



A Review of the Landau-Pomeranchuk-Migdal Effect

by

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Abstract

The Bethe-Heitler formula is the correct expression for the energy radiated in the form of bremsstrahlung when a charged particle interacts in isolation with the Coulomb field of a nucleus. When the effects of multiple scattering are taken into account, however, the formula needs modification. This is the Landau-Pomeranchuk-Migdal effect. We review here several approaches for the calculation of the revised spectrum and compare the results with experiment.

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Introduction

The process of bremsstrahlung has been well understood since Bethe and Heitler published their result in 1934 [1]. Their result is applicable to the case of a charged particle interacting in isolation with the Coulomb field of a nucleus. The emission of the bremsstrahlung photon does not occur instantaneously however. We can crudely argue that when an electron emits a photon there is a minimum distance which the photon must move away from the electron before we can distinguish the two particles from one another. This distance is of the order of the wavelength of the photon. If the electron is relativistic then the separation velocity between the electron and photon is very low, as seen from the laboratory frame. For very low energy (i.e. long wavelength) photons the pathlength of the electron over which the photon is "shaken off" can become very large compared to the mean distance between atoms in the medium through which the electron is travelling. The electron could thus have interactions with one or more other atoms before the initial photon is properly formed. These additional interactions will interfere destructively with the formation of the photon and so alter the radiation spectrum.

The first estimate of the effects of these subsequent interactions was made by Landau and Pomeranchuk in 1953 [10]. Their starting point is the classical electrodynamics expression for the energy radiated by an accelerating charged particle. Multiple scattering is included by inserting averaged quantities from scattering theory. Their calculation can only be considered an order of magnitude estimate. In the following year Migdal published a calculation based on the kinetic equation method [14]. Here distribution functions of scattered particles are derived and then used to perform the averaging. In 1956 he published a full quantum mechanical derivation of the effect [26]. The effect has thus subsequently become known as the Landau-Pomeranchuk-Migdal (LPM) effect.

Quantum electrodynamics is the best theory that we have to describe the interactions of electrons and photons. I therefore use QED in Chapter 1 to derive the Bethe-Heitler result. I then proceed in Chapter 2 to show that the classical expression for bremsstrahlung agrees with the QED expression in the limit of low frequency radiation. This serves to justify the use of classical theory for the treatment of the LPM effect. The full field theory derivation lies beyond the scope of this thesis. (It is, however, reassuring that Migdal's

quantum [26, 28, 29, 30] and classical derivations yield the same result.) I have calculated the effect in three separate ways. Firstly by the Landau-Pomeranchuk method, secondly by the functional integration method and lastly by means of the kinetic equation method. The results are then compared with recent experimental data.

Chapter 1

Bremsstrahlung Spectrum from a Static Coulomb Field

The process of bremsstrahlung involves the emission of a photon by an electron in the field of a nucleus [1, 2, 3, 4]. There are two Feynman diagrams that contribute to the process. The electron has a choice to either first interact with the Coulomb field and then emit the photon or vice versa. The amplitude for the process is a sum of these two contributions.

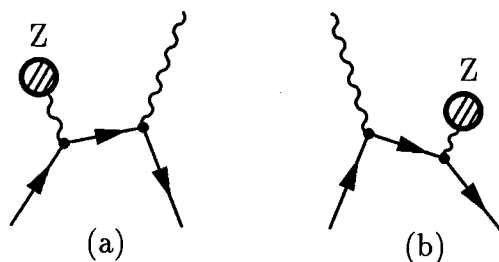


Figure 1.1: Bremsstrahlung diagrams

Both the diagrams of figure 1.1 have two vertices and so are second order. There can be no first order emission of radiation by an electron, i.e. no emission in a vacuum since it would be impossible to conserve both energy and momentum.

We denote the photon 4-momentum by k^μ and the polarization by ϵ^μ . We have $k^2 = 0$ (the photon is massless) and $\epsilon_\mu k^\mu = 0$ as electromagnetic waves are transversely polarized. In addition ϵ^μ is chosen to be normalized such that $\epsilon^2 = -1$. The electron has initial momentum p_i and spin α and final momentum p_f and spin β .

The external field is accounted for by replacing free spinors by solutions of the Dirac equation in the presence of the central field A^μ such that

$$A^0 = -\frac{Ze}{4\pi|\vec{x}|} = -Ze \int \frac{d^3x}{(2\pi)^3} e^{-i\vec{q}\cdot\vec{x}} \frac{1}{|\vec{q}|^2}, \quad \vec{A} = 0$$

where \vec{q} is the 3-momentum transferred to the nucleus, $\vec{q} = \vec{p}_f + \vec{k} - \vec{p}_i$.

The S-matrix element is

$$S_{fi} = -ie^2 \int d^4x d^4y \left[e^{ik\cdot x} \bar{\psi}_f(x) \not{\epsilon} S^F(x-y) \not{A}^{ext}(y) \psi_i(y) + e^{ik\cdot y} \bar{\psi}_f(x) \not{A}^{ext}(x) S^F(x-y) \not{\epsilon} \psi_i(y) \right] \quad (1.1)$$

The first term comes from figure 1.1 (a) and the second from (b). It is convenient to transform 1.1 into momentum space. This yields

$$S_{fi} = \frac{iZe^3}{|\vec{p}_f + \vec{k} - \vec{p}_i|^2} 2\pi \delta(p_i^0 - p_f^0 - k^0) \times \bar{u}(p_f, \beta) \left[\not{\epsilon} \frac{1}{\not{p}_f + \not{k} - m} \gamma^0 + \gamma^0 \frac{1}{\not{p}_i - \not{k} - m} \not{\epsilon} \right] u(p_i, \alpha) \quad (1.2)$$

We choose our normalization such that

$$\sum_{s=\alpha\text{ or } \beta} u(p, s) \bar{u}(p, s) = \frac{\not{p} + m}{2m} \quad (1.3)$$

It follows that

$$d\sigma = \frac{m}{p_i^0 |\mathbf{v}_i|} Z^2 e^6 \int 2\pi \delta(p_f^0 + k_f^0 - p_i^0) \left| \frac{[\dots]}{|\vec{q}|^2} \right|^2 \frac{m d^3 p_f d^3 k}{2p_f^0 k^0 (2\pi)^6}$$

The square bracket is the one in 1.2. We observe neither the final state spin nor the final polarization. So we sum over ϵ and β . We also do not know the initial spin state so we average over α . Let $\omega = k^0$ and Ω_γ, Ω_e be the directions of the outgoing photon and electron respectively. Thus

$$d\sigma = \frac{Z^2 \alpha^3 m^2 |\vec{p}_f|}{\pi^2 |\vec{p}_i| |\vec{q}|^4} F \omega d\omega d\Omega_\gamma d\Omega_e \quad (1.4)$$

$$F = \frac{1}{2} \sum_\epsilon Tr \left[\left(\not{\epsilon} \frac{\not{p}_f + \not{k} + m}{2p_f \cdot k} \gamma^0 - \gamma^0 \frac{\not{p}_i - \not{k} + m}{2p_i \cdot k} \not{\epsilon} \right) \left(\frac{\not{p}_i + m}{2m} \right) \times \left(\gamma^0 \frac{\not{p}_f + \not{k} + m}{2p_f \cdot k} \not{\epsilon} - \not{\epsilon} \frac{\not{p}_i - \not{k} + m}{2p_i \cdot k} \gamma^0 \right) \left(\frac{\not{p}_f + m}{2m} \right) \right] \quad (1.5)$$

F can be decomposed into three parts:

$$\begin{aligned}
F &= \frac{1}{2^5 m^2} (F_1 + F_2 + F_3) \\
F_1 &= \frac{1}{(p_f \cdot k)^2} \sum_{\epsilon} \text{Tr} \left[\not{\epsilon} (\not{p}_f + \not{k} + m) \gamma^0 (\not{p}_i + m) \gamma^0 (\not{p}_f + \not{k} + m) \not{\epsilon} (\not{p}_f + m) \right] \\
F_2 &= F_1(p_i \leftrightarrow -p_f) \\
F_3 &= -\frac{1}{(p_f \cdot k)(p_i \cdot k)} \sum_{\epsilon} \text{Tr} \{ [\gamma^0 (\not{p}_i - \not{k} + m) \not{\epsilon} (\not{p}_i + m) \gamma^0 \\
&\quad \times (\not{p}_f + \not{k} + m) \not{\epsilon} (\not{p}_f + m)] + (p_i \leftrightarrow -p_f) \}
\end{aligned}$$

Although ϵ is normalized we are still free to choose one of its components to be zero. We take $\epsilon^0 = 0$. We expand out all the brackets in the decomposition of F and use the properties of the γ matrices to calculate the trace. As an example we shall calculate the first term in F_1 .

$$\begin{aligned}
\text{Tr}[\not{\epsilon} \not{p}_f \gamma^0 \not{p}_i \gamma^0 \not{p}_f \not{\epsilon} \not{p}_f] &= \text{Tr}[\gamma^0 \not{p}_i \gamma^0 \not{p}_f \not{\epsilon} \not{p}_f (2\epsilon \cdot p_f - \not{p}_f \not{\epsilon})] \\
&= 2(\epsilon \cdot p_f) \text{Tr}[\gamma^0 \not{p}_i \gamma^0 \not{p}_f (2\epsilon \cdot p_f - \not{p}_f \not{\epsilon})] + m^2 \text{Tr}[\gamma^0 \not{p}_i \gamma^0 \not{p}_f] \\
&= (4(\epsilon \cdot p_f)^2 + m^2) (8p_i^0 p_f^0 - 4p_i \cdot p_f) + 8m^2 (\epsilon \cdot p_f) (\epsilon \cdot p_i)
\end{aligned}$$

Here we have used the fact that $\not{a} \not{b} + \not{b} \not{a} = 2a \cdot b$ and that $p_f^2 = p_i^2 = m^2$. The other contributions can be calculated in a similar fashion. The result of this lengthy calculation is

$$\begin{aligned}
F_1 &= \frac{8}{(k \cdot p_f)^2} \sum_{\epsilon} [2(\epsilon \cdot p_f)^2 (m^2 + 2p_i^0 p_f^0 + 2p_i^0 \omega - p_i \cdot p_f - k \cdot p_i) + \\
&\quad + 2\epsilon \cdot p_f \epsilon \cdot p_i k \cdot p_f + 2p_i^0 \omega k \cdot p_f - p_i \cdot k p_f \cdot k] \\
F_2 &= F_1(p_i \leftrightarrow -p_f) \\
F_3 &= \frac{16}{(p_f \cdot k)(p_i \cdot k)} \sum_{\epsilon} [\epsilon \cdot p_i \epsilon \cdot p_f (p_i \cdot k - p_f \cdot k + 2p_i \cdot p_f - 4p_i^0 p_f^0 - 2m^2) \\
&\quad + (\epsilon \cdot p_f)^2 k \cdot p_i - (\epsilon \cdot p_i)^2 k \cdot p_f + p_i \cdot k p_f \cdot k - m^2 \omega^2 \\
&\quad + \omega (\omega p_i \cdot p_f - p_i^0 p_f \cdot k - p_f^0 p_i \cdot k)]
\end{aligned}$$

Let θ_f be the angle that the photon makes with the final electron momentum, θ_i the angle of the photon and the initial momentum and ϕ the angle between the planes (\vec{k}, \vec{p}_f) and (\vec{k}, \vec{p}_i) as shown in figure 1.2.

It can be shown [5] that for any two conserved currents a^μ and b^μ (that is $k \cdot a = k \cdot b = 0$) that

$$\sum_{\epsilon} \epsilon_{\mu} a^{\mu} \epsilon_{\nu} b^{\nu} = -a \cdot b \quad (1.6)$$

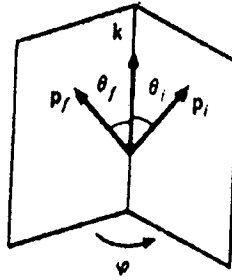


Figure 1.2: Kinematics of the bremsstrahlung process

p_f and p_i are not generally conserved currents. However, their projection into a plane normal to k is a conserved current. This projection has a magnitude $|\vec{p}| \sin \theta$. Therefore, with $\epsilon_0 = 0$

$$\sum_{\epsilon} (\vec{\epsilon} \cdot \vec{p}_f)^2 = |\vec{p}_f|^2 \sin^2 \theta_f \quad \sum_{\epsilon} (\vec{\epsilon} \cdot \vec{p}_i)^2 = |\vec{p}_i|^2 \sin^2 \theta_i$$

$$\sum_{\epsilon} (\vec{\epsilon} \cdot \vec{p}_f)(\vec{\epsilon} \cdot \vec{p}_i) = |\vec{p}_f| |\vec{p}_i| \sin \theta_f \sin \theta_i \cos \phi$$

At this point we shall make a small change in notation:

$$p_f^0 = E_f \quad p_i^0 = E_i \quad |\vec{p}_f| = p_f \quad |\vec{p}_i| = p_i \quad |\vec{q}^2| = q^2$$

The differential cross section 1.4 can then be written as

$$d\sigma = \frac{Z^2 \alpha^3}{(2\pi)^2} \frac{p_f}{p_i q^4} \frac{d\omega}{\omega} d\Omega_\gamma d\Omega_e$$

$$\times \left[\frac{p_f^2 \sin^2 \theta_f}{(E_f - p_f \cos \theta_f)^2} (4E_i^2 - q^2) + \frac{p_i^2 \sin^2 \theta_i}{(E_i - p_i \cos \theta_i)^2} (4E_f^2 - q^2) \right.$$

$$+ 2\omega^2 \frac{p_i^2 \sin^2 \theta_i + p_f^2 \sin^2 \theta_f}{(E_f - p_f \cos \theta_f)(E_i - p_i \cos \theta_i)} - 2 \frac{p_f p_i \sin \theta_i \sin \theta_f \cos \phi}{(E_f - p_f \cos \theta_f)(E_i - p_i \cos \theta_i)}$$

$$\left. \times (4E_i E_f - q^2 + 2\omega^2) \right] \quad (1.7)$$

This result was first derived by Bethe and Heitler in 1934 [1]. To get the total cross section we integrate 1.7 over both the electron and photon escape angles. The following integrals are required:

$$\int_0^\pi \frac{\sin^2 \theta}{1 - a \cos \theta} d\theta = \frac{\pi}{a^2} (1 - \sqrt{1 - a^2})$$

$$\int_0^\pi \frac{\sin^3 \theta}{(1 - a \cos \theta)^2} d\theta = \frac{2}{1 - a^2} + \frac{4}{a^2} \left[1 - \ln \frac{1 + a}{1 - a} \right]$$

The result then for the cross section for an electron scattering off a static potential is

$$\begin{aligned} \sigma = & \frac{Z^2 \alpha^3 p_f d\omega}{2\pi m^2 p_i \omega} \left\{ \frac{4}{3} - 2E_i E_f \frac{p_i^2 + p_f^2}{p_i^2 p_f^2} + m^2 \left(\frac{E_f}{p_i^3} \epsilon_i + \frac{E_i}{p_f^3} \epsilon_f - \frac{\epsilon_i \epsilon_f}{p_i p_f} \right) \right. \\ & + \left[\frac{8 E_i E_f}{3 p_i p_f} + \frac{\omega^2}{p_i^3 p_f^3} (E_i^2 E_f^2 + p_i^2 p_f^2) \right] \chi + \frac{\omega m^2}{2 p_i p_f} \left[\frac{E_i E_f + p_i^2}{p_i^3} \epsilon_i \right. \\ & \left. \left. - \frac{E_i E_f + p_f^2}{p_f^3} \epsilon_f + \frac{2\omega E_i E_f}{p_i^2 p_f^2} \right] \chi \right\} \end{aligned} \quad (1.8)$$

with

$$\begin{aligned} \epsilon_{i,f} &= \ln \frac{E_{i,f} + p_{i,f}}{E_{i,f} - p_{i,f}} \\ \chi &= \ln \frac{p_i^2 + p_i p_f - \omega E_i}{p_i^2 - p_i p_f - \omega E_i} \end{aligned}$$

In the context of the LPM effect the limiting behaviour of 1.8 as $\omega \rightarrow 0$ is of paramount importance. For convenience, and for later comparison purposes, we return to 1.2. It is easy to manipulate the matrix element into the form

$$\begin{aligned} S_{fi} &= \frac{iZe^3}{|\vec{p}_f + \vec{k} - \vec{p}_i|^2} 2\pi \delta(p_i^0 - p_f^0 - k^0) \\ &\quad \times \bar{u}(p_f, \beta) \left[\not{\epsilon} \frac{\not{p}_f + \not{k} + m}{(p_f + k)^2 - m^2} \gamma^0 + \gamma^0 \frac{\not{p}_i - \not{k} + m}{(p_i - k)^2 - m^2} \not{\epsilon} \right] u(p_i, \alpha) \\ &= \frac{iZe^3}{|\vec{p}_f + \vec{k} - \vec{p}_i|^2} 2\pi \delta(p_i^0 - p_f^0 - k^0) \\ &\quad \times \bar{u}(p_f, \beta) \left[\not{\epsilon} \frac{\not{p}_f + m}{2p_f \cdot k} \gamma^0 - \gamma^0 \frac{\not{p}_i + m}{2p_i \cdot k} \not{\epsilon} \right] u(p_i, \alpha) \end{aligned} \quad (1.9)$$

where we have discarded a factor of k in the numerator under the pretext $\omega \rightarrow 0$. We make use of the identity $\not{\epsilon} \not{b} + \not{b} \not{\epsilon} = 2a \cdot b$ once again to write 1.9 as

$$\begin{aligned} S_{fi} &= \frac{iZe^3}{|\vec{p}_f + \vec{k} - \vec{p}_i|^2} 2\pi \delta(p_i^0 - p_f^0 - k^0) \\ &\quad \times \bar{u}(p_f, \beta) \left[\frac{2\epsilon \cdot p_f - (\not{p}_f - m) \not{\epsilon}}{2p_f \cdot k} \gamma^0 - \gamma^0 \frac{2\epsilon \cdot p_i - \not{\epsilon} (\not{p}_i - m)}{2p_i \cdot k} \right] u(p_i, \alpha) \end{aligned}$$

From the Dirac equation we have that $\bar{u}(\not{p} - m) = (\not{p} - m)u = 0$ and so

$$S_{fi} = \frac{iZe^3}{|\vec{p}_f + \vec{k} - \vec{p}_i|^2} 2\pi \delta(p_i^0 - p_f^0 - k^0) \bar{u}(p_i, \beta) \gamma^0 u(p_i, \alpha) \left[\frac{\epsilon \cdot p_f}{k \cdot p_f} - \frac{\epsilon \cdot p_i}{k \cdot p_i} \right] \quad (1.10)$$

The factor $\bar{u}\gamma^0u$ is exactly the one which arises in the calculation of the elastic scattering cross section. It follows that

$$\frac{d\sigma}{d\Omega_{e,\omega\rightarrow 0}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} \frac{e^2}{4\pi^2\omega} \omega^2 d\omega d\Omega_\gamma \sum_\epsilon \left[\frac{\epsilon \cdot p_f}{k \cdot p_f} - \frac{\epsilon \cdot p_i}{k \cdot p_i} \right]^2 \quad (1.11)$$

where

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} = \frac{4Z^2\alpha^2m^2}{|\bar{q}|^4} |\bar{u}\gamma^0u|^2$$

Investigation of 1.11 shows that the photon energy spectrum behaves as $\frac{d\omega}{\omega}$ and so the probability for the emission of a photon with $\omega = 0$ is infinite. This is the "infrared catastrophe".

Let us also consider the behaviour of 1.8 where both the incident and emerging electrons are highly relativistic, i.e. $E_i, E_f \gg m$. The expression for the cross section simplifies to

$$\sigma = \frac{Z^2\alpha^3}{\pi m^2} \frac{E_f}{E_i} \frac{d\omega}{\omega} \left[\frac{E_i^2 + E_f^2}{E_i E_f} - \frac{2}{3} \right] \left[2 \ln \frac{2E_i E_f}{m\omega} - 1 \right] \quad (1.12)$$

Our entire derivation has thus far been based on the assumption that the field of the nucleus is a pure Coulomb field. For many-electron atoms this is not the case. The field is modified by the screening of the inner electrons by the outer ones. The effects of this screening are largely dependent on the energy of the electron. The photon which the electron exchanges with the nucleus is off mass shell and so the effective distance over which the bremsstrahlung interaction occurs (the impact parameter or formation zone) is determined by the uncertainty principle. The magnitude of the momentum transferred to the nucleus is

$$\begin{aligned} q &= \sqrt{E_i^2 - m^2} - \sqrt{E_f^2 - m^2} - \omega \\ &\simeq E_i \left(1 - \frac{m^2}{2E_i^2} \right) - E_f \left(1 - \frac{m^2}{2E_f^2} \right) - \omega \\ &\simeq \frac{\omega E_i}{\gamma^2(E_i - \omega)} \\ &\simeq \frac{\omega}{2\gamma^2} \quad , \omega \ll E_i \end{aligned} \quad (1.13)$$

The impact parameter is of the order of $\frac{1}{q}$. So for high electron energy and soft photon emission the impact parameter becomes large and can easily exceed the atomic radius. The electron then only sees the effective field of the atom. In this case we say that the screening is complete. The effective field is a sum of the Coulomb field and another term which is dependent on the atomic form factor. The form factor is just the Fourier transform of the charge distribution. For the case of heavy nuclei the Thomas-Fermi

model for the charge distribution gives accurate results. The calculation of the effects of the screening is carried out in detail in [1, 3]. The result is

$$\begin{aligned}\sigma &= \frac{Z^2 \alpha^3 E_f d\omega}{\pi m^2 E_i \omega} \left[\left(\frac{E_i^2 + E_f^2}{E_i E_f} - \frac{2}{3} \right) 2 \ln 183 Z^{-1/3} + \frac{2}{9} \right] \\ &\simeq \frac{Z^2 \alpha^3 8 d\omega}{\pi m^2 3 \omega} \ln 183 Z^{-1/3}\end{aligned}\quad (1.14)$$

From this we get the energy radiated in the interval $(\omega, \omega + d\omega)$ per unit time to be

$$\frac{dI}{d\omega_{BH}}(\omega) = \frac{e^2 E_s^2}{3\pi m^2 L} \quad (1.15)$$

where we have made the following definitions for E_s and the radiation length, L :

$$E_s = m\sqrt{4\pi 137}, \quad L^{-1} = \frac{4Z^2 e^4}{137m^2} n \ln 183 Z^{-1/3} \quad (1.16)$$

n is the density of the medium through which the electron is moving. We see that the radiation spectrum is a constant over ω in the highly relativistic limit.

Chapter 2

Bremsstrahlung in Classical Electrodynamics

The electromagnetic fields $F^{\alpha\beta}$ arising from an external source $J^\alpha(x)$ satisfy the inhomogeneous Maxwell equations [6]

$$\partial_\alpha F^{\alpha\beta} = 4\pi J^\beta$$

With the definition of the fields in terms of the potentials and the imposition of the Lorentz condition, $\partial_\alpha A^\alpha = 0$, this becomes

$$\square A^\beta = 4\pi J^\beta \quad (2.1)$$

In the case of an electron

$$J^\beta(x) = e \int d\tau V^\beta(\tau) \delta^4[x - r(\tau)]$$

V^β is the 4-velocity. The retarded solution of 2.1 is

$$A^0(\vec{x}, t) = \frac{e}{(1 - \vec{v}(\tau) \cdot \hat{n})R}, \quad \vec{A}(\vec{x}, t) = \frac{e\vec{v}(\tau)}{(1 - \vec{v}(\tau) \cdot \hat{n})R} \quad (2.2)$$

The potentials 2.2 are known as the Liénard-Wieckert potentials. $\vec{v}(\tau)$ and $\vec{r}(\tau)$ are the velocity and trajectory of the electron respectively. \hat{n} is a unit vector in the direction $\vec{x} - \vec{r}(\tau)$ and $R = |\vec{x} - \vec{r}(\tau_0)|$. These potentials generate the following electric field:

$$\vec{E}(\vec{x}, t) = e \frac{\hat{n} - \vec{v}}{\gamma^2(1 - \vec{v} \cdot \hat{n})^3 R^2} + e \frac{\hat{n} \times \{(\hat{n} - \vec{v}) \times \dot{\vec{v}}\}}{(1 - \vec{v} \cdot \hat{n})^3 R} \quad (2.3)$$

The first term of 2.3 describes the Coulomb field of the electron and the second describes the radiation field. We find that the energy radiated per unit solid angle per unit frequency interval is

$$\frac{d^2 I}{d\omega d\Omega} = 2|\vec{A}(\omega)|^2 \quad (2.4)$$

where $\vec{A}(\omega)$ is the Fourier transform of the quantity

$$\vec{A}(t) = \frac{1}{\sqrt{4\pi}} R \vec{E}$$

It follows that we can write 2.4 as

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi^2} \left| \int_{-\infty}^{\infty} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{v}) e^{i\omega(t - \hat{\mathbf{n}} \cdot \vec{r}(t))} dt \right|^2$$

Lemma 1 $\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{a}) \cdot \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{b}) = (\hat{\mathbf{n}} \times \vec{a}) \cdot (\hat{\mathbf{n}} \times \vec{b})$, $\forall \vec{a}, \vec{b}, \hat{\mathbf{n}} \in \mathfrak{R}^3$ and $\hat{\mathbf{n}}$ a unit vector.

Proof :

$$\begin{aligned} \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{a}) \cdot \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \vec{b}) &= \epsilon_{ijk} n_j (\hat{\mathbf{n}} \times \vec{a})_k \epsilon_{ilm} n_l (\hat{\mathbf{n}} \times \vec{b})_m \\ &= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) n_j n_l (\hat{\mathbf{n}} \times \vec{a})_k (\hat{\mathbf{n}} \times \vec{b})_m \\ &= n^2 (\hat{\mathbf{n}} \times \vec{a}) \cdot (\hat{\mathbf{n}} \times \vec{b}) - \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \vec{a}) \hat{\mathbf{n}} \cdot (\hat{\mathbf{n}} \times \vec{b}) \\ &= (\hat{\mathbf{n}} \times \vec{a}) \cdot (\hat{\mathbf{n}} \times \vec{b}) \quad \square \end{aligned}$$

Hence

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi^2} \left| \int_{-\infty}^{\infty} \hat{\mathbf{n}} \times \vec{v}(t) e^{i\omega(t - \hat{\mathbf{n}} \cdot \vec{r}(t))} dt \right|^2 \quad (2.5)$$

2.5 gives us the classical electrodynamics expression for the energy spectrum. If we know the electron's position and velocity as a function of time then we can calculate the energy radiated. As an example, and also to see how the classical expression compares with the Bethe-Heitler expression, we consider the case where an electron moves at a constant velocity \vec{v} until $t = 0$ where it has an abrupt collision with another particle and then moves at the velocity \vec{v}_1 for all $t > 0$. We assume that the electron is relativistic. The integral in 2.5 is then

$$\begin{aligned} \int_{-\infty}^{\infty} \hat{\mathbf{n}} \times \vec{v}(t) e^{i\omega(t - \hat{\mathbf{n}} \cdot \vec{r}(t))} dt &= \hat{\mathbf{n}} \times \left[\vec{v} \int_{-\infty}^0 e^{i\omega t(1 - \hat{\mathbf{n}} \cdot \vec{v})} dt + \vec{v}_1 \int_0^{\infty} e^{i\omega t(1 - \hat{\mathbf{n}} \cdot \vec{v}_1)} dt \right] \\ &= \frac{i}{\omega} \hat{\mathbf{n}} \times \left[\frac{\vec{v}_1}{1 - \hat{\mathbf{n}} \cdot \vec{v}_1} - \frac{\vec{v}}{1 - \hat{\mathbf{n}} \cdot \vec{v}} \right] \quad (2.6) \end{aligned}$$

Lemma 2 $(\hat{\mathbf{n}} \times \vec{a}) \cdot (\hat{\mathbf{n}} \times \vec{b}) = \sum_{\epsilon} (\vec{\epsilon} \cdot \vec{a})(\vec{\epsilon} \cdot \vec{b})$

Proof :

The ϵ^μ 's are 4-vectors that describe the state of polarization of the emitted photon. We choose $\epsilon^0 = 0$ and transform all our 3-vectors to 4-vectors with the time component equal to 0. The sum is over orthogonal polarization states. 1.6 gives

$$\sum_{\epsilon} (\vec{\epsilon} \cdot (\hat{n} \times \vec{a})) (\vec{\epsilon} \cdot (\hat{n} \times \vec{b})) = (\hat{n} \times \vec{a}) \cdot (\hat{n} \times \vec{b})$$

It is a property of the cross product that $\vec{\epsilon} \cdot (\hat{n} \times \vec{a}) = \vec{a} \cdot (\vec{\epsilon} \times \hat{n})$. \hat{n} is a unit vector in the direction of motion of the photon. $\vec{\epsilon}$ is by definition orthogonal to \hat{n} . The cross product $(\vec{\epsilon} \times \hat{n})$ serves only to rotate $\vec{\epsilon}$ by 90 degrees, but it remains in the plane perpendicular to \hat{n} . Since the sum over ϵ is over orthogonal polarizations, this leaves the sum invariant and so

$$\begin{aligned} \sum_{\epsilon} \vec{\epsilon} \cdot (\hat{n} \times \vec{a}) \vec{\epsilon} \cdot (\hat{n} \times \vec{b}) &= \sum_{\epsilon} (\vec{\epsilon} \cdot \vec{a}) (\vec{\epsilon} \cdot \vec{b}) \\ &= (\hat{n} \times \vec{a}) \cdot (\hat{n} \times \vec{b}) \quad \square \end{aligned}$$

So

$$\begin{aligned} \frac{d^2 I}{d\omega d\Omega} &= \frac{e^2}{4\pi^2} \left[\frac{(\hat{n} \times \vec{v}_1)^2}{(1 - \hat{n} \cdot \vec{v}_1)^2} + \frac{(\hat{n} \times \vec{v})^2}{(1 - \hat{n} \cdot \vec{v})^2} - 2 \frac{(\hat{n} \times \vec{v}_1) \cdot (\hat{n} \times \vec{v})}{(1 - \hat{n} \cdot \vec{v}_1)(1 - \hat{n} \cdot \vec{v})} \right] \\ &= \frac{e^2}{4\pi^2} \sum_{\epsilon} \left| \left(\frac{\vec{v}_1}{1 - \hat{n} \cdot \vec{v}_1} - \frac{\vec{v}}{1 - \hat{n} \cdot \vec{v}} \right) \cdot \vec{\epsilon} \right|^2 \\ &= \frac{e^2 \omega^2}{4\pi^2} \sum_{\epsilon} \left| \left(\frac{\vec{p}_1}{p_1 \cdot k} - \frac{\vec{p}}{p \cdot k} \right) \cdot \vec{\epsilon} \right|^2 \end{aligned} \quad (2.7)$$

It is convenient at this point to transform our energy spectrum to a differential cross section. The number of photons of energy ω emitted per unit frequency interval is $N = \frac{I}{\omega}$ and so

$$\frac{dN}{d\omega} = \frac{e^2}{4\pi^2 \omega} \omega^2 d\omega d\Omega \sum_{\epsilon} \left(\frac{\epsilon \cdot p_1}{k \cdot p_1} - \frac{\epsilon \cdot p}{k \cdot p} \right)^2$$

To get the cross section we multiply the above expression by the cross section for elastic (Mott) scattering [31]. Hence

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega} \right)_{\text{elastic}} \frac{e^2}{4\pi^2 \omega} \omega^2 d\omega d\Omega \sum_{\epsilon} \left(\frac{\epsilon \cdot p_1}{k \cdot p_1} - \frac{\epsilon \cdot p}{k \cdot p} \right)^2 \quad (2.8)$$

2.8 is identical to 1.11, i.e. the classical expression for bremsstrahlung agrees with the soft photon limit of the field theory expression. So we are justified in the use of classical electrodynamics for the investigation of the LPM effect.

Let us return to 2.7. We employ 1.6 to write

$$\begin{aligned}\frac{d^2 I}{d\omega d\Omega} &= -\frac{e^2 \omega^2}{4\pi^2} \left(\frac{p_1^\mu}{p_1 \cdot k} - \frac{p^\mu}{p \cdot k} \right)^2 \\ &= \frac{e^2 \omega^2}{4\pi^2} \left(\frac{2p_1 \cdot p}{(p_1 \cdot k)(p \cdot k)} - \frac{m^2}{(p_1 \cdot k)^2} - \frac{m^2}{(p \cdot k)^2} \right)\end{aligned}\quad (2.9)$$

We need to perform an integration over $d\Omega$ of 2.9.

$$\begin{aligned}\int d\Omega \frac{m^2}{(p \cdot k)^2} &= \int d\Omega \frac{m^2}{(p_1 \cdot k)^2} \\ &= \frac{m^2}{\omega^2 E^2} 2\pi \int d(\cos \theta) \frac{1}{(1 - v \cos \theta)^2} = \frac{4\pi}{\omega^2}\end{aligned}$$

To find $\int d\Omega \frac{2p_1 \cdot p}{(p_1 \cdot k)(p \cdot k)}$ we need to use Feynman's trick [7]

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2}$$

So

$$\begin{aligned}\int d\Omega \frac{2p_1 \cdot p}{(p_1 \cdot k)(p \cdot k)} &= \int_0^1 dx \int d\Omega \frac{2p_1 \cdot p}{[p_1 \cdot kx + p \cdot k(1-x)]^2} \\ &= \frac{2p_1 \cdot p}{\omega^2 E^2} \int_0^1 dx \int d\Omega \frac{1}{[1 - \hat{\mathbf{n}} \cdot (\vec{v}_1 x + \vec{v}(1-x))]^2} \\ &= \frac{2p_1 \cdot p}{\omega^2 E^2} \int_0^1 dx \int_{-1}^1 d(\cos \theta) \int_0^{2\pi} d\phi \frac{1}{(1 - |\vec{v}_1 x + \vec{v}(1-x)| \cos \theta)^2} \\ &= \frac{2p_1 \cdot p}{\omega^2 E^2} 4\pi \int_0^1 dx \frac{1}{1 - |\vec{v}_1 x + \vec{v}(1-x)|^2}\end{aligned}$$

The integral over x is of the form

$$\begin{aligned}\int_0^1 \frac{dx}{ax^2 + bx + c} &= \frac{1}{\sqrt{b^2 - 4ac}} \ln \left| \frac{2ax + b - \sqrt{b^2 - 4ac}}{2ax + b + \sqrt{b^2 - 4ac}} \right| \Big|_0^1 \\ &= \frac{2}{\sqrt{b^2 - 4ac}} \ln \left| \frac{b + \sqrt{b^2 - 4ac}}{b - \sqrt{b^2 - 4ac}} \right| \quad (a = -b \text{ here})\end{aligned}$$

If we denote the angle between \vec{v} and \vec{v}_1 by α , i.e. $\sin \alpha = \frac{|\vec{v}_1 - \vec{v}|}{v}$, and define $\xi = \gamma v \sin \frac{\alpha}{2}$ then we find

$$\int d\Omega \frac{2p_1 \cdot p}{(p_1 \cdot k)(p \cdot k)} = \frac{2p_1 \cdot p}{\omega^2 E^2} 4\pi \frac{\gamma^2}{2\xi \sqrt{\xi^2 + 1}} \ln \left| \frac{\xi + \sqrt{\xi^2 + 1}}{\xi - \sqrt{\xi^2 + 1}} \right|$$

Now

$$\begin{aligned}\gamma^2 \frac{p_1 \cdot p}{E^2} &= \gamma^2 (1 - v^2 \cos \alpha) \\ &= \gamma^2 \left(1 - v^2 + 2 \frac{\xi^2}{\gamma^2}\right) \\ &= 1 + 2\xi^2\end{aligned}$$

and

$$\begin{aligned}\left| \frac{\xi + \sqrt{\xi^2 + 1}}{\xi - \sqrt{\xi^2 + 1}} \right| &= \left| \frac{(\xi + \sqrt{\xi^2 + 1})^2}{(\xi - \sqrt{\xi^2 + 1})(\xi + \sqrt{\xi^2 + 1})} \right| \\ &= \left| \xi + \sqrt{\xi^2 + 1} \right|^2\end{aligned}$$

Therefore

$$\frac{dI}{d\omega} = \frac{2e^2}{\pi} \left[\frac{2\xi^2 + 1}{\xi\sqrt{\xi^2 + 1}} \ln \left| \xi + \sqrt{\xi^2 + 1} \right| - 1 \right] \quad (2.10)$$

The nature of the radiation spectrum is substantially different for the two extremes of ξ . $|\xi| \ll 1$:

$$\begin{aligned}\frac{dI}{d\omega} &\simeq \frac{2e^2}{\pi} \left[\left(1 + \frac{3}{2}\xi^2\right) \ln(1 + \xi)^{1/\xi} - 1 \right] \\ &\simeq \frac{3e^2}{\pi} \xi^2\end{aligned} \quad (2.11)$$

$|\xi| \gg 1$:

$$\frac{dI}{d\omega} \simeq \frac{4e^2}{\pi} \ln 2\xi \quad (2.12)$$

The spectrum 2.10 is plotted in figure 2.1.

We can easily modify 2.6 for the case of two collisions [8]. Suppose the electron moves at a constant velocity \vec{v}_0 until time t_1 where it has a collision and the velocity changes abruptly to \vec{v}_1 . Then at time t_2 another collision changes the velocity to \vec{v}_2 . 2.6 becomes

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{\mathbf{n}} \times \vec{v}(t) e^{i\omega(t - \hat{\mathbf{n}} \cdot \vec{r}(t))} dt &= \frac{i}{\omega} \left[\left(\frac{\vec{v}_1}{1 - \hat{\mathbf{n}} \cdot \vec{v}_1} e^{i\omega t_1 \hat{\mathbf{n}} \cdot (\vec{v}_0 - \vec{v}_1)} - \frac{\vec{v}_0}{1 - \hat{\mathbf{n}} \cdot \vec{v}_0} \right) e^{i\omega t_1 (1 - \hat{\mathbf{n}} \cdot \vec{v}_0)} \right. \\ &\quad \left. + \left(\frac{\vec{v}_2}{1 - \hat{\mathbf{n}} \cdot \vec{v}_2} e^{i\omega t_2 \hat{\mathbf{n}} \cdot (\vec{v}_1 - \vec{v}_2)} - \frac{\vec{v}_1}{1 - \hat{\mathbf{n}} \cdot \vec{v}_1} \right) e^{i\omega t_2 (1 - \hat{\mathbf{n}} \cdot \vec{v}_1)} \right]\end{aligned}$$

Since the angles of scattering are small, $|\vec{v}_j - \vec{v}_{j-1}| \ll 1$ and so

$$\begin{aligned}\int_{-\infty}^{\infty} \hat{\mathbf{n}} \times \vec{v}(t) e^{i\omega(t - \hat{\mathbf{n}} \cdot \vec{r}(t))} dt &= \frac{i}{\omega} \left[\left(\frac{\vec{v}_1}{1 - \hat{\mathbf{n}} \cdot \vec{v}_1} - \frac{\vec{v}_0}{1 - \hat{\mathbf{n}} \cdot \vec{v}_0} \right) e^{i\omega t_1 (1 - \hat{\mathbf{n}} \cdot \vec{v}_0)} \right. \\ &\quad \left. + \left(\frac{\vec{v}_2}{1 - \hat{\mathbf{n}} \cdot \vec{v}_2} - \frac{\vec{v}_1}{1 - \hat{\mathbf{n}} \cdot \vec{v}_1} \right) e^{i\omega t_2 (1 - \hat{\mathbf{n}} \cdot \vec{v}_1)} \right]\end{aligned}$$

Radiation Spectrum from a Single Collision

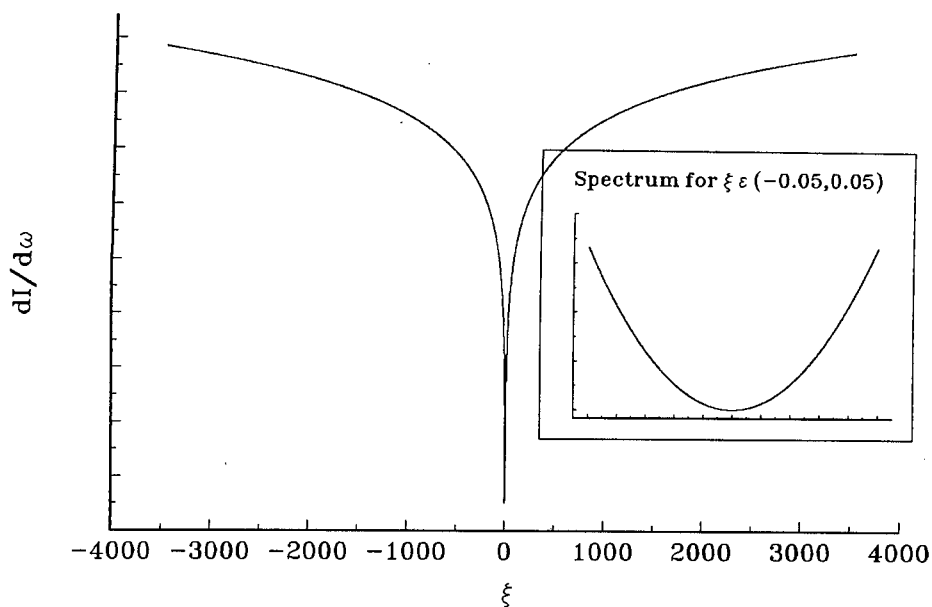


Figure 2.1:

For the case of N collisions this generalizes to

$$\int_{-\infty}^{\infty} \hat{\mathbf{n}} \times \vec{v}(t) e^{i\omega(t - \hat{\mathbf{n}} \cdot \vec{r}(t))} dt = \frac{i}{\omega} \sum_{j=1}^N \left[\frac{\vec{v}_j}{1 - \hat{\mathbf{n}} \cdot \vec{v}_j} - \frac{\vec{v}_{j-1}}{1 - \hat{\mathbf{n}} \cdot \vec{v}_{j-1}} \right] \exp \{i\omega(1 - \hat{\mathbf{n}} \cdot \vec{v}_{j-1})t_j\} \quad (2.13)$$

If small angle scattering occurs very rapidly, i.e. if $\omega(1 - \hat{\mathbf{n}} \cdot \vec{v}_{j-1})t_j \ll 1$ then 2.13 reduces to 2.6 with \vec{v}_1 replaced by \vec{v}_N . So in the low frequency regime if the scattering occurs very rapidly then the radiation from N scatterings is equivalent to the radiation from a single collision, i.e. *there is substantial suppression*.

Chapter 3

A qualitative estimate for bremsstrahlung intensity in condensed matter

This chapter follows closely the work of Galitsky and Gurevich [9]. When an electron emits a photon there is a minimum distance which the photon must move away from the electron before we can distinguish the two particles from one another. This distance is of the order of the wavelength of the photon. If the electron is relativistic then the separation velocity between the electron and photon is very low, as seen from the laboratory frame. For very low energy (i.e. long wavelength) photons the pathlength of the electron over which the photon is "shaken off" can become very large compared to the mean distance between atoms in the medium through which the electron is travelling. The electron could thus have interactions with one or more other atoms before the initial photon is properly formed. These additional interactions will interfere destructively with the formation of the photon and so alter the radiation spectrum.

The fields produced by the particle on traversing nearby points on the trajectory should differ little in phase and therefore add coherently. There is a maximum pathlength for which this remains true. We call this pathlength the coherence length l . We consider radiation emitted at an angle θ in the frequency interval $(\omega, \omega + d\omega)$. The coherence length must be a function of both ω and θ . The amplitude of the field radiated by the electron is proportional to the coherence length and so the intensity of radiation at (ω, θ) is proportional to the square of the coherence length.

$$\frac{d^2 I}{d\omega d\Omega}(\omega, \theta) = A l^2(\omega, \theta) \quad (3.1)$$

where A is the proportionality constant.

After the collision with the nucleus the electron moves with a velocity v . Suppose that

photons are emitted at times t and $t + (l/v)$. When the second photon is emitted the first has moved a distance l/v . From the definition of the coherence length the difference in the pathlengths of the two waves must be less than the wavelength, say $\lambda/2$. From this it follows (see Fig. 3) that

$$\frac{l}{v} - l \cos \theta = \frac{\lambda}{2} \quad (3.2)$$

or that ($v \simeq 1$)

$$l(\omega, \theta) = \frac{1}{2} \frac{\lambda}{1 - v \cos \theta} \quad (3.3)$$

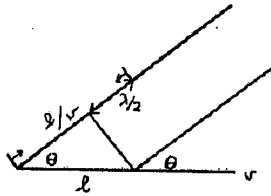


Figure 3.1:

If we substitute 3.3 into 3.1 and integrate over all angles we get the bremsstrahlung spectrum from a unit pathlength:

$$\frac{dI}{d\omega} = \int \frac{d^2 I}{d\omega d\Omega}(\omega, \theta) d\Omega = \frac{\pi}{2} A \frac{\lambda^2}{1 - v} \quad (3.4)$$

We can find the coefficient A by comparing 3.4 with the Bethe-Heitler result 1.15:

$$\frac{dI}{d\omega_{BH}}(\omega) = \frac{e^2}{3\pi} \frac{E_s^2}{m^2 L}$$

For a highly relativistic particle we can write

$$1 - v \simeq \frac{1}{2}(1 - v^2) \simeq \frac{1}{2} \frac{m^2}{E^2}$$

and so

$$A = \frac{1}{\pi \lambda^2} \frac{m^2}{E^2} \frac{dI}{d\omega_{BH}}(\omega)$$

The effects of multiple scattering on bremsstrahlung reduces to the fact that the electron instead of moving along a straight line moves along a winding trajectory of multiple scattering. Since the collision time is very short relative to the mean free time we can assume that only the direction of the electron velocity changes in a collision and not

the magnitude. We denote the angle of multiple scattering by θ_s , i.e. θ_s is the angle between the initial direction of motion and the direction of motion at time t . The velocity component in the initial direction v_{\parallel} is then, on average,

$$v_{\parallel} = v \langle \cos \theta_s \rangle \simeq v \left(1 - \frac{1}{2} \bar{\theta}_s^2\right)$$

Substituting v_{\parallel} for v in 3.2 we get

$$l_s(\omega, \theta) \simeq \frac{\lambda}{2} \left[1 - v \cos \theta \left(1 - \frac{1}{2} \bar{\theta}_s^2\right)\right]^{-1} \quad (3.5)$$

and

$$\begin{aligned} \frac{dI}{d\omega} &= \pi A \lambda^2 \left[1 - v^2 \left(1 - \frac{1}{2} \bar{\theta}_s^2\right)\right]^{-1} \\ &= \frac{dI}{d\omega_{BH}}(\omega) \frac{l_s(\omega, 0)}{l(\omega, 0)} \end{aligned} \quad (3.6)$$

It follows from 3.6 that the true bremsstrahlung spectrum can be found by evaluating the coherence length for zero angle photon emission, i.e. it is not necessary to calculate the entire radiation spectrum. The mean square scattering angle can be shown to be (see A.16, $v \simeq c = 1$)

$$\bar{\theta}_s^2 = \frac{E_s^2 l_s(\omega, 0)}{E^2 L}$$

$l_s(\omega, 0)$ is now implicitly defined in 3.5. We find

$$l_s(\omega, 0) = \frac{m^2 L}{2E_s^2} \left[\sqrt{1 + 4\pi \frac{E^2 E_s^2}{m^4 L \omega}} - 1 \right] \quad (3.7)$$

In the limit $\omega \ll E$, 3.7 reduces to

$$l_s(\omega, 0) = \frac{E}{E_s} \sqrt{\frac{L\pi}{\omega}}$$

and the radiation spectrum 3.6 becomes

$$\frac{dI}{d\omega}(\omega) \approx \frac{e^2 E_s}{3\pi E} \sqrt{\frac{\pi\omega}{L}} \quad (3.8)$$

The Bethe-Heitler spectrum approaches a constant value as $\omega \rightarrow 0$. 3.8 shows that when the effects of multiple scattering are included, then the spectrum approaches zero as $\omega \rightarrow 0$.

Chapter 4

Inclusion of Multiple Scattering by the Landau-Pomeranchuk Method

The first attempt to include the effects of multiple scattering on the bremsstrahlung spectrum were made by Landau and Pomeranchuk in 1953 [10]. Their starting point is the classical expression for the energy radiated 2.5:

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{4\pi^2} \left| \int_{-\infty}^{\infty} \hat{\mathbf{n}} \times d\vec{r} e^{i(\omega t - \vec{k} \cdot \vec{r}(t))} \right|^2$$

Or

$$dI = \frac{e^2 \omega^2}{4\pi^2} d\omega \iiint (\hat{\mathbf{n}} \times d\vec{r}_1) (\hat{\mathbf{n}} \times d\vec{r}_2) e^{i\omega(t_1 - t_2) - i\omega \hat{\mathbf{n}} \cdot (\vec{r}_1 - \vec{r}_2)} d\Omega \quad (4.1)$$

Lemma 3 $(\hat{\mathbf{n}} \times \vec{a}) \cdot (\hat{\mathbf{n}} \times \vec{b}) = \vec{a} \cdot \vec{b} - (\hat{\mathbf{n}} \cdot \vec{a})(\hat{\mathbf{n}} \cdot \vec{b})$

Proof:

$$\begin{aligned} (\hat{\mathbf{n}} \times \vec{a}) \cdot (\hat{\mathbf{n}} \times \vec{b}) &= \epsilon_{ijk} n_j a_k \epsilon_{ilm} n_l b_m \\ &= (\delta_{ji} \delta_{km} - \delta_{jm} \delta_{ki}) n_j n_l a_k b_m \\ &= \vec{a} \cdot \vec{b} - (\hat{\mathbf{n}} \cdot \vec{a})(\hat{\mathbf{n}} \cdot \vec{b}) \quad \square \end{aligned}$$

Hence 4.1 can be written as

$$dI = \frac{e^2 \omega^2}{4\pi^2} d\omega \iiint e^{i\omega(t_1 - t_2)} \{ d\vec{r}_1 \cdot d\vec{r}_2 - (\hat{\mathbf{n}} \cdot d\vec{r}_1)(\hat{\mathbf{n}} \cdot d\vec{r}_2) \} e^{-i\omega \hat{\mathbf{n}} \cdot (\vec{r}_1 - \vec{r}_2)} d\Omega \quad (4.2)$$

We now perform the integration over $d\Omega$ to get the total energy radiated in the frequency interval $(\omega, \omega + d\omega)$. We define $\vec{g} = \omega(\vec{r}_1 - \vec{r}_2)$. The integral over $d\Omega$ involves the following two integrals:

$$I_1 \equiv \int_{4\pi} d\Omega e^{-i\hat{n}\cdot\vec{g}}, \quad I_2 \equiv \int_{4\pi} (\hat{n} \cdot d\vec{r}_1)(\hat{n} \cdot d\vec{r}_2) e^{-i\hat{n}\cdot\vec{g}} d\Omega$$

\vec{g} is a fixed vector in the integration over $d\Omega$. So we are free to choose our axes such that $\vec{g} = (0, 0, g)$. I_1 is then

$$I_1 = 2\pi \int_{-1}^1 e^{-ig \cos \theta} d(\cos \theta) = 4\pi \frac{\sin g}{g} \quad (4.3)$$

Now $\frac{\partial}{\partial \vec{r}_1} I_1 = -i\omega \int \hat{n} d\Omega e^{-i\hat{n}\cdot\vec{g}}$ and $\frac{\partial}{\partial \vec{r}_2} I_1 = i\omega \int \hat{n} d\Omega e^{-i\hat{n}\cdot\vec{g}}$. So

$$I_2 = \frac{1}{\omega^2} \left(d\vec{r}_1 \cdot \frac{\partial}{\partial \vec{r}_1} \right) \left(d\vec{r}_2 \cdot \frac{\partial}{\partial \vec{r}_2} \right) 4\pi \frac{\sin g}{g} \quad (4.4)$$

4.2 can then be written as [11]¹

$$\begin{aligned} dI &= \frac{e^2 \omega^2}{\pi} d\omega \iint dt_1 dt_2 e^{i\omega(t_1 - t_2)} \left[\vec{v}_1 \cdot \vec{v}_2 - \frac{1}{\omega^2} \left(\vec{v}_1 \cdot \frac{\partial}{\partial \vec{r}_1} \right) \left(\vec{v}_2 \cdot \frac{\partial}{\partial \vec{r}_2} \right) \right] \frac{\sin g}{g} \\ &= \frac{e^2 \omega}{\pi} d\omega \iint dt_1 dt_2 e^{i\omega(t_1 - t_2)} \left[\vec{v}_1 \cdot \vec{v}_2 - \frac{1}{\omega^2} \frac{d}{dt_1} \frac{d}{dt_2} \right] \frac{\sin \omega |\vec{r}_1 - \vec{r}_2|}{|\vec{r}_1 - \vec{r}_2|} \end{aligned} \quad (4.5)$$

To proceed from here we need to apply a convention that is used in many problems where integrals like those found in 2.5 are involved. When we need to evaluate $e^{i\omega t}$ at $t = \pm\infty$ the convention is that we take it equal to zero. This makes physical sense as it eliminates the radiation produced when our electron is accelerated to its initial velocity at $t = -\infty$. We are not interested in how the electron reached its initial velocity, but only in what happens thereafter. Likewise we are not interested in the electron's deceleration at $t = \infty$.

We perform integration by parts over both t_1 and t_2 of the second term in 4.5. After applying our convention we obtain

$$dI = \frac{e^2 \omega}{\pi} d\omega \iint dt_1 dt_2 e^{i\omega(t_1 - t_2)} [\vec{v}_1 \cdot \vec{v}_2 - 1] \frac{\sin \omega |\vec{r}_1 - \vec{r}_2|}{|\vec{r}_1 - \vec{r}_2|} \quad (4.6)$$

We choose a system of axes such that the initial direction of motion of the electron defines the z -axis. We introduce a two-dimensional vector $\vec{\theta}$ which lies in the xy -plane and characterises the degree of multiple scattering. We denote the velocity at a time t_1 by \mathbf{v}_1 and by \mathbf{v}_2 at time t_2 . $|\vec{\theta}| = \theta$ is the angle of multiple scattering at the time t_2 .

$$\begin{aligned} \vec{r}_1 - \vec{r}_2 &= \int_0^{t_1} \vec{v}(\tau) d\tau - \int_0^{t_2} \vec{v}(\tau) d\tau = \int_{t_1}^{t_2} \vec{v}(\tau) d\tau \\ &= \int_0^{t_2 - t_1} \vec{v}(\tau + t_1) d\tau, \quad \tau \rightarrow \tau - t_1 \end{aligned} \quad (4.7)$$

¹At this point our calculation diverges from the original calculation of Landau and Pomeranchuk. They erroneously assumed that $\nabla_g \frac{1}{g}$ to be negligibly small.

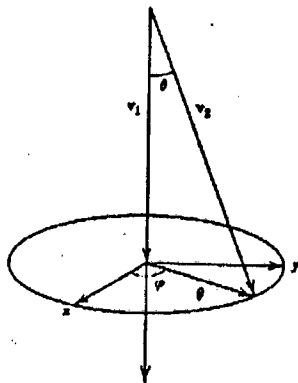


Figure 4.1: The two-dimensional vector $\vec{\theta}$ describing the scattering angle.

We assume that the electron interacts with particles in the medium over a very short time scale relative to its mean free time. We therefore assume that the speed of the electron does not change in the collision, only its direction changes. We also assume that the electron is moving relativistically so that the characteristic angles of scattering are small.

$$\begin{aligned}\vec{v}_2 = \vec{v}(t_1 + \tau) &= \hat{z}v \cos \theta + \hat{\theta}v \sin \theta \\ &\simeq \vec{v}_1 \left(1 - \frac{\theta^2}{2}\right) + v\vec{\theta}\end{aligned}\quad (4.8)$$

4.7 can thus be written as

$$\vec{r}_1 - \vec{r}_2 = \vec{v}_1(t_2 - t_1) - \frac{1}{2}\vec{v}_1 \int_0^{t_2-t_1} \theta^2(\tau) d\tau + v \int_0^{t_2-t_1} \vec{\theta} d\tau$$

We define $t = t_2 - t_1$:

$$\begin{aligned}|\vec{r}_1 - \vec{r}_2|^2 &= \left| \vec{v}_1 \left(t - \frac{1}{2} \int_0^t \theta^2 d\tau\right) + v \int_0^t \vec{\theta} d\tau \right|^2 \\ &= v^2 \left[t^2 - t \int_0^t \theta^2 d\tau + \left(\int_0^t \vec{\theta} d\tau \right)^2 \right] + O(\theta^4)\end{aligned}$$

And so

$$|\vec{r}_1 - \vec{r}_2| \simeq v \left[t - \frac{1}{2} \int_0^t \theta^2 d\tau + \frac{1}{2t} \left(\int_0^t \vec{\theta} d\tau \right)^2 \right]$$

and

$$\frac{1}{|\vec{r}_1 - \vec{r}_2|} \simeq \frac{1}{vt} \left[1 + \frac{1}{2t} \int_0^t \theta^2 d\tau - \frac{1}{2t^2} \left(\int_0^t \vec{\theta} d\tau \right)^2 \right]$$

From 4.8 we have

$$\vec{v}_1 \cdot \vec{v}_2 \simeq v^2 \left(1 - \frac{1}{2}\theta^2\right)$$

We are now in a position to rewrite 4.6 in a form where multiple scattering can be taken into account. We make a change of variables $T \equiv t_1, t \equiv t_2 - T$:

$$\begin{aligned} dI &= \frac{e^2\omega}{\pi} d\omega \int_{-\infty}^{\infty} dT \int_{-\infty}^{\infty} dt \frac{e^{-i\omega t}}{vt} \left[(v^2 - 1) - \frac{1}{2}v^2\theta^2 \right] \sin v\omega \left\{ t - \frac{1}{2} \int \theta^2 d\tau + \frac{1}{2t} \left(\int \vec{\theta} d\tau \right)^2 \right\} \\ &= -\frac{e^2\omega}{\pi v\gamma^2} d\omega \int_{-\infty}^{\infty} dT \int_{-\infty}^{\infty} \frac{dt}{t} e^{-i\omega t} \left[1 + \frac{1}{2}v^2\gamma^2\theta^2 \right] \sin v\omega \left\{ t - \frac{1}{2} \int \theta^2 d\tau + \frac{1}{2t} \left(\int \vec{\theta} d\tau \right)^2 \right\} \quad (4.9) \end{aligned}$$

We are really interested in the average energy radiated by the electron. It is difficult to perform the averaging of the integrand exactly in 4.9 because the scattering angle appears in the argument of the sine function. We can make an order of magnitude estimate for the energy radiated if we replace the average of the integrand by the average of each term separately. The averages that we require are calculated in Appendix A, equations A.17 and A.18. With this approximation the energy radiated per unit frequency interval per unit time is ($q \equiv \frac{E_s^2 v}{E^2 L}$):

$$\begin{aligned} \frac{dI}{d\omega} &= -\frac{e^2\omega}{\pi v\gamma^2} \int_{-\infty}^{\infty} \frac{dt}{t} e^{i\omega t} \left[1 + v\gamma^2 \frac{E_s^2 |t|}{2E^2 L} \right] \sin \omega v \left\{ t - \frac{E_s^2 t |t|}{12E^2 L} \right\} \\ &= \frac{e^2\omega}{\pi v\gamma^2} \int_0^{\infty} \frac{dx}{x} (e^{-i\omega x} + e^{i\omega x}) \left(1 + \frac{1}{2}\gamma^2 qx \right) \sin \left(\frac{1}{12}\omega qx^2 - v\omega x \right) \\ &= \frac{e^2\omega}{\pi v\gamma^2} 2 \int_0^{\infty} \frac{dx}{x} \left(1 + \frac{1}{2}\gamma^2 qx \right) \cos \omega x \sin \left(\frac{1}{12}\omega qx^2 - v\omega x \right) \\ &= \frac{e^2\omega}{\pi v\gamma^2} \int_0^{\infty} \frac{dx}{x} \left(1 + \frac{1}{2}\gamma^2 qx \right) \left[\sin \left(\frac{1}{12}\omega qx^2 - \omega(1+v)x \right) + \sin \left(\frac{1}{12}\omega qx^2 + \omega(1-v)x \right) \right] \quad (4.10) \end{aligned}$$

where in the last step we have used the trigonometric identity

$$\sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)]$$

We are interested in the effect of multiple scattering on the radiation emitted in the limit $\omega \rightarrow 0$. Hence we can restrict ourselves to the region of frequencies such that $\omega \ll q$. The terms linear in x in the arguments of the sine functions in 4.10 can then be neglected.

$$\frac{dI}{d\omega} = \frac{e^2\omega}{\pi v\gamma^2} 2 \int_0^{\infty} \frac{dx}{x} \left(1 + \frac{1}{2}\gamma^2 qx \right) \sin \left(\frac{1}{12}\omega qx^2 \right) \quad (4.11)$$

Now

$$\begin{aligned} \int_0^{\infty} \sin ax^2 dx &= \text{Im} \int_0^{\infty} e^{iax^2} dx \\ &= \text{Im} \left\{ \frac{1}{2} \sqrt{\frac{\pi}{-ia}} \right\} \\ &= \frac{1}{2} \sqrt{\frac{\pi}{a}} \text{Im} \sqrt{i} \\ &= \frac{1}{2} \sqrt{\frac{\pi}{2a}} \quad (4.12) \end{aligned}$$

and

$$\int_0^{\infty} \frac{dx}{x} \sin ax^2 = \frac{1}{2} \int_0^{\infty} \frac{\sin u}{u} du, \quad (u = ax^2)$$

This integral can be done by contour integration. We use the contour of figure 4.2. We integrate $f(z) = e^{iz}/z$ around γ . f is holomorphic except for a simple pole at 0, which the indented semicircle avoids.

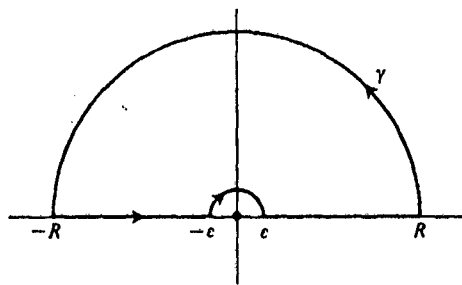


Figure 4.2: The contour γ

By Cauchy's theorem

$$\int_{-R}^{-\epsilon} f(u) du - \int_{\Gamma_{\epsilon}} f(z) dz + \int_{\epsilon}^R f(u) du + \int_{\Gamma_R} f(z) dz = 0$$

We calculate the contribution from Γ_{ϵ} in the limit as $\epsilon \rightarrow 0$. It gives

$$\lim_{\epsilon \rightarrow 0} \int_{\Gamma_{\epsilon}} f(z) dz = i\pi \text{res}\{f(z); 0\} = i\pi$$

If we let $R \rightarrow \infty$ then the contribution from $\Gamma_R \rightarrow 0$ and so

$$i\pi = \lim_{R \rightarrow \infty, \epsilon \rightarrow 0} \left\{ \int_{-R}^{-\epsilon} \frac{e^{iu}}{u} du + \int_{\epsilon}^R \frac{e^{iu}}{u} du \right\} = \lim_{R \rightarrow \infty, \epsilon \rightarrow 0} 2i \int_{\epsilon}^R \frac{\sin u}{u} du$$

Therefore

$$\frac{1}{2} \int_0^{\infty} \frac{\sin u}{u} du = \frac{1}{4} \pi \quad (4.13)$$

With the help of 4.12 and 4.13 we can rewrite 4.11 as

$$\begin{aligned} \frac{dI}{d\omega} &= \frac{2e^2\omega}{\pi v \gamma^2} \left\{ \frac{\pi}{4} + \frac{1}{4} \gamma^2 \sqrt{\frac{6\pi q}{\omega}} \right\} \\ &\simeq \frac{1}{2} \frac{e^2}{v} \sqrt{\frac{6q\omega}{\pi}} \end{aligned} \quad (4.14)$$

We follow Landau and Pomerancúk and define $E_0 = \frac{m^4 L}{E^2}$. With the definitions of q , L and E_s (defined in Appendix A) we can write 4.14 as

$$dI \simeq 8\sqrt{6\pi^3} \frac{d\omega}{L} \sqrt{\frac{\omega E_0}{E^2}} \quad (4.15)$$

The original result as quoted by Landau and Pomeranchuk is

$$dI \simeq \sqrt{\frac{2\pi}{3}} \frac{d\omega}{L} \sqrt{\frac{\omega E_0}{E^2}} \quad (4.16)$$

Our result is different only by a constant factor of 24π . It is also different by a factor of 2 from the calculations of Akhiezer and Shulga [8]. It appears they neglected to include the second term of the trigonometric identity used in 4.10.

Chapter 5

Inclusion of Multiple Scattering by Functional Integration

5.1 The Functional Integral

If we shoot a beam of electrons into an amorphous medium such that the initial direction of motion is along the z -axis, then it can be shown (Appendix A, Section 2) that the distribution of electrons after a time τ over the scattering angle $\vec{\theta}$ is [8, 12]

$$f(\tau, \theta) = \frac{1}{2\pi\sigma\tau} \exp\left\{-\frac{\theta^2}{2\sigma\tau}\right\}$$

In order to find the propability for having the scattering angle in the interval $(\vec{\theta}_N, \vec{\theta}_N + d\vec{\theta}_N)$ at the time τ we must consider all possible paths to this point. We define $\Delta = \frac{\tau}{N}$ where N is the number of distinct scattering intervals considered. After a time Δ the probability of the scattering angle being in the interval $(\vec{\theta}_1, \vec{\theta}_1 + d\vec{\theta}_1)$ is $p_1 = f(\Delta, \theta_1)d\theta_1$. The probability for the scattering angle to be about $\vec{\theta}_2$ after the next time step is $p_2 = p_1 f(\Delta, \theta_2 - \theta_1)d\theta_2$. So the probability of arriving at $\vec{\theta}_N$ at time τ for a particular path is

$$p_N = \frac{d\theta_1 d\theta_2 \dots d\theta_N}{(2\pi\sigma\Delta)^N} \exp\left\{-\frac{\theta_1^2}{2\sigma\Delta} - \frac{(\theta_2 - \theta_1)^2}{2\sigma\Delta} - \dots - \frac{(\theta_N - \theta_{N-1})^2}{2\sigma\Delta}\right\} \quad (5.1)$$

We use this weight to average the energy radiated. The result is a functional integral with respect to the Wiener measure $d_w \vec{\theta}$.

$$\begin{aligned} \left\langle \frac{dI}{d\omega} \right\rangle &= \int d_w \vec{\theta} \frac{dI}{d\omega} \\ &= \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p_N \frac{dI}{d\omega} \end{aligned} \quad (5.2)$$

We start with expression 4.9 for $\frac{dI}{d\omega}$:

$$\begin{aligned}\frac{dI}{d\omega} &= -\frac{e^2\omega}{\pi v\gamma^2} \int_{-\infty}^{\infty} dT \int_{-\infty}^{\infty} \frac{d\tau}{\tau} e^{-i\omega\tau} \left[1 + \frac{1}{2}v^2\gamma^2\theta^2\right] \sin v\omega \left\{ \tau - \frac{1}{2} \int_0^\tau \theta^2 dt + \frac{1}{2\tau} \left(\int_0^\tau \vec{\theta} dt \right)^2 \right\} \\ &= -\frac{e^2\omega}{\pi v\gamma^2} \int_{-\infty}^{\infty} dT \int_{-\infty}^{\infty} \frac{d\tau}{\tau} (\cos \omega\tau - i \sin \omega\tau) \left[1 + \frac{1}{2}v^2\gamma^2\theta^2\right] \\ &\quad \times \sin v\omega \left\{ \tau - \frac{1}{2} \int_0^\tau \theta^2 dt + \frac{1}{2\tau} \left(\int_0^\tau \vec{\theta} dt \right)^2 \right\}\end{aligned}\quad (5.3)$$

The imaginary part of 5.3 is an odd function in τ and so the integral reduces to

$$\begin{aligned}\frac{dI}{d\omega} &= -2\frac{e^2\omega}{\pi v\gamma^2} \int_{-\infty}^{\infty} dT \int_0^\infty \frac{d\tau}{\tau} \left[1 + \frac{1}{2}v^2\gamma^2\theta^2\right] \cos \omega\tau \sin v\omega \left\{ \tau - \frac{1}{2} \int_0^\tau \theta^2 dt + \frac{1}{2\tau} \left(\int_0^\tau \vec{\theta} dt \right)^2 \right\} \\ &= -\frac{e^2\omega}{\pi v\gamma^2} \int_{-\infty}^{\infty} dT \int_0^\infty \frac{d\tau}{\tau} \left[1 + \frac{1}{2}v^2\gamma^2\theta^2\right] \\ &\quad \times (e^{i\omega\tau} + e^{-i\omega\tau}) \operatorname{Im} e^{i\omega v\tau} \exp \left\{ i v\omega \left(-\frac{1}{2} \int_0^\tau \theta^2 dt + \frac{1}{2\tau} \left(\int_0^\tau \vec{\theta} dt \right)^2 \right) \right\}\end{aligned}\quad (5.4)$$

We make use of the fact that

$$\vec{\theta}^2 = \frac{\partial}{\partial \mu} \exp \left\{ \mu \vec{\theta}^2 \right\} \Big|_{\mu=0}$$

and the definition

$$Q_\omega = \int d_\omega \theta_j \exp \left\{ \mu \theta_j^2 - \frac{i\omega v}{2} \int_0^\tau \theta_j^2 dt + \frac{i\omega v}{2\tau} \left(\int_0^\tau \theta_j dt \right)^2 \right\} \quad , j = x \text{ or } y$$

to write 5.2 as

$$\left\langle \frac{dI}{d\omega} \right\rangle = -2\frac{e^2\delta}{\pi v} \int_{-\infty}^{\infty} dT \operatorname{Im} \int_0^\infty \frac{d\tau}{\tau} \left\{ e^{-i\delta\tau} \left[1 + \frac{1}{2}v^2\gamma^2 \frac{\partial}{\partial \mu}\right] Q_\omega^2 - e^{-2i\omega\tau} \left[1 + \frac{1}{2}v^2\gamma^2 \frac{\partial}{\partial \mu}\right] Q_{-\omega}^2 \right\} \Big|_{\mu=0}\quad (5.5)$$

$$\delta = \omega(1 - v).$$

5.2 Evaluation of the Functional Integral

Our goal now is to find Q_ω . We can write

$$Q_\omega = \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} \frac{dq}{\sqrt{\pi}} e^{-q^2} Q_\omega(N, q)\quad (5.6)$$

where $Q_\omega(N, q)$ is defined to be

$$Q_\omega(N, q) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{d\theta_1 \dots d\theta_N}{(2\pi\sigma\Delta)^{N/2}} \exp \left\{ - \sum_{n=0}^{N-1} \frac{(\theta_{n+1} - \theta_n)^2}{2\sigma\Delta} + \mu\theta_N^2 \right. \\ \left. - \frac{i}{2}\omega\Delta \sum_{n=1}^N \theta_n^2 + q\Delta \left(\frac{2i\omega}{\tau} \right)^{1/2} \sum_{n=1}^N \theta_n \right\}$$

θ_0 is taken to be 0. Functional integration is rather tricky. We only know how to do it for a small class of functions. One of these is the Gaussian case. Fortunately we can manipulate our functional integral into a Gaussian form. We start by making the change of variables $y_n = (2\sigma\Delta)^{-1/2}\theta_n$. Then

$$Q_\omega(N, q) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{dy_1 \dots dy_N}{\pi^{N/2}} \exp \left\{ - \sum_{n=0}^{N-1} (y_{n+1} - y_n)^2 + 2\sigma\Delta\mu y_N^2 \right. \\ \left. + b \sum_{n=1}^N y_n - i\omega\sigma\Delta^2 \sum_{n=1}^N y_n^2 \right\} \quad (5.7)$$

with $b \equiv 2q\Delta \left(\frac{i\omega}{N} \right)^{1/2}$. Now

$$2\sigma\Delta\mu y_N^2 - \sum_{n=0}^{N-1} (y_{n+1} - y_n)^2 - i\omega\sigma\Delta^2 \sum_{n=1}^N y_n^2 = \\ = 2\sigma\Delta\mu y_N^2 - \sum_{n=1}^N y_n^2 (1 + i\omega\sigma\Delta^2) - \sum_{n=0}^{N-1} y_n^2 + 2 \sum_{n=1}^{N-1} y_n y_{n+1} \\ = 2\sigma\Delta\mu y_N^2 - \sum_{n=1}^{N-1} y_n^2 (2 + i\omega\sigma\Delta^2) - y_N^2 (1 + i\omega\sigma\Delta^2) + 2 \sum_{n=0}^{N-1} y_n y_{n+1} \\ \equiv - \sum_{n,m=1}^N A_{nm} y_n y_m$$

The non-zero elements of A are:

$$A_{nn} = 2 + i\omega\sigma\Delta^2 \quad n = 1, \dots, N-1 \\ A_{NN} = 1 + i\omega\sigma\Delta^2 - 2\sigma\Delta\mu \\ A_{n,n+1} = A_{n+1,n} = -1 \quad n = 1, \dots, N-1$$

5.7 can thus be written as follows:

$$Q_\omega(N, q) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{dy_1 \dots dy_N}{\pi^{N/2}} \exp \left\{ b \sum_{n=1}^N y_n - \sum_{n,m=1}^N A_{nm} y_n y_m \right\} \quad (5.8)$$

Our result must be evaluated at $\mu = 0$ and in the limit as $N \rightarrow \infty$. $\Delta = \tau/N$ so $\Delta \rightarrow 0$. The imaginary components of A are of order Δ^2 . So let us for the moment consider the

matrix $A' = ReA$. The non-zero elements of A' are:

$$\begin{aligned} A'_{nn} &= 2 & n = 1, \dots, N-1 \\ A'_{NN} &= 1 \\ A'_{n,n+1} &= A'_{n+1,n} = -1 & n = 1, \dots, N-1 \end{aligned}$$

We can easily show that A' is a positive definite matrix. Choose any vector $\vec{x} \in \mathfrak{R}^N$, $\vec{x}^T = (x_1, x_2, \dots, x_N)$, ($\vec{x} \neq \vec{0}$).

$$\begin{aligned} \vec{x}^T A' \vec{x} &= \sum_{n=1}^{N-1} (2x_n^2 - 2x_n x_{n+1}) + x_N^2 \\ &= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \dots + (x_{N-1} - x_N)^2 + x_{N-1}^2 + x_N^2 > 0 \end{aligned}$$

Therefore A' is positive definite and A is positive definite to order Δ^2 . It is possible to reduce A to a diagonal form by a unitary transformation matrix U , i.e.

$$(U^{-1}AU)_{nm} = a_n \delta_{nm} \quad (5.9)$$

The a_n 's are the eigenvalues of A and since A is positive definite, all the a_n 's are positive. Let us now transform variables from y_n to z_n in accordance with the formula

$$y_n = \sum_{\lambda=1}^N U_{n\lambda} z_\lambda \text{ or } y = Uz$$

Then

$$\begin{aligned} \sum_{n,m=1}^N A_{mn} y_n y_m &= y^T A y \\ &= (Uz)^T A Uz \\ &= z^T U^{-1} A Uz & U^T = U^{-1}, U \text{ unitary} \\ &= z^T (a_n \delta_{nm}) z \\ &= \sum_{n=1}^N a_n z_n^2 \end{aligned}$$

The chain of differentials $dy_1 dy_2 \dots dy_N$ defines a volume element in N -dimensional space. Transform to spherical coordinates. $dy_1 dy_2 \dots dy_N = r^{N-1} dr d\Omega_N$ with $y_1^2 + y_2^2 + \dots + y_N^2 = r^2$ or $y^T y = r^2$. $dz_1 dz_2 \dots dz_N$ also defines an N -dimensional volume element. $dz_1 dz_2 \dots dz_N = r'^{N-1} dr' d\Omega_N$ with $z^T z = r'^2$. But $y^T y = (Uz)^T Uz = z^T U^{-1} Uz = z^T z$. Therefore $r = r'$ and $dy_1 dy_2 \dots dy_N = dz_1 dz_2 \dots dz_N$.

5.8 now becomes

$$Q_\omega(N, q) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{dz_1 dz_2 \dots dz_N}{\pi^{N/2}} \exp \left\{ b \sum_{n,\lambda=1}^N U_{n\lambda} z_\lambda - \sum_{n=1}^N a_n z_n^2 \right\}$$

$$= \frac{1}{(a_1 a_2 \dots a_N)^{1/2}} \int \dots \int_{-\infty}^{\infty} dz_1 dz_2 \dots dz_N \exp \left\{ b \sum_{n,\lambda=1}^N U_{n\lambda} z_\lambda \right\} \\ \times \prod_{j=1}^N \frac{1}{\sqrt{\pi}} (a_j)^{1/2} \exp \left\{ -a_j z_j^2 \right\}$$

The product of all the a_n 's is just $\det A$. Define $\sigma_j^2 = \frac{1}{2a_j}$. Then

$$Q_\omega(N, q) = (\det A)^{-1/2} \int \dots \int_{-\infty}^{\infty} dz_1 dz_2 \dots dz_N \exp \left\{ b \sum_{n,\lambda=1}^N U_{n\lambda} z_\lambda \right\} \prod_{j=1}^N (2\pi\sigma_j^2)^{-1/2} \exp \left\{ \frac{-z_j^2}{2\sigma_j^2} \right\}$$

Define $c_\lambda = \sum_{n=1}^N U_{n\lambda}$. Then $\sum_{n,\lambda=1}^N U_{n\lambda} z_\lambda = \sum_{\lambda=1}^N c_\lambda z_\lambda$ and

$$Q_\omega(N, q) = (\det A)^{-1/2} \int \dots \int_{-\infty}^{\infty} dz_1 dz_2 \dots dz_N \prod_{j=1}^N (2\pi\sigma_j^2)^{-1/2} \exp \left\{ bc_j z_j - \frac{z_j^2}{2\sigma_j^2} \right\} \\ = (\det A)^{-1/2} \prod_{j=1}^N \int_{-\infty}^{\infty} dz_j (2\pi\sigma_j^2)^{-1/2} \exp \left\{ bc_j z_j - \frac{z_j^2}{2\sigma_j^2} \right\} \\ = (\det A)^{-1/2} \prod_{j=1}^N (2\pi\sigma_j^2)^{-1/2} \int_{-\infty}^{\infty} dz_j e^{-\frac{1}{2\sigma_j^2} [z_j - b\sigma_j^2 c_j]^2 + \frac{1}{2} b^2 \sigma_j^2 c_j^2} \\ = (\det A)^{-1/2} \prod_{j=1}^N (2\pi\sigma_j^2)^{-1/2} e^{\frac{1}{2} b^2 \sigma_j^2 c_j^2} \int_{-\infty}^{\infty} dx_j e^{-\frac{x_j^2}{2\sigma_j^2}} \quad , x_j \equiv z_j - b\sigma_j^2 c_j \\ = (\det A)^{-1/2} \prod_{j=1}^N (2\pi\sigma_j^2)^{-1/2} e^{\frac{1}{2} b^2 \sigma_j^2 c_j^2} \sqrt{2\pi}\sigma_j \\ = (\det A)^{-1/2} \prod_{j=1}^N e^{\frac{1}{2} b^2 \sigma_j^2 c_j^2} \tag{5.10}$$

It will be useful to find the inverse of A . From 5.9 we have $A = U(a_n \delta_{nm})U^{-1}$ or

$$A^{-1} = U(a_n \delta_{nm})^{-1} U^{-1} \\ = U \begin{pmatrix} \frac{1}{a_1} & & & \\ & \frac{1}{a_2} & & \\ & & \ddots & \\ & & & \frac{1}{a_N} \end{pmatrix} U^T \\ = \begin{pmatrix} (\frac{1}{a_1} U_{11}^2 + \frac{1}{a_2} U_{12}^2 + \dots + \frac{1}{a_N} U_{1N}^2) & \dots & (\frac{1}{a_1} U_{11} U_{N1} + \frac{1}{a_2} U_{12} U_{N2} + \dots + \frac{1}{a_N} U_{1N} U_{NN}) \\ \vdots & \ddots & \vdots \\ (\frac{1}{a_1} U_{11} U_{N1} + \frac{1}{a_2} U_{12} U_{N2} + \dots + \frac{1}{a_N} U_{1N} U_{NN}) & \dots & (\frac{1}{a_1} U_{N1}^2 + \frac{1}{a_2} U_{N2}^2 + \dots + \frac{1}{a_N} U_{NN}^2) \end{pmatrix} \tag{5.11}$$

Define A_k as the determinant of the $k \times k$ matrix in the upper left corner of A . $A_0 = 1$, $A_1 = A_{11}$ and $A_N = \det A$. We claim that [13]

$$(A^{-1})_{km} = \begin{cases} \frac{1}{D_1} A_{k-1} D_{m+1} & m \geq k \\ \frac{1}{D_1} D_{k+1} A_{m-1} & m < k \end{cases} \quad (5.15)$$

This is easily shown by explicit evaluation of one of the rows. All the other rows are evaluated in an identical fashion and the result follows simply.

Lemma 4 $D_{k+1} A_k - A_{k-1} D_{k+2} = \det A = D_1 = A_N \quad k = 1, \dots, N$

Proof:

The lemma will be proved by induction. For the case $k = 1$: $LHS = D_2 A_1 - A_0 D_3 = a_{11} D_2 - D_3 = D_1 = RHS$. The last step has made use of 5.13. Now suppose that $D_{t+1} A_t - A_{t-1} D_{t+2} = D_1$ for some t . $D_{t+3} = a_{t+1, t+1} D_{t+2} - D_{t+1}$ and so

$$\begin{aligned} D_{t+2} A_{t+1} - A_t D_{t+3} &= D_{t+2} A_{t+1} - a_{t+1, t+1} A_t D_{t+2} + A_t D_{t+1} \\ &= D_{t+2} A_{t+1} - A_{t+1} D_{t+2} - A_{t-1} D_{t+2} + A_t D_{t+1} \quad (\text{from definition of } A_{t+1}) \\ &= D_1 \quad (\text{by induction hypothesis}) \end{aligned}$$

□

Proposition 1 $\Sigma_N^2 = \frac{1}{2} b^2 \sum_{n=1}^N (D_n D_{n+1})^{-1} \left(\sum_{k=n}^N D_{k+1} \right)^2$

Proof:

From the Lemma we have

$$\begin{aligned} A_k &= (D_1 + A_{k-1} D_{k+2})(D_{k+1})^{-1} \\ &= D_1 D_{k+2} (D_{k+1} D_{k+2})^{-1} + A_{k-1} D_{k+2} \frac{1}{D_{k+1}} \\ &= D_1 D_{k+2} (D_{k+1} D_{k+2})^{-1} + \frac{D_{k+2}}{D_k D_{k+1}} (D_k A_{k-1}) \\ &= D_1 D_{k+2} \left((D_{k+1} D_{k+2})^{-1} + (D_k D_{k+1})^{-1} \right) + A_{k-2} \frac{D_{k+2}}{D_k} \\ &= \dots = D_1 D_{k+2} \left((D_{k+1} D_{k+2})^{-1} + (D_k D_{k+1})^{-1} + \dots + (D_2 D_3)^{-1} \right) + \frac{D_{k+2}}{D_2} \\ &= D_1 D_{k+2} \sum_{i=1}^{k+1} (D_i D_{i+1})^{-1} \end{aligned}$$

Now

$$\Sigma_N^2 = \frac{1}{2} b^2 \sum_{k,m=1}^N (A^{-1})_{km}$$

$$\begin{aligned}
&= \frac{b^2}{2D_1} \sum_{k=1}^N \left(\sum_{m=k}^N A_{k-1} D_{m+1} + \sum_{m=1}^{k-1} D_{k+1} A_{m-1} \right) \quad (\text{by 5.15}) \\
&= \frac{b^2}{2D_1} \left(\sum_{k=1}^N D_1 D_{k+1} \sum_{i=1}^k (D_i D_{i+1})^{-1} \sum_{m=k}^N D_{m+1} + \sum_{k=1}^N D_{k+1} \sum_{m=1}^{k-1} D_1 D_{m+1} \sum_{j=1}^m (D_j D_{j+1})^{-1} \right) \\
&= \frac{1}{2} b^2 \left(\sum_{k=1}^N (D_k D_{k+1})^{-1} \sum_{l=k}^N \sum_{m=l}^N D_{l+1} D_{m+1} + \sum_{k=1}^{m-1} (D_k D_{k+1})^{-1} \sum_{l=k}^{N-1} \sum_{m=l+1}^N D_{l+1} D_{m+1} \right) \\
&= \frac{1}{2} b^2 \sum_{k=1}^N (D_k D_{k+1})^{-1} \left(\sum_{j=k}^N D_{j+1} \right)^2
\end{aligned}$$

□

5.10 now becomes

$$Q_\omega(N, q) = \frac{1}{D_1^{1/2}} \exp \left\{ \frac{b^2}{4} \sum_{n=1}^N (D_n D_{n+1})^{-1} \left(\sum_{k=n}^N D_{k+1} \right)^2 \right\} \quad (5.16)$$

We can now modify expression 5.16 for $Q_\omega(N, q)$ in the limit $N \rightarrow \infty$:

$$Q_\omega(N, q) = \frac{1}{D^{1/2}(\Delta)} \exp \left\{ \frac{q^2 \Delta^2 i \omega \sigma}{N} \sum_{n=1}^N (D(n\Delta) D(\Delta(n+1)))^{-1} \left(\sum_{k=n}^N D(k\Delta) \right)^2 \right\}$$

As $N \rightarrow \infty$, $\Delta \rightarrow 0$ and so we have approximated D as a continuous function of $t = n\Delta$.

5.6 becomes

$$\begin{aligned}
Q_\omega &= \lim_{N \rightarrow \infty} \int_{-\infty}^{\infty} \frac{1}{D^{1/2}(\Delta)} \frac{dq}{\sqrt{\pi}} \exp -q^2 \left\{ 1 - \frac{i\omega\sigma\Delta^3}{\tau} \sum_{n=1}^N (D(n\Delta) D(\Delta(n+1)))^{-1} \left(\sum_{k=n}^N D(k\Delta) \right)^2 \right\} \\
&= \frac{1}{\sqrt{\pi} D(0)} \int_{-\infty}^{\infty} dq \exp -q^2 \left\{ 1 - \frac{i\omega\sigma}{\tau} \int_0^\tau D^{-2}(t) dt \left(\int_t^\tau D(t') dt' \right)^2 \right\} \\
&= \left\{ D(0) \left\{ 1 - \frac{i\omega\sigma}{\tau} \int_0^\tau D^{-2}(t) dt \left(\int_t^\tau D(t') dt' \right)^2 \right\} \right\}^{-1/2} \quad (5.17)
\end{aligned}$$

$$\begin{aligned}
D'_n &\simeq \frac{D_{n+1} - D_n}{\Delta} \\
D''_n &\simeq \frac{D'_{n+1} - D'_n}{\Delta} \\
&\simeq \frac{1}{\Delta^2} (D_{n+2} - 2D_{n+1} + D_n) \\
&\simeq \frac{1}{\Delta^2} ((2 + i\omega\sigma\Delta^2) D_{n+1} - D_n - 2D_{n+1} + D_n) \quad (\text{by (5.8)}) \\
&\simeq i\omega\sigma D_{n+1}
\end{aligned}$$

$$N \rightarrow \infty, \quad \frac{d^2 D(t)}{dt^2} \simeq i\omega\sigma D(t) \quad (5.18)$$

Initial conditions: $D_N = 1 \Rightarrow D(N\Delta) = D(\tau) = 1$ and $D'(\tau) = 2\sigma\mu$ Solutions to 5.18 have the form

$$D(t) = A \sinh r(t - \tau) + B \cosh r(t - \tau) \quad , r^2 \equiv i\omega\sigma$$

In particular for our initial conditions

$$D(t) = \frac{2\sigma\mu}{r} \sinh r(t - \tau) + \cosh r(t - \tau) \quad (5.19)$$

When $\mu = 0$, $D(t) = \cosh r(t - \tau)$ and $D'(t) = r \sinh r(t - \tau)$. $\int_t^\tau \cosh r(t' - \tau) dt' = -\frac{1}{r} \sinh r(t - \tau)$ and so 5.17 is

$$\begin{aligned} Q_\omega &= \left\{ \cosh r\tau \left(1 - \frac{i\omega\sigma}{\tau r^2} \int_0^\tau \tanh^2 r(t - \tau) dt \right) \right\}^{-1/2} \\ &= \left\{ \cosh r\tau \left(1 - \frac{1}{\tau} \left(\tau - \frac{1}{r} \tanh r\tau \right) \right) \right\}^{-1/2} \\ &= \left(\frac{r\tau}{\sinh r\tau} \right)^{1/2} \end{aligned} \quad (5.20)$$

We shall also need to know $\frac{\partial Q_\omega}{\partial \mu}|_{\mu=0}$. From 5.17 we have

$$\begin{aligned} \frac{\partial Q_\omega}{\partial \mu}|_{\mu=0} &= -\frac{1}{2} Q_\omega^3 \left\{ -\frac{2\sigma}{r} \sinh r\tau \left(1 - \frac{i\omega\sigma}{\tau} \int_0^\tau D^{-2}(t) dt \left(\int_t^\tau D(t') dt' \right)^2 \right) \right. \\ &\quad \left. - \frac{i\omega\sigma}{\tau} \cosh r\tau \frac{\partial}{\partial \mu} \int_0^\tau D^{-2}(t) dt \left(\int_t^\tau D(t') dt' \right)^2 \right\} \end{aligned} \quad (5.21)$$

$$\begin{aligned} \frac{\partial}{\partial \mu} \int_0^\tau D^{-2}(t) dt \left(\int_t^\tau D(t') dt' \right)^2 &= \\ \int_0^\tau \left[-2D^{-3}(t) \frac{\partial D(t)}{\partial \mu} \left(\int_t^\tau D(t') dt' \right)^2 + 2D^{-2} \left(\int_t^\tau D(t') dt' \right) \left(\int_t^\tau \frac{\partial D(t'')}{\partial \mu} dt'' \right) \right] dt \\ &= -\frac{4\sigma}{r^3} \left\{ \int_0^\tau \tanh^3 r(t - \tau) dt + \int_0^\tau \frac{\sinh r(t - \tau)}{\cosh^2 r(t - \tau)} (1 - \cosh r(t - \tau)) dt \right\} \end{aligned} \quad (5.22)$$

The integrals in 5.22 are simple to evaluate. The results are

$$\int_0^\tau \tanh^3 r(t - \tau) dt = -\frac{1}{r} (\ln \cosh r\tau - \frac{1}{2} \tanh^2 r\tau)$$

and

$$\int_0^\tau \frac{\sinh r(t - \tau)}{\cosh^2 r(t - \tau)} (1 - \cosh r(t - \tau)) dt = -\frac{1}{r} (1 - \operatorname{sech} r\tau - \ln \cosh r\tau)$$

5.21 then becomes

$$\begin{aligned}
\frac{\partial Q_\omega}{\partial \mu} \Big|_{\mu=0} &= -\frac{1}{2} Q_\omega^3 \left\{ -\frac{2\sigma}{r} \sinh r\tau \left(1 - \frac{i\omega\sigma}{\tau r^2} \left(\tau - \frac{1}{r} \tanh r\tau \right) \right) + 4 \frac{i\omega\sigma^2}{\tau r^3} \cosh r\tau \times \right. \\
&\quad \left. \left(-\frac{1}{r} (\ln \cosh r\tau - \frac{1}{2} \tanh^2 r\tau) - \frac{1}{r} (1 - \operatorname{sech} r\tau - \ln \cosh r\tau) \right) \right\} \\
&= \sigma Q_\omega \frac{r\tau}{\sinh r\tau} \left(\frac{1}{r^2 \tau} \sinh r\tau \tanh r\tau + \frac{2}{r^2 \tau} \cosh r\tau (1 - \operatorname{sech} r\tau - \frac{1}{2} \tanh^2 r\tau) \right) \\
&= \frac{\sigma}{r} Q_\omega 2 \frac{\cosh r\tau - 1}{\sinh r\tau} \\
&= \frac{2\sigma}{r} Q_\omega \tanh \frac{r\tau}{2} \tag{5.23}
\end{aligned}$$

We can now return to 5.5 to find the average energy radiated. The first integral on the right of 5.5 is

$$\begin{aligned}
\operatorname{Im} \int_0^\infty d\tau e^{-i\delta\tau} \left(\frac{r}{\sinh r\tau} + \frac{2\sigma\gamma^2}{\sinh r\tau} \tanh \frac{r\tau}{2} \right) &= \\
\operatorname{Im} \int_0^\infty d\tau e^{-i\delta\tau} \left(\frac{r}{\sinh r\tau} + 2\sigma\gamma^2 \left(\frac{\cosh r\tau}{\sinh^2 r\tau} - \frac{1}{\sinh^2 r\tau} \right) \right) &= \\
= \operatorname{Im} \int_0^\infty d\tau e^{-i\delta\tau} \left(\frac{r}{\sinh r\tau} + \frac{2i\sigma\delta\gamma^2}{r} \left(\coth r\tau - \frac{1}{\sinh r\tau} \right) \right) &
\end{aligned}$$

Now $\frac{2i\sigma\delta\gamma^2}{r} = r$ and so

$$\operatorname{Im} \int_0^\infty d\tau e^{-i\delta\tau} \left(\frac{r}{\sinh r\tau} + \frac{2\sigma\gamma^2}{\sinh r\tau} \tanh \frac{r\tau}{2} \right) = \operatorname{Im} r \int_0^\infty d\tau e^{-i\delta\tau} \coth r\tau$$

The second integral on the right of 5.5 is the contribution of negative frequencies. This can have no bearing on the emission of radiation and so this term is discarded. 5.5 can thus be written as

$$\left\langle \frac{dI}{d\omega} \right\rangle = -2 \frac{e^2 \delta}{\pi} \int_{-\infty}^\infty dT \operatorname{Im} r \int_0^\infty d\tau e^{-i\delta\tau} \coth r\tau \tag{5.24}$$

The integral in 5.24 has a singularity at $\tau = 0$. We must have the energy radiated finite. The infinite contribution is a result of the instantaneous initial acceleration of the electron. We are interested only in the subsequent radiation so we remove the singularity. As $\tau \rightarrow 0$ $\coth r\tau \rightarrow \frac{1}{r\tau}$. So, with $z = r\tau$, 5.24 can be written as

$$\left\langle \frac{dI}{d\omega} \right\rangle = -2 \frac{e^2 \delta}{\pi} \int_{-\infty}^\infty dT \operatorname{Im} \int_0^\infty \exp \left\{ \frac{-i\delta z}{r} \right\} \left(\coth z - \frac{1}{z} \right) \tag{5.25}$$

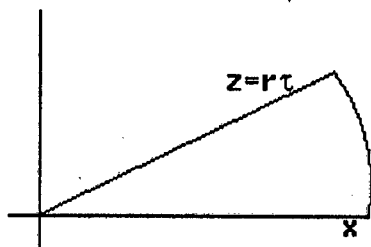


Figure 5.1: Contour used for the intergration in 5.25

Consider the contour in figure 5.1. The intergration over z is along the incline. The integrand vannishes on the arc at infinity. There are also no poles inside or on the contour so

$$\begin{aligned}
 \left\langle \frac{dI}{d\omega} \right\rangle &= -2 \frac{e^2 \delta}{\pi} \int_{-\infty}^{\infty} dT \operatorname{Im} \int_0^{\infty} dx \exp \left\{ \frac{-i\delta x}{r} \right\} \left(\coth x - \frac{1}{x} \right) \\
 &= 2 \frac{e^2 \delta}{\pi} \int_{-\infty}^{\infty} dT \int_0^{\infty} dx \exp \left\{ \frac{-\delta x}{\sqrt{2\omega\sigma}} \right\} \sin \frac{\delta x}{\sqrt{2\omega\sigma}} \left(\coth x - \frac{1}{x} \right) \\
 &= 2 \frac{e^2 \delta}{\pi} \int_{-\infty}^{\infty} dT \left(\int_0^{\infty} dx e^{-2sx} \sin 2sx \coth x - \frac{\pi}{4} \right) \quad (5.26)
 \end{aligned}$$

$s \equiv \frac{(1-v^2)}{4} \sqrt{\frac{\omega}{2\sigma}}$. We follow Migdal [14, 15] or [15, Appendix VI] and define the function

$$\Phi(s) = 24s^2 \left(\int_0^{\infty} dx e^{-2sx} \sin 2sx \coth x - \frac{\pi}{4} \right) \quad (5.27)$$

The average energy radiated can then be written as

$$\begin{aligned}
 \left\langle \frac{dI}{d\omega} \right\rangle &= \frac{2e^2 \delta T \Phi(s)}{\pi 24s^2} \\
 &= \frac{2e^2 \gamma^2}{3\pi} qT \Phi(s) \\
 &= 2 \left(\frac{dI}{d\omega} \right)_{BH} \Phi(s) \quad (5.28)
 \end{aligned}$$

$\left(\frac{dI}{d\omega} \right)_{BH}$ is the expression 1.15. It is the energy radiated in the low frequency regime by a particle moving through an amorphous medium where the effects of electron screening have been taken into account, but no consideration has been given to the effects of multiple scattering. The function $\Phi(s)$ can be approximated by (see figure 5.2) [16]

$$\Phi(s)^* = \left[1 + \left(\frac{1}{6s} \right)^2 \right]^{-1/2} \quad (5.29)$$

In the limit $s \ll 1$, $\Phi(s)^* \simeq 6s$ and so

$$\left\langle \frac{dI}{d\omega} \right\rangle \simeq \frac{16\pi}{\gamma^2 L} \sqrt{\frac{\omega}{q}} \quad (5.30)$$

which is just larger by a factor of about 35 than the expression obtained by Landau and Pomeranchuk, 4.16. Both expressions have the characteristic $\sqrt{\omega}$ behaviour.

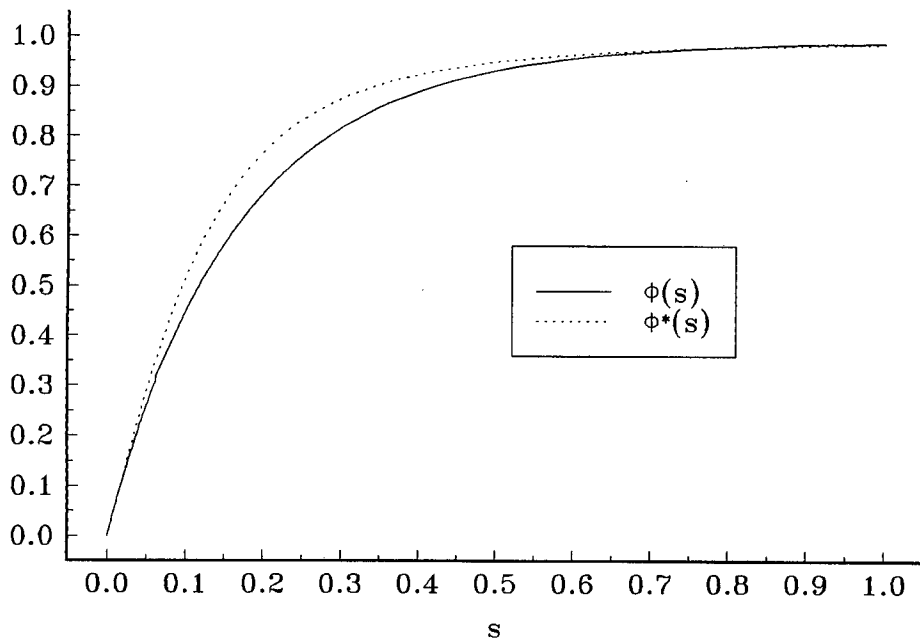


Figure 5.2: The function $\Phi(s)$ and its approximation $\Phi(s)^*$

Chapter 6

Inclusion of Multiple Scattering by the Kinetic Equation Method

We start again from our basic equation for the energy radiated from an accelerating charged particle 2.5 [8, 14, 17, 18, 19] or [15, p.158-164]

$$\begin{aligned} \frac{d^2 I}{d\omega d\Omega} &= \frac{e^2 \omega^2}{4\pi^2} \left| \int_{-\infty}^{\infty} dt \hat{\mathbf{n}} \times \vec{v}(t) e^{i\omega(t - \hat{\mathbf{n}} \cdot \vec{r}(t))} \right|^2 \\ &= \frac{e^2 \omega^2}{4\pi^2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 (\hat{\mathbf{n}} \times \vec{v}(t_1)) \cdot (\hat{\mathbf{n}} \times \vec{v}(t_2)) e^{i\omega(t_1 - t_2) + i\vec{k} \cdot (\vec{r}(t_2) - \vec{r}(t_1))} \quad (6.1) \end{aligned}$$

Let us split the integral in 6.1 into two parts. We divide the two dimensional space defined by t_1 and t_2 by the line $\tau = 0$ where $\tau \equiv t_2 - t_1$. For the integration over the upper half plane we define $t = t_1$ and for the lower half plane $t = t_2$. That is the upper integral is over $-\infty \leq t_1 \leq \infty, 0 \leq \tau \leq \infty$ and the lower integral over $-\infty \leq t_2 \leq \infty, -\infty \leq \tau \leq 0$. Then

$$\begin{aligned} \frac{d^2 I}{d\omega d\Omega} &= \frac{e^2 \omega^2}{4\pi^2} \int_{-\infty}^{\infty} dt \int_0^{\infty} d\tau \hat{\mathbf{n}} \times \vec{v}(t) \hat{\mathbf{n}} \times \vec{v}(t + \tau) e^{-i\omega\tau + i\vec{k} \cdot (\vec{r}(t + \tau) - \vec{r}(t))} \\ &\quad + \int_{-\infty}^{\infty} dt \int_0^{\infty} d\tau \hat{\mathbf{n}} \times \vec{v}(t + \tau) \hat{\mathbf{n}} \times \vec{v}(t) e^{i\omega\tau + i\vec{k} \cdot (\vec{r}(t) - \vec{r}(t + \tau))} \\ &= 2 \frac{e^2 \omega^2}{4\pi^2} \text{Re} \int_{-\infty}^{\infty} dt \int_0^{\infty} d\tau (\hat{\mathbf{n}} \times \vec{v}(t)) \cdot (\hat{\mathbf{n}} \times \vec{v}(t + \tau)) e^{-i\omega\tau + i\vec{k} \cdot (\vec{r}(t + \tau) - \vec{r}(t))} \quad (6.2) \end{aligned}$$

The two exponential terms are complex conjugates of one another, hence the factor of twice the real part of the integral.

Let us define $\vec{r}' = \vec{r}(t + \tau)$ and $\vec{v}' = \vec{v}(t + \tau)$. We include the effects of multiple scattering by averaging 6.2 by integration over weight functions or probability distributions. We define $W_1(\vec{r}', \vec{v}', t)$ to be the probability of the particle being in the neighbourhood of the

point (\vec{r}, \vec{v}) in phase space at the time t . We then define $W_2(\vec{r}', \vec{v}'; \vec{r}, \vec{v}; \tau)$ as the probability for the particle to be in the neighbourhood of (\vec{r}', \vec{v}') at the time $t + \tau$, given that it was in the neighbourhood of (\vec{r}, \vec{v}) at time t . W_1 and W_2 must satisfy the kinetic equation A.9

$$\frac{\partial W_{1,2}}{\partial t} + \vec{v} \cdot \frac{\partial W_{1,2}}{\partial \vec{r}} = n \int d\vec{v}' \sigma(\vec{v}' - \vec{v}) [W_{1,2}(\vec{r}, \vec{v}', t) - W_{1,2}(\vec{r}, \vec{v}, t)]$$

The initial condition on W_1 is $W_1(\vec{r}, \vec{v}, 0) = \delta(\vec{r})\delta(\vec{v} - \vec{v}_0)$ where v_0 is the initial velocity. For W_2 , $W_2(\vec{r}', \vec{v}'; \vec{r}, \vec{v}; 0) = \delta(\vec{r}' - \vec{r})\delta(\vec{v}' - \vec{v})$. We write 6.2 as

$$\frac{d^2 I}{d\omega d\Omega} = 2 \frac{e^2}{4\pi^2} Re \int_{-\infty}^{\infty} dt \int_0^{\infty} d\tau K \quad (6.3)$$

i.e.

$$K = \int d\vec{r} d\vec{v} d\vec{r}' d\vec{v}' e^{i\vec{k}(\vec{r}' - \vec{r}) - i\omega\tau} (\vec{k} \times \vec{v})(\vec{k} \times \vec{v}') W_1(\vec{r}, \vec{v}, t) W_2(\vec{r}', \vec{v}'; \vec{r}, \vec{v}; \tau) \quad (6.4)$$

6.4 contains the Fourier transform of W_2 . We define

$$W_k(\vec{v}', \vec{v}, \tau) = \int d(\vec{r}' - \vec{r}) e^{i\vec{k}(\vec{r}' - \vec{r}) - i\omega\tau} W_2(\vec{r}', \vec{v}'; \vec{r}, \vec{v}; \tau)$$

6.4 becomes

$$K = \int d\vec{r} d\vec{v} d\vec{v}' (\vec{k} \times \vec{v})(\vec{k} \times \vec{v}') W_1(\vec{r}, \vec{v}, t) W_k(\vec{v}', \vec{v}, \tau) \quad (6.5)$$

We can write the kinetic equation for W_2 in terms of W_k . We multiply A.9 by $\exp(i\vec{k}(\vec{r}' - \vec{r}) - i\omega\tau)$ and integrate over $d(\vec{r}' - \vec{r})$:

$$\begin{aligned} & \int d(\vec{r}' - \vec{r}) e^{i\vec{k}(\vec{r}' - \vec{r}) - i\omega\tau} \frac{\partial W_2}{\partial \tau} \\ & + \vec{v}' \cdot \int d(\vec{r}' - \vec{r}) e^{i\vec{k}(\vec{r}' - \vec{r}) - i\omega\tau} \frac{\partial W_2}{\partial \vec{r}'} = n \int d\vec{v}'' \sigma(\vec{v}'' - \vec{v}') [W_k(\vec{v}'', \vec{v}, \tau) - W_k(\vec{v}', \vec{v}, \tau)] \\ & \int d(\vec{r}' - \vec{r}) e^{i\vec{k}(\vec{r}' - \vec{r})} \left(\frac{\partial}{\partial \tau} (e^{-i\omega\tau} W_2) + i\omega e^{-i\omega\tau} W_2 \right) \\ & + \vec{v}' \cdot [e^{i\vec{k}(\vec{r}' - \vec{r}) - i\omega\tau} W_2|_{-\infty}^{\infty} - i\vec{k} W_k] = n \int d\vec{v}'' \sigma(\vec{v}'' - \vec{v}') [W_k(\vec{v}'', \vec{v}, \tau) - W_k(\vec{v}', \vec{v}, \tau)] \end{aligned}$$

For W_2 to be physically realistic we must have $\lim_{\vec{r}' - \vec{r} \rightarrow \pm\infty} W_2 = 0$ and so

$$\frac{\partial W_k}{\partial \tau} + i\omega W_k - i\vec{k} \cdot \vec{v}' W_k = n \int d\vec{v}'' \sigma(\vec{v}'' - \vec{v}') [W_k(\vec{v}'', \vec{v}, \tau) - W_k(\vec{v}', \vec{v}, \tau)] \quad (6.6)$$

We introduce the two-dimensional vectors $\vec{\theta}$ and $\vec{\theta}'$ which are perpendicular to the direction of escape of the photon, i.e. $\hat{n} \cdot \vec{\theta} = \hat{n} \cdot \vec{\theta}' = 0$. For a highly relativistic particle the magnitude of the scattering angles are small and we have

$$\vec{v} = v \left(1 - \frac{1}{2}\theta^2 \right) \hat{n} + v\vec{\theta} \quad (6.7)$$

$$\vec{v}' = v \left(1 - \frac{1}{2}\theta'^2 \right) \hat{n} + v\vec{\theta}'$$

So $\vec{k} \cdot \vec{v}' \simeq \omega v (1 - \frac{1}{2}\theta'^2)$. If we perform a Taylor expansion of the integrand of 6.6 we can transform the problem to finding the solution of a differential equation. From Lemma 5 we have

$$\cos \theta'' = \cos \theta' \cos \theta''' + \sin \theta' \sin \theta''' \cos \Psi \quad , \theta''' \ll 1 \quad (6.8)$$

If we define $\mu = \cos \theta'$ and rearrange 6.8 we have

$$\mu' \simeq \mu - \left(\frac{\mu}{2} \theta'''^2 - \theta''' \sqrt{1 - \mu^2} \cos \Psi \right) \quad (6.9)$$

Thus

$$W_k(\mu') - W_k(\mu) \simeq W_k(\mu) - \left(\frac{\mu}{2} \theta'''^2 - \theta''' \sqrt{1 - \mu^2} \cos \Psi \right) \frac{\partial W_k}{\partial \mu} + \frac{1}{2} (1 - \mu^2) \theta'''^2 \cos^2 \Psi \frac{\partial^2 W_k}{\partial \mu^2} - W_k(\mu)$$

and

$$\begin{aligned} & n \int d\vec{v}'' v [W_k(\vec{v}'') - W_k(\vec{v}')] \\ &= n \int d\Omega''' \int dv''' \delta(v''') \sigma \left[-\frac{\mu}{2} \theta'''^2 \frac{\partial W_k}{\partial \mu} + \frac{1}{2} (1 - \mu^2) \theta'''^2 \cos^2 \Psi \frac{\partial^2 W_k}{\partial \mu^2} \right] \\ &= n \int_0^{\theta_{max}} \theta''' d\theta''' \sigma \left[-\pi \mu \theta'''^2 \frac{\partial W_k}{\partial \mu} + \frac{\pi}{2} (1 - \mu^2) \theta'''^2 \cos^2 \Psi \frac{\partial^2 W_k}{\partial \mu^2} \right] \\ &= \frac{q}{4} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial W_k}{\partial \mu} \right] = \frac{q}{4} \Delta_{\theta'} W_k \end{aligned}$$

$q = 2\pi n \int_0^{\theta_{max}} \theta'''^3 d\theta''' \sigma$ is the mean square scattering angle per unit time. 6.6 can now be written as follows:

$$\frac{\partial W_k}{\partial \tau} + i\omega(1 - v + \frac{1}{2}v\theta'^2)W_k = \frac{q}{4} \Delta_{\theta'} W_k$$

For the relativistic case $1 - v \simeq \frac{1}{2}(1 - v^2)$ and so

$$\frac{\partial W_k}{\partial \tau} + \frac{i\omega}{2}(1 - v^2 + v\theta'^2)W_k = \frac{q}{4} \Delta_{\theta'} W_k \quad (6.10)$$

We propose that 6.10 has a solution of the form

$$W_k(\vec{\theta}', \vec{\theta}, \tau) = \exp \left\{ \alpha(\tau) \vec{\theta}'^2 + \beta(\tau) \vec{\theta}' \cdot \vec{\theta} + \gamma(\tau) \right\} \quad (6.11)$$

$W_k(\vec{\theta}', \vec{\theta}, 0) = \delta(\vec{\theta}' - \vec{\theta})$. Let us write $\vec{\theta} = (\theta_x, \theta_y)$ and $\vec{\theta}' = (\theta'_x, \theta'_y)$. Then $\Delta_{\theta'} = \frac{\partial^2}{\partial \theta_x'^2} + \frac{\partial^2}{\partial \theta_y'^2}$. 6.11 must satisfy 6.10.

$$\begin{aligned} \frac{\partial W_k}{\partial \tau} &= \left(\dot{\alpha} \vec{\theta}'^2 + \dot{\beta} \vec{\theta}' \cdot \vec{\theta} + \dot{\gamma} \right) W_k \\ \frac{\partial W_k}{\partial \theta_x} &= (2\alpha \theta'_x + \beta \theta_x) W_k \\ \frac{\partial^2 W_k}{\partial \theta_x'^2} &= 2\alpha W_k + (2\alpha \theta'_x + \beta \theta_x)^2 W_k \\ \text{likewise, } \frac{\partial^2 W_k}{\partial \theta_y'^2} &= 2\alpha W_k + (2\alpha \theta'_y + \beta \theta_y)^2 W_k \end{aligned}$$

If we equate the coefficients of $\vec{\theta}'^2$, $\vec{\theta}' \cdot \vec{\theta}$ and $\vec{\theta}'^0$ we get the following system of differential equations:

$$\dot{\alpha} + \frac{i\omega v}{2} = q\alpha^2 \quad (6.12)$$

$$\dot{\beta} = q\alpha\beta \quad (6.13)$$

$$\dot{\gamma} + \frac{i\omega}{2}(1-v^2) = q\alpha + \frac{1}{4}q\beta^2\theta^2 \quad (6.14)$$

6.12 is an equation in α alone:

$$\frac{1}{2}\left(\frac{i\omega v}{2}\right)^{-1/2} \int d\alpha \left[\frac{1}{\sqrt{q}\alpha - \sqrt{\frac{i\omega v}{2}}} - \frac{1}{\sqrt{q}\alpha + \sqrt{\frac{i\omega v}{2}}} \right] = \tau + \text{const.}$$

It follows that

$$\alpha(\tau) = \sqrt{\frac{i\omega v}{2q}} \left[\frac{C e^{\sqrt{2i\omega v q} \tau} + 1}{C e^{\sqrt{2i\omega v q} \tau} - 1} \right]$$

The initial condition is that at $\tau = 0$, W_k is a delta function in the angular variables. If we take $C = 1$ then we ensure that at $\tau = 0$, W_k is singular. Therefore

$$\alpha(\tau) = -\sqrt{\frac{i\omega v}{2q}} \coth \sqrt{\frac{i\omega v q}{2}} \tau \quad (6.15)$$

We can now solve for β in 6.13:

$$\int \frac{d\beta}{\beta} = - \int \sqrt{\frac{i\omega v q}{2}} \coth \sqrt{\frac{i\omega v q}{2}} \tau d\tau + \text{const.}$$

We get

$$\beta(\tau) = \sqrt{\frac{2i\omega v}{q}} \operatorname{cosech} \sqrt{\frac{i\omega v q}{2}} \tau$$

6.14 is now

$$\dot{\gamma} + \frac{i\omega}{2}(1-v^2) = -q\sqrt{\frac{i\omega v}{2q}} \coth \sqrt{\frac{i\omega v q}{2}} \tau + \frac{1}{4}q\theta^2 \frac{2i\omega v}{q} \operatorname{cosech}^2 \sqrt{\frac{i\omega v q}{2}} \tau$$

Hence

$$\gamma = -\frac{i\omega}{2}(1-v^2)\tau - \ln \sinh \sqrt{\frac{i\omega v q}{2}} \tau - \theta^2 \sqrt{\frac{i\omega v}{2q}} \coth \sqrt{\frac{i\omega v q}{2}} \tau + D$$

We can find the integration constant D by imposing the initial condition, i.e. that

$$\int_{\Omega} e^{\alpha(0)\vec{\theta}'^2 + \beta(0)\vec{\theta}' \cdot \vec{\theta} + \gamma(0)} d\vec{\theta}' = 1$$

This is true for any choice of $\vec{\theta}$. Let us take $\vec{\theta} = (\theta, 0)$. The magnitudes of $\vec{\theta}$ and $\vec{\theta}'$ are $\ll 1$ for a relativistic particle. W_k is a delta function for $\tau = 0$. For non-zero values of τ , W_k still drops off very rapidly from its peak as $\vec{\theta}'$ moves away from $\vec{0}$. In light of this we can replace the integration over the solid angle Ω by an integration over the entire xy plane.

$$\begin{aligned}
I &\equiv \int e^{\alpha(\tau)\vec{\theta}'^2 + \beta(\tau)\vec{\theta}' \cdot \vec{\theta} + \gamma(\tau)} d\vec{\theta}' \\
&= \int_{-\infty}^{\infty} d\theta_x \int_{-\infty}^{\infty} d\theta_y e^{\alpha(\tau)(\theta_x^2 + \theta_y^2) + \beta(\tau)\theta_x + \gamma(\tau)} \\
&= e^{\gamma} \sqrt{\frac{\pi}{-\alpha}} e^{-\frac{\beta^2 \theta^2}{4\alpha}} \int_{-\infty}^{\infty} d\theta_x e^{\alpha(\theta_x + \frac{\beta}{2\alpha}\theta)^2} \\
&= -\frac{\pi}{\alpha} \exp\left\{\gamma - \frac{\beta^2 \theta^2}{4\alpha}\right\} \\
&= -\frac{\pi}{\alpha} \operatorname{cosech} \sqrt{\frac{i\omega v q}{2}} \tau \exp\left\{D - \frac{i\omega}{2}(1-v^2)\tau\right. \\
&\quad \left. + \theta^2 \sqrt{\frac{i\omega v}{2q}} \operatorname{cosech} \sqrt{\frac{i\omega v q}{2}} \tau \left(\operatorname{sech} \sqrt{\frac{i\omega v q}{2}} \tau - \cosh \sqrt{\frac{i\omega v q}{2}} \tau\right)\right\} \\
&= -\pi \sqrt{\frac{2q}{i\omega v}} \cosh \sqrt{\frac{i\omega v q}{2}} \tau \exp\left\{D - \frac{i\omega}{2}(1-v^2)\tau - \theta^2 \sqrt{\frac{i\omega v}{2q}} \tanh \sqrt{\frac{i\omega v q}{2}} \tau\right\} \quad (6.16)
\end{aligned}$$

At $\tau = 0, I = 1$. That is $1 = -\pi \sqrt{\frac{2q}{i\omega v}} e^D$ or $D = \ln \sqrt{\frac{i\omega v}{2\pi^2 q}}$. So we now have completed the solution for W_k and can return to our ultimate goal of finding the energy radiated. In order to develop 6.3 we need to express $(\hat{n} \times \vec{v}) \cdot (\hat{n} \times \vec{v}')$ in terms of the angles $\vec{\theta}$ and $\vec{\theta}'$. We use 6.7 for this purpose:

$$\begin{aligned}
(\hat{n} \times \vec{v}) \cdot (\hat{n} \times \vec{v}') &= v^2 (\hat{n} \times \vec{\theta}) \cdot (\hat{n} \times \vec{\theta}') \\
&= v^2 \epsilon_{ijk} n_j \theta_k \epsilon_{ilm} n_l \theta'_m \\
&= v^2 [\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}] n_i n_l \theta_k \theta'_m \\
&= v^2 \vec{\theta} \cdot \vec{\theta}'
\end{aligned}$$

Hence 6.3 can be written as

$$\frac{d^2 I}{d\omega d\Omega} = \frac{e^2 \omega^2}{2\pi^2} \operatorname{Re} \int_{-\infty}^{\infty} dt \int_0^{\infty} d\tau \int d\vec{r} d\vec{v} W_1(\vec{r}, \vec{v}, t) \int d\vec{\theta}' v^3 \vec{\theta} \cdot \vec{\theta}' W_k(\vec{\theta}', \vec{\theta}, \tau) \quad (6.17)$$

$$\begin{aligned}
&\int \vec{\theta} \cdot \vec{\theta}' \exp\left\{\alpha\vec{\theta}'^2 + \beta\vec{\theta} \cdot \vec{\theta}' + \gamma\right\} d\vec{\theta}' = \\
&\quad \theta e^{\gamma} \int_{-\infty}^{\infty} \theta_x e^{\alpha\theta_x^2 + \beta\theta\theta_x} d\theta_x \int_{-\infty}^{\infty} d\theta_y e^{\alpha\theta_y^2} \\
&= \theta e^{\gamma} \sqrt{\frac{\pi}{-\alpha}} \exp\left\{-\frac{\beta^2 \theta^2}{4\alpha}\right\} \int_{-\infty}^{\infty} d\phi \left(\phi - \frac{\beta\theta}{2\alpha}\right) e^{\alpha\phi^2} \quad , \phi \equiv \theta_x + \frac{\beta\theta}{2\alpha}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\beta\theta^2}{2\alpha^2}\pi \exp\left\{\gamma - \frac{\beta^2\theta^2}{4\alpha}\right\} \\
&= \theta^2 \operatorname{sech}^2\sqrt{\frac{i\omega v q}{2}}\tau \exp\left\{-i\frac{\omega\tau}{2}(1-v^2)\right\} \exp\left\{-\theta^2\sqrt{\frac{i\omega v}{2q}}\tanh\sqrt{\frac{i\omega v q}{2}}\tau\right\} \\
&= \frac{2i}{\omega v} \exp\left\{-i\frac{\omega\tau}{2}(1-v^2)\right\} \frac{d}{d\tau} \exp\left\{-\theta^2\sqrt{\frac{i\omega v}{2q}}\tanh\sqrt{\frac{i\omega v q}{2}}\tau\right\}
\end{aligned}$$

We use this result in 6.17 and perform the integration over $d\Omega$.

$$\begin{aligned}
\frac{dI}{d\omega} &= \frac{e^2\omega^2}{2\pi^2} \int_{-\infty}^{\infty} dt \int d\vec{r} d\vec{v} W_1 \operatorname{Re} \frac{2i}{\omega} \int_0^{\infty} d\tau \exp\left\{-i\frac{\omega\tau}{2}(1-v^2)\right\} \frac{d}{d\tau} \pi \sqrt{\frac{2q}{i\omega}} \coth\sqrt{\frac{i\omega q}{2}}\tau \\
&= -\frac{e^2\omega}{\pi} \int_{-\infty}^{\infty} dt \operatorname{Im} \frac{q}{r} \int_0^{\infty} d\tau e^{-i\delta\tau} \frac{d}{d\tau} \coth r\tau \quad (6.18)
\end{aligned}$$

Here we have defined δ and r as in the previous chapter, i.e. $\delta = \omega(1-v) \simeq \frac{\omega}{2}(1-v^2)$ and $r^2 = i\omega\sigma = \frac{i\omega q}{2}$. We now perform an integration by parts in 6.18. The integrand has a pole at 0. We remove it as before by subtracting $\frac{1}{r\tau}$. As a result 6.18 can be written as

$$\frac{dI}{d\omega} = \frac{2e^2\delta}{\pi} \int_0^{\infty} dt \operatorname{Im} \left(-r \int_0^{\infty} d\tau e^{-i\delta\tau} \left(\coth r\tau - \frac{1}{r\tau} \right) \right) \quad (6.19)$$

With $z = r\tau$ 6.19 is identical to 5.25. So the energy radiated is also given by 5.28.

Chapter 7

Experimental Verification of the LPM Effect

In 1993 the first precise experiment on the LPM effect was carried out by the E146 collaboration at SLAC [20, 21, 22, 23]. They studied the emission of low energy (1-500 MeV) bremsstrahlung photons from high energy (25 GeV) electrons. The experimental setup is shown in figure 7.1.

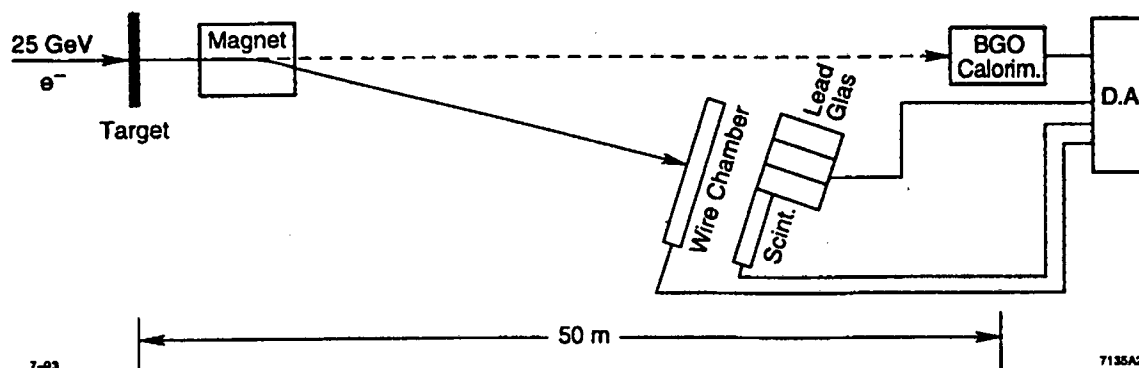


Figure 7.1: The layout of the E146 experiment at SLAC. Electrons enter End Station A and traverse a thin target. They are then bent downwards by a magnet and are detected by wire chambers and lead glass scintillators. Bremsstrahlung photons are detected in the BGO calorimeter.

The electron beam entered End Station A and traversed the thin target foils. Emerging electrons are bent downwards and are detected by wire chambers and lead glass scintillators. The bremsstrahlung photons are detected in a BGO calorimeter.

The targets were chosen to represent a wide range in atomic number (Z) and radiation

Target Material	Z	L (cm)	Thickness (% L)	LPM Threshold (MeV)
Carbon	6	18.8	2, 6	4.4
Aluminum	13	8.9	3, 6	9.2
Iron	26	1.76	3, 6	47
Tungsten	74	0.35	2, 6	235
Gold	79	0.34	0.1, 1, 6	240
Lead	82	0.56	2	147
Uranium	92	0.32	3, 5	265

Table 7.1: Targets used in the E146 experiment.

length L (defined in 1.16). The targets are listed in Table 7.1.

We can estimate the region where LPM suppression will be important as follows. A.16 gives the mean square value for the scattering angle of the electron for the case of elastic scattering. If the actual mean scattering angle exceeds this value then the electron has been disturbed in the formation zone and suppression should result. If we combine 1.13 with A.16 we find that suppression occurs if $\omega < E^2/E_{LPM}$ where $E_{LPM} = m^2 L \alpha / 4\pi$ is a medium dependant constant, ω is the photon energy and E is the electron energy. The target can be neither too thin such that the electron does not have multiple interactions, nor can it be too thick such that the electron has a chance to emit more than one bremsstrahlung photon.

If the theory we have developed is correct then we would expect to see that $dN/d\omega \propto 1/\sqrt{\omega}$ for low values of ω and that at higher ω values $dN/d\omega \propto 1/\omega$, the Bethe-Heitler spectrum. There is, however, another important effect at low ω that we have not considered. This is dielectric suppression. The bremsstrahlung photon which is being emitted also needs to be coherent. It is, however, moving in a medium with a non-unity index of refraction. Contributions to the photon from different parts of the formation zone will have a different phase. The criterion that the photon not interfere destructively with itself yields ([15], Chapter 3) suppression for $\omega < \gamma\omega_p$ where ω_p is the plasma frequency of the medium. This modifies the spectrum for very low ω to $dN/d\omega \propto \omega$. A sketch of the expected bremsstrahlung spectrum is given in figure 7.2.

Figure 7.3 shows the observed spectra for the two carbon targets; part (a) is for a target which is 4.1 mm $\simeq 2\%L$ thick and part (b) for a target which is 12.7 mm $\simeq 6\%L$ thick. The first thing one observes is that the predicted spectrum is not flat, even if the LPM and dielectric suppression effects are ignored. This is a result of multiple bremsstrahlung from a single electron. The solid line at the top of Figure 8.3(a) shows the Bethe-Heitler spectrum ignoring multi-photon pileup. The dashed line (largely hidden by the data points) is the spectrum predicted by our detailed calculation of the LPM effect. Clearly the data exhibits the small suppression at energies below 20 MeV.

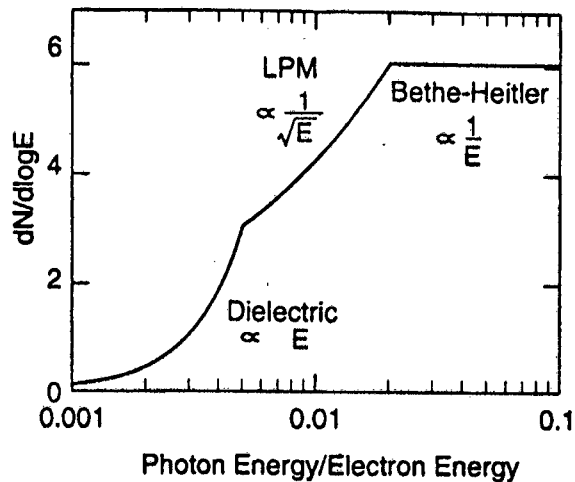


Figure 7.2: Sketch of the expected bremsstrahlung spectrum.

Figure 7.4 shows the corresponding spectra for the two uranium targets; part (a) for a $79 \mu m \simeq 3\%L$ target and part (b) for a $147 \mu m \simeq 5\%L$ target. The LPM suppression is clearly verified. The deviation from the expected LPM curve at very low energies is probably due to dielectric suppression.

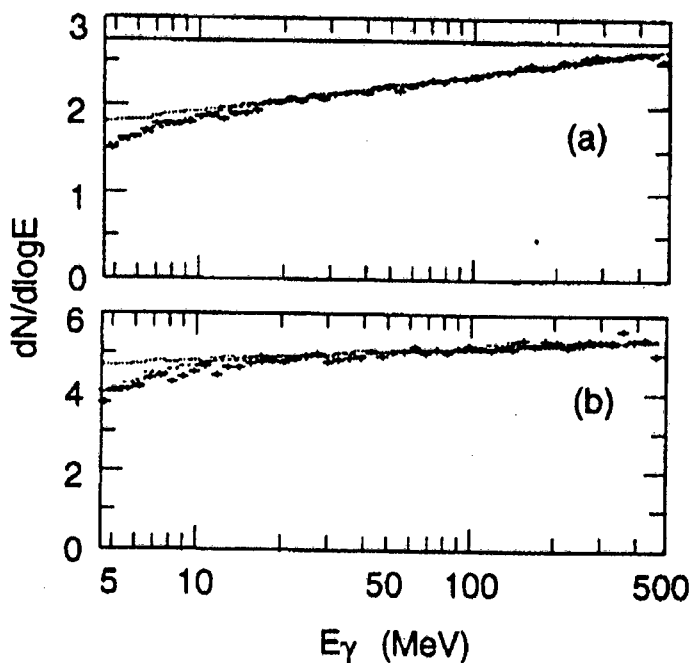


Figure 7.3: The observed bremsstrahlung spectrum (crosses) for two carbon targets, in units of photons per 1000 electrons, for a 2%L (a) and 6%L (b) thick target. The dotted line (top) shows the Bethe-Heitler Monte-Carlo expectations, whereas the dashed line (bottom) is the Monte Carlo expectations including the LPM effect. The solid line at the top in (a) is the Bethe-Heitler prediction ignoring multi-photon pileup.

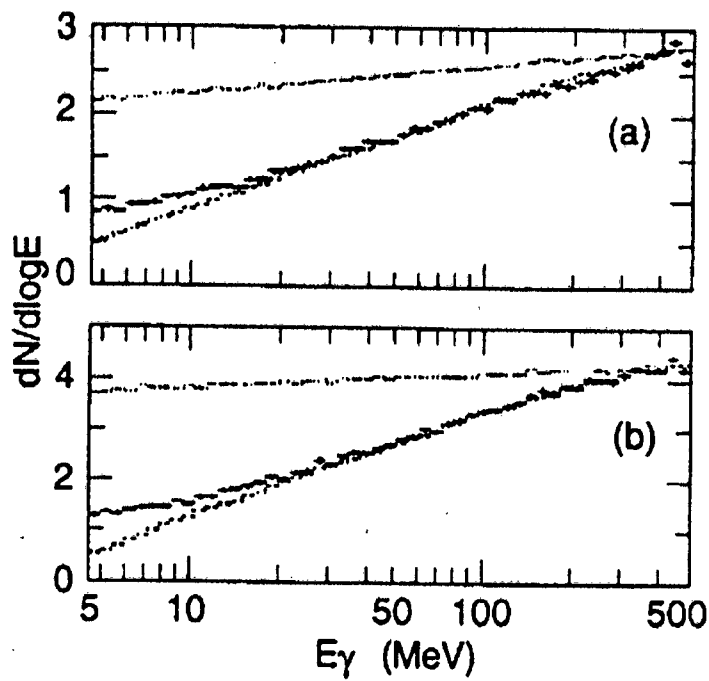


Figure 7.4: The observed bremsstrahlung spectrum (crosses) for two uranium targets, in units of photons per 1000 electrons, for a 2% L (a) and a 5% L (b) thick target. The dotted line (top) shows the Bethe-Heitler Monte-Carlo expectation, whereas the dashed line (bottom) is the Monte-Carlo expectation including the LPM effect.

Conclusion

The Bethe-Heitler result does not exactly reproduce the bremsstrahlung spectrum obtained from the passage of ultra-relativistic charged particles through an amorphous medium. The deviation is most pronounced for the soft part of the spectrum. The method originally employed by Landau and Pomeranchuk to include the effects of multiple scattering only qualitatively reproduces the experimental data with the $\sqrt{\omega}$ behaviour for small ω . The calculations based on the kinetic equation method and on the functional integration method yield the same result. This result is in excellent agreement with experiment.

Acknowledgements

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Appendix A

Elementary Kinetic Theory

A.1 The Boltzman equation

We begin by introducing a distribution function $f_N(x_1, \dots, x_{3N}; p_1, \dots, p_{3N}; t)$. f_N is defined such that $f_N d^{3N}x d^{3N}p$ is the probability of finding the system of N particles in the $6N$ -dimensional volume element about the point $(x_1, \dots, x_{3N}; p_1, \dots, p_{3N})$ in phase space at a time t [24]. f_N is normalized such that $\forall t$:

$$\int \dots \int_{\Gamma_{6N}} f_N d^{3N}x d^{3N}p = 1 \quad (\text{A.1})$$

Suppose the system is governed by the Hamiltonian $H(x_1, \dots, x_{3N}; p_1, \dots, p_{3N}; t)$. Consider the particle distribution at a time $t + dt$:

$$\begin{aligned} & f_N(x_1 + dx_1, \dots, x_{3N} + dx_{3N}; p_1 + dp_1, \dots, p_{3N} + dp_{3N}; t + dt) \\ & \simeq f_N + \frac{\partial f_N}{\partial x_1} dx_1 + \dots + \frac{\partial f_N}{\partial x_{3N}} dx_{3N} \\ & \quad + \frac{\partial f_N}{\partial p_1} dp_1 + \dots + \frac{\partial f_N}{\partial p_{3N}} dp_{3N} + \frac{\partial f_N}{\partial t} dt \\ & \simeq f_N + \frac{\partial f_N}{\partial x_1} \frac{\partial H}{\partial p_1} dt + \dots + \frac{\partial f_N}{\partial x_{3N}} \frac{\partial H}{\partial p_{3N}} dt - \frac{\partial f_N}{\partial p_1} \frac{\partial H}{\partial x_1} dt - \dots \\ & \quad - \frac{\partial f_N}{\partial p_{3N}} \frac{\partial H}{\partial x_{3N}} dt + \frac{\partial f_N}{\partial t} dt \end{aligned} \quad (\text{A.2})$$

where we have used the Hamilton equations of motion $dx_\alpha = \frac{\partial H}{\partial p_\alpha} dt$ and $dp_\alpha = -\frac{\partial H}{\partial x_\alpha} dt$.

If we rearrange (A.2) and let dt go to zero we get

$$\frac{df_N}{dt} = \frac{\partial f_N}{\partial t} + \sum_{\alpha=1}^{3N} \left(\frac{\partial f_N}{\partial x_\alpha} \frac{\partial H}{\partial p_\alpha} - \frac{\partial f_N}{\partial p_\alpha} \frac{\partial H}{\partial x_\alpha} \right) \quad (\text{A.3})$$

If there are no interactions between particles and if the system is in equilibrium then all changes in f_N must be of second order, i.e.

$$\frac{df_N}{dt} = 0 \quad (\text{A.4})$$

Ultimately we are only interested in the bulk properties of the distribution. These properties can be defined in terms of the behaviour of a single particle which is characteristic of the distribution. We define the 1-particle distribution function as

$$f(\vec{r}, \vec{p}, t) = N \int d^{N-1} \vec{r}' d^{N-1} \vec{p}' f_N(\vec{r}, \vec{r}'_1, \dots, \vec{r}'_{N-1}; \vec{p}, \vec{p}'_1, \dots, \vec{p}'_{N-1}; t) \quad (\text{A.5})$$

where we have fixed the coordinates and momentum of one particle and then integrated over the other $N - 1$ particles. The factor of N appears because we are free to choose any one of our N particles.

The most general Hamiltonian for a system of N particles with no interactions is

$$H(\vec{r}_1, \dots, \vec{r}_N; \vec{p}_1, \dots, \vec{p}_N, t) = \sum_{\alpha=1}^N \left(\frac{\vec{p}_\alpha^2}{2m} + U(\vec{r}_\alpha, t) \right)$$

In order to find a differential equation for $f(\vec{r}, \vec{p}, t)$ we start with (A.3,4):

$$\begin{aligned} & \frac{\partial}{\partial t} N \int d\vec{r}'^{N-1} d\vec{p}'^{N-1} f_N(\vec{r}, \vec{r}'_1, \dots, \vec{r}'_{N-1}; \vec{p}, \vec{p}'_1, \dots, \vec{p}'_{N-1}; t) \\ & + N \sum_{\alpha=1}^N \int d\vec{r}'^{N-1} d\vec{p}'^{N-1} \left(\frac{\partial f_N}{\partial \vec{r}_\alpha} \frac{\partial H}{\partial \vec{p}_\alpha} - \frac{\partial f_N}{\partial \vec{p}_\alpha} \frac{\partial H}{\partial \vec{r}_\alpha} \right) = 0 \end{aligned} \quad (\text{A.6})$$

$\vec{r}_N = \vec{r}$ Due to the form of our Hamiltonian $\frac{\partial H}{\partial \vec{p}_\alpha} = \frac{\vec{p}_\alpha}{m}$ which is independent of \vec{r} and $\frac{\partial H}{\partial \vec{r}_\alpha} = \frac{\partial U}{\partial \vec{r}_\alpha}$ which is independent of \vec{p} . Now

$$\begin{aligned} & \int d\vec{p}'^{N-1} \int d\vec{r}'^{N-1} \frac{\partial f_N}{\partial \vec{r}_\alpha} \frac{\partial H}{\partial \vec{p}_\alpha} \\ & = \int d\vec{p}'^{N-1} \frac{\vec{p}_\alpha}{m} \int d\vec{r}'^{N-1} \frac{\partial f_N}{\partial \vec{r}_\alpha} \end{aligned}$$

The integral over $d\vec{r}^{N-1}$ contains, $\forall \alpha \neq N$, the integral $\int_{-\infty}^{\infty} d\vec{r}_\alpha \frac{\partial f_N}{\partial \vec{r}_\alpha} = f_N|_{-\infty}^{\infty}$. For the normalization (A.1) to be sensible we must have $\lim_{|\vec{r}_\alpha|, |\vec{p}_\alpha| \rightarrow \infty} f_N = 0$. (A.6) can then be written as

$$\begin{aligned} \frac{\partial f(\vec{r}, \vec{p}, t)}{\partial t} + N \int d\vec{r}^{N-1} d\vec{p}^{N-1} \left(\frac{\partial f_N}{\partial \vec{r}} \frac{\vec{p}}{m} - \frac{\partial f_N}{\partial \vec{p}} \frac{\partial U}{\partial \vec{r}} \right) &= 0 \\ \implies \frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{r}} + \vec{F}(\vec{r}, t) \frac{\partial f}{\partial \vec{p}} &= 0 \end{aligned} \quad (\text{A.7})$$

where $\vec{F}(\vec{r}, t)$ is the external force. (A.7) is a statement of Liouville's Theorem.

We now allow pairs of particles to interact. We assume that the density is low enough that the probability of three or more particles interacting simultaneously is vanishingly small. The probability of a particle with momentum \vec{p} interacting with a particle with momentum \vec{p}_1 at the point \vec{r} in the time interval $(t, t + dt)$ is proportional to the product $f(\vec{r}, \vec{p}, t) f(\vec{r}, \vec{p}_1, t)$ with the proportionality constant $\omega(\vec{p}, \vec{p}_1; \vec{p}, \vec{p}_1)$. $\vec{p} \rightarrow \vec{p}'$ and $\vec{p}_1 \rightarrow \vec{p}'_1$. We can also have the reverse process where $\vec{p}' \rightarrow \vec{p}$ and $\vec{p}'_1 \rightarrow \vec{p}_1$. The interactions that concern us are electromagnetic and so we must have time reversal symmetry, i.e. $\omega(\vec{p}', \vec{p}'_1; \vec{p}, \vec{p}_1) = \omega(\vec{p}, \vec{p}_1; \vec{p}', \vec{p}'_1)$. (A.4) no longer holds. Instead we have

$$\frac{df}{dt} = \int d^3 \vec{p}_1 d^3 \vec{p}'_1 d^3 \vec{p}' \omega(\vec{p}', \vec{p}'_1; \vec{p}, \vec{p}_1) [f(\vec{p}', \vec{r}, t) f(\vec{p}'_1, \vec{r}, t) - f(\vec{p}, \vec{r}, t) f(\vec{p}_1, \vec{r}, t)] \quad (\text{A.8})$$

We can relate ω to the scattering cross section $\frac{d\sigma}{d\Omega}$ as follows:

$$\begin{aligned} \{\text{Scattering rate}\} &= \frac{d\sigma}{d\Omega} d\Omega \times \{\text{Incident flux}\} \times \{\text{Target density}\} \\ &= \frac{d\sigma}{d\Omega} d\Omega \times f(\vec{p}, \vec{r}, t) |\vec{v} - \vec{v}_1| \times f(\vec{p}_1, \vec{r}, t) d^3 \vec{p}_1 \end{aligned}$$

This must be equivalent to the loss term in (A.8). Hence

$$\omega(\vec{p}', \vec{p}'_1; \vec{p}, \vec{p}_1) d^3 \vec{p}' d^3 \vec{p}'_1 = \frac{d\sigma}{d\Omega} d\Omega |\vec{v} - \vec{v}'|$$

and so

$$\frac{\partial f}{\partial t} + \vec{v} \frac{\partial f}{\partial \vec{r}} + \vec{F}(\vec{r}, t) \frac{\partial f}{\partial \vec{p}} = \int d^3 \vec{p}_1 \int \frac{d\sigma}{d\Omega} d\Omega |\vec{v} - \vec{v}'| [f' f'_1 - f f_1] \quad (\text{A.9})$$

This is the Boltzman equation.

A.2 Distribution function for stochastic scattering

Let's consider the case of a particle beam incident on an amorphous solid. The target particles are assumed to be stationary both before and after a collision, i.e. we assume

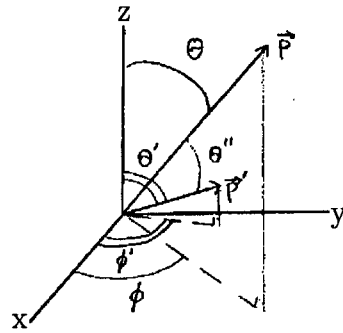
elastic, glancing collisions. If n is the number density of targets then

$$f(\vec{p}_1, \vec{r}, t) = f(\vec{p}'_1, \vec{r}, t) = n \delta^{(3)}(\vec{p}_1)$$

(A.9) reduces to

$$\frac{df}{dt} = nv \int \frac{d\sigma}{d\Omega} d\Omega [f(|\vec{p}'|, \cos \theta') - f(|\vec{p}|, \cos \theta)] \quad (\text{A.10})$$

θ and θ' are the angles \vec{p} and \vec{p}' make with the z -axis respectively. θ'' is the angle between \vec{p} and \vec{p}' in the plane formed by \hat{p} and \hat{p}' . $d\Omega$ is the element of solid angle into which the particle is scattered, i.e. $d\Omega = \sin \theta'' d\theta'' d\phi''$.



Lemma 5 $\cos \theta' = \cos \theta \cos \theta'' + \sin \theta \sin \theta'' \cos(\phi - \phi'')$

Proof: Rotate our system of axes s.t. \hat{p} now defines the new z -direction. Denote the old z -direction in the new coordinates as $\hat{\theta}_z$.

$$\hat{\theta}_z = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

and

$$\vec{p}' = |\vec{p}'| (\sin \theta'' \cos \phi'', \sin \theta'' \sin \phi'', \cos \theta'')$$

and so

$$\begin{aligned} \hat{\theta}_z \cdot \vec{p}' &= |\vec{p}'| \cos \theta' \\ &= |\vec{p}'| (\sin \theta \sin \theta'' \cos \phi \cos \phi'' + \sin \theta \sin \theta'' \sin \phi \sin \phi'' + \cos \theta \cos \theta'') \end{aligned}$$

$$\Rightarrow \cos \theta' = \cos \theta \cos \theta'' + \sin \theta \sin \theta'' \cos(\phi - \phi'') \quad \square$$

In our approximation $\theta'' \ll 1$. So $\sin \theta'' \simeq \theta''$, $\cos \theta'' \simeq 1 - \frac{1}{2}\theta''^2$ and

$$\cos \theta' - \cos \theta \simeq -\frac{1}{2}\theta''^2 \cos \theta + \theta'' \sin \theta \cos(\phi - \phi'') \quad (\text{A.11})$$

We can make a Taylor expansion of f about $\cos \theta$:

$$\begin{aligned} f(\cos \theta') &= f(\cos \theta) + (\cos \theta' - \cos \theta) \frac{\partial f}{\partial \cos \theta} + \frac{1}{2} (\cos \theta' - \cos \theta)^2 \frac{\partial^2 f}{\partial \cos^2 \theta} + \dots \\ &= f(\cos \theta) + \left[-\frac{1}{2} \theta''^2 \cos \theta + \theta'' \sin \theta \cos(\phi - \phi'') \frac{\partial f}{\partial \cos \theta} \right] \\ &\quad + \frac{1}{2} \theta''^2 \sin^2 \theta \cos^2(\phi - \phi'') \frac{\partial^2 f}{\partial \cos^2 \theta} + O(\theta''^3) \end{aligned}$$

The scattering we deal with has no ϕ dependence. $\int_0^{2\pi} d(\phi - \phi'') \cos(\phi - \phi'') = 0$ and so

$$\begin{aligned} \int_0^{2\pi} d(\phi - \phi'') [f(\cos \theta') - f(\cos \theta)] &= \frac{1}{2} \pi \theta''^2 \left[\sin^2 \theta \frac{\partial^2 f}{\partial \cos^2 \theta} - 2 \cos \theta \frac{\partial f}{\partial \cos \theta} \right] \\ &= \frac{1}{2} \pi \theta''^2 \left[(1 - z^2) \frac{\partial^2 f}{\partial z^2} - 2z \frac{\partial f}{\partial z} \right] \\ &= \frac{1}{2} \pi \theta''^2 \frac{\partial}{\partial z} \left((1 - z^2) \frac{\partial f}{\partial z} \right) \\ &= \frac{1}{2} \theta''^2 \Delta_{\theta} f \end{aligned}$$

where $z \equiv \cos \theta$ and Δ_{θ} is the 2-dimensional Laplacian. (A.10) now becomes

$$\begin{aligned} \frac{df}{dt} &= nv \int_{\theta_{min}}^{\theta_{max}} \sin \theta'' d\theta'' \frac{1}{2} \pi \theta''^2 \frac{d\sigma}{d \cos \theta''} \Delta_{\theta} f \\ &= v \frac{\langle \theta_s^2 \rangle}{4} \Delta_{\theta} f \end{aligned} \quad (\text{A.12})$$

where $\langle \theta_s^2 \rangle \equiv \int d\Omega'' \frac{d\sigma}{d\Omega''} n \theta''^2$. $\langle \theta_s^2 \rangle$ is the mean square change of angle per unit length. For the case of steady state flow $\frac{\partial f}{\partial t} = 0$ and if there are no external forces then the Boltzman equation gives

$$\vec{v} \cdot \nabla f = \frac{\langle \theta_s^2 \rangle}{4} v \Delta_{\theta} f \quad (\text{A.13})$$

If the beam is in the z -direction then an approximate solution of (A.13) is

$$f(z, \theta) = \frac{f(0)}{\pi z \langle \theta_s^2 \rangle} \exp \left(-\frac{\theta^2}{z \langle \theta_s^2 \rangle} \right)$$

So the distribution function for a charged particle moving through an amorphous solid is Gaussian. If we define the mean square scattering angle per unit time to be 2σ then

$$f(\tau, \theta) = \frac{f(0)}{2\pi\sigma\tau} \exp \left(-\frac{\theta^2}{2\sigma\tau} \right)$$

We can find $f(0)$ by normalizing to unit probability

$$\int f(\tau, \theta) d\Omega = \frac{f(0)}{2\pi\sigma\tau} \int_{-\infty}^{\infty} e^{-\frac{\theta_x^2}{2\sigma\tau}} d\theta_x \int_{-\infty}^{\infty} e^{-\frac{\theta_y^2}{2\sigma\tau}} d\theta_y = 1$$

The result is $f(0) = 1$ and so

$$f(\tau, \theta) = \frac{1}{2\pi\sigma\tau} \exp\left(-\frac{\theta^2}{2\sigma\tau}\right) \quad (\text{A.14})$$

It is a somewhat difficult matter to obtain an explicit expression for the mean square scattering angle per unit time, 2σ (often called q). We are primarily interested in relativistic scattering. We therefore have to consider cross sections which are corrected for electron screening, as is discussed in chapter 1. For the case of elastic (Mott) scattering with the Thomas-Fermi model of the atom assumed it can be shown that [25, 27]

$$q = \frac{E_s^2}{E^2 L} \quad (\text{A.15})$$

E_s and L are defined in 1.16. As a result the mean square scattering angle at a time τ is

$$\langle \theta^2 \rangle = \frac{E_s^2}{E^2 L} \tau \quad (\text{A.16})$$

E is the energy of the incident electron. It follows that

$$\left\langle \int_0^\tau \theta^2 dt \right\rangle = \frac{E_s^2}{2E^2 L} \tau |\tau| \quad (\text{A.17})$$

and

$$\begin{aligned} \left\langle \left(\int_0^\tau \vec{\theta} dt \right)^2 \right\rangle &= \left\langle \int_0^\tau \vec{\theta}_t dt \int_0^\tau \vec{\theta}_\nu d\nu \right\rangle \\ &= 2 \left\langle \int_0^\tau dt \int_0^t \vec{\theta}_t \cdot \vec{\theta}_\nu d\nu \right\rangle \\ &= 2 \int_0^\tau dt \int_0^t \langle \theta_\nu^2 \rangle d\nu \\ &= \frac{E_s^2}{3E^2} |\tau|^3 \end{aligned} \quad (\text{A.18})$$

Bibliography

- [1] H. Bethe and W. Heitler, Proc.Roy.Soc A, **146** (1934) 83
- [2] W. Heitler, Z.Physik, **84** (1933) 145
- [3] W. Heitler, The Quantum Theory of Radiation, Dover Book Company (1984) 242-250
- [4] C. Itzykson and J. Zuber, Quantum Field Theory, McGraw-Hill (1980) 238-241
- [5] J.D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics, McGraw-Hill (1964) 125
- [6] J.D. Jackson, Classical Electrodynamics, 2nd ed., John Wiley and Sons (1975) Chapter 14
- [7] M.D. Scadron, Advanced Quantum Theory, 2nd ed., Springer-Verlag (1991) 221-222
- [8] A.I. Akhiezer and N.F. Shul'ga, Sov.Phys.Usp, **30**(3) (1987) 197; Usp.Fiz.Nauk, **151** (1987) 385
- [9] V.M. Galitsky and I.I. Gurevich, Nuovo Cimento, **33**(2) (1964) 396
- [10] L.D. Landau and I.J. Pomeranchuk, Dokl.Akad.Nauk.SSSR, **92** (1953) 535; **92** (1953) 735. These papers are available in english in L. Landau, The Collected Papers of L.D. Landau, Pergamon Press (1965) 586-593
- [11] A.I. Akhiezer and N.F. Shul'ga, Sov.Phys.Usp, **25**(8) (1982) 541
- [12] N.V. Laskin et.al., Sov.Phys.JETP, **62**(3) (1985) 438
- [13] E.W. Montroll, Comm.Pure and Applied Math., **V** (1952) 415-453
- [14] A.B. Migdal, Dokl.Akad.Nauk.SSSR, **96** (1954) 49; Sov.Phys.JETP, **5** (1957) 527
- [15] M.I. Ter-Mikaelian, High-energy Electromagnetic Processes in Condensed Media, Wiley-Interscience (1972)
- [16] J. Cleymans, V.V. Goloviznin, K. Redlich, Phys.Rev.D, **47**(3) (1993)

- [17] I.I. Gold'man, *Sov.Phys.JETP*, **11**(6) (1960) 1341
- [18] M.I. Ryazanov, *Sov.Phys.Usp*, **17**(6) (1975) 815
- [19] J.S. Bell, *Nuc.Phys*, **8** (1958) 613
- [20] P. Chen and S. Klein, SLAC-PUB-6021 (1992)
- [21] S.R. Klein et.al., SLAC-PUB-6378 (1993)
- [22] R. Becker-Szendy et.al., SLAC-PUB-6400 (1993)
- [23] M.L. Perl, SLAC-PUB-6514 (1994)
- [24] N.P. Kalshnikov, V.S. Remizovich, M.I. Ryazanov, *Collision of Fast Charged Particles in Solids*, Gordon and Breach Science Publishers (1985)
- [25] Mott and Massey, *Theory of Atomic Collisions*, 3'rd ed., Oxford University Press (1965) 425
- [26] A.B. Migdal, *Phys.Review*, **103**(6) (1956) 1811
- [27] B. Rossi and K. Greisen, *Rev.Mod.Phys.*, **13** (1941) 240
- [28] A.B. Migdal, *Sov.Phys.JETP*, **5**(4) (1957) 527
- [29] E.L. Feinberg, *Sov.Phys.JETP*, **23**(1) (1966) 132
- [30] F.F. Ternovski, *Sov.Phys.JETP*, **12**(1) (1961) 123
- [31] V. Balek and N. Pisutowa, *Acta Physica Polonica*, **B21** (1990) 149