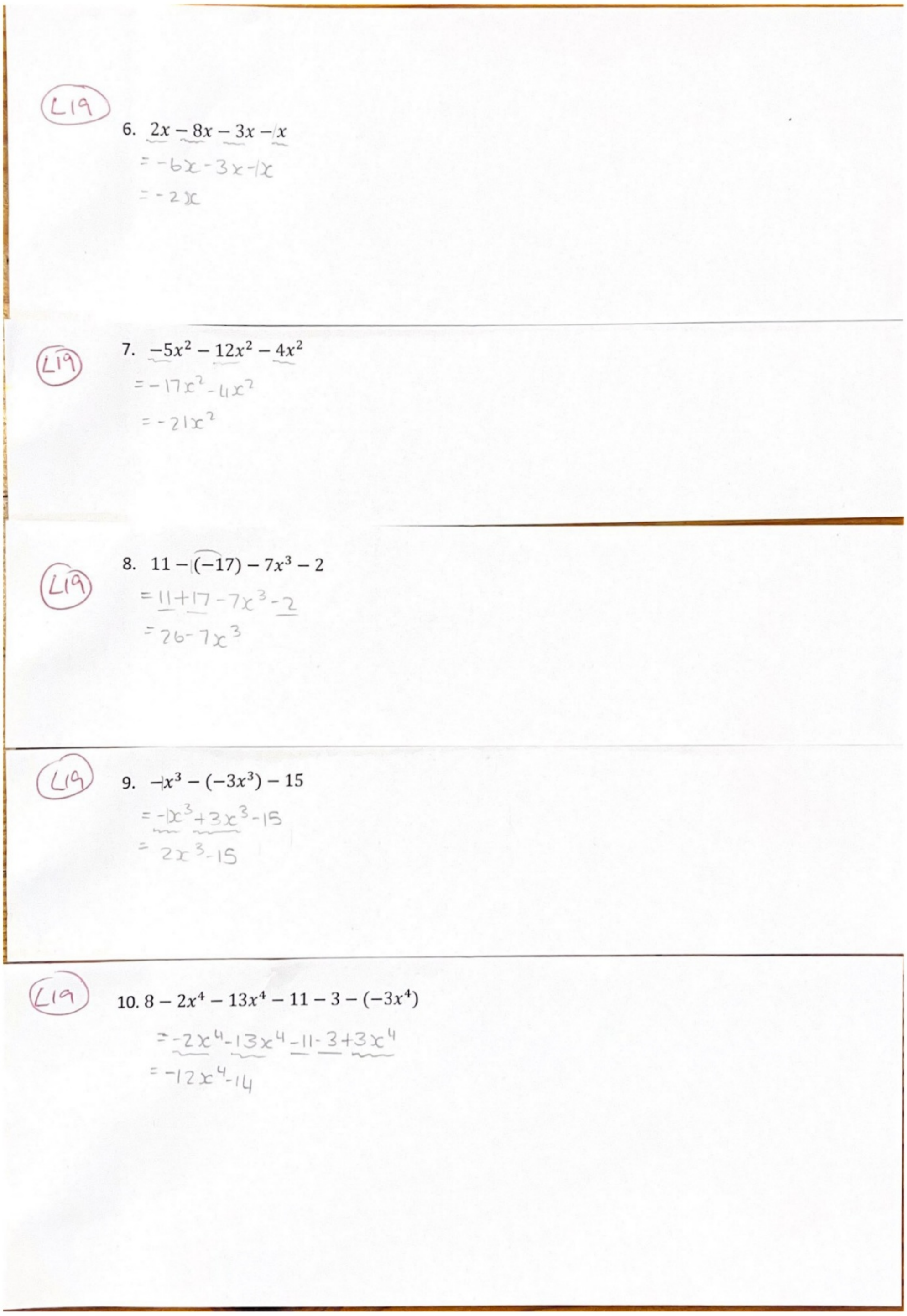
$= -7x - 14$

L19

5. $(3x - 5) - (3x - 5)$

$= 3x - 5 - 3x + 5$

$= x$



L19

6. $2x - 8x - 3x - 1x$

$= -6x - 3x - 1x$

$= -2x$

L19

7. $-5x^2 - 12x^2 - 4x^2$

$= -17x^2 - 4x^2$

$= -21x^2$

L19

8. $11 - (-17) - 7x^3 - 2$

$= 11 + 17 - 7x^3 - 2$

$= 26 - 7x^3$

L19

9. $-1x^3 - (-3x^3) - 15$

$= -1x^3 + 3x^3 - 15$

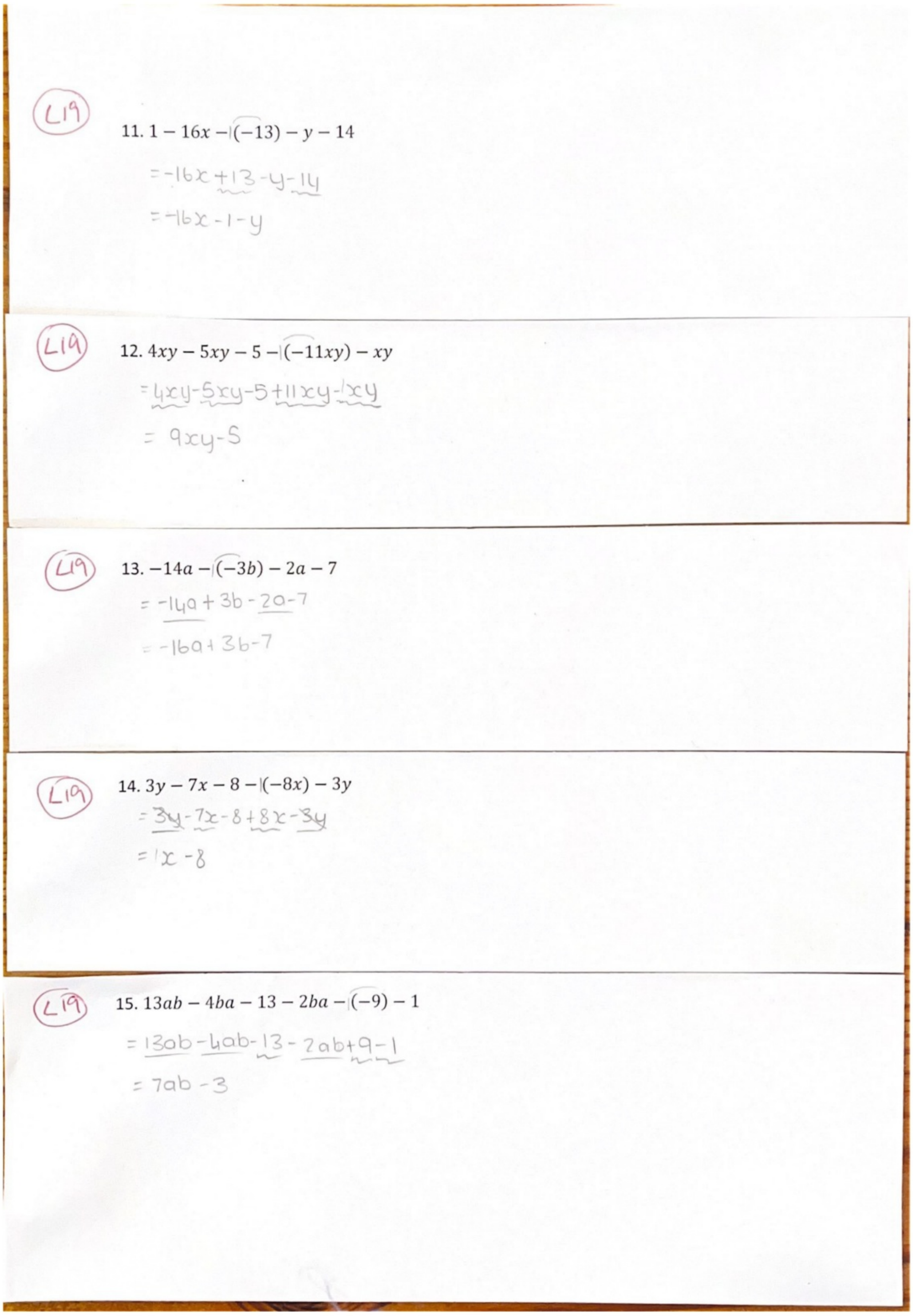
$= 2x^3 - 15$

L19

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$= -2x^4 - 13x^4 - 11 - 3 + 3x^4$

$= -12x^4 - 14$



L19

11. $1 - 16x - (-13) - y - 14$

$= -16x \utilde{+13} - y \utilde{-14}$

$= -16x - 1 - y$

L19

12. $4xy - 5xy - 5 - (-11xy) - xy$

$= \utilde{4xy} \utilde{-5xy} - 5 \utilde{+11xy} \utilde{-1xy}$

$= 9xy - 5$

L19

13. $-14a - (-3b) - 2a - 7$

$= \underline{-14a} + 3b - \underline{2a} - 7$

$= -16a + 3b - 7$

L19

14. $3y - 7x - 8 - (-8x) - 3y$

$= \underline{3y} - \utilde{7x} - 8 \utilde{+8x} - \underline{3y}$

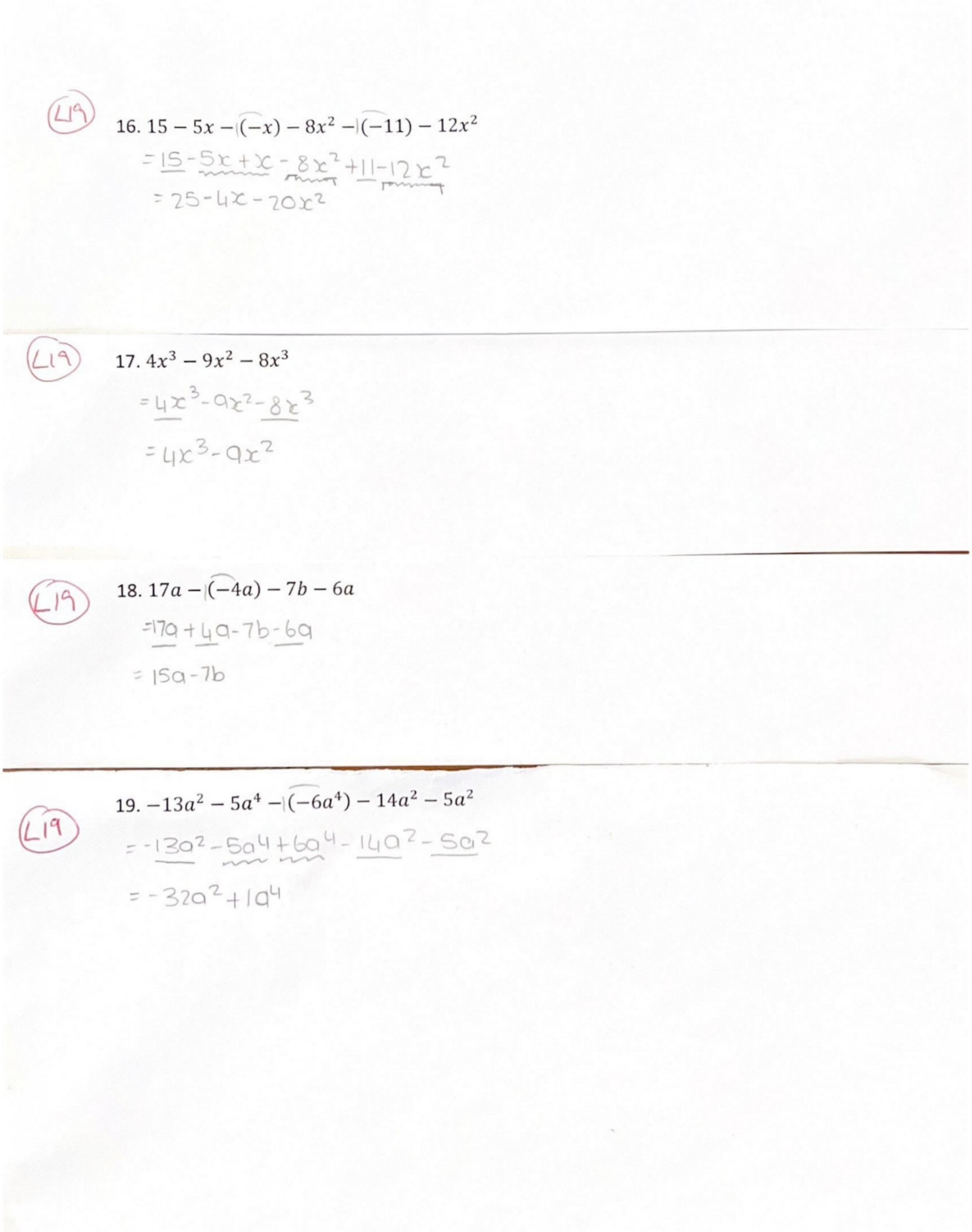
$= 1x - 8$

L19

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= \underline{13ab} \underline{-4ab} - \utilde{13} - \underline{2ab} \utilde{+9-1}$

$= 7ab - 3$



L19

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$= 15 - 5x + x - 8x^2 + 11 - 12x^2$

$= 25 - 4x - 20x^2$

L19

17. $4x^3 - 9x^2 - 8x^3$

$= 4x^3 - 9x^2 - 8x^3$

$= 4x^3 - 9x^2$

L19

18. $17a - (-4a) - 7b - 6a$

$= 17a + 4a - 7b - 6a$

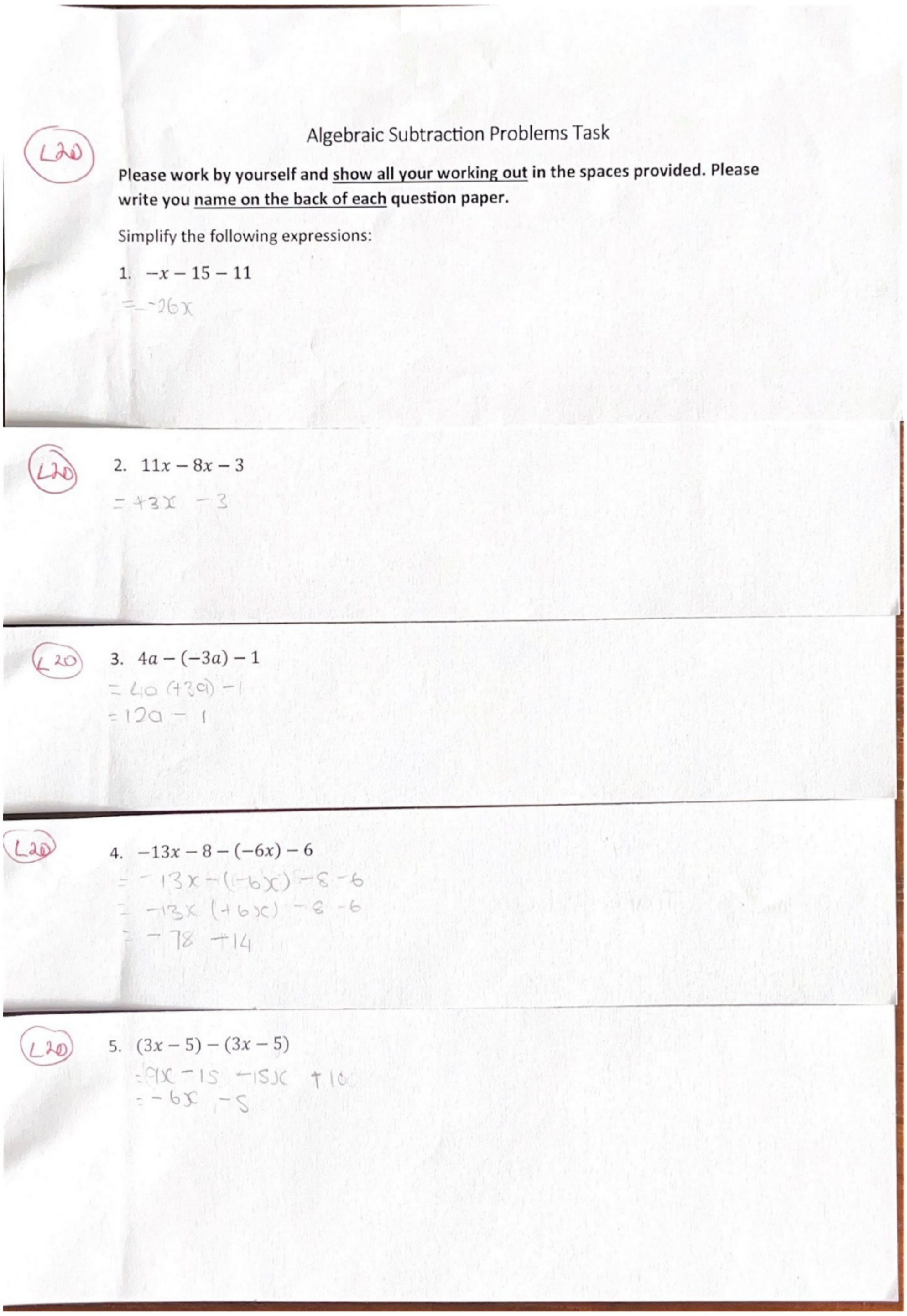
$= 15a - 7b$

L19

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$= -13a^2 - 5a^4 + 6a^4 - 14a^2 - 5a^2$

$= -32a^2 + 1a^4$



L20

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$= -26x$

L20

2. $11x - 8x - 3$

$= +3x \quad -3$

L20

3. $4a - (-3a) - 1$

$= 4a\ (+3a) - 1$

$= 12a - 1$

L20

4. $-13x - 8 - (-6x) - 6$

$= -13x + (-6x) - 8 - 6$

$= -13x\ (+6x) - 8 - 6$

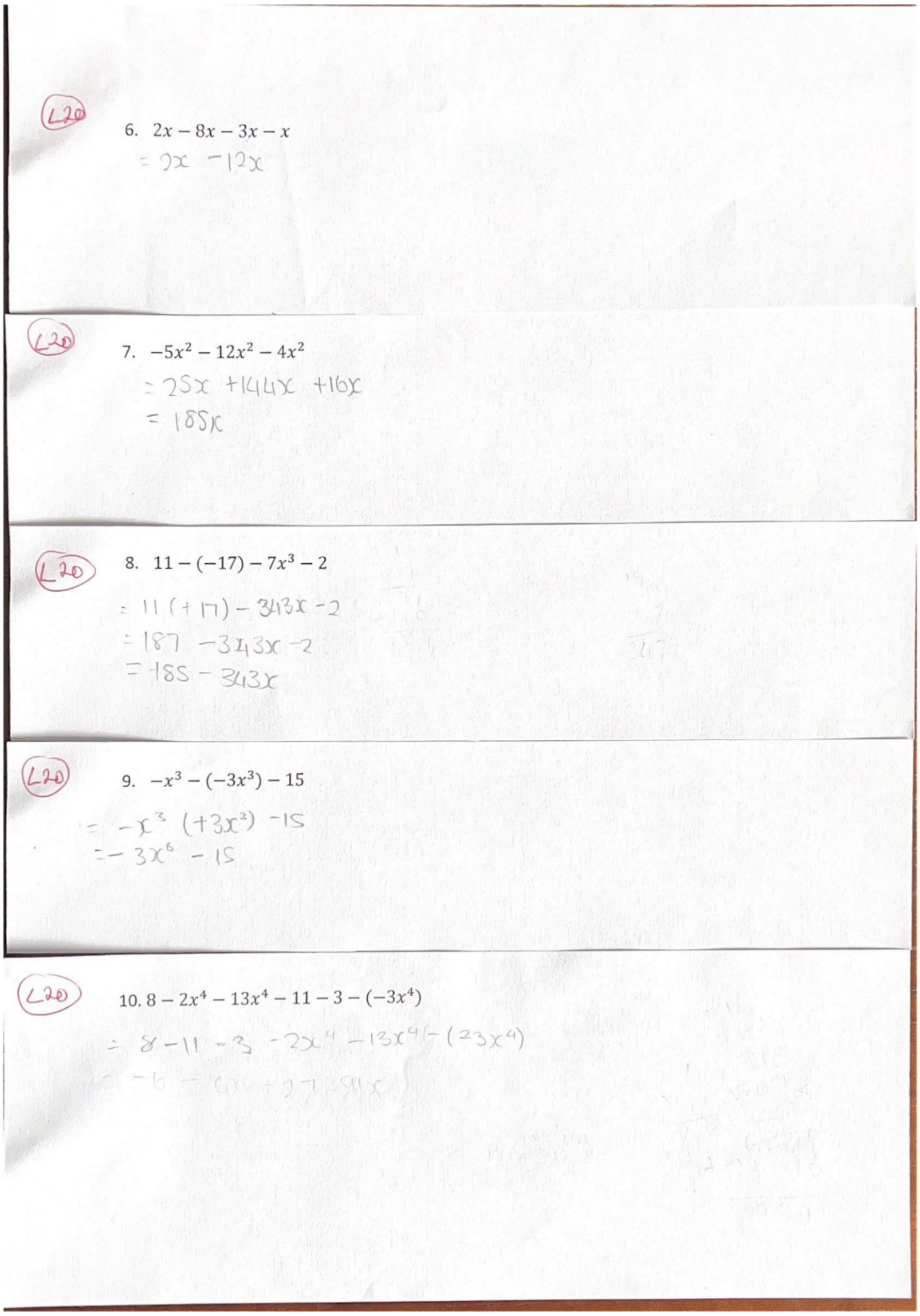
$= -78 \quad +14$

L20

5. $(3x - 5) - (3x - 5)$

$= 9x - 15 \quad -15x \quad +10$

$= -6x \quad -5$



L20

6. $2x - 8x - 3x - x$

$= 2x - 12x$

L20

7. $-5x^2 - 12x^2 - 4x^2$

$= 25x + 144x + 16x$

$= 185x$

L20

8. $11 - (-17) - 7x^3 - 2$

$= 11 (+17) - 343x - 2$

$= 187 - 343x - 2$

$= 185 - 343x$

L20

9. $-x^3 - (-3x^3) - 15$

$= -x^3 \ (+3x^3) - 15$

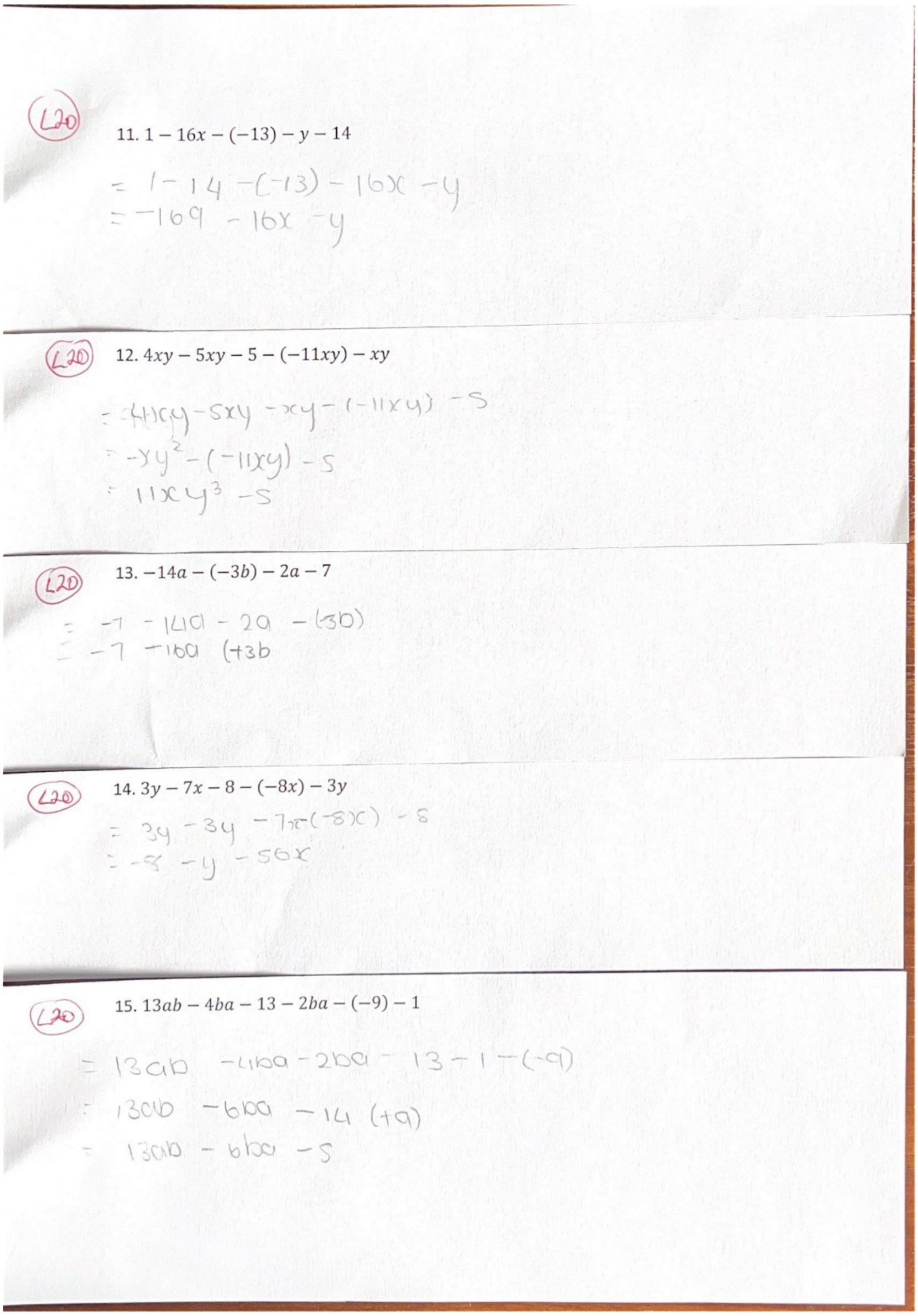
$= -3x^6 - 15$

L20

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$= 8 - 11 - 3 - 2x^4 - 13x^4 - (23x^4)$

$= -6 - (6x + 27 = 91x)$



L20

11. $1 - 16x - (-13) - y - 14$

$= 1 - 14 - (-13) - 16x - y$

$= -169 - 16x - y$

L20

12. $4xy - 5xy - 5 - (-11xy) - xy$

$= (4)xy - 5xy - xy - (-11xy) - 5$

$= -xy^2 - (-11xy) - 5$

$= 11xy^3 - 5$

L20

13. $-14a - (-3b) - 2a - 7$

$= -7 - 14a - 2a - (3b)$

$= -7 - 16a\ (+3b$

L20

14. $3y - 7x - 8 - (-8x) - 3y$

$= 3y - 3y - 7x - (-8x) - 8$

$= -8 - y - 56x$

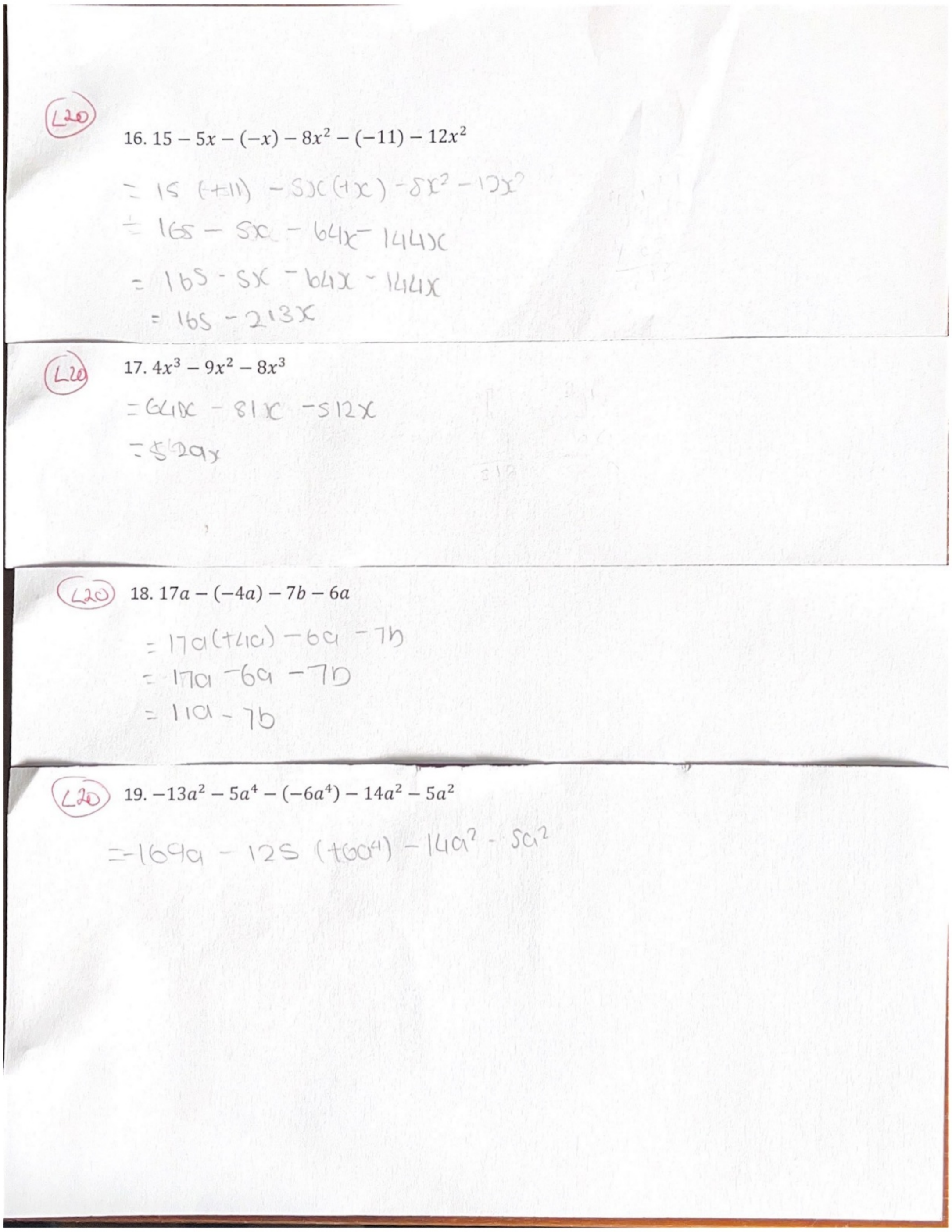
L20

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= 13ab - 4ba - 2ba - 13 - 1 - (-9)$

$= 13ab - 6ba - 14\ (+9)$

$= 13ab - 6ba - 5$



L20

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$= 15\ (+11) - 5x\ (+x) - 8x^2 - 12x^2$

$= 165 - 5x - 64x - 144x$

$= 165 - 5x - 64x - 144x$

$= 165 - 213x$

L20

17. $4x^3 - 9x^2 - 8x^3$

$= 64x - 81x - 512x$

$= 529x$

L20

18. $17a - (-4a) - 7b - 6a$

$= 17a(+4a) - 6a - 7b$

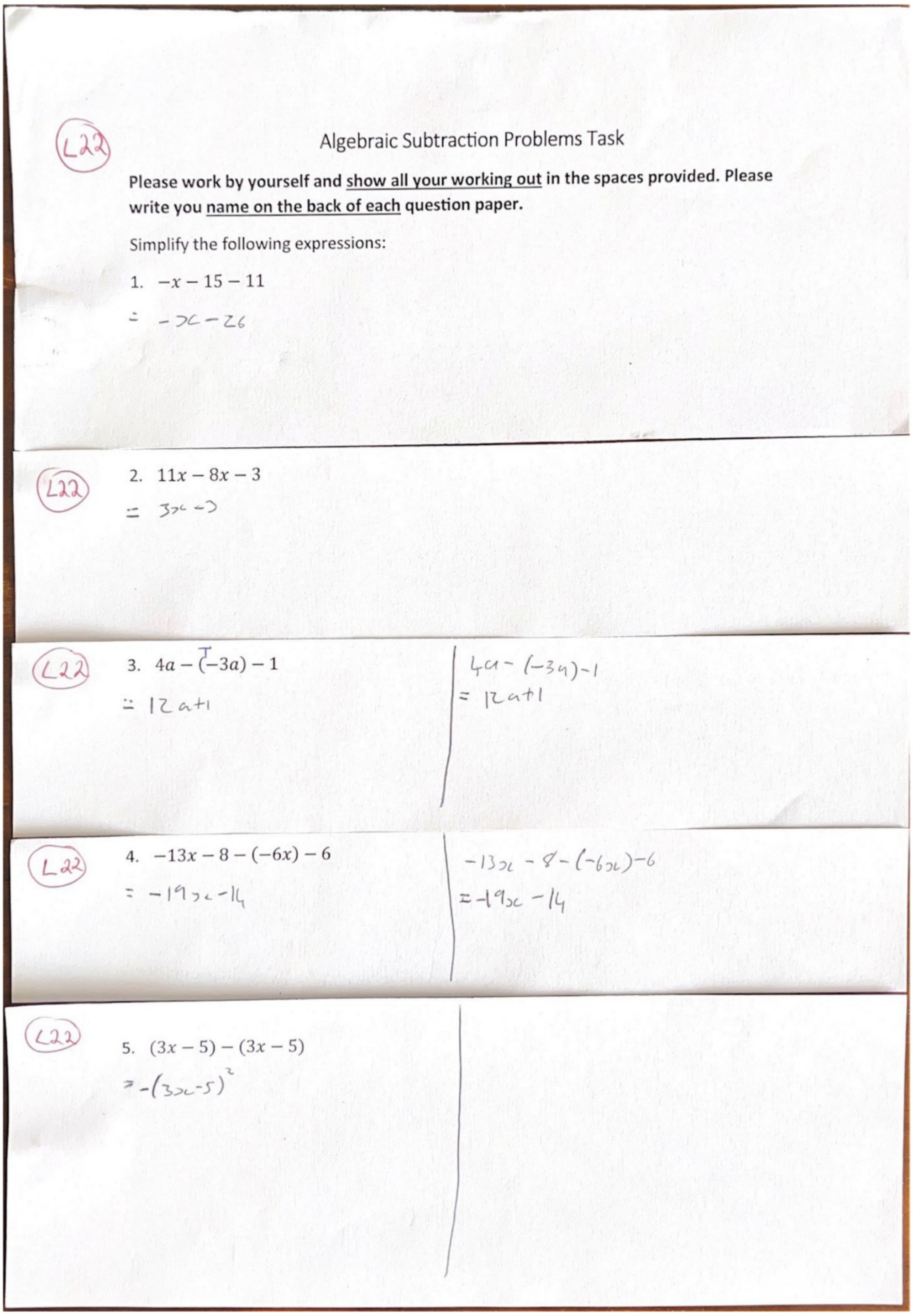
$= 17a - 6a - 7b$

$= 11a - 7b$

L20

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$= -169a - 125\ (+6a^4) - 14a^2 - 5a^2$



L22

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$= -x - 26$

L22

2. $11x - 8x - 3$

$= 3x - 3$

L22

3. $4a - \overset{T}{(}-3a) - 1$

$= 12a + 1$

$4a - (-3a) - 1$

$= 12a + 1$

L22

4. $-13x - 8 - (-6x) - 6$

$= -19x - 14$

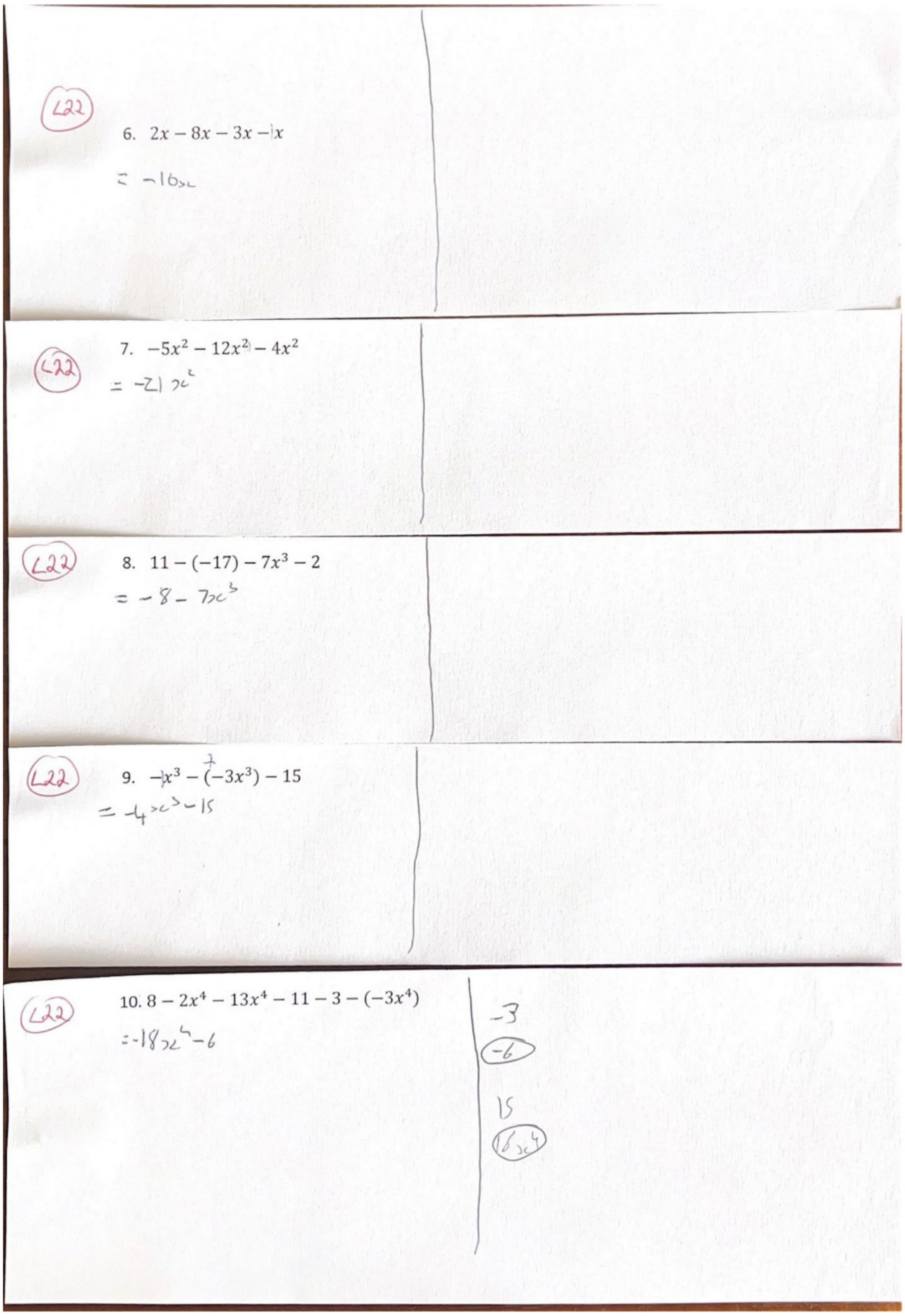
$-13x - 8 - (-6x) - 6$

$= -19x - 14$

L22

5. $(3x - 5) - (3x - 5)$

$= -(3x - 5)^2$



L22

6. $2x - 8x - 3x - 1x$

$= -16x$

L22

7. $-5x^2 - 12x^2 - 4x^2$

$= -21x^2$

L22

8. $11 - (-17) - 7x^3 - 2$

$= -8 - 7x^3$

L22

9. $-1x^3 - \overset{+}{(}-3x^3) - 15$

$= -4x^3 - 15$

L22

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

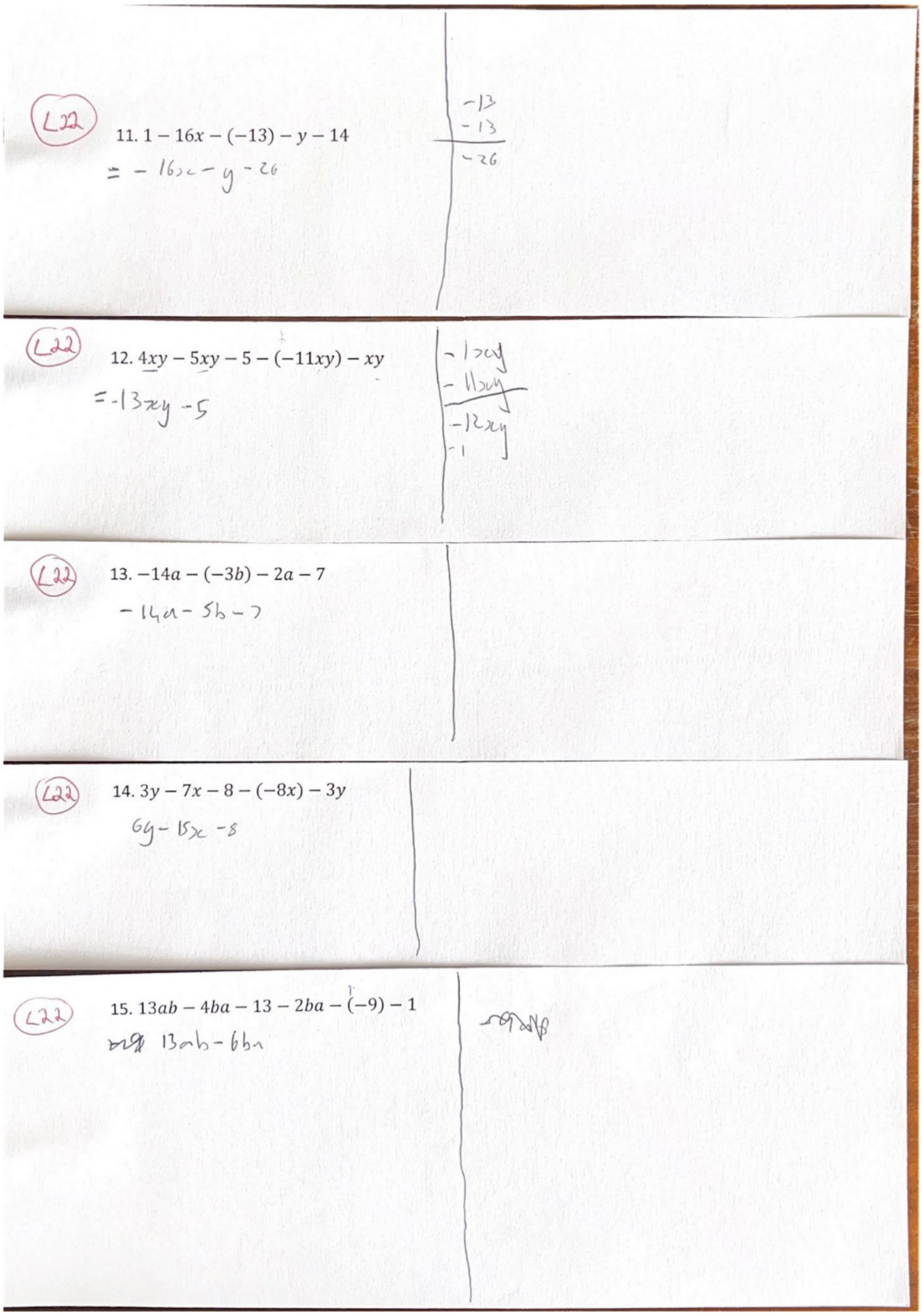
$= -18x^4 - 6$

-3

$\boxed{-6}$

15

$\boxed{16x^4}$



L22

11. $1 - 16x - (-13) - y - 14$

$= -16x - y - 26$

-13
-13
-26

L22

12. $4xy - 5xy - 5 - (-11xy) - xy$

$= -13xy - 5$

$-1xy$
$-11xy$
$-12xy$
-1

L22

13. $-14a - (-3b) - 2a - 7$

$-14a - 5b - 7$

L22

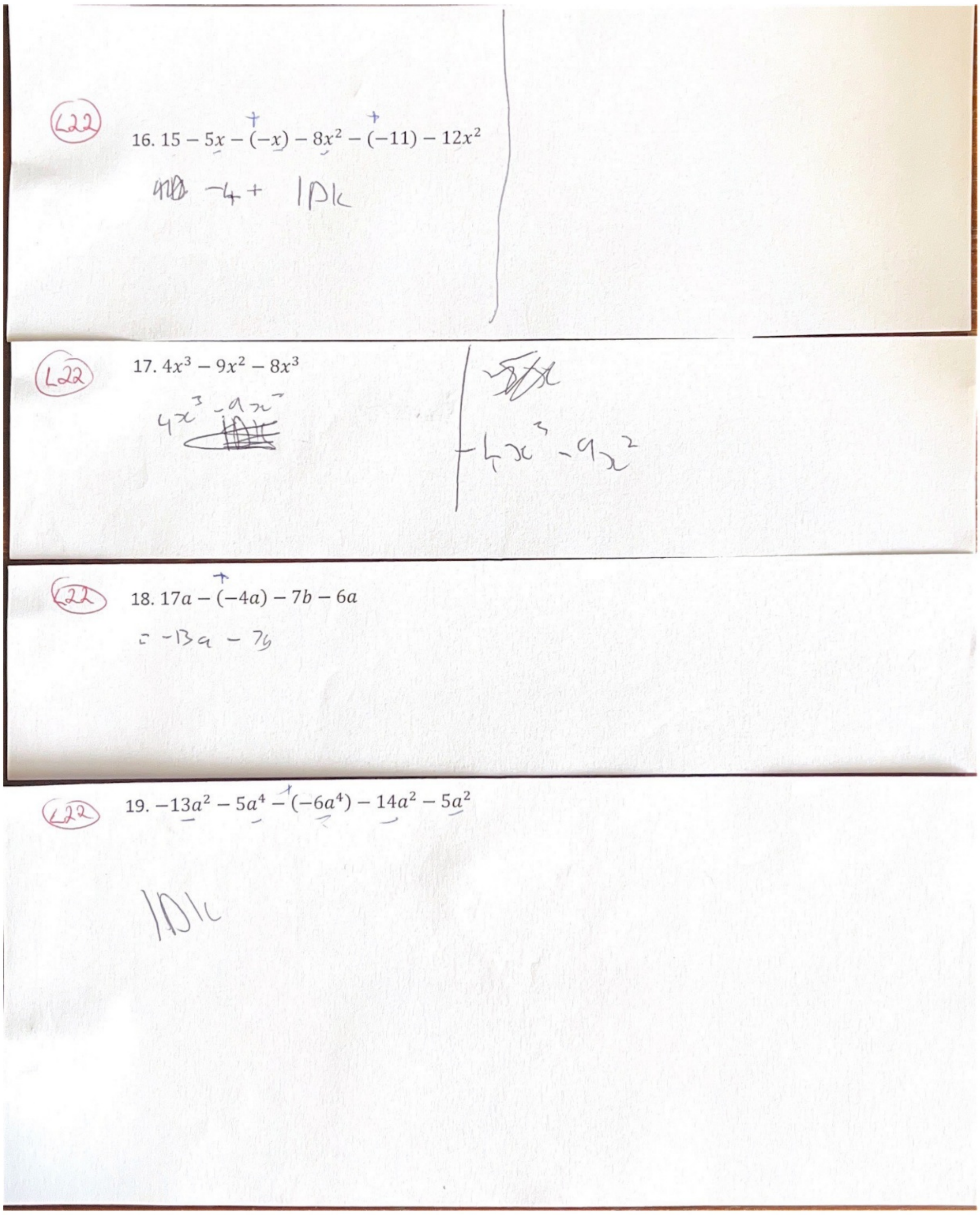
14. $3y - 7x - 8 - (-8x) - 3y$

$6y - 15x - 8$

L22

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$13ab - 6ba$



L22

16. $15 - 5x - \overset{+}{(}-x) - 8x^2 - \overset{+}{(}-11) - 12x^2$

[illegible] -4+ IDK

L22

17. $4x^3 - 9x^2 - 8x^3$

$4x^3 - 9x^2$

$-4x^3 - 9x^2$

L22

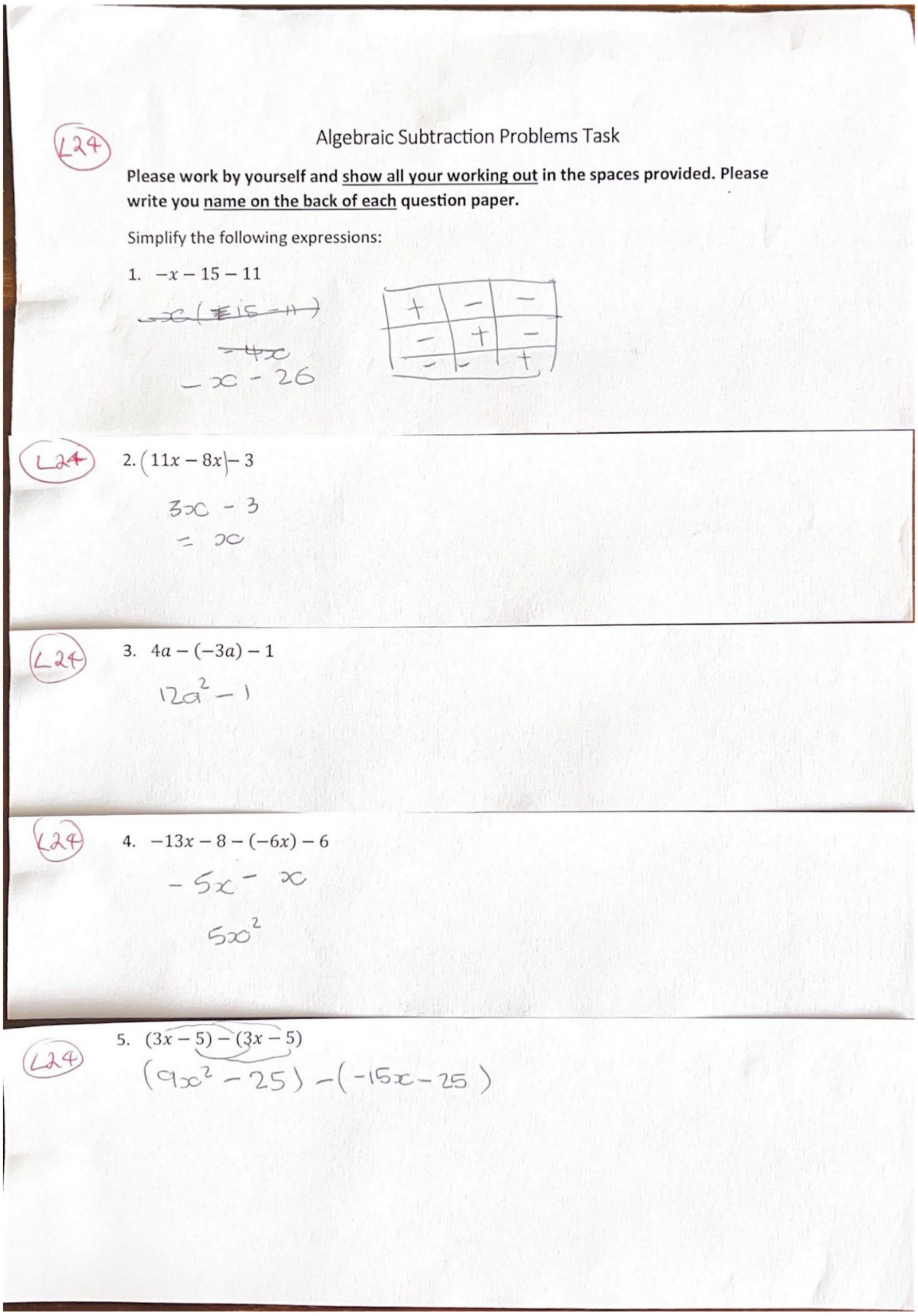
18. $17a - \overset{+}{(}-4a) - 7b - 6a$

$= -13a - 7b$

L22

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

IDK



L24

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

~~$-x(\pm 15 - 11)$~~

~~$-4x$~~

$-x - 26$

+	−	−
−	+	−
−	−	+

L24

2. $(11x - 8x) - 3$

$3x - 3$

$= x$

L24

3. $4a - (-3a) - 1$

$12a^2 - 1$

L24

4. $-13x - 8 - (-6x) - 6$

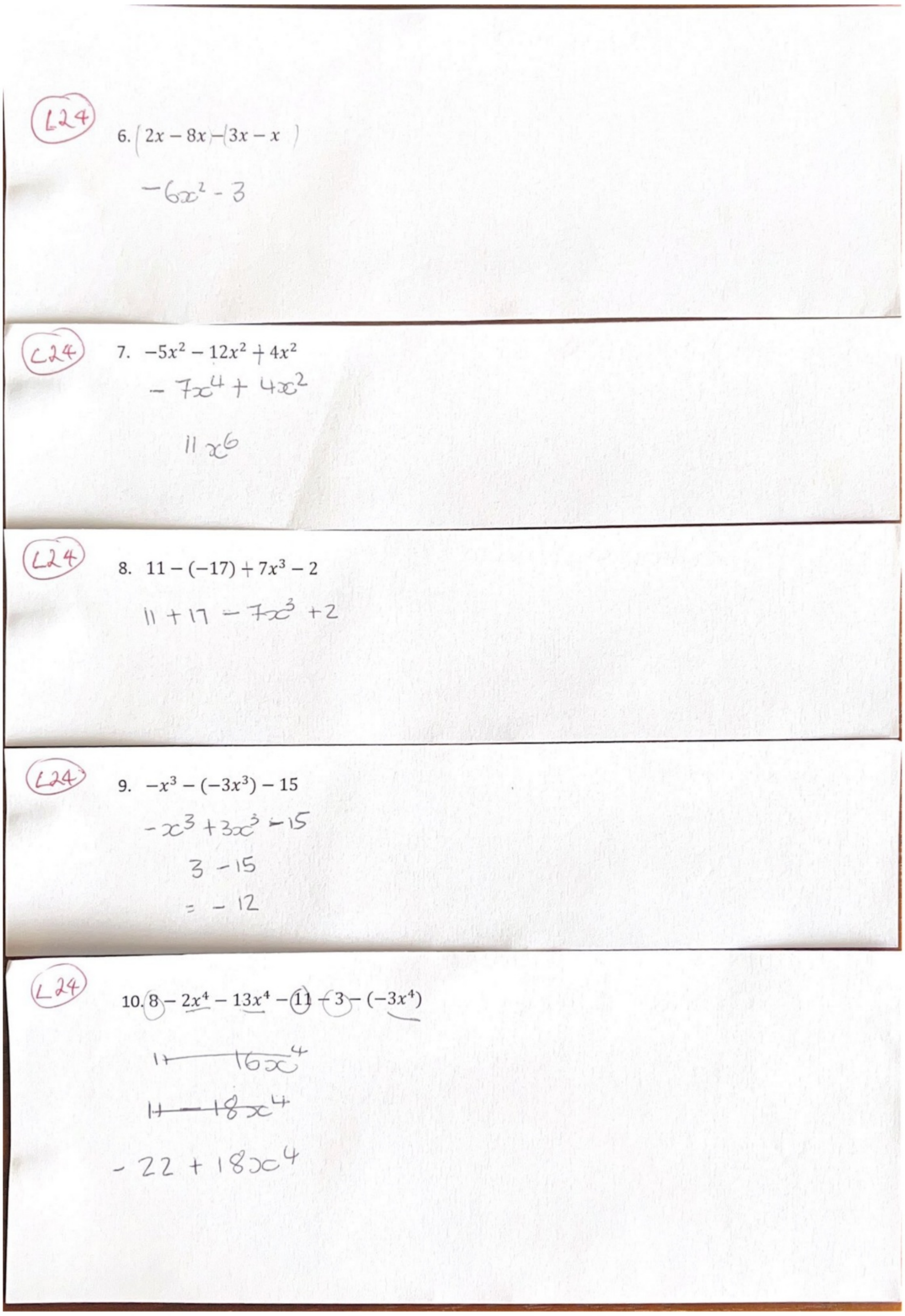
$-5x - x$

$5x^2$

L24

5. $(3x - 5) - (3x - 5)$

$(9x^2 - 25) - (-15x - 25)$



L24

6. $(2x - 8x) - (3x - x)$

$-6x^2 - 3$

L24

7. $-5x^2 - 12x^2 + 4x^2$

$-7x^4 + 4x^2$

$11x^6$

L24

8. $11 - (-17) + 7x^3 - 2$

$11 + 17 - 7x^3 + 2$

L24

9. $-x^3 - (-3x^3) - 15$

$-x^3 + 3x^3 - 15$

$3 - 15$

$= -12$

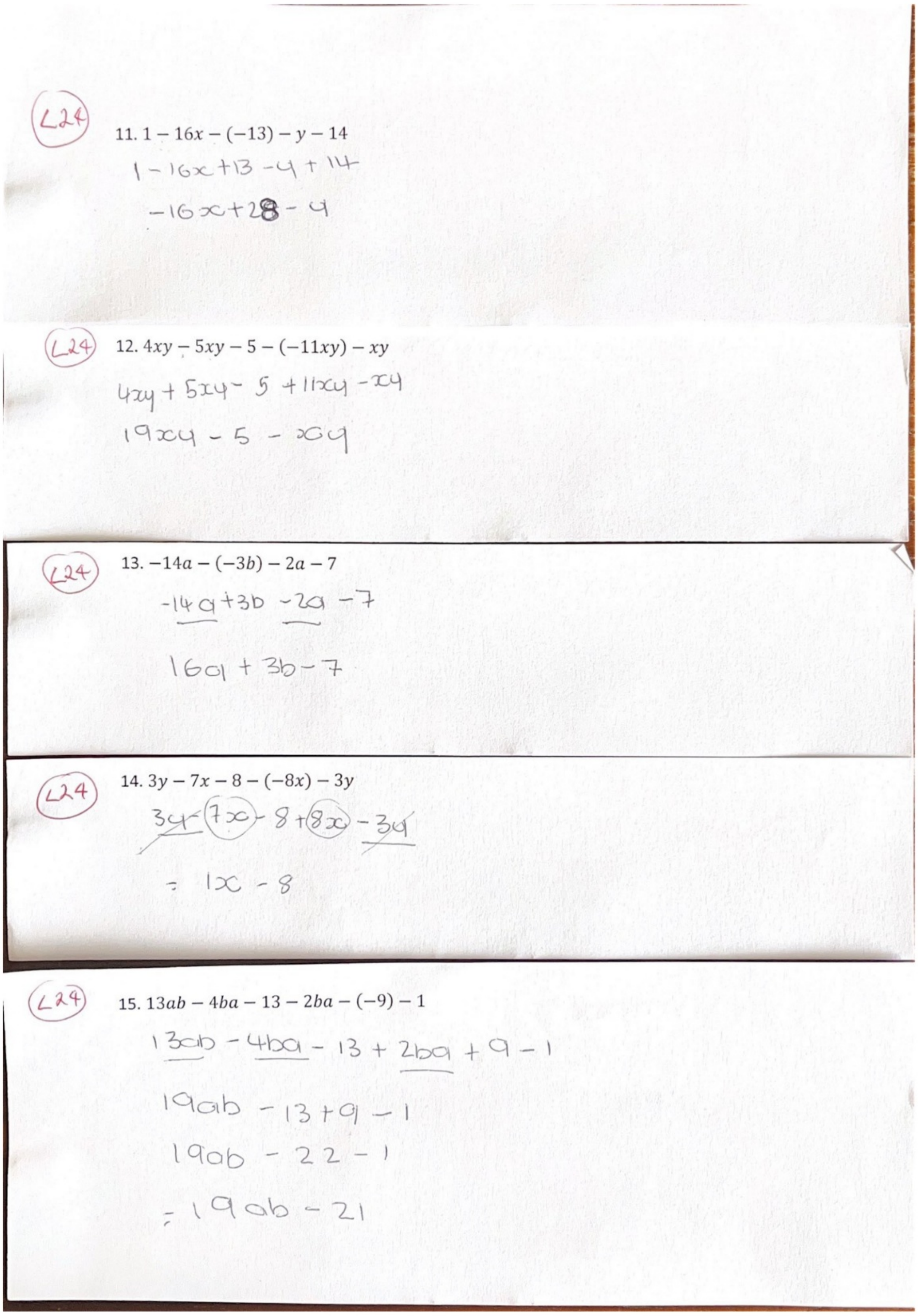
L24

10. $8 - \underline{2x^4} - \underline{13x^4} - 11 + 3 - (\underline{-3x^4})$

~~$11 - 16x^4$~~

~~$11 - 18x^4$~~

$-22 + 18x^4$



L24

11. $1 - 16x - (-13) - y - 14$

$1 - 16x + 13 - y + 14$

$-16x + 28 - y$

L24

12. $4xy - 5xy - 5 - (-11xy) - xy$

$4xy + 5xy - 5 + 11xy - xy$

$19xy - 5 - xy$

L24

13. $-14a - (-3b) - 2a - 7$

$\underline{-14a} + 3b \underline{-2a} - 7$

$16a + 3b - 7$

L24

14. $3y - 7x - 8 - (-8x) - 3y$

$\underline{\cancel{3y}} - 7x - 8 + 8x - \underline{\cancel{3y}}$

$= 1x - 8$

L24

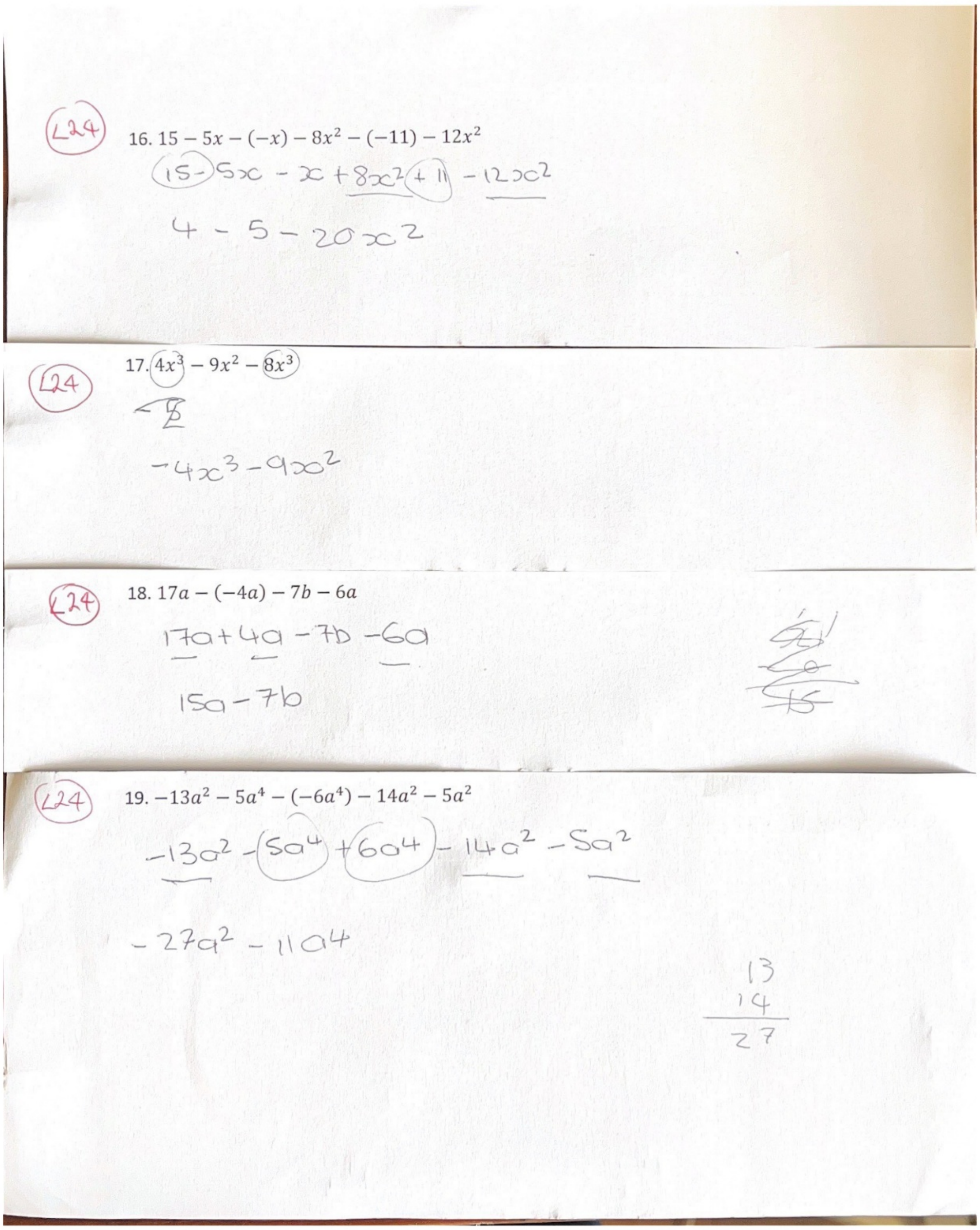
15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$1\underline{3ab} - \underline{4ba} - 13 + \underline{2ba} + 9 - 1$

$19ab - 13 + 9 - 1$

$19ab - 22 - 1$

$= 19ab - 21$



L24

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$15 - 5x - x + 8x^2 + 11 - 12x^2$

$4 - 5 - 20x^2$

L24

17. $4x^3 - 9x^2 - 8x^3$

$-4x^3 - 9x^2$

L24

18. $17a - (-4a) - 7b - 6a$

$17a + 4a - 7b - 6a$

$15a - 7b$

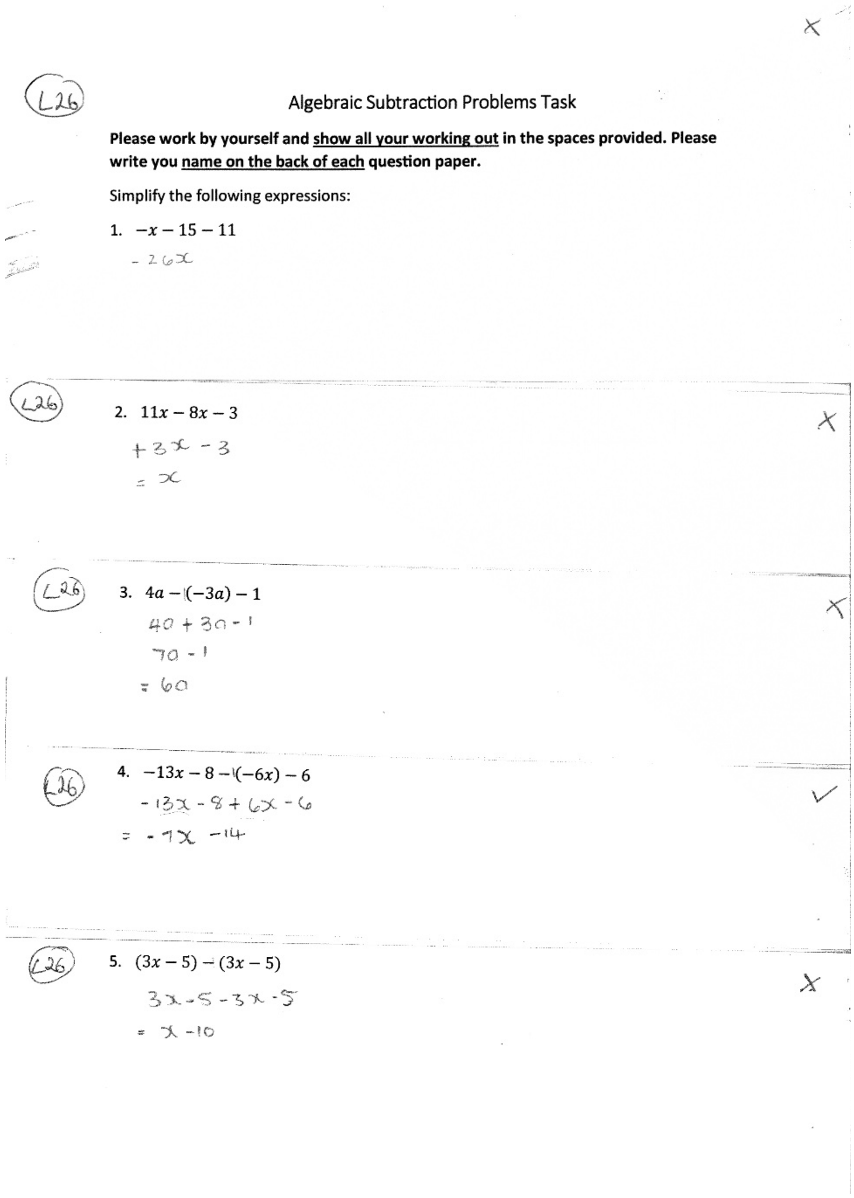
L24

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$-13a^2 - 5a^4 + 6a^4 - 14a^2 - 5a^2$

$-27a^2 - 11a^4$

$$\begin{array}{r} 13 \\ 14 \\ \hline 27 \end{array}$$



✗

L26

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$-26x$

L26

2. $11x - 8x - 3$ ✗

$+3x - 3$

$= x$

L26

3. $4a - (-3a) - 1$ ✗

$4a + 3a - 1$

$7a - 1$

$= 6a$

L26

4. $-13x - 8 - (-6x) - 6$ ✓

$-13x - 8 + 6x - 6$

$= -7x - 14$

L26

5. $(3x - 5) - (3x - 5)$ ✗

$3x - 5 - 3x - 5$

$= x - 10$

✓

L26 6. $2x - 8x - 3x - x$

$2x - 12x$

$= -10x$

L26 7. $-5x^2 - 12x^2 - 4x^2$ ✓

~~6x~~

$-21x^2$

L26 8. $11 - (-17) - 7x^3 - 2$ ✗

$11 + 17 - 7x^3 - 2$

$= 28 - 7x^3 - 2$

L26 9. $-x^3 - (-3x^3) - 15$ ✓

$-x^3 + 3x^3 - 15$

$2x^3 - 15$

L26 10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$ ✗

$8 - 2x^4 - 13x^4 - 11 - 3 + 3x^4$

$8 - 12x^4 - 14$

✓

L26 11. $1 - 16x - (-13) - y - 14$

$1 - 16x + 13 - y - 14$

$-16x - y$

L26 12. $4xy - 5xy - 5 - (-11xy) - xy$ ✓

$4xy - 5xy - 5 + 11xy - xy$

$9xy - 5$

L26 13. $-14a - (-3b) - 2a - 7$ ✓

$-14a + 3b - 2a - 7$

$-16a + 3b - 7$

L26 14. $3y - 7x - 8 - (-8x) - 3y$ ✓

$3y - 7x - 8 + 8x - 3y$

$1x - 8$

L26 15. $13ab - 4ba - 13 - 2ba - (-9) - 1$ ✗

$13ab - 4ba - 13 - 2ba + 9 - 1$

$7ba - 21$

✓

Ⓛ L26

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$15 - 5x + x - 8x^2 + 11 - 12x^2$

$26 - 4x - 20x^2$

L26

17. $4x^3 - 9x^2 - 8x^3$ ✗

$4x^3 - 9x^2$

L26

18. $17a - (-4a) - 7b - 6a$ ✗

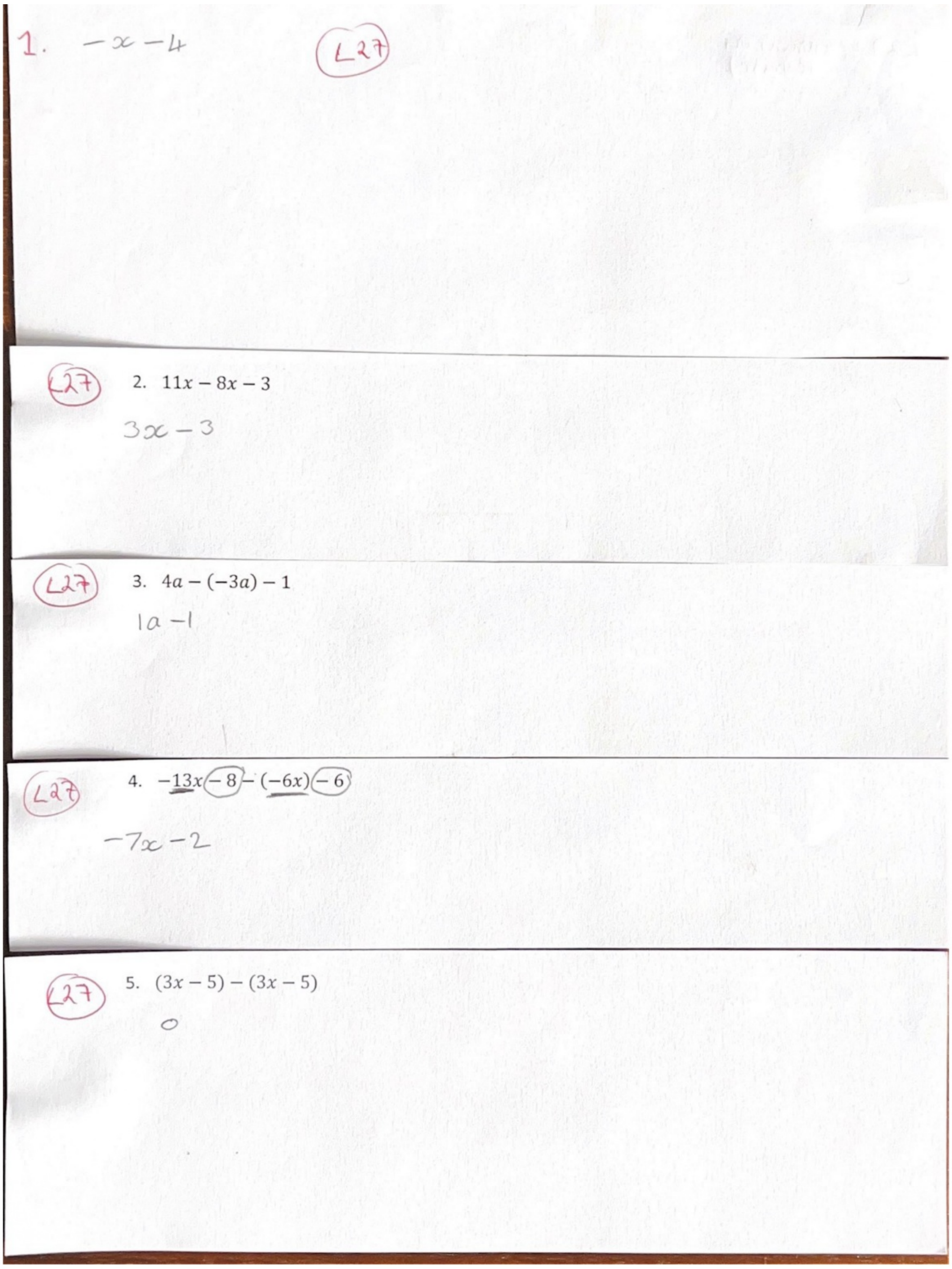
$17a + 4a - 7b - 7a$

$14a - 7b$

L26

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$ ✗

$-13a^2 - 5a^4 + 6a^4 - 14a^2 - 5a^2$



1. $-x-4$

L27

L27

2. $11x-8x-3$

$3x-3$

L27

3. $4a-(-3a)-1$

$1a-1$

L27

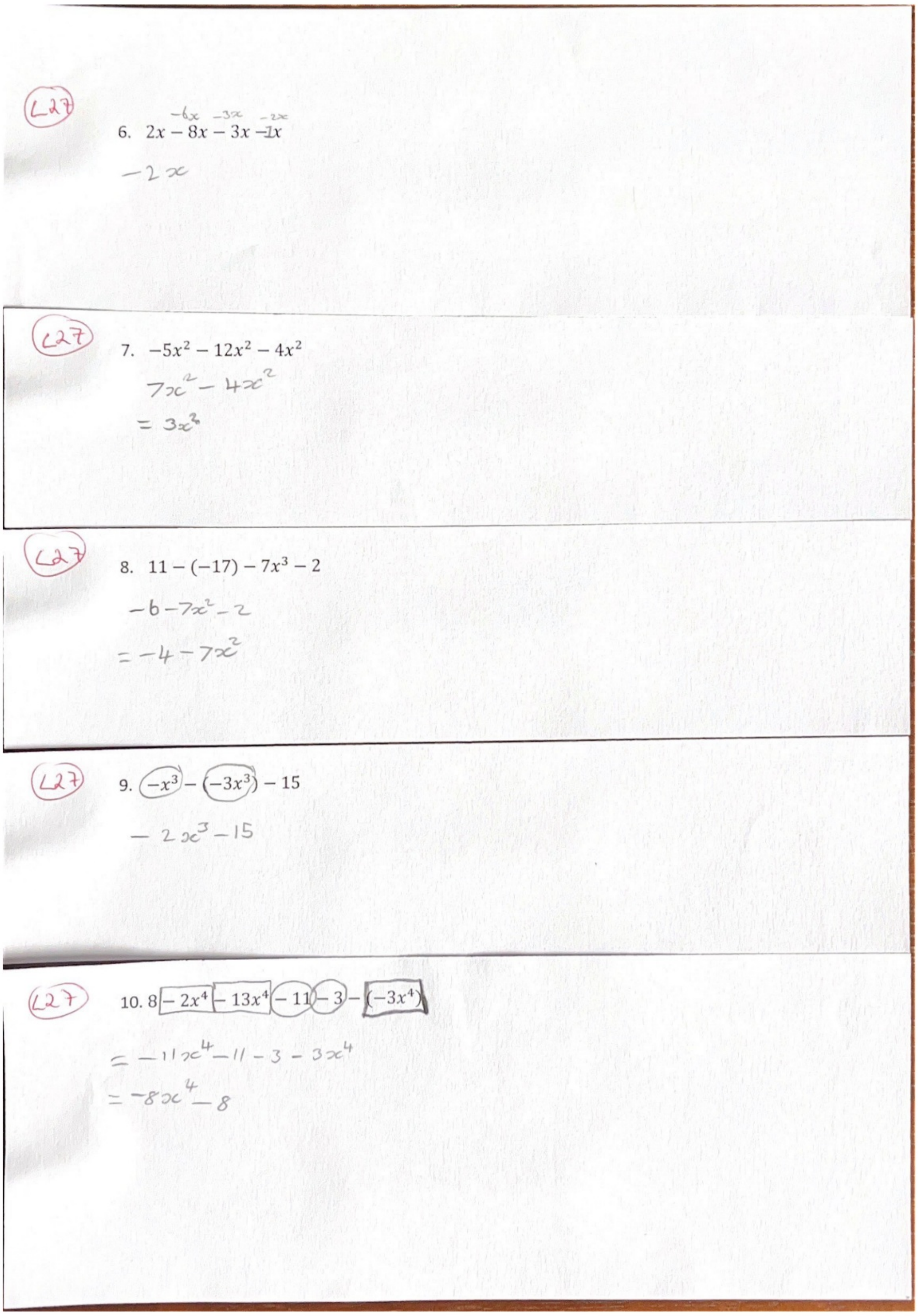
4. $-\underline{13x}\boxed{-8}-(\underline{-6x})\boxed{-6}$

$-7x-2$

L27

5. $(3x-5)-(3x-5)$

0



L27

6. $2x - \overset{-6x}{8x} - \overset{-3x}{3x} - \overset{-2x}{1x}$

$-2x$

L27

7. $-5x^2 - 12x^2 - 4x^2$

$7x^2 - 4x^2$

$= 3x^2$

L27

8. $11 - (-17) - 7x^3 - 2$

$-6 - 7x^2 - 2$

$= -4 - 7x^2$

L27

9. $-x^3 - (-3x^3) - 15$

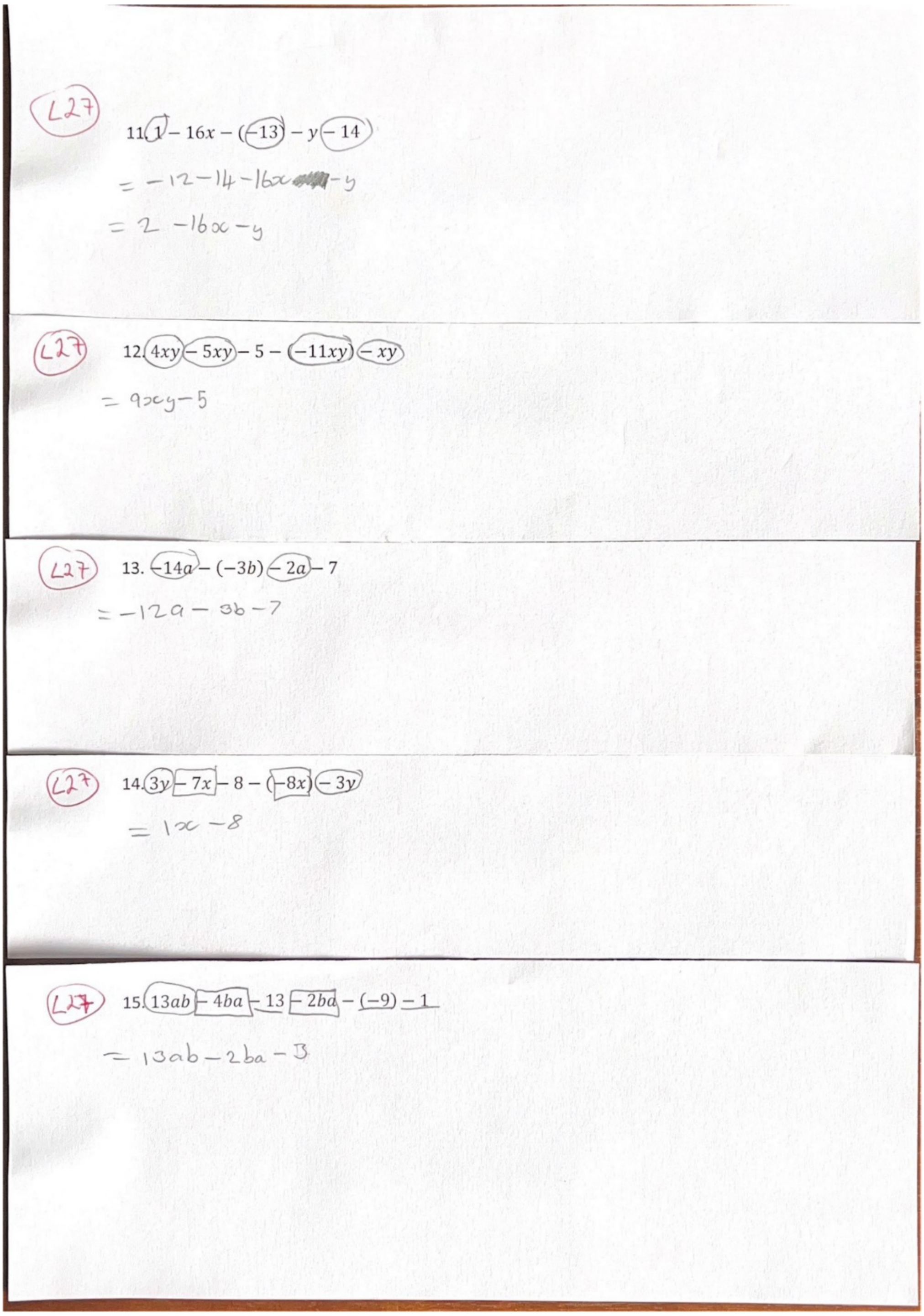
$-2x^3 - 15$

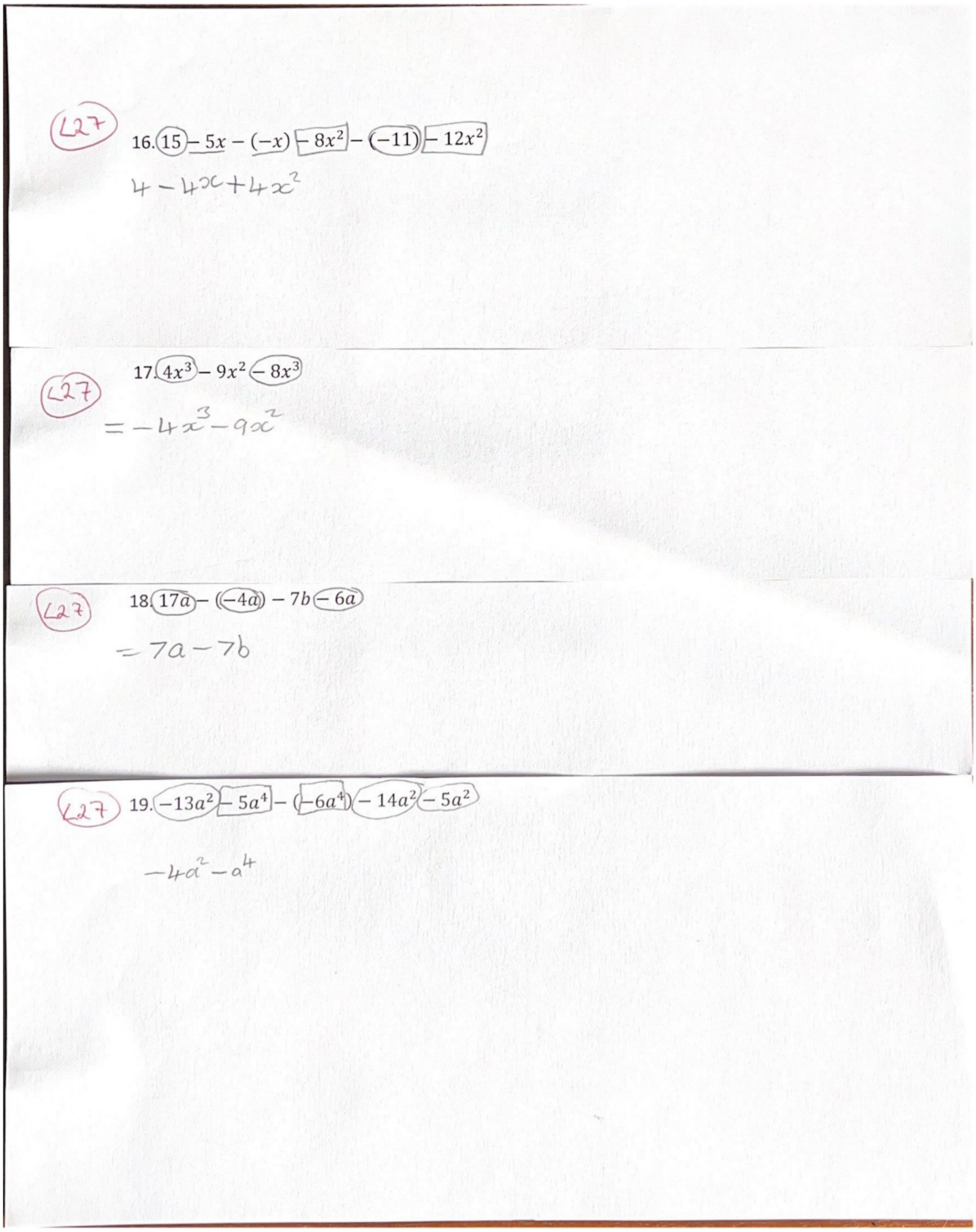
L27

10. $8 \boxed{-2x^4} \boxed{-13x^4} - 11 - 3 - \boxed{(-3x^4)}$

$= -11x^4 - 11 - 3 - 3x^4$

$= -8x^4 - 8$





L27

16. $15 - 5x - (-x) \boxed{- 8x^2} - (-11) \boxed{- 12x^2}$

$4 - 4x + 4x^2$

17. $4x^3 - 9x^2 - 8x^3$

L27

$= -4x^3 - 9x^2$

L27

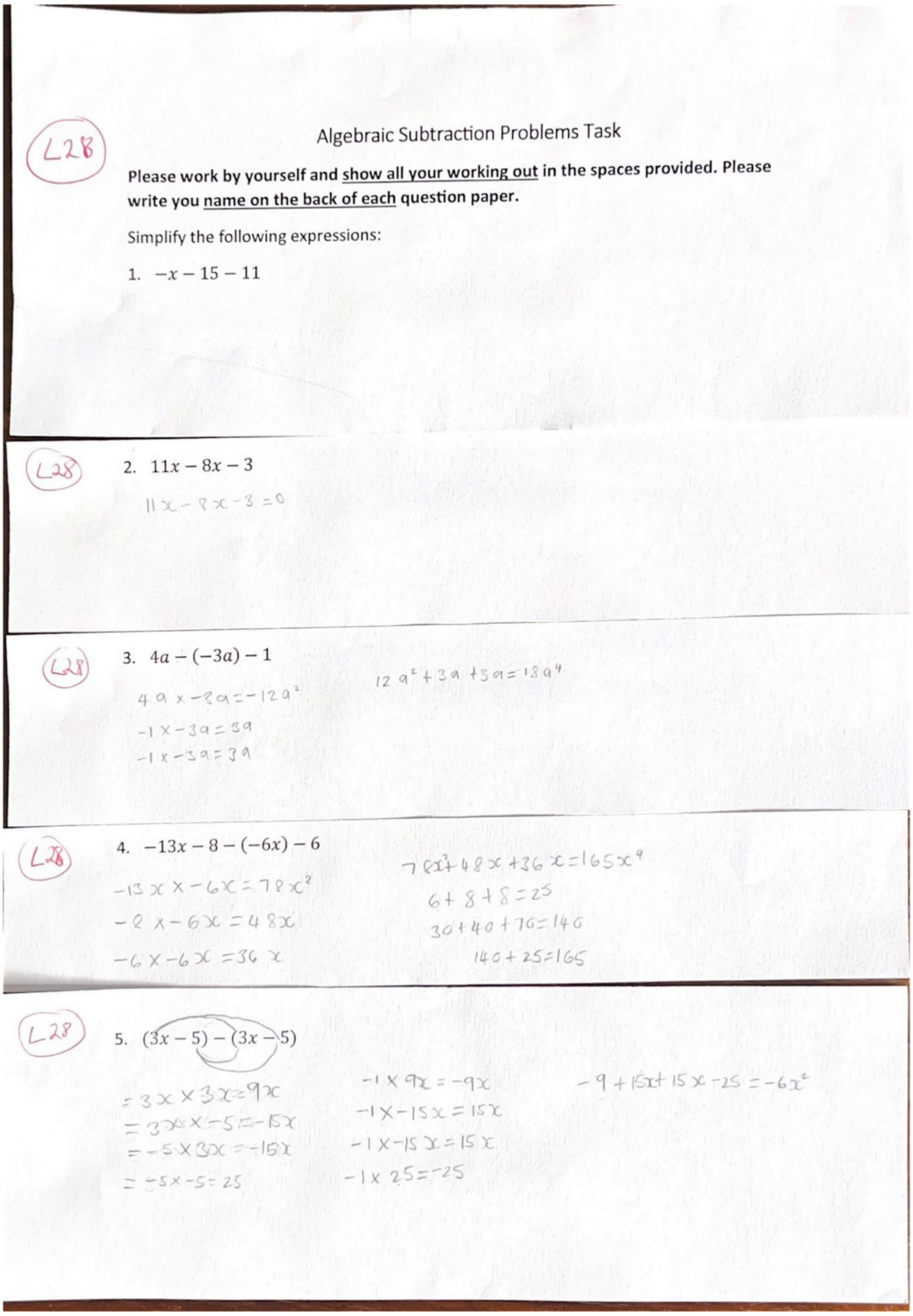
18. $17a - (-4a) - 7b - 6a$

$= 7a - 7b$

L27

19. $-13a^2 \boxed{- 5a^4} - (\boxed{-6a^4}) - 14a^2 - 5a^2$

$-4a^2 - a^4$



L28

Algebraic Subtraction Problems Task

Please work by yourself and show all your working out in the spaces provided. Please write you name on the back of each question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

L28

2. $11x - 8x - 3$

$11x - 8x - 3 = 0$

L28

3. $4a - (-3a) - 1$

$4a \times -3a = -12a^2$

$-1 \times -3a = 3a$

$-1 \times -3a = 3a$

$12a^2 + 3a + 3a = 18a^4$

L28

4. $-13x - 8 - (-6x) - 6$

$-13x \times -6x = 78x^2$

$-8 \times -6x = 48x$

$-6 \times -6x = 36x$

$78x^2 + 48x + 36x = 165x^4$

$6 + 8 + 8 = 25$

$30 + 40 + 70 = 140$

$140 + 25 = 165$

L28

5. $(3x - 5) - (3x - 5)$

$= 3x \times 3x = 9x$

$= 3x \times -5 = -15x$

$= -5 \times 3x = -15x$

$= -5 \times -5 = 25$

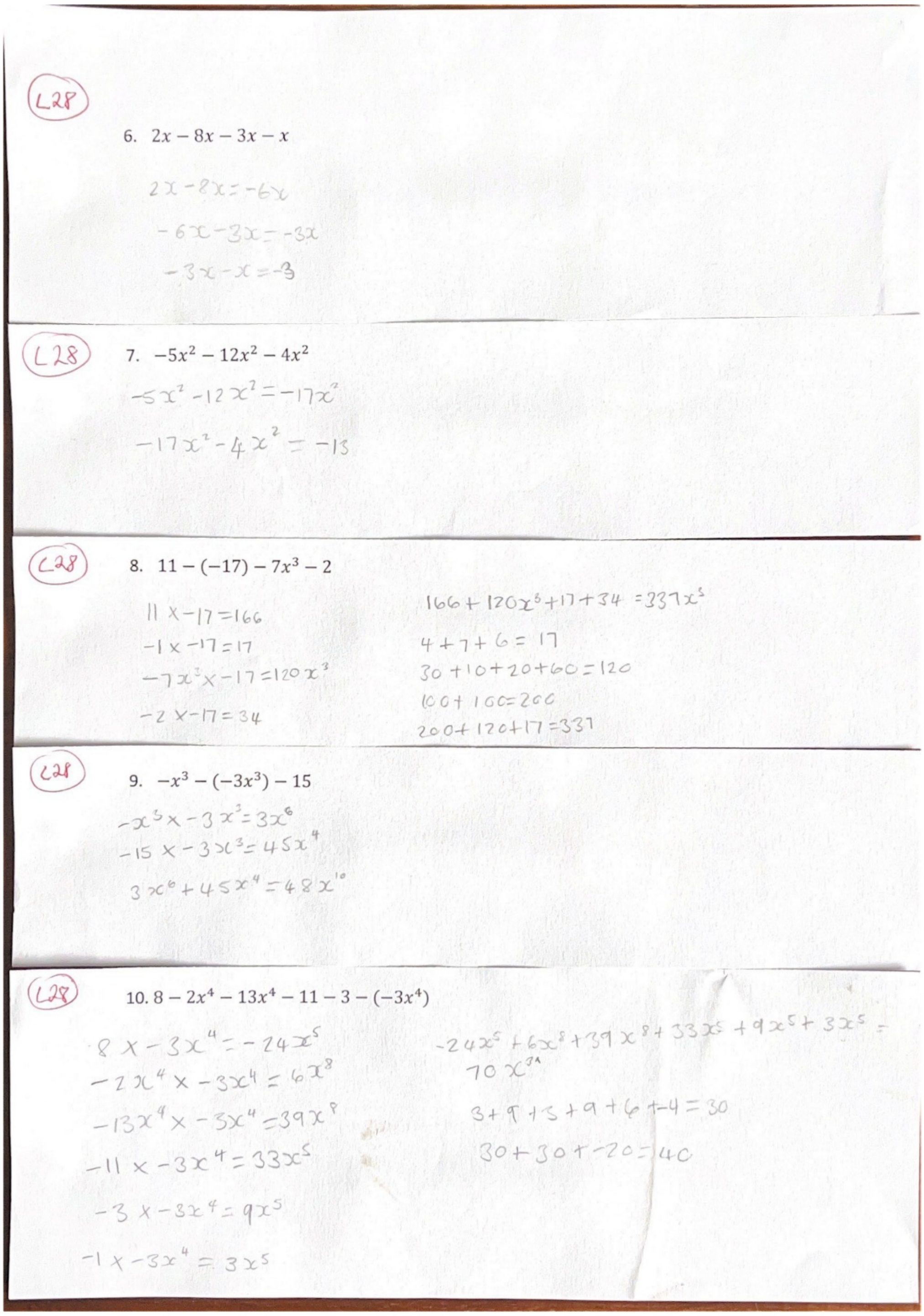
$-1 \times 9x = -9x$

$-1 \times -15x = 15x$

$-1 \times -15x = 15x$

$-1 \times 25 = -25$

$-9 + 15x + 15x - 25 = -6x^2$



L28

6. $2x - 8x - 3x - x$

$2x - 8x = -6x$

$-6x - 3x = -3x$

$-3x - x = -3$

L28

7. $-5x^2 - 12x^2 - 4x^2$

$-5x^2 - 12x^2 = -17x^2$

$-17x^2 - 4x^2 = -13$

L28

8. $11 - (-17) - 7x^3 - 2$

$11 \times -17 = 166$

$-1 \times -17 = 17$

$-7x^3 \times -17 = 120x^3$

$-2 \times -17 = 34$

$166 + 120x^3 + 17 + 34 = 337x^3$

$4 + 7 + 6 = 17$

$30 + 10 + 20 + 60 = 120$

$100 + 100 = 200$

$200 + 120 + 17 = 337$

L28

9. $-x^3 - (-3x^3) - 15$

$-x^3 \times -3x^3 = 3x^6$

$-15 \times -3x^3 = 45x^4$

$3x^6 + 45x^4 = 48x^{10}$

L28

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$8 \times -3x^4 = -24x^5$

$-2x^4 \times -3x^4 = 6x^8$

$-13x^4 \times -3x^4 = 39x^8$

$-11 \times -3x^4 = 33x^5$

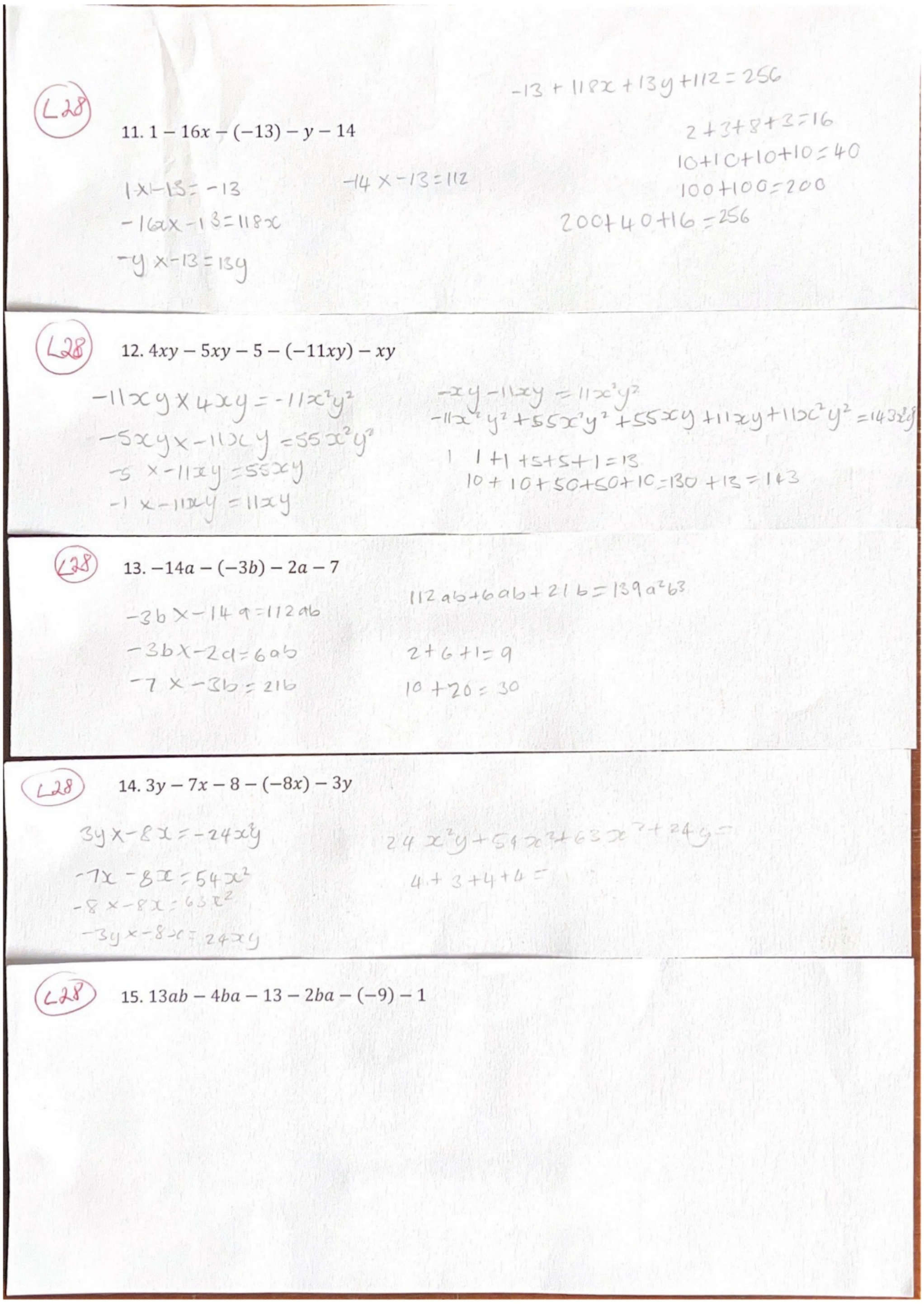
$-3 \times -3x^4 = 9x^5$

$-1 \times -3x^4 = 3x^5$

$-24x^5 + 6x^8 + 39x^8 + 33x^5 + 9x^5 + 3x^5 =$
$70x^{31}$

$3 + 9 + 3 + 9 + 6 + -4 = 30$

$30 + 30 + -20 = 40$



L28

$-13+118x+13y+112=256$

11. $1-16x-(-13)-y-14$

$2+3+8+3=16$

$10+10+10+10=40$

$100+100=200$

$1\times 1-13=-13$

$-14\times -13=112$

$-16x\times -13=118x$

$200+40+16=256$

$-y\times -13=13y$

L28

12. $4xy-5xy-5-(-11xy)-xy$

$-11xy\times 4xy=-11x^2y^2$

$-xy-11xy=11x^2y^2$

$-11x^2y^2+55x^2y^2+55xy+11xy+11x^2y^2=143x^2y$

$-5xy\times -11xy=55x^2y^2$

$-5\times -11xy=55xy$

$1 \quad 1+1+5+5+1=13$

$10+10+50+50+10=130+13=143$

$-1\times -11xy=11xy$

L28

13. $-14a-(-3b)-2a-7$

$112ab+6ab+21b=139a^2b3$

$-3b\times -14a=112ab$

$-3b\times -2a=6ab$

$2+6+1=9$

$-7\times -3b=21b$

$10+20=30$

L28

14. $3y-7x-8-(-8x)-3y$

$3y\times -8x=-24x^2y$

$24x^2y+54x^2+63x^2+24y=$

$-7x-8x=54x^2$

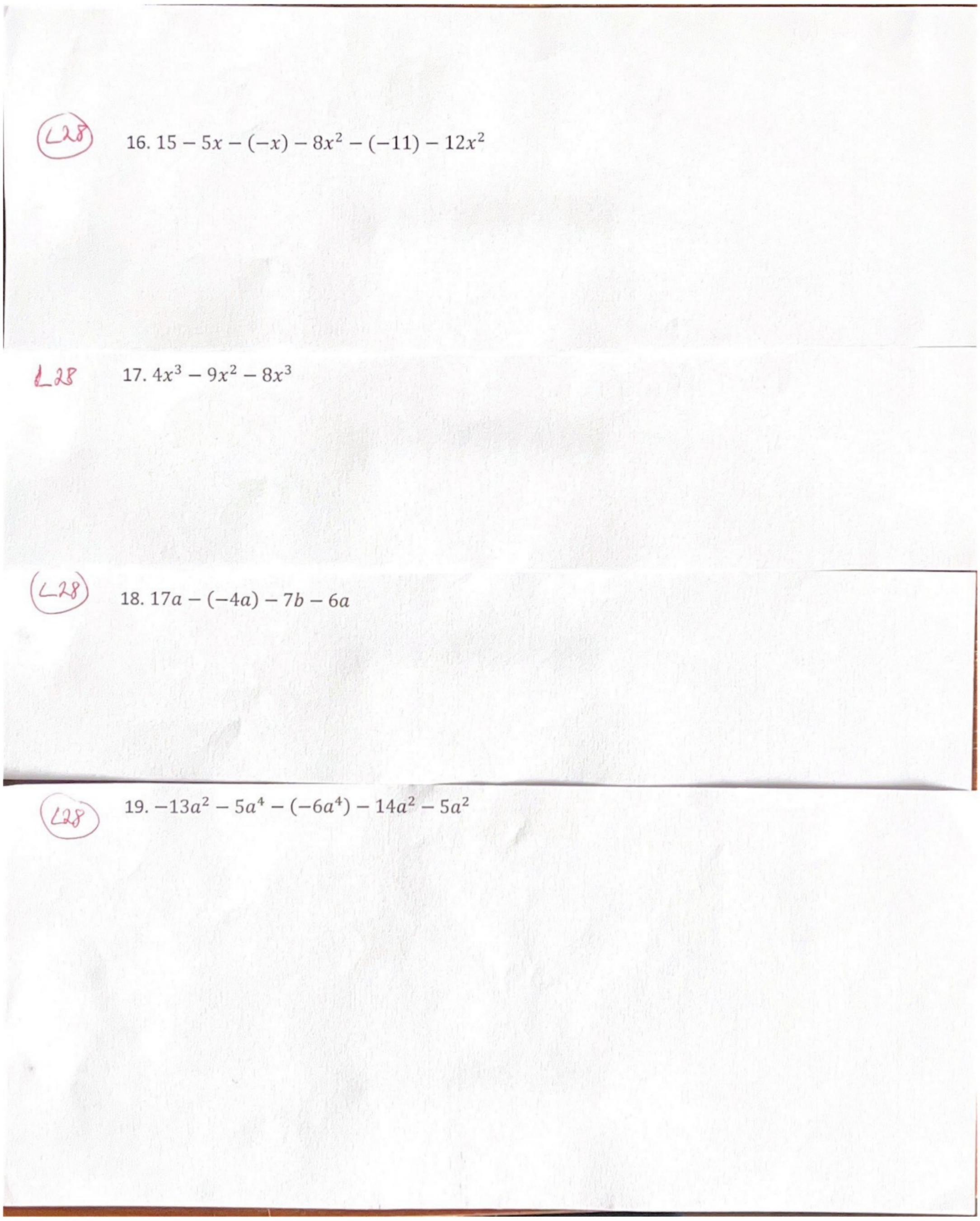
$4+3+4+4=$

$-8\times -8x=63x^2$

$-3y\times -8x=24xy$

L28

15. $13ab-4ba-13-2ba-(-9)-1$

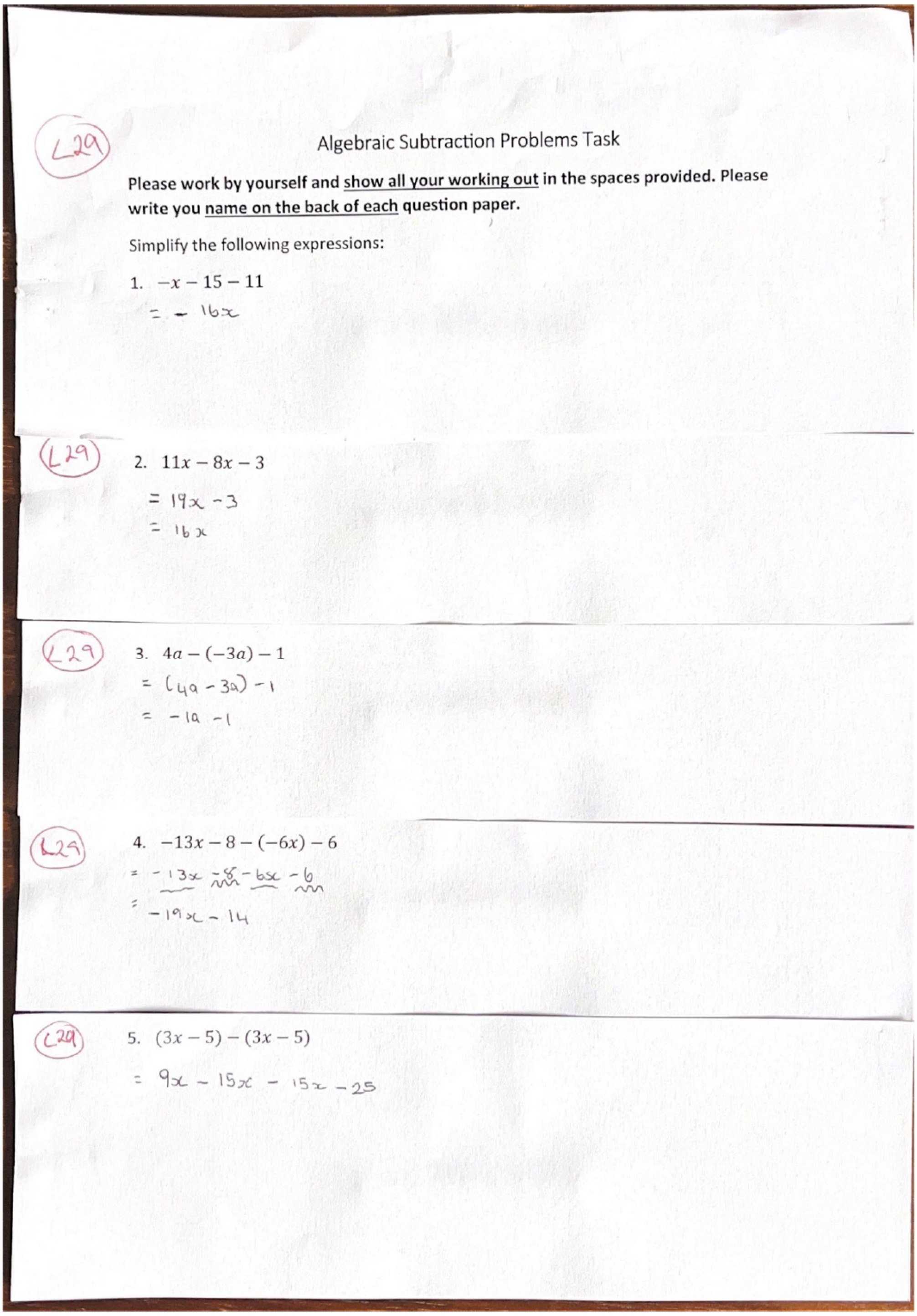


L28 16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

L28 17. $4x^3 - 9x^2 - 8x^3$

L28 18. $17a - (-4a) - 7b - 6a$

L28 19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$



L29

Algebraic Subtraction Problems Task

Please work by yourself and <u>show all your working out</u> in the spaces provided. Please write you <u>name on the back of each</u> question paper.

Simplify the following expressions:

1. $-x - 15 - 11$

$= - 16x$

L29

2. $11x - 8x - 3$

$= 19x - 3$

$= 16x$

L29

3. $4a - (-3a) - 1$

$= (4a - 3a) - 1$

$= -1a - 1$

L29

4. $-13x - 8 - (-6x) - 6$

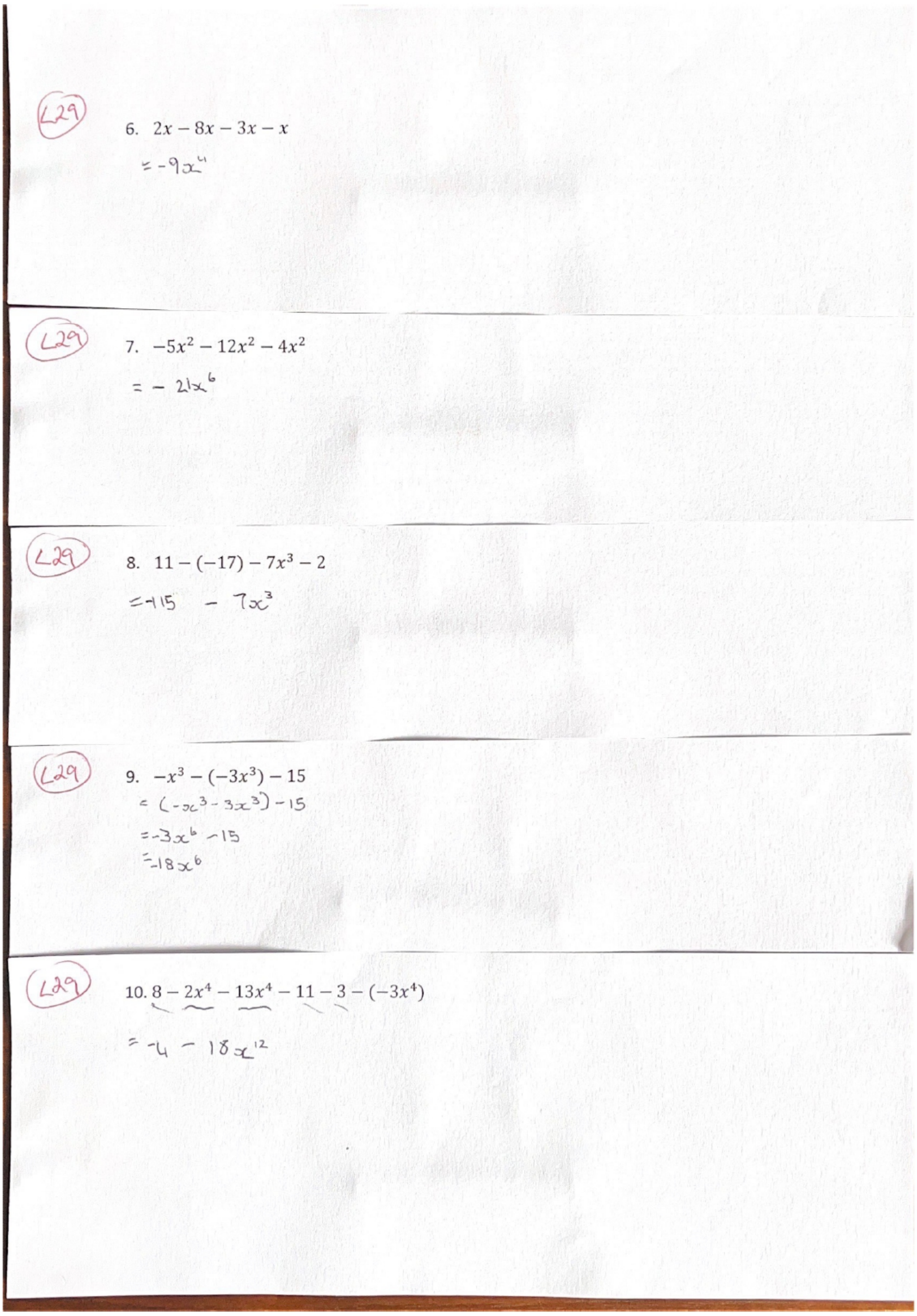
$= -13x - 8 - 6x - 6$

$= -19x - 14$

L29

5. $(3x - 5) - (3x - 5)$

$= 9x - 15x - 15x - 25$



L29

6. $2x - 8x - 3x - x$

$= -9x^4$

L29

7. $-5x^2 - 12x^2 - 4x^2$

$= -21x^6$

L29

8. $11 - (-17) - 7x^3 - 2$

$= +15 - 7x^3$

L29

9. $-x^3 - (-3x^3) - 15$

$= (-x^3 - 3x^3) - 15$

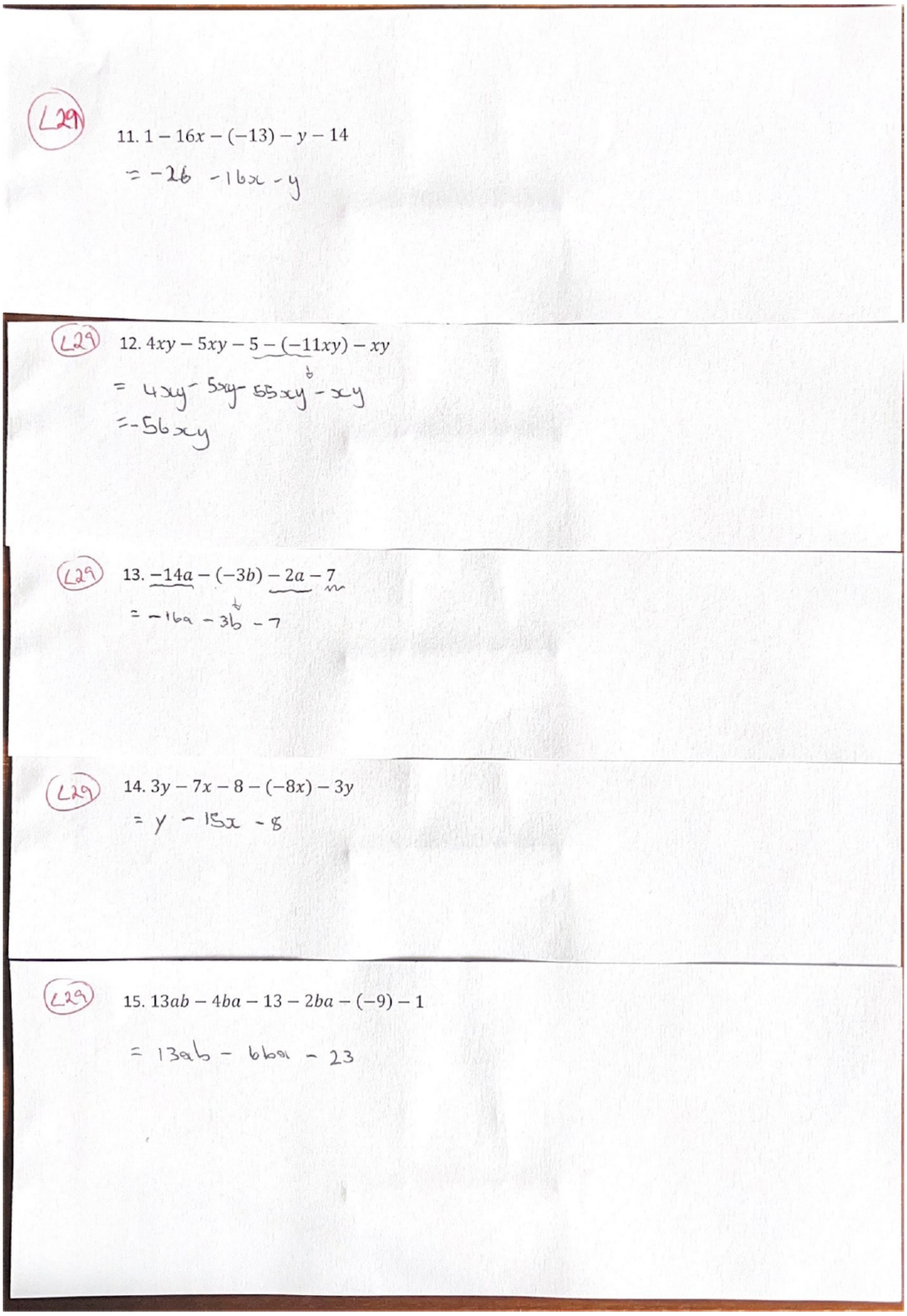
$= -3x^6 - 15$

$= -18x^6$

L29

10. $\underline{8} - \underline{2x^4} - \underline{13x^4} - \underline{11} - \underline{3} - (-3x^4)$

$= -4 - 18x^{12}$



L29

11. $1 - 16x - (-13) - y - 14$

$= -26 - 16x - y$

L29

12. $4xy - 5xy - \underline{5 - (-1}1xy) - xy$

$= 4xy - 5xy - 55\overset{\downarrow}{x}y - xy$

$= -56xy$

L29

13. $\underline{-14a} - (-3b) \underline{- 2a} - \underline{7}$

$= -16a - 3\overset{\downarrow}{b} - 7$

L29

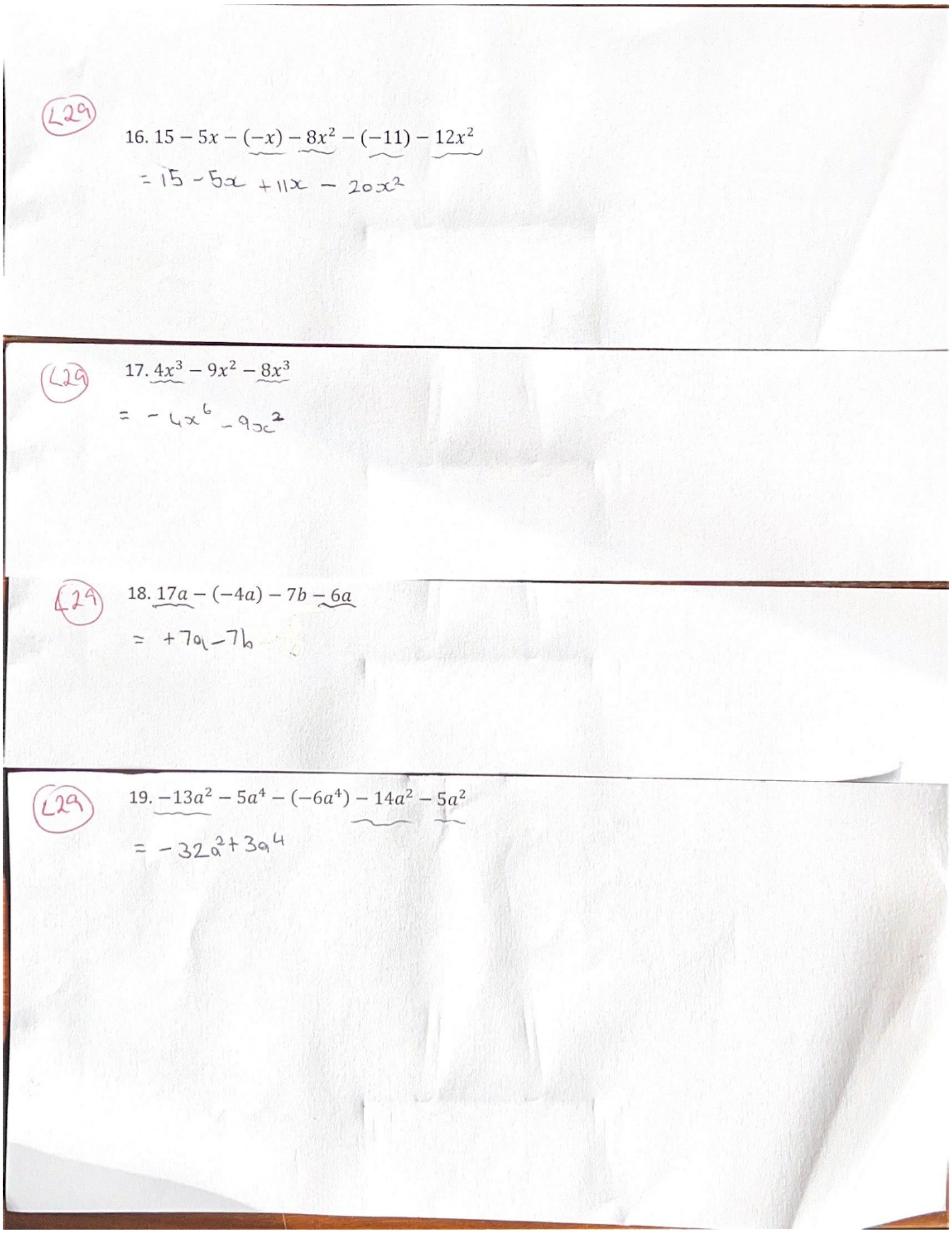
14. $3y - 7x - 8 - (-8x) - 3y$

$= y - 15x - 8$

L29

15. $13ab - 4ba - 13 - 2ba - (-9) - 1$

$= 13ab - 6ba - 23$



L29

16. $15 - 5x - (-x) - 8x^2 - (-11) - 12x^2$

$= 15 - 5x + 11x - 20x^2$

L29

17. $4x^3 - 9x^2 - 8x^3$

$= -4x^6 - 9x^2$

L29

18. $17a - (-4a) - 7b - 6a$

$= +7a - 7b$

L29

19. $-13a^2 - 5a^4 - (-6a^4) - 14a^2 - 5a^2$

$= -32a^2 + 3a^4$

Appendix 8 Interview notes (first round)

L1

$3-(-3)$ $-3-(-3)$ $(-3)-(-3)$

$3+3$ $-3+3$ $-3+3$

$=6$ $=0$

$(-4x)-(-4x)$ $(4x+1)-(4x+1)$

$-2x-3x-5x$ $13ba-4ba-13-2ba-(-9)-1$

~~9ba~~

$13ba\ \underline{-4ba-2ba}=6ba$

$5x-2$ $\underline{-13}+9\underline{-1}=-5$

$=3x$ $8x-3$ $6ba-5$

$11-5x^2$ $11x^4-5x^2$

x^3 $9-4x^2$ $=6x^2$

$-4-9x^2$

L4

$(3x-5)-(3x-5)$

$(3x-5)(-3x+5)$

$-9x^2-25$

$+\cancel{4x}+1-4x-1$

$=0$

$4x+1-(4x-1)$

$+4x+1\,\}+4x+1\,\}$

$=2$

$4x+1-(4x+1)$

$+\cancel{4x}+1-\cancel{4x}-1$

$=0$

$(4x+1)-(4x+1)$

$(4x+1\,\}-4x-1\,\}$

$=0$

$(3x-5)-(3x-5)$

$+\cancel{3x}-5-\cancel{3x}+5$

$=0$

$-3x-(-3x)$

$-3x(+3x)$

$-9x^2$

$17a-(-4a)$ ←

$2x-(-3x)$

$2x+3x$

$5x$

$-5+11xy-2xy$

$-5+9xy$

L6

$2x-2x=x$

$2-2=0$

$13b\overset{1}{a}-\overset{1}{4ba}-\overset{2}{13}-\overset{1}{2ba}-(\overset{2}{-9})-\overset{2}{1}$

$=7ba-13+9-1$

$=7ba-5$

$\square x^{\square}-\square x^{\square}$

$4a-(-3a)-1$

$=4a+3a-1$

$=7a-1$

$(-3x-5x)-(4x-2x)$

$-8x-6x$

$=-14x$

$3x^{3}-2x^{m}$

$1x^{2}$

$(4xy-5xy)-(55xy-1xy)$

$-1xy-56xy$

$=-57xy$

L7

$-3-(-3)$

$-3\ (3)$

$-\mathbf{8}$

$-12-8$ $\quad$ $-7-3$

4 $\quad$ 4

$-3-(-x)$

$-3\ (x)$

$-3x$

~~12~~ $9-11$ $\quad$ $15-13$

-2 $\quad$ 2

$7x-1-(7x-1)$

$7n-1 \quad -7x+1$

$7x-7x$

0

$\downarrow \quad \downarrow$

$7x+\underline{14}x-18^2+5n+7x^2$

$3x-5-1(3x-5)$

$3x-5-3x+5$

0

$(x+3)-(x+3)$

$x^2+3x+3x+9$

x^2+6x+9

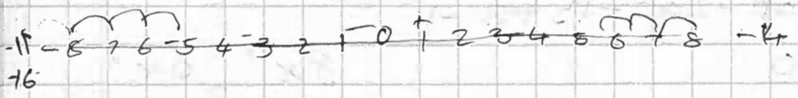
$-(x+3)-(x+3)$

$-8-3$ $\quad$ $-8+3$

$8-3$

$-11-5$

→ $\quad$ ⇐



$5x-x-1(-3x)$

$5x-x+3x$

$=9x$

$-2x-(-4x)$

$-2x+4x$

$+2x$

$8-2x^4-13x^4-11-3-(-3x^4)$

$8-2x^4-13x^4-11-3+3x^4$

$8-15x^4-11-3+3x^4$

$8-12x^4-11-3$

$8-14 \quad 8-12x^4-14$

$=-6 \quad \underline{-6-12x^4}$

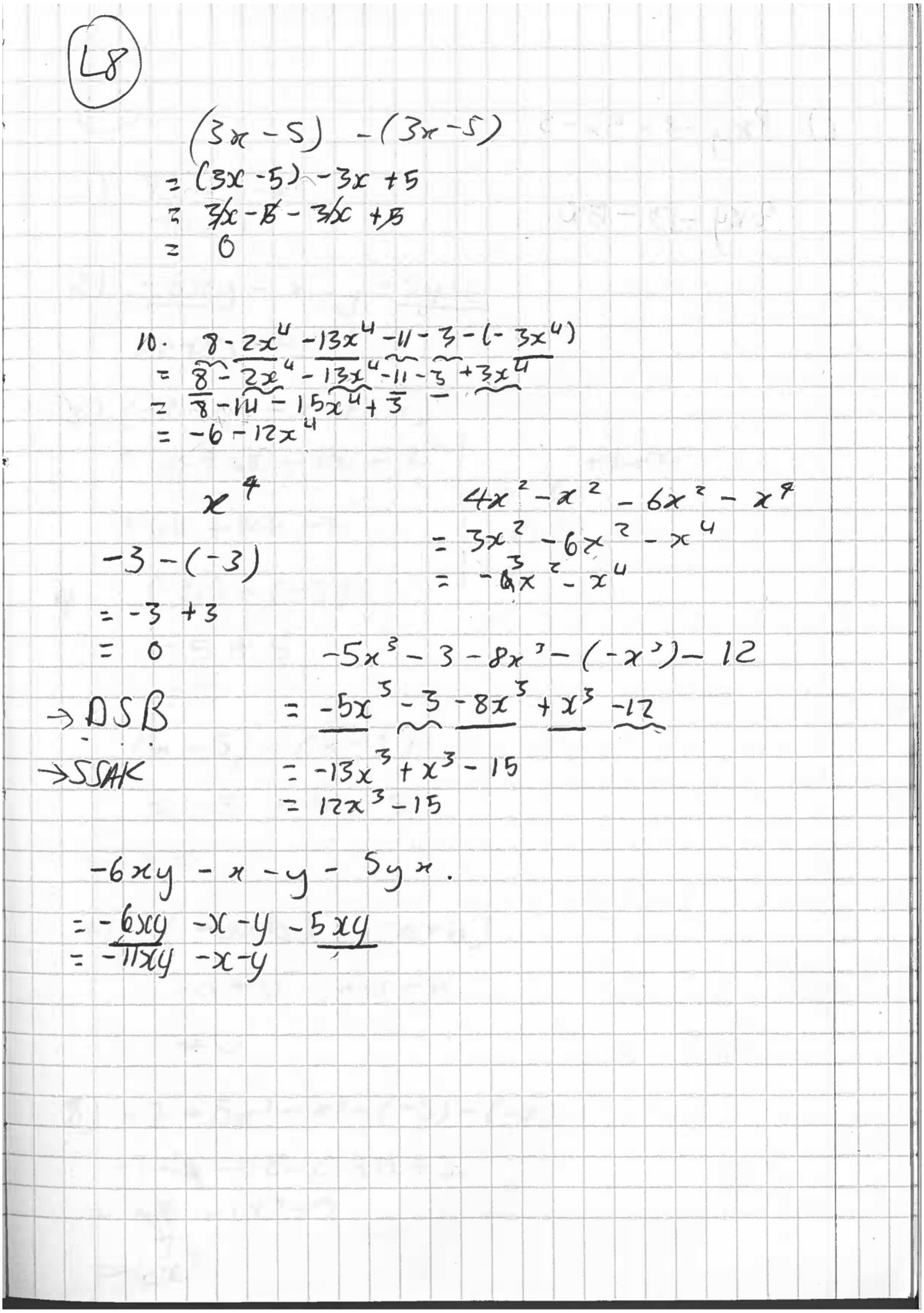
$\underline{17a}+\underline{4a}-7b-6\underline{a}$

$21a-7b-6a$

$13a-7b$

$1x^3-3x^3$

$-2x^3$



L8

$(3x - 5) - (3x - 5)$

$= (3x - 5) - 3x + 5$

$= \cancel{3x} - \cancel{5} - \cancel{3x} + \cancel{5}$

$= 0$

10. $8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$

$= \widetilde{8} - \overline{2x^4} - \overline{13x^4} \widetilde{-11} \widetilde{-3} + \overline{3x^4}$

$= \overline{8} - 14 - 15x^4 + \overline{3}$

$= -6 - 12x^4$

x^4

$4x^2 - x^2 - 6x^2 - x^4$

$= 3x^2 - 6x^2 - x^4$

$= -\overset{3}{\cancel{6}}x^2 - x^4$

$-3 - (-3)$

$= -3 + 3$

$= 0$

$-5x^3 - 3 - 8x^3 - (-x^3) - 12$

→ DSB $\quad = \underline{-5x^3} \widetilde{-3} \underline{-8x^3} + \underline{x^3} \widetilde{-12}$

→ SSAK $\quad = -13x^3 + x^3 - 15$

$= 12x^3 - 15$

$-6xy - x - y - 5yx.$

$= \underline{-6xy} - x - y \underline{-5xy}$

$= -11xy - x - y$

L10

$(3x-5)-(3x-5)=0$

$9x^2 - \cancel{15x} + \cancel{15x} + 25$

$= 9x^2 + 25$

$(3x-5)(3x-5)$

$4xy - 5xy - 5 - (-11xy) - xy$

$4xy - 5xy + 11xy - xy$
$= 10$

$10 - 5 = 5$

$5xy - 3xy - 7$

$\boxed{2xy - 7} \,/\, 2 - 7$

$6xy - 3xy - 2xy - 4$

$xy - 4$

$7xy - 5xy - xy$

$2xy - xy$
$= 2$

$3x - 3x = 0$

$-3x - (-3x) = 0$

$(-3x) - (-3x) = 0$

$-3x + 3x$

$(3x) - (3x) = 0$

$(4x+1) - (4x+1) = 0$

$2xy - 3yx$

$5x - 3x - 7 = 2x - 7$

$2 - 7$

$13ba - 4ba - 13 - 2ba - (-9) - 1$

$13ba - 4ba - 2ba$
$= 7ba$

$-13 + 9 - 1$

$7ba - 5$

L11

$-3x-(-3x)$

$-3x+3x$

0

$(-3x)-(-3x)$

$(-3x)(+3x)$

$=-9x^2$

$(-3)-(-3)$

$(-3)+3$

$=-9$

$(2)-(2)$

$(+2)-2$

-4

$2-(2)$ $+2-2$ $=0$

$2-(-2)$

$2+2$

$=4$

$2-2=0$

$-2-(-3)$

$-2+3$

$=+1$

$11-(-17)-7x^3-2$

$\underbrace{+17}_{}\underbrace{+11}_{}-7x^3\underbrace{-2}_{}$

$=26-7x^3$

$(-2)-(-3)$

$=(-2)+3$

$=-6$

$-8x-6x=-14x$

$-8x+6x=-2x$ $\quad$ $8-6$

$8x-6x=2x$

L13

$3 - (-3)$
~~-6~~ $3+3$
$= 6$

$-3 - (-3)$
$-3+3$
$= 0$

$(-3) - (-3)$
$-3+3$
$= 0$

$(-3x) - (-3x)$
$-3x + 3x$
$= 0$

$3x - 3x$
$= 0$

$(3x-5) - (3x-5)$
~~$3x - 5$~~
$= 0$

$5x - x$
$= 5$

$5x - 3x$
$= 2x$

$\underline{8x - 3x}, -x$
$=$ ~~$(-11x)$~~, $5x - x$

$3x - x$

$4x - 2x$
$= 2x$

$4ab - 3ba$

$13ab - 4ab - 13 - 2ab - (-8) - 1$
$= 13ab - 4ab - 2ab - 13 + 8 - 1$
$= 7ab - 20$

$8x^3 - 6x^3$
$= 2x^3$

$\underbrace{8x^3 - x^3} - 2x^2$

$-2x + 5x$
$= 3x$

$-x + 2x$
$= 2x$

$-x + 8x$
$= 8x$

$-x + 4x + x$
$= 4x$

$1 - (-2x) - 1$
$= 1 + 2x - 1$

$-x + 4x - x$

Ⓛ14

$7a - 1 = 6a$

$-13x - 8 - (-6x) - 6$

$-13x + 6x - 8 - 6$

$= -7x - 14$

$\underbrace{(3x-5)}\div\underbrace{(3x-5)}$

$\underbrace{-2x}\ \underbrace{+2x}$

~~$=-4x$~~

0

$10ab - 5ba$

$= 5ab$

$(-5x) - (-5x)$

$-5x + 5x$

x

$5x - x$

$= 4x$

$10x - 5y = 5xy$

$10x - 5$

$5x$

$-3 - (-3)$

$-3 + 3 = 0$

$(-3)(-3) = \underbrace{(-3) - (-3)}$ $\quad -3 + 3$

$(2) - (-2)$

$= 2 + 2$

$-5x^2 - 12x^2 - 4x^2$

$\underline{-5x^2} - \underline{16x^4}$

$-21x^6$

(L17)

$-3-(-3)$

$-3+3$

$=0$

$(-3)-(-3)$

$-3+3$

$-3-x-(-3x-x)$

$-3 \underline{-x} + 3x \underline{+x}$

$-3+3x$

$x-3-(x-3)$

$x-3(-x+3)$

$x-3-(x-3)$

$x-3\ (-x+3)$

$x-9$

$-x$

$x-x-3+3$

$x-x$

$3x-3x$

x

$5x-x-(-3x)$

$5x-x+3x$

$8x-bx \quad 7x$

$8x-2x$

$6x$

x^3

$3n$

$(-3a)-(-2a)$

~~L19~~ ~~L20~~ L19

$-3-x-1$

$-4x$

$3x-5-(3x-5)$

$3x-5-3x+5$

$-7x^2-3x^2-x^2-5x^2$

$-16x^2$

$8a-11a-(-3a)$

$8a-11a+3a$

$+11a-11a$

0

$8x-5x-3$

$3x-3$

$-3-(-3)$

$-3+3$

0

$x-3-(x-3)$

$x-3+x+3$

0

$2x-8x-3x-x$

$-6x-3x-1x$

$=-10x$

$8-2x^4-13x^4-11-3-(-3x^4)$

$8-15x^4-14+3x^4$

$-12x^4-6$

$x^3=x\times x\times x$

$\underbrace{-1(-x)}$

[illegible] 22

$3 - (-3)$

~~-9~~ 6

$(3x-5) - (3x-5)$

= ~~$9x^2+25$~~

0

$-3-(-3)$

0

$(-3)-(-3)$

0

= ~~$18x^4-6$~~

$-2x^4 - 13x^4 \overset{+}{-} (-3x^4)$ 1

$= -18x^4$

$8 - 11 - 3$

$= -6$

$= -12x^4 - 6$

-3
-6

18 = $17a + 4a - 6a$

= ~~2a~~ $15a - 7b$

3 = $4a + 3a - 1$

$= 7a - 1$

9) $-x^3 + 3x^3 - 15$

$= 2x^3 - 15$

12) $4xy - 8xy + 11xy - 2xy$

$= -6xy + 15xy$

$= 9xy - 5$

L24

$3x-(-3x)$ $\quad 2-(-2)$ $\quad (2)\overset{\downarrow}{-}(-2)$

$9x^2$ $\quad 4$ $\quad (2)(-2)$

$(-5)-(-5)$

$\overset{\downarrow}{5}-5=0$

25

$(5)-(5)=0$ $\quad (5)(5)$

$\rightarrow 0$

$(5)-5=-25$

$-5x^2-12x^2+4x^2$

$(5)x-5$

$-5-12=-7$

$13ab-\underline{4ba}-13-\underline{2ba}-(-9)-1$

$13ab-6b^2a^2-13+9-1$

$13ab-6b^2a^2-13+8$

$13ab-6b^2a^2-21$

$15-5x-(-x)-8x^2-(-11)-12x^2$

$15-5x+x\underline{-8x^2}+11\underline{-12x^2}$

$15-5x+x\overset{-}{\cancel{+}}20x^2+11$

$\underline{15}-6x^2-20x^2\underline{+11}$

$1-6x^2-20x^2$

$1-14x^2$

Appendix 9 Transcripts (first round)

Appendix 9.1 L1 First Interview

INT: OK, so there are a few questions I'm asking everyone that I want to go through and then there are just some other random questions depending on how the time goes. OK, let's go to question one first. It was minus x minus fifteen minus eleven [$-x-15-11$]. I can see here you've done the minus fifteen and the minus eleven and then you've done minus twenty-six minus x and then you put them together to make minus twenty-six x. So, can you just explain when you're getting to the twenty-six, what are you doing with the fifteen and the eleven?

L1: I'm, like, plussing them 'cause it's negative. OK, negative plus negative makes negative, so it's negative twenty-six.

INT: OK. So, you're doing fifteen plus eleven, and then you're putting the minus in front because they're both negative. And then you've got minus twenty-six and minus x and then you're putting them together?

L1: Yeah.

INT: OK, for this question [Q5] it was brackets, three x minus five, minus, brackets three x minus five [$(3x-5)-(3x-5)$]. Can you explain to me how you solved this problem?

L1: So, I put it in like three x times three x. But I was supposed to, I think, put the square over there, but I didn't.

INT: OK, that's fine.

L1: And there I did the same.

INT: Are you using the FOIL method?

L1: Yeah.

INT: OK, I just want to go through some other questions. So, what if I gave you a question like… let's say I give me something like this: three, minus, minus three [$3-(-3)$]. How would you simplify that?

L1: Probably like, dunno, don't exactly know what to do… oh, wait wait. So negative times negative is equal to positive. So, then I'll give three [writes 3 + 3] then equal to six.

INT: And then what if I do something like this? What if I do, let's say minus three minus, minus three [$-3-(-3)$]?

L1: So, there I do the same thing. It goes in [pointing to the negative symbol in front of the brackets], so there is plus three [writes $-3+3$], then it's equal to nothing.

INT: OK, so you can say zero [L1 writes $=0$]. So now if I put the first one in brackets, does that change the way that you would solve it? Or would you still solve it in the same way or simplify it in the same way?

L1: I don't think so. Just like, take the brackets away from this stuff and then I'll do the same thing [writes $-3+3$].

INT: And what if I do it with x? So, what if we did minus four x minus, minus four x [$-4x-(-4x)$]. Then would you still use the same method?

L1: Yeah, it's still the same.

INT: OK. And then what about if we do something like this… four x plus one and four x plus one [$(4x+1)-(4x+1)$]

L1: Umm… I would use the FOIL method.

INT: So, when there are two terms in the brackets you're going to use the FOIL method. But if there's only one term then you're going to do… you're going to make this addition [indicating to the two negative symbols] and do it like this.

L1: Yeah.

INT: OK, so as soon as this becomes an expression, then you're going to do multiplication with the brackets. Let's do another one [Q10]. So, eight minus two x to the power of four, minus thirteen x to the power four, minus eleven, minus three, minus, minus three x to the four [$8-2x^4-13x^4-11-3-(-3x^4)$]. I can see you've circled the numbers, and you've underlined the...

L1: Yeah. So, the circled ones are like the terms, like terms. And these are also like terms [indicating terms containing x^4] and that's also like.

INT: OK And then you changed this one [$-(-3x^4)$] to a plus.

L1: Yeah. Because negative and negative.

INT: So, then you've done the variables or the terms with x to the power of four in them and that's giving you minus eight x to the four.

L1: Yeah.

INT: And then you've done the numbers afterwards. So, do you often take the first two terms? Does it matter in which order? If I give you something like this [writes $-2x - 3x - 5x$], does it matter which ones you subtract from each other first?

L1: No…

INT: It doesn't? OK, so you could do those two first or these two or whichever. And then just to get to the minus eleven x. So, you've done the first two terms, so how do you get to the eleven from the two and the thirteen?

L1: So…I suppose this is actually supposed to be negative fifteen.

INT: OK, that's fine. So, you would change it now to negative fifteen x to the power of four?

L1: Yeah.

INT: And then plus the three x to the power of four.

L1: Yeah, and then it would be like…

INT: So minus fifteen plus three…

L1: So, it would be… twelve.

INT: Yeah, so negative twelve x to the power of four. OK. And then we've got eight minus eleven minus three. Ok, so the eight minus eleven you get to three. [L1 seems uncertain]. Would you change it?

L1: I think so.

INT: What would you change it to?

L1: I think it should be like negative three.

INT: OK. So negative three and negative three.

L1: And that would be equal to negative six.

INT: OK, so you would have a negative… what was it? Negative twelve x to the four minus twelve x to the four minus six [$-12x^4 - 6$] as your answer. Ok. There are two more that I want to check with everyone and then I can do some more of yours. OK, Question 15, we have thirteen ab, minus four ba, minus thirteen, minus two ab minus, minus nine, minus one [$13ab - 4ba - 13 - (-2ba) - (-9) - 1$]. Would you say that ab and ba are different?

L1: Um... ab…ba…Yeah.

INT: You would say they're not the same?

L1: Yeah.

INT: OK, you've left the ab by itself and then you've done minus four ba minus two ba to get minus six. So, what did you do? Did you do the same thing as previously? How do you get to six from the minus, from the four and the two?

L1: I did the same thing.

INT: So, you add them and then you put a minus?

L1: Yeah.

INT: And then minus thirteen, so you did a plus again… is that from…
L1: That's minus minus.

INT: OK and then minus five. And then you've just written thirteen ab. OK. And if I, if I gave you, if I changed it like this, if I said all of them would be ab, then…

L1: Then you would plus it.

INT: OK [writes $13ba - 4ba - 13 - 2ba - (-9) - 1$]. Can you just write down your answer?

L1: OK [writes working out and mumbles calculations to self]

INT: Thank you. So, you're doing it… they're all like terms now so then you can do it. But if this was ab then you're not going to do the same. Let's do this, Question 18 is seventeen a minus, minus four a, minus seven b, minus six a [$17a - (-4a) - 7b - 6a$]. You've done a plus here… [points at plus in working out]

L1: Yeah, it’s minus minus.

INT: Minus minus, OK. So, you're doing, so you would work out for the like terms first and then you just bring the other term down.

L1: Yeah.

INT: OK, let's go for this one. You've got eleven x minus eight x minus three. You've done eleven x minus eight x is three x, and then you’ve brought the three down, and then you get x. How did you get x from three x minus three?

L1: Yeah, I minused [pointing at the two threes in their working out $3x - 3$]

INT: The three from the three?

L1: Yeah.

INT: OK, if you've got the same number, do you, then is that when you cancel out the number? If I give you something like [writes $5x - 2$], would you do the same thing? Or how would you do this one? Would you simplify it at all? L1: I would probably minus the five. So, three x.

INT: OK, so you would say this would be equal to three x. So, I see just in this, I just went through the script quickly, so you've done the same thing here. You have seven a minus one and then you get the six a. So, you’re subtracting the number from that.

L1: [Nods yes]

INT: So, this one is kind of similar to what we were speaking about before, where I was asking about the order. So here you have underlined the first two terms and then the second two terms.

It was two x minus eight x…Oh ya…

INT: Why do you think it's a negative?

L1: Oh because of the eight. If the eight was there [pointing to the beginning of the expression] then it would be positive.

INT: OK, so if the bigger number, if you're subtracting a bigger number from a smaller number then it becomes negative. And then here you've done, you've underlined the three x and the x and you've got minus four x. So how do you get to the minus four?

L1: Because negative minus negative is plus.

INT: OK, you add your numbers in front and then you put the minus in front. Let's do this one: eleven minus, minus seventeen, minus seven x cubed, minus two [$11 - (-17) - 7x^3 - 2$].

L1: Umm. So that… I would put that one out [points to negative symbol inside brackets], make it like a plus, but it didn't. So, so eleven minus two is equal to nine and then the negative negative equals to positive seventeen, seventeen minus nine equals to eight, so eight minus that [points to $-7x^3$]. But I didn't do like what I normally do…

INT: But you… well you can do what you normally do. You can write your answer there.

L1: OK, so… [writes x^3]

INT: Oh, because it would be eight minus seven.

L1: Yeah.

INT: So it doesn't matter if the x is first or if the number is first, it's the… you're looking at these two numbers [points to the 8 and the 7, and then you're subtracting the one number from the other and then you get just the variable, or the term. OK. So, if I gave you, I don't know, let's say 11 minus five x squared, would you simplify that?

L1: Yeah.

INT: What would you do?

L1: [writes $6x^2$]

INT: OK. So, eleven minus the five gives you the six and then you've just got the x squared. But what if I give you like eleven x to the four minus five x squared [writes $11x^4 - 5x^2$?

L1: I don't think I would minus.

INT: You wouldn't. OK, so you would leave it the same. Would you just leave it like this? [L1 nods] OK. Is that because these two have different exponents? So, they're different?

L1: Yeah.

INT: Let's just go through this one. So, one minus sixteen x, minus, minus thirteen, minus y, minus fourteen [Q11: $1 - 16x - (-13) - y - 14$]. You've underlined the numbers again.

L1: Ya, so that's one, umm, and this became a plus because of the negative. And then that's just negative fourteen. So plus that too. So, I'll get fourteen. Then fourteen minus fourteen equals to zero.

INT: And then you've just brought down [pointing to y]

L1: Yeah.

INT: OK, so minus sixteen x minus y. Alright, let's look at this one. We've got four x y, minus five x y, minus five, minus, minus eleven x y, minus x y.

L1: Bring that down [points to $4xy$] and then that became plus eleven x y and minus x y, so then I put equal to nine x y. But I didn't bring this down [points to -5]

INT: Oh, you would have brought down the minus five.

L1: Yeah.

INT: OK. So, you would just change that. Ok, and Question 14. Three y minus seven x minus eight mins, minus eight x, minus three y. So, I can see you've underlined the three y and you've circled the x's.

L1: OK, so these are different things. So, three y minus three y is equal to 0 because it cancels out. And then negative seven x and that becomes a plus [pointing to the two negative symbols in front of $8x$], plus eight x, so that's equal to x. So, then I… well I don't know… I don't know why I wrote the [pointing to $3y$].

INT: Oh, so you would have written it just as x?

L1: No, no I would have written the eight there [pointing in place of the $3y$].

INT: Oh, I see, you would do x minus eight?

L1: Yeah.

INT: Ok we can do the answers to these three questions and then we can finish. So, Question 16 is fifteen, minus five x, minus, minus x, minus eight x squared, minus, minus eleven, minus twelve x squared. So, we have three different types of…

L1: Ya so fifteen, comes down and then plus because of that [pointing to $-(-11)$] equals twenty-six, and then negative five x…[mumbles] then that becomes plus x because of the negative and equal to negative four x. Then x squared [pointing at $-8x^2$] minus twelve x [pointing to $-12x^2$] becomes twenty x squared…

INT: You've done two, you've done two negatives here [pointing to $-8x^2 - 12x^2$], but then you've done positive.

L1: Oh, Oh yeah. I would, I would…

INT: Would you change it to a negative?

L1: Yeah.

INT: So, you would have minus twenty x squared [L1 agrees]. But then here you've put the minus.

L1: Yeah, I think so.

INT: So, you maybe forgot there. OK, so you did the twenty-six minus the four x, and then you would have put the minus twenty x squared. OK, Question 17 was… this one's got two different types of terms, so four x cubed, minus nine x squared, minus eight x cubed.

L1: So, I've got the like terms and four x minus eight x cubed equals to four x cubed. Then four x cubed minus nine x squared I just left.

INT: Ok you're saying these are unlike terms so you wouldn't simplify. But if this had been for example, minus four minus nine x squared [writes $-4 - 9x^2$] like that, would you do something differently?

L1: I wouldn't actually do something.

INT: You wouldn't? You would leave it the same? OK. If it's a negative? What if it was, um, nine minus four x squared? [writes $9 - 4x^2$]

L1: If this one had a x then I would minus it. But it doesn't.

INT: It doesn't. But then earlier you were saying…

L1: Yeah, I know.

INT: So, we're changing it now. OK, that's fine. So, you think you'd do differently. So, what if it was, um, eight x minus three?

L1: I'll leave it.

INT: OK, that's interesting. So now you're saying if it's got an x and it's a number [pointing to $8x - 3$] then they're also unlike terms. OK, Question 19: minus thirteen a squared, minus five a to the power of four, minus, minus six a to the four, minus fourteen a squared minus five a squared.

L1: So, I underlined the like terms again and then it was negative, minus negative which is… yeah those are all negative so I just plussed them, so I got negative thirty-two a squared. And then a four [pointing at $-5a^4$] plus six a four, then I just got a four. And then I just put them together [pointing at final answer $-32a^2 + a^4$].

INT: OK, I think that's fine.

Appendix 9.2 L4 First Interview

INT: OK, so basically I'm just going to give you one slip at a time and I just want you to just talk me through step by step what you've done and then if I have any other questions I might ask you to explain further. [Q1] So you had minus x, minus fifteen, minus eleven. OK, which you just rewrote. And then? [this was not done in the book]

L4: Then I added 'cause negative fifteen and negative eleven it's like terms, I added that and then you left with negative x, and then I added, and then negative twenty-six x.

INT: So, you added the fifteen and eleven, but then you put a minus in front. Is that what you mean?

L4: Yes.

INT: OK and then when did you get, what did you do to get to the minus twenty-six x?

L4: Because it's a negative and negative. I said twenty-six x.

INT: I see, so you did negative and negative, so you added them, but you kept the negative in front.

L4: I think it's wrong…

INT: Well, if you think it's wrong, that's OK. What do you think is the answer? If you think it's wrong.

L4: Negative…

INT: Just this one? [points to previous line of solution $-x - 26$]

L4: Ya.

INT: So you're saying you think instead of putting them together, you should have just left them.

L4: Ya, because it isn't like terms.

INT: Ya so in this question [Q5], you did do that. So, you maybe just went, after you did Question 1, you did it differently. OK, so this one is Question 5. So, you've got two brackets, three x minus five, minus, and then the bracket, three x minus five. [$(3x - 5) - (3x - 5)$] Do you want to just talk through what you did?

L4: I multiplied three x and three x, and three x times negative x… wait I first…put this [points to minus between the brackets] into. I took the negative because it's a negative one, I said negative one times three x and negative one times negative five. And then…

INT: So here you've brought the negative inside the brackets.

L4: Ya, but it's supposed to be a positive [points to -5].

INT: You would want to change to a five, so do you want to maybe just write down… I'll give you a pen… Maybe write down here what you think you would do instead.

L4: [writes new answer $-9x^2 - 25$] And then because it's a negative and a positive, then you don't have to do the middle term. Then there isn't the middle term.

INT: Oh, I see OK.

L4: And then it's just gonna be three x times minus three x… minus nine x squared and then negative twenty-five.

INT: So, do you think, whenever you see brackets like this, would you always multiply when you see brackets, or do you think… are there maybe other times where if there is a bracket you wouldn't multiply? [L4 hesitates to respond] What if I give you something like, OK, let's start with something else. Let's say I give you minus three x, minus, minus three x [$-3x - (-3x)$]. How would you work out that one? This is a different question.

L4: This is a negative one [pointing to negative symbol in front of bracket] and then I would first do the negative one into the bracket and then I'd do it like that.

INT: Do you mind just writing out your answer? [L4 writes in workbook, final answer $-9x^2$] I see, so you've got… because this one's in brackets you're bringing the negative inside and then that makes it positive because it's two negatives. And then because we've still got brackets, we're gonna do multiplication? OK. Let's look at Question 6. So, this one is all just a bunch of x's. So, two x minus eight x minus three x and minus x. [ ] Do you always use the same order? Does it matter which ones you work with first?

L4: I always work with the first one.

INT: So, you would do these two [points to first two terms] first and then you would do the other ones.

L4: Yes.

INT: But would it make a difference if you did these ones [points to last two terms] first?

L4: I think so.

INT: OK so talk me through your next step.

L4: So I took two x minus eight x and then it's negative six x. Then I took the remainders, minus three x minus x [pointing to terms while explaining]. Then negative nine x then the extra x and then negative ten x. [not in workbook, but on slip?]

INT: OK, ya so you did those two and then these two and then you just had two left. This one, Question 10, So this time we working with exponents and we've got to the power of four and then we're working with integers. OK, do you want to explain what you did?

L4: So first, because here's a bracket, you must always work out the bracket first. So then I said negative one times negative three x to the power four, then once it's there then I just put the rest in there and then I took like terms and I added the like terms.

INT: So in this one, here [points to $-3x-(-3x)$ in workbook] we've got minus three x, minus, minus three x. But then you did multiplication, but for this one [points to Q10 slip] you're doing addition, you're not doing multiplication. Is that because this one's in a whole expression that's got more terms in it? Or why would you… what are you doing differently in this one to just if you're working with this by itself? Or is it because they are two x's?

L4: Yes

INT: So, when there's a number then you'll do addition or whatever it is. But if it's if there was an x here then you would have multiplied, do you think? [L4 agrees] OK let's see… Question 15. We've got ab and ba and then we also have integers. OK, so just talk me through your answer.

L4: There's a bracket. So then I did the bracket again and then I took over like terms and then…I subtracted the like terms and subtracted the…

INT: The numbers. OK. And you said that ab and ba is the same.

L4: Ya.

INT: OK next we're going to do this, Question eighteen. So again we've got brackets…

L4: Brackets again… seventeen a and minus six a because it's like terms and then you left with seven b.

INT: OK, so again this one you've done it a little bit differently because here you've done seventeen a minus, minus four a. But in the other one that we did, we had, we had something minus, minus in brackets. So this time you've done addition [Q12], but this one [$-3x - (-3x)$] you did multiplication.

L4: Because the negative one… The number on the end will always times into the bracket. You won't take seventeen a like that.

INT: But this one, in this one [pointing to $-3x - (-3x)$ in workbook] it was the same. So if I just wrote it like this, like seventeen a, minus, minus four a [writes $17a - (-4a)$ in workbook]. If we just look at the first bit. So why? Why are you doing this one with addition? But in this case…

L4: Oh I see…

INT: Do you see what I mean? Do you think it you just did it differently depending on the question? Which of these do you think you would do if I asked you again? So if I give you a question like, um… let's say… let's take two two x minus, minus three x [writes $2x - (-3x)$]. How would you, if you're just looking at that one, how would you work out the answer for that?

L4: Is this a x?

INT: Ya, sorry.

L4: [L4 writes $2x + 3x$ then $5x$ underneath]

INT: So you would stick with this method rather than go in with that one.

L4: Yeah.

INT: Let’s do Question 7 as well. So we've got all of them are x squared, and again, just so you did…

L4: From order

INT: You do those two first and then did you get to the minus seven? Just so that it's got all the working out.

L4: Negative five minus twelve. Because negative five is less than twelve and negative sev… then I say… then twelve minus five but then you just add the negative.

INT: OK I see.

L4: And then because it’s x squared it stays x squared.

INT: And then here? How did you get to the minus eleven?

L4: Then negative and negative, then I said negative seven x squared… you basically don't have to worry about the x squared, you must just put it in the next number. Then negative seven minus four, because it's both negatives, you add it, so it'll stay negative and you know I got negative eleven.

INT: So you add the seven and the four and then you put the minus in front because both of them were negative. OK. Then Question 12. So this time we've got xy’s but all of them are xy’s, it's not like the other question which had ab and ba. OK, so just speak me through your working out.

L4: Because there’s a bracket again I said negative one times negative eleven xy then it gave me positive eleven xy.

INT: So now you're using this method again that we did just now.

L4: Yeah. And then I just took the rest of the numbers and I put it there and then I added it all.

INT: Ok so you’ve done four xy minus five xy gives you minus xy.

L4: And then negative five… and then I just added the rest of the terms and then that gave me negative six… no… because that’s unlike term then…

INT: So that one you’ve carried down…

L4: Ya.

INT: Here it looks like you've taken the…the eleven xy has come down but then these two you've taken away.

L4: Oh… ya… I’m not sure why I did it. Oh! Because it cancels each other.

INT: OK so why does it cancel? Because they're the same.

L4: If there isn't a number [pointing to $-xy$] then it will always be a negative one, so negative one x… oh no it doesn't cancel each other out, I thought it does. It's supposed to be negative two.

INT: OK so then you would have eleven xy minus two xy which would be… what would be the answer then if you added a minus two xy, do you want to write it on the paper? So at the moment we've got this line which is minus five plus eleven xy and then minus the two xy that you just said.

L4: [Writes $-5+11xy-2xy$] Then it would be: negative five plus nine xy.

INT: OK. So that so you would change your answer to that. OK, seeing as we worked through those so quickly, let me go through some more. I’m trying to think if there are specific questions that would be better to go through. We can go through this one: Question 3. OK, so we have the brackets again and this time in Question 3, you seem to have used this method again [pointing to $-3x-(-3x)$] instead of doing the multiplication.

L4: So, negative one times negative three, because you… the negative one and the a, so the negative one times negative three a will be positive three a. And then I'm just bringing down the terms. Then four a plus three a will be seven a, but because there’s no a here it’s not… it’s a negative one [final answer $7a-1$].

INT: It’s not a like term, ya. OK. Let's look at another question. This one's [Q11] also got brackets, let's do this one.

L4: There’s another bracket so, then I first did the bracket in negative one times negative thirteen because it's a negative times a negative, it cancels each other out and makes a positive. And then I just brought down again and then I added like terms. Then negative sixteen x, thirteen minus fourteen is negative one… and then negative one plus one cancel each other out and then negative y is left [final answer $-16x - y$].

INT: OK because the y and the x are not like terms. Let's see this one [Q13].

L4: OK, it's a bracket and then negative one times negative cancels each other out again, it becomes positive three b 3B. Then I brought down negative fourteen a and negative two a is like terms and then three b there isn't another like term, so we leave it as is and with negative seven. So then you can only do the negative fourteen a and the negative two a [final answer $-16a + 3b - 7$].

INT: Ya so you can't make this any simpler because you don't have any more like terms since. OK, let's do this one [Q14].

L4: Negative times negative cancel each other out, positive $8x$, bring down the other terms. Three y and three y cancels each other out because here is a positive [pointing to first term $3y$] but you don't have to write the positive if it is. Then negative seven x with a positive eight x give you negative… gives you one x and then you have here negative eight.

INT: OK, so you've taken the three y and then because we're minussing three y you've said that those cancel out.

L4: Yeah because the positive and negative cancels.

INT: OK, so what if I give you something like, um let's say four x plus one, minus, four x plus one [$4x + 1 - 4x - 1$]

L4: You can put it… four x and the negative four x, because here is a positive [draws $+$ in front of first $4x$] but you don't, obviously, have to write it, then that will cancel each other out and then positive one and negative one, because it's a positive and a negative it cancels each other also out. So then I would write zero [writes $= 0$].

INT: OK, you say the answer zero. OK, what if I write it like this: what if I say four x plus one minus, and then I say four x minus one like this [writes $4x + 1 - (4x - 1)$].

L4: Then because there's a bracket you must work out the bracket again and then four x… it will be four x plus one first and now I'm going to do the bracket… negative four x plus one [writing $4x + 1(-4x + 1)$]. Actually, you don't put the brackets [scratches out brackets] and then four x minus four x because there's a positive there [points to first $4x$] it cancels each other out and then it will be two because it will be one plus one [final answer $= 2$]

INT: This time you didn't… you brought the negative in, you multiplied that in but you then took away the brackets. If you're doing that then what about if I do it like this: four x plus one minus, and then I do the same thing, four x plus one [writes $4x + 1 - (4x + 1)$]. How would you do it? Would you do the same method?

L4: Ya, I think so.

INT: Just write down your own answer.

L4: [Writing $4x + 1 - 4x - 1$] And then four x and negative four x cancels, and then minus one [answer -1].

INT: Why are you getting minus one? Just explain so that I'm following.

L4: Positive one and the… so four x here because it's a plus four x and four x cancels each other and it gives you zero, then positive one and negative one… Oh that actually also gived you zero [scratches out -1 and writes 0].

INT: So now if I do the same thing but I put the first one in brackets, does that change? Let's say I write it like this [$(4x + 1) - (4x + 1)$]. It's the same but now I've put this one in brackets [pointing to first binomial]. What would you do now?

L4: If it was a negative then I would just take that. But there is going to be a middle term now, so then I would just say four x plus one [writes $(4x + 1)$] and then because there isn't another one there, then negative…

INT: Are you going to use the same method as this one [pointing to $4x + 1 - (4x + 1)$] or are you going to do it differently?

L4: I think I'm gonna take the brackets away [scratches out brackets around $4x + 1$] and then negative four x minus one, because then…ya. And then positive cancels each other out [pointing to $4x$ and $-4x$]… Oh no I'm going to put the brackets [draws brackets around $4x + 1$ and $-4x - 1$] and then four x times… No, no brackets [scratches out brackets] and then four x minus four x cancels each other out and then one and negative one gives zero [answer $= 0$].

INT: OK, so now we've done. Now you've been following the same method, so what if I give you this question again? [writes Q5: $(3x - 5) - (3x - 5)$]

L4: [Writes $3x - 5 - 3x + 5$] Then the three x and the three x cancels each other out cus there's a positive there and the negative five plus five, then it's a zero.

INT: OK! Let's do this one just to explain. We have a lot of different terms. You've got a squared and a to the power of four and we've got brackets, so just speak through your answer.

L4: So negative… first you work out the bracket because there is a one here but you don't have to write the one, then negative times six a us going to give you positive six a, and then I just brought everything down again. Then because this is like terms, then those three is like terms [pointing to a^4 terms] and this is like [pointing to a^2 terms] then this and this and this gave me negative thirty-two because it's all negatives and then I basically just add the numbers and you leave the term a squared as a squared. And then negative five a plus six a, it's just one more so than positive one a four. But you don't also have to write the positive one, you can just leave it as plus a four.

INT: OK, so with the with the minus thirty-two you did thirteen plus fourteen plus five and then you put the minus in front because they all had minuses.

L4: Ya.

INT: OK, I think that'll be fine. Thank you so much.

Appendix 9.3 L6 First Interview

INT: OK so question one we had minus x minus fifteen minus eleven, and then I just want you to explain to me what you did or what you think you did to get the answer.

L6: Um, so a negative… you minus a negative to a negative it will stay a negative, but if it was a times it would now make it a positive… so it's not a times it's a minus, so I added those two [points to -15 and -11] so they make a negative twenty-six and that's, what do you call it…it doesn't… you can't add it…

INT: Unlike terms.

L6: Unlike terms, yeah, so I'll just put it next to it so it will be negative twenty-six minus x.

INT: OK I see here you put them together and said minus twenty-six x, but then you said that's not right because they're not like terms.

L6: Not like terms, yeah.

INT: OK and you said you added the eleven and the fifteen to get twenty-six and then it was negative because they're both negative?

L6: Yeah

INT: OK, let's do another one. Question 5, we've got three x minus five in brackets and then minus and then brackets three x minus five. So just explain to me what you did.

L6: Um, I did the uh folio method that my miss taught me.

INT: OK

L6: So, these two… [pointing to -5 and $3x$] so I will like, how can I say now, I will swap it around.

INT: OK, you swap it?

L6: So, all the like terms I swap.

INT: OK and then you put them together.

L6: Yeah. So, I put that three over there so it will automatically become a negative and then five will go over, that will become a positive so it will be positive five minus five equals now zero and then three x minus three x equals zero. So, it will be x.

INT: OK so you kept them in brackets to show that these are the like terms and then you said five minus five is zero and then with the three x and the three x? How did you get to x?

L6: Uh, I minus the three.

INT: OK so if you've got three and three then you're cancelling them out?

L6: Yeah

INT: And then there will be x left. OK, what if I give you something simple like if I just give you two x minus two x? What about that, how would you simplify that? Would you do the same thing?

L6: Yes

INT: So, you would say that this is equal to x? But if I give you just two minus two…

L6: Equals zero

INT: OK so you're saying when it's just numbers then it's zero but if there's an x next to it then you’re cancelling the numbers but you're keeping the x there.

L6: Yeah

INT: OK let's do this one, Question 10. I see here you've written little ones above the numbers and then you've written twos above the terms…

L6: Like terms.

INT: OK so that’s for you to organise your terms into like terms and unlike terms. Do you want to just speak through it?

L6: So, I added the like terms, I numbered the like terms in order so I can see what I can minus and then I minus everything, it will equal to that, negative three…

INT: OK, so you did the eight and the minus eleven and the minus three.

L6: Yes. And then when I got my answer, I did the same with the two x, with the other like terms I did the same and I got my answer.

INT: OK so you've done eight minus eleven minus three to get minus three.

L6: Yes

INT: And you did minus two x to four minus thirteen x to the four, oh and here you've done a plus

L6: Yes

INT: So how does the plus work here? [Pointing to $+3x^4$]

L6: Um it's a negative times a negative equals a positive.

INT: OK and then you just took away the brackets.

L6: Yeah

INT: Then here you've put a minus in the exponent [points to $-12x^{-6}$ in final answer]. Can you just explain where that's coming from?

L6: Let me see… Um… Oh yeah, so I minus this here with that [points to $-13x^4$ and then $-2x^4$] then it gave me, um… negative… negative fifteen x.

INT: To the power of four?

L6: No just negative $15x$.

INT: OK

L6: And then I plussed that [points to $3x^4$], then it will equal to… a plus to a negative equals… yeah twelve… so you minus it so it will be twelve x to the power of four, but I don't know why I put that negative [points to superscript -4 in final answer $-3 - 12x^{-4}$].

INT: Don't know why you put a negative, that's fine.

L6: So, it will just be to the power of four.

INT: OK, but you said just now that if you do minus two x to the power of four minus thirteen x to the power of four then it would just be minus fifteen x, but not to the power of four.

L6: No because I minus, I deduct that.

INT: You're taking… Oh so if you're doing subtraction then you're also taking away the exponent, but then it's coming back because you've added the term here [points from $3x^4$ to $-13x^4$]

L6: Plus, yeah.

INT: I see, so if you minus then you're minusing the exponent and then it becomes negative. OK, let's do Question 15. I see you've done the like terms again for the numbers and then you've drawn a line to show that those two are the same. So, are you saying that ab and ba are different? [6:02]

L6: Yes

INT: OK just speak to the next step.

L6: So, I did these two and then that's a negative four ba and then a negative two ba, so I added those two and then the negative and the negative stays the same and I just added those two and it's equal to six ba, negative six ba. The thirteen says the same because there's no like terms with that so it will stay the same and the thirteen and the nine and the negative one will equal to negative five.

INT: So, if we change this, let's say I wrote the first term if I say thirteen ba minus four ba minus thirteen minus two ba minus minus nine minus one [writes $13ba - 4ba - 13 - 2ba - (-9) - 1$]. How would you change your answer? Would you change your answer now that I've changed ab to ba?

L6: Yeah, I would add this [points to $13ba, -4ba$ and $-2ba$] this three because they're like terms and then the negative and that [points to $-(-9)$] will, a negative times a negative is a positive, so it will be negative two b a plus nine minus one.

INT: Do you want to just write your next steps then we can see how your answer would be different for this question.

L6: OK [writes solution, final answer $7ba - 5$]

INT: OK, you've used the same system, so thirteen ba and then minus four ba and minus the two ba you've given seven ba 4, and then you've done the numbers and that's given you minus five. So, you're going to give a different answer if these are in the same order but if you have ab and ba then they're not like terms.

L6: No. [9:00]

INT: Let's do Question 18. We've got a minus and brackets minus and we have two different types of terms one with a's and one with b's, can you just explain your answer?

L6: So the same thing with the like terms that a's will add and the b's, there's only one b so it will stay the same. So, I used the same method that I did before, negative times a negative equals a positive and then I made it seventeen a plus four a minus seven b minus six a [$17a + 4a - 7b - 6a$] so I just simplify that and then I could add it so it will give me now fifteen a and minus seven b and the minus seven b just comes down.

INT: Let's do a few more, so here is a nice example from an earlier question where it shows what you were doing so I can see now that you were doing minus and minus times gives you positive. Now with this question [Q3] you've timesed the four and the three

L6: Yes

INT: So why are you doing it differently? In this question your timesing this number and this number, but in the other questions you're just adding them. So if we look at this one you've done four a times three a and in the other question [Q18] you did seventeen plus the four, so what do you think is different or do you think you would change one of your answers.

L6: Yeah, I would because I wasn't really thinking I was just writing.

INT: Yeah, this is Question 18 so it's much later on.

L6: So, I don't know what I did there, I just remembered about the negative times a negative is a positive then I forgot about adding those.

INT: So, what do you think is the right way to do it? You're saying seventeen a plus four a but in this question instead of saying four a plus three a you've done four a times a. Do you think it's times if you have minus minus or do you think it's a plus?

L6: It's a plus.

INT: Let's just rewrite the answer to Question 3 then, so the question is four a minus minus three a minus one [$4a-(-3a)-1$] and then just write your answer for that.

L6: [writes $= 4a + 3a - 1$ then $= 7a - 1$]

INT: So, this time you've done four a plus three a and you're not multiplying them you're just adding them together for seven a minus one. OK. In this one [Q2] you've shown again a lot of working out just by drawing around it. What are you using the brackets for here?

L6: Like terms

INT: OK because you have two x's?

L6: Ya, so that I just left that out [points to -1] because that would just come down because there's nothing that goes with that and then I minus those two and then equals three x.

INT: OK. In this one [Q6] you've also done brackets – is that to organise which terms you're going to subtract?

L6: Yes

INT: If you have something like this [points to Q6], does it matter it in which order you do the subtraction?

L6: No

INT: But would you use a specific order? Would you normally do the first two if they are like terms?

L6: Yes, I normally do the first two.

INT: And then the other two. So, you did these two first [pointing to $2x$ and $8x$] and then those two second [pointing to $3x$ and x], and then you've worked with those two numbers to bring it back down?

L6: Yes, just to make it more simple for me so I don't have to add all of it at once.

INT: OK, so here, by putting these brackets here you're saying you'll do two x minus eight x first to get minus six x, and then you're subtracting, and then you're doing three x minus x gives you two x, and you've just brought down the minus.

L6: Yes

INT: What if it was something different, let's say minus three x minus five x minus four x minus two x [$-3x - 5x - 4x - 2x$]? Can you just solve that one?

L6: [writes $= 8x - 6x$ and $= -14x$].

INT: OK, minus three x and minus five x you've said is minus eight x and then you've done…

L6: Four x minus two x but the negative will be there [points to negative in front of $4x$].

INT: OK so because they're both negative, you're going to add these two numbers and keep it negative.

L6: Yes

INT: And then have you done the same thing here.

L6: Yes. The eight and the six and then you've got the minus.

INT: OK, Question 7, I see you're doing the first two terms and then you're bringing the last one down. Here you were talking about this earlier, you're not including the two anymore [referring to exponent 2 that has been subtracted, pointing to term $-17x$], why is that?

L6: Because I minused it.

INT: Are you saying then that if you have x to the power of something let's say some number I'm just going to draw blocks because it doesn't matter what the numbers are [writing $\square x^{\square} - \square x^{\square}$]. Let's say that this is the same number like a two for example [writes 2 in each superscript box], when you're doing this when you have an exponent, are you saying that you would subtract these numbers [pointing to the coefficient boxes] from one another but if these exponents or the same then you would subtract we'll cancel them out.

L6: Yes, but I would cancel them out even if that was a three and this was a two then I would cancel it um and it would be a negative exponent one.

INT: OK, if we have something like three x cubed minus two x [writes $3x^3 - 2x$].

L6: Then I would minus those and I would have one x to the power squared [1x^2], because then I minused x with that.

INT: OK so you've done three minus one gets you the two. So, you're doing a subtraction here [draws around coefficients] and you're doing a subtraction of these ones [draws around superscripts].

L6: Yes

INT: Let's do this Question 8. You haven't written it, but it seems you've done a plus again [pointing to $11 - (-17)$ and 28 below it] and then you've drawn that those are like terms and this has given you….

L6: Twenty-six.

INT: OK, straightforward. Just talk through this Question 9.

L6: Negative times a negative gives a positive and then those two are like terms [points to $-x^3$ and $+3x^3$] and it will be… a positive and a negative so this will be like… I will like swap the those two around and it will be three x cubed minus x cubed, like I did there [points to previous question $3x^2 - 2x$] so it will give me two x.

INT: So, you've done this three minus the one [pointing to coefficients] and the three minus three [pointing to superscripts] gives you nothing so you've taken away the number.

L6: This is like, my Miss told me there's an invisible one there [points to $-x$] so I will minus that there so it will be two x.

INT: And this will just be the x because you've minused the threes?

L6: Yes, two x and then that's the unlike term.

INT: This one [Q12] has lots of different xy terms so let's go through that one. So here you've done [pointing to lines drawn above all xy terms] for all of them except for this one…

L6: All the like terms

INT: And you've changed here to a plus [points to $-(-11xy)$ and $+11xy$ below it].

L6: Yes, but I would do, now when I look at the question I would add those to that [pointing to -5 and $-(-11xy)$].

INT: What would you add?

L6: This, I would times this to that [points to -5 and $-11xy$].

INT: You would times it, OK, because there are brackets, is that why?

L6: Yes. Because those are not the same [pointing to -5 and $-11xy$]

INT: OK, if they're not the same would you times then?

L6: Like five times eleven would equal now fifty-five xy with a positive, but a positive and a negative equals now a negative [gesturing negative symbol with hand].

INT: OK, do you mind just writing out your answer? You can just do from here you don't have to do the first one, just what you would do next.

L6: [writes $4xy - 5xy - 55xy - xy$] That I would write there [points to second line of solution on paper slip]

INT: OK that would be your second step.

L6: Ya I would take that out because that's like…

INT: INT: OK let's carry on from there what would be your next step?

L6: Adding them. So now I would do my brackets like I did there [draws brackets around $4xy - 5xy$ and $55xy - xy$] I would bring that one down [points to minus in front of $55xy$] and I would add those and it would be negative one xy and those [pointing to $55xy - xy$] will be, this an invisible one, so fifty-four xy and I'll add those [points to $-1xy - 54xy$] negative fifty-five xy.

INT: So, you've separated them to work with them one at a time and then you bring that down and you bring this down.

L6: Oh, sorry this is a six [changes $-54xy$ to $-56xy$] and then that would be a seven [changes $-55xy$ to $-57xy$].

INT: OK. I think let's stop there because your bell just rang. Thank you so much.

Appendix 9.4 L7 Interview

INT: So, there are a few questions that I want to go through first which I ask everyone and then there are a few others we can go through depending on how much time. Basically, I want you to explain step by step what you did, or what you think you did, if you can remember, and if you think you want to change your answer now, then you can do that and show me what you are doing as your answer. Question 1 was minus x minus fifteen minus eleven. So, what did you do step by step? And how would you do it now, as well, if you would change it?

L7: I did this: minus eleven from fifteen and I got to that answer.

INT: OK, so that's how you came to the four. So, let's do a different example: minus twelve minus eight. How would you go about solving this? [Writes $-12-8$]

L7: [Writing 4 underneath] I will take twelve and I'll minus eight from twelve and I think I'll get positive four.

INT: When you're working with negative numbers, or minuses, do you just look at the numbers and work with the symbols afterwards, or do you visualise a number line, or…

L7: I first look at the symbols.

INT: The symbols, and then the numbers? So why do you choose to do a positive answer for this one?

L7: I think because they are both negatives, so the answer is probably positive.

INT: So, you're saying if you have two negatives then it's a positive answer? And what if we have a smaller number, the number itself is smaller first, something like minus seven minus three? Would you do it differently, or would you use the same method?

L7: I would use the same method.

INT: OK, would you mind just writing your answer out. [Writing $-7-3$ answer below is 4]

L7: Four.

INT: Would you also say this one is four? And, what if only one of them is negative: What if it is nine minus eleven?

L7: It will be negative two.

INT: And why is it now a negative?

L7: Because the first number is smaller.

INT: Oh OK, so the first number is smaller and then we go subtracting so you say it is negative. So, if it was like this [writing $15 - 13$]?

L7: Two

INT: Positive two, and why is it now positive?

L7: Fifteen minus thirteen is two.

INT: OK, let's do another one. This one, Question 5. So, in Question 5 we had three x minus five in brackets, minus three x minus five, also in brackets. So, first thing I want to ask is what do you think the brackets are telling you to do in this question.

L7: Multiply.

INT: OK, so would you always multiply even if there was a minus in between the brackets? What do you think this minus means?

L7: This is a negative, ya.

INT: So, are you still multiplying, are you saying the negative applies to what is inside the brackets?

L7: Yes.

INT: OK, and then how did you get to this answer?

L7: It's the same numbers, so I just squared it.

INT: So, you're just squaring it because we have two of the same brackets, is that why? And what happens to this minus that was in the middle, or did you ignore the minus?

L7: A negative times a negative is a positive, so it becomes a positive.

INT: Oh, you mean that's positive here because there's a negative but these … you're squaring a negative. Is that what you're saying?

L7: Hmmm. Something like that.

INT: OK, let's do some different examples. So, what if I give you something like minus three minus minus three [writes $-3-(-3)$]. If I give you something like this, what would you say these brackets are for?

L7: It's to multiply, but that and that [pointing to the negative symbols on either side of open bracket] becomes a positive.

INT: So, you're multiplying the negative signs? Do you mind writing out your answer for this one, how you would simplify it?

L7: OK, so I'd say negative three here and that would become a positive, but then I still multiply that and it will be negative six, no negative nine.

INT: OK, so you've multiplied these two together to get a positive and then that symbol goes away?

L7: Ya, and then it becomes a positive and I don't put the positive sign there, but then I multiply those two.

INT: And if I give you something like this: minus three minus minus x [writes $-3-(-x)$].

L7: It will still be a positive x, becomes positive x and it will be negative times positive will be negative three x [writes answer $-3x$.

INT: OK, here's a positive but you don't write it because you don't write the positive sign, and now you've multiplied these two together. What about if I give you this, how would you go about that? [Writes $7x-1-(7x-1)$]

L7: I actually don't know how I'd go about that… I think I'll take… there will be a negative one here, and I'll put that in there [pointing from negative symbol in front of open bracket to in front of $7x$], and it'll become negative and this will become positive. So, it'll be… it'll be… negative seven x plus one… but this will still be here… so it will be seven x minus one minus seven x minus one [$7x-1-7x+1$].

INT: So, this time you're not multiplying. Well, you said you've multiplied the minus in, but how would you simplify this, what would your next step be?

L7: So, there's no like terms besides this one, so the ones will cancel out, so it will be seven n minus seven x because of the like terms.

INT: Oh sorry, this is also an x.

L7: Oh. Then it'll be, I guess zero.

INT: OK, so you say these two cancel out. Oh, you just said about like terms, so when do you have like terms?

L7: So, I'll say… minus eighteen squared… [writing $7x + 14x - 18^2$] so the like terms will be that [pointing to $7x$ and $14x$] OK, I'm going to write another one, uh five n [writes $+5n + 7x^2$ after $18x^2$] so these will be like terms and there will be no other like terms, cause there's a square there.

INT: So, what is it that you are identifying here, with these two being like terms?

L7: These two?

INT: No, these two, sorry.

L7: We'll add that.

INT: But I'm saying why are they like terms?

L7: There's no reason why they are like terms, I don't think so.

INT: Do you mean maybe because they both have x's?

L7: Ya.

INT: OK. But then you said this one is a squared, so it's not the same.

L7: Ya.

INT: This one is an n so it's not the same. And then this one?

L7: It's squared, there's no x.

INT: OK. So, if I changed this question and I wrote it three x minus five minus three x minus five [$3x - 5 - (3x - 5)$]. Would you say that then if we compare the two… so this one, the first term is in brackets, would you say that these two are different or would you say that they are the same?

L7: I don't know, actually. I think they look the same because they both have those but there's just no brackets there.

INT: Hmmm. Do you think these brackets make a difference to the first two terms?

L7: I think it does.

INT: OK, so what do you think would be the difference? Or for example, this one you said you would multiply, so would you do this one differently, then?

L7: There'll be a one here.

INT: OK.

L7: Now I multiply the one in, so I'll get negative three x plus five… so it'll be three x minus five. Three x minus three x will cancel out and the answer is zero.

INT: OK. So, they look the same but the only difference is that there are brackets, but you're saying that if there are brackets then you would multiply, but if there are no brackets around the first one then you would not.

L7: Ya.

INT: OK, thank you. Let me just see … I don't have any special questions on that one. Let me just do one more example. So, let's say you have x plus three minus x plus three [$(x+3)-(x+3)$].

L7: My Miss, she gives us like a FOIL method. So, I would say… like that [drawing arcs for FOIL method] So, x multiplied by x will be x squared, x multiplied by positive three… of x, three x, plus nine, I add to the middle, six x plus nine [final answer x^2+6x+9].

INT: OK, what about this minus, would you do anything with this minus, or would you just leave it?

L7: I'd just leave it.

INT: What about if there were two minuses. So, it's minus x plus three minus x plus three [$-(x+3)-(x+3)$]. Do you think that would change your answer?

L7: I actually don't know. I don't think so, because we are working within the brackets.

INT: Are you saying when you have brackets then you're looking at the brackets, you're not worried so much about the symbols? So, do you think these two [pointing to $(x+3)-(x+3)$ and $-(x-3)-(x-3)$] would be the same?

L7: I think so.

INT: OK, let's check Question 6. So here we've got all x's. So, it's two x minus eight x minus three x minus x. The first thing I want to know is: Do you think it matters in which order you do the subtraction? I see up here you wrote it differently, you wrote minus x minus three x plus two x minus x. So, you wrote it in order from biggest to smallest. Normally, when you try to solve this kind of problem, would you change the order to be like this, or do you think it does not matter which way you go about it?

L7: I'd normally change it like that because it will be easier. Then I will minus each of it until I get nothing from this one.

INT: OK, so here you've done minus five x, so does that come from the first two terms [pointing to $-8x-3x$]?

L7: Yes.

INT: So, normally would you put the first two terms together?

L7: I would minus that, then I'll get that [point to $-5x$], then this, then I'll get that [points to $-x$].

INT: OK, so for the minus eight and the minus three you've done minus five. So how did you get to the five?

L7: Negative five… minus three from eight, then I got negative.

INT: How did you know to put a negative symbol in front?

L7: Negative three… so I'd make a number line [drawing number line and explaining on there]

INT: But what if you just had eight minus three and you do it on the number line, how would you do it then?

L7: Positive five.

INT: On this one you're going in this direction [left to right], and on this one you're going in this direction [right to left]. How did you know to go in this direction rather than to keep going in that direction if you're working with negatives?

L7: What do you mean?

INT: So here, when you do eight minus three, when you work with a positive number, you go from the right side to the left side, you are going down the number line, but then when you're doing minus eight minus three, then you're going from left to right. So, would you always go from left to right if you do subtraction with negative numbers?

L7: Most of the time, yes.

INT: What if, just to compare, say you had minus eight plus three.

L7: Then I'd go that way [right to left].

INT: OK, then you'd go in the other… So, what would your answer be?

L7: It would be… negative… wait, I'm rethinking now…

INT: That's OK.

L7: No, it would still be positive, because you're plussing, so you'd be adding more positive numbers. So, I'd still go that way, which means that this one would go that way [changes to left to right].

INT: OK, so you'd change this one to go the other direction. What would you change this one to?

L7: To be… three plus eight… eleven.

INT: OK, now you said three plus eight, but you're still keeping it negative.

L7: Hmmm.

INT: So you're saying that if you have two negative numbers you plus them but you keep the minus sign in front. Then, if we have something like minus eleven minus five…

L7: That would be negative sixteen.

INT: OK. Are you plussing the eleven and the five together?

L7: Ya.

INT: OK. I also wanted to do one more example where… let's say we have five x minus x minus, and then I'm bringing brackets in again, minus three x [writes $5x - x - (3x)$].

L7: It's always a one here, so I would be that… oh wait, no… that's a negative with a negative, so that would be a positive three x. So, five x minus x times positive three x. Wait, now I can remove the brackets. So, it's positive three, so it's plus. These are all like terms. Five x, one x, so that's six x. So, it'll be nine x.

INT: OK, I see what you've done there. You've taken the brackets away because you've done multiplication. What if we do something simple and we've only got two terms like minus two x minus minus four x [$-2x - (-4x)$]. Would you still do the same method?

L7: I'd still put the negative in there so that it would become positive x. So negative two x positive four x. And I'll take the x that side [indicates the $-2x$ to come after the $4x$], so I'll get negative, no wait, positive two x.

INT: When you said just now you'd take this x to that side, why would you do that?

L7: Because it's negative. I'll just take it over and I'll get positive two x.

INT: Is that just to make it easier for you to do the subtraction? Is it because of the difference between the numbers, because one of the numbers is bigger, or is it because this one has a minus in front?

L7: Because this one has a minus in front.

INT: So, you prefer to do the positive number first and then the negative number, if you are subtracting. OK. Question 10. So, we had eight minus two x to the power of four minus thirteen x to the power of four minus eleven minus three minus minus three x to the power of four [$8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$]. I can see here you've done squiggles there and you've underlined these.

L7: Because they are like terms. But for now, I'll do this different now.

INT: OK, you can write your answer.

L7: I'm going to write the question first [rewrites question]. Now, how I was doing here: I must still take that in there and it will become a positive [draws arc between negative symbols on either side of bracket]. I'd say eight, two x four, minus thirteen x to the four, minus eleven minus three and then plus three x to the four. Then I underline the like terms [underlining like terms]. So, negative and negative [pointing to $-2x^4$ and $-13x^4$] then I would…

INT: You can use your number line again if you want to.

L7: I'm thinking…

INT: So, what you were doing here before – you said you did minus eight minus three you went to minus eleven, so you went down. How would you do it over here?

L7: So, I'd probably add the two and I'll still get a negative. I'd say eight negative fifteen x minus eleven minus three [writes $8 - 15x^4 - 11 - 3 + 3x^4$]. So that's a positive, so now I go that way. I'd say eight minus five x and then I would do the other ones. I'd first do this because that's next to each other… negative eleven, move three, negative fourteen… so that's a positive, that's negative… that would be eight minus fourteen is six, negative six… then I'd say negative six minus twelve… negative twelve to the four [final answer $-6 - 12x^4$].

INT: Here, where you did the x minus fourteen to get to the minus six, how did you work out that it was six?

L7: I went on the number line.

INT: Did you visualise it?

L7: Same as I did here. I just said fourteen is here, so I'd go back [pointing to -14 on number line and moving right].

INT: OK, that makes sense. Just three more questions I want to go through. So, in this one you've got two variables next to each other: four xy minus five xy minus five minus minus eleven xy minus xy [$4xy - 5xy - 5 - (-11xy) - xy$].

L7: What I did is similar to what I did here… I did the like terms, then I calculated all the like terms… Oh I did this and that [pointing to $+11xy - xy$] and that and that [pointing to $4xy - 5xy$] then I got there [pointing to $-1xy$] I minused five from four and I got negative one xy. And here there was eleven x y, so I minused… so there was always a one there, so then I took one x minus eleven xy… no, eleven x y minus one x and I got ten xy, so the five is still in the middle. But then I did this too… I took negative one x and I minused ten x minus one x and then I got nine…

INT: OK, that's the explanation. This one we had thirteen a b minus four b a and minus two b a – those are the ones with letters, and then we've got minus thirteen and minus minus nine and minus one. I can see here you did the a b's and the b a's together, so are you saying a b and b a are the same?

L7: Yes.

INT: OK, so what was your next step, similar to the previous question?

L7: So, the a b and the b a are the same. So, I did the same… I underlined the like terms… OK, I see what I did… I minused four b a from thirteen a b and then I got a nine a b… and then I took thirteen… and here I minused two b a from seven a b and I minused nine from thirteen and then I still had the one… so I had four and one… so I took four and I minused one from four and I got negative three and I kept the seven a b.

INT: Here you've just said that minus four minus one gives you minus three, but in the other one when we were doing minus eight minus three you were going the other way and you added them and put the minus in front.

L7: I did not use the number line then.

INT: OK, so would you change this?

L7: I'd change this to a positive.

INT: What would you change it to?

L7: No, wait, I'll change it to negative five.

INT: OK, let's look at this one and then I just have some general questions I'm going to ask you and then we can finish. Oh, this one is seventeen a minus minus four a minus seven b and minus six a. You've kept the a's and b's separate…

L7: OK, but I'll probably do this different… I'll take seventeen a and I'll multiply it in then I'll get positive four a minus seven b minus six a and I'll do the like terms… so I'll add those two and I'll get twenty-one a minus seven b minus six a and I would minus that again so that would be twenty… no, thirteen a minus seven b.

INT: OK, so you did the twenty minus seven to get to the thirteen?

L7: No, six.

INT: Oh right, of course. OK, I just want to check some general questions with you… If I give you something like say x cubed minus three x cubed, what does the exponent tell you about the x?

L7: It's power to three. I wouldn't add this [exponent], I would just add this, so it would be negative two x three.

INT: OK, so you will leave the exponents as they are. Why is that?

L7: Because when you add numbers with exponents you don't add the exponents, but when you multiply you add the exponents.

INT: OK, so you only add the exponents when you multiply, not when you‘re doing addition or subtraction.

OK, I think that's fine, thank you so much, that was very helpful.

Appendix 9.5 L8 Interview

INT: There are, how many? Six questions, I think, that I've been doing with most people, and we can just see how the time goes. Maybe we can do some extras. I think it might run into your break a little bit, if that's OK, but I'll try not to take too long. OK, let's start with Question 1 and I just want you to talk me through your process when you're solving these kinds of problems. So, we've got minus x minus fifteen minus eleven. So how did you get your answer?

L8: First, I did the… what do you call the variables that are the same…

INT: The like terms.

L8: Yeah. And then I brought down the bigger… they have the same sign so I added them together and just brought them down and we wouldn't simplify more left it the same.

INT: OK. So, when you have two numbers which both have minuses in front, then you add them, but you keep the minus in front. Is that right?

L8: Yes.

INT: OK. This one, Question 5. We have three x minus five, minus, and then in brackets, three x minus five. Do you want to speak through what you've done there?

L8: So, I brought that, the first bracket, down Miss and then I timesed the negative into the whole bracket.

INT: OK. And then you've taken the brackets away because you've done a multiplication. OK. And are these brackets necessary? Or what do you think these brackets are telling you? What do you think the brackets are telling you to do?

L8: Oh, I just do it because like, so I don't get confused, Miss.

INT: OK, so like a separating?

L8: Yeah.

INT: OK and what did you do to get to this answer?

L8: Oh I think I multiplied it, ya, it wasn’t right though.

INT: What would you do now? Let's redo it. If I give you the question now, how would you do it?

L8: [Writes working out, answer 0]

INT: OK, so these brackets, you've just taken them away. They're not changing anything.

L8: No, Miss.

INT: OK, then Question 6. We have two x minus eight x minus three x minus x. Do you want to explain what you've done there? Or you can redo it if you want to.

L8: Oh, I just collected the like terms and then I first add the two negatives together to get eleven x and then I brought it down, Miss. Then I minussed two x from eleven x so then I brought just x down. Oh, ya, so then I also added eleven x and x together and it would be twelve x and then I minussed two x from twelve x would be negative ten x.

INT: OK. You've brought this one [$2x$] down each time and you said you're collecting the ones which are all negative first. Are you dealing with them first?

L8: Ya.

INT: So when you're doing subtraction problems, is there a specific order that you follow?

L8: Yes, Miss. I first do the negatives. I do them in groups.

INT: OK, you'll do the negative terms first and then you'll see where you are and then you end up with something like this and then you just get your answer. And when you're working with subtraction problems, do you just look at the numbers and then figure out which symbol to put in front? Or do you imagine like a number line?

L8: No, I just look at the numbers and the symbols.

INT: Just the numbers and the symbols, OK. Let's do Question 10. I see here you've done some underlining,

L8: Yes, Miss, I was underlining the like terms. Only when there's a lot of variables, I underline the like terms.

INT: And then do you want to just speak through what you did next?

L8: Can I write it?

INT: Ya you can write it on this page.

L8: [Writing working out, answer $-6 - 12x^4$]

INT: OK, because there are lots of things you've underlined them, is that right?

L8: Yes, Miss.

INT: And then I see you've shuffled things around here. So where is this fourteen coming from?

L8: Oh, fourteen is from the negative three and the negative eleven.

INT: And how did you get to minus fourteen from the minus eleven and minus three?

L8: I brought the negative sign down and I added the numbers.

INT: OK. Is that the same? Oh here you have a plus three and a minus thirteen, what did you do for this? Or did you do these two?

L8: Yeah, I did the two negatives [$-2x^4$ and $-13x^4$] so I brought the sign down and I added it together.

INT: OK. And then you're leaving your answer like this. And when you're working with exponents, what does the exponent mean [writes x^4]?

L8: It means it's being raised to the power of four, Miss.

INT: And what does that mean?

L8: That it's being multiplied four times by itself.

INT: By itself. OK… here [Q12] you've also done the underlining.

L8: Ya because there are a lot of variables.

INT: OK, and you're just grouping them together to make it easier for yourself, that makes sense. Let me check the other questions… I think I'll give you some extra questions to solve to see how you would do them. Oh, let me first give you something different like this [writes $-3 - (-3)$]. How would you simplify that?

L8: [Writes $-3 + 3$ then $= 0$] I first bring down this negative three and then I times the negative into the brackets and it would change to positive three and then they would just cancel each other out.

INT: OK, what about… [writes $4x^2 - x^2 - 6x^2 - x^4$]?

L8: [Writes working out, answer $-6x^2 - x^4$]

INT: OK, explain to me how you came to the minus six x squared?

L8: I brought down the bigger sign… oh, it's supposed to be three [scratches out 6 and writes 3 above it] so I brought down the bigger sign and minussed the three from the six.

INT: OK. When you say you brought down the biggest sign, what do you mean? Are you looking at the two numbers?

L8: Yes, Miss. So, in primary school we did something that was called DSB. So, we would like, DSB, so the difference, whichever one is bigger, then you subtract them and then you bring down the bigger sign.

INT: OK, let me write that down, DSB. So, different, subtract, the bigger one?

L8: Yes Miss, and then same sign, add, keep.

INT: Oh, same sign add keep, like this? [Writes SSK]

L8: Yes, but with an "A" Miss.

INT: Oh, SSAK [writes SSAK]. That's interesting. I haven't heard about that before. So that's what you did here, you looked at the difference, you said this one's bigger.

L8: Yes, Miss.

INT: And then you subtracted the smaller number from the big number and you kept the sign of the bigger number. OK. Interesting. Let's just do a few more, what about… minus five x cubed minus three minus eight x cubed minus, minus x cubed minus twelve [$-5x^3 - 3 - 8x^3 - (-x^3) - 12$].

L8: [Writes working out, answer $12x^3 - 15$]

INT: OK, you've done the underlining again. Is that like terms?

L8: Yes, Miss. I have different lines.

INT: I see, you use different ones to show you which ones are like. Like these are like terms [points to $-5x^3$ and $-8x^3$] and those are like terms [points to -3 and -12], but they're not the same.

L8: Yes, Miss.

INT: OK, then you've got three different terms of x cubed here and then you simplified them down to two. So do you normally go, if you have three for example, would you do a specific pair of them first?

L8: Yes, I first if there was more positives then I'd first do the positives, but if there was more negatives I'd first do the negatives.

INT: OK. Let's do one more [writes $-6xy - x - y - 5yx$].

L8: [Writes $-6xy - x - y - 5xy$ then $-11xy - x - y$]

INT: OK, just speak through what you've done. So I see you changed this to an xy. So those are the same, xy and yx.

L8: I rewrote it in alphabetical order. So, I just brought everything down except for the five yx I changed to five xy and then I grouped them and then I brought down the biggest sign. I brought down the sign because they had the same sign and I added them together and then I just put that minus, negative x and negative y because they weren't like terms.

INT: OK! I think that's fine. I don't have any other questions for you. Thank you!

Appendix 9.6 L10 First Interview

INT: So, let’s just start with Question 1. You can use my pen. So, first question, we’ve got minus x minus fifteen minus eleven. I just want you to explain how you would approach this question. What did you do here to get your final answer?

L10: I did minus fifteen minus eleven, so then I got minus twenty-six and then… um… I think I just took negative x there to make it minus twenty-six x.

INT: So, for the minus twenty-six, do you do minus fifteen minus eleven or is there another trick you've got like with two negatives? OK. So minus fifteen and minus eleven takes you to minus twenty-six.

L10: Yes.

INT: You've got minus twenty-six and you've got minus x. So, you've put them together to be minus twenty-six x. If there's a letter and a number, would you always put them next to each other if you're doing subtraction?

L10: No.

INT: Not normally, OK. Do you think you would rather leave your answer like this [points to $-x - 26$]?

L10: Yes.

INT: OK, that's fine. And it was the first question, so I'm sure your answers may change as we go on. OK, let's look at this question, Question 5. I this one, we've got brackets three x minus five and then a minus and then brackets three x minus five. Can you just explain what you did for this one?

L10: I did FOIL method, so it's like first and then inner, outer and last. I did three times three x, three x times three x …. Oh… it’s supposed to be nine x squared.

INT: OK, you can change your answer if you want to a different answer. Let's see.

L10: I did three x [times] negative five and then I got negative fifteen. And I did negative five times three x. Then I got plus fifteen x and then I got negative five times negative five is twenty-five, so those two cancel each other out. Then I got that [points to answer].

INT: OK, so you've got a negative nine here. Is that from this negative [points to negative symbol between brackets]? Where's this negative? How did you get to this negative in front of the nine?

L10: Oh, I probably got it from the middle.

INT: OK, so would you say that if you see brackets, then it implies that you must multiply?

L10: No. I made a mistake.

INT: I'm interested in seeing what you're working out. So let me rewrite the question and let's see if you would want to change your answer. If you think you've made a mistake, you can just say and then we can see what you would have done differently.

L10: Three x times three x that's nine x squared [writing]

INT: Now there's no minus in front of the nine x. Is that just because you've taken the three x and the three x? But what about this minus [points to negative symbol between brackets]? Is anything happening to this minus?

L10: Umm …

INT: Let me give you a different question. I just want to figure out your thinking. So, let's say I gave you something like three x minus three x [writes $3x - 3x$], how would you solve or simplify that?

L10: I'll get zero.

INT: OK, so you'd say this is the same as zero. Then what about something like minus three x minus minus three x [writes $-3x - (-3x)$]?

L10: Zero.

INT: Also zero. OK, what now, if I put the first one in brackets? [writes $(-3x) - (-3x)$]

L10: Negative three x, and it's plus three x is zero.

INT: Also zero. OK, so you're saying that if what's inside the brackets is the same and I'm minusing, then it's zero for this one?

L10: No, it's because this minus and a minus that make a plus and this is negative.

INT: So, are you saying that this would be minus three x plus three x and that’s why it's zero?

L10: Yes.

INT: OK, so what about this one? If you're doing multiplication, is that because there are two terms? Or would you do something different?

L10: I think multiplication is a bit different.

INT: OK, so would you still answer this question the same?

L10: Do zero?

INT: Would you make this zero or would you still do multiplication?

L10: I would still do multiplication.

INT: Why would you do multiplication? What makes this different?

L10: Ooh, I think I was supposed to minus it because I think if it was multiplication, there isn't supposed to be this.

INT: OK, so if there's something in between the brackets now you're saying that it wouldn't be multiplication. So, if it was like this three x minus five and then just immediately again three x minus five [writes $(3x-5)(3x-5)$], would you say this would be multiplication?

L10: Yes.

INT: But now because there’s a minus in between, we're not doing multiplication?

L10: I think so.

INT: OK, what would you say is the answer for this then, if you're not doing multiplication?

L10: I’m not sure, I’d say zero.

INT: Still zero? OK, so let's think … So, this is the three x minus five in brackets, and you've got the same thing in the brackets afterwards, which is similar to what we did here, right? Let's say I did it like this: three x minus three x. What would you say three x minus three x is, if both of them are in brackets?

L10: Three x minus three x will be zero.

INT: OK, so this is zero. So, would this be the same? Would this also be zero?

L10: I think so.

INT: OK, what about if I give you something like this? [writing] What would you say that is?

L10: Zero.

INT: OK. Why would you say it's zero?

L10: Because it's the same, but it's just minus.

INT: So we're minusing the same thing from the other thing. So, you would actually change it to being a zero if your brackets are the same and you're minusing one from the other?

L10: I think so.

INT: It seems that way. That's what you did here. Let's do Question 10. We have eight minus twelve x to the power of four minus thirteen x to the power of four minus eleven minus three minus minus three x to the power of four [Type equation here.]. I see here you've written a plus. Can you just explain where the plus is coming from?

L10: It's coming from two negatives make a plus. I did a negative and a negative so that … ohh no I didn't!

INT: So, these ones are the ones with the x to the power of four?

L10: Yes. And then I did the negative, negative twelve … plus …

INT: And then here you've done the eight and the eleven and the minus three. OK, let's do Question 12. We have four $x\ y$ minus five $x\ y$ minus five minus minus eleven $x\ y$ minus $x\ y$.

L10: So, I made plus $x\ y$ because of the minus. Then I just wrote I just wrote, like, I just wrote the [pointing to expression]…

INT: Just rewrote it? OK. Here you said four $x\ y$ minus five $x\ y$ plus eleven $x\ y$ equals two $x\ y$. Is that the first of those three terms?

L10: Oh, that's wrong. That's wrong.

INT: Why do you think it's wrong?

L10: Because the four isn't also negative.

INT: OK.

L10: But then that would have been negative $x\ y$, negative one $x\ y$. And that would have been ten $x\ y$. And zero two $x\ y$.

INT: So maybe then let's rewrite it because then we can see what you're thinking is. So we've got four $x\ y$ minus five $x\ y$ minus five minus… [rewriting Q12]. Let's do it again. How would you do it differently?

L10: Now, I would just write four $x\ y$ minus five $x\ y$, but then I'll just take that and plus eleven $x\ y$ and then I would see that and then I'll do this. So that will be negative $x\ y$ minus one $x\ y$, that will be ten $x\ y$. Then minus six ... Oh, then I think that would be ten, just then. Then there's minus five so ten minus five which equals five.

INT: So, you've done four $x\ y$ minus five $x\ y$ is minus $x\ y$ plus eleven $x\ y$ gives you ten $x\ y$? And then you're subtracting another $x\ y$, which gives you just the ten. So are you saying that because we have ten $x\ y$ minus $x\ y$ the $x\ y$'s are are cancelling?

L10: I'm pretty sure.

INT: Then you've just got ten and then because you've got a number, now you can minus the five.

L10: Yes.

INT: What if I give you five xy five minus three xy minus seven. How would you simplify that?

L10: Five xy minus three xy would be … yoh, this makes me think …

INT: OK, you can think about it, let's just see which way you would answer it.

L10: It's either two xy minus seven or just two minus seven [writing $2xy - 7$ / $2 - 7$].

INT: OK, so if it's two xy you've said it's five minus three, you've kept the xy. And for this one you're saying you would take the xy away like you did in this one [pointing to solution

$= 10$ for $4xy - 5xy - 5 - (-11xy) - xy$ in workbook]. So … what if I say five x minus three x minus seven—if I don't do two letters at a time, what would you say this one is?

L10: The same. Like either that or that [pointing to the two solutions $2xy - 7$ and $2 - 7$].

INT: So you would say this could be two x minus seven or it could be two minus seven [writes $2x - 7$ and $2 - 7$].

L10: Ya.

INT: So then if we're just saying it's two minus seven, then we're getting rid of the x's completely.

L10: Hmmm.

INT: OK, but let's look at a different question you did here. For example, let's look at Question 2. You said eleven x minus eight x minus three is three x minus three.

L10: Then I chose this one [points to $2xy - 7$].

INT: So you think you'd stick to that one.

L10: Yes.

INT: You would say this would be your answer? [Circles $2xy - 7$]

L10: Yes.

INT: So would you change your answer then for this question? [Points to previous solution $= 10$]

L10: Sure.

INT: You don't have to. I'm just curious. So if I give you something like… [writes $6xy - 3xy - 2xy - 4$], can you simplify this one for me?

L10: I can do it any way, so like… it would be three xy [pointing tp $6xy - 3xy$] and then that would just be xy [pointing to $-2xy$], because you don't have to write the one… xy minus four [writing $xy - 4$].

INT: OK, so that would be your final answer. In this one, you've kept the $x\ y$ now because that's all that was left over?

L10: Yes.

INT: And if I gave you seven $x\ y$ minus five $x\ y$ minus $x\ y$?

L10: Nothing. That would just be two.

INT: OK, can you just write it?

L10: Two $x\ y$ minus $x\ y$.

INT: I see. It seems whenever you're subtracting just $x\ y$, then you’re saying can cancel the letters out because you're taking the letters away?

L10: Hmm.

INT: Question 15. In this one we've got thirteen $a\ b$ minus four $b\ a$ minus thirteen minus two $b\ a$ minus minus nine minus one. And I think your final answer is thirteen $a\ b$ minus six $b\ a$ minus five. So would you say that $a\ b$ and $b\ a$ are different things then.

L10: Yes.

INT: So if I give you two $x\ y$ minus three $y\ x$, would you be able to simplify this?

L10: Not really.

INT: Are you saying they're not the same so you can't do anything further? You seem to have brought these ones down and then you simplified that first and you did the same thing for the numbers.

L10: Yes.

INT: And you've done a plus here.

L10: Plus because minus minus.

INT: And then you've got the minus five. So how would you change your answer then if it was thirteen $b\ a$ minus four $b\ a$ minus thirteen minus two $b\ a$ minus minus nine minus one?

L10: I would group all the things together that match. It's nine and seven $b\ a$ make those thirteen plus nine minus one will be negative five. Then the answer is seven $b\ a$.

INT: So you would only do this one if all of the letter combinations are the same, then you can subtract them from one another.

INT: Let's just go through Question 18 as well. So, seventeen a minus minus four a minus seven b minus six a. Just talk me through what you've done here. So you've got a plus …

L10: Because of minus minus.

INT: And so you've done the a's first.

L10: Yes.

INT: And then this is your final answer because you brought the …

L10: No, there's no, there's just one b.

INT: So there's nothing you can do with the b further.

INT: OK, let's see. [Q2] We've got four a minus minus three a, and you've given seven a minus one as your final answer.

L10: So four a plus three a is seven a.

INT: OK. Question 6. Two x minus eight x minus three x minus x. How would you approach this now? So you've just written the answer, but what do you think your steps would have been?

L10: Oh, I think that was wrong. Because two x minus eight x would be … um … negative six x, but then minus three will be negative nine x and minus x be probably just without the x.

INT: Oh, I see you're doing the same thing that you're doing before. So maybe just write out the answer there and I can see it.

L10: OK. [writing]

INT: So you've done these two first, then you've done the minus three and then you've done those two to get three. And then because we did, because there's a minus x at the end, you're saying that the letters then cancel out. So we just end up with the number.

L10: Hmm.

INT: OK, let’s just do these two and then you can go. Minus thirteen x minus eight minus minus six x minus six.

L10: Uhm, that’s negative thirteen x. And then I did, but then I did the plus six x, is seven x. Then the fourteen is negative eight negative six …

INT: OK, so you’ve done minus thirteen x plus six x gives you seven x and seven x minus fourteen. OK, and what if I give you minus ten x plus three x. How would you simplify that?

L10: Negative ten x plus three x would be negative seven x.

INT: In that one you're saying negative seven, but in this one you've said seven. So, it’s minus thirteen x plus six x.

L10: Oh, that's supposed to be a negative.

INT: Would you make it a negative?

L10: Yes.

INT: Last question: Minus x cubed minus minus three x cubed minus fifteen.

L10: That will stay negative, but then it will be positive three x to the power of three and then the negative would cancel out the x power three. So it will be just three minus fifteen.

INT: So it's again the same thing here - because we've got a minus x cubed and we're taking that from something with x cubed, you’re saying that we can cancel the x cubed completely and just end up with three and then we've just got three minus fifteen and that gives us minus twelve.

L10: Yes.

INT: OK, I think that's enough. Thank you so much for helping me.

L10: You're welcome.

Appendix 9.7 L11 First Interview

INT: OK, so for Question 1, it was minus x minus fifteen minus eleven. And then you've written equals minus sixteen minus x. So, can you just explain how you got to the minus sixteen?

L11: Negative eleven and negative fifteen are like terms. So, I first did that.

INT: OK, you can just tell me what the right answer is.

L11: Umm, the right answer is minus twenty-six minus x.

INT: OK. Let’s look at Question 5. In this one we did was brackets minus three x minus five brackets minus brackets three x minus five brackets. So just explain to me what you did for this question. You can take a second to say how you would answer it now, if you would answer differently.

L11: Oh. So, I timesed the negative inside the bracket. And then you get negative five and positive three x.

INT: So, this negative in here, you've brought inside the bracket to be in front of the three and then you've done the three and the minus three to get a minus nine. Is it like a FOIL method that you're using?

L11: Yes.

INT: So, would you say that whenever we see brackets, we think of multiplication?

L11: Yes, yes, yes.

INT: Do you think brackets can indicate anything else, or is it always going to be indicating multiplication?

L11: Probably will always be indicating multiplication?

INT: OK, and what if I give you something like, um, this [$-3x-(-3x)$ How would you simplify that?

L11: So, I’d first do the brackets and times the negative. Then they'll probably also cancel each other out seeing that that’s negative and this positive.

INT: So, you're saying this is equal to zero. In this one, you've multiplied the negative into the bracket and then you've added it because it's two negatives for a positive. How would you answer if I wrote it like this? [writing $(-3) - (-3)$] What would you do now? Would you do the same thing? Would you do something different?

L11: I think I would …

INT: Let's just see. I'm so curious what you're thinking.

L11: I shouldn't have written brackets here because I already times that.

INT: OK, so you wouldn't do it with the brackets.

L11: Yeah, so I'd times that inside.

INT: So, you're saying now again in this one that if there aren't brackets around the first one then you're just adding it, but if you put brackets on the first one, then you're going to multiply,

L11: Yeah.

INT: So, let's say I give you just numbers. How would you simplify that? Wait, let me put it in brackets: minus three in brackets minus minus three. [$(-3) - (-3)$]

L11: So, I'd times like I did the other times negative inside the bracket. Make positive three.

INT: So, we're saying brackets minus three minus minus three is going to give us minus nine. What if I give you something like this? [$(2) - (2)$]

L11: I'd probably like times like this?

INT: Are these brackets automatically telling you that you must multiply?

L11: Yes.

INT: Do you want to just write out the rest of your answer so I can see?

L11: [writing $(+2) - 2 = -4$]

INT: Now we're saying you've multiplied the two twos together and there's a negative. And if I wrote it like this? Does that change something? [$2 - (2)$]

L11: I guess.

INT: I mean, it doesn't have to. I'm just curious. How would you read this to yourself?

L11: I'm not really sure of it.

INT: What if it's just like this? $[2-2]$

L11: It will equal to zero.

INT: OK, so this is zero. But now when there's brackets involved, then you're not sure anymore whether it's going to be just two minus two, or if you have to multiply. Is that what you're saying? And what about if I did it like this? Type equation here.

L11: I think it will become positive. I think let me just do this. I thought I'd transfer the negative inside the bracket which will make negative two and this will be positive two and probably equal to zero. So, cancel each other out.

INT: So, now you're saying it's become two minus two, then it just becomes zero. Then would you change this one or would you keep this one the same?

L11: This one? Would I change that?

INT: I'm just wondering because now you've said you're bringing the zero in and then you're doing two minus two is zero, whereas in this one you were multiplying. So, I'm wondering if you would change this one as well so that it would just be two minus two which would be zero as well. Or would you leave it as? Multiplication.

L11: Probably leave it as multiplication.

INT: OK, so why do you think this one would be multiplication and not this one? I mean in terms of the two: you didn't multiply these two together but you did multiply those twos together. Is it maybe because of the brackets?

L11: I times the negative inside the bracket to make negative two, and then I brought down the positive two and then that's how we got to zero, because they cancel each other out. And then what I did here was, it's a negative two, so I times that inside the bracket make negative four.

INT: OK, I see. So, you're saying this minus you multiplying like this, but this minus you multiplying in that way to make the whole thing negative and then you multiplying the twos together.

L11: Yes. I times the negative inside the bracket, which will make positive two and then that's also positive, so probably just equal to four.

INT: OK, but if you just look at it like that, then you what would you say? Because we're adding now, hey, we're not doing multiplication anymore. So, you're saying once you've multiplied the negative in, you can get rid of the brackets and then it's a positive two and then you've got two plus two is four.

L11: Yes.

INT: But when there's a minus, are you doing two times two or two plus two?

L11: I'm timesing.

INT: OK, so that's two times two is giving you the four and this is negative giving you this one. What if it was minus two minus minus three. [$-2-(-3)$] How would you simplify that one?

L11: The answer would be positive. Times the negative inside the bracket, which will make positive three, bring down negative two and then that will be $-2+3= +1$.

INT: Ja, that makes sense. And you've not multiplied these two together. Would you change your answer if I did it like this? [writing] $(-2)-(-3)$]

L11: Oh, it's like exactly like that.

INT: Just think about. So, you've done minus two minus. You've brought the minus in to make it plus three. Then it's minus two plus three. Do these brackets make a difference? Does this imply multiplication? Or we don't need these brackets. What do you think?

L11: It implies multiplication. Like what I did there: I times negative inside the bracket because it's in front. What is this?

INT: Maybe write it down and see what you get. [L11 writes $(-2)+3= -6$]. What would you write down for the next step? OK, so you've done the plus three, you've brought the minus into the minus is a plus, and then you're multiplying the three and the two to get six.

You've got minus. OK, Question 6. Here we’ve got two x minus eight x minus three x minus x. I see you've drawn a one in there, that's just to show that it's one x. And would you say that it matters in which order you are subtracting? Like, does it matter if you do these two first or those two or the last two? Does it make a difference?

L11: I don't think it’ll make a difference because all of them are like terms, but I just did it because it's more easier.

INT: Yeah, yeah, it's just a method of doing it. If you didn't do it that way, you would have done it another way. So, you did two x minus eight x minus three x to get …

L11: First I did all the negatives and then …

INT: Oh, OK, oh, I see. So those two x’s are from there and you've done minus eight minus three minust one to get to minus twelve.

L11: And it's more easier.

INT: OK, that makes sense. So, you've done all the negative ones first, and then you've added the two from the beginning and then you got minus ten. OK, Question 8: Eleven minus minus seventeen minus seven x cubed minus two.

L11: I times the negative inside the bracket it made two plus seventeen.

INT: Yeah, you seem to have dropped the eleven.

L11: Yeah, I forgot about that.

INT: OK, so you would have just forgotten about it. So, how would you do the rest of? Or maybe start from the beginning.

L11: So, I’d times the negative inside the bracket which will make positive seventeen and then I’ll bring down all the other terms and then it’s like terms …

INT: Cool. So, this one you've brought down because there are no other like terms. Then you've decided these ones all together. Nice. OK, let’s go to Question 10: eight minus two x over four minus thirteen x to the power of four minus eleven minus three minus minus three to the four. I see you’ve underlined like terms again and you've changed this one to a plus.

I'm assuming that's the minus minus makes a plus. And then what did you do to get to the minus six? So, you've underlined all the like terms or the numbers.

L11: Oh, so negative eleven negative three which will equal to negative fourteen. And then I subtracted. it will equal to negative six.

INT: And then you did the same thing for the twelve. So, minus two x to the four minus thirteen x to the four plus the three to the four.

L11: Yeah. Yeah.

INT: Cool. Yeah, that's fine. OK, let’s look at this one. It looks like you brought these ones down. You changed this one to a plus, OK?

L11: Oh, I see. So, it's positive thirteen and minus fourteen and then I like subtracted that.

INT: OK. And what about this one? I see you brought these ones down.

L11: Oh, probably forgot.

INT: So, maybe just the same as another question. You would have put the one here. So, you would have had one minus all of this. How would that affect your final answer? If you had a one in front?

L11: Oh, then I would add the positive fourteen and negative fourteen. So, fourteen minus fourteen is equal to zero.

INT: So, you just have these two. OK, cool. Let's take this one. Question 12: So, these ones are with x and y. We've got four xy minus five xy minus five minus minus eleven xy minus xy. I see you started with plus xy. Is that similar to what you did before where you did the positives first and the negative second?

L11: Oh, I times, I just had the bracket, so I got to …

INT: OK, and then and then did you just, what did you do to bring down the other ones? You just brought this one down, and then I see you've underlined all the terms with xy again.

L11: Yes.

INT: Which order would you have done it in to get to the nine xy?

L11: Umm, so I added my positives, which will equal to fifteen xy, and then the negatives, ja, so it's negative six xy, fifteen minus six, which equal to … yeah.

INT: Let's see Question 13. I see you've drawn the little arch to show that it's minus, minus, minus, plus. And then have you brought down the other ones?

L11: Yes.

INT: How did you get to minus sixteen a?

L11: Negative, negative two a. So, I added that.

INT: If you have like minus fourteen a and minus two a, what do you do to get to the final answer? Do you add these two numbers together and put a minus in front or do you subtract minus two from minus fourteen?

L11: I added the two and the fourteen.

INT: And then how do you know whether or not you must put a minus?

L11: [unsure]

INT: If I could just give you something like let's say minus eight x minus six x, then how would you simplify that? [writes $-8x - 6x$]

L11: Probably negative fourteen x.

INT: How do you get to the fourteen is more what I'm asking.

L11: So probably I added the two negatives.

INT: You add the two numbers or the negative? What do you mean?

L11: The numbers.

INT: The eight and the six to get fourteen? And then what if I gave you minus eight x plus six x? [$-8x + 6x$]

L11: Then they will like cancel out. … I can't explain it …

INT: That's OK. So, let's just see … do you think minus eight x plus six x is going to be positive or negative?

L11: It's going to be negative because it's more negative than positive.

INT: OK, so you've got a bigger negative number than a positive number …

L11: [writes $-2x$.]

INT: So, you're taking the smaller number from the bigger number, but you're putting a minus in front, is that what you're saying?

L11: Yeah.

INT: So, if it was eight x minus six x.

L11: Oh that, then it would be positive. Yeah.

INT: You're saying this one is equal to two x, but this one you're saying is equal to minus two x. And then this one you said was minus fourteen x.

L11: Yes.

INT: So, to get to the fourteen, you did eight plus six, but how did you know it was negative? Just because these are both negative?

L11: Yes.

INT: So, it depends. If they're both negative, it's negative. Is that what you're saying? But if one of them is negative, then how do you know if it's gonna be negative or positive? Do you look at the numbers?

L11: Yes. Which one is bigger than?

INT: OK, so if you're subtracting the smaller number, oh, sorry, if the smaller number is positive, then it will be negative. Like your positive number here was six. Let's look at Question 15: thirteen ab minus four ba minus thirteen minus two ba plus nine minus one is what you've written. You've changed this minus minus to a plus. And then you've ended up with seven ab. Are the ab and the ba the same?

L11: Yes, because Ma'am has always told us you have to put it in alphabetical order.

INT: OK, so it doesn't matter if it's ba, it's the same thing.

L11: Yeah.

INT: OK. And then how did you get to seven?

L11: So, I underlined all the like terms: negative two ba and negative four ba which will equal to negative six ba. I subtracted that from thirteen to eight to seven.

INT: I think that's good. We’ve done twenty minutes. Thank you!

Appendix 9.8 L13 First Interview

INT: OK, so there's just a few questions that I want to go to that I did with the others as well. And then we can just see how much time we have left of the 20 minutes. Question 1 first I think: It was just minus x minus fifteen minus eleven. I just want you to explain how you got to minus four.

L13: Uhm, I just minused eleven from fifteen.

INT: What do you do when you have two minuses? Are you minusing the bigger number from the smaller number or are you adding them or what's your tactic?

L13: I minus the smaller number from the bigger number.

INT: OK. And then what? What makes you put a minus in front?

L13: Uhm, because it's a negative.

INT: OK, let's see Question 5: three x minus five in brackets minus brackets three x minus five. And you get minus two x plus two x. I just want you to talk through your answer, if you can remember,

L13: I don't even know what I did there.

INT: OK, in that case, how would you do this now?

L13: I was trying to think of one of the rules we're learning, but because there's a minus, I don't think I can do that.

INT: OK, so you are you talking about doing multiplication, but because there's a minus in the middle?

L13: Yeah.

INT: OK, so if there's a minus in between the two brackets, then you're not going to multiply?

L13: Yeah.

INT: OK, So, what do you think would be your next step then? If you're not going to multiply and if you are going to do subtraction?

L13: I think I will do FOIL method.

INT: But that's multiplication.

L13: Ohh.

INT: Well, I mean, what would you do? What do you think? So, let's maybe start with something else. What if I just give you something like this three minus minus three? [$3-(-3)$]

L13: Umm, negative six?

INT: OK. How did you get to negative six?

L13: No, it's not negative six.

INT: OK, so what do you think the minus brackets minus does? Or like a minus and a minus?

L13: Makes a positive.

INT: OK, so then what would it be?

L13: Would be three plus three. And it would be six.

INT: OK. And then what if I give you minus three minus minus three. So, all I've done now is change the first number to a minus three. [$-3-(-3)$]

L13: [writing $-3+3=0$]

INT: And then would you do the same thing if I put the first one in brackets? Does this make any difference if I put the first one in brackets? [$(-3)-(-3)$]

L13: Yeah.

INT: You think so?

L13: [writing $-3+3$] No, it'll still be negative. Negative three plus three …

INT: Which is also just zero.

L13: Yeah.

INT: OK, so these brackets don't actually make a difference to this minus three? So, then what if I give you … [writing $3x-3x$]] so, all I've done now is added an x next to the three.

L13: It would be the same [writing] … just be x with …

INT: What do you think? So, this one you said if you're doing minus three and three then it's zero. So now we have minus three x and three x.

L13: It'll just be zero.

INT: OK, so then now what about this one? These two things are the same inside the brackets and then you're minusing the one from the other, so it's the same thing here. So how would you do this one? [$(3x-5)-(3x-5)$]

L13: [writing $-3x+3x=0$]

INT: So, this time it's not a minus in front.

L13: I don't know what if I did.

INT: OK, let's do it like this. What if I say three … [writing $(3x-5)-(3x-5)$]

L13: Umm, it'll be zero. This one will also be zero.

INT: OK.

L13: Because they're the same.

INT: They're the same. They're subtracting the same thing from itself. Cool. Let's look at this one: If you have an expression like this where it's like two x minus eight x minus three x minus x. And you said minus six x minus three. Let me first ask you this: If you have an expression like this, would you do the minus in a specific order? Like a lot of people say they do the first two terms or they do all of the negatives or something like that. Does it matter in which order you subtract them?

L13: Yes.

INT: OK. So which order would you use?

L13: I go from here [pointing left to right].

INT: OK, so what are you saying? You do the first two terms and then subtract those and then do the next two terms?

L13: Mmmh.

INT: OK, So, you've done two minus eight x is minus six x. So how did you get to the minus six on the two and the eight?

L13: I do negative eight plus two negative six.

INT: OK, that makes sense. And then you've written minus three x minus x is minus three.

L13: I think I minus the x and I just leave this.

INT: OK, if I give you something like five x minus x [writing $5x - x$] how would you simplify that? Would you do the same thing or would you do differently?

L13: I think I just do the same.

INT: OK. [writing $5x - 3x$]

L13: [writing $2x$]

INT: OK, so in this case you're doing five minus three is two and then you're doing the x's. But if there's no number in front of the x, do you take the x's away? Is that what you're saying?

L13: Hmmm.

INT: Let's do another one, let's say we did eight x minus three x minus x. How would you do that one? [$8x - 3x - x$]

L13: [writing $-11x$]

INT: OK. How did you get to minus eleven?

L13: Wait … ohh … [writing $5x - x$]

INT: Well, how did you get to the five x?

L13: Eight x plus negative three x.

INT: OK, so that gives you the five. And then what about this x?

L13: I don't know.

INT: OK. Well, so the five x you've done eight and the three. Would you just ignore that minus x then or would you put it at the end?

L13: I'd put it at the end.

INT: Would you do it like this? [writing $5x - x$]

L13: Yeah.

L13: And then can you simplify this further?

L13: No.

INT: But then what about here? {both laughing] Why are you doing different things?

INT: OK. Well, let's say, what about if it's just three x minus, x for you? [$3x - x$] Would you say that it's just three or would you say it's something else?

L13: You can't simplify them more.

INT: You can't simplify this further. But what if it's four x minus two x? [$4x - 2x$] Can you simplify that?

L13: [writing $2x$]

INT: So, are you saying if there's a number in front of the second x, then you would subtract the numbers, but if there's no number in front of the x, then you would just leave it like this?

L13: Yeah.

INT: OK, cool. Now let's look at this one. So, you've underlined all of the terms with x to the power of four. What does that help you with?

L13: Like terms.

INT: OK, so all the terms that have the same like letter or like terms.

L13: Hmm.

INT: OK, so you've underlined them. So, here I see you've changed it to a plus. Instead of minus minus three x to the four, you've done plus three x to the four. You've changed this one.

L13: Because of the bracket. Because it's two negatives.

INT: Two negatives. OK. And then here you've put all of the like terms first and then you put the numbers afterwards.

L13: Hmm.

INT: OK. Does that make it easier for you to do all of the terms which are the same first and then all of the numbers which are …

L13: Hmm.

INT: OK. I see you did a minus twelve x to the four. So how did you get to the minus twelve x to the four?

L13: I did those. [pointing to writing] I just did those and then I did those …

INT: OK, so minus two x to the power four minus thirteen x to the power four plus the three x to the four, then you've got the minus twelve and then you did eight minus eleven minus three. OK. Let's look at this one: we've got ab and we've got ba. Are those different or the same?

L13: Different.

INT: OK.

L13: Wait. Hmm. I don't know.

INT: Well, if you're not looking at your answers, and if I give you something now like four ab minus three ba, would you simplify that or would you leave it as it is?

L13: I'll leave it as it is.

INT: You'd leave it so you'd say that the ab and ba are different.

L13: Hmm.

INT: OK. But here you've underlined them. Is that just because they're the ones with letters in them?

L13: Yeah.

INT: OK. So, I see, you've left the ba in ab separate. So, we've got minus four ba minus two ba. So, again, you've written the like terms together at the beginning and then plus thirteen

ab … and then minus thirteen, and then here you've changed to plus again. Is that from the minus minus?

L13: Hmm.

INT: OK. So you would leave your answer like this, you wouldn't simplify it any further.

L13: Hmm.

INT: OK. If we just go back to this question, how would you solve it they were all ab? Let’s say we have thirteen ab minus four ab minus thirteen minus two ab minus minus … [writing $13ab - 4ab - 13 - 2ab - (-8) - 1$] So, can you just solve that?

L13: [writing $13ab - 4ab - 2ab - 13 + 8 - 1 = 7ab - 20$]

INT: Thirteen ab minus four ab minus two ab , got seven ab and then minus thirteen plus eight minus one minus twenty. What did you do to get the seven? Did you do these two first and then that one?

L13: Yeah.

INT: OK. And then how did you get to the minus twenty?

L13: I did those two and then I did …

INT: Did you add these numbers to one another … did you do thirteen plus eight minus one to get to twenty?

L13: Yeah.

INT: OK. And what about this minus? So, it's minus thirteen, is that where this minus is coming from?

L13: Hmm.

INT: We might have to move rooms. I'm just going to see. I think let's just keep going and see if anyone comes … OK, let's just take this one as well: So here we just have two different letters, you know, next to each other. We've got seventeen a minus minus four a minus seven b minus six a.

L13: Hmm.

INT: So, again you've underlined the terms with a's and then … are those …

L13: Like terms.

INT: OK. So how do you get to the twenty-one here from the previous line?

L13: I did those two.

INT: OK. So, seventeen and four is twenty-one …

L13: And then I minused the six.

INT: OK, to give you fifteen.

L13: Hmm.

INT: OK, let me see … I actually just want to ask you a random question. So, what if I give you eight x cubed minus x cubed minus two x squared. $[8x^3 - x^3 - 2x^2]$ What would you do to simplify that? Can you simplify it?

L13: I don't think you can.

INT: You wouldn't simplify it? OK, so what if I give you eight x cubed minus six x cubed? $[8x^3 - 6x^3]$ Would you simplify that or would you leave it as it is?

L13: I'll simplify. $[2x^3]$

INT: OK. So, this one, you're saying you wouldn't simplify because again, there's no number in front of the x cubed, OK. So, there must be a number in front of the x cubed for you to subtract.

L13: Hmm.

INT: OK. Let's look at Question 14. So, we've got three y minus seven x minus eight minus eight minus eight x minus three one. OK, let's just go through the answer. So, you've got three y minus three y ...

L13: Hmm.

INT: Are they like terms again?

L13: Yeah, like terms.

INT: And then you've done the x's and then you've done the others. OK. And then you've got no more y's left.

L13: Yeah, so I left it out.

INT: OK, so you've got the three y and three y cancel each other out, so there's none of that left. And then you've got minus seven plus eight gives you the positive one. OK … let's just look at this one quickly: So, [Q9] we've got minus x cubed minus minus three x cubed minus fifteen.

L13: Hmm.

INT: So, you changed it to a plus here [pointing to $+3x^3$ in answer]

L13: Because it's negative and negative.

INT: You've got minus x cubed three x cubed minus fifteen. And then you've said three x cubed. So, what is happening with the minus x cubed?

L13: It's because the positive is higher.

INT: OK, so if you have something like minus x plus let's say two x [$-x + 2x$] would you simplify this?

L13: Mmh.

INT: OK, what would you simplify to?

L13: [writes $= 2x$]

INT: Just a two x? So, is that because there's no number in front of this and it's a negative, so you just take it away and leave the positive one?

L13: Hmm.

INT: OK, so now you've just left with so the minus x cubed falls away and you've got three x cubed minus fifteen. What if you have something like minus two x plus five x? [$-2x + 5$]

L13: [writing $3x$]

INT: OK. So, when there's the numbers in front, then we deal with the numbers, but we don't do … we don't … if it was let's say minus six …

L13: [writing]

INT: And what if I give you minus x plus four x plus x? $[-x + 4x + x]$

L13: [writing $4x$]

INT: OK, so are you saying that these two would then both fall away?

L13: Hmm.

INT: So, they cancel each other out?

L13: Hmm.

INT: OK, but what if we have minus x plus four x *minus* x? $[-x + 4x - x]$ Does that make it different? Would you give the same answer?

L13: Mmmh… I'd give the same answer.

INT: Same answer? Also four x?

L13: Ya.

INT: OK, so if there's no number in front of the x's then it doesn't matter. Then you just use the one that there is a number in front of.

L13: Mmmh.

INT: OK, let's do a couple more. Let's look at this one: fifteen minus five x minus minus x minus x squared minus minus eleven minus twelve x squared.

L13: [writing]

INT: OK. And so again, you seem to have underlined the ones with x's or x squares. And then you've done the $x's$ first and then you've done the x squareds. OK. So, the x and x squared are different terms.

L13: Yeah.

INT: OK. So minus five x plus x, then you've given minus four and minus eight minus twelve you're given minus twenty. So how do you get to minus twenty from the minus eight and the minus twelve?

L13: I added.

INT: You added the eight to twelve to get twenty and then why is there minus in front?

L13: Because they're both negative.

INT: OK and then with the fifteen and eleven you just add them to get twenty-six.

L13: Hmm.

INT: Cool. OK, let's just do these two and then I think we can stop: four x cubed minus nine x squared minus eight x cubed. So, this time you've only underlined the ones with x cubed. Is that because there are no numbers?

L13: I underlined them because they were like terms.

INT: OK. And there's no other term for the x squared.

L13: Uh-uh,

INT: OK. So, here you wrote twelve first and you scratched it out. Did you do four plus eight first?

L13: Yeah.

INT: And then you change it to a minus four.

L13: Hmm.

INT: Why did you change it to a minus four?

L13: Because the negative number is more than the positive.

INT: OK. So, the negative number is bigger, which tells you it's going to be a negative. And then how did you get the four, the actual four itself?

L13: Positive four minus eight.

INT: OK. So that gives you the minus four. OK, let me just give you the last one. So minus thirteen a squared minus five a to the four minus minus six a to the four minus fourteen a squared minus five a squared. OK, you've underlined these once again.

L13: Like terms.

INT: OK, umm, so you've got minus thirteen a squared minus fourteen a squared minus five a squared. And then you've just carried the other ones. And here you've changed it to a plus again,

L13: Because they're two negatives.

INT: It's two negatives.

INT: OK. And then the minus twenty-seven, where do you get the minus seven from? So, you had minus thirteen and minus fourteen and minus five.

L13: I just added them.

INT: So, you added thirteen and fourteen to get twenty-seven and then it's a negative.

L13: Yeah.

INT: Because? Where is the minus coming from?

L13: Because the thirteen is also negative.

INT: OK. So, they're both negative, then you put it as a negative. OK. And then this one, you've changed it to one. Is that the minus five and the six.

L13: Hmm.

INT: OK, cool. And then these ones you've got minus. Did you add them again to get thirty-two?

L13: Ya.

INT: OK. And then I just want to do one more. How would you simplify one minus minus two x minus one?

L13: [writing]

INT: Okay. Is that you final answer?

L13: Ya.

INT: Could you do anything with the ones or would you leave them as they are?

L13: Leave them.

INT: OK. Cool. OK. I think that's good enough for me. Thank you so much.

Appendix 9.9 L14 First Interview

INT: It's about going to be about 20 minutes and let's just see how things go. I might ask you more questions, but these are the ones I want to start with. For Question 2 it was eleven x minus eight x minus three and then you wrote three x minus three. So just explain to me quickly how did you get to the three x?

L14: So, what I did was, it's a x and then a x [pointing to like terms] and this one is just on the side. Eleven minus eight, three, and three is left over so it's three x minus three.

INT: OK and then for the x equals zero, is that your final answer?

L14: That's my final answer, but I didn't know if I was supposed to write it down.

INT: Would you leave it as this answer [pointing to $x = 0$] or would you leave it as this answer [pointing to $3x - 3$] now that you're…

L14: I think I'll do it same as this [pointing to $3x - 3$]

INT: As this one instead? As three x minus three, OK. Let's look at this question [Q2]. Here we have four a minus, minus three a minus one. What would you do here where we have a minus and a minus?

L14: Becomes a positive.

INT: Positive, OK, what would you say then?

L14: Four a plus three a minus one.

INT: OK, so that's what you did to get to the seven, so then you got seven a minus one, and then you wrote six a? What did you do to get the six a?

L14: So, seven a minus the one.

INT: OK, you minus the one from the seven, even though there's no a next to the one? Are you just subtracting it because it's the numbers, is that right?

L14: Mmmh.

INT: OK. This one's quite similar [Q4], we have minus thirteen x minus eight minus, minus six x minus six. What would be your first step?

L14: Um… like terms. So, thirteen and six. Negative thirteen minus…. Negative thirteen x minus six… no…. ya that's what I thought. Then thirteen minus the six… no you bring this over [gesturing arc from $-6x$ to $13x$] then it becomes a positive.

INT: would it be positive? OK.

L14: Ya 'cause you change this 'cause it comes over

INT: 'Cause it's a minus and a minus? OK.

L14: And you get negative thirteen plus six x, so seven.

INT: It's giving you seven [points to $-7x$], or minus seven?

L14: Ya.

INT: OK.

L14: And then negative eight minus six is negative two.

INT: OK.

L14: But that's not right…

INT: OK, maybe just rewrite the answer then. Say we've got the question here [writes Q4 in book]. How would you rewrite your answer now? If you're saying that that one was incorrect.

L14: [writes in book] And negative… and minus fourteen… like that [final answer $-7x - 14$].

INT: And then would leave your answer like this?

L14: Hmm…

INT: Seems that way because you left it now. You wouldn't want to simplify it further? [L14 hesitates to answer] Just because I want to compare it to the previous question. So, in the previous question… so we had minus seven x minus fourteen [$-7x - 14$] but in the other question [Q4] you were doing minus seven x minus two and then you did five. So, what's the difference in this one [points to $-7x - 14$] and with this one [points to $-7x - 2$], is it maybe the numbers that are different?

L14: The two isn't supposed to be there, because…

INT: Oh, oh, let me go back to a different question… let's say this one [Q3] instead, sorry. So here you had seven a minus one and then you said it was six a, but now you've got minus seven x minus fourteen…

L14: You *can* simplify it further.

INT: *Would* you simplify further?

L14: Negative… no… ya, negative twenty-one x.

INT: OK, but you say you're starting with like terms, so the x's and x's and the numbers, but then do you think once you've simplified it as far as you can with the x's and with the numbers, then would just keep simplifying it? Is that what you're saying?

L14: Hmm, ya, you can. You can simplify it or if you want to.

INT: But you don't have to, is that what you're saying?

L14: I mean if you, you know, if you wanna get an extra mark then you can. [4:43]

INT: OK, let's look at another question [Q5]. We have three x minus five minus three x minus five [$(3x-5)-(3x-5)$] and then just talk me through what you did here because here you've changed the signs to being plusses. I'm curious what the…

L14: Oh honestly, I don't know what I did there.

INT: That's fine. How would you answer it now if you just look at this?

L14: Um… [rewrites Q5]. I honestly don't know.

INT: Let's start with something different. Say I give you something like this… [writes $-3-(-3)$] If I just give you that with numbers, how would you do this one?

L14: So, negative three and a negative and a negative is a positive, so like that [writing $-3+3$].

INT: OK, so you'd have minus three plus three. Can you simplify that further?

L14: Ya, it's zero.

INT: OK, you say this is equal to zero. Then what if I write it like this instead? [Writes $(-3) - (-3)$] Do these brackets make a difference? [L14 agrees] They do. OK, what do you think that means?

L14: It's times.

INT: So, if I've got the brackets around the first one you're saying then it must be times.

L14: Ya the brackets mean times.

INT: And what if there's something in between the brackets? [L14 hesitates] So are you saying that these ones are equal [writing $(-3)(-3) =$ in front of $(-3) - (-3)$]?

L14: No.

INT: What's the difference?

L14: There's no minus in the middle.

INT: Ya. So, this one we would times [$(-3)(-3)$] and… [pointing to $(-3) - (-3)$]

L14: This one we just minus.

INT: How would you then simplify this one?

L14: Negative nine, no, three minus… but that negative times that negative is a positive… so it would basically be… with positive three plus, no…

INT: If you think about this one [$-3 - (-3)$], you had minus three minus, minus three gives you minus three plus three. Now you've got minus three minus, minus three.

L14: I'm confused about this one.

INT: So, when there's a bracket in front, then you're not sure if you're supposed to use multiplication or if it's still the same. [L14 agrees] And what if I gave you something like this? What if I did two minus, minus two, without the minuses [writing $(2) - (-2)$]?

L14: I would think that would be… two plus two.

INT: OK, so this you would say would be two plus two. Does that make it easier for you to do this one [points to $(-3) - (-3)$]? All you've done is you've taken what's in the bracket, you've ignored the brackets, and then you've done minus minus gives you plus.

L14: I would understand it, but I don't understand how the positive works here in front [points in front of (-3)]. Would it be minus three plus three again?

INT: [Writing $-3+3$]. That would make sense if you did it the same method as this one [points to $(2)-(-2)$]. So, because there's a positive here, then you could say it would be minus three plus three. What about with this [points to $(3x-5)-(3x-5)$], how would you go about this?

L14: First work in the brackets? So, three x minus five, is negative… negative two x, and then positive times a negative is… minus two x [writing $-2x-2x$]. Then… [writes $=4x$].

INT: OK, you've first tried to work out what's inside the bracket and then you've said that your answer is negative, and then you've worked out what's in this bracket and you've put the negative in the middle. OK, but this one you're saying that what's inside the brackets here [points to first set of brackets] is negative. Does that mean that this one would also be negative [points to second set of brackets]? Is that where this negative comes from?

L14: Yeah, because a negative times a positive is negative [gestures from first set of brackets to $2x$ then to minus in front of $2x$].

INT: But is this one positive [points to second set of brackets] if this one is negative [points to first set of brackets]? If they're the same?

L14: Um…

INT: So you said that this [underlining first $(3x-5)$] is equal to minus two x [underlining $-2x$], which means that it's the same thing [underlining second $(3x-5)$], right? Which has to be equal to minus two x [underlining $-2x$], but there's also this minus in the middle.

L14: Must change it to a positive.

INT: Plus, OK. So it would be minus two x plus two x.

L14: Becomes zero.

INT: Zero, OK. Let's do that this one [Q15]. We have thirteen ab minus four ba plus two ba, so you've written all the ones with a's and b's first and then you've done all the numbers so we have plus 9, minus thirteen and minus one. Would you say that ab and ba are the same or different?

L14: Different.

INT: If I just give you something like this, ten ab minus five ba [writing $10ab - 5ba$], can you simplify this or would you leave them separate?

L14: I would leave them separate because unlike terms.

INT: OK, here you'd leave them separate because there are letters but what about in this case… where did we write it… Oh in the other question, where you had seven a minus one equals six a [$7a - 1 = 6a$], if it's a number [points to -1] does that mean you can subtract it from the coefficient? But then if there's a different letter then is it different? [L14 hesitates to answer] Like if you had ten x minus five y [writing $10x - 5y$] versus ten x minus just five [writing $10x - 5$ underneath]. Would you be able to simplify either of those?

L14: OK so ten x minus five y would just be five xy [writes $= 5xy$ next to first expression], and I think this one would just be five x [writes $5x$ next to second expression]. And then maybe for this one it would just be five ab [writes $= 5ab$ underneath $10ab - 5ba$] because you're adding another one on to that [gestures from $5ba$ to $10ab$] but this is… a is first then b.

INT: OK, but then would you say maybe that ab and the ba are the same if you're getting five ab? Or do you still of them as different but you're minusing the numbers?

L14: I think of them still as different. [11:52]

INT: OK, so they're different but you can minus the numbers and then you just take whichever one comes first? [L14 nods] OK. So here [pointing to Q15 solution] you've done thirteen ab minus two ba, so you did minus four plus two [points to $4ba + 2ba$]?

L14: Ya

INT: OK, gave you minus I mean gave you eleven. And then you seem to have just pulled the numbers down and then we had, OK, now you've gone from eleven ab plus nine minus thirteen minus one to twenty ab minus thirteen minus one.

L14: [Pointing to $20ab + 9$] 'Cause of them, and just bring it down.

INT: So, it was the eleven and the nine to get twenty, OK. And then you brought these down, then how did you get to the seven ab?

L14: Twenty minus fourteen.

INT: OK, that gave you… [pointing to $6ab$]

L14: No, no, no, no, it's not, it's six I mean.

INT: Ya, that's how you got to the six by minus fourteen from twenty?

L14: Mmmh.

INT: OK. So, I want to do one more thing. What if you have [writes $-5x-(-5x)$], say you've got minus five x minus, minus five x.

L14: Um, so, negative five x. Negative and a negative is a positive so plus five x [writes $-5x+5x$] and the answer would be just x.

INT: Oh because, now the minus five and the five have cancelled and you just have an x left over. OK, let's look at some more questions, let's do this one [Q7]. We have minus five x to the power of two, minus twelve x to the power of two, minus four x to the power of two [$-5x^2-12x^2-4x^2$]. Can you just explain what you did with the next step?

L14: So negative five x two minus twelve, I first work with these two here [points to $-12x^2$ and $-4x^2$] and left the five. So, five x two then negative twelve minus four, but it's wrong… it supposed to be…

INT: Here, you can write it here.

L14: I would say negative five x two, minus twelve x two minus four x two [rewriting question]. Bring down this and take away four is going to be negative sixteen x to the power four [writing $-5x^2-16x^4$].

INT: OK, so how come you're putting this one to the power four? Can you explain that part?

L14: Because two and two, the two exponents add.

INT: OK, so we're adding the exponents, but we're subtracting the numbers.

L14: I guess sometimes I get confused by the exponents… Negative five and sixteen… is negative twenty-one, x to the power of six.

INT: OK, so you would say it doesn't matter if you're minusing the numbers. But then you would, because you're putting these together, you would add the exponents. So, then you've done the same thing here. You're minusing these two numbers from each other [underlining -5 and -16], but you are adding the exponents to one another. OK. Let's do this one: four xy minus five xy minus five minus, minus eleven xy minus xy [$4xy - 5xy - 5 - (-11xy) - xy$]. So, they've all, except for the five, they've all got xy in them. Where do you think the minus one xy is coming from?

L14: Look for the like terms. So it's four xy and then that [pointing to $5xy$] and that [pointing to $-11xy$]. What I did was, I didn't actually work with that one first. I first work with this [points to $4xy - 5xy$]. Four xy minus five xy is negative one xy plus that is plus sixteen xy.

INT: OK, you put the minus five and the plus eleven xy together.

L14: This wasn't supposed to be like that…

INT: OK. What would you do differently?

L14: I would change that [pointing to $-11xy$] so negative five plus eleven is… that'll be six… negative six, no… positive six ya.

INT: OK, would you say it's just positive six or would you say it's positive six xy?

L14: Positive six xy.

INT: OK, so this one you would change to positive six instead of sixteen.

L14: And then the xy is left over.

INT: OK.

L14: Then make this six xy [scratches out the 1 of 16]. Negative one, that will actually be five xy [scratches out 1 of 15] minus xy. Then four, no… ya it will be five again [scratches out 14 and writes 5 for answer $5xy$], because with the xy there's nothing there.

INT: Oh, so there's no number in front of the xy. So you're saying it's five minus nothing…

L14: I'm not sure if there's supposed to be like an invisible one in front of it or something.

INT: So, it could be the invisible one. What if I just said to you five x minus x [writes $5x - x$].

L14: I would put $4x$.

INT: You'd say four x. So, for this one would you say four xy then?

L14: Ya.

INT: OK. Let's do another one [Q13]: minus fourteen a minus, minus three b, minus two a minus seven [$-14a-(-3b)-2a-7$]. You've written a plus there [points to $+2b$].

L14: Mmmh because negative and a negative is a positive. So, negative fourteen a plus, this becomes a plus two [gesturing arc from $-2a$'s position to directly behind $-14a$] then that falls back [gesturing $-(-3b)$ movement to right] so it'll be plus three b minus the seven.

INT: OK, so this negative two you've brought over to be a plus. Just explain quickly again why it's a plus instead of a minus?

L14: Because you bring this over because of the like terms.

INT: So, is it because it's going over another negative number that it's now changing to a positive?

L14: Ya, because if it's a negative number and you bring it over then it becomes a positive.

INT: OK. That's where you got the minus twelve a from and then you've done three b minus seven is minus four b. So, even though this one [the 7] doesn't have a b, you've said that the the three and the minus seven gives you minus four.

L14: [Scratches out answer of $8ab$ and writes $-16ab$]

INT: OK minus twelve minus four. Let's do one more [Q17]: four x cubed minus nine x squared minus eight x cubed. So four x cubed you've brought down, brought down, brought down [question rewritten in second line]. Then you've said minus five x the power of five.

L14: I added that two numbers and then minus eight x three, and negative eight x is thirteen, negative thirteen x to the power of eight because I added the top ones.

INT: So, you added the top ones because you're putting them together, and then you've done minus thirteen because you've got minus and minus. So how did you know that it would be a minus thirteen? What did you do to get the thirteen and the minus?

L14: So, four x to the power three minus nine x to the power three, then you bring that down [pointing to $-8x^3$]. So, negative five x to the five minus eight x to the three becomes negative thirteen x to the eight.

INT: OK. Thank you so much!

Appendix 9.10 L17 Interview

INT: So, it will just be about twenty minutes. So, there are a few questions that I've been asking everyone. So, we'll do those first and then depending on how quickly we do those, we can see if there are any extra questions. So, Question 1 was minus x minus fifteen minus eleven, and I just want to know how you go about solving those problems.

L17: I add all the…all the same…

INT: The like terms?

L17: All the like terms together, I added those together.

INT: OK, so you added the fifteen and eleven. How did you know to put the minus in front?

L17: Because both of them are negative and there's no multiplication.

INT: OK. So, if you were multiplying, you would have changed it to a positive?

L17: Yes.

INT: OK, because they just subtracting and they both negative. So, what were these brackets for? [points to brackets $(-15-11)$] Is that to show that those two are like terms?

L17: Yes.

INT: Let's do this one. In this question it's three x minus five in brackets, minus, and then also in brackets three x minus five. So, what I'm also curious about is what do you think these brackets [points to brackets] … I see how you've drawn lines around it [points to lines drawn by L17 typical of FOIL method] … what do you think these brackets are telling you?

L17: We must multiply each term into it.

INT: OK, do you mind just talking through your solution? So here you've kept the brackets, but you've done multiplication, is that right? OK And what about if there's a minus sign in between the brackets? How does that change the answer?

L17: Everything will like swap to the opposite…

INT: to the opposite sign? OK, let's say I give you something different like… say we have just with numbers, minus three minus, minus three [$-3-(-3)$] how would you do that one? You can write your answer.

L17: I will change the sign because it’s in a bracket so it's multiplication you know like… [writes $-3+3$] and I will add it up and it will be equal to zero [writes $=0$].

INT: If I was to put the first one in brackets [writes $(-3)-(-3)$], would that change the way that you do it or would you do the same thing

L17: I would do the same thing [writes $-3+3$].

INT: OK, if I change it now to… say I add something in [writes $-3-x-(-3x-x)$].

L17: [writes $-3-x+3x+x$]

INT: So, what did you do now to get rid of those brackets?

L17: It's multiplication. So, like, I changed the sign because there was a negative sign outside.

INT: OK, so you're multiplying the negative sign in. And then I would you simplify this? [points to answer $-3-x+3x+x$]

L17: I would add the like terms. There’s a like term and there’s a like term [underlining $-x$ and $+x$]… hmmm… they would cancel each other out.

INT: The x’s?

L17: Yeah, positive x and negative x.

INT: OK, and you’re not going to simplify this further? [L17 shakes head] OK, so then let's do one more example. Let's say we have x minus three minus x minus three [writes $(x-3)-(x-3)$].

L17: [writes $x-3(x+3)$]

INT: OK, now you're keeping the brackets? So, what is different about… [points to previous expression $-3-x-(-3x-x)$] in this one you took the brackets away and you changed it to a plus, but now you're keeping the brackets.

L17: Because I multiplied the minus sign into the bracket.

INT: OK, so for this one you multiplied, you did a minus and a minus makes plus [points to $+$ signs in $-3x + 3x + x$]. But here you haven't put a minus in front of the x [points to $(x - 3)$. If there's nothing in front of the x here, do you only apply the minus to the other sign? [points to -3 in second $(x - 3)$] Or would you put a minus in front of this x? [points to second x in $x - 3(x + 3)$]

L17: I would put a minus.

INT: You would, OK, and would you keep the brackets?

L17: Must I multiply further on?

INT: I'm just curious, let's say now we've added in this minus [points to $-$ sign in front of second x in $x - 3(-x + 3)$], do you think you if there is a sign in front of this x now that you've done the multiplication, would you keep these brackets? And then how would you do it further? Or would you take the brackets away?

L17: I would take the brackets away and then multiply that into the brackets.

INT: OK, I just want to understand what you mean. So, x minus three minus x minus three [rewrites expression $x - 3 - (x - 3)$]. Let's just do it again, sorry, because I basically wrote that.

L17: [writes $x - 3(-x + 3)$ and I would multiply that into that bracket [writes $x - 9$].

INT: So now with this one [points to $-3 - x - (-3x - x)$] you brought the minus in, but then you took the brackets away and then you just added or subtracted the terms. But then in this one [points to $x - 3 - (x - 3)$] you've kept the brackets. And now, did you multiply these terms [points to the x's and then to the -3 and $+3$] together? [L17 agrees] So, what's the difference between this one [$-3 - x - (-3x - x)$] and this one [$x - 3 - (x - 3)$] that you carried on multiplying here? Whereas with this one you took the brackets away and just did plus and minus.

L17: Because I multiplied the minus sign into the brackets.

INT: OK… are you saying you're just bringing the minus in [points to x inside brackets]? But are you not multiplying the minus in here [points to -3 inside brackets] as well? Do you see what I mean? I'm getting confused. So, I'm curious why, if you've already brought in the minus, you're still keeping the brackets, versus in this one you multiplied in the minus, but then you took the brackets away. [L17 seems uncertain] So would you which one of these would you do rather? [Time 7:31]

L17: This one [points to [$-3 - x - (-3x - x)$] solution]

INT: This one? OK, let's say we follow this method, then these brackets would go away [draws lines through brackets in $x - 3(-x + 3)$] if you've used the same method, so then this would be x minus three minus x plus three [$x - 3 - x + 3$]. And then how would you simplify that? Would you still do multiplication, or would you do addition and subtraction?

L17: Addition and subtraction.

INT: OK, So, what would be your answer?

L17: [Pauses then writes $-x$]

INT: Minus x? OK, just explain to me where the minus x is coming from.

L17: Those two [points to the two x's in the expression].

INT: Those two? How did you get to the minus x? You've taken away the threes [L17 agrees], so your threes are cancelled, and you just had x minus x, then you get minus x. [L17 seems uncertain, stays quiet] Would you keep it as that? [L17 uncertain] Let's say I give it to you like this, rewrite it [writes $x - x - 3 + 3$]… minus three plus three. If we just rearranged it.

L17: That two would cancel each other and x minus x [writes $x - x$].

INT: Would you simplify that further? [L17 shakes head] You'd leave it like that, OK. What about if it was with numbers? What about if it was three x minus three x [writes $3x - 3x$]? Would you leave it like that as well, or would you do something else?

L17: Um… maybe it is x [writes x underneath $3x - 3x$]

INT: OK, what did you do now? You've taken the threes away. Is that because it's the same number? [L17 nods] So are you saying, here [points to $x - x$], you would leave it like this because there are no numbers in front of the x's? But when there are numbers, then you're working with the numbers [L17 nods]. OK, let's do another one. For Question 6, does it matter in which order you subtract the terms? If all of them are like terms. [L16 shakes head] No. But even if it doesn't matter, I see here you've used brackets. Would you normally put them in pairs to make it easier for yourself?

L17: Yes

INT: OK, so that's what the brackets are for. So, in this case these brackets are for you for grouping, but you're not going to multiply.

L17: No

INT: OK, let's just do another example question for this. Say we have five x minus x minus minus [writes $5x - x - (-3x)$]. So, this time *I'm* putting brackets. How would you simplify this?

L17: [writes $5x - x + 3x$ then $8x - x$]

INT: OK, so you changed this one to a plus. What did you do to get to the plus?

L17: I multiplied the minus sign into the bracket.

INT: OK, so you've done two minuses is giving a plus. And then how did you get to the eight?

L17: I added five x plus three x.

INT: And you would leave this answer like this, you're not going to simplify it further? [L17 shakes head] OK, what about if there was a number in front? Say we have we change it to eight x minus two x as a different thing [writes $8x - 2x$]. Would you simplify that?

L17: [writes $6x$ underneath]

INT: So, what if there was a one in front? [writes 1 in front of x in $8x - x$ for $8x - 1x$]

L17: I would subtract the one.

INT: And what would that give you?

L17: [writes $7x$]

INT: So if there's no number in front of the x, then you're going to leave it like it is. And that's what you did in this one [points to $x - x$]. [L17 nods]. OK, for Question 10, we've got exponents. We've got eight minus two x to the power of four minus thirteen x to the power of four minus eleven minus three minus minus three to the power of four. Do you want to explain what you think you did for this one? I see you've done some underlining and some bracketing, and you've rewritten it in a different order.

L17: All the like terms I put together.

INT: OK

L17: And this one doesn’t have a variable at the end, so I group that together [points to $(+8 - 11 - 3)$] and that are all the same [points to $-2x^4$ and $13x^4$.

INT: Yeah. This one I see also you've changed it to plus [points to $+3x^4$]. I'm assuming that's just the minus, the minus and then how did you get to the minus twelve?

L17: I added and subtracted the two, the thirteen and the three.

INT: And then the minus six is from the eight minus eleven minus three, OK. Let’s do this question [Q12]. We've got four xy minus five xy minus five minus minus eleven xy minus xy [$4xy - 5xy - 5 - (-11xy) - xy$]. Again, you've rewritten all the xy’s together, so that's your grouping. Here, I see you've done a one in front [points to 1 in $1xy$ term]. Is that so that when you minus, you can minus the numbers? [L17 nods] OK. Oh, I see it’s break time now… I just want to go through this [Q15]. Here, I see also you've done ab and you've rewritten all of them as ab, so are ba and ab the same?

L17: Yeah, I just wrote it in alphabetical order.

INT: OK, that makes sense. And then again, you've done the brackets to group them. Is that right?

L17: Yes

INT: OK I just have some general questions. I wanted to ask you, when you’re doing subtraction, do you visualise a number line or do you just work with the numbers and the symbols?

L17: Just the numbers and the symbols.

INT: And if we have like n cubed [writes n^3], what does the cube tell you about the n? Or what does this mean to you?

L17: n three times… n is three times

INT: Three times n? And what's the difference between three n [writes $3n$] and n cubed?

L17: The n is getting multiplied three times…

INT: By itself?

L17: By itself.

INT: Just one more question, if I ask you this [writes $-3a-(-2a)$], I've asked you this before but just to clarify, you were saying before that you would multiply this minus in [points to minus sign in front of brackets] then you would take the brackets away. Whereas if I give you something like this [draws brackets around $-3a$], do these brackets make a difference? Would you then multiply, or do you think these brackets don't matter?

L17: I would multiply.

INT: Multiply, OK. I think that's fine, I think it's going to be very loud now. Thank you so much.

Appendix 9.11 L19 Interview

INT: OK, so I just want to go through these questions first. I want you to explain your step-by-step thinking, your working out when you were answering the questions. If there's anything you would do differently now, you can say and then you can write it out. [Q1] So minus x minus fifteen minus eleven and then you came to minus twenty-six x. If you just explain that. You can just talk through it.

L19: So, I added these two together because it's both negative and I added the x together.

INT: So, you first did the numbers and then you put all of them together. OK. If I give you something like this [writes $-3 - x - 1$], then how would you do that one?

L19: I would add these two together, then I would do the variable.

INT: Do you mind just writing out your answer?

L19: [writes $-4x$]

INT: OK. Let's look at Question 5. This one we had, in brackets, three x minus five, and then also brackets three x minus five. So, my first question is, what do you think these brackets are for?

L19: Um… to like separate it.

INT: Ok, so do you think these brackets in the beginning are necessary?

L19: Hmmm… yes?

INT: Do you have a reason or is that just what you think off the top of your head?

L19: I don't think there's a reason.

INT: OK, so I can see here you did, you just rewrote it three x minus five, then you said minus three x plus five. So, what did you do with the negative that was in between the brackets?

L19: So, what I did was, there is a one in front [points to bracket] and then I timesed it in so it changed the sign.

INT: OK, and then you took the brackets away

L19: Yes

INT: And then I see you've crossed everything out. So, is that you saying that they're cancelling each other out?

L19: Yes.

INT: OK. So, what would you say is the final answer then, or would you just leave it as this?

L19: I'll leave it as x.

INT: As x?

L19: Yeah.

INT: OK. Can I give you another? Let's do some more similar ones just with numbers… something like that [writes $-3-(-8)$]. How would you simplify that?

L19: I would put a one there [points between negative symbol and bracket (-8)]

INT: You can write your answer.

L19: [writes $-3+3$] and then it cancels out again so I would say zero [writes 0].

INT: And what then, if we have something like this [writes $x-3-(x-3)$]?

L19: I'd also times it. Must I write?

INT: Yeah

L19: So… [writes $x-3-x+3$] and then… zero [writes 0]

INT: Also zero. So now, if I give you the other one again, let's say like this [writes $3x-5-(3x-5)$]. [L19 writes $3x-5-3x+5$] Would you simplify further? Would you leave it like that?

L19: Leave it like that.

INT: So how come with this one [points to $x-3-(x-3)$] you would simplify it further to 0, but then with this one you're leaving it the same?

L19: All the answers go to zero.

INT: Oh, are you saying this one's also going to zero [points to $3x-5-3x+5$]?

L19: Yeah

INT: OK, so your answer would actually be zero. OK. I wanted to ask you about this question then. So, for Question 6, you've underlined all of them and I'm curious what your process is for solving this type of problem. So why did you underline them, do you think?

L19: Um…

INT: If you can't remember that's also fine.

L19: Hmm… how did I get that answer…

INT: Or think about how you would do it now if you're given that question

L19: [writes down Q6] Oh… that's how I got it [writes answer, final line $-10x$].

INT: OK, I can see here you've just brought these two down. So when you're doing a subtraction where all of them are like terms. Do you do it in a specific order?

L19: Yes.

INT: What order do you use?

L19: So, if it's like… I don't know how to explain it… if it's all… if this two was positive [points to first two terms] then I would add that two and that two was negative [points to last two terms] and that two was negative then I would add that, and then I would bring it down and then I would simplify.

INT: So, kind of doing it in pairs of ones that are similar. Let's do another example. Let's say minus seven x squared minus three x squared minus x squared minus five x squared [writes $-7x^2 - 3x^2 - x^2 - 5x^2$]. Can you simplify that for me?

L19: [writes in book $-16x^2$]

INT: OK so you just did that in your head. How did you get to the sixteen from all four numbers? Did you pair them up or…?

L19: S0 I added that two together [points to $-3x^2$ and $-5x^2$] which gave me eight and then I added this one [points to $-7x^2$] which gave me ten and then I added that which give me fifteen and I added that one which gave me sixteen [refers to $-16x^2$].

INT: OK, so why are you adding the numbers together?

L19: Because it's um like terms.

INT: OK but all of them are negative. But you have kept a negative here. So, did you add these numbers together [points to number symbols] to get sixteen?

L19: Yes.

INT: And then how did you know to put the minus in front?

L19: Because all of these is negative.

INT: So, when you're when you're working with subtraction problems like this, do you just work with the numbers and then you figure out which symbol put in front? Or do you visualise like a number line?

L19: No, I don't visualise a number line

INT: OK just working with the numbers. So, let's just do another example. Let's say we've got, eight a minus eleven a… add in a bracket [writes $8a - 11a - (-3a)$] Yeah say you got something like this.

L19: [writes in book, mumbles to themself]

INT: What did you to get the positive eleven a?

L19: So, I added those two together [points to $8a$ and $-(-3a)$].

INT: OK, because they're both positive? Is that why?

L19: Yes.

INT: Okay. Thank you. Now let's do… let's do Question 10. So, this one is eight minus two x to the power of four minus thirteen x to the power of four minus eleven minus three minus minus three x to the power of four [$8 - 2x^4 - 13x^4 - 11 - 3 - (-3x^4)$].

L19: Can I write?

INT: Yes.

L19: [mumbles to themself, writes in book 0]

INT: OK, so it looks like in this one [holding up L19's task script for Q10] you just forgot the eight because you got a similar answer, but you've got a minus six there. OK, you changed this one to a plus [points to $+3x^4$], just explain what you've done there.

L19: So, I took that into the bracket and a negative and a negative makes a positive.

INT: OK, and then the minus fifteen you've done with these two…

L19: I added those [points to $-2x^4$ and $-13x^4$] and then I added those [points to -11 and -3] and then I took this down [points to 8] and then I did like terms, I added that and that [points to $-15x^4$ and $+3x^4$] to get twelve and then this and that [points to 8 and -14] to get that [points to -6].

INT: OK, let's do another one. I just wanted to also ask while we're on a question which has exponents. [Writes x^3] How would you describe this? What does the exponent mean?

L19: Cubed, it's cubed.

INT: OK, can you write what it is?

L19: [writes $x \times x \times x$]

INT: Let's do Question 12. So, we have four xy minus five xy minus five minus minus eleven xy minus xy [$4xy - 5xy - 5 - (-11xy) - xy$]. Do you want to just talk through it?

L19: So, what I would do first is… does it have to go to this answer?

INT: No, no, you can change the answer. If the answer is different once you've explained it, that's fine.

L19: So, I took this times that and then made it a positive [points to $-(-11xy)$]. Then I took everything down again. And then I added like terms.

INT: Is that what you've underlined, the like terms?

L19: Yes. And then I took it down and I simplified it.

INT: OK. And this one you leave like this; you're not going to simplify this any further [points to answer $9xy - 5$]?

L19: No.

INT: OK. Then if I give you… so previously we were saying that if you have minus 3 minus x minus 1, then you say it's minus four x. But in this one, if you have nine xy minus 5, then you leave it like this. What's the difference between these two that you wouldn't put the numbers together?

L19: Hmmm…

INT: Or would you change one of them?

L19: I wouldn't change this one [points to $9xy - 5$]. But I'm not sure about this one.

INT: OK, let's maybe look at another example. Let's say you've got eight x minus five x minus 3. If you want to try that.

L19: [writes $3x - 3$]

INT: OK, would you leave it like that then?

L19: Yes.

INT: This one [Q2] is similar to what we did earlier. So, you wrote x as your answer for this one. Do you want to just talk through it? And if you change your answer, or if not. Let's just see what you would do now.

L19: So, I'll take these two [points to $11x$ and $8x$] and simplify it, it would be… [counts on fingers] Oh, it's 3. So positive three x and then minus 3.

INT: And then would you just leave it like that? Like you've done with this one?

L19: Yes.

INT: Just one last thing. You were saying earlier that when you have something like this [writes $-(-x)$] then you imagine there's a one there [points between first negative symbol and bracket] and then you multiply.

L19: Yes.

INT: So, do you think in this case the brackets are telling you that you must multiply the two symbols together?

L19: No, the brackets are, like, telling me to this number and multiply it inside the brackets.

INT: Oh, I see. So, this which is now a minus one [writes 1 between first negative symbol and bracket], you're taking this whole number and you're multiplying it with whatever's inside.

L19: Yes.

INT: OK, I think that is fine. Thank you so much.

Appendix 9.12 L22 First Interview

INT: So, there are a few questions I just want to check with you first and then depending on time we can go through some more. So, Question 1: minus x minus fifteen minus eleven. I'm just curious what your tactic is for getting to the minus twenty-six.

L22: I added those two and the signs are negative.

INT: OK, why is the sign negative?

L22: Because there's two negatives.

INT: OK, so if it's both negative you add them but then you put a minus in front. Thanks. OK. So, for this one [Q5], it's three x minus five minus, three x minus five. OK And then you wrote minus three x minus five squared [$-(3x-5)^2$].

L22: Ya, because … no, this is wrong.

INT: Would you do it differently?

L22: Mmh.

INT: OK, you can do it. Let me see. I'm curious to see how you would answer it. So let me see what you would do. [Rewrites question in workbook]

L22: [writing $9x^2+25$] Like this.

INT: OK, you do it like this. OK, what if I gave you something like this just with numbers? [writing $3-(-3)$]

L22: [writing, hesitatingly, -9]

INT: OK, so you'd still multiply.

L22: No, no, no.

INT: OK.

L22: It's zero.

INT: Why is it zero?

L22: No it's positive… ya it's zero because it's three minus three.

INT: But there are two minuses …

L22: Which makes it positive six.

INT: OK, so you change it to six.

L22: Yeah.

INT: OK. So, three minus minus three becomes three plus three is six. What if I give you minus three minus minus three [$-3-(-3)$]?

L22: It becomes positive. It's zero.

INT: Then it's zero. OK, and then what if I write it like this: minus three minus minus three [$(-3)-(-3)$]? Did those brackets make a difference to these two or is it the same?

L22: It's gonna make it the same. This isn't supposed to be times. There's a minus … it's two terms.

INT: There's a minus in between. OK, so do this one and then you can do that one.

L22: There's supposed to be a positive that doesn't make a positive.

INT: I don't know, what do you think? Does it make a difference to put the brackets around this minus three? Does it change anything?

L22: No.

INT: OK. So, could you remove the brackets then, if it doesn't make a difference?

L22: Yes.

INT: Ya. OK.

L22: The answer is still the same.

INT: So, now we're doing minus three minus minus three. We're subtracting the same thing from itself.

L22: Yeah.

INT: OK, so how would you do this one?

L22: This one would be… also zero… because it's the same [gestures between the two sets of brackets].

INT: OK. Yes, it's the same thing that you just subtracting from now. OK, so you would change this one now. So when there's a minus in between the brackets, it's not going to be multiplication anymore. You're saying now it's just subtraction. It's just that we're putting things together. OK. Um, next one … OK, eight minus two x to the power of four minus thirteen x power four minus eleven minus three minus minus three x to the four [Q10]. I'm curious about this. If you remember, I'm not sure,

L22: I don't know how I got to this.

INT: OK no, that's fine, that's fine. So, here you said eighteen x to the four minus six. What would you do now if you do see this question?

L22: I would get all of the like terms, like this one, this one and this one. And I would add all of those together. And I would take others which is that one, that one, that one, I will get that answer.

INT: OK. Do you mind just writing it out on the on the page. You can just write your answer. That's right.

L22: [writes same answer as on slip]

INT: No, I mean your working out, like what's the next step that you would do or would you just write your answer like that.

L22: I would show calculations. So, that would be negative two x four minus thirteen x four, then minus minus three x four. Then I'll answer this one.

INT: Mmh.

L22: Then I'd still put them [the constants]…

INT: You'd put them afterwards. So, are you just collecting the like terms now?

L22: I usually just do it in my head … I don't usually do calculations.

INT: OK.

L22: And I get 18 x four.

INT: I just want to see where you're getting the eighteen from.

L22: Negative two minus thirteen is fifteen. Oh, it's a plus then [points to $-(-3x^4)$].

INT: OK.

L22: Starting from here … it's going to be twelve …

INT: Positive or negative twelve?

L22: Negative twelve

INT: Negative twelve x to the power of four.

L22: Yeah.

INT: OK. And then you've got the eight and the minus eleven and the minus three.

L22: Umm, I'd minus eleven … eight minus eleven equals to, you mind if I work it out …

INT: Ya, of course.

L22: [writing] It's negative three.

INT: OK.

L22: [writing on the side of the page] it's negative three, just negative three minus three is negative six.

INT: Oh, is that what you did [pointing to same working out on side of script]?

L22: Yeah.

INT: So, what did you do to get to the minus three from the eight and eleven?

L22: I minus eleven from eight, I go eight, seven, six, five, four…

INT: Oh, you go down. That makes sense. Yeah. OK, so you're going down from eight to under zero and then you're subjecting one. OK. So, what would your final answer be?

L22: My new answer is now going to be twelve x four minus six.

INT: So, another one [Q18]: seventeen a minus minus four a minus seven b minus six a. You probably just worked out in your head. How would you have gotten to the minus thirteen?

L22: OK, now that I realised that this is supposed to be a positive.

INT: OK.

L22: My answer would be… seventeen a plus four a, twenty-one a… oh no I forgot the six a. Then twenty-one minus six is… fifteen a, minus seven b.

INT: Cool. Alright, let's go through this one [Q2] quickly four a minus minus three a minus one.

L22: This one is, I see now, is also supposed to be possible now. So, it's four a plus three a minus one. And the answer would be three plus four, obviously seven.

INT: Yeah. OK. I just want to look at the order. So, these ones are all like terms. We've got two x minus eight x minus three x minus x. Do you do the subtraction in a particular order? Do you think it makes a difference?

L22: I'll do the couple of negatives first.

INT: OK. Interesting. OK. So, you would do the minus eight, minus three and the minus one.

L22: Yeah.

INT: OK, and then you would subtract that final answer from the two?

L22: Yeah.

INT: OK, cool. [going to next one] So, in the previous one you were multiplying when there was a bracket, but now in this one you do seem to have changed it. So, we've gone minus x cubed minus minus three x cubed [Q9]. You've written minus four x cubed, and then minus fifteen. Where is this minus coming from [points to minus of $-4x^3$]? Because these are both negative?

L22: Umm … negative four x ….this now is supposed to be a positive [draws plus over open bracket of $-(-3x^3)$]. There's an invisible one here [points in front of open bracket].

INT: OK.

L22: So, I would add it to that. And it would be negative four.

INT: Would be negative four. But now you've said that this is a plus.

L22: Yeah.

INT: So, it's minus x cubed plus three x cubed.

L22: Yeah, obviously that's gonna be different.

INT: OK, so you can just write it out.

L22: Minus x cubed plus three x cubed minus fifteen. The answer is going to be two x cubed minus fifteen.

INT: OK. [next one] four xy minus five xy minus five minus minus eleven xy minus xy.

L22: Again, I just group the, umm, the like terms. Yeah.

INT: So, you would do the four minus five and then?

L22: Four minus five and then … now it's different now because I can't really … [writing] So, it's four xy plus eleven xy minus xy. SO this would now be four xy and… wait both of them are negative so it's six xy, negative six xy and the positive that will be fifteen xy. It's going to be positive.

INT: OK. Why do you say so?

L22: Because this is more than that.

INT: OK, you're subtracting a smaller number from a bigger number.

L22: Yeah. OK, so this is gonna be positive nine xy.

INT: Cool.

L22: Minus five.

INT: Oh ya, minus five. Okay, now this one is a little bit different: thirteen ab minus four ba minus thirteen minus two ba minus minus nine minus one.

L22: Again just the like terms [writes $13ab - 4ab$]

INT: OK. So, you're saying now that the ba is the same as ab.

L22: Oh no that’s wrong.

INT: Would you say they’re different?

L22: Yes, because they're two different numbers.

INT: Why do you say so?

L22: Because a isn’t equal to b and b isn’t equal to a [pointing to a and b].

INT: OK, so you're saying that if you have a a next to b it’s not the same as b next to a?

L22: No.

INT: OK.

L22: But, I have to do the whole thing.

INT: OK, that's fine. You can do that.

L22: Now we do… so there’s only one ab?

INT: It's thirteen ab minus four ba.

L22: That was the four ba minus two ba, negative two ba [writes $4ba - 2ba$]. I’ll do that first. Then it will give me thirteen ab, negative six ba. Now I do the thirteen, negative thirteen. That should be a positive… plus nine minus one. Now I write thirteen ab minus six ba, now I calculate this. So that would be eight minus thirteen,… wait… thirteen, twelve, eleven… that would be five. It would be negative five.

INT: OK. So, what did you do there? Did you go down from eight by thirteen, or did you …

L22: I went right down to …. What did I do? [laughter]

INT: Would you say you did eight down to zero and then another five? Or did you do thirteen minus eight and put a minus in front?

L22: I said eight minus thirteen.

INT: But how did you get to the five?

L22: Oh. Eight, seven, six …

INT: OK, cool. Let's see this one [Q16] because I'm curious to see how you would do it now. OK. So we've got fifteen minus five x minus minus x minus eight x squared minus minus eleven minus four x squared.

L22: I just wrote "I don't know" because I was running out of time… So again, here I'll do the like terms.

INT: OK.

L22: So yeah, I can see these are like terms. This on its own. Those two are like terms and those two are like terms.

INT: OK.

L22: So, we do the x squared first. Negative eight x squared minus twelve x squared will be… twelve, thirteen, fourteen, fifteen,… will be negative twenty x squared [writing, answer $-20x^2$]. And then I would do negative five x plus x will give me… negative four x [writing $-4x$ next to $-20x^2$]. Then the like terms will be negative fifteen plus eleven will be positive twenty-six [final answer $-20x^2 - 4x + 26$].

INT: Cool. He does know. OK, so in this one [Q17], there are only two different types of terms, so four x cubed minus nine x squared minus eight x cubed.

L22: OK. So, for this one, I said eight minus four.

INT: So, why are you doing eight minus four not four minus eight? Here you've written a minus in front of the four here [pointing to side of script], but you didn't write a minus in front of the four here [pointing to final answer].

L22: I probably just forgot to add it.

INT: OK, so you'd say it's minus four x cubed and then just the minus one. Okay, try and do this one [Q19] now that you're on a roll.

L22: So this is like terms… [mumbles to self, inaudible, underlining like terms]. Thirteen a squared minus fourteen a squared minus five a squared… oh it's a negative thirteen [writing] [mumbles to self, inaudible]. It is negative twenty-seven… plus five, that's thirty-two…

INT: OK, so now you've added them together, but you know that they're gonna be negative.

L22: Yes. … negative thirty-two a square. Now I'll work out this one $[-(-6a^4)]$ it's positive, I mean it will plus these two, it will come out to a positive… let's just say negative four a squared plus six a… [writing $-32a + a^4$]

INT: OK. Are you saying it's gonna be a positive? But you've said minus. You said plus minus six a.

L22: Negative five.

INT: Then you've written plus minus. But here you just replaced a plus.

L22: Oh, I didn't… wait… ya it is supposed to be plus. It's supposed to be a positive a four.

INT: Oh you would just do a plus. OK, well, I think that's good enough for me. Thank you so much.

Appendix 9.13 L24 First Interview

INT: Let's see. There's a pen, you can write something down. OK, let's just start with question one. I just wanted you to explain the table a little bit to me just so that I could see what you're thinking is behind that. So, we had minus x minus fifteen minus eleven, and then you say it is minus x minus twenty-six. So, what did you, what is the table?

L24: If you multiply positive by negative, it's going to be negative. So yeah, that's what.

INT: So is that how you got a minus twenty-six here because you had a minus and a minus. I'm assuming that's where the minus is coming from.

L24: Yes.

INT: What about Question 5? I don't know if this is your final answer or if you're still busy working out, but let's just see. I'm curious to see what your thinking process is for this one. So, we've got brackets three x minus five brackets minus brackets three x minus five.

L24: I took the three x and I multiplied it by the three x. And then I multiplied again by five. I think it's the foil method. And then that's how I got to go on to this. And then I took the five or so and then I did the same thing.

INT: OK. So, you did five and five is twenty-five, and then you've done the five and the three is fifteen and five. Okay, if I ask you to do something like three x minus minus three x, how would you simplify this one?

L24: Anything outside the bracket where you supposed to work inside, is multiplication?

INT: Maybe you can write out what you would do then I can see.

L24: Wouldn’t it be zero?

INT: Would it be zero if …

L24: It’s supposed to end in a positive. I think it’s [writing]

INT: So, what did you do to get the six? Did you add the threes together? OK. And then the x squared, is that from adding the x’s or multiplying x’s?

L24: I think it's because I added it.

INT: You're adding. Okay, what if you have something like this just with numbers? How would you do that one? … OK, so how did you get to the four?

L24: Oh yeah, I'm multiplying. So, this actually … that’s nine…

INT: OK, so are you multiplying because there are brackets over here? Would you say that this is the same if I do it like this? Or is this different?

L24: It's the same.

INT: So, these brackets don't actually make a difference, you're saying? But you’re saying you'd still multiply?

L24: Yeah.

INT: And then what if we have it like this? So now the brackets are right next to each other.

L24: It means multiplication.

INT: But would you do something different than if there's something between the brackets? Or does this minus apply to one of the brackets specifically and you still multiply?

L24: Uhm … I just used the signs to say what the final answer is going to be.

INT: Okay, I see. So this sign, you're saying it's not minus. It's rather telling you that it's going to be negative or positive.

L24: Yeah.

INT: Is that what you mean? So, because you've got a minus and a minus, you're saying it's positive. But this minus isn't the same as subtraction. It's just to show negative or positive. Is that what you mean?

L24: Yes.

INT: And another one. I actually want to do one more thing … What if it was like this? So, we've got the same again, the same thing in the brackets, but now we're doing a negative and a negative. So, you're still saying this is multiplication?

L24: Yeah.

INT: How would you write your answer for that one?

L24: Hmmm ...

INT: What if I just did it like this?

L24: Isn’t that zero?

INT: Yes, this is zero, but now what if I do it like this?

L24: Then it’s still zero.

INT: Okay, so this is zero.

L24: Yeah.

INT: So, then we've got five minus five is zero.

L24: But what do the brackets mean though? It doesn't mean …

INT: Well, that's what I'm curious to find out from you. Do you think that these two brackets make a difference to this?

L24: Jaaa …

INT: If it's the first one, would you say that the brackets automatically imply that multiplication has to happen? Or can the brackets mean different things depending on the expression?

L24: I think it means different things depending on the expression.

INT: Okay, because what if I wrote it like this? [writing] So, only the first one has brackets. Do the brackets mean anything then? Or could I take them away?

L24: I think they mean something like … yeah …

INT: So then would you still say that this is equal to zero or would you do something differently?

L24: I would think it's multiplication.

INT: So, you would take this, you would multiply those two together? So, what would that give you?

L24: Twenty-five.

INT: And what about the negative? Would you do something with the negative?

L24: I would make it negative twenty-five.

INT: Okay, so you're saying this is the same?

L24: Wait! Yes. No, I don't know. Isn’t it when there’s brackets, doesn't it automatically mean multiplication?

INT: Well, I would say this automatically means multiplication.

L24: Oh yeah. Oh yeah. Because then there's nothing in between. Yeah.

INT: So, what do you think? Do you think if there is something in between the brackets that changes it? Or do you think that's still just multiplication?

L24: I think it does change that.

INT: Would you say that it is subtraction?

L24: Yeah.

INT: So just now you said that it's just equal to zero. So, whatever's inside this bracket you're taking away from whatever's inside that bracket.

L24: Uh-uh.

INT: So then if we go back to this one where we're taking away one bracket from another bracket - how would you simplify this then?

L24: I would say zero now.

INT: You would say zero, okay, because it's the same thing in both brackets. So, would you change your answer then for question 5? Or would you leave it the same? So again, we've got two brackets with the same thing. Would you still multiply because they've got maybe two terms? Or would you do the same as you did here and say it's equal to zero?

L24: I did the same as I did, yeah, and say it's equal to zero.

INT: OK, another one [Q7]… minus five x squared minus twelve x squared and… [pointing to $-4x^2$ that has been changed to $+4x^2$]

L24: I changed that to a positive because it's like, a negative, negative, positive.

INT: Are you saying that because this negative [pointing to $-4x^2$] followed this negative [pointing to $-12x^2$], it becomes positive?

L24: Ya, because there's a negative there [pointing to $-5x^2$] and there [pointing to $-12x^2$] and then…

INT: Oh, I see. So, it's minus minus and then it can be a plus, the next one. But what if it was just like this? Just that part? [Writes $-12x^2 - 4x^2$] Would you leave it as this or would you do a plus?

L24: I would leave it like that.

INT: But as soon as there's a minus in front [writes $-5x^2$ in front of $12x^2$]…

L24: Ya.

INT: Then you say you can change this one to a plus.

L24: Ya.

INT: OK, you've changed it to a plus and now you've got a minus seven x to the power of four. So, can you just explain where this one's coming from?

L24: Oh, I took the negative five x squared and I minused it. …. all squared … then I got seven x and then I added the exponents.

INT: So, how did you get the seven from the minus five and the minus twelve?

L24: So, if you minus five and twelve isn't it automatically negative because it's a small number …?

INT: Okay, so you took the five away from the twelve, but because they're both negative, you said that it's a minus seven?

L24: Maybe it should be positive …

INT: Well, what do you think if you did like this: if I just gave you minus five minus twelve, then you would say minus seven?

L24: Yeah.

INT: And then the changing the exponent to a four?

L24: I added.

INT: So, if you're working with like terms that have an exponent, it doesn't matter if you're subtracting or adding them, you'll add their exponents?

L24: Yes.

INT: And then you've done minus seven x to the power of four plus four x squared. And then you got positive eleven x to the power of six.

L24: Oh, that should have been negative because you always take this sign of the bigger sign.

INT: So, you've taken the sign of the biggest number, which is a negative, and you … what would the number be? Would it be negative eleven x to the power of six? And you've done the six because it's the four and two.

L24: Yeah.

INT: Another one. Question 9 is minus x cubed minus minus three x cubed minus fifteen.

L24: So, because of the negative and the negative, I made it positive, the three x cubed.

INT: So, now you've said three.

L24: Yeah, because I cancelled these.

INT: So, because it's a negative x cubed by itself, you cancel the letters.

L24: Yeah, because like, x cubed x cubed.

INT: OK, so then you've just got the three leftover and then you've done minus fifteen. Next one: So, eight minus two x to the four minus thirteen x to the four minus eleven minus three minus minus three x to the four. And it looks like you've circled the numbers and underlined the ones with letters.

L24: I circled the numbers because I added them all together. And then with underlining, I think I did the same.

INT: So, you got minus twenty-two from the eight minus eleven and minus three. And then the eighteen, did you do two plus thirteen plus three. Okay, what made it become positive?

L24: Probably because it was minuses …

INT: Question 12: Four xy minus five xy minus five minus minus eleven xy minus xy.

L24: The same thing here, because it was a negative, negative positive.

INT: And here you've changed this one to a plus, so you've got four xy plus five xy. Why do you think you would have changed it to a plus?

L24: I don't know if it was a mistake. Yeah, because the front was positive, wasn't negative.

INT: Okay, so you would have maybe left this as a minus rather. Or you would now maybe leave it as a minus. And then so let's say it is minus five xy … so you'd have four xy minus five xy plus eleven xy minus xy.

L24: Uh-uh.

INT: If the first two terms are like terms, would you do them first and then do the other ones?

L24: Yeah.

INT: You said you would change this one to a minus, but let's just suppose that you're leaving it as it is. Then you've got nineteen xy and minus xy. Would you do something with the xy's?

L24: No, because … Oh … I did in the other one.

INT: Maybe because it was x and y, you would do something different. In the other one you took the letters away completely when you did a minus. So, would you say it's just nineteen minus five?

L24: Yeah.

INT: Is that what you would go for?

L24: But I think what would have been right is if they're like as an example, if it was nineteen x minus five minus, and then it was another nineteen xy, then because it's the same then I'd cancel it out and then just get left with the negative, then negative five.

INT: Okay, that makes sense. Now this one: You've got thirteen ab minus four ba minus two ba. You've underlined them, also. Would ab and ba are the same?

L24: No. I probably didn't

INT: Would you not say that this time? So, you would say that ab is different to ba ?

L24: Yeah.

INT: If you think that they're different, would you leave the thirteen ab? What would you do differently? Previously you said thirteen ab minus four ba plus two ba gives you … nineteen. Maybe you can retry this question. So, let's just try thirteen ab minus four ba minus thirteen minus two ba minus nine minus one.

L24: [writing]

INT: This time you've gone for the ba’s and now you've written an exponent of two. Is that because you're putting them together?

L24: Yeah.

INT: Thirteen maybe you've left as it is. And then you did minus four and minus two to get minus six. And then you've changed this one to a plus. I'm assuming that's the minus minus, and then you've just simplified here to get an eight, and then you're leaving this one because you're saying these ones aren't the same.

L24: Yeah.

INT: Let's just do these last two questions. We've got fifteen minus five x minus minus x minus eight x squared minus minus eleven minus twelve x squared. So, you've brought the fifteen down and the minus five x and then

L24: I forgot to change that positive.

INT: So this one you would change to a plus.

L24: Yeah.

INT: But in this one you changed to a plus. Is that because it's after minus?

L24: I think so.

INT: Maybe we can do this one again as well just to see if you would do it differently. So, fifteen minus five x minus minus x minus eight x squared minus minus eleven minus twelve x squared. Let's see how you do it now.

L24: [writing]

INT: You've done the minus five x plus x minus eight x squared plus eleven. What about the fifteen?

L24: Oh.

INT: And then you said fifteen minus five x plus x … so you've done the minus eight x squared minus and the minus twelve x squared and you said that's plus twenty x squared. So how come you change it to a positive?

L24: Because it's negative out, but it's not multiplying.

INT: Would you leave it as a negative?

L24: Yeah.

INT: So, let's say this would be the negative. And you've still got two terms which are numbers and two terms which are x's. Would you simplify further or would you leave it like this?

L24: I would simplify further.

INT: OK. What would you simplify it to?

L24: [writing]

INT: So, the six, is that from the minus five and then you've got another one, so that makes it minus six and you've put the x's together?

L24: Yeah.

INT: And then you've done fifteen and eleven, so where did you get the one from?

L24: I subtracted the fifteen and the eleven.

INT: And then minus six x squared minus twenty x squared and then you've done the six and the twenty together. So, you subtracted the smaller number from the big number and then it's minus.

L24: Uh-uh.

INT: Okay, let's just do the last question and then you can go: Seventeen a minus minus four a minus seven b minus six a. This one's got less, it's got seven letters but they're not together. So, you've written seventeen a, then you say plus four a. Is that from minus minus?

L24: Yeah.

INT: …minus seven b and minus six a, and then you did fifteen a. Did you do seventeen plus four?

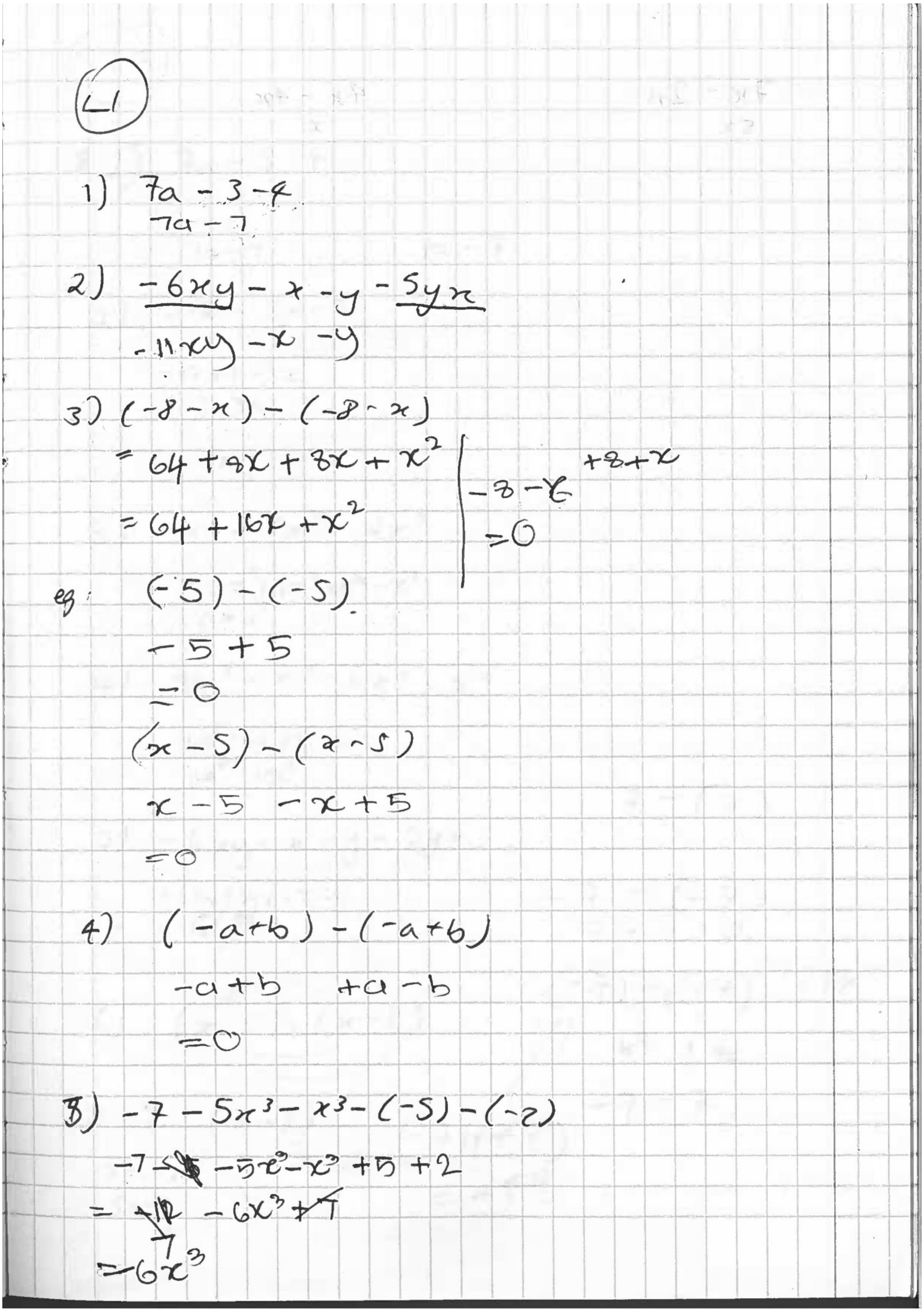
L24: Yeah.

INT: And then minus the six?

L24: Yeah.

INT: And then you left it like this because they're not like terms. I think that’s good. Thank you.

Appendix 10 Interview notes (follow-up)



L1

1) $7a - 3 - 4$

$7a - 7$

2) $\underline{-6xy} - x - y - \underline{5yx}$

$-11xy - x - y$

3) $(-8 - x) - (-8 - x)$

$= 64 + 8x + 8x + x^2$

$= 64 + 16x + x^2$

$-8 - x \quad +8 + x$

$= 0$

eg: $(-5) - (-5)$

$-5 + 5$

$= 0$

$(x - 5) - (x - 5)$

$x - 5 \quad - x + 5$

$= 0$

4) $(-a + b) - (-a + b)$

$-a + b \quad + a - b$

$= 0$

5) $-7 - 5x^3 - x^3 - (-5) - (-2)$

$-7 - \cancel{5} - 5x^3 - x^3 + 5 + 2$

$= \cancel{+12} - 6x^3 \cancel{+7}$

7

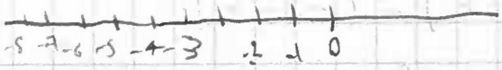
$= -6x^3$

6) $8xy - 8 - 5x - 5$

$8xy - 13 - 5x$

L4

←



2) $7a - 3 - 4$

$= 7a - 7$

eg $-3 - (-4)$

$-3 + 4$

$= 1$

$-8 - (-13)$

$-8 + 13$

$= 5$

4) $4x^2 - x^2 - 6x^2 - x^2$

$3x^2 - 6x^2 - x^2$

$-3x^2 - 1x^2$

$-4x^2$

$4x^2 - x^2$

$-7x^2 + \overset{3x^2}{\cancel{x}}$

$-4x^2$

10) $-7\cancel{x} - 5x^3 - x^3 - (-5) - (-2)$

$-7 - 5x^3 - x^3 + 5 + 2$

$-7 - 6x^3 + 7$

$-6x^3$

$(-8ab) - (-8ab)$

~~(-8ab) + 8ab~~

$-8ab + 8ab$

$= 0$

16) $(-8 - x) - (-8 - x)$

$-8 - x + 8 + x$

11) $-12ab - 8ab - 3 - (-2ba) - (-6)$

$-12ab - 8ab - 3 + 2ba + 6$

$20ab + 3 + 2ba$

14) $-(x - 1) - (-x + 1)$

$-x + 1 + x - 1$

0

7) $-6xy - x - y - 5yx$

$(3x^2 - 1) - (-3x^2) - x^4$

$3x^2 - 1 + 3x^2 - x^4$

$6x^2 - 1 - x^4$

12) $9a - (-3b) - (-10a) - b - 3$

$9a + 3b + 10a - b - 3$

$19a + 2b - 3$

$(-3x^2) - x^4$

$+ 3x^8$

$2x^2 - (2x^3)$

$2x^2 - 2x^3$

L6

2) $7a-3-4$

$=7a-7$

$-3-7=-10$

$-\underline{8}+\underline{3}=-5$

$8-3=5$

3) $-x^2-4x^3-2x^2$

$=-3x^2-4x^3$

4) $4x^2-x^2-\underbrace{6x^2-x^2}$

~~$=5x^2-7x^2$~~ $=7x^2-5x^2$

$=\cancel{12}x^2$ $=-2x^2$

10) $-7-5x^3-x^3-1(-5)-1(-2)$

$=-7-5x^3-x^3+5+2$

$=\cancel{8}-5x^3-x^3$

$=\cancel{8}-6x^3$

11) $-12ab-8ab-3-(-2ba)-1(-6)$

$=-12ab-8ab-3+2ba+6$

$=-20ab+3+2ba$

13) $(x-5)-1(x-5)$

$=x\underline{-5}-x\underline{+5}$

$=x$

$x-x=x$

$5-5=0$

$2x-2x=0x$

15) $(-a+b)-(-a+b)$

$=-a+b+a-b$

$=a\cancel{b}$

$-3x-1(-3x)$

$-3x+3x$

$=x$

16) $(-8-x)-(-8-x)$

$=-8-x+8+x$

$=x$

eg: $\overbrace{7x-3x}-\overbrace{5x-x}.$

$=4x-6x$

$=-2x$

eg. $\overbrace{8x-3x}-3x-2x.$

$=5x-5x$

$=x$

7) $-6xy-x-y-5yx.$

$7x-5x-2x$ $\qquad$ $7-5-2$

$\boxed{}x$ $\qquad$ 0

L10

2) $7a - 3 - 4$

$= 7a - 7$

$-1 - 1 = -2$

4) $4x^2 - x^2 - 6x^2 - x^2$

$-x^2 - x^2 = 2x^2$

$= -2x^2 - 2x^2$

$-1x - 1x = -2x^2$

$= -4x^2$

7) $-6xy - x - y - 5yx$

$-6xy - x - y - 5xy$

$= -11xy - x - y$

11) $-12ab - 8ab - 3 - (-2ba) - (-6)$

$-20ab - 3 + 2ba + 6$

$-20ab + 3 + 2ba$

$-7 - 3$

$7 + 3 \quad -10$

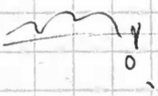
-10

16) $(-8 - x) - (-8 - x)$

$= -8 - x + 8 + x$

$= -x + x$

$= 0$



$-7 + 3$

$3 - 7 = -4$

15) $(-a+b)-(-a+b)$

$\cancel{-a}\cancel{+b}+\cancel{B}a\cancel{-b}$

$=0$

6) $8xy-8-5x-5$

$8xy\ -13\ -5x$

10) $-7-5\cancel{x^3}-x^3-(-5)-(-2)$

$-7\ \cancel{+5}\ \cancel{+5}\ +2$

$=-5$

$11x^3-2x^3\ =9x^3$

eg: $11x^3-1x^3\ =10x^3$

eg: $(-3a)-(-2a)$

$-3a+2a$

$=-a$

$=\cancel{-7}\ -6x^3+\overset{\nearrow}{5}+2$

$=-6x^3$

$\underline{-7}a+\underline{3}a$

$-7\ -6x^3+5+2.$

eg $5x-x-8x^2-x^2$

$4x\ -9x^2$

L11

2) $7a \underline{-3} \underline{-4}$

$= -7+7a.$

1) $\underline{-12} - 3x \underline{-9}$

$= -3x-21$

5) $-4x^4 - (-6x^4) - 6$

$= \underline{-4x^4} \underline{+6x^4} - 6$

$= 2x^4 - 6.$

$\underbrace{-7} + 3$

$-5 - (-7)$

$-5 + 7$

2

7) $\underset{\sim}{-6xy} - x - y - \underset{\sim}{5yx}.$

$= -11xy - x - y.$

10) $-7 - 5x^3 - x^3 - \overset{\frown}{(-5)} - \overset{\frown}{(-2)}$

$= \underline{-7} - \underset{\sim}{5x^3} \underset{\sim}{-x^3} \underline{+5+2}$

$= -6x^3$

eg: $(-4x) - \overset{\frown}{(-4x)}$

$= (-4x) + 4x$

$= -16x^2.$

$-4x - (-4x)$

$= -4x+4x = 0$

~~$x+2\ 2(x-2)$~~

$-(-2) - \overset{\frown}{(-4)}$

$= (-2) + 4$

$+2 + 4 \quad -6$

$+(-3) - (-5)$

$= -3+5.$

$= 2.$

13) $(x-5)\overset{\frown}{-(x}-5)$

$= \underline{x-5}\ \underline{-x+5}$

$= 0$

14) $-(x\mp1) - (-x+1)$

$= -x+1 +x-1$

$=0.$

15) $\overset{\frown}{+(-}a+b) \overset{\frown}{-(-a+}b)$

$= -a+b+a-b.$

$=0$

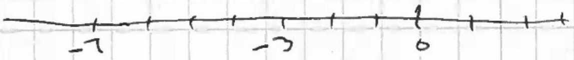
4) $\underline{4x^2}\ \underline{-1x^2 - 6x^2} - 1x^2$

$= 4x^2 - 8x^2$

$= -4x^2.$

413

2) $7a - 3 - 4$
$= 7a - 7$



$8 - 5$

$-8 + 5$

4) $4x^2 - x^2 - 6x^2 - x^2$ $\qquad$ $-2x^2 - x^2 - x^2$
$= 4x^2 - 6x^2 - x^2 - x^2$
$= -2x^2$

eg: $-12x^2 - 3x^2 - 3x^2$
$= -15x^2 - 3x^2$
$= -18x^2$

7) $-6xy - x - y - 5yx$
$= -6xy - 5xy - x - y$
$= -11xy - x - y$

10) $-7 - 5x^3 - x^3 - (-5) - (-2)$
$= -7 - 5x^3 - x^3 + 5 + 2$
$= -7 + 5 + 2 - 5x^3 - x^3$
$= -5x^3 - x^3$
$= -6x^3$

eg: $6x - x - 2x^2 - 3x^2 - x$
$= -2x^2 - 3x^2 + 6x - x - x$
$= -5x^2 + 6x$

13) $\{x-5\}-(x-5)$

$x-5-(x-5)$
$=x-x-5-5$
$=0$

$(-5x)-(-5x)$
$=-10x$

$-5x-(-5x)$
$=-10x$

$-(x-1)+(x-1)$
$=-x+1+x-1$
$=-x+x+1-1$
$=\cancel{2x}0$

$-(8+x)-(-8-x)$
$=-8-x+8+x$
$=-8+8-x+x$
$=0$

Dg $(a+b)-(a+b)$
$=a+b-a+b$
$=a-\cancel{b}a+b+b$
$=2b$

$(-a+b)-(-a+b)$
$=-a+b+a-b$
$=-a+a+b-b$
$=0$

eg $2ab-ab-3ab-2ab-ab$
$=2ab-2ab-3ab-ab-ab$
$=-3ab\cancel{-ab}$

L14

1) $7a - 3 - 4$

$7a - 7$ $a - 7$

2) $-12 - 3x - 9$

$-12 + 9 - 3x$

$3 - 3x$

x

3) $-x^2 - 4x^3 - 2x^2$

$-4x^3 + 2x^2 + x^2 - x^2$

$6x^7$

4) $4x^2 - x^2 - 6x^2 - x^2$

$4x^2 + 6x^2 + x^2 - x^2$

$10x^8$

5) $-6xy - x - y - 5yx$.

$-6xy + 5yx - x - y$

$= -1x^2y$

6) $(x-5) - (x-5)$

$x - 5 - (x - 5)$

$x - 5 - 5x$

$5x - 5x$

x

$7 - (7)$.

$-7 - (-7)$

$-7 + 7 = 0$.

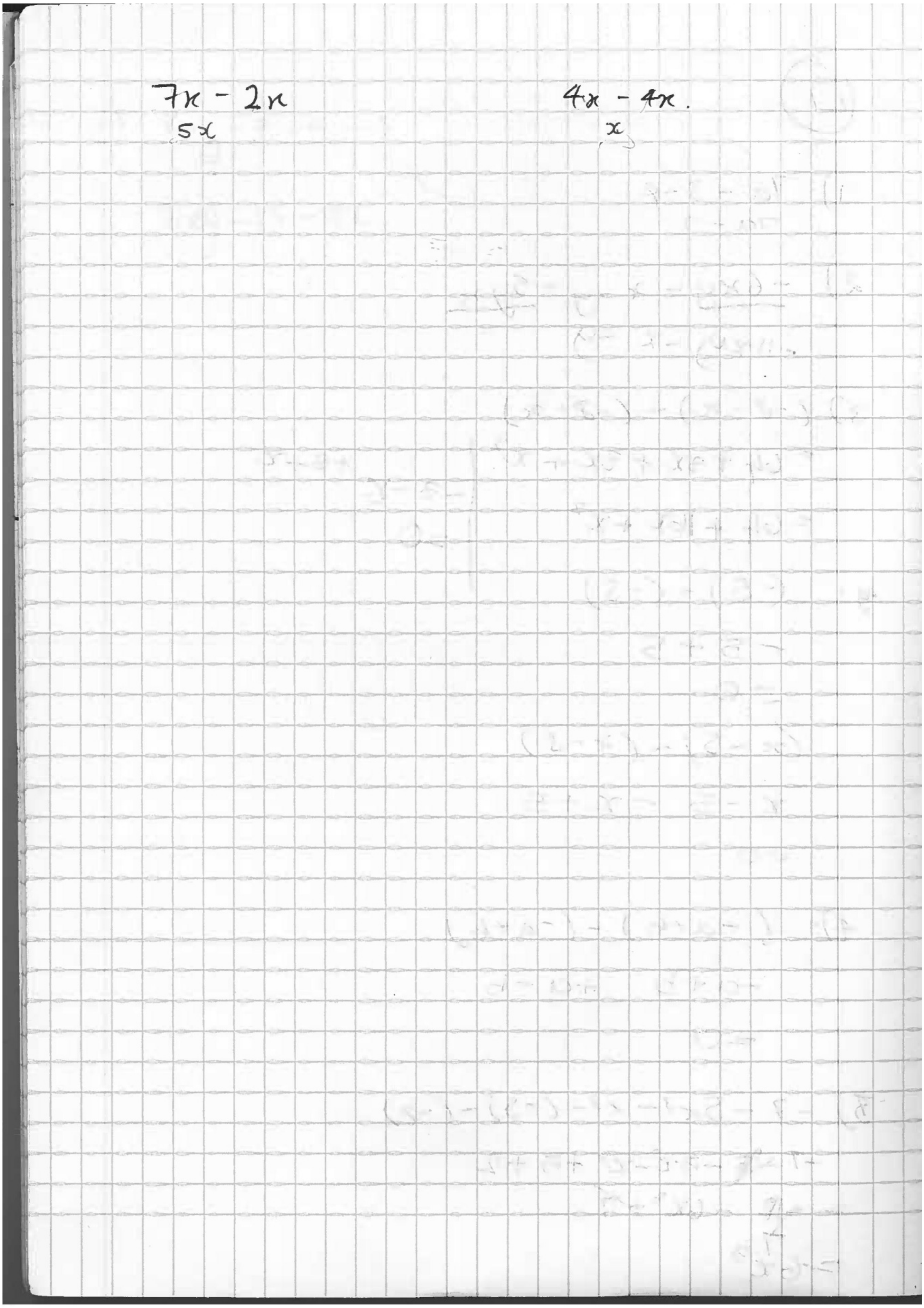
$(-7) - (-7)$. $= 14$

$49 + 14$

$-7 - 7$.

$(-7)(+7)$

$= -49$.



$7x - 2x$

$5x$

$4x - 4x.$

x

L22.

1) $7a - 3 - 4.$

$= 7a - 7$

2) $4x^2 - x^2 - 6x^2 - x^2$

$= 4x^2 - 8x^2$

$= -4x^2$

eg: $-9a^2 - (-7a^2) - a^2 - 2a^2$

$-12a^2 + 7a^2$

$= -5a^2$

3) $8xy - 8 - 5x - 5$

$= 8xy - 5x - 13$

4) $-6xy - x - y - 5yx.$

=

5) $-12ab - 8ab - 3 - (2ba) - (-6).$

+	−	−
−	+	−
−	−	+

$= -20ab - 2ba \overset{+3}{\cancel{\text{[illegible]}}}$

6) $(x - 5) - (x - 5).$

$= x - 5 - x + 5$

$= 0 + 0$

$= 0$

$x - 5 - (x - 5).$

$= x - 5 - x + 5$

$= \overset{0}{\cancel{2x}} + 0$

$= 0$

7) $(-8-x)-(-8-x)$

$= -8-x+8+x$

$= 0+0$

$= 0$

8) $-x^2-\underline{4x^3}-\underline{2x^2}-\underline{x^3}$

$= -5x^3-3x^2$

L24

2) $7a-3-4$

$7a-7$

$a=-7-7=0$

$a=0$

$7a-3-4$

$7a-7$

3) $-x^2-4x^3-2x^2$

-3

$-1x^2-4x^3$

eg: $-8a-13a-a^2-a$

$-21a-a^2-a$

$-21a+a+a^2$

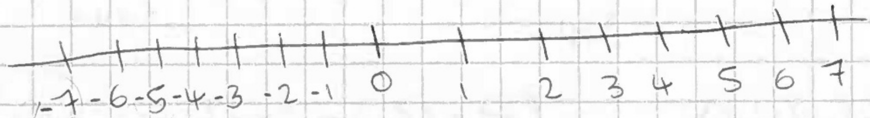
$-\overset{20}{\cancel{21}}a+a^2$

$-21+1$

-20

$-1-2=-3$

$-4--6=-10$



4) $4x^2-x^2-6x^2-x^2$

$=-2x^2$

$-(-=+$

$--=+$

$-3a-3a=0$

$-x^2-x^2-3a-(-3a)$

$9a^2$

$-x^2-x^2$

5) $-4x^4-(-6x^4)-6$

$24x^4-6$

$5a-3a-(-3a)$

$2a-(-3a)$

$6a^2$

$-4x^4--6x^4$

$-10x^4$

+	−	−
−	+	−
−	−	+

$8-7-(-5)$

$-1-(-5)$

5

4e 13) $(x - 5) - (x - 5)$

~~$x^2 - 5x - 5x$~~

$(x - 5) - (x - 5)$

$x^2 - 5x - \cancel{5x} - 5x$

~~x~~ x

$(x-5)(x-5)$

$(x - 5) - (x - 5)$

$x^2 - 5x - 5x + 25$

$x - 5 - (x - 5)$

$\underbrace{8 - 7}_{} - (-7)$

$\underbrace{-1 - (-7)}_{}$

7

Appendix 11 Transcripts (follow-up)

Appendix 11.1 L1 Follow-Up Interview

INT: I've written some questions and I'm just going to ask you to write your answers and then we can just talk about it. The first question is… [writes $7a - 3 - 4$] I just want you to simplify this.

L1: [Writes $7a - 7$]

INT: OK, just talk to me what you've done.

L1: I just like… did that [points to $-3 - 4$].

INT: You've left the seven a by itself.

L1: Yeah, I'm not sure if I should, like, minus it or… ya minus it or leave it.

INT: Do you feel like this is your final answer?

L1: Yeah.

INT: And then I just want to check with this one, last time, I don't remember if I asked you, when you're doing subtraction, do you just look at the numbers and the symbols and then figure out which symbol it's supposed to be and how the numbers work? Or do you imagine a number line?

L1: No, I just, like, look at the numbers.

INT: So, if you have two negatives, or if you're subtracting, like if you have a minus something minus something, then how would you go about finding which number it is?

L1: I'm not sure…

INT: So like how did you know now that it must be a seven?

L1: Oh, because they're both negatives and negative plus negative… negative negative together equals to negative.

INT: So I'm assuming what you did is you added the three and the four to get the seven and then you knew that it was a negative because you had two negative symbols. Is that what you're trying to say?

L1: Ya.

INT: OK, another one is: minus six xy minus x minus y minus yx [writes $-6xy - x - y - 5yx$].

L1: [Underlines $6xy$ and $5yx$] I underline the like terms then… [writes $-11xy - x - y$]. Ya…

INT: You're going to leave it like this, OK. You've underlined the xy and yx as like terms so…

L1: Oh no, it's actually not.

INT: Would you not make them the same?

L1: No, they're not the same.

INT: Why do you say so?

L1: Because that's a y there and an x [pointing to $-5yx$ then $-6xy$].

INT: So, are you saying that the matters which order your letters are in? That's going to make a difference.

L1: No, no, no, I don't think so. I think it is like terms.

INT: It is like terms, OK. But now you're saying these two aren't like terms because you've got an xy and then you've got an x and you've got y. So these two are like terms [pointing to $-6xy$ and $-5yx$], but you're saying these two are unlike terms. OK. And then again to get the minus eleven, what did you do with the six and the five? Or the minus six and the minus five.

L1: I just plussed them.

INT: OK, next one I wanted to do was this question: minus eight minus x, minus, minus eight minus x [$(-8 - x) - (-8 - x)$].

L1: Eight times eight is… sixty-four? Ya, sixty-four. And then… negative… eight x… [answer: $64 + 16x + x^2$].

INT: OK, so it seems like you've used the FOIL method here.

L1: Yeah.

INT: You've multiplied these numbers and then the last two and then outside inside. So, if you see brackets, or if you see an expression with two brackets close to each other, does that for you mean that you must multiply?

L1: Um… I don't actually know what to do with it…

INT: Because there's a minus in the middle.

L1: Yeah, I didn't… I didn't think about.

INT: Well, let's do a different example then. Let's say… [writes $(-5) - (-5)$ if I give you just this. How do you think you'd do it if it's just the numbers?

L1: I would do it differently. I'd take… so negative times negative is a positive. That'll be plus five, then that I just take it out the bracket then it will equal to zero.

INT: OK.

L1: I would do this one also differently [pointing to $(-8 - x) - (-8 - x)$]

INT: OK, you can do it here, next to this.

L1: So, I would do, negative and negative is positive so it's positive eight… actually I don't know…

INT: OK let's go from here then we'll go further down. We started off with minus five minus, minus five. Now let's say we've got x minus five, minus, x minus five [$(x - 5) - (x - 5)$].

L1: Um… [writes $x - 5 - x + 5$] Oh, this could cancel them all out. That's positive and it will cancel out the negative, that's negative and it will cancel out the positive, so equal to zero.

INT: OK, equal to zero. If you look inside the brackets, we have the same things inside the brackets, so I'm subtracting the same thing from itself. So then if you retry this one [$(-8 - x) = (-8 - x)$].

L1: It would be… positive eight and then that would be positive x and it would be… [writes $-8 - x + 8 + x$] and then they would cancel, ya, I would cancel them out.

INT: OK.

L1: And it would equal to zero.

INT: So, these brackets at the front, we can just take them away? They're not making a difference. But these ones, we first have to do something, because there is a symbol in front.

L1: Ya.

INT: Let's do another one. Let's try, for example, this [writes $(-a + b) - (-a + b)$].

L1: [Writes $-a + b + a - b$] Ya, those would all still cancel out.

INT: Still cancelling, OK. Let me do two more questions… What about something like this: minus seven, minus five x cubed, minus x cubed, minus, minus five, minus, minus two [$-7 - 5x^3 - x^3 - (-5) - (-2)$]. Go over to the next page if you want more space.

L1: [Writes $-7 - 5 - 5x^3 - x^3 + 5 + 2$ then $-12 - 6x^3 + 7$]

INT: So, this is the minus seven. Is this five [pointing to -5] that five?

L1: Oh, no. I thought that was… [scratches out -5].

INT: OK, now you've got minus seven minus six x cubed plus seven. So, are you leaving it like this?

L1: Ya.

INT: OK, you wouldn't simplify it further?

L1: Oh wait, yeah. I will cancel out the… [scratches out -7 and $+7$] and that will equal to… [writes $6x^3$].

INT: And now you're not… [pointing to negative symbol in front of $6x^3$ in previous line]

L1: Oh, I didn't see that [adds negative symbol in front of answer]

INT: OK minus six x cubed is your final answer. Let's do another one: eight xy minus eight, minus five x minus five [$8xy - 8 - 5x - 5$].

L1: [Writes $8xy - 13 - 5x$] Ya.

INT: OK you've taken the numbers, and you've subtracted…

L1: The like terms.

INT: The like terms, OK. And you've said xy and x are not like terms so you've left them. I think that's fine, thank you so much!

Appendix 11.2 L4 Follow-Up Interview

INT: I'll just give you some questions and then I just want you to write down your answer and then explain what it is. OK… Question 2 is the one I want to start with. I'll just write it in the book [writes $7x - 3 - 4$]

L4: Is this four Miss?

INT: Yes.

L4: [Writes $7x - 7$]

INT: OK. And then just give a little explanation of how you got to this seven.

L4: This and this [points to -3 and -4]. This two is like terms and this isn't, so we add the like terms they sum to be negative seven and then seven a it's not like terms so you can't add them.

INT: OK so you not going to add them. When you're doing the subtraction, are you looking just at the numbers and then the symbols, or do you imagine a number line when you're doing this?

L4: Yes, Miss.

INT: Which one?

L4: I imagine a number line.

INT: OK, so let's just draw it here. If you have a number line [draws number line], can you just try and draw or show me what you do on the number line.

L4: So with negative three here [points to -3 on number line], then because it's a negative [points to -4 in expression] … I feel like this is when you must add it. I don't say… OK, so negative three [points to -3 on number line] and then I add it like this [gestures right to left along number line].

INT: OK so you're going in this direction when you've got the negative and the negative…

L4: Ya, because I know already that if it's both negative then it must be a negative.

INT: What if then I give you… minus three minus minus four [writes $-3 - (-4)$]?

L4: Then when there is just a negative then it's going to be a negative one [gestures 1 between negative symbol and open bracket]. Then, OK, negative three stands on its own then negative one times negative four, a negative times a negative is a positive, so plus four [writing $-3 + 4$]. Then negative three plus four is going to give you one.

INT: And with this one do you also use the number line, or are you just using the numbers?

L4: I'm just like using… in my mind.

INT: OK, so for example how did you know to get to the one?

L4: Because if it if it was negative three plus three then it would be zero, but now you just add the one.

INT: OK, so you're splitting it up into different pairs. Let's look at some different numbers. Let's say we have minus eight minus minus thirteen [writes $-8 - (-13)$].

L4: Negative eight then negative one times negative three, plus thirteen [writing $-8 + 13$]. Negative eight plus thirteen… five.

INT: OK. And so just explain how you got to the five.

L4: Negative eight and then like positive eight and it will give you… so what I do is… because I know if the bigger number is positive the answer is going to be positive. Then I just, so there I got a positive [pointing to $+13$] so there already the imaginary positive is there so then I just work with negative eight and thirteen and thirteen minus eight is five.

INT: OK I see. So, if you're if your negative number is smaller and then you have a positive number you know the answer is positive and then you take the difference between the actual numbers themselves.

L4: Yes.

INT: OK, let's do this question [writes $4x^2 - x^2 - 6x^2 - x^2$]: four x squared minus x squared minus six x squared minus x squared.

L4: So, they're all like terms, but I will just go from left to right, so it's going to be three x squared, because you keep the terms because it's the same [writes working out, answer $-4x^2$].

INT: OK, you took the first two terms and you simplified.

L4: Yes and then I just put these [$-6x^2$ and $-x^2$] so that I don't get confused. And then three x squared minus six x squared, because this is bigger then it's going to be negative three x squared and then I brought this down, then negative… minus one x squared and then it's going to be negative four x squared because it's the same terms. And when you add the same terms, it will give you a negative term.

INT: OK, that makes sense. With this [pointing to $3x^2$], you said you were working from left to right. Does it matter if you did it in a different order, or do you think you should always work from left to right?

L4: I think I should always work from left to right.

INT: Is that because it makes it easier or because you think the answer will be different if you do a different direction? I mean not direction, a different order. For example, if you…

L4: I feel like it makes it easier.

INT: OK. If you did these two first [points to $-6x^2$ and $-x^2$] and brought those two [points to $4x^2$ and $-x^2$] down. Let's say we did, in the next line, four x squared minus x squared and you did these two first.

L4: [Writes $-7x^2 + 4x^2 - x^2$, answer $-4x^2$]

INT: So now you've simplified these two and then you took it to the front. Do you often, when you're working this kind of thing out, do you like to keep the things that you're simplifying you take to the front or you keep at the front?

L4: What do you mean Miss?

INT: So if we just looked at this and simplified it straight down, then the three x squared would be underneath here [pointing under $4x^2 - x^2$], right. And then the seven x squared would be this bit.

L4: Yes…

INT: But you just brought the minus seven x squared in front, or is that just because of the way that it was written?

L4: Because negative six x squared, even if I first had to put this here and then I had to put…

INT: I think I see what you mean, like you would have put three x squared and then minus seven x squared.

L4: Yes.

INT: OK. Another one [writes $-7-5x^3-x^3-(-5)-(-2)$]. So, it's minus seven minus five x cubed, minus three x cubed minus, minus five, minus, minus two.

L4: So first I must always work out the brackets. I'll write everything the same until I get to that one. And then this is negative one here and negative one here [pointing to negative symbols in front of brackets] so then negative times that it will be positive five and positive two and then I will just see what is all common. So, negative seven and then negative five minus one is going to give negative six x cubed, plus seven. Negative seven plus seven is going to be zero. OK, and then you're going to be left with negative six x cubed.

INT: So here you rewrote everything and then you just changed these two and you got rid of the brackets. If we just isolated that and we looked at something like… [writes $(-8ab)-(-8ab)$].

L4: Then… wait… because this stands on its own then the brackets can go away [pointing to first term].

INT: OK.

L4: And then this and then it becomes negative times a negative is positive, and then it's going to give you zero.

INT: So these brackets don't make a difference. It's just setting apart, OK.

L4: It's like to separate so you know.

INT: OK, well then let's do another question with brackets. Let's do Question 16 [writes $(-8-x)-(-8-x)$]. Now we've got two terms inside the brackets.

L4: [Writes $-8-x+8+x$] And then you take all the like terms and negative eight plus eight is going to give you zero, and then negative x plus x is also going to give you zero.

INT: OK, so it will just be zero. What about this similar question [writes $(-x-1)-(-x-1)$].

L4: [Writes $-x-1+x+1$] Then the negative and the positive cancel each other out, this two [draws line connecting $-x$ and x] cancel each other out and this two [draws line connecting -1 and $+1$] cancel each other out [writes 0 underneath].

INT: OK, so when you see the brackets, will you always… in this example the thing inside the brackets is exactly the same, right? You said this is equal to zero [pointing to previous solution] and this one [current solution]. But you'll still write it out first and then…

L4: Yes, just to double check.

INT: OK, let's do another question. I'll write it here. Question 11 [writing $-12ab-8ab-3-(-2ba)-(-6)$]. Minus twelve ab minus eight ab minus three minus, minus two ba minus, minus six.

L4: First do the brackets and I'll put this here so that I don't get confused, I'll add it there [writing terms not in brackets]. This is going to be negative one times negative two is positive two ba, plus six. And then… because it's still negative, it's going to stay negative and then it's going to be twenty ab then negative, because this is like terms, then negative three plus six is going to be positive three, plus two ba [answer $20ab+3+2ba$].

INT: And the ba and the ab you'll keep separately? Are they different? Are they unlike terms?

L4: Ya.

INT: OK, let's do another one like this. Minus six xy minus x minus y minus five yx [$-6xy-x-y-5yx$]. Would you simplify this? Can you simplify this? And if you can, how would you do it?

L4: No, I don't think I can

INT: You can't? So these are all unlike terms?

L4: No… yes.

INT: Let's do another question…. [writes $(-3x^2-1)-(-3x^2)-x^4$] what about this?

L4: I'll take it out of the brackets because it's one term, there's nothing in front or at the back. And then this [points to $(-3x^2)$] will be positive because negative times negative… three x squared. Then, three x squared… [writes answer $6x^2-1-x^4$].

INT: You've taken it all that the brackets and then you've got six x squared, is that from the three x squared?

L4: Ya [draws line connecting $3x^2$ and $+3x^2$].

INT: And then these you've just you've just brought them down because they're unlike terms. When you say like terms versus unlike terms, how do you recognise a like term, or terms that are alike?

L4: Was I supposed to put this one also times [gesturing arc between $-x^4$ and $(-3x^2)$]?

INT: What makes you think that you should do that?

L4: Because if… if this one wasn't there [pointing to $(3x^2 - 1)$] and that one was still there I would have timesed it that way [same arc gesture as before].

INT: What do you mean? If you had… [writes $(-3x^2) - x^4$]

L4: I would times it in there [gestures arc $-x^4$ to inside brackets] because it doesn't matter if it's on this side or that side I would still times it. And then… [writes $+3x^8$]

INT: OK, why are you… in this example we have minus eight ab minus, minus eight ab, is minus eight ab plus eight ab. And in this one we have minus three x squared minus x to the four. So why do you want to multiply with this one? What's different about this one that's not like this [$(-8ab) - (-8ab)$]?

L4: Hmm. I think I was supposed to leave it so [points to $6x^2 - 1 - x^4$ answer].

INT: You'd leave it like this. Is it the fact that there are brackets and there's something coming after the brackets that makes you want to multiply them?
L4: Yes.

INT: OK, then I'm just curious about this one here, you've changed it to x to the power of eight [pointing to $+3x^8$].

L4: I can't remember if you had to add it or times it.

INT: Well let's see, if I give you… [writes $2x^2 - (2x^3)$] If I give you this, would you multiply them or would you subtract?

L4: I’d write it like this… [writes $2x^2 - 2x^3$].

INT: And would you leave it like that, or would you simply it further?

L4: I'd leave it like that.

INT: OK, so these are unlike terms.

L4: Or I’d put it like this… I’d leave it like that.

INT: So, what makes these terms unlike?

L4: The exponents.

INT: The exponents are different. OK. Let me just double check my questions… Oh let’s just do this one question [writes $9a - (-3b) - (-10a) - b - 3$].

L4: [Writes working out, answer $19a + 2b - 3$].

INT: You’ve taken the brackets away again, I see?

L4: Yes.

INT: And you changed this to a positive [pointing to $(+3b)$…

L4: And I added all the like terms.

INT: And you’re leaving it like this because they’re unlike terms.

L4: Yes.

INT: OK, that’s good for me. Thank you so much.

Appendix 11.3 L6 Follow-Up Interview

INT: I'm going to write down some questions and then I just want you to show me your working out. I'm going to start with this one [writes $7a - 3 - 4$], and you can just do your working out and we'll see.

L6: [Writes $7a - 7$] That's all I can do.

INT: You're not going to simplify any further.

L6: No.

INT: And why?

L6: 'Cause it's, how I can say, it's not the same.

INT: Unlike terms.

L6: Unlike terms yes.

INT: OK, next one [writes $-x^2 - 4x^3 - 2x^2$]

L6: [Writes $-3x^2 - 4x^3$]

INT: OK just explain, you've drawn an arrow there so just explain what you've done.

L6: That's because the like terms, I minus it and the unlike term I just move it to the side.

INT: OK. And then you're just leaving it like this. And how did you know that this must be a negative?

L6: Because it's not like, if it was a times that would change but it's not, it's just minus so it will still be negative, it will be negative thee x.

INT: OK, if I give you something like, let's say… minus three minus seven [writes $-3 - 7$]

L6: Um, minus ten.

INT: Minus ten. So, when you do this, are you looking at just at the numbers and then at the symbols? Or do you imagine like a number line?

L6: No I look at the numbers and then I just add them and then I'll just put the…

INT: The symbol?

L6: The symbol, ya.

INT: So you’re adding the three and the seven to get ten, then how do you know to put a negative?

L6: Because they're both negative.

INT: OK. If it was minus eight plus three [writes $-8+3$]

L6: Then I would… how can I say… so the bigger number is a negative so it will automatically be a negative and the smaller numbers of positive. I will, like, subtract there and then it will be negative five.

INT: OK, negative five because the number [underlining the 8] itself is bigger, whereas if we did eight minus three…

L6: Then it will be… just five. Because it's a positive number, the bigger number.

INT: OK. Let’s do this one: four x squared minus x squared minus six x squared minus x squared [writes $4x^2 - x^2 - 6x^2 - x^2$].

L6: [Writes working out, answer $12x^2$]

INT: OK. You've grouped them.

L6: Yeah.

INT: And then how did you get to the five?

L6: I added this two [pointing to $4x^2 - x^2$]. I mean I… no, I’m wrong, I only see now [scratches out second line and writes $3x^2 - 5x^2$ then $= -2x^2$].

INT: You've done the four x squared minus x squared to get three x squared.

L6: Yes.

INT: And then how did you get to the minus five x squared?

L6: Minus x to six x will be five x.

INT: OK And then you bring the minus down.

L6: Ya.

INT: So you're working with this two [underlines $6x^2 - x^2$] and then this minus [in front of $6x^2$] is coming in front.

L6: Yeah.

INT: And then you did, here, you got minus two from the three and the five. Did you use the same method? Did you do that [pointing to $-8 + 3$]?

L6: Ya I did the same method.

INT: Well in this case, ya, I see what you mean. The bigger number is negative.

L6: Yeah.

INT: OK, minus two x squared. Then I want to do this one: minus seven minus five x cubed minus x cubed minus, minus five, minus, minus two [$-7 - 5x^3 - x^3 - (-5x^3) - (-2)$].

L6: [Writes working out, answer $-2 - 6x^3$]

INT: I see here you drew a one [points in front of (-5)].

L6: An invisible one, ya.

INT: What is that helping you to do?

L6: It will stay the same… but it will be a times so, like, times… the symbol will change. So negative times a negative is a positive and then the same [pointing to (-2)].

INT: The same thing.

L6: Ya. And then I added the like terms [pointing to constants] and that's also like terms [pointing to $-5x^3$ and x^3] and it got me that answer.

INT: Just explain how you came to the minus two. What did you do?

L6: I added those two, it was positive seven and negative seven, oh ya, it's zero [scratches out -2] and then my answer will be that [points to $-6x^3$].

INT: Again you've got two negatives, so I'm assuming you did the same thing as you did with those two negatives [points to $-3 - 7$], so you know it's negative and you add them. OK, then Question 11: minus twelve ab minus eight ab minus three, minus, minus two ba, minus, minus six [$-12ab - 8ab - 3 - (-2ba) - (-6)$]. That's another longer one.

L6: [Writes working out, answer $-20ab + 3 + 2a$]

INT: I see, again, you've done the same thing. You've changed to plusses [pointing to terms in brackets]

L6: Ya, I always do the brackets first.

INT: So, you've done the brackets first and then you've simplified, and then I saw you were pointing at the like terms. [L6 agrees] So you did those first? [L6 agrees] And ab and ba are different?

L6: ab and ba are different.

INT: OK, let's do this one [writes $(x - 5) - (x - 5)$]. So this one is x minus five, minus, x minus five, but the x minus five are both in brackets.

L6: Hmm...

INT: So, in this case [pointing to Question 10 terms in brackets] I saw you did a one and you said you were multiplying the symbols. So, if there had been a number in front, you would have multiplied the number in.

L6: Yeah.

INT: IN this case, there are no numbers…

L6: No numbers.

INT: We just have something in brackets and a minus and something else in brackets. So, what do these…

L6: It will be… there's a one there [gestures 1 in front of bracket], positive one, and it will be… x negative five, and the same here [writing $x - 5 - x + 5$, then $= x$].

INT: OK, so you just said zero, what were you referring to when you were saying zero?

L6: Those two [pointing to -5 and $+5$].

INT: OK, so you're saying that the minus five plus five gives you zero and then then you said x.

L6: Positive x and negative x.

INT: So why… if I just write it like this [writes $x - x$]

L6: Then, like, I'll just say x.

INT: Just x.

L6: Ya.

INT: OK, but if it's numbers [writes $5 - 5$], then you say it's zero?

L6: Zero, ya.

INT: What about if I give you something like this: two x minus two x?

L6: Then I'll write, uh… zero.

INT: Then you'll write zero. So what's the difference with these [pointing to $2x - 2x$] and with this one [pointing to $x - x$]? Why do you put x here?

L6: No, I'll write x, because I minused those two [pointing to the 2s].

INT: Oh, you minus the numbers, so it's like a zero and then x, but the zero you don't write.

L6: Ya, it's just the x.

INT: OK, what if I give you minus three x minus, minus three x [$-3x - (-3x)$], can you simplify that?

L6: [Writes $-3x + 3x$ then $= x$]

INT: OK, so the minus three and the three are cancelling out to get zero and then the x is…

L6: Ya, I just write it.

INT: You just leave the x. OK, let's do another one of these. Let's do this one [writes $(-a + b) - (-a + b)$]. This one doesn't have any numbers, so it's minus a plus b minus minus a plus b.

L6: [Writes $-a + b + a - b$ then ab]

INT: OK, just explain. You've done a similar thing here as there [pointing to $x - x$]…

L6: So a, a and the bs [pointing to letters then to solution]

INT: But now you're putting them next to each other.

L6: ab.

INT: Ya. Why is there not a sign in between them anymore?

L6: Let me just see… so it's a minus b [draws minus between letters].

INT: OK, so it was minus a plus a is a and b minus b is minus b and you're leaving it as a minus b. Let's do another one, let's do this one [writes $(-8 - x) - (-8 - x)$]. So now it's a similar thing but this time we're doing a number and a letter.

L6: [Writes $-8 - x + 8 + x$ then $= x$]

INT: OK, so the numbers are cancelled again, but the letter you're keeping there. I'm going to do another one [writes $7x - 3x - 5x - x$]. Will you just simplify that one.

L6: [Writes $4x - 6x$ then $= -2x$]

INT: So again, you've grouped them.

L6: Yeah.

INT: And then the minus six x, where does the minus six come from?

L6: Five x minus x equals minus six x.

INT: Let's do another one [writes $8x - 3x - 3x - 2x$].

L6: [Writes $5x - 5x$ then $= x$]

INT: OK, I'll do one more question and then I think we can probably finish there… [writes $-6xy - x - y - 5yx$] Can you simplify this? And if you can't…

L6: No.

INT: Can you explain why?

L6: Because they're not like terms, none of them.

INT: So the xy and the yx are different, and then you have just an x and just a y. OK. Then I just want to ask you: if you see something like seven x minus five x minus two x and seven minus five minus two, will you think about these differently?

L6: Yes.

INT: Ya? Because this one has letters? How would you simplify them?

L6: It would be the same answer, just with an x for this one and just with the number.

INT: If you see that they all have the same letter, then do you just think about the numbers like this [pointing to $7 - 5 - 2$]?

L6: Yeah. So this would be, like, zero [writes 0 under numbers] and this would be x [underneath $7x - 5x - 2x$].

INT: OK because there's just nothing there because the numbers have made zero.

L6: Ya.

INT: OK! Thank you!

Appendix 11.4 L10 Follow-Up Interview

INT: OK. We're going to start with Question 2, which is seven a minus three minus four. I just want you to right down your answer. You can use this pen.

L10: [Writes $7a - 7$]

INT: OK, seven a minus seven and you're not going to simplify that further?

L10: No.

INT: And just explain why.

L10: Because they not like terms.

INT: OK, so they're unlike terms, so they don't get simplified. Then this one: four x squared minus x squared minus six x squared minus x squared [$4x^2 - x^2 - 6x^2 - x^2$].

L10: [Writes $-2x^2 - x^2$]

INT: OK, just explain, where is the minus two coming from?

L10: The four x squared and six x squared.

INT: OK, so you did those two first to get minus two and then the minus x squared?

L10: Minus x squared minus x squared.

INT: OK, so you've said minus x squared minus x squared is just minus x squared [writing $-x^2 - x^2 = -x^2$].

L10: Doesn't look right…

INT: You can change it if you want… OK, well think about it like this. What if it was minus one minus one [writes $-1 - 1$]?

L10: Oh, then it will be negative two.

INT: OK, so that's negative two. So, if you have minus x squared minus x squared…

L10: Negative x to the power of four?

INT: OK… let's go through it like this. Let's do minus x minus x [writes $-x - x$].

L10: I don't know…

INT: OK, so we've said minus one minus one is minus two. Right? Now we have minus one x squared, minus one x squared [talking about x^2 but writing 1 in front of each x in $-x - x$]. So, if you just look at the numbers, then you said here minus one minus one is minus two plus two.

L10: Minus two x squared.

INT: OK [writes $= -2x^2$ next to $-x - x$] Why are you putting a square there?

L10: Oh, no, not a square.

INT: So just minus two x [scratches out superscript for answer $-2x$]. OK, so we said minus x minus x is minus two x. Then what about minus x squared minus x squared?

L10: Minus two x squared.

INT: Minus two x squared. So then would you change this one to minus two x there [pointing to previous solution $-2x^2 - x^2$]?

L10: Yes.

INT: OK [writes 2 between $-$ and x^2], so now we have minus two x squared minus two x squared. Can you simplify that further?

L10: [Writes $-4x^2$]

INT: OK. Then let's do Question 7. We've got minus six xy minus x minus y minus five yx [$-6xy - x - y - 5yx$]. So first I want to ask you, would you say that xy and yx are the same or different?

L10: Different.

INT: OK. So, if we have two letters next to each other, but they're not in the same order, then it's not the same. OK. So would you say, can you simplify this?

L10: Um…

INT: Or not?

L10: I don't think so.

INT: Is that because all of these are unlike terms?

L10: Yes.

INT: OK. If change it to xy for both of them [writes $-6xy - x - y - 5xy$], then how would you simplify it?

L10: Um… [writes $-11xy - x - y$]

INT: OK so these ones are staying like this because they're unlike terms.

L10: Ya.

INT: OK. Let's do this one: minus twelve ab minus eight ab minus three minus, minus two ba minus, minus six [$-12ab - 8ab - 3 - (-2ba) - (-6)$].

L10: Um… [writes working out, answer $-20ab + 3 + 2ba$]

INT: I see you've done minus twenty, I'm assuming that's coming from these two [points to $-12ab$ and $-8ab$]?

L10: Yes.

INT: And how do you know that it's, like, where does the minus and the twenty come from?

L10: Because with negative minus a negative you add them together but keep the negative.

INT: OK, so if they're both negative, then you add the numbers and then you keep the, you take the negative symbol.

L10: Yes.

INT: OK. So, when you're doing, I don't know, something, let's say like minus seven minus three [writes $-7-3$], do you look just at the numbers and the symbols, or do you, like, imagine a number nine?

L10: No, I look at the numbers.

INT: The numbers and the symbols. So here, what you just said was that if they're both negative, you'll add these two numbers [points to 7 and 3]. So, you'll do seven plus three equals ten and you'll know that there must be a minus in front because they're both negative.

L10: Yes.

INT: So, what if it's minus seven plus three [writes $-7+3$]?

L10: Then I would add the negative seven but because… then I think of a number line.

INT: OK.

L10: Yes, because negative seven if you add three it goes to the right, and then it will be negative four.

INT: OK. So, if they're both negative, then you don't worry about the number line because you can just add the numbers and bring down the symbol. But if you start with the negative number and you're adding, then you'll imagine a number line and you'll move up towards the zero.

L10: Yes.

INT: And what if you did…um… what if it was three minus seven [writes $3-7$]?

L10: Then it would be negative four.

INT: OK, and then again are you using a number line? Or are you just looking at the symbols?

L10: Symbols.

INT: OK, so when it's a negative number that you're adding to then you'll move towards the zero [gestures left to right], whereas if it's a positive number and you're subtracting from it, then you are just... how did you get to the four?

L10: I minus seven from three… or I just took three from seven and then I just kept the sign.

INT: OK, is that because the seven is bigger than three?

L10: Yes.

INT: And you're subjecting the bigger number. OK, Let's do… Question 16. It's minus eight minus x in brackets and then minus, brackets, minus eight plus x $[(-8-x)-(-8-x)]$.

L10: Umm…

INT: So first of all, what do you think the different brackets are doing? Like, are these brackets [pointing to the first set of brackets] necessary? And are these brackets [pointing to second set of brackets] necessary?

L10: Um… I think it is.

INT: OK, and what do you think these brackets are telling you?

L10: I think it's going to become positive. So, it'll be negative… [takes pen]

INT: OK, do you want to write it down?

L10: [Writes $-8-x+8+x$]

INT: Can you simplify them any further?

L10: [Writes $=-x+x$ then $=0$]

INT: OK, you've just taken these brackets away, so these brackets aren't actually making a difference.

L10: No.

INT: Is that because there's no symbol in front?

L10: Yes.

INT: Whereas with this one you have a minus in front and so you've changed, I'm assuming you've multiplied the minus…

L10: Yes.

INT: Is that what you're doing? OK. And then you cancelled the eights and then you also said that minus x and plus x it gives you zero.

L10: Ya.

INT: OK, let's do another one. Let's do one with different variables [writes $(-a+b)-(-a+b)$]. We have minus a plus b and minus, minus a plus b.

L10: [Writes $-a+b+a-b$ then draws lines through each term and writes 0]

INT: OK, again you've changed the symbol to a plus [points to $+a$]… and this one changes to a minus [points to $-b$]…

L10: Yes.

INT: And then you've got zero. OK. I'm going to go through a few other questions. Let's do these ones [writes $8xy - 8 - 5x - 5$].

L10: [Writes $8xy - 13 - 5x$]

INT: OK, where is the minus thirteen coming from?

L10: Minus eight minus five.

INT: And then with this one, again, did you look at the just the numbers [pointing to 8 and 5] and the symbols to get minus thirteen? [L10 agrees] OK. I want to do this question and then I think we might be fine. Minus seven minus five x cubed minus x cubed minus, minus five minus, minus two [$-7 - 5x^3 - x^3 - (-5) - (-2)$].

L10: [Writes working out, answer -5]

INT: OK, just explain your working out. So, is this minus seven [in first line] this one [in second line]?

L10: Ya.

INT: And then where is this minus five [pointing to -5 in second line] …

L10: I took the x three away from the five x three.

INT: OK, so then you've cancelled the actual letters, the x three and then you've got minus five. Then this [points to $-(-5)$] becomes plus five.

L10: Ya.

INT: OK, you've cancelled these two fives, I see?

L10: Ya.

INT: And then minus seven plus two gives you minus five. OK, so if we have… uh… [writes $11x^3 - x^3$] If I just give you eleven x cubed minus x cubed, how do you simplify that?

L10: Umm, I think *then* I'll take it away. So, I don't think that's right [points to $-5x^3$ and $-x^3$].

INT: OK, well, you can change your answer, or you can rewrite your answer.

L10: [Writes $-7 - 6x^3 + 5 + 2$] That will become zero [scratches through the constants] then… [answer $-6x^3$].

INT: So when you originally saw the five x cubed and the x cubed, you wanted to take out the x cubed.

L10: Yes.

INT: But then when I wrote them down by themselves then you wanted to work as if there's a one here [draws 1 in front of x^3 before minus], I'm assuming. So, what would you simplify this to [$11x^3 - x^3$]?

L10: Oh I would make that, I think… just eleven.

INT: Just eleven? OK so if it's by itself, now you're doing the same thing you were doing there [points to $-5x^3$ and $-x^3$]. You're saying you'll take away the x cubed. But what if I give you give you eleven x cubed minus two x cubed [writes $11x^3 - 2x^3$]. What would you do about that?

L10: Nothing.

INT: Nothing.

L10: Oh! Um… I think there will be just nine, or nine x. Oh… if it's the same base then I think you keep this [pointing to x]… so I think then this will just be nine x squared… x cubed. So, then this [pointing to $11x^3 - x^3$] will be ten x cubed.

INT: OK, and this one? [Pointing to $-5x^3$ and $-x^3$]

L10: Um, six x cubed.

INT: OK, like you did there [pointing to $= -6x^3$]. OK, this one, we would say it's going to be nine x cubed [writes $= 9x^3$ next to $11x^3 - 2x^3$] and this you said would be ten x cubed [wrties $= 10x^3$ next to $11x^3 - x^3$] and then this one up here… So maybe let's rewrite this one. Now you had minus seven, you've just said you would change that to minus six x cubed, and then you'd have the plus five and the plus two [writing $-7 - 6x^3 + 5 + 2$].

L10: So, then your answer will just be…

INT: Ya… minus six x cubed? OK. I just want to do another one like that. Let's just do one like this [writes $5 - x - 8x^2 - x^2$].

L10: [Writes $4x - 9x^2$]

INT: OK, now you've done again what you did there where… you said five minus one is four and minus eight minus one is minus nine. So, we're keeping this. Then I just want to ask you again for this one, this type of question [writes $(-3a) - (-2a)$]. If we have minus three a minus, minus two a, how will you simplify that?

L10: [Writes $-3a + 2a$ then $= a$] Or just one a, but you don't write the one.

INT: Ya, so again, you've said these brackets, you've taken them away, they don't matter [pointing to $(-3a)$]. But these brackets you've taken away after you've changed the sign to a plus.

L10: Ya.

INT: OK? And then you've got minus three a plus two a and you said it equals one a. How did you get to the positive one?

L10: Um… oh! Oh, then it’s negative a [writes minus in front of a].

INT: OK, so negative a. With this one, did you look at the symbols or did you think of a number line?

L10: I looked at the symbols.

INT: OK, so if you… because here, earlier, we were talking about symbols and number lines and stuff. For example, with this one [pointing to $-7 + 3$] you said for minus seven plus three, you would think of a number line and move up. Would you still do that if it was minus seven a plus three a? Or did the symbols make you think about it differently?

L10: It just makes you think differently.

INT: OK. So, with this one then, would you look at the numbers, just numbers and symbols [points to $-7a + 3a$ and underlines -7 and -3], addition or subtraction.

L10: Ya.

INT: OK. So, when there's a letter involved then it makes you think about it differently and you stop using a number line for this type of question.

L10: Ya.

INT: OK, I think that's enough. Thank you so much.

Appendix 11.5 L11 Follow-Up Interview

INT: OK, so first question I'm going to go through is seven a minus three minus four. And I just want you to simplify it.

L11: So like terms and then the letter is here [writes $-7 + 7a$].

INT: OK, so you did the... I mean, it doesn't make a difference if you write it down, minus seven plus seven a or seven a minus seven. When you did the subtraction here [pointing to -3 and -4], that was the first thing you did, so then you brought that to the front. Is that right?

L11: Yes, Miss.

INT: If I give you, hmm, let's say this one: minus twelve minus three x minus nine [$-12 - 3x - 9$]. Can you simplify that one?

L11: I'll do the like terms. I don't really think it matters if I put minus three x first…

INT: No exactly, ya. I'm just curious about the thinking that goes into your math, that's why.

L11: [Writes $-3x - 21$]

INT: I see, so this time you just brought that one down [$-3x$] and then you've done that. OK, next one I want to do is Question 5 which is minus four x to the power of four minus, minus six x to the power four minus six [$-4x^4 - (-6x^4) - 6$].

L11: Here I'm going to times the inside the bracket and it will be positive [writes working out, answer $2x^4 - 6$].

INT: OK, so how did you know that this was going to be positive?

L11: Because I subtracted that by that [pointing to $-4x^4$ and $+6x^4$].

INT: OK, so you did six minus four. If I give you something… say minus seven plus three [writes $-7 + 3$]. Do you imagine the number line or do you look at the numbers and the symbols?

L11: So this is the bigger number [pointing to -7]. The number with the bigger, no… how do I explain…

INT: Are you saying this number itself [underlines 7] is bigger? But it's negative…

L11: Ya, I guess so.

INT: Would this be positive or negative [pointing to $-7 + 3$]?

L11: Will be negative.

INT: OK, because the three is smaller than the seven?

L11: Yes.

INT: What if we have, let’s do it like this [writes $-5-(-7)$], how would you simplify that?

L11: I’ll probably take the negative and times it inside the bracket and make it positive seven. Then that's obviously going to be the bigger number then it would be positive too.

INT: OK, you would do minus five plus seven and then that's equal to two. And again, here, you're just working with five and seven, you're not thinking of like a number line when you're moving up or down?

L11: No.

INT: OK. Next one is Question 7 [writes $-6xy-x-y-5yx$].

L11: [Underlines $-6xy$ and $-5yx$] I’m not sure about those two.

INT: OK, what do you think?

L11: I think this is going to add, which will make it negative eleven xy, yx [writes $-11xy-x-y$].

INT: OK, so these two are like terms, doesn't matter if the letters are in a different order, but these two are unlike terms.

L11: Yes.

INT: OK. Next… minus seven minus five x cubed minus x cubed minus, minus five minus, minus, two [$-7-5x^3-x^3-(-5)-(-2)$].

L11: [Writing working out, answer $-6x^3$] This will equal to zero [pointing to -7 and $+5+2$] because, ya…

INT: OK, so you're saying the five and the two make seven and then minus seven and plus seven gives you zero. OK. And then you've got minus five x cubed and minus x cubed gives minus six x cubed. Just now we were looking at symbols and the subtraction just with numbers. If you're looking at an expression which has letters also in it, do you think about it differently to the way you would think just when you're working with numbers? Or do you just do you isolate the numbers? Or…

L11: Ya I isolate I guess like [uses finger to cover up x^3 of $-5x^3$ term].

INT: Ya so here I see you've underlined like terms. Would you then just think about the numbers that are in front of those letters?

L11: It has to be like terms and then I just [uses finger to cover x^3 again], ya…

INT: I see. You're not isolating numbers like that because those aren't like terms, the minus seven and the minus five x cubed.

L11: Yes, they're not like terms.

INT: OK. But these two [pointing to x^3 terms], and then those [pointing to constants] are obviously like terms because they're just numbers. I wanted to do another one. How would you simplify minus four x minus, minus four x [$(-4x) - (-4x)$].

L11: Should I simplify?

INT: Ya. [L11 hesitates] What do you think you'll do? What's making you hesitate?

L11: I think it's the negative. Ya, I think I have to like times it into the bracket.

INT: OK, well, what if I wrote it just like this? Is it the fact that there are two sets of brackets that's making you confused?

L11: That also.

INT: OK. So, if it's minus four x minus, minus, four x [$-4x - (-4x)$], how would you simplify that? Does that change the way that you look at it?

L11: It's still the same, but just without the brackets.

INT: OK. So, these brackets don't actually make a difference?

L11: Yeah, but this… [pointing to minus in front of $(-4x)$]

INT: Well, what did you do before, with this?

L11: I would times it inside the bracket. So, it will be positive x… [writes $-4x + 4x$] Am I supposed to simplify?

INT: Ya, how would you simplify that?

L11: I'll probably… I don't know how I'll do it…

INT: Well, just look at it like this by itself now as minus four x plus four x [using hand to cover $-4x - (-4x)$ so that only $-4x + 4x$ shows].

L11: There's lots of… you know there's lots… how do you get it? I think like… [writes $(x + 2)(x - 2)$ by first drawing brackets then signs then x's then numbers].

INT: Oh, OK. Are you trying to factorise it?

L11: Oh! Miss told me too… I thought Miss told me to simplify, OK no, no, no [scratches out $(x + 2)(x - 2)$].

INT: Well, I'm just curious. Writing it like this, what prompted you to want to write it like this [pointing to factorisation]? Is it because there were brackets and there was multiplication?

L11: Oh no I'm misunderstood. I thought Miss told me to factorise.

INT: Oh, OK. No, I just want to see if you can make this any simpler. Minus four x plus four x.

L11: Then ya I guess that will work… wait, but doesn't it cancel each other out? Ya.

INT: Then what is the answer?

L11: I think it's zero, it's zero [writes $= 0$] because they cancel each other out.

INT: OK, so then what about this one? [Points to $(-4x) - (-4)$]

L11: See that's the same so… I think the brackets actually does make a difference…

INT: Does it make a difference? Do we need these brackets? [Pointing to first term] Are they just there to separate or are they implying something different?

L11: [Hesitates] I think they imply something different… I'm not sure…

INT: Why do you think so?

L11: At the same time, no… let's me just try… [writes $(-4x) + 4x$] But then how is the brackets going to go away? I'm not sure…

INT: Let's just think about it. We've got these brackets here. Previously, when we were first looking at these two examples, you said you don't think these brackets make a difference. So we had two sets of brackets, but there's a symbol in between the brackets. And now again, now you've…

L11: Timesed it inside.

INT: Timesed it in so you can take those brackets away. So now you've got plus four x. If you've got brackets at the front and then you have a symbol that's a plus or a minus, do you think that these brackets [pointing to brackets in $(-4x) + 4x$] are implying that you then need to multiply? Or is it just a way of separating terms? Or is there another thing that it could be doing?

L11: Probably like to multiply… I'm not sure…

INT: Are you saying you think there might be a difference between it like this [$(-4x) + 4x$] and it's like this [$-4x + 4x$]?

L11: It's the same, but the bracket, so…

INT: And you're not sure if the brackets mean you have to approach it differently or not.

L11: I'm not sure if you have to times it inside…

INT: OK, if I tell you *have to* simplify this now and then what is your approach going to be? [L11 hesitates] Will you multiply or will you do minus and plus?

L11: Probably multiply, I guess… [writes $-16x^2$]… I'm not sure…

INT: Let's do another one ad then we can see. How about [writes $(-2) - (-4)$], just with numbers?

L11: Oh I see, OK wait [writes $(-2) + 4$], but I'm not… oh…

INT: Let’s look at this example here for Question 10 [points to $-(-5) - (-2)$], because here you essentially had the same thing. You had minus five minus, minus two, but there was a minus in front and then you changed it to plus five plus two, so you took that minus. So, for example, let's just say I added the minus in there [writes minus in front of $(-2) - (-4)$]

L11: Oh, then I would times it inside the bracket.

INT: You would times it in. So the fact that there wasn't a symbol in front of the bracket, is that what made you wonder whether you should do multiplication or whether you should do…

L11: Yes.

INT: Because it wasn't part of an expression like this where you were trying to change all of the signs. So now with the minus in front, what would you do differently?

L11: I would times that inside the bracket and the brackets fall away, which will make a positive.

INT: So it'll be a plus two and a plus four [writes $+2 + 4$ underneath], if I’m just copying that answer there [points to $+5 + 2$ in answer to Q10].

L11: Yes, then it would be six.

INT: If that's what you're doing when there's a minus in front, then if there is nothing in front [writing $(-3) - (-5)$], if it’s like an invisible plus [draws plus in front of (-3)] …

L11: Then, ya… probably still equal to negative though.

INT: Now that there's a plus instead of a minus, will you use the same method? Or will you do it differently?

L11: Probably use the same method.

INT: Do you mind just simplifying it there?

L11: [Writes $-3 + 5$]

INT: Can you simplify it further?

L11: [Writes $= 2$]

INT: OK, let's do another question similar to that, but maybe with more terms. Let's do Question 13: x minus five minus x minus five [$(x - 5) - (x - 5)$]. What would you do now? [L11 hesitates] So now again we have two sets of brackets and a minus, but this time they have two terms inside the brackets.

L11: [Writes $x - 5 - x + 5$] I’m not sure if you have to keep the brackets…

INT: Can you simplify this further? Or will you leave it like that?

L11: I’d leave it like… oh, wait… [underlines like terms then hesitates]

INT: So, you've underlined x and…

L11: [Writes $= 0$] Ya…

INT: Let’s do another one: minus x minus one minus, minus x plus one [$-(x-1)-(-x+1)$].

L11: [Writes $-x+1-x-1$] It’s equal to zero.

INT: OK, again [writes $= 0$]. And what about this one? [Writes $(-a+b)-(-a+b)$] So now both of them have letters.

L11: [Writes $-a+b+a-b$] But I’m not sure if you have to keep the brackets, I’m not really sure.

INT: Why do you think you’d keep the brackets?

L11: Because how are they going to get away?

INT: OK, are you saying that because there was a symbol here [pointing to minus between brackets] and you could bring that in, then you could take the brackets away. But here [pointing to first set of brackets] there's nothing in front, you're not changing anything about this, so can you take the brackets away or not?

L11: I don’t know.

INT: Is that what you mean?

L11: Yes.

INT: Well, if you imagine that there is a plus there, which there is [draws plus in front of first set of brackets], then do you think it’s fine to take the brackets away?

L11: Mmmh.

INT: So when there's a symbol in front and you can use that symbol to change what's inside or to bring in the symbol, then you drop the brackets.

L11: Yes. Oh, but I guess the bracket does fade away because of that because of that, the imaginary [pointing to plus sign in front of brackets]

INT: The imaginary plus. So you’re timesing in the plus with the minus which gives you the minus, and a plus and a plus gives you plus. Can you simplify this any further?

L11: [Writes 0].

INT: OK. Let’s do Question 4: four x squared minus x squared minus six x squared minus x squared [$4x^2-x^2-6x^2-x^2$].

L11: So they’re all like terms. [Writes working out, answer $-6x^2$].

INT: I see you've underlined these three terms [$-x^2 - 6x^2 - x^2$], why did you underline all three of them?

L11: Because they're negatives, so I do the negatives first.

INT: OK, so you did all the negatives to get minus eight x squared, and then you brought down the four and then you did the four and the minus eight. OK. And when you were doing this, what method did you use to get to the minus eight? What did you think?

L11: I added all the negatives together.

INT: Would you do one plus six plus one? Is that what you mean? What do you mean by add?

L11: Ya, negative one and negative six and so on.

INT: OK, and then that gives you minus eight and then to get to the minus four?

L11: Then I brought the positive four x squared down and then, ya [pointing to $4x^2$ and $-6x^2$].

INT: Do you do eight minus four and then put a negative in front? Or do you do four minus eight and just work backwards towards zero.

L11: Eight minus four.

INT: When you do the eight minus four, how did you know it was going to be negative?

L11: Because the bigger number.

INT: The bigger number is the negative number. OK. I think that we can finish there. Thank you!

Appendix 11.6 L13 Follow-Up Interview

INT: OK. So, I'm gonna start with Question 2, which is seven a minus three minus four. If you could just simplify that.

L13: [Writes $-8+5$]

INT: OK. So, you're not simplifying this any further.

L13: Uh-uh.

INT: And that's because?

L13: Because there's not an a here also.

INT: So, they're not like terms.

L13: They're not like terms.

INT: OK And with the minus three and the minus four, how did you know that it was going to be minus seven?

L13: Because it's two negatives.

INT: OK. And when you're doing subtraction problems, do you just look at the numbers and the symbols or do you imagine a number line that you're working on?

L13: I imagine a number line.

INT: OK. So, if we did the number line [drawing number line], can you try and draw what it is that you did on the number line.

L13: If this was negative and then I just add negative four.

INT: OK. And do you do that for any kind of subtraction problem? So, let's say you had like … eight minus five. Would you just work with the numbers in your head or would you still imagine the number line?

L13: I would work with the numbers.

INT: OK. And what if it's minus eight plus five.

L13: Then I'd use the number line.

INT: OK, So, if you're working with a negative number, then you'll use the number line. So, let's see this question: Four x squared minus x squared minus six x squared minus x squared. [$4x^2 - x^2 - 6x^2 - x^2$]

L13: [Writes $-2x^2$]

INT: OK. So, you've rewritten it. Is that because these two terms are the same, so you're going to put them next to each other?

L13: Hmm.

INT: OK. And then how did you get to the minus two x

L13: Minus four x two minus six two.

INT: OK, so that gives you the minus two. And what about these two?

L13: They cancel out.

INT: OK, what about this: minus twelve x squared minus three x squared minus three x squared? [$-12x^2 - 3x^2 - 3x^2$]

L13: [Writes $-18x^2$]

INT: OK, so here you've put these two and then you've subtracted minus three from minus twelve to get minus fifteen and then you go down to minus three.

L13: Yeah.

INT: So, in this one you had the same terms subtracted from itself. So, you had minus six squared minus x squared, which is this basically the same here minus three x squared minus three x squared. But in this one you said they cancelled each other out. And then this one, you've subtracted it. So, what if you said … minus two x squared minus x squared minus x squared. Would you use the same method here or …

L13: They would still cancel out, because there's no number.

INT: There's no number. OK. So, it's not the same as if there's a number in front. You can cancel them out because there's no number. Let's do another one: minus six xy minus x minus y minus five yx. [$-6xy - x - y - 5y^2$]

L13: [Writes $-11xy - x - y$]

INT: OK, so you've left the xy but the x and the yx you said are the same.

L13: Yeah.

INT: So, the order of the letters doesn't matter. OK, let's do another one: minus seven minus one x cubed minus x cubed minus minus five minus minus two. [$-7 - 5x^3 - x^3 -$ (-5) – (-2)]

L13: [Writes $-5x^3 - x^3$]

INT: So, you change these to pluses. Where is that coming from?

L13: Because brackets.

INT: So, you've got a negative and a negative with brackets, so you can change it to a plus. OK. And then I see you've grouped the numbers together here.

L13: Hmm.

INT: OK. So, is that just to make it easier for you to add them together?

L13: Yeah.

INT: So, now you've got minus five x cubed minus x cubed. Would you not simplify this any further?

L13: I think you could.

INT: OK. What would you simplify to?

L13: [Writes $-6x^3$]

INT: OK. And then for this one, did you use numbers again or did you use the number line?

L13: The number line.

INT: OK, going to another one: [$6x - x - 2x^2 - 3x^2 - x$]

L13: [Writes $-5x^2 + 6x$]

INT: OK. So, you put the like terms together and here you've written two x squared. Can you take this minus away?

L13: Ohh, I forgot to.

INT: Would you? So, would you put this minus in there?

L13: Hmm.

INT: OK. And then what? How would that change your answer here?

L13: The negative five x squared.

INT: OK, so negative five x two. And then you put a six x. And then here there's these x's. Are you doing the same thing that you did there where they cancelled? But if there had been a number in front then it would be different.

L13: Yeah.

INT: OK, cool. Let's go to another one and do Question 13: x minus five minus x minus five. $[x - 5 - (x - 5)]$

L13: [hesitant]

INT: What's making you hesitate with this? Is it the bracket?

L13: The minus.

INT: The minus? OK, well what if I give you something like … if you had five x minus five x. $[(-5x) - (-5x)]$ Are these brackets necessary and do you think if there's a symbol in between, what do these brackets …? What do brackets make you think of? Is actually what I'm asking.

L13: When I see the brackets, I think you must multiply.

INT: OK. And what about when there's a symbol in between? Or let's say we do minus five x minus five x. $[-5x - (-5x)]$ So, now we've got minus five x minus minus five x. What do you want to do with this symbol [pointing to middle minus]?

L13: I think you would just minus it…

INT: OK, so can you simplify this?

L13: [Writes $-10x$]

INT: OK, so how do you know that this must be negative?

L13: Because you minusing the two negatives.

INT: OK, so you've got minus five x and minus five x, but what about this minus in the middle? Is this telling you that you must minus there? OK. OK. And what about if I wrote it like this? [writing] Would that be different to you or is that the same?

L13: I think it's the same.

INT: OK. So, this would also be minus ten x? OK. So, now with this one … So, here what you said was that we had minus five x minus five x and we were subtracting them. So, now we just have x minus five and x minus five [$x - 5 - (x - 5)$] and we're subtracting them. So, do you think you'd be able to simplify this one? We can start like this. Let's say we take these brackets away, so it's actually just x minus five minus x minus five. Does that make it easier?

L13: I think so, if uh … I think it would just cancel out.

INT: OK, it would just be zero. What about … if I do this and I've got brackets in both of them? Um … you've got minus x minus one plus x minus one. [$-(x - 1) + (x - 1)$]

L13: [Writing] I don't really know what to do.

INT: OK, well, let's just look at it like this. What do you think that's minus is telling us to do with this expression.

L13: Multiply.

INT: So, you want to multiply. Do you mean multiply the minus in?

L13: Hmm.

INT: Well, you can start with that. So, how would you do that If you're just looking at those, that little part of the expression, how would you multiply them? What would you write to show that you've multiplied in the minus?

L13: Change the signs?

INT: OK. Are you only changing the sign, so just this sign is what you changing?

L13: Hmm.

INT: OK. And then what if you just looking at that part plus x minus one, does that change the signs or do you keep them the same?

L13: Keep the same.

INT: OK. And so, this you're just applying to whatever the sign is that's in the middle.

L13: Hmm.

INT: OK. And then how would you simplify this?

L13: [writes 0]

INT: OK, so then let's do another one. Let's do … what if I did minus eight plus x minus minus eight minus x? [$-(8+x)-(-8-x)$]

L13 : [writing 2 – changed to 0, see later]

INT : OK. So, here you've got the minus in. So, with this one, I'm just curious. Here you've said the minus goes in front of the eight and then to the x as well, changes the sign and then there's minus changes that sign and changes that sign. But here you didn't change the sign in front of this x.

L13: Ohh.

INT: Would you change the sign?

L13: Yes, I would.

INT: So, this would be minus x plus one plus x minus one. Then you'd have minus x plus x plus one minus one. And then what would your final answer be? Would it be different if it's minus x?

L13: It would cancel.

INT: OK. So, it would be zero.

L13: Hmm.

INT: OK, let's do another one. Let's say we have a plus b minus a plus b. [$(a+b)-(a+b)$] So, there's no negative in front of this one anymore, but there's still this negative in the middle.

L13: [writing $2b$]

INT: OK, so we've got a plus b minus a plus b. So, here again you brought this minus and you made it plus and plus, but this time you've only brought the minus in front of the a. You haven't changed the sign in front of the b.

L13: I just took away the brackets.

INT: OK, so, what's the difference with this one where you just took the brackets away and kept the minus versus this one where you multiply the symbol in?

L13: Because there's no minus by the a.

INT: OK. So, oh OK, with this one you just took the brackets away, and with this one you're just taking the brackets away. And then you've got a minus a and then b plus b gives you … OK. This was Question 15. Oh no. Let me do another version of this where it's minus a plus b and minus minus a plus b. [$(-a+b)-(a+b)$]

L13: [writing 0]

INT: OK, so these ones stayed the same, you just took the brackets away. So, these brackets make no difference, OK. And then these you change the signs, so you multiplied again. OK. Just take this one: two ab minus ab minus three ab minus two ab minus ab. [$2ab-ab-3ab-2ab-ab$] So, now we're working with two places at a time.

L13: [writing $-3ab$]

INT: Just minus $3ab$?

L13: Hmm.

INT: OK. So, again, I see you've rewritten it. So, two ab, minus two ab. I see you wrote those similar terms and then the three ab and then the two ab's and then you've just got three ab. So, are you saying that the two ab and minus two ab cancel, and the minus ab and the minus ab cancel?

L13: Ya.

INT: And these ones are a positive and a negative, but these ones are two negatives. But there are no numbers in front of these ones. Is that why?

L13: Hmm.

INT: So, it's the same as what we did before.

L13: Yeah.

INT: OK. Thank you.

Appendix 11.7 L14 Follow-Up Interview

INT: OK, so I'm going to write the questions on here and then I just want you to go through your answers. [Writes $7a - 3 - 4$] Will you just simplify that for me, please? Seven a minus three minus four.

L14: Yoh…

INT: Just try, I'm just curious about your thinking.

L14: Seven a minus three minus four… negative three minus four is… seven [writing $7a - 7$]. Isn't it… positive one?

INT: The first think you said was minus seven, so what makes you think it's minus seven?

L14: Negative three take away four…

INT: If you're doing subtraction like this and you have minus something minus something else, do you imagine a number line? Or do you just look at the two numbers and look at the symbols?

L14: Look at the symbols.

INT: Symbols and the numbers, OK. Have you done three plus four to get seven and then you've done a minus. Or have you done minus three and then move down another four? Which one?

L14: Second one.

INT: So, you started at minus three and then you went 'if I go four down then it gets to minus seven,' OK. And would you leave it like this? [L14 agrees]. OK, let's do another one.

L14: I don't know what I'm doing…

INT: No, it's fine. I'm just curious about your thinking process. Let's do another one. What about this [writes $-12 - 3x - 9$]? We've got minus twelve minus three x minus nine.

L14: Negative twelve minus three x minus nine… [writes $-12 + 9 - 3x$ then $3 - 3x$].

INT: OK, just explain what you're doing to get a plus nine?

L14: I brought that over [gestures arc from -9 to -12]

INT: OK, so you're bringing this over and then it changes to a plus. And then how did you get a plus three from the minus twelve and the nine?

L14: Negative twelve plus nine, it goes that way [gestures left to right along imaginary number line].

INT: OK. And then you're going to leave it like this. You left that one. So, I'm assuming you think this one as three minus three x.

L14: Or six x.

INT: Would you put six x? OK.

L14: Not six x, just x [writes x underneath], three minus three.

INT: OK, so three minus three cancels the threes, is that right? [L14 agrees] Then you just have the x. OK, let's do another one. Let's do… this one. Now we have minus x squared minus four x cubed minus two x squared. What about that?

L14: [Mumbles to self then starts writing answer in book. Final answer $7x^7$]

INT: OK, so you've changed this to a plus [points to $3x^2$]. Did you do the same thing you did in the previous question? [L14 nods] OK, so you're bringing it over and then it changes to a plus. But this one is staying a minus [points to $4x^3$].

L14: It's a plus.

INT: Oh, it's a plus as well. OK, so whenever you're bringing it over a negative, then it's changing to a plus. OK, so it's plus four x cubed plus two x squared, and this one now also becomes a plus.

L14: Oh it must be minus. [Crosses out $+x^2$ and writes $-x^2$ and changes final answer to $6x^7$]

INT: OK you keep it as a minus. And then for the six, is it four and the two? [L14 nods] OK. And here you've changed the exponent to seven?

L14: It's three plus two plus two.

INT: Three plus two plus two, OK, so you're adding the exponents together. Do you always add the exponents when you… if you have different variables with different exponents, or if you have the same variable, sorry, the same letter with different exponents you're putting them all together, then you'll add the exponents.

L14: Ya

INT: Ya, OK. Let's do another one. Let's do this one: four x squared minus x squared minus six x squared minus x squared.

L14: [Writes working out, answer $11x^8$]

INT: OK, four x squared you've brought down. And this one [points to $+x^2$], oh, which one has changed to a positive?

L14: This one goes [gestures arc from second $-x^2$ over $6x$]

INT: This one went over and that one just stayed, OK [pointing to $+x^2 - x^2$]. And where is the eleven coming from?

L14: Four plus six… oh, it's supposed to be ten [scratches out 11 and writes 10].

INT: OK, so it's ten because those are the numbers you're adding together [points to $4x^2$ and $6x^2$], and then the eight, is that two plus two plus two plus two? [L14 agrees] Let's do another one [writes $-6xy - x - y - 5yx$]. How would you simplify that?

L14: [Writes working out, answer $-1x^3y^3$]

INT: OK, the plus five yx, is that the bringing over? And then you've left these two the same [pointing to $-x$ and $-y$]. So, you've brought one thing over [pointing to $5yx$], are you saying these ones [pointing to $-6xy$ and $+5yx$] are like terms?

L14: Mmmh.

INT: Ok and then you've done minus six plus five, that's the minus 1 [pointing to $-1xy$]? And xy and yx and x and y [pointing to letters in expression], what did you do to get the exponents for the x's and y's?

L14: [Scratches out superscripts 3 and 3 and writes superscripts 2 and 2] x plus x is two minus that… [scratches out superscripts] it's supposed to be just like that, because x and x then you minus that is one.

INT: Oh, I see what you mean. So, you would have two and then you've taken one away, so it becomes just the one, x to the one, and then with the y's?

L14: Same thing.

INT: OK, same thing. So, the y is going to be to the… but now, here you've got an x and an x and then you're taking one x away, but now you've just got one… Oh, no, you've got three, sorry. So, it's a y and a y and a y and you take one away, OK. Let's do this one. What about this [writes $(x-5)-(x-5)$]? [L14 hesitates] What do you think? Let's start with something else, let's start with this [writes $(-7)-(-7)$]: minus seven, in brackets, minus, minus seven, also in brackets.

L14: [Writes 49] This means you must like, times [pointing to brackets].

INT: Do you think you must times? What if there's something in the middle? Do you still think it's times?

L14: 'Cause of the brackets… no…

INT: Are the brackets making you think that you should be doing multiplication?

L14: [Scratches out 49, mumbles to self] Yoh…

INT: What if we do it just the seven? Or what if I write it like this [writes $-7-(-7)$], how does that change what you would do?

L14: Um… seven minus, bracket… ya I would do seven times seven.

INT: Seven times, still. When you see the brackets you want to do times. What if I put a seven in brackets like that [writes $7-(7)$]? [L14 nods, implying they would multiply] If I give you [writes $-7-7$] without the brackets…

L14: Negative seven negative seven… negative fourteen.

INT: OK, so this is negative fourteen, but, this one, if there are brackets then you are multiplying.

L14: Wait, wait… I feel like…

INT: What do you think these brackets are telling you to do? Or what do you think the brackets are there for? So you have a minus and a minus.

L14: Means it makes it positive, so negative seven plus seven… equals zero.

INT: OK, so you're saying this one would be minus seven plus seven equals zero. OK, so if you've done that for this, what about this one?

L14: Minus seven… minus seven… minus positive fourteen.

INT: Positive fourteen for this one? OK, just explain that again.

L14: Wait, negative seven minus… um…

INT: For this one [$-7-(-7)$] we had negative seven, minus, negative seven. Now we have negative seven, minus, negative seven. So, they're the same, the only thing is that this one has brackets around it. So, do you think that these brackets change the way that you must simplify it?

L14: Mmmh. A negative times a negative is a positive, so I feel like it's a positive seven and then take the minus away, so it'll be times seven. So, positive seven times negative seven.

INT: OK, this one we're saying… [writes $(-7)(+7)$] like this? Is that what you mean? [L14 agrees] And that would be…

L14: Forty-nine. Negative forty-nine.

INT: OK, then with this one [$(x-5)-(x-5)$], because we have two sets of brackets, would you say this must also be multiplication?

L14: I would times that and that [pointing to x's] and that and that [pointing to -5's]

INT: Like the FOIL method? Where you do this [draws arcs joining terms for FOIL method]. What if I gave you x minus five minus x minus five [writes $x-5-(x-5)$]?

L14: x minus five… first work in the brackets [writes $x-5-5x$ then $5x-5x$].

INT: OK, you change it to $5x$. Just explain why you're putting the five and the x together now.

L14: x minus five…

INT: So, you bring them together, but here [pointing to $3 - 3x$], oh you did put them together. Well, like here for example you had seven a minus seven [pointing to $7a - 7$]. Is it because there's a number in front of the a that you're not putting these two numbers together? [L14 hesitates] Do you see what I mean? So here you have seven a minus seven and here you have x minus five and you're rewriting x minus five as five x. So, you're putting the number in front. If I gave you a minus seven [writes $a - 7$] would you leave it like that? Or would you change it?

L14: I'd leave it like that.

INT: So why are you changing this one [points to $(x - 5) - (x - 5)$]

L14: Um…

INT: What makes you want to change it? Is it because it's in brackets? [L14 agrees] So, you're wanting to simplify whatever is in the brackets first. [L14 agrees] OK. And then here, you've done x minus five and you've also changed that to five x. Is that because they're the same thing? OK, so you've got five x and five x, would you simplify that? Or would you just leave it?

L14: I just just put x. Oh, no… ya, just x.

INT: OK, so you've just put x, you've cancelled the fives, is that right? [L14 agrees] But then when you were working with these ones [pointing to $-6xy - x - y - 5yx$], you were making the exponents bigger when you had more x's. Is that because there's a plus here? Or because you only have to x's so you're not going to deal with the exponents? You're only looking at the numbers?

L14: [L14 very hesitant] Because that cancels out each other and it just becomes x.

INT: OK. If I give you seven x minus two x [writes $7x - 2x$] and I give you four x minus four x [writes $4x - 4x$]. How would you do both of those?

L14: [Writes $5x$ under $7x - 2x$ and x under $4x - 4x$]

INT: OK, so this one is five x and this one is just x. So, when it's the same number then you cancel the numbers, but when you have different numbers then you subtract the numbers from each other. [L14 agrees] OK. It's already been twenty minutes so we can stop there. Thank you so much!

Appendix 11.8 L22 Follow-Up Interview

INT: OK, I'm going to write the questions on here and then you can just write down your answer and then there might also be some questions about that. Writes [$7a - 3 - 4$]

L22: [writing $7a - 7$]

INT: OK, cool. Just quickly explain what you've done. So, you've got minus seven. You're leaving that a seven a minus seven. Where did the minus seven come from? Or what did you do to get minus seven?

L22: Because they're both negatives, so I just add those two [pointing to 3 and 4].

INT: OK, so you're adding them and then you're putting a minus in front because they're both negative. And when you're doing subtraction, do you just look at the numbers and then the symbols, or do you imagine a number line that you …

L22: No.

INT: Just numbers and symbols? OK, the next one I wanted to do was this one: four x squared minus x squared minus six x squared minus x squared. [$4x^2 - x^2 - 6x^2 - x^2$]

L22: [writes $-4x^2$]]

INT: OK. Just explain what you've got.

L22: First I did all the negatives … and then I did …. two and then I just subtracted positive four x squared.

INT: OK. So, you're saying you do all the negatives, so normally if the first number is positive and the others are negative, will you do the negatives first and then sort of …

L22: What was the most.

INT: Whatever makes the most sense? OK. Well, let's just do another one. Then let's say we started with a negative number [writing $-9a^2 - (-7a^2) - a^2 - 2a^2$] … I'm going to add in a bracket … ya, let's just do that.

L22: [writing $5a^2$, later changed to negative] So, I did all the negatives first and negative negative make a positive. I've got positive seven, I just subtracted.

INT: OK, so this one you've got minus twelve a squared plus seven a squared, and then you've got five a squared. And how did you know this must be a positive?

L22: Oh, it's supposed to be negative.

INT: [writing $8xy - 8 - 5x - 5$]

L22: [writing $8xy - 5x - 13$]

INT: OK, so you've left the xy and the x as separate terms and then you've brought the eight and the minus five.

L22: Yes, I subtracted those from one another.

INT: And you wouldn't simplify it further?

L22: No.

INT: Are you saying these are all unlike terms?

L22: Yes.

INT: OK, what about this one? Can you simplify this? [$-6xy - x - y - 5yx$]

L22: [no answer written]

INT: OK, so, if we have xy and we have yx then we've got this. They're unlike terms.

L22: They're unlike terms.

INT: OK, so you just leave this as it is. What if I do a similar question like this and It's kind of got ab's and ba's and numbers. [$-12ab - 8ab - 3 - (2ba) - (-6)$]

L22: [writes $-20ab - 2ba - 3$]

INT: OK, so the minus twenty ab, is that the first two terms?

L22: Yes.

INT: OK And then the minus two ba?

L22: It's that one.

INT: So, do you think the brackets make a difference here?

L22: They do.

INT: And then minus three. And what about this? Ohh, the minus three?

L22: Positive six.

INT: OK, so you've got minus three and positive six and you've written minus three.

L22: Yeah.

INT: So where is this minus three.

L22: Six minus three.

INT: Oh, six minus three. OK. But six minus three and you said minus three.

L22: Wait no … positive six minus … it's a positive three. [corrects in book]

INT: OK. So, it's minus three plus six or six minus three is positive. OK. And you're leaving the ab and ba are different just like you did in this one. OK, let's do another question: So, now we have two sets of brackets. You've got brackets x minus five, minus brackets x minus five [$(x - 5) - (x - 5)$].

L22: [pauses] I don't know how to do this.

INT: What do you think these brackets are for?

L22: I must simplify… oh no… but wouldn't that… no, I don't know.

INT: Well, what were you going to say there?

L22: I think this changes the sign [pointing to middle minus and second term $(x - 5)$]… Ya it must become negative.

INT: Let's just see. What if I write it like this without the first set of brackets and I just do [writing $x - 5 - (x - 5)$] … Do you say this minus changes the sign of the x?

L22: Yeah.

INT: Do you think it changes the sign of the five as well?

L22: Yeah.

INT: OK, so how would you simplify that?

L22: [writing $x-5-x+5=2x$]

INT: We've got x minus x and minus five plus five. So how did you get to the two x?

L22: x plus … oh wait … x minus … zero. So, zero plus zero.

INT: Zero. OK, so, now what do you think this would be if this one is x minus five minus brackets x minus five? [$x-5-(x-5)$]. Do you think this would be the same, or would you do something different here?

L22: Can I just take the brackets away?

INT: What do you think? Do you think these brackets make a difference?

L22: No.

INT: OK, let's do a different example. [writes $-8-x)-(-8-x)$] What about this: minus eight minus x, minus minus eight minus x.

L22: [writes 0]

INT: OK, so again zero. So, you've taken these brackets away and then you've changed these both to positive. Is that from taking this minus?

L22: Yes.

INT: OK. And then you say zero. OK, I think that should be fine. I'm just double check that I don't have any other questions for you. … Oh, let me just do one more question and then you can go. And what about something like this [writes $-x^2-4x^3-2x^2-x^3$]: minus x squared minus four x cubed minus two x squared minus x squared.

L22: [Writes $-5x^3-3x^2$]

INT: Where is the minus five coming from?

L22: It's a negative and a negative [pointing to $-4x^2$ and $-x^2$].

INT: OK, so these ones you're saying those are the like terms and these the like terms. OK. And then you've got negative and negative again. You want to simplify this any further.

L22: No.

INT: OK, I think that's fine.

Appendix 11.9 L24 Follow-Up Interview

INT: So, we're starting with Question 2, which is seven a seven a minus three minus four. $[7a - 3\ - 4]$

L24: [writes $a = 0$]

INT: OK, just explain what you've done.

L24: I see the negative three and the negative four which added is negative seven. And then I took this seven over, so it's positive and become negative. And then I said a equals negative seven minus seven which is zero.

INT: So, you've taken the seven a minus seven, and then you wanted to find what a was for this to be true. So, you've turned it into an equation.

L24: Yes.

INT: But if you're just simplifying the expression, then is this the value that you would give to a for this to be true, but this is the simplified version.

L24: Uh-uh.

INT: OK. Just now you said you added the three in the four to get and then you got to minus seven. So, do you, when you're doing subtraction, do you just look at the numbers and the symbols or do you imagine a number line that you're working up and down?

L24: A number line.

INT: Can you just explain a little bit how that works when you're imagining the number line, if you're moving, for example with minus three minus four. Could you sketch out the number line? It may be a bit difficult, but try and explain with a diagram what you do.

L24: [drawing number line and showing] I put the negative three and then I added the negative four.

INT: By adding the four you mean you're moving to the left.

L24: Ya, because it's negative.

INT: OK, that makes sense. Let's do Question 3, which is minus x squared minus four x cubed minus two x squared. $[-x^2 - 4x^3 - 2x^2]$

L24: [Writes $-1x^2 - 4x^3$]

INT: So, just talk me through it. You went from minus x squared to minus two x squared and then you've got a positive x squared. So, did you visualise the number line again? How did you do it when it's not numbers anymore? When you have a variable do you also use the number line?

L24: No. I just looked at the x squared on top, like the negative x squared and then I did this because they both like terms. And then it there's obviously a one here. So, I said negative one x squared minus two x squared. And then I just put it down because that's like different.

INT: That one's an unlike term because they've got different exponents. Okay. And then just getting to the minus one - so, you went from minus one and minus two and then that gives you the minus one.

L24: Uh-um.

INT: Let me just give you a different example. What about something like this? $[-8a - 13a - a^2 - a]$

L24: [writes $-20a + a^2$]

INT: OK, just again just speak through what you've done?

L24: So, the negative eight a minus thirteen I added it and then I got twenty-one a minus a squared minus a, and then… I took this [pointing to $-a^2 - a$] and I, like, I made it swap places so I changed the sign.

INT: OK. Is that, you're taking it over, so you're changing the sign?

L24: Ya, and then I added it with this [pointing to $+a$] which I got negative twenty-two a and then I plussed [pointing to a^2]

INT: The a squared. OK, so just here, with this swapping places of this one, so you're saying if you take this one [pointing to $-a$], like if you're taking it over there, the minus a squared then it becomes positive. And then why does this one also become positive? [Pointing to $-a^2$]

L24: Because isn't it also changing its place?

INT: Oh, I see what you mean, so like you're taking this one over, so it's becoming positive and this one's also going over so it’s also going to be positive.

L24: Ya.

INT: OK. And then here, with the minus twenty-two, the signs here are different but you’re still adding the numbers together.

L24: Ya.

INT: OK, so how do you know when to add the numbers and when to subtract one number from the other? [L24 hesitates] If that makes sense? Like if I give you, well if we just look at the numbers like this [writes $-21 + 1$ is 20]

L24: I’d add it.

INT: So, it would be minus twenty-two? Is that what you’re saying?

L24: Ya… no… hmmm. Well, if negative twenty-one was on here [pointing to number line] then you have to go this way [gestures left to right] because it’s positive. So, it will be negative twenty.

INT: So, would you change this to negative twenty, or leave it?

L24: No, change it.

INT: Going back to this one: we’ve had minus and minus, so it’s the same as minus one minus two if I’m just looking at the numbers. Before you were saying you would move to the left if they're both negative and then if you start with the negative and you’re adding one, then you go to the right. So, for this one will you go to the left?

L24: Yeah, I would.

INT: So, it would be minus three.

L24: Yeah.

INT: So, this would be minus three?

L24: Ya.

INT: OK. Another one. This is Question 4: [$4x^2 - x^2 - 6x^2 - x$^2]

L24: [writing $-2x^2$]

INT: So, you've drawn lines through these ones.

L24: Yeah, I cancelled them.

INT: OK And then you're doing four minus six to get to minus two. Why are you cancelling them? I just want you to explain your thinking.

L24: Because if I put them like that [writing $-(- = + \quad and \quad - - + = +$] - it’s cause you're taking away.

INT: OK, so you're taking away. But what if we have something like … well, let's do the same thing. Let's say we do this. [writing $-3a - (-3a)$] What if I add in an extra one, how does that work? Would you still be doing subtraction? How would you read this with the brackets?

L24: I think I'm multiplying.

INT: OK, why do you want to multiply? Because they're brackets?

L24: Yeah.

INT: OK, so what would you simplify this to?

L24: [writing $9a^2$]

INT: OK. And if I put it in a different expression – let’s say I add a number in front, would you still multiply now?

L24: Yeah.

INT: OK. So just simplify … [$5a - 3a - (-3a)$]

L24: [writing $60a^2$]

INT: Ohh OK I see. So now you're simplifying this first and then you're multiplying. So, five a minus minus three two a minus minus three a. And then how did you know it was positive?

L24: Because the negative and negative is positive.

INT: OK. Let’s do another one. What about if it's just numbers? [$8 - 7 - (15)$]

L24: [writing 5]

INT: So, you're doing the same method as you did with the other one. And then what about if I do, what if there's something afterwards? So, this is Question 5. What if I have minus four x to the power of four minus minus six x to the power of four minus six. [$-4x^4 - (-6x^4) - 6$]

L24: [writing $24x^4 - 6$]

INT: Just explain again how did you know to do positive.

L24: OK, this is like we used to do. [Drawing a grid with plus and minus signs.] It helps you to remember like a positive and a negative is a negative, and ya…

INT: OK, but in this case, we've got three negatives. We've got a minus four and then we've got a minus minus, then the number.

L24: So, so wait then it would it be like … would you actually subtract this from the four minus six.

INT: Well, what do you think? Because we've got two positives, what do you think the two positives are doing?

L24: Two positives?

INT: I mean negative. Sorry! We've got negative, negative. So, is it minus, minus or do you think it's a bracket in between? If you're doing minus, minus, like subtracting a negative, then we would normally change it to a positive. And if you have multiplying negatives, you also change it to a positive, right? But in this case, we had something and then we had a minus and then something else, and the something else in brackets is negative. And then you multiply, even though there's a minus separating the

L24: Ohh.

INT: So now you said for this one you would multiply, but you're multiplying a negative then by a negative, and there's another negative, so there must be a negative then that you're ignoring. If you're doing minus minus gives you plus because then there would be another minus and minus and plus gives you minus or let's say these brackets weren't there. Let's say it was just minus four x to the four minus six x to the four. How would you simplify that?

L24: [writing $-4x^4 - 6x^4 = -10x^4$]

INT: OK. So, are you adding the four and the six now?

L24: Uh-um.

INT: How do you know that you want to keep the negative?

L24: Because it's like three of them.

INT: OK. So, so, so, so it would be negative negative? … Let's do it again on the number line, let's say we have minus four minus minus six.

L24: Yeah, I added them.

INT: So you're saying minus ten. And then you're not multiplying these two. Is that because there's no bracket?

L24: No, it's not the same.

INT: They're not like terms. So, normally when you see an expression with brackets in it, if there's a symbol in front of the brackets, will you multiply that symbol into the brackets and then multiply the number?

L24: Ya.

INT: Ya. OK, I think there's another question. This also has brackets in it, so I'm just curious about this. It's x minus five minus x minus five. [$(x-5)-(x-5)$] Now again we're working with brackets. So, now I'm curious: if you see a bracket and then a symbol and then a bracket, are you wanting to multiply? How do you want to deal with this negative symbol?

L24: [writing $x^2 - 5x - 5x$]

INT: So initially I'm seeing you're going to multiply, is that right? You wanted to use the FOIL method.

L24: Uh-um.

INT: And then you're multiplying. OK, so first, outer, inner, last?

L24: Ya.

INT: You've done x squared minus five x minus five x and then another minus five x. OK, so you multiply these two. This, I'm assuming comes from those two. What about the minus five and the minus five?

L24: Ooooh, Oh, yeah! [writing $x^2 - 5x - 5x - 25$]

INT: OK, so I'm curious if you're doing anything with this negative symbol? Or are you just looking at the two brackets and not using the negative symbol?

L24: Ya, just looking at the brackets.

INT: OK. So, also with the minus twenty-five, how do you know it's negative?

L24: Because the five was negative and it's negative.

INT: But then in this one you said negative and negative makes positive.

L24: Oh, yeah. [changing to $+25$]

INT: So, you're changing it. So, basically we're doing the same thing as if it was written like this, right? So can we just ignore this minus?

L24: I think so.

INT: And what if I wrote it differently? What if I wrote it like this without the brackets around the first ones? How would you simplify that? [$x - 5 - (x - 5)$]

L24: I would multiply this into that.

INT: Would you multiply just the five or would you multiply the whole expression?

L24: The expression.

INT: OK, whole expression. So, you did the same thing?

L24: Uh-um.

INT: OK. Let me do one more. What if we didn't have two terms? What if we had three terms like this: eight minus seven minus minus seven. [$8 - 7 - (-7)$]

L24: [writes $-1 - (-7) = 7$]

INT: I see you're using the same method here, simplifying those two as you did in this one. OK, I think we can stop there.

Appendix 12 Computations involving superscript objects

None of the task items required learners to multiply terms together or to do any form of exponentiation[44], yet there were seven learners[45] who resorted to taking superscript objects as the arguments of transformations, like addition and/or subtraction (see Figure 54).

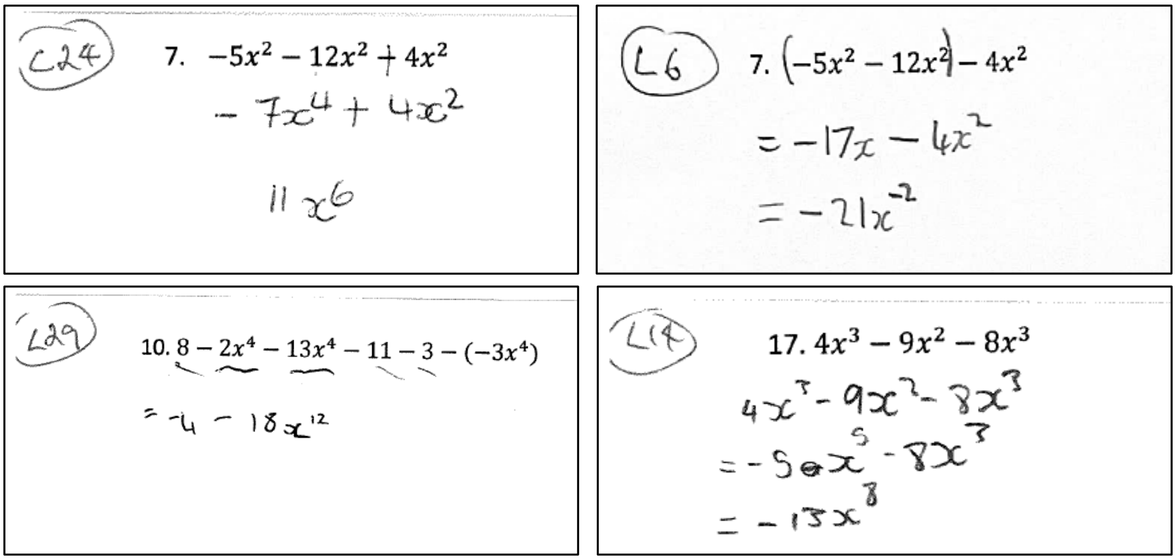


Figure 54: Examples of addition and/or subtraction of superscripts

The solutions in Figure 54 demonstrate a clear distinction between numbers as numeral objects and numbers as superscript objects, but the meanings ascribed to the superscript objects (and possibly the numeral objects) are not aligned with their mathematical denotation. In their attempts to simplify the expressions, learners seemingly formulated their own operations specifically for superscript objects. For example, in L14, L24 and L29's solution above the final superscript object is the sum of the superscript objects in the expression. Only L6 was prompted by the negative symbol to subtract superscript objects. Their solution to Q7 is represented diagrammatically in Figure 55 (p.419). When asked why they weren't including the superscript object /2/ in the term $-17x$, L6 responded: "Because I minused it." It seems that the superscript object Φ of the term $-17x$ was read as 0 or, more likely, "nothing." For L6 then, the negative symbol was taken as applying to both numeral objects and superscript objects, unlike for L14, L24 and L29 who applied it only to numeral objects. Notably, in their

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[44] Variables with superscripts were included in the task precisely to see if they would elicit any non-standard computations.

[45] L2, L6, L14, L20, L24, L28 and L29.

follow-up interview, L6 no longer manipulated superscripts objects, having adapted their approach.

$$\begin{tikzcd}
 & & /-5x^2-12x^2-4x^2/ \arrow[dl, "\mathrm{SUN}"] \arrow[ddr, "\mathrm{SUN}"] & \\
 & /-5x^2-12x^2/ \arrow[dl, "\mathrm{SEL}_{\sigma n}"'] \arrow[ddd, "\mathrm{SEL}_l"] \arrow[dr, "\mathrm{SEL}_s"'] & & \\
(/-5/,/-12/) \arrow[d, "\mathrm{NUM}"'] & & (/2/,/2/) \arrow[d, "\mathrm{NUM}"] & /-4x^2/ \arrow[dddd, "i_x"] \\
(-5,-12) \arrow[dd, "\mathrm{SSK}"'] & & 2,2 \arrow[d, "-"'] & \\
 & /x/ \arrow[dd, "i_x"] & 0 \arrow[d, "\mathrm{VOID}"] & \\
(-17) \arrow[d, "\mathrm{STR}"'] & & \Phi \arrow[dl, "i_x"'] & \\
/-17/ \arrow[r, "i_x"'] & /-17x/ \arrow[rr, "i_x"'] & & /-17x-4x^2/ \arrow[ddlll, "\mathrm{SEL}_{\sigma n}"'] \arrow[ddl, "\mathrm{SEL}_l"'] \arrow[d, "\mathrm{SEL}_s"] \\
 & & & (\Phi,/2/) \arrow[d, "\mathrm{NUM}"] \\
(/-17/,/-4/) \arrow[dd, "\mathrm{SSK}"'] & & /x/ \arrow[ddd] & 0,2 \arrow[d, "-"] \\
 & & & -2 \arrow[d, "\mathrm{STR}"] \\
/-21/ \arrow[drr] & & & /-2/ \arrow[dl] \\
 & & /-21x^{-2}/ &
\end{tikzcd}$$

Figure 55: Diagrammatic representation of computations likely used by L6 in Q7

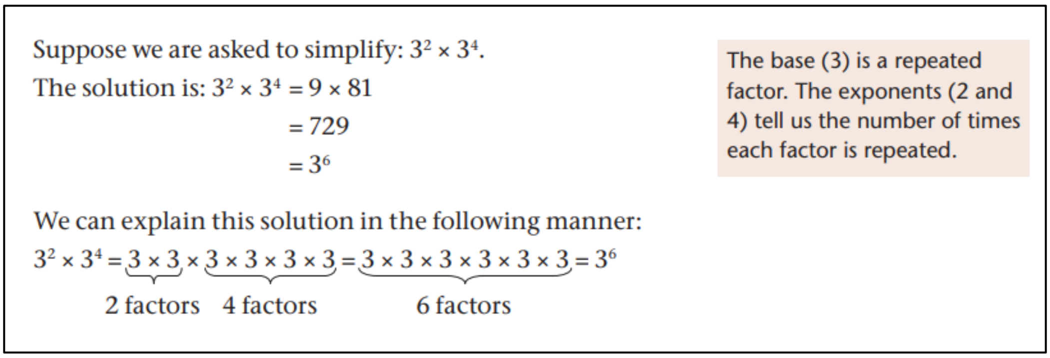
Left-to-right (or more generally “one-directional”) reading bias seemingly impeded on some learners’ computations for expressions containing superscript objects, which they tried to simplify by operating on the superscripts as independent to their bases. Given the typography of superscript objects—written smaller than, to the right of, and slightly raised above their bases—learners may identify superscript objects as belonging to a distinct *spatial level*, as shown in Figure 56. The *superscript level* may accommodate computations independently of those accommodated at the *base level*.

$$\text{base level}^{\text{superscript level}}$$

Figure 56: Spatial levels

Several learners read terms like x^2 as "x two," indicating a nescience regarding exponentiation. Corroborating this argument is L14's simplification of $4x^3 - 9x^2 - 8x^3$ to $-13x^8$ in which they added the superscripts despite doing subtraction at the base level. The choice of addition appears to be determined by the lack of sign objects at the superscript level, whilst the negative symbol at base level elicited subtraction of numeral objects. Only L6 was prompted to subtract superscripts by the presence of a negative symbol at base level. In both cases, learners were working at different levels notwithstanding the rules of exponents.

With exponents typically being introduced by repeated multiplication in school mathematics, as seen in Figure 57, it is easy for learners to think of whole number exponents as cardinal values indicating how many copies of the base object there are in the product.



Suppose we are asked to simplify: $3^2 \times 3^4$.

The solution is: $3^2 \times 3^4 = 9 \times 81$

$= 729$

$= 3^6$

The base (3) is a repeated factor. The exponents (2 and 4) tell us the number of times each factor is repeated.

We can explain this solution in the following manner:

$$3^2 \times 3^4 = \underbrace{3 \times 3}_{\text{2 factors}} \times \underbrace{3 \times 3 \times 3 \times 3}_{\text{4 factors}} = \underbrace{3 \times 3 \times 3 \times 3 \times 3 \times 3}_{\text{6 factors}} = 3^6$$

Figure 57: Extract from section on exponents in Grade 8 textbook (DBE, 2017, p.61)

The term x^n may therefore be thought of as a collection of n x's rather than as the product of x with itself n times (see Davis et al., 2022, pp.14-16). As L14's solution above suggests, learners may read an expression like $x^2 + x^2$ as two collections of two x's each which they can simplify to a collection of four x's denoted by x^4. The term x^2 would in that case represent the same collection, i.e. $(/x/,/x/)$, as the term $2x$, which would explain the assumption that $x^2 + x^2 = x^4$, since $2x + 2x = 4x$. Alternatively, the reading "x two" may generate a collection $(/x/,/^2/)$ instead which can easily be mapped to the collection $(/2/,/x/)$ generated by parsing $2x$ according to types. If an expression is read as a collection of objects rather than a composition of operations, then the relations between different types of objects, like superscript and base or coefficient and variable, are likely to be ignored.

Appendix 13 Considering a preference for monoidal structures

On numerous occasions, learners did manipulations that were seemingly reversible, but when presented with an expression in which one such reverse manipulation could be used, they instead opted for an entirely different one. For example, learners used the transformation $ax - x \mapsto a$ despite being aware that $ax + x = (a+1)x$. These learners knew that *adding* the term x meant that the coefficient of the other term ax would increase by 1, yet they did not recognise that therefore *subtracting* the term x should *decrease* the coefficient a by 1. It seems that when learners are working at Level 1 the notion of the "invisible 1" is at play, whereas at Level 2 the parsing of expressions by type prompts them to read a numeral object Φ as *nothing* rather than /1/. The lack of a written coefficient for the subtrahend term appears to have triggered the removal of the letter object. The expression $ax - x$ can be read as "a, x, minus x" and if one thinks of $/a/$ and $/x/$ as distinct objects that are simply placed next to one another, then taking away the $/x/$ does leave one with only $/a/$. Clearly then $/a/$ is not thought of as related to $/x/$ in any way. By contrast, the transformation $ax + x \mapsto (a+1)x$ portrays a different reading by which a *is* related to x, suggesting that it is also the operator symbol which determined the transformation learners chose when presented with an expression of the form $ax \pm x$.

For example, two learners (L10 and L13) who used the transformation $ax - x \mapsto ax$ also used the transformation $ax + x \mapsto (a+1)x$ elsewhere. One would think that if learners assume that subtracting x from ax makes no difference and produces ax, i.e. $ax \xrightarrow{-x} ax$, then they would automatically assume the reverse too, that adding x to ax makes no difference, i.e. $ax \xrightarrow{+x} ax$. Since $-x$ has no numeral object, learners opt instead for using the operator object on the letter objects by "subtracting" the $/x/$ from $/ax/$ leaving $/a/$ which seems a suitable answer. Although $+x$ does not have a numeral object either, learners are aware that $/axx/$ is not a suitable answer so they transition back to Level 1 and add 1 to a for a suitable solution $(a+1)x$. Some learners of course opted for $/ax^2/$, with the two x's being indicated by the superscript /2/. A similar argument can be made with regards to the transformation (iv) $ax - x \mapsto a$ discussed in Section 7.5 and Section 7.6, which some learners used despite recognising that $ax + x = (a+1)x$. Although learners seemed aware of subtraction as the inverse operation to addition at an elementary level, they did not seem aware that adding an object should have the inverse effect to subtracting the same object.

Reversibility is not common in real life; there are few processes that can truly be undone to return something to its original state. Neumann (2017, p.25) explains that a "process of computation is defined as reversible if the input can be restored from the output." Note that this distinguishes a reversible computation as one for which there exists a *distinct* reverse computation, i.e. an inverse. Given the irreversibility of most things in real life, the notion of mathematical inverses is not one that can easily be assimilated to existing mental structures that relate to everyday life. Neumann (2017, p.28) highlights that the irreversible nature of natural cognitive systems makes it difficult to model them mathematically because mathematical models are typically reversible.

In Chapter 6 we saw that some learners struggled to recognise additive inverses when letter objects were involved, making use of transformations like $ax - ax \mapsto a$ and $ax - ax \mapsto x$, where only one type of object was cancelled out, and attempting to use multiplication when additive inverses were in brackets (e.g. Q5). These and other computations that learners did suggest that learners privilege structures that do not require reversibility. A mathematical structure which need not be reversible that may indeed model the structures learners rely on in their computations is a *monoid*.

> A *monoid* is a set M equipped with an associative[46] binary operation $*: M \times M \mapsto M$ and an identity element[47] $1_M \in M$, denoted $(M, *, 1_M)$.

For example, the natural numbers under addition constitute a monoid, $(\mathbb{N}, +, 0)$, since for any $a, b, c \in \mathbb{N}$ we have $(a + b) + c = a + (b + c)$, i.e. associativity, and $a + 0 = a = 0 + a$, i.e. the identity element 0. The core domain of number equips humans with an innate capacity to grasp natural number concepts (Feigenson et al., 2004; Gelman, 2015; Spelke, 2017), which naturally puts the monoid $(\mathbb{N}, +, 0)$ in a privileged position as opposed to, for example, the set of real numbers under addition—which requires learners to have constructed mental mechanisms to deal with negative numbers—or the set of natural numbers under subtraction

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[46] Associativity means that for every $a, b, c \in M$, $(a * b) * c = a * (b * c)$, i.e. the order in which $*$ is applied does not matter.

[47] The identity element 1_M has the property that for every $a \in M$, $1_M * a = a = a * 1_M$.

which does not satisfy the property of closure. It is evident from the data that learners favoured addition over subtraction, particularly when dealing with letter objects with which they were less comfortable. This preference for addition may be explained by Chomsky's (2015) argument that Merge, which puts things together, is fundamental to computation.

The transformations mentioned above indicate that learners were comfortable with the idea that adding x to ax will increase ax, but they did not recognise that this reasoning should require subtracting x to decrease ax in the same way. The transformations $-x$ and $+x$ were not recognised as inverses, rather they were considered unrelated. This unawareness of inverse relations supports the argument here that learners' computations naturally favour monoidal structures as these are most compatible with their core domain mechanisms, like those relating to number.

Appendix 14 A look into equivalence classes

When simplifying an expression, learners typically begin by identifying like terms and parsing the expression accordingly. Let T be the set of terms constituting an expression, then parsing the expression according to like terms can be described mathematically as generating equivalence classes in T with respect to the equivalence relation[48] "is like to." For example, let the expression $11x - 8x - 3$ be represented by set $T = \{11x, -8x, -3\}$. The expression contains two x-terms and one constant. The equivalence classes in T with respect to the relation "is like to" are therefore as follows:

[x-terms]$= \{11x, -8x\}$

[constants]$= \{-3\}$

Notice that the equivalence classes constitute a partition of T[49]. The simplest form of the expression will contain only one of each type of term. To simplify like terms, such that there is only one of each type, the terms belonging to the same equivalence class can be added together:

x-term: $11x - 8x = 3x$

constant: -3

The final expression consists of the sum of these final terms, namely $3x - 3$. The equivalence relation "is like to" is applied to expressions when learners do Level 1 simplification.

Many learners went on to do Level 2 simplification in their attempts to deal with lack of closure, identifying like objects in the absence of like terms. Let O be the set of objects (e.g. numeral objects, letter objects) constituting an expression. Parsing an expression by types of objects therefore generates equivalence classes in O with respect to the equivalence relation "is the same type as." Let us denote the equivalence class for each type of object[50] as follows: $[\sigma]$ sign, $[n]$ numeral, $[l]$ letter, and $[s]$ superscript. The expression $3x - x$ can be represented by the set $O = \{/3/, /x/, /-/, /x/\}$ (assuming that the negative symbol is read as

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[48] See Cheng (2023, pp.82-94) for an interesting and easy-to-follow chapter on equivalence relations.

[49] A partition of T is a set of one or more non-empty subsets of T such that every element of T is in exactly one subset.

[50] Operator objects are not included here because they can be read as sign objects.

a sign object rather than an operator object), in which we have the following equivalence classes with respect to the equivalence relation “is the same type as”:

$$[\sigma] = \{-\}$$

$$[n] = \{3\}$$

$$[l] = \{x, x\}$$

At Level 2, learners wanted to simplify the expression such that there was only one of each type of object, similarly to the terms at Level 1. As discussed above, where an expression contained an operator object and only one numeral object, i.e. $[n]$ had only one element, several learners chose to “subtract” letter objects in their attempts to deal with lack of closure, i.e. turning to $[l]$ for the arguments of subtraction. At each level of simplification, learners wanted to reduce the expression such that every equivalence class had only one element. We see then that, fundamentally, learners were doing the same thing at both Level 1 and Level 2. Transitioning to Level 2, however, meant that the value of their final answer was no longer equivalent to the value of the original expression. This highlights the importance of developing learners’ understanding of the mathematical denotation of expressions and the components they are comprised of. Learners’ choices of computations demonstrated that they were not always aware of how transformations changed the value of the expressions.