

Construal Level Theory and Mathematics Education



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Abstract

A common complaint of mathematics students is that mathematics is highly abstract. Students often find it difficult to attach meaning to the mathematical concepts they are expected to master. In addition to coming to grips with the abstract nature of the subject, mathematical proficiency requires engagement at a more concrete level. Students must be able to perform step by step algorithmic procedures, detailed algebraic manipulations and master new symbol systems. Mathematical competence often requires thinking at high and low levels of abstraction almost simultaneously and this creates a tension which lies at the core of mathematics education. This tension has been addressed in the literature on procedural versus conceptual approaches to mathematics education and in the literature on cognitive and metacognitive mathematical demands. Construal level theory, and to a lesser extent dual process theory, are theories in cognitive and social psychology which provide a lens through which the difficulties of reasoning at multiple levels of abstraction can be viewed. Construal level theory posits that thinking about psychologically distant objects influences the extent to which we view possibly unrelated objects abstractly or concretely. Psychological distance and abstract thought are cognitively linked together and make up *Far Mode* thinking. Psychological proximity and concrete thinking are intrinsically linked together to form *Near Mode* thinking. It is argued that construal level theory forms a useful framework for interpreting much mathematics education research as well as helping to explain the difficulties students experience in implementing problem solving heuristic strategies. Evidence is presented suggesting that priming mathematics students to adopt either a Near or Far mental mode has an impact on their performance in solving conceptually challenging mathematical problems.

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1. Introduction

1.1. Overview

Mathematics is often regarded by students as a difficult subject to study. It is the most abstract subject that most students will ever encounter and the abstract, decontextualized nature of mathematics is often necessarily far removed from every day, common sense experience of the real world. As a result it resists intuitive, common sense understanding. Adding to this difficulty is the fact that mathematical proficiency also requires students to master content at a more concrete level. Students must learn algorithms, algebraic procedures and detailed symbolic manipulation. Conceptual, abstract understanding and concrete, procedural fluency are often required simultaneously in order to successfully solve a problem.

This tension inherent in mathematical proficiency has been addressed in multiple ways in the mathematics education literature, of which two are of particular relevance. There is a literature focusing on procedural versus conceptual approaches in mathematics education, and another closely related area of research which focuses on cognition and metacognition. In both cases, it is recognised that it is not possible to fully separate the different modes of thinking. Both are required in varying degrees for effective problem solving.

As a general matter, increased procedural proficiency benefits conceptual understanding and improved conceptual understanding helps in the development of procedural fluency. This does not imply, however, that these crossover effects are symmetrical in their importance and there is debate about where the emphasis in teaching should lie (Hiebert & Lefevre, 1986). Some evidence suggests that in the case of certain mathematical concepts, procedural fluency precedes conceptual understanding of the concept, but in other cases the reverse is true (Rittle-Johnson & Siegler, 1998).

Research into the problem solving efforts of typical mathematics students shows that while students often have more than enough appropriate mathematical content knowledge, they frequently fail to bring this knowledge to bear in novel problem situations (Schoenfeld, 1985). This failure to apply previously learned knowledge in new situations represents a failure of metacognition or a failure to successfully link lower order knowledge structures to higher level goal or problem states.

Studies concerning procedural and conceptual knowledge on the one hand and cognition and metacognition on the other both involve investigation into when different levels of abstraction are appropriate in particular situations. However, being conscious of the desired level of abstraction does not necessarily mean it is a simple matter to instil this awareness in students. There are thus both descriptive and prescriptive elements to this research; what students should be encouraged to do and how best to get them to do it

What is lacking in the research literature is an understanding of how the human mind is naturally predisposed to deal with concrete and abstract thinking and what causes us to instinctively adopt abstract or concrete mind-sets in different situations.

The capabilities of conscious, language based reasoning and abstract thought are relatively recent evolutionary developments (Evans, 2003), while more concrete and locally based cognition has been a feature of our development for millions of years. Several branches of psychology deal with this distinction and how these two modes of cognition interact. One example is *dual process theory* (Chaiken & Trope, 1999) in which the two types of cognition are broadly categorised as System 1 and System 2. Both types of cognition occur simultaneously, but in familiar or routine situations System 1 type cognition plays the dominant role. System 1 type cognition occurs automatically and almost effortlessly; it is intuitive and much of it is unconscious. System 1 type reasoning dates to human origins and is also prominent in other animals. System 2 is conscious, much slower and more effortful and is used when weighing evidence and in hypothetical reasoning. System 2 is typically implicated when System 1 type cognition breaks down and in important or unusual situations. System 2 reasoning is regarded by some theorists as being unique to humans and not shared by other animals (Evans, 2003), although there is growing recognition that some animals (chimps in particular) have several of the capacities associated with higher level System 2 cognition (Evans, 2008). Compared to System 1 cognition, System 2 is evolutionarily recent and has developed significantly since human origins. There has been a great deal of debate about the relative importance of each system in decision making under various circumstances (Chaiken & Trope, 1999; Haidt, 2001; Kahneman, 2011).

A lesser known but related branch of psychology known as *construal level theory* deals more specifically with the distinction between concrete and abstract thought. Construal level theory

suggests that various conceptions of *psychological distance* are associated with adopting either an abstract or a concrete cognitive stance (Trope & Liberman, 2003; Trope & Liberman, 2010). Thinking of physically distant objects or of events in the distant past or future causes us to think more abstractly about other objects and events even if the two are not related. Conversely, thinking about events or objects near in time or space causes us to adopt a more concrete mind-set. Two other dimensions of psychological distance have a similar effect; social distance and hypotheticality. Social distance refers to the degree of familiarity in social relations. Strangers are socially distant others and social distance can be increased with friends or family by adopting more formal modes of address. Hypotheticality concerns the plausibility or likelihood of possible scenarios; extremely unlikely events can be thought of as more psychologically distant.

To the extent that adopting an abstract or concrete mind-set is important in mathematical problem solving, construal level theory promises to be a useful tool in understanding why some approaches in mathematics education experience difficulties and also for helping students to adopt the desired level of abstraction.

A third substantial section of the mathematics education literature which can be viewed through the lens of construal level theory is the use of heuristic strategies in mathematical problem-solving. The use of heuristics or “rules of thumb” in mathematical problem-solving has seen a great deal of support in the mathematics education research literature, yet the classroom evidence suggests that students often find the simply-expressed strategies challenging to use in mathematical contexts (Schoenfeld, 1985, 2008). Considering the archetypal heuristic strategies from the viewpoint of construal level theory it seems apparent that different strategies require different construals, potentially at odds with the construal called for by the problem within which the heuristic strategy is being used.

1.2. Defining Abstraction

The current study concerns the relevance of construal level theory, a branch of cognitive and social psychology, for areas of mathematics education research investigating the need for subjects to apply different levels of abstraction in their mathematical reasoning. It is thus important be clear about how the term abstraction will be used and understood going forward.

Abstraction is used in a philosophical and mathematical sense, as a topic studied in metaphysics or ontology and it is also used in a psychological sense, as process that humans, and to a lesser degree other animals, perform naturally and automatically. In addition to automatic intuitive abstraction there are the distinctively human processes of abstraction such as the development of abstract scientific theories to understand and describe the natural world. The philosophical and psychological senses of abstraction are related because humans have to use their natural powers of abstraction to think, however imperfectly, about abstract mathematical objects and to be able to solve mathematical problems.

1.2.1. Mathematical and philosophical abstraction. Mathematics deals with abstract objects and processes and mathematical objects are often the products of these processes (Tall, 1991). Abstract objects, as opposed to concrete objects, have no tangible existence in the natural world. Philosophers debate whether abstract objects can meaningfully be said to exist but mathematicians take their existence for granted (Gowers, 2002) and so will we in this study. While all mathematical objects are equally abstract in the sense that they have no tangible existence, they also exist within a hierarchy and it makes sense to speak of superordinate and subordinate abstract objects. A number is purely abstract as is the notion of a function, which is defined as an ordered pair (Sfard, 1991). But since the ordered pairs of a function are often made up of numbers, numbers are inputs into functions and numbers thus exists at a lower level. In this sense, there are different levels of abstraction of mathematical objects. Pier Ferrari (2003) has argued that mathematical abstraction often involves generalisation and decontextualisation, but also the emergence new, intangible mathematical objects.

There is widespread agreement about the classification of some paradigmatic “objects” as being either abstract or concrete. For example numbers are regarded as purely abstract and physical objects like rocks are purely concrete (Lewis, 1986). There is less agreement however about the dividing line between the two cases, or even whether such a dividing line exists or can be specified with any precision. The letter “A” for example exists as a concept and sometimes as ink on paper arranged in a particular shape. Individual humans have a concrete biological existence but are typically thought of in more abstract terms as “persons” with various abstract qualities such as “kindness” or “intelligence”. The works of Shakespeare exist in physical form but in another sense they exist in a more enduring, timeless, abstract sense as well (Rosen, 2012).

1.2.2. Psychological abstraction. The process of abstraction involves generalising over objects or actions and classifying them into broader categories and is the basis of all reasoning (Wason, 1966). It typically involves decontextualisation and neglecting information deemed irrelevant in order to focus on more fundamental features. Jean Piaget identified three primary kinds of abstraction. *Empirical abstraction* derives from the identification of common properties of objects and allows for generalisation (Beth & Piaget, 1966). *Pseudo-empirical* abstraction refers to understanding the actions that can be performed on objects and the properties of those actions (Piaget, 1985; Dubinsky, 1991). *Reflective abstraction* occurs through mental actions on mental concepts. These actions themselves become new objects of thought (Piaget, 1972; Gray & Tall, 2007). All three psychological processes are important for developing an understanding of mathematical concepts and reflective abstraction is especially significant in advanced mathematics (Dubinsky, 1991).

Objects and events can be viewed in a great variety of different ways. A chair for example can be viewed as something to be sat upon, as firewood or, in the right set of circumstances, as a work of art. There are of course many other ways in which a particular chair can be viewed but every choice involves representing the chair at a particular level of abstraction. The selection of mental construal used to consider an object will typically be dictated by the motivations and goals for considering it in the first place. We employ different abstractions for different purposes and we employ abstraction automatically in all of our thinking and interactions with the world (Gray and Tall, 2007). Even visual perception requires abstraction. We automatically impose an interpretive apparatus on what we see, sometimes resulting in optical illusions such as seeing faces in the clouds or the Müller-Lyer illusion (Gregory, 1997). Of the various ways of viewing an object such as a chair, visual perception is a concrete, low level construal, whereas construing as chair as a work of art is a higher level abstraction.

In general, higher levels of abstraction involve neglecting more information and focusing on fewer core features of an object or event. Exactly what those core features are varies depending on the goal of the abstraction, and because each individual has different goals and motivations there is often great difficulty in generating agreement on what an appropriate abstraction should be. The fact that an abstraction serves some purpose implies that, in the words of Trope and Liberman (2010), “the process of abstraction involves not only a loss of

specific, idiosyncratic, and incidental information, but also ascription of new meaning deduced from stored knowledge and organized in structured representations.” (p. 441). One of the most important features of abstraction is that it allows us to compare different objects or events. Whether objects or events are similar to each other “depends on whether it has similar *relevant* features” (Hanson, 2008).

Actions and events also can be viewed more or less abstractly. Viewing an action abstractly typically involves considering the high level goal of the action. A concrete view of the action focuses more on how the action is performed. Broadly speaking, high level abstract construals consider why an action is performed and a low-level construal considers how it is performed (Trope and Liberman, 2010). A concrete object can have many abstractions depending on the purpose the abstraction is intended to serve. Conversely, an abstract object has many concrete representations. It is worth noting that often when we demonstrate our understanding of an abstract object or concept we are typically expected to present some sensible concrete representation of the object; but it is only one of many possible representations (Dreyfus, 1991). This demonstrates understanding, but even if this representation exists only in our thoughts it should not be confused with the abstraction itself (Sfard, 1991).

Much of science involves developing and systematising our natural ability to conceptualise things abstractly, as well as being an attempt to systematically categorise processes and objects as examples of more general concepts according to their properties. Both dogs and cats are classified into the more abstract category of mammals, which itself forms a subset of the still more abstract category of animals, and so on. In many areas of study it is notoriously difficult to unify events and objects within a widely agreed upon superordinate conceptual framework. The physical sciences such as biology, chemistry and physics have had more success, but mathematics is unique in the coherence and precision with which higher and lower order abstractions interact with one another. The concept of the number “3” is abstracted away from similar numbers of particular objects and this abstraction is universally applicable and agreed upon. This ability to abstract in a precise way enables us to separate powerful ideas from their concrete context in the here and now and apply them in novel circumstances and far away in space and time. This systematic use of abstraction is an enormously powerful tool for manipulating the world around us and shaping our future.

The limiting case of this formal process of abstraction is mathematics. Most sciences are involved in the project of developing abstract conceptual frameworks which are robust and generally applicable, but their subject matter is ultimately the stuff of the natural world. This is sometimes also the case in mathematics where concepts are readily applied to natural phenomena; number and calculus are examples. However many mathematical objects have no such real world analogue and are literally inconceivable. We can define them and learn the rules that govern these objects but we cannot visualise some concepts in the way we can visualise three apples to represent the abstract concept of the number “3”. Yet mathematicians are still capable of treating these inconceivable concepts as objects, and to “see” them in the mind’s eye in a way that is similar to seeing or visualising a concrete object (Sfard, 1991; Gray & Tall, 1994). There is a transition from viewing a mathematical object as a process or expression of generality to viewing it as an object which can be manipulated as a whole. In the words John Mason, abstraction refers to

An extremely brief moment which happens in the twinkling of an eye; a delicate shift of attention from seeing an expression as an expression of generality, to seeing the expression as an object or property. Thus, abstracting lies between the expression of generality and the manipulation of that expression while, for example, constructing a convincing argument.

(Mason, 1989, p. 2)

The language of the above quotation captures well the problem of mathematical abstraction. There is a gap between the type of abstraction we do naturally as humans and the kind we expect of mathematics students. This gap is elusive and subtle and often missed by both teachers, who take this shift for granted, and students who make this “delicate shift” too seldom.

Finally, it is important to note that thinking about abstract objects does not necessarily mean thinking in a highly abstract way. The numbers 3 and i are purely abstract objects, but once we have developed proficiency with using and manipulating these objects, they will be viewed at a low level of cognitive abstraction because they are merely building blocks for thinking about higher order objects or solving problems which require thinking about relationships at a much higher level of abstraction.

1.3. Research Question

Several branches of mathematics education research and the branch of psychology known as construal level theory are concerned with reasoning at different levels of abstraction. There has been little attempt to integrate the research programs into a coherent whole. This thesis is an attempt to explore possibilities for such integration. The research question that will be addressed in this study comes in two parts:

- To what extent can construal level theory and dual process theory be used as conceptual frameworks to inform existing branches of mathematics education research?
- Does the level of mental abstraction induced by manipulating psychological distance have an impact on mathematical performance? If level of mental abstraction does influence mathematical performance, what is the nature of this influence?

1.4. Structure of Dissertation

There is very little research investigating the relevance of construal level theory to established areas of mathematics education research. Chapter 2 of this dissertation reviews some of the existing literature concerning construal level theory, as well as dual process theory, which is closely related to construal level theory but is a more established branch of psychology with a larger literature. Chapter 2 also reviews three areas of the mathematics education research literature which are plausibly related to construal level theory and dual process theory. These three branches are research into procedural and conceptual knowledge in mathematics, research into cognitive and metacognitive mathematical knowledge and research into mathematical problem solving heuristics. Overviews of these lines of enquiry will serve to make the case that their apparent connections are more than just skin deep and that construal level theory in particular can inform and deepen understanding of mathematical learning. Chapter 2 lays the foundation for addressing the first part of the research question. Chapter 3 describes the investigation of the second part of the research question. First year engineering students enrolled in the first year engineering mathematics course at the University of Cape Town were primed to adopt either Near or Far mental modes. They then wrote a two short tests and their performance was assessed. The first test attempted to evaluate the students' conceptual understanding while the second test focused primarily on procedural fluency. Data from these two interventions is analysed and a positive hypothesis concerning construal level and performance is tested in a third data collection.

Chapter 4 will address the first part of the research question directly by exploring the connections between, and the potential application of, dual process theory and construal level theory to areas of mathematics education discussed in Chapter 2. In addition to suggesting that there is significant scope for cross-disciplinary integration of these areas of research, it is argued that construal level theory sheds light on some of reasons for student difficulty with mathematics. Finally, an explanation for the results of Chapter 3 is tentatively proposed. Chapter 5 provides conclusions to the current dissertation and suggests that there is significant scope for further research into the effects of construal level on mathematical performance.

2. Literature Review

The purpose of this chapter is to introduce the reader to several distinct but related areas of research. The first two sections describe two areas of research in cognitive and social psychology. The final three sections describe three branches of mathematics education research. Each research literature will be discussed largely separately in this chapter while the connections and applications between the different areas of research will be discussed in detail in Chapter 4.

2.1. Dual Process Theory

The idea that humans employ two distinct and often conflicting types of reasoning has been around for thousands of years. This conflict is familiar to anyone who has learned to drive, which initially demands intense concentration but later can be done without much focused attention, or has rationalised their desire to eat an extra slice of cake despite an explicit goal of eating more healthily. There is often a large difference between our conscious reasoning and goals on the one hand and more immediate intuitive and emotional concerns on the other.

The distinction has been framed in a variety of ways. Plato and the Stoics framed the two natures as a conflict between reason and emotion. Emotion was viewed as a source of error to be corrected and mastered by reason. Hume reversed the dichotomy by placing emotion at the centre of his system and famously claimed, “we speak not strictly and philosophically when we talk of the combat of passion and of reason. Reason is, and ought only to be the slave of the passions...” (Hume, 2003, p. 217). The contrast has also been noted in the context of mathematics. The great mathematician Henri Poincaré referred to two different kinds of mathematical minds (Poincaré, 1913) and mathematicians often describe their mathematical thinking in highly metaphorical and intuitive terms as well as formal and deductive (Sfard, 1994).

The dichotomy has also been stated in terms of implicit versus explicit reasoning (Evans, 2003) or intuitive (or experiential) versus analytic thinking (Leron & Hazzan, 2006; Amsel et al., 2008). Philosophers and psychologists continue to differ in their views on the relative importance and reliability of the different systems and on how they interact (Haidt, 2001; Kahneman, 2011), but despite some researchers arguing that the mind is better considered

within a single system framework (Osman, 2004) there is general agreement that some version of a dual process model is an appropriate model of the mind.

More recently, cognitive and social psychologists have investigated this distinction in much more depth resulting in the development of a large literature known as dual process theory (Chaiken & Trope, 1999; Evans, 2008). There are several versions of the theory, but common to all of them is the distinction between slow, effortful and analytic reasoning on the one hand and more automatic intuitive and emotional cognition on the other. In addition to the on-going theoretical work, there is now a large body of empirical evidence that there are two distinct cognitive systems, each originating in different parts of the brain and with distinct evolutionary origins. The terminology most commonly used to describe the two modes of thinking classifies them as System 1 and System 2 cognition (Stanovich & West, 2000; Kahneman, 2011).

Using the definitions adopted by Daniel Kahneman, System 1 type cognition “operates automatically and quickly, with little or no effort and no sense of voluntary control.” (Kahneman, 2011, p.20). Despite the terminology, System 1 is not a single unified system, but rather a collection of sub-systems with the aforementioned characteristics. To avoid the potential confusion caused by the implication that all System 1 cognition originates in the same part of the brain, some researchers have suggested using the term “Type 1” cognition to indicate that it is a type of cognition rather than a unified cognitive system (Stanovich, 2012; Evans & Stanovich, 2013). Some of these sub-systems are instinctive and do not require conscious control, for example breathing and blinking. Other systems comprise of domain-specific knowledge acquired through training. However the learning mechanisms for developing this mode of cognition are domain-general. In other words the same learning mechanism is capable of producing an almost endless variety of domain-specific System 1 type cognition (Evans, 2003). Some familiar examples of this type of System 1 cognition are driving a car or a skilled chess player playing against a novice opponent. As the above examples suggest, many System 1 processes occur in parallel and with a large amount of autonomy. Given that System 1 cognition operates in parallel and quickly, it has a very high capacity and operates largely independently of working memory. System 1 makes extensive use of heuristics and often attempts to replace a complex problem with a simple problem which can be dealt with by a System 1 process. System 1 offers the first cognitive response to any input and is holistic and context dependant in nature. System 1 is also more social and

emotional in nature. In benign environments System 1 is typically very reliable but is prone to bias and breakdowns in unfamiliar or hostile cognitive environments.

System 2 on the other hand, “allocates attention to the effortful mental activities that demand it, including complex computations. The operations of System 2 are often associated with the subjective experience of agency, choice, and concentration.” (Kahneman, 2011, p. 21).

Compared to System 1, System 2 is slow, effortful and has a much lower capacity. Due to its demanding nature, System 2 thinking operates serially and not in parallel. System 2 is responsible for rule based, systematic, step by step reasoning and is capable of running hypothetical simulations. System 2 thinking is responsible for playing an inhibiting role where appropriate on the quickly generated reactions of System 1. Chess can serve as an example of this type of interaction between System 1 and 2 thinking. When chess pieces are arranged in a familiar position (typically near the beginning of a game), a skilled player can safely let his moves be guided by System 1. However, when the pieces are in an unusual arrangement, or when playing a skilled opponent, System 2 has to take over and potentially reject intuitively appealing moves. A player will have to simulate various paths opened up by different moves and evaluate their merits before coming to a decision. Some researchers have suggested that System 2 thinking is primarily concerned with abstract reasoning, but others have warned against this (Evans, 2008). While it is true that some types of abstract thinking can only occur in System 2, much effortful, sequential reasoning is not especially abstract in nature. Additionally, depending on how the term is used, much of the heuristic reasoning of System 1 could be considered to involve abstraction. Since abstraction is central to the present study this possibility is considered in more detail.

Abstract thinking – which is not synonymous with analytic thinking – is usually considered to be a product of System 2 (Evans, 2003). The terms *abstract*, *abstraction* and *abstract thinking* need to be dealt with carefully however. The word abstraction is not ambiguous in the sense that it is typically used to imply decontextualisation and generalisation in order to highlight core features of an object (Ferrari, 2003), but the term is often used with a lack of precision and this lack of precision can lead to confusion. Abstract thinking does not imply difficult scientific or mathematical thinking.

Many types of cognition can be said to involve abstraction and much of this cognition is in fact a product of System 1 and not System 2. A well known example of reasoning that is both

abstract and automatic is of the ease with which humans adopt what Daniel Dennett has termed the *intentional stance* (Dennett, 1989). The intentional stance refers to our tendency to treat objects as agents; we can think that a thermostat “wants” to keep a room at a particular temperature and a chess program “wants” to defeat an opponent. Adopting the intentional stance is considerably more abstract than adopting the *physical* or *design stance*. When considering a chess playing program it is usually irrelevant and infeasible to consider the electrons in the computer or the complicated algorithms that generate particular moves. It is more appropriate to consider what moves a competent opponent would choose. Using the intentional stance “is *like* an idealized model in science - maximally abstract and stripped down to the essentials.” (Dennett, 2013, p. 76). Adopting the abstract, intentional stance is an automatic, intuitive mode of human cognition and is even adopted by babies as young as 12 months of age (Gergely, Nádasdy, Csibra, & Bíró, 1995).

Abstract concepts are often defined as apart from concrete existence, so abstract concepts in this sense certainly exist in mathematics. Negative numbers are a familiar example, but a more pure example is that of the number i , which is defined to be the square root of -1 . Unlike many other mathematical objects, the number 5 for example, there is literally no concrete real world object that can serve as an example to help understand what it “really means”. Once a person has developed some familiarity with imaginary numbers, however, basic arithmetic will not require System 2. In much the same way that $2 + 2$ is calculated automatically, so is $i + i$. This calculation is performed by System 1, no matter how abstract the mathematical object; and though this calculation is an instance of thinking about abstract objects it probably does not count as an example of abstract reasoning.

Abstraction is also used in a different sense where abstractions of objects or events are formed by reducing the information content of an object or concept while retaining the core information relevant for a particular purpose. Thus objects or activities can have many different abstractions depending on the goal of the abstraction. For example, the activity of playing tennis can be described in many different ways. It could be described as hitting a yellow ball, or having fun (Trope & Liberman, 2010) . Each of these descriptions exists at a different level of abstraction, with ball hitting being a low level description and having fun being a high level one because the detail of the activity is omitted; there are more social activities that could be classified as having fun than activities involving hitting yellow balls. Much abstract reasoning involves recognising that different objects or events have similar

features in common and can be recognised as being part of a group. Many different types of animal are part of the same group known as mammals for instance. Abstraction of this sort lies at the heart of the scientific method and can have considerable predictive power, as the removal of situation specific context helps to determine how the concept would operate in quite different contexts. This type of systematic use of abstraction is a product of System 2 and much of mathematics involves a more extreme version of this type of abstraction.

The use of abstraction in this looser sense often occurs very quickly and easily and is properly seen as being performed by System 1. In fact, it is something that many animals do; herbivores would act the same way if confronted by a tiger as a lion – even if lions were the predator that they had coevolved with – since tigers share many salient features with lions. This is an example of an automatic heuristic process and is not of systematic conscious abstract reasoning. Systematic, cognitively effortful abstract reasoning is often what is meant by abstract thought, but the more automatic processes are also examples of abstraction. Abstraction of this automatic sort is so ubiquitous in cognition that instances of System 1 abstraction vastly outnumber instances of System 2 abstraction. Abstraction of both types is important for mathematical proficiency and this distinction will be explored further.

2.1.1. Dividing System 2 in two. One of the pioneers (and primary critics) of dual-process theory, Keith Stanovich, has proposed that there are two types of reasoning typically classified as being a product of System 2 that are distinct enough that it would be useful to classify them separately. Stanovich has proposed what he calls the Tripartite Model of Mind. In this proposed framework System 1 retains the properties already discussed while System 2 is split into two parts according to the two primary roles it plays in “successful” cognition. The first role that System 2 plays is to inhibit the outputs suggested by System 1 when those outputs are inappropriate, as is often the case in novel or hostile environments. The second role is to generate alternative responses generated by hypothetical reasoning and cognitive simulation. In other words, System 2 has to monitor the automatic responses of System 1, override them where necessary and decide what type of cognitive process is likely to generate a better response; it then subsequently needs to carry out the reasoning required to produce the desired response. The cognitive work required to carry out hypothetical simulations is very taxing and suggests a reason why System 2 thinking is primarily serial in nature. Stanovich proposed that the part of System 2 responsible for monitoring and overriding

System 1 and deciding on a new process be termed the *reflective mind*, while the System 2 cognition responsible for the simulating be termed the *algorithmic mind* (Stanovich, 2012). Although this new model is new at time of writing and has not gained the widespread acceptance enjoyed by the more established dual-process theory it does present a promising way of conceptualising abstract reasoning which is the focus of the present study. The reflective mind is responsible for comparing and adjudicating between alternatives suggesting potentially fruitful ways forward. These tasks are primarily abstract in nature. The algorithmic mind is responsible for implementing complex algorithmic, step by step procedures suggested by the reflective mind. This process is extremely cognitively demanding but is typically much less abstract. This division of System 2 into two parts will be significant in the following chapters.

2.2. Construal Level Theory

An emerging branch of psychology which is related to dual process theory is construal level theory (Trope & Liberman, 2010). Construal level theory proposes that the subjective perception of distance is related to the level of abstraction that a person will adopt. If an object or event is perceived to be far away we are automatically primed to adopt an abstract mental construal of that object or event. In a similar way, if an object is perceived as being nearby then we automatically adopt a concrete mental construal of it. Because level of mental construal and perceived subjective distance are both products of our cognitive process, and they are intrinsically linked in our cognition, we can group subjectively perceived psychological distance and level of abstraction into distinct modes of thinking. Perceived closeness and concrete mental construals are cognitively inextricably linked and they can be grouped together as *Near Mode* cognition. In a similar way, distant objects and abstract construals are also cognitively linked with one another and they can be grouped together as *Far Mode* thinking (Hanson, 2009). This categorisation is similar to the System 1 and System 2 classification of the previous section.

In addition to the broad classification into two modes of thinking, there is a superficial correspondence between System 1 thinking and *Near Mode* on the one hand and System 2 thinking and *Far Mode* on the other. System 1 cognition is evolutionarily old and much of it is shared with other animals, especially mammals. While System 2 cognition is a more recent development evolutionarily and while some animals display certain aspects of System 2 cognition, it is primarily a human specific ability. In the same way, most animals operate

exclusively in Near Mode. The ability to consciously form abstract construals of objects far away, to think about the distant past and imagine the distant future and to consider hypothetical counterfactuals is an almost exclusively human ability. In fact, Daniel Gilbert has suggested that the ability to think about the future is the defining characteristic that separates humans from other animals (Gilbert, 2006). Another area of correspondence is the fact that System 1 has a very large processing capacity which can process and respond to large amounts of information which is always present in the immediate environment. We can only ever experience or act upon the immediate environment which is the here and now. As a result all of the information about our immediate surroundings is processed initially by System 1. Both Near Mode and System 1 are implicated in taking into account specific, context dependent detail. Despite these similarities there are important differences between System 1 processing and Near Mode on the one hand and System 2 type thinking and Far Mode on the other.

As mentioned in the previous section, some researchers have drawn an explicit equivalence between abstract thinking and System 2 thinking. Automatic and affective processes tend to be concrete in nature (Metcalfé & Mischel, 1999; Sloman, 1996) while conscious, effortful cognition is typically involves a higher level of abstraction (Trope, 2004). One reason for this is the limitations of working memory. Conscious thought can only operate on about seven objects at a time and, as a result, if one is reasoning in a deliberate effortful way much information from the context is necessarily ignored, which implies a process of abstraction. Much conscious thought is language based and the use of words also involves abstraction. Consider two ways of representing a room: a photograph and the word “gloomy”. The photograph contains a great deal of information while the word very little, but it is still extremely descriptive.

There are good reasons for caution in adopting an equivalence between System 1 and Near Mode or System 2 and Far Mode. These reasons will not be recapitulated here, but it is worth adding more examples to further weaken the idea that there is a necessary connection between System 2 type cognition and Far Mode. Emotional responses are typically involuntary and automatic and are thus a product of System 1. But emotional responses vary greatly in level of abstraction. Fear is a concrete emotion but anxiety is by its nature more abstract since it is typically not caused by stimuli in the immediate environment (Trope & Liberman, 2010). Regret is an emotion intrinsically linked to events that happened in the past,

sometimes a long time ago and is bound up with hypothetical, counterfactual reasoning since regret typically involves wishing events had happened other than the way they did. We can automatically judge a person's behaviour to be dignified, but dignity is an extremely abstract quality which is famously difficult for philosophers to pin down (Bostrom, 2009). Another difference between dual process theory and construal level theory is that the distinction between System 1 and System 2 cognition is very clear cut, whereas because distance exists on a continuum and objects can be viewed at many different levels of abstraction depending on how much information is neglected, there is no clear dividing line between Near and Far mental modes. In other words, it is sensible to talk about "middle" mode in construal level theory, but there is no System 1.5.

The framework of dual-process theory thus does not have a straightforward relationship to construal level theory despite the similarities. At the end of this section however I will use the work of Stanovich to tentatively propose a model for integrating the two theories.

2.2.1. Psychological distance. So far I have referred to subjective distance when describing construal level theory, but it is important for the concept of subjective distance to be developed more systematically in order to see how different conceptions of distance relate to each other and then to investigate how these conceptions of distance relate to the level of mental construal adopted. The concept of psychological distance arises from the fact that we do not directly experience the past or future, distant places, the mental states of others and the way the world could have been, but isn't (Trope & Liberman, 2010). However, being able to think about such things is an important part of being human. Fineas Gage famously survived an accident where a large iron rod was driven through his head resulting in much of his left frontal lobe being destroyed (Gilbert, 2006). The frontal lobe is responsible for our ability to think about the future and as a result Gage lost the ability to consider future plans or possible events (among other changes in his personality) and this had a large impact on the course of his life. Being unable to think about the future is so alien to a normal human life that it is almost literally impossible to imagine. Being able to think about faraway places, someone else's perspective or how your life would be different had you acted differently in the past is similarly central to human experience. The concept of psychological distance refers to the perceived distance of object or events from the self in the here and now, which is where all of our actual experiences are located (Trope & Liberman, 2010). This apprehension of distance

need not be the same as the actual distance from a person and as a result, is necessarily subjective.

There are at least four distinct types of psychological distance (Trope & Liberman, 2010). There are the central common sense notions of physical distance and time. Conceptions of physical distance underpin much of our thinking and infuse our language when discussing other things. For example, we often think about past events as being “long ago” where long is a spatial concept (Boroditsky, 2007). The other two dimensions of psychological distance are less intuitive. Social distance refers to familiarity with other people. Family and friends are socially close while strangers are distant. Social distance also has a relative status component. Rich and powerful people are socially distant from poorer people; a boss may interact with her employees every week for years while remaining socially distant from them. Social distance between two people can be very different for both people. Fans will often feel socially close to celebrities, but the vast majority of fans will be unknown to the celebrity and will thus be socially distant. The final dimension of psychological distance that will be considered is the “hypotheticality” dimension. The hypotheticality dimension of psychological distance refers to our ability to imagine the world as it could have been or could be in the future. This involves thinking about counterfactuals of past events, considering the potential consequences of an event or decision and thinking about improbable events. Hypotheticality concerns what we usually would term “imagination”. Examples of this type of thinking are imagining how history would have been different had John F. Kennedy not been assassinated, speculating how your life would change if you got married and trying to work out how to avoid a nuclear war. The answers to these questions cannot be arrived at by considering previous experience. Hypothetical thinking is the dimension of Far Mode that is most clearly associated with System 2 cognition.

One of the central tenets of construal level theory is that all the dimensions of psychological distance are mentally associated with each other and taken together, form the single concept of psychological distance (Trope & Liberman, 2010). Considering an object distant on one dimension automatically affects the perception of the distance of the object on the other dimensions. A great deal of evidence has accumulated to support this claim, even though much of it was collected well before the emergence of construal level theory.

Social psychologists have often equated physical distance with social distance; for example, choosing a seat far away from someone is a form of social distancing (Macrae, Bodenhausen, Milne, & Jetten, 1994; Mooney, Cohn, & Swift, 1992). People typically use more polite language when addressing strangers than when addressing friends and the use of polite language also causes people to believe that the target of the communication was more physically and temporally removed. Evidence has also suggested the reverse direction of causation (Stephan, Liberman, & Trope, 2010). Imagining remote locations brings to mind the distant future or past, other people rather than oneself and unlikely rather than likely events (Bar-Anan, Liberman, Trope, & Algom, 2007). Finally, hypothetical events also affect the perception of other distances. People expect unlikely events to happen far away in time and space and to happen to other people (Wakslak & Trope, 2008). It is no accident that many science fiction and fantasy novels and films take place in the far future or distant past and in space or on other planets. Perhaps the most familiar example is Star Wars. Alien species feature prominently, which induces great social distance and the events depicted are extremely improbable. Famously – there is a Wikipedia entry devoted only to the opening text (“*Star Wars* opening crawl”, 2013) – the movie begins with text moving ever further into space and beginning with the words: “A long time ago in a galaxy far, far away”. This is slightly unusual because unlike typical science fiction and despite the advanced technological development depicted, the events happened in the distant past. In terms of construal level theory, however, this detail is irrelevant to the effect; all that matters is that it happened in a time very distant from the present.

2.2.2. Construal level. The ability to adopt construals of varying degrees of abstraction is a necessary result of our need to think about distant objects. We want to think about the past and future in meaningful ways, but the vast majority of concrete details of these distant times is inaccessible to us and since our horizons broaden as distance increases it is computationally infeasible to consider detail on the same level as for near objects. As a result we piece together informal theories of these times based on limited information and general regularities and trends. When thinking of distant places, general characteristics such as whether it is hot or cold, rainy or dry, flat or mountainous play a larger role in our representations of those places than of our home city which we can imagine in more detail. The people of foreign countries will be conceptualised as speaking certain languages and as conforming to various cultural stereotypes, whereas thoughts about our own country or friends will include a great deal more context and idiosyncratic nuance. Hypothetical

reasoning is an extremely computationally expensive operation of System 2 (Stanovich, 2011) and since an abstraction is connected to the goals of the abstraction, which involves additional information, it is essential that information irrelevant to motivation and goals is omitted from the cognitive process.

Different levels of mental construal can be applied in different contexts. Categories, concepts, emotions and actions can be created or viewed in terms of different construals. In all cases however, higher levels of construal involve more abstraction, discounting information considered irrelevant to the purpose of the abstraction while retaining core characteristics and thinking in terms of superordinate rather than subordinate categories. Higher levels of construal are more stable and coherent. Since higher level construals need to neglect information considered irrelevant to the purpose of the construal, they are necessarily tied to goals and values which are themselves abstract. Concepts such as freedom, equality and fairness are highly abstract. It also means that different abstractions will be appropriate in serving different goals or values. If one is hungry and lost in the woods, a rabbit is much more likely to be conceptualised as food than when reading a child a bed time story, where cuddly cuteness is more appropriate. This is significant in the context of mathematics education since the contexts of learning mathematics and solving mathematical problems are so unusual and the motivations and goals of a student doing the mathematics are so varied and are frequently inaccessible to educators.

2.2.3. Construal level and psychological distance. We now come to the central premise of construal level theory: just as the different dimensions of psychological distance are automatically mentally associated with each other, psychological distance is mentally associated with level of mental construal adopted. Greater psychological distance is associated with higher level mental construals and psychological proximity is associated with low level construals. This link is well captured by the aphorism of being “unable to see the forest for the trees”. To see the forest as a whole it helps to take a step back and look from a distance, while if we want to examine individual trees we should get close up. As already indicated in this section, level of construal and psychological distance can be grouped into two categories: *Near* and *Far* mental modes.

2.2.4. Evidence for construal level theory. Evidence for the link between psychological distance and level of construal comes from a variety of sources.

Evidence from investigations into the way in which we process mental images helps demonstrate the link between psychological distance and construal. In a common task, people were presented with an image of a large letter created out of many smaller different letters (a large L made up of many small Hs) and were asked to confirm if one of the letters appears in the image (Navon, 1977). Here the large letter represents the higher level, superordinate structure and the small letters the lower level concrete components. Before completing this task, participants were primed to think in Near or Far Mode by considering either psychologically close or distant events. Participants primed with the distant construal were faster to identify the large letter and slower to identify the smaller letters (Liberman and Forster, 2009). Similarly, those primed with psychological closeness were quicker to identify the smaller letters and slower to identify the larger. The link has been established in the other direction within the same basic experimental setup. Participants primed to focus on the large letter estimated various events and objects to be further away in space and time and estimated typical life events to be less probable (Wakslak & Trope, 2009). Adopting a psychologically distant mind-set aids in the recognition of abstract shapes (Förster, Friedman, & Liberman, 2004; Wakslak, Trope, Liberman, & Alony, 2006; Smith & Trope, 2006) and it is essential to have a high level construal of objects in order to recognise them from a distance.

Pictures and words are two ways of representing an object. Pictures must physically resemble the object (otherwise it cannot be said to be a picture of that object), whereas a word carries little information about the details of the object other than its defining essence. The word chair refers to objects that look very different from one another, but to be a chair they must share the characteristic of being designed to be sat upon. Pictures are therefore low level construals which are processed in Near Mode, while words are high level construals and tend to invoke Far Mode. Studies have shown that subjects more rapidly identify words referring to distant objects and more rapidly identify pictures of proximal objects (Amit, Trope, & Algom, 2009).

An important instance of abstraction is categorisation. The broader and more inclusive the categorisation the more abstract it is. An obvious example is life on earth. The highest and most inclusive level is living things, with successively lower level categorisations referring to animals, mammals, human beings and then to individual people. With every step generality is lost, but many specific additional details become appropriate to use. As construal level theory

would predict, there is evidence that subjects who received priming for psychological distance will group a collection of objects into fewer categories than those with psychologically proximal priming (Liberman, Sagristano, & Trope, 2002; Wakslak et al., 2006). Again the reverse is also true. There is evidence that subjects presented with more narrowly defined categories deemed that particular future events were more likely than subjects receiving more abstract priming by way of more inclusive categorisation of objects (Wakslak & Trope, 2009).

The final major example of abstraction relevant to this study involves the different construals of actions. Broadly speaking, high level construals of actions involve considering the motivation behind the action, or why the action was carried out and low level construals involve how the action was carried out. The predictions of construal level theory with regard to action are by now clear. Psychologically distant actions are considered in more abstract terms. For example, participants in one study were more likely to describe studying in the distant future in terms of the goal of doing well in school. But if the studying was to be done soon it was more likely to be described in terms like “reading a textbook” (Fujita, Henderson, Eng, Trope, & Liberman, 2006). Conversely, subjects primed to adopt a high level construal of an event (to think about the event in terms of a “why” question) were more likely to think of the event as taking place far away in time (Liberman, Trope, McCrea & Sherman, 2007; McCrea et al., 2008) and involving socially distant others (Stephan et al., 2010). Using more abstract language to describe an event causes people to think the event took place in the more distant past (Semin & Smith, 1999).

2.3. Dual Process Theory and Construal Level Theory

Dual process theory and construal level theory should not be conflated. System 1 type reasoning is not synonymous with Near Mode reasoning and System 2 is not always an example of Far Mode thinking. Despite reasons to be cautious when comparing the two theories they are certainly related and it is worth investigating these relationships more closely.

One way in which System 2 type reasoning is linked with a Far mental mode stems from the strictly limited capacity of System 2. The limited capacity of System 2 means it needs to exclude lots of information from its reasoning processes and because abstraction involves

neglecting information deemed irrelevant, System 2 is at least partially intrinsically abstract and is to a certain extent a Far Mode process.

There is an extremely large literature on heuristics and biases in human reasoning and these heuristics and biases form a core part of the evidence for dual process theory (Evans, 2008). Daniel Kahneman has stated that he and his long term collaborator Amos Tversky always viewed their work on heuristics and biases as being a dual process theory (Kahneman, 2002). Many of these biases however are also well accounted for by construal level theory. Two examples of this are found in the common reasoning biases known as the fundamental attribution error and the conjunction fallacy.

The fundamental attribution error refers to the well documented tendency to explain other peoples' behaviour in terms of personality traits rather than situational factors. For example, someone stepping on your foot on a bus is more likely to be explained in terms of general clumsiness, but if we step on someone else's foot then we are more likely to explain it in terms of contextual factors in the situation, like the bus being crowded (Trope & Liberman, 2010). We explain our negative actions in terms of specific detail, which is processed in Near Mode, whereas the actions of a stranger, who is socially distant, are explained using a general category which is a function of Far Mode.

The conjunction fallacy concerns the tendency of people to consider a hypothetical situation *more* probable when additional information is provided, even though the additional detail makes the event strictly less likely. In an experiment (Tversky & Kahneman, 1983), professional analysts were asked to estimate the probability of one of the events described below. Each analyst saw only one of the scenarios.

A complete suspension of diplomatic relations between the USA and the Soviet Union, sometime in 1983...

A Russian invasion of Poland, and a complete suspension of diplomatic relations between the USA and the Soviet Union, sometime in 1983.
(Tversky & Kahneman, p. 307).

The second scenario is strictly less likely, after all there are many possible reasons for diplomatic relations between countries to break down, but the analysts considered it more likely. This is explained by construal level theory since the extra detail invokes Near Mode thinking and events considered in Near Mode are consistently estimated to be more likely. The work of Keith Stanovich suggests one avenue for a partial integration of the two theories. Stanovich has proposed a tripartite model of the mind which is intended to build on dual process accounts by suggesting that System 2 type reasoning is better conceptualised as involving two broadly distinct types of reasoning, each responsible for one of the two primary tasks of System 2: monitoring and overriding System 1 type responses and developing an alternative response when System 1 reasoning is deemed inappropriate. This model of cognition suggests that System 2 can be considered to comprise of the *reflective mind* and the *algorithmic mind* (Stanovich, 2011). In Stanovich's model, the algorithmic mind is responsible for the hypothetical reasoning that will formulate the alternative response, whereas the reflective mind is responsible for assessing the outputs of System 1 and initiating the override. Despite terming this model the tripartite model of the mind, it still fits within a general dual process model since both the reflective and algorithmic minds are instances of System 2 type thinking. The essence of the dual process model is the distinction between autonomous, high capacity processing and conscious, low capacity processing and this remains at the core of the tripartite model.

The algorithmic mind. In order to come up with a better response than System 1 offers we need to simulate a simplified model of the world in order to “test” the effects of possible actions in this model. To prevent this hypothetical model from being confused with the real world it needs to be cognitively decoupled from our models of the real world. Sustaining this decoupling and developing complex hypothetical models is extremely taxing and is one explanation for the relative limitations and slow speed of System 2 reasoning. It also helps to explain why multiple System 2 processes cannot be carried out in parallel. The effortful form of reasoning which overrides System 1 responses is carried out by the algorithmic mind and is general in nature. The algorithmic mind is responsible for detailed reasoning across many domains (Geary, 2005; Horn & Noll, 1997). It is the capacity of the algorithmic mind that is generally measured in intelligence tests (Stanovich, 2012).

Psychometricians distinguish between typical performance situations and optimal performance situations. Optimal performance situations measure performance on tasks where

the task interpretation and goals are determined by the examiner and not the subject and the subject is instructed to maximise performance. Optimal performance tasks measure efficiency of the processing of the algorithmic mind. Typical performance situations on the other hand are at least partly open ended. No instructions are given to maximise performance and the problem interpretation and goals pursued are at least partly left open to the individual. As a result, problem interpretation and prioritisation form crucial components of such tests. Typical performance tests are measures of the reflective mind.

The reflective mind. The reflective mind is responsible for monitoring the outputs of System 1 type thinking and initiating the override of such outputs to be carried out by the algorithmic mind. The reflective mind is responsible for assessing and interpreting problem situations and formulating appropriate goals and goal hierarchies. Given the role of the reflective mind, its operations exist at a higher level than the algorithmic mind. The reflective mind is reflected in a person's general thinking disposition (Stanovich, 2012). General thinking disposition or cognitive style are comprised of beliefs, epistemic worldview, values and self-regulation (Koedinger, Roll, Holyoak, & Morrison 2012). While algorithmic processing is general in nature and is used across domains, reasoning conducted by the reflective mind is much more specialised and is primarily involved in probabilistic, causal and scientific reasoning (Stanovich, 2009). The effectiveness of the reflective mind in performing its functions of monitoring System 1 type processes and coordinating the operation of the algorithmic mind is *not* assessed in any intelligence test in widespread usage (Stanovich, 2012) and, crucially, is not strongly correlated with the results of traditional intelligence tests assessing the efficiency of the subject's algorithmic reasoning. The surprising fact that those with high IQs do not on average have noticeably more efficient reflective minds shows the importance of this cautionary quote from the philosopher Nick Bostrom: "When headed the wrong way, the last thing needed is progress." (Bostrom, n.d.)

2.3.1. The tripartite model of mind and construal level theory. We are now in a position to tentatively suggest a partial integration of dual process theory and construal level theory:

- The reflective mind is primarily associated with Far Mode thinking
- The algorithmic mind is primarily associated with Near Mode thinking

The reflective mind has to oversee a huge amount of information processed by System 1 and cannot analyse this information in detail. Instead, it accesses general knowledge, higher order epistemological models and goals. For example, a trained scientist has the ability to assess information using a well-developed theoretical model from her discipline. These higher order belief systems are used to filter and evaluate systems and select the type of override to initiate. While the nature of the processing capabilities of the reflective mind are more specialised than those of the algorithmic mind, the reflective mind is responsible for processing everything that we think about. These specialised capabilities must be applied very broadly and since the reflective mind is a System 2 capability, most of the incidental detail available from perception and long term memory must be abstracted away. For these reasons the reflective mind is associated with Far Mode. The algorithmic mind on the other hand accesses cognitive microstrategies and production systems for processing sequences of detailed thoughts. The algorithmic mind will not wilfully exclude information relevant to its reasoning sequences. The processes of the algorithmic mind are far more fine grained and detailed than those of the reflective mind and are thus processed more in Near Mode. It is not possible to categorise System 1 type reasoning in a similar way. Whether a person has previously been primed to adopt an abstract or concrete construal will affect whether System 1 automatically processes in Near Mode or in Far Mode.

2.4. Procedural and Conceptual knowledge

There has been much debate about whether mathematics education should emphasise understanding of concepts or whether the so called “skill and drill” approach is more effective in developing mathematical proficiency. To the extent that mathematics education is ever the subject of public debate or political controversy, this is the topic that makes the news and that lay people and parents have strong feelings about. In fact, the debate has been so heated that the term “math wars” has been used to describe the political battle over the curriculum which has primarily focused on this issue (Schoenfeld, 2004; Star, 2005). One of the best known examples of this was the ill-fated attempted implementation of so called “New Math” in America in the 1960’s. New Math was the American government’s attempt to reverse the relative decline of mathematics attainment of American students relative to Russian students by revamping the curriculum to focus intensely on highly abstract mathematical concepts. The program was not popular with parents or teachers, many of whom did not understand the concepts introduced, and it was quickly abandoned. The experience of New Math has endured in the public imagination. A punk rock band adopted

the name New Math, it has been referenced on The Simpsons and there is an entry on Wikipedia keeping track of popular culture references (“New Math,” 2013). Another approach to mathematics education which is commonly emphasised focuses on procedural proficiency and is familiar from versions of the slogan “back to basics” or the more descriptive “skill and drill”. Procedural fluency tends to be more popular with traditionalists but less popular with mathematics education researchers (Star, 2005).

The problem of the need to reason at various levels of abstraction has been addressed in a variety of ways and in a number of different fields. Of particular interest here is the way it has been addressed in the mathematics education literature. The issue is sometimes presented as a dichotomy between abstract reasoning with emphasis on understanding mathematical concepts, and concrete detailed reasoning with the emphasis on procedural skills.

Increasingly however researchers are emphasising the relationships between the two types of reasoning with some researchers suggesting that the two types are inseparable and should be viewed in terms of duality rather than the starker either-or approach (Sfard, 1991; Gray & Tall, 1994).

The academic discussion over these two, apparently quite different, modes of thinking, and their relative importance in mathematical proficiency goes back over one hundred years. During that time, a great many terms have been used to describe each mode of thinking. As the distinction emerged as a topic of study, the two modes of reasoning were viewed as quite separate and the dichotomy was framed as a debate over whether mathematics education should focus primarily on understanding the meaning of concepts or whether it should focus on promoting skills in performing procedures. In 1895, John Dewey along with James Alexander McLellan, proposed a mathematics curriculum that focused more on student understanding (McLellan & Dewey, 1895). Thorndike argued on the other hand for a more skill based instruction (Thorndike, 1921). The debate has gone back and forth since with researchers emphasising different aspects of each approach but in broadly the same tradition. Of the many ways of framing the discussion just a few of the choices of classification for the two poles of mathematical thinking are included here. The contrast has been described as being between abstract and algorithmic (Halmos, 1985), dialectic and algorithmic (Henrici, 1974), declarative and procedural (Anderson, 1976), or instrumental and relational (Skemp, 1976) types of reasoning. The dichotomy has been highlighted from a psychological perspective as well and Piaget refers to figurative and operative modes of mathematical thinking (Piaget, 1972). Anna Sfard has used the similar conceptions of structural and

operational components but has stressed the duality of the different approaches rather than framing it as a dichotomy (Sfard, 1991). Sfard also more explicitly deals with the fact that we are dealing with the nature of mathematical entities, which is primarily a philosophical, ontological issue and the way in which human minds perceive such entities. This mode of analysis is especially relevant to the current study and will be dealt with in more detail in a separate section.

Each of these descriptions of reasoning can easily be classified into the more general sense of more abstract, conceptual ways of viewing mathematics and lower level, more concrete ways of viewing mathematics. Perhaps the most widely used and accepted terms classifying the different poles of reasoning in mathematics education research are the procedural and conceptual views of mathematical reasoning (Hiebert & Lefevre, 1986) and this is the terminology that will be used for the rest of this section. Procedural and conceptual approaches should not be assumed to map perfectly onto the other frameworks mentioned above, but the terms have resisted precise definition agreed upon by researchers. For our purposes here procedural can be considered an umbrella term containing the lower level more concrete aspects and conceptual, the higher level abstract aspects. Rather than attempting to adjudicate between the different ways of stating the distinction and arguing for one interpretation, all of the varieties will be taken to form a broad family of theories which can be viewed through the lens of construal level theory.

Hiebert and Lefevre define conceptual knowledge as “knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as important as the discrete pieces of information... a unit of conceptual knowledge cannot be an isolated piece of information” (Hiebert & Lefevre, 1986, p. 5-6). This view is consistent with that advanced by others; Baroody & Ginsburg (1986) assert that meaningful knowledge of mathematics (conceptual knowledge) is consistent with a schema theory of knowledge described by Anderson (1983): “A schema is an abstract structure of information. It is abstract in the sense that it summarizes information about many particular cases. A schema is structured in the sense that it represents the relationships among components” (p. 5). Conceptual understanding is improved when pieces of information that were previously assumed to be unrelated are seen to be related in some way. This realisation often happens suddenly and can result in a dramatic cognitive reorganisation (Sfard, 1991). Conceptual knowledge is sometimes described as being static and integrative in nature (de

Jong & Ferguson-Hessler, 1996; Sfard, 1991). It comprises the additional knowledge that a subject adds to a problem before set procedures are able to be brought to bear on it. If the pieces of information are recognised to be linked to the same, deeper principle then the information is being conceptualised in a more abstract way. Conceptual understanding is also illustrated in a case where new information is assimilated into an existing conceptual scheme in the sense used by Piaget, Inhelder, & Szeminska (1981). The information is recognised as being of a certain class or type and is grouped in that category and as such is also an example of abstraction. To assimilate a new piece of information as part of an established mental category involves abstracting incidental details away from the information and seeing what core qualities it has in common with other pieces of information in that category. This process of abstraction can come in two forms, primary or reflective. Information can be recognised and categorised within existing mental frameworks, in which case the abstraction is happening at the localised level between the piece of new information and the existing categories.

However, sometimes new or existing information is linked together at a new more abstract level and new superordinate categories are created. This is a deeper and more profound form of conceptual understanding and occurs at the reflective level (Hiebert & Lefevre, 1986). This type of conceptual understanding is similar to what Piaget termed reflective abstraction (Piaget, 1972). It is important to recognise that the structuring of relationships between information and concepts can happen at different levels of abstraction.

Hiebert and Lefevre define procedural knowledge as being comprised of two parts. One part is composed of the formal, symbolic representation system and the other is comprised of the rules and algorithms or step by step systematic procedures involved in mathematical tasks (Hiebert & Lefevre, 1986). Knowledge of the symbol representation system includes knowledge of the rules of syntax and symbol manipulation, but does not include deep understanding of the mathematical object the symbol represents, which is included as part of conceptual knowledge. At the superficial level, this type of knowledge need not go further than simply systematically manipulating marks on paper, though it almost always will involve deeper understanding than that (though where this deeper understanding should more properly be considered conceptual knowledge is unclear).

The second component of procedural knowledge involves the ability to complete a predetermined step by step set of operations and thus compared to conceptual knowledge is more dynamic in nature (de Jong & Ferguson-Hessler, 1996; Sfard, 1991). In order to start an input object is required for the procedure to operate. The procedure provides the rules for transforming the object in some way to produce the input for the next step and has been characterised as a “production system” (Anderson, 1983). The object can be a symbol as is typical in algebra or calculus or a concrete object or geometric image. The step by step nature of this type of procedural knowledge is usually linear in nature, though this aspect has been challenged by some researchers (Star, 2005). This type of procedural knowledge has been the primary type of mathematical knowledge that has been taught in western mathematics curricula. At the extreme end of procedural knowledge as defined in this way, and unfortunately common in reality, it is possible to perform procedures well enough to pass many assessments without knowing what the symbols being manipulated mean or why the procedures carried out work. This is procedural knowledge without any conceptual knowledge.

As a matter of observed reality however, the students most capable of carrying out procedural tasks usually do have somewhat developed conceptual understanding, because conceptual understanding is a valuable aid in helping students remember the procedures. Having a deeper conceptual understanding of a procedure means that fewer pieces of information need to be kept in working memory as forgotten steps can more easily be recreated (Hiebert & Lefevre, 1986). Additionally, conceptual understanding implies that procedures are not bound to specific contexts. There is thus recognition that the same or similar procedures are applicable in different situations. A student with primarily procedural knowledge will have to remember the details of many different detailed contexts and then hope that a question resembles one of the remembered scenarios sufficiently closely.

Just as concepts can be arranged hierarchically, so can procedures. Usually, a procedurally based solution method will involve sub-procedures in many of these steps. At a simple level, steps will often involve operations like multiplication or addition which have their own procedures for calculation and hopefully can be taken for granted when learning new algorithmic procedures. This hierarchical structuring provides one illustration of the connection between procedural and conceptual knowledge. The more procedures become taken for granted and embedded in super-procedures, the more the super-structure of nested

procedures resemble mathematical concepts and flexible use of these super-procedures entails conceptual understanding. This grey area between procedural and conceptual knowledge is primarily the area occupied by the branch of mathematics education research devoted to heuristic problem solving techniques. Heuristic problem solving techniques provide guides for dealing with mathematical problems in different domains. While heuristic techniques can come in linear step by step form, many do not. They are typically not of the form where one missed link results in failure; rather they are more rules of thumb and avenues that should be explored. Heuristics clearly have similarities with procedural knowledge, and usually call on procedural knowledge in the process of carrying them out, but effective use of heuristics requires a deeper conceptual understanding of the domain.

2.4.1. Relationships between conceptual knowledge and procedural knowledge.

Procedural and conceptual knowledge typically develop together as subjects develop mathematical maturity. When the distinction first emerged they were treated as distinct realms that could be studied in isolation. Increasingly the relationships between procedural and conceptual knowledge and the fact that they each depend on the other has come into focus. This fact, however, doesn't tell us anything about causality: do improvements in procedural fluency help spur improvements in conceptual understanding and vice versa? If the answer is yes in both cases, as is almost certainly true, is one direction of influence stronger or more important than in the other and if so, does this provide a guide for pedagogy and curriculum design? As has already been mentioned, these questions have been addressed for at least a hundred years and the discussion is on-going.

Rittle-Johnson and Alibali have argued that procedural and conceptual knowledge influence each other in an iterative fashion, with improvements in one domain allowing and spurring developments in the other (Rittle-Johnson, Siegler, & Alibali, 2001). But they have also provided some evidence that at least in the case of developing proficiency in arithmetic, conceptual understanding is a greater aide to spurring development in procedural knowledge than vice versa (Rittle-Johnson and Alibali, 1999). On the other hand, Jon Star has argued that conceptual knowledge has received too much attention in the mathematics education research literature and argues for a reconceptualization of procedural knowledge that should receive more research focus (Star, 2005). He argues that procedural knowledge can be of a more flexible and robust nature than that implied by the definitions typically used without deserving the classification of conceptual knowledge. Typically procedural is taken to mean

superficial and conceptual is taken to mean deep but this is not always the case. A child may have a concept for a dog, but this will be more superficial than that of an adult whose conception will be superficial compared with that of a biologist. So Star argues for classifications that would include deep procedural knowledge and shallow conceptual knowledge. In a similar way to Star, de Jong and Ferguson-Hessler suggest distinctions between types of knowledge on the one hand and qualities of knowledge on the other. In this classification, types of knowledge are either procedural or conceptual and qualities are deep or shallow (de Jong & Ferguson-Hessler, 1996). The potential relationships between procedural and conceptual knowledge are worth examining in more detail.

2.4.2. The effect of conceptual knowledge on procedural knowledge. Improving conceptual understanding has several benefits for procedural fluency. Proficiency with symbol manipulation is an example of procedural competence but this is often carried out with little or no understanding of the symbols being manipulated. Developing understanding of the symbols used means that more meaning is being attached to routine procedures (Schoenfeld, 1986). Developing conceptual understanding of procedures helps with the storing and retrieval of the procedures. If procedures are understood they become connected with structured networks of information rather than being stored as isolated examples tied to their original context. Information connected to such networks is remembered better than isolated information (Anderson, 1983; Skemp, 1976). The classification and organisation of information that is an inherent component of conceptual knowledge means that procedures are recognised as being of a particular type and similar to other learned procedures. If many procedures are remembered as being similar, then only the unique features of each procedure need be recalled rather than each procedure in its entirety which can reduce the demands on working memory when problem solving. Via similar mechanisms, improved conceptual knowledge can also aid in effective use of procedures. Better conceptual understanding leads to improved problem representation which helps to select an appropriate procedure when other less efficient procedures would also be applicable. Improved conceptual knowledge can also assist with preserving working memory by allowing subjects to reason directly on the problems rather than devoting attention to the symbols being manipulated in the procedure (Greeno, 1983; Chi, Feltovich, & Glaser, 1981). It also helps monitor whether a procedure is appropriate and whether the outputs of a procedure make sense (Schoenfeld, 1985). As well as helping subjects recall procedures, conceptual understanding helps subjects abstract the

use of procedures away from specific contexts and aides in transfer of procedures to different domains (Rittle-Johnson & Alibali, 1999; Carpenter, 1986).

2.4.3. The effect of procedural knowledge on conceptual knowledge. Though there has been less attention and research into this direction of causality it is generally true that improvements in procedural proficiency generally benefit conceptual understanding as well. The ability to manipulate mathematical symbols is a form of procedural knowledge and control of symbols is one way in which we achieve control over our thoughts (Skemp, 1971). This control over symbols is helpful in attaining conceptual understanding of the abstract entities those symbols represent and it aides in organising and operating on abstract ideas. Byers and Erlwanger (1984) have argued that in some cases notation systems and the operation of mathematical syntax are responsible for the development of mathematical concepts. In a similar way, the procedures themselves can promote conceptual understanding (Baroody & Ginsburg, 1986).

Importantly, procedures operate on mathematical concepts. Repeated use of procedures means repeated exposure to the correct use of mathematical objects and over time this assists in understanding those objects better. Finally, an indirect but very important way in which procedural fluency assists conceptual understanding is that if a procedure is highly routinized, then it takes up very little working memory which is then available to concentrate on higher level conceptual issues (Kotovsky, Hayes, & Simon, 1985).

2.5. Cognition and Metacognition

Like research into procedural and conceptual knowledge in, another branch of mathematics education research that can be viewed through the lens of dual process theory and construal level theory concerns the importance of metacognition in mathematical problem solving. Research into metacognition is motivated in part by the fact that many, if not most, mathematics students often fail to solve mathematical problems which they should easily be able to solve given their domain specific knowledge (Schoenfeld, 1985; Gillard, Van Dooren, Schaeken, & Verschaffel, 2009). Put another way, people are often capable of correctly carrying out detailed algorithmic procedures and understanding various mathematical concepts but fail to use these procedures in unfamiliar contexts where the procedure is nevertheless appropriate. The failure to apply mathematical content knowledge represents a failure of mathematical metacognition whereas forgetting content or making a mistake during

a routine procedure would be a failure of mathematical cognition. Roughly speaking, the distinction between cognition and metacognition is summed up by the description from Garofalo & Lester (1985): “Cognition is involved in doing, whereas metacognition is involved in choosing and planning what to do and monitoring what is being done” (p. 164). Metacognitive processes are involved in monitoring and coordinating cognitive processes so metacognitive processes exist at a higher, more abstract level than cognitive ones. Since cognition and metacognition operate at different levels of abstraction it is clear that construal level theory is relevant for research into mathematical cognition and metacognition. This relevance will be explored in detail in a later section. The focus of this section is to provide an overview of research into cognition and metacognition in the mathematics education research literature.

As in the case of procedural and conceptual knowledge there is no clear dividing line between what should be considered cognition and what should be considered metacognition and there are several complications that arise when using the term metacognition. One reason for this is simply that metacognition is arguably an example of cognition and that metacognition is always implicitly present during cognition. Another is that the term has not been uniformly defined in the literature and often functions as a “buzzword” with various, often overly general and inclusive meanings. An early and widely used definition comes from John Flavell (1976): “Metacognition refers to one’s knowledge of one’s own cognitive processes or anything related to them, e.g. the learning-relevant properties of information or data.” (p. 232), but research into metacognition has typically been narrower than this definition would indicate (Schoenfeld, 1985). When used with more precision, however, metacognition has several, almost distinct meanings with the most significant cleavage in its use being the distinction between knowledge about cognition and regulation of cognition. It is therefore important to be clear in which sense the word is being used. Finally, the intuitive definition of metacognition, often used by researchers, as simply “thinking about thinking” seems to rule out the more automatic and unconscious types of cognition that should count as examples of metacognition. One example is that of having an acquaintance’s name on the tip of your tongue. In this example you appear to have (almost certainly accurate) knowledge of information you possess but are not able to retrieve the actual information. In other words, you know that you know something, which makes it an example of metacognition.

2.5.1. Cognition. A central component of cognition is the *knowledge base* which can itself be split into two parts; what you actually know and how you access that knowledge. Schoenfeld (1992) has likened this distinction to a library. What you know is equivalent to the books contained in the library and the access to that knowledge is equivalent to how the books are catalogued. One could push the analogy further by suggesting that the knowledge base can be limited by considering a library that either has few books where some of those books could be in poor condition or missing pages, and a cataloguing system that routinely misfiles books or, equally damaging, is complicated and requires a great deal of attention to navigate.

The mathematical knowledge base includes intuitive knowledge, mathematical facts, definitions and axioms, algorithmic and routine procedures and relevant competencies and knowledge of the rules of mathematical discourse (Schoenfeld, 1985). This mathematical knowledge can itself be broadly split into two components, conceptual or declarative on the one hand and procedural on the other (Anderson, 1976; Hiebert & Lefevre, 1986). This division of knowledge types has been discussed at greater length in the previous section but it is worth observing that, though perhaps intuitively suggestive, cognition and metacognition do not map neatly onto procedural and conceptual knowledge since in general both procedural and conceptual knowledge would be classified as examples of cognition and not metacognition. This classification should not be taken to be definitive however as some conceptual knowledge is an essential component of metacognitive processes.

Accessing the knowledge that one has is a crucial task that does more to blur the distinction between cognition and metacognition. Information is usually recalled effortlessly and rapidly and is an example of routine cognition. We are also often aware that we have a piece of information and can undertake various actions in an attempt to remember it. One example has already been mentioned, that of momentarily forgetting an acquaintance's name where there is no obvious action that could help recall it. Another is misplacing a set of keys where we might mentally retrace our steps in order to find them. These are examples of metacognition. Consciously accessing memory contents is carried out in working memory, which is the arena where "the thinking gets done" (Schoenfeld, 1992, p. 350). Humans can store very large amounts of information in long-term memory, but the amount of information stored in working (or short-term) memory is strictly limited and very small; about seven "chunks" of information at a time (Miller, 1956). However, these "chunks" of information can represent

complicated objects which implicitly contain a large amount of information and not simply individual units like a simple shape or word. The so-called chunking of information is one way of overcoming the limitations of working memory. Chunks are configurations that are instantly familiar and recognisable. Examples of chunks are familiar images, words, sounds and also configurations unique to individuals with particular expertise such as common arrangement of chess pieces on a board or a complex mathematical concept. Experts in a domain will have a great many domain specific chunks of information in long term memory and this includes the domain of mathematics. Chunks form the “vocabulary” for, and are the basis of, competence in a domain. The most familiar example of this is language; words form the chunks and a typical adult’s vocabulary is about 50 000 words. Under ordinary conditions, words are recalled and used effortlessly to form sentences but writers and poets may spend an enormous amount of effort and attention on word selection and order when attempting to express a particular thought. The chunks have variously been described as scripts (Schank & Abelson, 1977), frames (Minsky, 1975) or schemata (Hinsley, Hayes, & Simon, 1977). These different terms express a similar idea: people abstract from their experience to form concepts and representations of these experiences which then influence how they experience and classify related situations (Schoenfeld, 1985). This style of abstraction happens naturally and automatically and is an indispensable component of normal human cognition (Gray & Tall, 2007).

The information stored and used in short term memory can originate from perceived sensory information or information accessed from long-term memory. There is continuous processing in working memory determining which perceptual information to discard and which information to use from long-term memory. The selection of an appropriate chunk from long-term memory often happens automatically and without conscious effort. During this process new chunks can be formed and committed to long-term memory for later recall, which in turn contributes to improved performance.

The sheer number of chunks typically necessary for competent performance and the time it takes, especially for more complex chunks to become embedded in long-term memory, provides a compelling reason why competence in a new domain is so difficult and takes so long to achieve. This is especially true in a domain like advanced mathematics because there is so little exposure to the abstract mathematical notions in everyday life that embedding of new chunks can only happen when consciously studying the subject.

A large number of mathematical concepts and procedures must be stored in chunks in long term memory for proficient mathematical problem solving. Knowledge of mathematical concepts, procedures and problem solving strategies form the knowledge base and are examples of mathematical cognition. However, while an adequate knowledge base is necessary for effective problem solving, it is not sufficient because in the absence of efficient metacognition, problems presented in unfamiliar context can prevent the knowledge from being used efficiently or at all.

2.5.2. Metacognition. Alan Schoenfeld, who has worked to define metacognition with more precision than is common in the research literature, divides metacognition into three related but distinct categories: “(1) individuals’ declarative knowledge about their cognitive processes, (2) self-regulatory procedures, including monitoring and “on-line” decision making, and (3) beliefs and affects and their effects on performance” (Schoenfeld, 1992, p. 347). This conception of metacognition is limited by its implication that metacognition is always conscious and deliberate whereas we have already seen examples of automatic processes which count as metacognition. It seems likely however that conscious metacognition is more teachable than the more automatic sort and is more relevant to the mathematics education literature (Schoenfeld, 1985; Lester, Garofalo & Kroll, 1989). As well as neglecting more intuitive forms of metacognition, metacognition in the sense of the second and third categories has been of the most interest to mathematics education researchers and will take up the majority of the current section.

Regulation of cognition. A description of the second category of metacognition is provided by Brown (1978). Effective “executive” functioning is able to

(1) Predict the system’s capacity limitations; (2) be aware of its repertoire of heuristic routines and their appropriate domain of utility; (3) identify and characterise the problem at hand; (4) plan and schedule appropriate problem-solving activities; (5) monitor and supervise the effectiveness of those routines it calls into service; and (6) dynamically evaluate these operations in the face of success or failure so that the termination of strategic activities can be strategically timed. These forms of executive decision making are perhaps the crux of efficient problem solving because the use of an appropriate piece of knowledge or routine

to obtain that knowledge at the right time and in the right place is the essence of knowledge.

(Brown, 1978, p. 82)

As a description of the regulation of cognition, the above quote is useful in articulating ideal metacognitive behaviour when facing a non-routine problem. It is far removed from most students' behaviour when facing unfamiliar problems however. A less formidable way of thinking about regulation of cognition is that when facing a problem in an unfamiliar context the problem must be *analysed*, an attempted solution should be *planned*, implementation of an attempted solution must be *monitored* and a potential solution should be *evaluated* (Schoenfeld, 1992).

Observation of students in problem solving situations reveals that conscious metacognitive behaviour is often almost completely absent. Typical behaviour will see students immediately classify the problem as being of a certain type through previously developed problem schemata (Hinsley et al., 1977) and move directly from reading the question to exploring the first avenue that occurs to them. If this avenue turns out not to be fruitful, students seldom pause to evaluate their progress and consider an alternative path. In fact they can often not provide *any* explanation of their actions at all. Schoenfeld (1985) refers to this automatic selection of a problem representation as schema-access which in the absence of effective regulation can have disastrous results. In other words in the absence of metacognition there is little of the analysis, planning or monitoring that is needed to stop continuing down an unproductive path. Another common manifestation of a lack of metacognition is when a local difficulty, such as an algebraic error while on a productive path, is interpreted as being a more global failure and the productive approach is abandoned. Finally, students will often present an answer that they arrive at, which even if correct in some technical sense, makes little sense in the problem context (Schoenfeld, 1987). This demonstrates an absence of any attempt at evaluation to determine if the problem has been sensibly solved. It is important to remember that this behaviour is the norm even when students have more than enough mathematical resources in the domain to solve the problem and yet they often fail to use this knowledge at all. In other words, if the problem had been framed in a specific way the students would be able to solve the problem without difficulty and phrase the answer in a sensible way. In fact in another context it would not occur to them to give such an answer. Schoenfeld provides as an example a division problem where the question is phrased, "how

many buses are required?” A large percentage of students provide an answer involving a fractional number of busses which makes no sense (Schoenfeld, 1987). They would likely only provide such an answer in the context of mathematical problem solving where students often assume that their calculations have no real world meaning.

Experts, on the other hand, spend more time analysing the problem after reading it. They consider many possible paths to make progress but are able to quickly abandon unproductive directions. They typically spend less time than students in calculation and more time in planning and thinking about the problem. Finally they also treat verifying that any conclusions make sense in the problem context as an important part of the solution. Crucially they have usually developed conscious mechanisms to guide them in their solution attempts and prevent them from spending too long on unproductive activities. They will often pause to evaluate their progress all the while keeping in mind the goal they are trying to achieve. Schoenfeld (1985) has observed that experts act as though they are having a conversation with themselves, suggesting alternative paths, counterfactuals or possible flaws in their argument. The significant ways in which expert behaviour differs from that of novice mathematicians, analysing the problem before attempting a solution and pausing to check if they are making progress relative to goals, seem to have one element in common. They all involve adopting a higher level construal, or taking a step back from the problem. In the language of construal level theory, expert problem solvers adopt a Far mental mode much of the time while students are often perpetually stuck in Near Mode.

Beliefs and affects. Individuals’ implicit and explicit beliefs about mathematics and their emotional responses to mathematical issues are important components of mathematical metacognition. Belief is a nebulous concept and it is often difficult to pin down exactly what a person believes because there are both cognitive and affective aspects contributing to a belief and these aspects can directly contradict each other. Many people have strongly held views against racism or sexism but then display racial bias on implicit association tests or manifest other discriminatory behaviours (Greenwald, McGhee, & Schwartz, 1998). Another example is that a person may consciously believe they are safe standing on the top of a tall building but still be very scared, implying a belief that they are not in fact safe. Tamar Gendler (2008) has coined the term “alief” for the affective component of belief in an attempt to distinguish between the two aspects. A crude way of summing this up is that people often do not know exactly what their beliefs are.

Beliefs and affects in mathematical problem solving do not have the immediate and dramatic impact on performance that regulation of cognition does but are still significant. Everyone has an epistemology or worldview which they use as a lens through which to view the world. Worldviews influence what information people choose to ignore, their motivations and their attitudes to challenges. Scientifically minded sceptics and conspiracy theorists who believe the moon landing was faked will have different reactions to evidence suggesting men really did walk on the moon. Not only will people in the two groups immediately form different opinions about new claims, their motivations will drive them in different directions in seeking to defend their views. Of course many people simply do not care one way or another about the moon landing and may not attend any evidence presented.

Similarly, people have beliefs about mathematics and mathematical problem solving that affect how they behave in class or approach problems. If a student believes that only geniuses can understand mathematical concepts, it is less likely that she will take constructive steps to understand them. A well-known instance of affective beliefs impacting negatively on mathematical performance is *stereotype threat* where women's performance on a mathematics test is hampered if the test is described as producing gender differences (Spencer, Steele & Quinn, 1999). People have a great many beliefs about mathematics and their own mathematical ability. Many of them are not held explicitly but they are important components of metacognitive behaviour.

“Makes sense” epistemology. One consistent source of error is that many students have what can be called a “makes sense” epistemology (Perkins, 1982). When removed from a formal, clearly demarcated context, people will have automatic responses to a problem which seem plausible and they will simply see no reason to implement the formal mathematical techniques they have learned (Schoenfeld, 1985). Metacognitive processes are engaged if an individual is surprised or experiences difficulty (Alter, Oppenheimer, Epley, & Eyre, 2007) but as Schoenfeld points out, being on a disastrous path is often not interpreted as experiencing a difficulty. One implication from the tendency to ignore mathematical resources when encountering an unfamiliar problem is that students implicitly believe that the formal mathematics that they have learnt has nothing to do with problem solving or discovery. A clear example of this phenomenon is found in physics students who even after completing a course in classical mechanics will answer questions in a way consistent with

“Aristotelian” or medieval “impetus” conceptions of physics (Champagne, Klopfer & Anderson, 1980; Halloun & Hestenes, 1985) and the more abstract the physical concept is the more prevalent and difficult to eradicate the misconception is. People develop common sense physics models based on the demands of their everyday experience and these models typically emerge when presented with a problem in an unfamiliar context. When these models conflict with formal, explicit instruction, they will experience substantial difficulty if their overarching worldview does not allow for the resolution of such conflicts. “Makes sense” or “naïve” epistemologies are a straightforward example of System 1 type reasoning and lack of metacognition correcting these mistaken conceptions is a failure of System 2 to perform its inhibitory function. This will be discussed in more detail in a future section.

Schoenfeld (1985) argues that there is a similar problem in mathematics which he terms “naïve empiricism” but that it can prove an even more intractable problem. Since much mathematics is only ever encountered in the classroom, students abstract meaning and develop models about the nature of mathematics almost exclusively from these experiences. This is bound to lead to a much more disparate variety of informal models of mathematics because students have many different teachers who between them may not present a coherent picture of the nature of mathematics (or even have one themselves). In addition the amount of time spent doing mathematics is small compared to the time “developing” common sense ideas about physics. As a result individual students are likely to develop highly idiosyncratic mental models of the nature of mathematics and while they may apply these with consistency (Schoenfeld, 1985); these models are likely to be flawed.

Many mathematical notions, such as proof, are almost never invoked for use in problem solving thus the “makes sense” attitude does not see “proof” as linked at all with “problem solving” and so will not call upon the learned resources. Schoenfeld (1992) has identified several beliefs a typical student holds about mathematics:

- Mathematics problems have one and only one right answer.
- There is only one correct way to solve any mathematics problem -- usually the rule the teacher has most recently demonstrated to the class.
- Ordinary students cannot expect to understand mathematics; they expect simply to memorize it, and apply what they have learned mechanically and without understanding.

- Mathematics is a solitary activity, done by individuals in isolation.
- Students who have understood the mathematics they have studied will be able to solve any assigned problem in five minutes or less.
- The mathematics learned in school has little or nothing to do with the real world.
- Formal proof is irrelevant to processes of discovery or invention.

(Schoenfeld, 1992, p. 69)

Another remarkable and commonly held belief, especially among students who struggle the most with mathematical problem solving, is that regulation of cognition in the form of analysing, planning and verification is a necessary component of problem solving but such students also explicitly believe that these are the tasks of the teacher (Vermunt, 1996).

2.5.3. Metacognitive models in problem solving. A slightly different approach to metacognition emphasises the importance of stable conceptual models in mathematical problem solving. Conceptual models of mathematical concepts or problems exist at different levels of abstraction. At lower levels of abstraction, conceptual models will often be of mathematical objects, while at higher levels of abstraction, conceptual models can refer to a person's conceptualisation of a mathematical problem. At intermediate levels of abstraction a conceptual model can refer to mathematical objects which coordinate other mathematical objects; for example, functions are mathematical objects which structure numbers or variables. Conceptual models are thus hierarchical; they function as "forms" for lower order structures and as "content" for higher order structures (Lesh, 1982, pp.9-10). The stability of a conceptual model involves the degree to which the model is well coordinated; stable models typically integrate more of information relevant to the problem and focus on core features rather than surface detail (Lesh, 1983). When a conceptual model refers to a mathematical object, its meaning is similar to that of a *reified* mathematical notion (Sfard, 1991), however in this context the focus is more on the importance of stable conceptual models to problem solving rather than the mathematical objects themselves.

Expert problem solvers quickly develop stable conceptual models of mathematical problems and these models enable them to access appropriate mathematical resources (Schoenfeld, 1985). Students however, typically have unstable conceptual models of mathematical

problems, especially in the beginning stages of a solution attempt, and this often means that the appropriate resources are not accessed. In the course of a solution attempt, students will often develop several different models of the problem and while there is no guarantee that each successive model will be better, sometimes these models will iteratively converge on an appropriate, stable model of the problem (Lesh, 1983).

The case where a student iteratively converges on a stable conceptualisation of a problem represents an ideal problem solving scenario. In such circumstances a student has the appropriate mathematical resources and the metacognitive tools to recognise flaws in their conceptualisation of the problem and make constructive changes to their conceptual model. In reality however, most students lack these metacognitive abilities, and a poorly coordinated conceptual model is more likely to lead to a wild goose chase than the problem being constructively reconceptualised.

We have already seen how instability in higher order models is damaging for problem solving performance. In addition, weaknesses of lower order content knowledge can also have negative effects on mathematical metacognition in several ways. If the conceptual models of relevant mathematical objects are poorly coordinated, it is likely that core information will be ignored and surface level detail will receive undue attention and this will lead to an incorrect reading of the problem. In addition, a lack of procedural fluency is also associated with various difficulties; if a procedure is cognitively demanding, System 2 type processes will be fully occupied and will not be available for monitoring global progress. Thus, if carrying out a procedure is not furthering global goals it is unlikely that this fact will be noticed. On the other hand, if the attention of System 2 is too focused on overall goals, it is more likely that mistakes will be made in executing the relevant procedures and implementing an appropriate procedure poorly may be mistakenly be interpreted as a broader failure of the selected strategy (Lesh, 1982; Matz, 1982).

The focus on iteratively developing conceptual models is contrasted by English et al. (2008) with the models of Schoenfeld (1985) and Artzt and Armour-Thomas (1992) which are more linear in nature and aim to assist in accessing the known and appropriate resources. The development of increasingly stable conceptual models relies on stable conceptual and procedural resources which can be drawn on in order to develop an integrated model while working on a problem. The iterative nature of this process by which a problem may be

reconceptualised many different times resonates with the observation noted by Schoenfeld (1992) that people's planning activities are opportunistic in nature and that local refinements of a conceptualisation can impact on more global conceptualisations (Hayes-Roth & Hayes-Roth, 1979)

The development of stable higher order conceptual frameworks is of critical importance for mathematical proficiency. Many teaching strategies, including instruction on the use of problem solving heuristic techniques, take for granted that such stable frameworks are in place; these strategies may therefore be inappropriate or even counterproductive for people with unstable conceptual frameworks (Lesh, 1983; Schoenfeld, 1985). If problem solving is viewed primarily as selecting and linking together stable learned procedures then these heuristic strategies can help students to select and sequence appropriate activities. However if problem solving involves iteratively modifying unstable conceptual models of problems then instruction that will help students detect deficiencies in these models and assist in the construction of more stable conceptual structures would be more appropriate.

Construal level theory is implicated in the development of stable conceptual models because these models exist at different levels of abstraction. Unstable models of problem situations are likely to focus unduly on irrelevant surface detail but these models will often change and develop in a short space of time during a solution attempt. If these developments are occurring in a constructive direction they will integrate more information relevant to the situation and focus on core features. The iterative cycle of reconceptualising a problem has been described by Lesh as,

a repeating cycle of "losing the forest when looking at the trees" and "losing the trees when looking at the forest." This sort of "mutation" from wholes to parts, to new wholes, to new parts sometimes led to productive ideas or interpretations; in other cases, it derailed promising solution efforts.

(Lesh, 1985, p 321)

This cycle clearly refers to shifting between Near and Far Modes. Constructive development of conceptual models involves developing higher order more abstract models. These higher order models suggest they should be considered in Far Mode, but they rely crucially on

having stable procedural and conceptual systems at lower orders, so both Near and Far Mode are relevant for successful problem solving.

2.6. Heuristics

The term *heuristic* has a quite different meaning in the context of mathematics education than it does in cognitive psychology. In cognitive psychology a heuristic is an automatic response to a stimulus and is an example of System 1 type cognition (Tversky & Kahneman, 1974; Evans, 2003). As has already been discussed, these responses tend to function well in benign environments but are often inappropriate in hostile or unfamiliar contexts and result in a number of well documented and systematic biases. Heuristics in the context of mathematics are problem solving strategies or rules of thumb that when applied in a problem solving context are likely to be helpful in accessing the mathematical resources necessary to solve the problem (Pólya, 1945; Schoenfeld, 1985). These rules of thumb may be applied automatically and unconsciously by experts but when applied by mathematical novices they are intended to be applied consciously, if nothing “comes to mind” when attempting a solution. Heuristics in a problem solving context then are associated with System 2 type reasoning and thus denote an entirely different type of reasoning compared to the term as used in cognitive psychology. Problem solving heuristics are closely related to both mathematical resources and to metacognition. They are more general and exist at a higher level of abstraction than routine procedures and algorithms learned by rote but they often are presented in a procedural form. In many cases they are procedures where the inputs are themselves procedures (Hiebert & Lefevre, 1986). Heuristics are also related to metacognition in that they are intended to assist with accessing relevant resources an individual may have, but they are more detailed and exist at a lower level of abstraction than the metacognitive functions already discussed. Metacognition usually refers to global decision making rather than the more detailed and specific form heuristics take. Heuristics are typically considered to be an aspect of cognition rather than metacognition (Schoenfeld, 1992). Given the similarities of heuristics to resources and to metacognition it is useful to think of heuristics as a bridge between resources and metacognition or as an attempt to “proceduralise” some metacognitive tasks. In a similar way, heuristics occupy a middle ground between procedural and conceptual forms of mathematical knowledge as they are not readily classified as an example of either type of knowledge (Hiebert & Lefevre, 1986).

2.6.1. Rationale for Heuristics. One reason for the study of heuristics is that experts, in their own development into proficient problem solvers, tend to develop similar problem solving strategies (Schoenfeld, 1985). In an individual, these strategies evolve over a long period of time and can be idiosyncratic but when many individuals develop similar strategies this is evidence for the effectiveness of these strategies. Teaching these strategies directly could, in principle, be a highly effective educational approach. Since certain strategies are an important component of expert behaviour, teaching them could prevent students from attempting many less fruitful strategies. As a result students can develop productive strategies more quickly which would not only save time, but could prevent students from getting frustrated when sincere effort bears relatively little fruit.

Heuristics are the result of the systematisation of these problem solving strategies (Schoenfeld, 1985). Research into the development and use of heuristics to aid problem solving has followed in the footsteps laid down by George Pólya in his book *How to Solve It* (1945). He proposed a number of techniques to help in the solution of genuine mathematical problems. It is important to stress that even though heuristics can sometimes have the appearance of a procedure as in the case of the suggestion to draw a diagram, they are not algorithms which can simply be learned by rote and then applied as has sometimes been implied by the use of the term (Schoenfeld, 1992). They are intended to be applied in novel contexts where there is genuine uncertainty about which resources are appropriate and how to proceed in general. Examples of these include exploiting analogy, drawing a figure, decomposing and recombining, induction, variation and working backwards, where many of these heuristics are related and contain elements taken from other heuristics (Pólya, 1945).

Exploiting analogy. Examining one of Pólya's heuristic strategies will serve to highlight the potential benefits of such strategies but also demonstrates why they are difficult for students to implement successfully. The heuristic strategy we shall examine, selected somewhat at random, is the technique of Exploiting Analogy (Pólya, 1945). Analogous objects or events are the same when viewed in certain ways. When a country acts in a militarily bellicose way, it could be compared to a bully picking a fight on a playground. Thinking and reasoning by analogy is an important and natural part of everyday, common sense thinking and it is employed with wildly varying levels of precision. An argument from analogy can be so imprecise as to completely lack meaning but it could also be sufficiently precise so as to constitute an example of formal mathematical proof (Pólya, 1945). One way

of exploiting analogy is to notice a mathematical problem is analogous to a simpler or better understood one which, when solved, can be used to solve the original problem. This process can involve essentially solving several simpler problems. Analogy can also be used to develop plausible forecasts about the nature of a solution. These forecasts should not be mistaken for mathematical proof but can they be helpful in formulating a solution to a problem.

This heuristic technique highlights both the benefits and some of the difficulties of using heuristics in problem solving. It is probably easy to recall from one's own experience examples where using analogies helped to see a problem in a new and productive way, possibly leading to the problem solution. It is a short step from remembering an example taken from past experience to seeing that a similar process can help solve mathematical problems if you had never consciously thought to employ this technique before. There is also the simple fact that people successful in the domain of mathematical problem solving frequently use this technique so there is a *prima facie* case for attempting to emulate it.

Several potential difficulties for successfully employing the technique are also apparent, however. When using analogies in everyday life one typically understands the objects or actions compared by the analogy, otherwise it is difficult to see why the analogy would be considered appropriate. In fact, in an apt use the analogy implies a deep, abstract understanding of core features two different things share because this is the basis for the analogy. Mathematical objects and concepts are typically not well understood by novice mathematicians. The relevant mathematical object may not be *reified*, (Sfard, 1991) so they often cannot “see” the objects in their minds eye. If this is the case an appropriate analogy will likely not be available because the core properties cannot then be related to core properties of *anything* else. Similarly, Pólya (1945) recommends “decomposing” the problem into several simpler ones provided the resulting problems are “intelligently connected” (p. 38), but novice mathematicians will often find it difficult to ensure that the problems are intelligently connected. Again, seeing that a simpler part is coherently related to the original problem which, it should be remembered, is not well understood, requires a deep understanding of the relevant mathematical content. Even being aware of what content is relevant cannot be taken for granted and is one of the difficulties heuristics are designed to address. Further difficulties are that even if an appropriate analogous problem has been solved, it may not be an easy task to relate the solution to the original problem and flawed

forecasts about the nature of the solution based on the analogous problem may be difficult to discern from appropriate forecasts.

Inherent Complexity of Heuristics. Difficulties of applying heuristic strategies have been noted by many mathematics education researchers (Schoenfeld, 1985; English, Lesh, & Fennewald, 2008). In the decades following the publication of *How To Solve It* there was a great deal of research into developing heuristic strategies in the hope that they would be effective in helping school and university students develop their problem solving skills. The research has not, in general, met with much success (Begle, 1979; Lester & Kehle, 2003). Schoenfeld (1985) has observed that while heuristics such as those proposed by Pólya may be accurate *descriptive* representations of successful problem solving, they are not sufficiently detailed or *prescriptive* to be of much help to students with relatively limited mathematical knowledge bases. Relatively simple-seeming strategies such as “consider special cases” mask their complexity and create an illusory sense of coherence. Short statements of the heuristic like this function as labels for a class of, sometimes only superficially related, sub-strategies which are carried out in several phases and require a great deal of subject knowledge (Schoenfeld, 1985).

In a detailed discussion of two of Pólya’s proposed heuristics, Schoenfeld (1985) demonstrates the complexity of these strategies and points out that even their correct use can lead to failure in the absence of sophisticated metacognitive oversight. The strategy of considering special cases is revealed to be a summary statement encompassing five sub-strategies (one of which is itself divided into two) which means the suggestion to consider a special case is far too general and provides no clues for which sub-strategy would be appropriate. The heuristic of establishing sub-goals which, when fulfilled, can be used to solve the original problem is also much more complex than a naïve reading might suggest. Five simple problems from the same subject domain which can be solved with the use of sub-goals are revealed to each require a different approach, each of which is not a trivial application of the heuristic.

Using heuristics to examine an idealised, successful solution to a problem included in *How to Solve It*, Schoenfeld identifies four distinct heuristic strategies that are applicable to the problem. After employing two strategies, including two variants of the same heuristic, the idealised student settles on working on a modified version of the problem which is tackled

using the heuristic of trying to solve an easier, related problem involving the use of a construction. However, once correctly deciding to construct a figure “similar to the one you need” (Schoenfeld, 1985, p. 86) there are several plausible figures one could construct and selecting the wrong one guarantees failure. Selecting the correct figure to construct is a significant metacognitive decision and is not an easy one to make. This is an example of an idealised student making efficient use of heuristic strategies and making smooth progress and yet applying these methods still turns out to be extremely complex even after assuming the student had a good understanding of the relevant resources. An attempt to demonstrate effective use of heuristics in an instructional setting will typically bypass these complexities (Schoenfeld, 1985), possibly because a teacher is unaware of them, and while students may be able to make sense of the solution procedure they will nevertheless be unprepared to effectively employ the strategies themselves. One final difficulty in the successful application of heuristic strategies is that, despite being specified in general ways without reference to particular subject domains, effective use requires extensive knowledge of a particular domain and they are difficult to successfully use in new domains (Schoenfeld, 1985).

One way of thinking about the difficulties in using heuristic strategies effectively is that experts know how to use these strategies *because they are experts* and have a deep conceptual understanding of the content knowledge in the relevant domain (English et al., 2008). Explicit instruction in the use of heuristic strategies has shown positive results in the past but often these interventions are provided by atypically qualified instructors and over a long period of time. It is difficult to tell if the improvement resulted from the heuristics or the deeper conceptual understanding obtained through the unusually good teaching over a relatively long period of time (English et al., 2008).

An illustration of the point that effective use of analogies assists in understanding advanced concepts comes from a Richard Feynman anecdote. Feynman reportedly said that if he understood a topic in physics he was researching then he could deliver a lecture appropriate to first year physics students (Reis, 1999). While the students may find such a lecture engaging and have the sense they understood what was covered, it is unlikely they will have developed an understanding of the concepts discussed. Rather, the lecture will likely employ analogies from every day experience that the students *do* understand but it is not likely that they will be able to transfer the understanding into the domain of physics. However, such a lecture is more likely to be understood and appreciated by *other physicists*. If the topic is one

of current interest to physicists and has not been digested into a standard form familiar to all experts, such simplified and general analogies may be helpful in providing a different, constructive way of looking at the original material. Daniel Dennett has proposed exploiting this phenomenon to help philosophers“ constructively engage in debate with one another. He suggests that philosophers can make themselves better understood in special sessions where they address an audience of undergraduate students (with other academics in attendance), rather than engaging other philosophers directly (Dennett, 2013). The point as it applies to mathematical problem solving is that analogies are most likely to be successfully employed by those who already possess a deep understanding of the mathematical content. The fact that an appropriate analogy may be drawn from common experience may be less useful to a mathematical novice in the problem solving situation than an expert may suppose.

Despite recognising the considerable complexity and difficulty of implementing heuristic strategies Schoenfeld has argued that heuristic strategies can be productively taught and has provided evidence of successful problem solving instruction which can be achieved, at least in some cases, by dealing with the complexity head on. Heuristic strategies can be rendered more *prescriptive* rather than *descriptive* by providing students with a larger number of more specific procedures that are individually comprehensively learned, making tacit processes explicit and emphasising conceptual understanding and metacognitive processes as well as specific procedures (Newell, 1966; Schoenfeld 1992).

However, even with this more comprehensive approach there are difficulties in successfully teaching heuristic problems solving strategies of the sort discussed above. Beyond the obvious difficulty of widely applying such a demanding pedagogic program some research suggests that heuristics still struggle to make an impact on problem solving in schools (Lester & Kehle, 2003). Heuristics that are effective for individuals with stable conceptual frameworks may actually be counterproductive in individuals with unstable frameworks, which is typically the default situation (Lesh, 1985). It has been suggested that heuristics of the sort described by Pólya and Schoenfeld are appropriate for those students with stable procedural knowledge but that effective heuristics for those with unstable frameworks are those that assist in identifying deficiencies in these conceptual frameworks (Lesh, 1985). In other words, different heuristic strategies are appropriate for different people.

Finally, heuristic strategies advocated by Pólya and Schoenfeld and those appropriate for students with unstable conceptual models vary in the level of abstraction they require. The

heuristic strategy of exploiting analogy is highly abstract. Recognising that a well understood phenomenon is conceptually similar to a novel mathematical problem involves neglecting all the irrelevant detail and context from the context where the core idea is understood, which means a high degree of abstraction. Exploiting analogy will likely invoke a Far mental mode. The heuristic strategy of drawing a diagram of the situation is helpful by rendering the problem in a more concrete way. Similarly, developing stable conceptual models of a problem also involves focusing on core features of a problem and neglecting irrelevant surface detail. Heuristics that assist with developing stable conceptual models thus assist in developing more abstract construals of the problem. As a result, construal level theory is relevant to both types of heuristic because of the importance of the level of abstraction adopted.

2.7. Conclusion

A theme running through all the branches of enquiry outlined above is that reasoning occurs at different levels of generality or abstraction. The research on procedural and conceptual knowledge in mathematics and mathematical cognition and metacognition highlights the importance of different levels of mental construal in different contexts. The psychological research on dual process theory and construal level theory seeks to explain how and why humans are able to reason using different levels of abstraction. The research into mathematical problem solving heuristics does not deal explicitly with abstraction but different heuristics require different levels of abstraction in their implementation and as such heuristic strategies implicitly also concern levels of abstraction.

The idiom of failing to see the forest for the trees features frequently in the mathematics education literature (Lesh, 1985; Hiebert & Lefevre, 1986; Schoenfeld, 2013). Sometimes, though less frequently, this idiom is reversed and researchers refer to the failure to see the trees for the forest (Lesh 1985; Alter et al., 2010). The idiom is mentioned so frequently in the literature on construal level theory (Wakslak, Trope, Liberman, & Alony, 2006; Trope, 2004; Dhar & Kim, 2007) that it almost functions as its unofficial slogan.

Investigation of these different areas of research demonstrates that the two theories of cognitive psychology can plausibly serve as lens through which to view several areas of mathematics education research. The following two chapters will investigate this possibility in more detail. Chapter 3 investigates the effect of level of mental construal on students“

mathematical performance in two distinct problem solving situations; one in which conceptual understanding is the focus and the other in which procedural fluency is emphasised. Chapter 4 investigates the connections between the theories of psychology and mathematics education in detail.

3. Case Study

3.1. Research Question

Given the discussions of Chapter 2 it is plausible that construal level will have an impact on mathematical performance. The second part of the research question as stated in the introduction is addressed in this chapter:

Does the level of mental abstraction induced by contextual priming have an impact on mathematical performance? If level of mental abstraction does influence mathematical performance what is the nature of this influence?

Many factors influence mathematical performance and construal level may impact on some and not others. In general, more conceptual approaches to mathematics education can be thought of as more abstract and more procedural approaches can be thought of as more concrete. As a result the focus of the present study is to discover the effects of construal level on performance relating to problems that are either primarily procedural or conceptual in nature. The results will then be viewed in the context of the procedural versus conceptual approaches to mathematics education discussed in Chapter 2.

Factors beyond strictly mathematical considerations are also relevant here. Previous research by McCrea, Wieber and Myers (2012) has suggested that construal level affects men and women differently when it comes to performance on certain mathematical tasks. The question of whether construal level influences men and women differently is also addressed.

3.2. Testing the Effect of Construal Level on Mathematical Performance

3.2.1. Methodology. In order to address the questions stated above the study was split into two stages. The null hypothesis during the first stage is that construal level has no effect on mathematical performance, regardless of problem type. The second stage is based on the results and analysis of the data collected during the first stage. A hypothesis of the effect of construal level is proposed and tested for confirmation. The priming tasks and the mathematical problems the students were required to complete are included in the appendices.

Due to practicality and ethical concerns this was a quasi-experimental experimental empiracle study. Students were not allocated to control groups at random and the study was not performed in laboratory conditions. The quazi-experimental design of the study means that it is not possible to properly account for any confounding variables which may arise, which means that it is not possible to draw firm conclusions from the data collected regards of any tests of statistical significance. On the other hand, the quazi-experimental design of the study means that the data was collected in conditions very similar to typical tutorial conditions and as a result any overt effects on performance which may arise from laboratory conditions and randomised assignments to groups was minimised.

3.2.2. Stage 1. The first stage of the study involved gathering data from two separate interventions designed to determine the effect of construal level on mathematical performance. The first intervention attempts to determine the effect on performance on mathematical problems where the emphasis is on conceptual understanding while the second intervention aims to determine if there is an effect on performance where the emphasis is on procedural fluency.

3.2.3. First intervention: Conceptual problem. The first intervention involved students enrolled in ASPECT, a foundation programme in engineering studies, and took place in the context of a routine mathematics tutorial for the course END1017F on the afternoon of the 12th of April 2012. The course was taught by Dr Tracy Craig. In order to test the effect of construal level, roughly half of the students were “primed” to adopt a concrete construal mind-set and half were primed to adopt an abstract construal mind-set. At the end of the tutorial, all of the students wrote the same short test. The students wrote a short test at the end of every tutorial for the course. Usually however they wrote the test in the same groups of three they were in for the entire tutorial period, but for the purposes of this study, they worked on the tutorial in groups of three but wrote the test individually. This is true for all the data collection sessions. The test results comprise the principle data for the study.

Priming. The priming of the students to adopt abstract or concrete construal styles was achieved by interventions at the beginning and end of the tutorial session. Students were randomly assigned to one of two groups at the beginning of the tutorial. Students in both groups were required to write a short paragraph describing an aspect of their studies in engineering. The second instance of the priming took place as part of the short test itself and

involved drawing on Construal Level Theory in order to alter apparently trivial details in the way the test problem is framed. Both stages of the priming were designed to induce students to adopt either an abstract or concrete construal style consistently throughout the tutorial and short test.

Initial priming intervention. The first priming task took place at the beginning of the tutorial and took the form of a question requiring students to write a short paragraph related to their studies in engineering. They were given ten minutes to complete the task.

Abstract construal group. The priming exercise for the abstract construal group was phrased as follows

*Write a short paragraph explaining **why** want to become an **engineer** (and not a member of some other profession)*

According to Construal Level Theory, this task encourages students to adopt an abstract construal style in several ways.

Asking a “why” question is a common method of inducing an abstract construal style (Trope & Liberman, 2010). Phrasing a question in this way requires the subject to think about the overarching goals of an activity while abstracting away from the concrete details of what specific actions are required to achieve those goals.

The question induces students to focus on the attributes of being an engineer that provide reasons to go through the costly and time consuming process of studying. This primes abstract construal in several ways. First, the student will only become an engineer in the reasonably distant future and Construal Level Theory suggests greater temporal distance primes a more abstract construal. Second, thinking about becoming an engineer is thinking of a hypothetical scenario which is also associated with an abstract construal.

In addition, framing of the question induces students to think of themselves as a member of a group, apprentice engineers. Emphasising the association of oneself as a member of a group rather than as a unique individual draws attention to general features that are believed to be

shared by all members of that group and thus requires abstracting away from the many specific ways in which every person is different.

The attempt to induce students to see themselves as a member of the engineering community (or possibly the community of engineering students) rather than as a member of some other group (gender and race are obvious alternative examples of group membership which are relevant to this study) is significant. A recent study suggested that inducing female students to adopt an abstract construal style caused them to self-stereotype according to the widely held negative stereotype of women i.e. “women are bad at mathematics” and hence to do worse than they otherwise would on a mathematics test (McCrea et al, 2012). The focus on engineering is an attempt to avoid a possibly detrimental self-stereotyping effect and make a positive stereotype more salient. Engineering is a high status profession associated with a high level of mathematical proficiency. By making engineering salient it is hypothesised that a positive self-stereotype will be induced.

Concrete construal group. The task at the beginning of the tutorial to induce a concrete construal style took the form of describing how the students planned to prepare for an upcoming class test.

Class Test 3 is coming up soon. Write up a rough study plan for the test. Your plan should include details about how much time you will spend working on problems and how much time you will spend studying in other ways. You should include how you plan to use the textbook, handbook, your own notes and any other resources.

Asking “how” questions is a common way of inducing a concrete construal style of thinking as it focuses attention on the concrete details of how some goal is to be achieved. The temporal proximity of the test is emphasised because thinking of events as near in time (past or future) also primes a concrete construal style.

Construal Level Theory suggests that adopting a concrete construal style is not associated with subjects employing stereotypes or (potentially detrimental) self-stereotyping. However this task does require the students to consider activities and behaviour appropriate for engineering students in a similar way to the abstract priming task. The engineering theme for both tasks is an attempt to maintain a level of consistency between the groups in order that

any differences between groups will be as far as possible due to the different construal styles adopted.

Second priming intervention: Abstract construal group. The second instance of priming involves altering the test question in subtle ways to either maintain, or (perhaps more likely) reinstate an abstract construal style. The diagram included in the problem simply includes the geometric shapes common to the questions for both groups without added detail or context. The question refers to Australia which is far away and the included company name is unfamiliar and intended to seem “foreign” to a typical South African. These details are trivial in terms of mathematical content and do not affect the problem solution in any way. Despite these apparently trivial changes however, similarly small changes have been shown to have demonstrable effects in analogous experimental settings (Trope & Liberman, 2010).

Concrete construal group. The second instance of priming for the concrete construal style group involves providing extra (irrelevant) detail in the diagram included in the problem. In this case, the image representing the cylinder involved in the question is a picture of a Coke can. Coke was chosen specifically because of its ubiquity and thus will be familiar and unambiguous to all the students. In addition the name used in the question is a familiar South African name and the action in the question is described in the present tense so as to be “near” in time. The context has also been chosen to be one that students will be familiar with; preparing food in a kitchen.

3.2.4. Data collection. The intervention was constructed with the ordinary operation of a tutorial in mind and disruptions to the familiar tutorial environment were kept to a minimum. The tutorial venue was the same for all the students participating in the study. The venue was a single large room with easily movable tables and chairs, frequently found in different configurations. Students typically work in groups of three of their own choosing and they also choose which tables to sit at. Before the beginning of the tutorial the tables were arranged to create a gap in the middle of the venue. Students sitting at tables at either side of the gap made up the two different construal groups. The tutors were made aware of the data collection and the gap in the venue separating the two groups and arrangements were made that ensured that students on either side of the venue received consistent priming throughout the intervention.

Students were not separated by gender. The gender of each student was determined by the name on the submitted script, confirmed by the lecturer if necessary. The choice not to split the students up by gender was deliberate as it is plausible that they would be more likely to self-stereotype if it were made clear to them that gender was a factor under consideration. It would also have caused unwanted disruption to the ordinary operation of the tutorial. In the event, students were divided more or less evenly by gender and in total numbers into the two different construal styles.

The tutorial session began with the first priming intervention after a brief announcement explaining that they would be beginning the tutorial with a short exercise relating to ASPECT, with tutors distributing the priming tasks. The responses to the priming task were gathered in after the ten allocated minutes. The responses were gathered immediately to prevent them from changing their answers later and to prevent students from delaying completing the task till later on in the tutorial which would have had the effect of delaying the priming of their construal state while attempting to complete the tutorial. Tutors distributed the tests at the end of the tutorial in a similar way with half an hour being allocated for the test.

70 students attended the tutorial and completed the test which was marked out of a total of 9. During the course of the tutorial, two students moved to a different part of the venue and as a result received a test with the opposite priming to the priming task at the beginning of the tutorial. The test results from these two students therefore were excluded from the analysis. Thus 68 students completed the process with consistent priming throughout the tutorial. Of the total, 36 students were primed to adopt a concrete construal style and 32 primed to adopt an abstract construal style.

Gender. Of the 68 students who maintained consistent priming throughout, 18 were female and 50 were male. Of the 18 female students, 11 received the concrete priming and 7 received the abstract priming. The 50 male students were evenly split with 25 students in each priming group.

3.2.5. Results. Since no hypothesis of the effect of construal level on performance is being advanced the null hypothesis is that construal level has no effect on mathematical

performance. As a result a two tailed t-test was used to determine whether there was a significant difference in mean performance between students in the two groups. The result for the group primed for concrete construal was a mean of 4.6 out of a possible 9. The mean result for the group primed for abstract construal was 4 out of 9. Comparing the results, the t-test gives $t_{(66)} = 1.35$ corresponding to $p = 0.19$. The results are thus not statistically significant at the 5% level. One reason why the difference between the mean results is not statistically significant is that the variance in the abstract construal group is very large.

The most striking feature of the results is the difference in the standard deviations of each group. The standard deviation for the group with priming to adopt a concrete construal was 1.6. The standard deviation for the abstract construal group was 2.3. Using Levene's Test the probability that the difference in the variance is due to chance is $p = 0.003$. This means that the difference in the variance of the two groups with different priming's is highly unlikely to be due to chance.

The results were also broken down by gender to determine if construal level impacted differentially according to gender. Of the 68 students who maintained consistent priming throughout the tutorial 50 were male students and 18 female. Both groups displayed similar patterns to each other and to the class as a whole.

The male group was split evenly into abstract and concrete construal groups with 25 students in each construal group. The group primed to adopt an abstract construal style achieved a mean score of 4 with a standard deviation of 2.4. The group primed to adopt a concrete construal style achieved a mean score of 4.8 with a standard deviation of 1.76. The two tailed t-test applied as above is associated with $p=0.23$, which is also not significant at the 5% level. Of the 18 female students who maintained consistent priming throughout the tutorial session, 11 were primed to adopt an abstract construal style and 7 a concrete construal style. The abstract construal group achieved mean score of 3.7 out of 9 with a standard deviation of 1.9. The concrete construal group achieved a mean score of 4.3 with a standard deviation of 1.9. Given the size of the sample, it is unsurprising that the difference is not statistically significant with a two tailed t-test giving $p=0.51$.

3.2.6. Discussion. Due to the relatively small size of the tutorial group, the apparently large differences between the mean scores and standard deviations for each group are not statistically significant. The results are nonetheless suggestive. The standard deviation for the concrete construal group is smaller than the standard deviation for the abstract construal group despite having a larger range and higher mean and median score. The difference in spread between the two different groups is demonstrated in the box and whisker plot as shown in Figure 1.

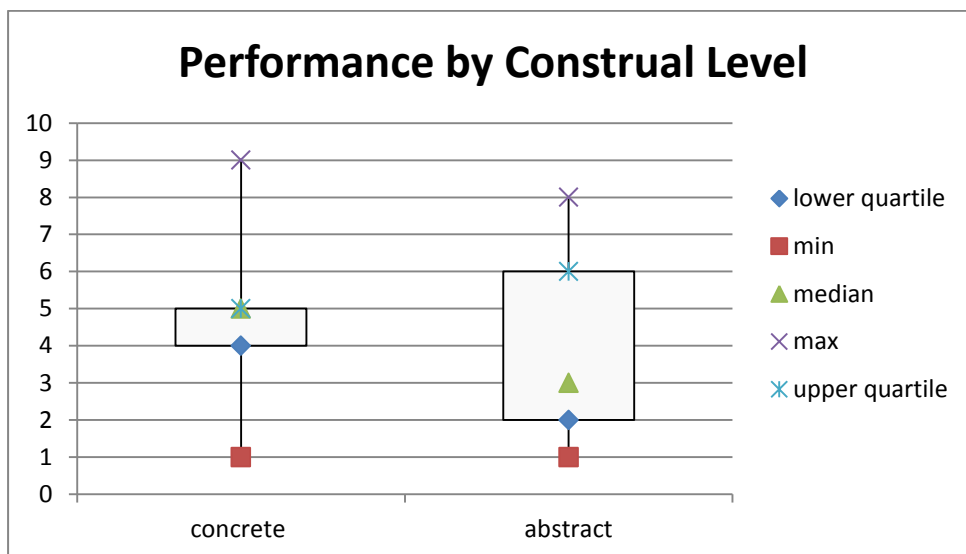


Figure 1. Performance on Conceptual Question

As discussed in the section outlining the intervention, the problem the students were required to work on was intended to be conceptually demanding. Conceptualising the problem correctly is essential before bringing more procedural knowledge to bear on the problem and procedural fluency will be of little use with an incorrect approach.

Students primed to adopt an abstract construal style, while performing more poorly on average were more likely to both perform very well *and* very poorly. The variance among the group with Far Mode priming was statistically significantly larger than the group primed to adopt a Near mental mode. This result is consistent with the hypothesis that students primed to think abstractly, are more likely to approach the problem in a high level, conceptual way, with the students performing well adopting a broadly correct approach and students performing badly having adopted an incorrect conceptual approach. Those students adopting

the wrong general approach were less likely to bring procedural knowledge to bear in an appropriate way, resulting in low marks.

Students primed to adopt a concrete construal style performed slightly better on average, yet despite this higher average mark on the test, their upper quartile was lower than students from the abstract construal group. Students in the abstract construal group were more likely to make substantial progress towards a solution. The results suggest that in general, students with concrete priming were less likely to have a developed idea of how the full solution should proceed, but also that they were less likely to have incorrect conceptualisations of the problem. This is consistent with the hypothesis that students would be more focused on bringing procedural knowledge to bear immediately rather than developing a more general conceptual plan to solving the problem.

Given the quazi-experimental nature of the study, these results cannot be viewed as conclusive despite the statistically different variance between the two groups. There are potentially many confounding influences which the study could not account for. The results should therefore be seen as suggestive of a possible effect that can be investigated in a different empirical study.

Gender. A previous study by McCrea et al (2012) of the effect of construal level on mathematical performance in psychology students, showed that, while the performance of both male and female students was similar when they were primed to adopt a concrete construal, adopting an abstract construal caused the results to diverge. The authors of that study attribute the difference to the effects of self-stereotyping in the students. Adopting an abstract construal causes people to resort more to stereotypes, both of others and oneself. There is a persistent stereotype (Spencer et al, 1999) of women having less mathematical ability than men. The authors suggest that the male students adopted the positive self-stereotype which improved their performance, while the female students adopted the negative self-stereotype which harmed performance.

Since construal level affected men and women differently in the McCrea et al (2012) study we are interested to see if that effect is replicated in this intervention. The effect of construal on performance was similar for both male and female students. Both groups performed better when primed to adopt a concrete construal rather than an abstract construal. The major

difference between the results of this intervention and the results of the previous study, is that male students did worse rather than better when primed to think abstractly.

One possible explanation of the difference between the two studies is the difference in what the students were studying. The students participating in the first study were psychology students and mathematical ability will have been of limited importance for advancement in their studies relative to other subjects. Students in the current study on the other hand are studying engineering, where performance in mathematics is central to academic advancement. To the extent that stereotyping and self-stereotyping is an obstacle to female students pursuing mathematics dependent subjects at school, students will often have had to consciously overcome this stereotype. It is unlikely that female engineering students will still see themselves as being bad at maths once they have reached university level, if anything the reverse is more likely as they have already proved their ability to get this far. If this is the case, then it should not be surprising that construal level did not have a different effect on male and female engineering students.

3.2.7. Second intervention: Procedural problem. The second intervention involved, with a few exceptions, the same students as the students from the first intervention. The students were enrolled in the course END 1018S, a continuation of first year engineering mathematics in the second semester. The intervention took place in the context of a routine mathematics tutorial on the afternoon of the 19th of September 2012. The second intervention was structured in much the same way as the first one. Depending on where students chose to sit in the tutorial venue they would receive priming to adopt either an abstract or concrete construal style and with the priming taking places using two interventions.

Priming. Once again the priming intervention took place in two instances; students in both groups were required to write a short paragraph relating to an aspect of their degree. The second stage of the priming took place as part of the short test itself and involved drawing on construal level theory in order to alter apparently trivial details in the way the test problem is framed to maintain consistent priming throughout the session.

Initial priming intervention. Students were given ten minutes at the beginning of the tutorial to complete the following short tasks.

Abstract Construal group. The priming exercise for the abstract construal group was phrased as follows

When you graduate, what are you expecting your financial situation to be? Will you be paying off student loans or working off a bursary? Do you have a job lined up? If not what will be your main considerations when looking for a job (salary, community development, location near to family or anything else you're looking for in a job)?

The choice of question was guided by similar considerations to those already discussed. The task requires considering a hypothetical and relatively distant future state of affairs both of which are psychologically distant. The phrasing of the question also emphasises overarching values to be taken into account when considering a job rather than practical or mundane considerations of the job which requires an effort in abstraction. Once again students are encouraged to identify themselves as members of the engineering profession.

Concrete construal group. The first priming task for the concrete construal group took the form of considering what stationary they would like ASPECT to provide if given the choice.

Assume ASPECT will be getting extra money next year. The money can only be spent on stationery for ASPECT students. Based on your experiences this year, what stationery should ASPECT buy that would be most helpful to next years' students (Include items up to the cost of a calculator. So things like pens, exams pads and folders etc., not iPads).

The task primes concrete thinking primarily by focusing, including using concrete examples of stationary, on practical details of students' day to day life at university. The END1017F lecturer had already started providing pens and pencils to students so the students were familiar with the project which would reduce social distance and cause the scenario to be less hypothetical. The task also refers to this year, which is psychologically near in time.

Second priming intervention: Abstract construal group. The second instance of priming involves altering the test question in subtle ways to either maintain, or (perhaps more likely) reinstate an abstract construal style. As in the previous data collection, the problem for

students in both abstract and concrete construal groups was mathematically identical and with exactly the same solution procedure.

The question was phrased using language and names taken from Lord of the Rings. Typical of much of the “fantasy” genre in film and novels, Lord of the Rings is very “far” on all the dimensions of psychological distance. The story takes place in a medieval setting in “Middle Earth” and by implication takes place in the distant past and somewhere other than Earth. There are numerous fictional species and the events depicted are highly improbable so the story involves large social distance and is psychologically distant on the hypotheticality dimension. Many (though not all) of the students were familiar with the Lord of the Rings movies or books, which would reduce the social distance induced, but the overall effect would still be to induce an abstract construal style.

Second priming intervention: Concrete construal group. The second instance of priming for the concrete construal style group involves phrasing the question in terms of the student completing the question (near socially) needing to cater for drinks for a party that night which is near on the temporal dimension. Pictures of well-known drinks were included in the question and would be familiar to the students to be near on the hypotheticality dimension.

3.2.8. Data collection. A different venue was used for this data collection. In this venue the desks could not be moved and it was not possible to arrange it in a way that would encourage students to split into roughly equal groups as in the first collection. As a result the class was not especially evenly spread with 23 of 67 students who maintained consistent priming over the whole session allocated to the concrete construal group and 44 allocated to the abstract construal group.

Gender. Of the 67 students who maintained consistent priming throughout the session 15 were female and 52 were male. Of the female students 10 were primed to adopt an abstract construal and 5 a concrete construal. Among the male students, 34 received the abstract priming and 18 received priming to adopt a concrete construal style.

3.2.9. Results. Once again the null hypothesis is that construal level has no effect on performance and a two tailed t-test was used to compare the results.

The mean mark for students primed to adopt a concrete construal stance was 7.9 out of a possible 10 marks with a standard deviation of 2.83. The mean mark for students primed to adopt an abstract construal was 8.4 out of a possible 10 with a standard deviation of 2.84. Comparing the results gives $p = 0.5$ which is not statistically significant.

Gender. Among the male students, the mean for those with concrete priming was 7.7 out of 10 compared with 8.3 for those with abstract priming. The standard deviations were almost identical being 2.81 and 2.8 respectively. The t-test again shows that this difference is not significant giving a p value of 0.45.

The mean mark for the female students who received the concrete priming was 8.6 with a standard deviation of 3.13 compared to 9.5 with a standard deviation of 1.58. Despite the large difference in mean result, unsurprisingly given the number of female students the difference is not significant with $p = 0.46$.

3.2.10. Discussion. This data collection examined the effect of construal level on performance on a mathematical problem that was primarily a test of procedural fluency and there appears to be no effect on performance. The similarity of the results for students in both construal states can be demonstrated visually in the box and whisker plot shown in Figure 2. For the procedural question, the standard deviation for both groups was almost identical, whereas there was a marked difference for the conceptual question. As Figure 2 shows, the interquartile range for the abstract construal group was actually smaller than the interquartile range for the concrete construal group.

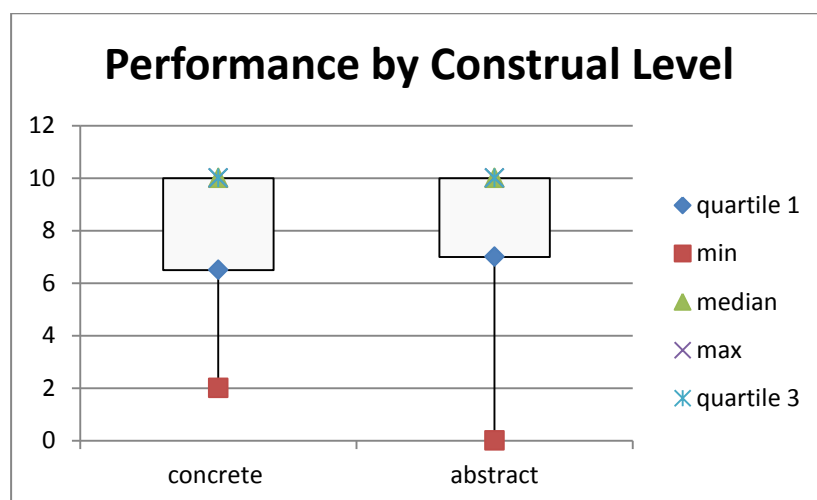


Figure 2. Performance on Procedural Question

This result stands in contrast to the first data collection which examined the effect of construal level on performance on conceptually challenging problems where construal level seems to have an effect.

The most striking difference between the two interventions seems to be that in the case of problems requiring conceptual understanding, being primed to adopt an abstract construal style resulted in a much larger range of results, whereas problems focusing on procedural fluency have an almost identical variance in results.

Consistent across both interventions is the fact that construal level does not seem to impact differentially by gender.

The results from both interventions suggest that a hypothesis can tentatively be put forward to be tested i.e. that construal level impacts performance on mathematical problems requiring conceptual understanding. Testing this hypothesis is addressed in stage 2 of the case study.

3.3. Testing Hypothesis on the Effects of Construal Level.

3.3.1 Stage 2. Stage 2 of the study involves the testing of the hypothesis arrived at after analysing the data discussed in stage 1. The new, positive hypothesis, being tested is that the construal style adopted impacts performance on mathematical problems with an emphasis on conceptual understanding. In particular it is predicted that the group of students primed to adopt Far mental mode would have a larger variance than the group primed to adopt a Near mental mode.

In order to test this hypothesis, the intervention from stage 1 of the study focusing on the effect of construal level on performance for conceptually demanding problems was repeated with the 2013 cohort of ASPECT first year students on the 17th April 2013. To aid the comparison the data was collected in as similar a way as possible to the intervention in 2012. As a result descriptions of the procedures and priming tasks are not repeated.

However, there are instances where the tutorial environment differed from that of 2012. The course END1017F was taught by Mrs Kalpana Nathoo and not Dr Tracy Craig. While the class had been exposed to related rates problems before the tutorial session, they had not been exposed to the specific type used in the data collection. The 2012 cohort had been exposed to problems of this type in lectures. In order to make up for their absence in classes, an example was covered towards the end of the tutorial. Specifically covering an example of this type during the tutorial period did not happen in the 2012 data collection. In addition, the tutor for the course was unable to attend the tutorial which resulted unavoidably in my presence in the venue during the data collection in the role of tutor. Finally, the venue was a lecture venue with fixed seating and “benches” as opposed to a tutorial venue with movable chairs and desks. As in the 2012 collection however students were free to choose their own places and they worked in groups of three as before.

38 students attended the tutorial and wrote the short test at the end of the tutorial period. Of these, 20 receive priming to adopt an abstract construal style and 18 received priming to adopt a concrete construal style. The 2013 cohort was considerably smaller than the 2012 cohort and a large majority of students were male, as a result the data for effects of construal level by gender are not considered in this case.

3.3.2. Results. In this case, the hypothesis is that construal level has an effect on mathematical performance. As a result a two tailed t-test, where the variances of the two groups are *not* assumed to be equal, is used to compare outcomes.

The result for the group primed for concrete construal was a mean of 4.33 out of a possible 9 and a standard deviation of 2.03. The mean result for the group primed for abstract construal was 4.75 with a standard deviation of 2.07. Comparing the results, the t-test gives $p = 0.54$. The results are thus not statistically significantly different to the 10% level.

Graphically, it can be seen that while the range of marks for students primed to adopt an abstract construal style is larger, the interquartile range is smaller than for the students primed to adopt a concrete construal style. This result fails to replicate the results from stage 1 of the study and does not confirm the hypothesis being tested.

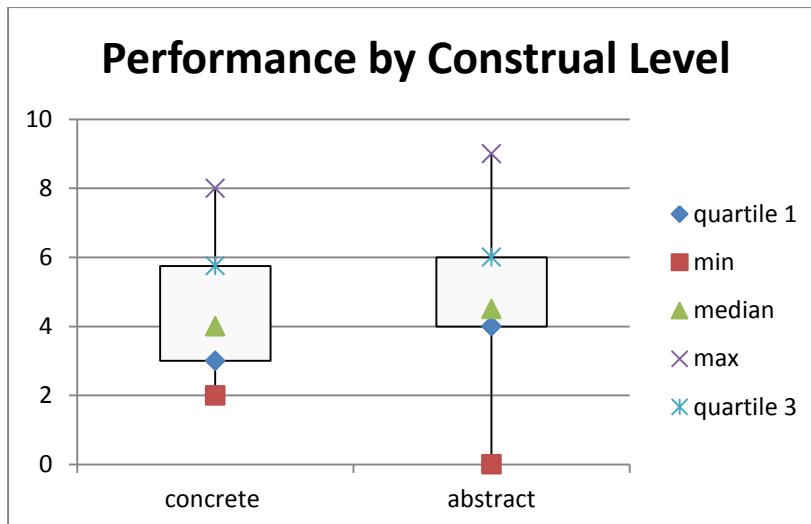


Figure 3. Performance on Conceptual Question

3.3.3. Discussion. The results of the data collection carried out in 2012 suggested that students primed to adopt a Far mental mode would have a larger variance of outcomes than those primed to adopt a Near mental mode. Analysis of the data showed that there was no statistically significant difference observed between the groups with different priming. One possible explanation for the failure to replicate the results obtained in the 2012 data collection is the reduced sample size in the 2013 collection. With such a small number of students in the 2013 cohort and the necessity of splitting the class further into 2 even smaller groups meant it was unlikely that any significant results would be observed.

It is also plausible that the difference in the tutorial environment or other confounding factors contributed to the failure to replicate the results. In particular, the fact that the lecturer covered an example similar to that in the test plausibly undermines the comparison between the two sets of results. The mean mark for the 2013 class was higher than the mean mark for the 2012 cohort despite not covering similar lectures in regular lectures. While the difference is not significant, it does suggest that the demonstration of such a problem so close to the test had an effect. Another possible mechanism affecting the outcome is that the demonstration of the problem so close to the test had the effect of priming students in Near mental mode since a similar example was available so Near in time.

In conclusion, the hypothesis that level of construal impacts performance on mathematical problems with an emphasis on conceptual understanding was not confirmed. Despite this

failure to replicate the findings from the first data collection the results from the first data collection remain suggestive and are consistent with what construal level theory would predict. Rather than abandoning the hypothesis that the level of mental construal has an effect on the variance in outcomes it should be tested again in a randomised trial and in laboratory conditions where the effects of construal level can be better isolated.

4. Discussion

The preceding chapters have dealt with branches of cognitive psychology and mathematics education largely separately. With the partial exception of construal level theory which is a relatively new branch of psychology, each branch has its own extensive literature with established bodies of theory and varying degrees of empirical support, but there has been little attempt to integrate these branches of psychology with the branches of mathematics education research. Recently, researchers have begun applying dual process theory to mathematics education but almost nothing has been written about the possible application of construal level theory to topics in mathematics education. In the absence of a body of research relating the fields it has been necessary to consider them separately in order that potential links between the fields are well grounded.

Theories of cognitive psychology apply to everything that people think about which includes how they reason about mathematical concepts and how they approach solving mathematical problems. Dual process theory and construal level theory appear particularly relevant in the case of mathematics because of the way they deal with systematic, step-by-step reasoning and abstract thinking which are particularly associated with mathematical reasoning.

In the case of dual process theory and construal level theory there is extensive empirical support for the central claims of each theory. Dual process theory asserts that there are two distinct types of thinking; fast, effortless, associative thinking which is the default mode of thinking and slow, effortful and systematic reasoning that oversees and sometimes overrides the automatic mode of thinking. Construal level theory asserts that thinking about psychologically distant object or events causes us to think in a more abstract way and vice-versa. Dual process theory deals with fast and slow thinking and construal level theory deals with Near and Far thinking. All four types of thinking are important for mathematical understanding and problem solving. The current chapter will investigate the ways in which dual process theory and construal level theory can be applied to existing research in mathematics education.

4.1. Procedural and Conceptual Knowledge

4.1.1. Procedural and conceptual knowledge and dual process theory. Despite the obvious parallels between this branch of mathematics education research and both dual process theory and construal level theory, there has been relatively little attempt to integrate these theories into a unified approach. The two theories of cognitive psychology are more general than research into procedural and conceptual knowledge as they apply to all forms of cognition across many domains and as a result, much of the research into the procedural and conceptual distinction in mathematics education can be viewed through the lens of both dual process theory and construal level theory. In addition the evidence accumulated in mathematics education research can be used to bolster the case for both psychological theories. Research into procedural and conceptual knowledge in mathematics is implicitly part of both dual process theory and construal level theory. Just as much of the evidence in favour of both theories predates the emergence of the formulation of the models and has since been integrated, data from mathematics education should similarly be integrated and recently some researchers have argued that this integration is worth doing (Leron & Hazzan, 2006; Gillard et al., 2009).

An important insight of dual process theory is that systematic step by step reasoning is extremely effortful and places great demands on working memory. Step by step, systematic reasoning is carried out by System 2 and thus it follows that routine mathematical procedures and step by step algorithmic procedures must be performed by System 2. However, since the processing capacity of System 2 is very limited, being occupied with carrying out the routine steps of a procedure mean it cannot simultaneously actively monitor progress and exert executive control. There is a strict upper limit to how many pieces of information can be kept in working memory at any one time. Teachers often require students to carry out a specified procedure and at the same time expect students to make connections at a higher, conceptual level. One instructional technique with the aim of developing conceptual understanding is to require students to work on many problems that are structurally similar. Students may be told that all problems embody the same principle, however if mastering the procedure and noticing the conceptual links between the problems both require conscious, systematic reasoning, dual process theory suggests that these reasoning processes cannot be carried out simultaneously. Teachers should not expect students to be able to attend to the demands of developing procedural fluency and to simultaneously be on the lookout for deeper conceptual

links. The serial nature of System 2 reasoning means that attention should be directed to one of these tasks at a time, perhaps in an iterative fashion. It is not being argued here that procedural and conceptual knowledge develop separately, rather there seems to be an instructional inference from the limited capacity of System 2 reasoning and working memory.

Another important insight of dual process theory is that novel tasks that must initially be carried out using System 2 cognitive processes can be carried out by System 1 processes through repetition and practice. System 1 has a much larger processing capacity, has access to the potentially vast amount of information stored in long term memory and can perform many operations simultaneously. If a procedure becomes familiar enough to be carried out by System 1, System 2 is freed up to focus on other important tasks and is more likely to be able to make the links so crucial to conceptual knowledge.

The example of novice and expert chess players can illustrate this point. To begin playing chess one first needs to learn the rules. However, even a beginner must play with the intention to checkmate the opponent. To play effectively one needs to consider strategy and possible developments resulting from each potential move. Initially, both of these are tasks for System 2 and if a subject is engaged in an actual game she simply will not have the cognitive resources available to devote much thought to strategy and the mental simulation required for effective moves if she constantly has to remember how each piece is able to move. However, with time and practice, knowledge of the rules becomes automatic and is incorporated into System 1 type reasoning. In addition, certain commonly occurring configurations and associated appropriate moves will become stored in long term memory available to System 1 type processing. This transfer of skills and knowledge to System 1 frees up the player to think about longer term consequences of moves and general strategy. In other words, the procedural knowledge of the rules is an essential prerequisite for developing a deeper conceptual understanding of chess strategies.

Dual process theory suggests that System 1 reasoning can never be shut down even with conscious effort. The best that can be done to “stop” System 1 cognition is to consciously focus attention elsewhere (Kahneman, 2011). System 1 type cognition is the bridge between perception and System 2 type reasoning. This means that subjects always have an intuitive reaction generated by System 1 to a given situation. System 2 can override this intuition (in fact that is one of its primary roles) but it cannot prevent it. If System 1 is operating in a

benign environment, its intuitive reactions are generally a good guide to action (Stanovich, 2012). However if the intuitive, automatic processes of System 1 are operating in a hostile environment it is prone to systematic errors and it is in this type of situation that System 2 thinking will ideally override the suboptimal outputs of System 1. Novice mathematicians are effectively perpetually in a hostile environment in this sense. System 1 is simply not designed to reason in a logical step by step manner or deal systematically with many pieces of information simultaneously. An example of this is the base-rate fallacy where subjects neglect the general frequency of an event when presented with new evidence relating to such an event. A notorious example is that doctors, who are all highly educated, typically neglect the general prevalence in the population of a condition and incorrectly interpret positive test results as being too high (Casscells, Schoenberger, & Graboys, 1978). Mathematical novices will have intuitive reactions to mathematical objects and problems, but appropriate action is often counterintuitive and requires specialised training to perform correctly. This results in a particularly insidious source of difficulty in solving mathematics problems because an individual may have acquired the appropriate resources to solve problem but may still be led astray by an incorrect intuitive reaction. Effective metacognition could overcome this difficulty but research suggests that metacognitive processes are typically triggered by cognitive conflict or surprise (Alter et al., 2007). However, inappropriate intuitions in a problem situation may generate neither conflict nor surprise meaning that System 2 does nothing to intervene. Conceptual understanding is essential if System 2 is to be able to exercise its function of overriding these inappropriate intuitive responses effectively.

One reason that System 2 may fail to perform its function of overriding intuitive responses is that System 2 is often engaged to try to justify the incorrect intuition, which simply compounds the original difficulty. It is important to remember that System 2 type reasoning abilities are a recent evolutionary phenomenon and evolved not to solve mathematical problems but to deal with problems of people's environment and many of the most important skills required to survive and thrive are social skills (Cosmides & Tooby, 1992; Stanovich & West, 2000). In other words, sometimes it would be important that System 2 processes "get it right", but in many social contexts, what is important is convincing others that you are right regardless of the truth. There is considerable evidence that much reasoning simply performs the role of providing justification for automatic intuitive responses (Haidt, 2001; Kahneman, 2011). The tendency of System 2 to justify knee jerk reactions may be helpful in social situations and even for mathematical problem solving for experts but it is disastrous for

novices for whom mathematical problem solving contexts are hostile environments, no matter how accommodating the instructor (Leron & Hazzan, 2006).

Another important point is that over an extended period of time, repetition and practice under effective instruction can effectively transform a hostile environment into a benign one because the System 1 type cognition responsible for intuitive judgments can be changed. Examples from learning to drive and chess can illustrate this point. Competent drivers may know in some sense that they are supposed to “steer into the skid” if they lose control of their car but this is deeply counterintuitive for most people and they are unlikely to be able to call upon this knowledge in an emergency. Professional drivers on the other hand will have experienced situations proving the wisdom of the advice many times and their knee jerk reaction is indeed to steer into the skid without recourse to conscious reflection. For professional drivers more situations occur in a benign environment.

In the domain of chess, even if novice players have reached the stage where knowledge of the rules has been transferred to System 1, they will have no way to quickly identify poor moves. System 2 has to evaluate all possible moves which implies that the computational limits of System 2 ensure that no move can be examined in depth. Expert players have a great deal of experience and are familiar with many board configurations, each of which is represented as a single “chunk” of information. As a result experts automatically reject poor moves freeing up attention for detailed analysis of better candidates. Again, transfer of knowledge to System 1 is essential for higher order conceptual connections to be made.

The role of intuition in mathematics has been addressed in the mathematics education research (Fischbein, 1987; Sfard, 1994)). Professional mathematicians often describe their thinking processes in highly metaphorical and intuitive terms. Mathematical knowledge is described as self-evident, coercive, global and extrapolative (Fischbein, 1987). Knowledge of this type is clearly System 1 and the description of this type of knowledge as coercive has particular resonance (Kahneman, 2011). Mathematical understanding is seldom described by the application of formal deductive steps or with appeal to axioms or properties. Often mathematicians will arrive at an important result before they develop a step by step logical proof. According to Halmos (1985), the prominent mathematician Lefschetz “saw mathematics not as logic but as pictures. His insights were great, but his “proofs” were almost always wrong.” (p. 87). Mathematicians have developed their powerful intuition over

the course of many years during which concepts could only be considered using System 2. The concepts which eventually become “self-evident” were initially processed according to definitions and formal rules of the discipline. In other words, mathematicians are able to understand mathematical concepts using System 1 rather than System 2 and this ability is profoundly important.

It is System 1 which automatically populates the mind and contextualises information present in a particular context, including mathematical problem solving (Stanovich, 2003). Not only will this automatic, intuitive process be more constructive for expert problem solvers since mathematical problem solving is a more benign environment for experts, but construal level theory is implicated here as well. Depending on whether an individual has been primed to adopted a Near or Far mental mode those contextualisations can involve abstract or concrete construals of the problem. Depending on the demands of a problem, even expert problem solvers can be helped or hindered by being in either an appropriate or inappropriate mental mode. The relevance of construal level theory for procedural and conceptual knowledge is discussed in the next section.

4.1.2. Procedural and conceptual knowledge and construal level theory. The relevance of construal level theory to research into procedural and conceptual knowledge in mathematics education is intuitively obvious as both are fundamentally concerned with need to be able to understand and reason about objects at different levels of abstraction. Construal level theory claims that for humans to be able to think about objects, events or people far removed from the here and now we have to form abstract construals of these things. The need for abstract construals stems from the fact that our processing capacity is limited and abstractions allow for the neglecting of information deemed irrelevant to particular goals. Despite the fact that our cognitive processing ability has expanded radically over the course of our evolutionary development, such abilities are extremely demanding of available resources and as a result, cognitive abilities that allow us to think effectively about distant objects without placing excessive demands on available resources are very valuable. Conceptual understanding and abstraction are intimately connected. Conceptual knowledge as defined by Rittle-Johnson and Siegler (1998) involves "understanding the principles that govern the domain and of the interrelations of pieces of knowledge in the domain (although this knowledge does not need to be explicit)" (p. 77). Appealing to governing principles and the emphasising links between pieces of knowledge are hallmarks of abstract thinking. Both

conceptual understanding and abstraction are associated with links and relationships between objects. Both conceptual understanding and abstraction are associated with removal from a particular context and the ability to apply core features in different contexts.

Procedural knowledge and low level, concrete detail are both more detailed and more closely tied to particular contexts and sensitive to local details. Procedural knowledge is associated with routine tasks, algorithms, action sequences and specific, detailed strategies (Rittle-Johnson and Siegler, 1998). Successful application of procedural knowledge depends on correct step by step sequences and errors in matters of detail will typically result in the breakdown of the process.

There is evidence that if a more abstract construal is required to complete a task then adopting a Far mental mode facilitates completion of this task. For example participants' performance on a Gestalt Completion Test is improved when primed to adopt Far mental mode (Förster, Friedman, & Liberman, 2004). In this case, performance is improved when the task requirements and mental mode are consistent. There is similar evidence that performance on tasks requiring concrete, detailed thinking is improved when subjects are primed to adopt a Near mental mode. One experiment suggests that a Far Mental mode undermined performance on analytic reasoning tasks requiring concrete, low-level processing (Förster et al., 2004) Unsurprisingly, when task requirements are inconsistent with mental mode, performance on those tasks suffers. The data collected for this study also suggests that individuals primed to adopt an abstract construal style were more likely to solve a challenging mathematical problem where conceptual understanding is required than those primed to adopt a concrete construal.

Since the link between psychological distance and level of abstraction is general across domains, it should apply in the case of mathematical understanding. The implications of construal level theory for mathematics education seem to be clear:

- If the aim is to teach for abstract, conceptual understanding, students should be primed to adopt Far mental mode.
- If the aim is to instil procedural proficiency then students should be primed to adopt Near mental mode.

Construal level theory also suggests a complication for this apparently straightforward recommendation. In the course of the evolution of Near and Far Modes of thinking, it was not typically important that the results of Near Mode reasoning and Far Mode reasoning were consistent (Hanson, 2009). Since, by definition, Far Mode reasoning considers distant events and objects these objects and events were typically less relevant than nearer objects to decision making. Robin Hanson has suggested that Far Mode type reasoning played an important social role in enabling individuals to present a good image to others in a group, rather than generating accurate answers to particular questions. In mathematics however, it is of central importance that the results of Near and Far Mode type reasoning are consistent. This tension between the evolutionary requirements of the different modes and the normative requirements impacts on the types of instruction that are appropriate in mathematics education.

Applications of construal level theory. Mathematics education research has tended to emphasise the importance of conceptual understanding for mathematical proficiency over procedural fluency (Star, 2005). Despite this emphasis, the pedagogic practice in high school mathematics education and perhaps undergraduate university courses too, tends to have a heavily procedural focus (Engelbrecht, Harding, & Potgieter, 2005). One manifestation of the predominance of the procedural approach is the phenomenon of "teaching to the test" where students are drilled in familiar questions and become highly familiar with previous examination papers for a particular course. Students who excel in these conditions can do so despite having relatively little understanding of the mathematical content being examined. The procedural emphasis in classroom practice has led some educators to assume that students' procedural knowledge is superior to their conceptual knowledge of the mathematics taught. To test this hypothesis, the relative proficiency of first year students on problems with either a procedural or conceptual emphasis was examined in a 2005 study (Engelbrecht et al., 2005). Contrary to the authors expectations the students performed roughly equally on tests of both types of knowledge. The study also examined students' beliefs about their procedural and conceptual abilities. This revealed that students were more confident of their conceptual understanding than the results of their performance on conceptual problems warranted. In other words, relative to their conceptions of their procedural competence, students were overconfident of their conceptual understanding. Construal level theory offers an explanation for why students may tend to believe they understand concepts better than they really do.

Overconfidence of one's understanding of a concept is an example of what psychologists call an *illusion of explanatory depth*. There is evidence that people often believe that they have a good understanding of many manmade or natural objects or phenomena. However when pressed to explain these things, they often do a poor job. In fact, they are often surprised by this failure to be able to explain more fully (Alter, Oppenheimer, and Zemla, 2010). Several studies have shown that the illusion of explanatory depth is more likely to occur when people adopt an inappropriately abstract construal when thinking about an object. These studies show that when subjects are induced to adopt a low level concrete construal of an object their estimations of their level of understanding of the object is much more accurate (Alter et al., 2010). It is suggested that one reason why abstract construals increase overconfidence about levels of understanding is that understanding what an object is *for* or the higher level attributes of a process are confused with understanding *how* that object or process works. The sensation of understanding is a single type of emotion but understanding exists at different levels of abstraction.

In mathematics, the properties of mathematical objects and the appropriate actions and processes that can be performed on them are detailed and specific. They are not merely incidental to the mathematical object; rather the object can be viewed as comprising exactly those properties and processes. However if a mathematical concept is considered in terms of what it is *for*, for example in justifying why it is being taught in class, focusing on this high level representation can impair the realisation that more work needs to be done to understand it appropriately. One of the more common complaints of mathematics students is that they do not understand the relevance of the concepts they are taught, but the very fact of understanding *why* a concept is important can lead them to believe they understand far more about that concept than they really do.

The potential dangers of inappropriately abstract construals resulting in the illusion of explanatory depth has not been applied in the mathematics education literature but examples of this phenomenon come readily to mind. A student may reflect on her understanding of functions by considering a simple parabola and the "vertical line test" and conclude that she understands the notion but still fail to understand the crucial concepts of "domain" and "range" as they apply to functions. Another phenomenon which will be familiar to many mathematics teachers and students is that students often have a strong sense that they understand the concepts as taught in class or as presented in a textbook, but then have

tremendous difficulty applying what they have supposedly learned to even very similar situations. This phenomenon is one reason behind many teachers' insistence that students practice many examples and not wait until they think they understand before doing so. Students simply fail to grasp that they need to master a great deal of detail in order to competently use a concept in a problem situation and this is plausibly attributed to them adopting an inappropriately abstract construal of the concept. In other words, they consider the situation in Far Mode when Near Mode would be more appropriate.

One important lesson to be drawn from the link between Far mental mode and overconfidence is that thinking abstractly is not always desirable even if one is thinking about abstract mathematical notions. This line of thought echoes the concerns of Jon Star who has argued for a richer conception of procedural knowledge and that the mathematics literature has focused too much on conceptual understanding (Star, 2005). While abstract construals may be appropriate when considering the meaning of a concept and how it is related to other concepts, adopting an abstract construal may often not be appropriate for manipulating mathematical objects or performing detailed calculations.

Research into the order of acquisition of procedural and conceptual knowledge has shown that there is no reliable pattern to which comes first. In their review investigating this question Rittle-Johnson and Siegler (1998) find that some concepts are acquired before procedural fluency but for others the reverse seems to be true. Rittle-Johnson has suggested that the two types of knowledge often develop in an iterative fashion but elsewhere has suggested that conceptual knowledge may be more important in developing procedural knowledge than the reverse. It is important to keep in mind however that mathematical concepts vary greatly in their relation to everyday experience. For example, Rittle-Johnson and Siegler suggest that conceptual knowledge will precede procedural knowledge when the concepts are familiar from everyday contexts. This effect will be accentuated when procedures associated with these concepts are typically not performed in these contexts (Rittle-Johnson and Siegler, 1998). When the concepts involved are numbers, addition, multiplication and equivalence it would make sense that people are exposed to the concepts frequently in everyday life. The content of university level mathematics courses on the other hand is often considerably removed from everyday experience. In many cases there is literally no real world analogue for the concept being taught. In these cases aiming for

conceptual understanding before knowledge of the properties or procedures associated with the concepts seems less likely to be fruitful.

4.2. Operational and Structural Conceptions of Mathematics and Concept Reification

While the trend in research into procedural and conceptual knowledge has increasingly been to emphasise their interconnectedness and mutual dependence, several theories attempt to go further and deny the dichotomy entirely. This is done by suggesting that the same mathematical notions can correctly be represented in both procedural and conceptual ways. The procedural and conceptual points of view can be applied from both philosophical and psychological perspectives. The process of *encapsulation* or *reification* of processes has been explored in theories developed by Piaget (1972), Greeno (1983), Dubinsky (1991) among others. Gray & Tall (1994) have introduced the notion of a *precept* whereby a mathematical notion can be viewed as process and as an object. The single term emphasises that the process and object perspectives are inextricably linked together. In this section, one of these theories developed by Anna Sfard (1991) will be analysed through the lenses of dual process theory and construal level theory with a particular emphasis on construal level theory.

Many mathematical notions can equally correctly be viewed as objects or as processes. A function can be viewed as a set of ordered pairs or as a computational process transforming inputs into outputs or as a “method of getting from one system to another” (Skemp, 1971, p. 246). Rational numbers can be viewed as objects in themselves and as the process of the division of two integers as is vividly illustrated by this anecdote:

I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is $\frac{134}{29}$ (and so forth). What a tremendous labor-saving device! To me, „134 divided by 29“ meant a certain tedious chore, while $\frac{134}{29}$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, $\frac{a}{b}$ and a divided by b are just synonyms. To him it was just a small variation in notation. (Thurston, 1990, p. 5)

From a philosophical or mathematical point of view, both functions and rational numbers can be viewed as objects with certain properties or as processes. This is true for most

mathematical notions (Sfard, 1991). Seeing a mathematical notion as an object is to treat it in a similar way as a tangible, real world object and like a real world object, recognising a mathematical object implies, “being able to recognise the idea “at a glance” and to manipulate it as a whole, without going into details.” (Sfard, 1991, p. 4). Viewed as an object, mathematical notions are static, timeless and structurally related to other objects. Viewed as a process they are more dynamic (the notion of time is implicit in them), detailed and more likely to be viewed in isolation. In Sfard’s terminology, viewing mathematical notions as objects is termed the *structural* conception and viewing them as processes is termed the *operational* conception. It should be clear that the structural conception of mathematics is consistent with the notion of conceptual knowledge as has already been discussed and the operational conception is consistent with idea of procedural knowledge, but Sfard is attempting to resist the idea of the two conceptions being fundamentally different and instead arguing that they are fundamentally similar; two sides of the same coin. Being less detailed and more integrative, the structural point of view is the more abstract mathematical perspective. Another crucial element of this point of view is that even though both perspectives are correct from a mathematical point of view they are extremely different from a psychological point of view.

Historically, mathematical notions have originated and been developed and used initially in purely operational ways. Division was carried out as a process long before fractions were recognised as numbers in their own right. The structural conception associated with the operational process often took centuries to be properly developed. In the case of functions a suitable structural conception was explicitly sought for centuries but eluded an agreed upon conception until relatively recently. It is suggested that there are psychological reasons why the transition to adopting a structural conception of mathematical notions is a difficult leap to make and that the operational conception is often a necessary basis for the development of the structural view.

Similarly to the historical development of structural conceptions, individuals will first understand a mathematical notion from an operational perspective and only after much time and effort, if at all, attain a structural conception of it. The operational conception of mathematical notions typically comes before the structural conception for deeply ingrained psychological reasons. A proposed model for the psychological development of the understanding was initially proposed by Piaget who stated (1970, p 16): “the abstraction is

not drawn from the object that is acted upon, but from the action itself". This clearly represents an operational approach in Sfard's terms and it is this approach that is developed in Sfard's model.

Sfard outlines three stages in the psychological process of developing a structural conception of mathematical notions. In other words, there are three stages in the process of moving from an operational approach to concepts to a structural approach. The stages are *interiorization*, *condensation* and *reification* respectively. The process of *interiorization* begins, at the very lowest level, on concrete objects. For example, the development of the concept of number begins with counting concrete objects (often fingers). At a higher level of abstraction, *interiorization* of negative numbers begins with the process of subtraction. The process of *interiorization* is complete when an individual becomes skilled at the process and can visualise the process without needing to carry it out in practice. The second stage of *condensation* involves the process of routinisation of procedures to the point where long sequences of calculation can instead be mentally represented in more compact units of information. Meaning can be attached to these units without the need to perform each step of calculation comprising the operational procedure. This condensing of many steps of calculation is where the structural conception of the object is first born. However, so long as these units remain linked to the processes and contexts where they first arose they cannot be viewed as fully realised abstract mathematical objects which can be viewed in their own right and applied in different contexts. When this process of de-contextualisation is complete and the concept is detached from the process that produced it the mathematical notion is said to be *reified*. The notion can be "seen" with the mind's eye as a static object which is a member of a certain category in a way similar to the way we can recognise an individual chair at a glance, and at this stage the notion can be said to be viewed in a *structural* way rather than only in an *operational* way. After time, the category itself becomes the focus of attention and different objects can be recognised as members of the same category and relations with other categories can be considered. *Reification* is usually experienced as a sudden, disruptive conceptual shift and involves viewing mathematical concepts at a higher level of abstraction. Once *reified*, these objects can be "recognised at a glance" and can be represented by simple images or symbols *instead of* detailed calculational processes. This frees up attention which can be brought to bear on other aspects important to problem solving.

Once a concept is viewed as an object it can be used in new processes as an input in a new cycle of *interiorization*, *condensation* and *reification* in the development of a structural conception of a new mathematical notion. Initially the operational conception of mathematical notions begins with action and processes performed on concrete objects, but once these notions are viewed as objects in their own right, the operational conception of new notions can begin to develop with actions and processes being performed on this new, higher level abstract object. An operational conception of this *new* higher level mathematical notion could be developed without the input of the reified object but this would be an extremely laborious process and as more proficiency with many mathematical notions is necessary for successful problem solving, maintaining operational conceptions exclusively is an unworkable strategy even though it would be philosophically sound. Since each time this process is undergone, the *reified* object exists at a higher level of abstraction than the objects that act as inputs into the process the structural conception of mathematics is much more hierarchical than the operational conception. It involves many different levels of abstraction whereas the operational conception involves increasingly long sets of actions and procedures at the same level of abstraction. In this sense, the structural conception is deeper and involves an enormous compression of information into an organised network of objects that is characteristic of conceptual knowledge.

Once again an example from chess can help illustrate this point. Relative to human masters, chess-playing computer programs have an enormous advantage in their computational capacity and their “understanding” of effective chess play is almost entirely operational in nature. Chess masters on the other hand compensate for their limited capacity to carry out long computational processes with a hierarchical, *structural* understanding of chess positions and strategy. Their superior ability relies less on mentally visualising individual configurations or simulating many moves ahead than might be imagined (de Groot, 1965, 1966). For a master a very large number of configurations are included as “chunks” of information with particular properties (Simon, 1980). In other words these configurations are seen as “objects” in their own right and can be assessed without the need to model the consequences of every possible move made from that position. At the time of writing, the best human and computer players are still more or less evenly matched.

The difficulty inherent in the process of reification and the need to use reified mathematical objects as inputs into the process of developing further structural conceptions of objects

suggests a large reason why mathematics is so difficult for many students because very often the required reification of lower order objects never takes place. This severely limits the prospects for reification of higher order objects ever happening and it is easy for students to get forever stuck with operational conceptions of mathematical notions which are carried out laboriously and without meaning. In addition to limiting the prospects for the development of structural conceptions, the inability to view notions as objects rather than processes greatly hampers the development of effective procedural fluency because if processes are used as inputs for still more processes they become extremely difficult to manage. Sfard has called this a “vicious circle” with the lack of structural and of operational competence negatively affecting prospects for the development of the other type of competence.

The “vicious circle” is especially difficult to escape from in cases where the objects in need of reification are used symbolically in the development of the operational conception of the notion. For example the symbol i representing the so-called imaginary component of complex numbers must be used in a procedural way in order for the notion to become reified but the lack of reification hampers procedural fluency.

The model outlined above explicitly deals with the psychological process of how we come to know about mathematical notions. Dual process theory and construal level theory have largely emerged since Sfard developed the model of the operational and structural conceptions of mathematical objects and the process of reification and they are promising tools for developing the model further. Many aspects of Sfard’s model already implicitly form part of dual process theory and construal level theory. For example Sfard suggests that one reason for the necessity of the development of the structural conception of mathematical objects is because of the effortful sequential nature of carrying out detailed step by step procedures and because of the extremely limited processing capacity of working memory as opposed to associative memory. This observation is an example of the central tenet of dual process theory. The model also discusses the difficulty in viewing a notion, familiar at the operational level as an object at a qualitatively higher level of abstraction. This is a phenomenon that construal level theory promises to illuminate.

4.2.1. Dual process theory and concept reification. The process of obtaining an operational conception of mathematical notions involves the time consuming and effortful processes of *interiorization* and *condensation*. Both of these stages involve systematic step by step reasoning which are paradigmatic examples of System 2 thinking. When processes have

been fully condensed it is a sign that these processes are able to be carried out largely by System 1 type cognition. Typically, only once the notion has been fully developed operationally can it be conceived structurally via the process of reification. Once a mathematical notion has been reified and can be recognised “at a glance” it is System 1 that is capable of this automatic recognition and not System 2. It is essential to be able to view notions as objects rather than processes because of the limited calculational capacities of System 2. If a notion has to be processed operationally then System 2 will be fully occupied and there is simply no capacity to consider broader relationships and links. If however the notion is represented as an object, one “chunk” of information, then links and relationships with several other objects can be considered at one time. The cognition necessary for manipulating several objects at once in the course of problem solving as well as cognition with the purpose of integrating these objects into a larger schema will again be a task of System 2.

For decades it has been recognised that the maximum number of “chunks” (Simon, 1980) of information that can be kept in working memory at any one time ranges from 5 to 9 (Miller, 1956). This limitation does not apply to System 1, which has a much higher capacity and operates in parallel. If more aspects of mathematical competency can be performed by System 1 style reasoning the amount of detail that can be taken into account and processed is increased enormously and System 2 can focus its processing capacity on the interaction between the reified mathematical objects. The research into chess performance suggests that chess masters have approximately 50 000 “chunks” of information related to different configurations stored in associative memory. Not only will players remember situations, they will immediately identify an appropriate move (Simon, 1980; Schoenfeld, 1985). This recognition and the instinctive response are activities of System 1. The suggested move can then be analysed and accepted or rejected in favour of a better one and this analysis is carried out by System 2. Similarly, mathematical competence likely depends heavily on memory of familiar situations acquired through practice over time and accessed by System 1 type reasoning (Schoenfeld, 1985).

4.2.2. Construal level theory and concept reification. Central to the process of developing a structural conception of mathematical notions is its hierarchical nature. The shift from an operational perspective to a structural perspective of a mathematical notion involves viewing the notion in a more abstract way. This shift to conceiving of a notion at a

higher level often happens rapidly and represents a dramatic ontological shift (Sfard, 1991; Sfard, 1994). Every time the process of reification has been completed successfully the newly conceived object can become an input into the process for the development of the structural conception of another notion which involves another increase in the level of abstraction adopted. The structural conception of mathematical objects is thus deeper and involves many different layers of abstraction whereas adopting the operational conception involves only one level of abstraction. Construal level theory is clearly relevant here because it concerns how we are psychologically primed to adopt different levels of abstraction in various situations. A major goal in mathematics education is to assist students in being able to adopt a structural conception of mathematical notions which students find extremely challenging. Construal level theory helps to explain why this is so and also provides tools for manipulating level of abstraction adopted which could assist in the process of reification described.

The first two stages of development described in the model proposed by Sfard are not only examples of System 2 thinking but are also examples of Near Mode reasoning. One of the earliest examples of this process is in the development of the concept of number which is developed through the action of counting *concrete* objects. Once the process of counting has been interiorised and condensed then the concept of number can become reified. This final stage involves a sudden disruptive shift in the level of abstraction required. When the three stage process begins again with the reified object as an input, a low level construal is once again appropriate.

As time goes by and a student's mind becomes populated with mathematical objects the connections between them and their organisation into appropriate categories becomes increasingly important. If this appropriate structuring does not take place then the problem of working memory being overloaded by the need to think about each object individually becomes overwhelming. The hierarchical structuring of reified objects enables the same condensing of information as in the condensation period of the structural development of one mathematical notion as objects are collected into still higher level categories which can themselves become reified and be "recognised at a glance".

Two important aspects of mathematical proficiency are the development of understanding of mathematical notions and then using those notions as tools in solving mathematical problems. Developing proficiency in both of these activities requires constant shifts in the appropriate

level of abstraction used and construal level theory provides a possible explanation for why it is difficult to make these shifts and suggests one way to make it slightly easier.

Poincaré once remarked, “Science is built up of facts, as a house is built of stones; but an accumulation of facts is no more a science than a heap of stones is a house.” (1905, p. 141). To be able to “see” and analyse features of the house one needs to abstract away from the concrete details of the stones and focus on higher order features of the house which involves adopting a high level, abstract construal. It is essential to remember that the objects represented by the stones in the metaphor themselves have each demanded a complicated shifting of levels of mental abstraction which has to be adopted. In the development of the structural conception of these objects, the achievement of seeing them as objects was one that required a shift to a higher level of abstraction. Viewing them as components of a larger structure then requires viewing them as lower order, subordinate objects.

4.2.3 The difficulty of reification. What follows is a speculative suggestion for how these sudden shifts in understanding and concept formation happen. There are many anecdotes from thinkers about how ideas would come to them in a familiar pattern where they would work intensely for a long period of time on a problem. This would involve immersing themselves in detail and making many attempts at a solution to the problem at hand however the solution or an idea would often come to them in an unexpected flash on a walk or in casual conversation (Hadamard, 1945; Kahneman 2011). Some even would include walks as part of their typical day. Now it is dangerous to generalise from anecdotes from people who are highly atypical since they well known enough to relate their anecdotes to large audiences, but a similar experience is probably familiar to many people when studying hard for an exam or grappling with a challenging problem.

I suggest that when engaged in effortful detailed reasoning one becomes effectively trapped in Near Mode as all cognition and priming is focused on detailed, lower order, step by step reasoning. However a transition to a higher level of abstraction is needed which implies that a shift to Far Mode is necessary. This shift can be achieved by increasing the level of psychological distance. It is suggested that the activities such as walking or conversation or listening to music can serve to, in some cases literally, broaden horizons and result in the unintended increase in psychological distance and causes one to think in Far Mode and adopt a more appropriate level of abstraction.

The fact that the processes of interiorisation and condensation result in much of the calculation being carried out by System 1 instead of System 2 suggests that this transfer of calculational ability between systems is also an essential component of being able to adopt a structural conception of mathematical objects. Once all the detail, especially if much of the detail has been encompassed into System 1 style thinking, can be viewed in Far Mode one is psychologically primed to adopt the more abstract, higher level construal necessary for the conceptual links and connection at the higher level to be made. The metaphor of not being able to see the forest for the trees seems to refer to exactly this problem of being stuck in Near Mode and being unable to see the bigger picture. In their work on procedural and conceptual knowledge, Hiebert and Lefevre speak of the necessity of “stepping back” in the process of reflection to assist in appreciating the connections involved in conceptual knowledge (1986). When describing how he comes to understand a concept the mathematician Jacques Hadamard claimed he uses “a schema, which always and must be of vague character, so as not to be deceptive.” (Hadamard, 1945, p. 77). Hadamard also refers to *cloudy imagery* in the search for understanding. Vagueness and cloudy imagery are characteristic of Far Mode thought and point to the dangers of prematurely imposing a specific interpretation on a mathematical concept, which is more likely in Near Mode. Construal level theory provides a more precise language with which to discuss what mathematics researchers have known for some time.

Construal level theory also sheds light on the problem of the “vicious circle” where lack of operational competence impedes the development of a structural conception of mathematical objects and vice versa. The process of reification involves a dramatic ontological shift, adding an entirely new, higher level of abstraction. However this newly conceived abstract object must then be used as an input into mathematical problem solving or in the development of a new mathematical concept, which means viewing this new abstract object in low level subordinate terms relative to other processes. In other words, this new object needs to be understood at different levels of abstraction at the same time, or in Near and Far Modes at once. Another slightly different way of describing the situation is provided by Gray and Tall (1994) with the notion of “procept”. Mathematical symbols can be seen as both processes and as objects and students must be able to switch effortlessly between these two different views. Gray and Tall suggest that mathematicians use symbols in an ambiguous way, continuously shifting between perspectives depending on the demands of the situation.

Viewing a notion as a process invokes a Near mental mode while viewing it as an object invokes a Far mental mode. Mathematicians may be able to do this effortlessly, but it does represent a significant shift in mental construal (Gray & Tall, 1992). It is no wonder that many individuals find this so baffling and fail to adopt a proceptual view of mathematical symbols.

The analysis in the discussion above can be summarised in a slightly expanded version of Sfard's model of reification. The three stages of developing a mathematical concept involve moving from a low level Near Mode of thinking to a higher level Far Mode of thinking. At the same time, it is necessary to successfully carry out processes using System 1 that initially must be carried out by System 2. In other words, successfully developing a concept involves two significant and parallel shifts in reasoning styles. This expanded model is represented in Figure 4. Sfard is concerned with the difficulty of the process of reification and advances the notion "vicious circle" as a partial explanation of this phenomenon. It is the contention of this section that construal level theory and dual process theory can shed further light on why this process is so challenging and that it can suggest possible ways of easing the transition.

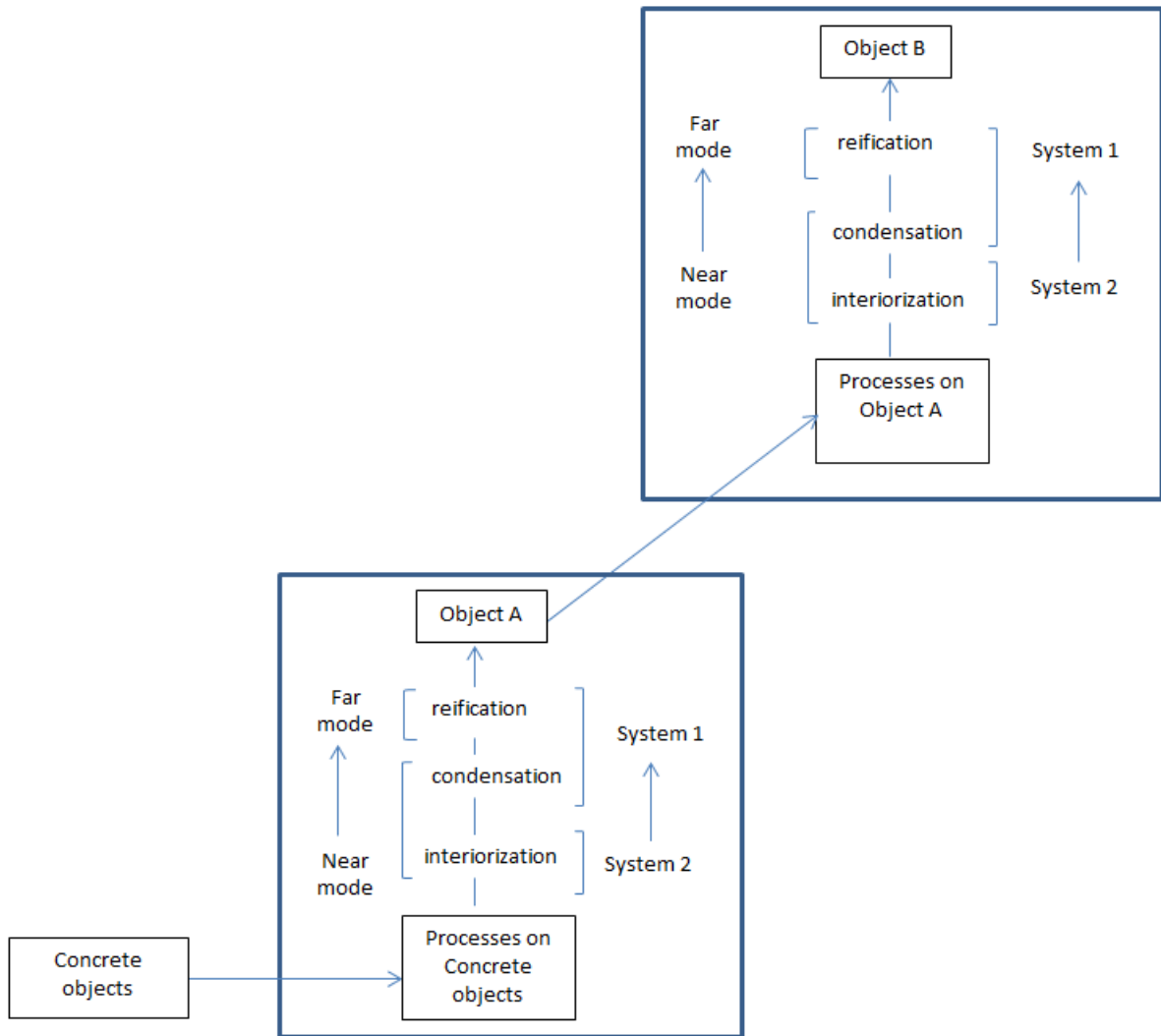


Figure 4. Reification and Construal Level

4.3. Cognition, Metacognition, Dual Process Theory and Construal Level Theory

4.3.1. Metacognition and dual process theory. Dual process theory is not just intimately connected to research into both cognitive and metacognitive processes it is, at least in part, a theory of cognition and metacognition. Most thinking is an example of "regular" cognition and this encompasses both System 1 and System 2 types of thinking. Metacognition is involved in monitoring cognition and is responsible for superordinate coordination of goals and strategies. These functions are carried out by System 2. However metacognitive processes are also implicated whenever the regular operation of cognition is interrupted and the decision is taken to override the default response of System 1 type processes. This type of interruption often occurs without conscious deliberation or effort and as a result is a function of System 1 and occurs whenever an activity is interrupted due to an intuition that "something has gone wrong" (Alter et al., 2007; Koriat & Levy-Sadot,1999).

Metacognition carried out by System 1 has not been studied extensively by mathematics education researchers relative to the conscious System 2 type of metacognition. Examples of System 1 metacognition are familiar from everyday experience. An illustration comes from the experience of a man whose job it was to manually operate a lift in a hotel. Several of the guests would regularly use this lift but he only remembered which floor a few of these guests stayed on which was always the first floor. The author relates that this recollection was accompanied by irritation since the guests had not simply used the stairs. In sum, an affective emotional response disrupted the normal procedure of operating the lift and was associated with knowledge that he did not typically retain (Koriat & Levy-Sadot,1999). Of more relevance here are examples of System 1 type metacognition in experts in domains other than mathematics that have been studied in more detail. Some examples of this have already been discussed as in the case of the fire fighter who instinctively knew that something was amiss and ordered his men out only to have the floor collapse soon afterwards (Kahneman, 2011). Other examples could include chess players who have a sense that a move that is theoretically strong doesn't "feel" right, perhaps because of a suspicion that it is the move that the opponent hoped would be played. In all of the above examples, emotional or affective factors played a role in regulating how they thought about particular situations and therefore count as examples of metacognition.

Intuitive, System 1 type metacognition is extremely important to effective problem solving, but this insight is probably of limited significance for general mathematics education practice as it is generally exhibited by mathematicians with thousands of hours' worth of experience. It is difficult to see how this valuable skill could be taught to students other than by emphasising continuous and prolonged practice of solving problems and attempts to provide a constructive space for such practice under supportive supervision.

Some metacognition is performed entirely by System 1 type processes but the most common examples of metacognition involve System 1 triggering the intervention of System 2 which is then implicated analysing the situation. The System 1 trigger for metacognitive processes typically comes in the form of discomfort or dysfluency (Alter et al., 2007). Discomfort and surprise are common everyday experiences just as the acts of reflecting on possible causes and solutions to these disturbances are. One does not have to be trained in a particular discipline to engage in these activities, though experience invariably helps. Such metacognitive processes are often far less natural in specialised domains however. One of the principal obstacles to inculcating effective metacognition in mathematics problem solving situations is that students will often not experience discomfort or surprise even if the solution attempt is going disastrously wrong (Schoenfeld, 1985). In other words, there are often no cues from System 1 type processes signifying that anything is amiss. Interestingly, when studying students' problem solving attempts, Schoenfeld would ask students why they had attempted the solution in the way that they had and students would often not be able to provide any answer. It was as if another mind was responsible from the one reflecting unsuccessfully on the attempt. This interpretation is clearly an implicit invocation of dual process theory which is often described as being about two different minds (Evans, 2003; Kahneman, 2011).

Three problems which have been extensively studied in the literature on dual process theory highlight the relationship between dual process theory and metacognition and potential applications to mathematics education; the Wason selection task (Wason, 1966; Cosmides, 1989), syllogistic reasoning items (Stanovich, 2011) and the Bat-and-Ball problem. The Wason selection task is a test of logical reasoning that has been extensively used by psychologists (Wason, 1966). The task comes in different forms (Cosmides & Tooby, 1992) but in general the task is presented in decontextualized, abstract form or in a contextualised, realistic form (Griggs & Cox, 1982). In the abstract version, participants are presented with

four cards each with a letter on one side and a number on the other. Two cards have the letter facing up and the other two have a number facing up as shown in Figure 5. Participants are then asked which cards must be turned over in order to verify or falsify the following rule: “If there is an A on one side of the card, then there is a 3 on the other side of the card” (Evans, 2003, p. 456).

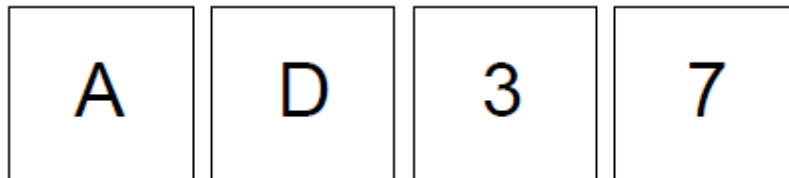


Figure 5. Decontextualized Task

In this formulation of the problem the generally accepted correct answer is that the cards showing A and 7 must be turned over, though this has been disputed (Nickerson, 1996). Between 10% and 20% of respondents give the correct answer when tested (Evans, 2003). The most commonly given answer however is that cards A and 3 should be turned over in order to confirm or falsify the statement. This suggests that turning cards A and 3 represents the intuitive, easily accessible answer.

In the realistic formulation, a task logically identical is presented in a contextualised and realistic form. Each of the four cards has a statement about what type of drink an individual is drinking on the one side and that individual's age on the other. Two cards with the information about the drink are facing up as well as two cards with the age as shown in Figure 6. In this formulation, participants are asked which cards need to be turned over in order to verify the following rule, “If a person is drinking beer, then that person must be over 18 years of age” (Evans, 2003, p456).

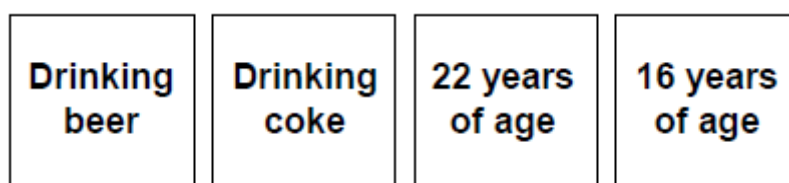


Figure 6. Realistic Task

In this formulation of the problem, the correct answer is that the “Drinking beer” and the “16 years of age” card and the large majority (approximately 75%) of those tested give the correct answer (Griggs & Cox, 1982). The problem in this form is relatively easy and the correct answer is intuitively accessible.

In both formulations there is an immediately intuitive answer for a majority of participants. However, the intuitive answer in the abstract condition is incorrect but when the task is framed in a realistic way the intuitive answer was correct. There is considerable evidence that the incorrect intuitive response to the abstract formulation is caused by the System 1 heuristic known as matching bias (Evans, 1998). Matching bias is a tendency to see as relevant, the lexical content of a statement rather than the logical relations. In this case the mentioning in the problem statement of A and 3. Matching bias is more pronounced in problems with more abstract content (Evans, 2003). There is evidence that the matching bias heuristic can be inhibited with training Allowing System 2 type reasoning to be brought to bear on the problem (Houdé et al, 2000). In the case of the “realistic” formulation formulation of the problem, the correct answer is strongly cued by relevant prior knowledge and as a result intuitive, System 1 cognition generates an appropriate answer.

An example of a syllogistic reasoning problem is as follows:

Premise 1: All living things need water.

Premise 2: Roses need water.

Therefore, Roses are living things.

(Stanovich, 2011, p. 40)

This argument is intuitively plausible and most people judge it to be valid, but the conclusion does not in fact logically follow. In this case the realistic contextualisation of the problem is what leads participants to give an incorrect answer. This is an example of another heuristic of System 1 known as belief bias (Gilbert, Krull & Malone, 1990). Belief bias is a tendency to judge the strength of an argument based on the plausibility of the conclusions rather than how

strongly the argument supports the conclusion. System 1 automatically believes new proposition and it is the task of System 2 to interrogate it and potentially discard it (Gilbert et al, 1990). Since System 1 cannot be turned off and System 2 requires conscious effort, this form of belief bias is difficult to overcome. Again, effective metacognition could prevent the acceptance of the intuitive conclusion but in most cases does not.

A third problem relating dual process theory and metacognition is the so called Bat-and-Ball problem:

A bat and ball cost \$1.10.

The bat costs one dollar more than the ball.

How much does the ball cost?

(Kahneman, 2011, p. 44)

The correct answer to the problem is 5 cents which can easily be calculated. For many people however, an answer of 10 cents presents itself intuitively and with little effort, hallmarks of System 1 reasoning. In fact, majority of American university students answer this question incorrectly (Kahneman, 2011). This is despite the ease with which the intuitive, incorrect solution can be checked. Metacognitive processes can easily prevent giving wrong answers but all too often System 2 will fail to do this in the absence of cognitive conflict.

System 2 type reasoning is computationally expensive and as a result is “lazy” (Kahneman, 2011). If the outputs of System 1 processes are plausible and do not cause cognitive conflict System 2 will deem the response appropriate and thus fail in its role of monitoring the automatic responses of System 1. This reluctance to override System 1 is deeply ingrained and recent evidence suggests that even when individuals are specifically primed for System 2 engagement, such as in mathematics tests, where they know correct procedures are difficult, intuitive answers are often not overridden by System 2 (Leron & Hazzan, 2006).

This lack of constructive System 2 involvement or lack of problem solving intuition is, at least partly, a function of the fact that most mathematical problem solving situations faced by students are highly unnatural so System 1 is functioning, in a highly hostile environment. Again it is difficult to see how to teach students to experience these intuitive, possibly even emotional, reactions when things are not progressing smoothly other than to encourage

prolonged problem solving practice and providing supportive environments in which such intuitions can develop. Such supportive problem solving environments are likely to overlap to a large extent with contexts where explicit metacognitive instruction is in fact possible. It is to this possibility to which we now turn our attention.

4.3.2. Dual process theory, construal level theory and metacognition. So far we have discussed classes of metacognition where the intuitive automatic processes of System 1 have had a leading role. Of more relevance to research concerning metacognition in a mathematics education context are those processes where System 2 type reasoning both initiates and carries out the overriding and intervening functions.

Research into the significance of metacognition in mathematical problem solving is exemplified by the work of Alan Schoenfeld (1985). The tripartite model of mind proposed by Keith Stanovich has particular resonance with Schoenfeld's work on metacognition to the point where they could plausibly be viewed as being part of the same theoretical model with metacognition as used by Schoenfeld and the type of cognition associated with the reflective mind being essentially the same construct. The role of metacognition in both models is to interrupt and override unproductive paths being pursued, either consciously or subconsciously. Both metacognition and the reflective mind reflect broader belief systems and epistemological worldviews. Perhaps the primary difference between the models is that the importance of the reflective mind is primarily in its role inhibiting System 1 type processes and then using available information in deciding which hypothetical simulations to consider where these hypothetical simulations are carried out by the algorithmic mind. Metacognition as conceived of by Schoenfeld on the other hand involves the interruption and coordination of computationally taxing solution attempts which are almost exclusively System 2 processes. This is primarily a difference of emphasis however as the reflective mind is also responsible for monitoring and overriding System 2 thinking where appropriate. Some researchers have noticed the similarities between System 2 and metacognition in mathematics education and hinted at the need for a "System 3":

In a way we might say that an operation of a "System 3" is needed here (to monitor S2), but in practice, this function is recursively assumed by S2 itself. While monitoring and critiquing S1 is one of the reasons S2 has evolved in the first place, monitoring the S1/S2 interaction seems to be what Geary (2002) has called *secondary*

cognitive ability, one which will not normally develop without explicit instruction.

(Leron & Hazzan, 2006, pp. 123-124)

This tentatively suggested “System 3” seems precisely that which Stanovich has called the reflective mind.

Since both the reflective mind and the algorithmic mind are System 2 types of reasoning, they cannot both operate at full capacity at the same time. It is suggested that maintaining cognitive decoupling is extremely demanding of cognitive resources and this is a task of the algorithmic mind. This implies that the operations of the reflective mind are not demanding on the same scale and have some scope to operate while the algorithmic mind is engaged. Nevertheless intense concentration can come at the expense of the reflective mind and could explain another problem impeding effective metacognition noted by Schoenfeld and Lesh, the problem of cognitive "tunnel vision" where the selected path is pursued at all costs, completely discounting new information and making effective regulation impossible.

Finally, since conscious metacognitive activity is associated with the reflective part of System 2 and it has been suggested that the reflective mind is associated with Far mental mode, it follows that conscious forms of metacognition are associated with Far mental mode. Studies by Schoenfeld on effective metacognition (Schoenfeld, 1985) reflect the importance of psychological distance. Transcripts of expert problem solvers approaching novel mathematical problems are replete with examples of these experts pausing to evaluate their actions, attempting to explain why they are doing this and reminding themselves of their goals. “Why” questions and considering ultimate goals of an activity are characteristic operations of a Far mental mode. Elsewhere in the mathematics education literature there is frequent use of distance metaphors. It is common to read of the need for students to “take a step back” from the details so that they can “see the forest for the trees” (Hiebert & Lefevre, 1986). Expert problem solvers seem to have mastered the technique of shifting the level of abstraction they adopt in mathematical problem solving. They are able to carry out detailed calculation and procedures in Near Mode, but are also able to adopt a more abstract construal of the situation by constantly shifting to a higher level of abstraction in which they can monitor progress relative to their goals and to avoid focusing too much on superficial detail and focus on core information instead. This constant shifting of levels of abstraction is

something most students fail to do and even explicit instruction is of limited effectiveness in instilling this habit (Schoenfeld, 1985; English et al., 2008).

Mathematical problem solving heuristics also require different levels of abstraction for successful use and like conscious metacognition it is difficult to teach. Dual process theory and construal level theory can shed some light on why heuristic strategies can be so difficult for novice mathematicians to master. The application of these two branches of psychology to research into problem solving heuristics is the subject of the next section.

4.4. Heuristics

4.4.1. Heuristics and construal level theory. The branches of mathematics education research concerning procedural and conceptual knowledge and cognition and metacognition have intuitively obvious parallels with construal level theory. All three research domains concern how knowledge and thinking occur at different levels and the suggestion of deeper connections between the domains calls out for further study. Research in mathematical problem solving heuristics on the other hand does not have an intuitively obvious relationship to construal level theory. However, in the process of solving mathematical problems different construal levels are called for at varying points in the problem solution. When confronting a new problem it is important to take into account all the information and get a general sense of the problem before particular solution methods are decided on. This involves getting a general overview of the problem which involves taking a Far view. Once a plan for solving the problem has been formulated however the plan must be implemented which will often involve detailed step by step calculations where one mistake could prove damaging. This stage of the problem solving process is conducted in Near Mode.

Problem solving heuristics have not met with the success that they seemed to promise mostly because they are usually far too general to provide specific guides to action for those who have no idea of how to proceed (Schoenfeld, 1985; English et al, 2008). They are descriptive in the sense of describing certain actions commonly performed by successful problem solvers but are insufficiently prescriptive to be an effective guide to action. This seems to suggest that commonly suggested heuristics are intended to be processed in Far Mode but they lack sufficient cues about how to successfully transition into the Near Mode mind-set required for much of the problem solving.

Alan Schoenfeld has attempted to address the difficulties involved with using problem solving heuristics by developing more detailed and explicit guidelines for problem solving. This more prescriptive strategy represents an attempt at breaking up more general heuristics into more formally articulated sub-categories in order to guide novice mathematicians when approaching a novel problem. Since the program has met with success it is worth examining through the lens of construal level theory to determine if the theory can shed any new light on the processes that students are encouraged to go through when applying the strategy.

At the broadest level the strategy suggests students approach each new problem by attempting to progress through a series of stages: analysis, design, exploration, implementation and verification. While these stages do have a natural order to them in that analysis should come before exploration and verification is the last stage, the strategy is not intended to be implemented in a linear or mechanical way, rather it serves as a guide for action for students uncertain of how to proceed.

Design in particular is not a distinct stage to be implemented in sequence but is rather intended to remind students to maintain a global, goal oriented mind set; “You should outline a solution of the problem at a rough and qualitative level and then elaborate it in detail as the solution process proceeds” (Schoenfeld, 1985, p.108). In terms of construal level theory, this can be viewed as an explicit suggestion to begin the solution process by adopting an abstract construal and then proceeding to more concrete, complex and detailed calculation. Schoenfeld’s description of design also suggests that no matter how demanding the concrete details become it is important to maintain a global perspective so that unproductive lines of calculation can be terminated and other avenues explored. This suggests that even though focus on concrete details may be required it is always necessary to maintain higher level awareness, in other words it is essential that a problem solver continuously shift between both high and low level construals while working on a problem.

Schoenfeld elaborates more fully the analysis, exploration and verification stages. Included in the analysis stage are parts that prime more concrete thinking and parts that encourage more abstract thinking. The first two steps in the analysis sub-section involve drawing a diagram or rough sketch and examining special cases. Both of these steps involve attempts to make the problem take a more concrete form; drawing a diagram is a literal attempt to give the problem a concrete form. Replacing variables with specific values also serves to make the problem

less abstract. The third suggested step in the analysis stage involves simplifying the problem by exploiting symmetry, scaling or using “without loss of generality” arguments before getting immersed in details. This involves the use of abstract mathematical techniques and involves an abstract mind-set.

The exploration stage involves three steps to be attempted sequentially and represents a progression from more concrete to more abstract and speculative. The first step is to consider essentially equivalent problems. This step involves several sub-categories and involves considering a variety of permutations and small changes to the problem as stated. It is detail specific and relatively unsystematic and thus invokes a relatively low level construal. The second and third steps are to consider slightly modified problems and then broadly modified problems respectively. These steps are only to be taken after the first step has failed and involve a deeper understanding of abstract mathematical principles and Schoenfeld recognises that each subsequent stage is more speculative and difficult than the last. In terms of construal level theory, the exploration stage becomes progressively more abstract and distant on the hypotheticality dimension.

The implementation stage should be the most straightforward of the stages and primarily involves procedural proficiency. This stage is the only one that primarily invokes a concrete mind-set. The final stage of verification is broken into two steps; specific tests and general tests. The specific tests involve a relatively low level of construal, but Schoenfeld stresses that at the general level, verifying the problem solution often suggests alternate solutions and connections and can promote conscious awareness of which parts of the process were useful and can be used in future. In other words, global verification helps to develop deeper conceptual understanding of the principles involved.

Most stages of implementing this heuristic strategy involve both high and low level construals. Broadly one starts and ends the process with an abstract construal and the heuristics act to guide problem solvers into areas where they can effectively marshal their concrete procedural knowledge. Construal level theory indicates that high and low level construals involve adopting fundamentally different mind-sets and thus suggests that this continuous shifting between different modes of thinking is intrinsically unintuitive and challenging.

4.4.2 Heuristics and dual process theory. Attempting to solve a novel mathematical problem is cognitively taxing and as a result it can be taken for granted that so long as a sincere attempt to solve the problem is being made System 2 will be fully engaged. A very common problem afflicting students in these solution attempts is, as Schoenfeld has termed it, “pursuing wild mathematical geese” (Schoenfeld, 1987, p. 193). Students will choose a strategy and pursue it single-mindedly no matter how little progress seems to get made. Thus, this knee jerk response to a problem is extremely significant as if it is inappropriate the failure of the attempt is practically guaranteed from the first second. The impulsive, intuitive reaction and subsequent direction adopted is a product of System 1 and because of the automatic nature of System 1 type processes it is not possible to suppress. The initial reaction can be overridden by System 2 but System 2 is frequently engaged in trying to find justifications for the intuitive response which in a mathematical context is wholly inappropriate. As a result, one simple heuristic strategy suggested by dual process theory is to consciously examine any intuitive responses even if the response is intuitively satisfying. The heuristic could also make explicit the danger of spending too much effort justifying this response and to delay solution attempts until the problem has been considered in more conscious detail.

The automatic processes of System 1 cannot be prevented from occurring and as described above and at length by Schoenfeld (1985) often lead to unreflective pursuit of the first potential solution path that occurs to them. This effect is so powerful that students will often spend considerable amounts of time on this path and yet be unable to provide *any* reasons why they are doing this. There is a powerful positive side to the automatic intuition in problem solving. It has been noted that experts in a domain will spontaneously come up with several appropriate possible responses as in the case of a chess master. The heuristics employed by mathematicians studied by Pólya and Schoenfeld are techniques that are *automatically* employed. Even proficient problem solvers will not arrive at a solution to a novel and challenging problem using only System 1 type processes (this is true by definition; if it were possible the problem would be classified as an exercise). They are able to automatically exclude a huge number of possible avenues and automatically apply appropriate heuristic techniques which will reliably guide them to a problem solution. There is another layer in this story of the importance of System 1; every time a heuristic strategy is employed there is additional information made available which is then acted on by System 1. An example of this is the strategy of drawing a diagram. Drawing a diagram representing

information contained in the question will require the attention of System 2, but once completed the concrete visual representation provides a new physical stimulus and will trigger further automatic intuitive responses. If the diagram is appropriately constructed and the subject has command of the appropriate mathematical resources, then the odds of this intuitive response being fruitful are increased. In addition, we can see the significance of System 1 type cognition in aiding the effective use of heuristics. It also helps to explain why students engage with explicitly stated heuristics in a very different way than do experts; they are literally using a different part of their brain to do so.

System 2 reasoning is also implicated in the effective use of heuristic strategies, but there is less to say about the relationship between the mathematics education research and dual process theory in this case. System 1 type cognition may set a subject off on a wild goose chase but System 2 is responsible for conducting the chase. In particular, the algorithmic part of System 2 is engaged in these situations but all too often the reflective part of System 2 is not. This lack of oversight during an attempted solution represents a failure of metacognition and is discussed in a previous section. Problem solving heuristics of the detailed sort proposed by Schoenfeld (1985) can be interpreted as schemes designed to remind problem solvers of the importance of engaging the reflective mind.

4.5. The Effect of Construal Level on Mathematical Performance

There is very little data explicitly examining the effect of construal level on mathematical performance though there is data concerning the dangers of stereotype threat and its effect on the performance of women on mathematics problems (Keller & Molix, 2008; Wout, Danso, Jackson, & Spencer, 2008). Evidence suggests that women who are primed to adopt an interdependent self-construal (to think of themselves in terms of their connections to a broader group, in this case their gender), perform worse on mathematics tests than those who received priming for an independent self-construal (priming to consider themselves as individuals and their unique characteristics). This negative effect is more pronounced the more participants identified with their gender. Priming individuals to adopt an interdependent self-construal is one way of priming a Far mental mode. McCrea et al (2012) conducted a similar experiment where psychology students were primed to adopt either abstract or concrete mind-sets but without specifically invoking gender. In this case too, women's performance on mathematics tests was adversely affected in the abstract priming condition

but not in the concrete priming condition. Consistent with the stereotype, men performed better in the abstract priming condition than in the concrete priming condition. Researchers attribute these results to the negative stereotype about mathematical competence applied to women and positive stereotypes applied to men.

This evidence suggests, in addition to the possibly harmful effects of an abstract construal for women, that a positive stereotype activated by adopting a Far mental mode can improve performance. If another, more positive stereotype is made salient in the process of priming it is possible that women primed in Far Mode would also improve their performance. This possibility was taken into account in the data collected for this dissertation. The priming tasks involved making salient their membership of the group engineers or aspiring engineers. Since the students were studying engineering and engineering is a high status profession it is plausible that the abstract construal would not have a negative impact on performance for the women in the study. The results, while hardly definitive, are consistent with this hypothesis since the results of the men and women in study were not impacted differently by each priming condition.

One possible result from the data collected for this dissertation is the difference in the variance of mathematical performance by priming condition, rather than any difference in overall performance. Students' performance on mathematical problems which focused on conceptual understanding seems to have been impacted by the priming they received. While the group receiving Near Mode priming did better on average than the group primed to adopt a Far mental mode, this difference was not statistically significant. The variance of results among students with Far mode priming however was larger than the variance in outcomes among students primed to think in Near Mode.

The explanation for the difference in variance according to level of construal must at this stage be speculative, but the results can be viewed in a way that is consistent with the analysis in the rest of Chapter 4. Students primed to think in Far Mode are more likely to attempt to consider the problem as a whole rather than focusing on particular details. They are thus more likely to develop a conceptual model taking more elements into account, or in the terminology of Richard Lesh (1985), a stable conceptual model of the problem. Those primed to adopt a Far mental mode are also more likely to develop a false conceptual model of the problem. This is consistent with illusions of explanatory depth being induced by

inappropriately abstract construals of a problem (Alter et al., 2010). Students primed to think in Far Mode may have mistakenly believed that they had an adequate model of the problem and proceeded with the solution attempt and failed to notice the inadequacy of their solution.

Students primed to think in Near Mode had a lower variance in their results. One potential implication of this is that few students either had a completely appropriate or inappropriate conceptual model of the problem. In Near mental mode feasibility considerations assume greater importance than the desirability of the end state (Liberman & Trope, 1998). In other words getting something constructive done towards solving the problem could be viewed as more pressing than fully solving the problem. In more concrete terms it is plausible that students with Near Mode priming were more concerned to apply a particular method or procedure recently covered in lectures or used in the tutorial session. The correct application of a procedure without proceeding to a full solution could result in getting several of the available marks.

There is a large scope for further evidence to be collected on the relationship between construal level and mathematical performance. It would be useful to investigate the effects of different types of priming tasks and their impact on different types of mathematical problems. In particular it would be of interest to isolate the effects of construal level on problems testing student understanding of mathematical concepts and on general problem solving proficiency. Another potential area of interest is to test if construal level impacts performance on short questions that require rapid answers compared to problems answered over a longer time period.

4.6. Construal Level Theory, Dual Process Theory and Mathematics Education

The research question for this dissertation is stated in two parts:

- To what extent can construal level theory and dual process theory be used as conceptual frameworks to inform existing branches of mathematics education research?
- Does the level of mental abstraction induced by contextual priming have an impact on mathematical performance? If level of mental abstraction does influence mathematical performance what is the nature of this influence?

This chapter has attempted to answer both aspects of the research question. As theories of cognitive psychology, construal level theory and dual process theory are both relevant to mathematics performance as they are to any cognitive processes but I have argued that this relevance, of construal level theory in particular, is not superficial. Mathematics is highly abstract and “unnatural” relative to what our minds have evolved to think about. There are deep, psychological reasons why many people experience difficulty with mathematical concepts and problem solving. In the search to find effective teaching methods to help students develop mathematical proficiency, construal level theory and dual process theory can help explain why some approaches experience difficulty and suggest new approaches or ways in which established practices can be improved.

In addition to compelling theoretical arguments for the relevance of construal level theory and dual process theory there is empirical support that construal level affects mathematical performance. Much of the evidence in the research literature centres on stereotype threat; the negative effects of self-stereotyping on women’s mathematical performance. The data collected for this dissertation adds to the evidence that the level of construal adopted has an impact on mathematical performance. In this case however the evidence suggests that adopting an abstract mind-set increases the likelihood of individuals solving a challenging conceptual problem. The data does not unambiguously favour adopting a Far mental mode however as this mode of thinking also appears to increase the likelihood of developing an incorrect model of a mathematical problem.

5. Conclusion

The subject of this dissertation has been the difficulty many students have with mathematical abstraction. These difficulties include the need to understand new, abstract mathematical concepts and the problem of applying the correct level of abstraction in mathematical problem solving. Mathematical proficiency requires that higher level, abstract views of mathematical concepts or problems must be integrated with the lower level, concrete details. Inadequate knowledge of concrete details can hinder higher level understanding and a lack of understanding robs lower level procedures of meaning. Two areas of mathematics education research have emerged, at least in part, to tackle the difficulty inherent in the need to constantly reason at different levels of abstraction. There is a literature concerning procedural and conceptual knowledge in mathematics as well as research into cognition and metacognition in mathematical proficiency. Conceptual knowledge and conscious forms of metacognition are higher level, abstract types of knowledge while procedural knowledge and cognition exist at a lower, more concrete level.

Reasoning at different levels of abstraction is something that humans do naturally every day. However the conscious, systematic type of abstract reasoning required in mathematics is a relatively new ability in our evolutionary development and, though important, it is effortful and humans have limited capacity to deploy it. Conscious, abstract reasoning can be broken into its components, each of which has been studied in separate areas of cognitive psychology. Construal level theory concerns the levels of abstraction in human reasoning while dual process theory concerns the distinction between automatic, intuitive reasoning and effortful, systematic reasoning. It has been argued in this dissertation that both construal level theory and dual process theory can inform research into mathematical proficiency.

Dual process theory suggests that our automatic intuitions and beliefs about mathematical concepts and problem solving stemming from System 1 type thinking are highly unlikely to be correct because the behaviours necessary for mathematical competence were unlikely to have been evolutionarily adaptive (Stanovich, 2003). It is possible over time, that mathematical concepts and procedures that initially require effortful, systematic reasoning (characteristics of System 2 thinking), can be intuitively understood. That is, System 2 type processes can be transferred to System 1 thereby freeing up System 2 for other tasks. Dual process theory is also intimately connected with mathematical metacognition. System 2

reasoning is responsible for monitoring the automatic, intuitive responses of System 1 and overriding them when necessary. System 2 type reasoning is intrinsically lazy however which partly explains the difficulty that students experience overriding their intuitions in problem situations where they have previously mastered the correct mathematical content. The metacognitive function served by System 2 type reasoning is postulated to be associated with a Far mental mode. This leads to the applications of construal level theory to mathematics education research.

Construal level theory proposes a link between how we think about distant objects and events and the level of abstraction we use to think about those object and events. Taken together, abstract thinking and psychological distance can be thought of together as Far Mode thinking while concrete thinking and psychological closeness can be combined to form Near Mode thinking. Importantly, the link between psychological distance and level of abstraction is overdetermined; thinking about a distant object or event primes a subject to think in Far Mode, that subject will automatically think about other objects or events more abstractly. Of relevance to this dissertation is the fact that students will think about mathematical concepts and problems at a higher, more abstract level if they have previously been primed to think in Far Mode and at a more detailed, concrete level if they have been primed to think in Near Mode.

It has been argued in this dissertation that proficient mathematical performance requires constant shifting between abstract and concrete construals. This is the case in developing understanding of new, abstract mathematical objects and in competent mathematical problem solving. Regarding procedural and conceptual knowledge, a model proposed in this dissertation for the process of developing understanding of mathematical notions involves operating on lower level objects in a procedural way which develops towards conceptual understanding. In other words, much of the work must be done in Near Mode but the shift to higher, conceptual understanding may involve a literal shift of mental construal into Far Mode. Once this process is complete, it can begin again with the newly understood mathematical object being operated on in Near Mode in the process of understanding a still higher order mathematical object.

The case of mathematical problem solving, which research into metacognition and heuristic strategies addresses, also requires repeated shifting of mental modes. When approaching a

novel problem a Far mental mode is desirable in order to consider the big pictures and develop overarching goals and a solution strategy. The problem solution however requires detailed attention to the procedures and calculation taking all relevant detail into account. A Near mental mode is better suited to these tasks. It is a crucial aspect of successful problem solving however that solution attempts be evaluated relative to the problem goals as the solution proceeds. This ability, which is second nature to experts, is one that many novice mathematicians lack and involves metaphorically stepping back from a particular path and assessing progress on a higher level. Thus successful problem solving involves constant shifting between high and low level mental construals; constant shifting between Near and Far mental modes. Since it was not always important that the results of Near and Far Mode thinking be consistent, the appropriate balance between Near and Far mental modes in problem solving is delicate and difficult to achieve. It is a characteristic feature of failed solution attempts that students get “trapped” in a solution attempt and fail to keep higher level goals in mind. In other words they get trapped in Near Mode while experts are able to shift between mental modes when appropriate.

Evidence suggests that adopting a Far mental mode can cause individuals to self-stereotype and that this can benefit or hinder performance on mathematics tests. This evidence typically has not been collected with mathematics research in mind however. The data collected for this dissertation specifically addresses the question of whether adopting a Near or Far mental mode affects performance on both procedural and conceptual mathematical questions. While the data does not suggest that construal level affects performance on procedurally oriented tasks, it does indicate an effect on conceptually demanding problems. It appears that adopting a Far mental mode increases the probability that the correct conceptual model of a problem will be used *and* that the problem will be conceptualised in an unconstructive way. While adopting a Far mental mode may be more desirable for full understanding of a problem, it also increases the likelihood that problem solvers will mistakenly believe they understand the problem.

Dual process theory and construal level theory in particular are well suited as theoretical frameworks for application to much research in mathematics education. They provide plausible explanations for several well-known phenomena in mathematics education. In particular construal level theory provides an explanation for why abstract mathematical thinking is so difficult for so many students; high level abstract thinking literally requires a

different mental mode than detailed concrete reasoning and both types are indispensable for mathematical proficiency. There is tremendous scope for construal level theory and dual process theory to be integrated both theoretically and empirically with several important branches of mathematics education research.

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Appendix A: First Priming Intervention for Conceptual Problem

A.1. Far Mode Priming Task

Write a short paragraph explaining **why** you want to become an **engineer** (and not a member of some other profession).

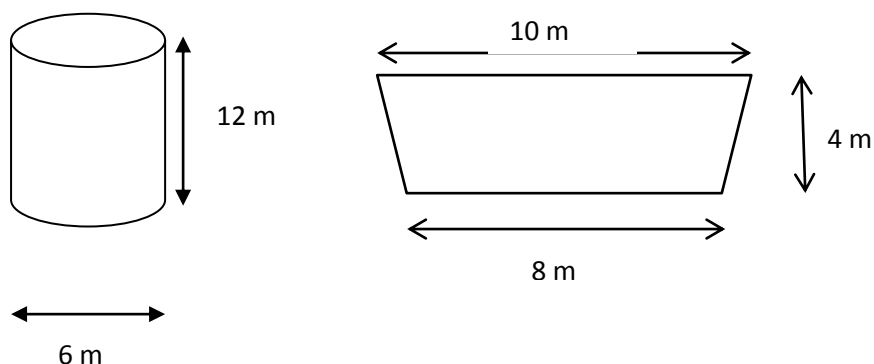
A.2. Near Mode Priming Task

Class Test 3 is coming up soon. Write up a rough study plan for the test. Your plan should include details about how much time you will spend working on problems and how much time you will spend studying in other ways. You should include how you plan to use the textbook, handbook, your own notes and any other resources.

Appendix B: Second Priming Intervention for Conceptual Problem

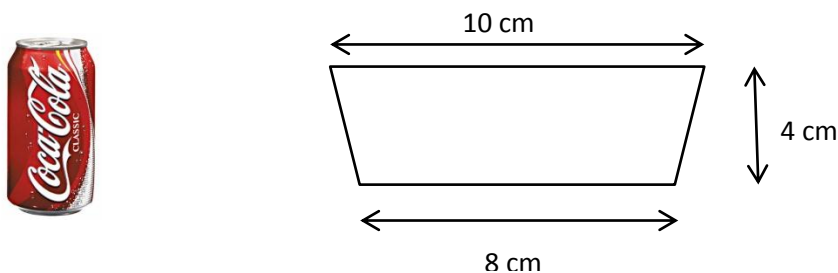
B.1. Far Mode Priming and Test Question

Siphiwe is attempting to make a Coca-Cola cake (yes, this is a real thing). The first thing he does is pour the coke from the can into a dish. The coke can is 6 cm across and 12 cm tall. The dish he pours it into has a trapezoidal cross-section, the base of the dish is 8 cm and the top is 10 cm as shown in the diagram below. The dish has a constant width of 5 cm. Find an equation relating the depth h of coke in the can and depth H of Coke in the dish. What is the rate of change of h with respect to H ?



B.2. Near Mode Priming and Test Question

McGuigan Simeon Wines is one of the largest wine companies in Australia. In preparation of one of their red wines, grapes must ferment for two years in a cylindrical tank 6 m across and 12 m high. It is then poured for mixing into a container with a trapezoidal cross-section, the base of the container is 8 m and the top is 10 m as shown in the diagram below. The container has a constant width of 5 m. Find an equation relating the depth h of wine in the tank and depth H of wine in the container. What is the rate of change of h with respect to H ?



Appendix C: First Priming Intervention for Procedural Problem

C.1. Far Mode Priming Task

When you graduate, what are you expecting your financial situation to be? Will you be paying of student loans or working off a bursary? Do you have a job lined up? If not what will be your main considerations when looking for a job (salary, community development, location near to family or anything else you're looking for in a job)?

C.2. Near Mode Priming Task

Assume ASPECT will be getting extra money next year. The money can only be spent on stationary for ASPECT students. Based on your experiences this year, what stationary should ASPECT buy that would be most helpful to next years' students (Include items up to the cost of a calculator. So things like pens, exams pads and folders etc., not iPads).

Appendix D: Second Priming Intervention for Procedural Problem

D.1. Far Mode Priming and Test Question



A long, long time ago in Middle Earth, Gimli the dwarf liked to keep track of the number of Wizards, Trolls and Orcs he had slain with his axe. By the time that the evil lord Sauron had been defeated, he had killed 560 villains in total. He had swung his axe 6200 times altogether because he needed twenty swings of the axe to kill each Wizard and Troll but only ten swings to kill each Orc. There are many more Trolls than evil Wizards, so he killed five times as many Trolls as Wizards.

If we let x , y and z be the number of wizards, trolls and orcs killed respectively, all the information in the above problem can be represented by the following system of equations.

$$\begin{aligned}x + y + z &= 560 \\20x + 20y + 10z &= 6200 \\5x &= y\end{aligned}$$

How many of each type of villain did Gimli kill?

D.2. Near Mode Priming and Test Question



You are giving a huge party tonight and need to buy some drinks. To keep things simple you're only buying three types of drink, Red Bull, Play and Coke. You must buy 560 drinks altogether. The Red Bull and Plays cost R20 each while each Coke costs R10 and you have R6200 to spend all together. You also know that five times as many of your friends drink Red Bull than drink Play.

If we let x , y and z be the number of Plays, Red Bulls and Cokes bought respectively, all the information in the above problem can be represented by the following system of equations.

$$x + y + z = 560$$

$$20x + 20y + 10z = 6200$$

$$5x = y$$

How many of each type of drink will you buy?