

University of Cape Town
Department of Mathematics and Applied Mathematics

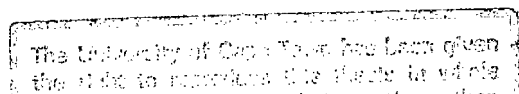
Congruences on Lattices (with Application to Amalgamation)

by

Lyneve Laing

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of Dr H Rose for the degree of
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Synopsis

We present some aspects of congruences on lattices. An overview of general results on congruence distributive algebras is given in Chapter 1 and in Chapter 2 we examine weak projections; including Dilworth's characterization of congruences on lattices and a finite basis theorem for lattices. The outstanding problem of whether congruence lattices of lattices characterize distributive algebraic lattices is discussed in Chapter 3 and we look at some of the partial results known to date. The last chapter (Chapter 6) characterizes the amalgamation class of a variety \mathcal{B} generated by a B -lattice, B , as the intersection of subdirect products of B , 2-congruence extendible members of \mathcal{B} and 2-chain limited members of \mathcal{B} . To this end we consider 2-congruence extendibility in Chapter 4 and n -chain limited lattices in Chapter 5. Included in Chapter 4 is the result that in certain lattice varieties the amalgamation class is contained in the class of 2-congruence extendible members of the variety. A final theorem in Chapter 6 states that the amalgamation class of a B -lattice variety is a Horn class.

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Introduction

The development of the study of congruences on lattices was given impetus by the paper of Funayama and Nakayama ([Fun42]) which proved one of the most important results about lattices namely that they are congruence distributive algebras. The fundamental paper of Jónsson providing powerful tools for the study of congruence distributive varieties and therefore lattice varieties, appeared 25 years later (see Chapter 1). Funayama and Nakayama's result together with the celebrated Grätzer-Schmidt characterization of an algebraic lattice as the congruence lattice of some algebra ([Gra62]) tells us that the congruence lattice of a lattice is a distributive algebraic lattice. Attempts to address the converse conjecture have created a large body of partial results of so-called "representation theorems" (Chapter 3). Despite the efforts of many this problem is *to our knowledge* still open.

Dilworth's characterization of congruences of lattices via weak projections initiated the use of weak projections for solving other problems of lattice theory. For example Herrmann's version of a finite basis theorem for lattices (Chapter 2).

In [Jon90] B. Jónsson characterizes $\text{Amal}(\mathcal{V})$ for finitely generated lattice varieties and in [Jip89] this result is generalized to residually small one-subalgebra congruence distributive varieties. A consequence of this characterization is that if \mathcal{V} is a residually small lattice variety then each member of $\text{Amal}(\mathcal{V})$ is congruence extensile. We show in Chapter 6 that for a variety \mathcal{B} generated by a B -lattice, B , $\text{Amal}(\mathcal{B})$ is the intersection of subdirect powers of B , 2-chain limited members of \mathcal{B} and 2-congruence extendible members of \mathcal{B} .

We begin in Chapter 0 with a summary of some basic results concerning varieties, filters, lattices and congruences to be used in subsequent chapters. We also introduce most of the notation and terminology used later.

Chapter 1 presents well-known general results concerning congruence-distributive varieties, including Jónsson's Lemma. We introduce the notion of a P -congruence extendible algebra (in a variety) where a congruence θ on an algebra A is called a P -congruence on A if $A/\theta \cong P$. We also prove some results concerning amalgamation and congruence extensile algebras.

Chapter 2 reviews the definition of a weak projection and presents Dilworth's characterization of congruences on lattices. Some results on modular and distributive lattices are discussed, most of which are referred to in subsequent chapters. We prove results about primitive subsets of lattices which are used in a proof of a finite basis theorem for lattices. The notions of an S -lattice and a S -variety are introduced and we show that in a S -variety, \mathcal{V} , subdirect products of non-modular subdirectly irreducible members of \mathcal{V} are closed under reduced products. A brief section on ideals and congruences is included in this chapter.

In Chapter 3 we consider the problem of representing a distributive algebraic lattice as the congruence lattice of a lattice. We also prove some results concerning the representation of *specific* distributive algebraic lattices as the congruence lattices of particular types of lattices. We present characterization theorems of lattices whose congruence lattices are Boolean lattices and lattices whose congruence lattices are Stone lattices.

In Chapter 4 we give a characterization of 2-congruence extendible lattices. An algebra A is said to be P -congruence extendible in a variety \mathcal{V} if whenever $A \leq C$ in \mathcal{V} and θ is a P -congruence on A then there exists a P -congruence on C extending θ . We consider 2-congruence extendibility in \mathcal{S} -varieties.

Chapter 5 introduces the notion of an n -chain limited lattice where a lattice is called n -chain limited if it does not have $(n+1)$ -congruences. We characterize 3-chain limited finite distributive lattices as semi-Boolean lattices.

Chapter 6 concerns B -lattices (lattices obtained by splitting an element a finite Boolean lattice). We characterize the amalgamation class of a variety \mathcal{B} generated by a B -lattice and prove that $\text{Amal}(\mathcal{B})$ is a Horn class.

Remarks:

- (i) We have attempted to keep this presentation self-contained, however the following standard texts on Universal Algebra and Lattice Theory can serve as a reference for concepts not developed here: [Bur81], [Cra73], [Gra78] and [McK87])
- (ii) To prevent the size of this thesis from becoming too unwieldy we omit details and simply outline the proofs of some long results (particularly those of Chapter 3).
- (iii) Sections 5.2 and 5.3 of Chapter 5 are neither in print nor in the form of a manuscript. They are the result of a collaboration of the student and her supervisor.
- (iv) We have tried to cover (in our opinion) the more important directions of research and have by no means attempted an exhaustive account of the topic this thesis.

Chapter 0

Notation, terminology and preliminary results

This chapter introduces the notation and terminology to be used in subsequent chapters. We include only the most general, introducing more specific terminology as it is needed. We review some well-known results in the broad areas of varieties, subdirectly irreducible algebras, filters, ultraproducts, algebraic lattices, congruences and homomorphisms. Included are precursory results to theorems appearing in later chapters. The important result of Funayama and Nakayama (see Theorem 0.4.1) proving that lattices are congruence distributive algebras is also presented here.

0.1 Varieties and subdirectly irreducible algebras

Let \mathcal{K} be a class of similar algebras. Then we use the following notation:

$A \in \mathbf{H}(\mathcal{K})$ if and only if A is a homomorphic image of some member of \mathcal{K} .

$A \in \mathbf{S}(\mathcal{K})$ if and only if A is isomorphic to a subalgebra of some member of \mathcal{K} .

$A \in \mathbf{P}(\mathcal{K})$ if and only if A is isomorphic to a direct product of members of \mathcal{K} .

Let Σ be a set of identities. Define $\text{Mod}(\Sigma)$ to be the class of algebras satisfying Σ . For an algebra A and set of identities Σ we write $A \models \Sigma$ to denote the A satisfies Σ and if \mathcal{K} is a class of algebras then $\mathcal{K} \models \Sigma$ means $A \models \Sigma$ for all $A \in \mathcal{K}$. A class \mathcal{K} of algebras is an *elementary class* if it is the class of all algebras which satisfy some set of first-order sentences.

A class \mathcal{K} of similar algebras is called a *variety* if it is closed under homomorphic images, subalgebras and direct products. We let \mathcal{K}^\vee denote the smallest variety containing \mathcal{K} . In

[Tar46] Tarski showed that $\mathcal{K}^{\mathcal{V}} = \mathbf{HSP}(\mathcal{K})$. A variety \mathcal{V} is *finitely generated* if $\mathcal{V} = \mathbf{HSP}(\mathcal{K})$ for some finite set \mathcal{K} of finite algebras. A class \mathcal{K} of algebras is an *equational class* if $\mathcal{K} = \mathbf{Mod}(\Sigma)$ for some set of identities Σ . One of the first significant results in the study of varieties is due to Birkhoff who showed that a class \mathcal{K} of algebras is an equational class if and only if \mathcal{K} is a variety ([Bir35]). Thus every variety is an elementary class.

An algebra A is a *subdirect product* of algebras $(A_i)_{i \in I}$ if there is an embedding $f : A \hookrightarrow \prod_{i \in I} A_i$ such that $\pi_i \circ f[A] = A_i$ for all $i \in I$, where π_i is the canonical projection onto A_i . For a class \mathcal{K} of similar algebras we write $A \in \mathbf{P}_S(\mathcal{K})$ if and only if A is isomorphic to a subdirect product of members of \mathcal{K} .

An algebra A is *subdirectly irreducible* if whenever A is a subdirect product of algebras $(A_i)_{i \in I}$, then A is isomorphic to A_i for some $i \in I$. A is *finitely subdirectly irreducible* if whenever A is a subdirect product of finitely many algebras A_1, \dots, A_n then A is isomorphic to A_i for some $i \in \{1, \dots, n\}$. For a variety \mathcal{V} , \mathcal{V}_{SI} is the class of all subdirectly irreducible members of \mathcal{V} and \mathcal{V}_{FSI} is the class of all finitely subdirectly irreducible members of \mathcal{V} .

We call a variety \mathcal{V} of algebras a *one-subalgebra variety* if every member of \mathcal{V} has a one element subalgebra. For example the variety of all lattices is a one-subalgebra variety.

Let \mathcal{K} be a class of algebras and let F be an algebra generated by a set $X \subseteq F$. Then F is *\mathcal{K} -freely generated* by X if any map $f : X \rightarrow A \in \mathcal{K}$ can be extended to a homomorphism $g : F \rightarrow A$. If $F \in \mathcal{K}$ then F is called the *\mathcal{K} -free algebra on $|X|$ generators* and is denoted by $F_{\mathcal{K}}(X)$ or $F_{\mathcal{K}}(|X|)$. For a cardinal $\beta \neq 0$ $F_{\mathcal{K}}(\beta)$ does not necessarily exist, however if \mathcal{K} is a variety then by [Bir35] the \mathcal{K} -free algebra on β generators always exists.

0.2 Filters and ultraproducts

Let L be a lattice. A *filter* in L is a sublattice F such that if $x \in F$ and $x \leq y$ then $y \in F$. A filter F is *proper* if $F \neq L$, is *principal* if $F = [x] = \{y \in L : x \leq y\}$ for some $x \in L$ and is *prime* if for any $x, y \in L$, $x \vee y \in F$ implies $x \in F$ or $y \in F$. A filter F is an *ultrafilter* if it is a maximal proper filter. In a distributive lattice a filter F is an ultrafilter if and only if it is a proper prime filter. For an arbitrary set I , \mathcal{F} is a *filter over I* if \mathcal{F} is a filter in $\mathcal{P}(I)$ (the lattice of subsets of I). Since $\mathcal{P}(I)$ is a Boolean algebra the following are equivalent:

- (a) \mathcal{F} is an ultrafilter over I .
- (b) \mathcal{F} is a proper prime filter over I .
- (c) Whenever I is partitioned into finitely many disjoint blocks then \mathcal{F} contains exactly one of these blocks.
- (d) For any $X \in \mathcal{P}(I)$ exactly one of $X, I \setminus X$ belongs to \mathcal{F} .

We use the following two lemmas in the proof of Proposition 1.3.3.

LEMMA 0.2.1 *If \mathcal{F} is a principal ultrafilter over I then $\mathcal{F} = \{\{i\}\}$ for some $i \in I$.*

PROOF. Let $\mathcal{F} = [J]$ for some $J \in \mathcal{P}(I)$. For all $i \in I$, either $\{i\} \in \mathcal{F}$ or $I \setminus \{i\} \in \mathcal{F}$. But $I \setminus \{i\} \in \mathcal{F}$ for all $i \in I$ implies $J = \emptyset$. This contradiction yields $J = \{\{i\}\}$ for some $i \in I$. \square

LEMMA 0.2.2 *If \mathcal{F} is a non-principal ultrafilter over I then $X \notin \mathcal{F}$ for any finite subset X of I .*

PROOF. Suppose $X \in \mathcal{F}$ for some finite subset X of I . we show that this implies that $\mathcal{F} = \{\{i\}\}$ for some $i \in X$. Proof is by induction on $|X|$. The case $|X| = 1$ is trivial. For $|X| > 1$ suppose the result holds for all $Y \in \mathcal{P}(I)$ with $|Y| < |X|$. Since $\mathcal{F} \neq [X]$ there exists an $A \in \mathcal{F}$ such that $A \subset X$ or A is non-comparable with X . In either case $A \cap X \in \mathcal{F}$ with $|A \cap X| < |X|$. By the inductive hypothesis $\mathcal{F} = \{\{i\}\}$ for some $i \in X \cap A$ completing the proof. \square

See [Bur81] page 134 for a proof of the following lemma.

LEMMA 0.2.3 *If F is a filter in a Boolean algebra B with $a \in B \setminus F$ then there is an ultrafilter U in B with $F \subseteq U$ and $a \notin U$.*

The corollary below is used in the proof of Proposition 1.3.6.

COROLLARY 0.2.4 *Let F be a filter in a Boolean algebra B . Then $F = \bigcap \mathcal{U}$ where \mathcal{U} is the set of all ultrafilters in B containing F .*

PROOF. Clearly $F \subseteq \bigcap \mathcal{U}$. Suppose $F \neq \bigcap \mathcal{U}$. Then there is an $a \in \bigcap \mathcal{U}$ such that $a \notin F$. By the previous lemma there is an ultrafilter U with $F \subseteq U$ and $a \notin U$. This contradiction completes the proof. \square

Let $C = \prod_{i \in I} C_i$ be a direct product of a family of algebras. Then if \mathcal{F} is a filter over I we can describe a congruence $\Phi_{\mathcal{F}}$ on C by $(a, b) \in \Phi_{\mathcal{F}}$ if and only if $\{i \in I : a_i = b_i\} \in \mathcal{F}$ where a_i is the i th coordinate of a . $\Phi_{\mathcal{F}}$ is called a *filtral congruence* in C . If \mathcal{F} is a proper filter/ultrafilter then the lattice $C/\Phi_{\mathcal{F}}$ is called a *reduced product/ultraproduct* of the algebras C_i . We use C/\mathcal{F} and $C/\Phi_{\mathcal{F}}$ interchangeably.

For any class of algebras \mathcal{K} let $P_U(\mathcal{K})$ denote the class of ultraproducts of members of \mathcal{K} .

0.3 Algebraic lattices

An element c of a lattice L is *compact* if and only if whenever $c \leq \bigvee S$ for some subset S of L , then $c \leq \bigvee S'$ for some finite subset S' of S . A lattice is *algebraic* if it is complete and

every element is the join of compact elements. For a lattice L we let $C(L)$ denote the set of compact elements of L and let $L^c = (C(L), \vee)$ denote the join-semilattice of compact elements of L .

An element a of a lattice L is *join irreducible* if $a = b \vee c$ implies $a = b$ or $a = c$; *completely join irreducible* if and only if whenever $a = \vee C$ for some subset C of L such that $\vee C$ exists in L then $a = c$ for some $c \in C$. Dually c is *meet irreducible* if $a = b \wedge c$ implies $a = b$ or $a = c$; *completely meet irreducible* if and only if whenever $a = \wedge C$ for some subset C of L such that $\wedge C$ exists in L then $a = c$ for some $c \in C$.

We denote the set of join irreducible elements of a lattice L by $J(L)$.

The following is a lattice-theoretic version of Birkhoff's subdirect representation theorem (see section 0.4 below).

THEOREM 0.3.1 *In an algebraic lattice L every element is the meet of a set of completely meet irreducible elements.*

PROOF. Let $a \in L$ and let $D = \{d \in L : d \text{ is completely meet irreducible and } a \leq d\}$. Let $b = \wedge D$. Clearly $a \leq b$. Let c be a compact element below b . Suppose $c \not\leq a$. Let $X = \{x \in L : a \leq x \text{ but } c \not\leq x\}$. Then $a \in X$ and, since c is compact, the join of any chain in X is also in X . So by Zorn's lemma, X has a maximal element m . Suppose $m = \wedge Y$ and $m < y \forall y \in Y$. Then $a \leq m < y \forall y \in Y$ implies, by maximality of m in X that $c \leq y \forall y \in Y$. So $m \vee c \leq y \forall y \in Y$ yielding $m \vee c \leq m$. This contradiction implies that m is completely meet irreducible and therefore $m \in D$. But then $b \leq m$ contradicting $c \not\leq m$. Thus for all compact elements c below b , we have $c \leq a$ and so $b \leq a$, completing the proof. \square

We use 0_L and 1_L respectively to denote the bottom and top of a lattice L (0 and 1 if L is understood).

We write $\text{At}(L)$ for the set of all atoms of a lattice L with 0 .

In a lattice L with $a, b \in L$ we write $a \prec b$ to denote that b covers a (i.e. $a < b$ and if $a \leq b \leq c$ then $a = c$ or $b = c$).

0.4 Congruences and homomorphisms

For an algebra A we let $\text{Con}(A)$ denote the lattice of all congruences on A . For $a, b \in A$ we let $\text{con}(a, b)$ denote the principal congruence generated by (a, b) and $\theta_{(a,b)}$ the largest congruence on A separating a and b . Then $\text{Con}(A)$ is algebraic with compact elements the finite joins of principal congruences.

Recall that for $\theta, \Psi \in \text{Con}(A)$ we have $\theta \wedge \Psi = \theta \cap \Psi$ and for $(\theta_i)_{i \in I} \subseteq \text{Con}(A)$ we have

$(a, b) \in \bigvee_{i \in I} \theta_i$ if and only if there is a sequence $a = e_0, e_1, \dots, e_n = b$ such that for all $i \in \{1, \dots, n\}$ $(e_{i-1}, e_i) \in \theta_{j_i}$ for some $j_i \in I$. If A is a lattice then there is a sequence $a \wedge b = e_0 \leq e_1 \leq \dots \leq e_n = a \vee b$ such that the above condition holds. (Since $(a, b) \in \bigvee_{i \in I} \theta_i \Rightarrow (a \wedge b, a \vee b) \in \bigvee_{i \in I} \theta_i \Rightarrow$ there is a sequence $(a \wedge b) = f_0, f_1, \dots, f_n = (a \vee b)$ such that for all $i \in \{1, \dots, n\}$ $(e_{i-1}, e_i) \in \theta_{j_i}$ for some $j_i \in I$. Let $e_0 = f_0, e_{i+1} = e_i \vee (f_{i+1} \wedge f_n)$ for $0 \leq i \leq n-1$. Then $(a \wedge b) = e_0 \leq e_1 \leq \dots \leq e_n = (a \vee b)$ is the desired sequence.)

We use the following notation. For an algebra A with $a \in A$ and $\theta \in \text{Con}(A)$:
 a/θ is the congruence class of a modulo θ . If A is a lattice with 0 then we let $K(\theta)$ denote the congruence class of 0 modulo θ ($0/\theta$).
 A/θ is the quotient algebra of A modulo θ .
 Δ_A is the bottom element of $\text{Con}(A)$.
 ∇_A is the top element of $\text{Con}(A)$ (we use Δ, ∇ if A is understood).

Let Ψ be a congruence on an algebra A . We call Ψ a *subdirectly irreducible congruence* if A/Ψ is a subdirectly irreducible algebra. If Ψ is a congruence on a lattice L then we call Ψ a *distributive congruence* if L/Ψ is a distributive lattice.

Let A, B be algebras with $f : A \rightarrow B$ a homomorphism. Then the kernel of f ($= \{(x, y) \in A : f(x) = f(y)\}$) is denoted by $\ker(f)$.

We will make frequent use of the following basic theorems:
Homomorphism Theorem : If $f : A \rightarrow B$ is a homomorphism then $A/\ker(f) \cong f[A]$.
Second Isomorphism Theorem : Let A be an algebra with $\Psi, \theta \in \text{Con}(A)$ and $\theta \subseteq \Psi$. Then $\Psi/\theta = \{(a/\theta, b/\theta) : (a, b) \in \Psi\}$ is a congruence on A/θ such that $A/\Psi \cong (A/\theta)/(\Psi/\theta)$.
Correspondence Theorem : For any algebra A and $\theta \in \text{Con}(A)$: $\text{Con}(A/\theta) \cong [\theta, \nabla]$.

The homomorphism theorem then yields the following result: An algebra A is a subdirect product of quotient algebras A/θ_i if and only if $\bigcap \theta_i = \Delta$.

We obtain the following useful characterization of subdirectly irreducible algebras.
An algebra A is subdirectly irreducible if and only if Δ_A is completely meet irreducible if and only if $\text{Con}(A)$ has a smallest non trivial congruence (a unique atom).
Note that by the correspondence theorem a congruence θ is subdirectly irreducible if and only if it is completely meet irreducible.

Birkhoff's subdirect representation theorem ([Bir44]) states that every algebra is a subdirect product of its subdirectly irreducible homomorphic images.

For a subdirectly irreducible algebra A we say that A is (a, b) -irreducible if $\text{con}(a, b)$ is the smallest non-trivial congruence on A . We call the set $\{a, b\}$, a *critical pair* of A . If A is a lattice then we may assume without loss of generality that $a > b$. In that case we shall refer to the quotient a/b as a *critical quotient* of A .

Let A and B be algebras with $f : A \rightarrow B$ a surjective homomorphism. Let $\theta \in \text{Con}(A)$ be such that $\theta = \ker(f)$. We then call θ is a B -congruence on A .

An algebra A is *congruence-distributive* if $\text{Con}(A)$ is a distributive lattice. A variety \mathcal{V} is congruence distributive if for all $A \in \mathcal{V}$, A is congruence-distributive.

THEOREM 0.4.1 [*Fun42*] *Lattices are congruence distributive algebras.*

PROOF. Let L be a lattice with $\theta, \Psi, \Phi \in \text{Con}(L)$. The inequality $(\theta \cap \Phi) \vee (\theta \cap \Psi) \leq \theta \cap (\Phi \vee \Psi)$ holds in any lattice. Suppose $(a, b) \in \theta \cap (\Phi \vee \Psi)$. Then $(a, b) \in \theta$ and there is a finite sequence $a \wedge b = e_0 \leq e_1 \leq \dots \leq e_n = a \vee b$ in L such that for each $i \in \{1, \dots, n\}$ either $(e_{i-1}, e_i) \in \Phi$ or $(e_{i-1}, e_i) \in \Psi$. Since $(a, b) \in \theta$ we also have $(a \wedge b, a \vee b) \in \theta$ and hence $(e_{i-1}, e_i) \in \theta$ for all $i \in \{1, \dots, n\}$. Thus $(e_{i-1}, e_i) \in \theta \vee \Phi$ or $(e_{i-1}, e_i) \in \theta \vee \Psi$. But this implies $(a, b) \in (\theta \cap \Phi) \vee (\theta \cap \Psi)$, proving $\theta \cap (\Phi \vee \Psi) \leq (\theta \cap \Phi) \vee (\theta \cap \Psi)$. \square

As a consequence of Theorem 0.4.1 every lattice variety is congruence distributive.

We say a system $(\phi_i)_{i \in I}$ of congruences of an algebra A is a *direct representation* of A if the canonical map $f : A \rightarrow \prod_{i \in I} A/\phi_i$ given by $f(a) = a/\phi_i$ is an isomorphism.

We use the following lemma in the proof of Theorem 1.3.6.

LEMMA 0.4.2 *A finite system (ϕ_0, \dots, ϕ_n) of congruences of an algebra A is a direct representation if and only if $\phi_0 \cap \phi_1 \dots \cap \phi_n = \Delta$ and for each $i \leq n$ $\phi_i \circ \bigcap_{j \neq i} \phi_j = \nabla$.*

PROOF. Let $f : A \rightarrow \prod_{i \in I} A/\phi_i$ be the mapping defined above.

For the forward implication suppose (ϕ_0, \dots, ϕ_n) is a direct representation of A and $\bigcap_{i=0}^n \phi_i \neq \Delta$. Then $\exists a, b \in A$ with $a \neq b$ such that $(a, b) \in \phi_i \forall i \in I$. But then $f(a) = f(b)$ implying $a = b$. This contradiction yields $\bigcap_{i=0}^n \phi_i = \Delta$. Let $a, b \in A$. Consider $(a/\phi_0, \dots, a/\phi_{j-1}, b/\phi_j, a/\phi_{j+1}, \dots, a/\phi_n) \in \prod_{i=0}^n A/\phi_i$. Then, by surjectivity of f , $\exists c \in A$ such that $(a, c) \in \bigcap_{i \neq j} \phi_i$ and $(c, b) \in \phi_j$ giving $(a, b) \in \phi_i \circ \bigcap_{i \neq j} \phi_j$.

For the reverse implication suppose that $\phi_0 \cap \phi_1 \dots \cap \phi_n = \Delta$ and for each $i \leq n$ $\phi_i \circ \bigcap_{j \neq i} \phi_j = \nabla$. Then f is injective since $\bigcap_{i=0}^n \phi_i = \Delta$. To show that f is surjective, suppose $c = (a_0/\phi_0, a_1/\phi_1, \dots, a_n/\phi_n) \in \prod_{i=0}^n A/\phi_i$. It remains to find an $a \in A$ such that $(a, a_i) \in \phi_i \forall i \in \{0, 1, \dots, n\}$. We proceed by induction on n . For $n = 1$ we have $\phi_0 \circ \phi_1 = \nabla$ and the result holds. Assume that for $n = k$ we have found such an a and replace k by $k+1$. By induction there is a $b \in A$ such that $(b, a_i) \in \phi_i \forall i \in \{0, 1, \dots, k\}$. Since $\bigcap_{j \neq k+1} (\phi_j \circ \phi_{k+1}) = \nabla$ there is an $a \in A$ such that $(b, a) \in \phi_j \forall j \leq k$ and $(a, a_{k+1}) \in \phi_{k+1}$. Thus $(a, a_i) \in \phi_i \forall i \in \{0, 1, \dots, k+1\}$. \square

Chapter 1

General results : congruence distributivity and amalgamation

Sections 1.1-1.3 of this chapter deal with general results on congruence distributive algebras (lattices are congruence distributive algebras (Theorem 0.4.1)). In Section 1.1 we present Jónsson's Lemma and its corollaries which describe some of the most important properties specific to congruence distributive varieties. As a result of Theorem 0.4.1 we can apply the powerful tools of Jónsson's Lemma to lattice varieties. Section 1.2 gives Jónsson's characterization of congruence distributive varieties in terms of Mal'cev conditions: the so-called Jónsson polynomials. In Section 1.3 we consider some results on congruences on products of congruence distributive algebras. Many of these results are referenced later on. Section 1.4 contains some basic results on the amalgamation class of a variety laying the ground for the development of Chapter 6. The notions of congruence extendible and congruence extensile algebras are defined and we examine results concerning the extension of congruences in congruence distributive varieties. Many of the results of this chapter are well known and are included for completeness.

1.1 Filtral congruences and Jónsson's Lemma

The following result is due to B. Jónsson. (Recall the definition of $\Phi_{\mathcal{U}}$ from Chapter 0.)

LEMMA 1.1.1 [Jon67] *If A is a congruence distributive subalgebra of a direct product $C = \prod_{i \in I} C_i$ and $\varphi \in \text{Con}(A)$ is subdirectly irreducible, then there exists an ultrafilter \mathcal{U} over I such that $\Phi_{\mathcal{U}}|_A \subseteq \varphi$.*

PROOF. If $\varphi = \nabla$ then any ultrafilter over I will do so assume $\varphi < \nabla$. Let $[J]$ denote the principal filter generated by a subset J of I and let $\sigma_J = \Phi_{[J]}|_A$. Let \mathcal{D} be the family of

all subsets J of I such that $\sigma_J \subseteq \varphi$. Then \mathcal{D} has the following properties:

- (i) $J \cup K \in \mathcal{D} \Rightarrow J \in \mathcal{D}$ or $K \in \mathcal{D}$
- (ii) $K \supseteq J \in \mathcal{D} \Rightarrow K \in \mathcal{D}$
- (iii) $I \in \mathcal{D}, \emptyset \notin \mathcal{D}$

For all $J, K \subseteq I$ we have $\sigma_{J \cup K} = \sigma_J \cap \sigma_K$. So if $J \cup K \in \mathcal{D}$ then $\varphi = (\varphi \vee \sigma_{J \cup K}) = (\varphi \vee \sigma_J) \cap (\varphi \vee \sigma_K)$ by distributivity of $\text{Con}(A)$. By meet irreducibility of φ in $\text{Con}(A)$ we have $\sigma_J \subseteq \varphi$ or $\sigma_K \subseteq \varphi$. Thus $J \in \mathcal{D}$ or $K \in \mathcal{D}$ and (i) is satisfied. For (ii) suppose $J \in \mathcal{D}$ and $J \subseteq K$. Then $\sigma_K \subseteq \sigma_J \subseteq \varphi$. Hence $K \in \mathcal{D}$. We have $\sigma_I = \Delta$ and $\sigma_\emptyset = \nabla$. Hence (iii) is satisfied. By Zorn's Lemma there is a maximal filter \mathcal{U} contained in \mathcal{D} . Then $\Phi_{\mathcal{U}}|_A = \bigcup_{J \in \mathcal{D}} \sigma_J \subseteq \varphi$. We claim \mathcal{U} is an ultrafilter over I . If not then there is a subset J of I such that neither J nor $I \setminus J$ belong to \mathcal{U} . Suppose $H \cap J \in \mathcal{D}$ for all $H \in \mathcal{U}$. Then by (ii) $\mathcal{U} \cup \{J\}$ would generate a filter contained in \mathcal{D} contradicting maximality of \mathcal{U} . Thus for some $H \in \mathcal{U}$ we have $J \cap H \notin \mathcal{D}$. Similarly there is a $K \in \mathcal{D}$ such that $(I \setminus J) \cap K \in \mathcal{D}$. We have $H \cap K \in \mathcal{D}$ and $H \cap K = J \cap (H \cap K) \cup ((I \setminus J) \cap (H \cap K))$ contradicting (i) and proving that \mathcal{U} is an ultrafilter. \square

COROLLARY 1.1.2 [Jon67] *Let \mathcal{K} be a class of algebras such that $\mathcal{V} = \mathcal{K}^{\mathcal{V}}$ is congruence distributive. Then*

- (i) $\mathcal{V}_{SI} \subseteq \mathcal{V}_{FSI} \subseteq \mathbf{HSP}_{\mathcal{U}}(\mathcal{K})$
- (ii) $\mathcal{V} = \mathbf{P}_S \mathbf{HSP}_{\mathcal{U}}(\mathcal{K})$.

PROOF. We always have $\mathcal{V}_{SI} \subseteq \mathcal{V}_{FSI}$. By Birkhoff's subdirect representation theorem every algebra in \mathcal{V} is isomorphic to a quotient algebra B/θ where B is a subalgebra of $C = \prod_{i \in I} C_i$ such that $C_i \in \mathcal{K}$ for all $i \in I$. If B/θ is finitely subdirectly irreducible then θ is meet irreducible and by Lemma 1.1.1 there exists an ultrafilter \mathcal{U} over I such that $\Phi_{\mathcal{U}}|_B \subseteq \theta$. By the second isomorphism theorem B/θ is a homomorphic image of $B/(\Phi_{\mathcal{U}}|_B)$. But $B/(\Phi_{\mathcal{U}}|_B)$ is isomorphic to a subalgebra of the ultraproduct $C/\Phi_{\mathcal{U}}$. Thus $B/\theta \in \mathbf{HSP}_{\mathcal{U}}(\mathcal{K})$.

(ii) follows directly from (i) and Birkhoff's subdirect representation theorem. \square

In Lemma 1.1.1 above we call \mathcal{U} a *Jónsson ultrafilter* and $\Phi_{\mathcal{U}}$ a *Jónsson congruence*.

LEMMA 1.1.3 *Let A be a congruence distributive algebra with $A \leq C = \prod_{i \in I} C_i$. Let $\theta \in \text{Con}(A)$ be subdirectly irreducible and suppose $\Omega_{\mathcal{F}}$ is a filtral congruence in C with $\Omega_{\mathcal{F}}|_A \subseteq \theta$. Then there exists a Jónsson congruence $\Psi \in \text{Con}(C)$ such that $\Omega_{\mathcal{F}} \subseteq \Psi$ and $\Psi|_A \subseteq \theta$.*

PROOF. Let Ψ be a Jónsson congruence on C such that $\Psi|_A \subseteq \theta$ and let \mathcal{C} be the Jónsson ultrafilter inducing Ψ . For $K \subseteq I$, let σ_K and \mathcal{D} be as in the proof of Lemma 1.1.1. Then \mathcal{C} is a maximal filter over I with respect to the property that $\mathcal{C} \subseteq \mathcal{D}$. Let $K \in \mathcal{F}$. For all $a, b \in A$ we have

$$\begin{aligned} (a, b) \in \sigma_K &\Rightarrow \{i \in I : a_i = b_i\} \supseteq K \in \mathcal{F} \\ &\Rightarrow (a, b) \in \Omega_{\mathcal{F}} \end{aligned}$$

Thus $\sigma_K \subseteq \Omega_{\mathcal{F}}|_A \subseteq \theta$. Consequently $\mathcal{F} \subseteq \mathcal{D}$ and so $\mathcal{F} \subseteq \mathcal{C}$. We then have $\Omega_{\mathcal{F}} \subseteq \Psi$. \square

LEMMA 1.1.4 [Fra63] *If \mathcal{K} is a finite set of finite algebras, then every ultraproduct of members of \mathcal{K} is isomorphic to a member of \mathcal{K} .*

PROOF. Let $A = \prod_{i \in I} A_i$ be a direct product of members of \mathcal{K} and let \mathcal{U} be an ultrafilter over I . Define an equivalence relation \sim on I by $i \sim j$ if and only if $A_i \cong A_j$. Then \sim partitions I into finitely many blocks I_0, I_1, \dots, I_m . Since \mathcal{U} is an ultrafilter over I , it must contain exactly one of the blocks I_n say. Let $X = A_i = A_j$ for $i, j \in I_n$. Then $\prod_{i \in I} A_i / \mathcal{U} \cong X^{I_n}$, the isomorphism given by $a / \mathcal{U} \mapsto (a_j)_{j \in I_n}$. Since X is finite we have $X^{I_n} \cong X = A_i \in \mathcal{K}$. \square

COROLLARY 1.1.5 *Let \mathcal{K} be a finite set of finite algebras such that $\mathcal{V} = \mathcal{K}^{\vee}$ is congruence distributive. Then*

- (i) $\mathcal{V}_{SI} \subseteq \mathcal{V}_{FSI} \subseteq \mathbf{HS}(\mathcal{K})$.
- (ii) \mathcal{V} has up to isomorphism only finitely many subdirectly irreducible members, each one finite.
- (iii) If $A, B \in \mathcal{V}_{SI}$ are non-isomorphic and $|A| \leq |B|$, then there is an identity which holds in A but not in B .

PROOF. (i) follows immediately from Corollary 1.1.2 and Lemma 1.1.4. (ii) follows from (i) since $\mathbf{HS}(\mathcal{K})$ has only finitely many members. $|A| \leq |B|$ implies $B \notin \mathbf{HS}(A)$. Hence $B \notin \{A\}^{\vee}$ and (iii) holds. \square

1.2 Jónsson polynomials: characterization of congruence distributivity

The classic theorem of Mal'cev [Mal54] showed that a variety \mathcal{V} of algebras has permutable congruences if and only if there exists a ternary polynomial p such that the identities

$$p(x, y, y) = x = p(y, y, x)$$

hold in \mathcal{V} .

An analogous characterization of arithmetical varieties (i.e. varieties in which every algebra is congruence distributive and congruence permutable) was given by Pixley [Pix63] : A variety \mathcal{V} is arithmetical if and only if there is a ternary polynomial p such that the following identities hold in \mathcal{V} :

$$p(x, y, y) = x = p(y, y, x) = p(x, y, x)$$

A. Day proved a similar result for congruence modular varieties (see [Day70]) :

A variety \mathcal{V} is congruence modular if and only if for some positive integer n , there exist quarternary poynomials p_0, p_1, \dots, p_n such that for $i = 0, 1, \dots, n - 1$, the following identities hold in \mathcal{V} :

$$p_0(x, y, z, w) = x, p_n(x, y, z, w) = w, p_i(x, y, y, x) = x$$

$$p_i(x, y, y, w) = p_{i+1}(x, y, y, w) \text{ for } i \text{ odd}$$

$$p_i(x, x, w, w) = p_{i+1}(x, x, w, w) \text{ for } i \text{ even.}$$

These results led to the general concept of a *Mal'cev type condition* being formulated by G. Gratzer in [Gra70] where conditions are presented for regularity and weak regularity. The following theorem due to Jónsson provides Mal'cev conditions for congruence distributivity.

THEOREM 1.2.1 [Jon67] *For any variety \mathcal{V} of algebras the following are equivalent:*

(i) \mathcal{V} is congruence distributive.

(ii) $F_{\mathcal{V}}(3)$ (the \mathcal{V} -free algebra on 3 generators) is congruence distributive.

(iii) For any $A \in \mathcal{V}$ and any $a, b, c \in A$: $(a, c) \in (\text{con}(a, c) \cap \text{con}(a, b)) \vee (\text{con}(a, c) \cap \text{con}(b, c))$.

(iv) For some positive integer n there exist ternary polynomials p_0, p_1, \dots, p_n such that for $i = 0, 1, \dots, n - 1$ the following identities hold in \mathcal{V} :

$$p_0(x, y, z) = x$$

$$p_n(x, y, z) = z$$

$$p_i(x, y, x) = x$$

$$p_i(x, x, z) = p_{i+1}(x, x, z) \text{ for } i \text{ even}$$

$$p_i(x, z, z) = p_{i+1}(x, z, z) \text{ for } i \text{ odd.}$$

PROOF. (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are obvious.

Assuming (iii), let $A = F_{\mathcal{V}}(\{a, b, c\})$ with $\theta = \text{con}(a, c)$, $\phi = \text{con}(a, b)$ and $\psi = \text{con}(b, c)$. Then there exist finitely many elements d_0, d_1, \dots, d_n in A such that

$$a = d_0 (\theta \cap \phi) d_1 (\theta \cap \psi) d_2 (\theta \cap \phi) d_3 \dots d_{n-1} (\theta \cap \phi) d_n = c.$$

Each d_i is of the form $p_i(a, b, c)$ and $p_0(a, b, c) = a, p_n(a, b, c) = c$. Also

$$p_i(a, b, a) \theta p_i(a, b, c) \theta p_{i+1}(a, b, c) \theta p_{i+1}(a, b, a).$$

But θ restricted to the subalgebra of A generated by $\{a, b\}$ is trivial. Thus $p_i(a, b, a) = p_{i+1}(a, b, a)$ and so $p_i(a, b, a) = a$.

For i even we have

$$p_i(a, a, c) \phi p_i(a, b, c) \phi p_{i+1}(a, b, c) \phi p_{i+1}(a, a, c).$$

But since ϕ restricted to the subalgebra of A generated by $\{a, c\}$ is trivial, $p_i(a, a, c) = p_{i+1}(a, a, c)$. Similarly for i odd $p_i(a, c, c) = p_{i+1}(a, c, c)$. The equations in (iv) are satisfied for $x = a, y = b, z = c$, and thus hold in \mathcal{V} .

Now assume (iv) and consider $A \in \mathcal{V}$ with $\theta, \phi, \psi \in \text{Con}(A)$. Let $\delta_k = \phi \circ \psi \circ \phi \circ \psi \cdots (k \text{ factors})$. Then $\theta \cap (\phi \vee \psi) = \bigcup_{k \in \omega} (\theta \cap \delta_k)$. To prove the distributivity of $\text{Con}(A)$ it is sufficient to show that $\theta \cap \delta_k \subseteq (\theta \cap \phi) \vee (\theta \cap \psi)$ for all $k \in \omega$. For $k = 1$ the inclusion is obvious. So assume it holds for a given value of k and replace k by $k + 1$. Then $\delta_{k+1} = \delta_k \circ \beta$ where $\beta = \phi$ or $\beta = \psi$. We claim:

$$\theta \cap \delta_{k+1} \subseteq (\theta \cap \delta_k^{-1}) \circ (\theta \cap \delta_k) \circ (\theta \cap \beta) \circ (\theta \cap \beta) \circ (\theta \cap \delta_k^{-1}) \cdots \quad (1.1)$$

with $2n$ factors on the right.

Assuming $a \theta c$ and $a \delta_k b \beta c$, let $d_i = p_{i+1}(a, b, c)$. Then $a = d_0, c = d_n$ and for $i \in \{0, 1, \dots, n\}$:

$d_i = p_i(a, b, c) \theta p_i(c, b, c) = c = p_i(c, c, c) \theta p_i(a, c, c)$ and $d_i = p_i(a, b, c) \theta p_i(a, b, a) = a = p_i(a, a, a) \theta p_i(a, a, c)$.

For i even $d_i = p_i(a, b, c) \delta_k^{-1} p_i(a, a, c) = p_{i+1}(a, a, c) \delta_k p_i(a, b, c) = d_{i+1}$. Consequently $d_i (\theta \cap \delta_k^{-1}) p_i(a, a, c) (\theta \cap \delta_k) d_{i+1}$. For i odd $d_i = p_i(a, b, c) \beta p_i(a, c, c) = p_{i+1}(a, c, c) \beta p_{i+1}(a, b, c) = d_{i+1}$. Thus $d_i (\theta \cap \beta) p_i(a, c, c) (\theta \cap \beta) d_{i+1}$. Hence 1.1 follows and consequently $\theta \cap \delta_k \subseteq (\theta \cap \phi) \vee (\theta \cap \psi)$ as required. \square

The polynomial p_0, p_1, \dots, p_n are known as Jónsson polynomials. Note that the above theorem offers an alternative proof of the congruence distributivity of lattices considering the polynomials $p_0(x, y, z) = x$, $p_1(x, y, z) = (x \wedge y) \vee (y \wedge z) \vee (z \wedge x)$ and $p_2(x, y, z) = z$.

1.3 Congruences on products

LEMMA 1.3.1 *Let B be a congruence distributive algebra with $\{\psi_i : i \in I\}$ a finite set of congruences on B such that $\bigcap \psi_i = \Delta$. Then if $\theta \in \text{Con}(B)$ is subdirectly irreducible we have $\psi_i \subseteq \theta$ for some $i \in I$.*

PROOF. We have $\theta = \theta \vee \bigcap_{i \in I} \psi_i$. Thus by distributivity of $\text{Con}(B)$ and meet irreducibility of θ , $\theta = \theta \vee \psi_i$ for some $i \in I$. \square

The following corollary is a direct consequence of Lemma 1.3.1 above.

COROLLARY 1.3.2 *Let A, B be algebras such that $A \times B$ is congruence distributive. If $\theta \in \text{Con}(A \times B)$ is subdirectly irreducible then either $\ker(\pi_A) \subseteq \theta$ or $\ker(\pi_B) \subseteq \theta$.*

A subalgebra A of a direct product of algebras $(A_i)_{i \in I}$ is called *discrete* if for all $a \neq b$ in A , a and b differ on only finitely many coordinates.

PROPOSITION 1.3.3 *Let $\prod_{i \in I} A_i$ be a product of congruence distributive algebras. Let A be a discrete subalgebra of $\prod_{i \in I} A_i$. If $\theta \in \text{Con}(A)$ is subdirectly irreducible then $\ker(\pi_i) \subseteq \theta$ for some $i \in I$.*

PROOF. Let \mathcal{U} be a Jónsson ultrafilter over I with $\Psi \in \text{Con}(\prod_{i \in I} A_i)$ the corresponding Jónsson congruence such that $\Psi|_A \subseteq \theta$. If \mathcal{U} is not principal then by Lemma 0.2.2 we have $X \notin \mathcal{U}$ for all finite subsets X of I . Let $a, b \in A$ with $a \neq b$. Then $\{i \in I : a_i \neq b_i\} \notin \mathcal{U}$ since this set is finite. Thus $\{i \in I : a_i = b_i\} \in \mathcal{U}$ and so $(a, b) \in \Psi|_A$. But then $\nabla_A = \Psi|_A \subseteq \theta$ and so $\ker(\pi_i) \subseteq \theta$ for all $i \in I$. If \mathcal{U} is principal then by Lemma 0.2.1 $\mathcal{U} = \{\{i\}\}$ for some $i \in I$. Thus $\ker(\pi_i) \subseteq \Psi|_A \subseteq \theta$. \square

In general if $B = \prod_{i \in I} B_i$ is a direct product of *finitely* many algebras then $\text{Con}(B)$ embeds in $\prod_{i \in I} \text{Con}(B_i)$. We make frequent use of the following proposition which shows that if B is congruence distributive then the embedding is an isomorphism.

PROPOSITION 1.3.4 [Jon90] *Let $B = \prod_{i \in I} B_i$ be the direct product of finitely many algebras. If B is congruence distributive then $\text{Con}(B)$ is isomorphic to $\prod_{i \in I} \text{Con}(B_i)$.*

PROOF. Define a map $\alpha : \prod_{i \in I} \text{Con}(B_i) \rightarrow \text{Con}(B)$ which takes $\theta = (\theta_i)_{i \in I}$ to $\bar{\theta}$, where for $x, y \in B$ we have $(x, y) \in \bar{\theta}$ if and only if $(x_i, y_i) \in \theta_i \forall i \in I$. Then $\bar{\theta}$ is a congruence relation on B . Now $\theta \subseteq \psi$ in $\prod_{i \in I} \text{Con}(B_i) \Leftrightarrow \theta_i \subseteq \psi_i \forall i \in I \Leftrightarrow \bar{\theta} \subseteq \bar{\psi}$.

Conversely, suppose $\bar{\theta} \subseteq \bar{\psi}$. Then $(a, b) \in \theta_i \Leftrightarrow (x, y) \in \bar{\theta}$ where $x_i = a, y_i = b$ and $x_j = y_j \forall i \neq j$. Thus $(x, y) \in \bar{\psi}$ and so $(a, b) \in \psi_i$. We have $\theta_i \subseteq \psi_i \forall i \in I$, giving $\theta \subseteq \psi$. Thus α is an injective homomorphism.

To show that α is a surjection let ρ_i be the kernel of the projection from B to B_i . For $\psi \in \text{Con}(B)$ we have $\psi = \psi \vee \Delta = \psi \vee \bigcap_{i \in I} \rho_i = \psi \vee \bigcap_{i \in I} (\psi \vee \rho_i)$, by distributivity of $\text{Con}(B)$. Now $B/\rho_i \cong B_i$ and $\rho_i \subseteq \psi \vee \rho_i$. Then for each $i \in I$ there exists a $\phi \in \text{Con}(B)$ such that $(x, y) \in \psi \vee \rho_i$ if and only if $(x_i, y_i) \in \phi_i$ (second isomorphism theorem).

$$\begin{aligned} \text{Consequently } (x, y) \in \psi &\Leftrightarrow (x, y) \in \bigcap_{i \in I} (\psi \vee \rho_i) \\ &\Leftrightarrow (x_i, y_i) \in \phi_i \forall i \in I. \end{aligned}$$

Thus $\psi = \bar{\phi}$ where $\phi = (\phi_i)_{i \in I}$ and so α is a surjection. \square

LEMMA 1.3.5 Let $\mathcal{U}_1, \dots, \mathcal{U}_n$ be distinct ultrafilters over I . Then for each $j \in \{1, \dots, n\}$ we have $\bigcap_{i \neq j} \mathcal{U}_i \not\subseteq \mathcal{U}_j$.

PROOF. Suppose $\bigcap_{i \neq j} \mathcal{U}_i \subseteq \mathcal{U}_j$ for some $j \in \{1, \dots, n\}$. Since the \mathcal{U}_i 's are distinct we have $\forall i \neq j \exists A_i \in \mathcal{U}_i - \mathcal{U}_j$. But then $\bigcup_{i \neq j} A_i \in \bigcap_{i \neq j} \mathcal{U}_i$ and $\bigcup_{i \neq j} A_i \notin \mathcal{U}_j$ (by primeness). \square

The following result of Bergman is used in the proofs of Lemma 5.1.2 and Theorem 5.3.1.

THEOREM 1.3.6 [Ber89] Let \mathcal{S} be a finite set of finite algebras generating a congruence distributive variety. Then every finite homomorphic image of a product of members of \mathcal{S} is isomorphic to a product of homomorphic images of those members of \mathcal{S} i.e. $\mathbf{HP}(\mathcal{S})_{fin} \cong \mathbf{PH}(\mathcal{S})$.

PROOF. Let $L = \prod_{i \in I} S_i$ be a product of members of \mathcal{S} and let $\theta \in \text{Con}(L)$ be such that L/θ is finite. Then $\theta = \alpha_1 \cap \dots \cap \alpha_n$ where each α_i is completely meet irreducible. Let $\Phi_{\mathcal{U}_i}$ be a Jónsson congruence (with corresponding Jónsson ultrafilter \mathcal{U}_i) such that $\Phi_{\mathcal{U}_i} \subseteq \alpha_i$ ($1 \leq i \leq n$). The \mathcal{U}_i 's may not be distinct, so for $\mathcal{U}_j = \mathcal{U}_k$ replace both α_j and α_k by $\alpha_j \cap \alpha_k$. Then we have pairwise distinct ultrafilters $\mathcal{U}_1, \dots, \mathcal{U}_m$ and congruences β_1, \dots, β_m such that $\theta = \beta_1 \cap \dots \cap \beta_m$ and $\Phi_{\mathcal{U}_j} \subseteq \beta_j$ for each $j = 1, \dots, m$. (The β_j 's are not necessarily completely meet irreducible.)

We claim that for each $k \in \{1, \dots, m\}$ we have $(\bigcap_{j \neq k} \Phi_{\mathcal{U}_j}) \circ \Phi_{\mathcal{U}_k} = \nabla$. To see this let $x, y \in L$ and put $\mathcal{V} = \bigcap_{j \neq k} \mathcal{U}_j$. Then \mathcal{V} is a filter over I and $\Phi_{\mathcal{V}} = \bigcap_{j \neq k} \Phi_{\mathcal{U}_j}$. Also, by Lemma 1.3.5 we cannot have $\mathcal{V} \subseteq \mathcal{U}_k$. Let $E \in \mathcal{V} \setminus \mathcal{U}_k$. Define $z \in L$ as follows:

$$z_i = \begin{cases} x_i & i \in E \\ y_i & i \in I \setminus E \end{cases}$$

Then $x \Phi_{\mathcal{V}} z \Phi_{\mathcal{U}_k} y$ proving the claim. Since $\Phi_{\mathcal{U}_k} \leq \beta_k \forall k \in \{1, \dots, m\}$ we have $(\bigcap_{j \neq k} \beta_j) \circ \beta_k = \nabla$ and so $(\bigcap_{j \neq k} (B_j/\theta)) \circ B_k/\theta = \nabla_{L/\theta}$. Lemma 0.4.2 and the second isomorphism theorem yield

$$L/\theta \cong \prod_{j=1}^m (L/\theta)/(\beta_j/\theta) \cong \prod_{j=1}^m (L/\beta_j).$$

Since $\Phi_{\mathcal{U}_j} \subseteq \beta_j$ L/β_j is a homomorphic image of $L/\Phi_{\mathcal{U}_j}$. By Lemma 1.1.4 $L/\Phi_{\mathcal{U}_j}$ is isomorphic to S_i for some $i \in I$. Thus $L/\theta \in \mathbf{PH}(\mathcal{S})$. \square

An algebra B is a *retract* of an algebra A if there exist homomorphisms $f : B \rightarrow A$ and $g : A \rightarrow B$ such that $g \circ f$ is the identity on B . Then f is an embedding and g is called a *retraction* of f .

We make use of the following lemma in the proof of Lemma 6.3.1.

LEMMA 1.3.7 [Jip89] *Let A and B be congruence distributive algebras, $a \in A$ and let $\{a\}$ be a subalgebra of A . Let $h_a : B \hookrightarrow A \times B$ be the embedding given by $h_a(b) = (a, b)$ for all $b \in B$. Then the projection $\Pi_B : A \times B \rightarrow B$ is the only retraction of h_a onto B .*

PROOF. Suppose $g : A \times B \rightarrow B$ is a retraction of h_a onto B . By Proposition 1.3.4 there are congruences $\theta_A \in \text{Con}(A)$ and $\theta_B \in \text{Con}(B)$ such that $\ker(g) = \theta_A \times \theta_B$. Since $g \circ h_a$ is an identity map on B we must have that θ_B is the trivial congruence on B . To show that $g = \Pi_B$ we need to show that $\theta_A = \nabla_A$. It suffices to show that for any $a_1 \in A$ we have $(a_1, a) \in \theta_A$. For any $b_1, b_2 \in B$ with $b_1 \neq b_2$ we have $g(a, b_1) = b_1 \neq b_2 = g(a, b_2)$. Now if $(a, a_1) \notin \theta_A$ for some $a_1 \in A$ then there exist $b_1, b_2 \in B$ such that $g(a, b_1) = b_1 \neq b_2 = g(a_1, b_1)$. Thus $g(a, b_2) = g(a_1, b_1)$ implying $(b_1, b_2) \in \theta_B$ and $(a, a_1) \in \theta_A$, a contradiction. \square

1.4 Extension of congruences and amalgamation

As an example of the connection between extensions of congruences and amalgamation we have for \mathcal{V} a residually small congruence distributive one-subalgebra variety (e.g. a lattice variety) that members of the amalgamation class of \mathcal{V} are congruence extensible in \mathcal{V} . (See Proposition 1.4.13). In this section we introduce the notions of congruence extensible and congruence extendible algebras and review rudimentary results concerning amalgamation and absolute retracts some of which will be used in Chapter 6. Also presented are results on the extension of congruences in congruence distributive varieties and Bergman's characterization of the congruence extensible members of an arbitrary variety ([Ber85]).

1.4.1 Congruence extendible/extensible algebras

For a positive integer n we denote by 'n' the n -element chain.

An algebra A in a variety \mathcal{V} is said to be *congruence extensible in \mathcal{V}* or satisfies the *congruence extension property* (CEP) if for any $C \in \mathcal{V}$ with $A \leq C$, every congruence on A can be extended to a congruence on C . Let \mathcal{V}_{CEP} denote the class of congruence extensible members of \mathcal{V} .

$A \in \mathcal{V}$ is *P -congruence extendible* in \mathcal{V} if whenever $C \in \mathcal{V}$ and $A \leq C$ then every P -congruence on A can be extended to a P -congruence on C . A is called *congruence extendible* in \mathcal{V} if A is P -congruence extendible for every P -congruence on A .

In \mathcal{N} , the variety generated by the pentagon, N , we have $A \in \mathcal{N}$ is 2-congruence extendible implies $A \in \mathcal{N}_{CEP}$ (see Proposition 4.2.11). However the converse does not hold since $\mathbf{2} \in \mathcal{N}_{CEP}$ but $\mathbf{2}$ is not 2-congruence extendible in \mathcal{N} . (Consider the embedding $\mathbf{2} \hookrightarrow N$ which takes $\mathbf{2}$ to the critical quotient of N .)

1.4.2 Absolute retracts and essential extensions

An algebra A in a variety \mathcal{V} is an *absolute retract* of \mathcal{V} if for every embedding $f : A \hookrightarrow B$ there is an epimorphism $g : B \rightarrow A$ such that $g \circ f$ is the identity map on A . Equivalently A is an absolute retract if and only if A is congruence extendible in \mathcal{V} . We denote the class of all absolute retracts in a variety \mathcal{V} by \mathcal{V}_{AR} .

THEOREM 1.4.1 [Jip89] *Let \mathcal{V} be a one-subalgebra congruence distributive variety. Then every direct product of absolute retracts in \mathcal{V} is an absolute retract in \mathcal{V} .*

Let \mathcal{V} be a variety. An extension C of an algebra A is said to be *essential* if every non-trivial congruence on C restricts to a non-trivial congruence on A . A is *maximally irreducible* in \mathcal{V} if A is subdirectly irreducible and A has no subdirectly irreducible proper essential extension in \mathcal{V} . The class of maximal irreducibles in \mathcal{V} is denoted by \mathcal{V}_{MI} .

A variety \mathcal{V} is *residually small* if there is an upper bound on the cardinality of the subdirectly irreducible members of \mathcal{V} , equivalently; if the subdirectly irreducible members of \mathcal{V} (up to isomorphism) form a set.

THEOREM 1.4.2 [Tay72] *\mathcal{V} is a residually small variety if and only if every member of \mathcal{V}_{SI} has an essential extension in \mathcal{V}_{MI} .*

LEMMA 1.4.3 *In a variety \mathcal{V} , $M \in \mathcal{V}_{MI}$ if and only if $M \in \mathcal{V}_{SI} \cap \mathcal{V}_{AR}$.*

1.4.3 Amalgamation

A *diagram* in a variety \mathcal{V} of algebras is a quintuple (A, f, C, g, D) where $f : A \hookrightarrow C$ and $g : A \hookrightarrow D$ are embeddings with $A, C, D \in \mathcal{V}$. An *amalgam* in \mathcal{V} of this diagram is a triple (E, f', g') where $f' : C \hookrightarrow E$ and $g' : D \hookrightarrow E$ are embeddings with $E \in \mathcal{V}$ and $f \circ f' = g \circ g'$. If such an amalgam exists, we say the diagram can be amalgamated in \mathcal{V} . An algebra $A \in \mathcal{V}$ is called an *amalgamation base* for \mathcal{V} if every diagram (A, f, C, g, D) can be amalgamated in \mathcal{V} . The class of all amalgamation bases for \mathcal{V} is called the *amalgamation class* of \mathcal{V} and is denoted by $\text{Amal}(\mathcal{V})$. If $\text{Amal}(\mathcal{V}) = \mathcal{V}$ then \mathcal{V} is said to satisfy the *amalgamation property*. By [Pie68] the variety of distributive lattices satisfies the amalgamation property. However for any non-distributive finitely generated lattice variety \mathcal{V} we have $\mathbf{2} \notin \text{Amal}(\mathcal{V})$ and consequently the amalgamation property does not hold in \mathcal{V} .

THEOREM 1.4.4 [Ber85] *Every absolute retract of a variety \mathcal{V} is an amalgamation base of \mathcal{V} .*

Note that Lemma 1.4.3 and Theorem 1.4.4 imply that for a variety \mathcal{V} , $\mathcal{V}_{MI} \subseteq \text{Amal}(\mathcal{V})$.

The following theorem generalizes Jónsson's characterization of the amalgamation class of a finitely generated lattice variety (see [Jon90]).

THEOREM 1.4.5 [Jip89] *Let \mathcal{V} be a residually small one-subalgebra congruence distributive variety. Then the following are equivalent:*

- (i) $A \in \text{Amal}(\mathcal{V})$.
- (ii) For any embedding $f : A \hookrightarrow C$ with $C \in \mathcal{V}$, and for any homomorphism $g : A \rightarrow M$ with $M \in \mathcal{V}_{MI}$, there is a homomorphism $h : C \rightarrow M$ such that $h \circ f = g$.
- (iii) Let $h : A \rightarrow M \in \mathcal{V}_{MI}$ be a homomorphism and $g : A \hookrightarrow A \times M$ be the embedding given by $g(a) = (a, h(a))$ for all $a \in A$. If $f : A \hookrightarrow B$ is an essential embedding then the diagram $(A, f, B, g, A \times M)$ can be amalgamated in \mathcal{V} .

Let $C = \prod_I C_i$ be a product of algebras and let $f : A \hookrightarrow C$ be an embedding. We shall say that f is *regular* if for any distinct $i, j \in I$ we have $\theta_i|_f[A] \neq \theta_j|_f[A]$, where θ_i and θ_j are the kernels of the canonical projections of C onto C_i and C_j respectively.

The following lemma is Corollary 1.1.3 of [Bru92].

LEMMA 1.4.6 *Let \mathcal{V} be a finitely generated, residually small one-subalgebra variety. Suppose that for any distinct $C, D \in \mathcal{V}_{MI}$ C is not an image of a subalgebra of D . Let A be a subdirect product of members of \mathcal{V}_{MI} . Then $A \in \text{Amal}(\mathcal{V})$ if and only if for any regular subdirect representation $f : A \rightarrow S$ where S is a product of members of \mathcal{V}_{MI} , and any homomorphism $g : A \rightarrow M$ where $M \in \mathcal{V}_{MI}$, there is a homomorphism $h : S \rightarrow M$ such that $g = h \circ f$.*

1.4.4 Extension of congruences

PROPOSITION 1.4.7 *Let \mathcal{V} be a residually small one-subalgebra congruence distributive variety with $M \in \mathcal{V}_{MI}$ and P a retract of M . Let $A \in \text{Amal}(\mathcal{V})$ and let $f : A \hookrightarrow C$ be an embedding with $C \in \mathcal{V}$. Then every P -congruence on A can be extended to a P -congruence on C .*

PROOF. Let θ be a P -congruence on A . Then, since P is a retract of M there is a homomorphism $g : A \rightarrow M$ such that $\theta = \ker(g)$. By Corollary 1.4.5 there is a homomorphism $h : C \rightarrow M$ such that $h \circ f = g$. Let $k : M \rightarrow P$ be a retraction and let $\Psi = \ker(h \circ k)$. Then Ψ is a P -congruence on C extending θ . \square

PROPOSITION 1.4.8 [Jon90] *Let $A \leq B = \prod_{i \in I} B_i$ be a subdirect product of finitely many algebras. If A is congruence distributive, then every congruence on A can be extended to a congruence on B .*

of M_j for $M_i \neq M_j$. Then every M_i -congruence on A can be extended to a M_i -congruence on $\prod_{i \in I} M_i$.

PROOF. Let θ be an M_i -congruence on A and let Ψ be a Jónsson congruence on $\prod M_i$ with $\Psi|_A \subseteq \theta$. Then by Lemma 1.1.4 $(\prod M_i)/\Psi \cong M_j$ for some $j \in I$. Since $A/\theta \hookrightarrow (\prod M_i)/\Psi$ we have $M_i \hookrightarrow M_j$ and since $M_j \in \mathcal{V}_{AR}$ (Lemma 1.4.3), M_i is an image of M_j . By assumption we then have $M_i = M_j$ and so Ψ is an M_i -congruence on $\prod M_i$. Since $A/\theta \hookrightarrow A/\Psi|_A \hookrightarrow \prod M_i/\Psi$ we have $A/\Psi|_A \cong M_i$ and hence $\Psi|_A = \theta$. \square

The following corollary is applied in Chapter 4 in the characterization of 2-congruence extendible lattices.

COROLLARY 1.4.12 *Let A and C be distributive lattices with $A \leq C$. Then every 2-congruence on A can be extended to a 2-congruence on C .*

PROOF. Since C is a distributive lattice it is a subdirect product of $\mathbf{2}$: $C \leq \mathbf{2}^I$. Thus $A \leq C \leq \mathbf{2}^I$ and by the above proposition every 2-congruence θ on A can be extended to a 2-congruence Ψ on $\mathbf{2}^I$. But then the restriction of Ψ to C is an extension of θ to C and $A/\theta \hookrightarrow C/\Psi|_C \hookrightarrow \mathbf{2}^I/\Psi$ implies that $\Psi|_C$ is a 2-congruence on C . \square

PROPOSITION 1.4.13 [Jon90] *Let \mathcal{V} be a residually small one-subalgebra congruence distributive variety. Then $\text{Amal}(\mathcal{V}) \subseteq \mathcal{V}_{CEP}$.*

PROOF. We use Proposition 1.4.10. Let $A \in \text{Amal}(\mathcal{V})$ and let B be an extension of A . Let θ be a subdirectly irreducible congruence on A with $v_\theta : A \rightarrow A/\theta$ the canonical homomorphism from A onto A/θ . By Theorem 1.4.2 there is an embedding $g : A/\theta \hookrightarrow M$ for some $M \in \mathcal{V}_{MI}$ and by Theorem 1.4.5 there is a map $h : B \rightarrow M$ such that $g \circ v_\theta = h|_A$. Then $\ker(h)$ extends θ since for $a, b \in A$ we have $(a, b) \in \ker(h)|_A \Leftrightarrow h(a) = h(b) \Leftrightarrow g \circ v_\theta(a) = g \circ v_\theta(b) \Leftrightarrow v_\theta(a) = v_\theta(b) \Leftrightarrow (a, b) \in \theta$. By Lemma 1.4.9 there is a subdirectly irreducible congruence $\Psi \in \text{Con}(B)$ such that $\Psi|_A = \ker(h)|_A = \theta$. Thus by Proposition 1.4.10 $A \in \mathcal{V}_{CEP}$. \square

Using Proposition 1.4.13 and the fact that the variety of distributive lattices satisfies the amalgamation property ([Pie68]) we can deduce the following well-known result.

COROLLARY 1.4.14 *The variety of distributive lattices satisfies (CEP).*

Chapter 2

Weak projections, congruences and ideals

In this chapter we focus on weak projections which play a crucial role in the study of congruences on lattices given Dilworth's characterization of lattice congruences via weak projections. In Section 2.1 we present this characterization and then in Section 2.2 we look at some basic results on modular and distributive lattices which are based on weak projections. We include a brief section (2.3) on primitive subsets of lattices, proving some results that are subsequently used in Herrmann's proof of a finite basis theorem for lattices (see Section 2.4) as well as in Chapter 6. In Section 2.5 we define a \mathcal{S} -lattice and a \mathcal{S} -variety and using weak projections we prove that the class of subdirect products of non-modular subdirectly irreducible members of a \mathcal{S} -variety is closed under reduced products. We conclude with a short section (2.6) on the link between congruences and ideals. Unrelated to the rest of the material of this chapter it is tacked on here for want of a better place.

2.1 Dilworth's Theorem

We begin with the Grätzer-Schmidt theorem which provides sufficient conditions for a binary relation on a lattice to be a congruence relation. These conditions are then used to prove Dilworth's theorem.

LEMMA 2.1.1 (Grätzer-Schmidt Criteria) [Gra58] *Let L be a lattice with θ a binary relation on L . Then θ is a congruence relation on L if and only if the following conditions are satisfied for all $x, y, z \in L$:*

- (i) θ is reflexive.
- (ii) $(x, y) \in \theta$ if and only if $(x \wedge y, x \vee y) \in \theta$.

- (iii) If $x \leq y \leq z$, $(x, y) \in \theta$ and $(y, z) \in \theta$ then $(x, z) \in \theta$.
(iv) If $x \leq y$ and $(x, y) \in \theta$ then $(x \wedge w, y \wedge w) \in \theta$ and $(x \vee w, y \vee w) \in \theta \forall w \in L$.

PROOF. If θ is a congruence relation on L then clearly (i), (ii), (iii) and (iv) hold. For the converse we have for $a, b \in L$ that $(a, b) \in \theta$ if and only if $(a \wedge b, a \vee b) \in \theta$. Since \vee and \wedge are commutative we then have $(b, a) \in \theta$ and so θ is symmetric. For transitivity, suppose $(a, b) \in \theta$ and $(b, c) \in \theta$. Then

$$\begin{aligned}
(a, b) \in \theta &\Rightarrow (a \wedge b, a \vee b) \in \theta && \text{(by (ii))} \\
&\Rightarrow ((a \wedge b) \vee a, (a \vee b) \vee a) \in \theta && \text{(by (iv))} \\
&\Rightarrow (a, a \vee b) \in \theta \\
&\Rightarrow (a \wedge (b \wedge c), (a \vee b) \wedge (b \wedge c)) \in \theta && \text{(by (iv))}
\end{aligned}$$

Similarly $((a \wedge b) \vee (b \vee c), a \vee b \vee c) \in \theta$. Now we have $a \wedge b \wedge c \leq (a \vee b) \wedge (b \wedge c) = b \wedge c \leq b \vee c = (a \wedge b) \vee (b \vee c) \leq a \vee b \vee c$ and the formulas resulting from replacing \leq by θ in the above inclusions are also true. So by (iii) we have $(a \wedge b \wedge c, a \vee b \vee c) \in \theta$. Also since $a \wedge b \wedge c \leq a \wedge c \leq a \vee c \leq a \vee b \vee c$ we have $(a \wedge c, a \vee c) \in \theta$ (by (iv)). For the substitution property suppose $(a, b) \in \theta$. Then $(a \wedge b, a \vee b) \in \theta$ (by (ii)). So by (iv) $((a \wedge b) \vee c, a \vee b \vee c) \in \theta$ for any $c \in L$. But $(a \wedge b) \vee c \leq a \vee c \leq a \vee b \vee c$. Thus $(a \vee c, a \vee b \vee c) \in \theta$ by (iv). Similarly $(b \vee c, a \vee b \vee c) \in \theta$ and so by transitivity and symmetry $(a \vee c, b \vee c) \in \theta$. So if $(a, b) \in \theta$ and $(e, f) \in \theta$ then $(a \vee e, b \vee e)$, $(b \vee e, b \vee f) \in \theta$ and by transitivity $(a \vee e, b \vee f) \in \theta$. Meets are handled in a similar way. Thus θ satisfies the substitution property and hence is a congruence relation on L . \square

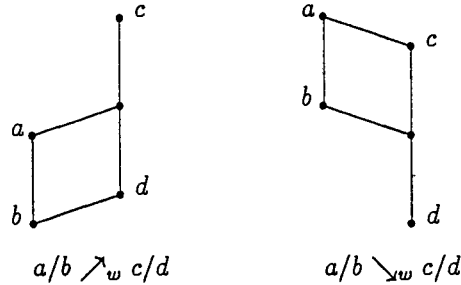


Figure 2.1

Let L be a lattice and let $a/b, c/d$ be quotients in L . Then a/b transposes up to c/d ($a/b \nearrow c/d$) if $c = a \vee d$ and $b = a \wedge d$. Dually a/b transposes down to c/d ($a/b \searrow c/d$) if $a = b \vee c$ and $d = b \wedge c$. We say a/b and c/d are transposes ($a/b \sim c/d$) if a/b transposes up or down to c/d ; and a/b and c/d are projective ($a/b \approx c/d$) if there is a finite sequence $a/b = e_0/f_0, e_1/f_1, \dots, e_n/f_n = c/d$ such that e_i/f_i and e_{i+1}/f_{i+1} are transposes for $i < n$. We say a/b transposes weakly up into c/d ($a/b \nearrow_w c/d$) if $b = a \wedge d$ and $a \leq c$. Dually a/b transposes weakly down into c/d ($a/b \searrow_w c/d$) if $a = b \vee c$ and $d \leq b$. Now a/b

transposes weakly into c/d ($a/b \sim_w c/d$) if a/b transposes weakly up or down to c/d and a/b is weakly projective into c/d ($a/b \approx_w c/d$) if there is a finite sequence as above such that e_i/f_i transposes weakly into e_{i+1}/f_{i+1} for $i = 0, \dots, n-1$.

We write $a/b \overset{k}{\approx}_w c/d$ if a/b is weakly projective into c/d in k steps.

THEOREM 2.1.2 [Dil93] *Let Q be a set of quotients in a lattice L . Let θ_Q denote the least congruence on L which collapses all the quotients in Q . Then for $a, b \in L$, $(a, b) \in \theta_Q$ if and only if there is a finite sequence*

$$a \wedge b = e_0 \leq e_1 \leq \dots \leq e_n = a \vee b$$

in L such that each e_i/e_{i-1} ($i = 1, \dots, n$) is weakly projective into some quotient in Q .

PROOF. We first show that θ_Q is a congruence relation by showing that it satisfies the Grätzer-Schmidt Criteria of Lemma 2.1.1.

Let $a \in L, c/d \in Q$. Then $a/a \nearrow_w (a \wedge c)/(a \wedge d) \searrow_w c/d$. Thus a/a projects weakly into every quotient of Q and so θ_Q is reflexive. Conditions (ii) and (iii) hold easily for θ_Q . For condition (iv) suppose we have $a \leq b$ and $(a, b) \in \theta_Q$. Then there is a sequence $a = e_0 \leq \dots \leq e_n = b$ such that for each $i \in \{0, \dots, n\}$ e_i/e_{i-1} is weakly projective into some quotient c_i/d_i of Q . Then for any $f \in L$, $a \wedge f = e_0 \wedge f \leq \dots \leq e_n \wedge f = b \wedge f$ and $(e_i \wedge f)/(e_{i-1} \wedge f) \nearrow_w e_i/e_{i-1}$. So for each $i \in \{1, \dots, n\}$ $(e_i \wedge f)/(e_{i-1} \wedge f) \approx_w c_i/d_i$. Thus $(a \wedge f, b \wedge f) \in \theta_Q$. Joins are treated similarly.

Let $\theta \in \text{Con}(L)$ be such that $(c, d) \in \theta \forall c/d \in Q$. let $(a, b) \in \theta_Q$. Then there is a finite sequence $a \wedge b = e_0 \leq e_1 \leq \dots \leq e_n = a \vee b$ such that for each $i \in \{1, \dots, n\}$ e_i/e_{i-1} is weakly projective into some quotient in Q . But then $(e_i, e_{i-1}) \in \theta$. Since $e/f \searrow_w c/d$ and $(c, d) \in \theta$ we have $(c \wedge e, d \wedge e) = (e, f) \in \theta$. Similarly $e/f \nearrow_w c/d$ and $(c, d) \in \theta$ imply $(e, f) \in \theta$. By transitivity of θ we have $(a \wedge b, a \vee b) \in \theta$, i.e. $(a, b) \in \theta$. Thus $\theta_Q \subseteq \theta$ and θ_Q is the least congruence on L which collapses all the quotients in Q . \square

The following characterization of principal congruences in a lattice is an immediate consequence of Theorem 2.1.2.

COROLLARY 2.1.3 *Let $a/b, c/d$ be quotients of a lattice L . Then $(c, d) \in \text{con}(a, b)$ if and only if there is a finite sequence*

$$a = e_0 \leq e_1 \leq \dots \leq e_n = b$$

such that $e_i/e_{i-1} \approx_w c/d$ for all $1 \leq i \leq n$.

2.2 Modular and distributive lattices

In this section we cover some results concerning modular and distributive lattices that make use of the notion of weak projectivity and that are of relevance to work developed in later chapters. For further details the reader is referred to texts such as [Gra78] and [McK87].

2.2.1 Modular lattices

A lattice L is said to have the *projectivity property* if whenever a/b is weakly projective into c/d then a/b is actually projective with a subinterval of c/d .

THEOREM 2.2.1 (Dedekind's Transposition Principle) *Let L be a modular lattice with $a, b \in L$. Then $\psi_b : x \mapsto x \wedge b$ is an isomorphism between $(a \vee b)/a$ and $b/(a \wedge b)$. The inverse isomorphism is $\phi_a : x \mapsto x \vee a$. Moreover the image under either of these maps of a subquotient is a transpose of that subquotient.*

PROOF. For $x \in (a \vee b)/a$ we have $\phi_a(\psi_b(x)) = (x \wedge b) \vee a = x \wedge (a \vee b) = x$. Dually for $x \in b/(a \wedge b)$ we have $\psi_b(\phi_a(x)) = x$. Let x/y be a quotient in $(a \vee b)/a$. Then $\psi_b[x/y] = (x \wedge b)/(y \wedge b)$ and $y \wedge (x \wedge b) = y \wedge b$. Also $x \leq x \wedge (b \vee a) \leq x \wedge (b \vee y) = (x \wedge b) \vee y \leq x$. Thus $(x \wedge b) \vee y = x$ and $(x \wedge b)/(y \wedge b)$ is a transpose of x/y . \square

COROLLARY 2.2.2 *Projective intervals in a modular lattice are isomorphic.*

PROOF. Suppose a/b and c/d are transposes and assume $a/b \nearrow c/d$. Then $a \wedge d = b$ and $a \vee d = c$. By Theorem 2.2.1 $a/(a \wedge d) \cong (a \vee d)/d$ i.e. $a/b \cong c/d$. \square

COROLLARY 2.2.3 *Every modular lattice has the projectivity property.*

PROOF. Let L be a modular lattice with $a/b, c/d$ quotients in L . Suppose a/b is weakly projective into c/d . The proof is by induction on the length n of the chain of weak projectivity. The case $n = 1$ is straightforward. Assume the result holds for $n = k$ and replace k by $k+1$. Then $a/b \overset{k}{\approx}_w s/t$ for some quotient s/t and $s/t \sim_w c/d$. By the inductive hypothesis a/b is actually projective with a subinterval u/v of s/t . Suppose $s/t \nearrow_w c/d$. Then $s \wedge d = t$ and $s \leq c$. By Theorem 2.2.1 $s/(s \wedge d) \cong (s \vee d)/d$ i.e. $s/t \cong (s \vee d)/d$ and $(u \vee d)/(v \vee d)$ is a transpose of u/v . Thus a/b is actually projective with a subinterval of c/d . The case $s/t \searrow_w c/d$ is similar. \square

LEMMA 2.2.4 *Let L be a modular lattice. If u/v is a prime quotient of L , then $\text{con}(u, v)$ is an atom of $\text{Con}(L)$.*

PROOF. Suppose $\Delta < \text{con}(r, s) \leq \text{con}(u, v)$ in $\text{Con}(L)$. Then by Theorem 2.1.2, Corollary 2.2.3 and the primeness of u/v there is a finite sequence

$$r \wedge s = e_0 \leq e_1 \leq e_2 \leq \cdots \leq e_n = r \vee s$$

such that e_{j+1}/e_j is projective with u/v . Then u/v is projective with a subinterval of $(r \wedge s)/(r \vee s)$. By Corollary 2.1.3 $\text{con}(u, v) \leq \text{con}(r \wedge s, r \vee s) = \text{con}(r, s)$ proving that $\text{con}(u, v)$ is an atom. \square

LEMMA 2.2.5 *Let L be a distributive algebraic lattice such that the top element of L is a join of atoms. Then L is a Boolean algebra. Dually, if the smallest element of L is a meet of coatoms, then L is a Boolean algebra.*

PROOF. Let $b \in L$. Consider $A = \{a \in \text{At}(L) : a \not\leq b\}$. Let $b^* = \bigvee A$. Then $b \wedge b^* = b \wedge \bigvee A = \bigvee_{a \in A} (b \wedge a) = \bigvee 0 = 0$. Let $B = \{a \in \text{At}(L) : a \leq b\}$. Then $\bigvee B \leq b$ hence $\bigvee B \vee b^* \leq b \vee b^*$. i.e. $1 \leq b \vee b^*$ and so $b \vee b^* = 1$. Thus b^* is the complement of b in L . \square

THEOREM 2.2.6 *Let L be a modular lattice. If L has finite length m then $\text{Con}(L)$ is isomorphic to a Boolean algebra 2^n where $n \leq m$.*

PROOF. Let $a_0 < a_1 < a_2 \cdots < a_m$ be a maximal chain in L . Then by Lemma 2.2.4 the principal congruences $\text{con}(a_i, a_{i+1})$ ($i = 0, 1, \dots, m-1$) are atoms (not necessarily distinct) of $\text{Con}(L)$ and their join collapses the whole of L . The result now follows from Lemma 2.2.5. \square

2.2.2 Distributive lattices

LEMMA 2.2.7 *Let L be a distributive lattice. If a/b and c/d are projective in L , then either these two intervals are transposes or there are intervals u/v and u'/v' such that*

$$\begin{array}{c} a/b \nearrow u/v \searrow c/d \quad \text{or} \\ a/b \searrow u'/v' \nearrow c/d \end{array}$$

PROOF. It is sufficient to show that if $a_0/b_0 \nearrow a_1/b_1 \searrow a_2/b_2$ then there is a a_3/b_3 such that $a_0/b_0 \searrow a_3/b_3 \nearrow a_2/b_2$ (since then we remove the 'kinks' as illustrated below).

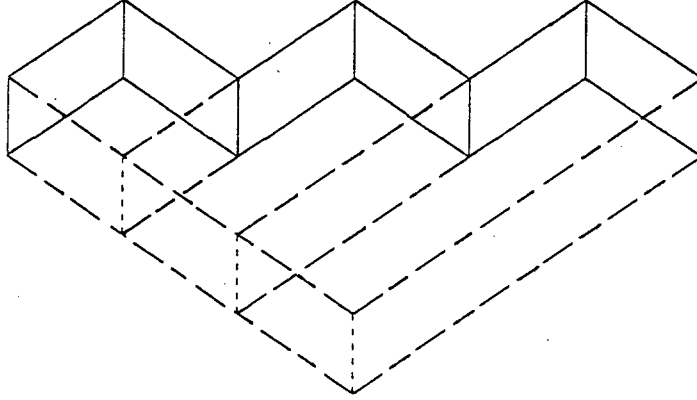


Figure 2.2

Let $a_3 = a_0 \wedge a_2, b_3 = a_0 \wedge b_2$. Then

$$b_0 \wedge a_3 = b_0 \wedge (a_0 \wedge a_2) = b_0 \wedge a_2 = (a_0 \wedge b_1) \wedge a_2 = a_0 \wedge (b_1 \wedge a_2) = a_0 \wedge b_2 = b_3$$

$$b_0 \vee a_3 = b_0 \vee (a_0 \wedge a_2) = (a_0 \wedge b_1) \vee (a_0 \wedge a_2) = a_0 \wedge (b_1 \vee a_2) = a_0 \wedge a_1 = a_0$$

$$b_2 \wedge a_3 = b_2 \wedge (a_0 \wedge a_2) = b_2 \wedge a_0 = b_3$$

$$b_2 \vee a_3 = b_2 \vee (a_0 \wedge a_2) = (b_1 \wedge a_2) \vee (a_0 \wedge a_2) = (b_1 \vee a_0) \wedge a_2$$

The above equalities complete the proof. \square

THEOREM 2.2.8 *Let L be a distributive lattice with $x, y, a, b \in L$ and $a \leq b$. Then $(x, y) \in \text{con}(a, b)$ if and only if $x \wedge a = y \wedge a$ and $x \vee b = y \vee b$.*

PROOF. Define a relation ϕ on L by $(x, y) \in \phi$ if and only if $x \wedge a = y \wedge a$ and $x \vee b = y \vee b$. Then ϕ is obviously an equivalence relation. To show that ϕ has the substitution property let $(x, y) \in \phi, z \in L$. Then $(x \wedge z) \wedge a = z \wedge x \wedge a = z \wedge y \wedge a = (y \wedge z) \wedge a$. Also $(x \wedge z) \vee b = (x \vee b) \wedge (z \vee b) = (y \vee b) \wedge (z \vee b) = (y \wedge z) \vee b$. Thus $(x \wedge z, y \wedge z) \in \phi$. Similarly $(x \vee z, y \vee z) \in \phi$ and so ϕ is a congruence relation on L with $(a, b) \in \phi$. To show that $\phi \subseteq \text{con}(a, b)$ let $(x, y) \in \phi$ and let θ be a congruence on L such that $(a, b) \in \theta$. We have $x = x \vee (x \wedge a) = x \vee (y \wedge a) = (x \vee y) \wedge (x \vee a) \theta (x \vee y) \wedge (x \vee b) = (x \vee y) \wedge (y \vee b) = y \vee (x \wedge b) \theta y \vee (x \wedge a) = y \vee (y \wedge a) = y$. Thus by transitivity $(x, y) \in \theta$ proving $\theta \subseteq \text{con}(a, b)$ and, since $(a, b) \in \theta$, $\theta = \text{con}(a, b)$. \square

The next result follows immediately from Theorem 2.2.8.

COROLLARY 2.2.9 *Let L be a distributive lattice with $x, y, a, b \in L$ and $x \leq y \leq a \leq b$ or $a \leq b \leq x \leq y$. Then $(x, y) \in \text{con}(a, b) \Rightarrow x = y$.*

LEMMA 2.2.10 *In a finite Boolean algebra L every prime quotient in L transposes onto $a/0$ for some atom $a \in L$. Dually every prime quotient transposes onto $1/c$ for some coatom c of L .*

PROOF. Let p/q be a prime quotient in L and let a be an atom below p but not below q . Then $a \wedge q = \mathbf{0}$ and $q \leq a \vee q \leq p \Rightarrow a \vee q = p$ proving $a/\mathbf{0} \searrow p/q$. \square

COROLLARY 2.2.11 *In a finite Boolean algebra L , every congruence is a principal congruence of the form $\text{con}(\mathbf{0}, b)$ for some $b \in L$.*

PROOF. Since L is finite, every congruence θ on L is a principal congruence $\theta = \text{con}(a, b) = \bigvee_I \text{con}(p_i, q_i)$ where p_i/q_i is a prime quotient in $(a \vee b)/(a \wedge b)$. By Lemma 2.2.10, for each $i \in I$ there is an atom a_i such that p_i/q_i is a transpose of $a_i/\mathbf{0}$. We then have $\theta = \text{con}(\mathbf{0}, \bigvee_I a_i)$. \square

LEMMA 2.2.12 *Let L be a distributive lattice with zero. Let a be an atom of L and let B be a finite set of atoms of L . Then $a \leq \bigvee B \Rightarrow a = b$ for some $b \in B$.*

PROOF. Assume $a \neq b$ for all $b \in B$. Then $a \leq \bigvee B \Rightarrow a \wedge \bigvee B = a \Rightarrow \bigvee_{b \in B} (a \wedge b) = a \Rightarrow a = \mathbf{0}$ and this contradiction completes the proof. \square

The following lemma is used in Chapter 5 to characterize $\mathbf{3}$ -chain limited finite distributive lattices. Recall from Chapter 0 that a congruence θ on a lattice L is a $\mathbf{2}$ -congruence if $L/\theta \cong \mathbf{2}$.

LEMMA 2.2.13 *Let D be a finite distributive lattice with*

$$\mathbf{0} = s_0 \prec s_1 \prec \cdots \prec s_k = \mathbf{1}$$

a maximal chain in D . Then:

(i) θ is an atom in $\text{Con}(D)$ if and only if $\theta = \text{con}(s_i, s_{i+1})$ for some $i \in \{0, 1, \dots, k-1\}$. Moreover $\text{con}(s_i, s_{i+1}) \neq \text{con}(s_j, s_{j+1})$ for $i \neq j$, and thus $\dim(D/\text{con}(s_i, s_{i+1})) = \dim(D) - 1$.

(ii) None of the distinct prime quotients of the above chain project onto each other.

(iii) If p/q is a prime quotient of D then there is a unique $i \in \{0, 1, \dots, k-1\}$ such that p/q projects onto s_{i+1}/s_i .

(iv) If θ is a $\mathbf{2}$ -congruence on D then for some $0 \leq i \leq k-1$ we have $D/\theta = \{s_i/\theta, s_{i+1}/\theta\}$.

(v) If $d \in D$ is a join of atoms of D then the principal ideal $(d]$ is a Boolean lattice.

(vi) If $d \in D$ is a meet of coatoms in D then the principal filter $[d)$ is a Boolean lattice.

PROOF. (i) We have $\text{con}(s_0, s_1) \vee \text{con}(s_1, s_2) \vee \cdots \vee \text{con}(s_{k-1}, s_k) = \nabla$ in $\text{Con}(D)$. By Lemma 2.2.4 $\text{con}(s_i, s_{i+1})$ is an atom of $\text{Con}(D)$ for every $i \in \{0, 1, \dots, k-1\}$. Suppose $\text{con}(a, b)$ is an atom of $\text{Con}(D)$. Then $(a, b) \in \text{con}(s_0, s_1) \vee \text{con}(s_1, s_2) \vee \cdots \vee \text{con}(s_{k-1}, s_k)$ and so by Lemma 2.2.12 $\text{con}(a, b) = \text{con}(s_i, s_{i+1})$ for some $i \in \{0, 1, \dots, k-1\}$.

Suppose $\text{con}(s_i, s_{i+1}) = \text{con}(s_j, s_{j+1})$ for some $i \neq j$. Assume without loss of generality that $s_i < s_j$. Then by Theorem 2.2.8 $s_i \wedge s_j = s_{i+1} \wedge s_j$ i.e. $s_i = s_{i+1}$ contradicting $s_i < s_{i+1}$.

(ii) Suppose s_{i+1}/s_i projects onto s_{j+1}/s_j for some $i < j$, $i, j \in \{0, 1, \dots, k-1\}$. Then by Theorem 2.1.2 $\text{con}(s_i, s_{i+1}) \leq \text{con}(s_j, s_{j+1})$ contradicting (i) above.

(iii) By Lemma 2.2.4 $\text{con}(q, p)$ is an atom of $\text{Con}(D)$. By (i) above $\text{con}(q, p) = \text{con}(s_i, s_{i+1})$ for some $i \in \{0, 1, \dots, k-1\}$. Thus p/q is projective with s_{i+1}/s_i for some unique $i \in \{0, 1, \dots, k-1\}$ (by Theorem 2.1.2, Corollary 2.2.3 and (i) above).

(iv) Suppose $(s_i, s_{i+1}) \in \theta$ for all $i \in \{0, 1, \dots, k-1\}$. Then $\theta = \nabla$ contradicting the fact that θ is a 2-congruence.

(v) Follows from Lemma 2.2.5.

(vi) This is the dual of (v). □

THEOREM 2.2.14 *Let L be a distributive lattice of finite length m . Then $\text{Con}(L)$ is isomorphic to 2^m .*

PROOF. By Lemma 2.2.5 $\text{Con}(L)$ is a Boolean algebra. By Lemma 2.2.13(i) above $\text{Con}(L)$ has m atoms and so $\text{Con}(L) \cong 2^m$. □

2.3 Primitive subsets of lattices

The definitions and results of this section will be used in the next section of this chapter to prove a finite basis theorem for lattices as well as in Chapter 6. Lemmas 2.3.1 - 2.3.4 are due to R. Wille.

A finite subset P of a lattice L is called *primitive* if there is a proper quotient a_P/b_P of L such that $a_P/b_P \approx_w (c \vee d)/d$ for all $c, d \in P$ with $c \not\leq d$. We say that P is a primitive subset of L with respect to a_P/b_P .

LEMMA 2.3.1 [Wil72] *Every finite subset of a subdirectly irreducible lattice is primitive.*

PROOF. Let L be a subdirectly irreducible lattice with $Q = \{x_1/y_1, \dots, x_n/y_n\}$ a finite set of proper quotients of L . We show by induction that for any positive integer n there

is a proper quotient a_n/b of L such that a_n/b generates the minimal non-zero congruence of L and $a_n/b \approx_w x_i/y_i$ for all $i \in \{1, \dots, n\}$. For the case $n = 1$ let a/b be a quotient in L generating the minimal non-zero congruence of L . Then by Corollary 2.1.3 there is an $a_1 \in L$ such that $b < a_1 \leq a$ and $a_1/b \approx_w x_1/y_1$. Since $b < a_1 \leq a$ we have $\text{con}(a_1, b) = \text{con}(a, b)$. Suppose the inductive hypothesis holds for $n = k$. Then there is a quotient a_k/b of L such that $a_k/b \approx_w x_i/y_i$ for all $i \in \{1, \dots, k\}$ and $\text{con}(a_k, b)$ is the smallest non-zero congruence on L . We then have $(a_k, b) \in \text{con}(x_{k+1}, y_{k+1})$ and again by Corollary 2.1.3 there exists an $a_{k+1} \in L$ with $b < a_{k+1} \leq a$ and such that $a_{k+1}/b \approx_w x_{k+1}/y_{k+1}$ and a_{k+1}/b generates the minimal non-zero congruence on L . Now we have $a_{k+1}/b \searrow_w a_k/b$ and hence $a_{k+1}/b \approx_w x_i/y_i$ for all $i \in \{1, \dots, n\}$. \square

For the next three lemmas let L and L' be lattices with $f : L \rightarrow L'$ an epimorphism.

LEMMA 2.3.2 [Wil72] *Let a'/b' be a quotient of L' and c/d a quotient of L with $a'/b' \approx_w^n f(c)/f(d)$. Then there is a quotient a/b of L with $a/b \approx_w^n c/d$ and $f(a)/f(b) = a'/b'$.*

PROOF. Suppose $a'/b' \searrow_w f(c)/f(d)$. Choose $a'' \in f^{-1}(a')$. Define $a = a'' \wedge c$, $b = a'' \wedge d$. Then $a/b \searrow_w c/d$ and $f(a)/f(b) = a'/b'$. Dually for $a'/b' \nearrow_w f(c)/f(d)$ there is a quotient a/b of L with $a/b \nearrow_w c/d$ and $f(a)/f(b) = a'/b'$. The assertion of the lemma then follows from this. \square

LEMMA 2.3.3 [Wil72] *Let $c_1/d_1, \dots, c_n/d_n$ be quotients of L with $f(c_i)/f(d_i) = f(c_j)/f(d_j)$ for $1 \leq i, j \leq n$. Then there is a quotient a/b of L with $a/b \approx_w^2 c_i/d_i$ and $f(a)/f(b) = f(c_i)/f(d_i)$ for $1 \leq i \leq n$.*

PROOF. Define $a = c_1 \wedge \dots \wedge c_n$. Then $a/(a \wedge d_i) \searrow_w c_i/d_i$ for $1 \leq i \leq n$. Define $b = (a \wedge d_1) \vee \dots \vee (a \wedge d_n)$. Then $a/b \nearrow_w a/(a \wedge d_i)$. Thus $a/b \nearrow_w a/(a \wedge d_i) \searrow_w c_i/d_i$ and $f(a) = f(c_1) \wedge f(c_2) \wedge \dots \wedge f(c_n) = f(c_i)$ for $1 \leq i \leq n$. Also $f(b) = (f(a) \wedge f(d_1)) \vee \dots \vee (f(a) \wedge f(d_n)) = (f(c_1) \wedge f(d_1)) \vee \dots \vee (f(c_n) \wedge f(d_n)) = f(d_1) \vee \dots \vee f(d_n) = f(d_i)$ for $1 \leq i \leq n$. \square

LEMMA 2.3.4 [Wil72] *Let P' be a primitive subset of L' with respect to a'/b' . If P is a subset of L such that the restriction of f to P is a poset isomorphism from P onto P' , then P is primitive in L with respect to a proper quotient a/b such that $f(a)/f(b) = a'/b'$.*

PROOF. For $c, d \in P$ with $c \not\leq d$ we have $f(c)/f(d) \in P'$ and $f(c) \not\leq f(d)$. Thus $a_{P'}/b_{P'} \approx_w f(c \vee d)/f(d)$. By Lemma 2.3.2 there is a quotient a/b of L with $a/b \approx_w (c \vee d)/d$ and $f(a)/f(b) = a_{P'}/b_{P'}$. Hence by Lemma 2.3.3 there is a quotient a_P/b_P of L such that $a_P/b_P \approx_w (c \vee d)/d$ for all $c, d \in P$ with $c \not\leq d$ and $f(a_P)/f(b_P) = a_{P'}/b_{P'}$. \square

2.4 Finite basis theorem

An equational basis for a variety \mathcal{V} is a collection Σ of identities such that $\mathcal{V} = \text{Mod}(\Sigma)$. Baker's finite basis theorem ([Bak77]) states that any finitely generated congruence distributive variety \mathcal{V} is finitely based i.e. $\mathcal{V} = \text{Mod}(\Sigma)$ for a finite set of identities Σ . This is not the case in general as can be seen by Lyndon's seven element counterexample ([Lyn54]). Since lattices are congruence distributive Baker's result holds for any finitely generated variety of lattices, an assertion made as early as 1945 by Schützenberger ([Sch45]). As proofs of Baker's theorem are available in many sources (e.g. [Bur81], [Jip92]) we present the finite basis theorem for lattice varieties. This was first proved by McKenzie in [McK70] where it is also shown that the lattice diagrammed in Figure 2.3 below is not finitely based. We follow the model-theoretic approach of Herrmann [Her73] which gives a sufficient condition for the existence of a finite basis without exhibiting an explicit set of identities. (In contrast to Baker's constructive proof.)

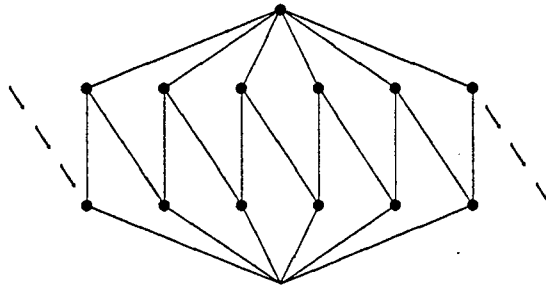


Figure 2.3

Bounded sets of quotients. A set of quotients of a lattice is called *k-bounded* if there is a proper quotient weakly projective in k steps into each of them. If any such k exists then we say the set is *bounded*.

LEMMA 2.4.1 *Every finite set of proper quotients of a subdirectly irreducible lattice is bounded.*

PROOF. Let L be a subdirectly irreducible lattice with $Q = \{x_1/y_1, \dots, x_n/y_n\}$ a finite set of proper quotients of L . Then $P = \{x_1, y_1, x_2, y_2, \dots, x_n, y_n\}$ is a finite subset of L . Hence by Lemma 2.3.1 there is a proper quotient a/b of L such that $a/b \approx_w (c \vee d)/d$ for all $c, d \in P$ with $c \not\leq d$. In particular $a/b \approx_w (x_i \vee y_i)/y_i = x_i/y_i$ for all $i \in \{1, \dots, n\}$ proving that Q is bounded. \square

LEMMA 2.4.2 *If $f : L \rightarrow L'$ is a lattice epimorphism with $c_1/d_1, \dots, c_n/d_n$ quotients of L such that $\{f(c_1)/f(d_1), \dots, f(c_n)/f(d_n)\}$ is k -bounded, then $\{c_1/d_1, \dots, c_n/d_n\}$ is $(k+2)$ -bounded.*

PROOF. Let p/q be a proper quotient of L such that $p/q \approx_w^k f(c_i)/f(d_i)$ for all $i \in \{1, \dots, n\}$. Then by Lemma 2.3.2 there are quotients a_i/b_i in L with $a_i/b_i \approx_w^k c_i/d_i$ and $f(a_i)/f(b_i) = p/q$. By Lemma 2.3.3 there is a quotient c/d of L with $c/d \approx_w^2 a_i/b_i$ for all $i \in \{1, \dots, n\}$ and $f(c)/f(d) = p/q$. Then c/d is a proper quotient of L such that $c/d \approx_w^{k+2} c_i/d_i$ for all $i \in \{1, \dots, n\}$ and so $\{c_1/d_1, \dots, c_n/d_n\}$ is $(k+2)$ -bounded. \square

Defining equations for the variety \mathcal{K}^\vee where \mathcal{K} is a class of lattices defined by a finite set of positive universal sentences. We first show that the notion of weak projectivity can be expressed as an open first order formula.

Let $x_i, y_i, u, v, x_i^j, y_i^j$ be variables and for any $n, k \in \omega$, let $z_{n,k}$ be the array $x_i^0, y_i^0, \dots, x_i^k, y_i^k$ ($i = 1, \dots, n$). Define inductively:

$$\gamma_{n,0}(u, v, z_{n,0}) \equiv \bigwedge_{i=1}^n u/v = x_i^0/y_i^0$$

$$\beta_{n,0}(x_1, y_1, \dots, x_n, y_n, u, v, z_{n,0}) \equiv \gamma_{n,0}(u, v, z_{n,0}) \wedge \bigwedge_{i=1}^n (x_i^0/y_i^0 = x_i/y_i).$$

$$\gamma_{n,k+1}(u, v, z_{n,k+1}) \equiv \gamma_{n,k}(u, v, z_{n,k}) \wedge \bigwedge_{i=1}^n (x_i^k/y_i^k \sim_w x_i^{k+1}/y_i^{k+1})$$

$$\beta_{n,k+1}(x_1, y_1, \dots, x_n, y_n, u, v, z_{n,k+1}) \equiv \gamma_{n,k+1}(u, v, z_{n,k+1}) \wedge \bigwedge_{i=1}^n (x_i^k/y_i^k = x_i/y_i).$$

Then for each $n, k \in \omega$, $\beta_{n,k}(x_1, y_1, \dots, x_n, y_n, u, v, z_{n,k})$ translates as:

For each $i \leq n$, $u/v \approx_w^k x_i/y_i$ by means of quotients $x_i^0/y_i^0, \dots, x_i^k/y_i^k$. i.e. $u/v = x_i^0/y_i^0 \sim_w x_i^1/y_i^1 \sim_w \dots \sim_w x_i^{k-1}/y_i^{k-1} \sim_w x_i^k/y_i^k = x_i/y_i$.

Let \mathcal{K} be a class of lattices defined by a finite set of identities Σ . Without loss of generality we may assume that Σ consists of universal disjunctions of equations:

$$\alpha \equiv \forall z_1 \dots z_{m_\alpha} \left(\bigvee_{i=1}^{n_\alpha} f_i^\alpha(z_1, \dots, z_{m_\alpha}) = g_i^\alpha(z_1, \dots, z_{m_\alpha}) \right)$$

such that the inequalities $f_i^\alpha(z_1, \dots, z_{m_\alpha}) \geq g_i^\alpha(z_1, \dots, z_{m_\alpha})$ ($1 \leq i \leq n_\alpha$) hold in any lattice. We will now construct the defining set of inequalities for the variety \mathcal{K}^\vee .

To each $\alpha \in \Sigma$ and every $k \in \omega$ we assign a universal Horn sentence α_k . Let $p_i = f_i^\alpha(z_1, \dots, z_{m_\alpha})$, $q_i = g_i^\alpha(z_1, \dots, z_{m_\alpha})$. Define

$$\alpha_k \equiv \forall z_1 \dots z_{m_\alpha} u v z_{n_\alpha, k} (\beta_{n_\alpha, k}(p_1, q_1, \dots, p_{n_\alpha}, q_{n_\alpha}, u, v, z_{n_\alpha, k}) \Rightarrow u = v)$$

Then α_k is valid in a lattice L if and only if there is no assignment of elements a_1, \dots, a_{m_α} to the variables z_1, \dots, z_{m_α} such that the set $\{f_1^\alpha(a_1, \dots, a_{m_\alpha})/g_1^\alpha(a_1, \dots, a_{m_\alpha}), \dots, f_{n_\alpha}^\alpha(a_1, \dots, a_{m_\alpha})/g_{n_\alpha}^\alpha(a_1, \dots, a_{m_\alpha})\}$ becomes k -bounded. Note that α_k is equivalent to a Horn formula since $\beta_{n_\alpha, k}$ is a conjunction of positive atomic formulas.

LEMMA 2.4.3 (i) α implies α_k and α_k implies α_l ($1 \leq l \leq k$) in any lattice.
(ii) In a subdirectly irreducible lattice, $\{\alpha_k : k \geq 1\}$ implies α .

- (iii) The validity of α_k is preserved under sublattices and direct products.
(iv) If L' is an image of L and α_k holds in L for any given $k \geq 3$ then α_{k-2} holds in L' .

PROOF. (i) Let L be a lattice in which α holds. let $k \in \omega$ and suppose there are $a_1, \dots, a_{m_\alpha}, u, v \in L$ such that $u/v \approx_w^k f_i^\alpha(a_1, \dots, a_{m_\alpha})/g_i^\alpha(a_1, \dots, a_{m_\alpha})$ for all $i \in \{1, \dots, n_\alpha\}$. Since α holds in L , there is a $j \in \{1, \dots, n_\alpha\}$ such that $f_j^\alpha(x_1, \dots, x_{m_\alpha}) = g_j^\alpha(x_1, \dots, x_{m_\alpha})$ for all $x_1, \dots, x_{m_\alpha} \in L$. In particular $f_j^\alpha(a_1, \dots, a_{m_\alpha}) = g_j^\alpha(a_1, \dots, a_{m_\alpha})$. But then we must have $u = v$ and so α_k holds in L .

Let $l, k \in \omega$ with $1 \leq l \leq k$. Then for any lattice L and quotients $a/b, c/d$ in L we have $a/b \approx_w^l c/d \Rightarrow a/b \approx_w^k c/d$. Hence if α_k holds in L then so does α_l .

(ii) Let L be a subdirectly irreducible lattice such that α_k holds in L for all $k \geq 1$. Let $a_1, \dots, a_{m_\alpha} \in L$. Then the set $\{f_i^\alpha(a_1, \dots, a_{m_\alpha})/g_i^\alpha(a_1, \dots, a_{m_\alpha}) : 1 \leq i \leq n_\alpha\}$ cannot be bounded. hence by Lemma 2.4.1 we must have $f_j^\alpha(a_1, \dots, a_{m_\alpha}) = g_j^\alpha(a_1, \dots, a_{m_\alpha})$ for some $j \in \{1, \dots, n_\alpha\}$ and so α holds in L .

(iii) This follows directly from the fact that α_k is a universal Horn sentence.

(iv) Let $\eta : L \rightarrow L'$ be a lattice epimorphism. Suppose α_k holds in L for some $k \geq 3$ but α_{k-2} does not hold in L' . Then for some $a_1, \dots, a_{m_\alpha} \in L'$ the set: $\{f_i^\alpha(a_1, \dots, a_{m_\alpha})/g_i^\alpha(a_1, \dots, a_{m_\alpha}) : 1 \leq i \leq n_\alpha\}$ is $(k-2)$ -bounded. But then by Lemma 2.4.2 the set $f_i^\alpha(\eta(a_1), \dots, \eta(a_{m_\alpha}))/g_i^\alpha(\eta(a_1), \dots, \eta(a_{m_\alpha})) : 1 \leq i \leq n_\alpha\}$ is k -bounded, contradicting the fact that α_k holds in L . Thus α_{k-2} holds in L' . \square

PROPOSITION 2.4.4 Let \mathcal{K} be a class of lattices defined by a finite set of identities Σ .

We then have:

- (i) \mathcal{K}^\vee is the class of all lattices satisfying $\{\alpha_k : \alpha \in \Sigma, k \geq 1\}$.
(ii) If $L \in \mathcal{K}^\vee$ is subdirectly irreducible then $L \in \mathcal{K}$.

PROOF. Suppose L is a lattice satisfying $\{\alpha_k : \alpha \in \Sigma, k \geq 1\}$. Let S be a subdirectly irreducible image of L . Then by Lemma 2.4.3(iv) $S \models \{\alpha_k : \alpha \in \Sigma, k \geq 1\}$. Thus by Lemma 2.4.3(ii) $S \models \Sigma$ and $S \in \mathcal{K}$. Hence $L \in \mathbf{P}_S(\mathcal{K}) \subseteq \mathcal{K}^\vee$ proving $\text{Mod}(\{\alpha_k : \alpha \in \Sigma, k \geq 1\}) \subseteq \mathcal{K}^\vee$. We have $\mathcal{K} \models \{\alpha_k : \alpha \in \Sigma, k \geq 1\}$. (Lemma 2.4.3(i)). Hence $\mathcal{K}^\vee = \mathbf{HSP}(\mathcal{K}) \models \{\alpha_k : \alpha \in \Sigma, k \geq 1\}$. (Lemma 2.4.3(iii) and (iv)). Thus $\mathcal{K}^\vee \subseteq \text{Mod}(\{\alpha_k : \alpha \in \Sigma, k \geq 1\})$ proving (i).

(ii) This follows directly from (i) and Lemma 2.4.3(ii). \square

Weak (projective) radius. The notion of a weak (projective) radius is used to provide conditions under which the variety generated by a class of lattices defined by a finite set of equations is finitely based.

A lattice L has a *weak n -radius $\leq s$* ($wr_n(L) \leq s$) if any bounded set of at most n quotients in L is s -bounded. We show that 'weak radius' is a first order concept. For any n, s, k define the sentence:

$$\rho_{n,s,k} \equiv \forall x_1 y_1 \dots x_n y_n (\exists uvz_{n,k} (u \neq v \wedge \beta_{n,k}(x_1, y_1, \dots, x_n, y_n, u, v, z_{n,k})) \Rightarrow \exists pqz_{n,s} (p \neq$$

$q \wedge \beta_{n,k}(x_1, y_1, \dots, x_n, y_n, p, q, z_{n,s}))$.

The sentence $\rho_{n,s,k}$ holds in a lattice L if and only if every k -bounded set of at most n quotients in L is s -bounded. Thus $wr_n(L) \leq s$ if and only if $\rho_{n,s,k}$ holds in L for all $k \geq 1$. Define $\rho_{n,s} \equiv \rho_{n,s,s+1}$.

LEMMA 2.4.5 *In any lattice $\rho_{n,s}$ implies $\rho_{n,s,k}$ for all $k \geq 1$.*

PROOF. Let L be a lattice. The proof is by induction on k . For $k = 1$ we have $s \geq k$ and so $\rho_{n,s,k}$ always holds in L . Assume the result holds for $k = j - 1$ i.e. $\rho_{n,s}$ implies $\rho_{n,s,j-1}$. Suppose L satisfies $\rho_{n,s}$ and $\{x_1/y_1, \dots, x_n/y_n\}$ is a j -bounded set of quotients of L . Let u/v be a proper quotient in L such that $u/v \approx_w^j x_i/y_i$ for all $i \in \{1, \dots, n\}$. Then there are quotients a_i/b_i in L such that $a_i/b_i \sim_w x_i/y_i$ and $u/v \approx_w^{j-1} a_i/b_i$. By assumption $\rho_{n,s,j-1}$ holds in L and so there is a proper quotient p/q in L such that $p/q \approx_w^s a_i/b_i$ for all $i \in \{1, \dots, n\}$. Hence $p/q \approx_w^{s+1} a_i/b_i$ for all $i \in \{1, \dots, n\}$. Thus since $\rho_{n,s}$ holds in L the set $\{x_1/y_1, \dots, x_n/y_n\}$ is s -bounded. Consequently L satisfies $\rho_{n,s,j}$ and the assertion of the lemma follows by induction. \square

COROLLARY 2.4.6 *In any lattice L , $wr_n(L) \leq s$ if and only if $\rho_{n,s}$ holds in L .*

PROOF. This follows from Lemma 2.4.5 since $wr_n(L) \leq s$ if and only if $L \models \forall k(\rho_{n,s,k})$ if and only if $L \models \rho_{n,s}$. \square

COROLLARY 2.4.7 *If $\rho_{n,s}$ and α_s are valid in a lattice L , then α_k is valid in L for all $k \geq 1$.*

PROOF. Suppose α_k does not hold in L for some $k \geq 1$. Then there exist $a_1, \dots, a_{m_\alpha} \in L$ such that the set $\{f_i^\alpha(a_1, \dots, a_{m_\alpha})/g_i^\alpha(a_1, \dots, a_{m_\alpha}) : 1 \leq i \leq n_\alpha\}$ is k -bounded. By Lemma 2.4.5 $\rho_{n,s,k}$ holds in L . Hence $\{f_i^\alpha(a_1, \dots, a_{m_\alpha})/g_i^\alpha(a_1, \dots, a_{m_\alpha}) : 1 \leq i \leq n_\alpha\}$ is s -bounded. But this contradicts the validity of α_s in L , proving that α_k is valid in L . \square

LEMMA 2.4.8 *If L is a subdirect product of lattices $(L_i)_{i \in I}$ such that $wr_n(L_i) \leq s$ for all $i \in I$, then $wr_n(L) \leq s + 2$.*

PROOF. Let $\{a_1/b_1, \dots, a_n/b_n\}$ be a set of quotients of L bounded by c/d . Then for some $i \in I$ $\pi_i(c)/\pi_i(d)$ is a proper quotient of L_i where π_i is the projection map onto L_i . Consequently $\{\pi_i(a_1)/\pi_i(b_1), \dots, \pi_i(a_n)/\pi_i(b_n)\}$ is bounded by $\pi_i(c)/\pi_i(d)$ and hence is s -bounded. By Lemma 2.4.2 $\{a_1/b_1, \dots, a_n/b_n\}$ is $(s + 2)$ -bounded. \square

PROPOSITION 2.4.9 *If \mathcal{K} is a class of lattices defined by a finite set Σ of at most n -termed universal disjunctions of equations (i.e. $n_\alpha \leq n \forall \alpha \in \Sigma$) and if for all subdirectly irreducibles $L \in \mathcal{K}$ there exists a positive integer s such that $wr_n(L) \leq s$, then $\mathcal{K}^\vee = \mathbf{HSP}(\mathcal{K})$ can be defined by a finite set of equations.*

PROOF. Let $\mathcal{V} = \mathbf{HSP}(\mathcal{K})$, $L \in \mathcal{V}$. Then L is a subdirect product of lattices $(L_i)_{i \in I}$ where $L_i \in \mathcal{V}_{SI}$. By Proposition 2.4.4(ii) $L_i \in \mathcal{K}$ for all $i \in I$. Thus $wr_n(L_i) \leq s$ for all $i \in I$ and hence by Lemma 2.4.8 $wr_n(L) \leq s + 2$. By Corollary 2.4.6, $\rho_{n,s+2}$ holds in L . Thus $\{\alpha_k : k \in \Sigma, k \geq 1\}$ implies $\rho_{n,s+2}$, by Proposition 2.4.4(i). By the Compactness Theorem there is a finite subset $T \subseteq \{\alpha_k : k \in \Sigma, k \geq 1\}$ such that T implies $\rho_{n,s+2}$. Let $j = \max\{s + 2, k : \exists \alpha \in \Sigma \text{ such that } \alpha_k \in T\}$. We claim that the finite set $\{\alpha_j : \alpha \in \Sigma\}$ defines \mathcal{V} . Since $s + 2 \leq j$ we have $\{\alpha_j : \alpha \in \Sigma\}$ implies $\{\alpha_{s+2} : \alpha \in \Sigma\}$. (Lemma 2.4.3(i)). Also for all $\alpha_k \in T$ we have $k \leq j$ and hence $\{\alpha_j : \alpha \in \Sigma\}$ implies T . Thus by Corollary 2.4.7 $\{\alpha_j : \alpha \in \Sigma\}$ implies $\{\alpha_k : \alpha \in \Sigma : k \geq 1\}$ and the result follows from Proposition 2.4.4(i). \square

We are now ready to prove the main theorem of this section:

THEOREM 2.4.10 (McKenzie's Theorem) *Let L be a finite lattice, then $\mathbf{HSP}(\{L\})$ has a finite equational base.*

PROOF. Let L be an n -element lattice. Let \mathcal{M} be the class of all lattices M such that M has at most n elements and $M \notin \mathbf{HSP}(\{L\})$. Choose σ_M to be an equation valid in L but not in M . Let \mathcal{V} be the variety determined by $\{\sigma_M : M \in \mathcal{M}\}$, and let $\Sigma = \{\bigvee_{1 \leq i < j \leq n} x_i = x_j\}$. Then the class \mathcal{K} of lattices in \mathcal{V} which satisfy Σ is the class of at most n -element lattices of \mathcal{V} . Now $L \in \mathcal{K}$ since $L \models \sigma_M$ for all $M \in \mathcal{M}$ and L has n elements. Thus $\mathbf{HSP}(\{L\}) \subseteq \mathbf{HSP}(\mathcal{K})$. Let $A \in \mathcal{K}$. Then A has at most n elements. Suppose $A \notin \mathbf{HSP}(\{L\})$. Then $A \in \mathcal{M}$ and $A \not\models \sigma_A$. This contradiction of $A \in \mathcal{K}$ implies that $A \in \mathbf{HSP}(\{L\})$. Hence $\mathbf{HSP}(\mathcal{K}) \subseteq \mathbf{HSP}(\{L\})$ and we have $\mathbf{HSP}(\mathcal{K}) = \mathbf{HSP}(\{L\})$. Now $wr_n(A) \leq n^2$ for any $A \in \mathcal{K}$. (Since A has at most n^2 quotients.) The assertion of the theorem now follows from Proposition 2.4.9. \square

2.5 Reduced products of S-lattices

In this section we introduce the notions of an S -lattice and a \mathcal{S} -variety and show that the class \mathcal{P} of subdirect products of non-modular subdirectly irreducibles of a \mathcal{S} -variety is closed under reduced products. This result (Proposition 2.5.5), which relies on a characterization of \mathcal{P} in terms of weak projections, is used in Chapter 6 to show that the amalgamation class of a special class of \mathcal{S} -varieties (i.e. B -lattice varieties) is elementary.

Let a/c be a quotient of a lattice L . We say that a/c is an N -quotient of L if, for some $b \in L$, the set $\{a, c, b\}$ generates a sublattice of L isomorphic to the pentagon N in which a/c is a critical quotient:

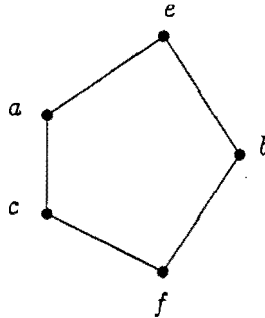


Figure 2.4

We use the notation $N(a/c, b)$ to denote that $\{a, c, b\}$ generates a sublattice of L isomorphic to the pentagon N with critical quotient a/c , or alternately to denote the sublattice of L consisting of the five elements $a, b, c, a \vee b, a \wedge c$, the meaning being clear from the context.

A lattice L is said to be *semidistributive* if it satisfies the following two implications for all $a, b, c, d \in L$:

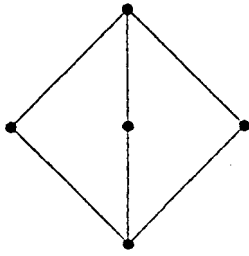
$$a = b \vee c = b \vee d \text{ implies } a = b \vee (c \wedge d) \text{ and dually}$$

$$a = b \wedge c = b \wedge d \text{ implies } a = b \wedge (c \vee d).$$

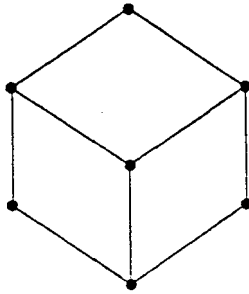
A lattice variety \mathcal{V} is semidistributive if each of its members is a semidistributive lattice. For further details regarding semidistributive varieties the reader is referred to [Jon79] and [Ros84]. A variety is semidistributive if and only if it does not contain the diamond (M_3) or the lattices L_1, L_2, L_3, L_4, L_5 illustrated in Figure 2.5. (We follow the notation used in [Jon79] and [Ros84].)

A subdirectly irreducible lattice L is an S -lattice if L is a finite non-modular, semidistributive, lattice such that none of the lattices L_{11} or L_{12} are embeddable into L . (See Figure 2.6.) We call a semidistributive variety \mathcal{V} a S -variety if \mathcal{V} is finitely generated and all non-modular members of \mathcal{V}_{SI} are S -lattices.

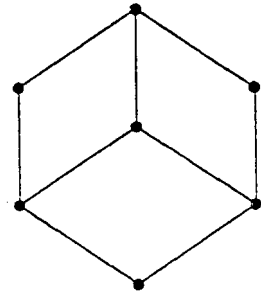
In [Ros84] it is shown that any S -lattice has a unique critical quotient which is an N -quotient. It is also shown that if a/c is the critical quotient of an S -lattice L , and if $b \in L$ is non-comparable with a (and c) then $N(a/c, b)$. An S -variety does not contain L_{11} and L_{12} since these are non-modular subdirectly irreducible lattices. See Figure 2.7 for examples of S -lattices.



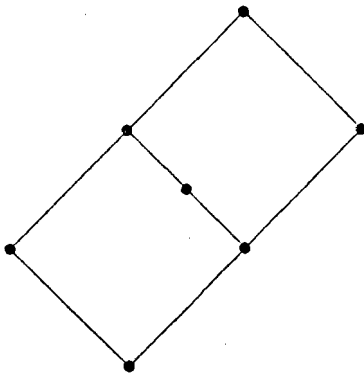
M_3 (Diamond)



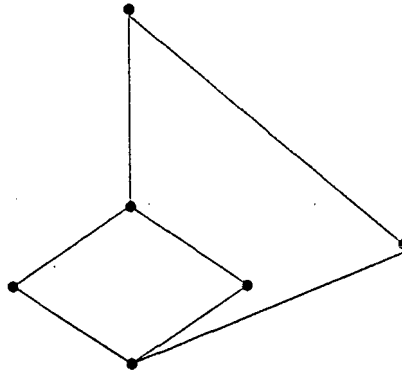
L_1



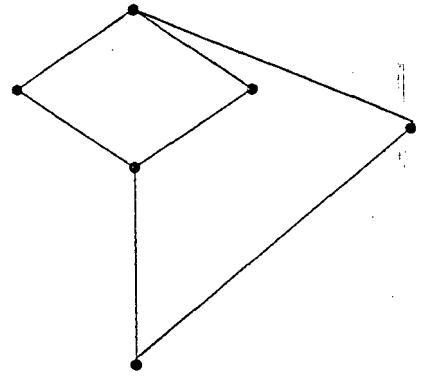
L_2



L_3

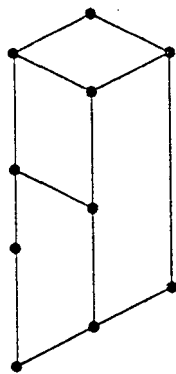


L_4

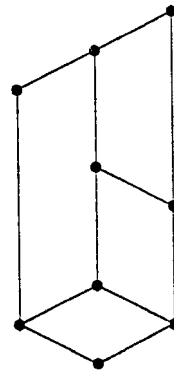


L_5

Figure 2.5



L_{11}



L_{12}

Figure 2.6

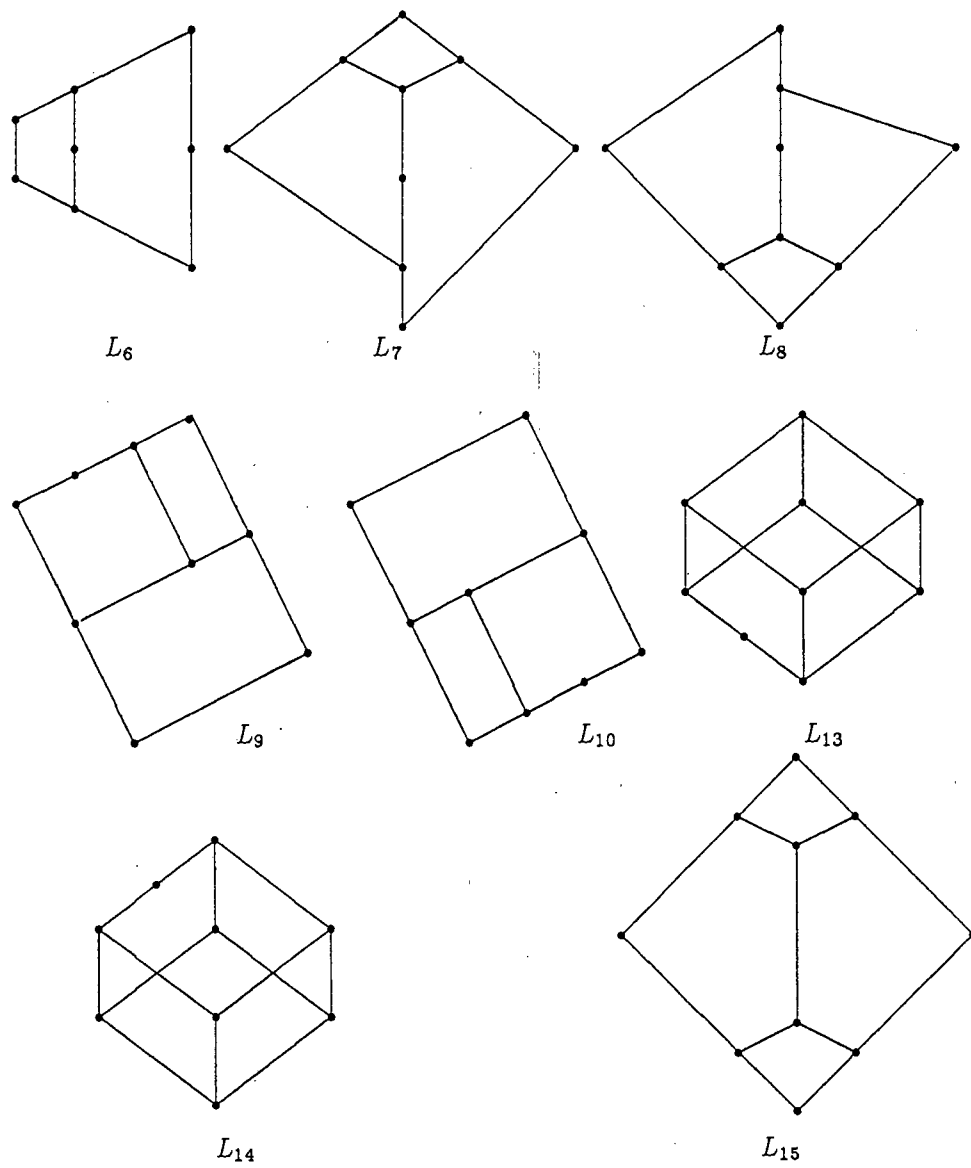


Figure 2.7. Examples of S -lattices

The reader is referred to [Ros84] for the proof of the following lemma.

LEMMA 2.5.1 [Ros84] *Let L be a semidistributive lattice that excludes L_{11} and L_{12} . If $a, b, c, u, v \in L$, with $N(a/c, b)$ and if u/v projects weakly into a/c , then $N(u/v, b)$. Hence if p/q is a quotient of L such that $(q, p) \in \text{con}(a, c)$ then $N(p/q, b)$.*

For the rest of this section let \mathcal{V} be an \mathcal{S} -variety and let \mathcal{S} denote the (finite) set of all non-modular subdirectly irreducible members of \mathcal{V} (i.e. \mathcal{S} is the set of all S -lattices of \mathcal{V}). Let \mathcal{P} denote the class of subdirect products of members of \mathcal{S} .

LEMMA 2.5.2 *Let $L \in \mathcal{V}$. Then the following statements are equivalent:*

- (i) $L \in \mathcal{P}$.
- (ii) *For every proper quotient p/q in L there is an N -quotient s/t such that $s/t \approx_w p/q$.*

PROOF. (i) \Rightarrow (ii) : Assume $L \in \mathcal{P}$ and let p/q be a proper quotient in L . Then there is a subdirect embedding $f : L \hookrightarrow \prod_{i \in I} L_i$ where $L_i \in \mathcal{S} \forall i \in I$. For each $i \in I$ let f_i denote the surjection $f \circ \pi_i : L \rightarrow L_i$. Let p/q be a proper quotient in L . Then there is an $i \in I$ such that $f_i(p)/f_i(q)$ is a proper quotient of L_i . Since L_i is an S -lattice, the (unique) critical quotient of L_i is an N -quotient and, since the pentagon is a projective lattice, there is an N -quotient u/v of L such that $f_i(u)/f_i(v)$ is the critical quotient of L_i . We have $f_i(u)/f_i(v) \approx_w f_i(p)/f_i(q)$ in L_i . Hence, by Lemma 2.3.2 there is a quotient x/y of L such that $x/y \approx_w p/q$ and $f_i(x)/f_i(y) = f_i(u)/f_i(v)$. By Lemma 2.3.3 there is a quotient s/t of L such that $s/t \approx_w x/y$ and $s/t \approx_w u/v$. Then by Lemma 2.5.1 s/t is an N -quotient of L and $s/t \approx_w x/y \approx_w p/q$.

(ii) \Rightarrow (i) : Assume (ii) holds. Let $\{p_i/q_i : i \in I\}$ be the set of all non-trivial quotients of L . Then, by assumption, for each $i \in I$ there is an N -quotient s_i/t_i of L such that $s_i/t_i \approx_w p_i/q_i$. Let θ_i be a subdirectly irreducible congruence on L such that $(s_i, t_i) \notin \theta_i$. Then θ_i is non-modular and $(p_i, q_i) \notin \theta_i$. Thus $\bigcap_I \theta_i = \Delta_L$ and hence L is a subdirect product of S -lattices in \mathcal{V} . \square

LEMMA 2.5.3 *The class \mathcal{P} is closed under ultraproducts.*

PROOF. Let $L = \prod_I L_i$ where for all $i \in I, L_i \in \mathcal{P}$. Let \mathcal{F} be an ultrafilter over I with Φ the congruence on L induced by \mathcal{F} . Let $(p/\Phi)/(q/\Phi)$ be a proper quotient in L/Φ . Then $X = \{i \in I : p_i > q_i\} \in \mathcal{F}$. Since $L_i \in \mathcal{P}$ for all $i \in I$, it follows that for all $i \in X$ there is an N -quotient s_i/t_i of L_i such that $s_i/t_i \approx_w p_i/q_i$ (Lemma 2.5.2). For $i \in I \setminus X$ let $s_i = t_i$ be an arbitrary element of L_i . Consider $s = (s_i)_{i \in I}, t = (t_i)_{i \in I}$. Then $(s/\Phi)/(t/\Phi)$ is an N -quotient in L/Φ such that $(s/\Phi)/(t/\Phi) \approx_w (p/\Phi)/(q/\Phi)$. By Lemma 2.5.2 $L/\Phi \in \mathcal{P}$. \square

As an alternative to the above proof recall from Section 2.4 that the notion of weak projectivity can be expressed as a first order formula (as can the concept of an N -quotient).

It then follows immediately from Lemma 2.5.2 that \mathcal{P} is an elementary class. That \mathcal{P} is closed under ultraproducts then follows from Łoś's Theorem (see for example [Cha78] Theorem 4.1.12).

LEMMA 2.5.4 *Suppose $A \in \mathcal{P}$. Let $(\theta_k)_{k \in K}$ be a class of congruences on A such that $A/\theta_k \in \mathcal{P}$ for all $k \in K$. If $\Psi = \bigcap_{k \in K} \theta_k$ then $A/\Psi \in \mathcal{P}$.*

PROOF. Since $A/\theta_k \in \mathcal{P}$ for each $k \in K$ we have a subdirect representation $A/\theta_k \leq \prod_{i \in I_k} (A/\theta_k)/(\Psi_i/\theta_k)$ where $(A/\theta_k)/(\Psi_i/\theta_k) \in \mathcal{S}$ for all $i \in I_k$ and $\bigcap_{i \in I_k} (\Psi_i/\theta_k) = \nabla_{A/\theta_k}$. Thus (by the second isomorphism theorem) $A/\theta_k \leq \prod_{i \in I_k} A/\Psi_i$ where $\bigcap_{i \in I_k} \Psi_i = \theta_k$ and $A/\Psi_i \in \mathcal{S} \forall i \in I_k$. Let $J = \bigcup_{k \in K} I_k$. Then $\Psi = \bigcap_{k \in K} \theta_k = \bigcap_{i \in J} \Psi_i$. Thus $A/\Psi \leq \prod_{i \in J} A/\Psi_i$ is a subdirect embedding with $A/\Psi_i \in \mathcal{S}$ for all $i \in J$. This proves $A/\Psi \in \mathcal{P}$. \square

PROPOSITION 2.5.5 *The class \mathcal{P} is closed under reduced products and therefore \mathcal{P} is a Horn class.*

PROOF. Let $L = \prod_I L_i$ where for all $i \in I, L_i \in \mathcal{P}$. Let \mathcal{F} be a filter over I and let \mathcal{K} be the class of all ultrafilters over I containing \mathcal{F} . By Corollary 0.2.4 $\mathcal{F} = \bigcap \mathcal{K}$ and $\Phi_{\mathcal{F}} = \bigcap_{\mathcal{U} \in \mathcal{K}} \Phi_{\mathcal{U}}$ where $\Phi_{\mathcal{F}}$ denotes the congruence on L induced by \mathcal{F} . By Lemma 2.5.3 $L/\Phi_{\mathcal{U}} \in \mathcal{P}$ for any $\mathcal{U} \in \mathcal{K}$. Hence by Lemma 2.5.4 $L/\Phi_{\mathcal{F}} \in \mathcal{P}$. That \mathcal{P} is a Horn class follows from [Cha78] and the fact that \mathcal{P} is elementary. (See the comment following Lemma 2.5.2.) \square

2.6 Ideals and congruences

Let L be a join semilattice. An *ideal* of L is a sub-semilattice of L that is closed downwards. We let $\mathcal{I}(L)$ denote the set of all ideals of L . Then $\mathcal{I}(L)$ is a complete lattice and for ideals $I, J, (I_s)_{s \in S}$ we have $I \wedge J = I \cap J$ and $\bigvee_{s \in S} I_s = \{i \in L : i \leq i_{s_0} \vee \dots \vee i_{s_n} \text{ for some } n \geq 1, s_j \in S \forall j = 0, \dots, n \text{ and } i_{s_j} \in I_{s_j}\}$.

Recall from Chapter 0 that for a lattice L , L^c denotes the join semilattice of compact elements of L .

LEMMA 2.6.1 *A lattice is algebraic if and only if it is isomorphic to the lattice of all ideals of a join-semilattice with 0.*

PROOF. Let H be a join-semilattice with 0. Then $\mathcal{I}(H)$ is complete and for all $a \in H$, $[a] (= \{x \in L : x \leq a\})$ is a compact element of $\mathcal{I}(H)$ and $I = \bigvee \{[a] : a \in I\}$ for all

$I \in \mathcal{I}(H)$. Thus $\mathcal{I}(H)$ is algebraic. On the other hand let L be an algebraic lattice. Then L^c is a join-semilattice with 0 and the map $f : a \mapsto \langle a \rangle$ is an isomorphism between L and $\mathcal{I}(L^c)$. \square

For an ideal I of a lattice L we let $\theta[I]$ denote the smallest congruence Ψ on L such that $(e, f) \in \Psi$ for all $e, f \in I$. An ideal I is called a *kernel* of a congruence relation θ if I is a θ -class.

The following theorem is an immediate consequence of Theorem 2.1.2.

THEOREM 2.6.2 *Let I be an ideal of a lattice L and let $a, b \in L, a \leq b$. Then $(a, b) \in \theta[I]$ if and only if for some integer n there exists a sequence $a = e_0 \leq e_1 \leq \dots \leq e_n = b$ such that for each i ($0 < i \leq n$) there exists a quotient $c_i/d_i \in I$ such that $e_i/e_{i-1} \approx_w c_i/d_i$.*

COROLLARY 2.6.3 *Let I be an ideal of a lattice L . Then I is a kernel of a congruence relation if and only if for $a \in L, b, c, d \in I, a/b \approx_w c/d \Rightarrow a \in I$.*

PROOF. Suppose I is a kernel of a congruence relation. Then I is a kernel of $\theta[I]$. Hence if $a/b \approx_w c/d$ for $b, c, d \in I, a \in L$ then by Theorem 2.6.2 $(a, b) \in \theta[I]$ and so $a \in I$. For the reverse implication suppose $(a, b) \in \theta[I]$ for some $a \in L, b \in I, a \leq b$. By Theorem 2.6.2 there is a sequence $a = e_0 \leq e_1 \leq \dots \leq e_n = b$ such that for each i ($0 < i \leq n$) there exist a quotient $c_i/d_i \in I$ such that $e_i/e_{i-1} \approx_w c_i/d_i$. By induction on n we have that $\{e_i : 0 \leq i \leq n\} \subseteq I$. Hence $a \in I$ and so I is a kernel of $\theta[I]$. \square

In a distributive lattice every ideal is the kernel of a congruence relation. The congruence relation $\theta[I]$ is characterized in the following theorem:

THEOREM 2.6.4 [Gra58] *Let I be an ideal of a distributive lattice L . Then $(x, y) \in \theta[I]$ if and only if $x \vee y = (x \wedge y) \vee i$ for some $i \in I$.*

PROOF. We have $\theta[I] = \bigvee \{\text{con}(a, b) : (a, b) \in I \times I\} = \bigcup \{\text{con}(a, b) : (a, b) \in I \times I \text{ and } a \leq b\}$. This follows from the fact that $\text{con}(a, b) \vee \text{con}(c, d) \subseteq \text{con}(a \wedge b \wedge c \wedge d, a \vee b \vee c \vee d)$. So $(x, y) \in \theta[I] \Rightarrow (x, y) \in \text{con}(a, b)$ for some $a, b \in I, a \leq b$. By Theorem 2.2.8 we have $x \vee b = y \vee b$. Now $(x \wedge y) \vee (b \wedge (x \vee y)) = ((x \wedge y) \vee b) \wedge ((x \wedge y) \vee (x \vee y)) = (x \vee b) \wedge (y \vee b) \wedge (x \vee y) = (x \vee b) \wedge (x \vee y) = (x \vee y \vee b) \wedge (x \vee y) = x \vee y$. Thus the condition of the theorem is satisfied with $i = b \wedge (x \vee y)$.

For the reverse implication suppose $(x \wedge y) \vee i = x \vee y$ for some $i \in I$. Then $(x \vee y)/(x \wedge y) \searrow i/(x \wedge y \wedge i)$ and so by Theorem 2.1.2 we have $(x \vee y, x \wedge y) \in \text{con}(x \wedge y \wedge i, i) \subseteq \theta[I]$. Thus $(x, y) \in \theta[I]$. \square

A lattice L is *relatively complemented* if every quotient a/b in L is complemented. A *generalized Boolean algebra* is a relatively complemented distributive lattice with 0 .

We prove below that generalized Boolean algebras are characterized by a one-to-one correspondence between congruences and ideals.

THEOREM 2.6.5 [Has52] *Let L be a lattice. There is a one-to-one correspondence between ideals and congruence relations of L under which the ideal corresponding to a congruence relation θ is a congruence class under θ if and only if L is a generalized Boolean algebra.*

PROOF. Consider the map $f : \text{Con}(L) \rightarrow \mathcal{I}(L)$ defined by $\theta \mapsto \mathbf{0}/\theta$. Then for any ideal I of L we have $I = \mathbf{0}/\theta[I]$. (Since by Theorem 2.6.4 above we have $a \in I \Rightarrow a \vee \mathbf{0} = (a \wedge \mathbf{0}) \vee a \Rightarrow (a, \mathbf{0}) \in \theta[I]$ and $(a, \mathbf{0}) \in \theta[I] \Rightarrow a \vee \mathbf{0} = (a \wedge \mathbf{0}) \vee i$ for some $i \in I \Rightarrow a = i \in I$.)

To show that f is one-to-one let $\theta \in \text{Con}(L)$ and suppose $a, b \in L$. Let c be the relative complement of $a \wedge b$ in $(a \vee b)/\mathbf{0}$. Then $(a, b) \in \theta \Leftrightarrow (a \wedge b, a \vee b) \in \theta \Leftrightarrow (a \wedge b \wedge c, (a \vee b) \wedge c) \in \theta \Leftrightarrow (\mathbf{0}, c) \in \theta$. Thus θ is completely determined by $\mathbf{0}/\theta$ and f is one-to-one. \square

Now let S be any partially ordered set. Then an *order ideal* of S is a subset of S that is closed downwards. Let $\mathcal{O}(S)$ denote the collection of order ideals of S partially ordered by set inclusion. Then $\mathcal{O}(S)$ is a complete sublattice of $\mathcal{P}(S)$ (the lattice of all subsets of S).

LEMMA 2.6.6 *Every distributive algebraic lattice is infinitely join-distributive.*

PROOF. Let L be a distributive algebraic lattice with $A \subseteq L$ and $b \in L$. Then $\bigvee_{a \in A} (b \wedge a) \leq b \wedge \bigvee_{a \in A} a$. Let $c \in C(L)$ with $c \leq b \wedge \bigvee_{a \in A} a$. Then $c \leq b$ and $c \leq \bigvee_{a \in A} a$. Since c is compact there is a finite subset $A' \subseteq A$ such that $c \leq \bigvee_{a \in A'} a$. But then $c \leq b \wedge \bigvee_{a \in A'} a = \bigvee_{a \in A'} (b \wedge a) \leq \bigvee_{a \in A} (b \wedge a)$. Thus every compact element below $b \wedge \bigvee_{a \in A} a$ is below $\bigvee_{a \in A} (b \wedge a)$ and so we have $\bigvee_{a \in A} (b \wedge a) = b \wedge \bigvee_{a \in A} a$. \square

LEMMA 2.6.7 *Let L be a distributive algebraic lattice in which every element is a join of completely join irreducible elements. Let P denote the partially ordered set of non-zero completely join irreducibles of L . Then $L \cong \mathcal{O}(P)$.*

PROOF. Define $f : L \rightarrow \mathcal{O}(P)$ by $f(a) = \{x \in P : x \leq a\}$ for $a \in L$. Then f is one-to-one and both f and its inverse are order preserving. To show that the map is surjective let $I \in \mathcal{O}(P)$. Then $x \in f(\bigvee I) \Rightarrow x = x \wedge \bigvee I \Rightarrow x = \bigwedge_{i \in I} (x \wedge i)$ (by Lemma 2.6.6). Since x is completely join irreducible we have $x = x \wedge i$ for some $i \in I$ proving $x \in I$ and thus $f(\bigvee I) \subseteq I$. The reverse inclusion follows easily proving that $f(\bigvee I) = I$ and hence that f is surjective. \square

An element a of a lattice L is *distributive* if for all $b, c \in L$ $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

THEOREM 2.6.8 [Ore35] *Let L be a lattice. Then the following are equivalent:*

(i) *b is a distributive element of L .*

(ii) *The relation σ_b on L defined by $(x, y) \in \sigma_b \Leftrightarrow x \vee b = y \vee b$ is a congruence relation.*

PROOF. (i) \Rightarrow (ii): σ_b is easily an equivalence relation and the substitution property follows from the distributivity of b .

(ii) \Rightarrow (i): Let $x, y \in L$. Then $x \vee b = (x \vee b) \vee b$ and so $(x, x \vee b) \in \sigma_b$. Similarly $(y, y \vee b) \in \sigma_b$. Thus $(x \wedge y, (x \vee b) \wedge (y \vee b)) \in \sigma_b$. Hence $b \vee (x \wedge y) = b \vee ((x \vee b) \wedge (y \vee b))$ and so $b \vee (x \wedge y) = (b \vee x) \wedge (b \vee y)$. \square

THEOREM 2.6.9 *Let L be a lattice with I a distributive ideal of L . Then $(x, y) \in \theta[I] \Leftrightarrow x \vee i = y \vee i$ for some $i \in I$.*

PROOF. For all $x, y \in L$ we have $(x, y) \in \theta[I] \Leftrightarrow ((x), (y)) \in \sigma_I$ where σ_I is the congruence on $\mathcal{I}(L)$ defined in Theorem 2.6.8. Thus $(x, y) \in \theta[I] \Rightarrow (x) \vee I = (y) \vee I \Rightarrow x \leq y \vee i$ and $y \leq x \vee j$ for some $i, j \in I$. Then $x \vee (i \vee j) = y \vee (i \vee j)$ and $i \vee j \in I$. On the other hand suppose $x \vee i = y \vee i$ for some $i \in I$. Then $(x) \vee I = (y) \vee I$ i.e. $((x), (y)) \in \sigma_I$ and so $(x, y) \in \theta[I]$. \square

Recall from Chapter 0 that for a congruence θ on a lattice with 0 , $K(\theta) = 0/\theta$.

THEOREM 2.6.10 *Let L be a lattice with 0 and I a non-empty distributive ideal of L . Then $I = K(\theta[I])$.*

PROOF. We use Theorem 2.6.9. Let $i \in I$. Then $i \vee i = 0 \vee i \Rightarrow (i, 0) \in \theta[I] \Rightarrow i \in K(\theta[I])$. Conversely $(i, 0) \in \theta[I] \Rightarrow i \vee j = 0 \vee j$ for some $j \in I$. Then $i \leq j$ and so $i \in I$. \square

Chapter 3

Congruence lattices of lattices : representation theorems

As a consequence of Funayama and Nakayama's 1942 result, the congruence lattice of a lattice is distributive and algebraic. (The congruence lattice of any algebra is algebraic.) The converse question as to whether these properties characterize congruence lattices of lattices is a long standing open problem. We will call a distributive algebraic lattice *representable* if it is isomorphic to the congruence lattice of a lattice. In this chapter we examine the partial results known to date:

- (i) Every distributive algebraic lattice in which every non-zero element is a join of completely join irreducibles is representable ([Cra73], [Gra62]). In particular every finite distributive lattice is representable.
- (ii) Every distributive algebraic lattice whose compact congruences form a lattice is representable ([Sch81], [Pud85]).
- (iii) Every distributive algebraic lattice with up to \aleph_1 compact elements is representable ([Huh89b]).

The longer proofs of (ii) and (iii) above are not presented in their full details in order to restrict the length of this chapter and because Tischendorf's theorem presented in Section 3.5 extends the above results and provides sufficient new conditions for a distributive algebraic lattice to be representable.

The results of Sections 3.6 - 3.8 concern the representation of particular lattices as the congruence lattices of specific types of lattices. We show that:

- (i) Every finite distributive lattice is the congruence lattice of a finitely generated modular lattice ([Fre75]).

- (ii) Every finite distributive lattice is the congruence lattice of a complemented modular lattice (which is a sublattice of the lattice of all subspaces of a countably infinite dimensional vector space over the two element field ([Sch84])).
- (iii) Every complete lattice is the lattice of complete congruences of a complete lattice ([Gra91]).

In Sections 3.9 and 3.10 we characterize those lattices whose congruence lattices are Boolean algebras and lattices whose congruence lattices are Stone lattices.

3.1 Preliminary definitions and results

Meet prime, join prime elements

An element a of a lattice L is *join prime* if for any $b, c \in L$ such that $a \leq b \vee c$ we have $a \leq b$ or $a \leq c$. Dually, a is *meet prime* if whenever $b \wedge c \leq a$ then $b \leq a$ or $c \leq a$.

LEMMA 3.1.1 *In a distributive lattice L , $a \in L$ is meet irreducible if and only if a is meet prime; join irreducible if and only if join prime.*

PROOF. In any lattice meet prime elements are meet irreducible. Let a be a meet irreducible element of L and suppose $b \wedge c \leq a$. Then $a \vee (b \wedge c) = a$ and so by distributivity $(a \vee b) \wedge (a \vee c) = a$. By meet irreducibility of a we have $b \leq a$ or $c \leq a$. The join statement follows dually. \square

Atomic, atomistic, weakly atomic lattices

We recall that a lattice L is *relatively complemented* if every quotient in L is complemented.

LEMMA 3.1.2 *Every complemented modular lattice is relatively complemented.*

PROOF. Let a/b be a quotient in a complemented modular lattice L with $x \in a/b$. Let x' be the complement of x in L . Then $x \wedge ((x' \wedge a) \vee b) = (x \wedge x' \wedge a) \vee b = (\mathbf{0} \wedge a) \vee b = b$ and $x \vee ((x' \wedge a) \vee b) = x \vee (x' \wedge a) = a \wedge (x' \vee x) = a \wedge \mathbf{1} = a$. Thus $(x' \wedge a) \vee b$ is the complement of x in a/b . \square

A lattice L with zero is *atomic* if for every non-zero $x \in L$, there is an atom $a \leq x$; *atomistic* if every non-zero element of L is the join of atoms below it.

A lattice L is *weakly atomic* if every proper quotient of L contains a prime quotient.

LEMMA 3.1.3 *Every algebraic lattice is weakly atomic.*

PROOF. Let a/b be a quotient of L . We first show that a/b is algebraic. Let $c \in C(L)$ be such that $c \leq a$ and let $S \subseteq a/b$ with $b \vee c \leq \bigvee S$. Then $c \leq \bigvee S$ and so there is a finite subset $S' \subseteq S$ such that $c \leq \bigvee S'$. Since $S' \subseteq a/b$ we have $b \leq \bigvee S'$ and so $b \vee c \leq \bigvee S'$. Thus $b \vee c \in C(a/b)$. For any $x \in a/b$ we have $x = \bigvee \{b \vee c : c \leq x \text{ and } c \in C(L)\}$. This proves that a/b is algebraic.

Now let a/b be a proper quotient of L with $c > b$ a compact element of a/b . Consider $P = \{x \in a/b : x < c\}$ as a subset of a/b . Then $b \in P$. Let M be a maximal chain in P with $q = \bigvee M$. Then $q \leq c$. Suppose $q = c$. Then there is a finite subset $M' \subseteq M$ such that $c \leq \bigvee M'$. But, since M' is a chain in P there is a $m' \in M'$ such that $c \leq m'$. This contradicts $m' \in P$ and so $q < c$. Now suppose $q < z < c$ for some $z \in L$. Then $z \in P$ and $M \cup \{z\}$ is a chain in P contradicting the maximality of M . Hence q/c is a prime quotient of a/b and L is weakly atomic. \square

LEMMA 3.1.4 *Every algebraic complemented modular lattice is atomic.*

PROOF. Let L be an algebraic complemented modular lattice with x a non-zero element of L . By Lemma 3.1.3 there exist $p, q \in L$ such that $0 \leq p \prec q \leq x$. By Lemma 3.1.2 L is relatively complemented. Let p' be the complement of p in $q/0$. Then $p'/0 \searrow q/p$ and since $p \prec q$ we have by Corollary 2.2.2 that $0 \prec p'$. Thus p' is an atom below x and L is atomic. \square

LEMMA 3.1.5 *Every algebraic complemented modular lattice is atomistic.*

PROOF. Let L be an algebraic complemented modular lattice. Then since every element of L is the join of compact elements below it we need only show that the statement of the lemma holds for all compact elements of L . Let $x \in C(L)$ and let A be the set of atoms below x . Then by Lemma 3.1.4 A is not empty. Let $b = \bigvee A$ and assume $b \neq x$. Consider the set $B = \{y \in L : b \leq y < x\}$. Then $b \in B$ and if C is a chain in B then $b \leq \bigvee C \leq x$. Suppose $x = \bigvee C$. Then since $x \in C(L)$ there is a finite subset $C' \subseteq C$ such that $x \leq \bigvee C'$. Since C' is a chain there is a $c \in C'$ such that $x \leq c$. But this contradicts $c \in B$. Hence $\bigvee C < x$ and so $\bigvee C \in B$. By Zorn's Lemma B has a maximal element m . By Lemma 3.1.2 L is relatively complemented. Let m' be the complement of m in $x/0$. Then $m'/0 \searrow x/m$. Since $m \prec x$, Corollary 2.2.2 yields that m' is an atom of L . Then $m' \leq b \leq m$ contradicting $m \wedge m' = 0$. We must then have $b = x$ and hence x is the join of all atoms below it. \square

Distributive semilattices

We will use the following characterization of distributive lattices:

LEMMA 3.1.6 *A lattice L is distributive if and only if*

$$x \leq y \vee z \Rightarrow \exists y' \leq y \exists z' \leq z (x = y' \vee z').$$

PROOF. Distributivity implies $x \leq y \vee z \Rightarrow x \wedge (y \vee z) = x \Rightarrow (x \wedge y) \vee (x \wedge z) = x$ with $x \wedge y \leq y$ and $x \wedge z \leq z$.

For the reverse implication let $x, y, z \in L$. We always have $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$. Now $x \wedge (y \vee z) \leq y \vee z$. So $\exists y' \leq y, z' \leq z$ such that $x \wedge (y \vee z) = y' \wedge z'$. But then $y' \leq x \wedge y$ and $z' \leq x \wedge z$. So $x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$ proving the distributivity of L . \square

For the rest of this chapter a semilattice will mean a join-semilattice and we call a semilattice S *distributive* if its ideal lattice is a distributive algebraic lattice. By Lemma 3.1.6 this is equivalent to saying that whenever we have elements $x, y, z \in S$ with $x \vee y \geq z$, there are elements $x', y' \in S$ such that $x' \leq x, y' \leq y$ and $x' \vee y' = z$.

Remarks : To investigate whether a given distributive algebraic lattice L can be represented as the congruence lattice of a lattice it is sufficient to show that the semilattice of compact elements of L is isomorphic to the compact congruences of a lattice K . Since then $L \cong I(L^c) \cong I(\text{Con}^c(K)) \cong \text{Con}(K)$. by Lemma 2.6.1.

Direct Limits

We consider some basic facts on direct limits of algebras.

Let (I, \trianglelefteq) be a partially ordered, upward directed set (i.e. for every $i, j \in I \exists k \in I$ such that $k \trianglerighteq i$ and $k \trianglerighteq j$). Consider $\mathcal{C} = \{C_i : i \in I\}$ a class of algebras of some fixed type \mathcal{F} together with homomorphism $\alpha_{i,j} : C_i \rightarrow C_j$ for every pair $i, j \in I$ with $i \trianglelefteq j$ and such that:

- (i) $\alpha_{i,i} = id_{C_i} \quad \forall i \in I$
- (ii) $\alpha_{j,k} \circ \alpha_{i,j} = \alpha_{i,k} \quad \forall i \trianglelefteq j \trianglelefteq k \in I$

Then the $\alpha_{i,j}$'s are said to form a *limit system* and $((I, \trianglelefteq), \mathcal{C}, (\alpha_{i,j})_{i \trianglelefteq j})$ is a directed family of sets.

Consider the set $\bigcup_{i \in I} C_i$ and define a relation ρ on $\bigcup_{i \in I} C_i$ by $x \rho y \Leftrightarrow x \in C_i, y \in C_j$ for some $i, j \in I$ and there exists a $z \in C_k$ such that $i \trianglelefteq k, j \trianglelefteq k$ and $\alpha_{i,k}(x) = z, \alpha_{j,k}(y) = z$. Then ρ is an equivalence relation on $\bigcup_{i \in I} C_i$ and $C_\infty = (\bigcup_{i \in I} C_i) / \rho$ is the *direct limit* of the class \mathcal{C} with respect to the homomorphisms $\alpha_{i,j}$.

The maps $\lambda_i : C_i \rightarrow C_\infty$ defined by $\lambda_i(c) = c/\rho$ are such that the following diagram commutes for all $i \trianglelefteq j$ in I :

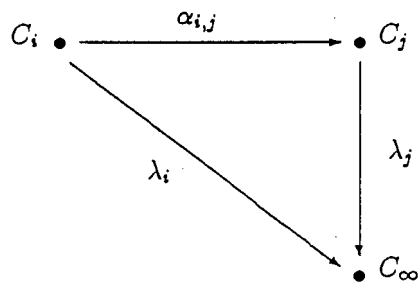


Figure 3.1

Note that if the $\alpha_{i,j}$'s are embeddings, then so are the λ_i 's.

Suppose we have maps $\gamma_i : C_i \rightarrow D$ such that the following diagram commutes for all $i, j \in I$ with $i \leq j$:

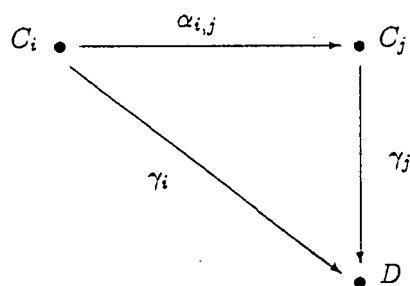


Figure 3.2

Then there is a unique homomorphism $\gamma : C_\infty \rightarrow D$ defined by $\gamma(c/\rho) = \lambda_i(c)$ such that the following diagram commutes for all $i \in I$:

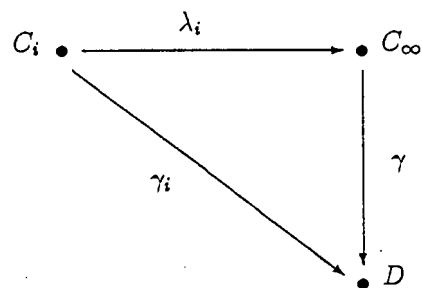


Figure 3.3

Note that if the γ_i 's are embeddings then so is λ .

3.2 Lattices in which every non-zero element is a join of join irreducibles

In this section we show that every distributive algebraic lattice L in which every non-zero element is the join of completely join irreducibles is the congruence lattice of a lattice K . This theorem has as a corollary that if L is finite then so is K . The proof is due to Dilworth and was first published by G. Grätzer and E.T. Schmidt [Gra62]. See also [Cra73].

We start with a few lemmas primarily concerning sectionally complemented lattices.

A lattice L is *sectionally complemented* if L has a zero and for every $a \in L$, $a/\mathbf{0}$ is complemented.

LEMMA 3.2.1 *Let L be a sectionally complemented lattice. Then for $\theta \in \text{Con}(L)$ we have $\theta = \bigvee \{\text{con}(\mathbf{0}, x) : (\mathbf{0}, x) \in \theta\}$.*

PROOF. Let a/b be a proper quotient in L with $(a, b) \in \theta$. Then b has a complement c in $a/\mathbf{0}$. We have $(a, b) \in \theta \Rightarrow (a \wedge c, b \wedge c) = (c, \mathbf{0}) \in \theta$ and $(c \vee b, \mathbf{0} \vee b) = (a, b) \in \text{con}(c, \mathbf{0})$. Thus $\theta \subseteq \bigvee \{\text{con}(\mathbf{0}, x) : (\mathbf{0}, x) \in \theta\}$. The reverse inclusion follows immediately. \square

Recall from Chapter 0 that if θ is a congruence on a lattice with $\mathbf{0}$ then we let $K(\theta)$ denote $\mathbf{0}/\theta$ (the congruence class of $\mathbf{0}$ modulo θ).

LEMMA 3.2.2 *In a sectionally complemented lattice L , $\theta \in \text{Con}(L)$ is precisely the least congruence which collapses all the quotients in $\{x/\mathbf{0} : x \in K(\theta)\}$.*

PROOF. Let Ψ be the least congruence of L collapsing all the quotients in $\{x/\mathbf{0} : x \in K(\theta)\}$. Then $\Psi \subseteq \theta$ follows easily. Let $(a, b) \in \theta$ with c the complement of $a \wedge b$ in $(a \vee b)/\mathbf{0}$. Then $(a, b) \in \theta \Leftrightarrow (a \wedge b, a \vee b) \in \theta \Rightarrow (a \wedge b \wedge c, (a \vee b) \wedge c) \in \theta \Rightarrow (\mathbf{0}, c) \in \theta \Rightarrow (\mathbf{0}, c) \in \Psi \Rightarrow (\mathbf{0} \vee (a \wedge b), c \vee (a \wedge b)) \in \Psi \Rightarrow (a \wedge b, a \vee b) \in \Psi \Leftrightarrow (a, b) \in \Psi$. Thus $\theta \subseteq \Psi$ proving the statement of the lemma. \square

LEMMA 3.2.3 *If p/q is a prime quotient in a lattice L , then $\text{con}(p, q)$ is completely join irreducible in L .*

PROOF. Suppose $\text{con}(p, q) = \bigvee_{i \in I} \Phi_i$ where $\Phi_i \in \text{Con}(L)$ for all $i \in I$. Then $(p, q) \in \bigvee_{i \in I} \Phi_i$ and, since p/q is a prime quotient of L , we have $(p, q) \in \Phi_i$ for some $i \in I$. Thus $\Phi_i = \text{con}(p, q)$ and so $\text{con}(p, q)$ is completely join irreducible. \square

We are now ready to prove the main result this section:

THEOREM 3.2.4 [Gra62] *Let L be a distributive algebraic lattice in which every non-zero element is a join of completely join irreducible elements of L . Then there is a lattice K such that $L \cong \text{Con}(K)$.*

PROOF. Let P be the set of non-zero join irreducible elements of L . For each $i \in \{0, 1\}$ define $H_i = \{p_i : p \in P\}$ where the mapping $p \mapsto p_i$ is a bijection between P and H_i . Set $p_0 = p_1$ if and only if p is maximal in P . Agree that if p' denotes any one of p_0, p_1 then p'' denotes the other one. Let $H = H_0 \cup H_1$.

Call a subset A of H closed if:

$$p \leq q \text{ and } p', q' \in A \text{ imply } p'' \in A.$$

Then H is closed and the intersection of arbitrarily many closed subsets is closed. Consequently for every A in H there is a least closed subset of H containing A . Denote this closed subset by \bar{A} . Let K be the lattice of finite closed subsets of H . Note that if A is finite then so is \bar{A} . Now for $A, B \in K$, $A \wedge B = A \cap B$ and $A \vee B = \overline{A \cup B}$. For every $p \in P$ the singleton $\{p'\}$ is closed and so is \emptyset , the zero of K . We associate p' with the singleton $\{p'\}$.

We first show that K is sectionally complemented. Let $A, B \in K$ with $A \subseteq B$. Define $A^* = (B \setminus A) \setminus C$ where C consists of all $p' \in A$ satisfying:

$$\text{there exists a } q \in P \text{ such that } p < q, q' \in B \setminus A \text{ and } p'' \in A.$$

We claim that A^* is the complement of A in $B/0$. Suppose $p < q$ in P and $p', q' \in A^*$. Then, since B is closed we have $p'' \in B$. If $p'' \in A$ then by the definition of C we have $p' \in C$ contradicting $p' \in A^*$. If $p'' \in C$ then $p' \in A$, again contradicting $p' \in A^*$. Thus $p'' \in A^*$ and A^* is closed. We have $A \cap A^* = A \cap (B \setminus A) \setminus C = \emptyset$. Also $A \cup A^* = A \cup (B \setminus A) \setminus C \subseteq B$. We need to show that $B \subseteq A \vee A^*$. Consider the set $D = \{z \in P : z' \in B \setminus (A \vee A^*)\}$. Suppose $D \neq \emptyset$. Then, by the finiteness of B , D has a maximal member p . Now $p' \notin A \vee A^* \Rightarrow p' \in C$. So there is a $q \in P$ such that $p < q, q' \in B \setminus A$ and $p'' \in A$. By the maximality of p in D we must have $q' \in A \vee A^*$. Then we have $p < q, q' \in A \vee A^*, p'' \in A \subseteq A \vee A^*$. Since $A \vee A^*$ is closed we must have $p' \in A \vee A^*$. This contradiction completes the proof that $B \subseteq A \vee A^*$. Hence $B = A \vee A^*$ and we conclude that K is sectionally complemented.

Fix $r \in P$ and define $\mathcal{A}_r = \{A \in K : p' \in A \Rightarrow p \leq r\}$. Then \mathcal{A}_r is an ideal of K . Since, if $A \in \mathcal{A}_r$ with $B \subseteq A$ then $p' \in B \Rightarrow p' \in A \Rightarrow p \leq r$ proving $B \in \mathcal{A}_r$. Furthermore if $A, B \in \mathcal{A}_r$ then $p' \in A \vee B \Rightarrow \exists q \in P$ such that $q', p'' \in A \cup B$ and $p \leq q$. Then $q' \in A$ or $q' \in B$. Hence $p \leq q \leq r$ and so $A \vee B \in \mathcal{A}_r$.

We show that \mathcal{A}_r is a distributive element of $\mathcal{I}(K)$. Let $A \in \mathcal{A}_r$ and $B, C \in K$. Define $A^\dagger = Q \cup R$ where $Q = \{p', p'' : p' \in A\}$ and $R = \cup\{p', p'' : p' \in B \cup C, p \leq r\}$. We claim that for any $D \in K$ with $D \subseteq B \cup C$ we have $A^\dagger \vee D = A^\dagger \cup D$. To show this let

$q', p' \in A^\dagger \cup D$ with $p \leq q$. We show $p'' \in A^\dagger \cup D$ proving that $A^\dagger \cup D$ is closed. If $p' \in A^\dagger$ then $p'' \in A^\dagger$ and we are done. Suppose that $p' \in D$. Then $q' \in D$ implies $p'' \in D$ as D is closed. Thus $p'' \in A^\dagger \cup D$. Otherwise if $q' \in A^\dagger$ then either (i) $q \in A$ in which case $p \leq q \leq r$ and so $p' \in R$ implying $p'' \in R$ and hence $p'' \in A^\dagger$, or (ii) $q' \in R$ in which case $p \leq q \leq r$ and $p' \in D$ implying $p' \in A^\dagger$ and so $p'' \in A^\dagger$. Thus $A^\dagger \cup D$ is closed and $A \cup D = \overline{A \vee D}$.

Now let $I, J \in \mathcal{I}(K)$ and $Z \in (\mathcal{A}_r \vee I) \wedge (\mathcal{A}_r \vee J)$. Then $Z \subseteq (A \vee B) \wedge (A \vee C)$ for some $A \in \mathcal{A}_r, B \in I, C \in J$. Thus $Z \subseteq (A^\dagger \vee B) \wedge (A^\dagger \vee C) = (A^\dagger \cup B) \cap (A^\dagger \cup C) = A^\dagger \cup (B \cap C) \in \mathcal{A}_r \wedge (I \vee J)$. Hence $(\mathcal{A}_r \vee I) \wedge (\mathcal{A}_r \vee J) = \mathcal{A}_r \wedge (I \vee J)$ proving that \mathcal{A}_r is a distributive ideal of K .

Let $\theta \in \text{Con}(K)$. Then we show that θ is completely join irreducible if and only if $\theta = \text{con}(\mathbf{0}, p')$ for some $p \in P$. For the forward implication let θ be a completely join irreducible element of $\text{Con}(K)$. Then by Lemma 3.2.1 $\theta = \bigvee \{\text{con}(\mathbf{0}, X) : (\mathbf{0}, X) \in \theta\}$. Since every element of K is the join of finitely many atoms we have $\theta = \bigvee \{\text{con}(\mathbf{0}, p') : (\mathbf{0}, p') \in \theta\}$. By join irreducibility of θ we have $\theta = \text{con}(\mathbf{0}, p')$ for some $p \in P$. For the reverse implication let $p \in P$. Then p' is an atom of K and hence $p'/\mathbf{0}$ is a prime quotient in K . The result now follows from Lemma 3.2.3.

We also have for each $p \in P$, $\text{con}(\mathbf{0}, p') = \text{con}(\mathbf{0}, p'')$. Since if $p' \neq p''$ then there is a $q \in P$ with $p \leq q$. We have $p'' \leq p' \vee q'$ and hence $p'' \vee q' = p' \vee q'$. Thus $p'/\mathbf{0} \searrow (p' \vee q')/q' \nearrow p''/\mathbf{0}$. By Corollary 2.1.3 we have $\text{con}(\mathbf{0}, p') = \text{con}(\mathbf{0}, p'')$. From now on we will denote $\text{con}(\mathbf{0}, p')$ by ϕ_p .

Let $p < q$ in P . Then $p''/\mathbf{0} \searrow_{\omega} (q' \vee p')/p' \nearrow q'/\mathbf{0}$. Thus $\phi_p \subseteq \phi_q$.

Let θ be a join irreducible congruence of K . Then $\theta = \phi_r$ for some $r \in P$. We show that $K(\phi_r) = \mathcal{A}_r$. Let $A \in \mathcal{A}_r, p' \in A$. Then $p \leq r$ and so $\phi_p \subseteq \phi_r$ and hence $(p', \mathbf{0}) \in \phi_r$. This is true for all $p' \in A$ proving that $(A, \mathbf{0}) \in \phi_r$ and consequently $\mathcal{A}_r \subseteq K(\phi_r)$. For the reverse inclusion it is sufficient to show that $K(\phi_r) \subseteq K(\theta[\mathcal{A}_r])$ (by Theorem 2.6.10). Suppose $(A, \mathbf{0}) \in \phi_r$. Then $(A, \mathbf{0}) \in \text{con}(r', \mathbf{0})$ and, since $r' \in \mathcal{A}_r$ we have $\text{con}(r', \mathbf{0}) \subseteq \theta[\mathcal{A}_r]$ (Theorem 2.6.10) and so $(A, \mathbf{0}) \in \theta[\mathcal{A}_r]$. We therefore have $A \in K(\theta[\mathcal{A}_r])$ proving $K(\phi_r) = \theta[\mathcal{A}_r]$.

Consider the map $r \mapsto \phi_r$ of P onto the poset of non-zero join irreducible elements of $\text{Con}(K)$. Then $\phi_r = \phi_p \Rightarrow \mathcal{A}_r = \mathcal{A}_p \Rightarrow r \leq p$ and $p \leq r \Rightarrow p = r$ proving that the map is one-to-one.

Also $r \leq p \Rightarrow \mathcal{A}_r \leq \mathcal{A}_p \Rightarrow K(\phi_r) \leq K(\phi_p) \Rightarrow \phi_r \leq \phi_p$ (by Lemma 3.2.2) and so the map is order preserving. To show that the inverse is order preserving let $\phi_r \leq \phi_p$. Then $K(\phi_r) \leq K(\phi_p)$ implying $\mathcal{A}_r \leq \mathcal{A}_p$ and so $r \leq p$. Hence, by Lemma 2.6.7 we have $L \cong \mathcal{O}(P) \cong \mathcal{O}(J(\text{Con}(K))) \cong \text{Con}(K)$. \square

COROLLARY 3.2.5 *If L is a finite distributive lattice then L is isomorphic to the congruence lattice of a finite lattice.*

PROOF. If L is finite then K described in the proof of Theorem 3.2.4 is finite. \square

In [Sch75] E.T. Schmidt has shown that if L is a finite distributive lattice with exactly one coatom, then there exists a finite lattice K such that $L \cong \text{Con}(K)$ and K has length 5. This is a generalization of a result of J. Berman which proves the same in the case that L is a finite chain ([Ber72]).

3.3 Lattices whose compact elements form a lattice

In this section we sketch E.T.Schmidt's proof of the result that a distributive algebraic lattice whose compact elements form a lattice is representable. We restate this theorem for later reference.

THEOREM 3.3.1 [Sch81] *The ideal lattice of a distributive lattice with 0 is the congruence lattice of a lattice.*

Recall the definition of a distributive semilattice from Section 3.1 of this chapter. Let S, T be two distributive semilattices. Then a homomorphism $f : S \rightarrow T$ is *weak-distributive* if $f(a) = f(b \vee c)$ and $b \vee c \leq a$ imply the existence of elements $b' \geq b, c' \geq c$ such that $f(b') = f(b), f(c') = f(c)$ and $b' \vee c' = a$.

The congruence relation induced by a weak-distributive homomorphism is called a *weak-distributive congruence*.

A congruence relation θ of a semilattice is called *monomial* if every congruence class of θ has a maximal element and θ is *distributive* if it is the join of weak-distributive monomial congruences. Note that a distributive congruence is weakly distributive (see Lemma 3.6.6. of [Sch82]).

We call a map h between semilattices monomial/distributive if $\ker(h)$ is a monomial/distributive congruence.

If (B, \wedge, \vee) is a generalized Boolean lattice then (B, \vee) will be called a *generalized Boolean semilattice*.

Let S be a distributive semilattice with $0, 1 \in S$ and let B be a Boolean semilattice with unit 1_B and zero 0_B . Then B is a *pre-skeleton* of S if there exists a mapping $f : B \rightarrow S$ such that the following conditions are satisfied:

- (i) f is a $\{0, 1\}$ -homomorphism.
- (ii) If $f(1_B) = x \vee y$ then there exist $x_1, y_1 \in B$ such that $x_1 \vee y_1 = 1_B, f(x_1) \leq x, f(y_1) \leq y$.

LEMMA 3.3.2 *Every bounded distributive lattice has a pre-skeleton.*

A partial sublattice H of a Boolean algebra B is called a *join-base* if the following conditions are satisfied:

- (i) $0, 1 \notin H$.
- (ii) There is a dimension function δ from H onto an ideal of ω such that $x \prec y$ in H if and only if $x \leq y$ and $\delta(x) = \delta(y) + 1$. The set of all $x \in H$ with $\delta(x) = i$ is denoted by H^i .
- (iii) For every finite subset $U = \{u_1, \dots, u_n\}$ of B there exists an $i \in \omega$ such that each H^k ($k \geq i$) has a finite subset $A_k(U)$ with the property that each $u \in U$ has a unique join representation as a join of elements of $A_k(U)$.
- (iv) If $a, b \in H$ and $a \wedge b \neq 0$ in B then $a \wedge b \in H$. If $a \vee b$ exists in H and a, b are incomparable then $a, b \in H^i, a \vee b \in H^{i-1}$ for some $i > 1$. Let $a \in H^i, c \in H^{i-1}, d \in H^j, j < i, a < c \leq d$. Then there exist unique elements $b \in H^i, a_0, b_0 \in B$ such that $a \vee b = c, a_0 \vee b_0 = d, a_0 \wedge (a \vee b_0) = a, b_0 \wedge (a \vee b) = b$.

Now let H be a join-base of a Boolean semilattice B and let $f : H \rightarrow L$ be a homomorphism into a distributive lattice with 0 . Then f can be extended in a natural way to a homomorphism $\varphi : B \rightarrow L$ as follows: For $a \in B, a \neq 0$ we have $a = h_1 \vee \dots \vee h_n$ with $h_i \in H \forall i \in \{1, \dots, n\}$. Define $\varphi(a) = f(h_1) \vee \dots \vee f(h_n)$ and set $\varphi(0) = 0$.

By condition (iii) in the definition of a join-base, this definition of φ is unique and φ is a $\{0\}$ -homomorphism of B into L . We call φ an *L -valued homomorphism of B induced by f* .

LEMMA 3.3.3 *Let L be a bounded distributive lattice and suppose $\varphi : B \rightarrow L$ is a weak-distributive homomorphism of a Boolean semilattice B generated by a homomorphism $f : H \rightarrow L$ of a join-base H . Then φ is a distributive homomorphism.*

Let \mathcal{D} be the class of all bounded distributive lattices and let $(L_i)_{i \in I}$ be lattices in \mathcal{D} . A lattice L in \mathcal{D} is called a *free $\{0, 1\}$ -distributive product* of the $(L_i)_{i \in I}$ if every L_i has an embedding ϵ_i into L such that :

- (i) L is generated by $\cup\{\epsilon_i(a) : a \in L_i, i \in I\}$.
- (ii) If K is any lattice in \mathcal{D} and $\varphi_i : L_i \rightarrow K$ is a $\{0, 1\}$ -homomorphism for $i \in I$, then there exists a $\{0, 1\}$ -homomorphism $\varphi : L \rightarrow K$ such that $\varphi_i = \varphi \circ \epsilon_i$ for all $i \in I$.

Let $(A_i)_{i \in I}$ be lattices with zero. The lower discrete product $\prod_d(A_i : i \in I)$ is the sublattice of the direct product $\prod_{i \in I} A_i$ consisting of elements a for which $a_i = 0$ for all but finitely many $i \in I$.

LEMMA 3.3.4 *Let L be a bounded distributive lattice with $(A_i)_{i \in I}$ Boolean semilattices. If $\varphi_i : A_i \rightarrow L$ ($i \in I$) are L -valued $\{0, 1\}$ -homomorphisms generated by $f_i : H_i \rightarrow L$, then the free $\{0, 1\}$ -distributive product $A = \prod^* A_i$ has a join-base H and a homomorphism $f : H \rightarrow L$ such that $H \cap A_i = H_i$ for each $i \in I$ and $f|_{\epsilon_i[A_i]} = \epsilon_i \circ f_i$ where ϵ_i is as in the definition of $\prod^* A_i$. Consequently the L -valued homomorphism φ induced by f satisfies $\varphi_i = \varphi \circ \epsilon_i$.*

LEMMA 3.3.5 *Let A_1, A_2 be Boolean semilattices and let $\varphi_i : A_i \rightarrow L$ be L -valued $\{0, 1\}$ -homomorphisms generated by homomorphisms $f_i : H_i \rightarrow L$ of the join bases $H_i \subseteq A_i$ ($i = 1, 2$). Then $H = H_1 \cup H_2$ is a join-base of $A_1 \times A_2$ and there exists a common extension f of f_1 and f_2 to H .*

Let L be a distributive lattice with 1. Then $a \in L$ has order n ($o(a) = n$), if n is the smallest positive integer such that there exists a sequence $a = x_0, x_1, x_2, \dots, x_n$ satisfying:

$$a < a \vee x_1 < a \vee x_2 < \dots < a \vee x_n = 1 \text{ and}$$

$a \vee x_1 \vee x_2 \vee \dots \vee x_{i-1}$ is incomparable with x_i ($i = 1, \dots, n$). If no such sequence exists then we define $o(a) = \infty$.

We are now ready to outline the proof of the following proposition:

PROPOSITION 3.3.6 *Let L be a distributive lattice with 0. Then there exists a generalized Boolean semilattice B and a distributive congruence θ such that $L \cong B/\theta$.*

We first assume that L is a bounded distributive lattice. Then by Lemma 3.3.2 there is a pre-skeleton D of L with a $\{0, 1\}$ -homomorphism $\varphi_0 : D \rightarrow L$ satisfying the following condition: If $\varphi_0(1_D) = x \vee y$ then there exist $x_1, y_1 \in D$ such that $x_1 \vee y_1 = 1_D, \varphi_0(x_1) \leq x, \varphi_0(y_1) \leq y$.

Now let $u \neq 0_D, 1_D$ be an element of D and let $a = \varphi_0(u)$. The principal ideal (a) of L is a bounded distributive lattice and again by Lemma 3.3.2, there is a pre-skeleton $B(a)$ of (a) with $\{0, 1\}$ -homomorphism $\varphi_a : B(a) \rightarrow (a)$. If u' denotes the complement of u in D then $D \cong (u) \times (u')$. We construct a new Boolean algebra $B[1_D, u] = ((u) * B(a)) \times (u')$. Then D and $B(a)$ are Boolean subalgebras of $B[1_D, u]$. (We identify elements x of D with $(x \wedge u, x \wedge u')$ and elements x of $B(a)$ with $(x, 0_D)$.) By Lemmas 3.3.4 and 3.3.5 there exists a common extension $\varphi : B[1_D, u] \rightarrow L$ of φ_0 and φ_a . Then φ is a $\{0, 1\}$ -homomorphism

with the following property:

If $r \in T = \{0_D, u\}$, $\varphi(r) = x \vee y$, then there exist $x_1, y_1 \in B[1_D, u]$ such that $x_1 \vee y_1 = r, \varphi(x_1) \leq x, \varphi(y_1) \leq y \dots\dots\dots (\dagger)$.

Using the same method for an element $v \in D \subseteq B[1_D, u]$ we obtain from $B[1_D, u]$ a Boolean algebra $B[1_D, u, v]$ satisfying (\dagger) for $T = \{1_D, u, v\}$.

Now let $u, v \in D$ with $u, v \neq 0_D, 1_D$. We show that $B[1_D, u, v]$ is isomorphic to $B[1_D, v, u]$. Let B_u denote the pre-skeleton $B(\varphi_0(u))$ of the principal ideal (u) . Similarly B_v will denote $B(\varphi_0(v))$. We have $D = (u \wedge v) \times (u \wedge v') \times (u' \wedge v) \times (u' \wedge v')$. Now $B[1_D, u, v] = ((u) * B_u) \times (u') = (((u \wedge v) \times (u \wedge v')) * B_u) \times (u' \wedge v) \times (u' \wedge v') = ((u \wedge v) * B_u) \times ((u \wedge v') * B_u) \times (u' \wedge v) \times (u' \wedge v') = (((u \wedge v) * B_u) \times (u' \wedge v)) \times (((u \wedge v') * B_u) \times (u' \wedge v'))$. Consequently we get $B[1_D, u, v] = ((u \wedge v) * B_u * B_v) \times ((u' \wedge v) * B_v) \times ((u \wedge v') * B_u) \times (u' \wedge v')$. This representation is symmetric in u and v , hence $B[1_D, u, v] \cong B[1_D, v, u]$.

Note that if $x \in L$ has order one then there exists an $a \in D$ such that $\varphi_0(a) = x$. Consequently each element of L which is the meet of elements of order one is the image of an element of D .

Continuing the construction above we get for arbitrary $u_1, u_2, \dots, u_n \in B$ ($u_i \neq 0_D, 1_D$) a Boolean semilattice $B[1_D, u_1, \dots, u_n]$ with a homomorphism of this Boolean semilattice into L such that the condition (\dagger) is satisfied for the set $T = \{1_D, u_1, \dots, u_n\}$. These Boolean semilattices form a direct family. Letting C_1 be the direct limit we have $D = C_0$ a subalgebra of C_1 and an L -valued homomorphism $\varphi_1 : C_1 \rightarrow L$, which satisfies (\dagger) with $T = D$. Let $x \in L$ be such that x is the meet of elements of order ≤ 2 . Then $x = \varphi_1(y)$ for some $y \in C_1$. Starting with C_1 (instead of $D = C_0$) we repeat the construction above to obtain a Boolean semilattice C_2 with C_1 a subalgebra of C_2 and corresponding L -valued homomorphism $\varphi_2 : C_2 \rightarrow L$. Then C_2 satisfies (\dagger) for $T = C_1$. If $x \in L$ is a meet of elements of order ≤ 3 , then there exists a $y \in C_2$ such that $\varphi_2(y) = x$. Similarly we get C_i ($i = 3, 4, \dots$). These C_i 's again form a direct family. The direct limit \bar{B} is a Boolean semilattice and the direct limit of the join-bases of the C_i 's induces an L -valued homomorphism $\bar{\varphi} : \bar{B} \rightarrow L$. By the construction of $\bar{\varphi}$, the image of each non-zero element of \bar{B} is the meet of elements of finite order. Since (\dagger) is satisfied for the whole of \bar{B} , we have that $\bar{\varphi}$ is a weak-distributive homomorphism. By Lemma 3.3.3 $\bar{\varphi}$ is distributive. Thus for every bounded distributive lattice L we have a Boolean semilattice B and a distributive homomorphism $\varphi : B \rightarrow L$ mapping B onto the set of all elements of L which have a meet representation of elements of finite order.

Now let L be a distributive lattice with 0 . Then for every $a \in L, a \neq 0$ the principal ideal (a) is a bounded distributive lattice. Thus for every (a) we have a Boolean semilattice B_a and a distributive homomorphism φ_a of B_a onto the set of all elements of (a) which have a meet representation of elements of finite order. Consider the lower discrete product $B = \prod_a (B_a : a \in L, a \neq 0)$. Then B is a generalized Boolean semilattice and the map

$h : B \rightarrow L$ defined by $h(x) = \bigvee \varphi_a(x_a)$ is a distributive homomorphism onto L proving Proposition 3.3.6. \square

Let B be a sublattice of a lattice A and let θ be a congruence of A . Then there is a smallest congruence relation $\theta^0 \in \text{Con}(A)$ such that $\theta^0|_B \geq \theta$. The correspondence $\theta \mapsto \theta^0$ is a homomorphism of $\text{Con}^c(B)$ into the semilattice $\text{Con}^c(A)$. If this homomorphism is onto then we call B a *strongly large sublattice* of A and A is a *strong extension* of B .

We sketch the proof of the following proposition.

PROPOSITION 3.3.7 *Let B be a generalized Boolean semilattice with θ a distributive congruence on B . Then there exists a lattice K such that the ideal lattice of B/θ is isomorphic to $\text{Con}(K)$.*

PROOF. We first show that if θ is a monomial distributive congruence of a generalized Boolean semilattice B , then there is a lattice L such that $\text{Con}(L) \cong \mathcal{I}(B/\theta)$. We consider the subset A of $B \times B \times B$ consisting of all elements (a, b, c) such that $a \wedge b = a \wedge c = b \wedge c$. Then A is a (modular) lattice and the elements $\{(a, \mathbf{0}_B, \mathbf{0}_B) : a \in B\}$ form a sublattice of A isomorphic to B . Let L be the subset of A consisting of all elements (a, b, c) such that a is a maximal element of a θ -class. For $a/\theta \in B/\theta$ let a_m denote the maximal element of a/θ . Then B/θ is isomorphic to the ideal I of L generated by $(\mathbf{1}_B, \mathbf{0}_B, \mathbf{0}_B)$, the isomorphism given by $a/\theta \mapsto (a_m, \mathbf{0}_B, \mathbf{0}_B)$. Furthermore L is a strong extension of I and a congruence relation of I has an extension to L if and only if it is of the form $\theta[I']$ where I' is an ideal of L . Hence $\text{Con}^c(L) \cong I \cong B/\theta$. Also the ideal J of L generated by $(\mathbf{0}_B, \mathbf{0}_B, \mathbf{1}_B)$ is isomorphic to B and $I \cap J = \{\mathbf{0}_B\}$.

Now let θ be an arbitrary distributive congruence of a generalized Boolean semilattice B . Then $\theta = \bigvee \{\theta_j : j \in \omega\}$ where θ_j is a distributive, monomial congruence on B . For every $i \in I$ consider the lattice L_i as defined above. Then L_i has two ideals I_i and J_i such that $I_i \cong B/\theta_i$, $J_i \cong B$ and we have $\text{Con}^c(L_i) \cong B/\theta_i$. Consider the direct product $\prod_\omega B$. Call an element t of $\prod_\omega B$ normal if for $i \neq j \neq k \neq i$ we have $t_i \wedge t_j = t_i \wedge t_k = t_j \wedge t_k$. Let M be the sublattice of $\prod_\omega B$ consisting of all normal elements t for which $\{t_i : i \in \omega\}$ is finite. Let J^i be the ideal of M consisting of all t for which $t_j = \mathbf{0}_B$ if $j \neq i$. Then $J^i \cong B$. Furthermore M is a strong extension of J^i and $\text{Con}^c(M) \cong \text{Con}^c(J^i) \cong \text{Con}^c(B)$. Let \overline{M} be the dual lattice of M . Then \overline{J}^i is a filter of \overline{M} , \overline{J}^i is a Boolean algebra and we have a natural isomorphism $\overline{J}^i \cong J^i$ given by $x \mapsto x'$. Using the Hall-Dilworth construction (described in Section 3.6 below) with \overline{M} and L_i , identifying \overline{J}^i and J_i (for each $i \in \omega$) we obtain a partial lattice P . Then M and L_i are sublattices of P and P is a meet semilattice. Letting K be the free lattice generated by P , we have $\text{Con}^c(K) \cong B/\theta$ proving Proposition 3.3.7. \square

The main theorem (3.3.1) now follows immediately from Propositions 3.3.6 and 3.3.7.

3.4 Lattices with at most \aleph_1 compact elements

In this section we show that every distributive algebraic lattice L with at most \aleph_1 compact elements is representable. The case that L has countably many compact elements was settled by Dobbertin ([Dob86]) and independently and by different means in [Huh89a]. Our proof (due to A.P Huhn) makes use of the proposition below:

PROPOSITION 3.4.1 [Pud85] *Every finite subset of a distributive semilattice D (with 0) is contained in a finite, distributive 0 -subsemilattice of D .*

An *order filter* of a partially ordered set P is a subset of P that is closed upwards.

THEOREM 3.4.2 [Huh89b] *Every distributive algebraic lattice L with at most \aleph_1 compact elements is the congruence lattice of a lattice.*

PROOF. Let D be a distributive semilattice with 0 and assume that $|D| = \aleph_1$. We define a directed family of finite subsets of D . Let $\alpha < \omega_1$ be an ordinal. For $\alpha = 0$ define $h_\alpha = \{0\}$. For $\alpha = n + 1$ ($n \in \omega$) let $h_\alpha = h_n \cup \{a\}$ where a is an arbitrary element of $D \setminus h_n$. Suppose $\alpha = \omega\beta + n$, $n \in \omega$. Then $\omega\beta$ has a cofinal ω -chain $\alpha_0 < \alpha_1 < \alpha_2 < \dots$. For $\alpha = \omega\beta$ let $h_\alpha = h_{\alpha_0} \cup \{a\}$ where $a \in D$ is such that $a \notin h_\gamma$ for $\gamma < \omega\beta$. For $\alpha = \omega\beta + n + 1$ let $h_\alpha = h_{\alpha_{n+1}} \cup h_{\alpha_{\omega\beta+n}} \cup \{a\}$ where $a \in D$ and $a \notin h_\gamma$: $\gamma \leq \alpha$. Let H be the set of all h_α : $\alpha \leq \omega_1$. The inclusion relation orders H diagrammed in Figure 3.4 below:

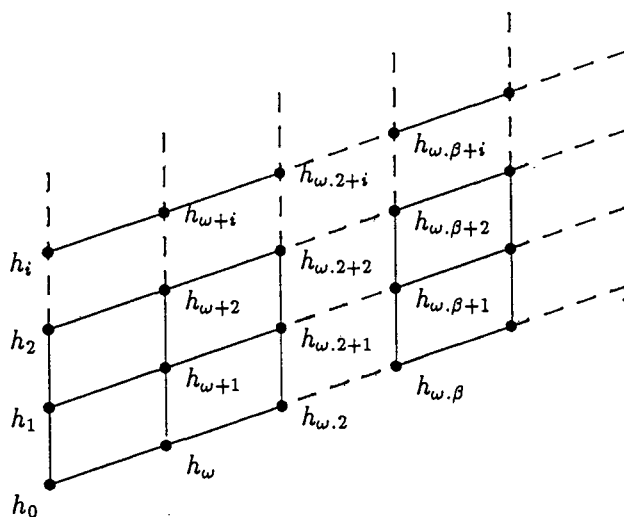


Figure 3.4 H

By Proposition 3.4.1, we can choose, by induction on α , for every $h \in H$, a finite distributive 0 -subsemilattice D_h of D containing h such that $h \leq k \Rightarrow D_h \subseteq D_k$. Then

D is the direct limit of the D_h 's. For $h \leq k$ (i.e. $D_h \subseteq D_k$) define $\varepsilon(h, k) : D_h \rightarrow D_k$ by $\varepsilon(h, k)(d) = d$.

For every $h, i \in H$ with $h \leq i$ let $D(h, i)$ be the finite distributive lattice with the following join irreducibles: $j \in J(D(h, i))$ if j is a mapping of an order filter P of the partially ordered set $[h, i]$ to $\bigcup_{x \in P} D_x$ such that for all $x \in P, j(x) \in J(D)$ (0 is not irreducible) and, whenever $x \leq y, x, y \in P$, then $j(y) \leq \varepsilon(x, y)(j(x))$. The join irreducibles of $D(h, i)$ are ordered componentwise: i.e. $(j(x) : x \in P) \leq (j'(x) : x \in Q)$ if and only if $Q \subseteq P$ and for all $x \in Q, j(x) \leq j'(x)$.

We now define 0 -preserving lattice embeddings: $\varphi(hg, i) : D(h, i) \rightarrow D(g, i)$ for $g \leq h \leq i$ and $\varphi(h, ij) : D(h, i) \rightarrow D(h, j)$ for $h \leq i \leq j$ such that diagrams (1), (2) and (3) of Figure 3.5 are commutative.

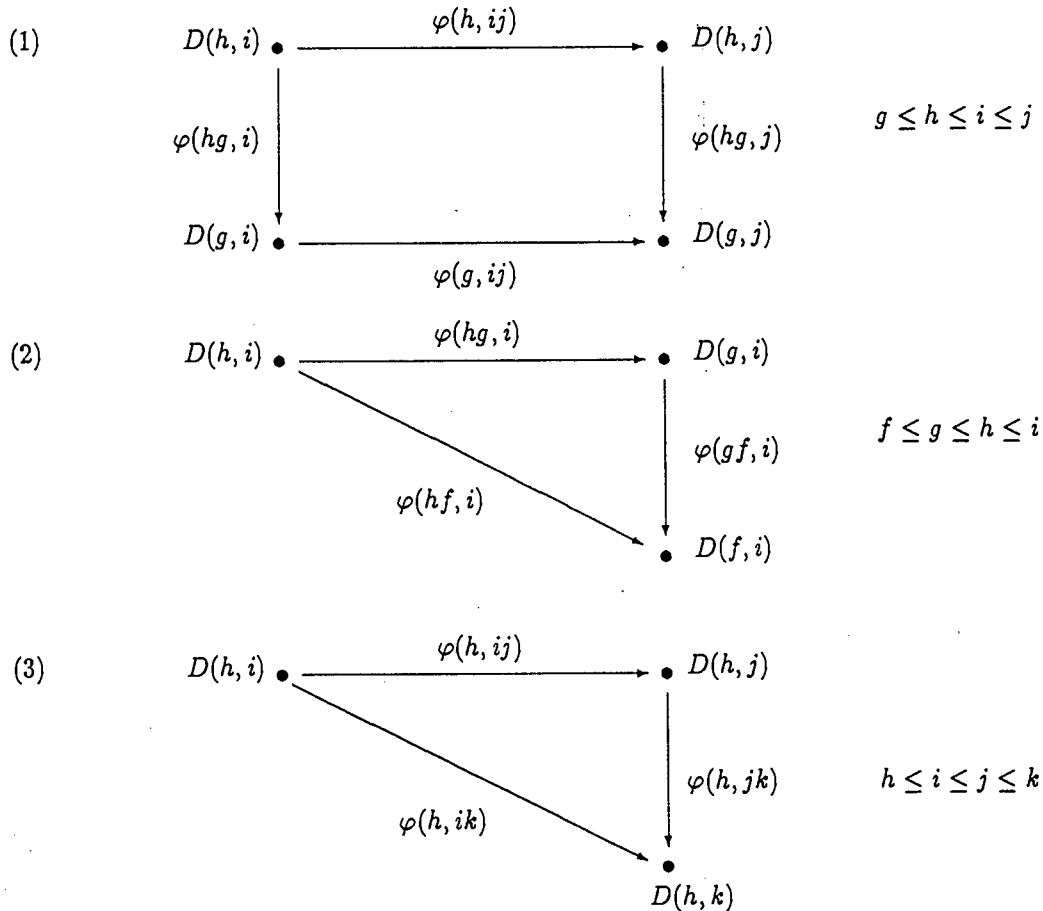


Figure 3.5

Note that if $g \leq h$ then the join irreducibles of $D(h, i)$ are join irreducibles of $D(g, i)$. We therefore get an embedding $\varphi(hg, i) : D(h, i) \rightarrow D(g, i)$ by mapping irreducibles $j \in$

$D(h, i)$ to $j \in D(g, i)$ and extending the mapping in such a way that joins are preserved. This mapping is one-to-one since the dual mapping by Priestley's duality $J(D(g, i)) \rightarrow J(D(h, i))$ maps $(j(x) : x \in P)$ to $(j(x) : x \in P \cap [h, i])$ and therefore is onto (see for example [Dav90] Theorem 10.26).

Consider $h \leq i \leq j$ and let $(j(x) : x \in P) \in J(D(h, i))$. Define $\varphi(h, ij)((j(x) : x \in P))$ to be the join of all $(j'(x) : x \in Q)$ such that Q is an order filter in $[h, j]$, $Q \cap [h, i] = P$, $j'(x) \leq j(x)$ for $x \in P$ and $(j'(x) : x \in Q) \in J(D(h, j))$. Extend this mapping to arbitrary elements of $D(h, i)$ in such a way that it preserves joins. To show that $\varphi(h, ij)$ is one-to-one we have to prove that its dual map by Priestley's duality is onto. First observe that we obtain $[h, j]$ from $[h, i]$ by adding finite chains $a_{11}, a_{12}, \dots, a_{1n_1}$; $a_{21}, a_{22}, \dots, a_{2n_2}$; \dots ; $a_{m1}, a_{m2}, \dots, a_{mn_m}$ successively to $[h, i]$. Now suppose that $(j(x) : x \in P)$ is a join irreducible in $D(h, i)$. We show that there is a $(j(x) : x \in Q) \in D(h, j)$ which gets mapped onto $(j(x) : x \in P)$ by the dual map. We simplify the proof into two cases illustrated in Figure 3.6 below.

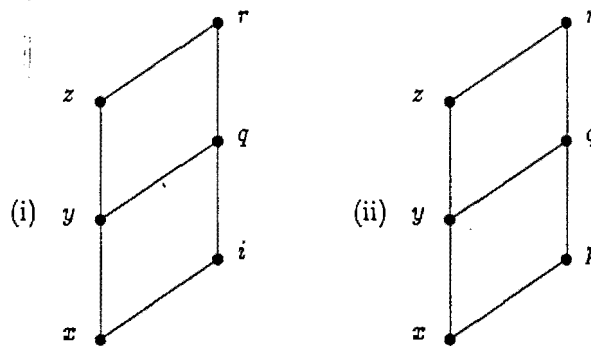


Figure 3.6

(i) Suppose that the adjoined elements of the chain are r and q and the chain consisting of the lower covers of elements i, q and r is x, y, z with $j(x), j(y)$ and $j(z)$ not defined. Then we may choose $j(q) \in J(D_q)$ such that $j(q) \leq \varepsilon(i, q)(j(i))$ and similarly choose $j(r) \in J(D_r)$ such that $j(r) \leq \varepsilon(q, r)(j(q))$.

(ii) Suppose that the adjoined elements are p, q, r with chain of lower covers x, y, z such that $j(x), j(y), j(z)$ are defined. We may choose $j(r) \in J(D_r)$ such that $j(r) \leq \varepsilon(z, r)(j(z))$ and $j(r) \leq \varepsilon(y, r)(j(y))$. Let $a = \varepsilon(y, q)(j(y))$. Then $j(r) \leq \varepsilon(q, r)(a)$ and a is a join of join irreducibles in D_q , $a = \bigvee \beta_\gamma$. Then $j(r) \leq \bigvee \varepsilon(q, r)(\beta_\gamma)$ and so by join-primeness of $j(r)$ (see Lemma 3.1.1) we have $j(r) \leq \varepsilon(q, r)(\beta_{\gamma_0})$ say. Put $j(q) = \beta_{\gamma_0}$. Define $j(p)$ similarly. Using these two cases we proceed inductively to obtain a vector $(j(x) : x \in Q)$ which gets mapped to $(j(x) : x \in P)$ by the dual map.

We now show that diagrams (1), (2) and (3) of Figure 3.5 are commutative.

(1) : Let $(j(x) : x \in P) \in J(D(h, i))$. Then $\varphi(hg, j) \circ \varphi(h, ij)((j(x) : x \in P)) = \bigvee S$ where $S = \{(j'(x) : x \in Q) : Q \text{ is an order filter in } [h, j], Q \cap [h, i] = P, (j'(x) : x \in Q) \in J(D(h, i)) \text{ and } j'(x) \leq j(x) \forall x \in P\}$. Also $\varphi(g, ij) \circ \varphi(hg, i)((j(x) : x \in P)) = \bigvee T$ where

$T = \{(j'(x) : x \in Q) : Q \text{ is an order filter in } [g, j], Q \cap [g, i] = P, (j'(x) : x \in Q) \in J(D(h, j)) \text{ and } j'(x) \leq j(x) \forall x \in P\}$. Let Q be an order filter in $[h, j]$ with $Q \cap [h, i] = P$. Then Q is an order filter in $[g, j]$ and $Q \cap [g, i] = Q \cap [h, i] = P$. Thus $S \subseteq T$. Also if Q is an order filter in $[g, j]$ such that $Q \cap [g, i] = P$ then $Q \subseteq [h, j]$ and $Q \cap [h, i] = Q \cap [g, i] = P$. Hence $T \subseteq S$ proving the commutativity of diagram (1).

(2) : This follows immediately from the fact that the $\varphi(hg, i)$'s ($h \leq g \leq i$) are embeddings.

(3) : Let $(j(x) : x \in P) \in J(D(h, i))$ and let $Z = \{(k(x) : x \in R) : R \text{ is an order filter in } [h, j], R \cap [h, i] = P, k(x) \leq j(x) \forall x \in P, (k(x) : x \in R) \in J(D(h, k))\}$. Then $\varphi(h, jk) \circ \varphi(h, ij)((j(x) : x \in P)) = \bigvee S$ where $S = \{(j'(x) : x \in Q) : Q \text{ is an order filter in } [h, k] \text{ with } Q \cap [h, j] = R, j'(x) \leq k(x) \forall x \in R \text{ where } (k(x) : x \in R) \in Z, (j'(x) : x \in Q) \in J(D(h, k))\}$. Also $\varphi(h, ik)((j(x) : x \in P)) = \bigvee T$ where $T = \{(j'(x) : x \in Q) : Q \text{ is an order filter in } [h, k] \text{ with } Q \cap [h, i] = P, j'(x) \leq j(x) \forall x \in P, (j'(x) : x \in Q) \in J(D(h, k))\}$. Let $(j(x) : x \in P) \in S$. Then $Q \cap [h, i] = Q \cap [h, j] \cap [h, i] = R \cap [h, i] = P$ and $j'(x) \leq k'(x) \forall x \in R, k(x) \leq j(x) \forall x \in P \Rightarrow j'(x) \leq k(x) \leq j(x) \forall x \in P$ and hence $(j'(x) : x \in Q) \in S$. Suppose $(j'(x) : x \in Q) \in T$. Then $R = Q \cap [h, j]$ is an order filter of $[h, j]$ and $R \cap [h, i] = Q \cap [h, j] \cap [h, i] = Q \cap [h, i] = P$. Hence $(j'(x) : x \in Q) \in S$ and diagram (3) commutes.

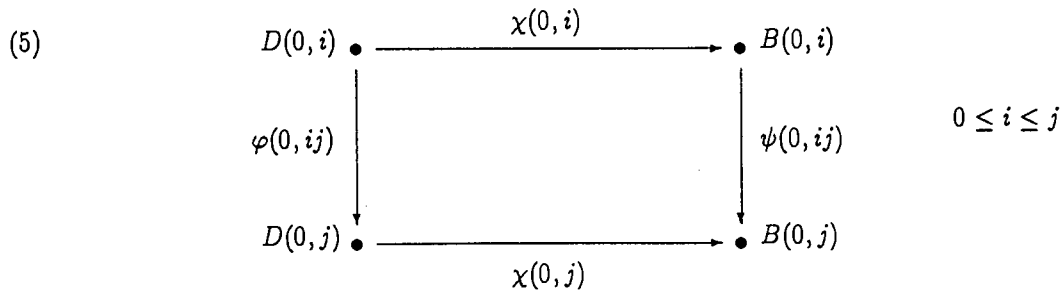
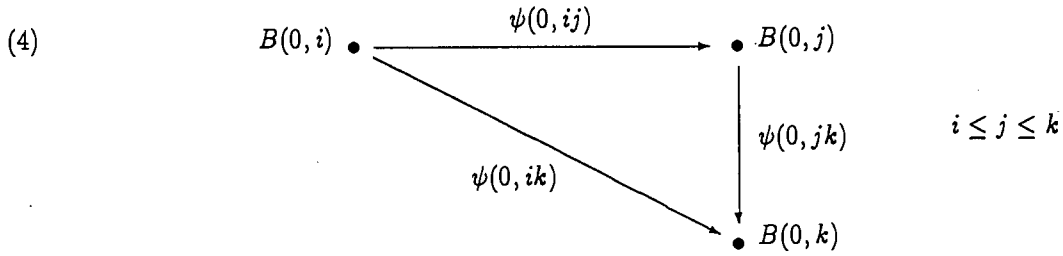


Figure 3.7:

We now define Boolean algebras $B(0, i)$ ($i \in H$) and 0-preserving lattice embeddings: $\psi(0, ij) : B(0, i) \rightarrow B(0, j)$ for $i \leq j$, $\chi(0, i) : D(0, i) \rightarrow B(0, i)$ such that diagrams (4) and (5) of Figure 3.7 commute. The atoms of $B(0, i)$ are $[(j(x) : x \in P)]$ where $(j(x) : x \in P) \in J(D(0, i))$ - the only difference being that in $B(0, i)$ they are not ordered. Define $\chi(0, i) : D(0, i) \rightarrow B(0, i)$ by $\chi(0, i)(x) = \bigvee \{[a] : a \leq x, a \in J(D(0, i))\}$. (Then

$D(0, i)$ can be considered as a $\{0, 1\}$ -sublattice of $B(0, i)$. Consider $[(j(x) : x \in P)] \in \text{At}(B(0, i))$. Then $\varphi(0, ij)((j(x) : x \in P)) = \bigvee S$ where $S = \{(j'(x) : x \in Q) : Q \text{ is an order filter in } [0, j], Q \cap [0, i] = P, (j'(x) : x \in Q) \in J(D(0, j)) \text{ and } j'(x) \leq j(x) \forall x \in P\}$. Define $\psi(0, ij)((j(x) : x \in P))$ as $\bigvee \{[(j'(x) : x \in Q)] : (j'(x) : x \in Q) \in S\}$. Then the commutativity of (4) follows from the commutativity of (3) and the commutativity of (5) follows from the definitions of χ and ψ . The dual map of ψ under Stone's duality is surjective. The proof is similar to that of the surjectivity of the dual map under Priestley's duality of $\varphi(h, ij)$. It then follows that ψ is a lattice embedding.

Let $B(0, -)$ be the direct limit of all the $B(0, i)$'s and for each $h \in H$ let $D(h, -)$ be the direct limit of the $D(h, i)$'s. Then there are embeddings $\psi(0, i-) : B(0, i) \rightarrow B(0, -)$ and $\varphi(h, i-) : D(h, i) \rightarrow D(h, -)$ such that diagrams (6) and (7) of Figure 3.8 commute.

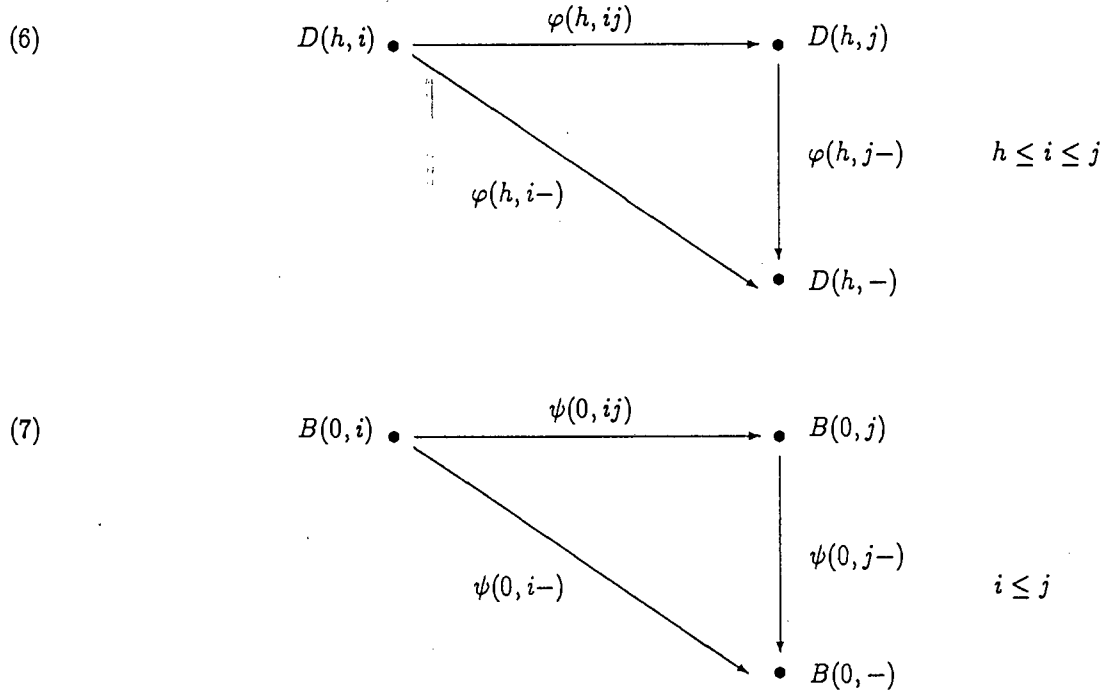


Figure 3.8

Let $g \leq h \leq i \leq j$. Then

$$\begin{aligned} \varphi(g, j-) \circ \varphi(hg, j) \circ \varphi(h, ij) &= \varphi(g, j-) \circ \varphi(g, ij) \circ \varphi(hg, i) && \text{(diagram (1))} \\ &= \varphi(g, i-) \circ \varphi(hg, i) && \text{(diagram (6))} \end{aligned}$$

Thus, since $D(h, -)$ is the direct limit of the $D(h, i)$'s, we have a unique embedding $\varphi(hf, -) : D(h, -) \hookrightarrow D(f, -)$ such that diagram (8) of Figure 3.9 commutes for all $i \leq j$. Similarly by the commutativity of diagrams (5) and (7) we have a unique embedding $\chi(0, -) : D(0, -) \hookrightarrow B(0, -)$ such that diagram (9) of Figure 3.10 commutes for all $i \in H$.

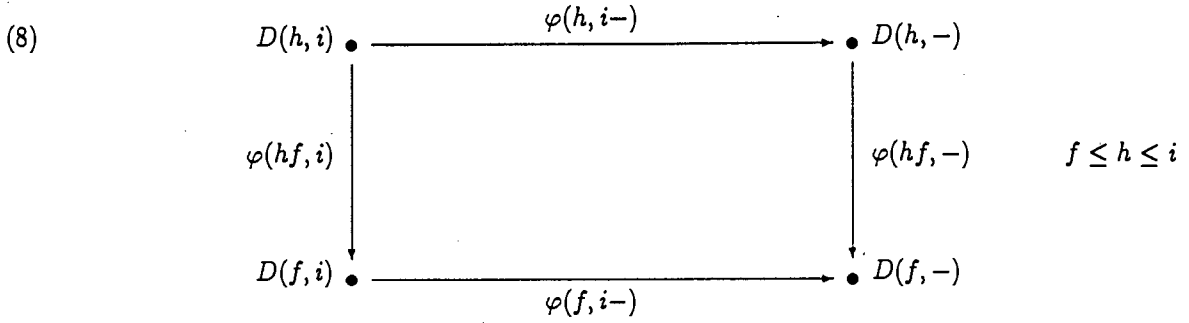


Figure 3.9:

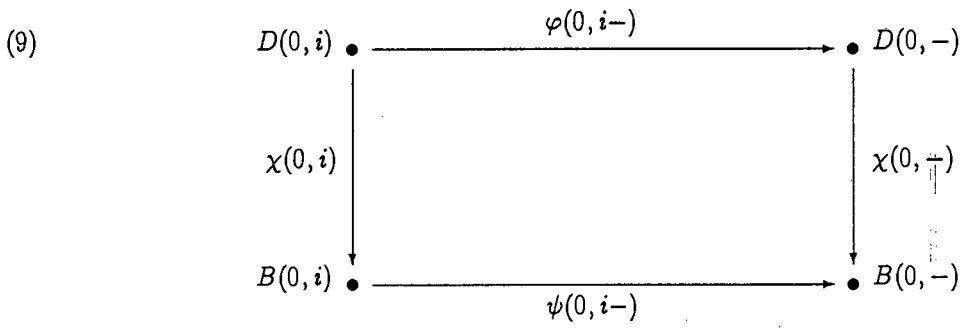


Figure 3.10:

Consider $f \leq g \leq h \leq i$. Then

$$\begin{aligned}
 \varphi(hf, -) \circ \varphi(h, i-) &= \varphi(f, i-) \circ \varphi(hf, i) && \text{(diagram (8))} \\
 &= \varphi(f, i-) \circ \varphi(gf, i) \circ \varphi(hg, i) && \text{(diagram (2))} \\
 &= \varphi(gf, -) \circ \varphi(g, i-) \circ \varphi(hg, i) && \text{(diagram (8))} \\
 &= \varphi(gf, -) \circ \varphi(hg, -) \circ \varphi(h, i-) && \text{(diagram (8))}
 \end{aligned}$$

Thus by the uniqueness of $\varphi(hf, -)$ in diagram (8) the following diagram commutes:

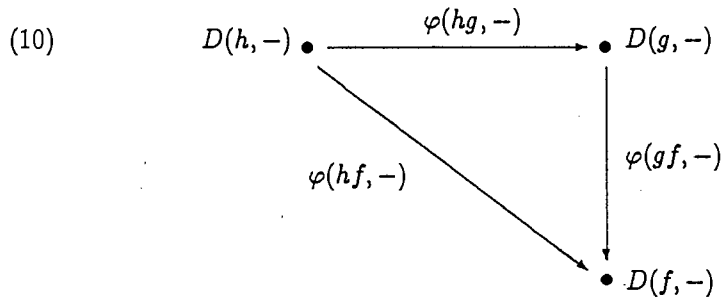


Figure 3.11

For $g \leq h \leq i$ define the maps $\varphi'(gh, i) : D(g, i) \rightarrow D(h, i)$ on the irreducibles of $D(g, i)$ by $\varphi'(gh, i)((j(x) : x \in Q)) = (j(x) : x \in Q \cap [h, i])$ and extend this map to the whole of

$D(g, i)$ in such a way that joins are preserved. Then $\varphi'(gh, i) \circ \varphi(hg, i) = id_{D(h, i)}$. Also, $\varphi'(gh, i)$ is a $\mathbf{0}$ -preserving join-homomorphism. By finiteness of the $D(h, i)$'s, $\varphi'(gh, i)$ is monomial. To show that it is weakly distributive let $(j(x) : x \in Q) \in J(D(g, i))$. Then $\varphi(hg, i) \circ \varphi'(gh, i)((j(x) : x \in Q)) = (j(x) : x \in Q \cap [h, i]) \geq (j(x) : x \in Q)$. Thus for all $z \in D(g, i)$ we have $\varphi(hg, i) \circ \varphi'(gh, i)(z) \geq z$. Now let $a, b, c \in D(g, i)$ with $\varphi'(gh, i)(b \vee c) = \varphi'(gh, i)(a)$ and $b \vee c \leq a$. Let $b' = \varphi(hg, i) \circ \varphi'(gh, i)(b) \wedge a$ and $c' = \varphi(hg, i) \circ \varphi'(gh, i)(c) \wedge a$. Then, since $\varphi'(gh, i) \circ \varphi(hg, i) = id_{D(h, i)}$ we have $\varphi'(gh, i)(c') = \varphi'(gh, i)(c)$ and $\varphi'(gh, i)(b') = \varphi'(gh, i)(b)$. Also $b' \vee c' = a$ follows from the distributivity of $D(g, i)$ and the fact that $\varphi(hg, i) \circ \varphi'(gh, i)(a) \geq a$. Thus $\varphi'(gh, i)$ is weakly distributive.

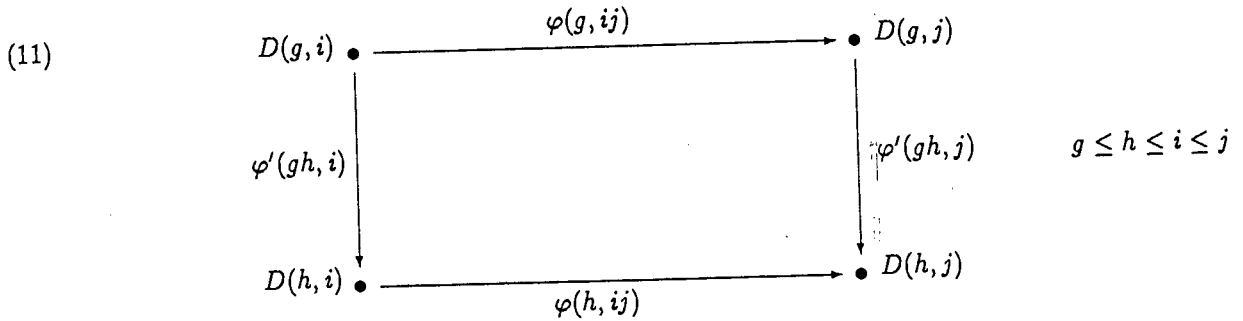


Figure 3.12

We now show that diagram (11) commutes. Consider $g \leq h \leq i \leq j$ and let $(j'(x) : x \in Q) \in J(D(g, i))$. Then $\varphi'(gh, j) \circ \varphi(g, ij)((j'(x) : x \in Q)) = \vee S$ where $S = \{(k(x) : x \in R \cap [h, j]) : R \text{ is an order filter in } [g, j], R \cap [g, i] = Q, k(x) \leq j'(x) \forall x \in Q, (k(x) : x \in R) \in J(D(g, i))\}$. And $\varphi(h, ij) \circ \varphi'(gh, i)((j'(x) : x \in Q)) = \vee T$ where $T = \{(k(x) : x \in R) : R \text{ is an order filter in } [h, j], R \cap [h, i] = Q \cap [h, i], k(x) \leq j'(x) \forall x \in Q \cap [h, i], (k(x) : x \in R) \in J(D(h, j))\}$. Let $(k(x) : x \in R \cap [h, j]) \in S$. Since R is an order filter in $[g, j]$, $R' = R \cap [h, j]$ is an order filter in $[h, j]$ and $Q \cap [h, i] = R \cap [h, i] = R \cap [h, j] \cap [h, i] = R' \cap [h, i]$. We then have $(k(x) : x \in R \cap [h, j]) \in T$ and so $\varphi'(gh, j) \circ \varphi(g, ij)((j'(x) : x \in Q)) \leq \varphi(h, ij) \circ \varphi'(gh, i)((j'(x) : x \in Q))$. For the reverse inequality let $(k(x) : x \in R) \in T$. Then $R \cap [h, i] = Q \cap [h, i] \neq \emptyset$ and so $[i, j] \subseteq R$. Let $R' = Q \cup [i, j]$. Then R' is an order filter in $[g, j]$, $R' \cap [h, j] = R$ and $R' \cap [g, i] = Q$. Let p be the least element of $Q \cap [h, i]$. Consider the sequence $((k'(x) : x \in R')$ defined by:

$$k'(x) = \begin{cases} k(x) & x \in R' \cap [h, i] \\ k(p) & x \in Q \cap [g, h]. \end{cases}$$

For all $x \in R$ we have $k(x) = k'(x)$. Consider $(k'(x) : x \in R' \cap [h, j]) = (k(x) : x \in R)$. Then this sequence is in S and hence $\varphi(h, ij) \circ \varphi'(gh, i)((j'(x) : x \in Q)) \leq \varphi'(gh, j) \circ \varphi(g, ij)((j'(x) : x \in Q))$ proving the commutativity of diagram (11).

The commutativity of diagram (12) below follows immediately from the definition of the map φ' .

$$(12) \quad \begin{array}{ccc} D(f, i) & \xrightarrow{\varphi'(fg, i)} & D(g, i) \\ & \searrow \varphi'(fh, i) & \downarrow \varphi'(gh, i) \\ & & D(h, i) \end{array} \quad f \leq g \leq h \leq i$$

Figure 3.13

For each $i \in H$ consider the map $\chi'(0, i) : B(0, i) \rightarrow D(0, i)$ defined by $\chi'(0, i)(a) = \bigvee \{(j(x) : x \in P) \in J(D(0, i)) : [(j(x) : x \in P)] \leq a\}$. Then $\chi'(0, i) \circ \chi(0, i) = id_{D(0, i)}$. It can be shown, in a similar way to $\varphi'(gh, i)$ that $\chi'(0, i)$ is a monomial $\mathbf{0}$ -preserving weakly distributive join-homomorphism. The commutativity of the following diagram then follows from the definitions of the maps involved.

$$(13) \quad \begin{array}{ccc} B(0, i) & \xrightarrow{\chi'(0, i-)} & D(0, i) \\ \psi(0, i) \downarrow & & \downarrow \varphi(0, ij) \\ B(0, j) & \xrightarrow{\chi'(0, j-)} & D(0, j) \end{array}$$

Figure 3.14

Let $g \leq h \leq i \leq j$. Then

$$\begin{aligned} \varphi(h, j-) \circ \varphi'(gh, j) \circ \varphi(g, ij) &= \varphi(h, j-) \circ \varphi(h, ij) \circ \varphi'(gh, i) && \text{(diagram (11))} \\ &= \varphi(h, i-) \circ \varphi'(gh, i) && \text{(diagram (6))} \end{aligned}$$

Then, since $D(g, -)$ is the direct limit of the $D(g, i)$'s there is a unique homomorphism $\varphi'(gh, -) : D(g, -) \rightarrow D(h, -)$ such that the following diagram commutes for all $i \in H$:

$$(14) \quad \begin{array}{ccc} D(g, i) \bullet & \xrightarrow{\varphi(g, i-)} & \bullet D(g, -) \\ \downarrow \varphi'(gh, i) & & \downarrow \varphi'(gh, -) \\ D(h, i) \bullet & \xrightarrow{\varphi(h, i-)} & \bullet D(h, -) \end{array} \quad h \leq g \leq i$$

Figure 3.15

Now for $g \leq h \leq i$ we have :

$$\begin{aligned} \varphi(h, i-) &= \varphi'(gh, i) \circ \varphi(hg, i) \\ &= \varphi'(gh, -) \circ \varphi(g, i) \circ \varphi(hg, i) && \text{(diagram (14))} \\ &= \varphi'(gh, -) \circ \varphi(hg, -) \circ \varphi(h, i-) && \text{(diagram (8))} \end{aligned}$$

Thus, since $D(h, -)$ is the direct limit of the $D(h, i)$'s we must have $\varphi'(gh, -) \circ \varphi(hg, -) = id_{D(h, -)}$. Now $\varphi'(gh, -)$ is a $\mathbf{0}$ -preserving monomial weakly distributive join-homomorphism and the proof of the commutativity of the following diagram follows from the commutativity of diagrams (12) and (14) in a similar way to the proof of the commutativity of diagram (10).

$$(15) \quad \begin{array}{ccc} D(f, -) \bullet & \xrightarrow{\varphi'(fg, -)} & \bullet D(g, -) \\ & \searrow \varphi'(fh, -) & \downarrow \varphi'(gh, -) \\ & & \bullet D(h, -) \end{array}$$

Figure 3.16

In a similar way we can define maps $\chi'(0, -) : B(0, -) \rightarrow D(0, -)$ such that $\chi'(0, -) \circ \chi(0, -) = id_{D(0, -)}$ and $\chi'(0, -)$ is a $\mathbf{0}$ -preserving monomial weakly distributive join homomorphism.

We consider the congruences θ_h associated with $\varphi'(0, h-) \circ \chi'(0, -) : B(0, -) \rightarrow D(h, i-)$. Then $\bigvee_{h \in H} \theta_h$ is a distributive congruence on $B(0, -)$. If we can show that $F = B(0, -) / \bigvee_{h \in H} \theta_h$ is isomorphic to D then we are done by Proposition 3.3.7. Now F is the direct limit of

the $D(h, -)$'s relative to the homomorphisms $\varphi(gh, -)$ and F has subsemilattices isomorphic to the D_h 's namely $D(h, h) \cong D_h$. Hence $\varphi(h, h-)[D(h, h)] \cong D_h$ and so $D \cong F$ completing our proof (see the remarks at the end of Section 3.1). \square

3.5 Sufficient conditions for the representability of an algebraic distributive lattice : Tischendorf's Theorem

In this section we present Tischendorf's theorem ([Tis94]) which gives sufficient conditions for the representability of a distributive semilattice as the lattice of compact congruences of a lattice. This result has as corollaries Theorems 3.2.4, 3.3.1 and 3.4.2.

We start with definitions and basic results on colimits of algebras. Let (I, \trianglelefteq) be a partially ordered, upward directed set. Consider $\mathcal{C} = \{C_i : i \in I\}$ a class of algebras of some fixed type \mathcal{F} together with a limit system of homomorphisms $\alpha_{i,j} : C_i \rightarrow C_j$. (Recall the definition of a limit system from Section 3.1.)

Let $K = \{c \in \prod_{i \in I} C_i : (\exists j \in I)(\forall k \triangleright j)(c_k = \alpha_{j,k}(c_j))\}$. The element j in this definition is not unique, since if $j \trianglelefteq j'$ and $k \triangleright j'$ we have $c_k = \alpha_{j,k}(c_j) = \alpha_{j',k} \circ \alpha_{j,j'}(c_j) = \alpha_{j',k}(c_{j'})$. Thus j' would also work. Define a binary relation \equiv on K by

$$a \equiv b \Leftrightarrow (\exists p \in I)(\forall q \trianglelefteq p) a_p = b_p$$

It follows from the upward directedness of I that \equiv is a congruence relation on K . Let $L = K / \equiv$. Then L is the *colimit* of the class \mathcal{C} with respect to the homomorphisms $\alpha_{i,j}$. We write $L = \varinjlim (\mathcal{C}, (\alpha_{i,j})_{i,j \in I})$ or $L = \varinjlim \mathcal{C}$. Note that $L \in \mathbf{HSP}(\mathcal{C})$. We will denote the \equiv -class of an element $u \in K$ by $[u]$.

We can define homomorphisms $\alpha_i : C_i \rightarrow L$ as follows. Let $x \in C_i$ and define $\mu_i(x) \in K$ by $(\mu_i(x))_j = \alpha_{i,j}(x)$ if $i \trianglelefteq j$, and choose $(\mu_i(x))_j$ arbitrarily otherwise. Let $\alpha_i(x) = [\mu_i(x)]$. We have $\alpha_j \circ \alpha_{i,j} = \alpha_i$ for all $i \trianglelefteq j$. Since for $x \in C_i$ and $k \triangleright j$, $(\mu_j(\alpha_{i,j}(x)))_k = \alpha_{j,k}(\alpha_{i,j}(x)) = \alpha_{i,k}(x) = (\mu_i(x))_k$.

LEMMA 3.5.1 *If the maps $\alpha_{i,j}$ defined above are one-to-one then so is every α_i .*

PROOF. $\alpha_i(x) = \alpha_i(y) \Rightarrow [\mu_i(x)] = [\mu_i(y)] \Rightarrow \exists j \in I$ such that $\mu_i(x)_k = \mu_i(y)_k \forall k \triangleright j$. Let $j' \in I$ be such that $j \trianglelefteq j', i \trianglelefteq j'$. Then $\mu_i(x)_{j'} = \mu_i(y)_{j'}$ which implies $\alpha_{i,j'}(x) = \alpha_{i,j'}(y)$ and so $x = y$. \square

LEMMA 3.5.2 *Let $[b] \in L$. Then $[b] = \alpha_j(y)$ for some $y \in C_j, j \in I$.*

PROOF. Let $j \in I$ be such that $\alpha_{j,k}(b_j) = b_k$ for all $k \supseteq j$. Then $\alpha_j(b_j) = [\mu_j(b_j)] = [b]$. \square

Let C be an algebra with a cofinal family $\mathcal{C} = \{C_i : i \in I\}$ of subalgebras of C such that $C = \bigcup\{C_i : i \in I\}$. Then the *direct union* of \mathcal{C} is the colimit $\varinjlim (\mathcal{C}, (\alpha_{i,j})_{i,j \in I})$ where $\alpha_{i,j}$ is the identity map embedding C_i into C_j whenever $C_i \subseteq C_j$. (The partial order on I is defined by $i \preceq j \Leftrightarrow C_i \subseteq C_j$). This colimit is isomorphic to C .

Recall from Section 3.1 that to investigate whether a given distributive algebraic lattice L can be represented as the congruence lattice of a lattice it is sufficient to show that the semilattice of compact elements of L is isomorphic to the compact congruences of a lattice. From now on let S be a distributive join-semilattice with 0 . We treat 0 as a nullary operation on S , ensuring that every sub-semilattice of S includes 0 . We say that the semilattice S is *representable* if it is isomorphic to the semilattice of compact congruences of some lattice.

Recall the definitions of a weak-distributive homomorphism, distributive homomorphism and a monomial congruence from Section 3.3.

A join-homomorphism $s : S \rightarrow S$ is called a *topological closure operator* if $x \leq s(x) = s(s(x))$ for all $x \in S$.

Note that a congruence θ on S is monomial if and only if θ is the kernel of a topological closure operator on S .

Let S_1, S_2 be distributive semilattices. Then by the above equivalence a map $h : S_1 \rightarrow S_2$ is distributive if and only if $\ker(h) = \sum \ker(s_i)$ where each s_i is a weak-distributive topological closure operator on S_1 .

A partially ordered set P is *locally countable* if every principal ideal of P is countable.

Let B be a generalized Boolean algebra with $h : B \rightarrow S$ a join-homomorphism onto S . Then by [Dob86] if B is locally countable then h is weakly distributive if and only if h is distributive.

Recall that a lattice with zero is atomistic if every non-zero element is the join of atoms below it.

LEMMA 3.5.3 *Let B be an atomistic generalized Boolean algebra with $h : B \rightarrow S$ a join-homomorphism onto S . Then h is distributive if and only if $h(a) \in J(S) \cup \{0\}$ for all $a \in \text{At}(B)$.*

PROOF. Suppose $h(a) \notin J(S) \cup \{0\}$ for some $a \in \text{At}(B)$. Then $h(a) = u \vee v$ with $u, v < h(a)$. Since h is onto there exist $b, c \in B$ such that $h(b) = u$ and $h(c) = v$. (We must then have b and c non-comparable with each other and with a .) Then $h(b \vee c) = u \vee v = h(a) = h(d)$ where $d = a \vee b \vee c$. Now (by Theorem 2.2.1) $(a \vee b \vee c)/(b \vee c) \cong a/(a \wedge (b \vee c)) = a/(a \wedge b) \vee (a \wedge c) = a/0$. Hence $a \vee b \vee c \succ b \vee c$, i.e. $b \vee c \prec d$. Suppose

there exist $b', c' \in B$ with $b' \vee c' = d$, $h(b) = h(b')$, $h(c) = h(c')$. Then $a \leq b' \vee c' \Rightarrow a \leq b'$ or $a \leq c'$ (by join-primeness of a). Assume without loss of generality that $a \leq b'$. Then $u = h(b') \geq h(a)$. This contradiction proves that h is not weakly distributive, and hence not distributive. For the reverse implication, suppose $h(a) \in J(S) \cup \{0\}$ for every $a \in \text{At}(B)$. Let $x, y, z \in B$ with $x \vee y < z$. Let a be an atom below z but not below $x \vee y$. To prove that h is weakly distributive it is sufficient to show that $h(a) \leq h(x)$ or $h(a) \leq h(y)$. Since if, for instance, $h(a) \leq h(x)$, we can let $x' = a \vee x, y' = y$. Then $x' \vee y' = z$ and $h(x') = h(x)$, $h(y') = h(y)$. If $h(a) = 0$ then this follows trivially. Suppose that $h(a) \in J(S)$. Then $h(a) \leq h(z) \Rightarrow h(a) \leq h(x) \vee h(y) \Rightarrow h(a) \leq h(x)$ or $h(a) \leq h(y)$ (by Lemma 3.1.1). Thus h is weakly distributive and, since B is locally countable, h is distributive.

We use the following result from [Sch81] (see Proposition 3.3.7).

THEOREM 3.5.4 *Let B be a generalized Boolean algebra with $h : B \rightarrow S$ a distributive join-homomorphism onto S . Then S is representable.*

In what follows let $\mathcal{S} = \{S_i : i \in I\}$ be a collection of finite distributive sub-semilattices of S whose direct union is isomorphic to S and which contains $\{0\}$ as its smallest element.

LEMMA 3.5.5 *For every $S_i \in \mathcal{S}$, let B_i be a finite Boolean algebra with $h_i : B_i \rightarrow S_i$ a weakly distributive join homomorphism onto S_i . If we have a limit system of 0 -preserving lattice embeddings $\beta_{i,j} : B_i \hookrightarrow B_j$ with the property :*

$$h_j \circ \beta_{i,j} = h_i$$

then there exists a generalized Boolean algebra L and a weakly distributive join-homomorphism $h : L \rightarrow S$ onto S .

PROOF. Let $L = \varinjlim B_i$, with $\beta_i : B_i \hookrightarrow L$ the embeddings such that $\beta_j \circ \beta_{i,j} = \beta_i$ for all $i \trianglelefteq j$. Then L is distributive and relatively complemented and hence is a generalized Boolean algebra. Let $x \in L$. Then by Lemma 3.5.2 we have $x = \beta_i(y)$ for some $i \in I, y \in B_i$. Let $h(x) = h_i(y)$. Then $h : L \rightarrow S$ is well-defined since, suppose $x = \beta_i(y) = \beta_j(z)$ for $i, j \in I, y \in B_i, z \in B_j$. Choose $k \in I$ with $k \triangleright i, k \triangleright j$. Then $\beta_i(y) = \beta_k(\beta_{i,k}(y)) = x = \beta_j(z) = \beta_k(\beta_{j,k}(z))$. Since the β_i 's are embeddings we have $\beta_{i,k}(y) = \beta_{j,k}(z)$. Then $h_i(y) = h_k(\beta_{i,k}(y)) = h_k(\beta_{j,k}(z)) = h_j(z)$. To show that h is onto let $s \in S$. Then $s \in S_i$ for some $i \in I$ and thus $s = h_i(b)$ for some $b \in S_i$. Then $\beta_i(b) \in L$ and $h(\beta_i(b)) = h_i(b) = s$. Now let $u = \beta_i(x), v = \beta_j(y)$. and let $k \in I$ be such that $k \triangleright i, j$. Set $x' = \beta_{i,k}(x), y' = \beta_{j,k}(y)$. Then $u = \beta_i(x) = \beta_k(\beta_{i,k}(x)) = \beta_k(x')$ and so $h(u) = h_k(x')$. Similarly $v = \beta_j(y')$ and $h(v) = h_k(y')$. Thus $h(u \vee v) = h(\beta_k(x') \vee \beta_k(y')) = h(\beta_k(x' \vee y')) = h_k(x' \vee y') = h_k(x') \vee h_k(y') = h(u) \vee h(v)$. Thus h is a join-homomorphism. To show that h is weakly distributive let $u \vee v \trianglelefteq w$ in L with $h(w) = h(u \vee v)$. As before choose $k \in I$ and $x, y, z \in L$

such that $u = \beta_k(x), v = \beta_k(y)$ and $z = \beta_k(w)$. Then $h(u) = h_k(x), h(v) = h_k(y)$ and $h(w) = h_k(z)$ implying $h_k(z) = h_k(x \vee y)$. Now $u \vee v \vee w = w \Rightarrow \beta_k(x \vee y \vee z) = \beta_k(z) \Rightarrow x \vee y \vee z = z \Rightarrow x \vee y \leq z$. Since h_k is weakly distributive there exist $x', y' \in B_k$ with $x' \vee y' = z$ and $h_k(x') = h_k(x), h_k(y') = h_k(y)$. Let $u' = \beta_k(x'), v' = \beta_k(y')$. Then $u' \vee v' = \beta_k(x' \vee y') = \beta_k(z) = w$ and $h(u') = h_k(x') = h_k(x) = h(u)$. Similarly $h(v') = h(v)$ proving the weak-distributivity of h . \square

For the rest of this section let $I = \mathcal{S}$ and identify S_i with $i \in I = \mathcal{S}$. By $[i, j]$ we denote the set of all $k \in I$ such that $i \trianglelefteq k \trianglelefteq j$. (i.e. the set of all $S_k \in \mathcal{S}$ such that $S_i \subseteq S_k \subseteq S_j$). For each $i \in I$ define F_i to be the set of all functions f such that :

- (F1) $\text{dom}(f)$ is an order filter in $[0, i]$.
- (F2) $f(j) \in J(S_j) \forall j \in \text{dom}(f)$.
- (F3) $j, k \in \text{dom}(f)$ and $j \trianglelefteq k \Rightarrow f(j) \leq f(k)$.

The elements of F_i are ordered by setting $f \geq g \Leftrightarrow \text{dom}(f) \subseteq \text{dom}(g)$ and $f(x) \geq g(x) \forall x \in \text{dom}(f)$. We consider the following two conditions on \mathcal{S} :

(E) (Extension Property) Let $i \trianglelefteq j \in I, V$ an order filter in $[0, i], W$ an order filter in $[0, j]$ such that for every element x in the order filter generated by V the set $\{y \in W : x \trianglelefteq y\}$ has a smallest element. If f is a function defined on $V \cup W$ satisfying (F2) and (F3), then there exists a function $g \in F_j$ such that $g|_{V \cup W} = f$.

(S) \subseteq is a lattice order on \mathcal{S} .

THEOREM 3.5.6 (Tischendorf's Theorem) *If \mathcal{S} satisfies (E) and (S) then \mathcal{S} is representable.*

PROOF. Denote the induced meet and join operations in I by \wedge and \vee . We define Boolean algebras $B_i : i \in I$ and mappings $h_i : B_i \rightarrow S_i, \beta_{i,j} : B_i \rightarrow B_j$ (for $i \trianglelefteq j$) satisfying the conditions of Lemma 3.5.5. Define $B_i = \mathcal{P}(F_i)$ and $h_i : B_i \rightarrow S_i$ by $h_i(T) = \sum\{f(i) : f \in T\}$. Then h_i is a join-homomorphism mapping B_i onto S_i . By (F2) and Lemma 3.5.3 h_i is distributive and hence weakly distributive. For $i \trianglelefteq j$ in I define $\beta_{i,j} : B_i \rightarrow B_j$ by $\beta_{i,j}(T) = \{f \in F_j : f|_{[0,i]} \in T\}$. Then $\beta_{i,j}$ is a lattice homomorphism. To show that $\beta_{i,j}$ is one-to-one suppose $\beta_{i,j}(T) = \beta_{i,j}(V)$. Then $\{f \in F_j : f|_{[0,i]} \in T\} = \{f \in F_j : f|_{[0,i]} \in V\}$. Let $f \in T$ and let $V = \text{dom}(f), W = \{j\}$. We have $f(i) \in J(S_i) \subseteq S_j$. Pick $a \in J(S_j)$ with $a \leq f(i)$. Consider the function g defined on $V \cup W$ as follows:

$$g(x) = \begin{cases} f(x) & x \in V \\ a & x \in W. \end{cases}$$

Then g satisfies the conditions of (E) and so g can be extended to a function $g' \in F_j$. We have $g'|_{[0,i]} = g|_{[0,i]} = f \in T$. Thus $g' \in \beta_{i,j}(T) = \beta_{i,j}(V)$ and so $f \in V$. Thus

$T \subseteq V$ and similarly $V \subseteq T$ proving that $\beta_{i,j}$ is an embedding. $\beta_{i,j}$ is obviously $\mathbf{0}$ -preserving and to show that the $\beta_{i,j}$'s form a limit system let $i \triangleleft j \triangleleft k$ with $T \in B_i$. Then $\beta_{j,k}(\beta_{i,j}(T)) = \{g \in F_k : g|_{[0,j]} \in F_j \text{ and } g|_{[0,i]} \in T\}$ and $\beta_{i,k}(T) = \{g \in F_k : g|_{[0,i]} \in T\}$. The inclusion $\beta_{j,k}(\beta_{i,j}(T)) \subseteq \beta_{i,k}(T)$ is immediate. For the reverse inclusion suppose $g \in F_k$ with $g|_{[0,i]} \in T$. Then $i \in \text{dom}(g)$ and $i \triangleleft j$ imply that $j \in \text{dom}(g)$. Thus $g|_{[0,j]} \in F_j$ proving $\beta_{i,k}(T) \subseteq \beta_{j,k}(\beta_{i,j}(T))$ and consequently the $\beta_{i,j}$'s form a limit system.

$$\begin{aligned} \text{Now } h_j(\beta_{i,j}(T)) &= \sum \{f(j) : f \in F_j \text{ and } f|_{[0,i]} \in T\} \\ &\leq \sum \{f(i) : f \in F_j \text{ and } f|_{[0,i]} \in T\} \text{ (by (F3))} \\ &\leq \sum \{f(i) : f \in T\} \\ &= h_i(T). \end{aligned}$$

For the reverse inclusion let $f \in T$. Then $f(i) = \sum_{p=1}^m a_p$ where $a_p \in J(S_j)$. Let $V = \text{dom}(f)$, $W = \{j\}$ and for each $p = 1, \dots, m$ define a map g_p on $V \cup W$ by:

$$g_p(x) = \begin{cases} f(x) & x \in V \\ a_p & x \in W. \end{cases}$$

As before g_p can be extended to a function $g'_p \in F_j$ with $g'_p \in \beta_{i,j}(T)$. Now $f(i) = \sum_{p=1}^m a_p = \sum_{p=1}^m g'_p(j)$. Thus $f(i) \leq h_j(\beta_{i,j}(T))$. This is true for every $f \in T$ and so $h_i(T) \leq h_j(\beta_{i,j}(T))$. Thus $h_j \circ \beta_{i,j} = h_i$ for all $i \triangleright j$ and we may apply Lemma 3.5.5 to obtain a generalized Boolean algebra B and a weak distributive join-homomorphism $h : B \rightarrow S$ mapping B onto S .

It remains to show that h is distributive. We will define topological closure operators $s_i : B \rightarrow B$ such that $\ker(h) = \sum_{i \in I} \ker(s_i)$. We recall from the proof of Lemma 3.5.5 that B is the colimit of the B_i 's with respect to the embeddings $\beta_{i,j}$ and the map $h : B \rightarrow S$ is defined as follows: if $x \in L$ and $x = \beta_i(y)$ for some $y \in B_i$, then $h(x) = h_i(y)$. In what follows let K be as in the definition of the colimit of the B_i 's. Let $[u] \in B$, $i \in I$ and $j \triangleright i$ be such that $u_k = \beta_{j,k}(u_j) \forall k \triangleright j$. Define $s_i^\circ : K \rightarrow K$ by:

$$(s_i^\circ(u))_k = \begin{cases} \{f \in F_k : f \leq g|_{[i,k]} \text{ and } g \in u_k\} & \text{if } k \triangleright i \\ F_k & \text{otherwise,} \end{cases}$$

and $s_i : B \rightarrow B$ by $s_i([u]) = [s_i^\circ(u)]$. Then s_i is well-defined, since $[u] = [v] \Rightarrow \exists j \in I, j \triangleright i$ such that $u_k = v_k \forall k \triangleright j$. Then for all $k \triangleright j$ we have $(s_i^\circ(u))_k = (s_i^\circ(v))_k$ proving $s_i([u]) = s_i([v])$.

We now show that $s_i^\circ(u) \in K$. Since $u \in K$ there exists a $j \in I, j \triangleright i$ such that $\beta_{j,k}(u_j) = u_k$ for all $k \triangleright j$. We show that $\beta_{j,k}((s_i^\circ(u))_j) = (s_i^\circ(u))_k$ for all $k \triangleright j$. In this case we have $\beta_{j,k}((s_i^\circ(u))_j) = \{e \in F_k : e|_{[0,j]} \in F_j, e|_{[0,j]} \leq g|_{[i,j]} \text{ for some } g \in u_j\}$ and $(s_i^\circ(u))_k = \{f \in F_k : f \leq g|_{[i,k]} \text{ for some } g \in u_k\}$.

Let $f \in (s_i^\circ(u))_k$. Then $g \in u_k \Rightarrow g \in \beta_{j,k}(u_j) \Rightarrow g|_{[0,j]} \in u_j \Rightarrow j \in \text{dom}(g)$. Also $f \leq g|_{[i,k]} \Rightarrow \text{dom}(g) \cap [i,k] \subseteq \text{dom}(f)$. Thus $f|_{[0,j]} \in F_j$. We have $f|_{[0,j]} \leq (g|_{[i,k]})|_{[0,j]} = (g|_{[0,j]})|_{[i,k]}$ and $g|_{[0,j]} \in u_j$. Hence $f \in \beta_{j,k}((s_i^\circ(u))_j)$ and so $(s_i^\circ(u))_k \subseteq \beta_{j,k}((s_i^\circ(u))_j)$. On the

other hand let $f \in \beta_{i,j}((s_i^o(u))_j)$. Then $f \in F_k, f|_{[0,j]} \leq g|_{[i,j]}$ for some $g \in u_j$. We will show that $f \leq g'|_{[i,k]}$ for some $g' \in u_k$. Let $V = \text{dom}(g), W = \{x \in [i, k] : x \supseteq y \in V, x \notin V\}$. Then $W \subseteq \text{dom}(f)$. Since if $x \in W$ then there exists a $y \in \text{dom}(g)$ such that $y \sqsubseteq x$ and $y \in \text{dom}(g) \Rightarrow y \sqsubseteq j \Rightarrow y \sqsubseteq x \wedge j \sqsubseteq j$. Thus $x \wedge j \in \text{dom}(g) \cap [i, j]$. Then, since $f|_{[i,k]} \leq g|_{[i,j]}$ we have $x \wedge j \in \text{dom}(f)$ implying $x \in \text{dom}(f)$.

We define a function g^* on $V \cup W$ as follows:

$$g^*(x) = \begin{cases} g(x) & x \in V \\ f(x) & x \in W. \end{cases}$$

We show that g^* satisfies the conditions of (E). Now V is an order filter in $[0, i]$. To show that W is an order filter in $[0, k]$ let $x_1 \in W$ with $x_2 \supseteq x_1$ in $[0, k]$. Then $x_1 \in [i, k]$ and $\exists y \in V$ such that $x_1 \sqsubseteq y$. Then $x_2 \in [i, k]$ and $x_2 \geq y$. Also $x_2 \in V \Rightarrow x_1 \in V$ proving that $x_2 \in W$ and hence that W is an order filter in $[0, k]$. Let x be in the filter generated by V and consider $\{y \in W : x \sqsubseteq y\} = Z$. We show that Z has a least element. There are three possible cases: (i) $x \in W$, (ii) $x \in V$, (iii) $x \notin V \cup W$.

For case (i) x is the least element of Z . In case (ii) let $y_1, y_2 \in Z$. Then $y_1 \wedge y_2 \supseteq x \in V, y_1 \wedge y_2 \in [i, k]$ and (since V is an order filter in $[0, i]$) $y_1 \wedge y_2 \notin V$. Thus $y_1 \wedge y_2 \in Z$ proving that Z has a least element. For case (iii) let $y_1, y_2 \in Z$. Now $x \supseteq z$ for some $z \in V$ and $y_1 \wedge y_2 \supseteq x \supseteq z$. The argument that Z has a least element is as in case (ii). It is obvious that g^* satisfies (F2). To prove that g^* satisfies (F3) let $x, y \in V \cup W$ with $x \sqsubseteq y$. Then $x \in W \Rightarrow y \in W$ and $W \subseteq \text{dom}(f)$ yields $g^*(x) = f(x) \geq f(y) = g^*(y)$. If $y \in V$ then $x \in V$ and $g^*(x) = g(x) \geq g(y) = g^*(y)$. Suppose then that $y \in W, x \in V$. Set $z = x \vee i$. Then $x \in V \Rightarrow x \in [0, j] \Rightarrow z \in [i, j]$ and $z \sqsubseteq y$. We also have $x \vee i \in V$. Thus (since $W \subseteq \text{dom}(f)$) we have $g^*(x) = g(x) \geq g(z) \geq f(z) \geq f(y) = g^*(y)$. This proves that g^* satisfies the conditions of (E). Then there exists a $g' \in F_k$ such that $g'|_{V \cup W} = g^*$. Since $g'|_{[0,j]} = g \in u_j$, we have $g' \in u_k$ (since $u_k = \beta_{j,k}(u_j)$). It remains to show that $f \leq g'|_{[i,k]}$. Let $x \in \text{dom}(g') \cap [i, k]$. Then $x \supseteq y$ for some $y \in V \cup W$. If $y \in W$ then, since $W \subseteq \text{dom}(f)$ and $x \supseteq y$, we have $x \in \text{dom}(f)$. Suppose then that $y \in V$. Then $x \supseteq y \vee i \in V \cap [i, j] = \text{dom}(g) \cap [i, j] \subseteq \text{dom}(f) \cap [0, j]$. Thus $y \vee i \in \text{dom}(f)$ and so $x \in \text{dom}(f)$. This proves $f \leq g'|_{[i,k]}$. Let $x \in \text{dom}(g') \cap [i, k]$. Then $x \in V \cup W$ and so $g'(x) = g^*(x)$. If $x \in W$ then $g^*(x) = f(x)$. If $x \in V$ then $x \in \text{dom}(g) \cap [i, k] = \text{dom}(g) \cap [i, j]$. Thus, since $f|_{[0,j]} \leq g|_{[i,j]}$ we have $g^*(x) = g(x) \geq f(x)$. Consequently $f \in (s_i^o(u))_k$ proving $s_i^o(u) \in K$.

We now show that s_i is a topological closure operator. We have $u_k \subseteq (s_i^o)_k \forall k \in I$. Thus $u \leq s_i^o(u)$ and hence $[u] \leq [s_i^o(u)] = s_i([u])$. Let $f \in (s_i^o(s_i^o(u)))_k$. If $k \geq i$ then $\exists g \in (s_i^o(u))_k$ such that $f \leq g|_{[i,k]}$. But then $\exists h \in u_k$ such that $g \leq h|_{[i,k]}$. Thus $f \leq g|_{[i,k]} \leq h|_{[i,k]}$ and so $f \in (s_i^o(u))_k$. If $k \not\geq i$ then $f \in F_k$ and so $f \in (s_i^o(u))_k$. Consequently $s_i^o(s_i^o(u)) = s_i^o(u)$ and so $s_i(s_i([u])) = s_i([u])$.

Let $[u], [v] \in B$ and $k \geq i$. Then

$$\begin{aligned} (s_i^o(u \vee v))_k &= \{f \in F_k : f \leq g|_{[i,k]} \text{ for some } g \in (u \vee v)_k\} \\ &= \{f \in F_k : f \leq g|_{[i,k]} \text{ for some } g \in u_k\} \cup \{f \in F_k : f \leq g|_{[i,k]} \text{ for some } g \in v_k\} \end{aligned}$$

$$\begin{aligned}
&= (s_i^o(u))_k \cup (s_i^o(v))_k \\
&= (s_i^o(u) \vee s_i^o(v))_k.
\end{aligned}$$

The case $k \not\geq i$ follows immediately since in this instance $(s_i^o(u \vee v))_k = F_k = (s_i^o(u))_k = (s_i^o(v))_k$. Thus $s_i^o(u \vee v) = s_i^o(u) \vee s_i^o(v)$ and so $s_i([u \vee v]) = s_i([u]) \vee s_i([v])$ proving that s_i is a topological closure operator.

It remains to show that $\ker(h) = \sum_{i \in I} \ker(s_i)$. Let $[u] \in B$ and suppose $j \geq i$ is such that $u_k = \beta_{j,k}(u_j) \forall k \geq j$. Then we have $(s_i^o(u))_k = \beta_{j,k}((s_i^o(u))_j)$ and

$$\begin{aligned}
h_j((s_i^o(u))_j) &= \sum\{f(j) : f \in F_j \text{ and } f \leq g|_{[i,j]} \text{ for some } g \in u_j\} \\
&\leq \sum\{g(j) : g \in u_j\} \\
&= h_j(u_j).
\end{aligned}$$

This together with the fact that $u_j \leq (s_i^o(u))_j$ gives $h_j(u_j) = h_j((s_i^o(u))_j)$ and so $h(s_i([u])) = h([s_i^o(u)]) = h_j((s_i^o(u))_j) = h_j(u_j) = h([u])$.

Now let $[u], [v] \in B$ with $s_i([u]) = s_i([v])$. Then $h([u]) = h(s_i([u])) = h(s_i([v])) = h([v])$. Thus $\ker(h) \geq \sum_{i \in I} \ker(s_i)$.

For the reverse inequality let $[u], [v] \in B$ with $h([u]) = h([v])$. Choose $j \in I$ such that $\forall k \geq j$ we have $\beta_{j,k}(u_j) = u_k$ and $\beta_{j,k}(v_j) = v_k$. Then $h([u]) = h_j(u_j)$ and $h([v]) = h_j(v_j)$. Thus $h_j(u_j) = h_j(v_j)$ i.e. $\sum\{f(j) : f \in u_j\} = \sum\{f(j) : f \in v_j\}$. Since $f(j)$ is join prime we have $\forall f \in u_j \exists g_f \in v_j$ such that $f(j) \leq g_f(j)$. Let $e \in (s_i^o(u))_j$. Then $e \leq f|_{[i,j]}$ for some $f \in u_j$. But $f(j) \leq g_f(j)$ for some $g_f \in v_j \Rightarrow e \leq f|_{[i,j]} \leq g_f|_{[i,j]} \Rightarrow e \in (s_i^o(v))_j$. Thus $(s_i^o(u))_j \subseteq (s_i^o(v))_j$ and by a similar argument $(s_i^o(v))_j \subseteq (s_i^o(u))_j$. Now $\forall k \geq j$ we have $(s_i^o(u))_k = \beta_{j,k}((s_i^o(u))_j) = \beta_{j,k}((s_i^o(v))_j) = (s_i^o(v))_k$. Thus $s_i([u]) = s_i([v])$ proving $\ker(h) \leq \ker(s_i) \leq \sum_{i \in I} \ker(s_i)$ and completing the proof. \square

COROLLARY 3.5.7 *Each of the following conditions implies that S is representable:*

- (i) *Every element of S is the join of completely join irreducibles.*
- (ii) *S is a lattice.*
- (iii) *$|S| \leq \aleph_1$.*

3.6 Finitely generated modular lattices

It is shown in [Sch74] that every finite distributive lattice D is isomorphic to the congruence lattice of a modular lattice L . This result was refined by R. Freese ([Fre75]) who showed that L can be chosen to be a *finitely generated* modular lattice. We present Freese's proof below.

In the following lemma let D be a $(0,1)$ -sublattice of 2^I for some set I . For any element $x \in 2^I$, let Π_x be the partition on I associated with x i.e. $\Pi_x = \{(i, j) \in I^2 : x_i = x_j\}$.

Let $U = \{(i, j) \in I^2 : d_i \leq d_j \text{ for all } d \in D\}$ and $D' = \{x \in 2^I : x_i \leq x_j \text{ for all } (i, j) \in U \text{ and } \Pi_{d^1} \wedge \Pi_{d^2} \wedge \cdots \wedge \Pi_{d^n} \leq \Pi_x \text{ for some } d^1, d^2, \dots, d^n \in D, n \in \omega\}$.

LEMMA 3.6.1 $D' = D$.

PROOF. Clearly $D \subseteq D'$. Also D' is a $(0,1)$ -sublattice of 2^I , since for $a, b \in D'$ we have $\Pi_a \wedge \Pi_b \leq \Pi_{a \wedge b}$ and $\Pi_a \vee \Pi_b \leq \Pi_{a \vee b}$. Suppose $D \neq D'$. Let $z \in D' \setminus D$ and let $\Pi_{d^1} \wedge \cdots \wedge \Pi_{d^n} \leq \Pi_z$ for some $d^1, \dots, d^n \in D$. Fix d^1, \dots, d^n and let ρ denote $\Pi_{d^1} \wedge \cdots \wedge \Pi_{d^n}$. Let $J = \{j_1, \dots, j_k\}$ be distinct representatives of the blocks of ρ and let $\Phi : 2^I \rightarrow 2^J$ be the projection map. Define $E' = \{d \in D' : \Pi_{d^1} \wedge \cdots \wedge \Pi_{d^n} \leq \Pi_d\}$ and let $E = E' \cap D$. We have $0, 1 \in E'$. Suppose $x, y \in E'$ with $\Phi(x) = \Phi(y)$, i.e. $x_j = y_j$ for all $j \in J$. Let $i \in I$. Then $(i, j) \in \rho$ for some $j \in J$. But then $(i, j) \in \Pi_x \wedge \Pi_y$ and $x_i = x_j = y_j = y_i$. Thus $x = y$ and $\Phi|_{E'}$ is one-to-one. Let $j \in J$. Define $a = \bigwedge \{x \in \Phi(E) : x_j = 1\}$ and $b = \bigvee \{x \in \Phi(E) : x_j = 0\}$. Then $a_j = 0$ and $b_j = 1$. If for some $k \in J, k \neq j$ we have $a_k = 1$, then $b_k = 1$. Since $(j, k) \notin \rho$ implies there is an $i \in \{1, \dots, n\}$ such that either $d_j^i = 1$ and $d_k^i = 0$ or $d_j^i = 0$ and $d_k^i = 1$. The first case contradicts the definition of a since in that case $a \leq d^i$ but $a_k = 1$ and $d_k^i = 0$. Similarly the second case contradicts the definition of b . Then a and $a \wedge b$ agree on $J \setminus \{j\}$ but disagree on j . Thus the embedding $\Phi(E) \subseteq 2^J$ is an irredundant subdirect embedding and so $\dim(\Phi[E]) = \dim(2^J) = k$. Note that $z \in E' - E$ and since $\Phi|_{E'}$ is an embedding, $\Phi(z) \notin \Phi[E]$. Let z^+ be the unique inverse image of the least element of $\Phi[E]$ above $\Phi(z)$ and let z^- be the unique inverse image of the greatest element of $\Phi[E]$ below z . We may assume that z is chosen in such a way that $\dim(\Phi(z^+)/\Phi(z^-))$ is minimal. Since $\Phi(z^-) < \Phi(z) < \Phi(z^+)$ we have $\dim(\Phi(z^+)/\Phi(z^-)) > 1$ in $\Phi[E]$. So there is an $x \in E$ such that $z^- < x < z^+$. By the minimality of $\dim(\Phi(z^+)/\Phi(z^-))$ we must have $x \vee z \in E$. Consequently $x \vee z = z^+$. Similarly $x \wedge z = z^-$. If y is another element of E such that $z^- < y < z^+$ then as before $y \vee z = z^+$ and $y \wedge z = z^-$. But then $y = y \wedge (y \vee z) = y \wedge (x \vee z) = (y \wedge x) \vee (y \wedge z) = (x \wedge y) \vee (x \wedge z) = x \wedge (y \vee z) = x \wedge (x \vee z) = x$. Hence $\dim(\Phi(z^+)/\Phi(z^-)) = 2$ in $\Phi[E]$. Thus $\Phi(z)$ and $\Phi(x)$ differ on exactly two coordinates, j and k say. We may assume $x_j = 0 = z_k, x_k = 1 = z_j$. Now $z \in D'$ implies that for some $d \in \{d^1, d^2, \dots, d^n\}$ we have $d_j \neq d_k$. We may assume $d_j = 1, d_k = 0$. We have $d \in E$ and $z = (d \wedge z^+) \vee z^- \in E$ contradicting $z \in E' - E$. Thus $D' \subseteq D$ and hence $D' = D$. \square

Recall that \prec denotes the covering relation in a lattice.

For a lattice L with $x, y \in L$ we write $x \preceq y$ if $x \prec y$ or $x = y$. If $x \preceq y$ then we define

$$y - x = \begin{cases} 1 & \text{if } x \prec y \\ 0 & \text{if } x = y. \end{cases}$$

Let $L = \prod_I L_i$ be a direct product of lattices. Then if $x, y \in L$ are such that $x_i \preceq y_i$ for all $i \in I$ then $y - x$ is the element of 2^I given by $(y - x)_i = y_i - x_i$.

LEMMA 3.6.2 *Let L be a modular lattice and let D be a distributive lattice with 0 and 1 . Then L is isomorphic to a sublattice of a modular lattice L^* (in the variety generated by L) such that for any prime quotient a/b in L , the sublattice a/b in L^* is isomorphic to D . Moreover $\text{con}(a,b)/\Delta_{L^*}$ in $\text{Con}(L^*)$ is isomorphic to $\text{Con}(D)$. If L is simple then $\text{Con}(L^*) = \text{Con}(D)$.*

PROOF. Let D be a distributive lattice with 0 and 1 . Then $D \leq \mathbf{2}^I$ for some set I . Let U and D' be as in Lemma 3.6.1 and define $L^* = \{x \in L^I : x_i \leq x_j \text{ for all } (i,j) \in U \text{ and } \prod_{d^1} \wedge \cdots \wedge \prod_{d^n} \leq \prod_x \text{ for some } d^1, \dots, d^n \in D\}$. Then L^* is a sublattice of L^I and L embeds in L^* , the embedding $f : L \hookrightarrow L^*$ given by $f(x)_i = x$ for all $i \in I$. For a prime quotient a/b of L , we define an isomorphism g from a/b in L^* to D' by

$$g(x)_i = \begin{cases} 0 & \text{if } x_i = b \\ 1 & \text{if } x_i = a. \end{cases}$$

By Lemma 3.6.1 a/b in L^* is isomorphic to D .

Consider θ in $\text{Con}(D)$ and define $\theta^* = (x, y) \in L^*$ such that there exists a finite sequence $x \wedge y = e^0 \leq e^1 \leq \cdots \leq e^n = x \vee y$ in L^* with $e_i^j \preceq e_i^{j+1}$ for all $i \in I, j \in \{0, \dots, n-1\}$ and $e^{j+1} - e^j \leq c^j - d^j$ for some $(c^j, d^j) \in \theta$. We show that θ^* is a congruence on L^* by showing that it satisfies the Grätzer-Schmidt criteria of Lemma 2.1.1. Conditions (i) - (iii) are immediate. For condition (iv) suppose $x \leq y, (x, y) \in \theta^*$ and $z \in L^*$. Then we have a sequence $x = e^0 \leq e^1 \leq \cdots \leq e^n = y$ such that $e_i^j \preceq e_i^{j+1}$ for all $i \in I, j \in \{0, \dots, n-1\}$ and $e^{j+1} - e^j \leq c^j - d^j$ for some $(c^j, d^j) \in \theta$. Consider the sequence $(x \wedge z) = e^0 \wedge z \leq e^1 \wedge z \leq \cdots \leq e^n \wedge z = y \wedge z$. Since $(e_i^{j+1} \wedge z_i)/(e_i^j \wedge z_i) \nearrow_w e_i^{j+1}/e_i^j$ and $e_i^j \preceq e_i^{j+1}$ we have by Corollary 2.2.3 that $(e_i^{j+1} \wedge z_i)/(e_i^j \wedge z_i) \nearrow e_i^{j+1}/e_i^j$ and hence (Corollary 2.2.2) $(e_i^{j+1} \wedge z_i)/(e_i^j \wedge z_i)$ and e_i^{j+1}/e_i^j are isomorphic. Thus $(e_i^{j+1} \wedge z_i) - (e_i^j \wedge z_i) = e_i^{j+1} - e_i^j$ and so $(e^{j+1} \wedge z) - (e^j \wedge z) \leq c^j - d^j$ proving $(x \wedge z, y \wedge z) \in \theta^*$. Joins are handled similarly proving condition (iv) of Lemma 2.1.1. Thus θ^* is a congruence on L^* .

Now let a/b be a prime quotient in L . If we identify D with the sublattice a/b in L^* , then for a congruence θ on D we have $\theta^* \cap (a/b)^2 = \theta$. To show this suppose $(c, d) \in \theta$ with $c \leq d$. Then for all $i \in I$ we have $c_i, d_i \in \{a, b\}$. Thus $c_i \preceq d_i$ and so $(c, d) \in \theta^* \cap (a/b)^2$ and $\theta \subseteq \theta^* \cap (a/b)^2$. For the reverse inclusion let $(c, d) \in \theta^* \cap (a/b)^2, c \leq d$. Then there is a sequence $c = e^0 \leq e^1 \leq \cdots \leq e^n = d$ such that $e^{j+1} - e^j \leq p^j - q^j$ for some $(p^j, q^j) \in \theta$. Suppose $p_i^j = e_i^j = b$. Then we must have $e_i^{j+1} = b$ since $e_i^{j+1} = a \Rightarrow e^{j+1} - e^j = 1 \Rightarrow p_i^j - q_i^j = 1$ contradicting $q^j \preceq p^j$. Thus $p^j \vee e^j = p^j \vee e^{j+1}$. Dually $q^j \wedge e^j = q^j \wedge e^{j+1}$. Thus $e_i^{j+1}/e_i^j \searrow (e_i^{j+1} \wedge p_i^j)/(e_i^j \wedge p_i^j) \nearrow ((e_i^{j+1} \wedge p_i^j) \vee q_i^j)/((e_i^j \wedge p_i^j) \vee q_i^j)$. Now $((e_i^{j+1} \wedge p_i^j) \vee q_i^j)/((e_i^j \wedge p_i^j) \vee q_i^j) \subseteq p_i^j/q_i^j$ and so $(e^{j+1}, e^j) \in \theta$. It follows by transitivity that $(c, d) \in \theta$.

To show that the quotient $\text{con}(a,b)/\Delta_{L^*}$ in $\text{Con}(L^*)$ is isomorphic to D , we define $\alpha : \text{Con}(D) \rightarrow \text{con}(a,b)/\Delta_{L^*}$ by $\alpha(\theta) = \theta^* \cap \text{con}(a,b)$ and $\beta : \text{con}(a,b)/\Delta_{L^*} \rightarrow \text{Con}(D)$ by $\beta(\Psi) = \Psi \cap (a/b)^2$. Then α and β are order preserving. Let $\theta \in \text{Con}(D)$. Then $(\beta \circ \alpha)(\theta) = \beta(\theta^* \cap \text{con}(a,b)) = \theta^* \cap \text{con}(a,b) \cap (a/b)^2 = \theta \cap \text{con}(a,b) = \theta$. Thus $\beta \circ \alpha$ is the identity

map on $\text{Con}(D)$. Let $\Psi \subseteq \text{con}(a, b)$ in $\text{Con}(L^*)$. Then $(\alpha \circ \beta)(\Psi) = \alpha(\Psi \cap (a/b)^2) = (\Psi \cap (a/b)^2)^* \cap \text{con}(a, b)$. Let $(x, y) \in \Psi$. Then $(x, y) \in \text{con}(a, b)$ and so there is a finite chain $x \wedge y = e^0 \leq e^1 \leq \dots \leq e^n = x \vee y$ such that e^{j+1}/e^j is projective with a subquotient c^j/d^j of a/b . Thus $(c^j, d^j) \in \Psi \cap (a/b)^2$ and (since e^{j+1}/e^j and c^j/d^j are isomorphic) $e^{j+1} - e^j = c^j - d^j$. It follows that $(x, y) \in (\Psi \cap (a/b)^2)^* \cap \text{con}(a, b)$ and $\alpha \circ \beta$ is the identity on $\text{con}(a, b)/\Delta_{L^*}$. \square

Note that L^* of Lemma 3.6.2 is such that all prime quotients in L become D in L^* . For the proof of the main theorem of this section we will require that in some situations a prime quotient a/b of L becomes D while other prime quotients of L remain prime. We construct a lattice satisfying this condition as follows: Let $a/b, c/d$ be prime quotients of L not projective to each other. Let $\theta = \text{con}(a, b)$ and $\Psi = \theta_{(a,b)}$. Then $(c, d) \in \Psi$ and since L has the projectivity property we have $\theta \cap \Psi = \Delta_L$. Thus $L \leq L/\Psi \times L/\theta$ is a subdirect representation. Applying Lemma 3.6.2 to L/Ψ we obtain a lattice $(L/\Psi)^*$ in which the quotient $(a/\Psi)/(b/\Psi)$ is isomorphic to D . As before L/Ψ can be embedded in $(L/\Psi)^*$ by the embedding $x/\Psi \mapsto (x/\Psi)^*$ where for $i \in I$, $(x/\Psi)_i^* = x/\Psi_i$. L can be embedded in $(L/\Psi)^* \times L/\theta$ by the embedding $x \mapsto ((x/\Psi)^*, x/\theta)$. Under this embedding a/b gets mapped to $((a/\Psi)^*, a/\theta)/((b/\Psi)^*, b/\theta) \cong (a/\Psi)^*/(b/\Psi)^* \cong D$ and c/d gets mapped to $((c/\Psi)^*, c/\theta)/((d/\Psi)^*, d/\theta) \cong (c/\theta)/(d/\theta) \cong 2$. In $\text{Con}((L/\Psi)^*)$ we have $\text{con}((a/\Psi)^*, (b/\Psi)^*)/\Delta \cong \text{Con}(D)$. And in $\text{Con}((L/\Psi)^* \times L/\theta) \cong \text{Con}((L/\Psi)^*) \times \text{Con}(L/\theta)$ (Proposition 1.3.4) we have $\text{con}(((a/\Psi)^*, a/\theta), ((b/\Psi)^*, b/\theta))/\Delta_{(L/\Psi)^* \times L/\theta} \cong \text{con}((a/\Psi)^*, (b/\Psi)^*)/\Delta_{(L/\Psi)^*} \times \text{con}((a/\theta), (b/\theta))/\Delta_{(L/\theta)} \cong \text{Con}(D) \times 1 \cong \text{Con}(D)$.

We obtain the desired lattice K by taking the sublattice of $L/\Psi \times L/\theta$ generated by L and a/b . Then $\text{Con}(K)$ is the lattice obtained from $\text{Con}(L)$ by replacing the prime quotients of the form $(\text{con}(a, b) \vee \sigma)/\sigma$ where $\sigma \leq \theta_{(a,b)}$ by $\text{Con}(D)$.

Theorem 3.6.3 below uses the following Hall-Dilworth construction ([Hal44]): Let L_1 and L_2 be two bounded lattices with F a principal filter of L_1 and I a principal ideal of L_2 such that $F \cong I$. Let $f : x \mapsto x'$ denote this isomorphism and let L be the set $L_1 \cup L_2$ where we identify x with x' for all $x \in F$. The elements of L are ordered as follows : $x \leq y$ has unchanged meaning if $x, y \in L_1$ or $x, y \in L_2$. For $x, y \notin I = F$, $x < y$ if and only if $x \in L_1, y \in L_2$ and $\exists z \in F$ such that $x < z$ in L_1 and $z < y$ in L_2 . Then L is a lattice - the Hall-Dilworth "glueing" of L_1 and L_2 . For the rest of this section K_1 will denote the four-generated modular lattice defined in [Day72] and illustrated in Figure 3.17.

In K_1 any two prime quotients are projective and if θ is the congruence generated by collapsing all prime quotients then $K_1/\theta \cong M_4$ (the six element lattice of length two). Thus $\text{Con}(K_1) \cong 3$. For any $n \in \omega$ we obtain a new modular lattice K_n by applying Lemma 3.6.2 to K_1 with $D = n + 1$, the $(n + 1)$ -element chain. In K_n the quotient $d/e : d = d_0 \succ d_1 \succ \dots \succ d_{n-1} \succ d_n = e$ is a chain of length n . From Lemma 3.6.2 it follows that $\text{Con}(K_n)$ is a 2^n element Boolean algebra with a new greatest element adjoined.

We are now ready to prove the main result of this section:

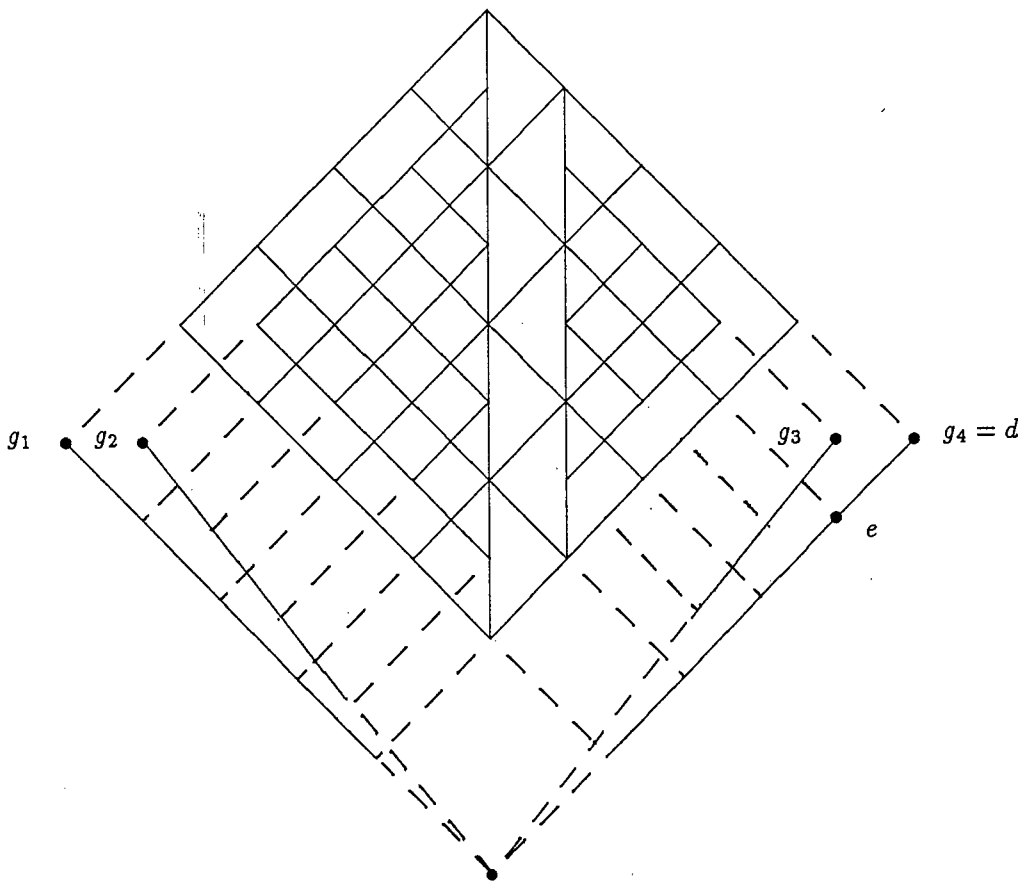


Figure 3.17 K_1

THEOREM 3.6.3 [Fre75] *Let D be a finite distributive lattice. Then there is a finitely generated modular lattice L such that $\text{Con}(L) \cong D$.*

PROOF. We prove a stronger result by induction: If D is a finite distributive lattice, then $D \cong \text{Con}(L)$ where L is a finitely generated modular lattice. Moreover there exists an $a \in L$ such that u/a is a chain, where u is the greatest element of L , and every congruence on L is determined by its restriction to u/a .

Let P be the partially ordered set of non-zero join irreducibles of D . The proof is by induction on the size of P . If $|P| = 1$ then $D \cong 2$ and we may take L to be 2 . Suppose $|P| \geq 2$. Choose a maximal element $p \in P$ and let $\{p_1, p_2, \dots, p_n\}$ be the set of co-covers of p in P . Let $D' = \mathcal{I}(P \setminus \{p\})$. We associate elements x of $P \setminus \{p\}$ with principal ideals $(x]$ of D' . By the inductive hypothesis there is a finitely generated modular lattice L' such that $\text{Con}(L') \cong D'$. Furthermore there exists an $a' \in L'$ satisfying the condition above. Let u' be the greatest element of L' . Now each $p_i, i \in 1, \dots, n$ is a join irreducible of D' hence p_i maps to a join irreducible $\text{con}(b'_i, c'_i)$ of $\text{Con}(L')$ under the isomorphism between D' and $\text{Con}(L')$. We may choose $u' \geq b'_i \geq c'_i \geq a'$ in L' . Since $\text{con}(b'_i, c'_i)$ is join irreducible for each $i \in \{1, \dots, n\}$ we can choose b'_i and c'_i such that $b'_i/c'_i \cap b'_j/c'_j$ has at most one element for $i \neq j$. For each $i \in \{1, \dots, n\}$ we will construct a modular lattice A_i such that A_i contains a sublattice isomorphic to the lattice diagrammed below:

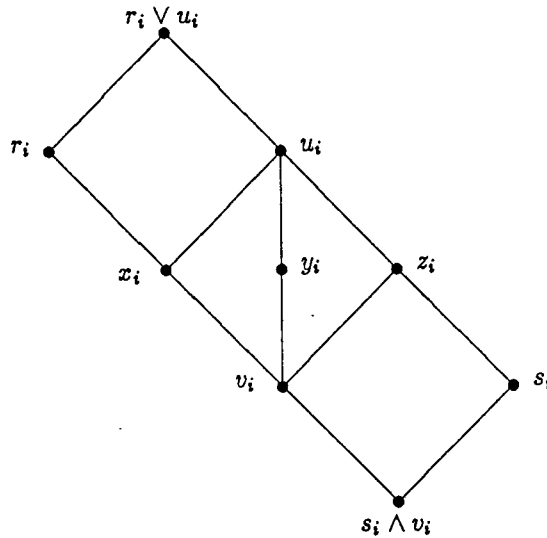


Figure 3.18 A_i

In Figure 3.18 we have $v_i/(s_i \wedge v_i) \cong c'_i/a'$, $x_i/v_i \cong b'_i/c'_i$ and $(r_i \vee u_i)/u_i \cong u'/b'_i$. We construct A_i in three steps. First apply Lemma 3.6.2 to M_3 taking the distributive lattice to be b'_i/c'_i to obtain a modular lattice B_i which contains M_3 as a sublattice with each prime quotient in M_3 isomorphic to b'_i/c'_i in B_i . Then let C_i be the direct product of b'_i/c'_i and u'/b'_i .

Apply the Hall-Dilworth construction to B_i and C_i by joining them at their common points to obtain a modular lattice D_i with sublattice as in Figure 3.19.

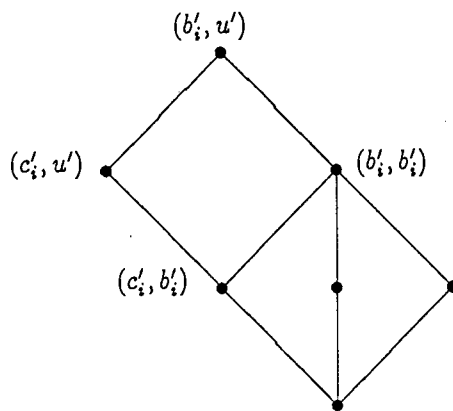


Figure 3.19 D_i

Repeat the above process with $B_i = b_i/c'_i \times c'_i/a'$ to obtain a modular lattice E_i with sublattice illustrated in Figure 3.20.

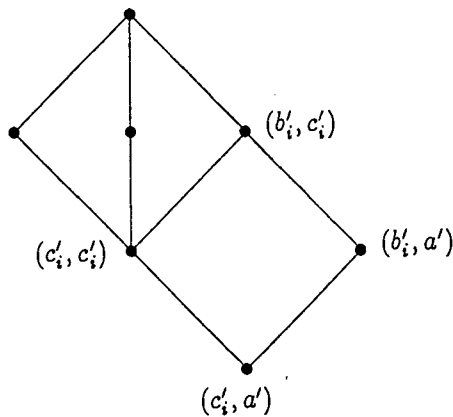


Figure 3.20 E_i

Finally apply the Hall-Dilworth construction to D_i and E_i to obtain A_i with the sublattice in Figure 3.18 above. We now apply the construction in the comments following Lemma 3.6.2 repeatedly to K_n in such a way that in the resulting lattice K'_n we have $d_{i-1}/d_i \cong b'_i/c'_i$ for $i = 1, \dots, n$. Let B be the lattice obtained by taking the direct product of u'/a' (in L') and $d_n/\mathbf{0} = e/\mathbf{0}$ (in K'_n). We then form the lattice L through repeated use of the Hall-Dilworth construction. First form L_0 from L' and B by identifying u'/a' and $(u', \mathbf{0})/(a', \mathbf{0})$. L_1 is formed from L_0 and A_n by identifying $(u', e)/(a', e)$ and $r_n/(s_n \vee v_n)$ (see Figure 3.22).

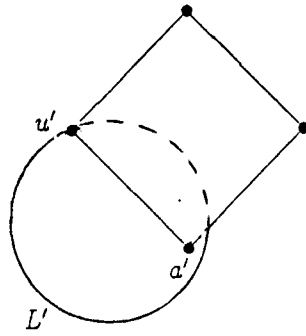


Figure 3.21 L_0

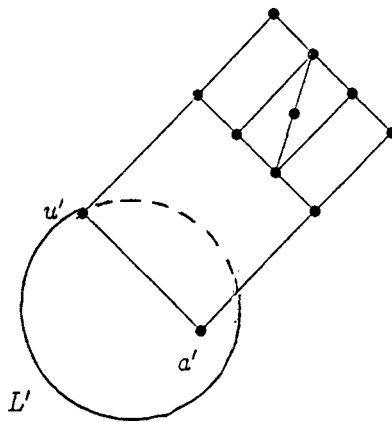


Figure 3.22: L_1

L_k is formed from L_{k-1} and A_{n-k+1} by identifying $(r_{n-k+2} \vee u_{n-k+2})/s_{n-k+2}$ and $r_{n-k+1}/(s_{n-k+1} \wedge v_{n-k+1})$. This is repeated until we get L_n , formed from L_{n-1} and A_1 by identifying $(r_2 \vee u_2)/s_2$ and $r_1/(s_1 \wedge v_1)$ (see Figure 3.23).

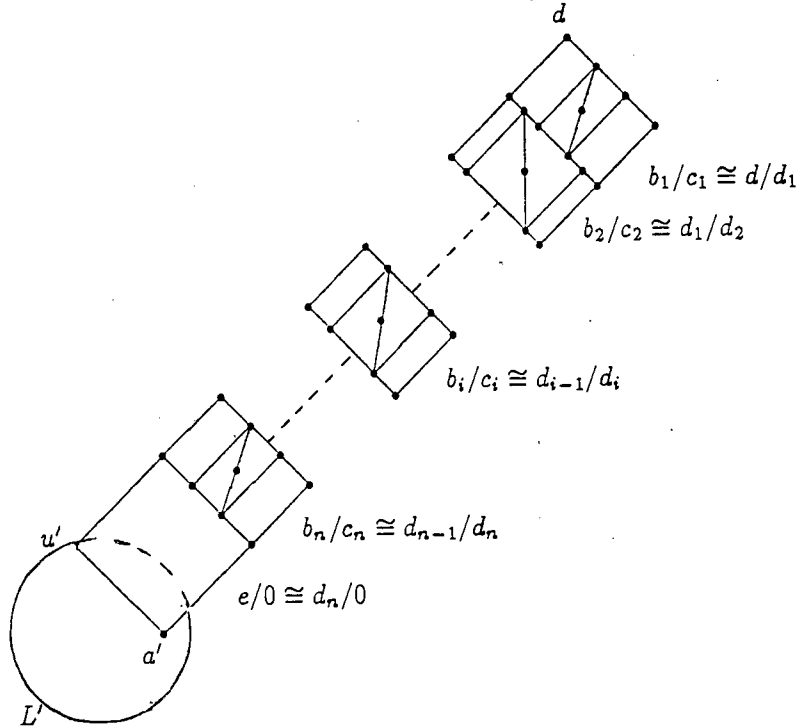


Figure 3.23 L_n

In L_n the quotient sublattice $(r_1 \vee u_1)/u'$ is isomorphic to $d/0$ in K'_n . Let L be the lattice obtained by identifying these quotients in the Hall-Dilworth glueing of L_n and K'_n . A schematic diagram of the sublattice d/a' of L is shown in Figure 3.24.

Now L is generated by the generators of L' , the four generators of K_n , $y_i : i = 1, \dots, n$ and a . Thus L is finitely generated. Let x/y be a quotient in L . Then $(x \vee d)/(y \vee d) \searrow_w x/y$, $(x \wedge u')/(y \wedge u') \nearrow_w x/y$ and $((x \wedge d) \vee u')/((y \wedge d) \vee u') \searrow_w (x \wedge d)/(y \wedge d) \nearrow_w x/y$. Thus $\text{con}(x \vee d, y \vee d) \vee \text{con}((x \wedge d) \vee u', (y \wedge d) \vee u') \vee \text{con}(x \wedge u', y \wedge u') \subseteq \text{con}(x, y)$. We also have $x/((x \wedge d) \vee y) = x/(x \wedge (y \vee d)) \nearrow (x \vee d)/(y \vee d)$, $(x \wedge d) \vee u')/((y \wedge d) \vee u') \searrow (x \wedge d)/((x \wedge d) \wedge ((y \wedge d) \vee u')) = (x \wedge d)/((x \wedge u') \vee (y \wedge d)) \nearrow ((x \wedge d) \vee y)/((x \wedge u') \vee y)$ and $(x \wedge u')/(y \wedge u') \nearrow ((x \wedge u') \vee y)/y$. Thus $(x, (x \wedge d) \vee y) \in \text{con}(x \vee d, y \vee d)$, $((x \wedge d) \vee y, (x \wedge u') \vee y) \in \text{con}((x \wedge d) \vee u', (y \wedge d) \vee u')$ and $((x \wedge u') \vee y, y) \in \text{con}(x \wedge u', y \wedge u')$. Hence $\text{con}(x, y) \subseteq \text{con}(x \vee d, y \vee d) \vee \text{con}((x \wedge d) \vee u', (y \wedge d) \vee u') \vee \text{con}(x \wedge u', y \wedge u')$ proving $\text{con}(x, y) = \text{con}(x \vee d, y \vee d) \vee \text{con}((x \wedge d) \vee u', (y \wedge d) \vee u') \vee \text{con}(x \wedge u', y \wedge u')$. Now $(x \vee d)/(y \vee d) \subseteq 1/d \subseteq 1/a$ where 1 is the top element of L (and of K'_n). Also d/u' is projective to $1/d$ and $((x \wedge d) \vee u')/((y \wedge d) \vee u') \subseteq d/u'$. Hence $\text{con}((x \wedge d) \vee u', (y \wedge d) \vee u')$ is determined by its restriction to $1/d$. Furthermore $(x \wedge u')/y \wedge u'$ lies in L' and hence $\text{con}(x \wedge u', y \wedge u')$ is determined by its restriction to u'/a' which is projective to d/a . Now

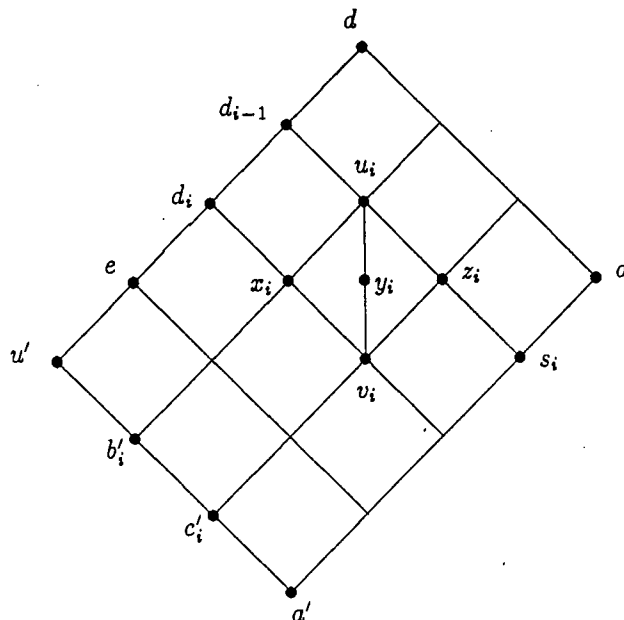


Figure 3.24

$\text{con}_{K'_n}(d_{i-1}, d_i)|_{d_{j-1}/d_j} = \Delta_{K'_n}$ ($i \neq j$) and so every congruence on L' has an extension to a congruence of L . Moreover every congruence on L , with the exception of $\text{con}(1, d)$ is an extension of a congruence on L' . In particular every congruence of L is a join irreducible congruence. Thus to show $\text{Con}(L) \cong D$ it is sufficient to show that the partially ordered set of join irreducibles of $\text{Con}(L)$ is isomorphic to P . Each $q \in P \setminus \{p\}$ is associated with a join irreducible θ'_q of $\text{Con}(L')$. We associate q with the extension of θ'_q to L . Note that if θ is the extension of a join irreducible congruence θ' of L' then, by the construction of L , $\theta \subseteq \text{con}(1, d)$ in L if and only if $\theta' \subseteq \text{con}(b'_i, c'_i)$ in $\text{Con}(L')$ for all $i = 1, \dots, n$. Finally $\text{con}(1, d) \neq \bigvee_{i=1}^n \text{con}(b'_i, c'_i)$ since $\text{con}(1, d) \supset \bigvee_{i=1}^n \text{con}(d_{i-1}, d_i)$ in K'_n . This establishes the isomorphism between D and $\text{Con}(L)$ and concludes the proof. \square

3.7 Complemented modular lattices

We outline the proof of the following theorem of by E.T. Schmidt. The reader is referred to [Sch84] for details.

THEOREM 3.7.1 [Sch84] *For every finite distributive lattice D there exists a complemented modular lattice K such that $\text{Con}(K) \cong D$ and K is a sublattice of the lattice of all subspaces of a countably infinite dimensional vector space over the two element field.*

An ideal I of a lattice L is *neutral* if for all $J, K \in \mathcal{I}(L)$:

$$(I \wedge J) \vee (J \wedge K) \vee (K \wedge I) = (I \vee J) \wedge ((J \vee K) \wedge (K \vee J)).$$

We call a modular lattice M a *locally finite complemented modular lattice* if for every finite subset X of M there exists a finite complemented sublattice M' such that $X \subseteq M'$.

Let L be a modular lattice with 0 . Then elements a_1, a_2, \dots, a_n and $c_{jk} : j \neq k, j, k = 1, \dots, n$ of L form a *normalized frame of order n* if the following hold:

- (i) $(a_i : i = 1, 2, \dots, n)$ is an independent sequence over 0 (i.e. the sublattice of L generated by a_1, a_2, \dots, a_n is a Boolean algebra with atoms a_1, a_2, \dots, a_n).
- (ii) $\{0, a_i, a_j, c_{ij}, a_i \vee a_j\}$ ($i \neq j$) is a diamond (i.e forms a sublattice of L isomorphic to M_3).
- (iii) $c_{ij} = c_{ji}$ and $c_{ij} = (c_{ik} \vee c_{jk}) \wedge (a_i \vee a_j)$ for $i, j, k = 1, \dots, n, i \neq j \neq k \neq i$.

We denote this normalized frame by $\mathcal{F} = (a_i, c_{ij})$. If $(a_i : i = 1, 2, \dots)$ is a denumerably infinite sequence and c_{ij} ($i \neq j, i, j = 1, 2, \dots$) are elements of L satisfying (ii) in the above definition, then $\mathcal{F} = (a_i, c_{ij})$ is a normalized frame of order ∞ .

For a vector space V we let $L(V)$ denote the lattice of all subspaces of V . Let \mathcal{V} denote the the infinite dimensional vector space over the two element field $\mathbf{2}$. Then $\mathcal{L} = L(\mathcal{V})$ is a modular lattice and the finite dimensional subspaces of \mathcal{V} form an ideal \mathcal{L}^f of \mathcal{L} . We let \mathcal{V}_n denote the n -dimensional vector space over $\mathbf{2}$ and $\mathcal{L}_n = L(\mathcal{V}_n)$.

Let L be a complemented sublattice of \mathcal{L} . Then a prime quotient a/b of L is not necessarily a prime quotient of \mathcal{L} . Let M be a $\{0, 1\}$ -sublattice of a/b . Then $L \cap M = \{a, b\}$ and $L \cup M$ is a relative sublattice of \mathcal{L} . We denote the sublattice of \mathcal{L} generated by $L \cup M$ by $L[M]$.

LEMMA 3.7.2 [Her74] *Let \mathcal{F} be a normalized frame of order n or ∞ in \mathcal{L} . Then the sublattice $\langle \mathcal{F} \rangle$ generated by \mathcal{F} is isomorphic to \mathcal{L}_n or \mathcal{L}^f respectively. The elements of \mathcal{F} are atoms of $\langle \mathcal{F} \rangle$.*

LEMMA 3.7.3 *Let L be a sublattice of \mathcal{L} isomorphic to \mathcal{L}^f or \mathcal{L}_n and let a/b be a prime quotient of L . If M is a locally finite complemented modular $\{0, 1\}$ -sublattice of a/b then $a/b \cap L[M] = M$. For every congruence relation θ of M there exists exactly one congruence relation $\bar{\theta}$ of $L[M]$ extending θ .*

We consider a specific normalized frame in \mathcal{L} . In \mathcal{V} we choose as a basis:

$$\begin{array}{cccc} e_{11} & e_{12} & e_{13} & \dots \\ e_{21} & e_{22} & e_{23} & \dots \\ e_{31} & e_{32} & e_{33} & \dots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

with infinitely many rows and columns. Let a_i be the subspace of \mathcal{L} spanned by the i th row ; $a_i = [e_{i1}, e_{i2}, \dots]$. Then $\{a_i : i = 1, 2, \dots\}$ is a denumerably infinite independent set. Let c_{ij} be the subspace $[e_{i1} + e_{j1}, e_{i2} + e_{j2}, \dots]$. Then $\{\emptyset, a_i, a_j, c_{ij}, a_i \vee a_j\}$ forms a diamond and so $\mathcal{F} = (a_i, c_{ij})$ is a normalized frame of order ∞ in \mathcal{L} . Furthermore the principal ideal (a_i) of \mathcal{L} is isomorphic to \mathcal{L} .

By replacing the lattice operations with their duals in the definitions of independent sequence and normalized frame we obtain the notion of a dual frame. Let a'_i be the subspace of \mathcal{V} spanned by $\{e_{jk} : j = 1, 2, \dots, k = 1, 2, \dots, j \neq k\}$. Then a'_i is a complement of a_i in \mathcal{L} . Let $c'_{ij} = c_{ij} \vee (a'_i \wedge a'_j)$. Then $\mathcal{F}' = (a'_i, c'_{ij})$ is a dual frame in \mathcal{L} . Let $L = \langle \mathcal{F} \rangle$ (the sublattice generated by \mathcal{F}) and let M be a locally finite complemented $\{0, 1\}$ -sublattice of (a_1) . Then every congruence θ of M is determined by the kernel $I = K(\theta)$ (see Theorem 2.6.5), which is a neutral ideal of M . By Lemma 3.7.3 θ has exactly one extension $\bar{\theta}$ to $L[M]$. Now $L[M]$ is a relatively complemented lattice and so $\bar{\theta}$ is determined by the kernel $\bar{I} = K(\bar{\theta})$ and $\bar{I} \cap (a_1) = I$. Now (a_1) and \bar{I} are ideals of $L[M]$ with $(a_1) \cap \bar{I}$ a neutral ideal of (a_1) . Let S be the sublattice of $L[M]$ generated by (a_1) and \bar{I} . Then it can be shown that every element s of S has a representation

$$s = a \vee x, a \leq a_1, x \in \bar{I}, a_1 \wedge s = a$$

and furthermore $\text{Con}(S) \cong \text{Con}(M)$.

Let $I' = \{x \in M : (x, 1) \in \theta\}$. Then I' is a filter in M . Consider the sublattice L' (of \mathcal{L}) generated by the dual frame \mathcal{F}' . Since $a_1/0$ and $1/a'_1$ are transposes, we can assume that M is a $\{0, 1\}$ -sublattice of $1/a'_1$. As before we can extend I' to a filter \bar{I}' of $L'[M]$. Let S' be the sublattice generated by $\{a'_1\}$ and \bar{I}' . Then $\text{Con}(S') \cong \text{Con}(M)$. Now $S \cap S' = \emptyset$ and $K = S \cup S'$ is a locally finite complemented modular lattice. This K is a sublattice of $L^*[M]$ where L^* is the sublattice of \mathcal{L} generated by $\mathcal{F} \cup \mathcal{F}'$.

Now let D be the finite distributive lattice we wish to represent. Then $\mathcal{O}(J(D)) \cong D$ (see Lemma 2.6.7). Thus to prove Theorem 3.7.1 it is sufficient to show that for every finite partially ordered set P , there exists a locally finite complemented modular lattice K_P such that $\text{Con}(K_P) \cong \mathcal{O}(P)$. Proof is by induction on $|P|$. If $|P| = 1$ then $K_P \cong \mathbf{2}$, which is a $\{0, 1\}$ -sublattice of \mathcal{L} .

Suppose $|P| \geq 2$. Let m be a maximal element of P with co-covers m_1, \dots, m_k in P . Let $Q = P \setminus \{m\}$. By induction there is a complemented modular lattice K_Q and an isomorphism $\varphi : \mathcal{O}(Q) \rightarrow \text{Con}(K_Q)$ with K_Q a $\{0, 1\}$ -sublattice of \mathcal{L} . Let (m_1, \dots, m_k) denote

the order ideal of Q generated by m_1, \dots, m_k . Then $\varphi((m_1, \dots, m_k)) = \theta \in \text{Con}(K_Q)$. Now K_Q is complemented and so θ is completely determined by $K(\theta)$ which is an ideal of K_Q . Let $I' = \{x \in K_Q : (x, \mathbf{1}) \in \theta\}$ and let $M = K_Q$. We may assume that M is a $\{0, 1\}$ -sublattice of $(a_1]$ where $\mathcal{F} = (a_i, c_{jk})$ is our fixed frame. Let $K_P = K$ be the lattice constructed previously from $\mathcal{F} \cup \mathcal{F}'$, M and I .

K is complemented, hence every principal congruence is of the form $\text{con}(\mathbf{0}, u)$. If $u \in S$ then $\text{con}(\mathbf{0}, u)$ is the extension of a congruence of $M = K_Q$. If $u \in S'$, then $u = a' \wedge x$ for some $a' \geq a'_1, x \in \bar{I}$ and $a' = u \vee a'_1$. Since $u/(u \wedge a'_1) \nearrow (u \vee a'_1)/a'_1$ we have $\text{con}(u \wedge a'_1, u) = \text{con}(a'_1, u \vee a'_1) = \text{con}(a'_1, a') = \text{con}(a_1, a)$. Thus $\text{con}(u \wedge a'_1, u)$ is the extension of a congruence on M . Now $\text{con}(\mathbf{0}, u) = \text{con}(\mathbf{0}, u \wedge a'_1) \vee \text{con}(u \wedge a'_1, u)$ implies that $\text{con}(\mathbf{0}, u)$ is the join of $\text{con}(\mathbf{0}, u \wedge a'_1)$ with a congruence relation which is the extension of a congruence of M . Moreover $\text{con}(\mathbf{0}, a'_1) = \text{con}(\mathbf{0}, a'_1)$ for arbitrary $u \in S'$, i.e. every join irreducible congruence relation of K_P is either the extension of a join irreducible congruence of M or it is $\text{con}(\mathbf{0}, a'_1)$. Then $\text{con}(\mathbf{0}, a') \geq \bar{\theta}$, for some $\theta \in \text{Con}(M)$ if and only if $\theta \leq \varphi((m_i))$. Consequently $J(\text{Con}(P))$ is isomorphic to P . \square

3.8 Complete congruences of complete lattices

The lattice of complete congruence relations of a complete lattice is a complete lattice. The converse question as to whether every complete lattice can be represented as the lattice of complete congruences of a complete lattice was raised by Wille in 1983 ([Wil83]). K.Reuter and R.Wille provided a partial solution to this problem in [Reu87] where it is shown that every distributive complete lattice in which every element is a join of join irreducibles has such a representation and S.-K. Teo solved this problem in the finite case ([Teo90]). A solution to Wille's question was first announced by Grätzer in 1988 (see [Gra88]) and we present the proof by Grätzer and Lakser in [Gra91]. In [Fre91] R. Freese, G.Grätzer and E.T. Schmidt refine the result presented below by showing that the every complete lattice L can be represented as the lattice of complete congruence relations of a complete modular lattice K . A further improvement is given in [Gra95] where it is shown that if η is a regular uncountable cardinal then every η -algebraic lattice can be represented as the lattice of η -complete congruence relations of an η -complete distributive lattice. In particular every complete lattice is isomorphic to the lattice of complete congruences of a complete distributive lattice.

An equivalence relation θ on a complete lattice L is a *complete congruence relation* if for any $I (x_i, y_i) \in \theta \forall i \in I$ implies $(\bigwedge_{i \in I} x_i, \bigwedge_{i \in I} y_i) \in \theta$ and $(\bigvee_{i \in I} x_i, \bigvee_{i \in I} y_i) \in \theta$.

For a quotient $p = a/b$ of a lattice K we will write $\text{con}(p)$ for $\text{con}(a, b)$ and for $\theta \in \text{Con}(K)$ we write $p \in \theta$ to mean $(a, b) \in \theta$. We let $\text{con}^c(a, b)$ denote the smallest compact congruence containing (a, b) and write $\text{Com}(K)$ for the lattice of complete congruences of a complete lattice K . For $n \in \omega$, n will denote the chain of length n .

Let L denote the complete lattice we wish to represent. For a lattice K let $Ip(K)$ denote the set of prime intervals of K . For $p_1 = x_1/y_1, p_2 = x_2/y_2 \in Ip(K)$ we consider four specific elements of $K \times K$:

$o(p_1, p_2) = (y_1, y_2), a(p_1, p_2) = (x_1, y_2), b(p_1, p_2) = (y_1, x_2), i(p_1, p_2) = (x_1, x_2)$. Then $\{o, a, b, i\}$ forms a sublattice of $L \times L$ isomorphic to 2×2 .

Let C be a chain. We define a *colouring* of C to be a map $\alpha : Ip(C) \rightarrow L \setminus \{0\}$. From a chain C and colouring α we construct a lattice $C(\alpha)$. This lattice is formed by taking all $p_1, p_2 \in Ip(C)$ with $\alpha(p_1) = \alpha(p_2)$ and augmenting $C \times C$ with new elements $m(p_1, p_2)$ in such a way that the elements $o(p_1, p_2), a(p_1, p_2), b(p_1, p_2), m(p_1, p_2), i(p_1, p_2)$ form a sublattice of $C(\alpha)$ isomorphic to M_3 . We denote this sublattice of $C(\alpha)$ by $M_3(p_1, p_2)$.

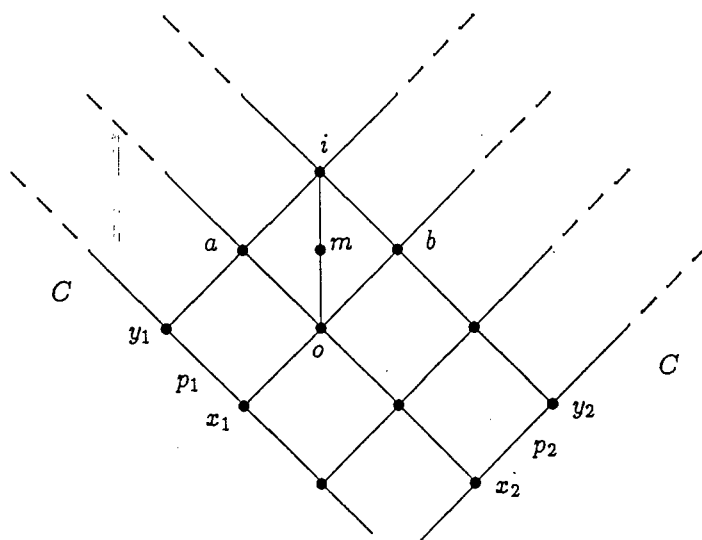


Figure 3.25: $M_3(p_1, p_2)$

The congruences of $C \times C$ are of the form $\theta_1 \times \theta_2$ where $\theta_1, \theta_2 \in \text{Con}(C)$ (see Proposition 1.3.4). Consider $\theta_1, \theta_2 \in \text{Con}(C)$ with the following property:

If $p_1, p_2 \in Ip(C)$ and $\alpha(p_1) = \alpha(p_2)$ then $p_1 \in \theta_1 \Leftrightarrow p_2 \in \theta_2$ (†)

and extend $\theta_1 \times \theta_2$ to $C(\alpha)$ as follows: Let $p_1, p_2 \in Ip(C)$ with $\alpha(p_1) = \alpha(p_2)$. Then if $o(p_1, p_2), a(p_1, p_2), b(p_1, p_2), i(p_1, p_2)$ are in the same congruence class modulo $\theta_1 \times \theta_2$ let $m(p_1, p_2)$ be in the same congruence class as these elements. Otherwise, let $m(p_1, p_2)/(\theta_1 \times \theta_2)$ be the singleton $\{m(p_1, p_2)\}$. Denote this extension of $\theta_1 \times \theta_2$ by $\theta_1 \times_\alpha \theta_2$. Note that θ_1 and θ_2 collapse exactly the same prime quotients of C . In the special case where $\theta_1 = \theta_2 = \theta$ we denote $\theta_1 \times_\alpha \theta_2$ by $\theta(\alpha)$.

We then have the following lemma.

LEMMA 3.8.1 *The congruences of $C(\alpha)$ are exactly the congruences of the form $\theta_1 \times_\alpha \theta_2$*

where θ_1, θ_2 satisfy property (†) above.

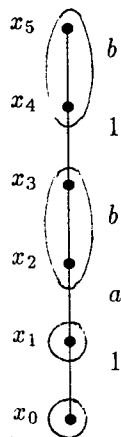


Figure 3.26: C

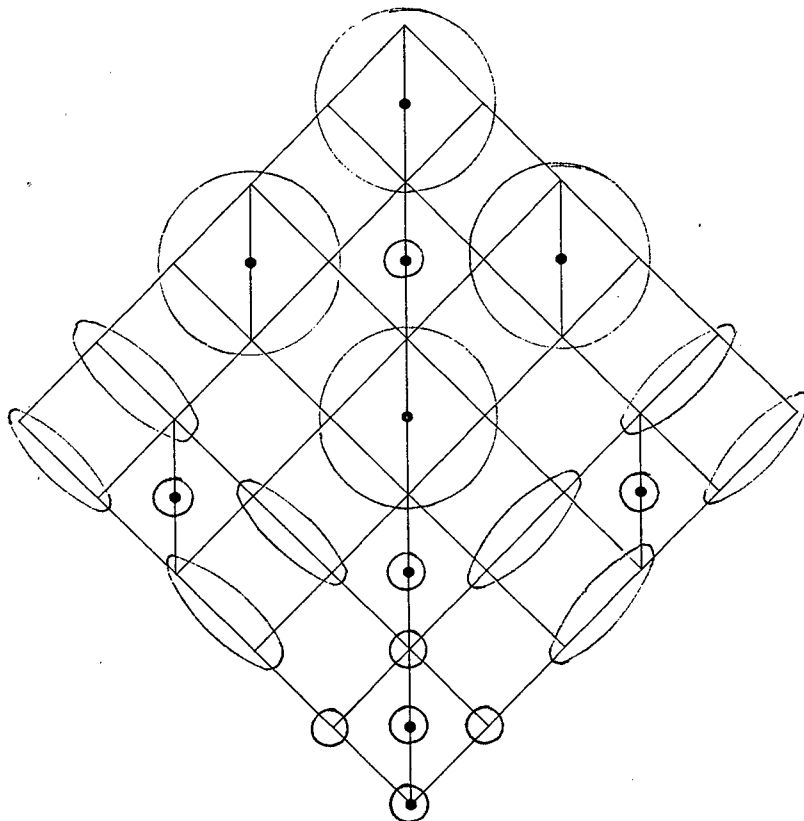


Figure 3.27

As an example consider the chain C in Figure 3.26 with colouring α written to the right of each prime quotient. Let θ be the congruence illustrated on C . Then Figure 3.27 depicts the congruence $\theta(\alpha)$ on $C(\alpha)$.

Let A be a complete lattice with $\theta \in \text{Con}(A)$. Define the *prime interior* of θ ($\text{pi}(\theta)$) by:

$$\text{pi}(\theta) = \bigvee \{ \text{con}^c(p) : p \in \text{Ip}(K) : \text{con}(p) \subseteq \theta \}.$$

A lattice A is *strongly atomic* if for any $w, z \in A$ such that $w < z$ there is a $p \in A$ such that $w \prec p \leq z$.

LEMMA 3.8.2 *In a strongly atomic complete lattice A , $\text{pi}(\theta) = \theta$ for any congruence θ of A .*

PROOF. The inclusion $\text{pi}(\theta) \subseteq \theta$ follows immediately from the definition. For the reverse inclusion let $x < y$ with $(x, y) \in \theta$. By completeness of $\text{pi}(\theta)$ there is a maximal $z \in y/x$ such that $(x, z) \in \text{pi}(\theta)$. If $z = y$ then we are done. Suppose not. Then, since A is strongly atomic, there is a $p \in A$ such that $z \prec p \leq y$. But then $(x, z) \in \text{pi}(\theta)$ and this contradiction of the maximality of z completes the proof. \square

Lemmas 3.8.1 and 3.8.2 yield the following result.

LEMMA 3.8.3 *Let C be a complete strongly atomic chain. Then the complete congruences of $C(\alpha)$ are congruences of the form $C(\theta)$ where θ is a complete congruence of C satisfying property (\dagger).*

We now construct the lattice K which is such that $\text{Com}(K)$ is isomorphic to L .

First consider non-empty subsets $X \subseteq L \setminus \{0\}$. Write $X = \{x_\gamma : \gamma \leq \zeta_X\}$ where $1 \leq \zeta_X \leq \zeta = |L \setminus \{0\}|$. Let $\{X^\delta : \delta \leq \chi\}$ denote the family of all such sets; the elements of X^δ are well-ordered: $X^\delta = \{x_\gamma^\delta : \gamma \leq \zeta_\delta\}$.

Define a chain $X^{\delta_1} = 1 + (\omega \times X^\delta) + 1$ where for lattices A and B , $A + B$ is the ordinal sum of A and B (i.e. we place B on top of A) and for ordinals α, β the ordinal product $\alpha \times \beta$ is the set $\{(\gamma, \delta) : \gamma \leq \alpha, \delta \leq \beta\}$ with the ordering $(\gamma_1, \gamma_2) \leq (\delta_1, \delta_2) \Leftrightarrow \gamma_1 \leq \delta_1$ or $(\gamma_1 = \delta_1 \text{ and } \gamma_2 \leq \delta_2)$.

The unit and zero of X^{δ_1} are denoted by 1^δ and 0^δ respectively, the other elements by :

$$j^\delta = (0, x_0^\delta) < (0, x_1^\delta) < \cdots < (i, x_0^\delta) < (i, x_1^\delta) < \cdots \text{ for } i < \omega.$$

Now define a colouring α^δ on X^{δ_1} as follows.

$$\begin{aligned} \alpha^\delta([0^\delta, j^\delta]) &= \bigvee X^\delta \\ \alpha^\delta([i, x_\gamma^\delta], u) &= x_\gamma \text{ for } i < \omega, \gamma \leq \zeta_\delta \text{ where} \end{aligned}$$

$$u = \begin{cases} (i, x_{\gamma+1}^\delta) & \text{if } \gamma + 1 < \zeta_\gamma \\ (i + 1, x_0^\delta) & \text{if } \gamma + 1 = \zeta_\gamma. \end{cases}$$

We then have the following important observation:

LEMMA 3.8.4 *The chain X^{δ^\dagger} is well-ordered, 1^δ is a limit of X^{δ^\dagger} . In X^{δ^\dagger} , for every $j^\delta \leq u < 1^\delta$ and every $\gamma \leq \zeta_\delta$ there is a $p \in Ip([u, 1^\delta])$ such that $\alpha^\delta(p) = x_\gamma^\delta$.*

We now construct a lattice \mathcal{M}_{X^δ} by first forming the lattice $X^{\delta^\dagger} \times 2$ and identifying $(x, 0)$ with x . Then X^{δ^\dagger} is a complete sublattice of $X^{\delta^\dagger} \times 2$. We form \mathcal{M}_{X^δ} by adding an element m^δ to $X^{\delta^\dagger} \times 2$ in such a way that $0^\delta < m^\delta < 1^\delta$. Then \mathcal{M}_{X^δ} is a complete lattice and X^{δ^\dagger} is a complete sublattice of \mathcal{M}_{X^δ} .

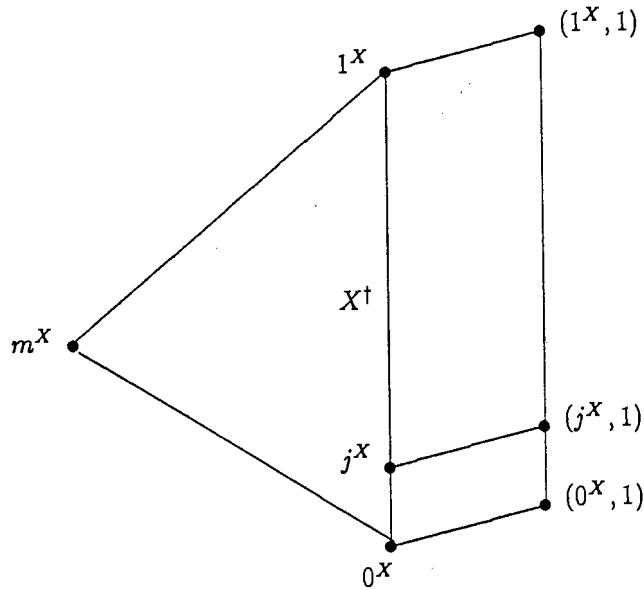


Figure 3.28 \mathcal{M}_{X^δ}

For each $X^\delta : \delta \leq \chi$, construct the chain X^{δ^\dagger} and form the ordinal sum :

$$C = \begin{cases} 1 + \sum_{\delta < \chi} (X^{\delta^\dagger}) + 1 & \text{if } \chi \text{ is limit} \\ 1 + \sum_{\delta < \chi} X^{\delta^\dagger} & \text{if } \chi \text{ is not limit.} \end{cases}$$

Then C is a well-ordered chain with zero and unit denoted by 0^C and 1^C respectively. Next we define a colouring α of C : For $p \in Ip(C)$:

$$\alpha(p) = \begin{cases} \alpha^\delta(p) & \text{if } p \in Ip(X^\delta) \text{ for some } \delta < \chi \\ 1 & \text{if } p = [0^C, 0^0] \\ 1 & \text{if } p = [1^\delta, 0^{\delta+1}] \text{ for some } \delta < \chi. \end{cases}$$

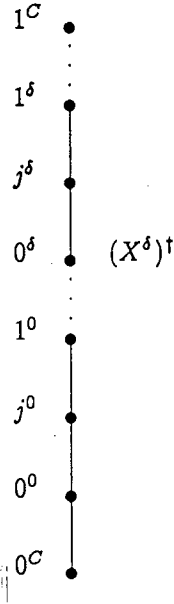


Figure 3.29 C

Finally define the lattice K as $C(\alpha)$ augmented with elements $m^\delta : \delta < \chi$. i.e. $K = C(\alpha) \cup \{m^\delta : \delta < \chi\}$ ordered as follows:

$$\begin{aligned} x \leq y & \text{ retains its meaning in } C(\alpha) \\ m^\delta < x & \text{ if and only if } (1^\delta, 0^C) \leq x \text{ in } C(\alpha) \text{ for } \delta < \chi \\ x < m^\delta & \text{ if and only if } x \leq (0^\delta, 0^C) \text{ in } C(\alpha) \text{ for } \delta < \chi. \end{aligned}$$

Then K is a complete lattice, $C(\alpha)$ is a complete $\{0, 1\}$ -sublattice of K and $(0^\delta, 0^C) \prec m^\delta \prec (1^\delta, 0^C)$ in K . Every chain in K is well-ordered and so K is weakly atomic. C is a complete sublattice of K (if we identify the element $(x, 0^C)$ with x in K).

It remains to determine the complete congruences of K .

Let $x \in L$. We define a binary relation Φ^x on C as follows:

$$\text{for } v, w \in C, v \leq w : (v, w) \in \Phi^x \Leftrightarrow \alpha(p) \leq x \text{ for every } p \in Ip([v, w]).$$

Then Φ^x is a complete congruence on C . We have $\Phi^0 = \Delta_C$ and $\Phi^1 = \nabla_C$. On $C(\alpha)$ define $\theta^x = \Phi^x(\alpha)$. To extend θ^x to K we need only make provision for $m^\delta : \delta < \chi$.

For $y, z \in \{0^\delta, m^\delta, 1^\delta\}$ $y \neq z$, let $(y, z) \in \theta^x$ if and only if $(0^\delta, 1^\delta) \in \theta^x$.

For $w \in C(\alpha) : w \notin \{0^\delta, m^\delta, 1^\delta\}$ let $(w, m^\delta) \in \theta^x$ if and only if $(0^\delta, 1^\delta) \in \theta^x$ and $(w, 1^\delta) \in \theta^x$ in $C(\alpha)$.

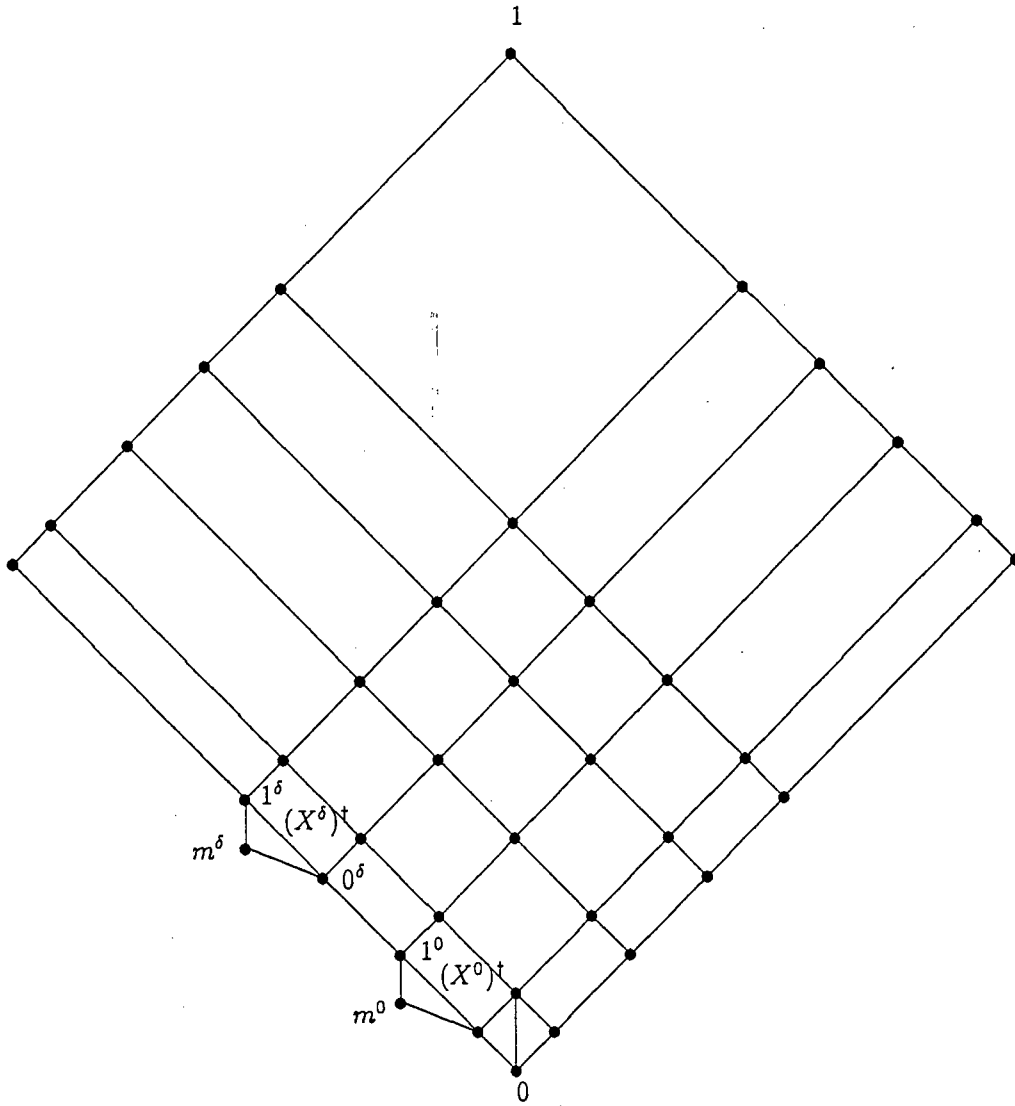


Figure 3.30 K

LEMMA 3.8.5 Suppose $m^\delta < w = \langle w_1, w_2 \rangle$ in K and $(m^\delta, w) \in \theta^x$ for some $x < 1$ and $\delta \in \chi$. Then $w \in C \times 0^C$.

PROOF. Suppose $w \notin C \times 0^C$. Then $w \geq (1^\delta, 0^0)$ and thus $m^\delta < (1^\delta, 0^C) < (1^\delta, 0^0) \leq w$. By definition of θ^x , $(m^\delta, w) \in \theta^x$ implies $(\langle 1^\delta, 0^C \rangle, w) \in \theta^x$ (in $C(\alpha)$). Then $(\langle 1^\delta, 0^C \rangle, \langle 1^\delta, 0^0 \rangle) \in \theta^x$ which implies $\alpha([0^C, 0^0]) \leq x$ contradicting $\alpha([0^C, 0^0]) = 1$. \square

LEMMA 3.8.6 For all $x \in L$, θ^x is a complete congruence relation on K .

PROOF. θ^x is an equivalence relation and by Lemma 3.8.3 $\theta^x|_{C(\alpha)}$ is a complete congruence on $C(\alpha)$. Consider $a < b$ in $C(\alpha)$ with $(a, b) \in \theta^x$ and $(m^\delta, w) \in \theta^x$ for some $\delta \in \chi, m^\delta < w$. By Lemma 3.8.5 $w = \langle w_1, 0^C \rangle$ for some $w_1 \in C$. We first restrict our attention to the case $a, b \in C \times C$. Let $a = \langle a_1, a_2 \rangle, b = \langle b_1, b_2 \rangle$ and consider $(a \vee m^\delta, w \vee b)$. There are three cases: (i) $a < m^\delta$, (ii) $m^\delta < a$, (iii) a and m^δ are non-comparable.

In case (i) we have $(a \vee m^\delta, w \vee b) = (m^\delta, \langle w_1 \vee b_1, b_2 \rangle)$. Since $a < m^\delta$ we have $a_2 = 0^C$ and since $(a, b) \in \theta^x$ we have $\alpha(p) \leq x$ for all $p \in Ip([0^C, b_2])$. We must have $b_2 = 0^C$ since otherwise $b_2 \geq 0^0$ and $\alpha([0^C, b_2]) \leq x$ contradicting $\alpha([0^C, b_2]) = 1$. Then

$$(a \vee m^\delta, w \vee b) = (m^\delta, \langle w_1 \vee b_1, 0^C \rangle) = \begin{cases} (m^\delta, w) & \text{if } b_1 \leq w_1 \\ (m^\delta, \langle b_1, 0^C \rangle) & \text{if } w_1 < b_1. \end{cases}$$

Now $(m^\delta, w) \in \theta^x$ by assumption and $(m^\delta, \langle b_1, 0^C \rangle) \in \theta^x$ if and only if $(0^\delta, 1^\delta) \in \Phi^x$ and $(1^\delta, b_1) \in \Phi^x$. Since $(m^\delta, w) \in \theta^x$ we have $(0^\delta, 1^\delta) \in \Phi^x$ and since $a \leq m^\delta$ we must have $a = \langle a_1, 0^C \rangle \leq \langle 0^\delta, 0^C \rangle$. Thus, since $(a, b) \in \theta^x$ and $\langle a_1, 0^C \rangle \leq \langle 0^\delta, 0^C \rangle \leq \langle 1^\delta, 0^C \rangle \leq \langle b_1, 0^C \rangle$ we have $(\langle 1^\delta, 0^C \rangle, \langle b_1, 0^C \rangle) \in \theta^x$. Consequently $(m^\delta, \langle b_1, 0^C \rangle) \in \theta^x$ completing the proof that $(a \vee m^\delta, w \vee b) \in \theta^x$.

For case (ii) we have

$$(a \vee m^\delta, w \vee b) = (a, \langle w_1 \vee b_1, b_2 \rangle) = (a, \langle w_1 \vee b_1, b_2 \rangle) = \begin{cases} (a, \langle w_1, b_1 \rangle) & \text{if } b_1 \leq w_1 \\ (a, b) & \text{otherwise.} \end{cases}$$

Now $(a, b) \in \theta^x$ by assumption, and if $b_1 \leq w_1$ then, since $1^\delta \leq a_1 \leq b_1 \leq w_1$ and $(1^\delta, w_1) \in \Phi^x$ we must have $(a_1, w_1) \in \Phi^x$ and so $(a, \langle w_1, b_2 \rangle) \in \theta^x$.

For case (iii) we have $(a \vee m^\delta, b \vee w) = (a \vee 1^\delta, w \vee b) = (\langle a_1 \vee 1^\delta, a_2 \rangle, \langle w_1 \vee b_1, b_2 \rangle)$. Now $(a_2, b_2) \in \Phi^x$ by assumption and so it remains to prove $(a_1 \vee 1^\delta, w_1 \vee b_1) \in \Phi^x$. We must have $1^\delta > a_1$ since $a_1 \geq 1^\delta \Rightarrow \langle a_1, a_2 \rangle \geq \langle 1^\delta, 0^C \rangle \Rightarrow a \geq m^\delta$. Thus $(a_1 \vee 1^\delta, w_1 \vee b_1) = (1^\delta, w_1 \vee b_1)$. We have $(1^\delta, w_1) \in \Phi^x$ and $(a_1, b_1) \in \Phi^x$. Thus, since $a_1 \leq 1^\delta$, we have $(1^\delta, b_1) \in \Phi^x$. This gives $(1^\delta, w_1 \vee b_1) \in \Phi^x$.

Now consider the case $a = m(p_1, p_2), b = o(p_1, p_2)$ for some $p_1 = x_1/y_1, p_2 = x_2/y_2 \in Ip(C)$ with $\alpha(p_1) = \alpha(p_2)$. Then θ^x collapses $M_3(p_1, p_2)$. We have two cases: (i) $m^\delta < m(p_1, p_2) = a$ and (ii) m^δ and $m(p_1, p_2)$ are non-comparable.

For case (i) $1^\delta \leq m(p_1, p_2) \leq i(p_1, p_2) = (y_1, y_2) \Rightarrow 1^\delta \leq y_1$. Now $(a \vee m^\delta, b \vee w) = (m(p_1, p_2), \langle w_1 \vee x_1, x_2 \rangle)$. If $w < x_1$ then this becomes $(m(p_1, p_2), i(p_1, p_2)) \in \theta^x$. If $x_1 \leq w_1$ then, since $1^\delta \leq y_1 < x_1 < w$ and $(1^\delta, w) \in \Phi^x$ we have $(y_1, w) \in \Phi^x$. We also have $(x_2, y_2) \in \Phi^x$ as θ^x collapses $M_3(p_1, p_2)$. Thus $(\langle y_1, y_2 \rangle, \langle w_1, x_2 \rangle) \in \theta^x$ which implies $(o(p_1, p_2), \langle w_1, x_2 \rangle) \in \theta^x$ and so $(m(p_1, p_2), \langle w_1, x_2 \rangle) \in \theta^x$.

For case (ii) $m^\delta \vee m(p_1, p_2) = 1^\delta \vee m(p_1, p_2) = 1^\delta \vee i(p_1, p_2) = \langle 1^\delta, 0^C \rangle \vee \langle x_1, x_2 \rangle = \langle 1^\delta \vee x_1, x_2 \rangle$. Now $\langle m^\delta \vee m(p_1, p_2), w \vee b \rangle = (\langle 1^\delta \vee x_1, x_2 \rangle, \langle w \vee x_1, x_2 \rangle) \in \theta^x \Leftrightarrow (1^\delta \vee x_1, w \vee x_1) \in \Phi^x$. If $1^\delta < x_1$ then

$$(1^\delta \vee x_1, w \vee x_1) = (x_1, w_1 \vee x_1) = \begin{cases} (x_1, x_1) & \text{if } w_1 < x_1 \\ (x_1, w_1) & \text{if } x_1 \leq w_1. \end{cases}$$

Now $(x_1, x_1) \in \Phi^x$ and since $(1^\delta, w_1) \in \Phi^x$ and $1^\delta \leq x_1 \leq w_1$ we have $(x_1, w_1) \in \Phi^x$. If $x_1 < 1^\delta$ then $(1^\delta \vee x_1, w_1 \vee x_1) = (1^\delta, w_1) \in \Phi^x$. Meets are handled similarly.

Finally to show that θ^x is a complete congruence on K it suffices to show that for $(m^{\delta_i}, w^i) \in \theta^x$ for $i \in I$ and $m^{\delta_i} \leq w^i$ we have $(\vee m^{\delta_i}, \vee w^i), (\wedge m^{\delta_i}, \wedge w^i) \in \theta^x$. Now either $\vee m^{\delta_i} = m^\delta$ for some $\delta = \delta_i, i \in I$ or $\vee m^{\delta_i} = 1$. In the first case $(\vee m^{\delta_i}, \vee w^i) = (m^\delta, \vee w^i) \in \theta^x \Leftrightarrow (1^\delta, 0^\delta) \in \Phi^x$ and $(1^\delta, \vee w^i) \in \theta^x$. We have $(1^{\delta_i}, w^i) \in \Phi^x$ for all $i \in I$, hence $(\vee 1^{\delta_i}, \vee w^i) \in \Phi^x$ by completeness of Φ^x . Thus $(1^\delta, \vee w^i) \in \Phi^x$ and $(1^\delta, 0^\delta) \in \Phi^x$ follows from the fact that $\delta = \delta_i$ and $(m^{\delta_i}, w^i) \in \theta^x$. In the case $\vee m^{\delta_i} = 1$ we must have $\vee w^i = 1$ and so $(\vee m^{\delta_i}, \vee w^i) \in \theta^x$. Similarly $(\wedge m^{\delta_i}, \wedge w^i) \in \theta^x$. \square

LEMMA 3.8.7 *Let $\theta \in \text{Con}(K)$ be such that θ collapses a quotient of $M_3(p_1, p_2)$ for some $p_1 = x_1/y_1, p_2 = x_2/y_2 \in \text{Ip}(C)$ with $\alpha(p_1) = \alpha(p_2)$. Then θ collapses p_1 and p_2 .*

PROOF. We have $(a(p_1, p_2), o(p_1, p_2)) \in \theta \Rightarrow (\langle x_1, y_2 \rangle, \langle y_1, y_2 \rangle) \in \theta \Rightarrow (\langle x_1, 0^C \rangle, \langle y_1, 0^C \rangle) \in \theta$ and so θ collapses p_1 . Also $(b(p_1, p_2), o(p_1, p_2)) \in \theta \Rightarrow (\langle y_1, x_2 \rangle, \langle y_1, y_2 \rangle) \in \theta \Rightarrow (\langle x_2, x_2 \rangle, \langle x_2, y_2 \rangle) \in \theta \Rightarrow (o(p_2, p_2), a(p_2, p_2)) \in \theta \Rightarrow (o(p_2, p_2), b(p_2, p_2)) \in \theta \Rightarrow (\langle x_2, x_2 \rangle, \langle y_2, x_2 \rangle) \in \theta \Rightarrow (\langle x_2, 0^C \rangle, \langle y_2, 0^C \rangle) \in \theta$ and so θ collapses p_2 . \square

LEMMA 3.8.8 *Let $\theta \in \text{Con}(K)$ be such that θ collapses $p_1 \in \text{Ip}(C)$ and $\alpha(p_1) = \alpha(p_2)$ for some $p_2 \in \text{Ip}(C), p_1 \neq p_2$. Then θ collapses $M_3(p_1, p_2)$.*

PROOF. Let $p_1 = x_1/y_1, p_2 = x_2/y_2$. Then $p_1 \in \theta \Rightarrow (\langle x_1, 0^C \rangle, \langle y_1, 0^C \rangle) \in \theta \Rightarrow (\langle x_1, y_2 \rangle, \langle y_1, y_2 \rangle) \in \theta \Rightarrow (a(p_1, p_2), o(p_1, p_2)) \in \theta \Rightarrow \theta$ collapses $M_3(p_1, p_2)$. \square

Now define a map $\Psi : L \rightarrow \text{Com}(K)$ by $\Psi(x) = \theta^x$ for all $x \in L$ (*)

We show that Ψ is the isomorphism between L and the lattice of complete congruences of the complete lattice K .

LEMMA 3.8.9 *The map Ψ defined in (\star) above is one-one and order preserving.*

PROOF. Let $v, w \in L$ with $v \leq w$. Then $\Phi^v \leq \Phi^w$ and so $\theta^v|_{C(\alpha)} \leq \theta^w|_{C(\alpha)}$. Then $\theta^v \leq \theta^w$ follows from the definition of these congruences extended to K . Thus Ψ is order-preserving. Let $v, w \in L \setminus \{0\}$ and suppose that $\theta^v = \theta^w$. Then by Lemma 3.8.4 there is a prime interval $p = x/y$ in C such that $\alpha(p) = v$. Thus $(x, y) \in \Phi^v \Rightarrow ((x, 0^C), (y, 0^C)) \in \theta^v \Rightarrow ((x, 0^C), (y, 0^C)) \in \theta^w \Rightarrow (x, y) \in \Phi^w \Rightarrow v = \alpha(p) \leq w$. Similarly $w \leq v$ and hence $w = v$. We have $\theta^0 = \Delta_K$ and since $\theta^x \neq \Delta_K$ for all $x \in L \setminus \{0\}$ it follows that Ψ is one-to-one. \square

In the following lemma for $\langle a, b \rangle \in C \times C$ we let $\langle a, b \rangle \times x$ denote $(\langle a, x \rangle, \langle b, x \rangle)$ in $C^2 \times C^2$.

LEMMA 3.8.10 *Let p be a prime interval of K . Then there exists a prime interval \bar{p} of C such that $\text{con}_K^c(p) = \text{con}_K^c(\bar{p})$.*

PROOF. Let p be a prime interval of K . Then p is in one of the forms (a)-(f) listed below where q_1, q_2 are prime intervals of C and $x \in C$. (\bar{p} is given on the right hand side):

- | | | | |
|----------------------------------|---------------|------------------------|---------------------------------|
| (a) $[o(q_1, q_2), m(q_1, q_2)]$ | \rightarrow | q_1 | for $\alpha(q_1) = \alpha(q_2)$ |
| (b) $[m(q_1, q_2), i(q_1, q_2)]$ | \rightarrow | q_1 | for $\alpha(q_1) = \alpha(q_2)$ |
| (c) $q_1 \times \{x\}$ | \rightarrow | q_1 | |
| (d) $\{x\} \times q_1$ | \rightarrow | q_1 | |
| (e) $[m^\delta, 1^\delta]$ | \rightarrow | $[0^\delta, j^\delta]$ | for $\delta < \chi$ |
| (f) $[0^\delta, m^\delta]$ | \rightarrow | $[0^\delta, j^\delta]$ | for $\delta < \chi$. |

We show that for each interval p on the left hand side of the above list, $\text{con}_K^c(p) = \text{con}_K^c(\bar{p})$ where \bar{p} is the corresponding interval on the right hand side.

This result holds for the intervals in (a) and (b) by Lemmas 3.8.7 and 3.8.8.

Let $q_1 = x_1/y_1$ in (c) and (d). Then $\langle x_1, x \rangle / \langle y_1, x \rangle \searrow \langle x_1, 0^C \rangle / \langle y_1, 0^C \rangle$ and so the two intervals in (c) are thus projective.

Let $\theta \in \text{Con}(K)$. Then by Lemmas 3.8.7 and 3.8.8, $q_1 \in \theta \Leftrightarrow M_3(q_1, q_1)$ is collapsed by $\theta \Leftrightarrow (a(q_1, q_2), i(q_1, q_2)) \in \theta \Leftrightarrow (\langle x_1, y_1 \rangle, \langle x_1, x_1 \rangle) \in \theta \Leftrightarrow \{x\} \times q_1 \in \theta$, proving the result for the intervals in (d).

For (e) we have $1^\delta / m^\delta \searrow j^\delta / 0^\delta$.

For (f) $(0^\delta, m^\delta) \in \text{con}(0^\delta, m^\delta) \Rightarrow (0^\delta \vee \langle 0^\delta, 0^0 \rangle, m^\delta \vee \langle 0^\delta, 0^0 \rangle) \in \text{con}(0^\delta, m^\delta) \Rightarrow (\langle 0^\delta, 0^0 \rangle, \langle 1^\delta, 0^0 \rangle) \in \text{con}(0^\delta, m^\delta) \Rightarrow (\langle 0^\delta, 0^0 \rangle \wedge j^\delta, \langle 1^\delta, 0^0 \rangle \wedge j^\delta) \in \text{con}(0^\delta, m^\delta) \Rightarrow (0^\delta, j^\delta) \in \text{con}(0^\delta, m^\delta)$. Thus $\text{con}_K(0^\delta, j^\delta) \leq \text{con}_K(0^\delta, m^\delta)$ and hence $\text{con}_K^c(0^\delta, j^\delta) \leq \text{con}_K^c(0^\delta, m^\delta)$. For the reverse inclusion we have $(0^\delta, j^\delta) \in \text{con}_K^c(0^\delta, j^\delta) \Rightarrow (0^\delta \vee m^\delta, j^\delta \vee m^\delta) \in \text{con}_K^c(0^\delta, j^\delta) \Rightarrow (m^\delta, 1^\delta) \in \text{con}_K^c(0^\delta, j^\delta)$. Let $u \in [j^\delta, 1^\delta]$. Then $(m^\delta \wedge u, 1^\delta \wedge u) \in \text{con}_K^c(0^\delta, j^\delta) \Rightarrow (0^\delta, u) \in \text{con}_K^c(0^\delta, j^\delta)$. Thus, by completeness of $\text{con}_K^c(0^\delta, j^\delta)$, $(0^\delta, 1^\delta) \in \text{con}_K^c(0^\delta, j^\delta)$. Consequently $\text{con}_K^c(0^\delta, m^\delta) \subseteq \text{con}_K^c(0^\delta, 1^\delta) \subseteq \text{con}_K^c(0^\delta, j^\delta)$, proving $\text{con}_K^c(0^\delta, m^\delta) = \text{con}_K^c(0^\delta, j^\delta)$. \square

It follows from Lemma 3.8.10 that in order to investigate the congruences $\text{con}^c(p)$ of K it is sufficient to consider prime intervals p of C .

The following lemma is a consequence of Lemmas 3.8.7 and 3.8.8.

LEMMA 3.8.11 *Let p_1, p_2 be prime intervals of C with $\alpha(p_1) = \alpha(p_2)$. Then $\text{con}_K^c(p_1) = \text{con}_K^c(p_2)$.*

LEMMA 3.8.12 *Let $X \subseteq L - \{0\}, X \neq \emptyset$. For each $x \in X \cup \{\vee X\}$ choose a prime interval p_x of $C \subseteq K$ such that $\alpha(p_x) = x$. Then*

$$\text{con}_K^c(p_{\vee X}) = \vee \{ \text{con}_K^c(p_x) : x \in X \}.$$

PROOF. Let $X = X^\delta$ for some $\delta < \chi$. By Lemma 3.8.11 the complete congruences are not affected by which prime intervals of a given colour we choose. So let $p_{\vee X} = [0^\delta, j^\delta]$ and $p_x = [(0, x^\delta), u_\gamma^\delta]$ for $x = x_\gamma^\delta$ where

$$u_\gamma^\delta = \begin{cases} (0, x_{\gamma+1}^\delta) & \text{if } \gamma + 1 < \zeta_\delta \\ (1, x_0^\delta) & \text{if } \gamma + 1 = \zeta_\delta. \end{cases}$$

Consider the well-ordered chain Q :

$$j^\delta = (0, x_0^\delta) < (0, x_1^\delta) < \cdots < (i, x_0^\delta) < (i, x_1^\delta) < \cdots < (i, x_\gamma^\delta) < \cdots$$

where $i < \omega$ and $\gamma < \zeta_\delta$.

Then each prime interval of this chain is of colour x for some $x \in X$ and so $\vee \{ \text{con}_K^c(p_x) : x \in X \}$ collapses all prime intervals of Q . Now $Q \cup \{1^\delta\}$ is a complete sublattice of K and 1^δ is a limit of Q . Thus $\vee \{ \text{con}_K^c(p_x) : x \in X \} = \text{con}_K^c(j^\delta, 1^\delta)$. We have

$$\begin{aligned} (j^\delta, 1^\delta) \in \text{con}_K^c(j^\delta, 1^\delta) &\Rightarrow (j^\delta \wedge m^\delta, 1^\delta \wedge m^\delta) \in \text{con}_K^c(j^\delta, 1^\delta) \\ &\Rightarrow (0^\delta, m^\delta) \in \text{con}_K^c(j^\delta, 1^\delta) \\ &\Rightarrow (0^\delta \vee \langle 0^\delta, 0^0 \rangle, m^\delta \vee \langle 0^\delta, 0^0 \rangle) \in \text{con}_K^c(j^\delta, 1^\delta) \\ &\Rightarrow (\langle 0^\delta, 0^0 \rangle, \langle 1^\delta, 0^0 \rangle) \in \text{con}_K^c(j^\delta, 1^\delta) \\ &\Rightarrow (\langle 0^\delta, 0^0 \rangle \vee j^\delta, \langle 1^\delta, 0^0 \rangle \vee j^\delta) \in \text{con}_K^c(j^\delta, 1^\delta) \\ &\Rightarrow (0^\delta, j^\delta) \in \text{con}_K^c(j^\delta, 1^\delta). \end{aligned}$$

Thus $\text{con}_K^c(0^\delta, j^\delta) \subseteq \text{con}_K^c(j^\delta, 1^\delta)$. We also have

$$\begin{aligned} (0^\delta, j^\delta) \in \text{con}_K^c(0^\delta, j^\delta) &\Rightarrow (m^\delta, 0^\delta) \in \text{con}_K^c(0^\delta, j^\delta) && \text{(Lemma 3.8.10)} \\ &\Rightarrow (0^\delta \vee m^\delta, j^\delta \vee 0^\delta) \in \text{con}_K^c(0^\delta, j^\delta) \\ &\Rightarrow (m^\delta, j^\delta) \in \text{con}_K^c(0^\delta, j^\delta) \\ &\Rightarrow (m^\delta \vee j^\delta, j^\delta \vee j^\delta) \in \text{con}_K^c(0^\delta, j^\delta) \\ &\Rightarrow (1^\delta, j^\delta) \in \text{con}_K^c(0^\delta, j^\delta). \end{aligned}$$

Hence $\text{con}_K^c(1^\delta, j^\delta) \subseteq \text{con}_K^c(0^\delta, j^\delta)$ and so $\bigvee \{\text{con}_K^c(p_x) : x \in X\} = \text{con}_K^c(j^\delta, 1^\delta) = \text{con}_K^c(0^\delta, j^\delta) = \text{con}_K^c(p_{\bigvee X})$. \square

LEMMA 3.8.13 *Suppose $\alpha(p_1) \leq \alpha(p_2)$ in K . Then $\text{con}_K^c(p_1) \leq \text{con}_K^c(p_2)$.*

PROOF. Suppose $\alpha(p_1) = a, \alpha(p_2) = b$. Let $X = \{a, b\}$ in Lemma 3.8.12. Then $\text{con}_K^c(p_b) = \text{con}_K^c(p_a) \vee \text{con}_K^c(p_b)$. Thus by Lemma 3.8.11 $\text{con}_K^c(p_2) = \text{con}_K^c(p_1) \vee \text{con}_K^c(p_2)$. \square

LEMMA 3.8.14 *Let p be a prime interval in $C \leq K$. Then $\text{con}_K^c(p) = \theta^{\alpha(p)}$.*

PROOF. We have $p \in \theta^{\alpha(p)}$ hence $\text{con}_K^c(p) \subseteq \theta^{\alpha(p)}$. For the reverse inclusion let u/v be a quotient of C with $(u, v) \in \Phi^{\alpha(p)}$. Then for all $q \in Ip([v, u])$ we have $\alpha(q) \leq \alpha(p)$. Hence by Lemma 3.8.13, $q \in \text{con}_K^c(p)$. Thus $(u, v) \in \text{con}_K^c(p)$. So $\text{con}_K^c(p)$ defines the relation $\Phi^{\alpha(p)}$ on C which was used to define $\theta^{\alpha(p)}$. The equality follows from this. \square

LEMMA 3.8.15 *All complete congruences $\neq \Delta_K$ of K are of the form θ^x for some $x \in L \setminus \{0\}$.*

PROOF. Let Φ be a complete congruence of K . Since K is strongly atomic (every chain in K is well-ordered), we have $\text{pi}(\Phi) = \Phi$ (Lemma 3.8.2). Let $X = \{\alpha(p) : p \in Ip(C) : \text{con}(p) \leq \Phi\}$. Then

$$\begin{aligned} \Phi &= \bigvee \{\text{con}_K^c(p) : p \in Ip(K) : \text{con}(p) \leq \Phi\} \\ &= \bigvee \{\text{con}_K^c(p) : p \in Ip(C) : \text{con}(p) \leq \Phi\} \quad (\text{Lemma 3.8.10}) \\ &= \bigvee \{\theta^x : x \in X\} \quad (\text{Lemma 3.8.14}) \\ &= \theta^{\bigvee X} \quad (\text{Lemmas 3.8.12 and 3.8.14}). \end{aligned} \quad \square$$

THEOREM 3.8.16 *The map $\Psi : L \rightarrow \text{Con}^c(K)$ defined in (\star) is an isomorphism between the complete lattice L and the lattice of complete congruences of the complete lattice K .*

PROOF. By Lemma 3.8.9 Ψ is one-to-one, by Lemma 3.8.12 it preserves joins. By Lemma 3.8.15 and the fact that $\Delta_K = \theta^0$, Ψ is surjective. \square

3.9 Lattices whose congruence lattices are Boolean algebras

In this section we present Crawley's characterization ([Cra60]) of lattices whose congruence lattices are Boolean algebras. These lattices are characterized in terms of their intrinsic

structure as opposed to the following two earlier characterizations:

(1) T. Tanaka ([Tan52]) : For a lattice L , $\text{Con}(L)$ is a Boolean algebra if and only if L is a discrete subdirect product of simple lattices.

(2) G. Gratzner and E.T. Schmidt ([Gra58]) : The congruence lattice of a lattice L is a Boolean algebra if and only if L has the projectivity property and the congruences of L are separable (where a congruence θ of a lattice L is separable if $\forall a, b \in L, a < b$, there is a sequence $a = e_0 < e_1 < \dots < e_n = b$ such that for all $0 \leq i < n$ either $(e_i, e_{i+1}) \in \theta$ or for all proper subquotients u/v of e_{i+1}/e_i , $(u, v) \notin \theta$).

A proper quotient a/b of a lattice L is said to be *minimal* if for every proper quotient c/d of L such that $c/d \approx_w a/b$ there is a finite sequence

$$b = e_0 \leq e_1 \leq \dots \leq e_n = a$$

such that $e_i/e_{i-1} \approx_w c/d$ for every $i \in \{1, \dots, n\}$.

THEOREM 3.9.1 (Characterization) [Cra60] *Let L be a lattice. Then $\text{Con}(L)$ is a Boolean algebra if and only if for every proper quotient a/b of L there is a finite chain $b = e_0 \leq e_1 \leq \dots \leq e_n = a$ such that each e_i/e_{i-1} ($i \in \{1, \dots, n\}$) is minimal.*

PROOF. We first show that for any proper quotient a/b of L , a/b is minimal if and only if $\text{con}(a, b)$ is an atom of $\text{Con}(L)$. Let a/b be a proper quotient of L such that $\text{con}(a, b)$ is an atom of $\text{Con}(L)$ and suppose c/d is a proper quotient of L weakly projective into a/b . Then $(c, d) \in \text{con}(a, b)$ and so $\text{con}(c, d) \subseteq \text{con}(a, b)$. But, since $\text{con}(a, b)$ is an atom, we must have $\text{con}(a, b) = \text{con}(c, d)$. Hence by Corollary 2.1.3 there is a finite sequence $b = e_0 \leq e_1 \leq \dots \leq e_n = a$ such that $e_{i-1}/e_i \approx_w c/d$ for all $i \in \{1, \dots, n\}$. Thus a/b is a minimal quotient in L . For the reverse implication suppose that a/b is a minimal quotient in L and let c/d be a proper quotient of L with $(c, d) \in \text{con}(a, b)$. Then there exists a finite sequence $d = e_0 \leq e_1 \leq \dots \leq e_n = c$ such that $e_{i-1}/e_i \approx_w a/b$ for all $i \in \{1, \dots, n\}$. Since a/b is minimal and $d/e_1 \approx_w a/b$ there is a finite sequence $a = f_0 \leq f_1 \leq \dots \leq f_k = b$ such that $f_{i-1}/f_i \approx_w d/e_1$ for all $i \in \{1, \dots, k\}$. But then $(f_{i-1}, f_i) \in \text{con}(d, e_1) \subseteq \text{con}(c, d)$ for all $i \in \{1, \dots, k\}$. By transitivity $(a, b) \in \text{con}(c, d)$ and hence $\text{con}(a, b) = \text{con}(c, d)$.

Now suppose $\text{Con}(L)$ is a Boolean algebra. Then by Lemma 3.1.5 $\text{Con}(L)$ is atomistic, hence every non-zero element of $\text{Con}(L)$ is a join of atoms. Let a/b be a proper quotient of L . Then $\text{con}(a, b)$ is a join of atoms $\text{con}(s_j, t_j)$ where s_j/t_j is a minimal quotient of L . Thus there is a finite sequence $b = e_0 \leq e_1 \leq \dots \leq e_n = a$ such that for each $i \in \{1, \dots, n\}$ $e_{i-1}/e_i \approx_w s_i/t_i$ for some minimal quotient s_i/t_i of L . Thus $\text{con}(e_{i-1}, e_i) \subseteq \text{con}(s_i, t_i)$. Hence $\text{con}(e_{i-1}, e_i)$ is an atom of $\text{Con}(L)$ (since $\text{con}(s_i, t_i)$ is) and so e_i/e_{i-1} is minimal.

For the converse let a/b be a proper quotient of L . Then there is a finite sequence $b = e_0 \leq e_1 \leq \dots \leq e_n = a$ in L such that each e_i/e_{i-1} is a minimal quotient. Then $\text{con}(e_i, e_{i-1})$

is an atom of $\text{Con}(L)$ for every $i \in \{1, \dots, n\}$. We have $\text{con}(a, b) = \bigvee_{i=1}^n \text{con}(e_i, e_{i-1})$ and since every member of $\text{Con}(L)$ is a join of principal congruences, every element of $\text{Con}(L)$ is a join of atoms. Hence by Lemma 2.2.5 $\text{Con}(L)$ is a Boolean algebra. \square

3.10 Lattices whose congruence lattices are Stone lattices

In [Jan68] M.F. Janowitz proved that the congruence lattice of a complete relatively complemented lattice is a Stone lattice and posed the problem of characterizing those lattice L for which $\text{Con}(L)$ is a Stone lattice. Such a characterization, presented in Theorem 3.10.5 below was provided by Iqbalunnisa in [Iqb71].

We begin by defining a Stone lattice and proving some preliminary results concerning pseudo-complements.

Let L be a lattice with a least element 0 . Let $a \in L$. Then $a^* \in L$ is the *pseudo-complement* of a in L if the following conditions hold:

- (a) $a \wedge a^* = 0$
- (b) $a \wedge x = 0 \Rightarrow x \leq a^*$ for all $x \in L$.

A lattice L is *pseudo-complemented* if every element has a pseudo-complement in L .

A pseudo-complemented bounded distributive lattice is a *Stone lattice* if for all $a \in L$, $a^* \vee a^{**} = 0$.

An element a of a bounded lattice is *simple* if a has a pseudo-complement a^* in L and $a \vee a^* = 1$.

LEMMA 3.10.1 *Every complete infinitely join-distributive lattice is a pseudo-complemented distributive lattice.*

PROOF. Let L be a complete infinitely join-distributive lattice with $a \in L$. Define $a^* = \bigvee \{x : x \in L, a \wedge x = 0\}$. Then $a \wedge a^* = \bigvee \{a \wedge x : x \in L, a \wedge x = 0\} = \bigvee 0 = 0$. Suppose $a \wedge x = 0$. Then, by definition, $x \leq a^*$ and so a^* is the pseudo-complement of a . \square

COROLLARY 3.10.2 *Every distributive algebraic lattice is pseudo-complemented, and hence for any lattice L , $\text{Con}(L)$ is pseudo-complemented.*

PROOF. Follows immediately from Lemma 2.6.6 and Lemma 3.10.1. \square

We say that a quotient c/d of a lattice is a *translate* of a quotient a/b if $c/d \approx_w a/b$.

LEMMA 3.10.3 [Iqb66] *Let θ be a congruence on a lattice L . Then the pseudo-complement of θ in $\text{Con}(L)$ is the congruence Ψ defined by : $(a, b) \in \Psi \Leftrightarrow$ no non-trivial translate of $(a \vee b)/(a \wedge b)$ is collapsed by θ .*

PROOF. We first show that Ψ is a congruence relation by showing that it satisfies the Grätzer-Schmidt Criteria of Lemma 2.1.1.

(i) Let $a \in L$ and suppose $c/d \nearrow_w a/a$. Then we have $c \leq a$ and $c \wedge a = d$. Hence $c = d$. Similarly $c/d \searrow_w a/a$ implies $c = d$. Thus $c/d \approx_w a/a \Rightarrow c = d$ and so $(a, a) \in \Psi$.

(ii) For $a, b \in L$, $(a, b) \in \Psi \Leftrightarrow (a \vee b, a \wedge b) \in \Psi$ follows directly from the definition of Ψ .

(iii) Suppose $a \leq b \leq c$ in L and $(a, b), (b, c) \in \Psi$. If $(p, q) \in \theta$ and $p/q \approx_w c/a$, then $(p, q) \in \text{con}(a, c) = \text{con}(a, b) \vee \text{con}(b, c)$. By Theorem 2.1.2 there is a finite sequence $q = e_0 \leq e_1 \leq \dots \leq e_n = p$ such that for each $i \in \{1, \dots, n\}$ either $e_{i-1}/e_i \approx_w b/a$ or $e_{i-1}/e_i \approx_w c/b$. Since $(p, q) \in \theta$ we have $(e_{i-1}, e_i) \in \theta$ for all $i \in \{1, \dots, n\}$, hence $e_i = e_{i-1}$ for all $i \in \{1, \dots, n\}$. Thus $p = q$ and $(a, c) \in \theta$.

(iv) Suppose $a \leq b$ in L and $(a, b) \in \Psi, c \in L$. Then $(b \vee c)/(a \vee c) \searrow_w b/a$. Hence if $c/d \approx_w (b \vee c)/(a \vee c)$ then $c/d \approx_w b/a$. Thus $(a \vee c, b \vee c) \in \Psi$. Similarly $(a \wedge c, b \wedge c) \in \Psi$. It follows from (i)-(iv) that Ψ is a congruence relation on L .

Suppose $(a, b) \in \theta \cap \Psi$. Then $(a \vee b, a \wedge b) \in \theta$ and $(a \vee b)/(a \wedge b) \approx_w (a \vee b)/(a \wedge b)$. Thus $a = b$ and $\theta \cap \Psi = \Delta$. Suppose $\theta \cap \Phi = \Delta$ for some $\Phi \in \text{Con}(L)$. Let $(a, b) \in \Phi$ and suppose c/d is a quotient in L with $(c, d) \in \theta$ and $c/d \approx_w (a \vee b)/(a \wedge b)$. Then by Theorem 2.1.2 $(c, d) \in \Phi$. Thus $(c, d) \in \Phi \cap \theta$ and so $c = d$. Consequently $(a, b) \in \Psi$ and so $\Phi \subseteq \Psi$. Hence Ψ is the pseudo-complement of θ in L . \square

LEMMA 3.10.4 [Iqb66] *A congruence on a lattice L is simple if and only if for any quotient a/b of L there is a finite sequence*

$$b = e_0 \leq e_1 \leq \dots \leq e_n = a$$

such that for each $i \in \{1, \dots, n\}$ either $(e_{i-1}, e_i) \in \theta$ or no non-trivial translate of e_{i-1}/e_i is collapsed by θ .

PROOF. For the forward implication let θ be a simple congruence on L . Then $\theta \vee \theta^* = \nabla$. Let a/b be a quotient of L . Then there is a finite sequence $b = e_0 \leq e_1 \leq \dots \leq e_n = a$ such that for each $i \in \{1, \dots, n\}$ either $(e_{i-1}, e_i) \in \theta$ or $(e_{i-1}, e_i) \in \theta^*$. The result now follows from Lemma 3.10.3.

For the converse let $a, b \in L$. Then by assumption there is a finite sequence $a \wedge b = e_0 \leq e_1 \leq \dots \leq e_n = a \vee b$ such that for each $i \in \{1, \dots, n\}$ either $(e_{i-1}, e_i) \in \theta$ or $(e_{i-1}, e_i) \in \theta^*$. (Lemma 3.10.3). Thus $(a, b) \in \theta \vee \theta^*$ and hence $\theta \vee \theta^* = \nabla$. \square

We are now ready to prove Iqbalunnisa's characterization:

THEOREM 3.10.5 [Iqb71] *Let L be a complete lattice. Then $\text{Con}(L)$ is a Stone lattice if and only if for any $\theta \in \text{Con}(L)$, there exists a finite sequence*

$$0 = e_0 \leq e_1 \leq \dots \leq e_n = 1$$

such that for each $i \in \{1, \dots, n\}$ either no non-trivial translate of e_i/e_{i-1} is collapsed by θ or every translate of e_i/e_{i-1} has a non-trivial translate collapsed by θ .

PROOF. To prove the forward implication let $\theta \in \text{Con}(L)$. Then θ^* is simple. By Lemma 3.10.4 there is a finite sequence $0 = e_0 \leq e_1 \leq \dots \leq e_n = 1$ such that for each $i \in \{1, \dots, n\}$ either $(e_{i-1}, e_i) \in \theta^*$ or no non-trivial translate of e_{i-1}/e_i is collapsed by θ^* . The result then follows from Lemma 3.10.3.

For the converse let $a \leq b$ in L and suppose $\theta \in \text{Con}(L)$. We will show that $(a, b) \in \theta^* \vee \theta^{**}$. By assumption and Lemma 3.10.3 there exists a finite sequence $0 = e_0 \leq e_1 \leq \dots \leq e_n = 1$ such that for each $i \in \{1, \dots, n\}$ either $(e_{i-1}, e_i) \in \theta^*$ or no non-trivial translate of e_i/e_{i-1} is collapsed by θ^* . For each $i \in \{0, \dots, n\}$ let $c_i = (e_i \wedge b) \vee a$. Then for $i \in \{1, \dots, n\}$ $(e_{i-1}, e_i) \in \theta^* \Rightarrow (c_{i-1}, c_i) \in \theta^*$ and $c_i/c_{i-1} \searrow_w (e_i \wedge b)/(e_{i-1} \wedge b) \nearrow_w e_i/e_{i-1}$. Thus c_i/c_{i-1} is a translate of e_i/e_{i-1} . So we have a sequence $a = c_0 \leq c_1 \leq \dots \leq c_n = b$ such that for each $i \in \{1, \dots, n\}$ either $(c_{i-1}, c_i) \in \theta^*$ or no non-trivial translate of c_i/c_{i-1} is collapsed by θ^* . By Lemma 3.10.4 θ^* is simple and hence $\text{Con}(L)$ is a Stone lattice. \square

Chapter 4

2-congruence extendibility

This chapter is motivated by Chapter 6 and lays much of the groundwork for results of that chapter where the amalgamation class of a B -lattice variety is characterized using 2-congruence extendibility. In Section 4.1 we present a characterization of 2-congruence extendible distributive lattices in terms of smallest distributive congruences. Section 4.2 considers 2-congruence extendibility in \mathcal{S} -varieties in preparation for Chapter 6 (B -lattices are \mathcal{S} -lattices). Amongst the results of this section we show that in a \mathcal{S} -variety \mathcal{V} , $\text{Amal}(\mathcal{V})$ is contained in the class of 2-congruence extendible members of \mathcal{V} (see Proposition 4.2.4 for a stronger version of this result), exemplifying the strong relationship between amalgamation and congruences. An interesting problem (not explored here) would be to investigate whether the class of 2-congruence extendible members of various lattice varieties are elementary. Most of the results of this chapter appear in [Lai96].

Recall that for a positive integer n , we denote the chain with n elements by \mathbf{n} . Let X be a sublattice of a lattice L . We say X is an n -sublattice of L if $X \cong \mathbf{n}$ and there is a retraction of L onto X . For example in the pentagon in Figure 2.4, Page 33 $\{e, f\}$ is a 2-sublattice but $\{a, c\}$ is not.

4.1 Distributive congruences and characterization of 2-congruence extendibility

Recall that a congruence Ψ on a lattice L is a distributive congruence if L/Ψ is a distributive lattice and that an algebra A in a variety \mathcal{V} is 2-congruence extendible in \mathcal{V} if whenever $C \in \mathcal{V}$ and $A \leq C$ then every 2-congruence on A can be extended to a 2-congruence on C .

LEMMA 4.1.1 *Let L be an arbitrary lattice. Then $\Psi \in \text{Con}(L)$ is the smallest distributive congruence on L if and only if $\Psi = \bigcap_I \theta_i$ where $\{\theta_i : i \in I\}$ is the set of all*

2-congruences on L.

PROOF. Since $\Psi \subseteq \theta_i \forall i \in I$ we have $\Psi \subseteq \bigcap_I \theta_i$. For the reverse inclusion let $a, b \in L$ be such that $(a, b) \notin \Psi$. Let $\Phi = \theta_{(a/\Psi, b/\Psi)}$. Then Φ is a subdirectly irreducible, distributive congruence on L/Ψ and hence is a 2-congruence. Let θ be the 2-congruence on L such that $\Psi \subseteq \theta$ and $\Phi = \theta/\Psi$. Then $(a, b) \notin \theta$ and hence $(a, b) \notin \bigcap_I \theta_i$. Thus $\bigcap_I \theta_i \subseteq \Psi$ proving $\Psi = \bigcap_I \theta_i$. \square

For the next lemma recall that a quotient a/c of a lattice L is an N -quotient of L if, for some $b \in L$, the set $\{a, c, b\}$ generates a sublattice of L isomorphic to the pentagon N in which a/c is a critical quotient.

LEMMA 4.1.2 *Let L be a lattice which does not have the diamond (M_3) as a sublattice. Then $\Psi \in \text{Con}(L)$ is the smallest distributive congruence on L if and only if $\Psi = \Sigma_S \text{con}(u_s, v_s)$ where $\{u_s/v_s : s \in S\}$ is the set of all N -quotients of L .*

PROOF. Since Ψ is distributive it collapses all N -quotients in L . Thus $\text{con}(u_s, v_s) \subseteq \Psi \forall s \in S$ and so $\Sigma_S \text{con}(u_s, v_s) \subseteq \Psi$. Furthermore, since L does not contain the diamond $\Sigma_S \text{con}(u_s, v_s)$ is a distributive congruence and therefore $\Psi \subseteq \Sigma_S \text{con}(u_s, v_s)$ proving the equality. \square

THEOREM 4.1.3 (Characterization of 2-congruence extendibility for arbitrary lattices) *Let A, C be arbitrary lattices with $A \leq C$. Let Ψ_A and Ψ_C be the smallest distributive congruences on A and C respectively. Then every 2-congruence on A can be extended to a 2-congruence on C if and only if $\Psi_A = \Psi_C|_A$.*

PROOF. Let X_C be the set of all 2-congruences on C and let $X_A = \{\Phi \in X_C : \Phi|_A \text{ is a 2-congruence on } A\}$. Assume that every 2-congruence on A can be extended to a 2-congruence on C . Let $\theta = \bigcap X_A$. Then by Lemma 4.1.1 $\Psi_C \subseteq \theta$ and so θ is a distributive congruence on A . We also have $\theta|_A = \Psi_A$. Thus $\Psi_C|_A \subseteq \theta|_A = \Psi_A$. Now $\Psi_C|_A$ is a distributive congruence on A (since $A/\Psi_C|_A$ embeds in C/Ψ_C) and so $\Psi_A \subseteq \Psi_C|_A$ whence $\Psi_A = \Psi_C|_A$. For the converse, assume that $\Psi_A = \Psi_C|_A$. Then the map $f : A/\Psi_A \rightarrow C/\Psi_C$ given by $f(a/\Psi_A) = a/\Psi_C$ is a well-defined embedding. Let θ be a 2-congruence on A . Then $\Psi|_A \subseteq \theta$ and so by the second isomorphism theorem θ/Ψ_A is a 2-congruence on A/Ψ_A . Let Ω be the congruence on $f[A/\Psi_A]$ corresponding to θ . i.e. the congruence generated by $\{(a/\Psi_C, b/\Psi_C) : (a, b) \in \theta\}$. Now $f[A/\Psi_A]$ and C/Ψ_C are distributive lattices with $f[A/\Psi_A] \leq C/\Psi_C$ and Ω is a 2-congruence on $f[A/\Psi_A]$. Thus by Corollary 1.4.12 there is a 2-congruence Φ on C with $\Psi_C \subseteq \Phi$ such that Φ/Ψ_C is a 2-congruence on C/Ψ_C extending Ω . But then Φ is a 2-congruence on C extending θ , since

$$\begin{aligned}
(a, b) \in \Phi|_A &\Leftrightarrow (a, b) \in \Phi \text{ and } (a, b) \in A \\
&\Leftrightarrow (a/\Psi_C, b/\Psi_C) \in \Phi/\Psi_C \text{ and } (a, b) \in A \\
&\Leftrightarrow (a/\Psi_A, b/\Psi_A) \in \Omega \\
&\Leftrightarrow (a, b) \in \theta.
\end{aligned}$$

□

COROLLARY 4.1.4 *Let \mathcal{V} be a variety which does not contain the diamond. Let $C \in \mathcal{V}$ and $A \leq C$. Suppose that every 2-congruence on A can be extended to a 2-congruence on C . Then for any distributive congruence θ on A and any N -quotient u/v of C , we have $\text{con}(u, v)|_A \subseteq \theta$.*

PROOF. Let Ψ_A, Ψ_C be the smallest distributive congruences on A and C respectively and let $\{u_s/v_s : s \in S\}$ be the set of all N -quotients in C . By Theorem 4.1.3 $\Psi_C|_A = \Psi_A$ and by Lemma 4.1.2 $\Psi_C = \Sigma_S \text{con}(u_s, v_s)$. Hence $\Psi_A = (\Sigma_S \text{con}(u_s, v_s))|_A = \Sigma_S \text{con}(u_s, v_s)|_A$ and the result follows from this equality. □

LEMMA 4.1.5 *Let A and C be arbitrary lattices and let $f : A \hookrightarrow C$ be an embedding. Suppose that every 2-congruence on $f[A]$ can be extended to a 2-congruence on C . Let $g : A \rightarrow X$ be a surjective homomorphism with X a finite Boolean lattice. Then there is a surjective homomorphism $h : C \rightarrow X$ with $h \circ f = g$.*

PROOF. We have $X \cong 2^n$ for some $n \in \omega$. For each $i \in \{1, \dots, n\}$ let $\pi_i : X \rightarrow 2$ be the i th projection and let $\theta_i = \ker(\pi_i \circ g)$. Then θ_i is a 2-congruence on A . Let $f(\theta_i)$ denote the congruence on C corresponding to θ_i under f . Then by assumption $f(\theta_i)$ can be extended to a 2-congruence $\bar{\theta}_i$ on C . Let $h_i : C \rightarrow 2$ be the canonical surjection with $\ker(h_i) = \bar{\theta}_i$. Then $h_i \circ f = \pi_i \circ g$. Define $h : C \rightarrow 2^n \cong X$ by $h(c)_i = h_i(c)$ for all $c \in C, i \in \{1, \dots, n\}$. Let $x \in 2^n$. Then there is an $a \in A$ such that $g(a) = x$ and $h(f(a))_i = h_i(f(a)) = \pi_i(g(a)) = x_i$, where x_i is the i th co-ordinate of x . Thus $h(a) = x$ proving that h is surjective and $h \circ f = g$ follows from $h_i \circ f = \pi_i \circ g$. □

4.2 2-congruence extendibility in \mathcal{S} -varieties

We now direct our study of 2-congruence extendibility specifically to \mathcal{S} -varieties. Most of the results of this section have their application in Chapter 6 in showing that for a variety \mathcal{B} generated by a B -lattice B , $\text{Amal}(\mathcal{B})$ is a Horn class. The result is based on the characterization of $\text{Amal}(\mathcal{B})$ as the intersection of the 2-congruence extendible members of \mathcal{B} , the 2-chain limited members of \mathcal{B} and subdirect powers of B . Recall that a subdirectly

irreducible lattice L is an S -lattice if L is a finite non-modular, semidistributive, lattice such that none of the lattices L_{11} or L_{12} are embeddable into L ; and that a S -variety is a finitely generated semidistributive variety \mathcal{V} such that all non-modular members of \mathcal{V}_{SI} are S -lattices.

LEMMA 4.2.1 *Let L be a S -lattice and let Φ be the smallest distributive congruence on L . Then $(a, 1) \notin \Phi$ for every $a \neq 1$ and $(b, 0) \notin \Phi$ for every $b \neq 0$.*

PROOF. By Lemma 4.1.2 $\Phi = \sum_S \text{con}(u_s, v_s)$ where $\{u_s/v_s : s \in S\}$ is the set of all N -quotients of L . Thus, if $(a, 1) \in \Phi$ then by Theorem 2.1.2 there is a sequence $a = e_0 \leq e_1 \leq \dots \leq e_n = 1$ such that for each $i \in \{1, \dots, n\}$, e_i/e_{i-1} is weakly projective into some N -quotient of L . But then by Lemma 2.5.1 e_i/e_{i-1} is an N -quotient for each $i \in \{1, \dots, n\}$. In particular $1/e_{n-1}$ is an N -quotient, and this contradiction completes the proof that $(a, 1) \notin \Phi$. The second half of the lemma is proved similarly. \square

COROLLARY 4.2.2 *Let \mathcal{V} be an S -variety. Then every non-trivial member of \mathcal{V} has 2-congruences.*

PROOF. Let $L \leq \prod L_i$ be a subdirect representation in \mathcal{V} . Then L_i is an S -lattice or is isomorphic to $\mathbf{2}$. Thus every member of \mathcal{V} has $\mathbf{2}$ or an S -lattice as an image. By Lemma 4.2.1, the smallest distributive congruence on an S -lattice K does not collapse the top and bottom of K . The statement of the corollary now follows from this. \square

Recall that for a lattice L , $N(a/c, b)$ in L means that $\{a, c, b\}$ generates a sublattice of L isomorphic to the pentagon N with critical quotient a/c .

THEOREM 4.2.3 *Let \mathcal{V} be an S -variety. Then every 2-congruence extendible member of \mathcal{V} is a subdirect product of S -lattices in \mathcal{V} (i.e. of non-modular subdirectly irreducibles in \mathcal{V}).*

PROOF. Let $A \in \mathcal{V}$ be 2-congruence extendible with $A \leq \prod_I L_i$ a subdirect representation. Let $J = \{i \in I : L_i \cong \mathbf{2}\}$ and suppose that $J \neq \emptyset$ and $j \in J$. Then $\ker(\pi_j)$ is a 2-congruence on A . Let L be an arbitrary S -lattice in \mathcal{V} with critical quotient a/c . Let $B = \prod_I B_i$ where

$$B_i = \begin{cases} L_i & i \in I \setminus J \\ L & i \in J. \end{cases}$$

Consider the embedding $f : \prod_I L_i \hookrightarrow B$ given by:

$$f(x)_i = \begin{cases} x_i & i \in I \setminus J \\ a & i \in J, x_i = 1 \\ c & i \in J, x_i = 0. \end{cases}$$

Let $\theta \in \text{Con}(B)$ be a 2-congruence extending $\ker(\pi_j)$ and let $u = (u_i)_{i \in I}, v = (v_i)_{i \in I}, z = (z_i)_{i \in I}$ where u_i/v_i is the (unique) critical quotient of B_i and $N(u_i/v_i, z_i)$. We then have $N(u/v, z)$ in B . Now $(u, v) \notin \theta$ implies that $N(u/v, z)$ cannot be collapsed by θ , contradicting the fact that θ is a 2-congruence. And $(u, v) \in \theta$ contradicts $\theta|_A = \ker(\pi_j)$. This completes our proof. \square

The following three results apply to \mathcal{S} -varieties by Corollary 4.2.2.

PROPOSITION 4.2.4 *Let \mathcal{V} be a residually small variety of lattices in which some member of \mathcal{V}_{MI} has a 2-congruence. Then all members of $\text{Amal}(\mathcal{V})$ are 2-congruence extendible.*

PROOF. Follows immediately from Proposition 1.4.7. \square

PROPOSITION 4.2.5 *Let \mathcal{V} be a residually small variety of lattices in which some member of \mathcal{V}_{MI} has a 2-congruence. Let $C \in \mathcal{V}$ and suppose that for every 2-congruence on C there exists an $A \in \text{Amal}(\mathcal{V})$ and an embedding $f : A \hookrightarrow C$ such that the restriction $\theta|_{f[A]}$ is a 2-congruence on A . Then C is 2-congruence extendible in \mathcal{V} .*

PROOF. Suppose $C \leq D \in \mathcal{V}$ and let θ be a 2-congruence on C . Let $A \in \text{Amal}(\mathcal{V})$ with $f : A \hookrightarrow C$ an embedding such that $\theta|_{f[A]}$ is a 2-congruence on A . By Proposition 1.4.7 there is a 2-congruence Ψ on D such that $\Psi|_{f[A]} = \theta|_{f[A]}$. Then $\Psi|_C = \theta$ proving that C is 2-congruence extendible in \mathcal{V} . \square

THEOREM 4.2.6 *Let \mathcal{V} be a residually small variety of lattices in which every non-trivial member of $\text{Amal}(\mathcal{V})$ has 2-congruences. Then direct products of members of $\text{Amal}(\mathcal{V})$ are 2-congruence extendible in \mathcal{V} .*

PROOF. Let $C = \prod_I A_i$ where $A_i \in \text{Amal}(\mathcal{V})$ for all $i \in I$. Let θ be a 2-congruence on C . Then there are $u, v \in C$ with $u > v$ such that $C/\theta = \{u/\theta, v/\theta\}$. Let Ψ be a Jónsson congruence on C such that $\Psi \subseteq \theta$ and let \mathcal{D} be the corresponding ultrafilter over I . Setting $X = \{i \in I : u_i > v_i\}$, we have $X \in \mathcal{D}$. For each $i \in I$ let $z_i \in A_i$ be fixed. There are two possible cases:

(i) There is a $j \in I$ such that $\{j\} \in \mathcal{D}$.

(ii) For all $j \in I, \{j\} \notin \mathcal{D}$.

For case (i) define an embedding $f : A_j \hookrightarrow A$ as follows. For all $x \in A_j$:

$$f(x)_i = \begin{cases} x & i = j \\ z_i & i \in I \setminus \{j\}. \end{cases}$$

Then $f(u_j)/\Psi = u/\Psi$ and $f(v_j)/\Psi = v/\Psi$. Hence $(f(u_j), f(v_j)) \notin \theta$ and $\theta|_{f[A_j]}$ is a 2-congruence on A_j .

For case (ii) we have $X \setminus \{j\} \in \mathcal{D}$ for all $j \in I$. Since we are assuming that C has at least one 2-congruence, (namely θ), there is a $j \in I$ such that A_j is non-trivial. By assumption there is a surjective homomorphism $g : A_j \rightarrow \mathbf{2}$. We define an embedding $f : A_j \hookrightarrow C$ as follows. For all $x \in A_j$:

$$f(x)_i = \begin{cases} x & i = j \\ u_i & i \in X \setminus \{j\} \text{ and } g(x) = 1 \\ v_i & i \in X \setminus \{j\} \text{ and } g(x) = 0 \\ z_i & i \in I \setminus \{X \setminus \{j\}\}. \end{cases}$$

Since g is a surjection there are $x, y \in A_j$ such that $g(x) = 1$ and $g(y) = 0$. Then $f(x)/\Psi = u/\Psi$ and $f(y)/\Psi = v/\Psi$. Thus $(f(x), f(y)) \notin \theta$ and so $\theta|_{f[A_j]}$ is a 2-congruence on A_j . It now follows from Proposition 4.2.5 that C is 2-congruence extendible in \mathcal{V} . \square

PROBLEM 4.2.7 *Can the above result be generalized to n -congruences?*

For a variety \mathcal{V} let $\mathbf{P}_{fin}(\mathcal{V})$ denote the class of direct products of finitely many members of \mathcal{V} .

Recall that an embedding $f : A \hookrightarrow C = \prod_I C_i$ is regular if for any distinct $i, j \in I$ we have $\ker(\pi_i)|_{f[A]} \neq \ker(\pi_j)|_{f[A]}$ where π_i is the canonical projection of C onto C_i .

LEMMA 4.2.8 *Let \mathcal{V} be a lattice variety with $A_i \in \mathcal{V}$ for all $i \in I$. Suppose $A \leq \prod_I A_i$ is a subdirect representation where $\prod_I A_i \in \mathbf{P}_{fin}(\mathcal{V})$. If every 2-congruence on A can be extended to a 2-congruence on B where $B \in \mathbf{P}_{fin}(\mathcal{V})$ and B is any regular subdirect representation of A , then every 2-congruence on A can be extended to a 2-congruence on $\prod_I A_i$.*

PROOF. The subdirect representation $A \leq \prod_I A_i$ can be restricted to a regular subdirect representation $A \leq \prod_J A_i$ for some $J \subseteq I$. For each $i \in I$ let e_i be an arbitrary element of A_i . Define an embedding $h : \prod_J A_i \rightarrow \prod_I A_i$ as follows: For all $a \in A$ let $h(a)_i = a_i$ for $i \in J$ and $h(a)_i = e_i$ for $i \in I \setminus J$. Let θ be a 2-congruence on A . Then by assumption θ can be extended to a 2-congruence θ_J on $\prod_J A_i$. By Proposition 1.3.4 $\theta_J = \prod_J \theta_j$ where $\theta_j \in \text{Con}(A_j)$ for all $j \in J$. Let $h(\theta_J)$ denote the congruence on $h[\prod_J A_i]$ corresponding to θ_J under h . Define a congruence $\Phi_I = \prod_I \Phi_i$ on $\prod_I A_i$ by $\Phi_i = \theta_i$ for $i \in J$ and $\Phi_i = \nabla_{A_i}$ for $i \in I \setminus J$. Then $\Phi_I|_{h[\prod_J A_i]} = h(\theta_J)$ and Φ_I is a 2-congruence on $\prod_I A_i$ extending θ . \square

THEOREM 4.2.9 *Let \mathcal{V} be an S -variety with $\{T_1, \dots, T_n\}$ the set of S -lattices of \mathcal{V}_{SI} . Let $B \in \mathcal{V}$ be an image of A where A is 2-congruence extendible in \mathcal{V} . Let $r : B \rightarrow T = T_1^{I_1} \times \dots \times T_n^{I_n}$ be a regular subdirect representation. Then every 2-congruence on B can be extended to a 2-congruence on T .*

PROOF. Let Ω be a congruence on A such that $A/\Omega \cong B$. Let $\{\varphi_{i_j} : i_j \in W_j\}$ be the set of all T_j -congruences on A for $j \in \{1, \dots, n\}$. Since the subdirect representation $r : A/\Omega \rightarrow T$ is regular, we may assume that for each $j \in \{1, \dots, n\}$, $I_j \subseteq W_j$ and $\varphi_{i_j} = \ker(\pi_{i_j} \circ r \circ k) \forall i_j \in I_j$ where $k : A \rightarrow A/\Omega$ is the canonical quotient map and $\pi_{i_j} : T_j^{I_j} \rightarrow T_j$ is the i_j th projection. Let $S = T_1^{W_1} \times \dots \times T_n^{W_n}$, $I = \bigcup_{j=1}^n I_j$, $W = \bigcup_{j=1}^n W_j$. Since $r : A/\Omega \hookrightarrow T$ is a subdirect representation $\bigcap_{i \in I} (\ker(\pi_i \circ r)) = \Delta_{A/\Omega}$ i.e. $\bigcap_{i \in I} \varphi_i = \Omega$. For each $j \in \{1, \dots, n\}$, let $z_j \in T_j$ be fixed. Let $h : T \hookrightarrow S$ be the embedding defined as follows: For all $x \in T$

$$h(x)_{i_j} = \begin{cases} x_{i_j} & i_j \in I \\ z_j & i_j \in W \setminus I. \end{cases}$$

Let $D = h(T)$ and $C = h \circ r(A/\Omega)$. Then $C \leq D \leq S$. For $\theta \in \text{Con}(A)$ with $\Omega \leq \theta$, let $\bar{\theta}$ denote the congruence on C corresponding to the congruence θ/Ω on A/Ω under $h \circ r$. For $a \in A$ let \bar{a} denote $h \circ r(a/\Omega)$.

We will show that for every $\theta \in [\Delta, \Omega]$ in $\text{Con}(A)$ such that θ/Ω is a 2-congruence on A/Ω , $\bar{\theta}$ can be extended to a 2-congruence on D . This then proves the statement of the theorem.

For each $i_j \in I$ let $\rho_{i_j} : A \rightarrow T_j$ be the surjective homomorphism given by $\rho_{i_j}(a) = r(a/\Omega)_{i_j}$. Then for all $i_j \in I$, $\varphi_{i_j} = \ker(\rho_{i_j})$. For $i_j \in W \setminus I$ let $\rho_{i_j} : A \rightarrow T_j$ be any surjective homomorphism with $\varphi_{i_j} = \ker(\rho_{i_j})$. Define $\rho : A \rightarrow S$ as follows: for $a \in A : \rho(a)_{i_j} = \rho_{i_j}(a) \forall i_j \in W$. Then ρ is an embedding since:

$$\begin{aligned} & \rho(a)_{i_j} = \rho(b)_{i_j} \quad \forall i_j \in W \\ \Rightarrow & (a, b) \in \varphi_{i_j} \quad \forall i_j \in W \\ \Rightarrow & a = b, \text{ since } A \text{ is a subdirect product of } T_j\text{'s (by Theorem 4.2.3)}. \end{aligned}$$

For every congruence ψ on A let $\rho(\psi)$ denote the congruence on $\rho[A]$ corresponding to ψ under ρ . Now for each $i \in I$, $\rho(\varphi_i) \in \text{Con}(\rho[A])$ can be extended to a congruence $\varphi'_i \in \text{Con}(S)$, namely $\varphi'_i = \ker(\pi_i)$. Each φ'_i is induced by a principal ultrafilter $\mathcal{F}_i (= \{\{i\}\})$ over W . Put $\Omega' = \bigcap_{i \in I} \varphi'_i$. Then $\rho(\Omega) = \Omega'|_{\rho[A]}$. Let $\mathcal{F} = \bigcap_{i \in I} \mathcal{F}_i$. Then \mathcal{F} is a filter over W which induces Ω' .

By Lemma 1.1.3 there is a Jónsson congruence $\Psi \in \text{Con}(S)$ such that $\Omega' \subseteq \Psi$ and $\Psi|_{\rho[A]} \subseteq \rho(\theta)$. Let \mathcal{D} be the corresponding Jónsson ultrafilter over W . Then $\mathcal{F} \subseteq \mathcal{D}$ and since $I \in \mathcal{F}$, we have $I \in \mathcal{D}$. Since $\bar{\theta}$ is a 2-congruence on C , θ is a 2-congruence on A . So there are $u, v \in A$ with $u > v$ such that $A/\theta = \{u/\theta, v/\theta\}$. Then $C/\bar{\theta} = \{\bar{u}/\bar{\theta}, \bar{v}/\bar{\theta}\}$. Since A is 2-congruence extendible in \mathcal{V} there is a 2-congruence Γ on S such that $\Gamma|_{\rho[A]} = \rho(\theta)$. Let $\Sigma \in \text{Con}(S)$ be a Jónsson congruence such that $\Sigma \subseteq \Gamma$. Then $\Sigma|_{\rho(\{A\})} \subseteq \Gamma|_{\rho(\{A\})} \subseteq \rho(\theta)$ and so $\Sigma = \Psi$.

We have $S/\Gamma = (\rho(u)/\Gamma, \rho(v)/\Gamma)$ and since $\Psi \subseteq \Gamma$ we have $(\rho(u), \rho(v)) \notin \Psi$. Now, for all $a \in A$ $\{i \in W : \rho(a)_i = \bar{a}_i\} \supseteq \{i \in I : \rho(a)_i = \bar{a}_i\} = \{i \in I : r(a/\Omega)_i = (h \circ r(a/\Omega))_i\} = I$ (by definition of h). Thus since $I \in \mathcal{D}$ we have $\rho(a)/\Psi = \bar{a}/\Psi$ and hence $\rho(a)/\Gamma = \bar{a}/\Gamma$. Consequently $\rho(u)/\Gamma = \bar{u}/\Gamma$, $\rho(v)/\Gamma = \bar{v}/\Gamma$ and $\Gamma|_D$ is a 2-congruence on D . Also $\Gamma|_D$ extends $\bar{\theta}$ since $(\bar{a}, \bar{b}) \in \bar{\theta} \Leftrightarrow (a, b) \in \theta \Leftrightarrow (\rho(a), \rho(b)) \in \Gamma|_{\rho[A]} \Leftrightarrow (\bar{a}, \bar{b}) \in \Gamma$. \square

COROLLARY 4.2.10 *Let B, r and T be as in Theorem 4.2.9. If $r : B \rightarrow T$ is any subdirect representation (not necessarily regular) then every 2-congruence on B can be extended to a 2-congruence on T .*

PROOF. Follows from Lemma 4.2.8 and Theorem 4.2.9. □

As a digression from this section of preliminaries to Chapter 6 we conclude with an interesting consequence of 2-congruence extendibility in \mathcal{N} , the variety generated by the pentagon.

PROPOSITION 4.2.11 *Let $A \in \mathcal{N}$. If A is 2-congruence extendible in \mathcal{N} then A is congruence extensile in \mathcal{N} .*

PROOF. Suppose $A \leq \prod_I S_i$ where $S_i \in \mathcal{N}_{SI} (= \{N, \mathbf{2}\}) \forall i \in I$. Let θ be a subdirectly irreducible congruence on A . If θ is a 2-congruence then θ can be extended to a 2-congruence on $\prod_I S_i$ by 2-congruence extendibility of A . Suppose that $A/\theta \cong N$. Let Ψ be a Jónsson congruence on $\prod_I S_i$ such that $\Psi|_A \subseteq \theta$. Now $(\prod_I S_i)/\Psi \cong N$ or $(\prod_I S_i)/\Psi \cong \mathbf{2}$. Since $A/\theta \cong N$ and $A/\theta \hookrightarrow A/\Psi|_A \hookrightarrow (\prod_I S_i)/\Psi$ we must have $(\prod_I S_i)/\Psi \cong N$. Also $(A/\Psi|_A)/(\theta/\Psi|_A) \cong A/\theta \cong N \cong A/\Psi|_A$. Thus Ψ is a subdirectly irreducible congruence on $\prod_I S_i$ extending θ and the result follows from Proposition 1.4.10. □

Note that the converse of Proposition 4.2.11 is false since $\mathbf{2}$ is congruence extensile in \mathcal{N} but it is not 2-congruence extendible in \mathcal{N} .

Chapter 5

2 and 3-chain limited lattices

We will call a lattice *n-chain limited* if it does not have $(n+1)$ -congruences. For example the lattices L_6^n (Figure 5.1) and their duals are 2-chain limited, L_7^n (Figure 5.2) and their duals are 3-chain limited and L_{10}^n are $(n+3)$ -chain limited.

For certain lattice varieties \mathcal{V} , $\text{Amal}(\mathcal{V})$ is a subclass of the n -chain limited members of \mathcal{V} . For example in Chapter 6 we show that for any variety \mathcal{B} generated by a B -lattice, $B \text{ Amal}(\mathcal{B})$ is the intersection of subdirect powers of B , 2-chain limited members of \mathcal{B} and 2-congruence extendible members of \mathcal{B} . This generalizes the result from [Bru92] that in \mathcal{N} , the variety generated by the pentagon, $\text{Amal}(\mathcal{N})$ is the intersection of 2-chain limited members of \mathcal{N} and 2-congruence extendible members of \mathcal{N} .

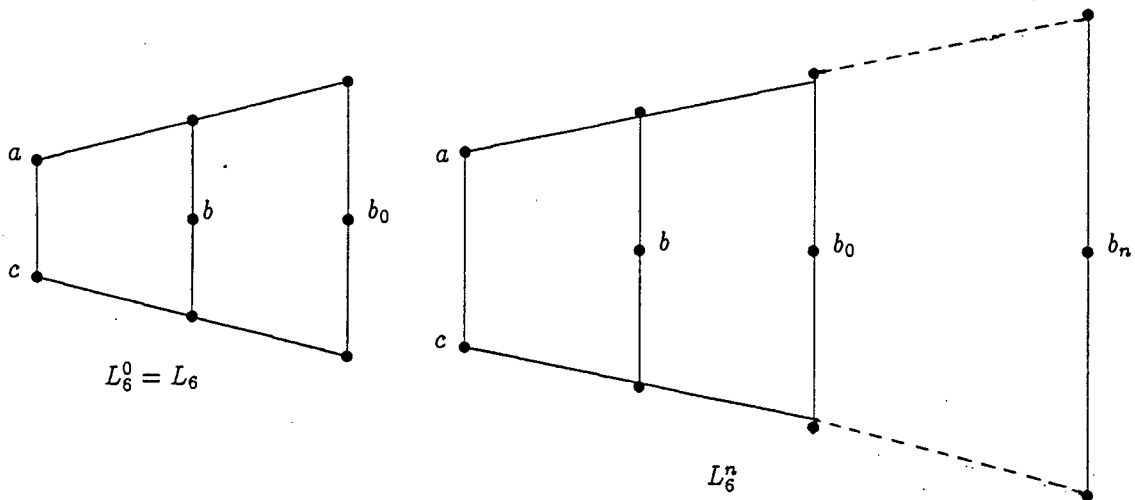


Figure 5.1

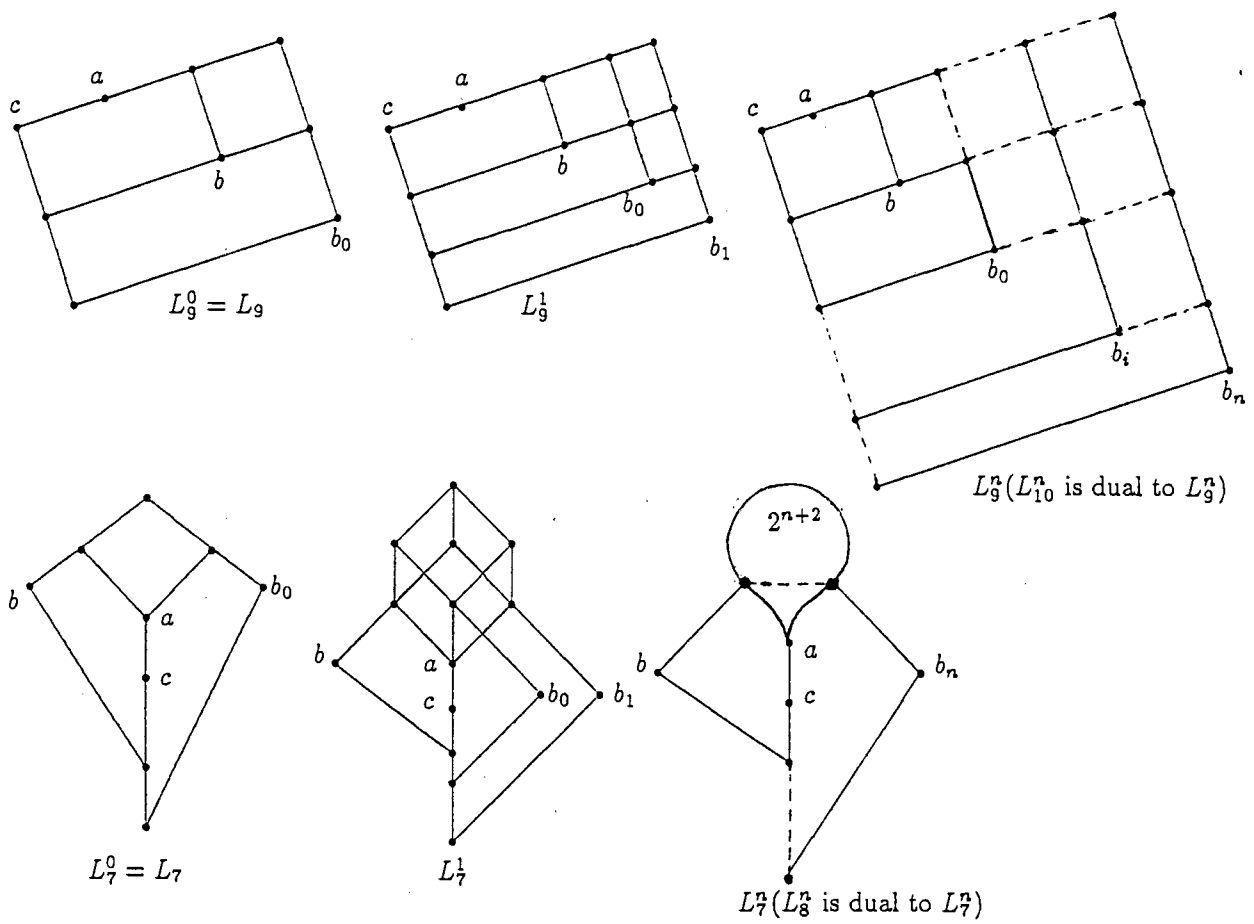


Figure 5.2.

5.1 \mathcal{S} -varieties with 2-chain limited members of the amalgamation class

We begin with some results concerning \mathcal{S} -varieties in which every maximal irreducible is 2-chain limited. Examples of such varieties include the varieties generated by L_3^n $n \in \omega$ (see Figure 5.1) and B -lattice varieties (see Chapter 6). We show that for such \mathcal{S} -varieties, \mathcal{V} , the amalgamation class of \mathcal{V} is contained in the class of 2-chain limited members of \mathcal{V} . Furthermore we prove that with further restrictions on \mathcal{V} products of members of $\text{Amal}(\mathcal{V})$ are also 2-chain limited (Theorem 5.1.5). Both these results are applied in Chapter 6 to B -lattices; Theorem 5.1.5 being used directly in the proof that the amalgamation class of a B -lattice variety is elementary. The results of this section appear in [Lai96].

LEMMA 5.1.1 *Let \mathcal{V} be a \mathcal{S} -variety. Let $A \leq C$ with $A \in \text{Amal}(\mathcal{V})$ and $C \in \mathcal{V}$. Let $M \in \mathcal{V}_{MI}$ with critical quotient a/c and let $g : A \rightarrow M$ be a homomorphism such that $g[A]$ is a distributive sublattice of M containing $\{a, c\}$. Suppose that $h : C \rightarrow M$ is a homomorphism with $h|_A = g$. Then $h[C] \subseteq (c) \cup (a)$ and so $h[C]$ is an ordinal sum of two distributive lattices.*

PROOF. Since $A \in \text{Amal}(\mathcal{V})$ every 2-congruence on A can be extended to a 2-congruence on C (by Proposition 4.2.4). Thus by Lemma 4.1.2 and Theorem 4.1.3 $\text{con}(u, v)|_A \subseteq \ker(g)$ for every N -quotient u/v of C . But then $\text{con}(u, v) \subseteq \ker(h)$ and so $h[C]$ is a distributive sublattice of M . By results from [Ros84] (see Chapter 2 Section 2.5) we must then have $h[C] \subseteq (c) \cup (a)$ (since $\{a, c\} \subseteq h[C]$ and if $b \in h[C]$ for some b which is non-comparable with a or c then $N(a/c, b) \subseteq h[C]$ contradicting distributivity of $h[C]$). \square

PROPOSITION 5.1.2 *Let \mathcal{V} be a \mathcal{S} -variety in which every member of \mathcal{V}_{MI} is 2-chain limited. Then every member of $\text{Amal}(\mathcal{V})$ is 2-chain limited.*

PROOF. Let $A \in \text{Amal}(\mathcal{V})$ and suppose that A has a 3-congruence. There is an embedding $f : A \hookrightarrow S = S_1^{I_1} \times \cdots \times S_n^{I_n}$ where $\{S_1, \dots, S_n\} \subseteq \mathcal{V}_{MI}$. Choose $S_k \in \{S_1, \dots, S_n\}$ and let a/c be the critical quotient of S_k . Then there is a homomorphism $g : A \rightarrow S_k$ with $g[A] \cong \mathbf{3}$ and $\{a, c\} \subseteq g[A]$. By Theorem 1.4.5 there is a homomorphism $h : S \rightarrow S_k$ such that $h \circ f = g$. Since $\mathbf{3}$ is distributive it follows from Lemma 5.1.1 that $h[S] \subseteq (c) \cup (a)$ i.e. $h[C]$ is an ordinal sum of distributive lattices. By Theorem 1.3.6 $h[S] \subseteq \text{PH}(\{S_1, \dots, S_n\})$. But since it is directly indecomposable $h[S]$ must be a homomorphic image of S_l for some $l \in \{1, \dots, n\}$. Then since $g[A] \subseteq h[S]$, $\mathbf{3}$ is an image of $h[S]$ contradicting the assumption that every member of \mathcal{V}_{MI} is 2-chain limited, and completing the proof. \square

LEMMA 5.1.3 *Let \mathcal{V} be a \mathcal{S} -variety such that every member of \mathcal{V}_{MI} is 2-chain limited. Let $C \in \mathcal{V}$ with θ_0, θ_1 two distinct 2-congruences on C . Put $\Phi = \theta_0 \cap \theta_1$ and let $p, q, r, s \in C$*

be such that $C/\theta_0 = \{p/\theta_0, q/\theta_0\}$ and $C/\theta_1 = \{r/\theta_1, s/\theta_1\}$. Let $A \in \text{Amal}(\mathcal{V})$ and let $f_0, f_1 : A \hookrightarrow C$ be two embeddings such that $\{p, q\} \subseteq f_0[A]$ and $\{r, s\} \subseteq f_1[A]$. Then Φ is not a 3-congruence on C .

PROOF. Let (D, g_0, g_1) be an amalgam of the diagram (A, f_0, C, f_1, C) in \mathcal{V} . Let $h = g_0 \circ f_0 = g_1 \circ f_1$. By Proposition 4.2.4 there are 2-congruences $\bar{\theta}_0$ and $\bar{\theta}_1$ on D such that $\bar{\theta}_0|_{h[A]}$ and $\bar{\theta}_1|_{h[A]}$ correspond under g_0 and g_1 respectively to $\theta_0|_{f_0[A]}$ and $\theta_1|_{f_1[A]}$. Let $\bar{\Phi} = \bar{\theta}_0 \cap \bar{\theta}_1$. Then $\bar{\Phi}$ is a 3-congruence on D and $\bar{\Phi}|_{h[A]}$ corresponds under h to a 3-congruence on A . This contradiction of Proposition 5.1.2 completes the proof. \square

PROPOSITION 5.1.4 *Let \mathcal{V} be a S -variety in which every member of \mathcal{V}_{MI} is 2-chain limited. Let $C \in \mathcal{V}$ and assume that for any two distinct 2-congruences θ_0 and θ_1 on C and some $p, q, r, s \in C$ with $C/\theta_0 = \{p/\theta_0, q/\theta_0\}$ and $C/\theta_1 = \{r/\theta_1, s/\theta_1\}$ there is an $A \in \text{Amal}(\mathcal{V})$ and there are embeddings $f_0, f_1 : A \hookrightarrow C$ such that $\{p, q\} \subseteq f_0[A]$ and $\{r, s\} \subseteq f_1[A]$. Then C is 2-chain limited.*

PROOF. Suppose Φ is a 3-congruence on C . Then (by the correspondence theorem) $\Phi = \theta_0 \cap \theta_1$ where θ_0, θ_1 are two distinct 2-congruences on C . It follows from the hypothesis of this proposition and Lemma 5.1.3 that Φ cannot be a 3-congruence on C and this contradiction completes the proof. \square

THEOREM 5.1.5 *Let \mathcal{V} be an S -variety in which every member of $\text{Amal}(\mathcal{V})$ is a subdirect product of members of \mathcal{V}_{MI} . Assume too that every member of \mathcal{V}_{MI} is 2-chain limited and has at least two distinct 2-congruences. Let $C = \prod_I A_i$ where $A_i \in \text{Amal}(\mathcal{V})$ for all $i \in I$. Then C is 2-chain limited.*

PROOF. Without loss of generality we may assume that each A_i is non-trivial. We use Proposition 5.1.4. Suppose θ_0 and θ_1 are two distinct 2-congruences on A . We will show that there is a $j \in I$ and embeddings $f_0, f_1 : A_j \hookrightarrow C$ such that for some $p, q, r, s \in C$ with $C/\theta_0 = \{p/\theta_0, q/\theta_0\}$ and $C/\theta_1 = \{r/\theta_1, s/\theta_1\}$ we have $\{p, q\} \subseteq f_0[A_j]$ and $\{r, s\} \subseteq f_1[A_j]$. Let $u > v, w > x \in C$ be such that $C/\theta_0 = \{u/\theta_0, v/\theta_0\}$ and $C/\theta_1 = \{w/\theta_1, x/\theta_1\}$. Let Ψ_0 and Ψ_1 be Jónsson congruences on C with $\Psi_0 \subseteq \theta_0$ and $\Psi_1 \subseteq \theta_1$. Let \mathcal{D}_0 and \mathcal{D}_1 be the corresponding Jónsson ultrafilters over I . Put $S = \{i \in I : u_i > v_i\}$ and $R = \{i \in I : w_i > x_i\}$. Then $S \in \mathcal{D}_0$ and $R \in \mathcal{D}_1$. For all $i \in I$ let $a_i \in A_i$ be fixed. There are three possible cases:

- (i) There is a $j \in I$ such that $\{j\} \in \mathcal{D}_0 \cap \mathcal{D}_1$.
- (ii) For all $j \in I$, $\{j\} \notin \mathcal{D}_0$ and $\{j\} \notin \mathcal{D}_1$.
- (iii) There is a $j \in I$ such that $\{j\} \in \mathcal{D}_0$, but $\{j\} \notin \mathcal{D}_1$ or vice-versa.

In case (i) define the embedding $f_0 : A_j \hookrightarrow C$ as follows. For all $x \in A_j$

$$f_0(x)_i = \begin{cases} x & i = j \\ a_i & i \in I \setminus \{j\}. \end{cases}$$

Let $f_1 = f_0$. Then $(f_0(u_j), u), (f_0(v_j), v), (f_0(w_j), w), (f_0(x_j), x) \in \theta_0 \cap \theta_1$. Let $p = f_0(u_j) = f_1(u_j)$, $q = f_0(v_j) = f_1(v_j)$, $r = f_0(w_j) = f_1(w_j)$, $s = f_0(x_j) = f_1(x_j)$.

Suppose that case (ii) holds. Pick any $j \in I$. Then $S \setminus \{j\} \in \mathcal{D}_0$ and $R \setminus \{j\} \in \mathcal{D}_1$. Observe that since A_j is non-trivial it is a subdirect power of members of \mathcal{V}_{MI} . Thus there are at least two distinct epimorphisms $k, t : A_j \rightarrow \mathbf{2}$. The embedding $f_0 : A_j \hookrightarrow C$ is defined as follows. For all $x \in A_j$

$$f_0(x)_i = \begin{cases} x & i = j \\ u_i & i \in S \setminus \{j\} \text{ and } t(x) = 1 \\ v_i & i \in S \setminus \{j\} \text{ and } t(x) = 0 \\ a_i & i \in I \setminus (S \cup \{j\}). \end{cases}$$

The embedding $f_1 : A_j \hookrightarrow C$ is defined similarly. For $x \in A_j$:

$$f_1(x)_i = \begin{cases} x & i = j \\ w_i & i \in R \setminus \{j\} \text{ and } k(x) = 1 \\ x_i & i \in R \setminus \{j\} \text{ and } k(x) = 0 \\ a_i & i \in I \setminus (R \cup \{j\}). \end{cases}$$

Since the maps k and t are epimorphisms there are $a, b, c, d \in A_j$ such that $t(a) = 1, t(b) = 0, k(c) = 1$ and $k(d) = 0$. Setting $p = f_0(a), q = f_0(b), r = f_1(c)$ and $s = f_1(d)$ we have $(p, u), (q, v) \in \theta_0$ and $(r, w), (s, x) \in \theta_1$.

The case (iii) is a combination of (i) and (ii). If there is a $j \in I$ with $\{j\} \in \mathcal{D}_0$ but $\{j\} \notin \mathcal{D}_1$ then $R \setminus \{j\} \in \mathcal{D}_1$. We then define f_0 as in case (i) and f_1 as in case (ii) and let $p = f_0(u_j), q = f_0(v_j), r = f_1(c)$ and $s = f_1(d)$ where c, d are as defined in case (ii). Otherwise, if there is a $j \in I$ with $\{j\} \in \mathcal{D}_1$ but $\{j\} \notin \mathcal{D}_0$ then $S \setminus \{j\} \in \mathcal{D}_0$. Thus define f_1 as in case (i) and f_0 as in case (ii) and let $p = f_0(a), q = f_0(b), r = f_1(v_j), s = f_1(w_j)$ where a, b are as in case (ii).

In all three cases we see that $C/\theta_0 = \{p/\theta_0, q/\theta_0\}$, $C/\theta_1 = \{r/\theta_1, s/\theta_1\}$ and $\{p, q\} \subseteq f_0[A_j]$, $\{r, s\} \subseteq f_1[A_j]$. By Proposition 5.1.4, C is 2-chain limited. \square

5.2 Characterization of 3-chain limited finite distributive lattices

In this section we characterize 3-chain limited finite distributive lattices as semi-Boolean lattices. The results appearing here and in Section 5.3 arose out of an investigation into 3-chain limited lattices and are neither in print nor in the form of a manuscript. However, as demonstrated by Theorem 5.3.1, they will undoubtedly assist in determining the amalgamation class of certain lattice varieties, for example varieties generated by K -lattices (see Chapter 6) or $\{L_7^n\}^\vee$ (see Figure 5.2).

An element d of a lattice L is said to be *Boolean* if the principal filter $[d]$ and the principal ideal $(d]$ are Boolean sublattices of L .

A finite distributive lattice L with Boolean elements is called *semi-Boolean*. We call a congruence θ on a lattice L semi-Boolean if L/θ is a semi-Boolean lattice.

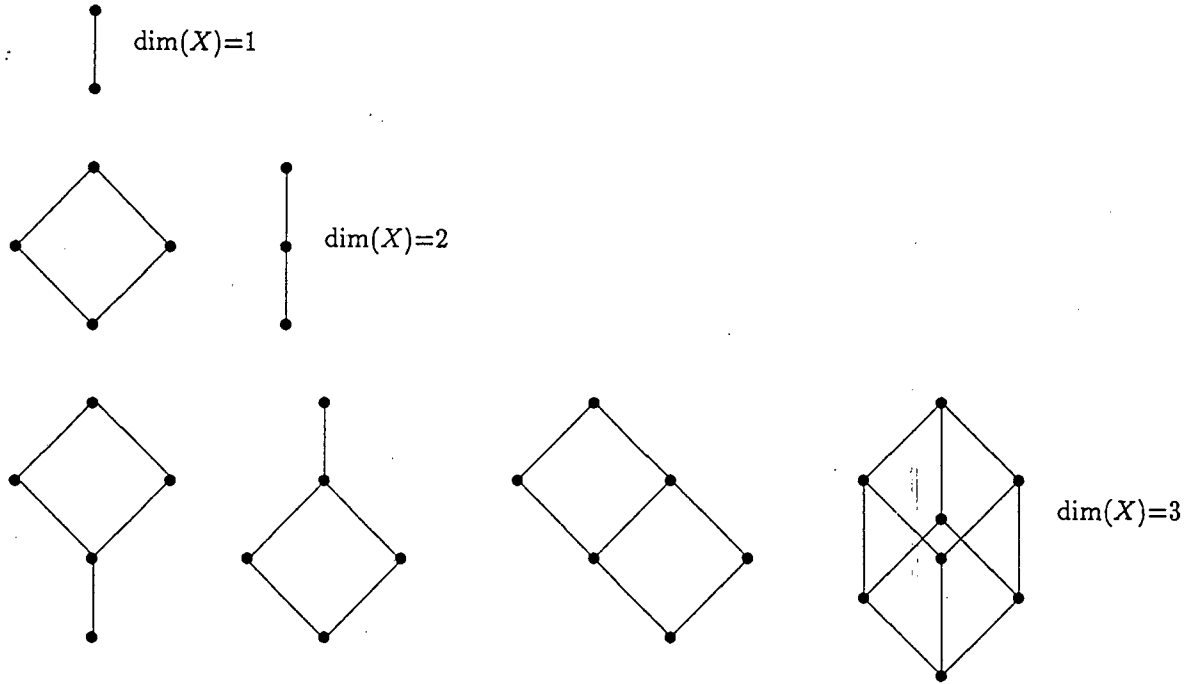


Figure 5.3 Semi-Boolean lattices of dimension ≤ 3

LEMMA 5.2.1 Let L be a semi-Boolean lattice with $\{x_1, \dots, x_m\}$ the atoms of L and $\{y_1, \dots, y_k\}$ the coatoms of L . Put

$$u = x_1 \vee x_2 \vee \dots \vee x_m$$

and

$$v = y_1 \wedge y_2 \wedge \dots \wedge y_k.$$

Then $v \leq u$ and d is a Boolean element of L if and only if $d \in u/v$.

PROOF. Since L has at least one Boolean element d , which is a join of atoms and meet of coatoms, we have $v \leq d \leq u$ proving the first part of the lemma. For the second part let $x \in L$ be Boolean. Then, as before $x \in u/v$. Conversely suppose $x \in u/v$. Then (x) is an interval in the Boolean lattice (u) (Lemma 2.2.13(v)), hence is Boolean. Similarly (x) is a Boolean lattice proving that x is a Boolean element of L . \square

LEMMA 5.2.2 Let L be a semi-Boolean lattice with d a Boolean element of L . Let p/q be a prime quotient in L , then exactly one of the following conditions hold:

- (a) There is a cover r of d such that p/q projects onto r/d in at most two steps.
- (b) There is a dual cover t of d such that p/q projects onto d/t in at most two steps.

PROOF. Let $\mathbf{0} = s_0 \prec s_1 \prec \cdots \prec s_i = d \prec \cdots \prec s_k = \mathbf{1}$ be a maximal chain in L through d and let p/q be a prime quotient of L . By Lemma 2.2.13(iii) p/q projects onto s_{j+1}/s_j for some $j \in \{0, 1, \dots, k-1\}$. But then $s_{j+1}/s_j \subseteq [d]$ or $s_{j+1}/s_j \subseteq (d]$. So by Lemma 2.2.10 s_{j+1}/s_j transposes onto r/d (d/t) for some cover r (co-cover t) of d . By Lemma 2.2.7 p/q projects onto r/d (d/t) in at most two steps. Suppose both (a) and (b) hold. Then r/d projects onto d/t , contradicting Lemma 2.2.13(ii) applied to a maximal chain through r , d and t . \square

LEMMA 5.2.3 *Let L be a semi-Boolean lattice with d a Boolean element of L . Let $x, y \in L$ be such that $x \leq d \leq y$. Then the quotient y/x is a semi-Boolean lattice and d is a Boolean element of y/x .*

PROOF. d/x and y/d are Boolean lattices as they are intervals in $(d]$ and $[d]$ respectively. Thus d is a Boolean element of y/x . \square

LEMMA 5.2.4 *Let L be a semi-Boolean lattice. Then L is 3-chain limited.*

PROOF. Suppose θ is a 4-congruence on L . Then by Theorem 2.2.14 and the correspondence theorem we have $[\theta, \nabla] \cong 2^3$ in $\text{Con}(L)$. Now let $\theta_1, \theta_2, \theta_3 \in \text{Con}(L)$ be such that $\theta = \theta_1 \cap \theta_2 \cap \theta_3$ and let $q_i \prec p_i$ be such that $L/\theta_i = \{p_i/\theta_i, q_i/\theta_i\}$ for $i = 1, 2, 3$. Then by Lemma 5.2.2 there is a cover (co-cover) r_i of d such that p_i/q_i projects onto r_i/d (d/r_i). Thus by Theorem 2.1.2 $(r_i, d) \notin \theta_i$. So there are at least two covers or co-covers r_j and r_k of d such that $(r_j, d) \notin \psi$ and $(r_k, d) \notin \Sigma$ where $\psi, \Sigma \in \{\theta_1, \theta_2, \theta_3\}$. Thus $(r_j, r_k) \notin \theta$. But r_j/θ and r_k/θ are non-comparable ($r_j/\theta < r_k/\theta \Rightarrow d/\theta = r_k/\theta$ or $d/\theta = r_j/\theta$) contradicting the fact that L/θ is a chain. \square

LEMMA 5.2.5 *Let L be a finite distributive lattice, such that every proper image of L is a semi-Boolean lattice. Then either L is isomorphic to $\mathbf{4}$ or L is semi-Boolean.*

PROOF. Let s_1, \dots, s_m be the atoms of L and t_1, \dots, t_k the coatoms of L . Put $u = s_1 \vee \cdots \vee s_m, v = t_1 \wedge \cdots \wedge t_k$. Assume that L is not isomorphic to $\mathbf{4}$ and L is not semi-Boolean. Then $v \not\leq u$, since otherwise (v) and (u) are Boolean being intervals in (u) and (v) respectively, contradicting the fact that L is not semi-Boolean. Thus either $u < v$ or u and v are non-comparable. If $u < v$ then we claim that $u \prec v$. To see this suppose $u \leq u_1 < v_1 \leq v$ and let $P = L/\text{con}(u_1, v_1)$. Then P is semi-Boolean and $u/\text{con}(u_1, v_1) = s_1/\text{con}(u_1, v_1) \vee \cdots \vee s_m/\text{con}(u_1, v_1)$ and so $u/\text{con}(u_1, v_1)$ is the largest Boolean element of P . Dually $v/\text{con}(u_1, v_1)$ is the smallest Boolean element of P . Thus $v/\text{con}(u_1, v_1) = u/\text{con}(u_1, v_1)$ and since $\dim(P) = \dim(L) - 1$, we have $u = u_1, v = v_1$, proving $u \prec v$. Now, since L is not semi-Boolean, $\mathbf{1}/v$ and $u/\mathbf{0}$ are non-trivial intervals in L . Let s be an atom in L and let $R = L/\text{con}(\mathbf{0}, s)$. Then, as before, $v/\text{con}(\mathbf{0}, s)$ is the

smallest Boolean element in R . Thus $(v/\text{con}(\mathbf{0}, s))/(s/\text{con}(\mathbf{0}, s))$ is a Boolean lattice in R and none of the prime quotients in v/s are collapsed by $\text{con}(\mathbf{0}, s)$ (Corollary 2.2.9). So $v/s \mapsto (v/\text{con}(\mathbf{0}, s))/(s/\text{con}(\mathbf{0}, s))$ and v/s is a Boolean sublattice of L for any atom s in L . Suppose $\dim(\text{At}(L)) \geq 2$. Then $\mathbf{0} = \bigwedge \text{At}(L)$ is a meet of dual covers of v and so $\{v\}$ is Boolean contradicting the assumption that L is not semi-Boolean. Thus L has exactly one atom, u , which is the only dual cover of v . Similarly v is the only coatom of L and the only cover of u . But then L is the chain $\mathbf{0} \prec u \prec v \prec \mathbf{1}$ contradicting our assumption that L is not isomorphic to $\mathbf{4}$.

In the case that u is non-comparable with v let p/q be a prime quotient in $(u \vee v)/(u \wedge v)$ and let $Q = L/\text{con}(p, q)$. As before $v/\text{con}(p, q)$ is the smallest Boolean element of Q and $u/\text{con}(p, q)$ is the largest. So $v/\text{con}(p, q) \leq u/\text{con}(p, q)$ giving $(u, v) \in \text{con}(p, q)$ and contradicting $\dim(Q) = \dim(L) - 1$. \square

THEOREM 5.2.6 *Let L be a finite distributive lattice. Then L is semi-Boolean if and only if L is 3-chain limited.*

PROOF. If L is semi-Boolean then L is 3-chain limited by Lemma 5.2.4. Suppose L is 3-chain limited. Our proof is by induction on the length of L . Clearly $\dim(L) = 1$ implies that L is semi-Boolean. Now assume that every 3-chain limited lattice of dimension $\leq k$ ($k \in \omega$) is semi-Boolean. Suppose $\dim(L) = k + 1$. Then every proper image of L is semi-Boolean and hence by Lemma 5.2.5 so is L . \square

COROLLARY 5.2.7 *A finite distributive lattice L is 3-chain limited if and only if the smallest distributive congruence on L is semi-Boolean.*

COROLLARY 5.2.8 *A lattice L is 3-chain limited if and only if every finite distributive congruence on L is semi-Boolean.*

5.3 \mathcal{S} -varieties with 3-chain limited members of the amalgamation class

We present a characterization of those \mathcal{S} -varieties in which the amalgamation class is a subclass of its 3-chain limited members. Examples of such \mathcal{S} -varieties include K -lattice varieties described in Chapter 6 (see Theorem 6.1.1(c)) and the varieties generated by L_7^n and their duals, as depicted in Figure 5.2.

THEOREM 5.3.1 *Let \mathcal{V} be an \mathcal{S} -variety. Then every member of $\text{Amal}(\mathcal{V})$ is 3-chain limited if and only if for every $M \in \mathcal{V}_{MI}$, M/Ψ is semi-Boolean where Ψ is the smallest distributive congruence on M .*

PROOF. The forward implication follows from the fact that $\mathcal{V}_{MI} \subseteq \text{Amal}(\mathcal{V})$ (Lemma 1.4.3 and Theorem 1.4.4) and Corollary 5.2.7.

For the reverse implication let $A \in \text{Amal}(\mathcal{V})$ and suppose that A has a $\mathbf{4}$ -congruence. Let $M \in \mathcal{V}_{MI}$ with critical quotient a/c . Then there is a homomorphism $g : A \rightarrow M$ such that $g[A] \cong \mathbf{4}$ and $a/c \subseteq g[A]$. We also have an embedding $f : A \hookrightarrow S = S_1^{I_1} \times S_2^{I_2} \times \cdots \times S_k^{I_k}$ where $S_i \in \mathcal{V}_{MI}$ for all $1 \leq i \leq k$. It follows from Theorem 1.4.5 that there is a homomorphism $h : S \rightarrow M$ such that $h \circ f = g$. Since $\mathbf{4}$ is a distributive lattice it follows from Lemma 5.1.1 that $h[S] \subseteq (c) \cup (a)$. By Theorem 1.3.6 $h[S] \subseteq \text{PH}(\{S_1, S_2, \dots, S_k\})$. But, since $h[S] \subseteq (c) \cup (a)$ it is directly indecomposable and so $h[S]$ is a homomorphic image of some $S_i \in \mathcal{V}_{MI}$. Now $g[A] \subseteq h[S]$ and so there is a retraction of $h[S]$ onto $g[A]$. Then $g[A] \cong \mathbf{4}$ is a homomorphic image of S_i contradicting the fact that the smallest distributive congruence on S_i is $\mathbf{3}$ -chain limited (Corollary 5.2.7). \square

PROBLEM 5.3.2 *Characterize those \mathcal{S} -varieties \mathcal{V} for which every member of $\text{Amal}(\mathcal{V})$ is n -chain limited.*

Chapter 6

Amalgamation in B-lattice varieties

A B -lattice is a non-modular, subdirectly irreducible lattice obtained from a finite Boolean lattice, X , by splitting an element d of X where d is neither the top nor bottom element of X . Given a B -lattice, B , we let \mathcal{B} denote the variety generated by B . The main result of this chapter (Theorem 6.3.6) tells us that $\text{Amal}(\mathcal{B})$ is closed under reduced products - hence is an elementary class determined by Horn sentences. This generalizes the [Bru92] result that $\text{Amal}(\mathcal{N})$ is elementary where \mathcal{N} is the variety generated by the pentagon. Theorem 6.3.6 is based on a characterization of $\text{Amal}(\mathcal{B})$ in terms of 2-chain limited and 2-congruence extendible members of \mathcal{B} (Theorem 6.3.4).

We begin in Section 6.1 with a short discussion of K -lattices as, by Proposition 6.2.1, the class of B -lattices is a subclass of the class of K -lattices. In section 6.2 we show that every B -lattice is an S -lattice and that every subdirectly irreducible member of \mathcal{B} is essentially embeddable into \mathcal{B} .

6.1 K-lattices

In [Day70] A. Day introduced the splitting of an interval in a lattice. Let L be a lattice and let $I = [u, v]$ be an interval in L . Construct a new lattice on the set $L[I] = (L \setminus I) \cup (I \times 2)$ with the order relation given by $x \leq y$ in $L[I]$ if and only if one of the following conditions hold:

- (a) $x, y \in L \setminus I$ and $x \leq y$ in L
- (b) $x = (a, i), y \in L \setminus I$ and $a \leq y$ in L
- (c) $x \in L \setminus I, y = (b, j)$ and $x \leq b$ in L
- (d) $x = (a, i), y = (b, j)$ and $a \leq b$ in L and $i \leq j$ in 2 .

Note that there is a natural epimorphism from $L[I]$ to L given by:

$$f(x) = \begin{cases} x & x \in L \setminus I \\ a & x = (a, i). \end{cases}$$

We will consider the special case in which the interval I consists of a single element d . In this case the new lattice is defined on the set $L[d] = (L \setminus \{d\}) \cup ((d, 0), (d, 1))$.

We define a K -lattice to be a non-modular, subdirectly irreducible lattice L , obtained by splitting an element d of a finite distributive lattice L' .

Let \mathcal{K} denote the class of all K -lattices. Then \mathcal{K} is contained in the class of all S -lattices.

Proofs of Theorems 6.1.1 and 6.1.2 can be found in [Ros84]. We use Theorem 6.1.1 to show that every B -lattice is a K -lattice.

THEOREM 6.1.1 *Let L' be a finite distributive lattice and let $d \in L'$ be neither the top nor bottom element of L' . Let L be obtained from L' by splitting d . Then $L \in \mathcal{K}$ (i.e. L is non-modular and subdirectly irreducible) if and only if all of the following conditions hold:*

- (a) *Every cover of d in L' is join-reducible.*
- (b) *Every co-cover of d in L' is meet-reducible.*
- (c) *Every prime quotient in L' projects onto a prime quotient p/q with $p = d$ or $q = d$.*

For example in the finite distributive lattice in Figure 6.1 below, splitting the element d yields a K -lattice, but splitting the element e does not as a is a meet irreducible co-cover of e .

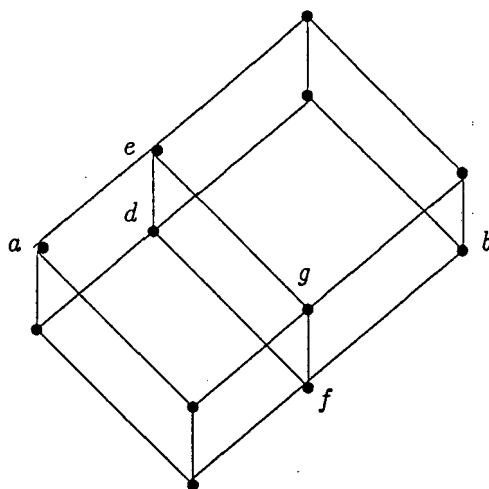


Figure 6.1

THEOREM 6.1.2 *Let $L \in \mathcal{K}$ be obtained by splitting an element d of a finite distributive lattice L' . Let $c = (d, 0), a = (d, 1)$. Then the following statements hold:*

- (a) *a/c is the unique N -quotient of L and it is a critical quotient with $L/\text{con}(a, c) \cong L'$.*
- (b) *Every non-distributive, subdirectly irreducible member of the variety generated by L belongs to \mathcal{K} .*
- (c) *The principal filter $[d]$ and the principal ideal $(d]$ of L' form non-trivial Boolean sublattices of L .*

6.2 B-lattices

The results of this section and Section 6.3 appear in [Lai96]. We define a B -lattice and show that every B -lattice is an S -lattice (Corollary 6.2.2) enabling us to apply the results of Section 2.5 of Chapter 2 to B -lattice varieties. We show for a variety \mathcal{B} generated by a B -lattice, B , that B is the only maximal irreducible of \mathcal{B} . We also prove some preliminary results which are applied in Section 6.3 in characterizing the amalgamation class of a B -lattice variety and in showing that the amalgamation class is elementary.

A lattice L is a B -lattice if it is obtained by splitting an element d of a finite Boolean lattice L' in which d is neither the top nor the bottom of L' . Examples of B -lattices include the pentagon and L_{13} and L_{14} of Figure 2.7 on Page 35.

The next proposition allows us to apply the results stated in Section 6.1 to B -lattices.

PROPOSITION 6.2.1 *Every B -lattice is a member of \mathcal{K} .*

PROOF. Let L' be a finite Boolean lattice with $d \in L'$ neither the top nor bottom of L' . Let L be the lattice obtained from L' by splitting d and let z be a cover of d in L' . Then since d is not the bottom of L' , the ideal (z) in L' is a Boolean lattice consisting of at least four elements. Hence z is join reducible and condition (a) of Theorem 6.1.1 holds. Condition (b) holds dually. For condition (c), let u/v be a prime quotient in L' . Then by Lemma 2.2.10, u/v transposes onto $a/0$ for some atom a of L' . Now, either $a \leq d$ or a and d are non-comparable. If $a \leq d$ then, since the ideal $(d]$ is a Boolean lattice, there is some co-cover x of d such that $a/0$ transposes onto d/x (by Lemma 2.2.10). If a is non-comparable with d then $a \vee d$ is a cover of d and $a/0$ transposes onto $(a \vee d)/d$. In both instances u/v projects onto a prime quotient p/q with either $p = d$ or $q = d$. \square

COROLLARY 6.2.2 *Every B -lattice is an S -lattice.*

PROOF. This follows from Proposition 6.2.1 and the fact that \mathcal{K} is contained in the class of all S -lattices. \square

For the rest of this chapter, let B be an arbitrary B -lattice and let a/c be the unique N -quotient of B which is also the unique critical quotient. (See Theorem 6.1.2 and the discussion of S -lattices in Chapter 2.) Let B' be the finite Boolean lattice from which B is obtained and let d be the splitting element of B' . Denote by \mathcal{B} the variety generated by B .

The next proposition shows that every subdirectly irreducible member of \mathcal{B} is essentially embeddable into B .

PROPOSITION 6.2.3 *B is the only member of \mathcal{B}_{MI} (up to isomorphism).*

PROOF. We need to show that every member of \mathcal{B}_{SI} is essentially embeddable in B . Let $S \in \mathcal{B}_{SI}$. If S is isomorphic to $\mathbf{2}$ then the embedding which takes $\mathbf{2}$ to a/c is the required essential embedding. If S is not isomorphic to $\mathbf{2}$ then S is non-distributive and so by Proposition 6.2.1 and Theorem 6.1.2(b), S is a K -lattice. Let u/v be the unique critical quotient of S and let S' be the finite distributive lattice from which S is obtained by splitting an element w . Since $S \in \mathbf{HS}(B)$ (Corollary 1.1.5), $\dim(S') \leq \dim(B')$ and the height and dual height of w in S' must be less than or equal to the height and dual height respectively of d in B' . It follows that there is an embedding $f' : S' \hookrightarrow B'$ such that $f'(w) = d$. Hence there is an embedding $f : S \hookrightarrow B$ such that $f(u) = a$ and $f(v) = c$ and f is the required essential embedding. \square

We say an algebra A is *automorphic* in an algebra C , if for any two embeddings $f, g : A \hookrightarrow C$ there are automorphisms $h : C \rightarrow C$ and $k : A \rightarrow A$ such that $g = h \circ f \circ k$.

We will show that there is no B -lattice in \mathcal{B} smaller than B which is automorphic in B ; a result required for Lemma 6.3.1. We first need the following lemma:

LEMMA 6.2.4 *Let L and M be finite Boolean algebras with $|L| \leq |M|$. Let $x \in L$ and $y \in M$ be such that*

- (a) *The height of x in L is less than or equal to the height of y in M .*
- (b) *The dual height of x in L is less than or equal to the dual height of y in M .*
- (c) *$x, y \neq 0, 1$.*

Then

- (i) *There is a Boolean algebra embedding $g : L \hookrightarrow M$ with $g(x) = y$.*
- (ii) *There is a lattice embedding $f : L \hookrightarrow M$ such that $f[L]$ is a quotient in M and $f(x) = y$.*

PROOF. (i) Considering the duality between finite Boolean algebras and finite sets, (i) amounts to the following: Given finite sets X and Y with $|X| \leq |Y|$, and given subsets $Z \subseteq X$ and $W \subseteq Y$ with $|Z| \leq |W|$ and $|X - Z| \leq |Y - W|$ there is a surjective map $g : Y \rightarrow X$ with $g[W] = Z$. This is clearly true.

(ii) There is a chain C in M such that the height of y in C is equal to the height of x in L

and the dual height of y in C is equal to the dual height of x in L . Let u and v be the top and bottom elements of C respectively. Then u/v is a sublattice of M , hence is a Boolean algebra with the same dimension as L . Thus u/v is isomorphic to L . We also have $y \in u/v$ and y has the same height in u/v as x has in L . Thus there is a lattice isomorphism from L to u/v mapping x to y . \square

PROPOSITION 6.2.5 *Let $A \in \mathcal{B}$ be a B -lattice which is not isomorphic to B . Then A is not automorphic in B .*

PROOF. Let A' be the finite Boolean algebra from which A is obtained by splitting an element e . Let $u = (e, 1), v = (e, 0)$ in A . By Proposition 6.2.3 A embeds in B . Any embedding of A into B must take u/v to a/c . Thus $|A'| \leq |B'|$ and the height of e in A' is less than or equal to the height of d in B' and the dual height of e in A' is less than or equal to the dual height of d in $|B'|$. Treating A' and B' as Boolean algebras and applying Lemma 6.2.4, we have a Boolean algebra embedding $g' : A' \hookrightarrow B'$ such that $g'(e) = d$ and a lattice embedding $f' : A' \hookrightarrow B'$ such that $f'[A']$ is a quotient in B' and $f'(e) = d$. Since A' is not isomorphic to B' , $f'[A']$ is a proper sublattice of B' . Define embeddings $f, g : A \hookrightarrow B$ by $f(u) = g(u) = a$, $f(v) = g(v) = c$ and $f(x) = f'(x) \forall x \notin \{u, v\}$, $g(x) = g'(x) \forall x \notin \{u, v\}$. Note that $g[A]$ contains both the top and bottom elements of B while $f[A]$ contains at most one of these. Hence there are no automorphisms $h : B \rightarrow B$ and $k : A \rightarrow A$ such that $g = h \circ f \circ k$. \square

The following four lemmas are necessary for the characterization of the members of $\text{Amal}(\mathcal{B})$ in Section 6.3.

LEMMA 6.2.6 *Every B -lattice is 2-chain limited.*

PROOF. Let A be a B -lattice with A' the finite Boolean algebra from which A is obtained. If Ψ is the smallest distributive congruence on A then $A/\Psi \cong A'$ - a Boolean algebra. The statement of the lemma then follows from this and the fact that $A \not\cong \mathbf{3}$. \square

LEMMA 6.2.7 *Let $A \in \mathcal{B}$ and suppose that A is 2-chain limited. Then*

- (i) *Every non-distributive subdirectly irreducible image of A is isomorphic to a B -lattice.*
- (ii) *If $A \leq B$ and A is non-distributive then A is a B -lattice.*

PROOF. (i) Let X be a non-distributive subdirectly irreducible image of A . Then X is a K -lattice by Theorem 6.1.2(b). Let X' be the finite distributive lattice from which X is obtained by splitting an element e . Then, since X is non-distributive, e is neither the top nor the bottom element of X' . Since A is 2-chain limited and X' is an image of A , X' is 2-chain limited. Thus X' is a Boolean lattice and so X is a B -lattice.

(ii) Since A is non-distributive it must contain a/c as an N -quotient. Thus there is a sublattice A' of B' containing d and such that A is obtained from A' by splitting d where d is neither the top nor bottom element of A' . Since A' is an image of A and A is 2-chain limited, so is A' . Thus A' is Boolean, whence A is a B -lattice. \square

Recall that a finite subset P of a lattice L is primitive if there is a proper quotient a_P/b_P of L such that $a_P/b_P \approx_w (c \vee d)/d$ for all $c, d \in P$ with $c \not\leq d$. We say that P is a primitive subset of L with respect to a_P/b_P .

LEMMA 6.2.8 *Let $A \in \mathcal{B}$ and $X \in \mathcal{K}$. Let p/q be the unique N -quotient of X . Suppose that $f : A \rightarrow X$ is an epimorphism and that u/v is a quotient in A with $f(u) = p$ and $f(v) = q$. Then there is a primitive subset P of A with respect to u/v such that $f|_P : P \rightarrow X$ is a poset isomorphism and $u, v \in P$.*

PROOF. We define a poset embedding $h : X \rightarrow A$ such that $f \circ h$ is the identity map on X . This embedding is defined inductively on the height of elements of X . If x is the bottom element of X (the unique element of height 0) then define $h(x) = y$ where y is some element of A such that $f(y) = x$. Now suppose that h has been defined for all elements of X of height less than or equal to n for some $n \in \omega$. Let $x \in X$ be of height $n + 1$, and let $y \in A$ be such that $f(y) = x$. Let x_1, \dots, x_k be all the elements of X strictly less than x and let $z = y \vee h(x_1) \vee \dots \vee h(x_k)$. Then $z \in A$. Define $h(x)$ as follows:

$$h(x) = \begin{cases} u & x = p \\ v & x = q \\ z \wedge v & x < q \\ z \vee u & x > p \\ z & \text{otherwise.} \end{cases}$$

Having defined h put $P = h[X]$. Then $f|_P$ is a poset isomorphism from P onto X . Also X is a primitive subset of X with respect to p/q and hence by Lemma 2.3.4 P is primitive in A with respect to u/v . \square

LEMMA 6.2.9 *If A is a subdirect power of B , then every non-distributive subdirectly irreducible image of A is isomorphic to B .*

PROOF. Let X be a non-distributive subdirectly irreducible image of A . Let $g : A \rightarrow X$ be a surjection and $f : A \rightarrow B^I$ a subdirect representation. By Theorem 6.1.2(b) X is a K -lattice. Let p/q be the unique N -quotient of X . Let $b \in X$ be such that $N(p/q, b)$. Since the pentagon is a projective lattice there are $u, v, w \in A$ such that $g(u) = p$, $g(v) = q$, $g(w) = z$ and $N(u/v, w)$ in A . Let $\pi_i : B^I \rightarrow B$ be the i th projection onto B for all $i \in I$. Then there is a $j \in I$ such that $\pi_j \circ f(u) \neq \pi_j \circ f(v)$. But then $\pi_j \circ f(x) \neq \pi_j \circ f(y)$ for any quotient

x/y in $N(u/v, w)$. Thus, since a/c is the unique N -quotient of B and $\pi_j \circ f(u)/\pi_j \circ f(v)$ is an N -quotient in B we must have $\pi_j \circ f(u) = a$ and $\pi_j \circ f(v) = c$. By Lemma 6.2.8 there is a primitive subset S of A with respect to u/v such that $u, v \in S$ and $(\pi_j \circ f)|_S : S \rightarrow B$ is an isomorphism. Then $g|_S : S \rightarrow X$ is an embedding. Since, suppose $g(d) = g(e)$ for some $d \neq e$ in S . Then as $u/v \approx_w (d \vee e)/e$ we have $g(u)/g(v) \approx_w g(d \vee e)/g(e)$. But this implies $p/q \approx_w g(e)/g(e)$ and this contradiction proves our claim. We have $|B| = |S| \leq |X|$ and since X is embeddable in B (Proposition 6.2.3), we have $|X| \leq |B|$ and so $X \cong B$. \square

6.3 Amalgamation

In this section we characterize the members of $\text{Amal}(\mathcal{B})$ in terms of the 2-chain limited and 2-congruence extendible members of \mathcal{B} . We first show that every member of $\text{Amal}(\mathcal{B})$ is a subdirect power of B (Theorem 6.3.3). We conclude with the main theorem of this chapter which states that $\text{Amal}(\mathcal{B})$ is an elementary class closed under reduced products.

Let A be an algebra with X a subdirectly irreducible image of A . We call X a *subfactor* of A if and only if there is a subdirect representation $e : A \rightarrow \prod_I X_i$ satisfying the following:

- (a) For all $i \in I, X_i$ is subdirectly irreducible.
- (b) There are distinct elements $x, y \in A$ such that $e(x)_i = e(y)_i$ for all $i \in I$ such that $X_i \not\cong X$.

The following technical lemma is needed to show that $\text{Amal}(\mathcal{B})$ is contained in the class of subdirect powers of B .

LEMMA 6.3.1 *Let $A \in \text{Amal}(\mathcal{B})$ and let X be a non-distributive subfactor of A . Then $X \cong B$.*

PROOF. By Proposition 5.1.2 and Lemma 6.2.7(i), X is isomorphic to a B -lattice. We will show that X is automorphic in B . Let $f, g : X \rightarrow B$ be two embeddings and let $e : A \rightarrow \prod_I X_i$ be as in the definition of subfactor. Without loss of generality we may assume that for all $i \in I$, if $X_i \cong X$ then $X_i = X$. Let $J = \{i \in I : X_i = X\}$. Then there is a $j \in J$ such that $e(x)_j \neq e(y)_j$. Let $\pi_i : \prod_I X_i \rightarrow X_i$ denote the i th projection for each $i \in I$. Let $k = \pi_j \circ e : A \rightarrow X$ and let p/q be the unique N -quotient of X . Then, since k is surjective, there is a quotient u/v of A such that $k(u) = p$ and $k(v) = q$. By Lemma 6.2.8 there is a primitive subset P of A with respect to u/v such that $k|_P : P \rightarrow X$ is a poset isomorphism. By Proposition 6.2.3, there is an embedding $\lambda_i : X_i \hookrightarrow B$ for all $i \in I \setminus J$. For $i \in J$ put $\lambda_i = g : X \hookrightarrow B$. Define an embedding $\lambda : \prod_I X_i \hookrightarrow B^I$ by $\lambda(x)_i = \lambda_i(x_i)$ for all $i \in I$. Let $\alpha = \lambda \circ e : A \hookrightarrow B^I$. Define another embedding $\beta : A \hookrightarrow A \times B$ by $\beta(a) = (a, f \circ k(a))$ for all $a \in A$. Then the diagram $(A, \alpha, B^I, \beta, A \times B)$ has an amalgam (C, α', β') in \mathcal{B} . Let $\pi_B : A \times B \rightarrow B$ be the projection onto B . Define an embedding $\mu : B \hookrightarrow A \times B$ by $\mu(b) = (u, b)$ for all $b \in B$. By Lemma 1.4.3 B is an absolute retract in

B so there is a retraction $\rho : C \rightarrow B$ of $\beta' \circ \mu$. Then, since $\rho \circ \beta' \circ \mu$ is the identity map on B , $\rho \circ \beta' : A \times B \rightarrow B$ is a retraction of μ . By Lemma 1.3.7 we must have $\rho \circ \beta' = \pi_B$. Now $\rho \circ \alpha' \circ \alpha = \rho \circ \beta' \circ \beta = \pi_B \circ \beta = f \circ k$ and $f \circ k(u) = f(p) = a$ and $f \circ k(v) = f(q) = c$. Thus $f \circ k[A]$ contains a/c as an N -quotient, hence so does $\rho \circ \alpha' \circ \alpha[A] \subseteq \rho \circ \alpha'[B^I]$. By Theorems 1.4.1 and 1.4.4, $B^I \in \text{Amal}(\mathcal{B})$. Thus by Proposition 5.1.2 B^I is 2-chain limited and hence so is $\rho \circ \alpha'[B^I]$. Now $\rho \circ \alpha'[B^I] \subseteq B$ and $\rho \circ \alpha'[B^I]$ is non-distributive. Thus by Lemma 6.2.7(ii) $\rho \circ \alpha'[B^I]$ is a B -lattice. Let Ψ be a Jónsson congruence on B^I with $\Psi \subseteq \ker(\rho \circ \alpha')$. Then Ψ is a B -congruence and $\rho \circ \alpha'[B^I]$ must be an image of B . (Since $B^I/\ker(\rho \circ \alpha')$ is an image of B^I/Ψ .) Since the only non-distributive image of B is B itself we have $\rho \circ \alpha'[B^I] \cong B$ and $\Psi = \ker(\rho \circ \alpha')$. Let \mathcal{D} be the Jónsson ultrafilter on B^I corresponding to Ψ . Let $F = \{i \in I : \alpha(u)_i \neq \alpha(v)_i\}$. We have $\rho \circ \alpha' \circ \alpha(u) = f \circ k(u) = a \neq c = f \circ k(v) = \rho \circ \alpha' \circ \alpha(v)$. Thus $(\alpha(u), \alpha(v)) \notin \Psi$ and so $F \in \mathcal{D}$.

Let $G = \{i \in I : \alpha(x)_i \neq \alpha(y)_i\}$. Now $\rho \circ \alpha' \circ \alpha(x) = f \circ k(x) = f \circ \pi_j \circ e(x) = f(e(x)_j) \neq f(e(y)_j) = \rho \circ \alpha' \circ \alpha(y)$. Thus $G \in \mathcal{D}$. Note that $G \subseteq J$ and so $J \in \mathcal{D}$.

There is a map $q : P \rightarrow X$ and a set $\{T_p \in \mathcal{D} : p \in P\}$ such that for all $p \in P$, $T_p \subseteq J$ and $e(p)_i = q(p)$ for all $i \in T_p$. To see this let $p \in P$. For each $i \in J$ define $J_i = \{j \in J : e(p)_j = e(p)_i\}$. Then the distinct J_i 's form a partition on J . Since X is finite there are finitely many distinct J_i 's and since \mathcal{D} is an ultrafilter and $J \in \mathcal{D}$ we must have $J_{i_p} \in \mathcal{D}$ for some $i_p \in J$. Let $T_p = J_{i_p}$ and define $q(p) = e(p)_{i_p}$.

Put $T = \bigcap_P T_p$. Since P is finite we have $T \in \mathcal{D}$. Let $E = \{i \in I : (B, f \circ k(p))_{p \in P} \cong (B, \alpha(p)_i)_{p \in P}\}$. Since B is finite, the structure $(B, f \circ k(p))_{p \in P}$ is definable by a first order sentence and so, recalling that $\rho \circ \alpha' \circ \alpha = f \circ k$ and $\rho \circ \alpha'[B^I] = B$ it follows that $E \in \mathcal{D}$.

Let $H = E \cap F \cap T$ then $H \in \mathcal{D}$. Thus $H \neq \emptyset$. Let $h \in H$. Then for all $p \in P$ we have $\pi_h \circ e(p) = e(p)_h = q(p)$ since $h \in \bigcap_P T_p$. Thus $q = \pi_h \circ e|_P$. Since $h \in J$ we have $\alpha(p)_h = (\lambda \circ e(p))_h = \lambda_h(e(p))_h = g \circ q(p)$. Thus, since $h \in E$ we have $(B, f \circ k(p))_{p \in P} \cong (B, \alpha(p)_h)_{p \in P} \cong (B, g \circ q(p))_{p \in P}$. Also $u, v \in P$ and $h \in F$ implies $\alpha(u)_h \neq \alpha(v)_h$. Thus $g \circ q(u) \neq g \circ q(v)$ and so $q(u) \neq q(v)$. Since u/v is the critical quotient of P , q is an embedding. Thus q is bijective, since $|X| = |P|$ and X is finite and so $q : P \rightarrow X$ is a poset isomorphism. Let $p : X \rightarrow P$ be the inverse of q and put $r = k|_P \circ p : X \rightarrow X$. Then r is a poset automorphism and hence a lattice automorphism. We have $(B, f \circ r(b))_{b \in X} \cong (B, f \circ k \circ p(b))_{b \in X} \cong (B, g \circ q \circ p(b))_{b \in X} \cong (B, g(b))_{b \in X}$. Thus there is an automorphism $t : B \rightarrow B$ such that $t \circ f \circ r = g$. X is therefore automorphic in B as desired and hence by Proposition 6.2.5 $X \cong B$. \square

LEMMA 6.3.2 *Every member of $\text{Amal}(\mathcal{B})$ is a subdirect product of non-modular subdirectly irreducibles in \mathcal{B} .*

PROOF. The result follows from Corollary 6.2.2, Corollary 4.2.2, Proposition 4.2.4 and Theorem 4.2.3. \square

THEOREM 6.3.3 *Every member of $\text{Amal}(\mathcal{B})$ is a subdirect power of B .*

PROOF. Let $A \in \text{Amal}(\mathcal{B})$. Then by Lemma 6.3.2, A is a subdirect product of its non-modular subdirectly irreducible images. Let $\{\theta_i : i \in I\}$ be the set of non-modular subdirectly irreducible congruences on A . Then $\bigcap_I \theta_i = \Delta_A$. Assume that A is not a subdirect power of B . Let $J_0 = \{i \in I : A/\theta_i \cong B\}$. Then there are distinct elements x_0 and y_0 of A such that $(x_0, y_0) \in \bigcap_{J_0} \theta_i$. There exists a non-modular subdirectly irreducible image X_0 of A such that for some $i \in I$, $A/\theta_i \cong X_0$ and $(x_0, y_0) \notin \theta_i$. Suppose that for some positive integer n we have a subset J_n of I such that $J_0 \subseteq J_n$, distinct elements x_n, y_n of A with $(x_n, y_n) \in \bigcap_{J_n} \theta_i$ and an X_n such that for some $i \in I$, $A/\theta_i \cong X_n$ and $(x_n, y_n) \notin \theta_i$. Let $J_{n+1} = J_n \cup \{i \in I : A/\theta_i \cong X_n\}$. Since $X_n \not\cong B$ we have by Lemma 6.3.1 that X_n is not a subfactor of A . Suppose for all $u \neq v$ in A we have $(u, v) \notin \bigcap_{J_{n+1}} \theta_i$. Then $A \leq \prod_{J_{n+1}} A/\theta_i$ is a subdirect representation. Let $x \neq y$ in A . Then, since X_n is not a subfactor of A , there is some $i \in J_{n+1}$ such that $(x, y) \notin \theta_i$ and $A/\theta_i \not\cong X_n$. but then $\bigcap_{J_n} \theta_i = \Delta$ (since $\{i \in J_{n+1} : A/\theta_i \not\cong X_n\} = J_n$). This contradicts $x_n \neq y_n$ and $(x_n, y_n) \in \bigcap_{J_n} \theta_i$. Hence there are distinct elements x_{n+1}, y_{n+1} of A such that $(x_{n+1}, y_{n+1}) \in \bigcap_{J_{n+1}} \theta_i$. Thus there is a non-modular subdirectly irreducible image X_{n+1} of A such that for some $i \in I$ with $A/\theta_i \cong X_{n+1}$ we have $(x_{n+1}, y_{n+1}) \notin \theta_i$. By induction we obtain an infinite sequence of non-modular subdirectly irreducible members of \mathcal{B} which are distinct up to isomorphism. This contradiction completes the proof. \square

THEOREM 6.3.4 *Let A be a subdirect power of B . Then the following are equivalent:*

- (a) $A \in \text{Amal}(\mathcal{B})$.
- (b) A is 2-chain limited and 2-congruence extendible in \mathcal{B} .
- (c) A is 2-chain limited and for any (regular) subdirect representation $f : A \rightarrow B^I$, every 2-congruence on $f[A]$ can be extended to a 2-congruence on B^I .
- (d) For any embedding $f : A \hookrightarrow C$ with $C \in \mathcal{B}$ and any homomorphism $g : A \rightarrow B$, there is a homomorphism $h : C \rightarrow B$ with $h \circ f = g$.
- (e) For any (regular) subdirect representation $f : A \rightarrow B^I$ and any homomorphism $g : A \rightarrow B$, there is a homomorphism $h : B^I \rightarrow B$ with $h \circ f = g$.

PROOF. (a) \Rightarrow (b): Assume (a). By Corollary 6.2.2, Corollary 4.2.2 and Proposition 4.2.4 A is 2-congruence extendible in \mathcal{B} . By Proposition 6.2.3, Lemma 6.2.6 and Proposition 5.1.2, A is 2-chain limited and (b) holds.

Clearly (b) \Rightarrow (c).

(c) \Rightarrow (e): Assume (c). Let $f : A \rightarrow B^I$ be a subdirect representation and $g : A \rightarrow B$ a homomorphism. We have two cases:

Case (i): $g[A]$ is distributive. Then since $g[A]$ is finite and 2-chain limited it must be a Boolean lattice. Also, since every 2-congruence on $f[A]$ can be extended to a 2-congruence on B^I , we have by Lemma 4.1.5 that there is a homomorphism $h : B^I \rightarrow B$ with $h \circ f = g$.

Case (ii): $g[A]$ is non-distributive. By Lemma 6.2.7(ii) $g[A]$ is a B -lattice. By Lemma 6.2.9

$g[A]$ is isomorphic to B . Thus g is surjective. Let $\theta = \ker(g)$ and let $f(\theta)$ denote the congruence on $f[A]$ corresponding to θ under f . Then $f[A]/f(\theta) \cong B$. Let Ψ be a Jónsson congruence on B^I with $\Psi|_{f[A]} \subseteq f(\theta)$. Then Ψ is a B -congruence on B^I . Since $f[A]/\Psi|_{f[A]} \leq B^I/\Psi \cong B$ we have $f[A]/\Psi|_{f[A]} \cong B$ and hence $f(\theta) = \Psi|_{f[A]}$. Let $k : B^I \rightarrow B$ be a surjective homomorphism with $\ker(k) = \Psi$. Then $\ker(k \circ f) = \ker(g)$ and $k \circ f$ is surjective. Thus there is an automorphism t of B such that $t \circ k \circ f = g$. Let $h = t \circ k : B^I \rightarrow B$. Then $h \circ f = g$ and (e) holds.

(e) \Rightarrow (a) by Lemma 1.4.6.

(a) \Leftrightarrow (d) by Theorem 1.4.5. □

LEMMA 6.3.5 *Let $A_i, i \in I$ be algebras with $f_i : A_i \rightarrow \prod_{j \in I} B_{(i,j)}$ a subdirect representation. Then there is a subdirect representation of $A = \prod_I A_i$ in $\prod_{j \in I} B_{(i,j)}$.*

PROOF. Let $J = \cup J_i$ (disjoint union). Define a map $h : A \rightarrow \prod_J B_{(i,j)}$ as follows. For all $a \in A$ let $h(a)_{(i,k)} = f_i(a_i)_{(i,k)}$. Then h is the desired subdirect representation. □

We are now ready to prove that the amalgamation class of \mathcal{B} is elementary.

THEOREM 6.3.6 (i) *If C is both an image of $A \in \text{Amal}(\mathcal{B})$ and a subdirect power of B then $C \in \text{Amal}(\mathcal{B})$.*

(ii) *$\text{Amal}(\mathcal{B})$ is an elementary class. It is closed under reduced products and is therefore definable by Horn sentences.*

PROOF. (i) Let $f : C \rightarrow B^I$ be a subdirect representation. Then by Corollary 4.2.10 every 2-congruence on $f[C]$ can be extended to a 2-congruence on B^I . Also, since A is 2-chain limited (Theorem 6.3.4) and C is an image of A , C is 2-chain limited. Hence by Theorem 6.3.4(c) $C \in \text{Amal}(\mathcal{B})$.

(ii) We first show that $\text{Amal}(\mathcal{B})$ is closed under products. Let $A = \prod_I A_i$ where $A_i \in \text{Amal}(\mathcal{B})$ for all $i \in I$. Then by Corollary 4.2.2 and Theorem 4.2.6 A is 2-congruence extendible in \mathcal{B} . Since B is non-modular, B^I must be a Boolean algebra with at least four elements. Thus B has at least two distinct 2-congruences. By Lemma 6.2.6, Proposition 6.2.3, Theorem 6.3.3 and Theorem 5.1.5 we have that A is 2-chain limited. Thus by Theorem 6.3.3, Lemma 6.3.5 and Theorem 6.3.4(b), $A \in \text{Amal}(\mathcal{B})$ and so $\text{Amal}(\mathcal{B})$ is closed under products. Now let \mathcal{U} be a proper filter over I and let $C = \prod_{\mathcal{U}} A_i$. By Proposition 2.5.5 and Theorem 6.3.3 C is a subdirect product of non-modular subdirectly irreducible members of \mathcal{B} . Now C is an image of $A \in \text{Amal}(\mathcal{B})$ and so by Lemma 6.2.9 every non-distributive homomorphic image of A and hence of C is isomorphic to B . Thus by (i) $C \in \text{Amal}(\mathcal{B})$ proving that $\text{Amal}(\mathcal{B})$ is closed under reduced products. In particular it is closed under ultraproducts. By [Yas74] any amalgamation class of a variety which is closed under ultraproducts is elementary. Lastly, any elementary class closed under reduced products is definable by Horn sentences. (See [Cha78].) □

Bibliography

- [Bak77] K. Baker. *Finite equational bases for finite algebras in congruence distributive equational classes*. *Advances in Mathematics* **24** (1985), 207–243.
- [Ber85] C. Bergman. *Amalgamation classes of some distributive lattices*. *Algebra Universalis* **20** (1985), 143–166.
- [Ber89] C. Bergman. *Non-axiomatizability of the amalgamation class of modular lattice varieties*. *Order* (1989), 49–58.
- [Ber72] J. Berman. *On the length of the congruence lattice of a lattice*. *Algebra Universalis* **2** (1972), 18–19.
- [Bir35] G. Birkhoff. *On the structure of abstract algebras*. *Proc. Camb. Phil. Soc.* **31** (1935), 433–454.
- [Bir44] G. Birkhoff. *Subdirect unions in universal algebra*. *Bull. Amer. Math. Soc.* **50** (1944), 764–768.
- [Bru92] P. Bruyns, C. Naturman, and H. Rose. *Amalgamation in the pentagon variety*. *Algebra Universalis* **29** (1992), 303–322.
- [Bur81] S. Burris and H.P. Sankappanavar. *A course in Universal Algebra*. Springer-Verlag, New York, 1981.
- [Cha78] C.C. Chang and H.J. Keisler. *Model Theory*. *Studies in Logic and the Foundations of Mathematics* 73. North-Holland, Amsterdam, London, 1978.
- [Cra73] P. Crawley and R.P. Dilworth. *Algebraic Theory of Lattices*. Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1973.
- [Cra60] P. Crawley. *Lattices whose congruences form a Boolean algebra*. *Pacific J. Math.* **10** (1960), 787–795.
- [Dav90] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, 1990.

- [Day70] A. Day. *A simple solution to the word problem for lattices*. Canadian Math. Bull. **2** (1970), 253–254.
- [Day72] A. Day, C. Herrmann, and R. Wille. *On modular lattices with four generators*. Algebra Universalis **2** (1972), 317–323.
- [Dil33] R.P. Dilworth. *The structure of relatively complemented lattices*. Annals of Math. **51** (1950), 348–359.
- [Dob86] H. Dobbertin. *Vaught measures and their applications in lattice theory*. J. of Pure and Applied Algebra **43** (1986), 27–51.
- [Fra63] T. Frayne, A.C. Morel, and D.S. Scott. *Reduced direct products*. Fund. Math., **51** (1962/63), 195–228.
- [Fre75] R. Freese. *Congruence lattices of finitely generated modular lattices*. Proc. of the Lattice Theory Conference, Ulm (1975), 62–70.
- [Fre91] R. Freese, G. Grätzer and E.T. Schmidt. *On complete congruence lattices of complete modular lattices*. Internat. J. Algebra Computat., **1**(2) (1991), 147–160.
- [Fun42] N. Funayama and T. Nakayama. *On the distributivity of a lattice of lattice-congruences*. Proc. Imp. Acad. Tokyo **18** (1942), 553–554.
- [Gra58] G. Grätzer and E.T. Schmidt. *Ideals and congruence relations in lattices*. Acta Math. Acad. Sci. Hung. **9** (1958), 137–175.
- [Gra62] G. Grätzer and E.T. Schmidt. *On congruence lattices of lattices*. Acta Math. Acad. Sci. Hung. **13** (1962), 179–185.
- [Gra63] G. Grätzer and E.T. Schmidt. *Characterization of congruence lattices of abstract algebras*. Acta Sci. Math. Szeged. **24** (1963), 34–59.
- [Gra70] G. Grätzer. *Two Mal'cev type theorems in universal algebra*. Journal of Combinatorial Theory **8** (1970), 334–342.
- [Gra78] G. Grätzer. *General Lattice Theory*. Academic Press, New York, 1978.
- [Gra88] G. Grätzer. *On the automorphism group and the complete congruence lattice of a complete lattice*. Abstracts Amer. Math. Soc. **9** (1988), Abstract 88T-06-215.
- [Gra91] G. Grätzer and H. Lakser. *On complete congruence lattices of complete lattices*. Trans. Amer. Math. Soc. **327** (1) (1991), 385–405.
- [Gra95] G. Grätzer and E.T. Schmidt. *Complete congruence lattices of complete lattices*. Journal of Algebra **171** (1995), 204–229.

- [Hal44] M. Hall and R.P. Dilworth. *The embedding problem for modular lattices*. Annals of Math. **45** (1944), 450–456.
- [Has52] J. Hashimoto. *Ideal theory for lattices*. Math. Japon. **2** (1952), 149–186.
- [Her73] C. Herrmann. *Weak (projective) radius and finite equational bases for classes of lattices*. Algebra Universalis **3** (1973), 51–58.
- [Her74] C. Herrmann, and A.P. Huhn. *Lattices of normal subgroups which are generated by frames*. Colloquia Math. Soc. János Bolyai, 14. Lattice Theory, Szeged (1974), 97–136.
- [Huh89a] A.P. Huhn. *On the representation of distributive algebraic lattices. II*. Acta. Sci. Math. **53** (1989), 3–10.
- [Huh89b] A.P. Huhn. *On the representation of distributive algebraic lattices. III*. Acta. Sci. Math. **53** (1989), 11–18.
- [Iqb66] Iqbalunnisa. *Normal, simple and neutral congruences on lattices*. Illinois J. Math. **10** (1966), 227–234.
- [Iqb71] Iqbalunnisa. *On lattices whose lattices of congruences are Stone lattices*. Fund. Math. **70** (1971), 315–318.
- [Jan68] M.F. Janowitz. *Perspective properties of relatively complemented lattices*. J. Nat. Sci. and Math. Lahore, W. Pakistan **8** (1968), 193–210.
- [Jip89] P. Jipsen and H. Rose. *Absolute retracts and amalgamation in certain congruence distributive varieties*. Canadian Math. Bull. **32** (1989), 309–313.
- [Jip92] P. Jipsen and H. Rose. *Varieties of Lattices*. Lecture Notes in Mathematics Volume 1533. Springer-Verlag, Berlin, Heidelberg, 1992.
- [Jon67] B. Jónsson. *Algebras whose congruence lattices are distributive*. Math. Scand. **21** (1967), 110–121.
- [Jon79] B. Jónsson and I. Rival. *Lattice varieties covering the smallest non-modular variety*. Pacific J. Math. **82** (1979), 463–478.
- [Jon90] B. Jónsson. *Amalgamation in small varieties of lattices*. J. of Pure and Applied Algebra **68** (1990), 195–208.
- [Lai96] L. Laing, C. Naturman and H. Rose. *Amalgamation in B-lattice varieties*. Preprint (1996).
- [Lyn54] R. Lyndon. *Identities in finite algebras*. Proc. Amer. Math. Soc. **5** (1954), 8–9.

- [Mal54] A.I. Malcev. *On the general theory of algebraic systems*. Mat. Sb. (N.S.) **35** (1954), 3–20.
- [McK70] R. McKenzie. *Equational bases for lattice theories*. Math. Scand. **27** (1970), 24–38.
- [McK87] R. McKenzie, G. McNulty and W. Taylor. *Algebras, Lattices, Varieties Volume I*. Wadsworth, Inc., Belmont, California, 1987.
- [Ore35] O. Ore. *On the foundation of abstract algebra (I)*. Ann. of Math. **36** (1935), 406–437.
- [Pie68] R.S. Pierce. *Introduction to the theory of abstract algebras*. Holt, Rinehart and Winston, 1968.
- [Pix63] A.F. Pixley. *Distributivity and permutability of congruence relations in equational classes of algebras*. Proc. Amer. Math. Soc. **14** (1963), 105–109.
- [Pud76] P. Pudlák. *A new proof of the congruence lattice representation theorem*. Algebra Universalis **6** (1976), 269–275.
- [Pud85] P. Pudlák. *On congruence lattices of lattices*. Algebra Universalis **20** (1985), 96–114.
- [Reu87] K. Reuter and R. Wille. *Complete congruence relations of concept lattices*. Acta. Sci. Math. **51** (1987), 319–327.
- [Ros84] H. Rose. *Nonmodular lattice varieties*. Memoirs of Amer. Math. Soc, **47(292)** (1984).
- [Sch45] M.P. Schützenberger. *Sur certaines axiomes de la théorie des structures*. C.R. Acad. Sci. Paris **221** (1945), 218–220.
- [Sch74] E.T. Schmidt. *Every finite distributive lattice is the congruence lattice of some modular lattice*. Algebra Universalis **4** (1974), 49–57.
- [Sch75] E.T. Schmidt. *On the length of a the congruence lattice of a lattice*. Algebra Universalis **5** (1975), 98–100.
- [Sch81] E.T. Schmidt. *The ideal lattice of a distributive lattice with 0 is the congruence lattice of a lattice*. Acta Sci. Math. **43** (1981), 153–168.
- [Sch82] E.T. Schmidt. *A survey on Congruence Lattice Representations*. Teubner Texts in Mathematics, 42 BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1982.
- [Sch84] E.T. Schmidt. *Congruence lattices of complemented modular lattices*. Algebra Universalis **18** (1984), 386–395.

- [Tan52] T. Tanaka. *Canonical subdirect factorizations of lattices*. J. Sci. Hiroshima Univ., Ser. A **16** (1952), 239–246.
- [Tar46] A. Tarski. *A remark on functionally free algebras*. Ann. of Math.(2) **47** (1946), 163–165.
- [Tay72] W. Taylor. *Residually small varieties*. Algebra Universalis **2** (1972), 33–53.
- [Tis94] M. Tischendorf. *On the representation of distributive semilattices*. Algebra Universalis **31** (1994), 446–455.
- [Teo90] S.-K. Teo. *Representing finite lattices as complete congruence lattices of complete lattices*. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **33** (1990), 177–182.
- [Wil72] R. Wille. *Primitive subsets of lattices*. Algebra Universalis **2** (1972), 95–98.
- [Wil83] R. Wille. *Subdirect decompositions of concept lattices*. Algebra Universalis **17** (1983), 275–287.
- [Yas74] M. Yashuhara. *The Amalgamation property, the universal-homogeneous models and generic models*. Math. Scand. **34** (1974), 5–36.