Rho-Meson Propagator at Two-Loop Order In The Kroll-Lee-Zumino Quantum Field Theory

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Abstract

The Kroll-Lee-Zumino theory, an Abelian renormalizable quantum field theory of charged pions and a neutral ρ meson, provides a framework to compute corrections to the tree-level Vector-Meson-Dominance model[1, 2]. Despite a large $\rho\pi\pi$ coupling ($g_{\rho\pi\pi} \approx 5$), the loop suppresion factor of $1/(4\pi)^2$ helps give reasonably small NLO (one-loop) corrections. When it comes to describing hadronic physics, one might say, these one-loop results achieve moderate to excellent success [3, 4, 5]. Where the one-loop results show less than impressive agreement with experiment, it seems plausible that the NNLO (two-loop) corrections would remedy the situation. In this thesis the two-loop contribution to the ρ^0 meson is calculated. It turns out to be larger than the one-loop result and so one must conclude that perturbation theory breaks down at this level.

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May all beings be happy and free…

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Introduction

The theory of the strong force, Quantum chromodynamics (QCD), has the peculiar properties of assymptotic freedom and confinement. This peculiarity makes it notoriously challenging to calculate the low-energy consequences in any meaningful way. It is not surprising then to find that a great deal of research on the strong force involves the use of effective models; this thesis is no exception.

To give some context we revisit the old notion of vector meson dominance (VMD) [1, 2, 6]; an idea is inspired by the observation that, in strong interactions, the virtual photon behaves like a hadron.The vector meson dominance principle, in its narrowest form, asserts that in the interaction of hadronic matter with electromagnetism, the electromagnetic current is identical the neutral rho-meson (ρ^0) field. In other words, the virtual photon in such interactions couples directly to a neutral rho meson, as shown in figure 1 for example. A more general form of the principle asserts that the electromagnetic current is identical to a linear combination of neutral vector meson fields. For the purposes of this thesis, it suffices to consider the narrow formulation VMD which

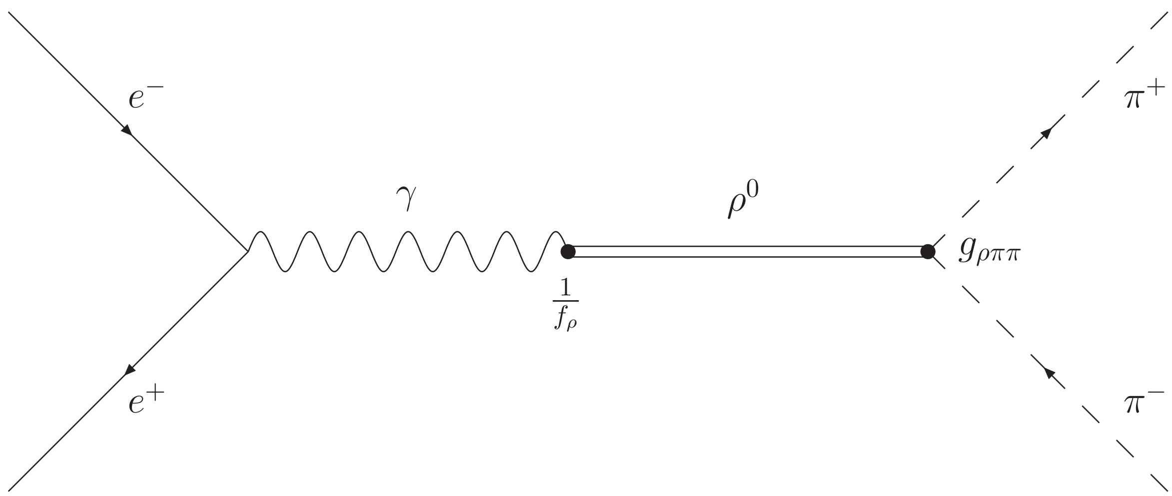


Figure 1: The process $e^+e^- \to \pi^+\pi^-$ according to VMD.

maybe expressed via the following current-field-identity (CFI)[2]:

$$\left(J_\mu{}^\gamma\right)_{\text{isovector}} = -\left(\frac{M^2}{f_\rho}\right) V_\mu \ , \tag{0.0.1}$$

where V_μ represents the ρ^0 field, M is the mass of ρ^0 and f_ρ determines the decay rate $\Gamma(\rho^0 \to e^+e^-)$. In 1967 Kroll, Lee and Zumino (KLZ) formulated this idea in terms of a renormalizable Abelian Quantum Field Theory[7], and the following is the part of their Lagrangian that describes ρ-π dynamics:

$$\mathcal{L}_{\text{KLZ}} = \partial_\mu\pi\partial^\mu\pi^* - m^2\pi\pi^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 V_\mu V^\mu + g_{\rho\pi\pi}V_\mu J^\mu_\pi + g^2_{\rho\pi\pi}V_\mu V^\mu \pi\pi^* \ , \tag{0.0.2}$$

where π is a complex pseudoscalar field representing the charged pions and m is the pion mass. We also have

$$F_{\mu\nu} = \partial_\mu \mu V_\nu - \partial_\nu V_\mu \ , \tag{0.0.3}$$

$$J^\mu_\pi = i\pi^* \overleftrightarrow{\partial}^\mu \pi \ . \tag{0.0.4}$$

The KLZ Lagrangian was shown to be renormalizable, despite the explicit mass term of the ρ^0 meson, since the ρ^0 field couples to the conserved current [7]. This means that there are no new free parameters needed at each order in perturbation theory, thereby giving the KLZ theory more predictive power than an unrenormalizable effective field theory. The relatively mild $\rho\pi\pi$ coupling ($g_{\rho\pi\pi} \approx 5$) along with loop suppression factors of $1/(4\pi)^2$ gives hope for a sensible perturbative expansion in the coupling. This theory then, allows for systematic calculation of corrections to tree-level Vector Meson Dominance. In this regard it has a clear advantage over ad-hoc tree level models.

Over the years these favourable properties of the theory have been exploited to calculate the pion's electromagnetic form factor and radius [3], as well as it's scalar form factor and radius [5, 4]. In [3] the electromagnetic form factor in the space-like region is calculated using the formula

$$F_\pi(q^2) = \frac{M^2 + F_{\text{vac}}(0)}{M^2 - q^2 + F_{\text{vac}}(q^2)} + \frac{M^2}{M^2 - q^2}\left(G(q^2) - G(0)\right) \ , \tag{0.0.5}$$

where $F_{\text{vac}}(q^2)$ was calculated from the one-loop self-energy diagrams of figure 2 and $G(q^2)$ involves the three-point diagram shown in figure 3. The results for the form factor in the space-like region are in spectacular agreement with experiment as seen in figure 4. The electromagnetic radius of the pion was calculated to be $\langle r_\pi^2\rangle_{\text{em}} = 0.46\,\text{fm}^2$, which compares much more favourably with experiment $\langle r_\pi^2\rangle_{\text{em}} = 0.439 \pm 0.008\,\text{fm}^2$ than tree-level VMD which predicts $\langle r_\pi^2\rangle_{\text{em}} = 0.39\,\text{fm}^2$. However, there is still room for improvement with regards to the electromagnetic radius; the same goes for the form factor in the time-like region as seen in figure 5. The authors suggest that the main source of systematic uncertainty in this determination stems from the NNLO (two-loop) contribution.

In the case of the scalar radius which plays an important role in chiral perturbation theory [8, 9], the NLO (one-loop) calculation in the KLZ theory gives a value of $\langle r_\pi^2\rangle_s = 0.40\,\text{fm}^2$ [4] which lies below the range of values from determinations using $\pi\pi$ scattering as well as Lattice QCD [10, 11, 12, 13, 14] $\langle r_\pi^2\rangle_s \approx 0.5 - 0.7\,\text{fm}^2$. In this case it is yet again suggested that the NNLO contribution is the main source of uncertainty.

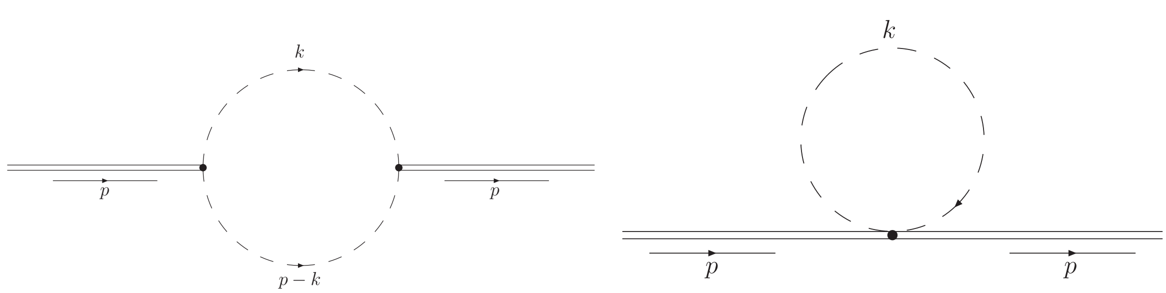
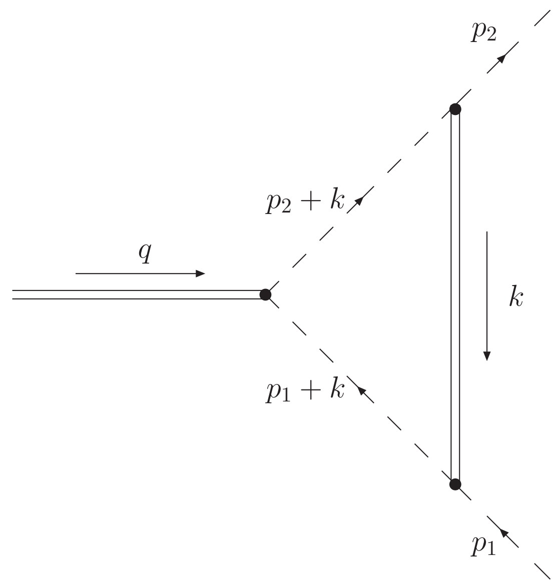


Figure 2: One-loop contribution to the ρ^0 self-energy. The double lines represent the ρ^0 meson and the dashed lines are pions.

Figure 3



Another interesting application of the KLZ theory was carried out by Gale and Kapusta [15], who used it to calculate the thermal production rate of lepton pairs in a hot pion gas, but in end they find that temperature effects are rather modest. In the conclusion of their paper it is suggested that two-loop contributions to the ρ^0 self-energy are required to determine the thermal production rate of real photons.

Looking at all the applications of the KLZ theory just recounted, one sees a strong case for pursuing NNLO (two-loop) calculations. This has the potential to extend the work of [15] to include new phenomana viz. thermal production of photons. In the case of the electromagnetic form factor in the time-like region as well as the work done in [3, 4], there's the potential to reduce systematic uncertainty and improve agreement with experiment. The prospect that a simple model, in the form of a renormalizable Abelian quantum field theory, can describe low-energy hadronic physics with a precision matching Lattice QCD and effective theories such as Chiral perturbation theory, is exciting.

This thesis pursues such a NNLO computation, and in particular, the two-loop contribution to the ρ^0 self energy is calculated; the results disabuse us of any hope of improving the afore mentioned results. It will be shown that while the loop suppression factor $1/(4\pi)^2$, mentioned earlier, gives reasonable NLO corrections, at NNLO the loop suppression factor (now $1/(4\pi)^4$) cannot quell the growth of the two-loop integrals. The NNLO contribution is unreasonably large and so perturbation theory breaks down.

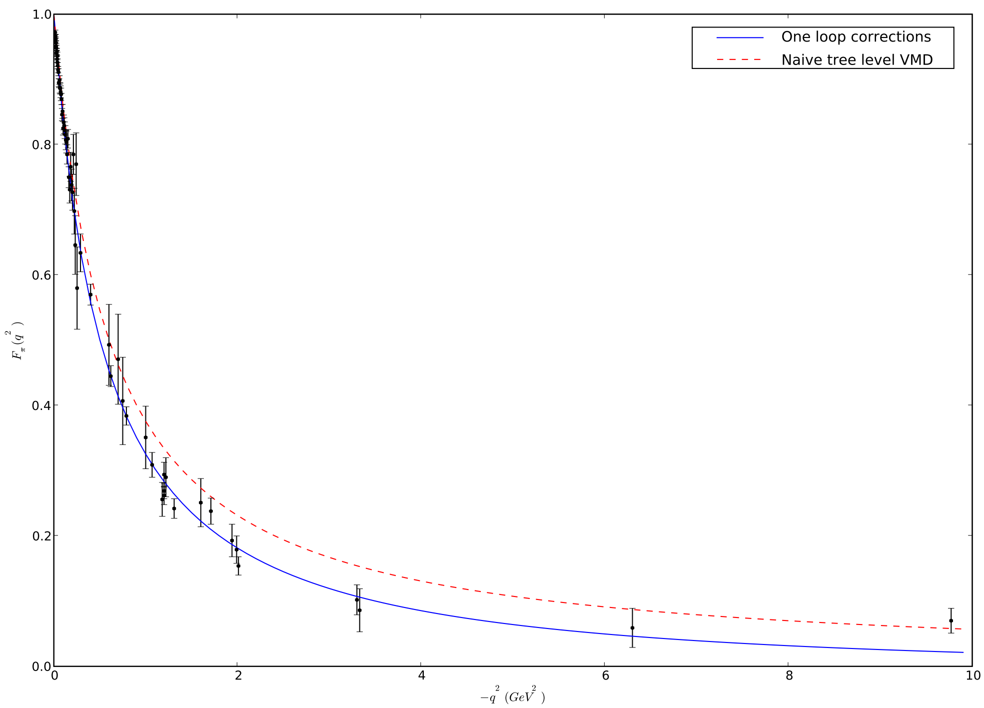


Figure 4: KLZ theory one-loop correction for the pion's electromagnetic form factor(solid blue line) compared to the tree level VMD prediction (red dashed line) in the space-like region [3].

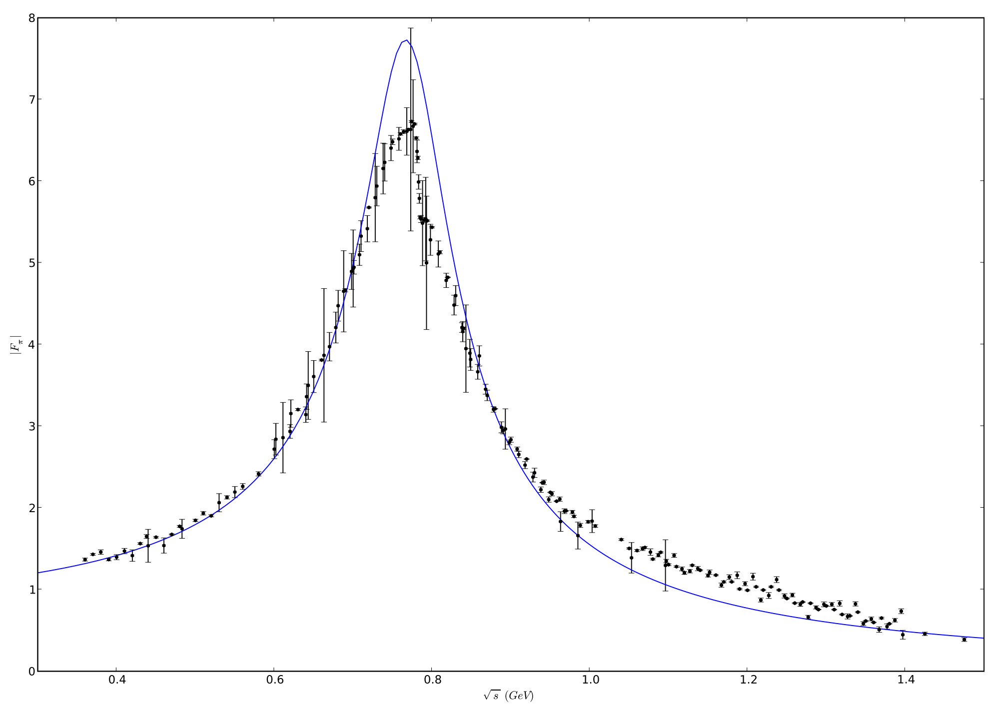


Figure 5: KLZ theory one-loop correction for the pion's in the time-like region [16].

1

KLZ-Lagrangian

Let's take a closer look at the Lagrangian of Kroll, Lee and Zumino which may be expressed as:

$$\mathcal{L}_{\text{KLZ}} = \underbrace{|D_\mu \pi|^2 - m^2 \pi^* \pi}_{\mathcal{L}_\pi} \underbrace{- \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} M^2 V^\mu V_\mu}_{\mathcal{L}_V} \tag{1.0.1}$$

where again π, π^* are charged pion fields, V^μ is the ρ^0 field, m and M are the pion and rho-meson mass parameters respectively.

We also have:

$$F^{\mu\nu} = \partial^\mu V^\nu - \partial^\nu V^\mu \tag{1.0.2}$$

$$D_\mu = \partial_\mu - igV_\mu \ . \tag{1.0.3}$$

Where $g = g_{\rho\pi\pi}$ is the coupling constant for ρ^0 to pions.

1.1 Hidden Gauge Symmetry

The KLZ Lagriangian does not, at face-value, seem gauge invariant because of the vector mass-term. However, it was shown by [17] that its gauge invariance is revealed using Stückelberg's trick [18, 19, 20]. We give a summary of how this is demonstrated. We focus on the non-gauge-invariant part of the Lagrangian $\mathcal{L}_V$. Consider the slightly modified version, the Stückelberg Lagrangian:

$$\mathcal{L}_{\text{st}} = \mathcal{L}_V + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + M\phi\partial_\mu V^\mu \tag{1.1.1}$$

where we have introduced a dynamic scalar field ϕ (Stückelberg's ghost). Under the following local transformations:

$$\begin{aligned} V_\mu &\to V_\mu + \partial_\mu\chi \\ \phi &\to \phi + M\chi \end{aligned} \tag{1.1.2}$$

the Stückelberg Lagrangian transforms as

$$\mathcal{L}_{\text{st}} \to \mathcal{L}_{\text{st}} + \partial_\mu\left(M\phi\partial^\mu\chi + M^2\chi V^\mu + M^2\chi\partial^\mu\chi\right) \tag{1.1.3}$$

where χ is an arbitrary function of space-time. The total derivative in (1.1.3) vanishes when we integrate over space time so the contribution of $\mathcal{L}_{\text{st}}$ to the action is invariant under the transformations (1.1.2). This means if we take the following modified version of the KLZ lagrangian (1.0.1):

$$\mathcal{L}_{\text{inv}} = |D_\mu\pi|^2 - m^2\pi^*\pi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}M^2V^\mu V_\mu + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + M\phi\partial_\mu V^\mu \ , \tag{1.1.4}$$

then the action $S_{\text{inv}} = \int d^4x\mathcal{L}_{\text{inv}}$, is invariant under the following gauge transformations:

$$\begin{aligned} \pi &\to e^{ig\theta(x)}\pi \ , & V_\mu &\to V_\mu + \partial_\mu\theta(x) \ , \\ \pi^* &\to e^{-ig\theta(x)}\pi^* \ , & \phi &\to \phi + M\theta(x) \ . \end{aligned} \tag{1.1.5}$$

with this gauge-invariant action in hand, quantization is done in the path-integral formalism. A standard requirement for perturbation theory in the path-integral formalism is gauge fixing; the following class of gauge conditions is considered:

$$F\left[V_\mu\,,\phi\right] - C(x) = 0 \ , \tag{1.1.6}$$

where $C(x)$ is an arbitrary function and

$$F\left[V_\mu\,,\phi\right] = \partial_\mu V^\mu + \xi M \phi \ . \tag{1.1.7}$$

This has the effect of adding a gauge-fixing term to the Lagrangian

$$\mathcal{L}_{\text{fix}} = -\frac{1}{2\xi}\left(\partial_\mu V^\mu + \xi M\phi\right)^2 \ , \tag{1.1.8}$$

accompanied by a contribution from the Fadeev-Popov determinant:

$$\mathcal{L}_{\text{FP}} = (\partial_\mu \eta^*)(\partial^\mu \eta) - \xi M^2 \eta^* \eta \ , \tag{1.1.9}$$

where η and η^* are Grassman fields (Fadeev-Popov Ghosts). Perturbation theory can then be done using the following effective Lagrangian

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \mathcal{L}_{\text{inv}} + \mathcal{L}_{\text{fix}} + \mathcal{L}_{\text{FP}} \\ &= \left|D_\mu \pi\right|^2 - m^2 \pi^* \pi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} M^2 V^\mu V_\mu - \frac{1}{2\xi}\left(\partial_\mu V^\mu\right)^2 \\ &+ \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}\xi M^2 \phi^2 + (\partial_\mu \eta^*)(\partial^\mu \eta) - \xi M^2 \eta^* \eta \ .\end{aligned} \tag{1.1.10}$$

Note how the choice of gauge-fixing functional (1.1.7), has eliminated the ϕ-V mixing term and given the Stückelberg ghost a gauge-dependent mass. The above Lagrangian (1.1.10) describes a massive vector (V_μ) interacting with charged scalars (π^*, π), the Stückelberg ghost as well as the Fadeev-Popov ghosts are just free fields, spectators. The true dynamics are captured by the first part of the

Lagrangian

$$\mathcal{L}_{\text{KLZ}} = |D_\mu \pi|^2 - m^2 \pi^* \pi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} M^2 V^\mu V_\mu - \frac{1}{2\xi} \left(\partial_\mu V^\mu\right)^2 \ . \tag{1.1.11}$$

The gauge parameter ξ can be changed arbitrarily with no bearing on the physical implications of the theory. With the gauge choice $\xi \to \infty$, we recover (1.0.1). Thus the hidden gauge symmetry is revealed.

1.2 Ward-Takahashi Identities

Following [17], we will use the gauge invariance of the theory to derive Ward Identities. First we introduce the Generating functional for full n-point Green's Functions:

$$W[\mathbf{J}] = \int \mathcal{D} \left(V_\mu \pi^* \pi \phi \eta^* \eta\right) \exp\left(i S_{\text{inv}} + i S_{\text{fix}} + i S_{\text{FP}} + i S_J\right) \tag{1.2.1}$$

where

$$S_{\text{fix}} = -\frac{1}{2\xi} \int d^4 x \left(\partial_\mu V^\mu - M \xi \phi\right)^2 \ , \tag{1.2.2}$$

$$S_{\text{FP}} = \int d^4 x \left\{(\partial_\mu \eta^*)(\partial^\mu \eta) - \xi M^2 \eta^* \eta + J_{\eta^*} \eta^* + J_\eta \eta\right\} \ , \tag{1.2.3}$$

$$S_J = \int d^4 x \left(j^\mu V_\mu + J_\pi \pi + J_{\pi^*} \pi^* + J_\phi \phi\right) \ . \tag{1.2.4}$$

Let's consider the behaviour of $W[J]$ under infinitesimal gauge transformations:

$$\begin{aligned} \pi &\to \pi + ig\delta\lambda(x)\, \pi \ , & V_\mu &\to V_\mu + \partial_\mu \delta\lambda(x) \\ \pi^* &\to \pi^* - ig\delta\lambda(x)\, \pi^* \ , & \phi &\to \phi + M\delta\lambda(x) \ . \end{aligned} \tag{1.2.5}$$

The only terms that break gauge-invariance in (1.2.1) are S_{fix} and S_J, each of which transforms as follows:

$$S'_{\text{fix}} = S_{\text{fix}} - \int d^4x \frac{\delta\lambda(x)}{\xi}\left(\Box + \xi M^2\right)\left(\partial_\mu V^\mu + M\xi\phi\right) \ , \tag{1.2.6}$$

$$S'_J = S_J + \int d^4x \left[-\partial_\mu j^\mu + ig(\pi J_\pi - \pi^* J_{\pi^*}) + M J_\phi\right]\delta\lambda(x) \ ,$$

when working to first order in $\delta\lambda(x)$. Since a gauge transformation inside the path-integral amounts to a change of integration variable, the value of the path-integral doesn't change, and so

$$\delta W[\mathbf{J}] = 0$$

$$\Longrightarrow\ 0 = \int \mathcal{D}\left(V_\mu \pi^* \pi \phi \eta^* \eta\right)\left\{\exp\left(iS'_{\text{inv}} + iS'_{\text{fix}} + iS'_{\text{FP}} + iS'_J\right) - \exp\left(iS_{\text{inv}} + iS_{\text{fix}} + iS_{\text{FP}} + iS_J\right)\right\}$$

$$\therefore 0 = \int d^4x \int \mathcal{D}\left(V_\mu \pi^* \pi \phi \eta^* \eta\right)\exp\left(iS_{\text{inv}} + iS_{\text{fix}} + iS_{\text{FP}} + iS_J\right)\left\{-\frac{1}{\xi}\left(\Box + \xi M^2\right)\left(\partial_\mu V^\mu + M\xi\phi\right)\right.$$
$$\left. - \partial_\mu j^\mu + ig(\pi J_\pi - \pi^* J_{\pi^*}) + M J_\phi\right\}\delta\lambda(x) \tag{1.2.7}$$

Since $\delta\lambda(x)$ was arbitrary the integrand in the preceding equation (1.2.7) vanishes identically and so we arrive at the Ward identities:

$$-\frac{(\Box + \xi M^2)}{\xi}\left(\partial_\mu \frac{\delta W}{i\delta j_\mu(x)} + M\frac{\delta W}{i\delta J_\phi(x)}\right) - \partial_\mu J^\mu(x) W + M J_\phi(x) W + ig\left(J_\pi \frac{\delta W}{i\delta J_\pi} - J_{\pi^*}\frac{\delta W}{i\delta J_{\pi^*}}\right) = 0 \ . \tag{1.2.8}$$

We introduce two more generating functionals: first is the generating functional for connected Green's Functions $Z[J]$ for which we have:

$$W[J] = \exp\left(iZ[J]\right) \tag{1.2.9}$$

$$\langle 0|\, \mathrm{T}\varphi_{i_1}(x_1)\ldots\varphi_{i_n}(x_n)\,|0\rangle = \left(\frac{1}{i}\right)^{n-1} \left.\frac{\delta^n Z[J]}{\delta J_{i_1}(x_1)\ldots\delta J_{i_n}}\right|_{J\equiv 0}$$
$$= G^{(n)}_{\text{conn}}\left(x_1, \ldots, x_n\right) \ , \tag{1.2.10}$$

secondly we have the generating functional, $\Gamma[\Phi]$, for proper vertex functions. It is related to $Z[J]$ in the following way:

$$\Gamma\left[V_{\mathrm{c}\mu}, \pi_{\mathrm{c}}^*, \pi_{\mathrm{c}}, \phi_{\mathrm{c}}\right] = Z[J] - \int d^4x \left(j_\mu V_{\mathrm{c}}{}^\mu + J_\pi \pi_{\mathrm{c}} + J_{\pi^*}\pi_{\mathrm{c}}^* + J_\phi \phi_{\mathrm{c}}\right) \ , \tag{1.2.11}$$

with

$$\begin{aligned}
&\frac{\delta\Gamma[\Phi]}{\delta V_{\mathrm{c}\mu}(x)} = -j^\mu(x) \ , \qquad \frac{\delta\Gamma[\Phi]}{\delta \pi_{\mathrm{c}}^*(x)} = -J_{\pi^*}(x) \ , \qquad \frac{\delta\Gamma[\Phi]}{\delta \pi_{\mathrm{c}}(x)} = -J_{\pi}(x) \ , \qquad \frac{\delta\Gamma[\Phi]}{\delta \phi_{\mathrm{c}}(x)} = -J_{\phi}(x) \ , \\
&\frac{\delta Z[J]}{\delta j_\mu(x)} = V_{\mathrm{c}}{}^\mu(x) \ , \qquad \frac{\delta Z[J]}{\delta J_{\pi^*}(x)} = \pi_{\mathrm{c}}^*(x) \ , \qquad \frac{\delta Z[J]}{\delta J_{\pi}(x)} = \pi_{\mathrm{c}}(x) \ , \qquad \frac{\delta Z[J]}{\delta J_{\phi}(x)} = \phi_{\mathrm{c}}(x) \ .
\end{aligned} \tag{1.2.12}$$

We can then write the Ward identities (1.2.8) in terms of these generating functionals giving:

$$-\frac{(\Box + \xi M^2)}{\xi}\left(\partial_\mu \frac{\delta Z}{\delta j_\mu(x)} + M\frac{\delta Z}{\delta J_\phi(x)}\right) - \partial_\mu j^\mu(x) + MJ_\phi(x) + ig\left(J_\pi \frac{\delta Z}{\delta J_\pi} - J_{\pi^*}\frac{\delta Z}{\delta J_{\pi^*}}\right) = 0 \ , \tag{1.2.13}$$

$$-\frac{(\Box + \xi M^2)}{\xi}\left(\partial_\mu V_{\mathrm{c}}{}^\mu + M\phi_{\mathrm{c}}\right) + \partial_\mu \frac{\delta\Gamma}{\delta V_{\mathrm{c}\mu}(x)} - M\frac{\delta\Gamma}{\delta\phi_{\mathrm{c}}(x)} - ig\left(\pi_{\mathrm{c}}\frac{\delta\Gamma}{\delta\pi_{\mathrm{c}}(x)} - \pi_{\mathrm{c}}^*\frac{\delta\Gamma}{\delta\pi_{\mathrm{c}}^*(x)}\right) = 0 \ . \tag{1.2.14}$$

Full rho-meson propagator

Let's discuss some quantities that are of great interest in this thesis. The full momentum space propagator of the rho meson, denoted $D^{\mu\nu}(p^2)$,is defined by the following equation:

$$\frac{1}{i}\left.\frac{\delta^2 Z[J]}{\delta j^\mu(x)\delta j^\nu(y)}\right|_{j=0} = \int d^4p \, e^{ip\cdot(x-y)} D^{\mu\nu}(p^2) \tag{1.2.15}$$

and for our Lagrangian (1.3.10), it takes the general form:

$$D^{\mu\nu}(p^2) = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\left(\frac{-i}{p^2 - M^2 - \Pi_{\mathrm{T}}(p^2)}\right) + \frac{p^\mu p^\nu}{p^2}\left(\frac{-i}{\frac{p^2-\xi M^2}{\xi} - \Pi_{\mathrm{L}}(p^2)}\right) \ , \tag{1.2.16}$$

which involves the ρ^0 self-energy:

$$\Pi_\rho^{\mu\nu}(p^2) = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Pi_{\mathrm{T}}(p^2) + \frac{p^\mu p^\nu}{p^2}\Pi_{\mathrm{L}}(p^2) \ , \tag{1.2.17}$$

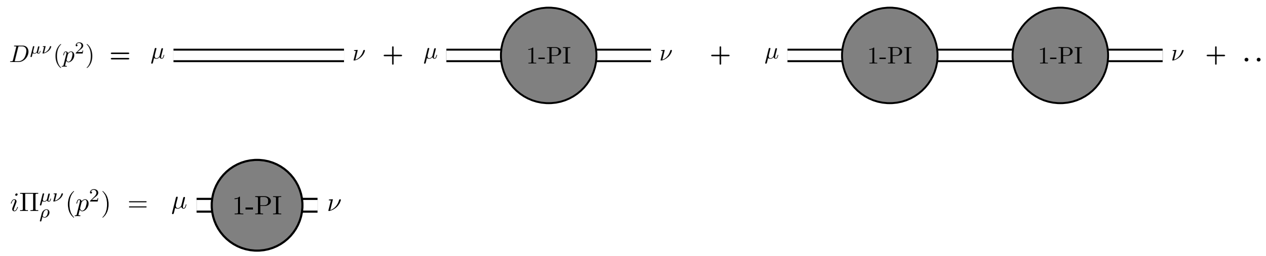


Figure 1.1: Momentum space self-energy and full propagator of the rho meson. Here the rho meson is represented by a double line.

a quantity obtained from the one-particle-irreducible Feynman diagrams as shown in figure 1.1. A related quantity is the ρ-ρ proper vertex function in momentum space, $\Gamma^{\mu\nu}_{\rho}(p^2)$, defined as follows:

$$\left.\frac{\delta^2\Gamma[\Phi]}{\delta V_{c\mu}(x)\delta V_{c\nu}(y)}\right|_{V_c=0} = \int d^4p\, e^{ip\cdot(x-y)}\Gamma^{\mu\nu}_{\rho}(p^2) \ . \tag{1.2.18}$$

To relate $\Gamma^{\mu\nu}_{\rho}(p^2)$ to $D^{\mu\nu}(p^2)$ we use the following relations from (1.2.12)

$$\frac{\delta\Gamma[\Phi]}{\delta V_{c\mu}(x)} = -j^{\mu}(x) \ , \tag{1.2.19}$$

$$\frac{\delta Z[J]}{\delta j_{\mu}(x)} = V_c{}^{\mu}(x) \ . \tag{1.2.20}$$

Taking the functional derivative of (1.2.20) with respect to $V_{c\nu}(y)$ and then using (1.2.19) gives:

$$-\int d^4v \frac{\delta^2\Gamma[\Phi]}{\delta V_{c\mu}(y)\delta V_{c\alpha}(v)}\frac{\delta^2 Z[J]}{\delta j^{\alpha}(v)\delta j_{\nu}(x)} = g^{\mu\nu}\delta(x-y) \ , \tag{1.2.21}$$

if we now set all fields to zero and go to momentum space, (1.2.21) becomes

$$\Gamma^{\mu\alpha}_{\rho}(p^2)\, D_{\alpha\nu}(p^2) = ig^{\mu}{}_{\nu} \ . \tag{1.2.22}$$

We can then immediately deduce the form of $\Gamma^{\mu\nu}_{\rho}$ to be:

$$\Gamma^{\mu\nu}_{\rho}(p^2) = \left(g^{\mu\nu} - \frac{p^{\mu}p^{\nu}}{p^2}\right)\left(M^2 - p^2 + \Pi_{\mathrm{T}}(p^2)\right) + \frac{p^{\mu}p^{\nu}}{p^2}\left(\frac{-p^2+\xi M^2}{\xi} + \Pi_{\mathrm{L}}(p^2)\right) \ . \tag{1.2.23}$$

To connect to the ward identites, we take the functional derivative of (1.2.14) with respect to $V_{c\nu}(y)$ and then set all sources to zero yielding:

$$-\frac{\Box_x + \xi M^2}{\xi}\partial_{x\mu}\delta(x-y)g^{\mu\nu} + \partial_{x\mu}\left.\frac{\delta^2\Gamma[\Phi]}{\delta V_{c\mu}(x)V_{c\nu}(y)}\right|_{V_c=0} = 0 \ , \tag{1.2.24}$$

and in momentum space this reads:

$$-\frac{-p^2 + \xi M^2}{\xi} ip^\nu + ip_\mu \Gamma_\rho^{\mu\nu} = 0 \ , \tag{1.2.25}$$

if we now use (1.2.23), we get:

$$\Pi_{\mathrm{L}}(p^2) = 0 \ . \tag{1.2.26}$$

In other words $\Pi_\rho^{\mu\nu}(p^2)$ is transverse, and so we may write this self-energy as

$$\Pi_\rho^{\mu\nu}(p^2) = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) F_{\text{vac}}(p^2) \ , \tag{1.2.27}$$

where we have adopted the notation $F_{\text{vac}} = \Pi_{\mathrm{T}}$ to be consistent with [15].

1.3 Renormalized Feynman Rules

In preparation for perturbative calculations, we work in $n = 4 - 2\epsilon$ dimensions. This has the effect of introducing the renormalization scale μ which has dimensions of mass. The effective Lagrangian for ρ-π dynamics is the following

$$\begin{aligned}\mathcal{L}_{\text{KLZ}} &= \partial_\mu {\pi_0}^* \partial^\mu \pi_0 - m_0^2 {\pi_0}^* \pi_0 - \frac{1}{4} {F_0}^{\mu\nu} F_{0\mu\nu} + \frac{1}{2} M^2 {V_0}^\mu V_{0\mu} \\ &- \frac{1}{2\xi_0}\left(\partial_\mu {V_0}^\mu\right)^2 + i g_0 \mu^\epsilon {V_0}^\mu \left({\pi_0}^* \overleftrightarrow{\partial_\mu} \pi_0\right) + g_0^2 \mu^{2\epsilon} {V_0}^\mu V_{0\mu} \pi_0 {}^* \pi_0 \ ,\end{aligned} \tag{1.3.1}$$

The fields and parameters in the Lagrangian have subscripts "o" to emphasise that the quantities are unrenormalized. They can be related to their physical(renormalized) counterparts by introducing renormalization constants in the following way

$$\pi_0 = \sqrt{Z_2}\pi\,, \qquad V_0^\mu = \sqrt{Z_3}V^\mu\,, \tag{1.3.2}$$

$$Z_2 m_0^2 = m^2 + \delta m^2\,, \qquad Z_3 M_0^2 = M^2 + \delta M^2\,, \tag{1.3.3}$$

$$g_0 Z_2 \sqrt{Z_3} = g Z_1\,, \qquad {g_0}^2 Z_2 Z_3 = g^2 Z_1'\,, \tag{1.3.4}$$

$$\frac{1}{\xi_0} Z_3 = \frac{1}{\xi} + \delta\left(\frac{1}{\xi}\right)\,, \tag{1.3.5}$$

and for each of the Z-factors we have:

$$Z_1 = 1 + \delta Z_1\,, \qquad Z_2 = 1 + \delta Z_2\,, \qquad Z_3 = 1 + \delta Z_3\,, \qquad Z_1' = 1 + \delta Z_1'\,. \tag{1.3.6}$$

The renormalization constants above, are understood to be an expansion over the renormalized coupling g:

$$\delta Z_1 = \sum_{j=1}^{\infty} \delta Z_1^{(j)} \alpha^j\,, \qquad \delta Z_2 = \sum_{j=1}^{\infty} \delta Z_2^{(j)} \alpha^j\,, \qquad \delta Z_3 = \sum_{j=1}^{\infty} \delta Z_3^{(j)} \alpha^j\,, \qquad \delta Z_1' = \sum_{j=1}^{\infty} \delta {Z_1'}^{(j)} \alpha^j\,, \tag{1.3.7}$$

$$\delta m^2 = \sum_{j=1}^{\infty} \delta^{(j)} m^2 \alpha^j\,, \qquad \delta M^2 = \sum_{j=1}^{\infty} \delta^{(j)} M^2 \alpha^j\,, \qquad \delta\left(\frac{1}{\xi}\right) = \sum_{j=1}^{\infty} \delta^{(j)}\left(\frac{1}{\xi}\right)\alpha^j\,, \tag{1.3.8}$$

where

$$\alpha = \frac{g^2}{16\pi^2}\,. \tag{1.3.9}$$

Each of the coefficients $\delta Z_1^{(j)}$, $\delta^{(j)} m^2$ etc. is to be determined order-by-order in perturbation theory. If we substitute the renormalization constants into (1.3.1) the Lagrangian takes the following form:

$$\begin{aligned}
\mathcal{L}_{\text{KLZ}} = {} & \partial_\mu \pi^* \partial^\mu \pi - m^2 \pi^* \pi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} M^2 V^\mu V_\mu \\
& - \frac{1}{2\xi} \left(\partial_\mu V^\mu\right)^2 + i g \mu^\epsilon V^\mu \left(\pi^* \overleftrightarrow{\partial_\mu} \pi\right) + g^2 \mu^{2\epsilon} V^\mu V_\mu \pi^* \pi \\
& + \delta Z_2 \partial_\mu \pi^* \partial^\mu \pi - \delta m^2 \pi^* \pi - \frac{1}{4} \delta Z_3 F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \delta M^2 V^\mu V_\mu \\
& - \frac{1}{2\xi} \delta\left(\frac{1}{\xi}\right) \left(\partial_\mu V^\mu\right)^2 + i g \delta Z_1 \mu^\epsilon V^\mu \left(\pi^* \overleftrightarrow{\partial_\mu} \pi\right) + g^2 \delta Z_1' \mu^{2\epsilon} V^\mu V_\mu \pi^* \pi\,,
\end{aligned} \tag{1.3.10}$$

from which the Feynman rules shown in figure 1.2 are obtained. They are written in arbitrary gauge, but we will take the Feynman gauge ($\xi = 1$) to simplify calculations. We can distinguish two parts in the Lagrangian: the renormalized Lagrangian:

$$\begin{aligned}\mathcal{L} &= \partial_\mu \pi^* \partial^\mu \pi - m^2 \pi^* \pi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} M^2 V^\mu V_\mu \\ &- \frac{1}{2\xi} \left(\partial_\mu V^\mu\right)^2 + ig\mu^\epsilon V^\mu \left(\pi^* \overleftrightarrow{\partial_\mu} \pi\right) + g^2 \mu^{2\epsilon} V^\mu V_\mu \pi^* \pi \ ,\end{aligned} \tag{1.3.11}$$

which has the same form as the unrenormalized Lagrangian, and the counter-term Lagrangian:

$$\begin{aligned}\mathcal{L}_\otimes &= \delta Z_2 \partial_\mu \pi^* \partial^\mu \pi - \delta m^2 \pi^* \pi - \frac{1}{4} \delta Z_3 F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \delta M^2 V^\mu V_\mu \\ &- \frac{1}{2\xi} \delta \left(\frac{1}{\xi}\right) \left(\partial_\mu V^\mu\right)^2 + ig\delta Z_1 \mu^\epsilon V^\mu \left(\pi^* \overleftrightarrow{\partial_\mu} \pi\right) + g^2 \delta Z_1' \mu^{2\epsilon} V^\mu V_\mu \pi^* \pi \,,\end{aligned} \tag{1.3.12}$$

which is tasked with absorbing any divergences from perturbative calculations.

It follows from (1.3.4) that Z_1, Z_2 and Z_1' are related by:

$$Z_1^2 = Z_1' Z_2 \ , \tag{1.3.13}$$

expanding both sides in α gives:

$$\begin{aligned}1 + 2\alpha \delta Z_1^{(1)} + O(\alpha^2) &= 1 + \alpha \delta {Z_1'}^{(1)} + \alpha \delta Z_2^{(1)} + O(\alpha^2) \\ \implies \delta {Z_1'}^{(1)} &= 2\delta Z_1^{(1)} - \delta Z_2^{(1)} \ .\end{aligned} \tag{1.3.14}$$

Renormalized Self-energy

Note that in the previous section about Ward-Takahashi identities, we were working in terms of unrenormalized quantities. So equation (1.2.26) should, strictly speaking be written as

$$\Pi_{0\mathrm{L}}(p^2) = 0 \ , \tag{1.3.15}$$

and so it is in fact the unernormalized self-energy that is transverse. However, the unrenormalized self-energy is related to the renormalized one, and for the longitudinal parts in particular, the relationship is the following:

$$\Pi_{0\mathrm{L}}(p^2) = \frac{\Pi_{\mathrm{L}}(p^2) + \delta M^2 - p^2 \delta\left(\frac{1}{\xi}\right)}{Z_3} \ . \tag{1.3.16}$$

We can now see that the renormalized self-energy will also be transverse provided the gauge parameter and the rho-meson mass don't get renormalized in the theory, that is provided:

$$\delta\left(\frac{1}{\xi}\right) = 0 = \delta M^2 \ . \tag{1.3.17}$$

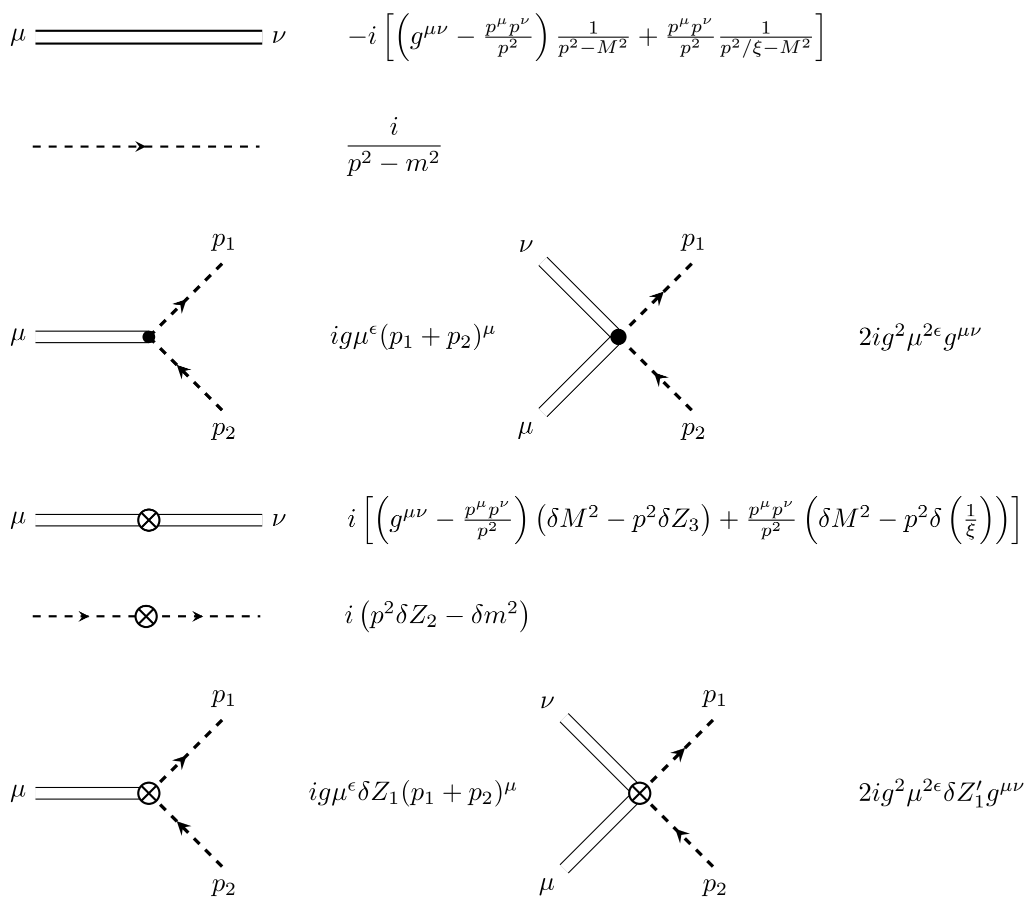


Figure 1.2: KLZ Feynman rules in arbitrary gauge.

1.3.1 R-ratio

The R-ratio, defined by

$$R(s) = \frac{\sigma\left(e^+e^- \to \text{hadrons}\right)}{\sigma\left(e^+e^- \to \mu^+\mu^-\right)}$$

is a well-measured observable [21]. It is related to the current correlator

$$\begin{aligned}\Pi^{\mu\nu}|_{\text{isovector}} &= i\int d^4x e^{-ip\cdot x}\left\langle 0\right| \mathrm{T} J^\mu_{\text{em}}(x) J^\nu_{\text{em}}(x)\left|0\right\rangle_{\text{isovector}} \\ &= \left(-g^{\mu\nu}p^2 + p^\mu p^\nu\right)\Pi_{\text{em}} \ , \end{aligned} \tag{1.3.18}$$

as follows

$$R(s) = 12\pi \, \text{Im}\,\Pi_{\text{em}} \ . \tag{1.3.19}$$

The current-field identity[2]

$$(J^\mu_{\text{em}})_{\text{isovector}} = -\frac{M}{f_\rho}V^\mu \ , \tag{1.3.20}$$

provides a connection between current correlator (1.3.18) and the full propagator of the ρ^0-meson in the KLZ theory:

$$\begin{aligned} i\int d^4x e^{-ip\cdot x}\left\langle 0\right| \mathrm{T} J^\mu(x) J^\nu(x)\left|0\right\rangle_{\text{isovector}} &= i\frac{M^4}{f_\rho^2}\int d^4x e^{-ip\cdot x}\left\langle 0\right| \mathrm{T}\, V^\mu(x) V^\nu(x)\left|0\right\rangle \\ &= \frac{M^4}{p^2 f_\rho^2}\left(-g^{\mu\nu}p^2 + p^\mu p^\nu\right)\frac{1}{M^2 - p^2 + F_{\text{vac}}} \ . \end{aligned} \tag{1.3.21}$$

Comparing (1.3.21) to (1.3.18) we immediately see the following relation

$$\Pi_{\text{em}}(p^2) = \frac{M^4}{p^2 f_\rho^2}\frac{1}{M^2 - p^2 + F_{\text{vac}}} \ , \tag{1.3.22}$$

or in real-imaginary form:

$$\Pi_{\text{em}}(p^2) = \frac{M^4}{p^2 f_\rho^2} \frac{M^2 - p^2 + \text{Re}\, F_{\text{vac}}}{\left(M^2 - p^2 + \text{Re}\, F_{\text{vac}}\right)^2 + \left(\text{Im}\, F_{\text{vac}}\right)^2} - i \frac{M^4}{p^2 f_\rho^2} \frac{\text{Im}\, F_{\text{vac}}}{\left(M^2 - p^2 + \text{Re}\, F_{\text{vac}}\right)^2 + \left(\text{Im}\, F_{\text{vac}}\right)^2} \ . \tag{1.3.23}$$

Equation (1.3.19) above then becomes:

$$R(s) = \frac{M^4}{s f_\rho^2} \frac{-12\pi\, \text{Im}\, F_{\text{vac}}(s)}{\left(M^2 - s + \text{Re}\, F_{\text{vac}}(s)\right)^2 + \left(\text{Im}\, F_{\text{vac}}(s)\right)^2} \ . \tag{1.3.24}$$

The above equation (1.3.24) provides the link between the KLZ theory and experiment, all we have to do is determine F_{vac} (or equivalently, the ρ^0 self-energy).

There is no royal road to science, and only those who do not dread the fatiguing climb of its steep paths have a chance of gaining its luminous summits.

Karl Marx

2

Reduction of the ρ^0 Self-Energy

The self-energy of the ρ^0 meson is obtained from the one-particle-irreducible diagrams shown in figure 2.1. Adding all these diagrams up (including symmetry factors) gives us the self-energy up to order α^2

$$I^{\mu\nu} + J^{\mu\nu} + \frac{1}{2}\xi_1^{\mu\nu} + \frac{1}{2}\xi_2^{\mu\nu} + \Omega^{\mu\nu} + 2Z^{\mu\nu} + 4X^{\mu\nu} + W^{\mu\nu} + A^{\mu\nu} + 2C_1^{\mu\nu} + C_2^{\mu\nu} + 2C_3^{\mu\nu} + C_4^{\mu\nu} + C_5^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) F_{\text{vac}} \,, \tag{2.0.1}$$

which should be transverse, while the individual diagrams are not necessarily transverse. Also note that p is the external momentum in each diagram.

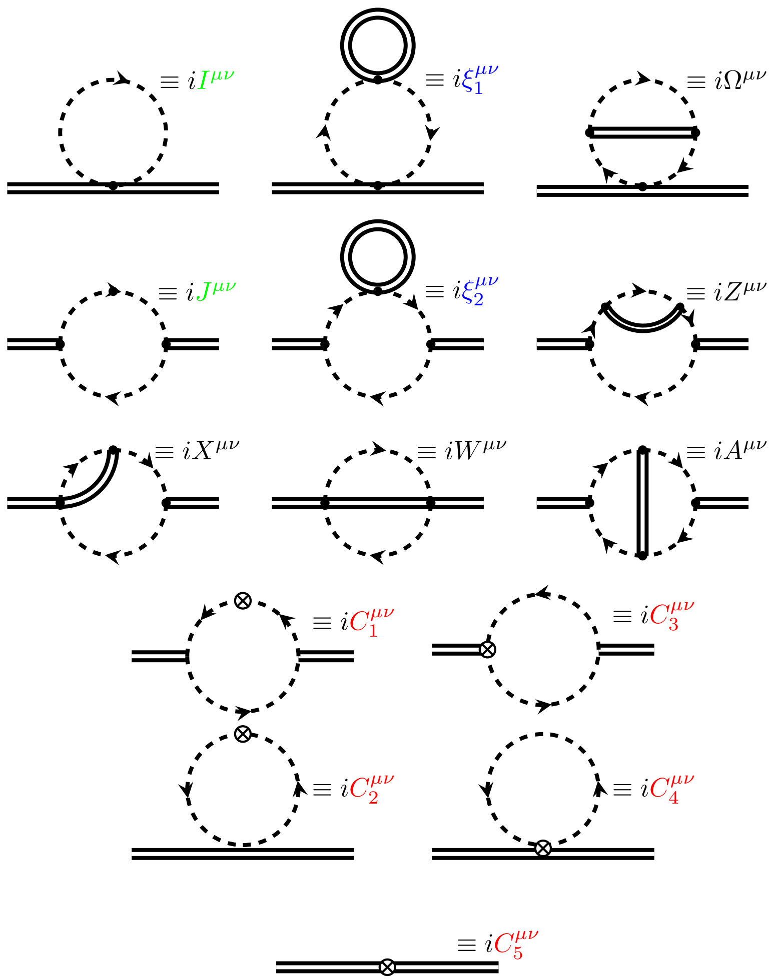


Figure 2.1: All diagrams contributing to the self energy to two-loop order.

To prepare for calculations, we adopt the following notation

$$\alpha = \frac{g^2}{16\pi^2}\,, \tag{2.0.2}$$

$$\mathfrak{D}^n k = \frac{d^n k}{i\pi^2\,(2\pi\mu)^{n-4}}\ . \tag{2.0.3}$$

This means we can make the following replacement in our Feynman diagram calculations

$$g^2\mu^{2\epsilon}\frac{d^n k}{(2\pi)^n} = i\alpha\mathfrak{D}^n k\,. \tag{2.0.4}$$

Applying the Feynman rules to the one-loop diagrams in figure 2.1 yields:

$$
\begin{aligned}
iI^{\mu\nu} &= -2g^2\mu^{2\epsilon}g^{\mu\nu}\int \frac{d^nk}{(2\pi)^n}\frac{1}{k^2-m^2} \\
&= 2i\alpha g^{\mu\nu}\int \mathfrak{D}^nk\frac{1}{k^2-m^2} \\
\implies I^{\mu\nu} &= -2\alpha g^{\mu\nu}\int \mathfrak{D}^nk\frac{1}{k^2-m^2} \ , \qquad (2.0.5)
\end{aligned}
$$

$$
\begin{aligned}
iJ^{\mu\nu} &= g^2\mu^{2\epsilon}\int \frac{d^nk}{(2\pi)^n}\frac{(2k+p)^\mu(2k+p)^\nu}{(k^2-m^2)\left((k+p)^2-m^2\right)} \\
&= i\alpha\int \mathfrak{D}^nk\frac{(2k+p)^\mu(2k+p)^\nu}{(k^2-m^2)\left((k+p)^2-m^2\right)} \\
\implies J^{\mu\nu} &= \alpha\int \mathfrak{D}^nk\frac{(2k+p)^\mu(2k+p)^\nu}{(k^2-m^2)\left((k+p)^2-m^2\right)} \ . \qquad (2.0.6)
\end{aligned}
$$

Proceeding similarly with the remaining diagrams we get:

$$\xi_1^{\mu\nu} = 4n\alpha^2g^{\mu\nu}\int \mathfrak{D}^nk\mathfrak{D}^nq\frac{1}{(k^2-m^2)^2\left(q^2-M^2\right)} \ , \qquad (2.0.7)$$

$$\xi_2^{\mu\nu} = 2n\alpha^2\int \mathfrak{D}^nk\mathfrak{D}^nq\frac{(2k+p)^\mu(2k+p)^\nu}{(k^2-m^2)^2\left((k+p)^2-m^2\right)\left(q^2-M^2\right)} \ , \qquad (2.0.8)$$

$$\Omega^{\mu\nu} = -2\alpha^2g^{\mu\nu}\int \mathfrak{D}^nk\mathfrak{D}^nq\frac{(2k+q)^2}{(k^2-m^2)^2\left((q-k)^2-M^2\right)\left(q^2-m^2\right)} \ , \qquad (2.0.9)$$

$$Z^{\mu\nu} = \alpha^2\int \mathfrak{D}^nk\mathfrak{D}^nq\frac{(k+q)^2(2k+p)^\mu(2k+p)^\nu}{(k^2-m^2)^2\left((k+p)^2-m^2\right)\left((q-k)^2-M^2\right)\left(q^2-m^2\right)} \ , \qquad (2.0.10)$$

$$X^{\mu\nu} = -2\alpha^2\int \mathfrak{D}^nk\mathfrak{D}^nq\frac{(k+q)^\mu(2k+p)^\nu}{(k^2-m^2)\left((k+p)^2-m^2\right)\left((q-k)^2-M^2\right)\left(q^2-m^2\right)} \ , \qquad (2.0.11)$$

$$W^{\mu\nu} = 4\alpha^2g^{\mu\nu}\int \mathfrak{D}^nk\mathfrak{D}^nq\frac{1}{\left((k+p)^2-m^2\right)\left((q-k)^2-M^2\right)\left(q^2-m^2\right)} \ , \qquad (2.0.12)$$

$$A^{\mu\nu} = \alpha^2\int \mathfrak{D}^nk\mathfrak{D}^nq\frac{(k+q)\cdot(k+q+p)(2k+p)^\mu(2q+p)^\nu}{(k^2-m^2)\left((k+p)^2-m^2\right)\left((q-k)^2-M^2\right)\left(q^2-m^2\right)\left((q+p)^2-m^2\right)} \ . \qquad (2.0.13)$$

We now consider the counter-term diagram C_1, applying the Feynman rules yields

$$iC_1^{\mu\nu} = -i\alpha\int \mathfrak{D}^nk\frac{(2k+p)^\mu(2k+p)^\nu}{(k^2-m^2)\left((k+p)^2-m^2\right)}\left(k^2\delta Z_2-\delta m^2\right) \ .$$

Recall that $\delta Z_2 = \sum_{j=1}^{\infty} \delta Z_2^{(j)} \alpha^n$, but we are only working to two-loop order(i.e order α^2) so we only require $\delta Z_2^{(1)} \alpha$ as the other factor of α comes from the Feynman diagram. Similarly we will take $\delta m^2 \equiv \delta^{(1)} m^2 \alpha$. We may then express the above equation as

$$
\begin{aligned}
iC_1^{\mu\nu} &= -i\alpha^2 \int \mathfrak{D}^n k \frac{(2k+p)^\mu (2k+p)^\nu}{(k^2-m^2)\left((k+p)^2-m^2\right)} \left(k^2 \delta Z_2^{(1)} - \delta^{(1)} m^2\right) \\
\implies C_1^{\mu\nu} &= -\alpha^2 \int \mathfrak{D}^n k \frac{(2k+p)^\mu (2k+p)^\nu}{(k^2-m^2)\left((k+p)^2-m^2\right)} \left(k^2 \delta Z_2^{(1)} - \delta^{(1)} m^2\right) \ . \qquad (2.0.14)
\end{aligned}
$$

Proceeding similarly for the remaining counter-term diagrams we get:

$$
C_2^{\mu\nu} = 2\alpha^2 g^{\mu\nu} \int \mathfrak{D}^n k \frac{k^2 \delta Z_2^{(1)} - \delta^{(1)} m^2}{\left(k^2-m^2\right)^2} \ , \qquad (2.0.15)
$$

$$
C_3^{\mu\nu} = \alpha^2 \delta Z_1^{(1)} \int \mathfrak{D}^n k \frac{(2k+p)^\mu (2k+p)^\nu}{(k^2-m^2)\left((k+p)^2-m^2\right)} \ , \qquad (2.0.16)
$$

$$
C_4^{\mu\nu} = -2\alpha^2 \delta {Z'_1}^{(1)} g^{\mu\nu} \int \mathfrak{D}^n k \frac{1}{k^2-m^2} \ , \qquad (2.0.17)
$$

$$
\begin{aligned}
C_5^{\mu\nu} &= \alpha \left\{ \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \left(\delta^{(1)} M^2 - p^2 \delta Z_3^{(1)}\right) + \frac{p^\mu p^\nu}{p^2} \left(\delta^{(1)} M^2 - p^2 \delta^{(1)} \left(\frac{1}{\xi}\right)\right) \right\} \\
&\quad + \alpha^2 \left\{ \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) \left(\delta^{(2)} M^2 - p^2 \delta Z_3^{(2)}\right) + \frac{p^\mu p^\nu}{p^2} \left(\delta^{(2)} M^2 - p^2 \delta^{(2)} \left(\frac{1}{\xi}\right)\right) \right\} \ . \qquad (2.0.18)
\end{aligned}
$$

2.1 T-Integrals

A one-particle-irreducible diagram— call it $i\Sigma^{\mu\nu}$— that contributes to the ρ^0 self-energy can be decomposed as follows

$$i\Sigma^{\mu\nu}(p^2) = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) i\Sigma_T(p^2) + \frac{p^\mu p^\nu}{p^2} i\Sigma_L(p^2)$$

into a transverse part Σ_T and a longitudinal part Σ_L, both of which are scalars. It will be our strategy to focus on calculating these scalars which projected out in the standard way

$$\begin{aligned}\Sigma_L(p^2) &= \frac{p^\mu p^\nu}{p^2}\Sigma_{\mu\nu}(p^2) \\ \Sigma_T(p^2) &= \frac{1}{n-1}\left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Sigma_{\mu\nu}(p^2)\,.\end{aligned} \tag{2.1.1}$$

At one-loop level it was shown by Passarino and Veltman[22], that Σ_T and Σ_L can be written as linear combinations the following scalar integrals

$$A_0(m_i^2) = \int \mathfrak{D}^n k \frac{1}{k^2 - m_i^2} \;\;, \tag{2.1.2}$$

$$B_0(p^2; m_i^2, m_j^2) = \int \mathfrak{D}^n k \frac{1}{(k^2 - m_i^2)((k+p)^2 - m_j^2)} \;\;. \tag{2.1.3}$$

It turns out that analogous basis integrals, called T-integrals, can be found in the two-loop case[23]. As the T-integrals will be so useful to us, we will now review them extensively. First note how in all our diagrams, we were able to assign to each propagator, one of the following momenta

$$k_1 = k\,, \qquad k_2 = k + p\,, \qquad k_3 = q - k\,, \qquad k_4 = q\,, \qquad k_5 = q + p\,. \tag{2.1.4}$$

The T-integrals are defined as follows

$$T_{i_1 i_2 \dots i_r}\left(p^2; m_{j_1}^2, m_{j_2}^2, \dots m_{j_r}^2\right) = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{(k_{i_1}^2 - m_{j_1}^2)(k_{i_2}^2 - m_{j_2}^2)\dots(k_{i_r}^2 - m_{j_r}^2)} \;,$$

where each $i_l \in \{1,2,3,4,5\}$ labels an internal momentum and the j_z's are arbitrary indices labelling masses. This means for example:

$$\begin{aligned} T_{234}(p^2; m_a^2, m_b^2, m_c^2) &= \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{(k_2^2 - m_a^2)(k_3^2 - m_b^2)(k_4^2 - m_c^2)} \\ &= \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{((k+p)^2 - m_a^2)\left((q-k)^2 - m_b^2\right)(q^2 - m_c^2)} \;, \\ \text{or} \qquad W^{\mu\nu} &= 4\alpha^2 g^{\mu\nu} \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{((k+p)^2 - m^2)\left((q-k)^2 - M^2\right)(q^2 - m^2)} \\ &= 4\alpha^2 g^{\mu\nu} T_{234}(p^2; m^2, M^2, m^2)\,. \end{aligned}$$

In the intermediate steps of our self-energy calculation, we will encounter integrals of the form

$$Y_{i_1 i_2 \dots i_r}^{l_1 l_2 \dots l_s}\left(p^2; m_{j_1}^2, m_{j_2}^2, \dots m_{j_r}^2\right) = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{k_{l_1}^2 k_{l_2}^2 \cdots k_{l_s}^2}{(k_{i_1}^2 - m_{j_1}^2)(k_{i_2}^2 - m_{j_2}^2)\dots(k_{i_r}^2 - m_{j_r}^2)} \;,$$

for example:

$$Y_{234}^{1}\left(p^2; m_2^2, m_3^2, m_4^2\right) = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{k^2}{((k+p)^2 - m_2^2)\left((q-k)^2 - m_3^2\right)(q^2 - m_4^2)} \;.$$

It turns out that these "Y-integrals" can be reduced to T-integrals [23].

Looking specifically at our ρ^0 self-energy calculation note the appearance of the following factors, exclusively, in the propagators

$$D_1 = k_1^2 - m^2 \qquad D_2 = k_2^2 - m^2 \qquad D_3 = k_3^2 - M^2 \qquad D_4 = k_4^2 - m^2 \qquad D_5 = k_5^2 - m^2$$

where, remember, m and M are the masses of the pion and ρ^0 respectively. This allows us to drop the arguments in the T-integral notation, writing

$$T_{i_1 i_2 \ldots i_r} = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{D_{i_1} D_{i_2} \cdots D_{i_r}} \ . \tag{2.1.5}$$

So, for example, T_{12345} is understood to mean

$$\begin{aligned} T_{12345} &= \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{D_1 D_2 D_3 D_4 D_5} \\ &= \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{(k^2 - m^2)\left((k+p)^2 - m^2\right)\left((q-k)^2 - M^2\right)(q^2 - m^2)\left((q+p)^2 - m^2\right)} \ , \\ \text{and} \qquad T_{1124} &= \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{(k^2 - m^2)^2 \left((k+p)^2 - m^2\right)(q^2 - m^2)} \ . \end{aligned}$$

We similarly extend this notation to Y-integrals eg.

$$Y^1_{2345} = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{k_1^2}{D_2 D_3 D_4 D_5}$$

2.1.1 Properties of T-Integrals

Index shuffle

Keeping with the argument-free notation above, consider the effect of the following variable changes on D_1, D_2, D_3, D_4, D_5

$$k \to -k - p \qquad\qquad q \to -q - p \tag{2.1.6}$$

$$k \rightleftarrows q \ . \tag{2.1.7}$$

From (2.1.6) we get $D_1 \rightleftarrows D_2, D_4 \rightleftarrows D_5$ and $D_3 \to D_3$. From (2.1.7) we get $D_1 \rightleftarrows D_4, D_2 \rightleftarrows D_5$ and $D_3 \to D_3$; and performing (2.1.6) and (2.1.7) sequentially gives $D_1 \rightleftarrows D_5, D_2 \rightleftarrows D_4$ and $D_3 \to D_3$. Since variable changes do not alter the value of an integral, we conclude that all T and Y-Integrals (in the argument-free notation) are invariant under the following index swaps

$$(1 \rightleftarrows 2)(4 \rightleftarrows 5) \ , \tag{2.1.8}$$

$$(1 \rightleftarrows 4)(2 \rightleftarrows 5) \ , \tag{2.1.9}$$

$$(1 \rightleftarrows 5)(2 \rightleftarrows 4) \ . \tag{2.1.10}$$

So we have for example

$$T_{1234} = T_{1235} = T_{1345} = T_{2345} \ . \tag{2.1.11}$$

Integration-by-parts identities and Master Integrals

There are, in principle, infinitely many T-integrals we can construct since repeated indices are allowed. However, some of them are just products of the one-loop basis integrals A_0 and B_0, and those that are bona fide two-loop integrals are all reducible to a handful master integrals, viz. $T_{134}, T_{1134}, T_{234}, T_{1234}, T_{11234}$, and T_{12345} [23]. One of the ways of doing this reduction makes use of the following property of Feynman Integrals in dimensional regularization

$$\int \mathfrak{D}^n k \frac{\partial}{\partial k^\mu} f^\mu(k) = 0 \ ,$$

which gives rise to the so-called integration-by-parts identities [24, 25]. We can use integration-by-parts to reduce the integral

$$A_1(m^2) = \int \mathfrak{D}^n k \frac{1}{(k^2 - m^2)^2}$$

in the following way

$$\int \mathfrak{D}^n k \frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2-m^2)} = 0$$
$$\implies \int \mathfrak{D}^n k \left\{ \frac{n}{D_1} - \frac{2k^2}{D_1^2} \right\} = 0$$
$$\int \mathfrak{D}^n k \left\{ \frac{n}{D_1} - \frac{2D_1 + 2m^2}{D_1^2} \right\} = 0$$
$$\therefore nA_0(m^2) - 2A_0(m^2) - 2m^2 A_1(m^2) = 0$$
$$\implies A_1(m^2) = \frac{n-2}{2m^2} A_0(m^2) \ , \tag{2.1.12}$$

Another commonly occurring integral is

$$B_{10}(p^2; m^2, m^2) = \int \mathfrak{D}^n k \frac{1}{(k^2-m^2)^2 \left((k+p)^2 - m^2\right)} \ ,$$

which we reduce similarly using

$$\int \mathfrak{D}^n k \frac{\partial}{\partial k^\mu} \frac{k^\mu}{(k^2-m^2)\left[(k+p)^2-m^2\right]} = 0$$
$$\implies (4m^2 - p^2) B_{10}(p^2; m^2, m^2) = (n-3) B_0(p^2; m^2, m^2) - A_1(m^2) \ . \tag{2.1.13}$$

The following T-integral reduction formulas are direct consequences of (2.1.12) and (2.1.13)

$$T_{1133} = \frac{1-\epsilon}{M^2} T_{113} \ , \tag{2.1.14}$$
$$T_{1144} = \frac{1-\epsilon}{m^2} T_{114} \ , \tag{2.1.15}$$
$$(4m^2 - p^2) T_{1123} = (n-3) T_{123} - T_{113} \ , \tag{2.1.16}$$
$$(4m^2 - p^2) T_{1124} = (n-3) T_{124} - T_{114} \ , \tag{2.1.17}$$

these will come in handy in the reduction of the ρ^0 self-energy diagrams to T-integrals. The T-integrals in the formulas above are just products of one-loop integrals, we now turn our attention to true two-loop integrals. It is interesting (and will later prove useful) to consider what happens to the master integrals when we set $p^2 = 0$; The vacuum integrals T_{134} and T_{1134} will be unaffected since they don't depend on p^2. For the rest of the master integrals we easily deduce the following

$$T_{234}|_{p^2=0} = T_{134} \ , \qquad T_{1234}|_{p^2=0} = T_{1134} \ ,$$
$$T_{11234}|_{p^2=0} = T_{11134} \ , \qquad T_{12345}|_{p^2=0} = T_{11344} \ . \tag{2.1.18}$$

The last two equations above involve T_{11134} and T_{11344} which may be reduced with the help of the following integration-by-parts identities

$$\int \mathfrak{D}^n k \mathfrak{D}^n q \frac{\partial}{\partial k^\mu} \left(\frac{k^\mu}{(k^2 - m^2)\left((k-q)^2 - M^2\right)(q^2 - m^2)} \right) = 0$$
$$\implies M^2 T_{1334} = (1 - 2\epsilon) T_{134} - 2m^2 T_{1134} \ , \tag{2.1.19}$$

$$\int \mathfrak{D}^n k \mathfrak{D}^n q \frac{\partial}{\partial q^\mu} \left(\frac{k^\mu}{(k^2 - m^2)\left((k-q)^2 - M^2\right)(q^2 - m^2)} \right) = 0$$
$$\implies (2m^2 - M^2) T_{1134} + T_{113} - T_{114} - M^2 T_{1334} = 0 \ , \tag{2.1.20}$$

$$\int \mathfrak{D}^n k \frac{\partial}{\partial k^\mu} \left(\frac{k^\mu}{(k^2 - m^2)^2} \right) = 0$$
$$\implies 2T_{1114} - 2T_{1113} = \frac{\epsilon}{m^2} \left(T_{113} - T_{114} \right) \ , \tag{2.1.21}$$

$$\int \mathfrak{D}^n k \mathfrak{D}^n q \frac{\partial}{\partial k^\mu} \left(\frac{(k+q)^\mu}{(k^2 - m^2)^2 \left((k-q)^2 - M^2\right)(q^2 - m^2)} \right) = 0$$
$$\implies 2(4m^2 - M^2) T_{11134} = -2(\epsilon + 1) T_{1134} - 2T_{1334} + 2T_{1133} + 2T_{1114} - 2T_{1113} \ , \tag{2.1.22}$$

$$\int \mathfrak{D}^n k \mathfrak{D}^n q \frac{\partial}{\partial q^\mu} \left(\frac{(k+q)^\mu}{(k^2 - m^2)^2 \left((k-q)^2 - M^2\right)(q^2 - m^2)} \right) = 0$$
$$\implies (4m^2 - M^2) T_{11344} = -2\epsilon T_{1134} + T_{1144} - 2T_{1133} + 2T_{1334} \ . \tag{2.1.23}$$

We can then write the following reduction formulas for T_{11134} and T_{11344}

$$2(4m^2 - M^2)T_{11134} = -2(\epsilon+1)T_{1134} - 2T_{1334} + \frac{2(1-\epsilon)}{M^2}T_{113} + \frac{\epsilon}{m^2}\left(T_{113} - T_{114}\right) \ , \tag{2.1.24}$$

$$(4m^2 - M^2)T_{11344} = -2\epsilon T_{1134}\frac{1-\epsilon}{m^2}T_{114} - \frac{2(1-\epsilon)}{m^2}T_{113} + T_{1334} \ . \tag{2.1.25}$$

Associated scalar diagrams

Each master integral involves an assignment of propagator momenta with which we can associate a scalar Feynman diagram that gives rise to the same combination of propagators. In figure 2.2 one can see the scalar diagram (also known as a topology) associated with each master integral. This enables us to study the analyticity of each master integral through the physical thresholds of the associated diagrams. This tells us the values of p^2 for which each T-integral may acquire an imaginary part. We see, for example, that $T_{12345}(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2)$ has physical thresholds at the following values of p^2: $(m_1+m_2)^2$, $(m_1+m_3+m_5)^2$, $(m_2+m_3+m_4)^2$ and $(m_4+m_5)^2$. This can be used as a consistency-check for our rather long calculation.

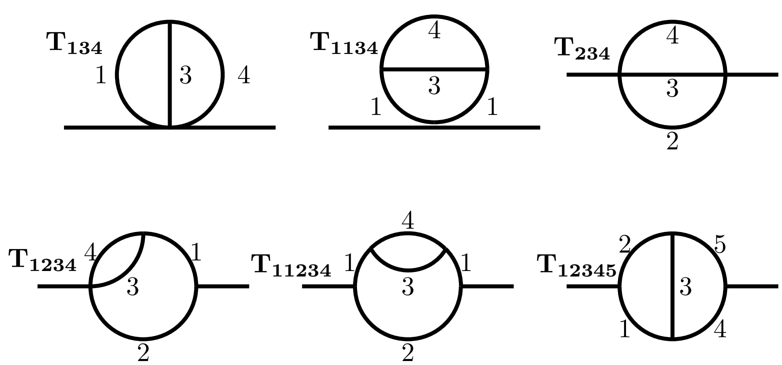


Figure 2.2: Associated scalar diagrams of master integrals.

2.1.2 Evaluating T-inegrals

Since, as we have claimed, the ρ^0 self-energy calculation hinges on the master integrals , it is well worth our time to take a closer look at them. There are six of them and I write them explicitly below

$$T_{134} = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{(k^2 - m^2)((q-k)^2 - M^2)(q^2 - m^2)} \ , \tag{2.1.26}$$

$$T_{1134} = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{(k^2 - m^2)^2((q-k)^2 - M^2)(q^2 - m^2)} \ , \tag{2.1.27}$$

$$T_{234} = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{((k+p)^2 - m^2)((q-k)^2 - M^2)(q^2 - m^2)} \ , \tag{2.1.28}$$

$$T_{1234} = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{(k^2 - m^2)((k+p)^2 - m^2)((q-k)^2 - M^2)(q^2 - m^2)} \ , \tag{2.1.29}$$

$$T_{11234} = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{(k^2 - m^2)^2((k+p)^2 - m^2)((q-k)^2 - M^2)(q^2 - m^2)} \ , \tag{2.1.30}$$

$$T_{12345} = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{(k^2 - m^2)((k+p)^2 - m^2)((q-k)^2 - M^2)(q^2 - m^2)((q+p)^2 - m^2)} \ . \tag{2.1.31}$$

The vacuum integrals

The vacuum integrals T_{134} and T_{1134} can be found in the literature, calculated using a variety of methods [26, 27, 28, 29, 30]. They maybe expressed in the following way

$$\begin{aligned} T_{134} = & \frac{1}{2\epsilon^2}(2m^2 + M^2) + \frac{1}{\epsilon}\left\{\frac{3}{2}(2m^2 + M^2) - 2m^2 L_m - M^2 L_M\right\} \\ & + \left(\frac{7}{2} + \frac{\zeta(2)}{2}\right)(2m^2 + M^2) + 2m^2(L_m^2 - 3L_m) + M^2(L_M^2 - 3L_M) \\ & - \frac{1}{2}M^2 \ln^2\left(\frac{m^2}{M^2}\right) + \frac{1}{2}\lambda\left\{-4\mathrm{Li}_2\left(\frac{1-\lambda}{2}\right) + 2\ln^2\left(\frac{1-\lambda}{2}\right) - \ln^2\left(\frac{m^2}{M^2}\right) + \frac{\pi^2}{3}\right\} \end{aligned} \tag{2.1.32}$$

$$\begin{aligned} T_{1134} = & \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\left(\frac{1}{2} - L_m\right) + \frac{1}{2} + \frac{1}{2}\zeta(2) + L_m^2 - L_m - \frac{1}{2\lambda}\left\{-4\mathrm{Li}_2\left(\frac{1-\lambda}{2}\right)\right. \\ & \left. + 2\ln^2\left(\frac{1-\lambda}{2}\right) - \ln^2\left(\frac{m^2}{M^2}\right) + \frac{\pi^2}{3}\right\} \ , \end{aligned} \tag{2.1.33}$$

where

$$\lambda = \sqrt{1 - \frac{4m^2}{M^2}} \ . \tag{2.1.34}$$

Note that T_{1134} and T_{134} are actually related. The relation can be easily deduced from (2.1.19) and (2.1.20) to be the following

$$(4m^2 - M^2)T_{1134} = (1 - 2\epsilon)T_{134} + T_{114} - T_{113} \ . \tag{2.1.35}$$

Kreimer's double integral representation of T_{12345}

Of all the master integrals, T_{12345} is the only one that remains finite as $n \to 4$. It will be evaluated numerically using the following integral representation, first derived by Kreimer [31]:

$$\begin{aligned} T_{12345}(p^2; m_1^2, m_2^2, m_3^2, m_4^2, m_5^2) &= -\frac{1}{\pi^4} \int d^4k d^4q \frac{1}{(k^2 - m_1^2)\left((k+p)^2 - m_2^2\right)\left((q-k)^2 - m_3^2\right)(q^2 - m_4^2)\left((q+p)^2 - m_5^2\right)} \\ &= -\frac{4}{p^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{w_1^2 - w_2^2} \frac{1}{w_4^2 - w_5^2} \ln\left(\frac{(w_1 + w_3 + w_4)(w_2 + w_3 + w_5)}{(w_1 + w_3 + w_5)(w_2 + w_3 + w_4)}\right) \ , \end{aligned} \tag{2.1.36}$$

where

$$w_1 = \sqrt{x^2 - \frac{m_1^2}{p^2} + i\varepsilon} \ , \tag{2.1.37}$$

$$w_2 = \sqrt{(x+1)^2 - \frac{m_2^2}{p^2} + i\varepsilon} \ , \tag{2.1.38}$$

$$w_3 = \sqrt{(x+y)^2 - \frac{m_3^2}{p^2} + i\varepsilon} \ , \tag{2.1.39}$$

$$w_4 = \sqrt{y^2 - \frac{m_4^2}{p^2} + i\varepsilon} \ , \tag{2.1.40}$$

$$w_5 = \sqrt{(y-1)^2 - \frac{m_5^2}{p^2} + i\varepsilon} \ . \tag{2.1.41}$$

We note that this representation holds for arbitrary masses, and to recover T_{12345} (in $n = 4$-dimensions) we simply set $m_1 = m_2 = m_4 = m_5 = m$ and $m_3 = M$.

Following [30], we map this integral to a unit square in the first quadrant by first mapping the whole x-y plane to the first quadrant:

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy F(x,y) = \int_{0}^{\infty} dx \int_{0}^{\infty} dy \left[F(x,y) + F(-x,y) + F(x,-y) + F(-x,-y)\right] \ , \tag{2.1.42}$$

then using the transformation

$$x = \frac{x'}{1-x'}$$
$$y = \frac{y'}{1-y'} \ .$$

The integration over x' and y' was done using the VEGAS [32] Monte Carlo integration program. We used three iterations with 50000 points to train the grid and three iterations with 500000 points to compute the final answer. The results are listed in Table2.1.

p^2(GeV^2)	ReT_{12345}	ImT_{12345}
0.1521	-13.46012(26)	-23.05974(25)
0.1764	-9.91831(23)	-22.57568(23)
0.1936	-7.95558(21)	-22.04279(21)
0.2209	-5.48979(19)	-21.07034(21)
0.25	-3.48594(18)	-19.98935(18)
0.2704	-2.35966(17)	-19.24347(18)
0.3025	-0.92800(15)	-18.11935(16)
0.3364	0.24604(14)	-17.01236(16)
0.359256	0.88998(14)	-16.31476(14)
0.37271	1.22365(14)	-15.92181(14)
0.39753	1.76577(13)	-15.23153(14)
0.410253	2.01108(12)	-14.89378(14)
0.434967	2.43323(12)	-14.26719(13)
0.44957	2.65364(12)	-13.91484(12)
0.472656	2.96345(11)	-13.38250(12)
0.49	3.16865(11)	-13.00191(12)
0.504768	3.32695(11)	-12.69005(13)
0.53325	3.59531(10)	-12.11783(10)
0.573958	3.906132(96)	-11.36228(12)
0.598751	4.061144(94)	-10.934604(97)
0.601555	4.077032(97)	-10.88758(11)
0.604506	4.093675(93)	-10.83868(10)

Table 2.1: Real and imaginary parts of $T_{12345}(p^2; m^2, m^2, M^2, m^2, m^2)$ (including the statistical error from the Monte Carlo Integration) in units of GeV^{-2}, with $m = .13957$ GeV and $M = 0.7755$ GeV.

The Semi-Numerical Algorithm

The remainder of the p^2-dependent integrals, $T_{234}, T_{1234}, T_{11234}$, will be evaluated using the semi-numerical algorithm described in [30], in conjunction with some analytically calculated integrals from [33]. The goal of the algorithm is to write an integral T as a sum $T_A + T_N$, where T_A is an integral involving massless propagators and so can be expressed analytically in a relatively simple way, and T_N is finite in $n = 4$-dimensions and is therefore amenable to numerical evaluation. Let's consider the case of T_{234} with arbitrary masses

$$T_{234}(p^2; m_2^2, m_3^2, m_4^2) = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{((k+p)^2 - m_2^2)\,((k-q)^2 - m_3^2)\,(q^2 - m_4^2)} \ . \tag{2.1.43}$$

The algorithm entails a substitution using the following simple identities

$$\frac{1}{(k-q)^2 - m_3^2} = \frac{1}{(k-q)^2} + \frac{m_3^2}{(k-q)^2\,((k-q)^2 - m_3^2)} \ , \tag{2.1.44}$$

$$\frac{1}{q^2 - m_4^2} = \frac{1}{q^2} + \frac{m_4^2}{q^2(q^2 - m_4^2)} \ . \tag{2.1.45}$$

On the right-hand side of each of the above equations, note how the first term replaces a massive propagator with a massless one and how the second term— by contributing an additional factor of momentum-squared to the denominator— decreases the degree of divergence of the integral. Equation (2.1.43) then becomes:

$$\begin{aligned}
T_{234}(p^2; m_2^2, m_3^2, m_4^2) &= \int \mathfrak{D}^n k \mathfrak{D}^n q \left\{ \frac{1}{((k+p)^2 - m_2^2)\,(k-q)^2 q^2} + \frac{m_3^2}{((k+p)^2 - m_2^2)\,((k-q)^2 - m_3^2)\,(k-q)^2 q^2} \right. \\
&\quad \left. + \frac{m_4^2}{((k+p)^2 - m_2^2)\,(k-q)^2 (q^2 - m_4^2) q^2} + \frac{m_3^2 m_4^2}{((k+p)^2 - m_2^2)\,((k-q)^2 - m_3^2)\,(k-q)^2 (q^2 - m_4^2) q^2} \right\} \\
&= T_{234}(p^2; m_2^2, 0, 0) + m_3^2 T_{2334}(p^2; m_2^2, m_3^2, 0, 0) + m_4^2 T_{2344}(p^2; , m_2^2, 0, m_4^2, 0) \\
&\quad + m_3^2 m_4^2 T_{23344}(p^2; m_2^2, m_3^2, 0, m_4^2, 0) \ ,
\end{aligned}$$

which simplifies to

$$T_{234}(p^2;m_2^2,m_3^2,m_4^2) = T_{234}(p^2;m_2^2,m_3^2,0) + T_{234}(p^2;m_2^2,0,m_4^2) - T_{234}(m_2^2,0,0) + m_3^2 m_4^2 T_{23344}(p^2;m_2^2,m_3^2,0,m_4^2,0)\ . \tag{2.1.46}$$

The first three terms on the right-hand side of (2.1.46) have analytic expressions that can be found in the literature [30, 33], and the last term is finite as $n \to 4$. So the goal of the algorithm is achieved in the case of T_{234} and we write:

$$T_{234A}(p^2;m_2^2,m_3^2,m_4^2) = T_{234}(p^2;m_2^2,m_3^2,0) + T_{234}(p^2;m_2^2,0,m_4^2) - T_{234}(m_2^2,0,0)\ , \tag{2.1.47}$$

$$T_{234N}(p^2;m_2^2,m_3^2,m_4^2) = m_3^2 m_4^2 T_{23344}(p^2;m_2^2,m_3^2,0,m_4^2,0)\,.$$

The remaining two integrals

$$T_{1234}(p^2;m_1^2,m_2^2,m_3^2,m_4^2) = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{(k^2-m_1^2)\,((k+p)^2-m_2^2)\,((k-q)^2-m_3^2)\,(q^2-m_4^2)}\ , \tag{2.1.48}$$

$$T_{11234}(p^2;m_1^2,m_1^2,m_2^2,m_3^2,m_4^2) = \int \mathfrak{D}^n k \mathfrak{D}^n q \frac{1}{(k^2-m_1^2)^2\,((k+p)^2-m_2^2)\,((k-q)^2-m_3^2)\,(q^2-m_4^2)}\ , \tag{2.1.49}$$

can be treated in the same way using (2.1.44) and (2.1.45), to yield:

$$T_{1234A}(p^2;m_1^2,m_2^2,m_3^2,m_4^2) = T_{1234}(p^2;m_1^2,m_2^2,0,0)\ , \tag{2.1.50}$$

$$T_{1234N}(p^2;m_1^2,m_2^2,m_3^2,m_4^2) = m_3^2 T_{12334}(p^2;m_1^2,m_2^2,m_3^2,0,0) + m_4^2 T_{12344}(p^2;m_1^2,m_2^2,0,m_4^2,0) + m_3^2 m_4^2 T_{123344}(p^2;m_1^2,m_2^2,m_3^2,0,m_4^2,0)\ , \tag{2.1.51}$$

as well as

$$T_{11234A}(p^2;m_1^2,m_1^2,m_2^2,m_3^2,m_4^2) = T_{11234}(p^2;m_1^2,m_1^2,m_2^2,0,0)\ , \tag{2.1.52}$$

$$T_{11234N}(p^2;m_1^2,m_1^2,m_2^2,m_3^2,m_4^2) = m_3^2 T_{112334}(p^2;m_1^2,m_1^2,m_2^2,m_3^2,0,0) + m_4^2 T_{112344}(p^2;m_1^2,m_1^2,m_2^2,0,m_4^2,0) + m_3^2 m_4^2 T_{1123344}(p^2;m_1^2,m_1^2,m_2^2,m_3^2,0,m_4^2,0)\ . \tag{2.1.53}$$

Analytic expressions for (2.1.50) and (2.1.52) are derived in [33] using a combination of: Cutkosky's cutting rules to extract the imaginary part of the integrals; and dispersion relations to recover their respective real parts. Equations (2.1.51) and (2.1.53) only involve finite integrals in the limit $n \to 4$ and so can be treated numerically. Berends and Tausk show us how to evaluate these integrals by adapting Kreimer's method to obtain analogous double-integral representations [30, 31]:

$$T_{234N}(p^2; m_2^2, m_3^2, m_4^2) = -4p^2 \int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} dy \ln\left(\frac{(w_2 + w_3 + w_4)(w_2 + \tilde{w}_3 + \tilde{w}_4)}{(w_2 + \tilde{w}_3 + w_4)(w_2 + w_3 + \tilde{w}_4)}\right) , \tag{2.1.54}$$

$$T_{1234N}(p^2; m_1^2, m_2^2, m_3^2, m_4^2) = 4 \int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} dy \frac{1}{w_1^2 - w_2^2} \ln\left(\frac{(w_1 + w_3 + w_4)(w_2 + \tilde{w}_3 + \tilde{w}_4)}{(w_2 + w_3 + w_4)(w_1 + \tilde{w}_3 + \tilde{w}_4)}\right) , \tag{2.1.55}$$

$$\begin{aligned} T_{11234N}(p^2; m_1^2, m_1^2, m_2^2, m_3^2, m_4^2) = \frac{4}{p^2} \int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} dy \frac{1}{(w_1^2 - w_2^2)^2} \Bigg\{ &\ln\left(\frac{(w_1 + w_3 + w_4)(w_2 + \tilde{w}_3 + \tilde{w}_4)}{(w_2 + w_3 + w_4)(w_1 + \tilde{w}_3 + \tilde{w}_4)}\right) \\ &- \frac{(w_1^2 - w_2^2)(\tilde{w}_3 + \tilde{w}_4 - w_3 - w_4)}{2w_1(w_1 + w_3 + w_4)(w_1 + \tilde{w}_3 + \tilde{w}_4)} \Bigg\} . \end{aligned} \tag{2.1.56}$$

Note the appearance of the new terms

$$\tilde{w}_3 = \sqrt{(x + y)^2 + i\epsilon} , \tag{2.1.57}$$

$$\tilde{w}_4 = \sqrt{y^2 + i\epsilon} . \tag{2.1.58}$$

To evaluate the above integrals numerically on, we again have to map the integration region to a unit square. In the case of (2.1.54) and (2.1.55) we use exactly the same transformation as we used for T_{12345}. In the case of (2.1.56) we map the x-y plane to the first quadrant as in the previous cases, then, following [30], we split the x-integral over two intervals: $[0, x_0)$ and $[x_0, \infty)$, where $x_0 = m_1/\sqrt{p^2}$. The first integral is mapped to a unit square using the transformation $x = x_0(1 - x'^2)/(1 + x'^2)$, $y = y'/(1 - y')$, and the same is done for the second integral using $x = x_0(1 + x'^2)/(1 - x'^2)$, $y = y'/(1 - y')$. The resulting Jacobians are respectively $4x_0x'/((1 + x'^2)(1 - y'))^2$ and $4x_0x'/((1 - x'^2)(1 - v))^2$.

To check my integration code, I computed the integrals (2.1.54),(2.1.55), and (2.1.56) using VEGAS with special values for the masses and compared the results to [30]. Tables 2.2,2.3 and 2.4 below contain the results, which show agreement between our results and [30] to within the statistical error reported by VEGAS.

Table 2.2: The imaginary part of $\frac{1}{p^2}T_{234N}(p^2; m_2^2, m_3^2, m_4^2)$ with $m_1 = 3$,$m_3 = 5$, and $m_4 = 7$. A: Result I calculated independently using VEGAS (including statistical error reported by VEGAS); B: Result from VEGAS reported by [30]; C: Absolute statistical error in B.

p^2 (GeV^2)	A	B	C
30.0	-0.294550(78)	-0.294209	-0.0003824
45.0	-0.496377(69)	-0.496721	-0.0002433
60.0	-0.641151(38)	-0.641142	-0.0001731
75.0	-0.741584(24)	-0.741894	-0.0001261
90.0	-0.791391(16)	-0.791552	-0.0001029
105.0	-0.808574(12)	-0.808654	-0.0000889
120.0	-0.799040(13)	-0.799139	-0.0000695
135.0	-0.7689001(83)	-0.769102	-0.0000592
150.0	-0.726279(11)	-0.726426	-0.0000508
165.0	-0.676523(11)	-0.676688	-0.0000453
180.0	-0.6235225(85)	-0.623591	-0.0000392
195.0	-0.5692108(64)	-0.569248	-0.0000347
210.0	-0.5150881(58)	-0.515157	-0.0000314
225.0	-0.4619109(78)	-0.461933	-0.0000273
240.0	-0.4138200(68)	-0.413841	-0.0000232

Table 2.3: $T_{1234N}(p^2; m_1^2, m_2^2, 0, m_4^2)$ with $m_1 = 1 = m_4, m_2 = 2$. A: Result I calculated independently using VEGAS (including statistical error reported by VEGAS); B: Result from VEGAS reported by [30]; C: Absolute statistical error in B.

	Real Part			Imaginary		
p^2 (GeV^2)	A	B	C	A	B	C
0.01	-0.98748(10)	-0.998834	0.005194			
0.1	-0.993406(72)	-0.993464	0.0004769			
1.0	-1.0559(21)	-1.05622	0.001690			
2.0	-1.146778(75)	-1.14637	0.0002063			
4.0	-1.412163(47)	-1.41243	0.0001836	0.000049(43)	0.00003568	0.0001891
4.1	-1.431323(38)	-1.43119	0.0001861	-0.000088(42)	-0.0000108	0.0002052
4.2	-1.451380(39)	-1.45113	0.0001741	-0.000233(42)	-0.0001076	0.0001937
4.4	-1.495018(43)	-1.49518	0.0001794	-0.001784(45)	-0.001814	0.0001814
4.6	-1.543093(45)	-1.54299	0.0001697	-0.005799(44)	-0.0056238	0.0001800
4.8	-1.595237(46)	-1.59521	0.0001595	-0.012840(47)	-0.0129236	0.0001809
5.0	-1.651404(46)	-1.65176	0.0001619	-0.023861(45)	-0.0238376	0.0001859
7.0	-2.376032(63)	-2.376	0.0001568	-0.455645(57)	-0.455946	0.0001778
9.0	-3.3756(12)	-3.3751	0.006413	-3.44392(37)	-3.44189	0.002581
10.0	-1.051572(72)	-1.05144	0.0001577	-2.864938(56)	-2.8651	0.0001604
100.0	0.1687905(72)	0.168776	0.00001857	-0.2262435(34)	-0.226256	0.00001290
1000.0	0.03432256(93)	0.0343229	3.776×10^{-6}	-0.02825224(40)	-0.0282523	2.232×10^{-6}

Table 2.4: $p^2 T_{11234N}(p^2; m_1^2, m_1^2, m_2^2, 0, m_4^2)$ with $m_1 = 1 = m_4, m_2 = 2$. A: Result I calculated independently using VEGAS (including statistical error reported by VEGAS); B: Result from VEGAS reported by [30]; C: Absolute statistical error in B.

	Real Part			Imaginary		
p^2 (GeV^2)	A	B	C	A	B	C
0.1	0.022143(76)	0.0222252	0.0000244477			
0.9	0.2155(43)	0.22454	0.00123497			
1.0	44130(23652)	5.7×10^6	3.477×10^6			
1.1	0.2846(31)	0.279069	0.00167442			
2.0	0.59348(92)	0.590465	0.000307042			
3.0	1.072426(63)	1.07237	0.000171579			
4.0	1.81875(49)	1.81909	0.000272864			
4.1	1.91855(39)	1.91854	0.00028778	0.00036(21)	0.0002918	0.0004377
4.4	2.26161(82)	2.26285	0.000633597	0.00768(22)	0.0077592	0.000364682
4.8	2.83860(62)	2.83883	0.000539378	0.06288(30)	0.0627915	0.000621636
5.0	3.18242(91)	3.18413	0.000700509	0.12206(47)	0.121994	0.000536774
7.0	9.3748(49)	9.38086	0.00140713	3.9752(20)	3.9742	0.00206658
8.0	15.8856(86)	15.9048	0.00318097	13.7584(36)	13.7599	0.00316477
8.9	31.887(51)	31.8909	0.0220047	102.281(16)	102.299	0.0204597
9.0	-1.403(50)$\times 10^7$	-2.9×10^8	1.537×10^8	1.47135(25)$\times 10^7$	1.8×10^8	1.26×10^8
9.001	-1277.69(53)	-1277.27	0.75359	-27.48(23)	-27.5309	1.10124
9.01	-374.578(90)	-374.544	0.134836	-23.696(55)	-23.7722	0.175914
10.0	-15.524(12)	-15.543	0.00341946	-6.0237(42)	-6.01643	0.00318871
100.0	-1.7409(38)	-1.74756	0.000681549	-0.0020(12)	-0.001904	0.00039984
1000.0	-1.6509(47)	-1.65614	0.000679018	-0.00020(95)	0.0009338	0.00056028

We end this section by presenting analytical parts of the T-integrals for mass-values relevant to our calculation, but first we adapt our index-free notation to the semi-numerical algorithm:

$$T_{234A} = T_{234}(p^2; m^2, M^2, 0) + T_{234}(p^2; m^2, m^2, 0) - T_{234}(m_2^2, 0, 0) \ , \tag{2.1.59}$$

$$T_{234N} = M^2 m^2 T_{23344}(p^2; m^2, M^2, 0, m^2, 0) \, , \tag{2.1.60}$$

$$T_{1234A} = T_{1234}(p^2; m^2, m^2, 0, 0) \ , \tag{2.1.61}$$

$$\begin{aligned} T_{1234N} &= M^2 T_{12334}(p^2; m^2, m^2, M^2, 0, 0) + m^2 T_{12344}(p^2; m^2, m^2, 0, m^2, 0) \\ &\quad + M^2 m^2 T_{123344}(p^2; m^2, m^2, M^2, 0, m^2, 0) \ , \end{aligned} \tag{2.1.62}$$

$$T_{11234A} = T_{11234}(p^2; m^2, m^2, m^2, 0, 0) \ , \tag{2.1.63}$$

$$\begin{aligned} T_{11234N} &= M^2 T_{112334}(p^2; m^2, m^2, m^2, M^2, 0, 0) + m^2 T_{112344}(p^2; m^2, m^2, m^2, 0, m^2, 0) \\ &\quad + M^2 m^2 T_{1123344}(p^2; m^2, m^2, m^2, M^2, 0, m^2, 0) \ . \end{aligned} \tag{2.1.64}$$

Below are the analytical parts of the T-integrals we obtain using results from [30, 33]

$$
\begin{aligned}
T_{234A} =& \frac{1}{2\epsilon^2}(2m^2+M^2)+\frac{1}{\epsilon}\left\{3m^2+\frac{3}{2}M^2-2m^2L_m-M^2L_M-\frac{1}{4}p^2\right\}+2m^2\left(L_m^2-3L_m\right)\\
&+M^2\left(L_M^2-3L_M\right)+\frac{1}{2}L_{|p|}+3(2m^2+M^2)+\frac{1}{2}M^2\zeta(2)-\frac{1}{4}(m^2+M^2)\ln^2\left(\frac{m^2}{M^2}\right)\\
&+\frac{1}{2}(m^2-M^2)\left\{\mathrm{Li}_2\left(\frac{m^2-M^2}{m^2}\right)-\mathrm{Li}_2\left(\frac{M^2-m^2}{M^2}\right)\right\}-\frac{1}{4}p^2\left\{\ln\left|\frac{p^2}{m^2}\right|+\ln\left|\frac{p^2}{M^2}\right|+\frac{13}{2}\right\}\\
&+\frac{1}{4}p^2\left\{\left(\frac{m^2}{p^2}\right)^2-\left(\frac{M^2}{p^2}\right)^2\right\}\ln\left(\frac{m^2}{M^2}\right)+\frac{1}{2}m^2\left(\frac{m^2}{p^2}-\frac{p^2}{m^2}\right)\ln\left(1-\frac{p^2}{m^2}\right)\\
&-\frac{1}{2}(p^2+2m^2)R-\frac{1}{2}(p^2+m^2+M^2)\tilde{R}\\
&-m^2\mathrm{Li}_2\left(\frac{p^2}{m^2}\right)+2m^2\left(1-\frac{m^2}{p^2}\right)\left\{\mathrm{Li}_2\left(1-r_1\right)+\mathrm{Li}_2\left(1-r_2\right)\right\}\\
&+m^2\left(1-\frac{M^2}{p^2}\right)\left\{\mathrm{Li}_2\left(\frac{1-\tilde{r}_1}{-\tilde{r}_1}\right)+\mathrm{Li}_2\left(\frac{1-\tilde{r}_2}{-\tilde{r}_2}\right)-\mathrm{Li}_2\left(\frac{m^2-M^2}{m^2}\right)\right\}\\
&+M^2\left(1-\frac{m^2}{p^2}\right)\left\{\mathrm{Li}_2(1-\tilde{r}_1)+\mathrm{Li}_2(1-\tilde{r}_2)-\mathrm{Li}_2\left(\frac{M^2-m^2}{M^2}\right)\right\}\ ,
\end{aligned}
\tag{2.1.65}
$$

$$
\begin{aligned}
T_{1234A} =& \frac{1}{2\epsilon^2}+\frac{1}{\epsilon}\left\{\frac{5}{2}-L_m+R\right\}+\frac{19}{2}+\frac{3}{2}\zeta(2)+L_m^2-(5+2R)L_m\\
&+\left(\frac{m^2}{p^2}-1\right)\ln\left(1-\frac{p^2}{m^2}\right)+\mathrm{Li}_2\left(\frac{p^2}{m^2}\right)+4R\\
&+\frac{m^2(r_1-r_2)}{2p^2}\left\{\ln^2(1+r_2)-\ln^2(1+r_1)+2\mathrm{Li}_2\left(\frac{1}{1+r_2}\right)-2\mathrm{Li}_2\left(\frac{1}{1+r_1}\right)-\mathrm{Li}_2\left(1-r_1\right)+\mathrm{Li}_2\left(1-r_2\right)\right.\\
&\left.-\mathrm{Li}_2\left(r_2(1-r_2)\right)-\eta\left(1-\frac{p^2}{m^2},r_2\right)\ln\left(r_2(1-r_2)\right)+\mathrm{Li}_2\left(r_1(1-r_1)\right)+\eta\left(1-\frac{p^2}{m^2},r_1\right)\ln\left(r_1(1-r_1)\right)\right\}\ ,
\end{aligned}
\tag{2.1.66}
$$

$$
\begin{aligned}
T_{11234A} =& \left\{\frac{1}{\epsilon}-2L_m+\frac{p^2}{m^2}-2m^2\right\}\frac{R}{4m^2-p^2}-\frac{1}{2p^2}\mathrm{Li}_2\left(\frac{p^2}{m^2}\right)+\frac{1}{m^2}\left(\frac{m^2}{p^2}-1\right)\ln\left(1-\frac{p^2}{m^2}\right)\\
&+\frac{m^2(r_1-r_2)}{2p^2(4m^2-p^2)}\left\{\ln^2(1+r_2)-\ln^2\left(1+r_1\right)+2\mathrm{Li}_2\left(\frac{1}{1+r_2}\right)-2\mathrm{Li}_2\left(\frac{1}{1+r_2}\right)-\mathrm{Li}_2(1-r_1)+\mathrm{Li}_2(1-r_2)\right.\\
&\left.-\mathrm{Li}_2\left(r_2(1-r_2)\right)-\eta\left(1-\frac{p^2}{m^2},r_2\right)\ln\left(r_2(1-r_2)\right)+\mathrm{Li}_2\left(r_1(1-r_1)\right)+\eta\left(1-\frac{p^2}{m^2},r_1\right)\ln\left(r_1(1-r_1)\right)\right\}\ .
\end{aligned}
\tag{2.1.67}
$$

where

$$L_M = \gamma_{\rm E} + \ln\left(\frac{M^2}{4\pi\mu^2}\right) \ , \tag{2.1.68}$$

$$L_m = \gamma_{\rm E} + \ln\left(\frac{m^2}{4\pi\mu^2}\right) \ , \tag{2.1.69}$$

$$L_{|p|} = \gamma_{\rm E} + \ln\left(\frac{|p^2|}{4\pi\mu^2}\right) \ , \tag{2.1.70}$$

and $\gamma_{\rm E}$ is Euler's constant, defined by:

$$\gamma_{\rm E} = \lim_{s\to\infty}\left(\sum_{j=1}^{s}\frac{1}{j} - \ln s\right) \ .$$

We also have the following definitions

$$x = \frac{p^2}{M^2} \tag{2.1.71}$$

$$y = \frac{m^2}{M^2} \tag{2.1.72}$$

$$R = \frac{m^2(r_1 - r_2)}{p^2}\ln r_1 \ , \tag{2.1.73}$$

$$\tilde{R} = \frac{1}{2}\frac{\tilde{r}_1 - \tilde{r}_2}{x}\left(\ln\tilde{r}_1 - \ln\tilde{r}_2\right) \ , \tag{2.1.74}$$

with

$$r_1 = \frac{1}{2}\left\{2 - \frac{p^2}{m^2} + \sqrt{\left(2 - \frac{p^2}{m^2}\right)^2 - 4}\right\} \ , \tag{2.1.75}$$

$$r_2 = \frac{1}{2}\left\{2 - \frac{p^2}{m^2} - \sqrt{\left(2 - \frac{p^2}{m^2}\right)^2 - 4}\right\} \ , \tag{2.1.76}$$

$$\tilde{r}_1 = \frac{1}{2}\left\{1 + y - x + \sqrt{(x-y-1)^2 - 4y}\right\} \ , \tag{2.1.77}$$

$$\tilde{r}_2 = \frac{1}{2}\left\{1 + y - x - \sqrt{(x-y-1)^2 - 4y}\right\} \ . \tag{2.1.78}$$

Note that all the above T_A's are invariant under the exchange of r_1 and r_2 (similarly $\tilde{r}_1$ and $\tilde{r}_2$). They also involve the dilogarithm, defined by

$$\mathrm{Li}_2(x) = -\int_0^x dz \frac{\ln(1-z)}{z} \ . \tag{2.1.79}$$

It has a branch cut along the real axis for $z > 1$ inherited from the principal branch of the logarithm. Ambiguities about values on the branch cut are resolved by taking p^2 to have a small positive imaginary part. We also define the η-function which helps in the addition of logarithms with complex arguments

$$\begin{aligned}\eta(a,b) &= \ln(ab) - \ln a - \ln b \\ &= 2\pi i \left[\theta(-\mathrm{Im}a)\theta(-\mathrm{Im}b)\theta(\mathrm{Im}(ab)) - \theta(\mathrm{Im}a)\theta(\mathrm{Im}b)\theta(-\mathrm{Im}(ab))\right] \ .\end{aligned} \tag{2.1.80}$$

2.2 Reduction

We will now take the expressions obtained from the Feynman diagrams and write them in terms of the basis integrals discussed in the previous section. We start with the one-loop diagrams $I^{\mu\nu}$ and $J^{\mu\nu}$. One can decompose $I^{\mu\nu}$ into a transverse part and a longitudinal part as follows

$$I^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) I_\mathrm{T} + \frac{p^\mu p^\nu}{p^2} I_\mathrm{L} \ ,$$

with

$$I_\mathrm{T} = -2\alpha A_0(m^2) \ , \tag{2.2.1}$$

$$I_\mathrm{L} = -2\alpha A_0(m^2) \ . \tag{2.2.2}$$

We can decompose $J^{\mu\nu}$ in the same way

$$J^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) J_\mathrm{T} + \frac{p^\mu p^\nu}{p^2} J_\mathrm{L} \ ,$$

and the longitudinal part can be projected out as follows

$$\begin{aligned}
\frac{p^\mu p^\nu}{p^2} J_{\mu\nu} &= \frac{\alpha}{p^2} \int \mathfrak{D}^n k \frac{(2k\cdot p + p^2)(2k\cdot p + p^2)}{(k^2-m^2)\left((k+p)^2 - m^2\right)} \\
&= \frac{\alpha}{p^2} \left\{ \int \mathfrak{D}^n k \frac{2k\cdot p + p^2}{k^2 - m^2} - \int \mathfrak{D}^n k \frac{2k\cdot p + p^2}{(k+p)^2 - m^2} \right\} \\
&= 2\alpha \int \mathfrak{D}^n k \frac{1}{k^2 - m^2} \\
\implies \; J_{\rm L} &= 2\alpha A_0(m^2) \; . \qquad (2.2.3)
\end{aligned}$$

We similarly have

$$\begin{aligned}
g_{\mu\nu} J^{\mu\nu} &= \alpha \int \mathfrak{D}^n k \frac{4k^2 + 4k\cdot p + p^2}{(k^2 - m^2)\left((k+p)^2 - m^2\right)} \\
&= \alpha \left\{ 4A_0(m^2) + (4m^2 - p^2) B_0(p^2; m^2, m^2) \right\}
\end{aligned}$$

which we use to get the transverse part of $J^{\mu\nu}$

$$J_{\rm T} = \frac{\alpha}{n-1} \left[(4m^2 - p^2) B_0(p^2; m^2, m^2) + 2A_0(m^2) \right] \; . \qquad (2.2.4)$$

Proceeding similarly with the rest of the diagrams we get*:

$$\begin{aligned}
\xi_1^{\mu\nu} &= \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \xi_{1\rm T} + \frac{p^\mu p^\nu}{p^2} \xi_{1\rm L} \; , \\
\xi_{1\rm T} &= 4n\alpha^2 T_{113} \; , \qquad (2.2.5) \\
\xi_{1\rm L} &= 4n\alpha^2 T_{113} \; . \qquad (2.2.6)
\end{aligned}$$

$$\begin{aligned}
\xi_2^{\mu\nu} &= \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \xi_{2\rm T} + \frac{p^\mu p^\nu}{p^2} \xi_{2\rm L} \; , \\
\xi_{2\rm T} &= -2\alpha^2 n T_{123} \; , \qquad (2.2.7) \\
\xi_{2\rm L} &= -2\alpha^2 n T_{113} \; . \qquad (2.2.8)
\end{aligned}$$

* Each of the reductions involves straight-foward completing-the-square in the numerator to cancel factors in the denominator. To see this done explicitly refer to [34].

$$\Omega^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Omega_\mathrm{T} + \frac{p^\mu p^\nu}{p^2}\Omega_\mathrm{L} \ ,$$

$$\Omega_\mathrm{T} = \alpha^2 \left\{-4T_{134} - 4T_{113} + 2T_{114} - 2(4m^2 - M^2)T_{1134}\right\} \ , \tag{2.2.9}$$

$$\Omega_\mathrm{L} = \alpha^2 \left\{-4T_{134} - 4T_{113} + 2T_{114} - 2(4m^2 - M^2)T_{1134}\right\} \ . \tag{2.2.10}$$

$$W^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)W_\mathrm{T} + \frac{p^\mu p^\nu}{p^2}W_\mathrm{L} \ ,$$

$$W_\mathrm{T} = 4\alpha^2 T_{234} \ , \tag{2.2.11}$$

$$W_\mathrm{L} = 4\alpha^2 T_{234} \ . \tag{2.2.12}$$

$$X^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)X_\mathrm{T} + \frac{p^\mu p^\nu}{p^2}X_\mathrm{L} \ ,$$

$$X_\mathrm{T} = \frac{\alpha^2}{n-1}\big\{ - 3T_{234} - T_{134} + T_{124} - T_{123} - Y^1_{2345} - (7m^2 - 2M^2 - 2p^2)T_{1234}\big\} \ , \tag{2.2.13}$$

$$X_\mathrm{L} = -2\alpha^2 T_{234} \ . \tag{2.2.14}$$

$$Z^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)Z_\mathrm{T} + \frac{p^\mu p^\nu}{p^2}Z_\mathrm{L} \ ,$$

$$\begin{aligned} Z_\mathrm{T} = \frac{\alpha^2}{n-1}\Bigg\{&-\frac{2m^2 - M^2 - 4p^2}{p^2}T_{234} + \frac{4m^2 - M^2 + 2p^2}{p^2}T_{134} - (n-1)T_{124} \\ &+ 2(n-1)T_{123} + (4m^2 - M^2)T_{1134} + 2(8m^2 - M^2 - p^2)T_{1234} \\ &+(4m^2 - p^2)(4m^2 - M^2)T_{11234} + \frac{2}{p^2}T_{13} - \frac{2}{p^2}Y^1_{234}\Bigg\} \ , \end{aligned} \tag{2.2.15}$$

$$\begin{aligned} Z_\mathrm{L} = \alpha^2 \Bigg\{&-\frac{4m^2 - M^2 - 2p^2}{p^2}T_{134} + 2T_{113} - T_{114} + (4m^2 - M^2)T_{1134} \\ &-\frac{2}{p^2}T_{13} + \frac{2}{p^2}Y^1_{234} + \frac{2m^2 - M^2}{p^2}T_{234}\Bigg\} \ . \end{aligned} \tag{2.2.16}$$

$$A^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)A_\mathrm{T} + \frac{p^\mu p^\nu}{p^2}A_\mathrm{L} \ ,$$

$$\begin{aligned} A_\mathrm{T} = \frac{\alpha^2}{n-1}\Bigg\{&4Y^1_{2345} + \frac{4}{p^2}Y^1_{234} - 8T_{124} - (8m^2 - 2M^2 - 4p^2)T_{1245} + 4T_{123} \\ &+ \frac{2(2m^2 - M^2)}{p^2}T_{234} - \frac{2(4m^2 - M^2 - 2p^2)}{p^2}T_{134} + 4(7m^2 - 3M^2 - 3p^2)T_{1234} \\ &-\frac{4}{p^2}T_{13} + (4m^2 - 2M^2 - p^2)(4m^2 - M^2 - 2p^2)T_{12345}\Bigg\} \ , \end{aligned} \tag{2.2.17}$$

$$A_{\rm L} = \alpha^2 \left\{ \frac{4}{p^2} T_{13} - \frac{4}{p^2} Y^1_{234} + \frac{2(4m^2 - M^2)}{p^2} T_{134} - \frac{2(2m^2 - M^2 - 2p^2)}{p^2} T_{234} \right\} \ . \tag{2.2.18}$$

$$C_1^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) C_{1\rm T} + \frac{p^\mu p^\nu}{p^2} C_{1\rm L} \ ,$$

$$C_{1\rm T} = -\frac{\alpha^2}{n-1} \left\{ (n-1) C_\pi^{(1)} B_0(p^2; m^2, m^2) + 2\delta Z_2^{(1)} A_0(m^2) + (4m^2 - p^2) \delta Z_2^{(1)} B_0(p^2; m^2, m^2) \right\} \ , \tag{2.2.19}$$

$$C_{1\rm L} = -\alpha^2 \left\{ C_\pi^{(1)} \frac{n-2}{2m^2} A_0(m^2) + 2\delta Z_2^{(1)} A_0(m^2) \right\} \ , \tag{2.2.20}$$

where

$$C_\pi^{(1)} = m^2 \delta Z_2^{(1)} - \delta^{(1)} m^2 \ . \tag{2.2.21}$$

$$C_2^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) C_{2\rm T} + \frac{p^\mu p^\nu}{p^2} C_{2\rm L} \ ,$$

$$C_{2\rm T} = 2\alpha^2 \left\{ 2\delta Z_2^{(1)} A_0(m^2) + 2C_\pi^{(1)} \frac{n-2}{2m^2} A_0(m^2) \right\} \ , \tag{2.2.22}$$

$$C_{2\rm L} = 2\alpha^2 \left\{ 2\delta Z_2^{(1)} A_0(m^2) + 2C_\pi^{(1)} \frac{n-2}{2m^2} A_0(m^2) \right\} \ . \tag{2.2.23}$$

$$C_3{}^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) C_{3\rm T} + \frac{p^\mu p^\nu}{p^2} C_{3\rm L} \ ,$$

$$C_{3\rm T} = \frac{\alpha^2}{n-1} \delta Z_1^{(1)} \left\{ (4m^2 - p^2) B_0(p^2; m^2, m^2) + 2A_0(m^2) \right\} \ , \tag{2.2.24}$$

$$C_{3\rm L} = 2\alpha^2 \delta Z_1^{(1)} A_0(m^2) \ . \tag{2.2.25}$$

$$C_4{}^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) C_{4\rm T} + \frac{p^\mu p^\nu}{p^2} C_{4\rm L} \ ,$$

$$C_{4\rm T} = -2\alpha^2 \delta {Z'_1}^{(1)} A_0(m^2) \ , \tag{2.2.26}$$

$$C_{4\rm L} = -2\alpha^2 \delta {Z'_1}^{(1)} A_0(m^2) \ . \tag{2.2.27}$$

$$C_5{}^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) C_{5\mathrm{T}} + \frac{p^\mu p^\nu}{p^2} C_{5\mathrm{L}} \ ,$$

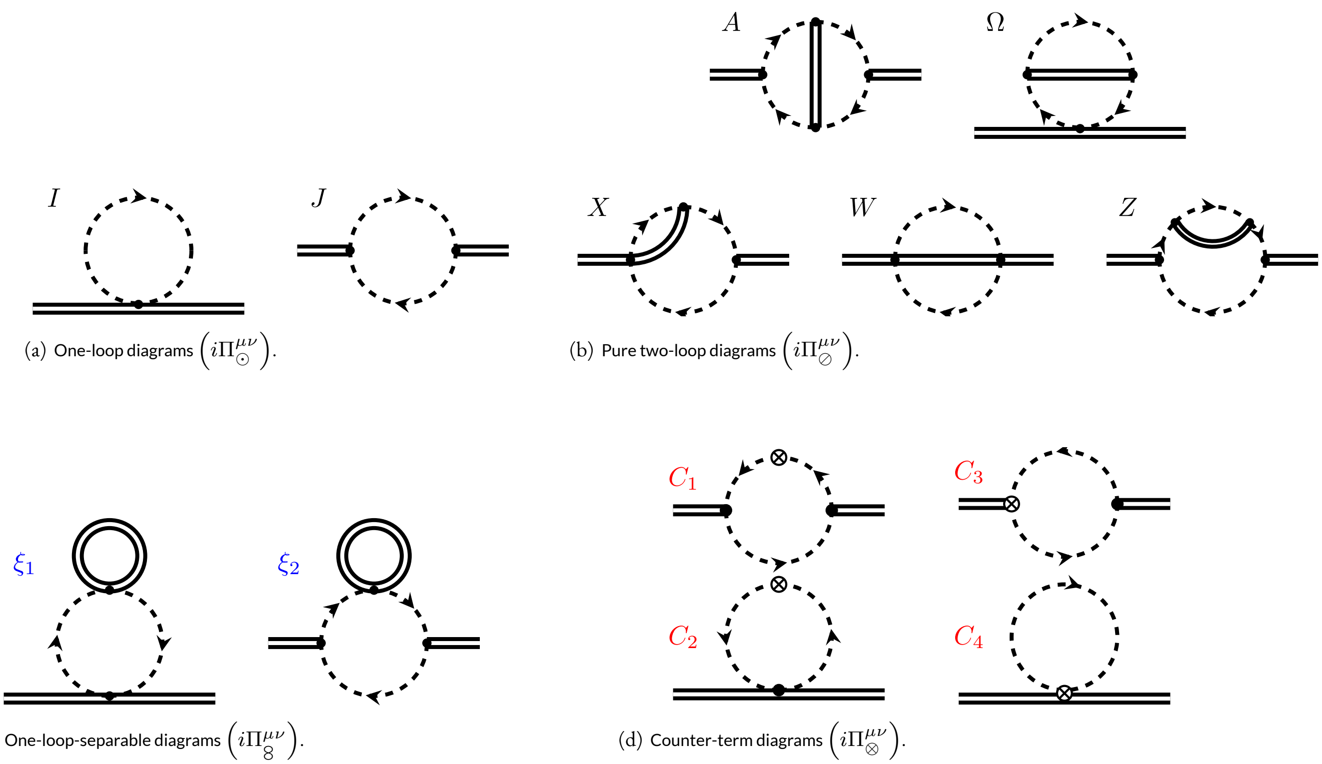
$$C_{5\mathrm{T}} = \alpha(\delta^{(1)}M^2 - p^2\delta Z_3^{(1)}) + \alpha^2(\delta^{(2)}M^2 - p^2\delta Z_3^{(2)}) \ , \tag{2.2.28}$$

$$C_{5\mathrm{L}} = \alpha\left(\delta^{(1)}M^2 - p^2\delta^{(1)}\left(\frac{1}{\xi}\right)\right) + \alpha^2\left(\delta^{(2)}M^2 - p^2\delta^{(2)}\left(\frac{1}{\xi}\right)\right) \ . \tag{2.2.29}$$

Transverse groupings

We will now show that when summing each of the groups of diagrams shown in figure 2.3, the longitudinal parts cancel out i.e. each class of diagrams is transverse. First consider the one-loop diagrams of figure 2.3a, for which we have:

$$\begin{aligned}\Pi_\odot^{\mu\nu} &= I^{\mu\nu} + J^{\mu\nu} \\ &= \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)(I_\mathrm{T} + J_\mathrm{T}) + \frac{p^\mu p^\nu}{p^2}(I_\mathrm{L} + J_\mathrm{L}) \ .\end{aligned}$$



(a) One-loop diagrams $\left(i\Pi_\odot^{\mu\nu}\right)$.

(b) Pure two-loop diagrams $\left(i\Pi_\oslash^{\mu\nu}\right)$.

(c) One-loop-separable diagrams $\left(i\Pi_8^{\mu\nu}\right)$.

(d) Counter-term diagrams $\left(i\Pi_\otimes^{\mu\nu}\right)$.

Figure 2.3: Transverse groups of diagrams.

From (2.2.2) and (2.2.3) we see that the longitudinal part of this sum is given by

$$\begin{aligned} I_{\mathrm{L}} + J_{\mathrm{L}} &= -2\alpha A_0(m^2) + 2\alpha A_0(m^2) \\ &= 0 \ . \end{aligned} \tag{2.2.30}$$

This means $\Pi_{\odot}^{\mu\nu}$ is transverse:

$$\Pi_{\odot}^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Pi_{\odot} \ , \tag{2.2.31}$$

$$\text{with} \qquad \Pi_{\odot} = \frac{\alpha}{n-1}\left\{(4m^2 - p^2)B_0(p^2; m^2 m^2) - 2(n-2)A_0(m^2)\right\} \ . \tag{2.2.32}$$

We call the diagrams of figure 2.3c "one-loop separable diagrams" because their associated integrals may be expressed as products of one-loop integrals. In this case we have:

$$\begin{aligned} \Pi_{8}^{\mu\nu} &= \frac{1}{2}\xi_1{}^{\mu\nu} + 2 \times \left(\frac{1}{2}\right)\xi_2{}^{\mu\nu} \\ &= \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\left(\frac{1}{2}\xi_{1\mathrm{T}} + \xi_{2\mathrm{T}}\right) + \frac{p^\mu p^\nu}{p^2}\left(\frac{1}{2}\xi_{1\mathrm{L}} + \xi_{2\mathrm{L}}\right) \ . \end{aligned}$$

Using (2.2.6) and (2.2.8), we see that the longitudinal part vanishes since:

$$\begin{aligned} \frac{1}{2}\xi_{1\mathrm{L}} + \xi_{2\mathrm{L}} &= \frac{1}{2}\left(4n\alpha^2 T_{113}\right) + \left(-2n\alpha^2 T_{113}\right) \ , \\ &= 0 \iff \Pi_{8}^{\mu\nu} \text{ is transverse} \ , \end{aligned} \tag{2.2.33}$$

$$\therefore \qquad \Pi_{8}^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\Pi_{8} \ ,$$

$$\text{with} \qquad \Pi_{8} = 2\alpha^2 n(T_{113} - T_{123}) \ . \tag{2.2.34}$$

In contrast with the one-loop-separable diagrams, figure 2.3b shows pure two-loop diagrams given by the following sum:

$$\begin{aligned} \Pi_{\oslash}^{\mu\nu} &= \Omega^{\mu\nu} + W^{\mu\nu} + 4X^{\mu\nu} + 2Z^{\mu\nu} + A^{\mu\nu} \\ &= \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)(\Omega_{\mathrm{T}} + W_{\mathrm{T}} + 4X_{\mathrm{T}} + 2Z_{\mathrm{T}} + A_{\mathrm{T}}) + \frac{p^\mu p^\nu}{p^2}(\Omega_{\mathrm{L}} + W_{\mathrm{L}} + 4X_{\mathrm{L}} + 2Z_{\mathrm{L}} + A_{\mathrm{L}}) \ . \end{aligned}$$

Substituting (2.2.10),(2.2.12),(2.2.14),(2.2.16), and (2.2.18) in the longitudinal part above gives:

$$
\begin{aligned}
\Omega_{\mathrm{L}} + W_{\mathrm{L}} + 4X_{\mathrm{L}} + 2Z_{\mathrm{L}} + A_{\mathrm{L}} = \alpha^2 \{ & -4T_{134} - 4T_{113} + 2T_{114} - 2(4m^2 - M^2)T_{1134} + 4T_{234} \\
& - 8T_{234} - \frac{2(4m^2 - M^2 - 2p^2)}{p^2} T_{134} + 4T_{113} - 2T_{114} \\
& + 2(4m^2 - M^2)T_{1134} - \frac{4}{p^2}T_{13} + \frac{4}{p^2}Y^1_{234} + \frac{2(2m^2 - M^2)}{p^2}T_{234} \\
& + \frac{4}{p^2}T_{13} - \frac{4}{p^2}Y^1_{234} + \frac{2(4m^2 - M^2)}{p^2}T_{134} - \frac{2(2m^2 - M^2 - 2p)}{p^2}T_{234} \Big\} \\
& = 0 \iff \Pi^{\mu\nu}_{\oslash} \text{ is transverse .}
\end{aligned}
$$

Using (2.2.9),(2.2.11),(2.2.13),(2.2.15), and (2.2.17) we can find the transverse part to be:

$$
\begin{aligned}
\Pi_{\oslash} &= \Omega_{\mathrm{T}} + W_{\mathrm{T}} + 4X_{\mathrm{T}} + 2Z_{\mathrm{T}} + A_{\mathrm{T}} \\
&= \frac{\alpha^2}{n-1} \{ 2(n-1)(T_{114} - T_{124}) - 4(n-1)(T_{113} - T_{123}) - 4(n-2)T_{134} + 4(n-2)T_{234} \\
&- 2(n-2)(4m^2 - M)T_{1134} + 4(8m^2 - 2M^2 - 2p^2)T_{1234} - (8m^2 - 2M^2 - 4p^2)T_{1245} \\
&+ 2(4m^2 - p^2)(4m^2 - M^2)T_{11234} + (4m^2 - 2M^2 - p^2)(4m^2 - M^2 - 2p^2)T_{12345} \} \; .
\end{aligned}
\tag{2.2.35}
$$

We now turn to the counter-term diagrams of figure 2.3d. Here we have

$$
\begin{aligned}
\Pi^{\mu\nu}_{\otimes} &= 2C_1^{\mu\nu} + C_2^{\mu\nu} + 2C_3^{\mu\nu} + C_4^{\mu\nu} \\
&= \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) (2C_{1\mathrm{T}} + C_{2\mathrm{T}} + 2C_{3\mathrm{T}} + C_{4\mathrm{T}}) + \frac{p^\mu p^\nu}{p^2} (2C_{1\mathrm{L}} + C_{2\mathrm{L}} + 2C_{3\mathrm{L}} + C_{4\mathrm{L}}) \; .
\end{aligned}
$$

Substituting (2.2.20),(2.2.23),(2.2.25), and (2.2.27) into the longitudinal part we get the following:

$$
2C_{1\mathrm{L}} + C_{2\mathrm{L}} + 2C_{3\mathrm{L}} + C_{4\mathrm{L}} = -2\alpha^2 \left\{ \delta Z_1'^{(1)} - 2\delta Z_1^{(1)} + \delta Z_2^{(1)} \right\} A_0(m^2) \tag{2.2.36}
$$

which clearly vanishes if we recall that the renormalization constants that appear above are related by

$$
Z_1'^{(1)} = 2\delta Z_1^{(1)} - \delta Z_2^{(1)} \; .
$$

So the counter term diagrams are transverse

$$\Pi_{\otimes}^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \Pi_{\otimes} \ ,$$

and using (2.2.19),(2.2.22),(2.2.24), and (2.2.26) we get:

$$\begin{aligned}
\Pi_{\otimes} &= 2C_{1\mathrm{T}} + C_{2\mathrm{T}} + 2C_{3\mathrm{T}} + C_{4\mathrm{T}} \\
&= \frac{\alpha^2}{n-1} \left\{ 2(n-1)C_{\pi}^{(1)} \left(\frac{n-2}{2m^2} A_0(m^2) - B_0(p^2; m^2, m^2) \right) \right. \\
&\left. + 2\left(\delta Z_2^{(1)} - \delta Z_1^{(1)} \right) \left(2(n-2)A_0(m^2) - (4m^2 - p^2) B_0(p^2; m^2, m^2) \right) \right\} \ . \qquad (2.2.37)
\end{aligned}$$

Having shown that the classes of diagrams in figure 2.3 are transverse we may express the full self energy to two-loop order as:

$$\begin{aligned}
\Pi_{\rho}^{\mu\nu} &= \Pi_{\odot}^{\mu\nu} + \Pi_{8}^{\mu\nu} + \Pi_{\oslash}^{\mu\nu} + \Pi_{\otimes}^{\mu\nu} + C_5{}^{\mu\nu} \\
&= \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left(\Pi_{\odot} + \Pi_{8} + \Pi_{\oslash} + \Pi_{\otimes} + C_{5\mathrm{T}} \right) + \frac{p^\mu p^\nu}{p^2} C_{5\mathrm{L}} \ . \qquad (2.2.38)
\end{aligned}$$

We note that the self-energy above should be finite, and we also note that it's longitudinal part(as it appears above) is made entirely of counter terms, the primary purpose of which is the absorption of divergences. Working to two-loop order, this longitudinal part, $C_{5\mathrm{L}}$, is finite if and only if each of δM^2 and $\delta\left(\frac{1}{\xi}\right)$ is finite to order α^2, meaning we do not need these renormalization constants. Setting both to zero i.e. $\delta^{(1)} M^2 = \delta^{(2)} M^2 = \delta^{(1)}\left(\frac{1}{\xi}\right) = \delta^{(2)}\left(\frac{1}{\xi}\right) = 0$, we get $C_{5\mathrm{L}} = 0$, which tells us that the self-energy is transverse up to two-loop order. By this very fact we may gain confidence that our reduction of the self-energy Feynman diagrams is correct. The self-energy may now be written in the following way:

$$\Pi_{\rho}^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \left(\Pi_{\odot} + \Pi_{8} + \Pi_{\oslash} + \Pi_{\otimes} - \alpha p^2 \delta Z_3^{(1)} - \alpha^2 p^2 \delta Z_3^{(2)} \right) \ . \qquad (2.2.39)$$

Almost all the diagrams have been expressed in terms of basis integrals, we just have express the counter terms that appear in (2.2.37) in terms of known quantities. To do this we need to calculate some one-loop diagrams.

2.3 One-Loop Diagrams

As already seen, the ρ^0 self-energy at two-loop order involves renormalization constants that must be determined from one-loop results. We need the constants $\delta Z_1^{(1)}$,$\delta Z_2^{(1)}$, $C_\pi^{(1)}$ and we obtain them by evaluating the pion self-energy and the $\rho\pi\pi$-vertex-function.

Pion Self-Energy

The pion self energy (shown in figure 2.4) is given by:

$$-i\Pi_{\pi(1)} = \frac{1}{2} iP_1 - iP_2 - iP_3$$

where

$$\begin{aligned} -iP_1 &= 2in\alpha \int \mathfrak{D}^n k \frac{1}{(k^2 - M^2)} \\ -iP_2 &= -i\alpha \int \mathfrak{D}^n k \frac{(k+2p)^2}{(k^2 - m^2)[(k+p)^2 - M^2]} \ , \\ -iP_3 &= i\alpha \left(p^2 \delta Z_2^{(1)} - \delta^{(1)} m^2 \right) + O(\alpha^2) \,. \end{aligned}$$

These reduce to

$$P_1 = -2n\alpha A_0(M^2) \ , \tag{2.3.1}$$

$$P_2 = \alpha \left\{ 2A_0(M^2) - A_0(m^2) + (2m^2 - M^2 + 2p^2) B_0(p^2; M^2, m^2) \right\} \ , \tag{2.3.2}$$

$$P_3 = -\alpha \left(p^2 \delta Z_2^{(1)} - \delta^{(1)} m^2 \right) \ , \tag{2.3.3}$$

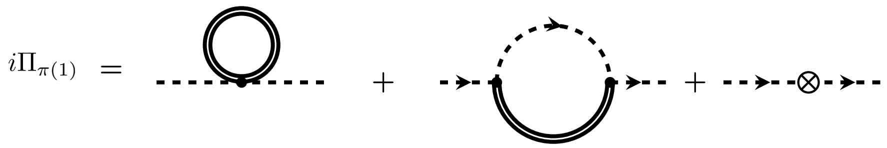


Figure 2.4: Pion self-energy to 1-loop order

from which the following result is obtained

$$\Pi_{\pi(1)} = \alpha\left[-(n-2)A_0(M^2) - A_0(m^2) + (2m^2 - M^2 - 2p^2)B_0(p^2; m^2, M^2)\right] - \alpha\left(p^2\delta Z_2^{(1)} - \delta^{(1)}m^2\right)$$

Imposing the on-shell renormalization conditions

$$\begin{aligned} \Pi_\pi\big|_{p^2=m^2} &= 0 \\ \frac{d}{dp^2}\,\Pi_\pi\big|_{p^2=m^2} &= 0\ , \end{aligned}$$

means that m, which was arbitrary up to this point, now represents the experimentally measured value of the pion mass, we get:

$$C_\pi^{(1)} = m^2\delta Z_2 - \delta m^2 \tag{2.3.4}$$

$$= (2-n)A_0(M^2) - A_0(m^2) + (4m^2 - M^2)B_0(m^2; M^2, m^2) \tag{2.3.5}$$

and

$$\delta Z_2^{(1)} = 2B_0(m^2; M^2, m^2) + (4m^2 - M^2)B_0'(m^2; M^2, m^2)\ , \tag{2.3.6}$$

where

$$B_0'(m^2; M^2, m^2) = \frac{\partial}{\partial p^2}B_0(p^2; M^2, m^2)\bigg|_{p^2=m^2}\ . \tag{2.3.7}$$

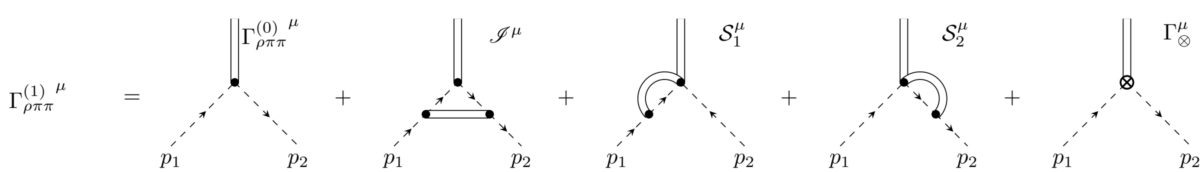


Figure 2.5: The one-loop vertex function

$\rho\pi\pi$-Vertex-Function

The $\rho\pi\pi$-vertex-function to one-loop order is the sum of the one-particle-irreducible diagrams that are seen in figure 2.5. Here the external pions are placed on-shell i.e. $p_1^2 = m^2$ and $p_2^2 = m^2$ leaving the diagrams dependant on the single momentum scale $q^2 = (p_1 - p_2)^2$. We write

$$\Gamma^{\mu}_{\rho\pi\pi} = \Gamma^{(0)\ \mu}_{\rho\pi\pi} + \mathscr{I}^{\mu} + \mathcal{S}^{\mu}_1 + \mathcal{S}^{\mu}_2 + \Gamma^{\mu}_{\otimes} \ , \tag{2.3.8}$$

where, according to the Feynman rules, we have:

$$\Gamma^{(0)\ \mu}_{\rho\pi\pi} = ig\mu^{\epsilon}(p_1 + p_2)^{\mu} \ , \tag{2.3.9}$$

$$\Gamma^{\mu}_{\otimes} = ig\mu^{\epsilon}(p_1 + p_2)^{\mu}\alpha\delta Z_1^{(1)} \ , \tag{2.3.10}$$

$$\mathscr{I}^{\mu} = ig\mu^{\epsilon}\alpha \int \mathfrak{D}^n k \frac{(k+2p_1)\cdot(k+2p_2)(2k+p_1+p_2)^{\mu}}{(k^2-M^2)\left((k+p_1)^2-m^2\right)\left((k+p_2)^2-m^2\right)} \ , \tag{2.3.11}$$

$$\mathcal{S}^{\mu}_1 = -2ig\mu^{\epsilon}\alpha \int \mathfrak{D}^n k \frac{(k+2p_1)^{\mu}}{(k^2-M^2)\left((k+p_1)^2-m^2\right)} \ , \tag{2.3.12}$$

$$\mathcal{S}^{\mu}_2 = -2ig\mu^{\epsilon}\alpha \int \mathfrak{D}^n k \frac{(k+2p_2)^{\mu}}{(k^2-M^2)\left((k+p_2)^2-m^2\right)} \ . \tag{2.3.13}$$

The integrals in (2.3.11),(2.3.12),(2.3.13) must now be reduced to known scalar integrals. Let's begin by evaluating $\mathscr{I}^{\mu}$, as a first step we prove the following result:

Claim 1. $(p_1 - p_2)_{\mu}\mathscr{I}^{\mu} = 0$

Proof.

$$(p_1 - p_2)_{\mu}\mathscr{I}^{\mu} = ig\mu^{\epsilon}\alpha \int \mathfrak{D}^n k \frac{(k+2p_1)\cdot(k+2p_2)[(2k+p_1+p_2)\cdot(p_1-p_2)]}{(k^2-M^2)\left((k+p_1)^2-m^2\right)\left((k+p_2)^2-m^2\right)}$$

We introduce the following notation

$$\Delta_0 = k^2 - M^2$$

$$\Delta_1 = (k + p_1)^2 - m^2$$

$$\Delta_2 = (k + p_2)^2 - m^2 \ ,$$

So that the numerator in the above integral becomes

$$(k + 2p_1) \cdot (k + 2p_2)[(2k + p_1 + p_2) \cdot (p_1 - p_2)] = (k^2 + 2k \cdot p_1 + 2k \cdot p_2 + 4p_1 \cdot p_2)(\Delta_1 - \Delta_2)$$

$$\implies (p_1 - p_2)_\mu \mathscr{I}^\mu = ig\mu^\epsilon \alpha \left\{ \int \mathfrak{D}^n k \frac{(k + 2p_1) \cdot (k + 2p_2)}{\Delta_0 \Delta_2} - \int \mathfrak{D}^n k \frac{(k + 2p_1) \cdot (k + 2p_2)}{\Delta_0 \Delta_1} \right\}$$

and the integral reduces to

$$\begin{aligned}
(p_1 - p_2)_\mu \mathscr{I}^\mu &= ig\mu^\epsilon \alpha \left\{ \int \mathfrak{D}^n k \frac{2k \cdot p_1}{(k^2 - M^2)\left((k + p_2)^2 - m^2\right)} - \int \mathfrak{D}^n k \frac{2k \cdot p_2}{(k^2 - M^2)\left((k + p_1)^2 - m^2\right)} \right\} \\
&+ ig\mu^\epsilon \alpha \left\{ (4p_1 \cdot p_2 + m^2 - p_2^2) B_0(p_2^2; M^2, m^2) - (4p_1 \cdot p_2 + m^2 - p_1^2) B_0(p_1^2; M^2, m^2) \right\} \\
&= ig\mu^\epsilon \alpha \frac{p_1 \cdot p_2}{p_2^2} \left\{ A_0(M^2) - A_0(m^2) + (m^2 - M^2 - p_2^2) B_0(p_2^2; M^2, m^2) \right\} \\
&- ig\mu^\epsilon \alpha \frac{p_1 \cdot p_1}{p_2^2} \left\{ A_0(M^2) - A_0(m^2) + (m^2 - M^2 - p_1^2) B_0(p_1^2; M^2, m^2) \right\} \\
&+ ig\mu^\epsilon \alpha \left\{ (4p_1 \cdot p_2 + m^2 - p_2^2) B_0(p_2^2; M^2, m^2) - (4p_1 \cdot p_2 + m^2 - p_1^2) B_0(p_1^2; M^2, m^2) \right\}
\end{aligned}$$

when we put the pions on-shell i.e. setting $p_1^2 = p_2^2 = m^2$, the right hand side clearly vanishes. □

A consequence of this result is that $\mathscr{I}^\mu$ has the following form

$$\mathscr{I}^\mu = (p_1 + p_2)^\mu \mathscr{I} \ ,$$

with

$$\mathscr{I} = \frac{1}{4m^2 - q^2}(p_1 + p_2)_\mu \mathscr{I}^\mu \ ,$$

$$q^2 = (p_1 - p_2)^2 \, .$$

A straight-foward reduction shows that

$$\begin{aligned}
\mathscr{I} &= \frac{ig\mu^\epsilon \alpha}{4m^2 - q^2} \int \mathfrak{D}^n k \frac{(k + 2p_1) \cdot (k + 2p_2)[(2k + p_1 + p_2) \cdot (p_1 + p_2)]}{(k^2 - M^2)\,((k + p_1)^2 - m^2)\,((k + p_2)^2 - m^2)} \\
&= ig\mu^\epsilon \alpha \left\{ \frac{1}{m^2}\big(A_0(M^2) - A_0(m^2)\big) + \left(2 - \frac{M^2}{m^2}\right) B_0(m^2; M^2, m) \right. \\
&\left. + \frac{4m^2 - 2q^2 - M^2}{4m^2 - q^2}\left[2B_0(m^2; M^2, m^2) - 2B_0(q^2; m^2, m^2) + (4m^2 - q^2 - 2M^2)C_0(q^2)\right]\right\} \qquad (2.3.14)
\end{aligned}$$

and so

$$\begin{aligned}
\mathscr{I}^\mu &= ig\mu^\epsilon (p_1 + p_2)^\mu \alpha \left\{ \frac{1}{m^2}\big(A_0(M^2) - A_0(m^2)\big) + \left(2 - \frac{M^2}{m^2}\right) B_0(m^2; M^2, m) \right. \\
&\left. + \frac{4m^2 - 2q^2 - M^2}{4m^2 - q^2}\left[2B_0(m^2; M^2, m^2) - 2B_0(q^2; m^2, m^2) + (4m^2 - q^2 - 2M^2)C_0(q^2)\right]\right\} \qquad (2.3.15)
\end{aligned}$$

where the scalar integral

$$C_0(q^2) = \int \mathfrak{D}^n k \frac{1}{(k^2 - M^2)\,((k + p_1)^2 - m^2)\,((k + p_2)^2 - m^2)} \ , \qquad (2.3.16)$$

remains finite in 4-dimensions; it will be studied later.

We now look at the diagram $\mathcal{S}_1^\mu$, which must be of the form (Lorenz-invariance demands it):

$$\mathcal{S}_1^\mu = p_1^\mu \mathcal{S}_1 \ , \qquad (2.3.17)$$

$$\text{with} \qquad \mathcal{S}_1 = \frac{1}{m^2} p_{1\mu} \mathcal{S}_1^\mu \ . \qquad (2.3.18)$$

Carrying out the contraction with p_1 shown above, we get

$$\begin{aligned}
\mathcal{S}_1 &= -\frac{2ig\mu^\epsilon\alpha}{m^2}\int \mathfrak{D}^n k \frac{k\cdot p_1 + 2p_1^2}{(k^2-M^2)((k+p_1)^2-m^2)} \\
&= -\frac{ig\mu^\epsilon\alpha}{m^2}\int \mathfrak{D}^n k \frac{\Delta_1-\Delta_0+4p_1^2-M^2}{\Delta_0\Delta_1} \\
&= -ig\mu^\epsilon\alpha\left\{\frac{1}{m^2}\big(A_0(M^2)-A_0(m^2)\big)+\frac{4m^2-M^2}{m^2}B_0(p_1^2;M^2,m^2)\right\}
\end{aligned} \tag{2.3.19}$$

$$\mathcal{S}_1^\mu = -ig\mu^\epsilon\alpha p_1^\mu\left\{\frac{1}{m^2}\big(A_0(M^2)-A_0(m^2)\big)+\frac{4m^2-M^2}{m^2}B_0(m^2;M^2,m^2)\right\}\ . \tag{2.3.20}$$

By exchanging p_2 for p_1 we get a similar reduction for the diagram $\mathcal{S}_2^\mu$:

$$\mathcal{S}_2^\mu = -ig\mu^\epsilon\alpha p_2^\mu\left\{\frac{1}{m^2}\big(A_0(M^2)-A_0(m^2)\big)+\frac{4m^2-M^2}{m^2}B_0(m^2;M^2,m^2)\right\}\ . \tag{2.3.21}$$

From (2.3.20) and (2.3.21) it follows that:

$$\mathcal{S}_1^\mu + \mathcal{S}_2^\mu = -ig\mu^\epsilon(p_1+p_2)^\mu\alpha\left\{\frac{1}{m^2}\big(A_0(M^2)-A_0(m^2)\big)+\frac{4m^2-M^2}{m^2}B_0(m^2;M^2,m^2)\right\}\ . \tag{2.3.22}$$

With all the reduction of diagrams complete, we can express the one-loop $\rho\pi\pi$-vertex-function as

$$\begin{aligned}
\Gamma^\mu_{\rho\pi\pi} &= ig\mu^\epsilon(p_1+p_2)^\mu\Gamma_{\rho\pi\pi}(q^2) \\
\Gamma_{\rho\pi\pi}(q^2) &= 1+\alpha\left\{-2B_0(m^2;M,m)+\frac{4m^2-2q^2-M^2}{4m^2-q^2}\Big[2B_0(m^2;M^2,m^2)\right. \\
&\quad \left. -2B_0(q^2;m^2,m^2)+(4m^2-q^2-2M^2)C_0(q^2)\Big]\right\}+\alpha\delta Z_1^{(1)}\ .
\end{aligned} \tag{2.3.23}$$

Imposing the renormalization condition $\Gamma_{\rho\pi\pi}(q^2=0)=1$ we determine the charge renormalization constant to be:

$$\begin{aligned}\delta Z_1^{(1)} &= 2B_0(m^2;M^2,m^2) - \frac{4m^2-M^2}{4m^2}\Big(2B_0(m^2;M^2,m^2) - 2B_0(0;m^2,m^2) + (4m^2-2M^2)C_0(0)\Big)\\ &= 2B_0(m^2;M^2,m^2) - \frac{4m^2-M^2}{2m^2}\left(B_0(m^2;M^2,m^2) - \frac{n-2}{2m^2}A_0(m^2) + (2m^2-M^2)C_0(0)\right) . \end{aligned} \tag{2.3.24}$$

In the last line we made use of the integration-by-parts identity 2.1.12. That's the last of the renormalization constants.

2.4 Further Reduction

In (2.3.6) and (2.3.24), there appear terms that can be reduced further, viz. $C_0(0)$ and $B'(m^2,M^2,m^2)$. We begin by paying special attention to $C_0(0)$. It is easy to show that $C_0(q^2)$ has the following Feynman-parameter representation:

$$C_0(q^2) = -\int_0^1 dx \int_0^{1-x} \Gamma(1+\epsilon)\left(\frac{\Delta_c}{4\pi\mu^2}\right)^{-\epsilon} \tag{2.4.1}$$

where

$$\Delta_c = (1-x)^2m^2 + q^2y(x+y-1) + xM^2 \ . \tag{2.4.2}$$

Setting $q^2=0$, this representation reduces to the following:

$$C_0(0) = -\int_0^1 dx \frac{1-x}{\Delta_c|_{q^2=0}}\Gamma(1+\epsilon)\left(\frac{\Delta_c|_{q^2=0}}{4\pi\mu^2}\right)^{-\epsilon} \ , \tag{2.4.3}$$

$$\Delta_c|_{q^2=0} = (1-x)^2 + xM^2 \ . \tag{2.4.4}$$

Compare this to the Feynman-parameter representation of $B_0(p^2,M^2,m^2)$:

$$B_0(p^2;M^2,m^2) = \int_0^1 dx\Gamma(\epsilon)\left(\frac{\Delta_b}{4\pi\mu^2}\right)^{-\epsilon} \ , \tag{2.4.5}$$

$$\Delta_b = (1-x)(m^2-xp^2) + xM^2 \ . \tag{2.4.6}$$

Taking the derivative with respect to m^2 yields

$$\frac{\partial}{\partial m^2} B_0(p^2; M^2, m^2) = -\int_0^1 dx \frac{1-x}{\Delta_b} \Gamma(1+\epsilon) \left(\frac{\Delta_b}{4\pi\mu^2}\right)^{-\epsilon} \quad , \tag{2.4.7}$$

for $p^2 = m^2$ this coincides exactly with 2.4.3, so we have just shown the following

$$C_0(0) = \left(\frac{\partial}{\partial m^2} B_0(p^2; M^2, m^2)\right)_{p^2 \to m^2} \quad . \tag{2.4.8}$$

The analytical expression for the right hand side above can be found in [33] substituting this into (2.3.24) gives:

$$\begin{aligned} \delta Z_1^{(1)} &= 2B_0(m^2; M^2, m^2) - \frac{4m^2 - M^2}{2m^2} \Bigg(B_0(m^2; M^2, m^2) - \frac{n-2}{2m^2} A_0(m^2) \\ &+ (2m^2 - M^2) \left(\frac{\partial}{\partial m^2} B_0(p^2; M^2, m^2)\right)_{p^2 \to m^2} \Bigg) \ . \end{aligned} \tag{2.4.9}$$

Next we turn to the derivative that appears in (2.3.6), and to do this we note that a derivative with respect to p^2 can be expressed in the following way:

$$\frac{\partial}{\partial p^2} = \frac{1}{2p^2} p^\mu \frac{\partial}{\partial p^\mu} \ , \tag{2.4.10}$$

$$\begin{aligned} \therefore \ \frac{\partial}{\partial p^2} B_0(p^2; M^2, m^2) &= \frac{p^\mu}{2p^2} \int \mathfrak{D}^n k \frac{\partial}{\partial p^\mu} \left(\frac{1}{(k^2 - M^2)\left((k+p)^2 - m^2\right)} \right) \\ &= -\frac{1}{2p^2} \int \mathfrak{D}^n k \frac{2k \cdot p + 2p^2}{(k^2 - M^2)\left((k+p)^2 - m^2\right)^2} \\ &= -\frac{1}{2p^2} \int \mathfrak{D}^n k \frac{\left((k+p)^2 - m^2\right) - (k^2 - M^2) + p^2 + m - M^2}{(k^2 - M^2)\left((k+p)^2 - m^2\right)^2} \ , \end{aligned} \tag{2.4.11}$$

the integral on the right splits into three terms: the first is

$$\int \mathfrak{D}^n k \frac{1}{(k^2 - M^2)\left((k+p)^2 - m^2\right)} = B_0(p^2; M^2, m^2) \ , \tag{2.4.12}$$

the second is

$$\int \mathfrak{D}^n k \frac{1}{\left((k+p)^2 - m^2\right)^2} = \frac{1}{\left(k^2 - m^2\right)^2}$$
$$= \frac{n-2}{2m^2} A_0(m^2) \ , \tag{2.4.13}$$

and lastly

$$\int \mathfrak{D}^n k \frac{p^2 + m - M^2}{\left(k^2 - M^2\right)\left((k+p)^2 - m^2\right)^2} = (p^2 + m - M^2) \frac{\partial}{\partial m^2} B_0(p^2; M^2, m^2) \ . \tag{2.4.14}$$

(2.4.11) therefore becomes:

$$\frac{\partial}{\partial p^2} B_0(p^2; M^2, m^2) = -\frac{1}{2p^2} \left\{ B_0(p^2; M^2, m^2) - \frac{n-2}{2m^2} A_0(m^2) + (p^2 + m^2 - M^2) \frac{\partial}{\partial m^2} B_0(p^2; M^2, m^2) \right\} \ . \tag{2.4.15}$$

Setting $p^2 = m^2$ we arrive at the desired derivative

$$B_0'(m^2; M^2, m^2) = -\frac{1}{2m^2} \left\{ B_0(m^2; M^2, m^2) - \frac{n-2}{2m^2} A_0(m^2) + (2m^2 - M^2) \left(\frac{\partial}{\partial m^2} B_0(p^2; M^2, m^2) \right)_{p^2 \to m^2} \right\} , \tag{2.4.16}$$

substituting this into (2.3.6) we get the following expression for the wave function renormalization constant:

$$\delta Z_2^{(1)} = 2B_0(m^2; M^2, m^2) - \frac{(4m^2 - M^2)}{2m^2} \left\{ B_0(m^2; M^2, m^2) - \frac{n-2}{2m^2} A_0(m^2) \right.$$
$$\left. + (2m^2 - M^2) \left(\frac{\partial}{\partial m^2} B_0(p^2; M^2, m^2) \right)_{p^2 \to m^2} \right\} \tag{2.4.17}$$

and comparing this to (2.4.9) we immediately see that:

$$\delta Z_1^{(1)} = \delta Z_2^{(1)} \ . \tag{2.4.18}$$

This greatly simplifies the counter-term contribution to the self energy (2.2.37), giving

$$\Pi_{\otimes} = 2\alpha^2 C_{\pi}^{(1)} \left(\frac{n-2}{2m^2} A_0(m^2) - B_0(p^2; m^2, m^2) \right) \ , \tag{2.4.19}$$

and now substituting (2.2.21), we get

$$\begin{aligned} \Pi_{\otimes} &= 2\alpha^2 \Big((2-n) A_0(M^2) - A_0(m^2) + (4m^2 - M^2) B_0(m^2; M^2, m^2) \Big) \left(\frac{n-2}{2m^2} A_0(m^2) - B_0(p^2; m^2, m^2) \right) \\ &= \alpha^2 \left\{ 2(2-n)(T_{113} - T_{123}) - 2(T_{114} - T_{124}) + T_{\otimes} \right\} \ , \end{aligned} \tag{2.4.20}$$

where

$$T_{\otimes} = 2(4m^2 - M^2) B_0(m^2; M^2, m^2) \left(\frac{n-2}{2m^2} A_0(m^2) - B_0(p^2; m^2, m^2) \right) \ , \tag{2.4.21}$$

and we have replaced some products of one-loop integrals by their T-integral form. This is the last piece we need in order to express the ρ^0 self-energy in terms of known basis integrals. Recall that the ρ^0 self-energy is given by:

$$\Pi_{\rho}^{\mu\nu} = \left(g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) F_{\text{vac}} \ , \tag{2.4.22}$$

and it follows from (2.2.39) that

$$F_{\text{vac}} = \Pi_{\odot} + \Pi_{8} + \Pi_{\oslash} + \Pi_{\otimes} - \alpha p^2 \delta Z_3^{(1)} - \alpha^2 p^2 \delta Z_3^{(2)} \ , \tag{2.4.23}$$

when working to two-loop order. Substituting (2.2.32),(2.2.34),(2.2.35), and (2.4.20) into (2.4.23) above yields:

$$F_{\text{vac}} = \alpha F_{\text{vac}}^{(1)} + \alpha^2 F_{\text{vac}}^{(2)} \ , \tag{2.4.24}$$

with

$$F^{(1)}_{\text{vac}} = \frac{1}{3-2\epsilon}\left\{(4m^2-p^2)B_0(p^2;m^2,m^2) - 4(1-\epsilon)A_0(m^2)\right\} - p^2\delta Z_3(1)\ , \tag{2.4.25}$$

$$\begin{aligned} F^{(2)}_{\text{vac}} = &\frac{1}{3-2\epsilon}\left\{-8(1-\epsilon)T_{134} + 8(1-\epsilon)T_{234} - 4(1-\epsilon)(4m^2-M^2)T_{1134} + 4(8m^2-2M^2-2p^2)T_{1234}\right. \\ &\left. -(8m^2-2M^2-4p^2)T_{1245} + 2(4m^2-p^2)(4m^2-M^2)T_{11234} + (4m^2-2M^2-p^2)(4m^2-M^2-2p^2)T_{12345}\right\} \\ &+ T_{\otimes} - p^2\delta Z_3^{(1)}\ . \end{aligned} \tag{2.4.26}$$

The self-energy at $p^2 = 0$

The ρ^0 self-energy can be written as:

$$\Pi^{\mu\nu}_{\rho} = g^{\mu\nu}F_{\text{vac}} - p^{\mu}p^{\nu}\left(\frac{F_{\text{vac}}}{p^2}\right)\ , \tag{2.4.27}$$

yet we know it has cannot have a pole at $p^2 = 0$, since that would imply the existence of a massless intermediate state [35]. A look at the one-particle irreducible diagrams that contribute to $\Pi^{\mu\nu}_{\rho}$ in figure 2.3 reveals the absence of such a state. In fact the lowest physical threshold is the one required for the creation of two pions, $p^2 = 4m^2$. The second term on the right-hand side of (2.4.27) should be regular at $p^2 = 0$ and so the following condition must hold true:

$$F_{\text{vac}}|_{p^2\to 0} = 0\ . \tag{2.4.28}$$

For consistency, this should be true of the two-loop result we arrived at in (2.4.24),(2.4.25),(2.4.26). We will now show that this is indeed the case.

Claim 2. $F^{(1)}_{vac}|_{p^2=0} = 0 = F^{(2)}_{vac}|_{p^2=0}$

Proof. From (2.4.25), we see that $F^{(1)}_{\text{vac}}|_{p^2=0}$ is proportional to

$$4m^2\left(B_0(0;m^2,m^2) - \frac{1-\epsilon}{m^2}A_0(m^2)\right) = 4m^2\left(A_1(m^2) - \frac{1-\epsilon}{m^2}A_0(m^2)\right)$$

which vanishes by the identity (2.1.12). In the case of $F^{(2)}_{\text{vac}}|_{p^2=0}$ first note that $T_{\otimes}$ vanishes for $p^2 = 0$ in a way similar to $F^{(1)}_{\text{vac}}|_{p^2=0}$.

This means that $F_{\text{vac}}^{(2)}|_{p^2=0}$ is proportional to

$$\begin{aligned}
&-8(1-\epsilon)T_{134}+8(1-\epsilon)T_{134}-4(1-\epsilon)(4m^2-M^2)T_{1134}+4(8m^2-2M^2)T_{1134}\\
&-(8m^2-2M^2)T_{1144}+8m^2(4m^2-M^2)T_{11134}+(4m^2-2M^2)(4m^2-M^2)T_{11344}\\
=&-4(1-\epsilon)(4m^2-M^2)T_{1134}+4(8m^2-2M^2)T_{1134}-(8m^2-2M^2)T_{1144}\\
&+8m^2(4m^2-M^2)T_{11134}+(4m^2-2M^2)(4m^2-M^2)T_{11344}\ .
\end{aligned}$$

Here we made use of (2.1.18). As soon as we substitute (2.1.15),(2.1.24), and (2.1.25), the expression above vanishes. □

We have finally reduced the self-energy to basis integrals, and consistency checks show that this reduction is correct. All the scalar integrals that contribute to the self-energy above ((2.4.24),(2.4.25),(2.4.26)) are known either analytically or numerically, and they can be found collected in the next section.

2.5 Scalar integral repository

This section is devoted to collecting all the scalar integrals that are required for evaluating ρ^0 self-energy. We will give some analytical expressions where possible, and numerical results otherwise.

Definitions

Let's begin by collecting some commonly occurring terms.

$$L_M = \gamma_{\mathrm{E}} + \ln\left(\frac{M^2}{4\pi\mu^2}\right) \ , \tag{2.5.1}$$

$$L_m = \gamma_{\mathrm{E}} + \ln\left(\frac{m^2}{4\pi\mu^2}\right) \ , \tag{2.5.2}$$

$$L_{|p|} = \gamma_{\mathrm{E}} + \ln\left(\frac{|p^2|}{4\pi\mu^2}\right) \ , \tag{2.5.3}$$

and γ_{E} is Euler's constant, defined by:

$$\gamma_{\mathrm{E}} = \lim_{s\to\infty}\left(\sum_{j=1}^{s}\frac{1}{j} - \ln s\right) \ .$$

Define

$$x = \frac{p^2}{M^2} \tag{2.5.4}$$

$$y = \frac{m^2}{M^2} \tag{2.5.5}$$

$$R = \frac{m^2(r_1 - r_2)}{p^2}\ln r_1 \ , \tag{2.5.6}$$

$$\tilde{R} = \frac{1}{2}\frac{\tilde{r}_1 - \tilde{r}_2}{x}\left(\ln\tilde{r}_1 - \ln\tilde{r}_2\right) \ , \tag{2.5.7}$$

with

$$r_1 = \frac{1}{2}\left\{2 - \frac{p^2}{m^2} + \sqrt{\left(2 - \frac{p^2}{m^2}\right)^2 - 4}\right\} \ , \tag{2.5.8}$$

$$r_2 = \frac{1}{2}\left\{2 - \frac{p^2}{m^2} - \sqrt{\left(2 - \frac{p^2}{m^2}\right)^2 - 4}\right\} \ , \tag{2.5.9}$$

$$\tilde{r}_1 = \frac{1}{2}\{1 + y - x + \sqrt{(x-y-1)^2 - 4y}\} \ , \tag{2.5.10}$$

$$\tilde{r}_2 = \frac{1}{2}\{1 + y - x - \sqrt{(x-y-1)^2 - 4y}\} \ . \tag{2.5.11}$$

In addition, we make the following definitions:

$$w = \frac{1-\tilde{r}_1}{\tilde{r}_2 - \tilde{r}_1} \qquad \tilde{w} = \frac{1-\tilde{r}_2}{\tilde{r}_1 - \tilde{r}_2} \ , \tag{2.5.12}$$

$$z = \frac{\tilde{r}_1(1-\tilde{r}_2)}{\tilde{r}_1 - \tilde{r}_2} \qquad \tilde{z} = \frac{\tilde{r}_2(1-\tilde{r}_1)}{\tilde{r}_2 - \tilde{r}_1} \ , \tag{2.5.13}$$

$$w_0 = \frac{1-r_1}{r_2 - r_1} \qquad \tilde{w}_0 = \frac{1-r_2}{r_1 - r_2} \ , \tag{2.5.14}$$

$$z_0 = \frac{r_1(1-r_2)}{r_1 - r_2} \qquad \tilde{z}_0 = \frac{r_2(1-r_1)}{r_2 - r_1} \ . \tag{2.5.15}$$

We will make use of the dilogarithm, defined by

$$\mathrm{Li}_2(x) = -\int_0^x dz \frac{\ln(1-z)}{z} \ . \tag{2.5.16}$$

It has a branch cut along the real axis for $z > 1$ inherited from the principal branch of the logarithm. Ambiguities about values on the branch cut are resolved by taking p^2 to have a small positive imaginary part. We also define the η-function which helps in the addition of logarithms with complex arguments

$$\begin{aligned}\eta(a,b) &= \ln(ab) - \ln a - \ln b \\ &= 2\pi i\left[\theta(-\mathrm{Im}a)\theta(-\mathrm{Im}b)\theta(\mathrm{Im}(ab)) - \theta(\mathrm{Im}a)\theta(\mathrm{Im}b)\theta(-\mathrm{Im}(ab))\right] \ .\end{aligned} \tag{2.5.17}$$

One-loop Scalars Integrals

$$
\begin{aligned}
A_0(m^2) &= \int \mathfrak{D}^n k \frac{1}{k^2 - m^2} \\
&= \frac{m^2}{\epsilon} + m^2(1 - L_m) + \epsilon m^2 \left\{ \frac{1}{2}\zeta(2) + \frac{1}{2} L_m^2 - L_m + 1 \right\} \ . \qquad (2.5.18)
\end{aligned}
$$

$$
\begin{aligned}
B_0(p^2; m^2, M^2) &= \int \mathfrak{D}^n k \frac{1}{(k^2 - m^2)\left((k+p) - M^2\right)} \\
&= \frac{1}{\epsilon} - \frac{1}{2}(L_m + L_M) + 2 - \frac{y-1}{2x} \ln y + \tilde{R} \\
&+ \frac{\epsilon}{2} \left\{ \zeta(2) + 8 + \frac{1}{4}(L_m + L_M)^2 + \frac{1}{4}\ln^2 y + 4\tilde{R} \right. \\
&+ \left(L_m + L_M\right) \left(-2 + \frac{y-1}{2x} \ln y - \tilde{R} \right) - \frac{2(y-1)}{x} \ln y \\
&\left. + \frac{\tilde{r}_1 - \tilde{r}_2}{x} \left[\ln w \ln z - \ln \tilde{w} \ln \tilde{z} + \mathrm{Li}_2(z) - \mathrm{Li}_2(\tilde{z}) + \mathrm{Li}_2(w) - \mathrm{Li}_2(\tilde{w}) \right] \right\} . \qquad (2.5.19)
\end{aligned}
$$

$$
\begin{aligned}
B_0(p^2; m^2, m^2) &= \frac{1}{\epsilon} + R + 2 - L_m + \frac{\epsilon}{2} \left\{ \zeta(2) + 8 + L_m^2 - 2(R+2) + 4R \right. \\
&\left. + \frac{m^2(r_1 - r_2)}{p^2} \left[\ln w_0 \ln z_0 - \ln \tilde{w}_0 \ln \tilde{z}_0 + \mathrm{Li}_2(z_0) - \mathrm{Li}_2(\tilde{z}_0) + \mathrm{Li}_2(w_0) - \mathrm{Li}_2(\tilde{w}_0) \right] \right\} \ . \qquad (2.5.20)
\end{aligned}
$$

Two-loop Scalars Integrals

We now write the scalar integrals in a way that is most useful for calculation. Let's introduce the following subscript notation for the T-integrals:

$$
T_{i_1 \ldots i_j} = T_{i_1 \ldots i_j D} + T_{i_1 \ldots i_j F}
$$

where $T_{i_1 \ldots i_j D}$ contains:

i. The divergent part of $T_{i_1 \dots i_j}$

ii. Terms dependent on the renormalization scale μ.

$T_{i_1 \dots i_j F}$ contains the remainder of the finite (order ϵ^0) terms of $T_{i_1 \dots i_j}$. For some of the T-integrals, $T_{i_1 \dots i_j F}$ will be, at least partially, calculated numerically using Kreimer's double integral representation.

Fully analitically evaluated integrals

$$T_\otimes = T_{\otimes D} + T_{\otimes F} = 2(4m^2 - M^2)B_0(m^2; m^2, M^2)\left(\frac{1-\epsilon}{m^2}A_0(m^2) - B_0\left(p^2; m^2, m^2\right)\right) , \tag{2.5.21}$$

where

$$T_{\otimes D} = -\frac{2}{\epsilon}(4m^2 - M^2)(R+2) + (4m^2 - M^2)(R+2)(3L_m + L_M) \tag{2.5.22}$$

$$\begin{aligned} T_{\otimes F} = &-\frac{m^2}{p^2}\left(4m^2 - M^2\right)(r_1 - r_2)\left\{\ln w_0 \ln z_0 - \ln \tilde{w}_0 \ln \tilde{z}_0 + \mathrm{Li}_2(z_0) - \mathrm{Li}_2(\tilde{z}_0) + \mathrm{Li}_2(w_0) - \mathrm{Li}_2(\tilde{w}_0)\right\} \\ &+ \left(4m^2 - M^2\right)(R+2)\left\{\left(1 - \frac{M^2}{m^2}\right)\ln\left(\frac{m^2}{M^2}\right) - 8 - \tilde{R}\Big|_{p^2=m^2}\right\} . \end{aligned} \tag{2.5.23}$$

Similarly we have

$$T_{1245D} = \frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(2R + 4 - 2L_m\right) + 2L_m^2 - 4(R+2)L_m , \tag{2.5.24}$$

$$T_{1245F} = R^2 + 8R + 12 + \zeta(2) + \frac{m^2}{p^2}(r_1 - r_2)\left\{\ln w_0 \ln z_0 - \ln \tilde{w}_0 \ln \tilde{z}_0 + \mathrm{Li}_2(z_0) - \mathrm{Li}_2(\tilde{z}_0) + \mathrm{Li}_2(w_0) - \mathrm{Li}_2(\tilde{w}_0)\right\} , \tag{2.5.25}$$

$$T_{134D} = \frac{1}{2\epsilon^2}(2m^2 + M^2) + \frac{1}{\epsilon}\left(3m^2 + \frac{3}{2}M^2 - 2m^2 L_m - M^2 L_M\right) + 2m^2(L_m^2 - 3L_m) + M^2(L_M^2 - 3L_M) , \tag{2.5.26}$$

$$\begin{aligned} T_{134F} = &\left(\frac{7}{2} + \frac{\zeta(2)}{2}\right)(2m^2 + M^2) - \frac{1}{2}M^2 \ln^2\left(\frac{m^2}{M^2}\right) + \frac{1}{2}M^2\lambda\left\{-4\mathrm{Li}_2\left(\frac{1-\lambda}{2}\right) + 2\ln^2\left(\frac{1-\lambda}{2}\right) - \ln^2\left(\frac{m^2}{M^2}\right)\right. \\ &\left.+\frac{\pi^2}{3}\right\} , \end{aligned} \tag{2.5.27}$$

$$T_{1134D} = \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\left(\frac{1}{2} - L_m\right) + L_m^2 - L_m\,, \tag{2.5.28}$$

$$T_{1134F} = \frac{1}{2} + \frac{1}{2}\zeta(2) - \frac{1}{2\lambda}\left\{-4\mathrm{Li}_2\left(\frac{1-\lambda}{2}\right) + 2\ln^2\left(\frac{1-\lambda}{2}\right) - \ln^2\left(\frac{m^2}{M^2}\right) + \frac{\pi^2}{3}\right\}\,, \tag{2.5.29}$$

where $\lambda = \sqrt{1 - \frac{4m^2}{M^2}}$.

Semi-numerical integrals

$$T_{234D} = \frac{1}{2\epsilon^2}(2m^2 + M^2) + \frac{1}{\epsilon}\left(3m^2 + \frac{3}{2}M^2 - 2m^2L_m - M^2L_M - \frac{1}{4}p^2\right) + 2m^2(L_m^2 - 3L_m) + M^2(L_M^2 - 3L_M)$$
$$+ \frac{1}{2}p^2L_{|p|}\,, \tag{2.5.30}$$

$$T_{234F} = 3(2m^2 + M^2) + \frac{1}{2}M^2\zeta(2) - \frac{1}{4}(m^2 + M^2)\ln^2\left(\frac{m^2}{M^2}\right) + \frac{1}{2}(m^2 - M^2)\left\{\mathrm{Li}_2\left(\frac{m^2 - M^2}{m^2}\right) - \mathrm{Li}_2\left(\frac{M^2 - m^2}{M^2}\right)\right\}$$
$$- \frac{1}{4}p^2\left\{\ln\left|\frac{p^2}{m^2}\right| + \ln\left|\frac{p^2}{M^2}\right| + \frac{13}{2}\right\} + \frac{1}{4}p^2\left\{\left(\frac{m^2}{p^2}\right)^2 - \left(\frac{M^2}{p^2}\right)^2\right\}\ln\left(\frac{m^2}{M^2}\right) + \frac{1}{2}m^2\left(\frac{m^2}{p^2} - \frac{p^2}{m^2}\right)\ln\left(1 - \frac{p^2}{m^2}\right)$$
$$- \frac{1}{2}\left(p^2 + 2m^2\right)R - \frac{1}{2}(p^2 + m^2 + M^2)\tilde{R} - m^2\mathrm{Li}_2\left(\frac{p^2}{m^2}\right) + 2m^2\left(1 - \frac{m^2}{p^2}\right)\left\{\mathrm{Li}_2\left(1 - r_1\right) + \mathrm{Li}_2(1 - r_2)\right\}$$
$$+ m^2\left(1 - \frac{M^2}{p^2}\right)\left\{\mathrm{Li}_2\left(\frac{1 - \tilde{r}_1}{-\tilde{r}_1}\right) + \mathrm{Li}_2\left(\frac{1 - \tilde{r}_2}{-\tilde{r}_2}\right) - \mathrm{Li}_2\left(\frac{m^2 - M^2}{m^2}\right)\right\}$$
$$+ M^2\left(1 - \frac{m^2}{p^2}\right)\left\{\mathrm{Li}_2\left(1 - \tilde{r}_1\right) + \mathrm{Li}_2(1 - \tilde{r}_2) - \mathrm{Li}_2\left(\frac{M^2 - m^2}{M^2}\right)\right\}$$
$$- 4p^2\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\ln\left(\frac{(w_2 + w_3 + w_4)(w_2 + \tilde{w}_3 + \tilde{w}_4)}{(w_2 + \tilde{w}_3 + w_4)(w_2 + w_3 + \tilde{w}_4)}\right)\,, \tag{2.5.31}$$

$$T_{1234D} = \frac{1}{2\epsilon^2} + \frac{1}{\epsilon}\left\{\frac{5}{2} - L_m + R\right\} + L_m^2 - (5 + 2R)L_m \tag{2.5.32}$$

$$T_{1234F} = \frac{19}{2} + \frac{3}{2}\zeta(2) + \left(\frac{m^2}{p^2} - 1\right)\ln\left(1 - \frac{p^2}{m^2}\right) + \frac{1}{2}\mathrm{Li}_2\left(\frac{p^2}{m^2}\right) + 4R + \frac{m^2(r_1 - r_2)}{2p^2}\left\{\ln^2(1 + r_2) - \ln^2(1 + r_1)\right.$$
$$+ 2\mathrm{Li}_2\left(\frac{1}{1 + r_2}\right) - 2\mathrm{Li}_2\left(\frac{1}{1 + r_1}\right) + \mathrm{Li}_2(1 - r_2) - \mathrm{Li}_2(1 - r_1)$$
$$\left.-\mathrm{Li}_2(r_2(1 - r_2)) - \eta\left[1 - \frac{p^2}{m^2}, r_2\right]\ln(r_2(1 - r_2)) + \mathrm{Li}_2(r_1(1 - r_1)) + \eta\left[1 - \frac{p^2}{m^2}, r_1\right]\ln(r_1(1 - r_1))\right\}$$
$$+ 4\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\frac{1}{w_1^2 - w_2^2}\ln\left(\frac{(w_1 + w_3 + w_4)(w_2 + \tilde{w}_3 + \tilde{w}_4)}{(w_2 + w_3 + w_4)(w_1 + \tilde{w}_3 + \tilde{w}_4)}\right)\,, \tag{2.5.33}$$

$$T_{11234D} = \frac{1}{\epsilon}\left(\frac{R}{4m^2-p^2}\right) - \left(\frac{2R}{4m^2-p^2}\right) L_m \ , \tag{2.5.34}$$

$$\begin{aligned} T_{11234F} = & \left(\frac{p^2}{m^2}-2\right)\frac{R}{4m^2-p^2} - \frac{1}{2p^2}\mathrm{Li}_2\left(\frac{p^2}{m^2}\right) + \frac{1}{m^2}\left(\frac{m^2}{p^2}-1\right)\ln\left(1-\frac{p^2}{m^2}\right) \\ & + \frac{m^2(r_1-r_2)}{2p^2(4m^2-p^2)}\Bigg\{\ln^2(1+r_2) - \ln^2(1+r_1) + 2\mathrm{Li}_2\left(\frac{1}{1+r_2}\right) - 2\mathrm{Li}_2\left(\frac{1}{1+r_1}\right) - \mathrm{Li}_2(1-r_1) + \mathrm{Li}_2(1-r_2) \\ & - \mathrm{Li}_2(r_2(1-r_2)) - \eta\left[1-\frac{p^2}{m^2}, r_2\right]\ln(r_2(1-r_2)) + \mathrm{Li}_2(r_1(1-r_1)) + \eta\left[1-\frac{p^2}{m^2}, r_1\right]\ln(r_1(1-r_1))\Bigg\} \\ & + \frac{4}{p^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{(w_1^2-w_2^2)^2}\left\{\ln\left(\frac{(w_1+w_3+w_4)(w_2+\tilde{w}_3+\tilde{w}_4)}{(w_2+w_3+w_4)(w_1+\tilde{w}_3+w_4)}\right) - \frac{(w_1^2-w_2^2)(\tilde{w}_3+\tilde{w}_4-w_3-w_4)}{2w_1(w_1+w_3+w_4)(w_1+\tilde{w}_3+\tilde{w}_4)}\right\} . \end{aligned} \tag{2.5.35}$$

Last is T_{12345}, the only finite one among the scalar integrals considered. It is evaluated fully numerically using Kreimer's double integral representation

$$T_{12345} = -\frac{4}{p^2}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{1}{(w_1^2-w_2^2)(w_4^2-w_5^2)} \ln\left(\frac{(w_1+w_3+w_4)(w_2+w_3+w_5)}{(w_2+w_3+w_4)(w_1+w_3+w_5)}\right) . \tag{2.5.36}$$

For each of the T-integrals above, the imaginary part remains zero below the lowest physical threshold. In the case of T_{234} the only threshold is $(2m+M)^2 = 1.11$ GeV^2, accordingly the real part is zero in the range $p^2 \in [0, 1\,\text{GeV}^2]$ as seen in figure 2.6. The lowest threshold is $4m^2 = 0.0779\text{GeV}^2$, for each of T_{1234},T_{11234}, and T_{12345}; this is in agreement with figures 2.7,2.8, and2.9, in each of which the imaginary part is zero below the threshold. We also note the similar behaviour of T_{11234F} and T_{12345} to that given by the independent calculation of [36] albeit with different mass values.

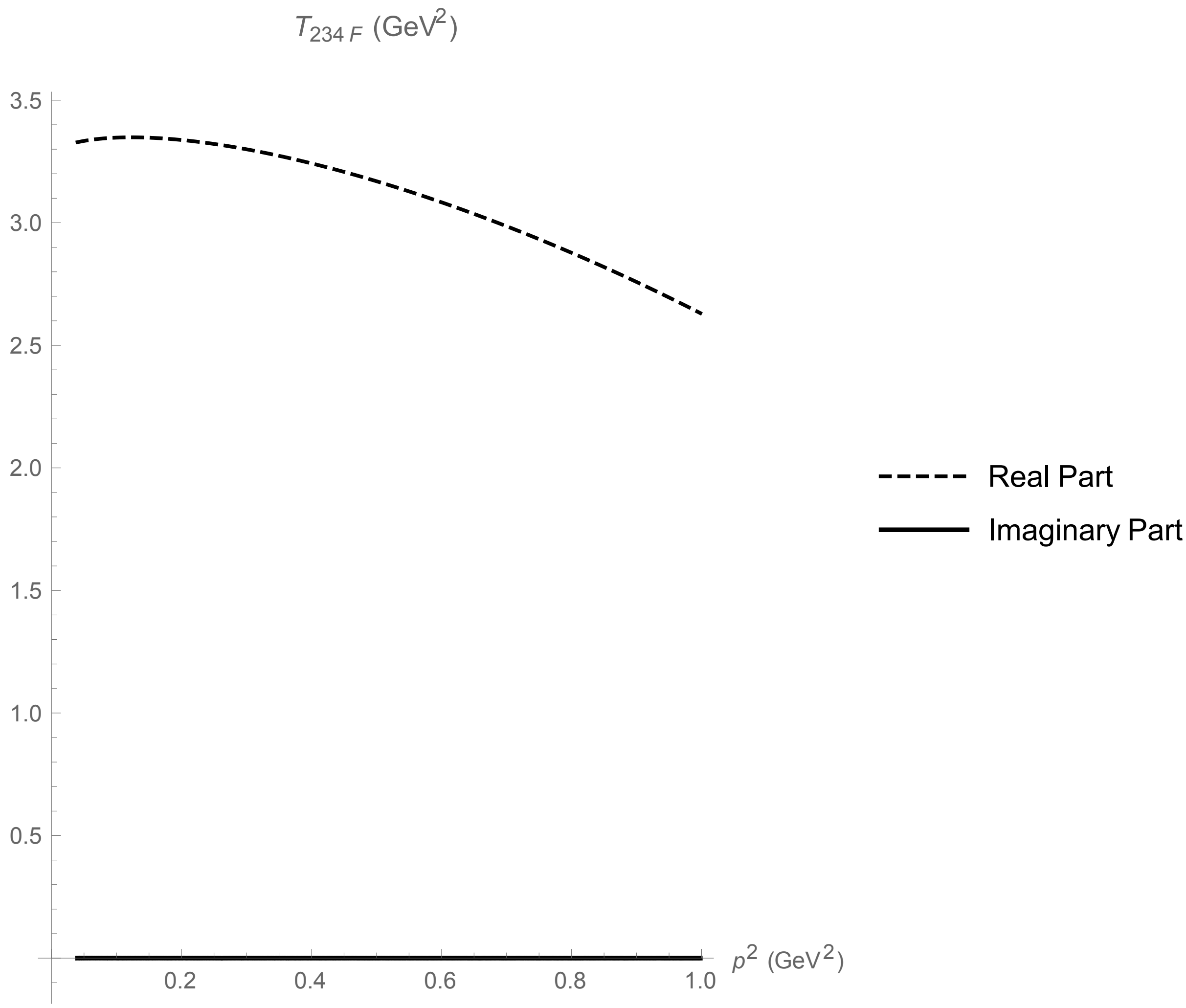


Figure 2.6: Real and imaginary parts of T_{234F}.

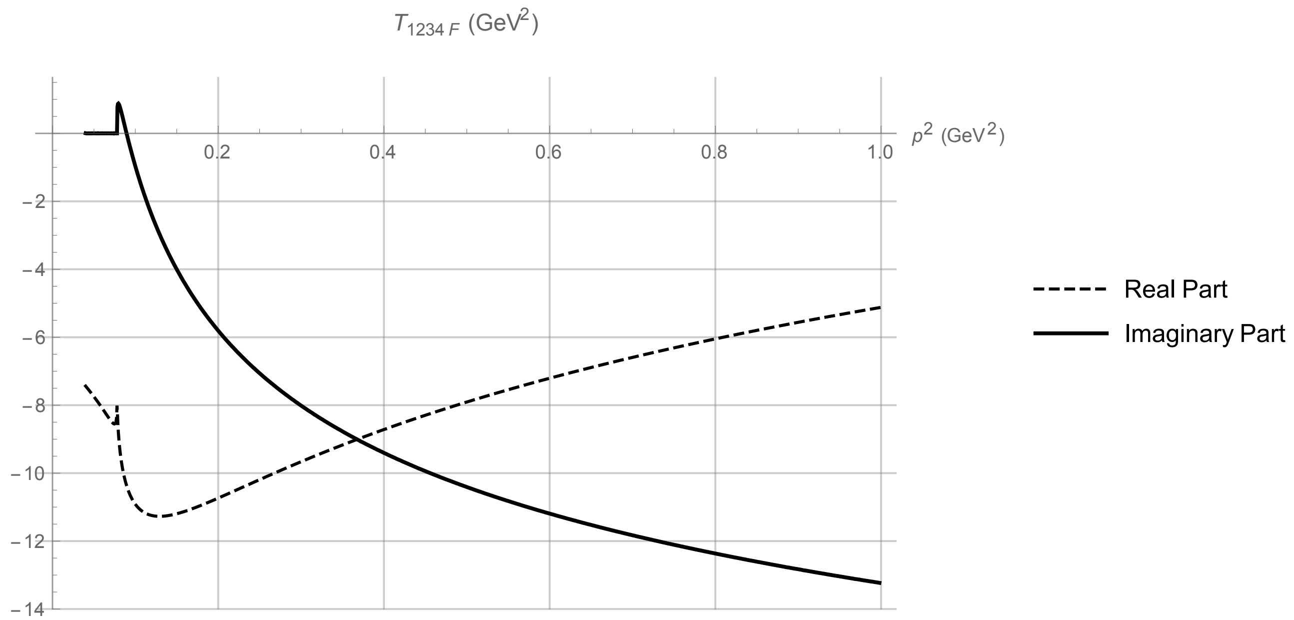


Figure 2.7: Real and imaginary parts of T_{1234F} with $m = 0.13957$ GeV^2 and $M = 0.7755$ GeV^2.

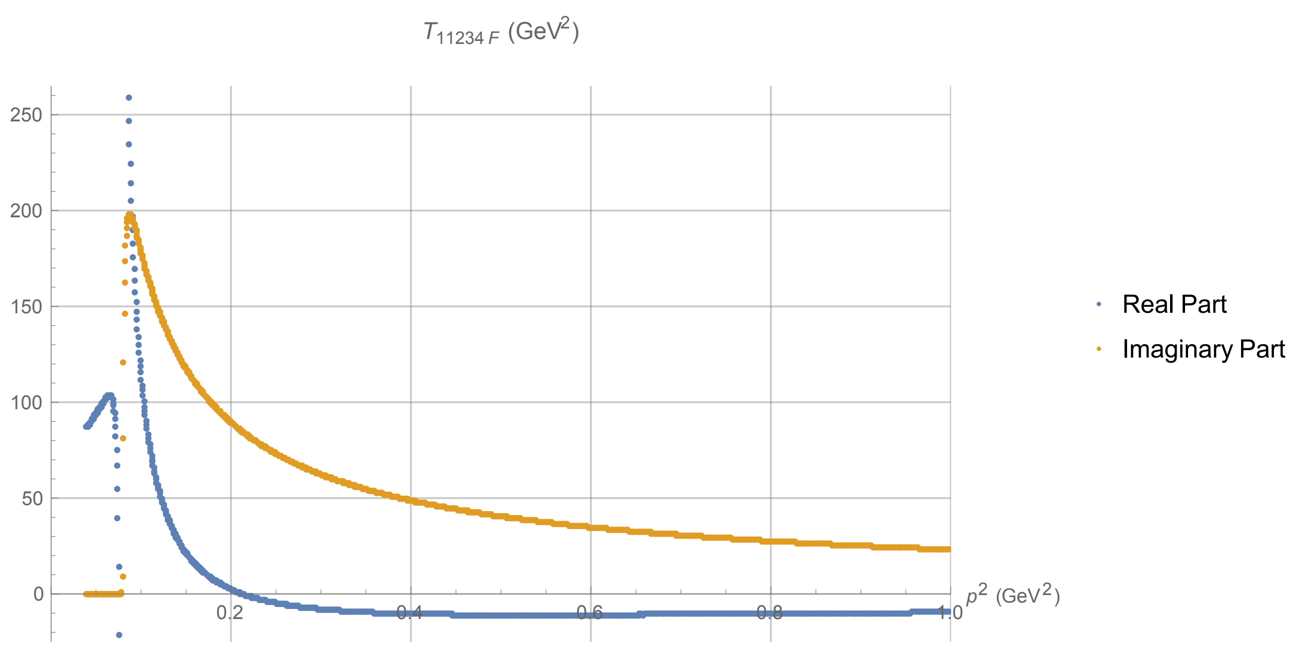


Figure 2.8: Real and imaginary parts of T_{11234F} with $m = 0.13957$ GeV^2 and $M = 0.7755$ GeV^2.

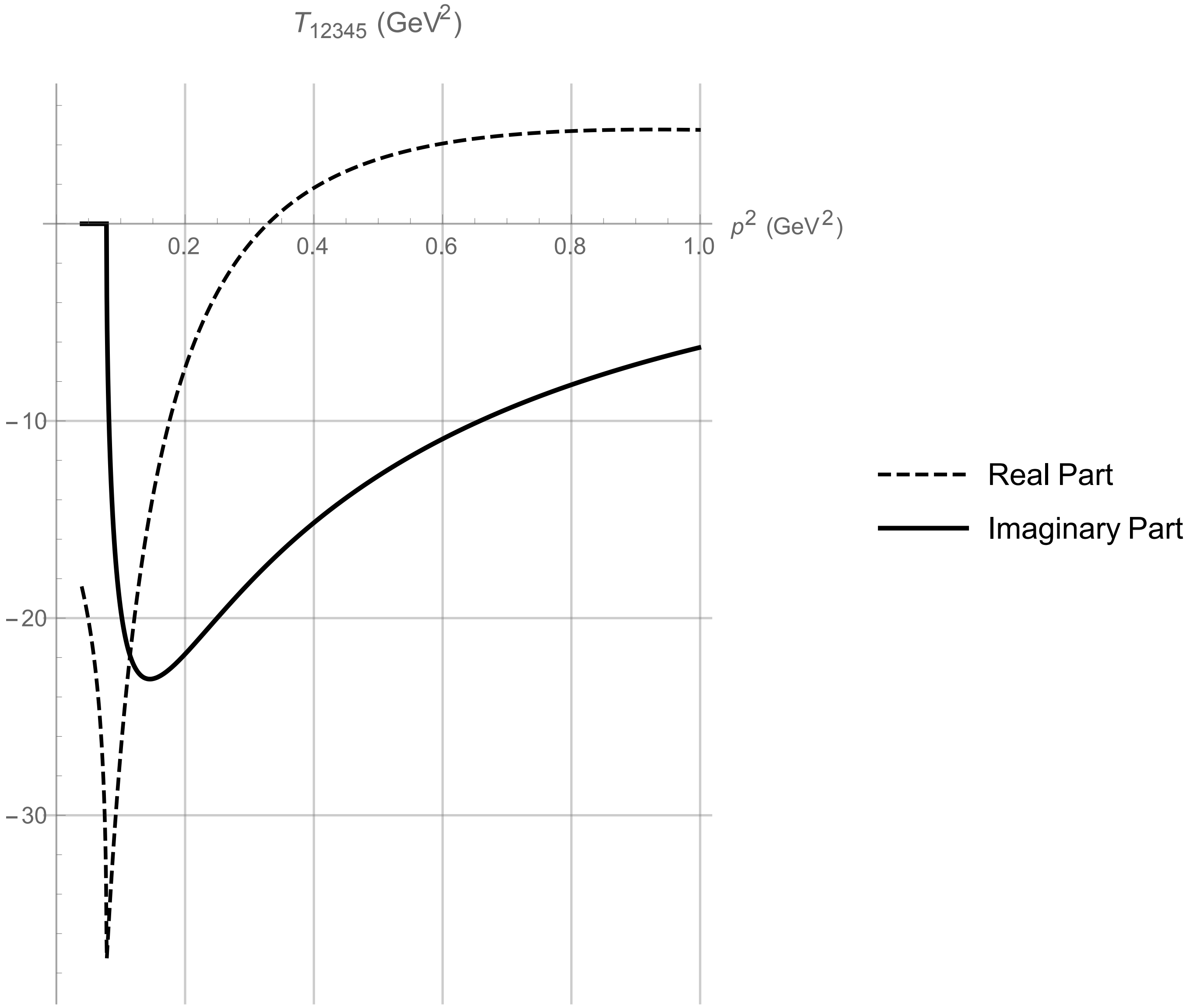


Figure 2.9: Real and imaginary parts of T_{12345} with $m = 0.13957$ GeV^2 and $M = 0.7755$ GeV^2.

3

Evaluating The Self-energy

Now that we have fully reduced the ρ^0 self-energy and collected all the required scalar integrals required for its evaluation, we will go ahead and evaluate it. We will proceed systematically: first considering the case of the one-loop contribution $F_{\text{vac}}^{(1)}$, then turning to the more complicated two-loop contribution $F_{\text{vac}}^{(2)}$.

To have any hope of a meaningful result, we need to ensure that the self-energy calculated is indeed finite, and to this end we impose the on-shell renormalization condition:

$$\text{Re}\, F_{\text{vac}}\big|_{p^2=0} = 0\ , \tag{3.0.1}$$

which according to (2.4.24), translates to the following:

$$\mathrm{Re}\, F^{(1)}_{\text{vac}}\Big|_{p^2=0} = 0 \ , \tag{3.0.2}$$

$$\mathrm{Re}\, F^{(2)}_{\text{vac}}\Big|_{p^2=0} = 0 \ . \tag{3.0.3}$$

3.1 One-loop Contribution

We turn our attention to the one-loop contribution to the self energy which is given by

$$F^{(1)}_{\text{vac}} = \frac{1}{3}\left(1+\frac{2\epsilon}{3}\right)\Big\{(4m^2-p^2)B_0(p^2;m^2,m^2) - 4(1-\epsilon)A_0(m^2)\Big\} - p^2\delta Z_3^{(1)} + O(\epsilon) \ . \tag{3.1.1}$$

Substituting in (2.5.18) and (2.5.20), then expanding to $O(\epsilon^0)$ the above equation becomes:

$$F^{(1)}_{\text{vac}} = \frac{1}{3}\left\{-\frac{p^2}{\epsilon} + p^2 L_m + 8m^2 - \frac{8p^2}{3} + (4m^2-p^2)R\right\} - p^2\delta Z_3^{(1)} + O(\epsilon) \ , \tag{3.1.2}$$

Note that R, the function of p^2 given in (2.5.6), can be expressed in real-imaginary form as follows:

$$\begin{aligned} R = &\left(1-\frac{4m^2}{p^2}\right)^{\frac{1}{2}}\left\{\ln\left|\frac{1-\sqrt{1-\frac{4m^2}{p^2}}}{1+\sqrt{1-\frac{4m^2}{p^2}}}\right| + i\pi\theta\left(p^2-4m^2\right)\right\} \\ &+\left(\frac{4m^2}{p^2}-1\right)^{\frac{1}{2}}\cos^{-1}\left(1-\frac{p^2}{2m^2}\right)\left[\theta(p^2-4m^2)-\theta(p^2)\right] \ , \end{aligned} \tag{3.1.3}$$

and so (3.1.2) becomes:

$$\begin{aligned} F^{(1)}_{\text{vac}} = \frac{1}{3}\Bigg\{&-\frac{p^2}{\epsilon} + p^2 L_m + 8m^2 - \frac{8p^2}{3} + p^2\left(1-\frac{4m^2}{p^2}\right)^{\frac{3}{2}}\left[\ln\left|\frac{1+\sqrt{1-\frac{4m^2}{p^2}}}{1-\sqrt{1-\frac{4m^2}{p^2}}}\right| - i\pi\theta\left(p^2-4m^2\right)\right] \\ &+p^2\left(\frac{4m^2}{p^2}-1\right)^{\frac{3}{2}}\cos^{-1}\left(1-\frac{p^2}{2m^2}\right)\Big[\theta(p^2-4m^2)-\theta(p^2)\Big] - p^2\delta Z_3^{(1)} + O(\epsilon) \ , \end{aligned}$$

the on-shell renormalization condition

$$\text{Re}\; F^{(1)}_{\text{vac}}\Big|_{p^2=M^2} = 0 \ ,$$

gives us the one-loop contribution to the ρ^0 wave-function renormalization constant:

$$\delta Z_3^{(1)} = \frac{1}{3}\left\{-\frac{1}{\epsilon} + L_m + \frac{8m^2}{M^2} - \frac{8}{3} + \left(1-\frac{4m^2}{M^2}\right)^{3/2} \ln\left|\frac{1+\sqrt{1-\frac{4m^2}{M^2}}}{1-\sqrt{1-\frac{4m^2}{M^2}}}\right|\right\} + O(\epsilon) \ , \tag{3.1.4}$$

and the one loop contribution to the self-energy (taking $\epsilon \to 0$):

$$\begin{aligned} F^{(1)}_{\text{vac}} = \frac{p^2}{3}&\left\{\left(1-\frac{4m^2}{p^2}\right)^{\frac{3}{2}}\left[\ln\left|\frac{1+\sqrt{1-\frac{4m^2}{p^2}}}{1-\sqrt{1-\frac{4m^2}{p^2}}}\right| - i\pi\theta\left(p^2-4m^2\right)\right] + \frac{8m^2}{p^2}\right. \\ &\left. + \left(\frac{4m^2}{p^2}-1\right)^{\frac{3}{2}} \cos^{-1}\left(1-\frac{p^2}{2m^2}\right)\left[\theta(p^2-4m^2)-\theta(p^2)\right] + C\right\} \end{aligned} \tag{3.1.5}$$

where

$$C = -\frac{8m^2}{M^2} + \left(1-\frac{4m^2}{M^2}\right)^{3/2} \ln\left|\frac{1-\sqrt{1-\frac{4m^2}{M^2}}}{1+\sqrt{1-\frac{4m^2}{M^2}}}\right| \ . \tag{3.1.6}$$

Recall that F_{vac} is expressed as

$$F_{\text{vac}} = \alpha F^{(1)}_{\text{vac}} + O(\alpha^2) \tag{3.1.7}$$

and so F_{vac} to one-loop order is given by:

$$\begin{aligned} F_{\text{vac}} = \frac{g^2 p^2}{48\pi^2}&\left\{\left(1-\frac{4m^2}{p^2}\right)^{\frac{3}{2}}\left[\ln\left|\frac{1+\sqrt{1-\frac{4m^2}{p^2}}}{1-\sqrt{1-\frac{4m^2}{p^2}}}\right| - i\pi\theta\left(p^2-4m^2\right)\right] + \frac{8m^2}{p^2}\right. \\ &\left. + \left(\frac{4m^2}{p^2}-1\right)^{\frac{3}{2}} \cos^{-1}\left(1-\frac{p^2}{2m^2}\right)\left[\theta(p^2-4m^2)-\theta(p^2)\right] + C\right\} \ . \end{aligned} \tag{3.1.8}$$

Comparing this result to [15], where the following formula for F_{vac} at one-loop is given:

$$F_{\text{G\&K}}(p^2) = \frac{g^2 p^2}{48\pi^2}\left\{\left(1-\frac{4m^2}{p^2}\right)^{3/2}\left[\ln\left|\frac{\sqrt{\left(1-\frac{4m^2}{p^2}\right)}+1}{\sqrt{\left(1-\frac{4m^2}{p^2}\right)}-1}\right| - i\pi\theta(p^2-4m^2)\right] + \frac{8m^2}{p^2} + C\right\}, \qquad (3.1.9)$$

we see a difference in the region $0 < p^2 < 4m^2$, otherwise the formulas are identical. Taking $g = 4\pi$, for the moment, we get the plots shown in figure 3.1 and figure 3.2.

Looking at the graphs, it is clear that our formula (3.1.8) gives regular behaviour as $p^2 \to 0$ and $F_{\text{vac}}|_{p^2=0} = 0$ as expected. The formula in [15], on the other hand, has a jump discontinuity at $p^2 = 0$ and is therefore incorrect in this region.

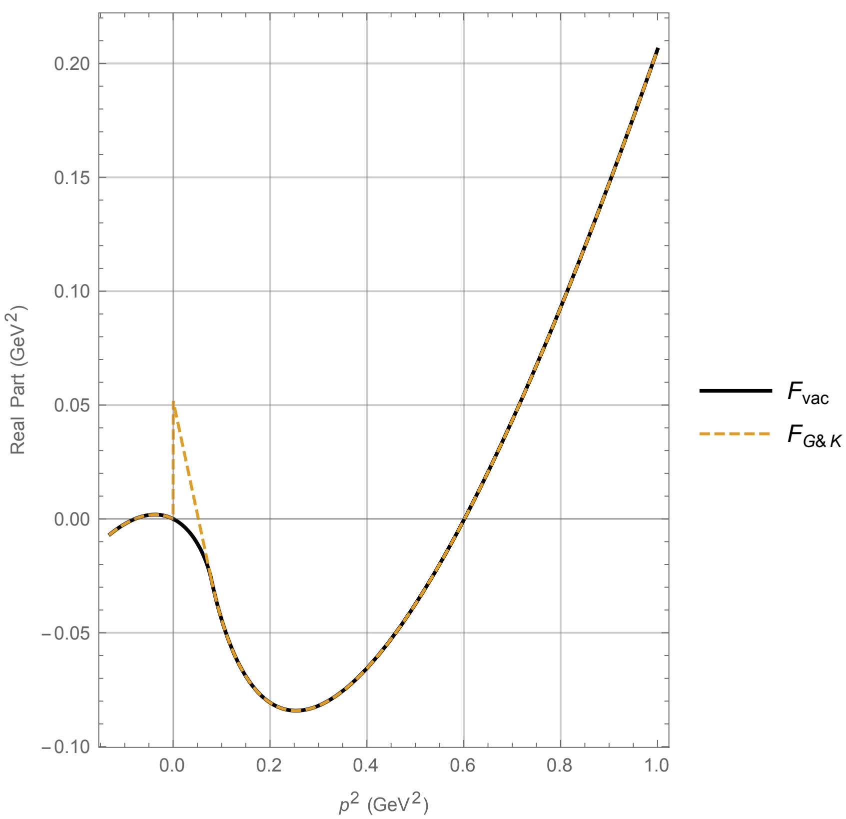


Figure 3.1: Comparison of the real parts of (3.1.8) and (3.1.9).

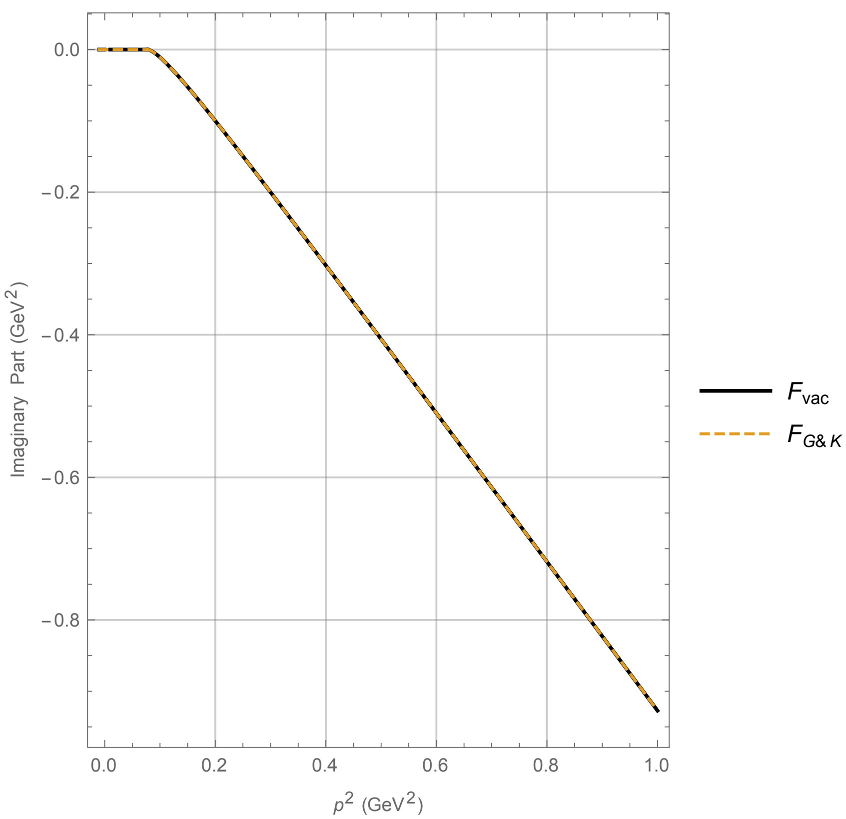


Figure 3.2: Comparison of the imaginary parts of (3.1.8) and (3.1.9).

3.2 Two-loop Contributuion

To analyse the divergences of the two-loop contribution to the self-energy, $F^{(2)}_{\text{vac}}$, we write the self energy in the following way

$$\begin{aligned}
F^{(2)}_{\text{vac}} &= \frac{1}{3}\frac{1}{\left(1-\frac{2}{3}\epsilon\right)}\Big\{ -8(1-\epsilon)(T_{134D}+T_{134F}) + 8(1-\epsilon)(T_{234D}+T_{234F}) \\
&- 4(1-\epsilon)(4m^2-M^2)(T_{1134D}+T_{1134F}) + 4(8m^2-2M^2-2p^2)(T_{1234D}+T_{1234F}) \\
&- (8m^2-2M^2-4p^2)(T_{1245D}+T_{1245F}) + 2(4m^2-p^2)(4m^2-M^2)(T_{11234D}+T_{11234F}) \\
&+ (4m^2-2M^2-p^2)(4m^2-M^2-2p^2)T_{12345}\Big\} + T_{\otimes D} + T_{\otimes F} - p^2\delta Z_3^{(2)} \\
&= F_D^{(2)} + F_F^{(2)} - p^2\delta Z_3^{(2)} \ ,
\end{aligned} \tag{3.2.1}$$

where

$$\begin{aligned}
F_D^{(2)} &= \frac{1}{3}\left(1+\frac{2}{3}\epsilon\right)\Big\{ -8(1-\epsilon)T_{134D} + 8(1-\epsilon)T_{234D} - 4(1-\epsilon)(4m^2-M^2)T_{1134D} \\
&+ 4(8m^2-2M^2-2p^2)T_{1234D} - (8m^2-2M^2-4p^2)T_{1245D} + 2(4m^2-p^2)(4m^2-M^2)T_{11234D}\Big\} + T_{\otimes D} \ ,
\end{aligned} \tag{3.2.2}$$

and

$$\begin{aligned}
F_F^{(2)} &= \frac{1}{3}\Big\{ -8T_{134F} + 8T_{234F} - 4(4m^2-M^2)T_{1134F} + 4(8m^2-2M^2-2p^2)T_{1234F} - (8m^2-2M^2-4p^2)T_{1245F} \\
&+ 2(4m^2-p^2)(4m^2-M^2)T_{11234F} + (4m^2-2M^2-p^2)(4m^2-M^2-2p^2)T_{12345}\Big\} + T_{\otimes F} \ .
\end{aligned} \tag{3.2.3}$$

Note that the formulas (3.2.2) and (3.2.3) above only hold when we work to order ϵ^0. This is adequate for our purposes since in the end we will set ϵ to zero, ridding ourselves of all terms proportional to ϵ.

Now let's study the divergent terms of $F^{(2)}_{\text{vac}}$, that means looking at $F_D^{(2)}$ and $p^2\delta Z_3^{(2)}$; $F_F^{(2)}$ is completely finite. The great promise of renormalization is that the divergent terms in $F_D^{(2)}$ be cancelled out by the appropriate choice of the constant $\delta Z_3^{(2)}$, which has the general form

$$\delta Z_3^{(2)} = \frac{a_k}{\epsilon^k} + \frac{a_{k-1}}{\epsilon^{k-1}} + \cdots + \frac{a_1}{\epsilon} + a_0 + O(\epsilon)\,, \quad k \in \mathbb{N}\,.$$

Here each of the a_k's is a constant (doesn't depend on p^2). This places a severe restriction on $F_D^{(2)}$, all divergent terms in $F_D^{(2)}$ must be of the form $p^2 \frac{a_j}{\epsilon^j}$, anything else cannot be cancelled out by $p^2 \delta Z_3^{(2)}$. We proceed to show that $F_D^{(2)}$ does indeed have the required form. $F_D^{(2)}$ in (3.2.2) appears as a linear combination of the following terms:

$$T_{\otimes D} = -\frac{2}{\epsilon}(4m^2 - M^2)(R+2) + (4m^2 - M^2)(R+2)(3L_m + L_M)\,, \tag{3.2.4}$$

$$\begin{aligned} T_{234D} &= \frac{1}{2\epsilon^2}(2m^2 + M^2) + \frac{1}{\epsilon}\left(3m^2 + \frac{3}{2}M^2 - 2m^2 L_m - M^2 L_M - \frac{1}{4}p^2\right) \\ &\quad + 2m^2(L_m^2 - 3L_m) + M^2(L_M^2 - 3L_M) + \frac{1}{2}p^2 L_{|p|}\,, \end{aligned} \tag{3.2.5}$$

$$\begin{aligned} T_{134D} &= \frac{1}{2\epsilon^2}(2m^2 + M^2) + \frac{1}{\epsilon}\left(3m^2 + \frac{3}{2}M^2 - 2m^2 L_m - M^2 L_M\right) \\ &\quad + 2m^2(L_m^2 - 3L_m) + M^2(L_M^2 - 3L_M)\,, \end{aligned} \tag{3.2.6}$$

$$\begin{aligned} -4(1-\epsilon)(4m^2 - M^2)T_{1134D} &= \frac{1}{\epsilon^2}\left(-8m^2 + 2M^2\right) + \frac{1}{\epsilon}L_m(16m^2 - 4M^2) \\ &\quad + L_m^2(-16m^2 + 4M^2) + 8m^2 - 2M^2 \end{aligned} \tag{3.2.7}$$

$$\begin{aligned} 4(8m^2 - 2M^2 - 2p^2)T_{1234D} &= \frac{1}{\epsilon^2}(16m^2 - 4M^2 - 4p^2) + \frac{1}{\epsilon}\Big\{80m^2 - 20M^2 - 20p^2 \\ &\quad + L_m(-32m^2 + 8M^2 + 8p^2) + R(32m^2 - 8M^2 - 8p^2)\Big\} \\ &\quad + L_m^2(32m^2 - 8M^2 - 8p^2) + L_m\big\{ - 160m^2 + 40M^2 + 40p^2 \\ &\quad + R(-64m^2 + 16M^2 + 16p^2)\big\}\,, \end{aligned} \tag{3.2.8}$$

$$\begin{aligned} -(8m^2 - 2M^2 - 4p^2)T_{1245D} &= \frac{1}{\epsilon^2}(4p^2 + 2M^2 - 8m^2) + \frac{1}{\epsilon}\Big\{R(8p^2 + 4M^2 - 16m^2) + 16p^2 + 8M^2 \\ &\quad - 32m^2 + L_m(-8p^2 - 4M^2 + 16m^2)\Big\} + L_m^2(8p^2 + 4M^2 - 16m^2) \\ &\quad + L_m\big\{R(-16p^2 - 8M^2 + 32m^2) - 32p^2 - 16M^2 + 64m^2\big\}\,, \end{aligned} \tag{3.2.9}$$

$$2(4m^2 - p^2)(4m^2 - M^2)T_{11234D} = \frac{1}{\epsilon}R(8m^2 - 2M^2) + L_m R(-16m^2 + 4M^2)\,. \tag{3.2.10}$$

In some of the above terms, the divergences appear with rather complicated coefficients such as $R(32m^2 - 8M^2 - 8p^2)$ (recall that R is a complicated function of p^2), so it is not at all obvious that $F_D^{(2)}$ will have the required form. Putting the above terms into equation(3.2.2) yields (after some spectacular cancellations) the following result

$$F_D^{(2)} = -\frac{2p^2}{\epsilon} + \frac{8p^2}{3}L_m + \frac{4p^2}{3}L_{|p|} + \left(4m^2 - M^2\right)\left(\ln\left(\frac{M^2}{m^2}\right) + \frac{4}{3}\right)R + 2\left(4m^2 - M^2\right)\left(\ln\left(\frac{M^2}{m^2}\right) + \frac{5}{3}\right) - \frac{2p^2}{3}\,, \tag{3.2.11}$$

which has just the right form of divergence to be cancelled by $p^2\delta Z_3^{(2)}$ and render the whole self energy finite. Also note that all terms proportional to $\frac{1}{\epsilon^2}$ have also vanished.

We will now determine $\delta Z_3^{(2)}$ explicitly and we will do so by imposing a renormalization condition on the self energy. We may now write (3.2.1) as

$$\begin{aligned} F_{\text{vac}}^{(2)} &= -\frac{2p^2}{\epsilon} + \frac{8p^2}{3}L_m + \frac{4p^2}{3}L_{|p|} + \left(4m^2 - M^2\right)\left(\ln\left(\frac{M^2}{m^2}\right) + \frac{4}{3}\right)R + 2\left(4m^2 - M^2\right)\left(\ln\left(\frac{M^2}{m^2}\right) + \frac{5}{3}\right) \\ &\quad - \frac{2p^2}{3} + F_F^{(2)} - p^2\delta Z_3^{(2)}\,, \end{aligned} \tag{3.2.12}$$

and imposing the mass-shell renormalization condition

$$\begin{aligned} &\text{Re}\left[\left.F_F^{(2)}\right|_{p^2=M^2}\right] = 0 \\ &\implies \delta Z_3^{(2)} = -\frac{2}{\epsilon} + \frac{8}{3}L_m + \frac{4}{3}L_M + \left(\frac{4m^2}{M^2} - 1\right)\left(\ln\left(\frac{M^2}{m^2}\right) + \frac{4}{3}\right)\text{Re}\left[\left.R\right|_{p^2=M^2}\right] \\ &\qquad + 2\left(\frac{4m^2}{M^2} - 1\right)\left(\ln\left(\frac{M^2}{m^2}\right) + \frac{5}{3}\right) - \frac{2}{3} + \frac{1}{M^2}\text{Re}\left[\left.F_F^{(2)}\right|_{p^2=M^2}\right] \end{aligned}$$

but

$$\text{Re}\left[\left.R\right|_{p^2=M^2}\right] = \sqrt{1 - \frac{4m^2}{M^2}}\ln\left|\frac{1 - \sqrt{1 - \frac{4m^2}{M^2}}}{1 + \sqrt{1 - \frac{4m^2}{M^2}}}\right|\,,$$

so we get

$$\begin{aligned} \delta Z_3^{(2)} &= -\frac{2}{\epsilon} + \frac{8}{3}L_m + \frac{4}{3}L_M - \left(1 - \frac{4m^2}{M^2}\right)^{3/2}\left(\ln\left(\frac{M^2}{m^2}\right) + \frac{4}{3}\right)\ln\left|\frac{1 - \sqrt{1 - \frac{4m^2}{M^2}}}{1 + \sqrt{1 - \frac{4m^2}{M^2}}}\right| \\ &\quad + 2\left(\frac{4m^2}{M^2} - 1\right)\left(\ln\left(\frac{M^2}{m^2}\right) + \frac{5}{3}\right) - \frac{2}{3} + \frac{1}{M^2}\text{Re}\left[\left.F_F^{(2)}\right|_{p^2=M^2}\right]\,. \end{aligned} \tag{3.2.13}$$

Here $F_F^{(2)}$ has yet to be determined. Putting (3.2.13) back into (3.2.12) gives the following

$$F_{\text{vac}}^{(2)} = f(p^2) + F_F^{(2)}\,, \tag{3.2.14}$$

where

$$f(p^2) = \frac{4p^2}{3}\ln\left|\frac{p^2}{M^2}\right| + \left(4m^2 - M^2\right)\left(\ln\left(\frac{M^2}{m^2}\right) + \frac{4}{3}\right)\left\{R - \frac{p^2}{M^2}\sqrt{1-\frac{4m^2}{M^2}}\ln\left|\frac{1-\sqrt{1-\frac{4m^2}{M^2}}}{1+\sqrt{1-\frac{4m^2}{M^2}}}\right|\right\}$$
$$+ 2\left(4m^2 - M^2\right)\left(\ln\left(\frac{M^2}{m^2}\right) + \frac{5}{3}\right)\left(1 - \frac{p^2}{M^2}\right) - \frac{p^2}{M^2}\text{Re}\left[F_F^{(2)}\Big|_{p^2=M^2}\right]. \tag{3.2.15}$$

We notice that $F_{\text{vac}}^{(2)}$ is free of divergences and that, after imposing the mass-shell condition, dependence on the renormalization scale μ has vanished. The mass-shell condition also ensures that M is the physical mass of the ρ^0 meson i.e. $M = 775.5$ MeV. All that remains is $F_F^{(2)}$, and we will have complete knowledge of the two-loop self energy. Fortunately this is easily done using the scalar integrals collected in the "repository section". In the figures that follow, we plot all the terms that contribute to $F_{\text{vac}}^{(2)}$ as functions of p^2 (except for $T_{134\text{F}}$ and $T_{1134\text{F}}$ which are constants), and in figure 3.11 is a graph of $F_{\text{vac}}^{(2)}$ itself. In all the contributing terms we note that the imaginary parts always vanish below $p^2 = 4m^2 = 0.0779$ GeV^2, hence this behaviour is displayed by $F_{\text{vac}}^{(2)}$ itself. This is to be expected since this is the lowest threshold of all the two-loop diagrams contributing to the ρ^0 self-energy.

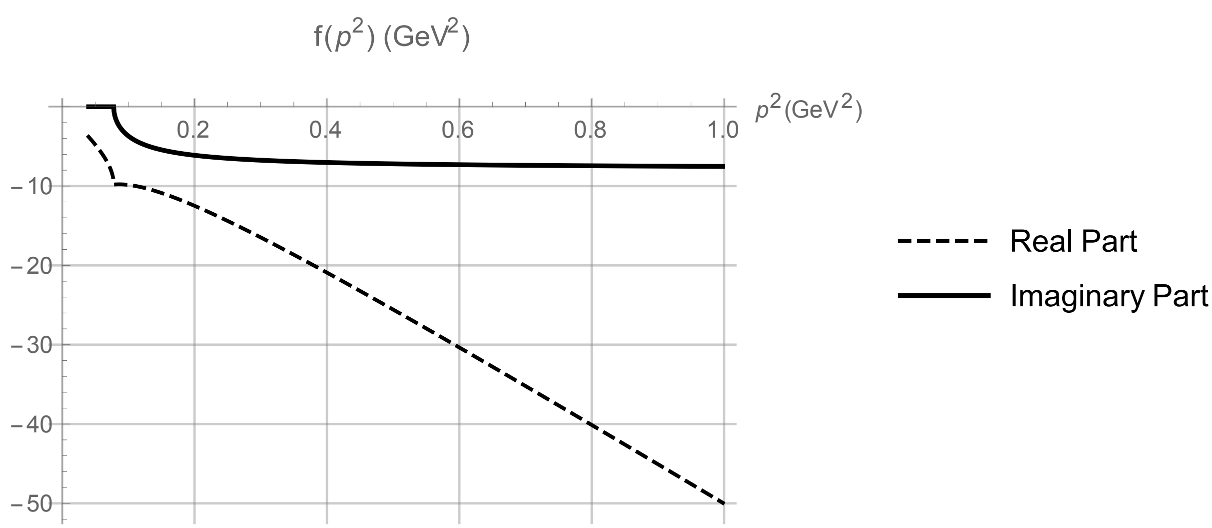


Figure 3.3: Real and imaginary parts of $f(p^2)$.

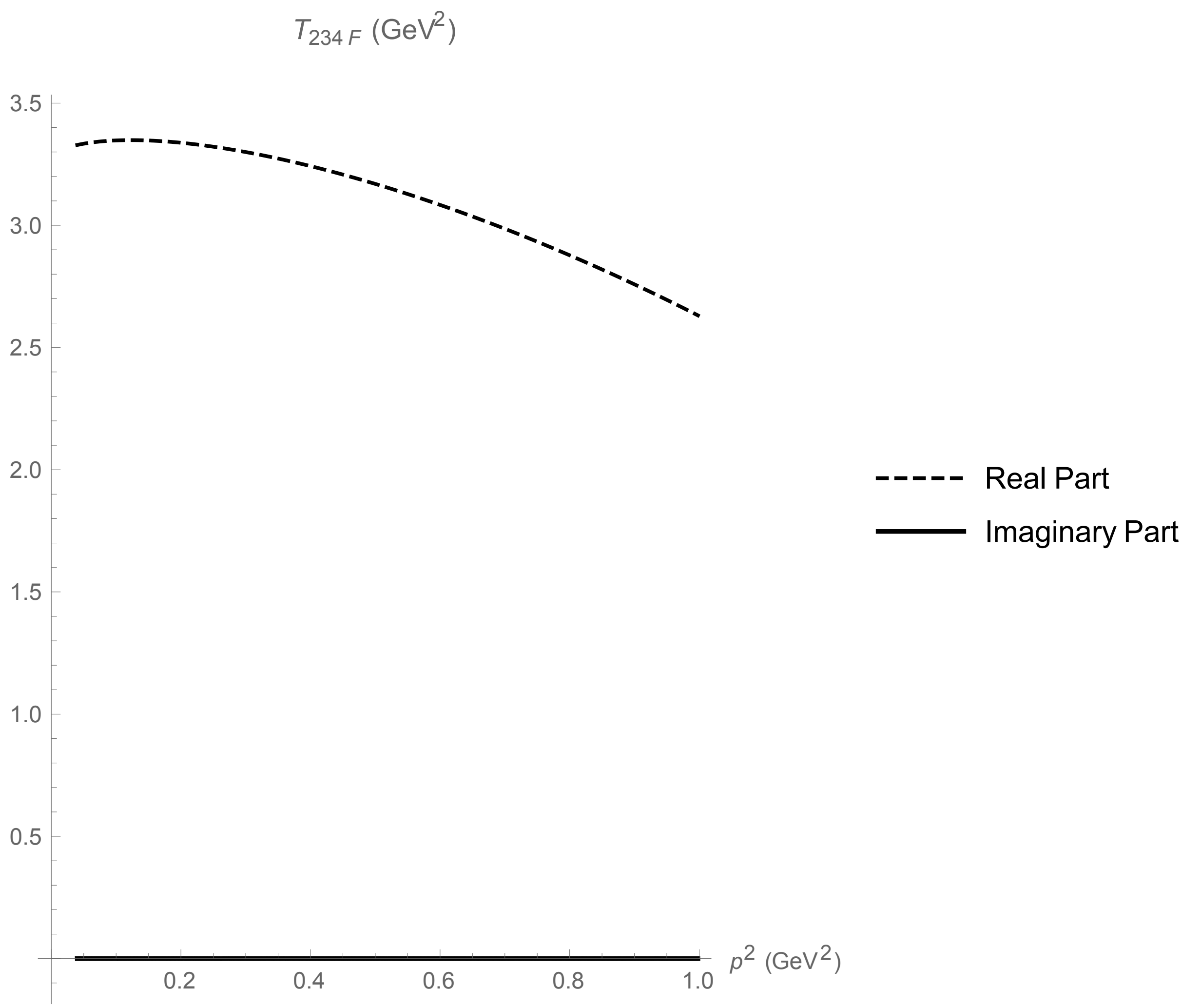


Figure 3.4: Real and imaginary parts of $T_{234\text{F}}$.

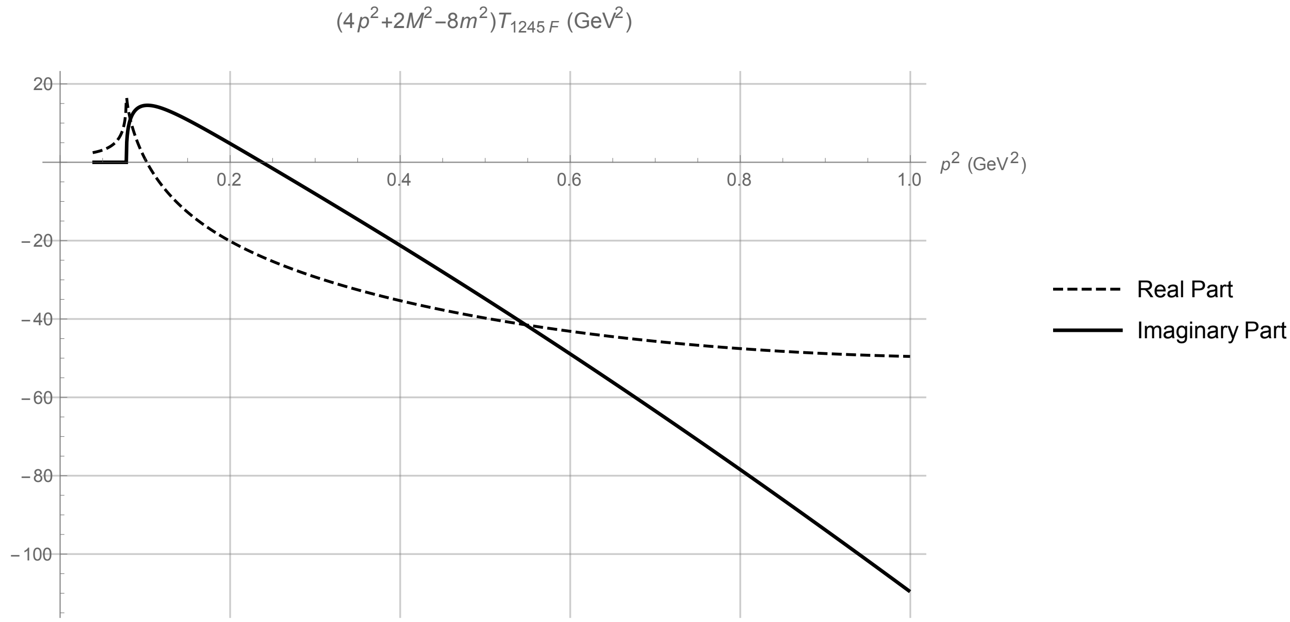


Figure 3.5: Real and imaginary parts of $T_{1245\mathrm{F}}$.

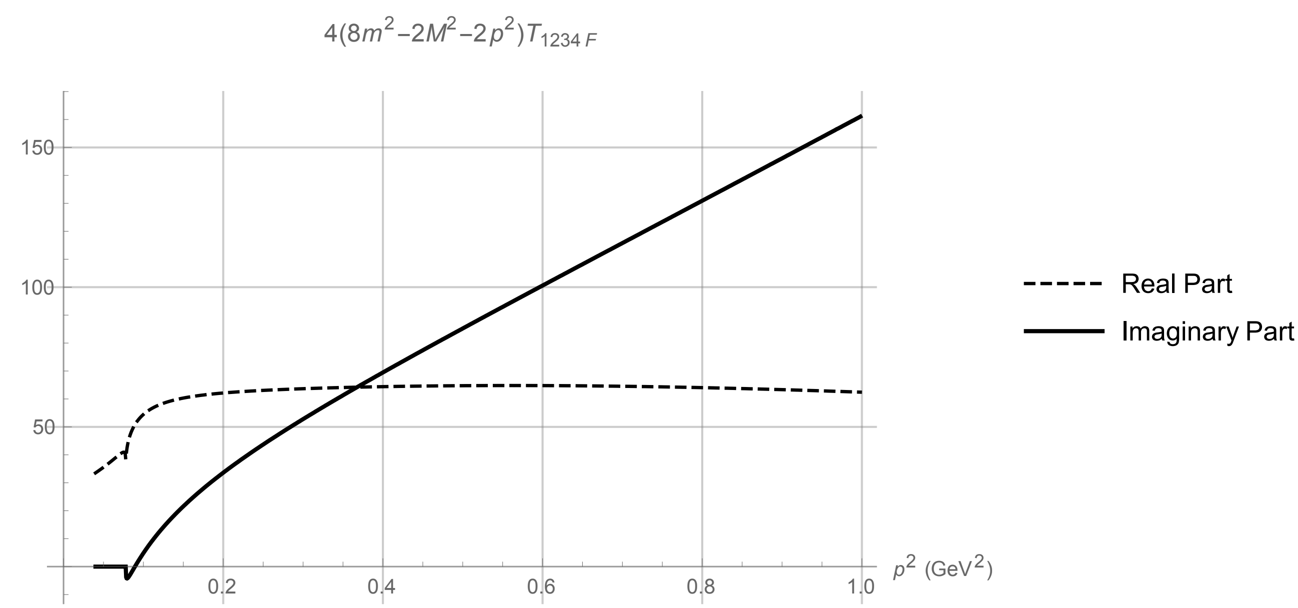


Figure 3.6: Real and imaginary parts of $T_{1234\mathrm{F}}$.

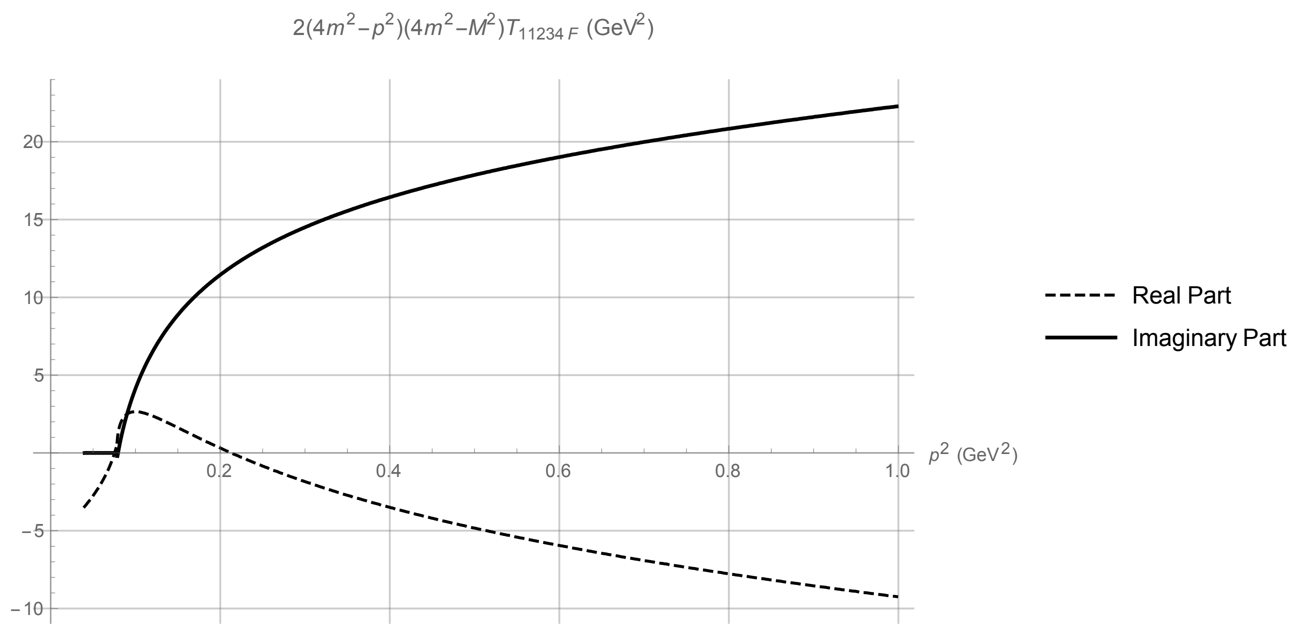


Figure 3.7: Real and imaginary parts of $2(4m^2 - p^2)(4m^2 - M^2)T_{11234\text{F}}$.

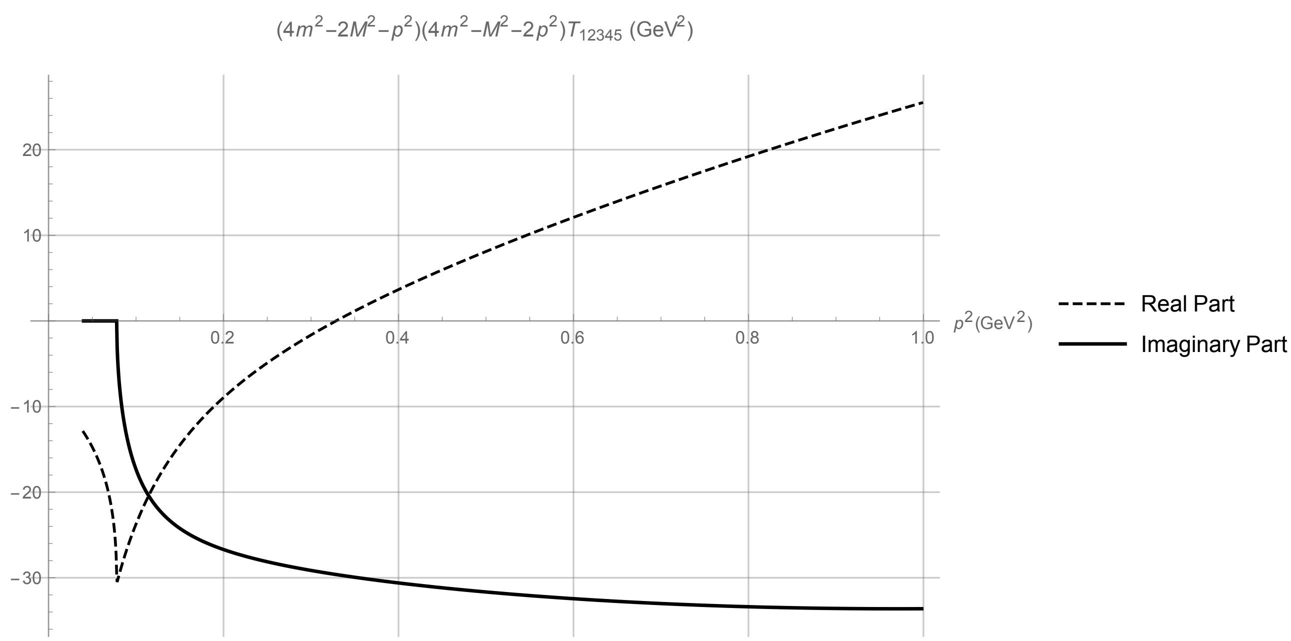


Figure 3.8: Real and imaginary parts of $(4m^2 - 2M^2 - p^2)(4m^2 - M^2 2p^2)T_{12345\text{F}}$.

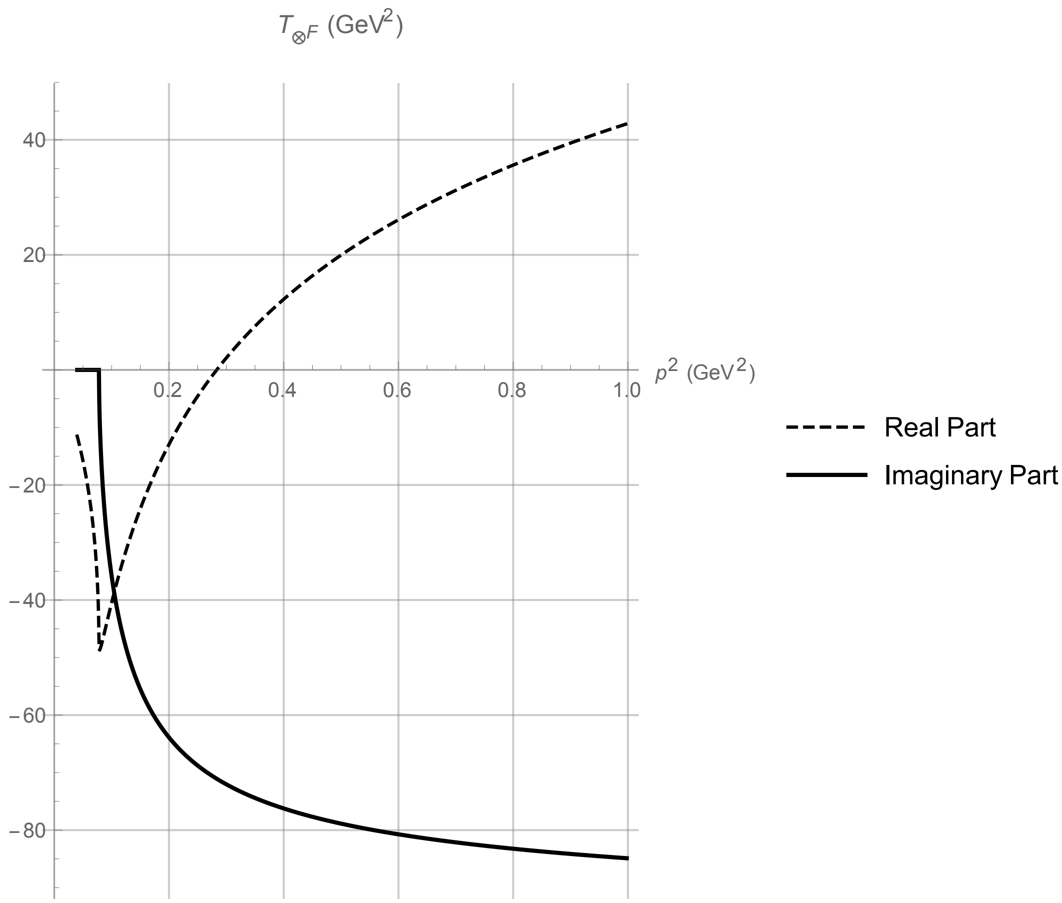


Figure 3.9: Real and imaginary parts of $T_{\otimes \mathrm{F}}$.

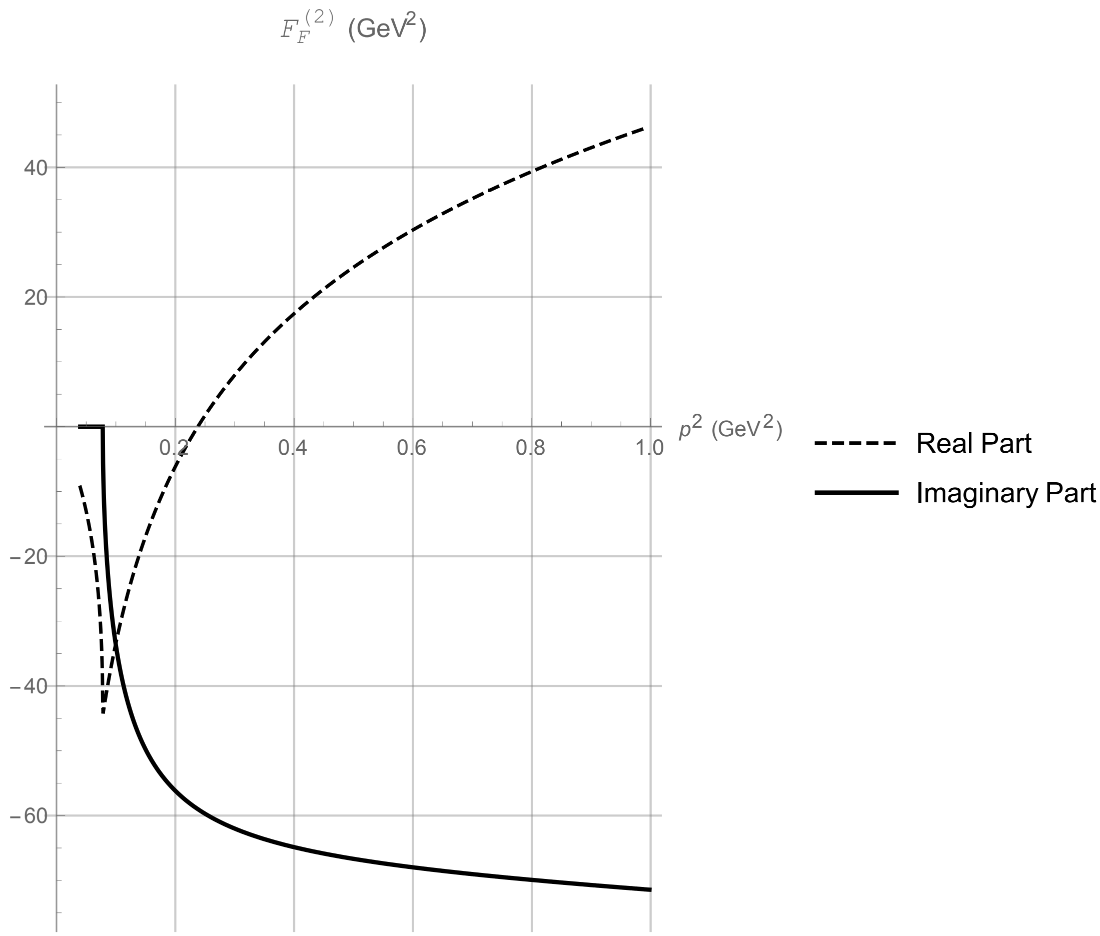


Figure 3.10: Real and imaginary parts of $F_F^{(2)}$.

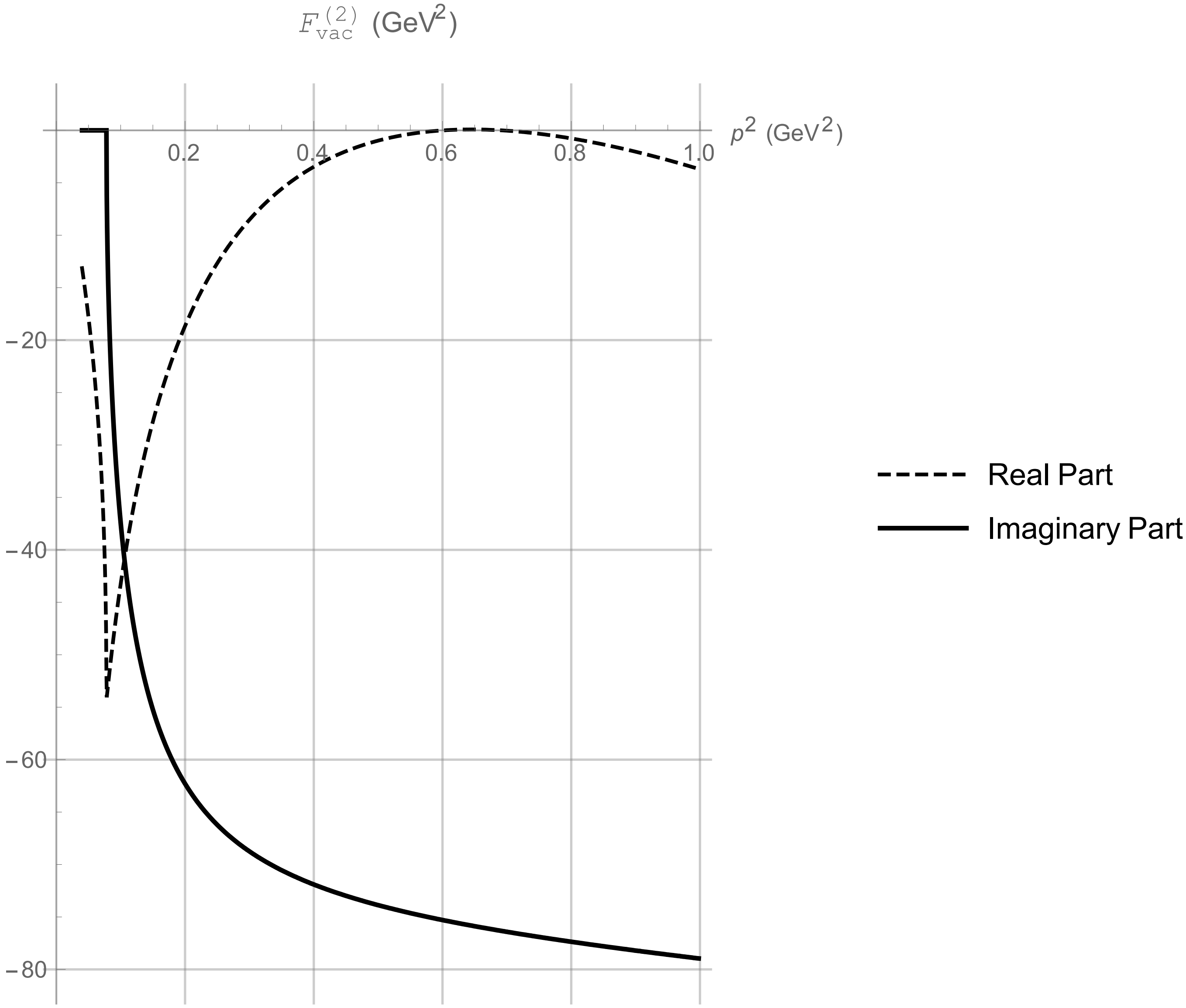


Figure 3.11: Real and imaginary parts of $F_{\mathrm{vac}}^{(2)}$.

...when you can measure what you are speaking about, and express it in numbers, you know something about it...

Lord Kelvin

4

Contact with experiment

We have determined F_{vac} to be:

$$F_{\text{vac}} = \alpha F_{\text{vac}}^{(1)} + \alpha^2 F_{\text{vac}}^{(2)}$$

and we have full knowledge of both $F_{\text{vac}}^{(1)}$ and $F_{\text{vac}}^{(2)}$. However, the constant $\alpha = \left(\frac{g_{\rho\pi\pi}}{4\pi}\right)^2$ has yet to be determined, to fix it we use an experimentally measured observable such as the width of the neutral ρ-meson, $\Gamma_\rho = 0.149$ MeV [21]. Now, according to the vector meson dominance model, the width of the ρ^0 meson is given by:

$$\Gamma_\rho = -\frac{1}{M} \, \text{Im}F_{\text{vac}}\big|_{p^2=M^2} \quad . \tag{4.0.1}$$

This allows for α to be found using the linear equation

$$\Gamma_\rho = -\frac{\alpha}{M}\ \mathrm{Im}F^{(1)}_{\text{vac}}\Big|_{p^2=M^2}\ , \tag{4.0.2}$$

when working to one-loop order, and the following quadratic equation

$$\Gamma_\rho = -\frac{\alpha}{M}\ \mathrm{Im}F^{(1)}_{\text{vac}}\Big|_{p^2=M^2} - \frac{\alpha^2}{M}\ \mathrm{Im}F^{(2)}_{\text{vac}}\Big|_{p^2=M^2}\ , \tag{4.0.3}$$

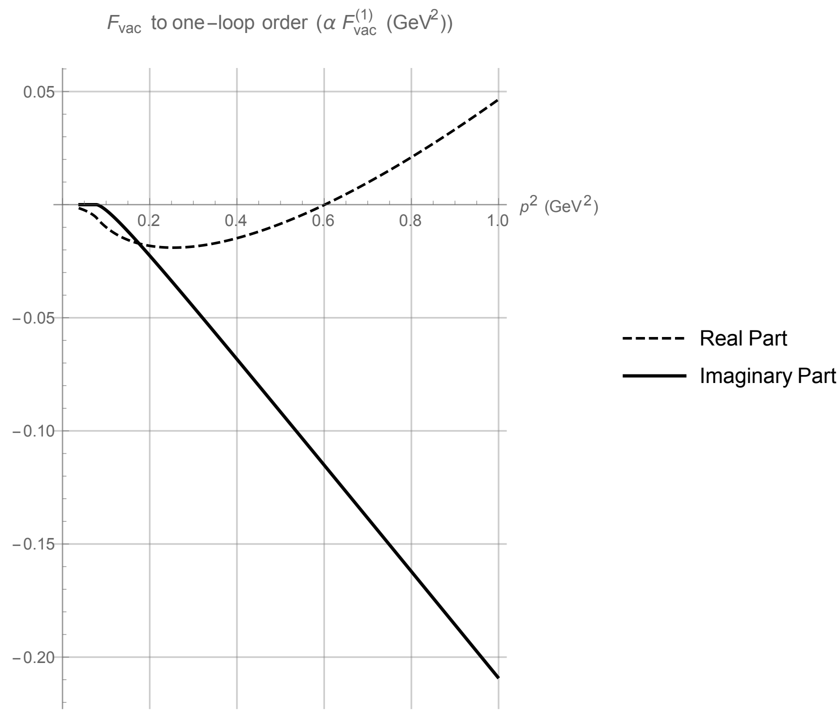


Figure 4.1: The real and imaginary parts of F_{vac} when working to one-loop order i.e. $F_{\text{vac}} = \left(\frac{g_{\rho\pi\pi}}{4\pi}\right)^2 F^{(1)}_{\text{vac}}$, with $g_{\rho\pi\pi} = 5.97$

for a two-loop order truncation. The above equations involve the following easily evaluated factors:

$$\text{Im}F^{(1)}_{\text{vac}}\Big|_{p^2=M^2} = -\pi\frac{M^2}{3}\left(1-\frac{4m^2}{M^2}\right)^{3/2}$$
$$= -0.5114\,\text{GeV}^2\ , \tag{4.0.4}$$

$$\text{Im}F^{(2)}_{\text{vac}}\Big|_{p^2=M^2} = -\pi M^2\left(\ln\left(\frac{M^2}{m^2}\right)+\frac{4}{3}\right)\left(1-\frac{4m^2}{M^2}\right)^{3/2} + \text{Im}F^{(2)}_{F}\Big|_{p^2=M^2}$$
$$= -75.31\,\text{GeV}^2\ . \tag{4.0.5}$$

The solution for the one-loop case (4.0.2) is $\alpha = 0.2259$ or equivalently $g_{\rho\pi\pi} = 5.97$. In the full two-loop case (4.0.3) we find $\alpha = 0.03592$ which is equivalent to $g_{\rho\pi\pi} = 2.38$. This big change in $g_{\rho\pi\pi}$, when going from one-loop to two-loops, means that the two-loop contribution does not produce a small modification to the one-loop result, suggesting that perturbation theory has broken down. The fact that the order α^2 term is indeed larger (in magnitude) than the order α term is clearly visible in figure 4.3.

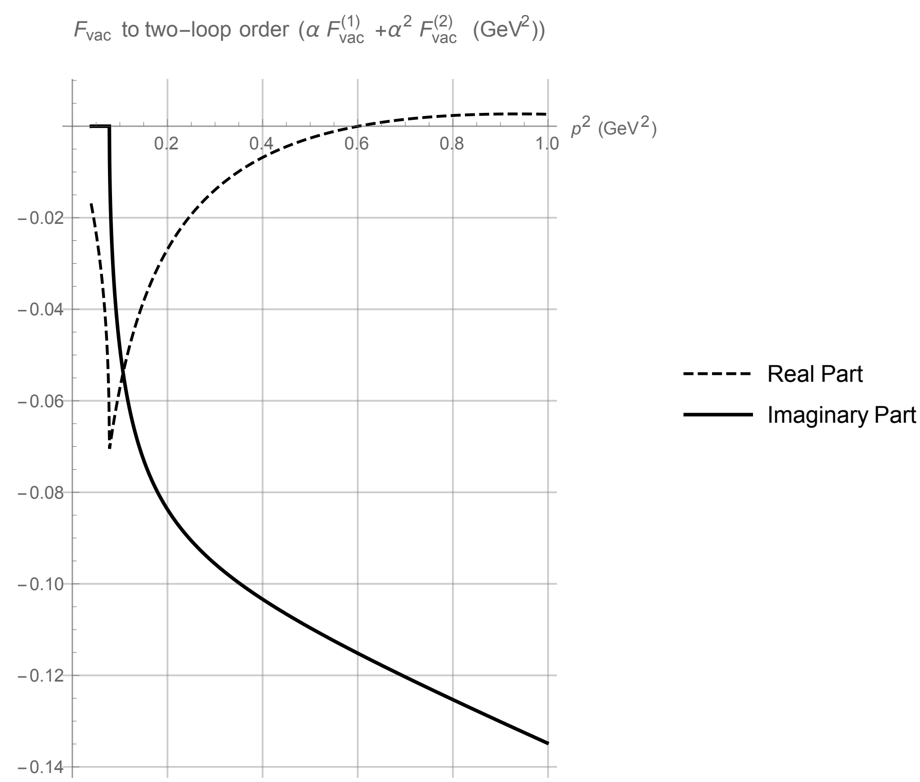
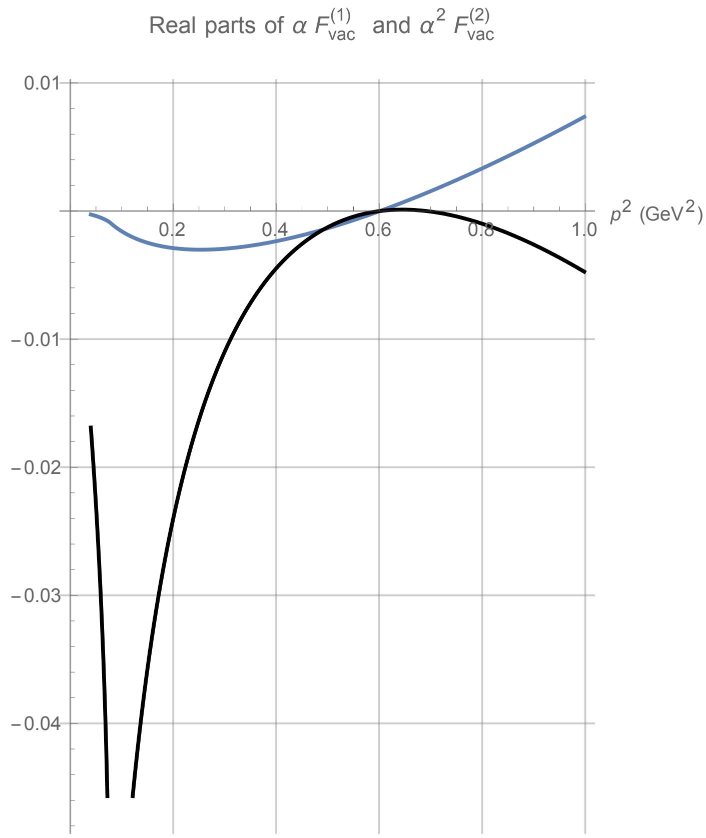
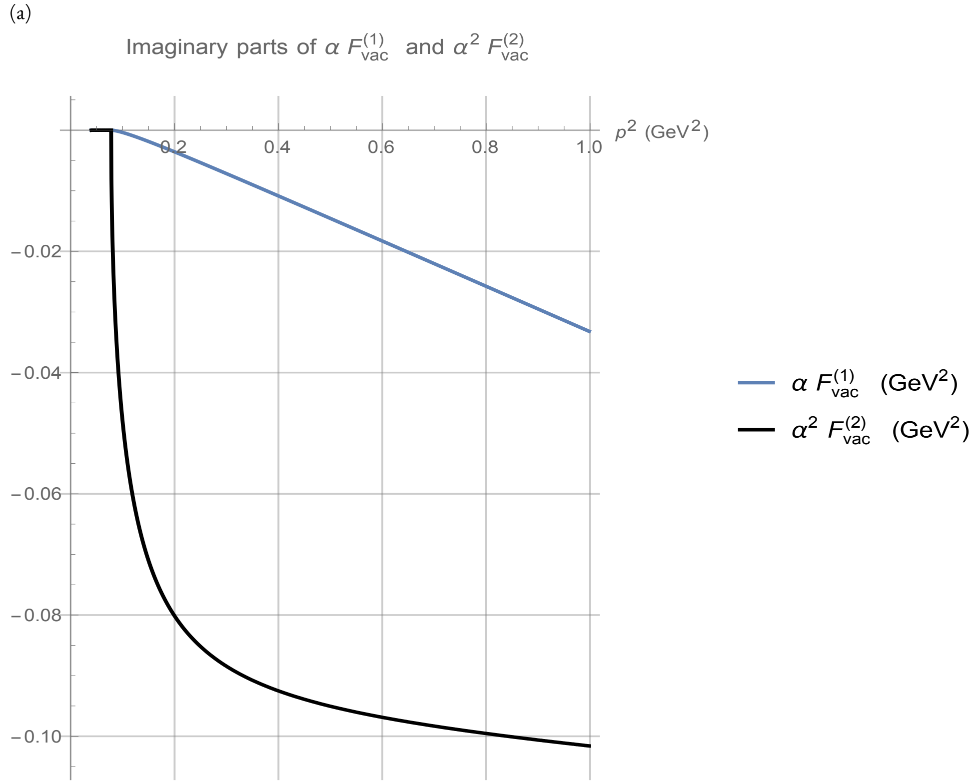


Figure 4.2: The real and imaginary parts the full two-loop $F_{\text{vac}} = \left(\frac{g_{\rho\pi\pi}}{4\pi}\right)^2 F^{(1)}_{\text{vac}} + \left(\frac{g_{\rho\pi\pi}}{4\pi}\right)^4 F^{(2)}_{\text{vac}}$, with $g_{\rho\pi\pi} = 2.38$.



(a)



(b)

Figure 4.3: Comparison of order $\left(\frac{g_{\rho\pi\pi}}{4\pi}\right)^2$ and order $\left(\frac{g_{\rho\pi\pi}}{4\pi}\right)^4$ contributions to the full two-loop F_{vac}, with $g_{\rho\pi\pi} = 2.38$.

We can now compare our results to experiment using the quantity

$$R(s) = \frac{M^2}{s}\frac{M^2}{f_\rho^2}\frac{-12\pi \text{Im}\, F_{\text{vac}}(s)}{\left(s - M^2 + \text{Re}\, F_{\text{vac}}(s)\right)^2 + \left(\text{Im}\, F_{\text{vac}}(s)\right)^2}\ , \tag{4.0.6}$$

where $f_\rho = 4.97 \pm 0.07$ [3]. The predicted $R(s)$ above (4.0.6), is compared to experiment in figure 4.4. Here we see the one-loop result is in much better agreement with experiment than the two-loop result.

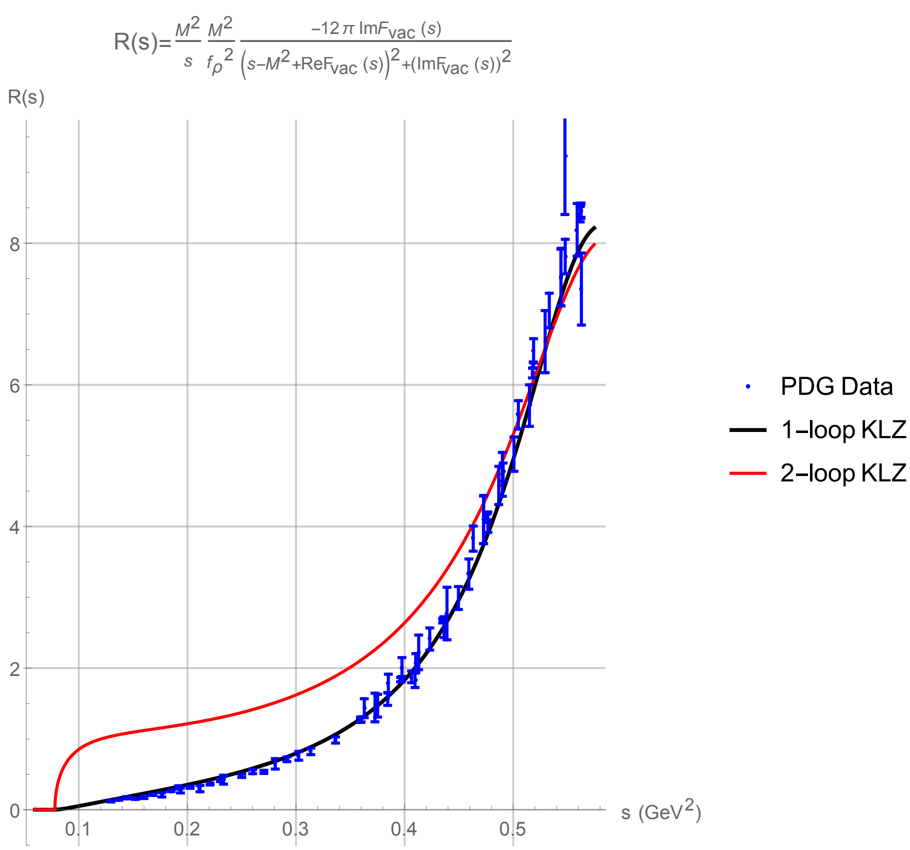


Figure 4.4: $R(s)$ from one and two-loop KLZ theory compared to experiment[21]

The proof of the pudding is in the eating.

English Proverb

5

Conclusion

The self-energy (or equivalently the propagator) of the neutral rho meson was calculated within the framework of the Kroll-Lee-Zumino quantum field theory and the result behaves as expected: Firstly, the self energy comes out transverse, a result consistent with the theory's Ward Identities; Secondly, the self-energy acquires an imaginary part only at values of p^2 greater than the threshold for pion-pair-production; Lastly, at $p^2 = 0$ the self-energy is regular, in other words $F_{\text{vac}}|_{p^2=0}$ vanishes as expected. By all indications then, the result obtained for the NNLO contribution to the ρ^0 self-energy is correct. This contribution, however, is larger than the NLO result despite the smallness of the expansion parameter α. So at this order, perturbation theory cannot be trusted. To drive this point home we compared theory against experiment for the observable $R(s)$, it was shown that the two-loop result compares poorly experiment, a far cry from one-loop prediction. This result does not come as a complete shock as quantum field theories are notorious for producing asymptotic series, as in the case of QED for example [37].

This is a disappointing outcome, considering the encouraging NLO results. Nevertheless, regarding the question of whether NNLO corrections can improve agreement between the KLZ theory and experiment... the proverbial pudding has been eaten.

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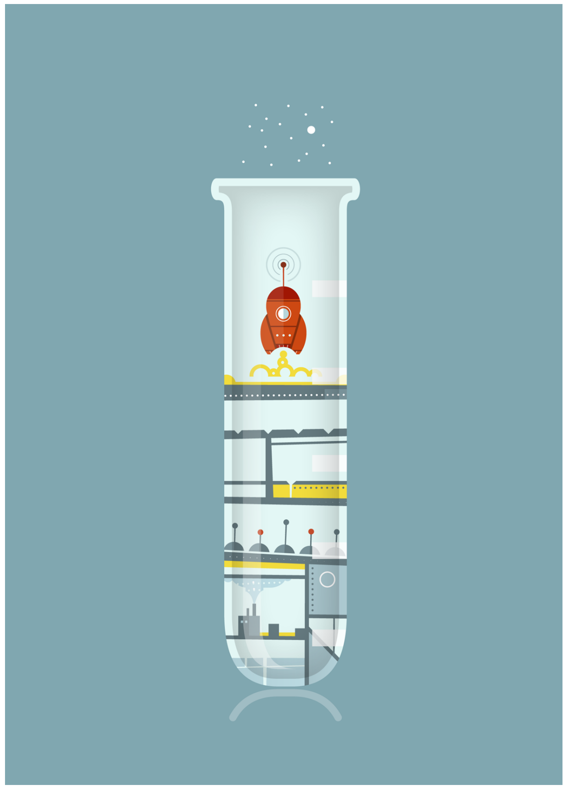

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