The Lattice of Quasi-Uniformities

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Abstract

Over the last thirty years much progress has been made in the investigation of the lattice of uniformities on a set X. In particular, Pelant, Reiterman, Rödl and Simon have published several articles concerning anti-atoms and complements in this lattice. The aim of this dissertation is to begin a similar investigation into the lattice of quasi-uniformities $\Theta(X)$ on a set X. It starts off with a summary of results obtained for the lattice of topologies on X, which, having been studied in great detail in the past, is intended as an example as to what may be achieved with $\Theta(X)$. An exposition of the lattice of uniformities is then given. We conclude by commencing an investigation into the lattice of quasi-uniformities on X. Where possible, results obtained for the lattice of uniformities are generalized to $\Theta(X)$, and some original results for $\Theta(X)$ are also presented.

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Chapter 1

Introduction

In his 1936 paper [5], Garett Birkhoff saw that an important aspect of the study of topologies on a set X is that of the comparison of two topologies on X. He hence ordered the collection of all topologies on a given set X by set inclusion, and studied the resulting lattice. Since then, much progress has been made in the investigation of this lattice. All the atoms and anti-atoms have been identified and characterized, and the lattice has been shown to be both atomic and anti-atomic. The cardinality of this lattice has been established for infinite X, it has been proven to be complemented, the lattice structure has been intensely studied and adjacent topologies have been investigated. In addition to all of this, sublattices and subcollections of this lattice, minimal and maximal topologies, intervals and many other aspects have been and still are under investigation.

Even though the above is not nearly an exhaustive list of what has been achieved with the lattice of topologies, it makes it clear that it has been studied in much detail. What is the motivation for this? It turns out that the results obtained thus far not only concern the lattice of topologies, but also other areas of mathematics. For example, in [32] it is mentioned that a certain result on the automorphisms of this lattice allows one to identify the set of topologies possessing a given topological property by simply observing the structure of the lattice of topologies on X (see Theorem 3.7.3 and Remark 3.7.4). In this article it is also shown that for any lattice L there exists a set X such that L may be embedded into the lattice of topologies on X (see Theorem 3.7.1).

The above-mentioned article is already an indication that the role of the lattice of topologies in general lattice theory cannot be easily ignored.

Valent and Larson [34] and Rosický [52] have in fact elaborated on this, and by combining some results in these two papers it can be shown that any finite lattice can be realized as an interval of T_1-topologies if and only if it is distributive.[1]

The above comments suggest that the study of lattices of structures on a set X has the potential to benefit many areas of mathematics. Hence, an investigation into the lattice of uniformities on X (ordered by set inclusion) commenced. In 1975, Pelant and Reiterman published a paper [44] investigating the anti-atoms of this lattice. In particular, they found that there is a strong relationship between the anti-atoms in this lattice and ultrafilters on X, and that the nature of this relationship depends on whether or not the anti-atom in question is proximally discrete (a uniformity is called proximally discrete if it induces the discrete proximity: see Propositions 4.3.9 and 4.3.21). In particular, in [46, Proposition 2.1] they show that every proximally non-discrete anti-atom is a copy of a certain type of anti-atom $\mathcal{J}_{\mathcal{F}}$ for an ultrafilter $\mathcal{F}$ on X. It is interesting to note that the properties of this ultrafilter $\mathcal{F}$ also appears to determine whether or not the anti-atom $\mathcal{J}_{\mathcal{F}}$ is proximally fine (i.e. the finest uniformity inducing its proximity). This can be seen from their result cited in Theorem 4.3.24.

Based on their above-mentioned results, Pelant and Reiterman conjecture in their paper [46] that uniformities can be used for the classification and investigation of ultrafilters. This claim is supported by the fact that the interaction of ultrafilters with certain other structures on a given set has been successfully employed for this purpose in the past. For example, given a set X, the ultrafilters on X can be regarded as the points of the Stone-Čech compactification $\beta(X)$ of the discrete topological space on X.[2] As Simon notes in [56], this in turn suggests that uniform anti-atoms may also be able to tell us something about the properties of the points in the Stone-Čech compactification of a discrete space.

The investigations into the lattice of uniformities continued, and in 1976 Reiterman and Rödl gave an example of a non-transitive anti-atom in [51].

[1] Valent and Larson proved that any finite distributive lattice can be realized as an interval of T_1 topologies in [34]. Rosický proved that every finite interval between T_1-topologies must be distributive in [52].

[2] In general, one considers the z-ultrafilters on a completely regular Hausdorff topological space X. These are then used to define the Stone-Čech compactification $\beta(X)$ of X. See [9, Section 2] for more details on this approach.

These anti-atoms are complicated structures, and there had been speculation as to whether they existed at all.[3] Diverging from the subject of anti-atoms, Pelant and Reiterman in 1981 published [45] in which they proved that the lattice of uniformities on a set X is complemented if and only if X is finite (see Corollary 4.5.2). The paper furthermore contains some very in-depth results regarding complements in this lattice.

In light of the above, it is natural to start an investigation into the lattice of quasi-uniformities on a set X, which has been done in this dissertation. We start with the atoms, which turn out to be relatively simple structures – in Corollary 5.2.10 we show that the atoms are exactly those quasi-uniformities generated by a special kind of pre-order on X. We also show that a quasi-uniformity is atomic if and only if it is transitive and totally bounded (Proposition 5.2.15). The anti-atoms, though, have proven to be somewhat more intricate. Theorem 5.3.20 does however completely characterize those anti-atoms which are proximally non-discrete in terms of ultrafilters on X and ultrafilters on $X \times X$. The proximally fine anti-atoms are characterized in Theorem 5.3.32. We also study the question of whether between two distinct but comparable uniformities there is always a non-symmetric quasi-uniformity, and obtain some interesting partial solutions to this problem in Section 5.4.3. Complementation in this lattice resembles complementation in the lattice of uniformities, and in Corollary 5.5.4 we show that the lattice of quasi-uniformities on a set X is complemented if and only if X is finite. The property of having a complement is also shown to be preserved by several operations one can perform on or between quasi-uniformities (see Section 5.5.1).

It is interesting to note the various resemblances between the lattice of uniformities and the lattice of quasi-uniformities on a given set X. We have already mentioned that the results regarding complementation for the two lattices are particularly similar. The atoms of the lattice of uniformities are also just the symmetric counterparts of the atoms in the lattice of quasi-uniformities (see Corollary 5.2.12). Upon comparing the anti-atoms, we see that in both cases there is a strong relationship between the anti-atoms of these lattices and ultrafilters on X. The nature of this link is also for each of them determined by whether or not the anti-atom in question induces the discrete proximity. In addition to this, it is interesting to note that the same

[3] See the explanation at the beginning of Section 4.3.4.

condition that is both necessary and sufficient for an anti-atom of the lattice of uniformities to be proximally discrete is also necessary and sufficient for an anti-atom of the lattice of quasi-uniformities to be proximally discrete (Propositions 4.3.9 and 5.3.15).

We bring it to the reader's attention that some of the most interesting new results on the lattice of quasi-uniformities obtained during this investigation are collected in [11] for possible publication. The proofs given in this dissertation and in [11] respectively may sometimes differ, but the reader will be made aware of this if this is the case.

This dissertation starts with some preliminary definitions and notation, which are given in the next chapter. The aforementioned chapter contains two separate sections on quasi-uniformities and quasi-proximities respectively, each of which lists some often used basic definitions and results needed to understand this dissertation. Chapter 3 consists of a short summary of the lattice of topologies on a set X, and is intended as an example as to what may be achieved with the lattice of quasi-uniformities. Chapter 4 is on the lattice of uniformities. It lists the most important results obtained for this lattice thus far, though in a more in-depth manner than was done in the previous chapter. The majority of the results in this chapter have been translated from using the covering definition of a uniformity to the entourage definition. Finally then Chapter 5 starts an investigation into the lattice of quasi-uniformities, and this dissertation ends with the Conclusion (Chapter 6) in which a few unsolved problems are listed.

In this dissertation, the axiom of choice (AC) is assumed to hold throughout, and its use need not be explicitly indicated. However, the reader will be made aware of the use of the continuum hypothesis via the symbol (CH), which will be placed at the beginning of the result in question. We have also abbreviated "if and only if" by "iff". References for results will be placed at the beginning of the proofs, unless only the statement of the result was obtained from the referenced document. In this case, the reference will be at the beginning of the statement, and the proof given here may differ from the one in the cited document. A list of the most important symbols defined and used throughout this dissertation is given at the end of the Conclusion (Chapter 6), just before the Bibliography. It includes references to where in this dissertation these symbols were first introduced.

Chapter 2

Preliminaries

In this chapter we summarize the most often used general definitions and notation used throughout this dissertation. We also summarize some often used definitions and fundamental results relating to (quasi-) uniformities and (quasi-) proximities on a set X.

2.1 General Definitions and Notation

In this section we introduce some general definitions and notation used in this dissertation.

Special Sets and Cardinalities

For any set X, $|X|$ denotes the cardinality of X. $\wp(X)$ denotes the powerset of X, i.e. $\wp(X) = \{A \mid A \subseteq X\}$, and if $A \subseteq X$, $X - A$ denotes the complement of A. The symbol ω denotes the set of natural numbers, and, as usual, its cardinality is denoted by $\aleph_0$. $\mathbb{R}$ is the set of real numbers, and c denotes the cardinality of $\mathbb{R}$ (i.e. $c = 2^{\aleph_0}$).

Relations

If R is a relation on a set X, R is called a *pre-order* if it is reflexive and transitive, and is called a *(partial) order* if it is also anti-symmetric. R is an *equivalence relation* if it is reflexive, transitive, and symmetric. By the *transitive closure* of R is meant the smallest transitive relation on X containing R.

If $A \subseteq X$, the *diagonal* of A is defined as $\Delta_A = \{(x, x) \mid x \in A\}$. If $A = X$ and there is no danger of confusion, we write Δ_X simply as Δ.

If $A \subseteq X \times X$ and $B \subseteq X \times X$, then $A \circ B = \{(x, z) \mid \exists y \in X$ such that $(x, y) \in A$ and $(y, z) \in B\}$. A^2 denotes $A \circ A$, A^3 denotes $A \circ A \circ A$, and so on. If U is any subset of $X \times X$, then $U^{-1} = \{(x, y) \mid (y, x) \in U\}$. If $x \in X$ then $U(x) = \{y \in X \mid (x, y) \in U\}$, and if $A \subseteq X$ we denote $U(A) = \{y \in X \mid (x, y) \in U$ for some $x \in A\}$.

If $f : X \to Y$ is a function, then $f \times f : (X \times X) \to (Y \times Y)$ is the function defined by $(f \times f)(x, y) = (f(x), f(y))$. If $A \subseteq X$, then $f|_A$ denotes the restriction of f to the set A.

Filters

Given a set X, a non-empty family $\mathcal{S}$ of subsets of X is said to have the *finite intersection property* provided that the intersection of every non-empty finite subfamily of $\mathcal{S}$ is non-empty. If $\mathcal{S}$ has the finite intersection property, the *filter generated by* $\mathcal{S}$ is given by $\mathrm{fil}(\mathcal{S}) = \{S \subseteq X \mid \bigcap \mathcal{H} \subseteq S$ for some non-empty finite subset $\mathcal{H}$ of $\mathcal{S}\}$. If S is a subset of X, then $\mathrm{fil}(S)$ denotes the filter generated by the base $\{S\}$.

An *ultrafilter* on X is a maximal filter, i.e. a filter that is not properly contained by any other filter on X. Note that if X is an infinite set, then the number of ultrafilters on X is $2^{2^{|X|}}$.

Lattices

If $\leq$ is a partial order on X, $(X, \leq)$ is called a *partially ordered set*. In any such partially ordered set, $a \in X$ is called a *maximal* element if $x \in X$ and $a \leq x$ implies $a = x$. *Minimal* elements are defined similarly. *Zorn's Lemma* states that if $(X, \leq)$ is a non-empty partially ordered set such that every non-empty chain of X has an upper bound, then X has a maximal element. The greatest lower bound $x \wedge y$ of two members x and y of X is called the *meet* of x and y, and the least upper bound $x \vee y$ of x and y is called their *join*. If $\{x_i \mid i \in I\}$ is any collection of members of L, then $\bigvee_{i \in I} x_i$ and $\bigwedge_{i \in I} x_i$ are defined similarly.

A partially ordered set $L = (X, \leq)$ such that the meet and join of any two elements of L always exist is called a *lattice* (note that by $x \in L$ we mean $x \in X$). It is called a *semilattice* with respect to sup (or inf) if $x \vee y$ (respectively $x \wedge y$) exists in L for all $x, y \in X$. L is *complete* if arbitrary meets and joins always exist in L. In this case the *top* (greatest) element of L is denoted by $\mathbf{1}$ and the *bottom* (least) element is denoted by $\mathbf{0}$. If A is a subset of X, $\leq|_A = \leq \cap (A \times A)$ and $K = (A, \leq|_A)$, then K is called a *sublattice* of L if for every $x, y \in K$, $x \vee y \in K$ and $x \wedge y \in K$.

If $x, y \in L$, then x is said to *cover* y if $y < x$ and there is no $z \in L$ such that $y < z < x$. If x covers y, then x is said to be an *immediate successor* of y, and y an *immediate predecessor* of x. An element a of L is called an *atom* if it covers $\mathbf{0}$, and an *anti-atom* if it is covered by $\mathbf{1}$.[1] a is said to be *atomic* if it can be written as the join of atoms, and *anti-atomic* if it can be written as the meet of anti-atoms. L is called *atomic* if every element of L other than $\mathbf{0}$ is atomic, and an *anti-atomic* lattice is defined similarly.

If $x \in L$, then $y \in L$ is called a *complement* of x if $x \vee y = \mathbf{1}$ and $x \wedge y = \mathbf{0}$. L is called *complemented* if every $x \in L$ has at least one complement.

A map from a lattice L to a lattice K is called a *lattice homomorphism* if it preserves finite meets and joins. The map is called a *complete homomorphism* if it preserves arbitrary meets and joins. A homomorphism is called an *embedding* if it is one-to-one, and a *(lattice) isomorphism* if it is one-to-one and onto. f is called an *automorphism* if it is an isomorphism mapping a lattice L into itself. A lattice $(X, \leq)$ is called *self-dual* if it is lattice isomorphic to $(X, \geq)$ (here $\geq = (\leq)^{-1}$), and $\geq$ is called the *dual order* of $\leq$.

L is called *modular* if $x, y, z \in L$ and $x \leq z$ implies $x \vee (y \wedge z) = (x \vee y) \wedge z$. L is said to be *distributive* if for every $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ (and $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$). Every distributive lattice is modular.

Proposition 2.1.1. *A lattice is*

1. *non-modular if and only if N_5 can be embedded into it, and*

2. *non-distributive if and only if N_5 or M_3 can be embedded into it.*

Proof. See [10, pg. 89]. The lattices N_5 and M_3 are as shown in the below figure. □

A lattice L is called *upper semi-modular* if for distinct x and y in L such that x and y cover an element $z \in L$, $x \vee y$ covers both x and y. A *lower semi-modular* lattice is defined dually. If L is a complete atomic lattice with its set of atoms denoted by A, L is said to be *tall* if for every $S \subseteq A$, with $s = \bigvee\{x \mid x \in S\}$, we have $\{x \in A \mid x \leq s\} = \bigcap\{B \mid S \subseteq B \subseteq A,\ x, y \in B \text{ and } z \leq x \vee y \text{ implies } z \in B\}$.

[1] Note that these definitions allow $\mathbf{0}$ to be an anti-atom and $\mathbf{1}$ to be an atom.

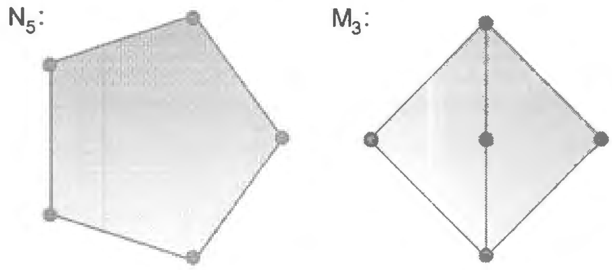

Figure 2.1: Diagrams for the lattices N_5 and M_3.

2.2 Uniformities and Quasi-Uniformities

In this section we mention some basic definitions and fundamental results pertaining to (quasi-) uniformities used in this dissertation. However, proofs will be mostly omitted. Readers that are familiar with uniformities and quasi-uniformities may opt to skip this section. The notation and conventions used mostly correspond to those used in [14].

Definition 2.2.1. A *quasi-uniformity* on a set X is a non-empty family $\mathcal{U}$ of subsets of $X \times X$ such that

1. $U \in \mathcal{U} \implies \Delta \subseteq U$,
2. $U \in \mathcal{U} \implies$ there is a $K \in \mathcal{U}$ such that $K \circ K \subseteq U$,
3. $U \subseteq A$ and $U \in \mathcal{U} \implies A \in \mathcal{U}$, and
4. $U, V \in \mathcal{U} \implies U \cap V \in \mathcal{U}$.

A *uniformity* is a quasi-uniformity that satisfies the following additional symmetry condition:

5. $U \in \mathcal{U}$ implies $U^{-1} \in \mathcal{U}$.

If $\mathcal{U}$ is a (quasi-) uniformity on X, then $(X,\mathcal{U})$ (or, sometimes, just X) is called a *(quasi-) uniform space.* Members of a quasi-uniformity are called *entourages.* A quasi-uniform space is called T_1*-separated* iff

$$\bigcap_{U \in \mathcal{U}} U = \Delta.$$

A quasi-uniformity will be called *non-symmetric* if it is not a uniformity.

Notation 2.2.2. If $\mathcal{U}$ is a quasi-uniformity, then $\mathcal{U}^{-1}$ is the quasi-uniformity given by $\{U^{-1} \mid U \in \mathcal{U}\}$, and is called the *conjugate* of $\mathcal{U}$. It is easily seen that a quasi-uniformity is a uniformity iff $\mathcal{U} = \mathcal{U}^{-1}$.

Definition 2.2.3. If $(X,\mathcal{U})$ is a quasi-uniform space and $A \subseteq X$ is non-empty, let $\mathcal{U}|_A = \{U \cap (A \times A) \mid U \in \mathcal{U}\}$. $\mathcal{U}|_A$ is called the *restriction* of $\mathcal{U}$ to A, and $(A,\mathcal{U}|_A)$ is called a *subspace* of $(X,\mathcal{U})$.

Definition 2.2.4. If $\mathcal{U}$ is a quasi-uniformity on X, $\mathcal{B}$ is called a *base* for $\mathcal{U}$ if $\mathcal{B} \subseteq \mathcal{U}$ and for every $U \in \mathcal{U}$ there is a $B \in \mathcal{B}$ such that $B \subseteq U$. $\mathcal{B}$ is called a *subbase* for $\mathcal{U}$ if $\{\bigcap \mathcal{H} \mid \mathcal{H} \subseteq \mathcal{B}, \mathcal{H}$ is non-empty and finite$\}$ is a base for $\mathcal{U}$.

Notation 2.2.5. If $\mathcal{U}$ and $\mathcal{V}$ are quasi-uniformities on X, $\mathcal{U}$ is said to be *coarser* than $\mathcal{V}$ (and $\mathcal{V}$ *finer* than $\mathcal{U}$) if $\mathcal{U} \subseteq \mathcal{V}$. The *discrete uniformity* $\mathcal{D}_X$ on X is the filter generated by the base $\{\Delta\}$, and the *indiscrete uniformity* $\mathcal{I}_X$ on X is $\{X \times X\}$. If there is no danger of confusion, $\mathcal{D}_X$ and $\mathcal{I}_X$ will be denoted by $\mathcal{D}$ and $\mathcal{I}$ respectively. It is clear that every quasi-uniformity is coarser than $\mathcal{D}$ and finer than $\mathcal{I}$.

Definition 2.2.6. If $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are (quasi-) uniform spaces, then a map $f : X \to Y$ is called *(quasi-) uniformly continuous* if for every $V \in \mathcal{V}$, $(f \times f)^{-1}(V) \in \mathcal{U}$. f is called a *(quasi-) uniform isomorphism* if it is one-to-one, onto, and f as well as its inverse are (quasi-) uniformly continuous. If there exists a (quasi-) uniform isomorphism between $\mathcal{U}$ and $\mathcal{V}$ we write $\mathcal{U} \simeq \mathcal{V}$.

Definition 2.2.7. Let $\{(X_i,\mathcal{U}_i) \mid i \in I\}$ be a family of (quasi-) uniform spaces. Then the *product (quasi-) uniformity* on $\prod_{i\in I} X_i$, denoted by $\prod_{i\in I} \mathcal{U}_i$, is defined to be the coarsest (quasi-) uniformity which makes all the projections $\pi_i : \prod_{i\in I} X_i \to X_i$ (for each $i \in I$) (quasi-) uniformly continuous. The product of two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ is also denoted by $\mathcal{U} \times \mathcal{V}$.

Definition 2.2.8. Let $\mathcal{S}$ be a collection of quasi-uniformities on a set X. Then we define the *supremum* of $\mathcal{S}$ to be

$$\bigvee \mathcal{S} = \bigvee_{\mathcal{U}\in\mathcal{S}}' \mathcal{U} = \text{fil}(\{U_1 \cap \ldots \cap U_n \mid \text{for each } 1 \le i \le n,\ U_i \in \mathcal{U}_i \text{ for some } \mathcal{U}_i \in \mathcal{S}, n \in \omega\}).$$

$\bigvee \mathcal{S}$ is the coarsest quasi-uniformity finer than each $\mathcal{U} \in \mathcal{S}$.

Pseudo-Metrics and Quasi-Pseudo-Metrics

Definition 2.2.9. A *quasi-pseudo-metric* on a set X is a function $d : X \times X \to [0, \infty)$ such that

1. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$, and
2. $d(x, x) = 0$ for all $x \in X$.

d is called a *pseudo-metric* if it also satisfies the following symmetry condition:

3. $d(x, y) = d(y, x)$ for all $x, y \in X$.

A pseudo-metric is called a *metric* if the second condition above is replaced by

$$d(x, y) = 0 \text{ iff } x = y.$$

Definition 2.2.10. Let ρ be a quasi-pseudo-metric on X. The collection of all sets of the form $U_\epsilon^\rho = \{(x, y) \mid \rho(x, y) < \epsilon\}$ (also denoted simply by U_ϵ if there is no danger of confusion) form a base for a quasi-uniformity called the *quasi-uniformity generated* (or *induced*) *by* ρ, which is denoted by $\mathcal{U}_\rho$. If ρ is a pseudo-metric, $\mathcal{U}_\rho$ is a uniformity.

The connection between quasi-uniformities and quasi-pseudo-metrics becomes clear below.

Lemma 2.2.11. *Let $(U_n)_{n\in\omega}$ be a sequence of reflexive relations on a set X such that for each $n \in \omega$, $U_{n+1} \circ U_{n+1} \circ U_{n+1} \subseteq U_n$. Then there is a quasi-pseudo-metric d on X such that $U_{n+1} \subseteq \{(x, y) \mid d(x, y) < (\frac{1}{2})^n\} \subseteq U_n$ for each $n \in \omega$. If each of the U_n are symmetric, then d can be found to be a pseudo-metric.*

Proof. See for example [23, Lemma 6.12]. Here the proof is done for symmetric U_n and pseudo-metrics, but by leaving out the symmetry conditions one immediately obtains a proof for this lemma. ☐

Corollary 2.2.12. *Given a set X, every (quasi-) uniformity $\mathcal{U}$ on X can be written in the form*

$$\mathcal{U} = \bigvee_{i\in I} \mathcal{U}_{d_i}$$

for some family of (quasi-) pseudo-metrics $\{d_i \mid i \in I\}$ on X.

Definition 2.2.13. A quasi-uniformity is called *quasi-pseudo-metrizable* if there exists a quasi-pseudo-metric ρ such that $\mathcal{U}_\rho = \mathcal{U}$. A uniformity $\mathcal{U}$ is similarly called (pseudo-) metrizable if there exists a (pseudo-) metric ρ such that $\mathcal{U}_\rho = \mathcal{U}$.

Corollary 2.2.14. *A quasi-uniformity $\mathcal{U}$ is quasi-pseudo-metrizable if and only if $\mathcal{U}$ has a countable base.*

Transitive Quasi-Uniformities

Definition 2.2.15. A quasi-uniformity $\mathcal{U}$ on a set X is called *transitive* if it has a base consisting of transitive relations on X.

Proposition 2.2.16. *If $\{\mathcal{U}_i \mid i \in I\}$ is a collection of transitive quasi-uniformities on a set X, then $\bigvee_{i \in I} \mathcal{U}_i$ is transitive too.*

Proof. This follows from the fact that the intersection of transitive relations is transitive. □

Topologies Induced by Quasi-Uniformities

Given any quasi-uniformity $\mathcal{U}$ on a set X, we can obtain a topology on X as follows:

Definition 2.2.17. Let $(X, \mathcal{U})$ be a quasi-uniform space. The *topology induced by $\mathcal{U}$* is given by

$$\mathcal{T}(\mathcal{U}) = \{G \subseteq X \mid \text{for every } x \in G \text{ there is a } U \in \mathcal{U} \text{ such that } U(x) \subseteq G\}.$$

If $\mathcal{T}$ is a topology on X, then $\mathcal{U}$ is said to be *compatible* with $\mathcal{T}$ if $\mathcal{T}(\mathcal{U}) = \mathcal{T}$, and $(X, \mathcal{T})$ is said to *admit* $\mathcal{U}$.

Several useful facts concerning these quasi-uniformly induced topologies are listed below.

Proposition 2.2.18. *If $\mathcal{U}$ and $\mathcal{V}$ are quasi-uniformities on a set X such that $\mathcal{U} \subseteq \mathcal{V}$, then $\mathcal{T}(\mathcal{U}) \subseteq \mathcal{T}(\mathcal{V})$.*

Proof. [14, Proposition 1.29]. □

Proposition 2.2.19. *Suppose $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$ is quasi-uniformly continuous. Then $f : (X, \mathcal{T}(\mathcal{U})) \to (Y, \mathcal{T}(\mathcal{V}))$ is continuous.*

Proof. [14, Proposition 1.14]. □

Proposition 2.2.20. *For any family of quasi-uniformities $\{\mathcal{U}_i \mid i \in I\}$ on a set X, $\mathcal{T}(\prod_{i\in I}\mathcal{U}_i) = \prod_{i\in I}\mathcal{T}(\mathcal{U}_i)$.*

Proof. [14, Section 1.16]. □

Proposition 2.2.21. *If $\{\mathcal{U}_i \mid i \in I\}$ is a collection of quasi-uniformities on a set X, then $\mathcal{T}(\bigvee_{i\in I}\mathcal{U}_i) = \bigvee_{i\in I}\mathcal{T}(\mathcal{U}_i)$.*

Proof. [14, Section 2.3]. □

Totally Bounded Quasi-Uniformities

Definition 2.2.22. Let $\mathcal{U}$ be a quasi-uniformity on a set X. $\mathcal{U}$ is said to be *totally bounded* if for every $U \in \mathcal{U}$ there is a finite cover $\{A_i \mid 1 \leq i \leq n\}$ (for some $n \in \omega$) of X such that for each $1 \leq i \leq n$, $A_i \times A_i \subseteq U$.

Proposition 2.2.23. *Let $\mathcal{U}$ be a totally bounded quasi-uniformity and $\{\mathcal{U}_i \mid i \in I\}$ a collection of totally bounded quasi-uniformities on a set X. Then the following holds:*

1. *Any quasi-uniformity coarser than $\mathcal{U}$ is totally bounded, and*

2. *$\bigvee_{i\in I}\mathcal{U}_i$ is totally bounded.*

Proof. [14, Section 1.32]. □

Given a topology $\mathcal{T}$ on X, we can always find a quasi-uniformity compatible with it, namely the *Pervin quasi-uniformity.* This is proven in the below proposition:

Proposition 2.2.24. *Let $(X, \mathcal{T})$ be a topological space and let $\mathcal{S} = \{(X \times X) - (G \times (X - G)) \mid G \in \mathcal{T}\}$. Then $\mathcal{S}$ is a subbase for a totally bounded transitive quasi-uniformity $\mathcal{P}$ compatible with $\mathcal{T}$, called the* Pervin quasi-uniformity *for $(X, \mathcal{T})$.*

Proof. [14, Proposition 2.1]. □

Proposition 2.2.25. *Let $(X, \mathcal{T})$ be a topological space. The Pervin quasi-uniformity is the finest totally bounded quasi-uniformity compatible with $\mathcal{T}$.*

Proof. [14, Section 2.2]. □

A concept that is closely related to total boundedness is that of precompactness.

Definition 2.2.26. Let $\mathcal{U}$ be a quasi-uniformity on a set X. Then $\mathcal{U}$ is said to be *precompact* iff for each $U \in \mathcal{U}$ there is a finite $F \subseteq X$ such that $U(F) = X$.

Remark 2.2.27. It is not too difficult to see that every totally bounded quasi-uniformity is precompact, and that a uniformity is totally bounded if and only if it is precompact.

2.3 Proximities and Quasi-Proximities

In this section we mention some basic results on (quasi-) proximities used in this dissertation. Again, readers that are familiar with proximities and quasi-proximities may opt to skip this section. The notation and conventions used mostly correspond to those used in [14].

Definition 2.3.1. A relation δ on $\wp(X)$ is called a *quasi-proximity* on X iff it satisfies the following conditions (here $A\delta B$ means $(A, B) \in \delta$, and $A\bar{\delta}B$ means $(A, B) \notin \delta$):

1. $X\bar{\delta}\emptyset$ and $\emptyset\bar{\delta}X$,
2. $C\delta(A \cup B)$ if and only if $C\delta A$ or $C\delta B$, and
   $(A \cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$.
3. $\{x\}\delta\{x\}$ for each $x \in X$, and
4. if $A\bar{\delta}B$, there exists a $C \subseteq X$ such that $A\bar{\delta}C$ and $(X - C)\bar{\delta}B$.

A *proximity* is a quasi-proximity satisfying the following additional symmetry condition:

5. $\delta = \delta^{-1}$.

If δ is a (quasi-) proximity on X, (X, δ) is called a *(quasi-) proximity space*. If there is no danger of confusion, we will simply refer to X as a (quasi-) proximity space. A quasi-proximity space is called T_1*-separated* if the third condition above is replaced by

$$\{x\}\delta\{y\} \text{ iff } x = y.$$

Definition 2.3.2. If A and B are subsets of X in the quasi-proximity space (X, δ), A is said to be *near* B if $A\delta B$ and A is said to be *far from* B if $A\bar{\delta}B$. If δ is a proximity, then A is near B iff B is near A. In this case, we simply say that A and B are *proximal*.

Definition 2.3.3. If δ and ρ are two quasi-proximities on a set X, then we say that δ is *finer* than ρ (or ρ is *coarser* than δ) if $\delta \subseteq \rho$. The *discrete proximity* is defined by letting $A\delta B$ iff $A \cap B \neq \emptyset$. This is the finest quasi-proximity on X. There is also a *coarsest* quasi-proximity on X, defined by letting $A\delta B$ iff $A \neq \emptyset$ and $B \neq \emptyset$.

Topologies Induced by Quasi-Proximities

Definition 2.3.4. Let (X, δ) be a quasi-proximity space. Then the function $\mathrm{cl}_\delta : \wp(X) \to \wp(X)$ defined by $\mathrm{cl}_\delta(A) = \{x \mid \{x\}\delta A\}$ for $A \subseteq X$ is a closure operator for a topology on X. The topology $\mathcal{T}(\delta)$ generated by this closure operator is called the *topology induced by* δ.

If $(X, \mathcal{T})$ is a topological space, it is said to *admit* a quasi-proximity δ (and δ is said to be *compatible with* $\mathcal{T}$) if δ induces $\mathcal{T}$.

Quasi-Proximities Induced by Quasi-Uniformities

So far we have seen that a quasi-uniformity can induce a topology, and that a quasi-proximity can induce a topology. Now we show how a quasi-uniformity can induce a quasi-proximity.

Definition 2.3.5. If $(X, \mathcal{U})$ is a quasi-uniform space, the *quasi-proximity induced by* $\mathcal{U}$ is the quasi-proximity $\delta_{\mathcal{U}}$ defined by letting $A\delta_{\mathcal{U}}B$ iff for each $U \in \mathcal{U}$, $(A \times B) \cap U \neq \emptyset$ (or otherwise put, $U(A) \cap B \neq \emptyset$). Note that if $\mathcal{U}$ is a uniformity, then $\delta_{\mathcal{U}}$ is a proximity. If $A\delta_{\mathcal{U}}B$, we sometimes say that A is near B in (or with respect to) $\mathcal{U}$.

Given a quasi-proximity δ on X, a quasi-uniformity $\mathcal{U}$ on X is said to be *compatible* with δ if $\delta_{\mathcal{U}} = \delta$. The class of all quasi-uniformities that are compatible with a quasi-proximity δ will be denoted by $\pi(\delta)$, and is called the *quasi-proximity class* of δ. If δ is a proximity, $\pi(\delta)$ is also called a *proximity class*.

Definition 2.3.6. A quasi-uniformity is called *proximally fine* iff it is the finest among all quasi-uniformities inducing its quasi-proximity, i.e. iff it is the finest member of its quasi-proximity class. Note that not every quasi-proximity class need have a finest member. When speaking in the context of uniformities only, as in Chapter 4, a uniformity is called proximally fine iff it is the finest uniformity inducing its proximity.

Quasi-uniformities that induce the same proximity as the discrete uniformity (namely the discrete proximity) are called *proximally discrete*. If a quasi-uniformity is not proximally discrete, it is called *proximally non-discrete*. It is easily seen that the indiscrete uniformity induces the coarsest quasi-proximity on X – see Definition 2.3.3 (the indiscrete uniformity is in fact the unique member of its proximity class).

Proposition 2.3.7. *If $\mathcal{U}$ and $\mathcal{V}$ are quasi-uniformities on a set X such that $\mathcal{U} \subseteq \mathcal{V}$, then $\delta_{\mathcal{V}} \subseteq \delta_{\mathcal{U}}$.*

Proof. [14, Proposition 1.29]. □

Proposition 2.3.8. *Let $\mathcal{U}$ be a quasi-uniformity on a set X. Then $\mathcal{T}(\mathcal{U}) = \mathcal{T}(\delta_{\mathcal{U}})$. Hence, if $\mathcal{U}$ is compatible with the quasi-proximity δ, then $\mathcal{T}(\mathcal{U}) = \mathcal{T}(\delta)$.*

Proof. Suppose $G \in \mathcal{T}(\mathcal{U})$. We show $X - G$ is closed in $\mathcal{T}(\delta_{\mathcal{U}})$. Suppose $x \in G$, then for some $U \in \mathcal{U}$ we have $U(x) \subseteq G$. Hence $(\{x\} \times (X - G)) \cap U = \emptyset$, so $\{x\}\bar{\delta}_{\mathcal{U}}(X - G)$ and $x \notin \mathrm{cl}_{\delta_{\mathcal{U}}}(X - G)$. Therefore, $X - G$ is closed in $\mathcal{T}(\delta_{\mathcal{U}})$, and hence $G \in \mathcal{T}(\delta_{\mathcal{U}})$.

Now suppose that $G \in \mathcal{T}(\delta_{\mathcal{U}})$. Then $X - G$ is closed in $\mathcal{T}(\delta_{\mathcal{U}})$. Suppose $x \in G$. Then $\{x\}\bar{\delta}_{\mathcal{U}}(X - G)$, and therefore there is a $U \in \mathcal{U}$ such that $(\{x\} \times (X - G)) \cap U = \emptyset$. Hence $U(x) \subseteq G$, proving that $G \in \mathcal{T}(\mathcal{U})$.

Hence $\mathcal{T}(\mathcal{U}) = \mathcal{T}(\delta_{\mathcal{U}})$. If $\mathcal{U}$ is compatible with δ, then $\delta_{\mathcal{U}} = \delta$, so $\mathcal{T}(\mathcal{U}) = \mathcal{T}(\delta_{\mathcal{U}}) = \mathcal{T}(\delta)$. □

Quasi-Uniformities Induced by Quasi-Proximities

Definition 2.3.9. Given a quasi-proximity δ on X, the *quasi-uniformity induced by δ*, denoted by $\mathcal{U}_\delta$, is generated by the subbase $\{(X \times X) - (A \times B) \mid A\bar{\delta}B\}$. If δ is a proximity, then $\mathcal{U}_\delta$ is a uniformity.

We have now obtained both a quasi-proximity from a quasi-uniformity and a quasi-uniformity from a quasi-proximity, i.e. we have a map of the form

$$\mathcal{U} \rightarrow \delta_{\mathcal{U}} \rightarrow \mathcal{U}_{\delta_{\mathcal{U}}}.$$

In light of this, a natural question to ask is whether the correspondence between quasi-uniformities and quasi-proximities on a set X is one-to-one. This is not exactly the case, but the below theorem does show that there is in fact a one-to-one correspondence between quasi-proximities and totally bounded quasi-uniformities.

Theorem 2.3.10. *Let (X, δ) be a quasi-proximity space. Then $\mathcal{U}_\delta$ is the unique totally bounded quasi-uniformity compatible with δ, and $\mathcal{U}_\delta$ is the coarsest quasi-uniformity compatible with δ.*

Proof. [14, Theorem 1.33]. □

Definition 2.3.11. If $\mathcal{U}$ is a quasi-uniformity, we let $\mathcal{U}_\omega$ denote the totally bounded quasi-uniformity compatible with $\delta_\mathcal{U}$. In other words, we denote $\mathcal{U}_{\delta_\mathcal{U}}$ by $\mathcal{U}_\omega$.

Remark 2.3.12. It is worth noting that

$$\mathcal{U}_\omega = \text{fil}(\{(X \times X) - (A \times B) \mid \exists U \in \mathcal{U} \text{ such that } U \cap (A \times B) = \emptyset\}).$$

It is easily seen that $\mathcal{U}_\omega$ is the finest totally bounded quasi-uniformity on X that is coarser than $\mathcal{U}$. This follows from Theorem 2.3.10 and because any quasi-uniformity $\mathcal{V}$ such that $\mathcal{U}_\omega \subseteq \mathcal{V} \subseteq \mathcal{U}$ is compatible with $\delta_\mathcal{U}$, since both $\mathcal{U}$ and $\mathcal{U}_\omega$ are (see Proposition 2.3.7).

Proposition 2.3.13. *Two quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X induce the same quasi-proximity if and only if $\mathcal{U}_\omega = \mathcal{V}_\omega$.*

Proof. [14, Section 1.37]. □

Proposition 2.3.14. *For each quasi-uniformity $\mathcal{U}$, $(\mathcal{U}^{-1})_\omega = (\mathcal{U}_\omega)^{-1}$.*

Proof. [14, Section 1.40]. □

Proposition 2.3.15. *For any quasi-uniformity $\mathcal{U}$, $\mathcal{T}(\mathcal{U}_\omega) = \mathcal{T}(\mathcal{U})$.*

Proof. Since $\mathcal{U}_\omega = \mathcal{U}_{\delta_\mathcal{U}}$ and hence $\mathcal{U}_\omega$ is compatible with $\delta_\mathcal{U}$, we have $\mathcal{T}(\mathcal{U}_\omega) = \mathcal{T}(\delta_\mathcal{U})$. But $\mathcal{T}(\delta_\mathcal{U}) = \mathcal{T}(\mathcal{U})$ by Proposition 2.3.8, and hence $\mathcal{T}(\mathcal{U}_\omega) = \mathcal{T}(\mathcal{U})$. □

Chapter 3

The lattice of Topologies

The lattice of topologies on a set X has already been studied in detail, and much has become known about this lattice. In this chapter we give a summary of the most important discoveries. The intent of this chapter is mainly to illustrate what can be achieved when studying lattices of structures on a set X in general. Hence, proofs of theorems are mostly omitted, and we do not go into too much detail. A large portion of this chapter was obtained from [32].

3.1 Introduction

Notation 3.1.1. We will denote the collection of all topologies on a set X, ordered by set inclusion $\subseteq$, by $\Sigma(X)$.

Theorem 3.1.2. *For any non-empty set X, $\Sigma(X)$ is a complete lattice, with bottom element the indiscrete topology and top element the discrete topology on X. If $\mathcal{C}$ is a collection of topologies on X, their join $\bigvee \mathcal{C}$ in $\Sigma(X)$ is the topology generated by the base $\{G_1 \cap ... \cap G_n \mid$ for each $1 \leq i \leq n$, $G_i \in \mathcal{T}_i$ for some $\mathcal{T}_i \in \mathcal{C}$, $n \in \omega\}$, and their meet is given by $\bigcap \mathcal{C}$.*

Proof. See [5]. □

There have been extensive efforts to determine the exact cardinality of $\Sigma(X)$ for any given X. In the case that X is infinite, this goal has certainly been achieved – it turns out to be the set-theoretic maximum.

Theorem 3.1.3. *If X is infinite, then $|\Sigma(X)| = 2^{2^{|X|}}$.*

Proof. See [15]. □

However, finding the cardinality of $\Sigma(X)$ when X is finite is not quite so straightforward. There is no known formula, but some partial results have been obtained, a few of which are listed below.

Theorem 3.1.4. *Suppose that X is finite. If $|X| = 1, 2, 3, 4, 5, 6$ or 7, then $|\Sigma(X)| = 1, 4, 29, 355, 6942, 209527$ and 9535241 respectively. If $|X| = n \neq 1$, then $2^n \leq |\Sigma(X)| \leq 2^{n(n-1)}$.*

Proof. This theorem was proved piecewise in several different papers. See [32, Theorem 1.5] for the list. □

3.2 Atoms in $\Sigma(X)$

Atoms in $\Sigma(X)$ turn out to be relatively simple structures, as we shall see below. Relevant questions to ask would be how many atoms there are in $\Sigma(X)$ for a given set X, whether the lattice is atomic and, if not, which topologies will be the atomic members of $\Sigma(X)$. These questions are all answered below without too much effort.

Theorem 3.2.1. [60] *A topology $\mathcal{T}$ is an atom in $\Sigma(X)$ if and only if it has the form $\{\emptyset, G, X\}$, where $\emptyset \subsetneq G \subsetneq X$.*

Proof. Clearly each topology $\mathcal{T}$ having the above form is an atom. Conversely, if $\mathcal{T}$ does not have the above form, it is either the indiscrete topology or contains at least two sets $G_1 \neq X$ and $G_2 \neq X$ that are not empty. Then $\{\emptyset, G_1, X\} \subsetneq \mathcal{T}$, so $\mathcal{T}$ is not an atom. □

Corollary 3.2.2. [60] *$\Sigma(X)$ is an atomic lattice.*

Proof. For any non-indiscrete topology $\mathcal{T}$, write $\mathcal{T} = \{G_i \mid i \in I\} = \bigvee_{i\in I}\{\emptyset, G_i, X\}$ for some index set I. □

Corollary 3.2.3. [60] *If X is finite, non-empty and $|X| = n$, then $\Sigma(X)$ has $2^n - 2$ atoms. If X is infinite, $\Sigma(X)$ has $2^{|X|}$ atoms.*

Proof. The number of atoms in $\Sigma(X)$ is $|\wp(X)| - 2$ for any non-empty set X. □

3.3 Anti-Atoms in $\Sigma(X)$

As was the case with atoms, all the obvious questions regarding anti-atoms in $\Sigma(X)$ have been answered. We show in particular that $\Sigma(X)$ is anti-atomic, and give a full description of the anti-atoms in $\Sigma(X)$.

We will need the following definition.

Definition 3.3.1. Suppose that $\mathcal{F}$ is an ultrafilter on X and that $x \notin \bigcap \mathcal{F}$. Denote $\mathcal{T}(x, \mathcal{F}) = \{G \subseteq X \mid x \notin G \text{ or } G \in \mathcal{F}\}$. $\mathcal{T}(x, \mathcal{F})$ is called an *ultratopology* on X, and it is said to be *principal* or *non-principal* depending on whether $\mathcal{F}$ is principal or non-principal.

Theorem 3.3.2. *A topology $\mathcal{T}$ is an anti-atom in $\Sigma(X)$ if and only if it is an ultratopology.*

Proof. See [15]. □

Corollary 3.3.3. $\Sigma(X)$ *is an anti-atomic lattice.*

Proof. See [15]. □

Corollary 3.3.4. *If X is finite, non-empty and $|X| = n$, $\Sigma(X)$ has $n(n-1)$ anti-atoms. Otherwise, if X is infinite, $\Sigma(X)$ has $2^{2^{|X|}}$ anti-atoms.*

Proof. See [15]. □

3.4 Adjacent Topologies in $\Sigma(X)$

In this section we present some results regarding which topologies have immediate successors and which have immediate predecessors in $\Sigma(X)$.

Before we continue we note that, in general, any two adjacent topologies are related in the following manner:

Proposition 3.4.1. *Suppose that $\mathcal{T}$ and $\mathcal{O}$ are adjacent topologies on X, say $\mathcal{T} \subsetneq \mathcal{O}$. Then $\mathcal{O}$ is the topology generated by the subbase $\mathcal{T} \cup \{A\}$ for some $A \notin \mathcal{T}$.*

Proof. [35, Lemma I.1.4] Pick $A \in \mathcal{O} - \mathcal{T}$, and let $\mathcal{H}$ be the topology generated by the subbase $\mathcal{T} \cup \{A\}$. Then $\mathcal{T} \subsetneq \mathcal{H} \subseteq \mathcal{O}$. Since $\mathcal{O}$ and $\mathcal{T}$ are adjacent, we must conclude that $\mathcal{H} = \mathcal{O}$. □

3.4.1 Immediate Successors

We start by mentioning some topologies which will always have immediate successors in $\Sigma(X)$. We will then also give examples of topologies which do not have immediate successors in $\Sigma(X)$.

Theorem 3.4.2. *Let $\mathcal{T}$ be a (completely) regular but non-T_1 topology on X. Then $\mathcal{T}$ has an immediate successor in $\Sigma(X)$.*

Proof. [35, Theorem I.2.9, Corollary I.2.10]. □

Corollary 3.4.3. *Every pseudo-metric topology which is not a metric topology has an immediate successor in $\Sigma(X)$.*

Proof. [35, Theorem I.2.11] Every pseudo-metric topology $\mathcal{T}$ is regular. If in addition to this $\mathcal{T}$ is not a metric topology, it is not T_1, and hence this result follows from Theorem 3.4.2. □

The condition that $\mathcal{T}$ be non-T_1 in the above results cannot be omitted. In [36, Example 4.1] and [35, Example I.4.2] it is proven that there exists a completely regular topology that has no immediate successor.

Theorem 3.4.4. *Let $\mathcal{T}$ be a T_1, first countable, completely normal topology on X. Then $\mathcal{T}$ does not have an immediate successor in $\Sigma(X)$.*

Proof. [35, Theorem I.4.3]. □

Corollary 3.4.5. *No metric topology on X has an immediate successor in $\Sigma(X)$.*

Proof. [35, Theorem I.4.7] A metric topology is T_1, completely normal and first countable, so this result follows from the above theorem. □

3.4.2 Immediate Predecessors

We now summarize some results regarding which topologies will have immediate predecessors in $\Sigma(X)$.

Theorem 3.4.6. *Let $\mathcal{T}$ be a non-indiscrete topology on X that is either regular, completely regular or Hausdorff. Then $\mathcal{T}$ has an immediate predecessor in $\Sigma(X)$.*

Proof. [35, Theorem II.1.8, Theorem II.1.10, Corollary II.1.11]. □

Corollary 3.4.7. *Let $\mathcal{T}$ be a non-indiscrete pseudo-metric topology on X. Then $\mathcal{T}$ has an immediate predecessor in $\Sigma(X)$.*

Proof. [35, Theorem II.1.12] This follows from the above theorem, as every pseudo-metric topology is regular. □

It clearly follows from the above corollary that every non-indiscrete metrizable topology has an immediate predecessor in $\Sigma(X)$. In the light of Corollary 3.4.5, this is somewhat surprising, seeing as no metric topology has an immediate successor. On the other hand, it is interesting that every pseudo-metric topology that is not a metric topology will always have an immediate successor *and* predecessor (Corollaries 3.4.3 and 3.4.7).

3.5 Complements in $\Sigma(X)$

The questions of whether $\Sigma(X)$ is complemented and of how many complements a given topology can have have been answered to a large extent already. We list the most important of these results here.

Proposition 3.5.1. *For any set X, $\Sigma(X)$ is complemented.*

Proof. See [61] and [58]. □

Theorem 3.5.2. *If X is infinite, every topology in $\Sigma(X)$ (except the discrete and indiscrete topologies) has at least $|X|$ complements.*

Proof. See [55]. □

Theorem 3.5.3. *If X is infinite, there exists a subset of $\Sigma(X)$ of cardinality $|X|$ such that any two elements in this subset are complements of each other.*

Proof. See [2]. □

3.5.1 AT Topologies and Complements in $\Sigma(X)$

The idea of an AT topology has been very prominent in the area of complementation of topologies. Results illustrating the role of AT topologies in complementation are given throughout [62]. We will not elaborate too much on the subject here; however, this notion will prove to be useful in the chapters to come, and hence we give a small introduction to these topologies below.

Definition 3.5.4. Let $(X, \leq)$ be a pre-ordered set. The *Alexandroff-Tucker (AT) topology* for $(X, \leq)$ is the topology generated by the base $\{\{y \in X \mid y \geq x\} \mid x \in X\}$.

Lemma 3.5.5. *If $\mathcal{T}$ is the AT topology for the pre-ordered set $(X, \leq)$,*

$$x \leq y \quad \Leftrightarrow \quad (\forall U \in \mathcal{T},\ x \in U \Rightarrow y \in U) \quad \Leftrightarrow \quad y \in U_x,$$

where U_x denotes the intersection of all open neighbourhoods of x in $\mathcal{T}$. Hence $U_x = \{y \mid y \geq x\}$.

Proposition 3.5.6. *A topology is an AT topology if and only if arbitrary intersections of open sets are open.*

Proof. Suppose first that $\mathcal{T}$ is a topology on X such that arbitrary intersections of open sets are open. Define the pre-order $\leq$ on X by letting $x \leq y$ iff for all $U \in \mathcal{T}$, $x \in U \implies y \in U$. Then $\mathcal{T}$ is the AT topology for $(X, \leq)$.

Conversely, arbitrary intersections of open sets in a given AT topology on X are open, since by the above lemma, U_x is open for each $x \in X$. □

Proposition 3.5.7. [62, Section 2] *The set of AT topologies on X under inclusion and the set of pre-orders on X under reverse inclusion are isomorphic.*[1]

Proof. (Sketch) If S and R are pre-orders on X, and the collection of all pre-orders on X is ordered by reverse inclusion, then $S \wedge R = \bigcup_{n\in\omega}(S \cup R)^n$ (the transitive closure of $S \cup R$) and $S \vee R = S \cap R$.

Using the above lemma it is easy to prove that, if $AT(R)$ denotes the AT topology generated by the pre-order R, $AT(R \wedge S) = AT(R) \wedge AT(S)$ and

[1] Note that [62, Section 2] claims that the set of AT topologies on X ordered by inclusion is isomorphic to the set of pre-orders on X ordered by inclusion.

$AT(R \vee S) = AT(R) \vee AT(S)$. It also follows easily that the relationship between pre-orders on X and AT topologies on X is 1-1 – no two distinct pre-orders induce the same AT topology. □

Proposition 3.5.8. *Every topology in $\Sigma(X)$ has a complement that is an AT topology.*

Proof. See [62, Proposition 3]. □

3.6 Lattice Structure of $\Sigma(X)$

In this section we look at the lattice structure of $\Sigma(X)$, i.e. properties like distributivity, tallness and self-dualness.

The structure of $\Sigma(X)$ turns out to be not at all simple, as the following theorem illustrates.

Theorem 3.6.1. *If $|X| > 2$, then $\Sigma(X)$ is non-distributive, non-modular and not even upper or lower semi-modular.*

Proof. [58], [60] and [33]. □

From the sections on atoms and anti-atoms we see that if X has more than three elements, $\Sigma(X)$ has more atoms than anti-atoms. This leads to the following observation.

Theorem 3.6.2. *If $|X| > 3$, then $\Sigma(X)$ is not self-dual.*

Proof. [58] This follows immediately from Corollaries 3.2.3 and 3.3.4. □

Finally, we have below yet another theorem demonstrating the complexity of $\Sigma(X)$.

Theorem 3.6.3. *$\Sigma(X)$ is tall iff X is finite.*

Proof. See [18] □

3.7 Morphisms of $\Sigma(X)$

In this section we take a quick look at a few theorems concerning the number of homomorphisms of $\Sigma(X)$, and the group of automorphisms of $\Sigma(X)$. These results have important implications for the topological properties of members of $\Sigma(X)$, which will be mentioned as well.

Theorem 3.7.1. *For every lattice L, there exists a set X such that L may be embedded in $\Sigma(X)$.*

Proof. [32, Theorem 1.9]. □

Theorem 3.7.2. *If $|X| \neq 2$, $\Sigma(X)$ has only trivial lattice homomorphisms, i.e. every lattice homomorphism of $\Sigma(X)$ onto a lattice L is either a lattice isomorphism, or L consists of a single element.*

Proof. See [18] □

Theorem 3.7.3. *If X contains one or two elements or is infinite, the group of lattice automorphisms of $\Sigma(X)$ is isomorphic to the symmetric group on X.*[2] *Otherwise, if X is finite and contains more than two elements, the group of lattice automorphisms of $\Sigma(X)$ is isomorphic to the direct product of the symmetric group on X with the two-element group.*

Proof. [15], [18]. □

Remark 3.7.4. [32, pg. 181] Suppose X is an infinite set. It follows from the above theorem that any automorphism of $\Sigma(X)$ simply permutes the elements of X. Hence, every automorphism of $\Sigma(X)$ would map a topology in $\Sigma(X)$ onto a homeomorphic image. Consequently, the topological properties of a member of $\Sigma(X)$ must be determined by its position in $\Sigma(X)$. In other words, the above theorem has the following consequence:

> Let X be an infinite set and P a topological property. Then the set of topologies in $\Sigma(X)$ possessing property P may be identified simply from the lattice structure of $\Sigma(X)$.

---

[2]The symmetric group on X is the group $S_X = \{f : X \to X \mid f \text{ is a bijection}\}$. Hence, S_X is essentially a group of permutations on X, as the members of S_X just permute the elements of X.

The following theorem illustrates the concept mentioned in the above remark.

Theorem 3.7.5. *If $\mathcal{T}$ is an anti-atom in $\Sigma(X)$, then $\mathcal{T}$ is T_1 iff it has no maximum complement in $\Sigma(X)$.*

Proof. See [54]. □

3.8 Maximal and Minimal Topologies

A concept that is closely related to atoms and anti-atoms in $\Sigma(X)$ is that of topologies that are maximal or minimal with respect to a given topological property. A topology $\mathcal{T}$ on X is said to be *maximal* with respect to a topological property P if $\mathcal{T}$ has property P and $\mathcal{T} \subsetneq \mathcal{H}$ for any topology $\mathcal{H}$ on X implies that $\mathcal{H}$ does not have property P. It is clear that such a maximal topology need not be the greatest topology that has property P. If, however, this is the case, then $\mathcal{T}$ is called *maximum* with respect to property P. Topologies that are *minimal* and *minimum* with respect to a topological property are defined similarly.

As we shall see, topologies that are maximal or minimal with respect to a given topological property can often be characterized very specifically in terms of their underlying topological structure.

It may also be the case that a property possessed by a topological space can be characterized in terms of a certain property the space is minimum or maximum with respect to. For example:

Theorem 3.8.1. *Let $(X, \mathcal{T})$ be a topological space. The following conditions are equivalent:*

1. *Every one-to-one map of $(X, \mathcal{T})$ onto itself is a homeomorphism.*
2. *$\mathcal{T}$ is minimum with respect to a certain topological property P_1.*
3. *$\mathcal{T}$ is maximum with respect to a certain topological property P_2.*

Proof. See [31, Theorem 1]. In the proof of [31, Theorem 1] a space $(Y, \mathcal{T}_0)$ is defined to have property P_1 if there exists a one-to-one, onto, continuous map $f : (Y, \mathcal{T}_0) \to (X, \mathcal{T})$. It is defined to have property P_2 if there exists a one-to-one, onto, continuous map $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_0)$. □

A result similar to the above involving minimal and maximal topologies and continuous one-to-one maps can be found in [49].

Before continuing, we note that there need not be a maximal topology for every topological property:

Example 3.8.2. [32, pg. 190] Let X be an uncountable set. Then it is easy to see that neither the property of separability nor the property of second countability has a maximal topology. On the other hand, if X were countable, the discrete topology would be the maximum separable and second countable topology. As [32, pg. 190] notes, it is clear that the discrete topology will be maximum with respect to a number of topological properties. For example, it is maximum T_0, T_1, T_2, regular, normal and disconnected, to name but a few.

[32, pg. 191-192] continues by giving two tables of characterizations of maximal and minimal topologies respectively in terms of a number of well-known topological properties. We list only a few of these here as an illustration. For the references, see [32, pg 191 - 192].

Property	Characterization of Maximal Topologies
Compact	maximal iff the closed subsets are precisely the compact subsets
Countably Compact	maximal iff the closed subsets are precisely the countably compact sets
Sequentially Compact	maximal iff the closed subsets are precisely the sequentially compact sets
Lindelöf	maximal iff the closed subsets are precisely the Lindelöf subsets
Non-T_0	maximal iff it has the form $\mathcal{T}(x, \text{fil}(\{y\})) \cap \mathcal{T}(y, \text{fil}(\{x\}))$
Non-T_1	maximal iff it is a principal ultratopology

Table 3.1: Characterizations of topologies that are maximal with respect to certain topological properties.

Property	Characterization of Minimal Topologies
T_0	minimal iff it is T_0, nested (a chain), and the complements of the point closures generate the topology
T_1	minimal iff the closed proper subsets of the topology are precisely the finite subsets
T_2	minimal iff it is T_2 and every open filter with a unique cluster point converges
T_3	minimal iff it is T_3 and every regular filter with a unique cluster point converges
Metrizable	minimal iff it is metrizable and compact

Table 3.2: Characterizations of topologies that are minimal with respect to certain topological properties.

3.9 Subcollections of Members of $\Sigma(X)$

Many subfamilies of $\Sigma(X)$ which are also lattices have been investigated. Among these are the lattices of T_1-topologies, partition topologies, regular topologies, completely regular topologies, and many more (see [32, pg. 181 - 189]). Such subcollections can be studied in much the same way as $\Sigma(X)$ has been studied. In this section, however, we will concentrate on the lattice of T_1-topologies.

Notation 3.9.1. The lattice of T_1-topologies on a set X ordered by set inclusion $\subseteq$ will be denoted by $\Lambda(X)$.

It is easy to see that $\Lambda(X)$ is a (complete) *sublattice* of $\Sigma(X)$. The table below gives a short summary and comparison of these two lattices.[3] References given in this table are for the results for $\Lambda(X)$, as the corresponding results for $\Sigma(X)$ have already been cited throughout this chapter.

Most of the other lattices mentioned at the beginning of this section, though, are not sublattices of $\Sigma(X)$. Take for example the lattices of regular topologies and completely regular topologies. We know that the join in $\Sigma(X)$ of (completely) regular topologies is (completely) regular, but that this need not be the case for the meet of (completely) regular topologies.

[3]See [32, Section II] for more details on $\Lambda(X)$.

Lattice Property	$\Sigma(X)$	$\Lambda(X)$	References
Complete	✓	✓	[5]
Atomic	✓	×	[4], [60]
Anti-Atomic	✓	✓	[15]
Complemented	✓	×	[57]
Modular/Distributive	×	×	[4]
Upper/Lower Semi-Modular	×	✓	[33]
Cardinality for Finite X	Generally > 1	1	[15]
Cardinality for Infinite X	$2^{2^{\|X\|}}$	$2^{2^{\|X\|}}$	[15]

Table 3.3: A comparison of $\Sigma(X)$ with its sublattice $\Lambda(X)$. ✓ indicates that the lattice possesses the given property, and × indicates that it does not.

The below table lists some topological properties, indicating which are preserved under finite meets, arbitrary meets, finite joins and arbitrary joins respectively. From this table it is possible to identify which subcollections of $\Sigma(X)$ will be sublattices of $\Sigma(X)$, and which will be complete sublattices. This table was obtained from [32, pg. 184].

Topological Property	$\wedge$	$\bigwedge$	$\vee$	$\bigvee$	$\subseteq$	$\supseteq$
T_1	✓	✓	✓	✓	×	✓
T_0, T_2, Totally Disconnected	×	×	✓	✓	×	✓
T_3, Regular, Completely Regular, Zero-Dimensional	×	×	✓	✓	×	×
First and Second Countable	×	×	✓	×	×	×
Principal	✓	✓	✓	×	×	×
Compact, Lindelöf, Connected, Separable	✓	✓	×	×	✓	×
Locally Connected	✓	✓	×	×	×	×
T_4, T_5, (Completely) Normal, Paracompact, Locally Compact	×	×	×	×	×	×

Table 3.4: A table of topological properties preserved under meets and joins. ✓ indicates that the property is preserved, × indicates that it is not.

We will not investigate any of these collections of topologies any further – for some examples discussed in detail see [32].

Chapter 4

The Lattice of Uniformities

In this chapter we study the lattice of uniformities on a set X, which, in this presentation, will be ordered by set inclusion. A large proportion of the results presented are due to J. Pelant, J. Reiterman, V. Rödl and P. Simon, and the main references are [45], [46] and [47]. It is fair to warn that all of the above use *reverse* inclusion as their order on the uniformities of X. Hence, for example, their meets will be our joins and their atoms will be our anti-atoms. They also define uniformities in terms of covers of X, whereas we have used entourages to define uniformities.[1] All of the results in this chapter are presented in terms of entourage uniformities.

In order to avoid repetition it will sometimes happen that we state a result in this chapter but delay its proof until Chapter 5. There it is then either done for the more general case of quasi-uniformities, or derived from other results obtained there.

4.1 Introduction

Notation 4.1.1. Consider the collection of all uniformities on a set X. When this collection is equipped with the partial order $\subseteq$ (set inclusion), one obtains a lattice which will be denoted by $\Psi(X)$.

It is clear that the top element of $\Psi(X)$ will be the discrete uniformity and the bottom element the indiscrete uniformity. Given a collection $\mathcal{S}$ of

[1] See for example [20] and [13, Section 8.1]. The former uses uniformities defined in terms of covers throughout their exposition, and the latter gives a proof of the equivalence of the two definitions.

uniformities on X, the join of all the uniformities in $\mathcal{S}$ is given by $\bigvee \mathcal{S}$, which was defined in Definition 2.2.8. It is also clear that, if we write $\mathcal{S} = \{\mathcal{U}_i \mid i \in I\}$, then

$$\begin{aligned} \bigwedge \mathcal{S} &\subseteq \operatorname{fil}(\{\bigcup_{i \in I} U_i \mid U_i \in \mathcal{U}_i \text{ for each } i \in I\}) \\ &= \bigcap \mathcal{S}. \end{aligned}$$

Hence, we can summarize $\Psi(X)$ as follows:

Theorem 4.1.2. *For any set X, $\Psi(X)$ is a lattice with top element $\mathcal{D}$ and bottom element $\mathcal{I}$. If $\mathcal{S}$ is any non-empty collection of uniformities on X, then $\bigvee \mathcal{S} = \operatorname{fil}(\{U_1 \cap U_2 ... \cap U_n \mid \text{each } U_i \in \mathcal{U}_i \text{ for some } \mathcal{U}_i \in \mathcal{S}, n \in \omega\})$. Hence, since $\Psi(X)$ is bounded below, it is a complete lattice and $\bigwedge \mathcal{S}$ is given by the join of all the uniformities coarser than $\bigcap \mathcal{S}$.*

4.2 Atoms in $\Psi(X)$

Most of the results regarding the atoms of $\Psi(X)$ can be deduced immediately from a set of corresponding results obtained for quasi-uniformities in the next chapter. Hence we will leave the proofs until we have dealt with the quasi-uniform case (see Section 5.2). We do mention the most important results here, though, with references to where the proofs are given.

Example 4.2.1. The simplest example of an atom in $\Psi(X)$ is what is called a *trivial atom*. Such an atom has the form

$$\operatorname{fil}(((X - \{x\}) \times (X - \{x\})) \cup \Delta)$$

for any $x \in X$. It is clear that this is in fact an atom in $\Psi(X)$, since if U is a symmetric relation on X such that $((X - \{x\}) \times (X - \{x\})) \cup \Delta \subsetneq U$, then $U \circ U = X \times X$.

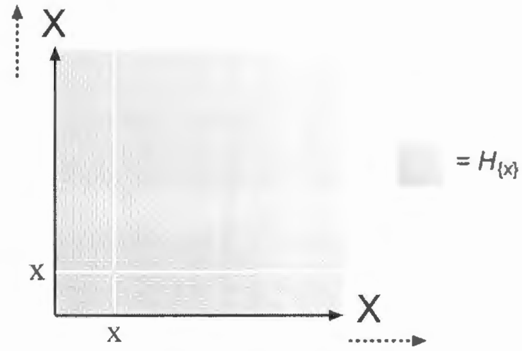

Figure 4.1: Diagrammatical representation of $H_{\{x\}} = ((X - \{x\}) \times (X - \{x\})) \cup \Delta$.

We obtain the general form of an atom in $\Psi(X)$ by replacing the singleton $\{x\}$ in the above example by any non-empty proper subset A of X. We have the following notation for these atoms:

Notation 4.2.2. We will denote, for any non-empty proper subset A of X,

$$H_A = (A \times A) \cup ((X - A) \times (X - A)).$$

Note that this is an equivalence relation on X, and hence we can define a corresponding uniformity $\mathcal{H}_A = \text{fil}(H_A)$.

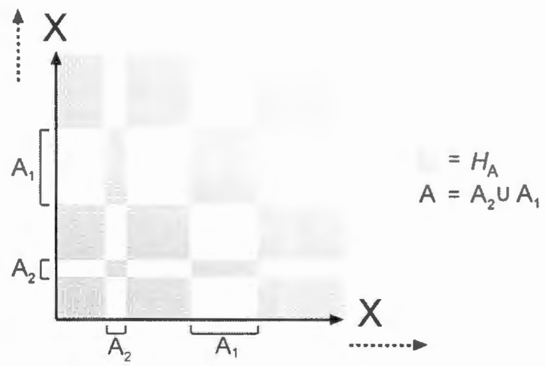

Figure 4.2: Diagrammatical representation of the set H_A.

Proposition 4.2.3. *$\mathcal{A}$ is an atom in $\Psi(X)$ if and only if $\mathcal{A} = \mathcal{H}_A$ for some non-empty proper subset A of X.*

Proof. See Corollary 5.2.12. □

Corollary 4.2.4. *If X is finite, non-empty and $|X| = n \neq 1$, $\Psi(X)$ has $2^{(n-1)} - 1$ atoms. If X is infinite, $\Psi(X)$ has $2^{|X|}$ atoms.*

Proof. See Corollary 5.2.13. □

Proposition 4.2.5. *All atoms in $\Psi(X)$ are transitive and totally bounded, and a uniformity is an atomic member of $\Psi(X)$ if and only if it is transitive and totally bounded.*

Proof. See Corollary 5.2.16. □

Finally there is still a question that has not yet been considered. Is $\Psi(X)$ atomic? The answer is *no*. In fact, we will give an example of a uniformity that does not contain a single atom of $\Psi(X)$. However, as this example answers the same question for the quasi-uniform case, we will delay it until the next chapter. See Example 5.2.18.

4.3 Anti-Atoms in $\Psi(X)$

We start this section with some basic results on anti-atoms in $\Psi(X)$. We will then proceed to study the proximally discrete and proximally non-discrete anti-atoms of $\Psi(X)$ separately.[2]

Our first basic result uses Zorn's Lemma to prove the existence of anti-atoms in $\Psi(X)$.

Proposition 4.3.1. *For any set X and every non-discrete uniformity $\mathcal{U}$ on X, $\mathcal{U}$ is contained by an anti-atom in $\Psi(X)$.*

Proof. This proof is done for the more general case of quasi-uniformities in Proposition 5.3.1. □

[2]A good percentage of the results stated on anti-atoms in $\Psi(X)$ can be found in [46] and [47].

Example 4.3.2. As a concrete example of an anti-atom in $\Psi(X)$, consider the uniformity $\mathcal{A}$ generated by the pre-order $\{(x,y),(y,x)\} \cup \Delta$, for $x \neq y$. It is clear that $\mathcal{A}$ is an anti-atom, and $\mathcal{A}$ is called a *trivial anti-atom* for similarly obvious reasons.

Notation 4.3.3. Suppose that $x, y \in X$ such that $x \neq y$. The trivial anti-atom $\text{fil}(\{(x,y)\} \cup \{(y,x)\} \cup \Delta)$ of $\Psi(X)$ will be denoted by $\mathcal{K}_{(x,y)}$.

The below proposition shows that trivial anti-atoms are an exceptional case, as they are the only anti-atoms that are not T_1-separated and do posses a non-isolated point. From these results it will also follow that $\Psi(X)$ is not anti-atomic.

Proposition 4.3.4. [46, Proposion 2.4] *All anti-atoms of $\Psi(X)$, except the trivial ones, are T_1-separated and all their points are isolated.*

Proof. Suppose that $\mathcal{U}$ is an anti-atom that is not T_1-separated. Then we can find a point $(x,y) \in \bigcap \mathcal{U}$ such that $x \neq y$. Then $\mathcal{U} \subseteq \mathcal{K}_{(x,y)}$ and hence $\mathcal{U} = \mathcal{K}_{(x,y)}$, so $\mathcal{U}$ is trivial.

Now suppose that $x \in X$ is not isolated with respect to $\mathcal{T}(\mathcal{U})$, where $\mathcal{U}$ is a non-trivial anti-atom in $\Psi(X)$. Suppose also that for some $V \in \mathcal{U}$, $V \subseteq (\{x\} \times X) \cup (X \times \{x\}) \cup \Delta$, and let $U \in \mathcal{U}$ be symmetric such that $U^2 \subseteq V$. If $(x,y) \in U$ and $(x,z) \in U$ where $y, z \neq x$, then $(z,y) \in U^2 \subseteq V$, so $z = y$. So there is at most one $y \neq x$ such that $(x,y) \in U$. Hence, since $U \subseteq (\{x\} \times X) \cup (X \times \{x\}) \cup \Delta$ and $\mathcal{U} \neq \mathcal{D}$, $U = \{(x,y)\} \cup \{(y,x)\} \cup \Delta$ for some $y \neq x$, contradicting that $\mathcal{U}$ is non-trivial. Hence, every member of $\mathcal{U}$ contains an element of $((X - \{x\}) \times (X - \{x\})) - \Delta$. Therefore, if $\mathcal{V}$ is the uniformity generated by the equivalence relation $((X-\{x\}) \times (X-\{x\})) \cup \Delta$, $\mathcal{U} \vee \mathcal{V}$ is not discrete, and it is strictly finer than $\mathcal{U}$ since x is isolated with respect to $\mathcal{T}(\mathcal{U} \vee \mathcal{V})$. We have contradicted the fact that $\mathcal{U}$ is an anti-atom, and the proof is complete. □

Corollary 4.3.5. *If X is infinite, $\Psi(X)$ is not anti-atomic.*

Proof. [46, Corollary 2.4] Suppose $\mathcal{U}$ is a T_1-separated uniformity on an infinite set X such that there is at least one $x \in X$ that is not isolated with respect to $\mathcal{T}(\mathcal{U})$. Let $\mathcal{U}' = \mathcal{U} \vee \text{fil}(((X - \{x\}) \times (X - \{x\})) \cup \Delta)$. Then $\mathcal{U}'$ is strictly finer than $\mathcal{U}$ because $((X - \{x\}) \times (X - \{x\})) \cup \Delta$ is not in $\mathcal{U}$. Note that $\mathcal{U}$ cannot be contained by a trivial anti-atom, as it is T_1-separated. Hence, if $\mathcal{A}$ is an anti-atom such that $\mathcal{U} \subseteq \mathcal{A}$, $((X-\{x\}) \times (X-\{x\})) \cup \Delta \in \mathcal{A}$

by the above proposition, and hence $\mathcal{U}' \subseteq \mathcal{A}$. Consequently $\mathcal{U}$ can not be the meet of anti-atoms. $\square$

For the purpose of studying the anti-atoms of $\Psi(X)$ further, we have divided them into two categories: the proximally discrete and the proximally non-discrete anti-atoms. We will see that every anti-atom of $\Psi(X)$ is related to an ultrafilter on X in a certain way, and that the category to which an anti-atom belongs is determined by (and determines) the nature of this relationship.

4.3.1 Proximally Discrete Anti-Atoms

In this section we will give a necessary and sufficient condition for an anti-atom of $\Psi(X)$ to be proximally discrete in terms of ultrafilters on X.

Before we start, we make a note on the existence of proximally discrete anti-atoms in $\Psi(X)$.

Remark 4.3.6. Suppose that X is infinite. Then since $\mathcal{D}_\omega$ is totally bounded and $\mathcal{D}$ is not, $\mathcal{D}_\omega \subsetneq \mathcal{D}$. Hence, if X is infinite, there is always at least one non-discrete uniformity inducing the discrete proximity. By Proposition 4.3.1, $\mathcal{D}_\omega$ must be contained by an anti-atom of $\Psi(X)$, which then has to be proximally discrete because $\mathcal{D}_\omega$ is. Therefore, $\Psi(X)$ will always have at least one proximally discrete anti-atom.

However, if X is finite, all uniformities are totally bounded and hence unique in their proximity classes. Hence, since $\mathcal{D}$ will be unique in its proximity class, there will be no proximally discrete anti-atoms.

The first step in establishing the relationship between the proximally discrete anti-atoms of $\Psi(X)$ and ultrafilters on X, is to make use of a filter on X to define a uniformity on X.

Definition 4.3.7. [47, Section 1.1] Suppose $\mathcal{F}$ is a filter on X. Then $\mathcal{U}_\mathcal{F}$ is defined to be the uniformity generated by the base consisting of all sets of the form

$$(F \times F) \cup \Delta,$$

where $F \in \mathcal{F}$. $\mathcal{U}_\mathcal{F}$ is called the *filter uniformity* with respect to $\mathcal{F}$.

Remark 4.3.8. If $\mathcal{F}$ is an ultrafilter on X, $\mathcal{U}_\mathcal{F}$ is proximally discrete. For suppose that A and B are two non-empty proper subsets of X such that $A \cap B = \emptyset$. Then either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$. If we assume the former, then $(A \times A) \cup \Delta \in \mathcal{U}_\mathcal{F}$, and since $((A \times A) \cup \Delta) \cap (A \times B) = \emptyset$, A is far from B in $\mathcal{U}_\mathcal{F}$. The latter case is similar.

Proposition 4.3.9. *An anti-atom $\mathcal{A}$ of $\Psi(X)$ is proximally discrete iff $\mathcal{U}_\mathcal{F} \subseteq \mathcal{A}$ for some non-principal ultrafilter $\mathcal{F}$ on X. If $\mathcal{A}$ is a proximally discrete anti-atom, then the ultrafilter $\mathcal{F}$ such that $\mathcal{U}_\mathcal{F} \subseteq \mathcal{A}$ is unique.*

Proof. See [46, Proposition 2.2] and [47, Proposition 1.3]. The proof is also done for the more general case of quasi-uniformities in Proposition 5.3.15. □

Corollary 4.3.10. *If X is finite, there are no proximally discrete anti-atoms in $\Psi(X)$. If X is infinite, the number of proximally discrete anti-atoms is $2^{2^{|X|}}$.*

Proof. Suppose X is infinite. Every proximally discrete anti-atom must contain $\mathcal{U}_\mathcal{F}$ for some unique ultrafilter $\mathcal{F}$ on X (by the above proposition), and every $\mathcal{U}_\mathcal{F}$ must be contained by at least one anti-atom (by Proposition 4.3.1). Hence there must be at least as many anti-atoms in $\Psi(X)$ as there are ultrafilters on X. Since the number of ultrafilters on X is $2^{2^{|X|}}$, and because set-theoretically there can be at most $2^{2^{|X|}}$ uniformities on X, the result is proven. □

Corollary 4.3.11. *If X is infinite, there are $2^{2^{|X|}}$ uniformities on X.*

A natural question to ask is whether $\mathcal{U}_\mathcal{F}$ will ever be an anti-atom in $\Psi(X)$, and if so, when. If X is countable, it can be established exactly when $\mathcal{U}_\mathcal{F}$ will be an anti-atom. To do this we need the following definition.

Definition 4.3.12. Let X be a countable set. A non-principal ultrafilter $\mathcal{F}$ on X is called *selective* (or *Ramsey*) if for every partition $\mathcal{P}$ of X into $\aleph_0$ pieces, either $\mathcal{F} \cap \mathcal{P} \neq \emptyset$ or there is an $F \in \mathcal{F}$ such that $F \cap P$ has one element for every $P \in \mathcal{P}$.

Remark 4.3.13. The existence of selective ultrafilters on a countable set X can be proved using the continuum hypothesis (CH). It is however consistent with set theory that selective ultrafilters do not exist. See [22, pg. 76, Theorem 7.8].

Proposition 4.3.14. *Suppose that X is countable and that $\mathcal{F}$ is a non-principal ultrafilter on X. Then $\mathcal{U}_{\mathcal{F}}$ is an anti-atom of $\Psi(X)$ iff $\mathcal{F}$ is selective.*

Proof. [46, Proposition 2.3]. □

Another reasonable question to ask is whether the anti-atom containing $\mathcal{U}_{\mathcal{F}}$ will be unique. This need not be the case, as the below result shows. But first we need to define the following order on ultrafilters.

Definition 4.3.15. For ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on ω, we write $\mathcal{F} \succ \mathcal{G}$ iff there exists a map $f : \omega \to \omega$ such that $f_*(\mathcal{F}) = \mathcal{G}$ (here $f_*(\mathcal{F}) = \{A \subseteq \omega \mid f^{-1}(A) \in \mathcal{F}\}$). This order is called the *Rudin-Keisler order* for ultrafilters. If f is finite-to-one (i.e. $f^{-1}(x)$ is finite for every $x \in \omega$), then $\mathcal{F}$ is called a *finite-to-one lift* of $\mathcal{G}$.

Theorem 4.3.16. (CH) *Let $1 \leq s < \aleph_0$ and let $\mathcal{G}$ be an ultrafilter on ω. Then there is an ultrafilter $\mathcal{F} \succ \mathcal{G}$ on ω such that there are precisely s distinct anti-atoms finer than $\mathcal{U}_{\mathcal{F}}$. All of these anti-atoms are transitive, and each of them has a base of the form $\{V \cap U \mid U \in \mathcal{U}_{\mathcal{F}}\}$ where $V = \bigcup_{P \in \mathcal{P}}(P \times P)$ for some partition $\mathcal{P}$ of ω.*

Proof. See [47, Theorem 4.1]. □

In contrast to the above theorem, we have the following almost opposite case, where $\mathcal{U}_{\mathcal{F}}$ is shown to be contained by a unique anti-atom. In fact, it shows more – that there are uniformities that have exactly the same form as the anti-atoms constructed above that will never be anti-atoms.

Theorem 4.3.17. (CH) *Let $\mathcal{G}$ be an ultrafilter on ω. Then there exists an ultrafilter $\mathcal{F} \succ \mathcal{G}$ on ω such that for each partition $\mathcal{P}$ of ω, the uniformity with the base $\{V \cap U \mid U \in \mathcal{U}_{\mathcal{F}}\}$, where $V = \bigcup_{P \in \mathcal{P}}(P \times P)$, is never an anti-atom of $\Psi(X)$. Moreover, there exists exactly one anti-atom $\mathcal{A}$ such that $\mathcal{U}_{\mathcal{F}} \subseteq \mathcal{A}$, and this anti-atom is transitive.*

Proof. [47, Theorem 4.2]. □

Finally we mention that certain parts of Theorem 4.3.16 can be extended considerably, proving that sometimes the number of distinct anti-atoms containing $\mathcal{U}_{\mathcal{F}}$ may even be the set-theoretic maximum.

Theorem 4.3.18. (CH) *Let $\mathcal{G}$ be an ultrafilter on ω. Then there exists an ultrafilter $\mathcal{F} \succ \mathcal{G}$ on ω such that there are $2^{\mathfrak{c}}$ distinct anti-atoms finer than $\mathcal{U}_{\mathcal{F}}$. (Note that $2^{\mathfrak{c}}$ is the cardinality of the set of all uniformities on ω.)*

Proof. [47, Theorem 4.3]. □

4.3.2 Proximally Non-Discrete Anti-Atoms

We now take a closer look at the proximally non-discrete anti-atoms of $\Psi(X)$. We will show that each proximally non-discrete anti-atom is related to an ultrafilter on X.

First we aim to find an example of a proximally non-discrete anti-atom in $\Psi(X)$. It is clear that every trivial anti-atom is proximally non-discrete. To find a non-trivial example, we need the following construction.

Definition 4.3.19. [46, Section 2.1] Let $\mathcal{F}$ be a filter on a set X. Then the collection of all sets of the form

$$U_F = \{((x,1),(x,2)) \mid x \in F\} \cup \{((x,2),(x,1)) \mid x \in F\} \cup \Delta$$

for $F \in \mathcal{F}$ forms a base for a uniformity on $X \times \{1,2\}$, which we denote by $\mathcal{J}_\mathcal{F}$.

Proposition 4.3.20. [46, Claim 2.1] *If $\mathcal{F}$ is an ultrafilter on X, then $\mathcal{J}_\mathcal{F}$ is a proximally non-discrete anti-atom in $\Psi(X \times \{1,2\})$.*

Proof. Let $\mathcal{V}$ be another non-discrete uniformity on $X \times \{1,2\}$ such that $\mathcal{V} \not\subseteq \mathcal{J}_\mathcal{F}$. Then there is a symmetric $V \in \mathcal{V}$ such that $V \notin \mathcal{J}_\mathcal{F}$. Define

$$F = \{x \mid ((x,1),(x,2)) \in V \text{ and } ((x,2),(x,1)) \in V\}.$$

Clearly $F \notin \mathcal{F}$ because otherwise $V \in \mathcal{J}_\mathcal{F}$. Hence $X - F \in \mathcal{F}$. Since $V \cap U_{X-F} = \Delta$, $\mathcal{V} \vee \mathcal{J}_\mathcal{F} = \mathcal{D}$. So $\mathcal{J}_\mathcal{F}$ is an anti-atom as claimed.

It is clear that $\mathcal{J}_\mathcal{F}$ is proximally non-discrete, since $X \times \{1\}$ and $X \times \{2\}$ have an empty intersection but are proximal in $\mathcal{J}_\mathcal{F}$. □

The above example of a proximally non-discrete anti-atom is quite significant. Anti-atoms of this form can in fact be used to find a general description of the proximally non-discrete anti-atoms in $\Psi(X)$.

Proposition 4.3.21. *Suppose X is infinite and let $\mathcal{A}$ be a uniformity on X. Then $\mathcal{A}$ is a proximally non-discrete anti-atom in $\Psi(X)$ iff $\mathcal{A}$ is uniformly isomorphic to $\mathcal{J}_\mathcal{F}$ for some ultrafilter $\mathcal{F}$ on X. Hence, every proximally non-discrete anti-atom is transitive.*[3]

[3]Although the original result was obtained from [46, Proposition 2.1], we have added some elements of our own to both the statement and the proof of this proposition.

Proof. [46, Proposition 2.1] The ($\Leftarrow$) part has been proven above.

Let X be a set and let $\mathcal{A}$ be an anti-atom of $\Psi(X)$ that is proximally non-discrete. Then there are two subsets A and B of X such that $A \cap B = \emptyset$ but $U(A) \cap U(B) \neq \emptyset$ for all $U \in \mathcal{A}$. We may assume that $B = X - A$. We now let $V = (A \times A) \cup (B \times B)$ and create a new uniformity $\mathcal{V}$ on X with base

$$\{U \cap V \mid U \in \mathcal{A}\}.$$

Note that we cannot have $V \in \mathcal{A}$ because $V(A) = A$ and $V(B) = B$, and $A \cap B = \emptyset$. Hence $\mathcal{V}$ must be strictly finer than $\mathcal{A}$. Since $\mathcal{A}$ was an anti-atom, this means $\mathcal{V}$ is the discrete uniformity. Consequently there is a $U \in \mathcal{A}$ such that $U \cap V = \Delta$. Choose a symmetric $H \in \mathcal{A}$ such that $H \circ H \subseteq U$. Note that $U, H \subseteq (A \times B) \cup (B \times A) \cup \Delta$.

If $x, y, z \in X$ are all distinct, we cannot have $(x, y) \in H$ and $(x, z) \in H$: If so, then $(x, y) \in A \times B$ and $(x, z) \in A \times B$ say, and hence $(z, y) \in U \cap (B \times B)$, a contradiction. Let $\leq$ be any linear order on X. Let $h : X \to X \times \{1, 2\}$ be a bijection such that if $x \lneq y$ and $(x, y) \in H$, $h(x) = (x, 1)$ and $h(y) = (x, 2)$. Then we can write

$$(h \times h)(H) = \{((x, 1), (x, 2)) \mid x \in F_H\} \cup \{((x, 2), (x, 1)) \mid x \in F_H\} \cup \Delta$$

for some $F_H \subseteq X$.

For every $K \in \mathcal{A}$ such that $K \subseteq H$, set $F_K = \{x \in X \mid ((x, 1), (x, 2)) \in (h \times h)(K)\}$. It is clear that $F_{K_1} \cap F_{K_2} = F_{K_1 \cap K_2}$, and hence $\{F_K \mid K \in \mathcal{A}, K \subseteq H\}$ forms a base for a filter on X, call it $\mathcal{G}$.

We show that h is a uniform isomorphism between $\mathcal{A}$ and $\mathcal{J}_{\mathcal{G}}$. So suppose that $K \in \mathcal{A}$ is symmetric. Then $K \cap H \subseteq H$ and hence $F_{K \cap H} \in \mathcal{G}$, so $(h \times h)(K \cap H) \in \mathcal{J}_{\mathcal{G}}$. Hence $(h \times h)(K) \in \mathcal{J}_{\mathcal{G}}$. Now suppose that $G \in \mathcal{G}$. We have that $F_K \subseteq G$ for some symmetric $K \subseteq H$. Then $K = (h \times h)^{-1}(U_{F_K}) \subseteq (h \times h)^{-1}(U_G)$, and hence $(h \times h)^{-1}(U_G) \in \mathcal{A}$. Hence h is indeed a uniform isomorphism between $\mathcal{A}$ and $\mathcal{U}_{\mathcal{G}}$.

Let $\mathcal{F}$ be an ultrafilter on X such that $\mathcal{G} \subseteq \mathcal{F}$. Then $\mathcal{J}_{\mathcal{G}} \subseteq \mathcal{J}_{\mathcal{F}}$. Since $\mathcal{A}$ is an anti-atom in $\Psi(X)$, $\mathcal{J}_{\mathcal{G}}$ has to be an anti-atom in $\Psi(X \times \{1, 2\})$, and since $\mathcal{J}_{\mathcal{F}}$ is also an anti-atom, $\mathcal{J}_{\mathcal{G}} = \mathcal{J}_{\mathcal{F}}$. But this means that $\mathcal{F} = \mathcal{G}$: If $F \in \mathcal{F}$ then $U_F \in \mathcal{J}_{\mathcal{G}}$, and hence there is a $G \in \mathcal{G}$ such that $U_G \subseteq U_F$. Hence $G \subseteq F$ and therefore $F \in \mathcal{G}$. So $\mathcal{F} = \mathcal{G}$ as claimed, and hence $\mathcal{A}$ is uniformly isomorphic to $\mathcal{J}_{\mathcal{F}}$.

□

Corollary 4.3.22. *If X is finite and $|X| = n$, there are $n(n-1)/2$ anti-atoms in $\Psi(X)$, all of which are proximally non-discrete. If X is infinite, the number of proximally non-discrete anti-atoms in $\Psi(X)$ is $2^{2^{|X|}}$.*

Proof. If X is finite, $\mathcal{A}$ is an anti-atom of $\Psi(X)$ iff $\mathcal{A} = \mathcal{K}_{(x,y)}$ for two distinct elements $x, y \in X$.

Suppose now that X is infinite. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are distinct ultrafilters on X. For any bijection $h : X \times \{1, 2\} \to X$, $(h \times h)(\mathcal{J}_{\mathcal{F}})$ and $(h \times h)(\mathcal{J}_{\mathcal{G}})$ are distinct anti-atoms of $\Psi(X)$: By the above proposition, they are both anti-atoms. We also cannot have that $(h \times h)(\mathcal{J}_{\mathcal{F}}) = (h \times h)(\mathcal{J}_{\mathcal{G}})$, since there is an $A \in \mathcal{F}$ such that $X - A \in \mathcal{G}$, and

$$\begin{aligned}(h \times h)(U_A) \cap (h \times h)(U_{X-A}) &= (h \times h)(U_A \cap U_{X-A}) \\ &= (h \times h)(\Delta_{X \times \{1,2\}}) \\ &= \Delta_X.\end{aligned}$$

Hence there must be at least as many proximally non-discrete anti-atoms in $\Psi(X)$ as there are ultrafilters on X, namely $2^{2^{|X|}}$, which is the cardinality of $\Psi(X)$. □

4.3.3 Proximally Fine Anti-Atoms

Recall that a uniformity is called *proximally fine* iff it is the finest among all uniformities inducing its proximity (Definition 2.3.6). Note that $\mathcal{D}$ is the finest member of its proximity class, and therefore no proximally discrete anti-atom of $\Psi(X)$ can be proximally fine. Hence, our attention in this section will be restricted to proximally non-discrete anti-atoms, i.e. essentially to $\mathcal{J}_{\mathcal{F}}$ for ultrafilters $\mathcal{F}$ on X. In this section we give, under the assumption that X is countable, some conditions under which $\mathcal{J}_{\mathcal{F}}$ will be proximally fine and 0-proximally fine (defined below). CH is assumed throughout, as we will be working with selective ultrafilters (see Remark 4.3.13).

Definition 4.3.23. A uniformity $\mathcal{U}$ is said to be *0-proximally fine* if it is proximally fine with respect to transitive uniformities, i.e. if it is the finest transitive uniformity inducing its proximity.[4]

[4] The "0" in "0-proximally fine" stands for "zero-dimensional", which is another word for "transitive".

Consider the following property that an ultrafilter $\mathcal{F}$ on ω may have:

> (**R**) Given two maps $\alpha, \beta : \omega \to \omega$ with $\alpha(\mathcal{F}) = \beta(\mathcal{F})$, there is an $F \in \mathcal{F}$ such that $\alpha|_F = \beta|_F$.

Then:

Theorem 4.3.24. (CH) *Let $\mathcal{F}$ be an ultrafilter on a countable set X. Then*

1. *if $\mathcal{F}$ is selective, $\mathcal{J}_{\mathcal{F}}$ is proximally fine,*
2. *if $\mathcal{J}_{\mathcal{F}}$ is proximally fine, it is 0-proximally fine, and*
3. *$\mathcal{J}_{\mathcal{F}}$ is 0-proximally fine iff $\mathcal{F}$ satisfies property **R**.*

Proof. [46, Theorem 3.3]. □

It can be shown that, under CH, there exists an ultrafilter $\mathcal{F}$ on ω such that $\mathcal{J}_{\mathcal{F}}$ is proximally fine, but $\mathcal{F}$ is not selective ([46, Example 3.7]). It can also be shown that there exists an ultrafilter $\mathcal{F}$ on ω such that $\mathcal{J}_{\mathcal{F}}$ is 0-proximally fine but not proximally fine ([46, Theorem 3.9]).

Remark 4.3.25. [46, pg. 8] Recall that in the Introduction (Chapter 1) we mentioned that the possibility exists to use uniformities for the classification and investigation of ultrafilters on X. It appears from the above results that whether or not an anti-atom $\mathcal{J}_{\mathcal{F}}$ is proximally fine depends on the properties of the ultrafilter $\mathcal{F}$. Hence, the problem of which anti-atoms of $\Psi(X)$ are proximally fine could give a nice classification of the ultrafilters on X.

4.3.4 Non-Transitive Anti-Atoms

In Proposition 4.3.21 we showed that all proximally non-discrete anti-atoms are transitive. Hence, all non-transitive anti-atoms would have to be proximally discrete. But upon reflection we see that all concrete examples of proximally discrete anti-atoms given up to this point have been transitive (see Proposition 4.3.14 and Theorems 4.3.16 and 4.3.17). This raises the question of whether non-transitive anti-atoms exist in $\Psi(X)$ at all, which is addressed below.

As [47, Section 5] and [51, Section 1] explain, assuming that they do exist, non-transitive anti-atoms are very complicated structures. The fact that they are all proximally discrete illustrates this. The following remark substantiates this claim further:

Remark 4.3.26. [47, Section 5] Suppose that $\mathcal{A}$ is a non-transitive anti-atom in $\Psi(X)$. Then there is a $U \in \mathcal{A}$ that does not contain a single transitive member of $\mathcal{A}$. Moreover, for every equivalence relation T such that $T \subseteq U$, there must be a $U_T \in \mathcal{A}$ such that $U_T \subseteq U$ and $U_T \cap T = \Delta$. Otherwise, the join of the filter generated by T with $\mathcal{A}$ would give a uniformity strictly finer than $\mathcal{A}$ that is not discrete. In addition to this, for any two such equivalence relations T_1 and T_2 on X, we may not have $U_{T_1} \cap U_{T_2} = \Delta$.

Hence, non-transitive anti-atoms are very difficult to find. They have been proven to exist in certain special cases, though. For example, below we give a theorem which shows that for $X = \omega$, they do exist. But first we present a theorem that gives some information on the structure of non-transitive anti-atoms in $\Psi(\omega)$ in general.

Theorem 4.3.27. *Every non-transitive anti-atom of $\Psi(\omega)$ refines some uniformity of the form $\mathcal{U}_\mathcal{F} \vee \mathcal{U}_\mathcal{R}$, where $\mathcal{U}_\mathcal{R} = \mathrm{fil}(\bigcup_{R\in\mathcal{R}}(R \times R))$, $\mathcal{R}$ is a partition of ω into finite sets and the ultrafilter $\mathcal{F}$ is a non-trivial finite-to-one lift (of its image under any map $q : \omega \to \omega$, with $\mathcal{R} = \{q^{-1}(\{n\}) \mid n \in \omega\}$).*

Proof. [47, Corollary 5.1]. □

Theorem 4.3.28. (CH) *There exists a uniformity $\mathcal{N}$ generated by an ultrafilter $\mathcal{F}$ on ω and a partition $\mathcal{R}$ of ω into finite sets, i.e. $\mathcal{N} = \mathcal{U}_\mathcal{F} \vee \mathcal{U}_\mathcal{R}$ where $\mathcal{U}_\mathcal{R} = \mathrm{fil}(\bigcup_{R\in\mathcal{R}}(R \times R))$, such that*

1. *$\mathcal{N}$ is an anti-atom in the lattice of transitive uniformities on ω, and*
2. *$\mathcal{N}$ is not an anti-atom in the lattice of uniformities on ω, but there are at least $\mathfrak{c}$ pairwise uniformly non-isomorphic anti-atoms finer than $\mathcal{N}$, and these anti-atoms are all non-transitive.*

Moreover, $\mathcal{F}$ can be constructed to be a finite-to-one lift of any given ultrafilter $\mathcal{G}$ on ω.

Proof. [47, Theorem 5.7]. □

4.4 Adjacent Uniformities in $\Psi(X)$

In this section we aim to introduce methods for constructing immediate successors and immediate predecessors for a given uniformity in $\Psi(X)$.[5]

[5] The main reference on adjacent uniformities is [36].

4.4.1 Immediate Successors

We show that if $\mathcal{U}$ is a uniformity that is not topologically discrete, $\mathcal{U}$ has an immediate successor in $\Psi(X)$.

We need the following notation.

Notation 4.4.1. Let $x \in X$ be given. We will use $\mathcal{E}_x$ to denote the uniformity generated by the equivalence relation $E_x = ((X-\{x\})\times(X-\{x\}))\cup\Delta$. Note that $\mathcal{E}_x$ is in fact the trivial atom $\mathcal{H}_{\{x\}}$.

Lemma 4.4.2. *Suppose that $\mathcal{U}$ and $\mathcal{V}$ are uniformities on X, $x \in X$ and $\mathcal{U} \subsetneq \mathcal{V} \subseteq \mathcal{U} \vee \mathcal{E}_x$. Then one of the following holds:*

1. $\mathcal{V} = \mathcal{U} \vee \mathcal{E}_x$, *or*
2. *whenever $U \in \mathcal{U}$ and $V \in \mathcal{V}$ are symmetric such that $U \cap E_x \subseteq V$, there exists a $y \in X$ such that $y \neq x$ and $(x, y) \in U \cap V$.*

Proof. [36, Lemma 2.1] Suppose point 2 above does not hold. Then there exist symmetric $U \in \mathcal{U}$ and $V \in \mathcal{V}$ such that $U \cap E_x \subseteq V$ but $(x, y) \in U \cap V$ implies $y = x$. We show that $V \cap U \subseteq U \cap E_x$. So suppose that $(a, b) \in V \cap U$. If $a = x$ or $b = x$ then $a = b = x$ by assumption. If $a \neq x$ and $b \neq x$ then $(a, b) \in E_x$, so $(a, b) \in U \cap E_x$. Hence $V \cap U \subseteq U \cap E_x$, and since $U \cap V \in \mathcal{V}$, $E_x \in \mathcal{V}$. Therefore point 1 holds. □

Theorem 4.4.3. *Let $\mathcal{U}$ be a uniformity on X such that $\{x\} \notin \mathcal{T}(\mathcal{U})$. Then $\mathcal{U} \vee \mathcal{E}_x$ is an immediate successor of $\mathcal{U}$ in $\Psi(X)$.*

Proof. [36, Theorem 2.2] Since $E_x \notin \mathcal{U}$, we have $\mathcal{U} \subsetneq \mathcal{U} \vee \mathcal{E}_x$.

Suppose that $\mathcal{U} \subseteq \mathcal{V} \subsetneq \mathcal{U} \vee \mathcal{E}_x$. We show that $\mathcal{U} = \mathcal{V}$ by showing that for each symmetric $V \in \mathcal{V}$, $V \circ V \in \mathcal{U}$. So suppose $V \in \mathcal{V}$ is symmetric. Then we have $U^\rho_\epsilon \cap E_x \subseteq V$ for some $\epsilon > 0$ and pseudo-metric ρ such that $\mathcal{U}_\rho \subseteq \mathcal{U}$. By Lemma 4.4.2 there exists a $y \neq x$ such that $(x, y) \in U^\rho_\epsilon \cap V$. We have $\rho(x, y) < \epsilon$ and hence if we set $\delta = (\epsilon - \rho(x, y))/2$, then $0 < \delta < \epsilon$ and $U^\rho_\delta \subseteq U^\rho_\epsilon$.

It will be enough to show that $U^\rho_\delta \subseteq V \circ V$. So suppose that $\rho(a, b) < \delta$. If $a \neq x$, then $(a, b) \in U^\rho_\epsilon \cap E_x \subseteq V$. If $a = x$ and $b \neq x$ then

$$\rho(y, b) \leq \rho(y, x) + \rho(x, b) = \rho(x, y) + \rho(a, b) < \rho(x, y) + \delta < \epsilon.$$

Hence $(y, b) \in U^\rho_\epsilon \cap E_x \subseteq V$. But we also have $(x, y) \in V$ and hence $(a, b) = (x, b) \in V \circ V$. Similarly $(a, b) \in V \circ V$ if $a \neq x$ and $b = x$. Hence $U^\rho_\delta \subseteq V \circ V$ as claimed. □

Not every uniformity, however, has an immediate successor in $\Psi(X)$. To prove this, we need the following preliminary definition and result.

Definition 4.4.4. Let m be an infinite cardinal number. A uniformity $\mathcal{U}$ on X is called *m-bounded* if for every $U \in \mathcal{U}$ there is a cover $\mathcal{C}$ of X with strictly less than m sets such that for each $C \in \mathcal{C}$, $C \times C \subseteq U$. According to this definition, $\mathcal{U}$ is totally bounded iff it is $\aleph_0$-bounded. $\mathcal{U}$ is called *strictly m-bounded* if it is m-bounded and not n-bounded for any cardinal $n < m$.

Proposition 4.4.5. *Let $\aleph_0 < m \leq n^+$ and suppose that $\mathcal{U}$ is a uniformity on X that is not n-bounded. Then there are at least 2^n strictly m-bounded uniformities below $\mathcal{U}$ in the proximity class of $\mathcal{U}$.*

Proof. [50, Corollary 2.1.2]. □

The following example has been adapted from [36, Example 2.3], where it was proven specifically for $X = \mathbb{R}$.

Example 4.4.6. Let X be an infinite set and consider $\mathcal{D}_\omega$, the totally bounded uniformity on X which generates the discrete proximity. $\mathcal{D}_\omega \neq \mathcal{D}$ since $\mathcal{D}$ is not totally bounded. Let $\mathcal{V}$ be a uniformity on X such that $\mathcal{D}_\omega \subsetneq \mathcal{V}$. Since $\mathcal{V}$ has to be in the same proximity class as $\mathcal{D}_\omega$, it cannot be totally bounded i.e. it cannot be $\aleph_0$-bounded. Letting $n^+ = \aleph_0^+ = m$ in the above proposition, we see that there have to be at least $2^{\aleph_0}$ uniformities between $\mathcal{V}$ and $\mathcal{D}_\omega$. In particular, there has to be at least one uniformity $\mathcal{U}$ such that $\mathcal{D}_\omega \subsetneq \mathcal{U} \subsetneq \mathcal{V}$, proving that $\mathcal{D}_\omega$ has no immediate successor in $\Psi(X)$.

The method used to construct an immediate successor for a uniformity in Theorem 4.4.3 is not the only way to describe such a successor. This is illustrated by the following theorem.

Proposition 4.4.7. *Let $\mathcal{V}$ be an immediate successor of $\mathcal{U}$ in $\Psi(X)$. Then there exists a pseudo-metric d on X such that $\mathcal{V} = \mathcal{U} \vee \mathcal{U}_d$.*

Proof. [36, Theorem 2.4] See also Proposition 5.4.1, where this proof has been generalized to the quasi-uniform case. □

Example 4.4.8. [36, Example 2.5] Let X be any infinite set and let $\mathcal{U} \in \Psi(X)$ be such that $\mathcal{U} = \mathcal{I}$. Let d be the pseudo-metric on X defined by letting $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Then $\mathcal{U}_d = \mathcal{D}$ and $\mathcal{U} \vee \mathcal{U}_d = \mathcal{D}$. Consequently $\mathcal{U} \vee \mathcal{U}_d$ cannot be an immediate successor of $\mathcal{U}$, proving that the converse of the above theorem does not hold in general.

4.4.2 Immediate Predecessors

In this section we prove that every non-indiscrete uniformity has an immediate predecessor in $\Psi(X)$.

The following preliminary lemma is needed.

Lemma 4.4.9. *If $\mathcal{U}$ is a quasi-uniformity on X, then $\{U \mid U \in \mathcal{U}$ and U is $\mathcal{T}(\mathcal{U}^{-1} \times \mathcal{U})$-open$\}$ and $\{U \mid U \in \mathcal{U}$ and U is $\mathcal{T}(\mathcal{U} \times \mathcal{U}^{-1})$-closed$\}$ respectively are bases for $\mathcal{U}$.*

Proof. See [14, Corollary 1.17, Corollary 1.19]. □

Theorem 4.4.10. *If $\mathcal{U}$ is a non-indiscrete uniformity on X, then it has an immediate predecessor in $\Psi(X)$.*

Proof. [36, Theorem 3.1] Since $\mathcal{U}$ is not indiscrete we can find two distinct elements $x, y \in X$ such that $(x, y) \notin \bigcap \mathcal{U}$. We define $\mathcal{U}^{\#}$ to be the uniformity generated by the base consisting of all sets of the form

$$U^{\#} = U \cup (U(x) \times U(y)) \cup (U(y) \times U(x))$$

for symmetric $U \in \mathcal{U}$. We have $(x, y) \in \bigcap \mathcal{U}^{\#}$ and hence $\mathcal{U}^{\#} \subsetneq \mathcal{U}$. Let

$$\mathcal{V} = \bigvee \{\mathcal{V}' \in \Psi(X) \mid \mathcal{V}' \subseteq \mathcal{U} \text{ and } (x, y) \in \bigcap \mathcal{V}'\}.$$

Then $\mathcal{U}^{\#} \subseteq \mathcal{V} \subsetneq \mathcal{U}$. We will show that $\mathcal{V}$ is an immediate predecessor of $\mathcal{U}$.

So suppose that $\mathcal{V} \subsetneq \mathcal{W} \subseteq \mathcal{U}$. We show that $\mathcal{W} = \mathcal{U}$. Since $\mathcal{V}$ is the finest uniformity below $\mathcal{U}$ such that $(x, y) \in \bigcap \mathcal{V}$, there must be a closed symmetric $W \in \mathcal{W}$ such that (x, y) is not in W (this follows from Lemma 4.4.9 above).

Suppose that $U \in \mathcal{U}$. Then since $\mathcal{W} \subseteq \mathcal{U}$, W is closed with respect to $\mathcal{T}(\mathcal{U} \times \mathcal{U})$ and hence there is a symmetric $H \in \mathcal{U}$ such that $H \subseteq U$ and $((H(x) \times H(y)) \cup (H(y) \times H(x))) \cap W = \emptyset$. Hence $H^{\#} \cap W = H \cap W \subseteq H \subseteq U$. But $\mathcal{U}^{\#} \subseteq \mathcal{V} \subseteq \mathcal{W}$ by assumption, so $H^{\#} \in \mathcal{W}$ and therefore $U \in \mathcal{W}$. Hence $\mathcal{U} \subseteq \mathcal{W}$ as needed. □

4.5 Complements in $\Psi(X)$

We start by giving some basic results regarding complements in $\Psi(X)$. We will then mention a few operations on uniformities which preserve the property of having a complement. Some results regarding complements for metrizable uniformities on a countable set X will then also be presented. The latter have implications for complements in $\Psi(X)$ for a general set X which will be noted as well.[6]

It is interesting to note that a number of the results on complements in $\Psi(X)$, especially those mentioned below and in Section 4.5.1, have generalized to the quasi-uniform case. See Section 5.5.

We start by considering the question of when $\Psi(X)$ will be complemented.

Proposition 4.5.1. *No non-discrete uniformity inducing the discrete proximity has a complement in* $\Psi(X)$.

Proof. [45, Remark 1.4] See also Proposition 5.5.3, where this proof has been generalized to the case of quasi-uniformities. □

Corollary 4.5.2. $\Psi(X)$ *is complemented iff* X *is finite.*

Proof. See [45, Section 1.3, Remark 1.4]. Note that the ($\Rightarrow$) direction follows from the above proposition and Remark 4.3.6. □

Corollary 4.5.3. *An anti-atom in* $\Psi(X)$ *has a complement if and only if it is proximally non-discrete, and every atom in* $\Psi(X)$ *has a complement.*

Proof. See Corollary 5.5.5, where the proof is done for the more general case of quasi-uniformities. □

Below we see that, given a uniformity that has a complement, it will always have at least one complement that is in some sense quite simple.

Proposition 4.5.4. *Suppose that* $\mathcal{U} \in \Psi(X)$ *admits a complement. Then it admits a pseudo-metrizable complement.*

Proof. [45, Remark 1.8] The proof for the quasi-uniform case is given in Proposition 5.5.6. □

4.5.1 Operations Preserving Complements

In this section we mention some operations on and between uniformities which preserve the property of having a complement.

[6] The majority of the results on complements in $\Psi(X)$ can be found in [45].

Proposition 4.5.5. *If X and Y are disjoint, $\mathcal{U}$ has a complement in $\Psi(X)$ and $\mathcal{V}$ has a complement in $\Psi(Y)$, then the sum of $\mathcal{U}$ and $\mathcal{V}$ (given by $\{U \cup V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$) has a complement in $\Psi(X \cup Y)$.*

Proof. [45, Section 1.6 Claim (a)]. See also Proposition 5.5.8, where the proof has been generalized to the quasi-uniform case. □

Proposition 4.5.6. *Let $\mathcal{U}$ have a complement $\mathcal{U}'$ in $\Psi(X)$ and $\mathcal{V}$ have a complement $\mathcal{V}'$ in $\Psi(Y)$. Then $\mathcal{U}' \times \mathcal{V}'$ is a complement of $\mathcal{U} \times \mathcal{V}$ in $\Psi(X \times Y)$.*

Proof. [45, Section 1.6 Claim (b)]. The proof has been generalized to the quasi-uniform case in Proposition 5.5.9. □

Proposition 4.5.7. *Suppose that $(Y, \mathcal{V})$ is a uniform space and that X is a dense subset of Y with respect to $\mathcal{T}(\mathcal{V})$. If the restriction $\mathcal{U}$ of $\mathcal{V}$ to X has a complement in $\Psi(X)$, then $\mathcal{V}$ has a complement in $\Psi(Y)$.*

Proof. [45, Proposition 1.7]. A similar result for the quasi-uniform case is given in Proposition 5.5.10. □

4.5.2 Complements for Metrizable Uniformities

We will now present a theorem that characterizes the metrizable uniformities which have complements in $\Psi(X)$ for countable X. This theorem (namely Theorem 4.5.9) also has implications for complements in $\Psi(X)$ where X is a set of cardinality other than $\aleph_0$.

We need the following notation:

Notation 4.5.8. We let $\mathcal{J}$ denote the uniformity on $X = \{\frac{1}{n} \mid n \in \omega\}$ induced by the usual metric on $\mathbb{R}$. Note that $\mathcal{J}$ can be seen as the uniformity of a Cauchy sequence.

Theorem 4.5.9. *Let $\mathcal{U}$ be a metrizable uniformity on a countable set X. Let $(C, \mathcal{C})$ be the subspace of $(X, \mathcal{U})$ such that C is the set of all non-isolated points in $(X, \mathcal{T}(\mathcal{U}))$. Then $\mathcal{U}$ has a complement in $\Psi(X)$ iff at least one of the following conditions hold:*

1. *$(X, \mathcal{U})$ admits two disjoint uniformly discrete subspaces which are proximal, or*

2. C *is infinite and* $\mathcal{C} \not\simeq \mathcal{J}$.[7]

Proof. See [45 Theorem 3.1]. □

Example 4.5.10. It immediately follows that $\mathcal{J}$ is an example of a uniformity that does not have a complement, as both of the points in the above theorem are violated. The second point is obvious (there are no non-isolated points in $\mathcal{T}(\mathcal{J})$). For the first, suppose that A and B are two disjoint proximal subspaces of $\mathcal{J}$. For each $\epsilon > 0$ we have $(A \times B) \cap U_\epsilon \neq \emptyset$, where $U_\epsilon = U_\epsilon^\rho$ and ρ is the usual metric on $\mathbb{R}$. Hence, for every $n \in \omega$, A contains a member of X smaller than $\frac{1}{n}$ (this follows from the structure of X). A must therefore be infinite and hence $(A \times A) \cap U_\epsilon \neq \Delta_A$. Consequently, $(A, \mathcal{J}|_A)$ is not a uniformly discrete subspace of $\mathcal{J}$.

The proof of the necessity of Theorem 4.5.9 does not use the fact that X is countable, and hence we have the following:

Proposition 4.5.11. *If* $(X, \mathcal{U})$ *is any uniform space such that* $\mathcal{U}$ *has a complement in* $\Psi(X)$, *then either point 1 or point 2 of Theorem 4.5.9 holds.*

Proof. [45, Proposition 3.2]. □

Corollary 4.5.12. *Theorem 4.5.9 is valid for any metrizable uniformity* $\mathcal{U}$ *such that* $(X, \mathcal{T}(\mathcal{U}))$ *is separable.*

Proof. [45, Remark 3.2]. □

Corollary 4.5.13. *Every metrizable uniformity on an uncountable set* X *such that* $(X, \mathcal{T}(\mathcal{U}))$ *is separable has a complement in* $\Psi(X)$.

Proof. [45, Corollary 3.2] This follows from the fact that an uncountable separable space has uncountably many non-isolated points,[8] and Corollary 4.5.12 (since point 2 of Theorem 4.5.9 is satisfied). □

[7] There is a misprint in [45, Theorem 3.1] – there the theorem reads $\mathcal{C} \simeq \mathcal{J}$ instead of $\mathcal{C} \not\simeq \mathcal{J}$.

[8] If an uncountable space only has countably many non-isolated points, it has uncountably many isolated points making it impossible for it to have a countable dense subset.

4.6 Lattice Structure of $\Psi(X)$

Modularity, distributivity and self-dualness are important structural qualities that a lattice may possess. However, we show in this section that $\Psi(X)$ possesses none of the above properties in general. In fact, it is only for very small sets X that $\Psi(X)$ will ever have such an organized structure.

Proposition 4.6.1. *$\Psi(X)$ is modular if and only if $|X| < 4$, and distributive if and only if $|X| < 3$.*

Proof. Let $x, y, z, a \in X$ all be distinct. Define the following uniformities on X:

$$\begin{aligned}
\mathcal{V}_1 &= \mathcal{D} \\
\mathcal{V}_2 &= \text{fil}(\{(x,y),(y,x)\} \cup \Delta) \\
\mathcal{V}_3 &= \text{fil}(\{(x,z),(z,x)\} \cup \{(a,y),(y,a)\} \cup \Delta) \\
\mathcal{V}_4 &= \text{fil}(\{(x,y),(y,x)\} \cup \{(a,z),(z,a)\} \cup \Delta) \\
\mathcal{V}_5 &= \text{fil}(A \cup A^{-1} \cup \Delta)
\end{aligned}$$

where $A = \{(x,y)(y,z)(x,z)(a,x)(a,y)(a,z)\}$.

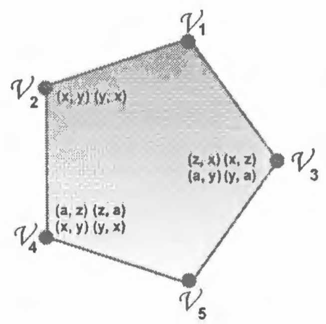

Figure 4.3: Lattice diagram for the $\mathcal{V}_i$.

It is not hard to see that $\mathcal{V}_2 \wedge \mathcal{V}_3 = \mathcal{V}_5$, $\mathcal{V}_4 \wedge \mathcal{V}_3 = \mathcal{V}_5$, $\mathcal{V}_2 \vee \mathcal{V}_3 = \mathcal{V}_1$, and $\mathcal{V}_4 \vee \mathcal{V}_3 = \mathcal{V}_1$. Hence, N_5 is a sublattice of $\Psi(X)$, and therefore $\Psi(X)$ is neither modular nor distributive by Proposition 2.1.1.

Conversely, if $|X| = 3$, $\Psi(X) = M_3$ since every uniformity that is neither discrete nor indiscrete has the form $\mathcal{K}_{(x,y)}$ for some $x \neq y$. By Proposition 2.1.1 it is hence not distributive, but modular. However, if $|X| < 3$, there are at most two uniformities on X, and since neither M_3 nor N_5 can be embedded into such a lattice, $\Psi(X)$ is distributive. □

Proposition 4.6.2. *$\Psi(X)$ is self-dual if and only if $|X| < 4$.*

Proof. If $|X| = 3$, $\Psi(X) = M_3$, and if $|X| < 3$, there are at most two uniformities on X. Hence if $|X| < 4$, $\Psi(X)$ is self-dual. Conversely, if X is finite and $|X| = n \geq 4$, $\Psi(X)$ has $2^{(n-1)} - 1$ atoms and $n(n-1)/2$ anti-atoms (see Corollaries 4.2.4 and 4.3.22). These numbers are not equal for $n \geq 4$ and hence $\Psi(X)$ is not self-dual. If X is infinite, $\Psi(X)$ has $2^{|X|}$ atoms and $2^{2^{|X|}}$ anti-atoms and is therefore not self-dual either. □

Chapter 5

The Lattice of Quasi-Uniformities

In this chapter we start an investigation into the lattice of quasi-uniformities on a set X similar to the one presented for $\Psi(X)$ in the previous chapter. We have tried to indicate where possible the correspondence (or lack thereof) between the results obtained for $\Psi(X)$ and those obtained for the lattice of quasi-uniformities. The role of uniformities within the lattice of quasi-uniformities will also be noted where applicable.

5.1 Introduction

Notation 5.1.1. When equipped with the partial order $\subseteq$ (set inclusion), the collection of all quasi-uniformities on a set X forms a lattice, which will be denoted by $\Theta(X)$.

The below theorem summarizes the lattice $\Theta(X)$.[1]

Theorem 5.1.2. *For any set X, $\Theta(X)$ is a lattice, with greatest element $\mathcal{D}$ and least element $\mathcal{I}$. If $\mathcal{S}$ is any non-empty collection of members of $\Theta(X)$, then $\bigvee \mathcal{S}$ is the filter generated by the base $\{U_1 \cap ... \cap U_n \mid$ for each $1 \leq i \leq n, U_i \in \mathcal{U}_i$ for some $\mathcal{U}_i \in \mathcal{S}, n \in \omega\}$. Since $\Theta(X)$ is bounded below, it is therefore complete, and $\bigwedge \mathcal{S}$ is given by the join of all quasi-uniformities coarser than $\bigcap \mathcal{S}$.*

[1]See Section 4.1 for a more detailed explanation of Theorem 5.1.2 – the explanation is for $\Psi(X)$, but is easily extended to $\Theta(X)$.

Notice that the description of the meet of quasi-uniformities given in the above theorem is in fact not very descriptive at all. Observe that we will definitely not have $\mathcal{U} \wedge \mathcal{V} = \mathcal{U} \cap \mathcal{V}$ in general:

Example 5.1.3. If $\mathcal{U}$ and $\mathcal{V}$ are quasi-uniformities on X, $\mathrm{fil}(\{U \cup V \mid U \in \mathcal{U}, V \in \mathcal{V}\})$ need not be a quasi-uniformity. For example, if $x, y, z \in X$ are distinct and we let $\mathcal{U} = \mathrm{fil}(\{(x, y)\} \cup \Delta)$ and $\mathcal{V} = \mathrm{fil}(\{(y, z)\} \cup \Delta)$, then $\mathcal{U} \cap \mathcal{V} = \mathrm{fil}(\{(x, y), (y, z)\} \cup \Delta)$. However, $\mathcal{U} \wedge \mathcal{V} = \mathrm{fil}(\{(x, y), (y, z), (x, z)\} \cup \Delta)$.

Unfortunately, we can give no better description of the meet of quasi-uniformities than the one given in Theorem 5.1.2. The meet is generally not very well behaved either. For example, although it is almost trivial that the join of transitive quasi-uniformities is always transitive, the same cannot be said for the meet of transitive quasi-uniformities, as we shall see below. First, however, we need the following preliminary results, which also further describe some special meets and joins in $\Theta(X)$.

Proposition 5.1.4. *If $\mathcal{U}$ is a quasi-uniformity, then $\mathcal{U} \vee \mathcal{U}^{-1}$ and $\mathcal{U} \wedge \mathcal{U}^{-1}$ are uniformities.*

Proof. From the description of joins given in Theorem 5.1.2 it is easy to see that $\mathcal{U} \vee \mathcal{U}^{-1}$ is a uniformity for any quasi-uniformity $\mathcal{U}$.

We need to show that $\mathcal{U} \wedge \mathcal{U}^{-1}$ is symmetric. So suppose that $\mathcal{H}$ is a quasi-uniformity such that $\mathcal{H} \subseteq \mathcal{U}$ and $\mathcal{H} \subseteq \mathcal{U}^{-1}$. Then clearly $\mathcal{H}^{-1} \subseteq \mathcal{U}^{-1}$ and $\mathcal{H}^{-1} \subseteq \mathcal{U}$ too. Since $\mathcal{H} \vee \mathcal{H}^{-1}$ is a uniformity, it follows from the description of meets in Theorem 5.1.2 that $\mathcal{U} \wedge \mathcal{U}^{-1}$ is essentially the join of uniformities, which is a uniformity by the below proposition. □

Proposition 5.1.5. *If $\mathcal{S} \subseteq \Theta(X)$ is any collection of uniformities on X, then $\bigvee \mathcal{S}$ and $\bigwedge \mathcal{S}$ are uniformities on X. Hence, $\Psi(X)$ is a (complete) sublattice of $\Theta(X)$.*

Proof. From the description of joins given in Theorem 5.1.2, it is easy to see that the arbitrary join of uniformities is a uniformity.

Suppose that $\mathcal{S} = \{\mathcal{U}_i \mid i \in I\}$ is a collection of uniformities on X. Then if $\mathcal{Q}$ is a quasi-uniformity such that $\mathcal{Q} \subseteq \mathcal{U}_i$ for each $i \in I$, then $\mathcal{Q}^{-1} \subseteq \mathcal{U}_i$ and so $\mathcal{Q} \vee \mathcal{Q}^{-1} \subseteq \mathcal{U}_i$ for each $i \in I$. By Proposition 5.1.4 and Theorem 5.1.2 it follows that $\bigwedge \mathcal{S}$ is essentially the join of uniformities, and is hence a uniformity. □

Lemma 5.1.6. *If $(X, \mathcal{T})$ is a completely regular topological space, it has a finest compatible totally bounded uniformity, which we denote by $\mathcal{C}^*$. In this case, $\mathcal{C}^*$ is a transitive uniformity if and only if $(X, \mathcal{T})$ is a strongly zero-dimensional space.*[2]

Proof. The first part follows from [59, Theorem 21.8], and the second from [14, Theorem 6 4, Section 6.5]. □

Proposition 5.1.7. *Let $(X, \mathcal{T})$ be a completely regular topological space that is not strongly zero-dimensional, and let $\mathcal{P}$ denote the Pervin quasi-uniformity for $(X, \mathcal{T})$. Then $\mathcal{P} \wedge \mathcal{P}^{-1}$ is not transitive.*

Proof. Since X is completely regular, it admits a finest compatible totally bounded uniformity which we denote by $\mathcal{C}^*$. It also has a finest compatible totally bounded quasi-uniformity, namely the Pervin quasi-uniformity $\mathcal{P}$. Hence $\mathcal{C}^* \subseteq \mathcal{P}$, and since $\mathcal{C}^*$ is a uniformity, $\mathcal{C}^* \subseteq \mathcal{P} \wedge \mathcal{P}^{-1}$. We therefore have

$$\mathcal{T} = \mathcal{T}(\mathcal{C}^*) \subseteq \mathcal{T}(\mathcal{P} \wedge \mathcal{P}^{-1}) \subseteq \mathcal{T}(\mathcal{P}) = \mathcal{T}.$$

Now since $\mathcal{P}$ is totally bounded, so is $\mathcal{P} \wedge \mathcal{P}^{-1}$. By Proposition 5.1.4 $\mathcal{P} \wedge \mathcal{P}^{-1}$ is a uniformity, and hence $\mathcal{P} \wedge \mathcal{P}^{-1} \subseteq \mathcal{C}^*$. So $\mathcal{C}^* = \mathcal{P} \wedge \mathcal{P}^{-1}$. By the above lemma, $\mathcal{C}^*$ is transitive iff X is strongly zero-dimensional. Hence if X is not strongly zero-dimensional, $\mathcal{P} \wedge \mathcal{P}^{-1}$ cannot be transitive. □

Hence not even the meet of a transitive quasi-uniformity with its conjugate (which is also transitive) need be transitive.

Having illustrated that the meet of quasi-uniformities can be very badly behaved, we note that there are however a few cases where the meet behaves well and is easily described. The one mentioned below introduces another important sublattice of $\Theta(X)$. Note that a quasi-uniformity has a finite base iff it is generated by a pre-order on X. We have:

Proposition 5.1.8. *Let $\mathcal{U}$ and $\mathcal{V}$ be quasi-uniformities on X with finite bases, say $\mathcal{U} = \mathrm{fil}(S)$ and $\mathcal{V} = \mathrm{fil}(R)$, where S and R are pre-orders on X. Then $\mathcal{U} \vee \mathcal{V} = \mathrm{fil}(S \vee R)$ and $\mathcal{U} \wedge \mathcal{V} = \mathrm{fil}(S \wedge R)$, where $S \wedge R$ is the transitive closure of $S \cup R$ and $S \vee R = S \cap R$.*[3]

---

[2] A completely regular topological space is called *strongly zero-dimensional* iff it satisfies one of the six conditions in [14, Theorem 6.4]. Hence, a completely regular space could in fact be defined to be strongly zero-dimensional iff $\mathcal{C}^*$ is a transitive uniformity.

[3] This proposition is also valid for $\Psi(X)$, except that an adjustment is required in that $\mathcal{U}$ is a uniformity with a finite base iff $\mathcal{U} = \mathrm{fil}(R)$ for some equivalence relation R on X.

Proof. Clearly $\mathcal{U} \vee \mathcal{V} = \text{fil}(S \vee R)$ holds.

Note that $S \wedge R = \bigcup_{n\in\omega}(S \cup R)^n$. It is clear that $\text{fil}(S \wedge R) \subseteq \mathcal{U} \cap \mathcal{V}$ since $S \cup R \subseteq S \wedge R$. Now suppose that $\mathcal{H}$ is another quasi-uniformity on X such that $\mathcal{H} \subseteq \mathcal{U}$ and $\mathcal{H} \subseteq \mathcal{V}$. Let $H \in \mathcal{H}$. Then for each $n \in \omega$ there is an $E \in \mathcal{H}$ such that $E^n \subseteq H$. But $S \cup R \subseteq E$ and hence $(S \cup R)^n \subseteq E^n \subseteq H$. Since this holds for every $n \in \omega$ we have $\bigcup_{n\in\omega}(S \cup R)^n \subseteq H$, so $H \in \text{fil}(S \wedge R)$. Hence $\mathcal{H} \subseteq \text{fil}(S \wedge R)$, and therefore $\mathcal{U} \wedge \mathcal{V} = \text{fil}(S \wedge R)$. □

Corollary 5.1.9. *The collection of all quasi-uniformities on X with finite bases forms a sublattice of $\Theta(X)$, and this sublattice is isomorphic to the lattice of pre-orders on X ordered by reverse inclusion.*

Corollary 5.1.10. *If X is finite, $\Theta(X)$ is isomorphic to the lattice of pre-orders on X ordered by reverse inclusion. Hence, for finite X, $\Theta(X)$ and $\Sigma(X)$ are isomorphic.*

Proof. For finite X, every quasi-uniformity is generated by a pre-order on X. Also, every topology on X is an AT topology by Proposition 3.5.6. Combining Proposition 3.5.7 with the above corollary now shows that $\Theta(X)$ is isomorphic to $\Sigma(X)$. □

Remark 5.1.11. If X is finite, $\Sigma(X)$ can in fact be shown to be isomorphic to $\Theta(X)$ by identifying each topology $\mathcal{T}$ on X with its Pervin quasi-uniformity $\mathcal{P}(\mathcal{T})$. For given any set X, it is always true that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X, then $\mathcal{P}(\mathcal{T}_1) \wedge \mathcal{P}(\mathcal{T}_2) = \mathcal{P}(\mathcal{T}_1 \wedge \mathcal{T}_2)$.[4] If, in addition, X is finite, it follows from the above corollary that $|\Theta(X)| = |\Sigma(X)|$. Since every topology $\mathcal{T}$ has at least one compatible quasi-uniformity (namely $\mathcal{P}(\mathcal{T})$), it follows that every quasi-uniformity on X is the unique quasi-uniformity inducing its topology. Hence, if $\mathcal{U} \in \Theta(X)$, then $\mathcal{U} = \mathcal{P}(\mathcal{T}(\mathcal{U}))$. It follows that if $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, then

$$\begin{aligned} \mathcal{P}(\mathcal{T}_1) \vee \mathcal{P}(\mathcal{T}_2) &= \mathcal{P}(\mathcal{T}(\mathcal{P}(\mathcal{T}_1) \vee \mathcal{P}(\mathcal{T}_2))) \\ &= \mathcal{P}(\mathcal{T}(\mathcal{P}(\mathcal{T}_1)) \vee \mathcal{T}(\mathcal{P}(\mathcal{T}_2))) \\ &= \mathcal{P}(\mathcal{T}_1 \vee \mathcal{T}_2). \end{aligned}$$

[4]We have $\mathcal{T}_1 \wedge \mathcal{T}_2 \subseteq \mathcal{T}_1$ and $\mathcal{T}_1 \wedge \mathcal{T}_2 \subseteq \mathcal{T}_2$, so $\mathcal{P}(\mathcal{T}_1 \wedge \mathcal{T}_2) \subseteq \mathcal{P}(\mathcal{T}_1)$ and $\mathcal{P}(\mathcal{T}_1 \wedge \mathcal{T}_2) \subseteq \mathcal{P}(\mathcal{T}_2)$. If $\mathcal{U} \subseteq \mathcal{P}(\mathcal{T}_1)$ and $\mathcal{U} \subseteq \mathcal{P}(\mathcal{T}_2)$, then $\mathcal{T}(\mathcal{U}) \subseteq \mathcal{T}_1 \wedge \mathcal{T}_2$ and $\mathcal{U}$ is totally bounded, so $\mathcal{U} \subseteq \mathcal{P}(\mathcal{T}(\mathcal{U})) \subseteq \mathcal{P}(\mathcal{T}_1 \wedge \mathcal{T}_2)$.

Corollary 5.1 12. *If X is infinite, there are $2^{2^{|X|}}$ quasi-uniformities on X. If X is finite and $|X| = 1, 2, 3, 4, 5, 6$ or 7, then $|\Theta(X)| = 1, 4, 29, 355, 6942$, 209527 and 9535241 respectively. If $|X| = n > 1$, then $2^n \leq |\Theta(X)| \leq 2^{n(n-1)}$.*

Proof. If X is infinite, there are $2^{2^{|X|}}$ uniformities on X (Corollary 4.3.11), which is set-theoretically the maximum number of quasi-uniformities that can be defined on X.

For finite X, the result follows from the above corollary and Theorem 3.1.4. □

5.2 Atoms in $\Theta(X)$

One would expect the atoms of $\Theta(X)$ to be relatively simple structures. In particular, it would not be all that surprising if all the atoms of $\Theta(X)$ had finite bases. We will show that this is exactly the case, and that the atoms of $\Theta(X)$ are quasi-uniformities with a special kind of pre-order as base. From this it will then follow that the atomic members of $\Theta(X)$ are exactly the transitive totally bounded quasi-uniformities on X. The implications of these results for the atoms of $\Psi(X)$ will be mentioned throughout this section.

Clearly, if $\mathcal{A}$ is an atom in $\Theta(X)$, then $\mathcal{A}^{-1}$ is too. One would also expect every atom $\mathcal{A}$ to be totally bounded, seeing as $\mathcal{A}_\omega \subseteq \mathcal{A}$. To prove this, we need the following lemma.

Lemma 5.2.1. *$\mathcal{I}$ is the unique quasi-uniformity in its proximity class. In other words, if $\mathcal{U}$ is a quasi-uniformity such that $\mathcal{U} \neq \mathcal{I}$, then $\mathcal{U}_\omega \neq \mathcal{I}$.*

Proof. Suppose that $\mathcal{U} \neq \mathcal{I}$. Then for some $U \in \mathcal{U}$, there is an $x \in X$ such that $U(x) \neq X$. Let $A = \{x\}$ and $B = X - U(x)$. Then $(A \times B) \cap U = \emptyset$ and hence A is far from B. Therefore $(X \times X) - (A \times B) \in \mathcal{U}_\omega$, so $\mathcal{U}_\omega \neq \mathcal{I}$. □

Corollary 5.2.2. *If $\mathcal{A}$ is an atom of $\Theta(X)$, $\mathcal{A}_\omega = \mathcal{A}$. Hence all atoms are totally bounded*

Proof. This follows immediately from Lemma 5.2.1, since if $\mathcal{A}$ is an atom of $\Theta(X)$, $\mathcal{A} \neq \mathcal{I}$, $\mathcal{A}_\omega \neq \mathcal{I}$ and $\mathcal{A}_\omega \subseteq \mathcal{A}$. □

An indication of what an atom of $\Theta(X)$ could look like in general comes from the following example.

Example 5.2.3. Consider the quasi-uniformity generated by a pre-order of the form

$$((X - \{x\}) \times X) \cup \Delta \text{ or } (X \times (X - \{x\})) \cup \Delta$$

for some $x \in X$. Such quasi-uniformities are called *trivial atoms.* They are easily seen to be atoms in $\Theta(X)$, because if, for example, V is any subset of $X \times X$ such that $((X - \{x\}) \times X) \cup \Delta \subsetneq V$, then $V \circ V = X \times X$.

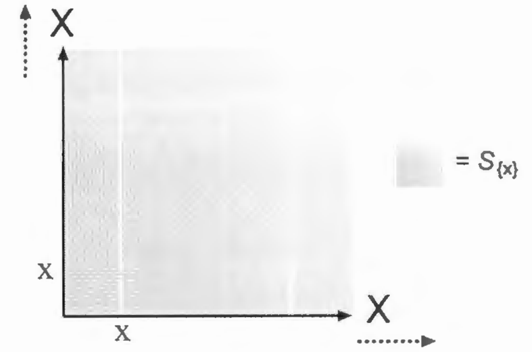

Figure 5.1: Diagrammatical representation of $S_{\{x\}} = ((X - \{x\}) \times X) \cup \Delta$.

Notice that another way of writing $((X-\{x\}) \times X) \cup \Delta$ is $(X \times X) - (\{x\} \times (X - \{x\}))$, or $(X \times X) - (A \times (X - A))$ where $A = \{x\}$. We will show that a quasi-uniformity is an atom exactly when it is generated by a pre-order of the above form for any $\emptyset \subsetneq A \subsetneq X$. We will be using the following notation:

Notation 5.2.4. Given a non-empty proper subset A of X, we shall denote

$$S_A = (X \times X) - (A \times (X - A)).$$

Since S_A is a pre-order on X, we can define a corresponding quasi-uniformity $\mathcal{S}_A = \text{fil}(S_A)$. Note that a different way of writing S_A is $S_A = (X \times A) \cup ((X - A) \times X)$, or $S_A = (A \times A) \cup ((X - A) \times X)$. Hence $(S_A)^{-1} = S_{X-A}$ and $(\mathcal{S}_A)^{-1} = \mathcal{S}_{X-A}$.

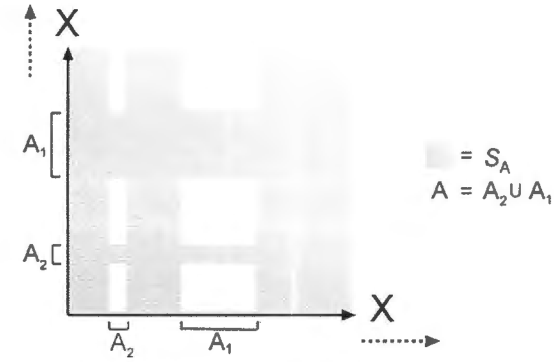

Figure 5.2: Diagrammatical representation of the set S_A.

Remark 5.2.5. Recall Notation 4.2.2 where $\mathcal{H}_A$ is defined for $\emptyset \subsetneq A \subsetneq X$. It is useful to note that $\mathcal{H}_A = \mathcal{S}_A \vee (\mathcal{S}_A)^{-1}$.

Proposition 5.2.6. *$\mathcal{S}_A$ is a (transitive) atom in $\Theta(X)$.*

Proof. We show that if R is any relation on X such that $S_A \subsetneq R$, then $R^3 = X \times X$. It will follow immediately that if $\mathcal{U} \subsetneq \mathcal{S}_A$, $\mathcal{U} = \mathcal{I}$.

So suppose $(x, y) \in X \times X$. We need to show that $(x, y) \in R^3$. There are two cases to consider:

1. $(x, y) \in S_A$. Since $S_A \subseteq R$, $(x, y) \in R$ as needed.

2. $(x, y) \notin S_A$. Then $(x, y) \in A \times (X - A)$. Since $S_A \subsetneq R$ there is an $(a, b) \in R$ such that $(a, b) \in A \times (X - A)$. Then $(x, a) \in A \times A \subseteq S_A \subseteq R$ and $(b, y) \in (X - A) \times X \subseteq S_A \subseteq R$. Hence $(x, y) \in R^3$ as needed.

□

Lemma 5.2.7. *Let $\mathcal{U}$ be a quasi-uniformity on X and let U be a transitive entourage of $\mathcal{U}$. Then $U \subseteq S_{U(A)}$ whenever $A \subseteq X$. Hence, if $\mathcal{U}$ contains a transitive entourage other than $X \times X$, $\mathcal{S}_B \subseteq \mathcal{U}$ for some $\emptyset \subsetneq B \subsetneq X$.*

Proof. Let U be a transitive entourage of $\mathcal{U}$ and $A \subseteq X$. Suppose $(x, y) \in U$. There are two cases:

1. $x \in U(A)$. Then $y \in U(U(A))$, but since U is transitive, $U(U(A)) = U(A)$. Hence $(x, y) \in U(A) \times U(A) \subseteq S_{U(A)}$.

2. $x \in X - U(A)$. Then $(x, y) \in (X - U(A)) \times X \subseteq S_{U(A)}$.

Hence $U \subseteq S_{U(A)}$. If $U \neq X \times X$, then for some non-empty $A \subseteq X$ we have $U(A) \neq X$. If we write $B = U(A)$, we have $\mathcal{S}_B \subseteq \mathcal{U}$. □

Corollary 5.2.8. *If $\mathcal{A}$ is a transitive atom in $\Theta(X)$, then $\mathcal{A} = \mathcal{S}_A$ for some non-empty proper subset A of X.*

Proof. If $\mathcal{A}$ is transitive, it contains $\mathcal{S}_A$ for some $\emptyset \subsetneq A \subsetneq X$ by the above lemma. Since both $\mathcal{S}_A$ and $\mathcal{A}$ are atoms, $\mathcal{A} = \mathcal{S}_A$. □

Hence $\mathcal{A}$ is a transitive atom in $\Theta(X)$ iff $\mathcal{A} = \mathcal{S}_A$ for some non-empty proper subset A of X. As we mentioned before, we want to show that $\mathcal{A}$ is an atom in $\Theta(X)$ iff $\mathcal{A} = \mathcal{S}_A$ for some non-empty proper subset A of X. This is achieved by means of the below proposition:

Proposition 5.2.9. *There are no non-transitive atoms in $\Theta(X)$.*

Proof. Suppose we had a non-transitive atom $\mathcal{U} \in \Theta(X)$. We have $\mathcal{U} = \mathcal{U}_\omega$ by Corollary 5.2.2, and hence $\{(X \times X) - (A \times B) \mid A \text{ is far from } B\}$ is a subbase for $\mathcal{U}$.

Now let A be far from B, and suppose there is a $C \subseteq X$ such that $A \subseteq C \subseteq X - B$ where C is far from $X - C$. Then $S_C \in \mathcal{U}$. Since $A \times B \subseteq C \times (X - C)$, it follows that $S_C \subseteq (X \times X) - (A \times B)$. Now S_C is transitive, so if for all subsets A and B of X such that A is far from B we could find such a C, $\mathcal{U}$ would be transitive, a contradiction. Hence we can find at least one pair of sets A and B such that A is far from B and there is no such C.

For this A and B we can find a $W \in \mathcal{U}$ such that $W^3(A) \cap B = \emptyset$ (because A is far from B). Set

$$D = W^2(A) - W(A).$$

We show that D is non-empty. We have $W(A) \cap B = \emptyset$. Hence $A \subseteq W(A) \subseteq X - B$. By the assumption on A and B, $W(A)$ has to be near $X - W(A)$. Hence we can find a point $(x, y) \in W \cap (W(A) \times (X - W(A)))$. Then $(x, y) \in W$, $(a, x) \in W$ for some $a \in A$ and $y \in X - W(A)$. It follows that $y \in W^2(A) - W(A) = D$.

A is far from D in $\mathcal{U}$ because $W(A) \cap D = \emptyset$. We now create a new quasi-uniformity on X, call it $\mathcal{U}^{\#}$, having a base consisting of all entourages of the form

$$U^{\#} = U \cup (U^{-1}(A) \times U(D))$$

for $U \in \mathcal{U}$. Clearly $\mathcal{U}^{\#}$ is strictly coarser than $\mathcal{U}$, as A is near D in $\mathcal{U}^{\#}$.

In $\mathcal{U}^{\#}$, D is far from B since $W(D) \cap B \subseteq W^3(A) \cap B = \emptyset$ and hence

$$\begin{aligned} & (D \times B) \cap W^{\#} \\ = & ((D \times B) \cap W) \cup ((D \times B) \cap (W^{-1}(A) \times W(D))) \\ = & \emptyset. \end{aligned}$$

Therefore $\mathcal{U}^{\#}$ cannot be the indiscrete uniformity. Since $\mathcal{U}^{\#} \subsetneq \mathcal{U}$, this contradicts that $\mathcal{U}$ is an atom. □

Corollary 5.2.10. *$\mathcal{A}$ is an atom of $\Theta(X)$ iff $\mathcal{A} = \mathcal{S}_A$ for some non-empty proper subset A of X.*

Corollary 5.2.11. *For any infinite set X, $\Theta(X)$ has $2^{|X|}$ atoms. If X is finite, non-empty and $|X| = n$, then $\Theta(X)$ has $2^n - 2$ atoms.*

Proof. If A and B are distinct non-empty proper subsets of X, then S_A and S_B and hence $\mathcal{S}_A$ and $\mathcal{S}_B$ are distinct. Hence this result follows from the above corollary – the number of atoms in $\Theta(X)$ must be $|\wp(X)| - 2$, since the empty set and X are the only members of $\wp(X)$ which cannot be used to construct an atom of $\Theta(X)$. □

Corollary 5.2.12. *$\mathcal{A}$ is an atom of $\Psi(X)$ iff $\mathcal{A} = \mathcal{H}_A$ for some non-empty proper subset A of X.*

Proof. The proof that $\mathcal{H}_A$ is an atom in $\Psi(X)$ is the symmetric analogue of the proof of Proposition 5.2.6.

Conversely, suppose that $\mathcal{A}$ is an atom in $\Psi(X)$. A proof that follows the same lines as that of Proposition 5.2.9 shows that there are no non-transitive atoms in $\Psi(X)$ (use symmetric entourages only and set $U^{\#} = U \cup (U(A) \times U(D)) \cup (U(D) \times U(A))$). Hence $\mathcal{A}$ is transitive, and so by Lemma 5.2.7 contains $\mathcal{S}_A$ for some $\emptyset \subsetneq A \subsetneq X$. $\mathcal{A}$ is symmetric, so it contains $(\mathcal{S}_A)^{-1}$ too, and hence by Remark 5.2.5, $\mathcal{H}_A \subseteq \mathcal{A}$. Therefore $\mathcal{A} = \mathcal{H}_A$. □

Corollary 5.2.13. *For any infinite set X, $\Psi(X)$ has $2^{|X|}$ atoms. If X is finite and $|X| = n \geq 1$, $\Psi(X)$ has $2^{(n-1)} - 1$ atoms.*

Proof. For A and B non-empty proper subsets of X, $\mathcal{H}_A = \mathcal{H}_B$ iff $B = A$ or $B = X - A$. Hence the number of atoms in $\Psi(X)$ will be $(|\wp(X)| - 2)/2$, since X and $\emptyset$ are the only two members of $\Psi(X)$ which cannot be used to construct an atom of $\Psi(X)$. □

Note that from Corollary 5.2.10 it follows that the atoms of $\Theta(X)$ are in fact the Pervin quasi-uniformities of the atoms in $\Sigma(X)$ (compare Theorem 3.2.1). This, together with Remark 5.1.11, leads to the question as to whether for any set X, $\Sigma(X)$ can be embedded into $\Theta(X)$ by identifying each topology on X with its Pervin quasi-uniformity. Remark 5.1.11 shows that for finite X this is certainly always the case. The below example, however, shows that this need not be the case for infinite X.

Example 5.2.14. Let $\mathcal{P}(\mathcal{T})$ denote the Pervin quasi-uniformity of the topology $\mathcal{T}$. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X, it need not be the case that $\mathcal{P}(\mathcal{T}_1) \vee \mathcal{P}(\mathcal{T}_2) = \mathcal{P}(\mathcal{T}_1 \vee \mathcal{T}_2)$: Take $X = \mathbb{R}$, let $\mathcal{T}$ be the usual uniformity on $\mathbb{R}$, $\mathcal{T}_1 = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, x) \mid x \in \mathbb{R}\}$, $\mathcal{T}_2 = \{\emptyset, \mathbb{R}\} \cup \{(x, \infty) \mid x \in \mathbb{R}\}$ and $G = \bigcup\{(n, n+1) \mid n \in \omega \cup \{0\}\}$. Note that $\mathcal{T}_1 \vee \mathcal{T}_2 = \mathcal{T}$ and $G \in \mathcal{T}$. If $G_1 \in \mathcal{T}_1$ and $G_2 \in \mathcal{T}_2$ are non-empty proper subsets of X, it is however easily seen that $((X - G_1) \cap G_2) \times G_2$ will always contain points of $G \times (X - G)$. From this observation it follows that $S_G \notin \mathcal{P}(\mathcal{T}_1) \vee \mathcal{P}(\mathcal{T}_2)$,[5] and hence $\mathcal{P}(\mathcal{T}_1) \vee \mathcal{P}(\mathcal{T}_2) \neq \mathcal{P}(\mathcal{T}_1 \vee \mathcal{T}_2)$.

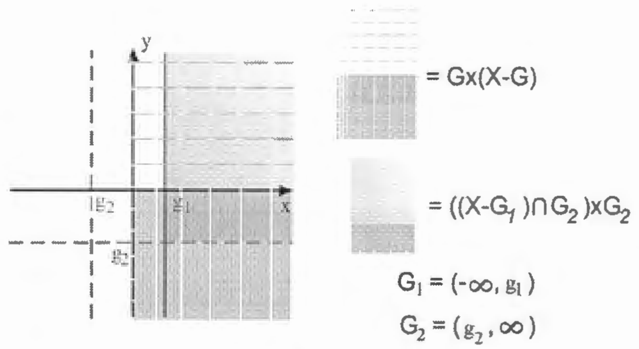

Figure 5.3: Diagrammatical representation of $((X - G_1) \cap G_2) \times G_2$ and $G \times (X - G)$.

We have already shown that every atom of $\Theta(X)$ is transitive and totally bounded. Below we see that the atomic members of $\Theta(X)$ are exactly the transitive totally bounded quasi-uniformities on X:

[5]If $\{U_1, ..., U_n\} \subseteq \mathcal{T}_1$ ($n \in \omega$) and $\{H_1, ..., H_m\} \subseteq \mathcal{T}_2$ ($m \in \omega$) are collections of non-empty proper subsets of X, set $G_1 = U_1 \cup ... \cup U_n$ and $G_2 = H_1 \cap ... \cap H_m$. Then $((X - G_1) \cap G_2) \times G_2 \subseteq (S_{U_1} \cap ... \cap S_{U_n}) \cap (S_{H_1} \cap S_{H_m})$, so $(S_{U_1} \cap ... \cap S_{U_n}) \cap (S_{H_1} \cap S_{H_m}) \nsubseteq S_G$.

Proposition 5.2.15. *A quasi-uniformity $\mathcal{U} \neq \mathcal{I}$ on X is an atomic member of $\Theta(X)$ iff it is transitive and totally bounded.*

Proof. We show that $\mathcal{U}$ is transitive and totally bounded iff it has the form

$$\bigvee_{E \in \mathcal{E}} \mathcal{S}_E$$

for some collection $\mathcal{E}$ of non-empty proper subsets of X. Clearly $\bigvee_{E\in\mathcal{E}} \mathcal{S}_E$ will be transitive and totally bounded for any such $\mathcal{E}$, since it is the join of transitive and totally bounded quasi-uniformities.

Now suppose that $\mathcal{U} \neq \mathcal{I}$ is a transitive and totally bounded quasi-uniformity on X. For each $U \in \mathcal{U}$ there is a finite cover $\mathcal{A}^U$ of X such that for each $A \in \mathcal{A}^U$, $A \times A \subseteq U$. Let $\mathcal{B}$ be a base of transitive entourages not containing $X \times X$ for $\mathcal{U}$, and set

$$\mathcal{E} = \{U(A) \mid A \in \mathcal{A}^U \text{ and } U \in \mathcal{B}\}.$$

We show $\mathcal{U} = \bigvee_{E\in\mathcal{E}} \mathcal{S}_E$.

By Lemma 5.2.7 we have $\bigvee_{E\in\mathcal{E}} \mathcal{S}_E \subseteq \mathcal{U}$, since $U \subseteq S_{U(A)}$ for each $A \in \mathcal{A}^U$ whenever $U \in \mathcal{B}$.

Now we show that $\mathcal{U} \subseteq \bigvee_{E\in\mathcal{E}} \mathcal{S}_E$ by showing that $\mathcal{B} \subseteq \bigvee_{E\in\mathcal{E}} \mathcal{S}_E$. Suppose $U \in \mathcal{B}$. We show that $S_{U(A_1)} \cap ... \cap S_{U(A_n)} \subseteq U$, where $\mathcal{A}^U = \{A_1, ..., A_n\}$ ($n \in \omega$). Suppose that $(x, y) \notin U$. Since $\mathcal{A}^U$ is a cover of X, there must be an $A \in \mathcal{A}^U$ such that $x \in A \subseteq U(A)$. But then $y \notin U(A)$: Otherwise $(a, y) \in U$ for some $a \in A$. Since $A \times A \subseteq U$, we have $(x, a) \in U$ and hence $(x, y) \in U^2 = U$, a contradiction. Therefore $x \in U(A)$ and $y \in X - U(A)$, so $(x, y) \notin S_{U(A)}$ as needed. □

It is clear that given any topology on X, the associated Pervin quasi-uniformity $\mathcal{P}$ is atomic. This follows from the above proposition, but is also clear from the definition of $\mathcal{P}$ (see Proposition 2.2.24).

Corollary 5.2.16. *All atoms in $\Psi(X)$ are transitive and totally bounded, and a uniformity is an atomic member of $\Psi(X)$ if and only if it is transitive and totally bounded.*

Proof. Clearly all atomic members of $\Psi(X)$ are transitive and totally bounded, as they are the join of transitive and totally bounded uniformities (see Corollary 5.2.12).

Let $\mathcal{U}$ be a uniformity on X that is either not transitive or not totally bounded. Then it is not the join of atoms in $\Theta(X)$ (see Proposition 5.2.15). Suppose it was the join of atoms in $\Psi(X)$. Since for each $\mathcal{H}_A \subseteq \mathcal{U}$ we have $\mathcal{S}_A \vee (\mathcal{S}_A)^{-1} = \mathcal{H}_A$, this would mean $\mathcal{U}$ is the join of atoms in $\Theta(X)$, a contradiction. □

Corollary 5.2.17. $\Theta(X)$ *(respectively* $\Psi(X)$*) is atomic iff* X *is finite.*

Proof. If X is finite, each quasi-uniformity (uniformity) on X is generated by a pre-order (respectively, equivalence relation) on X, and is hence transitive and totally bounded. If X is infinite, however, it is clear that $\mathcal{D}$ is not totally bounded. □

Whereas some quasi-uniformities are not atomic, it need not even be the case that every quasi-uniformity contains an atom of $\Theta(X)$. The following example illustrates this, and it also proves the corresponding result for $\Psi(X)$.

Example 5.2.18. Consider the usual uniformity $\mathcal{U}$ on $X = \mathbb{R}$, with ρ the usual metric on $\mathbb{R}$. Let A be any non-empty proper subset of X and let $\epsilon > 0$ be given. It is easily seen that A is near its complement with respect to ρ, and hence we can find an $x \in A$ and a $y \in X - A$ such that the distance between x and y is less than ϵ, i.e. $(x, y) \in U_\epsilon^\rho$ (this follows by the connectedness of $\mathbb{R}$, since either A or $X - A$ is not open). But $(x, y) \in A \times (X - A)$ and therefore $U_\epsilon^\rho \nsubseteq S_A$. Hence $S_A \notin \mathcal{U}$ and $\mathcal{S}_A \nsubseteq \mathcal{U}$, proving that $\mathcal{U}$ does not contain any atoms of $\Theta(X)$. We also have, for the same reasons, that $U_\epsilon^\rho \nsubseteq H_A$ and hence $\mathcal{U}$ contains no atoms of $\Psi(X)$ either.

From the above example it is clear that whenever $(X, \mathcal{V})$ is a quasi-uniform space such that $(X, \mathcal{T}(\mathcal{V}))$ is connected, $\mathcal{V}$ will not contain any atoms of $\Theta(X)$ or $\Psi(X)$.

The reason that the usual uniformity on $\mathbb{R}$ contains no atoms of $\Theta(\mathbb{R})$ appears to be a consequence of the fact that every set is near its complement with respect to the usual metric on $\mathbb{R}$. This observation is generalized as follows:[6]

Remark 5.2.19. A quasi-uniformity $\mathcal{U}$ will contain an atom $\mathcal{S}_A$ of $\Theta(X)$ if and only if A is far from its complement with respect to $\delta_{\mathcal{U}}$. Hence, the atoms below a quasi-uniformity $\mathcal{U}$ are determined by the quasi-proximity it induces. However, it is clear that the atoms below $\mathcal{U}$ by no means completely describe $\delta_{\mathcal{U}}$, as can be seen from Example 5.2.18.

[6]This remark can be adjusted for $\Psi(X)$ in the obvious way.

5.3 Anti-Atoms in $\Theta(X)$

In this section we investigate the anti-atoms of the lattice of quasi-uniformities on a set X. We will start with some basic results, and we will then show that there is a strong relationship between the anti-atoms of $\Theta(X)$ and ultrafilters on X. The proximally discrete and proximally non-discrete anti-atoms will then be described separately, and we will also discuss the proximally fine anti-atoms. The reader is warned that the approach taken in this section on anti-atoms of $\Theta(X)$, including the subsections on proximally discrete, proximally non-discrete and proximally fine anti-atoms, differs from the one taken in [11].

Clearly, if $\mathcal{A}$ is an anti-atom of $\Theta(X)$, $\mathcal{A}^{-1}$ is an anti-atom too. Also, as was the case with the lattice of uniformities, Zorn's Lemma guarantees the existence of anti-atoms in $\Theta(X)$:

Proposition 5.3.1. *Every non-discrete quasi-uniformity $\mathcal{U}$ is contained by an anti-atom in $\Theta(X)$.*

Proof. Let $\mathcal{S} = \{\mathcal{V} \in \Theta(X) \mid \mathcal{U} \subseteq \mathcal{V}, \mathcal{V} \neq \mathcal{D}\}$. $\mathcal{S}$ is non-empty since $\mathcal{U} \in \mathcal{S}$. We show that every non-empty chain in $\mathcal{S}$ has an upper bound in $\mathcal{S}$. So let $\mathcal{C}$ be a non-empty chain in $\mathcal{S}$. Then clearly $\bigvee \mathcal{C}$ is an upper bound of this chain. Now suppose that $\bigvee \mathcal{C} = \mathcal{D}$. Then there are entourages $V_1, ..., V_n$ ($n \in \omega$), each $V_i \in \mathcal{V}_i$ for some $\mathcal{V}_i \in \mathcal{C}$, such that $V_1 \cap ... \cap V_n = \Delta$. Since $\mathcal{C}$ is a chain, we may assume that $\mathcal{V}_1 \subseteq ... \subseteq \mathcal{V}_n$. Hence $V_i \in \mathcal{V}_n$ for $1 \leq i \leq n$, implying that $\mathcal{V}_n = \mathcal{D}$. This contradicts the definition of $\mathcal{S}$.

So $\bigvee \mathcal{C} \neq \mathcal{D}$ and hence $\bigvee \mathcal{C} \in \mathcal{S}$. By Zorn's Lemma $\mathcal{S}$ has a maximal element, and this maximal element is clearly an anti-atom of $\Theta(X)$. □

Naturally, the above existence proof is entirely non-constructive – it does not provide any examples. Hence, some examples are provided below:

Example 5.3.2. The simplest examples of anti-atoms in $\Theta(X)$ are called *trivial anti-atoms*. They have the form $\text{fil}(\{(x, y)\} \cup \Delta)$ for any $x, y \in X$ such that $x \neq y$.

Notation 5.3.3. Let $x, y \in X$ such that $x \neq y$. The trivial anti-atom fil $(\{(x, y)\} \cup \Delta)$ of $\Theta(X)$ will be denoted by $\mathcal{G}_{(x,y)}$.

We now give another example of an anti-atom in $\Theta(X)$. It is based on the idea of a trivial anti-atom.

Example 5.3.4. Let $x \in X$ and let $\mathcal{F}$ be an ultrafilter on X such that $\mathcal{F} \neq \mathrm{fil}(\{x\})$. Let $\mathcal{U} = \mathrm{fil}(\{(\{x\} \times F) \cup \Delta \mid F \in \mathcal{F}\})$. We show that $\mathcal{U}$ is an anti-atom of $\Theta(X)$. Suppose that $\mathcal{U} \subseteq \mathcal{V} \subsetneq \mathcal{D}$. Then since $\mathcal{U} \subseteq \mathcal{V}$, if we choose any $F \in \mathcal{F}$, $\{((\{x\} \times F) \cup \Delta) \cap V \mid V \in \mathcal{V}\}$ is a base for $\mathcal{V}$. Hence, if we write $\mathcal{G} = \mathrm{fil}(\{F \cap V(x) \mid V \in \mathcal{V}\})$, then $\{(\{x\} \times G) \cup \Delta \mid G \in \mathcal{G}\}$ is a base for $\mathcal{V}$. Since $\mathcal{U} \subseteq \mathcal{V} \subsetneq \mathcal{D}$, we have $\mathcal{F} \subseteq \mathcal{G} \neq \mathrm{fil}(\{x\})$ and therefore $\mathcal{F} = \mathcal{G}$. Consequently $\mathcal{U} = \mathcal{V}$, and hence $\mathcal{U}$ is an anti-atom of $\Theta(X)$ as claimed.

It is clear that if $\mathcal{F} = \mathrm{fil}(\{y\})$ for some $y \neq x$, $\mathcal{U}$ is just the trivial anti-atom $\mathcal{G}_{(x,y)}$. Similarly, $\mathcal{U}$ would also be an anti-atom of $\Theta(X)$ if $\mathcal{U} = \mathrm{fil}(\{(F \times \{x\}) \cup \Delta \mid F \in \mathcal{F}\})$.

Definition 5.3.5. Let $x \in X$, and let $\mathcal{F}$ be an ultrafilter on X such that $\mathcal{F} \neq \mathrm{fil}(\{x\})$. If $\mathcal{U}$ is the quasi-uniformity that has either

$$\{(\{x\} \times F) \cup \Delta \mid F \in \mathcal{F}\} \text{ or } \{(F \times \{x\}) \cup \Delta \mid F \in \mathcal{F}\}$$

as base, it is called a *semi-trivial anti-atom* (of $\Theta(X)$).

Remark 5.3.6. Recall Proposition 4.3.4, which says that every non-trivial anti-atom in $\Psi(X)$ is T_1-separated and has no non-isolated points. In the case of a semi-trivial anti-atom $\mathcal{U}$ of $\Theta(X)$, though, it is clear that x will not be isolated with respect to $\mathcal{T}(\mathcal{U})$ for whatever ultrafilter $\mathcal{F} \neq \mathrm{fil}(\{x\})$ we choose. Hence Proposition 4.3.4 cannot be extended to the quasi-uniform case. It is however true that an anti-atom of $\Theta(X)$ contains a non-isolated point if and only if it is semi-trivial.[7] It is also still the case that every non-trivial anti-atom of $\Theta(X)$ is T_1-separated.

Note that a consequence of the above comment is that the proof of Corollary 4.3.5, where $\Psi(X)$ is proven to be non-anti-atomic for infinite X, cannot be extended to $\Theta(X)$.

The above examples of anti-atoms in $\Theta(X)$ are all non-symmetric. Of course, one might ask whether a uniformity could ever be an anti-atom in $\Theta(X)$. The below proposition shows that this will never be the case.

Proposition 5.3.7. *Suppose $\mathcal{U}$ is a non-discrete uniformity on X. Then there is a non-symmetric quasi-uniformity $\mathcal{Q}$ on X such that $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{D}$.*

[7] If $\mathcal{U} \in \Theta(X)$ and x is a non-isolated point with respect to $\mathcal{T}(\mathcal{U})$, then $\mathcal{U} \subseteq \mathrm{fil}(\{(\{x\} \times F) \cup \Delta \mid F \in \mathcal{F}\})$ for some ultrafilter $\mathcal{F} \neq \mathrm{fil}(\{x\})$. Note also that if x is isolated, then $\mathcal{U} \not\subseteq \mathrm{fil}(\{(\{x\} \times F) \cup \Delta \mid F \in \mathcal{F}\})$ for any ultrafilter $\mathcal{F}$ on X other than $\mathrm{fil}(\{x\})$.

Proof. Let W be any linear order on X and let $\mathcal{Q} = \mathcal{U} \vee \mathrm{fil}(W)$. Clearly $\mathcal{U} \subseteq \mathcal{Q} \subseteq \mathcal{D}$. First, we show that $\mathcal{Q} \neq \mathcal{D}$. So let $V \in \mathcal{U}$, and let $U \in \mathcal{U}$ be symmetric such that $U \subseteq V$. Then there are $x, y \in X$ such that $x \neq y$, $(x, y) \in U \subseteq V$ and $(y, x) \in U \subseteq V$. Since W is linear, either $(x, y) \in V \cap W$ or $(y, x) \in V \cap W$, which proves that $V \cap W \neq \Delta$. Hence $\mathcal{Q} \neq \mathcal{D}$.

We have $W \in \mathcal{Q}$, and W contains no symmetric relation on X other than Δ since it is anti-symmetric. Hence $\mathcal{Q}$ is non-symmetric, and therefore $\mathcal{Q} \neq \mathcal{U}$.

So $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{D}$ as claimed. □

Corollary 5.3.8. *No uniformity can be an anti-atom of* $\Theta(X)$.

Recall that there is a strong relationship between the anti-atoms of $\Psi(X)$ and ultrafilters on X (see Propositions 4.3.9 and 4.3.21). There is a similar link between the anti-atoms of $\Theta(X)$ and ultrafilters on X, which we now describe.

We need the following definitions:

Definition 5.3.9. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are filters on X. We let

$$\mathcal{F} \times \mathcal{G} = \mathrm{fil}(\{F \times G \mid F \in \mathcal{F}, G \in \mathcal{G}\}).$$

Definition 5.3.10. Let $\mathcal{F}$ and $\mathcal{G}$ be filters on X. We define $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ to be the quasi-uniformity generated by the base

$$\{(F \times G) \cup \Delta \mid F \in \mathcal{F}, G \in \mathcal{G}\}.$$

If $\mathcal{F} = \mathcal{G}$, we simply denote $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ by $\mathcal{U}_{\mathcal{F}}$. $\mathcal{U}_{\mathcal{F}}$ as defined here is the same uniformity $\mathcal{U}_{\mathcal{F}}$ defined in Definition 4.3.7. Note that if $\mathcal{F}$ is principal, then $\mathcal{U}_{\mathcal{F}} = \mathcal{D}$.

Remark 5.3.11. Let $\mathcal{F}$ and $\mathcal{G}$ be ultrafilters on X such that $\mathcal{F} \neq \mathcal{G}$. Then:

1. $\mathcal{F} \times \mathcal{G}$ has a base $\mathcal{B}$ such that $B \cap \Delta = \emptyset$ for each $B \in \mathcal{B}$. This follows because there is an $A \subseteq X$ such that $A \in \mathcal{F}$ and $X - A \in \mathcal{G}$, and hence

   $$\mathcal{B} = \{(F \times G) \cap (A \times (X - A)) \mid F \in \mathcal{F}, G \in \mathcal{G}\}$$

   is such a base for $\mathcal{F} \times \mathcal{G}$.

2. If $\mathcal{U}$ is any filter on $X \times X$ such that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{U}$, then $\mathcal{U}$ has a transitive base. For if we pick $A \subseteq X$ as in point 1 above, then $\mathcal{B} = \{U \cap ((A \times (X - A)) \cup \Delta) \mid U \in \mathcal{U}\}$ is a transitive base for $\mathcal{U}$, since A and $X - A$ are disjoint.

Lemma 5.3.12. *Let $\mathcal{A}$ be an anti-atom in $\Theta(X)$. Then there are unique ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on X such that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$.*

Proof. Suppose $\mathcal{A}$ is an anti-atom. Let $\mathcal{G}$ be the filter generated by $\{(U - \Delta)(X) \mid U \in \mathcal{A}\}$ and let $\mathcal{F}$ be the filter generated by $\{(U^{-1} - \Delta)(X) \mid U \in \mathcal{A}\}$.[8]

Both $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters on X. For the sake of contradiction, suppose that $\mathcal{F}$ is not. Then for some $A \subseteq X$, $A \notin \mathcal{F}$ and $X - A \notin \mathcal{F}$. Then $(A \times X) \cup \Delta \notin \mathcal{A}$ and $((X - A) \times X) \cup \Delta \notin \mathcal{A}$. Since both of these are pre-orders on X and $\mathcal{A}$ is an anti-atom of $\Theta(X)$, there must be a $U \in \mathcal{A}$ and a $V \in \mathcal{A}$ such that $U \cap ((A \times X) \cup \Delta) = \Delta$ and $V \cap (((X - A) \times X) \cup \Delta) = \Delta$. Then $U \subseteq ((X - A) \times X) \cup \Delta$ and $V \subseteq (A \times X) \cup \Delta$. Therefore $U \cap V = \Delta$, contradicting that $\mathcal{A} \neq \mathcal{D}$. Hence $\mathcal{F}$ must be an ultrafilter after all, and similarly $\mathcal{G}$ is an ultrafilter. Clearly $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$.

Now suppose that $\mathcal{U}_{\mathcal{H}\times\mathcal{L}} \subseteq \mathcal{A}$ for ultrafilters $\mathcal{H}$ and $\mathcal{L}$ on X such that either $\mathcal{H} \neq \mathcal{F}$ or $\mathcal{L} \neq \mathcal{G}$. Assume the former. Then there is a subset A of X such that $A \in \mathcal{H}$ and $X - A \in \mathcal{F}$, so $(A \times X) \cup \Delta \in \mathcal{A}$ and $((X - A) \times X) \cup \Delta \in \mathcal{A}$. Consequently $\Delta \in \mathcal{A}$, a contradiction. So $\mathcal{H} = \mathcal{F}$ and similarly $\mathcal{L} = \mathcal{G}$. Therefore the ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on X such that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$ are unique. □

As we did with the anti-atoms of $\Psi(X)$, we will now deal with the proximally discrete and proximally non-discrete anti-atoms of $\Theta(X)$ separately. We will see that if $\mathcal{A}$ is an anti-atom, the ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on X such that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$ determine (and are determined by) whether $\mathcal{A}$ is proximally discrete or proximally non-discrete. It will in fact be shown that $\mathcal{A}$ is proximally discrete iff $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ is a uniformity.

5.3.1 Proximally Discrete Anti-Atoms

In this section we will give a necessary and sufficient condition for an anti-atom of $\Theta(X)$ to be proximally discrete in terms of ultrafilters on X.

[8] $\mathcal{F}$ and $\mathcal{G}$ can be seen as the left and right projections of $\mathcal{A}$ respectively.

We start with a simple fact regarding proximally discrete anti-atoms in $\Theta(X)$.

Remark 5.3.13. $\mathcal{A}$ is a proximally discrete anti-atom of $\Theta(X)$ iff $\mathcal{A}^{-1}$ is. For since $\mathcal{D}$ is a uniformity, $\mathcal{D}_\omega$ is a uniformity too. If $\mathcal{A}$ is a proximally discrete anti-atom, $\mathcal{D}_\omega \subseteq \mathcal{A}$ and hence $\mathcal{D}_\omega \subseteq \mathcal{A}^{-1}$, so $\mathcal{A}^{-1}$ is proximally discrete as well.

In Remark 4.3.6 we noted that $\Psi(X)$ contains a proximally discrete anti-atom iff X is infinite. The same is true for $\Theta(X)$:

Remark 5.3.14. Suppose that X is infinite. Then since $\mathcal{D}_\omega$ is totally bounded and $\mathcal{D}$ is not, $\mathcal{D}_\omega \subsetneq \mathcal{D}$. Hence, there is always at least one non-discrete quasi-uniformity inducing the discrete proximity (namely $\mathcal{D}_\omega$). By Proposition 5.3.1, $\mathcal{D}_\omega$ must be contained by an anti-atom of $\Theta(X)$. This anti-atom has to be proximally discrete because $\mathcal{D}_\omega$ is. Hence $\Theta(X)$ will possess at least one proximally discrete anti-atom.

However, if X is finite, all quasi-uniformities are totally bounded and hence unique in their quasi-proximity classes. Since $\mathcal{D}$ will be unique in its proximity class, there will be no proximally discrete anti-atoms in $\Theta(X)$.

We are immediately in a position to give a characteristic of the proximally discrete anti-atoms in $\Theta(X)$ that follows from Lemma 5.3.12. It is interesting to note that the relationships between the proximally discrete anti-atoms of $\Theta(X)$ and $\Psi(X)$ respectively and ultrafilters on X are exactly the same (compare Proposition 4.3.9).

Proposition 5.3.15. *An anti-atom $\mathcal{A}$ of $\Theta(X)$ is proximally discrete if and only if $\mathcal{U}_\mathcal{F} \subseteq \mathcal{A}$ for some non-principal ultrafilter $\mathcal{F}$ on X. If $\mathcal{A}$ is a proximally discrete anti-atom, then the ultrafilter $\mathcal{F}$ such that $\mathcal{U}_\mathcal{F} \subseteq \mathcal{A}$ is unique.*

Proof. Suppose $\mathcal{A}$ is an anti-atom of $\Theta(X)$. Let $\mathcal{U}_\mathcal{F} \subseteq \mathcal{A}$ for some non-principal ultrafilter $\mathcal{F}$ on X. $\mathcal{U}_\mathcal{F}$ is proximally discrete – see Remark 4.3.8. Hence $\mathcal{A}$ is proximally discrete.

For the converse, suppose $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$ for ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on X such that $\mathcal{F} \neq \mathcal{G}$ (Lemma 5.3.12). Then there is a non-empty proper subset A of X such that $A \in \mathcal{F}$ and $X - A \in \mathcal{G}$. Then $(A \times (X - A)) \cup \Delta \in \mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$. Since $\mathcal{A} \neq \mathcal{D}$, this means that A is near $X - A$ in $\mathcal{A}$. Therefore $\mathcal{A}$ is not proximally discrete. □

Corollary 5.3.16. *If X is finite, there are no proximally discrete anti-atoms in $\Theta(X)$. If X is infinite, the number of proximally discrete anti-atoms in $\Theta(X)$ is $2^{2^{|X|}}$.*

Proof. Suppose X is infinite. By Proposition 5.3.1, for each non-principal ultrafilter $\mathcal{F}$ on X, $\mathcal{U}_{\mathcal{F}}$ is contained by an anti-atom of $\Theta(X)$. This anti-atom is necessarily proximally discrete by the above proposition. The above proposition also states that every proximally discrete anti-atom contains $\mathcal{U}_{\mathcal{F}}$ for some *unique* non-principal ultrafilter $\mathcal{F}$ on X. Hence, there must be at least as many proximally discrete anti-atoms in $\Theta(X)$ as there are non-principal ultrafilters on X. This number is $2^{2^{|X|}} - |X| = 2^{2^{|X|}}$, which is the cardinality of $\Theta(X)$. □

In Theorem 4.3.17 we showed that it is possible for $\mathcal{U}_{\mathcal{F}}$ to be contained by a unique anti-atom in $\Psi(X)$. The question is whether this could happen in $\Theta(X)$. The below remark answers this question:

Remark 5.3.17. It will *never* be the case that there is a unique anti-atom in $\Theta(X)$ containing $\mathcal{U}_{\mathcal{F}}$ for an ultrafilter $\mathcal{F}$ on X. For let $\mathcal{A}$ be an anti-atom containing $\mathcal{U}_{\mathcal{F}}$, which is of course a uniformity. By Corollary 5.3.8, $\mathcal{A}$ has to be a non-symmetric quasi-uniformity, and hence $\mathcal{U}_{\mathcal{F}} \subsetneq \mathcal{A}$ and $\mathcal{U}_{\mathcal{F}} \subsetneq \mathcal{A}^{-1}$. Since $\mathcal{A}^{-1}$ is also an anti-atom, $\mathcal{U}_{\mathcal{F}}$ is contained by at least two anti-atoms in $\Theta(X)$.

5.3.2 Proximally Non-Discrete Anti-Atoms

In this section we show that Lemma 5.3.12 leads to a satisfying characterization of the proximally non-discrete anti-atoms in $\Theta(X)$ in terms of ultrafilters on X and ultrafilters on $X \times X$. This characterization is given in Theorem 5.3.20.

We start this section with an immediate consequence of Lemma 5.3.12:

Proposition 5.3.18. *If $\mathcal{A}$ is an anti-atom of $\Theta(X)$, it is proximally non-discrete if and only if $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$ for two ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on X such that $\mathcal{F} \neq \mathcal{G}$. If $\mathcal{A}$ is a proximally non-discrete anti-atom, then the ultrafilters $\mathcal{F}$ and $\mathcal{G}$ such that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$ are unique. Hence, every proximally non-discrete anti-atom is transitive.*

Proof. This fol ows directly from Lemma 5.3.12 and Proposition 5.3.15. The fact that $\mathcal{A}$ is transitive follows from Remark 5.3.11(2). ☐

Corollary 5.3.19. *If X is finite and $|X| = n$, there are $n(n-1)$ (proximally non-discrete) anti-atoms in $\Theta(X)$. If X is infinite, the number of proximally non-discrete anti-atoms in $\Theta(X)$ is $2^{2^{|X|}}$.*

Proof. If X is finite, $\mathcal{A}$ is an anti-atom of $\Theta(X)$ iff $\mathcal{A} = \mathcal{G}_{(x,y)}$ for some $x \neq y$.

Suppose X is infinite. Then by Proposition 5.3.1, for each two distinct ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on X, $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ is contained by an anti-atom which is necessarily proximally non-discrete (by the above proposition). Since the above proposition also states that every proximally non-discrete anti-atom contains $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ for two *unique* but distinct ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on X, it follows that the number of proximally non-discrete anti-atoms in $\Theta(X)$ is at least

$$2^{2^{|X|}} \times (2^{2^{|X|}} - 1) = 2^{2^{|X|}},$$

which is the cardinality of $\Theta(X)$. ☐

As a result of the above proposition, we have the following neat characterization of the proximally non-discrete anti-atoms of $\Theta(X)$.

Theorem 5.3.20. *A quasi-uniformity $\mathcal{A}$ on X is a proximally non-discrete anti-atom of $\Theta(X)$ if and only if it has the form $\text{fil}(\{H \cup \Delta \mid H \in \mathcal{H}\})$ for some ultrafilter $\mathcal{H}$ on $X \times X$ such that $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{H}$ for two distinct ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on X.*

Proof. In this proof we will use the following notation: If $\mathcal{U}$ is a non-discrete quasi-uniformity on X, we denote

$$\mathcal{H}_{\mathcal{U}} = \text{fil}(\{U - \Delta \mid U \in \mathcal{U}\}),$$

which is a filter on $X \times X$. If, on the other hand, $\mathcal{H}$ is a filter on $X \times X$, we denote

$$\mathcal{U}_{\mathcal{H}} = \text{fil}(\{H \cup \Delta \mid H \in \mathcal{H}\}).$$

Note that $\mathcal{U} = \mathcal{U}_{\mathcal{H}_{\mathcal{U}}}$ for every quasi-uniformity $\mathcal{U}$ on X, and $\mathcal{H} \subseteq \mathcal{H}_{\mathcal{U}_{\mathcal{H}}}$.

If $\mathcal{A}$ is a proximally non-discrete anti-atom of $\Theta(X)$, $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$ for two distinct ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on X by Proposition 5.3.18. By Remark 5.3.11(1), $\mathcal{F} \times \mathcal{G}$ has a base consisting only of sets not intersecting the diagonal, and therefore $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{H}_{\mathcal{A}}$. If $\mathcal{H}$ is another filter on $X \times X$ such that

$\mathcal{H}_\mathcal{A} \subseteq \mathcal{H}$, then $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{H}$. Note that $\Delta \notin \mathcal{H}$, since $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{H}$ implies that $\mathcal{H}$ also has a base consisting of sets not intersecting the diagonal, and hence $\mathcal{U}_\mathcal{H} \neq \mathcal{D}$. Since we also have $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{U}_\mathcal{H}$, it follows by Remark 5.3.11(2) that $\mathcal{U}_\mathcal{H}$ is a quasi-uniformity. But $\mathcal{A} = \mathcal{U}_{\mathcal{H}_\mathcal{A}} \subseteq \mathcal{U}_\mathcal{H}$, and hence $\mathcal{A} = \mathcal{U}_\mathcal{H}$. Therefore $\mathcal{H}_\mathcal{A} = \mathcal{H}_{\mathcal{U}_\mathcal{H}} \supseteq \mathcal{H}$, proving that $\mathcal{H}_\mathcal{A}$ is an ultrafilter on $X \times X$.

Conversely, suppose that $\mathcal{H}$ is an ultrafilter on $X \times X$ and that $\mathcal{F}$ and $\mathcal{G}$ are distinct ultrafilters on X such that $\mathcal{F}\times\mathcal{G} \subseteq \mathcal{H}$. Then $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{U}_\mathcal{H}$, $\mathcal{U}_\mathcal{H} \neq \mathcal{D}$ by Remark 5.3.11(1) and $\mathcal{U}_\mathcal{H}$ is a quasi-uniformity by Remark 5.3.11(2). If $\mathcal{V}$ is another non-discrete quasi-uniformity such that $\mathcal{U}_\mathcal{H} \subseteq \mathcal{V}$, then $\mathcal{H} \subseteq \mathcal{H}_{\mathcal{U}_\mathcal{H}} \subseteq \mathcal{H}_\mathcal{V}$, so $\mathcal{H} = \mathcal{H}_\mathcal{V}$ and $\mathcal{U}_\mathcal{H} = \mathcal{U}_{\mathcal{H}_\mathcal{V}} = \mathcal{V}$. Therefore $\mathcal{U}_\mathcal{H}$ is an anti-atom, and it is proximally non-discrete since $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{U}_\mathcal{H}$. □

A natural question to ask is whether, if $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters on X, $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ could ever be an anti-atom in $\Theta(X)$. In light of the above theorem, the answer is quite intuitive:

Corollary 5.3.21. *Let $\mathcal{F}$ and $\mathcal{G}$ be ultrafilters on X. Then $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ is an anti-atom of $\Theta(X)$ iff $\mathcal{F} \times \mathcal{G}$ is an ultrafilter on $X \times X$ and $\mathcal{F} \neq \mathcal{G}$.*

Proof. Suppose first that $\mathcal{F} \neq \mathcal{G}$ and $\mathcal{F} \times \mathcal{G}$ is an ultrafilter on $X \times X$. Then it follows immediately from Theorem 5.3.20 that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ is an anti-atom of $\Theta(X)$.

Note that if $\mathcal{F} = \mathcal{G}$, $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} = \mathcal{U}_\mathcal{F}$ and hence cannot be an anti-atom because it is a uniformity (see Corollary 5.3.8).

Now suppose that $\mathcal{F}\times\mathcal{G}$ is not an ultrafilter on $X\times X$. Suppose that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ was a (proximally non-discrete) anti-atom. Then by Theorem 5.3.20, $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} = \text{fil}(\{H \cup \Delta \mid H \in \mathcal{H}\})$ for some ultrafilter $\mathcal{H}$ on $X \times X$ such that $\Delta \notin \mathcal{H}$. By Remark 5.3.11(1), $\mathcal{F} \times \mathcal{G}$ has a base of sets not intersecting the diagonal, and since $\Delta \notin \mathcal{H}$, $(X \times X) - \Delta \in \mathcal{H}$. Hence $\mathcal{F} \times \mathcal{G} = \text{fil}(\{H - \Delta \mid H \in \mathcal{H}\}) = \mathcal{H}$. But then $\mathcal{F} \times \mathcal{G}$ is an ultrafilter, a contradiction. □

Suppose that X is countable. Recall Proposition 4.3.14, which states if $\mathcal{F}$ is an ultrafilter on X, $\mathcal{U}_\mathcal{F}$ is an anti-atom in $\Psi(X)$ iff $\mathcal{F}$ is selective. We now show that, provided that X is countable, $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ is an anti-atom in $\Theta(X)$ if and only if the ultrafilters $\mathcal{F}$ and $\mathcal{G}$ are distinct and either $\mathcal{F}$ or $\mathcal{G}$ is principal. The following preliminaries are needed to do this. They also introduce an example of a proximally non-discrete anti-atom in $\Theta(X)$ that is not (in general) semi-trivial.

Definition 5.3.22. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are filters on X. We have already defined a type of product between filters, namely $\mathcal{F} \times \mathcal{G}$. We now introduce another – define

$$\mathcal{F} \cdot \mathcal{G} = \{A \subseteq X \times X \mid \{x \mid A(x) \in \mathcal{G}\} \in \mathcal{F}\}.$$

$\mathcal{F} \cdot \mathcal{G}$ is a filter on $X \times X$ such that $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{F} \cdot \mathcal{G}$. It is also not too difficult to check that if $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters on X, then $\mathcal{F} \cdot \mathcal{G}$ is an ultrafilter on $X \times X$.[9]

Example 5.3.23. Suppose that $\mathcal{F}$ and $\mathcal{G}$ are distinct ultrafilters on X. Since $\mathcal{F} \cdot \mathcal{G}$ is an ultrafilter and $\mathcal{F} \times \mathcal{G} \subsetneq \mathcal{F} \cdot \mathcal{G}$, $\text{fil}(\{H \cup \Delta \mid H \in \mathcal{F} \cdot \mathcal{G}\})$ is a proximally non-discrete anti-atom in $\Theta(X)$ by Theorem 5.3.20.

Lemma 5.3.24. *If $\mathcal{F}$ and $\mathcal{G}$ are countably incomplete ultrafilters on a set X, $\mathcal{F} \times \mathcal{G}$ is not an ultrafilter.*

Proof. (Adapted from [9, Corollary 7.24]) Suppose neither $\mathcal{F}$ nor $\mathcal{G}$ are countably complete. Then there exist chains $(F_n)_{n\in\omega}$ and $(G_n)_{n\in\omega}$ in $\mathcal{F}$ and $\mathcal{G}$ respectively such that

$$F_1 \supsetneq F_2 \supsetneq F_3 \supsetneq \ldots,$$
$$G_1 \supsetneq G_2 \supsetneq G_3 \supsetneq \ldots,$$

and $\bigcap_{n\in\omega} F_n = \bigcap_{n\in\omega} G_n = \emptyset$.[10] For each $n \in \omega$, let $A_n = F_n - F_{n+1}$. Let $A = \bigcup_{n\in\omega}(A_n \times G_n)$. Then $\{x \mid A(x) \in \mathcal{G}\} = F_1 \in \mathcal{F}$, so $A \in \mathcal{F} \cdot \mathcal{G}$. We show that $A \notin \mathcal{F} \times \mathcal{G}$.

Suppose we could find an $F \in \mathcal{F}$ and a $G \in \mathcal{G}$ such that $F \times G \subseteq A$. Suppose that for each $n \in \omega$ there is an $x_n \in F_n$ such that $x_n \in F$. Since $A(x_n) \subseteq G_n$ for each $n \in \omega$, it follows that $G \subseteq G_n$ for each $n \in \omega$. So $G = \emptyset$, a contradiction. Hence there must be a maximum $n \in \omega$ such that there is an $x \in F_n \cap F$. Therefore, $F \cap F_{n+1} = \emptyset$, contradicting that $F \in \mathcal{F}$.

Hence $A \notin \mathcal{F} \times \mathcal{G}$, so $\mathcal{F} \times \mathcal{G} \subsetneq \mathcal{F} \cdot \mathcal{G}$ and therefore $\mathcal{F} \times \mathcal{G}$ is not an ultrafilter. □

Corollary 5.3.25. *Let X be countable and $\mathcal{F}$ and $\mathcal{G}$ be ultrafilters on X. Then $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ is a (proximally non-discrete) anti-atom of $\Theta(X)$ if and only if either $\mathcal{F}$ or $\mathcal{G}$ is principal and $\mathcal{F} \neq \mathcal{G}$.*

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[9]See [9, Lemma 7.20] for the details on $\mathcal{F} \cdot \mathcal{G}$.

[10]There must be a decreasing chain $(E_n)_{n\in\omega}$ in $\mathcal{F}$ such that $E = \bigcap_{n\in\omega} E_n \notin \mathcal{F}$. Hence $X - E \in \mathcal{F}$, and if we set $F_n = E_n \cap (X - E)$ $\forall n \in \omega$, then $(F_n)_{n\in\omega}$ is such a chain in $\mathcal{F}$.

Proof. Suppose that both $\mathcal{F}$ and $\mathcal{G}$ are not principal. Then by the above lemma $\mathcal{F} \times \mathcal{G}$ is not an ultrafilter, and hence by Corollary 5.3.21, $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ is not an anti-atom. It also follows from Corollary 5.3.21 that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ cannot be an anti-atom if $\mathcal{F} = \mathcal{G}$.

For the converse, if $\mathcal{F} \neq \mathcal{G}$ and either $\mathcal{F}$ or $\mathcal{G}$ is principal, then $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ is a semi-trivial anti-atom. □

5.3.3 Proximally Fine Anti-Atoms

Recall that a quasi-uniformity is called *proximally fine* iff it is the finest quasi-uniformity inducing its quasi-proximity, i.e. iff it is the finest member of its quasi-proximity class (Definition 2.3.6). Note that not all quasi-proximity classes need have a finest member. Note also that since $\mathcal{D}$ is the finest member of its proximity class, no proximally discrete anti-atom will be proximally fine. Hence, our attention in this section will be restricted to the proximally non-discrete anti-atoms of $\Theta(X)$. Our goal is to give a characterization of the proximally fine anti-atoms, which is achieved in Theorem 5.3.32.

First, a simple fact regarding proximally fine anti-atoms:

Remark 5.3.26. If $\mathcal{A}$ is a proximally fine anti-atom of $\Theta(X)$, then so is $\mathcal{A}^{-1}$. For suppose that $\mathcal{V}$ is a quasi-uniformity such that $\mathcal{V}_\omega = (\mathcal{A}^{-1})_\omega$. Then $(\mathcal{V}^{-1})_\omega = \mathcal{A}_\omega$ (by Proposition 2.3.14). Hence, since $\mathcal{A}$ is proximally fine, $\mathcal{V}^{-1} \subseteq \mathcal{A}$, so $\mathcal{V} \subseteq \mathcal{A}^{-1}$.

In order to give the promised characterization of the proximally fine anti-atoms of $\Theta(X)$, the following preliminary results are needed.

Lemma 5.3.27. *If $\mathcal{F}$ and $\mathcal{G}$ are filters on a set X such that $\mathcal{F} \subsetneq \mathcal{G}$, there is an ultrafilter $\mathcal{H}$ on X such that $\mathcal{F} \subseteq \mathcal{H}$ and $\mathcal{G} \not\subseteq \mathcal{H}$.*

Proof. [9, Lemma 7.17] Choose $A \in \mathcal{G} - \mathcal{F}$. Then $\mathcal{F} \cup \{X - A\}$ is a subbase for a filter on X. For suppose not. Then there exists a $B \in \mathcal{F}$ such that $B \cap (X - A) = \emptyset$. But then $B \subseteq A$, a contradiction. Hence, if $\mathcal{H}$ is an ultrafilter containing the filter generated by $\mathcal{F} \cup \{X - A\}$, we have $\mathcal{F} \subseteq \mathcal{H}$ but $\mathcal{G} \not\subseteq \mathcal{H}$ (since $A \in \mathcal{G}$ and $X - A \in \mathcal{H}$). □

Corollary 5.3.28. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are ultrafilters on X. Then if $\mathcal{F} \times \mathcal{G}$ is not an ultrafilter on $X \times X$, there are at least two distinct anti-atoms of $\Theta(X)$ containing $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$.*

Proof. If $\mathcal{F} = \mathcal{G}$, this has already been proven in Remark 5.3.17.

Now suppose that $\mathcal{F} \neq \mathcal{G}$. Suppose $\mathcal{F} \times \mathcal{G}$ is not an ultrafilter on $X \times X$. Then $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ is not an anti-atom by Corollary 5.3.21. Let $\mathcal{A}$ be an anti-atom containing $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$. Since $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$, $\mathcal{A}$ is proximally non-discrete. Hence we can write $\mathcal{A} = \text{fil}(\{H \cup \Delta \mid H \in \mathcal{H}\})$ for $\mathcal{H}$ an ultrafilter on $X \times X$ such that $\mathcal{J} \times \mathcal{L} \subseteq \mathcal{H}$ for two distinct ultrafilters $\mathcal{J}$ and $\mathcal{L}$ on X (see Theorem 5.3.20). $\mathcal{F} = \mathcal{J}$ and $\mathcal{G} = \mathcal{L}$ since otherwise $\mathcal{A} = \mathcal{D}$. $\mathcal{F} \times \mathcal{G} \neq \mathcal{H}$ and hence by Lemma 5.3.27 there exists an ultrafilter $\mathcal{K}$ on $X \times X$ such that $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{K}$ but $\mathcal{H} \neq \mathcal{K}$. By Remark 5.3.11(1), $\mathcal{H}$ and $\mathcal{K}$ both have bases consisting of sets none of which contain any points on the diagonal, because $\mathcal{F} \times \mathcal{G}$ does. Hence, if we write $\mathcal{E} = \text{fil}(\{K \cup \Delta \mid K \in \mathcal{K}\})$, $\mathcal{E}$ is an anti-atom of $\Theta(X)$ such that $\mathcal{A} \neq \mathcal{E}$. Since $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{E}$, the proof is complete. □

Remark 5.3.29. Note that in Remark 5.3.17 we showed that $\mathcal{U}_{\mathcal{F}}$ will never be contained by a unique anti-atom in $\Theta(X)$. It follows by the above result and Corollary 5.3.21 that if $\mathcal{F} \neq \mathcal{G}$, neither will $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$, unless it is itself an anti-atom.

Lemma 5.3.30. *Suppose $\mathcal{U}$ is a non-discrete quasi-uniformity such that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{U}$ for ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on X. Then $\mathcal{U}$ and $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ induce the same quasi-proximity.*

Proof. Suppose that A and B are non-empty proper subsets of X. Clearly, if A is far from B with respect to $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$, then A is also far from B with respect to $\mathcal{U}$.

Now suppose that A is far from B with respect to $\mathcal{U}$. Then $A \cap B = \emptyset$. We have that either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$. If $X - A \in \mathcal{F}$ then $((X - A) \times X) \cup \Delta \in \mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ and hence A is far from B with respect to $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$. If $A \in \mathcal{F}$, we cannot have $B \in \mathcal{G}$ since otherwise $(A \times B) \cup \Delta \in \mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{U} \neq \mathcal{D}$, contradicting that A is far from B in $\mathcal{U}$. Therefore $X - B \in \mathcal{G}$, $(A \times (X - B)) \cup \Delta \in \mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ and hence A is far from B in $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$.

We have shown that A is far from B in $\mathcal{U}$ iff A is far from B in $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$. Hence the quasi-proximities induced by $\mathcal{U}$ and $\mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ respectively are the same. □

Corollary 5.3.31. *Suppose that $\mathcal{A}$ is a proximally non-discrete anti-atom in $\Theta(X)$ and that $\mathcal{F}$ and $\mathcal{G}$ are the distinct ultrafilters on X such that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$. Then $\mathcal{A}$ is proximally fine iff $\mathcal{F} \times \mathcal{G}$ is an ultrafilter on $X \times X$.*

Proof. Suppose first that $\mathcal{F} \times \mathcal{G}$ is an ultrafilter on $X \times X$. Then, by Corollary 5.3.21, $\mathcal{A} = \mathcal{U}_{\mathcal{F}\times\mathcal{G}}$. Suppose that $\mathcal{V}$ is a quasi-uniformity that induces the same quasi-proximity as $\mathcal{A}$. We need to show that $\mathcal{V} \subseteq \mathcal{A}$. Suppose not. Then there is a $V \in \mathcal{V}$ such that $V \cap ((A \times B) \cup \Delta) = \Delta$ for some $A \in \mathcal{F}$ and $B \in \mathcal{G}$, since $\mathcal{A} = \mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ and $\mathcal{A}$ is an anti-atom. Since $\mathcal{F}$ and $\mathcal{G}$ are distinct ultrafilters, we may assume that $A \cap B = \emptyset$, and hence $V \cap (A \times B) = \emptyset$. Hence A is far from B in $\mathcal{V}$, but clearly A is near B in $\mathcal{A}$. Therefore $\mathcal{A}$ and $\mathcal{V}$ induce different quasi-proximities, a contradiction. Hence $\mathcal{V} \subseteq \mathcal{A}$ as claimed, proving that $\mathcal{A}$ is proximally fine.

Now suppose that $\mathcal{F} \times \mathcal{G}$ is not an ultrafilter on $X \times X$. Then $\mathcal{A} \neq \mathcal{U}_{\mathcal{F}\times\mathcal{G}}$ by Corollary 5.3.21, and hence by Corollary 5.3.28 there is at least one other anti-atom $\mathcal{E}$ of $\Theta(X)$ such that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{E}$. By Lemma 5.3.30, $\mathcal{A}$ and $\mathcal{E}$ induce the same quasi-proximity, and hence $\mathcal{A}$ is not proximally fine. □

Theorem 5.3.32. *Suppose that $\mathcal{A}$ is a proximally non-discrete anti-atom of $\Theta(X)$, and let $\mathcal{F}$ and $\mathcal{G}$ be the distinct ultrafilters on X such that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$. Then the following are equivalent:*

1. *$\mathcal{A}$ is proximally fine.*
2. *$\mathcal{F} \times \mathcal{G}$ is an ultrafilter on $X \times X$.*
3. *$\mathcal{A} = \mathcal{U}_{\mathcal{F}\times\mathcal{G}}$.*

Proof. Combine Corollaries 5.3.31 and 5.3.21 □

Suppose that X is countable, and let $\mathcal{F}$ be an ultrafilter on X. Recall that in Theorem 4.3.24, even though no necessary condition was given for the uniform anti-atom $\mathcal{J}_{\mathcal{F}}$ to be proximally fine, [46] was able to give a sufficient condition, namely that $\mathcal{F}$ be selective. For $\Theta(X)$, however, we are able to give the following characterization of the proximally fine anti-atoms for countable X. Note its simplicity compared to the corresponding results for the lattice of uniformities - as we noted in Remark 4.3.13, it is even consistent with set theory that there exist no selective ultrafilters, and we need to assume a condition such as CH to ensure their existence.

Corollary 5.3.33. *Suppose that X is countable. A proximally non-discrete anti-atom $\mathcal{A}$ of $\Theta(X)$ is proximally fine iff it is semi-trivial.*

Proof. Combine Theorem 5.3.32 with Corollary 5.3.25. □

5.4 Adjacent Quasi-Uniformities in $\Theta(X)$

In this section we aim to establish which quasi-uniformities will have immediate predecessors and which will have immediate successors in $\Theta(X)$. An important area of study that also falls into this section is that of the distribution of uniformities in $\Theta(X)$. In particular, we will be considering the question of whether between any two distinct uniformities there is always a non-symmetric quasi-uniformity.

5.4.1 Immediate Successors

In this section we show that not every non-discrete quasi-uniformity has an immediate successor in $\Theta(X)$.

Before we give an example of a quasi-uniformity without an immediate successor, we give a general description of immediate successors in $\Theta(X)$. The below proposition is the quasi-uniform analogue of Proposition 4.4.7 for uniformities.

Proposition 5.4.1. *Let $\mathcal{V}$ be an immediate successor of $\mathcal{U}$ in $\Theta(X)$. Then there exists a quasi-pseudo-metric d on X such that $\mathcal{V} = \mathcal{U} \vee \mathcal{U}_d$.*

Proof. Since $\mathcal{U} \subsetneq \mathcal{V}$ there must be a quasi-pseudo-metric d on X such that $\mathcal{U}_d \subseteq \mathcal{V}$ but $\mathcal{U}_d \not\subseteq \mathcal{U}$. Hence $\mathcal{U} \subsetneq \mathcal{U} \vee \mathcal{U}_d \subseteq \mathcal{V}$. Since $\mathcal{V}$ is an immediate successor of $\mathcal{U}$ it follows that $\mathcal{V} = \mathcal{U} \vee \mathcal{U}_d$. □

Example 4.4.8 shows that the converse of the above proposition need not hold.

Recall that in Example 4.4.6 we showed that if X is infinite, then $\mathcal{D}_\omega$ has no immediate successor in $\Psi(X)$. We now prove that for infinite X, $\mathcal{D}_\omega$ does not have an immediate successor in $\Theta(X)$ either.

Lemma 5.4.2. *Let $\mathcal{U}$ be a quasi-uniformity on X. Then $\mathcal{U}$ is totally bounded if and only if both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are hereditarily precompact.*

Proof. [27, Lemma 1.1]. □

Example 5.4.3. Let X be an infinite set. Let $\mathcal{V}$ be a quasi-uniformity on X such that $\mathcal{D}_\omega \subsetneq \mathcal{V}$. Because $\mathcal{D}_\omega$ is a uniformity, we also have $\mathcal{D}_\omega \subsetneq \mathcal{V}^{-1}$. Since

$\mathcal{V}$ belongs to the proximity class of $\mathcal{D}_\omega$, it is not totally bounded. Hence, by the above lemma, either $\mathcal{V}^{-1}$ or $\mathcal{V}$ is not hereditarily precompact. Suppose that it is $\mathcal{V}^{-1}$. We will construct a quasi-uniformity $\mathcal{Q}$ such that $\mathcal{D}_\omega \subsetneq \mathcal{Q} \subsetneq \mathcal{V}$. By symmetry it will then follow that if it is $\mathcal{V}$ that is not hereditarily precompact, there is a quasi-uniformity $\mathcal{Q}$ such that $\mathcal{D}_\omega \subsetneq \mathcal{Q} \subsetneq \mathcal{V}^{-1}$ and hence $\mathcal{D}_\omega \subsetneq \mathcal{Q}^{-1} \subsetneq \mathcal{V}$.

So suppose that $\mathcal{V}^{-1}$ is not hereditarily precompact. Then for some $V_0 \in \mathcal{V}$ it is possible to construct a sequence $(x_n)_{n\in\omega}$ such that

$x_p \notin V_0^{-1}(x_k)$ whenever $k < p < \aleph_0$, or otherwise put,
$x_k \notin V_0(x_p)$ whenever $k < p < \aleph_0$. $(\star)$

For each $n \in \omega$ set

$$A_n = \{x_s \mid n^2 \leq s < (n+1)^2\}.$$

Note that the A_n are pairwise disjoint. Also, for each $V \in \mathcal{V}$, define

$$M_V = V \cup \bigcup\{V^{-1}(A_n) \times V(A_k) \mid n \leq k < \aleph_0\},$$

and set $\mathcal{H} = \text{fil}(\{M_V \mid V \in \mathcal{V}\})$.

We show that $\mathcal{H}$ is a quasi-uniformity on X. Let $V \in \mathcal{V}$ be given and choose $H \in \mathcal{V}$ such that $H^2 \subseteq V_0 \cap V$. Note first that $H(A_k) \cap H^{-1}(A_n) = \emptyset$ whenever $n < k < \aleph_0$, since otherwise $x_p \in H^2(x_r) \subseteq V_0(x_r)$ for some $p < r < \aleph_0$, contradicting $(\star)$. With this fact in mind, a straightforward computation now shows that $(M_H)^2 \subseteq M_V$. Hence $\mathcal{H}$ is a quasi-uniformity on X, as claimed.

Set $\mathcal{Q} = \mathcal{D}_\omega \vee \mathcal{H}$. It is clear that $\mathcal{D}_\omega \subseteq \mathcal{Q} \subseteq \mathcal{V}$. We have that $(x_{k^2}, x_{n^2}) \notin M_{V_0}$ whenever $n < k < \aleph_0$: Suppose not. Then $(x_{k^2}, x_{n^2}) \in V_0^{-1}(A_p) \times V_0(A_q)$ for some $p \leq q < \aleph_0$. Now $x_{n^2} \in V_0(A_q)$ implies that $(x_m, x_{n^2}) \in V_0$ for some m such that $q^2 \leq m < (q+1)^2$ and $n^2 \geq m$, and hence $n \geq q$. Similarly $p \geq k$ and hence $n \geq q \geq p \geq k$, so $n \geq k$.

It follows that $\{x_{k^2} \mid k \in \omega\}$ is not a precompact subspace of $(X, \mathcal{Q}^{-1})$. But it is a precompact subspace of $(X, (\mathcal{D}_\omega)^{-1})$, because $\mathcal{D}_\omega$ is totally bounded. Hence $\mathcal{Q} \neq \mathcal{D}_\omega$

We now show that $\mathcal{Q} \neq \mathcal{V}$. For the sake of contradiction, suppose that $V_0 \in \mathcal{Q}$. Then there is a $U \in \mathcal{D}_\omega$ and $H \in \mathcal{V}$ such that $U \cap M_H \subseteq V_0$. Since $\mathcal{D}_\omega$ is totally bounded, there is a finite cover $\{D_i \mid i < n\}$ of X for some $n \in \omega$ such that $D_i \times D_i \subseteq U$ whenever $i < n$. Then since A_n has more than n elements, there is a $j < n$ such that D_j contains two distinct members of A_n, say x_s and x_r for $s < r < \aleph_0$. Since $A_n \times A_n \subseteq M_H$, we have $(x_r, x_s) \in U \cap M_H$. Since $(x_r, x_s) \notin V_0$ by $(\star)$, this is a contradiction. Hence $V_0 \notin \mathcal{Q}$ and therefore $\mathcal{Q} \neq \mathcal{V}$.

Hence $\mathcal{D}_\omega \subsetneq \mathcal{Q} \subsetneq \mathcal{V}$, proving that $\mathcal{D}_\omega$ has no immediate successor in $\Theta(X)$.

5.4.2 Immediate Predecessors

We will show that, as was the case in $\Psi(X)$, every non-indiscrete *uniformity* has an immediate predecessor in $\Theta(X)$. Unfortunately, every *quasi*-uniformity need not, and we will give an example of such a quasi-uniformity.

We start by mentioning certain types of non-indiscrete quasi-uniformities which will always have immediate predecessors in $\Theta(X)$. The first result is a basic consequence of Zorn's Lemma.

Proposition 5.4.4. *Suppose that $\mathcal{U}$ is a quasi-uniformity generated by a pre-order $T \neq X \times X$ on X. Then it has an immediate predecessor in $\Theta(X)$.*

Proof. Let $\mathcal{S} := \{\mathcal{V} \in \Theta(X) \mid \mathcal{V} \subseteq \mathcal{U}, T \notin \mathcal{V}\}$. $\mathcal{S}$ is non-empty because $\mathcal{I} \in \mathcal{S}$. Let $\mathcal{C}$ be a non-empty chain in $\mathcal{S}$. Then $\bigvee \mathcal{C}$ is an upper bound for $\mathcal{C}$. $T \notin \bigvee \mathcal{C}$ since otherwise $T \in \mathcal{V}$ for some $\mathcal{V} \in \mathcal{C}$,[11] and hence $\bigvee \mathcal{C} \in \mathcal{S}$. By Zorn's Lemma it follows that $\mathcal{S}$ must have a maximal element $\mathcal{V}$. Suppose $\mathcal{W}$ is a quasi-uniformity such that $\mathcal{V} \subsetneq \mathcal{W} \subseteq \mathcal{U}$. Then since $\mathcal{W}$ contains T and $\mathcal{U}$ is the smallest quasi-uniformity containing T, $\mathcal{W} = \mathcal{U}$. Hence $\mathcal{V}$ is an immediate predecessor of $\mathcal{U}$. □

Note that if $T = \Delta$, then the immediate predecessor found above is an immediate predecessor of $\mathcal{D}$, and hence an anti-atom (compare Proposition 5.3.1).

Another type of non-indiscrete quasi-uniformity which will always have an immediate predecessor in $\Theta(X)$, namely the doubly point-symmetric quasi-uniformity, is defined below. Directly below this definition we show every such quasi-uniformity has an immediate predecessor, and from this it will follow that every uniformity has an immediate predecessor in $\Theta(X)$.

[11] For a detailed proof, see for example Proposition 5.3.1, where this is done for the special case of $T = \Delta$.

Definition 5.4.5. A quasi-uniformity $\mathcal{U}$ is called *point-symmetric* if $\mathcal{T}(\mathcal{U}) \subseteq \mathcal{T}(\mathcal{U}^{-1})$. It is called *doubly point-symmetric* if both $\mathcal{U}$ and $\mathcal{U}^{-1}$ are point-symmetric, i.e. if $\mathcal{T}(\mathcal{U}) = \mathcal{T}(\mathcal{U}^{-1})$.

Lemma 5.4.6. *Each doubly point-symmetric quasi-uniformity $\mathcal{U} \neq \mathcal{I}$ on a set X has an immediate predecessor in $\Theta(X)$.*

Proof. Suppose that $\mathcal{U} \neq \mathcal{I}$. Then there exists a $V \in \mathcal{U}$ such that for some $x, y \in X$, $(x, y) \notin V$. We show that

$$\mathcal{U} \wedge \mathcal{G}_{(x,y)} = \text{fil}(\{H \cup (H^{-1}(x) \times H(y)) \mid H \in \mathcal{U}\}). \ (\star)$$

It is clear that $\text{fil}(\{H \cup (H^{-1}(x) \times H(y)) \mid H \in \mathcal{U}\}) \subseteq \mathcal{U} \wedge \mathcal{G}_{(x,y)}$. Suppose that $U \in \mathcal{U} \wedge \mathcal{G}_{(x,y)}$, and choose $W \in \mathcal{U} \wedge \mathcal{G}_{(x,y)}$ such that $W^3 \subseteq U$. Since $W \in \mathcal{U} \cap \mathcal{G}_{(x,y)}$ there must be a $P \in \mathcal{U}$ such that $P \cup \{(x, y)\} \subseteq W$. Hence $(P^{-1}(x) \times P(y)) \cup P \subseteq W^3 \subseteq U$, and therefore $U \in \text{fil}(\{H \cup (H^{-1}(x) \times H(y)) \mid H \in \mathcal{U}\})$. So $\mathcal{U} \wedge \mathcal{G}_{(x,y)} = \text{fil}(\{H \cup (H^{-1}(x) \times H(y)) \mid H \in \mathcal{U}\})$ as claimed.

Clearly $\mathcal{U} \wedge \mathcal{G}_{(x,y)} \subsetneq \mathcal{U}$. Suppose now that $\mathcal{Q}$ is a quasi-uniformity such that $\mathcal{U} \wedge \mathcal{G}_{(x,y)} \subsetneq \mathcal{Q} \subseteq \mathcal{U}$. Since $\mathcal{Q} \not\subseteq \mathcal{G}_{(x,y)}$ there must be a $Q_0 \in \mathcal{Q}$ such that $(x, y) \notin Q_0$. Choose $Q_1 \in \mathcal{Q}$ such that $(Q_1)^3 \subseteq Q_0$. Since $(x, y) \notin Q_0$ it follows that $(Q_1(x) \times Q_1^{-1}(y)) \cap Q_1 = \emptyset$. (†)

Consider any $U \in \mathcal{U}$. Since $Q_1 \in \mathcal{U}$ and $\mathcal{U}$ is doubly point-symmetric, there is a $V \in \mathcal{U}$ such that $V \subseteq (Q_1 \cap U)$, $V^{-1}(x) \subseteq Q_1(x)$ and $V(y) \subseteq Q_1^{-1}(y)$. Then since $\mathcal{U} \wedge \mathcal{G}_{(x,y)} \subsetneq \mathcal{Q} \subseteq \mathcal{U}$, we have $V \cup (V^{-1}(x) \times V(y)) \in \mathcal{Q}$ (see ($\star$)). But $(V \cup (V^{-1}(x) \times V(y))) \cap Q_1 \subseteq (V \cup (Q_1(x) \times Q_1^{-1}(y))) \cap Q_1 = V \subseteq U$ from (†). Hence $U \in \mathcal{Q}$ and therefore $\mathcal{U} = \mathcal{Q}$.

Consequently $\mathcal{U} \wedge \mathcal{G}_{(x,y)}$ must be an immediate predecessor of $\mathcal{U}$. □

Corollary 5.4.7. *If $\mathcal{U}$ is a uniformity other than $\mathcal{I}$ on a set X, it has an immediate predecessor in $\Theta(X)$.*

Proof. This follows directly from the above lemma. □

The above corollary can be seen as a partial generalization of Theorem 4.4.10, which says that every non-indiscrete uniformity has an immediate predecessor in the lattice of uniformities. Of course, the next question is whether this result generalizes completely, i.e. whether every quasi-uniformity has an immediate predecessor in the lattice of quasi-uniformities. As was mentioned earlier, though, this need unfortunately not be the case:

Example 5.4.8. Let $X = \mathbb{R}$, $a, b \in \mathbb{R}$ and set

$$A_{a,b} = ([a, \infty) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, b)).$$

Then $\mathcal{V} = \text{fil}(\{A_{a,b} \mid a, b \in \mathbb{R}, a < b\})$ is a quasi-uniformity on $\mathbb{R}$: we show that $(A_{a,\frac{a+b}{2}} \cap A_{\frac{a+b}{2},b})^2 \subseteq A_{a,b}$.

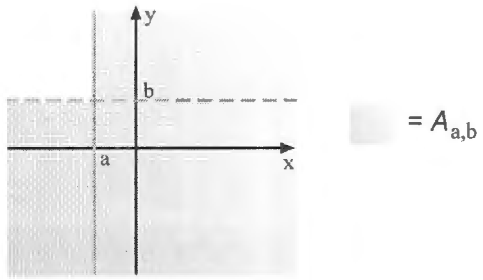

Figure 5.4: Diagrammatical representation of the subset $A_{a,b}$ of $\mathbb{R}^2$.

For the sake of contradiction, suppose that

$$(x, y) \in (A_{a,\frac{a+b}{2}} \cap A_{\frac{a+b}{2},b}) \text{ and } (y, z) \in (A_{a,\frac{a+b}{2}} \cap A_{\frac{a+b}{2},b}) \; (\star)$$

but $(x, z) \notin A_{a,b}$. Then $x < a$ and $z \geq b$. By the assumptions at $(\star)$ this means that $y < \frac{a+b}{2}$ and $y \geq \frac{a+b}{2}$, a contradiction. So $\mathcal{V}$ is in fact a quasi-uniformity on $\mathbb{R}$.

Suppose that $\mathcal{U}$ is a quasi-uniformity on $\mathbb{R}$ that is strictly coarser than $\mathcal{V}$. Then there are $a, b \in \mathbb{R}$ such that $a < b$ and $A_{a,b} \notin \mathcal{U}$. Let $\mathcal{H} = \mathcal{U} \vee \mathcal{W}$ where $\mathcal{W}$ is the quasi-uniformity on $\mathbb{R}$ with base $\{A_{c,d} \mid a \leq c < d \leq \frac{a+b}{2}\}$. We have $A_{a,\frac{a+b}{2}} \in \mathcal{W}$ and $A_{a,\frac{a+b}{2}} \subseteq A_{a,b}$, and therefore $A_{a,b} \in \mathcal{W} - \mathcal{U}$. This shows that $\mathcal{U} \subsetneq \mathcal{H}$.

We aim to show that $A_{\frac{a+b}{2},b} \notin \mathcal{H}$ in order to obtain $\mathcal{H} \subsetneq \mathcal{V}$. To do this we need the following fact:

> Suppose that for some $n \in \omega$ and for each $1 \leq i \leq n$, $c_i < d_i$, and $\bigcap_{i=1}^{n} A_{c_i,d_i} \subseteq A_{c,d}$ for some $c, d \in \mathbb{R}$. Then there is an $i \in \{1, ..., n\}$ such that $c \leq c_i < d_i \leq d$.

A quick proof: Assume the contrary. Set $h = \max\{c_i \mid d_i \leq d, i = 1, ..., n\}$ (if the set is empty, set $h = -\infty$). We always have $c_i < d_i$. Hence, if $h = c_i$, then $h < c$ by assumption, and if $h = -\infty$, $h < c$ too. We have $\bigcap_{i=1}^{n} A_{c_i,d_i}^{-1}(d) \supseteq (h, \infty)$. But since $A_{c,d}^{-1}(d) = [c, \infty)$, this is a contradiction because $h < c$. Hence the claim is verified.

Suppose now that $A_{\frac{a+b}{2},b} \in \mathcal{H}$. Then by the above claim there is an $A_{c,d} \in \mathcal{U} \cup \mathcal{W}$ such that $\frac{a+b}{2} \leq c < d \leq b$, because $\mathcal{H} = \mathcal{U} \vee \mathcal{W}$. By definition of $\mathcal{W}$ it follows that $A_{c,d} \notin \mathcal{W}$, so $A_{c,d} \in \mathcal{U}$. Since $A_{c,d} \subseteq A_{a,b}$ it follows that $A_{a,b} \in \mathcal{U}$, a contradiction.

Hence we must have $\mathcal{U} \subsetneq \mathcal{H} \subsetneq \mathcal{V}$ and it follows that $\mathcal{V}$ has no immediate predecessor in $\Theta(\mathbb{R})$.

5.4.3 The Distribution of Uniformities in $\Theta(X)$

In this section we attempt to answer the following question:

Problem 1. *Given any two uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X such that $\mathcal{U} \subsetneq \mathcal{V}$, is there a non-symmetric quasi-uniformity $\mathcal{Q}$ on X such that $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{V}$?*

A related problem is of course whether it is possible for two uniformities to be adjacent in the lattice of quasi-uniformities. A full answer to the above question has not yet been obtained, but in this section we present some partial solutions.

One approach to solving the above problem is as follows: Suppose $\mathcal{U} \subsetneq \mathcal{V}$. Then there is a pseudo-metric ρ on X such that $\mathcal{U}_\rho \not\subseteq \mathcal{U}$ and $\mathcal{U}_\rho \subseteq \mathcal{V}$. What we need is to find a non-symmetric quasi-uniformity $\mathcal{Q}$ on X such that $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{U} \vee \mathcal{U}_\rho$. The following related result applies this idea to equivalence relations (as opposed to pseudo-metrics).

Proposition 5.4.9. *Let $\mathcal{U}$ be a uniformity and R an equivalence relation on X such that $R \notin \mathcal{U}$. Then there is a non-symmetric quasi-uniformity $\mathcal{Q}$ on X such that $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathrm{fil}(\mathcal{U} \cup \{R\})$.*

Proof. By the axiom of choice, we may assume that the set of equivalence classes of R are linearly ordered by $\lhd$. If $x, y \in X$, we set $x \leq y$ iff $R(x) \lhd R(y)$. Clearly $\leq$ is a pre-order on X, and if $x \leq y$ and $y \leq x$ then $R(x) = R(y)$. Therefore $\leq \cap \geq = R$. Since $\lhd$ is a linear order, we must always have either $x \leq y$ or $y \leq x$, and hence $\leq \cup \geq = X \times X$.

Set $\mathcal{Q} = \mathcal{U} \vee \text{fil}(\leq)$. Since $R \subseteq \leq$ we have $\mathcal{U} \subseteq \mathcal{Q} \subseteq \mathcal{U} \vee \text{fil}(R)$. We show that $\mathcal{Q}$ is non-symmetric. Suppose not. Then $\mathcal{Q} \vee \mathcal{Q}^{-1} = \mathcal{Q}$, and hence there is a $U \in \mathcal{U}$ such that $(U \cap \leq) \subseteq \geq$. Now $\geq \notin \mathcal{U}$, since otherwise $\leq \in \mathcal{U}$ and therefore $R \in \mathcal{U}$, a contradiction. Hence there is a point $(x, y) \in U - \geq$. But then $x \leq y$ since $\leq \cup \geq = X \times X$, and consequently $(x, y) \in (U \cap \leq) \subseteq \geq$, a contradiction.

So $\mathcal{Q}$ cannot be a uniformity, and therefore $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{U} \vee \text{fil}(R)$ as needed. □

Suppose that $R = \Delta$ in the above proposition. Then $R(x) = \{x\}$ for each $x \in X$, and hence the linear order $\triangleleft$ chosen in the proof may in fact be any linear order on X. The proof of Proposition 5.3.7 is therefore just a special case of the above proof.

More special cases in which Problem 1 has been solved are given below. It follows that, if it *is* possible for two distinct but comparable uniformities to have no non-symmetric quasi-uniformity between them, they will belong to the same proximity class, and neither of them will be the totally bounded member of that proximity class.

Proposition 5.4.10. *Let $\mathcal{U}$ and $\mathcal{V}$ be two uniformities on X belonging to distinct proximity classes such that $\mathcal{U} \subsetneq \mathcal{V}$. Then there is a non-symmetric quasi-uniformity $\mathcal{Q}$ on X such that $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{V}$.*

Proof. Since $\mathcal{U}$ and $\mathcal{V}$ belong to distinct proximity classes, there must be sets $A, B \subseteq X$ such that A is near B with respect to $\mathcal{U}$ but A is far from B with respect to $\mathcal{V}$. Let

$$\mathcal{Q} = \mathcal{U} \vee \text{fil}(\{V \cup (V^{-1}(A) \times V(B)) \mid V \in \mathcal{V}\}).$$

Clearly $\mathcal{U} \subseteq \mathcal{Q} \subseteq \mathcal{V}$. We show that $\mathcal{Q}$ is not symmetric. Clearly A is near B with respect to $\mathcal{Q}$. We show that B is not near A in $\mathcal{Q}$. Since $\mathcal{V}$ is a uniformity, there is a symmetric $V \in \mathcal{V}$ such that $V \cap (B \times A) = V(B) \cap A = \emptyset$. Hence $(V \cup (V^{-1}(A) \times V(B))) \cap (B \times A) = \emptyset$. Therefore B is far from A with respect to $\mathcal{Q}$.

The above shows that $\mathcal{Q}$ is non-symmetric, and hence $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{V}$. □

Corollary 5.4.11. *No uniformity on X can be an atom of $\Theta(X)$.*

Proof. If $\mathcal{U}$ is a uniformity such that $\mathcal{U} \neq \mathcal{I}$, then $\mathcal{U}_\omega \neq \mathcal{I}$ (Lemma 5.2.1). The result now follows from the above proposition. □

Definition 5.4.12. Let $\mathcal{U}$ be a quasi-uniformity on X and let A be a non-empty subset of X. Then if $U \in \mathcal{U}$ and $(A \times A) \cap U = \Delta_A$, A is called *U-discrete.* If there is a $U \in \mathcal{U}$ such that A is U-discrete, A is also called *$\mathcal{U}$-discrete*, or if there is no danger of confusion, simply *discrete*. Note that A is a $\mathcal{U}$-discrete set if and only if $(A, \mathcal{U}|_A)$ is a discrete subspace of $(X, \mathcal{U})$.

Lemma 5.4.13. *Let $\mathcal{U}$ be a uniformity and $\mathcal{V}$ a quasi-uniformity on X such that $\mathcal{U} \subseteq \mathcal{V}$, and suppose there is a $\mathcal{V}$-discrete set $A \subseteq X$ that is not $\mathcal{U}$-discrete. Then there is a non-symmetric quasi-uniformity $\mathcal{Q}$ on X such that $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{V}$.*

Proof. By assumption there is a $V_0 \in \mathcal{V}$ and an injective sequence $\{x_\beta \mid \beta < \alpha\}$ for some cardinal number α such that $x_{\beta'} \notin V_0(x_\beta)$ for all $\beta, \beta' < \alpha$ such that $\beta \neq \beta'$, but $\{x_\beta \mid \beta < \alpha\}$ is not $\mathcal{U}$-discrete.

For each $V \in \mathcal{V}$ we define

$$M_V = V \cup \bigcup \{V^{-1}(x_\beta) \times V(x_{\beta'}) \mid \beta < \beta' < \alpha\}$$

and set $\mathcal{H} = \mathrm{fil}(\{M_V \mid V \in \mathcal{V}\})$. We show that $\mathcal{H}$ is a quasi-uniformity on X.

Let $V \in \mathcal{V}$ and choose $H \in \mathcal{V}$ such that $H^2 \subseteq V_0 \cap V$. We show $(M_H)^2 \subseteq M_V$. Note that whenever $\beta \neq \beta' < \alpha$, then

$$H^{-1}(x_\beta) \cap H(x_{\beta'}) = \emptyset. \ (\star)$$

since otherwise $x_\beta \in H^2(x_{\beta'}) \subseteq V_0(x_{\beta'})$, a contradiction. Suppose $(x, y) \in M_H$ and $(y, z) \in M_H$. There are four cases to consider.

1. $(x, y) \in H$ and $(y, z) \in H$.

2. $(x, y) \in H$, $(y, x_\beta) \in H$ and $(x_{\beta'}, z) \in H$ for some $\beta < \beta' < \alpha$. Then $(x, x_\beta) \in H^2 \subseteq V$ so $(x, z) \in V^{-1}(x_\beta) \times V(x_{\beta'}) \subseteq M_V$.

3. $(y, z) \in H$, $(x, x_\beta) \in H$ and $(x_{\beta'}, y) \in H$ for some $\beta < \beta' < \alpha$. Then $(x_{\beta'}, z) \in H^2 \subseteq V$ so $(x, z) \in V^{-1}(x_\beta) \times V(x_{\beta'}) \subseteq M_V$.

4. $(x, y) \in H^{-1}(x_\beta) \times H(x_{\beta'})$ and $(y, z) \in H^{-1}(x_\gamma) \times H(x_{\gamma'})$ for some $\beta < \beta' < \alpha$ and $\gamma < \gamma' < \alpha$. If $\beta' = \gamma$ then $(x, z) \in H^{-1}(x_\beta) \times H(x_{\gamma'})$ where $\beta < \gamma' < \alpha$, and therefore $(x, z) \in M_V$. If $\beta' \neq \gamma$, this case is impossible because it contradicts $(\star)$.

Now set $\mathcal{Q} = \mathcal{U} \vee \mathcal{H}$. Then $\mathcal{U} \subseteq \mathcal{Q} \subseteq \mathcal{V}$. Next we show that $\{x_\beta \mid \beta < \alpha\}$ is $\mathcal{Q} \vee \mathcal{Q}^{-1}$-discrete but not $\mathcal{Q}$-discrete.

So suppose that $(x_\beta, x_{\beta'}) \in M_{V_0} \cap (M_{V_0})^{-1}$ and $\beta \neq \beta'$. Then we must have $(x_\beta, x_{\beta'}) \in V_0^{-1}(x_\gamma) \times V_0(x_{\gamma'})$ and $(x_\beta, x_{\beta'}) \in V_0(x_{\kappa'}) \times V_0^{-1}(x_\kappa)$ for some $\gamma < \gamma' < \alpha$ and $\kappa < \kappa' < \alpha$. Hence $\gamma = \beta = \kappa'$ and $\kappa = \beta' = \gamma'$. But then $\beta < \beta'$ and $\beta' < \beta$, a contradiction. Hence $\{x_\beta \mid \beta < \alpha\}$ is indeed $\mathcal{Q} \vee \mathcal{Q}^{-1}$-discrete.

Now suppose that $Q \in \mathcal{Q}$, and choose an $H \in \mathcal{V}$ and a symmetric $U \in \mathcal{U}$ such that $U \cap M_H \subseteq Q$. Then since $\{x_\beta \mid \beta < \alpha\}$ is not $\mathcal{U}$-discrete and U is symmetric, there are $\beta < \beta' < \alpha$ such that $(x_\beta, x_{\beta'}) \in U$. Hence $(x_\beta, x_{\beta'}) \in U \cap M_H \subseteq Q$ and therefore $\{x_\beta \mid \beta < \alpha\}$ is not $\mathcal{Q}$-discrete.

Since $\{x_\beta \mid \beta < \alpha\}$ is $\mathcal{V}$-discrete, we hence know that $\mathcal{Q} \neq \mathcal{V}$. Also, since $\{x_\beta \mid \beta < \alpha\}$ is not $\mathcal{Q}$-discrete but is $\mathcal{Q} \vee \mathcal{Q}^{-1}$-discrete, $\mathcal{Q}$ is not a uniformity and therefore $\mathcal{Q} \neq \mathcal{U}$. Hence $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{V}$, as needed. □

Lemma 5.4.14. *Let $\mathcal{U}$ and $\mathcal{V}$ be two uniformities on an infinite set X. Suppose that for some infinite cardinal number m, $\mathcal{U}$ is m-bounded but $\mathcal{V}$ is not. Then there is a non-symmetric quasi-uniformity $\mathcal{Q}$ on X such that $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{V}$.*[12]

Proof. We find a $\mathcal{V}$-discrete set that is not $\mathcal{U}$-discrete. Since $\mathcal{V}$ is not m-bounded, there is a pseudo-metric ρ on X such that $\mathcal{U}_\rho \subseteq \mathcal{V}$ and an $\epsilon > 0$ such that no collection $\mathcal{C}$ of strictly less than m subsets of X such that for each $C \in \mathcal{C}$, $x, y \in C \Rightarrow \rho(x, y) < \epsilon$ will cover X. Write $B = U^\rho_{\epsilon/2}$. Choose an arbitrary $x_1 \in X$, and for every other $\alpha < m$, choose $x_\alpha \in X - \bigcup_{\beta<\alpha} B(x_\beta)$. If $A = \{x_\alpha \mid \alpha < m\}$, then clearly $(A \times A) \cap B = \Delta_A$. Hence A is B-discrete and therefore $\mathcal{V}$-discrete.

The set A, however, is not $\mathcal{U}$-discrete: Let $U \in \mathcal{U}$ be given, and let $\mathcal{C}$ be a cover of X with strictly less than m sets such that for each $C \in \mathcal{C}$, $C \times C \subseteq U$. Since $|A| = m$, some $C \in \mathcal{C}$ must contain more than one member of A, and hence there is a point $(x, y) \in (C \times C) \cap (A \times A) \subseteq U \cap (A \times A)$ such that $x \neq y$.

The result now follows from Lemma 5.4.13. □

Corollary 5.4.15. *Let X be infinite. If $\mathcal{U}$ is a totally bounded uniformity on X and $\mathcal{V}$ is any uniformity on X such that $\mathcal{U} \subsetneq \mathcal{V}$, then there is a non-symmetric quasi-uniformity $\mathcal{Q}$ on X such that $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{V}$.*

[12] See Definition 4.4.4 for the definition of an m-bounded uniformity.

Proof. If $\mathcal{V}$ is totally bounded, the result follows from Proposition 5.4.10. So suppose that $\mathcal{V}$ is not totally bounded. Then it is not $\aleph_0$-bounded, whereas $\mathcal{U}$ is. The result now follows from Lemma 5.4.14. □

5.5 Complements in $\Theta(X)$

In this section we discuss complementation in $\Theta(X)$. We have generalized several results obtained in [45] for complements in $\Psi(X)$ to complements in $\Theta(X)$. For example, we will show that $\Theta(X)$ is complemented iff X is finite, and also that most of the operations preserving complements for uniformities do so for quasi-uniformities as well. We will finally also construct complements for certain biresolvable quasi-uniformities in $\Theta(X)$.

Recall that $\Psi(X)$ is complemented iff X is finite (Corollary 4.5.2). Below the same is proved for $\Theta(X)$.

Proposition 5.5.1. *Every member of $\Theta(X)$ generated by a pre-order on X has a complement that is also generated by a pre-order on X.*

Proof. Combining Corollary 5.1.9 with Proposition 3.5.7 shows that the sublattice of $\Theta(X)$ consisting of all quasi-uniformities generated by a pre-order on X is isomorphic to the lattice of AT topologies on X. By Proposition 3.5.8, every AT topology has a complement that is an AT topology, and hence every quasi-uniformity generated by a pre-order has a complement also generated by a pre-order. □

Corollary 5.5.2. *If X is finite, $\Theta(X)$ is complemented.*

Proof. If X is finite, $\mathcal{U}$ is a quasi-uniformity on X iff $\mathcal{U} = \text{fil}(R)$ for some pre-order R on X. The result now follows from the above proposition. □

The below proposition is the quasi-uniform analogue of Proposition 4.5.1 for uniformities.

Proposition 5.5.3. *If X is infinite, $\Theta(X)$ is not complemented. In fact, no non-discrete quasi-uniformity inducing the discrete proximity has a complement in $\Theta(X)$.*

Proof. Let $\mathcal{U} \neq \mathcal{D}$ be any proximally discrete quasi-uniformity on X, and let $\mathcal{V}$ be a complement of $\mathcal{U}$. Since $\mathcal{U}$ is proximally discrete, $\mathcal{V}_\omega \subseteq \mathcal{U}_\omega$. Now since $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$, $\mathcal{U}_\omega \wedge \mathcal{V}_\omega = \mathcal{I}$ and hence $\mathcal{V}_\omega = \mathcal{I}$. By Lemma 5.2.1 it follows that $\mathcal{V} = \mathcal{I}$, which is impossible.

If X is infinite, Remark 5.3.14 shows that there is at least one non-discrete proximally discrete quasi-uniformity on X. □

Corollary 5.5.4. *$\Theta(X)$ is complemented iff X is finite.*

Proof. This fol ows directly from Corollary 5.5.2 and Proposition 5.5.3. □

Corollary 5.5.5. *An anti-atom of $\Theta(X)$ has a complement if and only if it is not proximally discrete. All atoms of $\Theta(X)$ have complements.*

Proof. If an anti-atom $\mathcal{A}$ is proximally discrete, the above proposition shows that it does not have a complement. If it is not proximally discrete, there must be a set $\emptyset \subsetneq A \subsetneq X$ that is near its complement with respect to $\mathcal{A}$, and hence $S_A \notin \mathcal{A}$. It is clear that $\mathcal{S}_A$ will be a complement of $\mathcal{A}$, since $\mathcal{S}_A \nsubseteq \mathcal{A}$, $\mathcal{S}_A$ is an atom and $\mathcal{A}$ is an anti-atom. If, on the other hand, $\mathcal{A}$ is an atom, $\mathcal{A} = \mathcal{S}_A$ for some non-empty proper subset A of X (by Corollary 5.2.10). By Proposition 5.5.1, it has a complement in $\Theta(X)$. □

It is easy to see that complements in $\Theta(X)$ need not be unique. Consider for example the atom $\mathcal{S}_A$ for $\emptyset \subsetneq A \subsetneq X$ such that A has two or more elements. Then we can find two points $(x, y) \in A \times (X - A)$ and $(a, b) \in A \times (X - A)$ such that $(x, y) \neq (a, b)$. Clearly $\mathcal{G}_{(x,y)}$ and $\mathcal{G}_{(a,b)}$ are both complements of $\mathcal{S}_A$.

Proposition 5.5.6. *If $\mathcal{U}$ has a complement $\mathcal{U}'$ in $\Theta(X)$, it has a quasi-pseudo-metrizable complement.*

Proof. Let $U \in \mathcal{U}$ and $U' \in \mathcal{U}'$ be such that $U \cap U' = \Delta$. There must be a quasi-pseudo-metric d on X such that $\mathcal{U}_d \subseteq \mathcal{U}'$ and $U^d_\epsilon \subseteq U'$ for some $\epsilon > 0$. Then clearly $\mathcal{U}_d \vee \mathcal{U} = \mathcal{D}$, and since $\mathcal{U}_d \subseteq \mathcal{U}'$, $\mathcal{U}_d \wedge \mathcal{U} = \mathcal{I}$. □

5.5.1 Operations Preserving Complements

In this section we present some results regarding operations on or between quasi-uniformities which preserve the property of having a complement. Most of these results have been generalized (to differing degrees) from corresponding results for $\Psi(X)$, originally due to [45] (see Section 4.5.1).

The main results in this section are all consequences of the following lemma, which was originally stated in [45, Section 1.2] for the uniform case:

Lemma 5.5.7. *If $\mathcal{U}, \mathcal{V} \in \Theta(X)$, then*

1. *$\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$ iff the only quasi-pseudo-metric ρ on X such that $\mathcal{U}_\rho \subseteq \mathcal{U}$ and $\mathcal{U}_\rho \subseteq \mathcal{V}$ is $\rho = \mathbf{0}$ (i.e. $\rho(x, y) = 0$ for all $x, y \in X$).*

2. *$\mathcal{U} \vee \mathcal{V} = \mathcal{D}$ iff there are quasi-pseudo-metrics ρ and σ on X and a $K > 0$ such that $\mathcal{U}_\rho \subseteq \mathcal{U}$, $\mathcal{U}_\sigma \subseteq \mathcal{V}$ and $\rho(x, y) + \sigma(x, y) \geq K$ for all $x \neq y$.*

Proof.

1. First suppose that $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$, and let ρ be a quasi-pseudo-metric on X such that $\mathcal{U}_\rho \subseteq \mathcal{U}$ and $\mathcal{U}_\rho \subseteq \mathcal{V}$. Then $\mathcal{U}_\rho = \mathcal{I}$, and hence $\rho = \mathbf{0}$.

   Now suppose that if ρ is a quasi-pseudo-metric on X such that $\mathcal{U}_\rho \subseteq \mathcal{U}$ and $\mathcal{U}_\rho \subseteq \mathcal{V}$, then $\rho = \mathbf{0}$. If $\mathcal{H}$ is a quasi-uniformity such that $\mathcal{H} \subseteq \mathcal{U}$ and $\mathcal{H} \subseteq \mathcal{V}$, and ρ is a quasi-pseudo-metric such that $\mathcal{U}_\rho \subseteq \mathcal{H}$, then $\mathcal{U}_\rho \subseteq \mathcal{U}$ and $\mathcal{U}_\rho \subseteq \mathcal{V}$. Hence $\rho = \mathbf{0}$, and therefore $\mathcal{H} = \mathcal{I}$.

2. Suppose first that $\mathcal{U} \vee \mathcal{V} = \mathcal{D}$. Suppose that for all quasi-pseudo-metrics ρ and σ on X such that $\mathcal{U}_\rho \subseteq \mathcal{U}$ and $\mathcal{U}_\sigma \subseteq \mathcal{V}$, and for all $K > 0$, there exist elements $x \neq y$ of X such that $\rho(x, y) + \sigma(x, y) < K$. Then $\sigma(x, y) < K$ and $\rho(x, y) < K$, so $U_K^\sigma \cap U_K^\rho \neq \Delta$. This contradicts the fact that $\mathcal{U} \vee \mathcal{V} = \mathcal{D}$.

   Now suppose that ρ and σ are quasi-pseudo-metrics on X such that $\mathcal{U}_\rho \subseteq \mathcal{U}$ and $\mathcal{U}_\sigma \subseteq \mathcal{V}$ respectively, and that there is a $K > 0$ such that for all $x \neq y$, $\rho(x, y) + \sigma(x, y) \geq K$. Note that if $\rho(x, y) < \frac{K}{2}$ and $\sigma(x, y) < \frac{K}{2}$, then $\rho(x, y) + \sigma(x, y) < K$. Hence $U_{\frac{K}{2}}^\rho \cap U_{\frac{K}{2}}^\sigma = \Delta$, and therefore $\mathcal{U} \vee \mathcal{V} = \mathcal{D}$.

□

Proposition 5.5.8. *Suppose that X and Y are disjoint, $\mathcal{U}$ has a complement in $\Theta(X)$ and $\mathcal{V}$ has a complement in $\Theta(Y)$. Then the sum of $\mathcal{U}$ and $\mathcal{V}$ (given by $\{U \cup V \mid U \in \mathcal{U}, V \in \mathcal{V}\}$) has a complement in $\Theta(X \cup Y)$.*

Proof. We will denote the sum of $\mathcal{U}$ and $\mathcal{V}$ by $\mathcal{U} + \mathcal{V}$. Let $\mathcal{U}'$ and $\mathcal{V}'$ be complements of $\mathcal{U}$ and $\mathcal{V}$ respectively. Let $\mathcal{H}$ be the quasi-uniformity on $X \cup Y$ generated by all quasi-pseudo-metrics ρ such that:

1. $\mathcal{U}_{\rho|_X} \subseteq \mathcal{U}'$ and $\mathcal{U}_{\rho|_Y} \subseteq \mathcal{V}'$, and

2. if $x \in X$ and $y \in Y$, then

   (a) $\rho(x, y) = \rho(x, x_0) + \rho(y_0, y)$, and

   (b) $\rho(y, x) = \rho(y, y_0) + \rho(x_0, x)$

   where $x_0 \in X$ and $y_0 \in Y$ are arbitrary but fixed points.

We show that $\mathcal{H}$ is a complement of $\mathcal{U} + \mathcal{V}$.

Given two quasi-pseudo-metrics δ on X and γ on Y respectively such that $\mathcal{U}_\delta \subseteq \mathcal{U}'$ and $\mathcal{U}_\gamma \subseteq \mathcal{V}'$, it is easy to construct a quasi-pseudo-metric ρ on $X \cup Y$ satisfying the two conditions above. One defines, for any $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$,

$$\begin{aligned} \rho(x_1, x_2) &= \delta(x_1, x_2), \\ \rho(y_1, y_2) &= \gamma(y_1, y_2), \\ \rho(x, y) &= \delta(x, x_0) + \gamma(y_0, y), \text{ and} \\ \rho(y, x) &= \gamma(y, y_0) + \delta(x_0, x). \end{aligned}$$

Hence there is at least one such quasi-pseudo-metric ρ, and $\mathcal{H}$ is well-defined.

First we show that $\mathcal{H} \wedge (\mathcal{U} + \mathcal{V}) = \mathcal{I}_{X \cup Y}$. Suppose that d is a quasi-pseudo-metric on $X \cup Y$ such that $\mathcal{U}_d \subseteq \mathcal{H}$ and $\mathcal{U}_d \subseteq \mathcal{U} + \mathcal{V}$. Then $\mathcal{U}_d|_X \subseteq \mathcal{U}$ and $\mathcal{U}_d|_Y \subseteq \mathcal{V}$. By the way that $\mathcal{H}$ was defined, we have that $\mathcal{H}|_X \subseteq \mathcal{U}'$ and hence $\mathcal{U}_d|_X \subseteq \mathcal{U}'$. Similarly $\mathcal{U}_d|_Y \subseteq \mathcal{V}'$. Therefore $\mathcal{U}_d|_X = \mathcal{I}_X$ and $\mathcal{U}_d|_Y = \mathcal{I}_Y$, and hence $d(x_1, x_2) = 0$ and $d(y_1, y_2) = 0$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ (by Lemma 5.5.7(1)).

For every $x \in X$ and $y \in Y$ we have $d(x, y) \leq d(x, x_0) + d(x_0, y_0) + d(y_0, y) = d(x_0, y_0)$. We note that for every quasi-pseudo-metric ρ described in points 1 and 2 above, we must have $\rho(x_0, y_0) = 0$. For each $\epsilon > 0$ there must be a ρ satisfying points 1 and 2 above and a $\delta > 0$ such that $U^\rho_\delta \subseteq U^d_\epsilon$, since $\mathcal{U}_d \subseteq \mathcal{H}$. Since $(x_0, y_0) \in U^\rho_\delta$, we conclude that $(x_0, y_0) \in U^d_\epsilon$ for every $\epsilon > 0$. Therefore $d(x_0, y_0) = 0$ and hence $d(x, y) = 0$. Similarly $d(y, x) = 0$ and hence $d = \mathbf{0}$. By Lemma 5.5.7(1) this means that $\mathcal{H} \wedge (\mathcal{U} + \mathcal{V}) = \mathcal{I}_{X \cup Y}$.

Now we must show that $\mathcal{H} \vee (\mathcal{U} + \mathcal{V}) = \mathcal{D}_{X \cup Y}$. Let ρ be a given quasi-pseudo-metric on X such that $\mathcal{U}_\rho \subseteq \mathcal{U}'$. We can extend ρ to a quasi-pseudo-metric ρ' on $X \cup Y$ by defining $\rho'(y_1, y_2) = 0$ for all $y_1, y_2 \in Y$, and if $x \in X$ and $y \in Y$ we define $\rho'(x, y) = \rho(x, x_0)$ and $\rho'(y, x) = \rho(x_0, x)$. Each such ρ' now also satisfies the conditions 1 and 2 above, and hence $\mathcal{U}' \subseteq \mathcal{H}|_X$. Therefore $\mathcal{H}|_X \vee \mathcal{U} = \mathcal{D}_X$ and similarly $\mathcal{H}|_Y \vee \mathcal{V} = \mathcal{D}_Y$. Hence there are

$H_1 \in \mathcal{H}$ and $U \in \mathcal{U}$ such that $H_1 \cap U = \Delta_X$, and $H_2 \in \mathcal{H}$ and $V \in \mathcal{V}$ such that $H_2 \cap V = \Delta_Y$. So $(H_1 \cap H_2) \cap (U \cup V) = \Delta_{X \cup Y}$, as needed. $\square$

Proposition 5.5.9. *Let $\mathcal{U}$ have a complement $\mathcal{U}'$ in $\Theta(X)$ and $\mathcal{V}$ have a complement $\mathcal{V}'$ in $\Theta(Y)$. Then $\mathcal{U}' \times \mathcal{V}'$ is a complement of $\mathcal{U} \times \mathcal{V}$ in $\Theta(X \times Y)$.*

Proof. Write $\mathcal{H} = \mathcal{U} \times \mathcal{V}$, and let $\mathcal{H}' = \mathcal{U}' \times \mathcal{V}'$. We show that $\mathcal{H}'$ is a complement of $\mathcal{H}$.

First we show $\mathcal{H} \wedge \mathcal{H}' = \mathcal{I}_{X \times Y}$. So suppose that ρ is a quasi-pseudo-metric on $X \times Y$ such that $\mathcal{U}_\rho \subseteq \mathcal{H}$ and $\mathcal{U}_\rho \subseteq \mathcal{H}'$. Let $x \in X$ be fixed. Define a quasi-pseudo-metric d on Y by letting $d(y_1, y_2) = \rho((x, y_1), (x, y_2))$ for all $y_1, y_2 \in Y$. We show $\mathcal{U}_d \subseteq \mathcal{V}$. So suppose $\epsilon > 0$. Then there is an $H \in \mathcal{H}$ such that $H \subseteq U^\rho_\epsilon$. We may assume that $H = (\pi_X \times \pi_X)^{-1}(U) \cap (\pi_Y \times \pi_Y)^{-1}(V)$ for some $U \in \mathcal{U}$ and $V \in \mathcal{V}$ (here π_X and π_Y denote the projections of $X \times Y$ onto X and Y respectively). Then $V \subseteq U^d_\epsilon$: For suppose $(a, b) \in V$. Then since $(x, x) \in U$, we have $((x, a), (x, b)) \in H \subseteq U^\rho_\epsilon$, so $d(a, b) = \rho((x, a), (x, b)) < \epsilon$ and hence $(a, b) \in U^d_\epsilon$. Hence $\mathcal{U}_d \subseteq \mathcal{V}$ as claimed. Similarly we have $\mathcal{U}_d \subseteq \mathcal{V}'$ and therefore $d = \mathbf{0}$, by Lemma 5.5.7(1). This holds for whichever $x \in X$ we fix, and similarly $\rho((x_1, y), (x_2, y)) = 0$ for all $x_1, x_2 \in X$ whenever $y \in Y$. Hence, if $x_1, x_2 \in X$ and $y_1, y_2 \in Y$,

$$\rho((x_1, y_1), (x_2, y_2)) \leq \rho((x_1, y_1), (x_1, y_2)) + \rho((x_1, y_2), (x_2, y_2)) = 0,$$

proving that $\rho = \mathbf{0}$, as needed.

Now we show $\mathcal{H} \vee \mathcal{H}' = \mathcal{D}_{X \times Y}$. Since $\mathcal{U} \vee \mathcal{U}' = \mathcal{D}_X$, we can find a $U \in \mathcal{U}$ and a $U' \in \mathcal{U}'$ such that $U \cap U' = \Delta_X$. Similarly there is a $V \in \mathcal{V}$ and a $V' \in \mathcal{V}'$ such that $V \cap V' = \Delta_Y$. It is not hard to see that

$$\begin{aligned} &((\pi_X \times \pi_X)^{-1}(U) \cap (\pi_Y \times \pi_Y)^{-1}(V)) \\ \cap &((\pi_X \times \pi_X)^{-1}(U') \cap (\pi_Y \times \pi_Y)^{-1}(V')) \quad = \quad \Delta_{X \times Y}. \end{aligned}$$

$\square$

Proposition 5.5.10. *Suppose that $(Y, \mathcal{V})$ is a quasi-uniform space and that X is a dense subset of Y with respect to $\mathcal{T}(\mathcal{V})$ and $\mathcal{T}(\mathcal{V}^{-1})$. If the restriction $\mathcal{U}$ of $\mathcal{V}$ to X has a complement $\mathcal{U}'$ in $\Theta(X)$, then $\mathcal{V}$ has a complement in $\Theta(Y)$.*

Proof. Let $\mathcal{V}'$ be the quasi-uniformity generated by all quasi-pseudo-metrics ρ on Y such that $\mathcal{U}_\rho|_X \subseteq \mathcal{U}'$, $\rho(x, y) \leq 1$ for all $x, y \in Y$ and $\rho(x, y) = 1$ if $x \neq y$ and either x or y is in $Y - X$. Then $\mathcal{V}'$ is a complement of $\mathcal{V}$.

First we show $\mathcal{V} \vee \mathcal{V}' = \mathcal{D}_Y$. We know there are $U \in \mathcal{U}$ and $U' \in \mathcal{U}'$ such that $U \cap U' = \Delta_X$. Now we have that $U = V \cap (X \times X)$ for some $V \in \mathcal{V}$. Also, we may assume that $U' = U_\epsilon^\rho$ for some $\epsilon > 0$ and quasi-pseudo-metric ρ on X such that $\mathcal{U}_\rho \subseteq \mathcal{U}'$. ρ can be extended to a quasi-pseudo-metric $\bar{\rho}$ on Y by letting $\bar{\rho}(y, y) = 0$ for all $y \in Y$ and $\bar{\rho}(x, y) = 1$ whenever $x \neq y$ and either x or y is in $Y - X$. We may assume that $\epsilon < 1$. If we write $V' = U_\epsilon^{\bar{\rho}}$, then $V \cap V' = \Delta_Y$, since $V' = U' \cup \Delta_Y$.

Now we show $\mathcal{V} \wedge \mathcal{V}' = \mathcal{I}_Y$. So suppose ρ is a quasi-pseudo-metric on Y such that $\mathcal{U}_\rho \subseteq \mathcal{V}$ and $\mathcal{U}_\rho \subseteq \mathcal{V}'$. Then $\mathcal{U}_\rho|_X \subseteq \mathcal{U}$. From the definition of $\mathcal{V}'$ it also follows that $\mathcal{U}_\rho|_X \subseteq \mathcal{U}'$: We have $\mathcal{U}_\rho \subseteq \mathcal{V}'$. Hence, for each $U \in \mathcal{U}_\rho$ there is a quasi-pseudo-metric d on Y and an $\epsilon > 0$ such that $\mathcal{U}_d|_X \subseteq \mathcal{U}'$ and $U_\epsilon^d \subseteq U$. Therefore $U \cap (X \times X) \in \mathcal{U}'$ and hence $\mathcal{U}_\rho|_X \subseteq \mathcal{U}'$.

Hence $\rho(x_1, x_2) = 0$ for all $x_1, x_2 \in X$. Now suppose that $y_1, y_2 \in Y$ and let $\epsilon > 0$ be given. Then, since X is dense in $(Y, \mathcal{T}(\mathcal{V}))$, there is an $x_1 \in X$ such that $\rho(y_1, x_1) < \frac{\epsilon}{2}$. Since X is dense in $(Y, \mathcal{T}(\mathcal{V}^{-1}))$, there is also an $x_2 \in X$ such that $\rho(x_2, y_2) < \frac{\epsilon}{2}$. Hence $\rho(y_1, y_2) \leq \rho(y_1, x_1) + \rho(x_1, x_2) + \rho(x_2, y_2) < \epsilon$. This shows that $\rho(y_1, y_2) = 0$, and hence $\rho = \mathbf{0}$. □

5.5.2 Complements for Biresolvable Quasi-Uniformities

We have seen that there are some quasi-uniformities which will never have complements in $\Theta(X)$. We have, however, constructed complements for a certain class of quasi-uniformities on X, and this class is a subclass of the biresolvable quasi-uniformities on X. In this section this construction is presented.

We have defined biresolvable quasi-uniformities as follows:

Definition 5.5.11. Let $\mathcal{U}$ be a quasi-uniformity on a set X. Then the quasi-uniform space $(X,\mathcal{U})$, or just the quasi-uniformity $\mathcal{U}$, is called *biresolvable* if there exists a subset D of X such that D is dense in $\mathcal{T}(\mathcal{U})$ and $X - D$ is dense in $\mathcal{T}(\mathcal{U}^{-1})$. If this is the case and $\mathcal{U}$ is a uniformity, the uniform space $(X, \mathcal{U})$ is simply called *resolvable* (since $\mathcal{T}(\mathcal{U}) = \mathcal{T}(\mathcal{U}^{-1})$).

Lemma 5.5.12. *Let $(X,\mathcal{U})$ be a biresolvable quasi-uniform space such that $\mathcal{U} \wedge \mathcal{U}^{-1} \neq \mathcal{I}$. Then $\mathcal{U}$ has a complement in $\Theta(X)$.*

Proof. Since $\mathcal{U} \wedge \mathcal{U}^{-1} \neq \mathcal{I}$, it contains an entourage $U \neq X \times X$. Choose a symmetric entourage $V \in \mathcal{U} \wedge \mathcal{U}^{-1}$ such that $V^9 \subseteq U$.

Suppose that $(x,y) \in V^3$ – then $V^3(x) \cup V^3(y) \neq X$, since otherwise $X \times X = (V^3(x) \cup V^3(y)) \times (V^3(x) \cup V^3(y)) \subseteq V^9 \subseteq U$.

Set $R = ((X \times X) - V^3) \cup \Delta$. We have that $R^2 = X \times X$: Suppose that $(x,y) \in V^3$. By the above comment we can find an $a \notin V^3(x) \cup V^3(y)$ – then $(x,a) \notin V^3$ and $(a,y) \notin V^3$. Hence $(x,a) \in R$ and $(a,y) \in R$, so $(x,y) \in R^2$. Hence $R^2 = X \times X$ as claimed.

Since $(X,\mathcal{U})$ is biresolvable we can find a $D \subseteq X$ such that D is dense in $(X,\mathcal{T}(\mathcal{U}))$ and $X - D$ is dense in $(X,\mathcal{T}(\mathcal{U}^{-1}))$. Set $D_1 = D$ and $D_2 = X - D$. Let $T = ((D_1 \times D_2) - V) \cup \Delta$. Then T is transitive, because D_1 and D_2 are disjoint. Let $\mathcal{V}$ be the quasi-uniformity generated by T. Then since $V \cap T = \Delta$, $\mathcal{U} \vee \mathcal{V} = \mathcal{D}$.

We show that $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$. Let $L \in \mathcal{U} \wedge \mathcal{V}$ and choose $M \in \mathcal{U} \wedge \mathcal{V}$ such that $M^6 \subseteq L$. Then there is a $W \in \mathcal{U}$ such that $T \cup W \subseteq M$, and $R \subseteq W \circ T \circ W$ as we now prove. Suppose $(x,y) \in R - \Delta$. Then there is a $d_1 \in D_1 \cap (V \cap W)(x)$ and a $d_2 \in D_2 \cap (V \cap W)^{-1}(y)$, since $V, W \in \mathcal{U}$ and by the denseness of D_1 and D_2 in $\mathcal{T}(\mathcal{U})$ and $\mathcal{T}(\mathcal{U}^{-1})$ respectively. For the sake of contradiction, suppose that $(d_1,d_2) \in V$. Then since $(x,d_1) \in V$ and $(d_2,y) \in V$, $(x,y) \in V^3$. By the definition of R this means that $(x,y) \notin R$, a contradiction. Hence $(d_1,d_2) \in T$ by definition of T. It follows that $(x,y) \in W \circ T \circ W$ and therefore $R \subseteq W \circ T \circ W$ as claimed.

Hence $X \times X = R^2 \subseteq (W \circ T \circ W) \circ (W \circ T \circ W) \subseteq M^6 \subseteq L$. So $\mathcal{U} \wedge \mathcal{V} = \mathcal{I}$ and $\mathcal{V}$ is a complement of $\mathcal{U}$. □

Corollary 5.5.13. *Let $\mathcal{U}$ be a uniformity on a set X such that $(X,\mathcal{U})$ is resolvable. Then $\mathcal{U}$ has a complement in $\Theta(X)$.*

Proof. This follows immediately from the preceding lemma. □

5.6 Lattice Structure of $\Theta(X)$

In this section we study some important aspects of the lattice structure of $\Theta(X)$, namely modularity, distributivity and self-dualness.

We start by proving that $\Theta(X)$ is in general neither modular nor distributive. It is in fact only for very small sets X that $\Theta(X)$ will ever have such an organized structure. The same is true for self-dualness.

Proposition 5.6.1. $\Theta(X)$ *is modular (distributive) if and only if* $|X| < 3$.

Proof. Let $a, m, n \in X$ all be distinct. Define the following quasi-uniformities on X:

$$
\begin{aligned}
\mathcal{V}_1 &= \mathcal{D} \\
\mathcal{V}_2 &= \text{fil}(\{(a,m)\} \cup \Delta) \\
\mathcal{V}_3 &= \text{fil}(\{(m,n)\} \cup \Delta) \\
\mathcal{V}_4 &= \text{fil}(\{(a,m)\} \cup \{(a,n)\} \cup \Delta) \\
\mathcal{V}_5 &= \text{fil}(\{(a,m)\} \cup \{(a,n)\} \cup \{(m,n)\} \cup \Delta)
\end{aligned}
$$

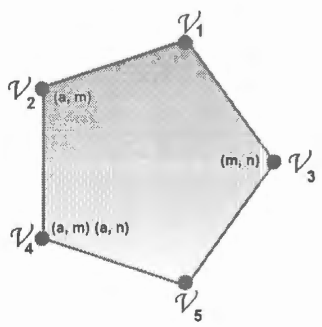

Figure 5.5: Lattice Diagram for the $\mathcal{V}_i$.

It is not hard to see that $\mathcal{V}_2 \wedge \mathcal{V}_3 = \mathcal{V}_5$, $\mathcal{V}_4 \wedge \mathcal{V}_3 = \mathcal{V}_5$, $\mathcal{V}_2 \vee \mathcal{V}_3 = \mathcal{V}_1$, and $\mathcal{V}_4 \vee \mathcal{V}_3 = \mathcal{V}_1$. Hence, N_5 is a sublattice of $\Theta(X)$, and by Proposition 2.1.1 this means that $\Theta(X)$ is neither modular nor distributive.

Conversely, if X has two or less elements, there are at most four possible quasi-uniformities on X, and hence neither N_5 nor M_3 can be embedded into $\Theta(X)$. Therefore $\Theta(X)$ will be distributive, by Proposition 2.1.1. □

Proposition 5.6.2. *If* $|X| > 3$, $\Theta(X)$ *is not self-dual.*

Proof. Recall that if X is finite, non-empty and $|X| = n$, $\Theta(X)$ has $2^n - 2$ atoms and $n(n-1)$ anti-atoms (see Corollaries 5.2.11 and 5.3.19). If $n > 3$, these numbers are not equal and hence $\Theta(X)$ will not be self-dual. If X is infinite, $\Theta(X)$ has $2^{|X|}$ atoms and $2^{2^{|X|}}$ anti-atoms and will therefore not be self-dual either. □

Chapter 6

Conclusion

In this dissertation many aspects of the lattice of quasi-uniformities on a set X have been described, such as the atoms, anti-atoms, lattice structure and complementation. The main results are listed, with references to where in this dissertation they were proved, at the end of the Introduction (Chapter 1).

However, a number of questions regarding this lattice still remain unanswered, some of which we now briefly discuss. Firstly, as was mentioned in Section 5.4.3, it is still unknown whether two uniformities can be adjacent in the lattice of quasi-uniformities. More generally:

Problem 1. *Given any two uniformities $\mathcal{U}$ and $\mathcal{V}$ on a set X such that $\mathcal{U} \subsetneq \mathcal{V}$, is there a non-symmetric quasi-uniformity $\mathcal{Q}$ on X such that $\mathcal{U} \subsetneq \mathcal{Q} \subsetneq \mathcal{V}$?*

Continuing with the idea of uniformities in $\Theta(X)$, there also remains open a question regarding their complements. Recall Example 4.5.10, where it is mentioned that the uniformity $\mathcal{J}$ of a Cauchy sequence has no complement in $\Psi(X)$. The immediate next question is of course whether $\mathcal{J}$ has a complement in $\Theta(X)$. If not, it seems natural to ask whether there does exist a uniformity not having a uniform complement that does have a quasi-uniform complement. In other words, the question is:

Problem 2. *If a uniformity has a quasi-uniform complement, does it necessarily have a uniform complement?*

Despite the numerous similarities (and differences) between $\Psi(X)$ and $\Theta(X)$ noted throughout this dissertation, there also still remain some questions that have been answered for the former but not for the latter. Two of these stand out, one of which is the following:

Problem 3. *Is $\Theta(X)$ anti-atomic? If not, which members of $\Theta(X)$ can be written as the meet of anti-atoms?*

The first part of this question has been answered for $\Psi(X)$ in the negative (Corollary 4.3.5). It was however noted in Remark 5.3.6 that the approach used there cannot be extended to prove the same for $\Theta(X)$. Since we do not as yet have a complete characterization of the proximally discrete anti-atoms of $\Theta(X)$ (or $\Psi(X)$), the second part of the above problem seems particularly non-trivial.

Similarly, the complicated structure of proximally discrete anti-atoms makes finding a non-transitive anti-atom in $\Theta(X)$ difficult, since all non-transitive anti-atoms are proximally discrete.[1] Whereas in Theorem 4.3.28 we cited a result which proves the existence of non-transitive anti-atoms in $\Psi(\omega)$, their existence in $\Theta(X)$ has not been proven for any X as yet. This then is another problem solved in the case of $\Psi(X)$ which remains open in $\Theta(X)$.

Problem 4. *Does there exist a non-transitive anti-atom in $\Theta(X)$?*

Although these and many other questions remain to be answered for $\Theta(X)$, the results obtained thus far are an indication of the significance of $\Theta(X)$ in the theory of quasi-uniformities. Proposition 5.2.15, for example, shows that the transitive totally bounded quasi-uniformities on X are exactly the atomic members of $\Theta(X)$. This suggests that the position of a quasi-uniformity in $\Theta(X)$ can be an indication of the properties it possesses.

Other areas of mathematics, however, can benefit as well. For example, Künzi mentions in [25, Section 1] that some results obtained in [7] relating to the semi-lattice of compatible totally bounded quasi-uniformities on a topological space X can be used to study certain lattice-theoretical questions using the theory of quasi-uniformities.[2]

More in the scope of this dissertation, though, is the link with ultrafilters on X. In Proposition 5.3.15 and Theorem 5.3.20 we found that there is a

[1]See Section 4.3.4. Here more reasons are given as to why such an anti-atom would be difficult to find. Although mentioned in the context of $\Psi(X)$, these comments are also valid for $\Theta(X)$.

[2]The result in question shows that every core-compact topological space X admits a coarsest (totally bounded) quasi-uniformity, i.e. for every core-compact space X, the semi-lattice of compatible (totally bounded) quasi-uniformities is a lattice. See [7, Lemma 5].

strong connection between the anti-atoms of $\Theta(X)$ and ultrafilters on X. This may mean that Pelant et. al.'s suggestion (in [46]) to use uniformities for the investigation and classification of ultrafilters could be extended to quasi-uniformities (see Chapter 1).

Of course, since quasi-uniformities are more general than uniformities, the anti-atoms of $\Theta(X)$ may have less intricate structures than those of $\Psi(X)$. It is hence possible that, in some cases, they may not provide as much information as in the uniform case. Our characterization of the proximally fine anti-atoms of $\Theta(X)$ found in Theorem 5.3.32 substantiates this claim. Concentrating specifically on the case where X is countable (Corollary 5.3.33), we saw that a necessary and sufficient condition for an anti-atom $\mathcal{A}$ of $\Theta(X)$ to be proximally fine is that it be semi-trivial. Upon comparison, the proximally fine anti-atoms of $\Psi(X)$ seem significantly more complicated. Not only does a characterization of these anti-atoms not exist as yet, but in the case of countable X, selective ultrafilters were used to find examples (Theorem 4.3.24). The relationship between the properties of an ultrafilter $\mathcal{F}$ and the proximal fineness of the uniform anti-atom $\mathcal{J}_{\mathcal{F}}$ hence seems to be more in-depth than the relationship between ultrafilters and anti-atoms in the quasi-uniform case.

The above, however, is not reason enough to believe that the relationship between ultrafilters on X and anti-atoms of $\Theta(X)$ is insignificant, as can be deduced for example from Lemma 5.3.30. This proposition essentially states that for an anti-atom $\mathcal{A}$ of $\Theta(X)$, the ultrafilters $\mathcal{F}$ and $\mathcal{G}$ on X such that $\mathcal{U}_{\mathcal{F}\times\mathcal{G}} \subseteq \mathcal{A}$ completely determine the quasi-proximity induced by $\mathcal{A}$. Hence it is clear that the ultrafilters $\mathcal{F}$ and $\mathcal{G}$ do play some role in determining the properties of the anti-atom $\mathcal{A}$.

Through this dissertation it has become clear that, besides the obvious, $\Theta(X)$ also has the potential to provide insight into the properties of quasi-uniformities on X, lattice theory in its own right and ultrafilters on X. This, of course, need not be (and probably is not) an exhaustive list. So although this dissertation has come to an end, the future of The Lattice of Quasi-Uniformities is secured.

Symbols

$X - A$	The complement of the set A	§ 2.1
$\wp(X)$	$\{A \mid A \subseteq X\}$ (the powerset of X)	§ 2.1
$U \circ V$	$\{(x, z) \mid \exists y \in X$ such that $(x, y) \in U$ and $(y, z) \in V\}$	§ 2.1
U^2	$U \circ U$	§ 2.1
U^{-1}	$\{(y, x) \mid (x, y) \in U\}$	§ 2.1
Δ_A	$\{(x, x) \mid x \in A\}$ (the diagonal of A, where $A \subseteq X$)	§ 2.1
Δ	Δ_X	§ 2.1
$\mathrm{fil}(\mathcal{C})$	Filter generated by the subbase $\mathcal{C}$ (where $\mathcal{C} \subseteq \wp(X)$)	§ 2.1
$\mathrm{fil}(C)$	Filter with base $\{C\}$ (where $C \in \wp(X)$)	§ 2.1
ω	The set of natural numbers	§ 2.1
$\aleph_0$	Cardinality of ω	§ 2.1
$\mathbb{R}$	The set of real numbers	§ 2.1
c	Cardinality of $\mathbb{R}$ (i.e. $c = 2^{\aleph_0}$)	§ 2.1
$\|A\|$	Cardinality of the set A	§ 2.1
$\wedge$	Greatest lower bound	§ 2.1
$\vee$	Least upper bound	§ 2.1
$\mathbf{0}$	Bottom element	§ 2.1
$\mathbf{1}$	Top element	§ 2.1
$\mathcal{I}$	(or $\mathcal{I}_X$) Indiscrete uniformity (on X)	2.2.5
$\mathcal{D}$	(or $\mathcal{D}_X$) Discrete uniformity (on X)	2.2.5
$\mathcal{U}^{-1}$	$\{U^{-1} \mid U \in \mathcal{U}\}$ (conjugate of the quasi-uniformity $\mathcal{U}$)	2.2.2
$\mathcal{U}\|_A$	$\{U \cap (A \times A) \mid U \in \mathcal{U}\}$ ($\mathcal{U}$ a quasi-uniformity)	2.2.3
$\mathcal{U} \simeq \mathcal{V}$	$\mathcal{U}$ and $\mathcal{V}$ are (quasi-) uniformly isomorphic	2.2.6
$\mathcal{T}(\mathcal{U})$	$\{G \subseteq X \mid \forall x \in G, \exists U \in \mathcal{U}$ such that $U(x) \subseteq G\}$	2.2.17

U^ρ_ϵ	$\{(x,y) \mid \rho(x,y) < \epsilon\}$ (ρ a quasi-pseudo-metric)	2.2.10
$\mathcal{U}_\rho$	$\text{fil}(\{U^\rho_\epsilon \mid \epsilon > 0\})$ (ρ a quasi-pseudo-metric)	2.2.10
$A\delta B$	$(A,B) \in \delta$, i.e. A is near B (δ a quasi-proximity)	2.3.1
$A\bar{\delta}B$	$(A,B) \notin \delta$, i.e. A is far from B (δ a quasi-proximity)	2.3.1
$\delta_{\mathcal{U}}$	$\{(A,B) \mid \forall U \in \mathcal{U}, U \cap (A \times B) \neq \emptyset\}$ ($\mathcal{U}$ a quasi-uniformity)	2.3.5
$\pi(\delta)$	$\{\mathcal{U} \in \Theta(X) \mid \delta_{\mathcal{U}} = \delta\}$ (δ a quasi-proximity)	2.3.5
$\mathcal{T}(\delta)$	$\{X - A \mid A = \{x \mid \{x\}\delta A\}\}$ (δ a quasi-proximity)	2.3.8
$\mathcal{U}_\delta$	$\text{fil}(\{(X \times X) - (A \times B) \mid A\bar{\delta}B\})$ (δ a quasi-proximity)	2.3.9
$\mathcal{U}_\omega$	$\mathcal{U}_{\delta_{\mathcal{U}}}$	2.3.11
$\Sigma(X)$	The lattice of topologies on X	3.1.1
$\Lambda(X)$	The lattice of T_1-topologies on X	3.9.1
$\Psi(X)$	The lattice of uniformities on X	4.1.1
$\Theta(X)$	The lattice of quasi-uniformities on X	5.1.1
$\mathcal{F} \prec \mathcal{G}$	Rudin-Keisler order for ultrafilters	4.3.15
U_F	$\{((x,1),(x,2)) \mid x \in F\} \cup \{((x,2),(x,1)) \mid x \in F\} \cup \Delta$	4.3.19
$\mathcal{J}_\mathcal{F}$	$\text{fil}(\{U_F \mid F \in \mathcal{F}\})$ ($\mathcal{F}$ a filter)	4.3.19
$\mathcal{U}_\mathcal{F}$	$\text{fil}(\{(F \times F) \cup \Delta \mid F \in \mathcal{F}\})$ ($\mathcal{F}$ a filter)	4.3.7
$\mathcal{J}$	The uniformity of a cauchy sequence	4.5.8
H_A	$(A \times A) \cup ((X - A) \times (X - A))$	4.2.2
$\mathcal{H}_A$	$\text{fil}(H_A)$	4.2.2
S_A	$(A \times A) \cup ((X - A) \times X)$	5.2.4
$\mathcal{S}_A$	$\text{fil}(S_A)$	5.2.4
$\mathcal{K}_{(x,y)}$	$\text{fil}(\{(x,y)\} \cup \{(y,x)\} \cup \Delta)$	4.3.3
$\mathcal{G}_{(x,y)}$	$\text{fil}(\{(x,y)\} \cup \Delta)$	5.3.3
E_x	$H_{\{x\}}$	4.4.1
$\mathcal{E}_x$	$\mathcal{H}_{\{x\}}$	4.4.1
AC	Axiom of Choice is used	Ch 1
CH	Continuum Hypothesis is used	Ch 1
iff	if and only if	Ch 1

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