

Monotone and Pseudomonotone Operators with Applications to Variational Problems

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Abstract

This work is primarily concerned with investigating how monotone and pseudomonotone operators between Banach spaces are used to prove the existence of solutions to nonlinear elliptic boundary value problems. A well-known approach to solving nonlinear elliptic boundary value problems is to reformulate them as equations of the form $A(u) = f$, where A is a monotone or pseudomonotone operator from a Sobolev space to its dual. We seek to study the abstract theory which underpins this approach and proves the existence of a solution to the equation $A(u) = f$, implying the existence of a weak solution to the elliptic boundary value problem. Further, we examine properties of monotone and pseudomonotone operators, with an emphasis on a characterization, which involves the latter, and establishes a connection between the operator and the principal part of a partial differential equation. In addition, results relating monotone and pseudomonotone operators with variational inequalities are explored.

Plagiarism Declaration

I, Byron Joseph Alexander, know the meaning of plagiarism and declare that all of the work in the document, save for that which is properly acknowledged, is my own.

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Introduction

The theory of nonlinear monotone operators, an active area of current research, has its initial development rooted in the 1960s, and has been applied in various fields, from abstract analysis to variational problems.

If $(V, \|\cdot\|)$ is a Banach space with dual V^* , and K is a nonempty subset of V , then an operator $A : K \rightarrow V^*$ is monotone if

$$\langle A(u) - A(v), u - v \rangle \geq 0 \quad \text{for all } u, v \in K,$$

while a set-valued mapping $A : K \rightarrow 2^{V^*}$ is monotone if

$$\langle u^* - v^*, u - v \rangle \geq 0 \quad \text{for any } u, v \in K, \quad u^* \in A(u), \quad v^* \in A(v).$$

The first systematic results on monotone operators were obtained by G. Minty (see [55]), on monotone operators on Hilbert spaces, and F.E. Browder (see [13]), on monotone operators on Banach spaces.

Quasilinear elliptic and parabolic partial differential equations form the basis of mathematical models of various steady-state phenomena and processes in mechanics, physics and many other areas of science. The properties of monotone operators were studied systematically by F.E. Browder in order to obtain existence theorems for quasilinear elliptic and parabolic partial differential equations. Those existence theorems were then later extended to a more general class of quasilinear elliptic differential equations by P. Hartmann and G. Stampacchia (see [38]).

Since there are several quasilinear partial differential equations involving operators which are not monotone, one could not solve the corresponding boundary or initial boundary value problems through the theorems by Browder ([20]). Therefore, Brézis introduced, in [6], a vast class of operators, which he classified as *pseudomonotone*, thereby extending Browder's existence theorem. Precisely, an operator $A : K \rightarrow V^*$ is pseudomonotone, in the sense of Brézis, if for every sequence $(u_n)_n$, such that

$$u_n \rightharpoonup u \in K \quad \text{and} \quad \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0,$$

it follows that

$$\langle A(u), u - v \rangle \leq \liminf_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle \quad \text{for every } v \in V.$$

A set-valued mapping $A : K \rightarrow 2^{V^*}$ is pseudomonotone if for every sequence $(u_n)_n$, such that $u_n \rightharpoonup u \in K$, and every sequence $(u_n^*)_n$, such that $u_n^* \in A(u_n)$, for which $\limsup_{n \rightarrow \infty} \langle u_n^*, u_n - u \rangle \leq 0$, it follows that for every $v \in K$ there exists a $u^* \in A(u)$, such that

$$\langle u^*, u - v \rangle \leq \liminf_{n \rightarrow \infty} \langle u_n^*, u_n - v \rangle.$$

This class of operators, widely applied in variational problems, has also been extensively researched, and still plays an important role in the investigation of the solvability of operator equations.

On the other hand, since the pioneering and celebrated work of Hartmann and Stampacchia (see [38]), the theory of variational inequalities (VIs), has become an important research topic, used as a mathematical model for dealing with problems arising in different fields, such as optimization theory, partial differential equations, mechanics (contact and obstacle problems, elasto-plasticity) and economic equilibrium. The theory of monotone and pseudomonotone operators has also been applied to the solvability of VIs.

After the definition of pseudomonotonicity by Brézis, another definition was given, some years later, by Karamardian in [45]. Precisely, an operator is K -pseudomonotone if

$$\langle A(v), u - v \rangle \geq 0 \implies \langle A(u), u - v \rangle \geq 0$$

for each $u, v \in K$. A set-valued mapping $A : K \rightarrow 2^{V^*}$ is K -pseudomonotone if, for each $u, v \in K$, $u^* \in A(u)$ and $v^* \in A(v)$, it follows that

$$\langle u^*, u - v \rangle \geq 0 \implies \langle v^*, u - v \rangle \geq 0.$$

Several existence results for VIs in connection with this new definition have been obtained in the literature. Note that both Brézis' and Karamardian's definitions of pseudomonotonicity were introduced with the main purpose of studying the solvability of VIs.

Our interest in this thesis is mainly on single valued monotone and pseudomonotone operators and their applications in nonlinear elliptic boundary value problems and variational inequalities.

In Chapter 1, we will present some important results in functional analysis and measure theory, which are essential for the development of the topic in later chapters. Reference to this chapter will be made throughout the thesis. Sobolev spaces play a fundamental role in the topic, as it serves as a fitting domain for the monotone (or pseudomonotone) operators described in Chapter 2. The introduction of Sobolev spaces, together with definitions, properties and important theorems will also be presented.

In Chapter 2, we will present the abstract theory of monotone and pseudomonotone operators mentioned above, and show how it can be applied to second order partial differential equations to show the existence of a weak solution. Here, we focus in particular on two existence results introduced by Minty, Browder and Brézis. Having two different definitions of pseudomonotone operators, it becomes natural to investigate the connection between the two or to establish some comparison ([26], [58], [63] and [67] are the main references for this chapter).

In Chapter 3, we look to apply the abstract theory introduced in Chapter 2 to prove the existence of weak solutions for second order quasilinear partial differential equations with boundary conditions. We further investigate another application of this theory in proving the existence of weak solutions to some nonlinear elliptic equations, governed by an anisotropic operator ([23], [39] and [58] are the main references for this chapter).

Seeing as pseudomonotone operators play such a fundamental role in the development of this topic, Chapter 4 is used to present a characterization of pseudomonotone operators. In particular, we exhibit a connection between a pseudomonotone operator and the principal part of a second order partial differential equation ([4] and [9] are the main references for this chapter).

There is a diverse range of real world applications which involve variational inequalities, hence, it is an appreciable tool in applied science. In Chapter 5, we examine some ideas and results with regard to variational inequalities and the use of monotone and pseudomonotone operators in existence results thereof ([9], [37], [44], [48], [61] and [68] are the main references for this chapter).

Chapter 1

Preliminary Material in Analysis

This chapter comprises of a selection of theorems, corollaries, lemmas and definitions, which will be used throughout the development of the topic. Additional results will be presented within subsequent chapters if they are of particular relevance. More specifically, we will present results in measure theory and functional analysis which are required. Lastly, we introduce Sobolev spaces.

1.1 Basic Definitions

Definition 1.1.1. A sequence $\{u_n\}_n$ in a normed space $(V, \|\cdot\|_V)$ is called a *Cauchy sequence*, if for every $\epsilon > 0$, there exists an integer N , such that $\|u_m - u_n\|_V < \epsilon$ holds whenever $m, n > N$. We say that V is *complete* and, therefore, a *Banach space*, if every Cauchy sequence in V converges to some limit in V .

Definition 1.1.2. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed spaces over the same field. We say operator $A : V \rightarrow W$ is *bounded* if it maps bounded sets in V to bounded sets in W .

Definition 1.1.3. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed spaces, then operator $A : V \rightarrow W$ is called *compact* if

- A is continuous,
- A maps bounded sets $U \subset V$ into relatively compact sets, i.e., $\overline{A(U)}$ is compact.

Definition 1.1.4. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be two normed spaces over the same field. We say that linear mapping $L : V \rightarrow W$ is bounded if there exists a constant $M > 0$ such that

$$\|L(v)\|_W \leq M\|v\|_V \quad \forall v \in V. \quad (1.1)$$

Remark 1. It is easy to show that a linear mapping $L : V \rightarrow W$ is bounded if and only if it is continuous. Let $\mathcal{L}(V, W)$ denote the vector space of linear bounded mappings from V to W . For every $L \in \mathcal{L}(V, W)$, let $\|L\|_{\mathcal{L}(V, W)} := \inf\{c \geq 0 : \|L(v)\|_W \leq c\|v\|_V \text{ for all } v \in V\}$.

Proposition 1.1.5. $\|\cdot\|_{\mathcal{L}(V, W)}$ is a norm on $\mathcal{L}(V, W)$. Furthermore, if $(W, \|\cdot\|_W)$ is a Banach space, then $(\mathcal{L}(V, W), \|\cdot\|_{\mathcal{L}(V, W)})$ is also a Banach space.

Definition 1.1.6. Let V be a normed space. We define the *dual space* of V (also a normed space, denoted by V^*) as the space of all bounded linear functionals on V such that

$$f : V \rightarrow \mathbb{R} \text{ is linear and } \|f\|_{V^*} := \sup_{\|x\| \leq 1} \{|f(x)|\} < \infty.$$

We can see that $V^* := (\mathcal{L}(V, \mathbb{R}))$, where $f : (V, \|\cdot\|_V) \rightarrow (\mathbb{R}, |\cdot|)$.

Definition 1.1.7. Let V be a normed space and let V^* be its dual space. The *double dual* is the dual of V^* , with norm

$$\|g\|_{V^{**}} := \sup_{f \in V^*, \|f\|_{V^*} \leq 1} \{|\langle g, f \rangle|\} \quad \forall g \in V^{**}.$$

Definition 1.1.8. A natural linear injection of a normed space V into its double dual space $V^{**} = (V^*)^*$ is provided by the mapping J (called *the evaluation map*) whose value Ju at $u \in V$ is given by

$$Ju(u^*) = u^*(u) \quad \forall u^* \in V^*.$$

Since $|Ju(u^*)| \leq \|u^*\|_{V^*} \|u\|_V$ we have

$$\|Ju\|_{V^{**}} \leq \|u\|_V.$$

By the Hahn-Banach Extension Theorem, we have that for any $u \in V$, we can find $u^* \in V^*$ such that $\|u^*\|_{V^*} = 1$ and $u^*(u) = \|u\|_V$. Therefore, J is an isometry. If J is surjective, then we say that V is *reflexive*.

Definition 1.1.9. The *weak topology* is the initial topology of a normed space V with respect to its continuous dual V^* . In other words, it is the coarsest topology in V such that each member of V^* remains continuous. We will refer to subsets of a topological vector space as weakly closed (respectively, weakly compact, etc.) if they are closed (respectively, weakly compact, etc.) with respect to the weak topology.

Theorem 1.1.10. (*Kakutani's Theorem*) Let V be a Banach space. Then V is reflexive if and only if $B_V = \{u \in V : \|u\| \leq 1\}$ is compact in the weak topology.

Theorem 1.1.11. (*Eberlein-Šmulian Theorem*) A Banach space V is reflexive if and only if every bounded sequence in V has a subsequence which converges weakly to an element of V .

Definition 1.1.12. Let H be a vector space. A *scalar product* $[u, v]$ is a bilinear form $H \times H \rightarrow \mathbb{R}$ such that

- $[u, v] = [v, u] \quad \forall u, v \in H,$
- $[u, u] \geq 0 \quad \forall u \in H,$
- $[u, u] \neq 0 \quad \forall u \neq 0.$

Remark 2. It follows from the *Cauchy-Schwarz inequality*, which is stated as

$$|[u, v]| \leq [u, u]^{\frac{1}{2}} [v, v]^{\frac{1}{2}} \quad \forall u, v \in H,$$

that we can define a norm

$$\|u\| := [u, u]^{\frac{1}{2}}.$$

Definition 1.1.13. A *Hilbert space* is a vector space H equipped with a scalar product, such that H is complete with respect to the norm $\|u\| := [u, u]^{\frac{1}{2}}$.

1.2 Functional Analysis Toolbox

In this section, a collection of results required to develop the subject of this thesis is presented. Below, we introduce one of the most fundamental and important theorems in functional analysis, credited to Austrian mathematician Hans Hahn and Polish mathematician Stefan Banach.

Theorem 1.2.1. (*The Hahn-Banach Extension Theorem*) *Let V be a normed space and let $g : V \rightarrow \mathbb{R}$ be a function satisfying*

$$g(tu) = t \cdot g(u), \quad g(u + v) \leq g(u) + g(v) \quad \forall t > 0 \quad u, v \in V.$$

Let $Y \subset V$ be a vector subspace and $f : Y \rightarrow \mathbb{R}$ be a linear map such that $f(u) \leq g(u) \quad \forall u \in Y$. Then there exists a linear map $\varphi : V \rightarrow \mathbb{R}$ such that for any $v \in Y$, $\varphi(v) = f(v)$ and $\varphi(u) \leq g(u) \quad \forall u \in V$.

Corollary 1.2.2. *Let V be a normed space and f be a linear functional defined on a subspace $Y \subset V$ with*

$$\|f\|_{Y^*} = \sup_{u \in Y, \|u\|_V \leq 1} \{|f(u)|\}. \quad (1.2)$$

Then, f can be extended to $g \in V^$ on Y with $\|g\|_{V^*} = \|f\|_{Y^*}$.*

Theorem 1.2.3. (*Banach-Steinhaus*) *Let V be a Banach Space and Y a normed space. Let $(T_i)_{i \in I}$ be an arbitrary family of elements of $\mathcal{L}(V, Y)$ such that*

$$\sup_{i \in I} \|T_i(u)\|_Y < \infty \quad \forall u \in V.$$

Then,

$$\sup_{i \in I} \|T_i\|_{\mathcal{L}(V, Y)} < \infty. \quad (1.3)$$

Definition 1.2.4. Let V be a vector space. A set K in V is said to be *convex* if for all $u, v \in K$ and all $t \in [0, 1]$, it follows that $(1 - t)u + tv \in K$.

Theorem 1.2.5. *Let V be a Banach space, with K a convex subset of V . Then, K is closed in the weak topology if and only if it is closed in the strong topology.*

Definition 1.2.6. A Banach space V is said to be *uniformly convex* if $\forall \epsilon > 0 \exists \delta > 0$ such that

$$(u, v \in V, \|u\| \leq 1, \|v\| \leq 1 \text{ and } \|u - v\| > \epsilon) \implies \left(\left\| \frac{u + v}{2} \right\| < 1 - \delta \right).$$

Examples of uniformly convex spaces include all Hilbert spaces and L^p spaces for $1 < p < \infty$.

Theorem 1.2.7. *Let V be a uniformly convex Banach space, then $u_n \rightharpoonup u$, and $\|u_n\| \rightarrow \|u\| \implies u_n \rightarrow u$.*

The following theorems play an important role in the study of partial differential equations.

Theorem 1.2.8. (*Brouwer Fixed Point Theorem*) Assume that

$$f : B(0, 1) \rightarrow B(0, 1)$$

is continuous, where $B(0, 1)$ denotes the closed unit ball in \mathbb{R}^N . Then f has a fixed point; that is, there exists a point $x \in B(0, 1)$, such that $f(x) = x$.

Theorem 1.2.9. (*Schauder's Fixed Point Theorem*) Suppose $K \subset V$ is compact and convex and assume

$$A : K \rightarrow K$$

is continuous. Then A has a fixed point in K .

Lemma 1.2.10. Let V be a normed space. If $(u_n)_n$ is a sequence in V such that every subsequence has a further subsequence, which all converge to the same limit $u \in V$, then the whole sequence also converges to this limit. i.e. $u_n \rightarrow u$.

Definition 1.2.11. A topological space V is called *separable* if it contains a countable, dense subset.

Proposition 1.2.12. If V^* is separable, then so is the normed space V .

Theorem 1.2.13. Let V be a Banach space such that the dual space V^* is separable, then all bounded sets are metrizable in the weak topology of V .

Theorem 1.2.14. (*Banach-Alaoglu*) Let V be a normed space. The closed unit ball of V^* is compact with respect to the weak* topology.

Theorem 1.2.15. (*Riesz-Frechet Representation Theorem*) Consider Hilbert space H . Given any $\psi \in H^*$, there exists a unique $f \in H$ such that

$$\langle \psi, u \rangle = [f, u] \quad \forall u \in H \quad \text{and} \quad \|f\|_H = \|\psi\|_{H^*}.$$

1.3 Measure Theory

This section includes results derived from measure theory, which will be utilized in the construction of proofs in later chapters.

1.3.1 The Lebesgue Measure on \mathbb{R}^N

The Lebesgue outer measure of $E \subset \mathbb{R}^N$ is given as

$$\lambda^*(E) := \inf \left\{ \sum_{k=0}^{\infty} \text{vol}(Q_k) : \{Q_k\}_k \text{ are rectangles such that } E \subset \bigcup_{k=0}^{\infty} Q_k \right\}.$$

The following properties are satisfied:

- $\lambda^*(E) \geq 0 \quad \forall E \subset \mathbb{R}^N$,
- $\lambda^*(\emptyset) = \lambda^*({x}) = 0 \quad \forall x \in \mathbb{R}^N$,

- If $E \subset F$ then, $\lambda^*(E) \leq \lambda^*(F)$.

Definition 1.3.1. A subset E of \mathbb{R}^N is *Lebesgue measurable*, if for every subset F of \mathbb{R}^N , we have

$$\lambda^*(F) = \lambda^*(E \cap F) + \lambda^*(F \setminus E).$$

We then let \mathcal{M} denote the family of Lebesgue measurable subsets of \mathbb{R}^N .

Theorem 1.3.2. \mathcal{M} is a σ -algebra on \mathbb{R}^N ($\lambda := \lambda^*|_{\mathcal{M}}$ is called the *N-dimensional Lebesgue measure*).

Definition 1.3.3. Let (X, ξ) and (Y, Σ) be measurable spaces, then a function $f : X \rightarrow Y$ is said to be *measurable* if the preimage of E under f is in ξ for every $E \in \Sigma$. The *preimage* of a set $A \subset Y$ under f is the subset of X defined by $f^{-1}(A) = \{x \in X : f(x) \in A\}$.

Definition 1.3.4. (*The integral of a positive simple function*) Let (X, ξ, λ) be a measure space. Denote by $S_+(X)$ the class of positive simple functions defined on X . If $\phi \in S_+(X)$ with $\phi(x) = \sum_{i=1}^N \alpha_i \chi_{E_i}(x)$, $E_i \cap E_j = \emptyset$ for $i \neq j$, then set

$$\int_X \phi d\lambda = \sum_{i=1}^N \alpha_i \lambda(E_i) \quad (1.4)$$

with the convention that $\alpha_i \lambda(E_i) = 0$ if $\alpha_i = 0$ and $\lambda(E_i) = \infty$.

Definition 1.3.5. (*The integral of a positive measurable function*) Let (X, ξ, λ) be a measure space and let $f : X \rightarrow [0, +\infty)$ be a measurable function. The λ -integral of f is the non-negative number (eventually ∞)

$$\int_X f d\lambda = \sup \left\{ \int_X \phi d\lambda : \phi \text{ simple and } 0 \leq \phi \leq f \right\}. \quad (1.5)$$

Definition 1.3.6. Let (X, ξ, λ) be a measure space. A function $f : X \rightarrow \mathbb{R}$ is said to be λ -*integrable*, if f is measurable and at least one of the λ -integrals $\int_X f^+ d\lambda$ and $\int_X f^- d\lambda$ is finite. If f is λ -integrable, then we define the λ -integral of f by

$$\int_X f d\lambda = \int_X f^+ d\lambda - \int_X f^- d\lambda. \quad (1.6)$$

Definition 1.3.7. Let (X, ξ, λ) be a measure space. A function $f : X \rightarrow \mathbb{R}$ is said to be λ -*summable*, if f is λ -integrable and $\int_X f d\lambda$ is finite.

Theorem 1.3.8. (*Fatou's Lemma*) Let (X, ξ, λ) be a measure space. Consider a sequence of λ -measurable functions $f_n : X \rightarrow [0, +\infty)$, then,

$$\int_X \liminf_{n \rightarrow \infty} f_n d\lambda \leq \liminf_{n \rightarrow \infty} \int_X f_n d\lambda. \quad (1.7)$$

Theorem 1.3.9. (*Reverse Fatou's Lemma*) Let (X, ξ, λ) be a measure space and let $\{f_n\}_n \subset L^1(X)$, $f_n : X \rightarrow \mathbb{R}$ be a sequence of λ -summable functions. Suppose that there exists a summable $g : X \rightarrow \mathbb{R}$ such that for all $n \in \mathbb{N}$, $f_n \leq g$ pointwise. Then,

$$\limsup_{n \rightarrow \infty} \int_X f_n d\lambda \leq \int_X \limsup_{n \rightarrow \infty} f_n d\lambda. \quad (1.8)$$

The following theorem is an important result in measure theory and one which we will use often.

Theorem 1.3.10. (*Lebesgue Dominated Convergence Theorem*) Let (X, ξ, λ) be a measure space and $f_n : X \rightarrow \mathbb{R}$ a sequence of λ -measurable functions. Assume that $(f_n)_n$ converges pointwise to a function $f : X \rightarrow \mathbb{R}$, and that there exists a λ -summable function $g : X \rightarrow \mathbb{R}$ such that $|f_n| \leq g$ for any $n \in \mathbb{N}$. Then, f is λ -summable and

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| d\lambda = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \int_X f_n d\lambda = \int_X f d\lambda.$$

Corollary 1.3.11. (*Dominated Convergence in L^p , $1 \leq p < \infty$*) Let (X, ξ, λ) be a measure space and $(f_n)_n$ a sequence of λ -measurable functions $f_n : \Omega \rightarrow \mathbb{R}$. Assume that $(f_n)_n$ converges pointwise to a function $f : X \rightarrow \mathbb{R}$, and that there exists a $g \in L^p(X)$ such that $|f_n| \leq g$ for any $n \in \mathbb{N}$. Then, f_n and f are in $L^p(X)$ and

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{L^p(X)} = \lim_{n \rightarrow \infty} \left(\int_X |f_n - f|^p d\lambda \right)^{\frac{1}{p}} = 0.$$

Theorem 1.3.12. (*Lebesgue's Differentiation Theorem*) Let $f : \mathbb{R}^N \rightarrow \mathbb{R}$ be locally summable,

(i) Then, for a.e. point $x_0 \in \mathbb{R}^N$,

$$\int_{B(x_0, r)} f dx \rightarrow f(x_0) \quad \text{as } r \rightarrow 0.$$

(ii) Also, for a.e. point $x_0 \in \mathbb{R}^N$,

$$\int_{B(x_0, r)} |f(x) - f(x_0)| dx \rightarrow 0 \quad \text{as } r \rightarrow 0. \quad (1.9)$$

A point x_0 which satisfies (1.9) is called a *Lebesgue point* of f .

Definition 1.3.13. Let (X, ξ, λ) be a measure space and $f, f_n : X \rightarrow \mathbb{R}$ measurable functions. The sequence $(f_n)_n$ is said to *converge in measure* if for every $\epsilon > 0$ it follows that

$$\lim_{n \rightarrow \infty} \lambda(\{x \in X : |f(x) - f_n(x)| \geq \epsilon\}) = 0. \quad (1.10)$$

Theorem 1.3.14. (*Markov's Inequality*) Let (X, ξ, λ) be a measure space. If f is a measurable function and $\epsilon > 0$, then

$$\lambda(\{x \in X : |f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_X |f| d\lambda. \quad (1.11)$$

1.4 Function Spaces

Consider the mapping $u : \Omega \rightarrow \mathbb{R}$ on domain $\Omega \subset \mathbb{R}^N$. The following notational conventions will be adopted, $\alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{Z}_+^N$ denotes a multi-index, $|\alpha| = \alpha_1 + \dots + \alpha_N$ so that

$$\nabla^\alpha u(x) := \frac{\partial^{|\alpha|} u(x)}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}},$$

and if m is a non-negative integer then

$$\nabla^m u(x) := \{ \nabla^\alpha u(x) : |\alpha| = m \}.$$

Additionally, we have $\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N} \right)$, which is the *gradient vector*.

For any non-negative integer m , we let $C^m(\Omega)$ denote the vector space, which is comprised of functions ψ , with its partial derivatives $\nabla^\alpha \psi$ of orders $|\alpha| \leq m$, which are continuous on Ω . The convention is to denote $C^0(\Omega)$ as $C(\Omega)$ and to take $C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$. The subspaces $C_c(\Omega)$ and $C_c^\infty(\Omega)$ consist of the functions in $C(\Omega)$ and $C^\infty(\Omega)$ that have compact support in Ω .

1.4.1 Bounded, Continuous Functions

We consider $C_B^m(\Omega)$ to be the vector space which consists of functions $\psi \in C^m(\Omega)$ for which $\nabla^\alpha \psi$ is bounded on Ω for $0 \leq |\alpha| \leq m$. Furthermore, we have that $C_B^m(\Omega)$ is a Banach space when considering it as a normed space endowed with the norm

$$\|\psi\|_{C_B^m(\Omega)} := \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} |\nabla^\alpha \psi(x)|.$$

1.4.2 Bounded, Uniformly Continuous Functions

We know that if $\psi \in C(\Omega)$ is bounded and uniformly continuous on Ω , then it has a unique bounded continuous extension to the closure $\overline{\Omega}$ of Ω . So we define the vector space $C^m(\overline{\Omega})$ as consisting of the functions $\psi \in C^m(\Omega)$, for which $\nabla^\alpha \psi$ is bounded and uniformly continuous on Ω , where $0 \leq |\alpha| \leq m$. It follows then that when considering the same norm as previously,

$$\|\psi\|_{C^m(\overline{\Omega})} := \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} |\nabla^\alpha \psi(x)|,$$

that this is also a Banach space.

1.4.3 Hölder Continuous Functions

Let $0 < \beta \leq 1$, we define $C^{m,\beta}(\overline{\Omega})$ as the subspace of $C^m(\overline{\Omega})$ which contains the functions ψ , such that $\nabla^\alpha \psi$ satisfies a Hölder condition for $0 \leq |\alpha| \leq m$ with exponent β , i.e., there exists a constant $K \geq 0$ such that

$$|\nabla^\alpha \psi(x) - \nabla^\alpha \psi(y)| \leq K|x - y|^\beta \quad \forall x, y \in \Omega.$$

Then, $C^{m,\beta}(\overline{\Omega})$ is a Banach space with norm defined as

$$\|\psi\|_{C^{m,\beta}(\overline{\Omega})} := \|\psi\|_{C^m(\overline{\Omega})} + \max_{0 \leq |\alpha| \leq m} \sup_{x, y \in \Omega, x \neq y} \frac{|\nabla^\alpha \psi(x) - \nabla^\alpha \psi(y)|}{|x - y|^\beta}.$$

Definition 1.4.1. Let (V, d_V) and (W, d_W) be two metric spaces and F a family of functions from (V, d_V) to (W, d_W) . The family F is *equicontinuous* at a point $x_0 \in V$, if for every $\epsilon > 0$, there exists a $\delta > 0$, such that $d_W(f(x_0), f(x)) < \epsilon$ for all $f \in F$ and all x such that $d_V(x_0, x) < \delta$. The *family is equicontinuous* if it is equicontinuous at each point in V .

Theorem 1.4.2. (*Arzela-Ascoli Theorem*) If a sequence $\{f_n\}_n$ in $C(V)$ is bounded and equicontinuous, then it has a uniformly convergent subsequence.

Lemma 1.4.3. (*Mac Shane's Lemma*) Let (V, d_V) be a metric space.

Let $f : A \subset (V, d_V) \rightarrow \mathbb{R}$ be a Lipschitz continuous function, i.e. $\exists K > 0$ such that

$$|f(x) - f(y)| \leq K d_V(x, y) \quad \forall x, y \in A,$$

then $\exists \bar{f} : V \rightarrow \mathbb{R}$ such that $\text{lip}(\bar{f}) \leq K$, where $\text{lip}(\bar{f})$ denotes the Lipschitz constant of \bar{f} .

Theorem 1.4.4. (*Rademacher's Theorem*) If we have an open set $A \subset \mathbb{R}^N$ and $f : A \rightarrow \mathbb{R}^M$ is Lipschitz continuous, then f is differentiable almost everywhere on A .

1.4.4 Lebesgue Spaces

Lebesgue spaces and its associated results, will play a prominent role in this thesis, as it is an essential concept in the framework of Sobolev spaces. This will be demonstrated in the following section.

Definition 1.4.5. Let (Ω, ξ, λ) denote a measure space with $\Omega \subset \mathbb{R}^N$, and let λ denote the Lebesgue measure. For $1 \leq p < \infty$, set

$$L^p(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{R} : f \text{ is measurable and } \left(\int_{\Omega} |f|^p d\lambda \right)^{\frac{1}{p}} < \infty \right\}.$$

With norm defined by

$$\|f\|_{L^p(\Omega)} := \left(\int_{\Omega} |f|^p d\lambda \right)^{\frac{1}{p}}$$

and

$$L^\infty(\Omega) := \left\{ f : \Omega \rightarrow \mathbb{R} : f \text{ is measurable and } \exists C \text{ such that } |f(x)| \leq C \text{ a.e. on } \Omega \right\},$$

with norm defined by

$$\|f\|_\infty := \inf \left\{ \alpha > 0 : \lambda(\{x \in \Omega : |f(x)| \geq \alpha\}) = 0 \right\}.$$

It can be shown that $L^p(\Omega)$ and $L^\infty(\Omega)$ are normed vector spaces

Theorem 1.4.6. (*Fischer-Riesz*) $L^p(\Omega)$ is a Banach space for any p , $1 \leq p \leq \infty$.

Theorem 1.4.7. $L^p(\Omega)$ is reflexive for any p , $1 < p < \infty$.

Theorem 1.4.8. $L^p(\Omega)$ is separable for any p , $1 \leq p < \infty$.

Theorem 1.4.9. (*Riesz representation theorem*) Let $\phi \in (L^p(\Omega))^*$, where $1 < p < \infty$. Then, there exists a unique function $u \in L^{p'}(\Omega)$ (where p' is the conjugate exponent defined as $\frac{1}{p} + \frac{1}{p'} = 1$) such that

$$\langle \psi, f \rangle = \int_{\Omega} u f d\lambda \quad \forall f \in L^p(\Omega) \quad \text{and} \quad \|u\|_{L^{p'}(\Omega)} = \|\psi\|_{(L^p(\Omega))^*}.$$

Remark 3. This result can be used to show the existence of an isometric isomorphism between $(L^p(\Omega))^*$ and $L^{p'}(\Omega)$, thus allowing us to identify these two normed spaces $((L^p(\Omega))^* = L^{p'}(\Omega))$. Using the Riesz representation theorem, we can similarly identify $(L^1(\Omega))^*$ with $L^\infty(\Omega)$ $((L^1(\Omega))^* = L^\infty(\Omega))$.

Proposition 1.4.10. (*Modes of Convergence*)

- Any sequence converging a.e. converges also in measure.
- Any sequence converging in $L^1(\Omega)$ converges in measure.
- Any sequence converging in measure, admits a subsequence converging a.e.

Proposition 1.4.11. Every convergent sequence $u_n \rightarrow u$ in $L^p(\Omega)$ has a pointwise convergent subsequence $u_{n_k}(x) \rightarrow u(x)$ a.e. $x \in \Omega$. Furthermore, there exists an $h \in L^p(\Omega)$, where $h(x) \geq 0$ and $|u_{n_k}(x)| \leq h(x)$ a.e. $x \in \Omega$.

Definition 1.4.12. Let (X, ξ, λ) be a measure space. A set $Y \subset L^1(X)$ is called *uniformly integrable* if for each $\epsilon > 0$ there corresponds a $\delta > 0$ such that

$$\left| \int_E f d\lambda \right| < \epsilon, \quad (1.12)$$

whenever $f \in Y$ and $\lambda(E) < \delta$.

Theorem 1.4.13. (*Vitali's Convergence Theorem*) Let (X, ξ, λ) be a measure space. If

- $\lambda(X) < \infty$,
- $\{f_n\}_n$ is uniformly integrable,
- $f_n(x) \rightarrow f(x)$ a.e. as $n \rightarrow \infty$,
- $|f(x)| < \infty$ a.e.,

then $f \in L^1(X)$ and

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| d\lambda = 0. \quad (1.13)$$

1.5 Sobolev Spaces

Sobolev spaces, presented below, have major significance in the development of the approach for proving the existence of weak solutions to elliptic boundary value problems which we will describe. Consider the domain as being an open connected set $\Omega \subset \mathbb{R}^N$. In the next chapters, for brevity, we will adopt the convention of $\Gamma := \partial\Omega$, while later in Chapter 4, when dealing with boundaries of several domains at once, we will refer to the boundary simply as $\partial\Omega$.

1.5.1 Weak Derivatives

Consider the multi-index α . Suppose that $u, v \in L^1_{loc}(\Omega)$ and

$$\int_{\Omega} u(x) \nabla^{\alpha} \psi(x) dx = (-1)^{|\alpha|} \int_{\Omega} v(x) \psi(x) dx \quad \text{for all } \psi \in C_c^{\infty}(\Omega).$$

Then v is the α^{th} – *weak derivative* of u in Ω and we denote it as $\nabla^{\alpha} u$.

To show that the classical derivative of u is also the weak derivative, we assume u is smooth. Then by integration by parts we get

$$\int_{\Omega} u(x) \nabla^{\alpha} \psi(x) dx = \int_{\Omega} (-1)^{|\alpha|} \nabla^{\alpha} u(x) \psi(x) dx \quad \text{for every } \psi \in C_c^{\infty}(\Omega).$$

The weak derivative has properties of both uniqueness and linearity. We are now in a position to define Sobolev spaces.

1.5.2 Sobolev Spaces

Below, a general definition of Sobolev spaces is provided, following this, we will consider only the case for $k = 1$.

Definition 1.5.1. For $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, we define a *Sobolev space* as:

$$W^{k,p}(\Omega) := \left\{ u \in L^p(\Omega) : \nabla^{\alpha} u \in L^p(\Omega) \text{ for all } |\alpha| \leq k \right\}. \quad (1.14)$$

If $u \in W^{k,p}(\Omega)$ then we define its norm to be

$$\|u\|_{W^{k,p}(\Omega)} := \begin{cases} \left(\sum_{|\alpha| \leq k} \|\nabla^{\alpha} u\|_{L^p(\Omega)}^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < +\infty, \\ \max_{|\alpha| \leq k} \|\nabla^{\alpha} u\|_{L^{\infty}(\Omega)} & \text{if } p = +\infty. \end{cases} \quad (1.15)$$

At this point, we define an important subspace in $W^{k,p}(\Omega)$, that of $W_0^{k,p}(\Omega)$.

Definition 1.5.2. The space $W_0^{k,p}(\Omega)$ is the closure of $C_c^{\infty}(\Omega)$ in $W^{k,p}(\Omega)$, with respect to the norm $\|\cdot\|_{W^{k,p}}$, where $1 \leq p < \infty$.

Remark 4. If $p = 2$, then the notation $H^k(\Omega) := W^{k,2}(\Omega)$ is used.

The following are some properties of the weak derivative:

Suppose that $u, v \in W^{k,p}(\Omega)$ and $|\alpha| \leq k$. It follows then that

- $\nabla^{\alpha} u \in W^{k-|\alpha|,p}(\Omega)$ and $\nabla^{\beta}(\nabla^{\alpha} u) = \nabla^{\alpha}(\nabla^{\beta} u) = \nabla^{\alpha+\beta} u$ for all multi-indices α and β , where $|\alpha| + |\beta| \leq k$.
- For each $\lambda_1, \lambda_2 \in \mathbb{R}$ it follows that $\lambda_1 u + \lambda_2 v \in W^{k,p}(\Omega)$ and further, $\nabla^{\alpha}(\lambda_1 u + \lambda_2 v) = \lambda_1 \nabla^{\alpha} u + \lambda_2 \nabla^{\alpha} v$, where $|\alpha| \leq k$.
- If O is an open subset of Ω then, $u \in W^{k,p}(O)$.

- If $\psi \in C_c^\infty(\Omega)$, then $\psi u \in W^{k,p}(\Omega)$ and

$$\nabla^\alpha(\psi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \nabla^\beta \psi \nabla^{\alpha-\beta} u \quad (\text{Leibniz Formula}), \quad \text{where} \quad \binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}.$$

From this point forth, we will consider only the case where $k = 1$.

Definition 1.5.3. The space $W_0^{1,\infty}(\Omega)$ is a subset of $W^{1,\infty}(\Omega)$, consisting of $f \in W^{1,\infty}(\Omega)$, such that for every $a \in \Gamma$ it follows that $\lim_{x \rightarrow a} f(x) = 0$.

Some important Sobolev space results will be presented in the next section.

Theorem 1.5.4. Sobolev space $W^{1,p}(\Omega)$ is a Banach space for $1 \leq p \leq \infty$.

Theorem 1.5.5. Sobolev space $W^{1,p}(\Omega)$ is uniformly convex (hence reflexive) for $1 < p < \infty$.

Theorem 1.5.6. Sobolev space $W^{1,p}(\Omega)$ is separable for $1 \leq p < \infty$.

Smooth functions play an important role in approximating functions in Sobolev spaces, used both as density results and as a local approximation. For local approximations, we consider the tool of mollifications¹.

Theorem 1.5.7. For $u \in W^{1,p}(\Omega)$ and $1 \leq p < \infty$, consider

$$u^\epsilon := \eta_\epsilon * u \quad \text{in } \Omega_\epsilon,$$

where $\Omega_\epsilon := \{x \in \Omega : \text{dist}(x, \Gamma) > \epsilon\}$. Then we have that

- $u^\epsilon \in C^\infty(\Omega_\epsilon)$ for each $\epsilon > 0$,
- $u^\epsilon \rightarrow u$ in $W_{loc}^{1,p}(\Omega)$ as $\epsilon \rightarrow 0$.

It is sometimes useful to approximate $u \in W^{1,p}(\Omega)$ by smooth functions. We now discuss the results from which these approximations derive.

Theorem 1.5.8. Take $u \in W^{1,p}(\Omega)$, where $1 \leq p < \infty$, and suppose that Γ is bounded. It follows that there exist functions $u_n \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$ such that

$$u_n \rightarrow u \quad \text{in } W^{1,p}(\Omega).$$

Further, we look at presenting an approximation for $W^{1,p}(\Omega)$ functions by $C^\infty(\overline{\Omega})$ functions.

Theorem 1.5.9. Take $u \in W^{1,p}(\Omega)$, where $1 \leq p < \infty$, and suppose that Ω is bounded and boundary Γ is C^1 . It follows that there exist functions $u_n \in C^\infty(\overline{\Omega})$ such that

$$u_n \rightarrow u \quad \text{in } W^{1,p}(\Omega).$$

We now look at an extension result. The result considers the extension of functions in the Sobolev space $W^{1,p}(\Omega)$ to functions in $W^{1,p}(\mathbb{R}^N)$. The result holds for $1 \leq p \leq \infty$.

¹See section 7.2 in the appendix for a discussion of mollifications and their properties.

Theorem 1.5.10. *Assume that Ω is bounded and Γ is C^1 . It follows then that there exists a linear extension operator*

$$S : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^N)$$

for all $u \in W^{1,p}(\Omega)$.

As will become clear, when using Sobolev spaces as our setting for investigating solutions to partial differential equations, it is important to define how a function behaves on the boundary of the given domain. By the definition of a Sobolev space, functions are defined only almost everywhere and are not necessarily continuous. Assuming that Γ is Lipschitz or C^1 (we thus have that Γ has Lebesgue measure zero), we seek to establish a meaning for $u \in W^{1,p}(\Omega)$ on the boundary Γ . The theorem which follows is useful in this regard.

Theorem 1.5.11. *There exists exactly one linear continuous operator $T : W^{1,p}(\Omega) \rightarrow L^1(\Gamma)$, such that for any $u \in C^1(\overline{\Omega})$ it follows that $T(u) = u|_{\Gamma}$. Furthermore, T is continuous (respectively compact) as the mappings*

$$T : W^{1,p}(\Omega) \rightarrow L^{p^\#}(\Gamma),$$

respectively,

$$T : W^{1,p}(\Omega) \rightarrow L^{p^\#-\epsilon}(\Gamma), \quad \epsilon \in (0, p^\# - 1], \quad (1.16)$$

where,

$$p^\# := \begin{cases} \frac{Np-p}{N-p} & \text{if } p < N, \\ \text{an arbitrarily large real} & \text{if } p = N, \\ +\infty & \text{if } p > N, \end{cases} \quad (1.17)$$

so that it holds that $\|T(u)\|_{L^{p^\#}(\Gamma)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for each $u \in W^{1,p}(\Omega)$, where the constant C depends on p and Ω . This is called the trace operator.

Finally, we look at some embeddings of Sobolev spaces. There are a number of Sobolev space related inequalities, depending on the defined parameters such as the value of p in relation to the dimension N . We look at some examples of these inequalities, restricting ourselves essentially to those which are used in the development of the topic.

Theorem 1.5.12. *The continuous embedding*

$$W^{1,p}(\Omega) \subset L^{p^*}(\Omega), \quad (1.18)$$

holds for the exponent p^* defined as

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ \text{an arbitrarily large real} & \text{if } p = N, \\ +\infty & \text{if } p > N. \end{cases} \quad (1.19)$$

Theorem 1.5.13. *The compact embedding (the Rellich-Kondrachev Theorem)*

$$W^{1,p}(\Omega) \Subset L^{p^*-\epsilon}(\Omega), \quad \epsilon \in (0, p^* - 1] \quad (1.20)$$

holds for p^* .

Theorem 1.5.14. (*Poincarè type inequality*) Suppose that $\Omega \subset \mathbb{R}^N$ is open and bounded. For each $q \in [1, p^*]$, there exists a constant $C_p < \infty$ such that

$$\|u\|_{W^{1,p}(\Omega)} \leq C_p (\|\nabla u\|_{L^p(\Omega; \mathbb{R}^N)} + \|u\|_{L^q(\Omega)}), \quad (1.21)$$

Theorem 1.5.15. (*Morrey's inequality*) Assume $N < p \leq \infty$. There exists a constant C , depending on p and N , such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^N)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^N)}, \quad (1.22)$$

for all $u \in C^1(\mathbb{R}^N)$, where $\gamma := 1 - \frac{N}{p}$.

Theorem 1.5.16. Assume $N < p \leq \infty$. Let Ω be a bounded, open subset of \mathbb{R}^N , where Γ is C^1 , and $u \in W^{1,p}(\Omega)$. Then, u has a continuous representative $u^* \in C^{0,\gamma}(\overline{\Omega})$ for $\gamma = 1 - \frac{N}{p}$, with the estimate

$$\|u^*\|_{C^{0,\gamma}(\overline{\Omega})} \leq C \|u\|_{W^{1,p}(\Omega)}, \quad (1.23)$$

where the constant C depends only on p, N and Ω .

Theorem 1.5.17. Consider $f : \Omega \rightarrow \mathbb{R}$, then f is locally Lipschitz in Ω if and only if $f \in W_{loc}^{1,\infty}(\Omega)$.

1.5.3 Anisotropic Sobolev Spaces

Definition 1.5.18. Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Take $p_0 > 1$ and $\vec{p} = (p_1, \dots, p_N)$ as a vector consisting of real numbers with $p_i > 1$. We define an *anisotropic Sobolev space* as

$$W^{1,\vec{p}}(\Omega) := \left\{ u \in L^{p_0}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega) \text{ for all } i \in \{1, \dots, N\} \right\}. \quad (1.24)$$

For $u \in W^{1,\vec{p}}(\Omega)$, we define its norm to be

$$\|u\|_{1,\vec{p},p_0} := \left(\int_{\Omega} |u(x)|^{p_0} dx \right)^{\frac{1}{p_0}} + \sum_{i=1}^N \left(\int_{\Omega} \left| \frac{\partial u(x)}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}}. \quad (1.25)$$

Theorem 1.5.19. The anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ endowed with the norm $\|\cdot\|_{1,\vec{p},p_0}$ is a separable reflexive Banach space, for $1 < p_i < \infty$, where $i \in \{0, \dots, N\}$.

Definition 1.5.20. The space $W_0^{1,\vec{p}}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$, with respect to the norm $\|\cdot\|_{1,\vec{p},p_0}$.

Theorem 1.5.21. The space $W_0^{1,\vec{p}}(\Omega)$ is a reflexive Banach space for any $\vec{p} = (p_1, \dots, p_N)$, with $1 < p_i < \infty$, where $i \in \{1, \dots, N\}$.

Theorem 1.5.22. The function space $C_c^\infty(\Omega)$ is dense in $W_0^{1,\vec{p}}(\Omega)$.

Remark 5. We denote $W^{-1,\vec{p}'}(\Omega)$ as the dual space of the anisotropic Sobolev space $W_0^{1,\vec{p}}(\Omega)$, where $\vec{p}' = \{p'_1, \dots, p'_N\}$.

Theorem 1.5.23. *The norm $\|u\| := \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}$ defines a norm on $W_0^{1, \vec{p}}(\Omega)$, which is equivalent to the norm $\|\cdot\|_{1, \vec{p}, p_0}$.*

Theorem 1.5.24. *The compact embedding*

$$W_0^{1, \vec{p}}(\Omega) \Subset L^q(\Omega), \quad (1.26)$$

holds for $1 \leq q < p^\#$, where $p^\# := \frac{N\bar{p}}{N-\bar{p}}$ and $\frac{1}{\bar{p}} := \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i}$.

We have now established sufficient preliminary material to begin the development of the abstract arguments.

Chapter 2

Monotone and Pseudomonotone Operators

2.1 Abstract Theory

The abstract theory of monotone type operators will serve as the foundation for our later arguments involving existence results of boundary value problems. Thus, it becomes important to gain a strong understanding of these abstract ideas and the nature of the monotone type operators. We investigate these results in this chapter. Theorem 2.1.11 and Theorem 2.1.15 are results of particular importance for the development of the thesis topic.

2.1.1 Various Types of Monotonicity

Let $(V, \|\cdot\|)$ be a reflexive Banach space (we will assume this is the case throughout the chapter, unless stated otherwise). Let V^* denote its dual space, with norm denoted by $\|\cdot\|_{V^*}$, while $\langle \cdot, \cdot \rangle$ denotes the duality brackets between V^* and V .

Definition 2.1.1. Consider the operator $A : V \rightarrow V^*$, then,

- A is said to be *monotone* if $\langle A(u) - A(v), u - v \rangle \geq 0 \quad \forall u, v \in V$.
- A is said to be *strictly monotone* if $\langle A(u) - A(v), u - v \rangle > 0 \quad \forall u, v \in V, \quad u \neq v$.
- A is said to be *d -monotone with respect to the seminorm $|\cdot|$* , where $d : \mathbb{R}^+ \rightarrow \mathbb{R}$ is an increasing function, if

$$\langle A(u) - A(v), u - v \rangle \geq (d(|u|) - d(|v|))(|u| - |v|) \quad \forall u, v \in V. \quad (2.1)$$

If $|\cdot| = \|\cdot\|$ is the norm on V , then A is simply *d -monotone*.

- A is said to be *uniformly monotone* if for some increasing continuous function $\gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ it follows that

$$\langle A(u) - A(v), u - v \rangle \geq \gamma(\|u - v\|)\|u - v\|. \quad (2.2)$$

If $\gamma(r) = \delta r$ for some $\delta > 0$, then we say that A is *strongly monotone*.

Remark 6. A strongly monotone $\implies A$ strictly monotone $\implies A$ is monotone.

Consider the function $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$ defined by $f(z) = |z|^{p-2}z$. We seek to show that this function is uniformly monotone by proving the following proposition:

Proposition 2.1.2. $\exists C > 0$ such that $[f(\zeta) - f(\eta), \zeta - \eta] \geq C\|\zeta - \eta\|^p$.

Proof: Consider the components of the function f ,

$$\phi_i(\zeta) := |\zeta|^{p-2}\zeta_i = \left(\sum_{j=1}^N \zeta_j^2 \right)^{\frac{p-2}{2}} \zeta_i,$$

the partial derivative of which is

$$\begin{aligned} \frac{\partial \phi_i}{\partial \zeta_j}(\zeta) &= \frac{(p-2)}{2} \left(\sum_{j=1}^N \zeta_j^2 \right)^{\frac{p-2}{2}-1} (2\zeta_j) \zeta_i + \left(\sum_{j=1}^N \zeta_j^2 \right)^{\frac{p-2}{2}} \delta_{ij} \\ &= (p-2)|\zeta|^{p-4}\zeta_i\zeta_j + |\zeta|^{p-2}\delta_{ij}. \end{aligned}$$

We now define the function $g(t) := \phi_i(\eta + t(\zeta - \eta))$. Then, using the fundamental theorem of calculus yields

$$g(1) = g(0) + \int_0^1 g'(t) dt,$$

where $g'(t)$ denotes the derivative of g with respect to t , hence,

$$\begin{aligned} g'(t) &= \nabla \phi_i(\eta + t(\zeta - \eta)) \cdot (\zeta - \eta) = \sum_{j=1}^N \frac{\partial \phi_i}{\partial \zeta_j}(\eta + t(\zeta - \eta))(\zeta_j - \eta_j) \\ &= \sum_{j=1}^N (p-2)|\eta + t(\zeta - \eta)|^{p-4}(\eta_j + t(\zeta_j - \eta_j))(\zeta_j - \eta_j)(\eta_i + t(\zeta_i - \eta_i)) \\ &\quad + |\eta + t(\zeta - \eta)|^{p-2}(\zeta_i - \eta_i). \end{aligned}$$

Therefore,

$$\begin{aligned} \phi_i(\zeta) - \phi_i(\eta) &= (p-2) \int_0^1 \sum_{j=1}^N |\eta + t(\zeta - \eta)|^{p-4}(\eta_j + t(\zeta_j - \eta_j))(\zeta_j - \eta_j)(\eta_i + t(\zeta_i - \eta_i)) dt \\ &\quad + \int_0^1 |\eta + t(\zeta - \eta)|^{p-2}(\zeta_i - \eta_i) dt. \end{aligned}$$

This results in

$$\begin{aligned} [f(\zeta) - f(\eta), \zeta - \eta] &= \sum_{i=1}^N (\phi_i(\zeta) - \phi_i(\eta))(\zeta_i - \eta_i) \\ &= (p-2) \int_0^1 |\eta + t(\zeta - \eta)|^{p-4} ((\eta + t(\zeta - \eta)) \cdot (\zeta - \eta))^2 dt \\ &\quad + \int_0^1 |\eta + t(\zeta - \eta)|^{p-2} |\zeta - \eta|^2 dt, \end{aligned}$$

where $[\cdot, \cdot]$ denotes the scalar product. Since

$$(p-2) \int_0^1 |\eta + t(\zeta - \eta)|^{p-4} ((\eta + t(\zeta - \eta)) \cdot (\zeta - \eta))^2 dt \geq 0,$$

ignoring this term yields,

$$[f(\zeta) - f(\eta), \zeta - \eta] \geq \int_0^1 |\eta + t(\zeta - \eta)|^{p-2} |\zeta - \eta|^2 dt.$$

Now use the result that $\forall \alpha > 0, \exists C > 0, \forall a_1, \dots, a_N \geq 0$ such that

$$\left(\sum_{i=1}^N a_i \right)^\alpha \geq C \sum_{i=1}^N a_i^\alpha.$$

Applying this result yields,

$$\int_0^1 |\eta + t(\zeta - \eta)|^{p-2} |\zeta - \eta|^2 dt \geq C \sum_{j=1}^N \left(\int_0^1 |\eta_j + t(\zeta_j - \eta_j)|^{p-2} dt \right) |\zeta_j - \eta_j|^2.$$

We now want to prove a lemma which shows the existence of a constant $D > 0$, such that

$$\int_0^1 |\eta_j + t(\zeta_j - \eta_j)|^{p-2} dt \geq D |\zeta_j - \eta_j|^{p-2}.$$

This would result in

$$\sum_{j=1}^N \left(\int_0^1 |\eta_j + t(\zeta_j - \eta_j)|^{p-2} dt \right) |\zeta_j - \eta_j|^2 \geq D \sum_{j=1}^N |\zeta_j - \eta_j|^p.$$

By equivalence of norms on \mathbb{R}^N , it then follows that for some constant $C' > 0$

$$\sum_{j=1}^N \left(\int_0^1 |\eta_j + t(\zeta_j - \eta_j)|^{p-2} dt \right) |\zeta_j - \eta_j|^2 \geq C' \|\zeta - \eta\|^p,$$

where we take $\|\cdot\| := |\cdot|$ as the usual norm on \mathbb{R}^N . The desired result follows. We need the following:

Lemma 2.1.3. *For all $i \in \{1, \dots, N\}$, $\exists D > 0$ depending only on p , such that*

$$\int_0^1 |\eta_i + t(\zeta_i - \eta_i)|^{p-2} dt \geq D |\zeta_i - \eta_i|^{p-2}.$$

Proof: This follows easily if $\zeta_i = \eta_i$. Assume that $\zeta_i - \eta_i \neq 0$. Set $d = \frac{\eta_i}{\zeta_i - \eta_i}$, we want to prove that

$$\int_0^1 |d + t|^{p-2} dt \geq D.$$

Three cases will be considered:

First assume that $0 < -d < 1$, then

$$\begin{aligned} \int_0^1 |d+t|^{p-2} dt &= \int_0^{-d} (-d-t)^{p-2} dt + \int_{-d}^1 (d+t)^{p-2} dt \\ &= \left(-\frac{1}{p-1} (-d-t)^{p-1} \right)_0^{-d} + \left(\frac{1}{p-1} (d+t)^{p-1} \right)_{-d}^1 \\ &= \frac{(-d)^{p-1} + (d+1)^{p-1}}{p-1} \geq \frac{1}{(p-1)2^{p-2}} > 0. \end{aligned}$$

The penultimate inequality follows from the convexity of the function $q(t) := t^\alpha$, where $\alpha > 1$.

For the second case assume that $d \geq 0$, then

$$\int_0^1 |d+t|^{p-2} dt = \int_0^1 (d+t)^{p-2} dt \geq \int_0^1 t^{p-2} dt = \frac{1}{p-1} > 0.$$

Lastly, assume that $d \leq -1$, therefore $d+t \leq 0$,

$$\int_0^1 |d+t|^{p-2} dt = \int_0^1 (-d-t)^{p-2} dt \geq \int_0^1 (1-t)^{p-2} dt.$$

Using the change of variable $u = 1-t$ yields

$$-\int_1^0 u^{p-2} du = \int_0^1 u^{p-2} du = \frac{1}{p-1}.$$

We find that

$$\int_0^1 |d+t|^{p-2} dt \geq \min \left\{ \frac{1}{p-1}, \frac{1}{(p-1)2^{p-2}} \right\} = \frac{1}{(p-1)2^{p-2}} = D > 0.$$

In conclusion, we have obtained a constant $D > 0$ (depending only on p) such that

$$[f(\zeta) - f(\eta), \zeta - \eta] \geq D \sum_{i=1}^N |\zeta_i - \eta_i|^p \geq C' \|\zeta - \eta\|^p \quad \forall \zeta, \eta \in \mathbb{R}^N.$$

□

Remark 7. $[f(\zeta) - f(\eta), \zeta - \eta] \geq h(\|\zeta - \eta\|)\|\zeta - \eta\|$, where $h(t) := C't^{p-1}$. Since h is an increasing, continuous functions, it follows that f is uniformly monotone.

Next, we consider a collection of definitions regarding the types of continuity of the operator $A : V \rightarrow V^*$.

2.1.2 Continuity Modes

- $A : V \rightarrow V^*$ is said to be *hemicontinuous* if A is directionally weakly continuous (i.e. $\forall u, v, w \in V : t \mapsto \langle A(u+tv), w \rangle$ is continuous). If this holds only when $v = w$ (i.e. $\forall u, v \in V : t \mapsto \langle A(u+tv), v \rangle$ is continuous), then A is said to be *radially continuous*.

- $A : V \rightarrow V^*$ is said to be *demicontinuous* if A is continuous as the operator $A : (V, \text{norm}) \rightarrow (V^*, \text{weak}^1)$ (i.e. $\forall v \in V$ the functional $u \mapsto \langle A(u), v \rangle$ is continuous).
- $A : V \rightarrow V^*$ is said to be *weakly continuous* if $\forall w \in V$ the functional $u \mapsto \langle A(u), w \rangle$ is weakly continuous (i.e. A is continuous as an operator $A : (V, \text{weak}) \rightarrow (V^*, \text{weak})$).
- $A : V \rightarrow V^*$ is said to be *strongly continuous* if it is continuous as an operator $A : (V, \text{weak}) \rightarrow (V^*, \text{norm})$.

Remark 8. A strongly continuous $\implies A$ demicontinuous $\implies A$ hemicontinuous.

2.1.3 Brouwer's Fixed-Point Theorem

We briefly discuss Brouwer's fixed point theorem and provide a concise proof. The proof relies on a topological result known as the Negative Retract Principle.

Negative Retract Principle: Let $\overline{B^N}$ be a closed unit ball in \mathbb{R}^N . There is no continuous mapping, say, $r : \overline{B^N} \rightarrow \partial B^N$, from the closed ball onto its boundary, such that the boundary is left pointwise fixed i.e $r(x) = x \quad \forall x \in \partial B^N$.

Theorem 2.1.4. *Every continuous mapping $f : \overline{B^N} \rightarrow \overline{B^N}$ has a fixed point.*

Proof: Assume that f is continuous and for every $x \in \overline{B^N}$, it follows that $f(x) \neq x$. We can construct a continuous mapping $r : \overline{B^N} \rightarrow \partial B^N$ which leaves the boundary fixed and thereby sets a contradiction. For any $x \in \overline{B^N}$, follow the directed line segment from $f(x)$ through x to its intersection with ∂B^N , let this point of intersection be $r(x)$. The continuity of this map follows from the continuity of f and satisfies $r(x) = x$ for all $x \in \partial B^N$ as follows:

Since translations are continuous, we can assume that the closed unit ball $\overline{B^N}$ is centred at the origin. Again, denote the scalar product as $[\cdot, \cdot]$ and consider

$$r(x) := f(x) + \lambda_x(x - f(x)),$$

with $\lambda_x \geq 0$, such that

$$\|f(x) + \lambda_x(x - f(x))\| = 1.$$

It follows then that $\|f(x) + \lambda_x(x - f(x))\|^2 = 1$, and it expands to

$$\|x - f(x)\|^2 \lambda_x^2 + 2[f(x), x - f(x)] \lambda_x + \|f(x)\|^2 - 1 = 0.$$

Then, applying the quadratic formula yields a non-negative λ_x , such that

$$\lambda_x = \frac{-[f(x), x - f(x)] + \sqrt{[f(x), x - f(x)]^2 - \|x - f(x)\|^2(\|f(x)\|^2 - 1)}}{\|x - f(x)\|^2}.$$

It can be observed that the denominator is never zero by our assumption on f . The term inside the square root is always non-negative and greater than $[f(x), x - f(x)]^2$, so $\lambda_x \geq 0$. It also follows easily that λ_x depends continuously on x and so $r(x)$ is also continuous. This contradicts the Negative Retract Principle. \square

¹Note that since V is reflexive, the weak and weak* topology on V^* coincide.

Corollary 2.1.5. *Let $M \subset \mathbb{R}^N$ be homeomorphic to the closed unit ball $\overline{B^N}$. It follows then that every continuous map $f : M \rightarrow M$ has a fixed point.*

Proof: Let $\phi : \overline{B^N} \rightarrow M$ be a homeomorphism. Consider the continuous map defined by

$$g := \phi^{-1} \circ f \circ \phi : \overline{B^N} \rightarrow \overline{B^N}.$$

By Brouwer's fixed-point theorem, we get $g(x_*) = x_*$. Applying ϕ to both sides of the equation yields

$$f(\phi(x_*)) = \phi(x_*),$$

thus, $\phi(x_*) \in M$ is a fixed point. \square

Remark 9. Examples of sets which are homeomorphic to the closed unit ball in \mathbb{R}^N , are non-empty, convex, compact sets in \mathbb{R}^N .

An important application of the Brouwer fixed-point theorem is that it can be used to show that there exists solutions to a system of equations:

$$g_i(x) = 0, \quad x \in \mathbb{R}^N, \quad i \in \{1, \dots, N\}, \quad (2.3)$$

where $g_i : \mathbb{R}^N \rightarrow \mathbb{R}$ are taken as continuous nonlinear mappings such that

$$\exists R > 0 : \sum_{i=1}^N g_i(x)x_i \geq 0 \quad \forall x \in \mathbb{R}^N \text{ with } |x| = R. \quad (2.4)$$

Lemma 2.1.6. *Let $g_i : \mathbb{R}^N \rightarrow \mathbb{R}$ be continuous functions for $i \in \{1, \dots, N\}$. Let g_i also satisfy (2.4). There then exists a solution $x_o \in \mathbb{R}^N$ of (2.3) with $|x_o| < R$.*

Proof: Assume that $g(x) = (g_1(x_1), \dots, g_N(x_N)) \neq 0$ for all $x \in \overline{B(0, R)} \subset \mathbb{R}^N$. Define a new function

$$f(x) := -R \frac{g(x)}{|g(x)|}.$$

It is easy to observe that f is a continuous mapping from a convex and compact set $\overline{B(0, R)}$ into itself. Corollary 2.1.5 implies the existence of a fixed point, i.e., $f(x_*) = x_*$. Thus, we have $|x_*| = R$. Furthermore, we have,

$$0 \leq \sum_{i=1}^N g_i(x_*)x_{*i} = -\frac{|g(x_*)|}{R} \sum_{i=1}^N f_i(x_*)x_{*i} = -\frac{|g(x_*)|}{R} |x_*|^2 = -R|g(x_*)| < 0,$$

which is a contradiction. The existence of a solution follows. \square

2.1.4 Monotone Operators

In this section, we present a theorem by Browder and Minty (1963) (see [19] and [55]), which aims to prove the existence of a solution to the functional equation $A(u) = f$, where $f \in V^*$ and operator $A : V \rightarrow V^*$ is assumed monotone, coercive and hemicontinuous. This result preceded another existence result involving $A : V \rightarrow V^*$, assumed in this case as a pseudomonotone operator, this result was by Brézis (1968) (see [10]). The result by Brézis will be examined in detail in the next section.

We will look at some results and properties of $A : V \rightarrow V^*$ on reflexive Banach space V , before proving these existence results.

The following result establishes a connection between the concept of a monotone operator and the more familiar notion of a real-valued monotone increasing function.

Proposition 2.1.7. *Let V be a Banach space with operator $A : V \rightarrow V^*$. Let*

$$f(t) = \langle A(u + tv), v \rangle \quad \forall t \in \mathbb{R}.$$

Then the following are equivalent: (i) The operator A is monotone. (ii) The function $f : [0, 1] \rightarrow \mathbb{R}$ is monotone increasing for all $u, v \in V$.

Proof: Assume A is monotone, for $0 \leq s < t$ it follows that

$$f(t) - f(s) = (t - s)^{-1} \langle A(u + tv) - A(u + sv), (t - s)v \rangle \geq 0.$$

Conversely, assume f is as in (ii), then for $u, w \in V$, let $v = w - u$, we then have

$$\langle A(w) - A(u), w - u \rangle = \langle A(u + v) - A(u), v \rangle = f(1) - f(0) \geq 0.$$

□

The next three lemmas serve as important tools, which will be used in our main existence proof involving monotone operators.

Lemma 2.1.8. *Let V be a reflexive Banach space. Let $\{u_n\}_n$ be a bounded sequence in V . If all convergent subsequences of the sequence $\{u_n\}_n$ converges weakly to the same limit u , then the whole sequence $\{u_n\}_n$ converges weakly to the same limit, i.e., $u_n \rightharpoonup u$ in V .*

Lemma 2.1.9. *Let V be a reflexive Banach space and $A : V \rightarrow V^*$ a monotone and hemicontinuous operator. Then*

(i) if A is maximal monotone, i.e. if $u \in V$ and $f \in V^$ are given such that*

$$\langle f - A(v), u - v \rangle \geq 0 \quad \forall v \in V,$$

then $A(u) = f$ in V^ .*

(ii) it follows from

$$u_n \rightharpoonup u \text{ in } V, \quad A(u_n) \rightharpoonup f \text{ in } V^* \text{ and } \langle A(u_n), u_n \rangle \rightarrow \langle f, u \rangle,$$

that $A(u) = f$ in V^ .*

(iii) it follows from either

$$u_n \rightharpoonup u \text{ in } V \text{ and } A(u_n) \rightarrow f \text{ in } V^*$$

or

$$u_n \rightarrow u \text{ in } V \text{ and } A(u_n) \rightharpoonup f \text{ in } V^*,$$

that $A(u) = f$ in V^ .*

Proof: (i) Suppose $u \in V$ and $f \in V^*$ are given as in the statement of (i). Take any $w \in V$ and set $v = u - tw$ for $t > 0$. Then note that

$$\langle f - A(v), u - v \rangle \geq 0 \implies \langle f - A(u - tw), w \rangle \geq 0.$$

Consider a sequence $t_n > 0$, such that $t_n \rightarrow 0$. Using the hemicontinuity of A yields

$$\langle f - A(u - t_n w), w \rangle \rightarrow \langle f - A(u), w \rangle \geq 0.$$

Now taking $-w$ instead, we obtain the reverse inequality, and the result $\langle f - A(u), w \rangle = 0$ for any $w \in V$ follows.

(ii) From the monotonicity of A it follows that

$$\langle A(u_n) - A(v), u_n - v \rangle = \langle A(u_n), u_n \rangle - \langle A(v), u_n \rangle - \langle A(u_n) - A(v), v \rangle \geq 0$$

for all $v \in V$ and all $n \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} (\langle A(u_n), u_n \rangle - \langle A(v), u_n \rangle - \langle A(u_n) - A(v), v \rangle) = \langle f, u \rangle - \langle A(v), u \rangle - \langle f - A(v), v \rangle \geq 0,$$

this gives

$$\langle f - A(v), u - v \rangle \geq 0 \quad \forall v \in V.$$

Since A is maximal monotone, the result follows from (i).

(iii) Since $\langle A(u_n), u_n \rangle \rightarrow \langle f, u \rangle$ follows from either assumption of (iii), we obtain $A(u) = f$ in V^* by structuring the proof in the same way as the proof of (ii). \square

Lemma 2.1.10. *Let V be a reflexive Banach space with operator $A : V \rightarrow V^*$. Then the following is valid:*

(i) *If A is strongly continuous, then A is compact.*

(ii) *If A is demicontinuous, then A is locally bounded, i.e., for each $x \in V$, there exists a neighbourhood $M \subset V$ of x such that $A(M) \subset V^*$ is bounded.*

(iii) *If A is monotone, then A is locally bounded.*

(iv) *If A is monotone and hemicontinuous, then A is demicontinuous.*

Proof: (i) Suppose that A is strongly continuous. Since sequential compactness and compactness coincide on metric spaces, it is enough to show that A takes a bounded set $B \subset V$ to a relatively sequentially compact set $A(B) \subset V^*$. Take an arbitrary sequence $\{A(u_n)\}_n$ in $A(B)$. Then since B is bounded, we also have that $\{u_n\}_n$ is bounded in V . From Theorem 1.1.11 (Eberlein-Šmulian Theorem), it follows that there exists a weakly convergent subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \rightharpoonup u$ in V . From the strong continuity it follows that $A(u_{n_k}) \rightarrow A(u)$ in V^* . This shows that $A(B)$ is relatively sequentially compact and therefore A is compact.

(ii) Assume that A is not locally bounded, then there exists a point $u \in V$ such that if we consider the open balls $B(u, \frac{1}{n})$, then $\forall n \in \mathbb{N}, \forall R > 0$ there exists a $v \in A(B(u, \frac{1}{n}))$ such that $\|v\|_{V^*} > R$. We can construct a sequence $u_m \rightarrow u$ by taking $u_m \in B(u, \frac{1}{m})$ for each $m \in \mathbb{N}$. We can choose the sequence $\{u_m\}_m$ such that $\|A(u_m)\|_{V^*} > m$. Then as $u_m \rightarrow u$, it follows that $\|A(u_m)\|_{V^*} \rightarrow \infty$. But since A is demicontinuous, $u_m \rightarrow u$ implies that $A(u_m) \rightharpoonup A(u)$ in V^* . So we have that $\{A(u_m)\}_m$ is bounded, thus we have a contradiction.

(iii) Assume that A is monotone but not locally bounded. As previously observed, there exists a $u \in V$ and a sequence $u_n \rightarrow u$ in V , such that $\|A(u_n)\|_{V^*} \rightarrow \infty$. Without loss of generality, we can take $u = 0$. Consider the sequence $s_n := (1 + \|A(u_n)\|_{V^*} \|u_n\|)^{-1}$.

From the monotonicity of the operator A , we have that both $\langle A(u_n) - A(v), u_n - v \rangle \geq 0$ and $\langle A(u_n) - A(-v), u_n - (-v) \rangle \geq 0$ hold, this leads to

$$\begin{aligned} \pm s_n \langle A(u_n), v \rangle &\leq s_n (\langle A(u_n), u_n \rangle - \langle A(\pm v), u_n \mp v \rangle) \\ &\leq s_n (\|A(u_n)\|_{V^*} \|u_n\| + \|A(\pm v)\|_{V^*} (\|u_n\| + \|v\|)) \\ &\leq 1 + \|A(\pm v)\|_{V^*} (D + \|v\|), \end{aligned}$$

where we use $\|u_n\| \leq D$, for some constant $D > 0$. Therefore,

$$\sup_n |\langle s_n A(u_n), v \rangle| < \infty \quad \forall v \in V.$$

According to Theorem 1.2.3, there exists a constant $C > 0$ such that

$$\sup_n \|s_n A(u_n)\|_{V^*} \leq C.$$

We obtain $s_n \|A(u_n)\|_{V^*} \leq C$, therefore $\|A(u_n)\|_{V^*} \leq \frac{C}{s_n} = C(1 + \|A(u_n)\|_{V^*} \|u_n\|_V)$. It follows, after expanding the expression, that

$$\|A(u_n)\|_{V^*} \leq \frac{C}{1 - C\|u_n\|},$$

this shows that $\|A(u_n)\|_{V^*}$ is bounded for some $n \geq n_0$, since $\|u_n\| \rightarrow 0$. Thus, we have a contradiction.

(iv) Suppose that $u_n \rightarrow u$ in V . Since $\{u_n\}_n$ is bounded, it is easy to show that the sequence $\{A(u_n)\}_n$ is also bounded, this follows from (iii). By the assumed reflexivity of V (and hence the reflexivity of V^*), there exists an element $f \in V^*$ and a subsequence $\{u_{n_k}\}_k$, such that $A(u_{n_k}) \rightharpoonup f$. From Lemma 2.1.9(iii), we get $A(u) = f$. We can state further that every subsequence of $\{A(u_n)\}$ has a subsequence which converges weakly. All these subsequences must converge weakly to $A(u)$. Assume we can find a subsequence $\{A(u_{n_l})\}_l$ such that $A(u_{n_l}) \rightharpoonup c \neq f$. Then applying 2.1.9(iii) again, yields $A(u) = c$, this gives a contradiction. So from Lemma 1.2.10, it follows that the whole sequence $\{A(u_n)\}_n$ converges weakly to $A(u)$. Hence we have that A is demicontinuous. \square

We are now in a position to prove the main theorem involving monotonicity. The following result is credited mathematicians Felix Browder and George Minty from the United States of America.

Theorem 2.1.11. *(Browder, Minty 1963) Let V be a separable, reflexive Banach space and $A : V \rightarrow V^*$ a monotone, coercive, hemicontinuous operator. Then for every $f \in V^*$, there exists a solution $u \in V$, such that*

$$A(u) = f \text{ in } V^*. \tag{2.5}$$

The set of solutions is bounded, convex and closed. If A is strictly monotone, then the solution is unique.

Proof:

Step 1: The theorem will be proved by using the Galerkin method. Take a sequence $(e_k)_k$ of linearly independent vectors in V , such that setting

$$V_n := \text{span}\{e_1, \dots, e_n\},$$

yields $V = \overline{\bigcup_n V_n}$. We are looking for a solution $u_n \in V_n$, which is of the form

$$u_n = \sum_{k=1}^n c_k^n e_k, \quad (2.6)$$

and which solves the Galerkin equations

$$\langle A(u_n) - f, e_k \rangle = 0 \quad \text{for } k \in \{1, \dots, n\}. \quad (2.7)$$

We first show that the solution of the Galerkin equations exists. This is a nonlinear system of equations

$$g(c^n) = 0 \quad \text{in } \mathbb{R}^n, \quad (2.8)$$

where $c^n := (c_1^n, \dots, c_n^n) \in \mathbb{R}^n$ and $g := (g_1, \dots, g_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are given by

$$g_k : \mathbb{R}^n \rightarrow \mathbb{R} : c^n \mapsto g_k(c^n) := \langle A(u_n) - f, e_k \rangle, \quad \text{where } k \in \{1, \dots, n\}.$$

Using a lemma proved previously, it follows that since A is monotone and hemicontinuous, it is also demicontinuous. It can then be shown that $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous, using the implied demicontinuity (consider a convergent sequence $x^l \rightarrow x$ in \mathbb{R}^n , then let $y_l = \sum_{k=1}^n x_k^l e_k$ and $y = \sum_{k=1}^n x_k e_k$, it follows that $y_l \rightarrow y$ in V . The demicontinuity of A implies that $\langle A(y_l), e_k \rangle \rightarrow \langle A(y), e_k \rangle$ for each $k \in \{1, \dots, n\}$, thus, each g_k is continuous). Since A is coercive, we have

$$\lim_{\|u\|_V \rightarrow \infty} \frac{\langle A(u), u \rangle}{\|u\|_V} \rightarrow \infty,$$

hence also,

$$\lim_{\|u\|_V \rightarrow \infty} \frac{\langle A(u) - f, u \rangle}{\|u\|_V} \rightarrow \infty,$$

then $\forall d > 0, \exists R > 0$, such that $\|u\|_V \geq R$ implies that

$$\frac{\langle A(u) - f, u \rangle}{\|u\|_V} > d.$$

We can then conclude that there exists a constant $R_1 > 0$, such that

$$\langle A(u) - f, u \rangle > 0 \quad \text{for all } u \in V : \|u\|_V \geq R_1. \quad (2.9)$$

From the equivalence of norms on finite-dimensional space V_n , there exists a pair of real numbers $0 < D_1 \leq D_2$ such that

$$D_1 \left\| \sum_{k=1}^n x_k e_k \right\|_V \leq |x| \leq D_2 \left\| \sum_{k=1}^n x_k e_k \right\|_V,$$

where $|x| := \left(\sum_{k=1}^n (x_k)^2 \right)^{\frac{1}{2}}$. Therefore, for any $x \in \mathbb{R}^n$, which satisfies $|x| \geq D_2 R_1$, it follows that $\left\| \sum_{k=1}^n x_k e_k \right\|_V \geq R_1$, hence,

$$\sum_{k=1}^n g_k(x) x_k = \left\langle A \left(\sum_{k=1}^n x_k e_k \right) - f, \sum_{k=1}^n x_k e_k \right\rangle > 0.$$

By Lemma 2.1.6, there exists a solution $c^n \in \mathbb{R}^n$ of (2.8), with $|c^n| \leq D_2 R_1$. Hence, we have

$$\|u_n\|_V \leq R_1, \quad (2.10)$$

where we have that $R_1 > 0$ is independent of $n \in \mathbb{N}$.

Step 2: We now show that $\{A(u_n)\}_n$ is bounded. Lemma 2.1.10 implies that A is locally bounded. This means that

$$\exists \alpha, \delta > 0 : \|v\|_V \leq \alpha \implies \|A(v)\|_{V^*} \leq \delta.$$

From the assumed monotonicity of A , we have

$$\langle A(u_n) - A(v), u_n - v \rangle \geq 0 \quad \forall v \in V.$$

The monotonicity of A and the Galerkin equation (2.7), which yields $\langle A(u_n), u_n \rangle = \langle f, u_n \rangle$ for all $n \in \mathbb{N}$, leads to

$$\begin{aligned} \|A(u_n)\|_{V^*} &= \sup_{\|v\|_V = \alpha} \alpha^{-1} \langle A(u_n), v \rangle \\ &\leq \sup_{\|v\|_V = \alpha} \alpha^{-1} (\langle A(v), v \rangle + \langle A(u_n), u_n \rangle - \langle A(v), u_n \rangle) \\ &\leq \alpha^{-1} (\delta \alpha + R_1 \|f\|_{V^*} + \delta R_1). \end{aligned}$$

Hence, $\{A(u_n)\}_n$ is bounded.

Step 3: We seek to pass to the limit in the final step. Since we have that V is reflexive and the sequence $\{u_n\}_n$ is bounded, using Theorem 1.1.11 (Eberlein-Šmulian Theorem), implies the existence of a weakly convergent subsequence and $u \in V$, such that

$$u_n \rightharpoonup u \quad \text{in } V.$$

For simplicity of notation, we have again denoted the subsequence by $\{u_n\}_n$. Taking any $v \in \bigcup_{n=1}^{\infty} V_n$ implies that there exists a $m \in \mathbb{N}$, such that $v \in V_m$. Since we have that u_n is a solution of (2.7), for all $n \geq m$, it follows that

$$\langle A(u_n), v \rangle = \langle f, v \rangle.$$

This implies that

$$\lim_{n \rightarrow \infty} \langle A(u_n), v \rangle = \langle f, v \rangle \quad \forall v \in \bigcup_{n=1}^{\infty} V_n. \quad (2.11)$$

Note that V^* is a reflexive Banach space. Since we have shown that $\{A(u_n)\}_n$ is bounded in Step 2, by the Eberlein-Šmulian Theorem, we have the existence of a subsequence $\{A(u_{n_k})\}_k$, where $A(u_{n_k}) \rightharpoonup \psi$ in V^* , therefore, by the uniqueness of limits and $\bigcup_{n=1}^{\infty} V_n$ being dense in V leads to $\psi = f$ in V^* . Using Lemma 2.1.8, we obtain the weak convergence of the whole sequence $\{A(u_n)\}_n$, therefore

$$A(u_n) \rightharpoonup f \text{ in } V^*.$$

Considering that u_n is a solution to (2.7), we have that $\langle A(u_n), u_n \rangle = \langle f, u_n \rangle$, this yields

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n \rangle = \lim_{n \rightarrow \infty} \langle f, u_n \rangle = \langle f, u \rangle.$$

After applying Lemma 2.1.9(ii), the desired result, $A(u) = f$ is obtained.

Now, we investigate some properties of the solution set for some fixed $f \in V^*$. Let $W := \{u \in V : A(u) = f \text{ in } V^*\}$ be the solution set.

- It is clear that $W \neq \emptyset$ from what we have shown.
- W is bounded. Assume to the contrary that for all $R > 0$ there exists a $u \in W$ such that $\|u\|_V \geq R$. Consider $R := R_1$, then a contradiction follows from (2.9), since

$$\|u\|_V \geq R_1 \implies \langle A(u) - f, u \rangle > 0, \text{ but } u \in W \implies \langle A(u) - f, u \rangle = 0.$$

- W is convex. Assume that $u_1, u_2 \in W$. Then consider $u = tu_1 + (1-t)u_2$, where $0 \leq t \leq 1$; we have the following inequality (using the assumed monotonicity of A),

$$\begin{aligned} \langle f - A(v), u - v \rangle &= \langle f - A(v), tu_1 + (1-t)u_2 - tv - v + tv \rangle \\ &= \langle f - A(v), t(u_1 - v) \rangle + \langle f - A(v), (1-t)(u_2 - v) \rangle \geq 0 \quad \forall v \in V. \end{aligned}$$

From Lemma 2.1.9(i), we get $A(u) = f$.

- W is closed. We want to show that $W = \overline{W}$. We have $W \subset \overline{W}$ trivially. Assume that $u \in \overline{W}$, then there exists a sequence such that $u_n \rightarrow u$, where $\{u_n\}_n \subset W$.

$$\begin{aligned} \langle f - A(v), u - v \rangle &= \lim_{n \rightarrow \infty} \langle f - A(v), u_n - v \rangle \\ &= \lim_{n \rightarrow \infty} \langle A(u_n) - A(v), u_n - v \rangle \geq 0 \quad \forall v \in V. \end{aligned}$$

Again by Lemma 2.1.9, we obtain $A(u) = f$, so $u \in W$.

Finally, assume that A is strictly monotone. Consider distinct elements u_1 and u_2 of the solution set W . Consequently,

$$0 < \langle A(u_1) - A(u_2), u_1 - u_2 \rangle = \langle f - f, u_1 - u_2 \rangle = 0.$$

We clearly have a contradiction. Hence, the solution to (2.5) must be unique if A is strictly monotone.

□

2.1.5 Pseudomonotone Operators

There is a large class of quasilinear partial differential equations, which do not fall in the assumptions of the Browder Minty Theorem. Therefore, a weaker notion of monotonicity, called pseudomonotonicity, was introduced with the aim of proving the existence of solutions for these quasilinear partial differential equations.

Definition 2.1.12. An operator $A : V \rightarrow V^*$ is *pseudomonotone* if

$$A \text{ is bounded,} \tag{2.12}$$

and

$$u_n \rightharpoonup u \text{ and } \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0$$

implies

$$\langle A(u), u - v \rangle \leq \liminf_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle \quad \forall v \in V. \tag{2.13}$$

Lemma 2.1.13. *Let V be a reflexive Banach space, then every pseudomonotone operator $A : V \rightarrow V^*$ is demicontinuous.*

Proof: Suppose $u_n \rightharpoonup u$. By definition of a pseudomonotone operator, $\{A(u_n)\}_n$ is bounded, therefore any subsequence of $\{A(u_n)\}_n$ is also bounded. Since we assume V is a reflexive Banach space, it follows that V^* is a reflexive Banach space. We can then apply Theorem 1.1.11 (Eberlein-Šmulian Theorem) and obtain a further weakly convergent subsequence, which I will again call $\{A(u_n)\}_n$ for simplicity. By Theorem 1.1.11 we obtain $A(u_n) \rightharpoonup f$, for some $f \in V^*$. Then, $\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = \langle f, u - u \rangle = 0$ (since we can show that $0 \leq |\langle A(u_n), u_n - v \rangle - \langle f, u - u \rangle| \leq \|A(u_n) - f\|_{V^*} \|u_n - u\|$, where $\|A(u_n) - f\|_{V^*}$ is bounded and $\|u_n - u\| \rightarrow 0$) and therefore, by (2.13), we get

$$\langle A(u), u - v \rangle \leq \liminf_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle = \langle f, u - v \rangle \tag{2.14}$$

for any $v \in V$.

Since $\langle A(u), u - v \rangle \leq \langle f, u - v \rangle$, taking $v = u - w$ results in $\langle A(u), w \rangle \leq \langle f, w \rangle$. Similarly, taking $v = u + w$ yields the reverse inequality $\langle A(u), w \rangle \geq \langle f, w \rangle$. Consequently, $A(u) = f$.

We showed that any subsequence of the sequence $\{A(u_n)\}_n$ has a weakly convergent subsequence, which converges to $A(u) = f$, therefore by Lemma 1.2.10 it follows that the whole sequence $\{A(u_n)\}_n$ converges weakly to $A(u)$. This proves that A is demicontinuous. \square

Definition 2.1.14. An operator $A : V \rightarrow V^*$ is *coercive* if $\exists \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ : \lim_{t \rightarrow +\infty} \varphi(t) = +\infty$ and $\langle A(u), u \rangle \geq \varphi(\|u\|)\|u\|$. Stated in a different form:

$$\lim_{\|u\| \rightarrow \infty} \frac{\langle A(u), u \rangle}{\|u\|} = +\infty. \tag{2.15}$$

Remark 10. Coercivity can be obtained from strong monotonicity:

$\langle A(u), u \rangle = \langle A(u) - A(0), u \rangle + \langle A(0), u \rangle \geq c\|u\|_V^2 - \|A(0)\|_{V^*}\|u\|_V$, it therefore follows that $\frac{\langle A(u), u \rangle}{\|u\|_V} \geq c\|u\|_V - \|A(0)\|_{V^*} \rightarrow \infty$ for $\|u\|_V \rightarrow \infty$.

Using a similar method, involving Galerkin approximations, as in Theorem 2.1.11 (Browder, Minty), we will prove the main existence result involving pseudomonotone operators. The following result is credited to French mathematician Hiam Brézis

Theorem 2.1.15. (*Brézis 1968*) *Let V be a separable, reflexive Banach space. Any operator $A : V \rightarrow V^*$, which is pseudomonotone and coercive is surjective. Stated differently, for any $f \in V^*$, there is at least one solution to the equation*

$$A(u) = f. \quad (2.16)$$

Proof:

Step 1: (*Galerkin Approximation*) As in the proof of Theorem 2.1.11 (Browder, Minty), we will prove using the Galerkin method. Take a sequence $(e_k)_k$ of linearly independent vectors in V , such that setting

$$V_n := \text{span}\{e_1, \dots, e_n\},$$

yields $V = \overline{\bigcup_n V_n}$. We are looking for a solution $u_n \in V_n$, which is of the form

$$u_n = \sum_{k=1}^n c_k^n e_k,$$

and which solves the Galerkin equations

$$u_n \in V_n : \langle A(u_n) - f, e_k \rangle = 0, \quad k \in \{1, \dots, n\}. \quad (2.17)$$

We first show that the solution of the Galerkin equations exists. This is a nonlinear system of equations

$$g(c^n) = 0 \quad \text{in } \mathbb{R}^n,$$

where $c^n := (c_1^n, \dots, c_n^n) \in \mathbb{R}^n$ and $g := (g_1, \dots, g_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by

$$g_k : \mathbb{R}^n \rightarrow \mathbb{R} : c^n \mapsto g_k(c^n) := \langle A(u_n) - f, e_k \rangle, \quad k \in \{1, \dots, n\}.$$

Claim 1: *Each g_k is continuous.*

Consider a convergent sequence $x^l \rightarrow x$ in \mathbb{R}^n , let $y_l = \sum_{k=1}^n x_k^l e_k$ and $y = \sum_{k=1}^n x_k e_k$, then $y_l \rightarrow y$ in V . Since A is bounded, it follows that $\{A(y_l)\}_{l \in \mathbb{N}}$ is bounded in V^* . With V^* being reflexive, it follows that $A(y_l) \rightharpoonup \psi$ (up to a subsequence, using the Eberlein-Šmulian Theorem). Hence, $\langle A(y_l), y_l - y \rangle \rightarrow 0$, and by the pseudomonotonicity of A we get

$$\langle A(y), y - v \rangle \leq \liminf_{l \rightarrow \infty} \langle A(y_l), y_l - v \rangle = \lim_{l \rightarrow \infty} \langle A(y_l), y_l - v \rangle = \langle \psi, y - v \rangle,$$

that is $\langle A(y) - \psi, y - v \rangle \leq 0 \quad \forall v \in V$. Take $w \in V$ and set $v = y - w$, this yields $\langle A(y) - \psi, w \rangle \leq 0 \quad \forall w \in V \Leftrightarrow \langle A(y) - \psi, w \rangle = 0 \quad \forall w \in V \Leftrightarrow A(y) = \psi$. Hence,

$$g_k(x^l) = \langle A(y_l) - f, e_k \rangle \rightarrow \langle A(y) - f, e_k \rangle = g_k(x).$$

Therefore, the mapping $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous.

Claim 2: *The Galerkin equations (2.17), has a solution $u_n \in V_n$, such that $\|u_n\|_V \leq R_1$, where $R_1 > 0$ is a constant independent of $n \in \mathbb{N}$.*

This follows from the same arguments as in Step 1 of the proof of Theorem 2.1.11 (Browder, Minty), using the continuity of g and the assumed coercivity of A . We then obtain a sequence of Galerkin solutions $\{u_n\}_n \subset V$, such that $u_n \in V_n$ and $\|u_n\| \leq R_1$ (this also implies that $\|A(u_n)\|_{V^*} \leq M$, for some constant $M > 0$, since A is assumed bounded). Thus, we have

$$\langle A(u_n) - f, v \rangle = 0 \quad \forall v \in V_n. \quad (2.18)$$

Step 2: Since $\{u_n\}_n$ was shown to be bounded, it follows that there is a subsequence $(u_{n_j})_j \in V$ such that $u_{n_j} \rightharpoonup u$ in V . Therefore,

$$\langle A(u_{n_j}), u_{n_j} \rangle = \langle f, u_{n_j} \rangle \rightarrow \langle f, u \rangle.$$

Take $v \in V$ and $\epsilon > 0$. By the density of $\bigcup_n V_n$ in V , there exists $w^\epsilon \in \bigcup_n V_n$ such that $\|v - w^\epsilon\| < \epsilon$, we also have $w^\epsilon \in \bigcup_n V_n \Leftrightarrow \exists n_\epsilon \in \mathbb{N}$ such that $w^\epsilon \in V_{n_\epsilon}$. Take j large enough so that $n_j > n_\epsilon$, this implies that $w^\epsilon \in V_{n_\epsilon} \subset V_{n_j}$, which implies that $\langle A(u_{n_j}) - f, w^\epsilon \rangle = 0$. Then,

$$\begin{aligned} \langle A(u_{n_j}) - f, v \rangle &= \langle A(u_{n_j}) - f, v - w^\epsilon \rangle \\ &\leq (M + \|f\|_{V^*})\epsilon. \end{aligned}$$

For $v = u_{n_j} - u$, we have

$$\langle A(u_{n_j}), u_{n_j} - u \rangle = \langle A(u_{n_j}) - f, u_{n_j} - u \rangle + \langle f, u_{n_j} - u \rangle.$$

Therefore,

$$\limsup_{j \rightarrow \infty} \langle A(u_{n_j}), u_{n_j} - u \rangle \leq (M + \|f\|_{V^*})\epsilon,$$

by arbitrariness of ϵ we have $\limsup_{j \rightarrow \infty} \langle A(u_{n_j}), u_{n_j} - u \rangle \leq 0$, then the pseudomonotonicity of A yields

$$\langle A(u), u - v \rangle \leq \liminf_{j \rightarrow \infty} \langle A(u_{n_j}), u_{n_j} - v \rangle.$$

If we take $v \in \bigcup_n V_n$, then for large j it follows that $v \in V_{n_j}$, and so

$$\langle A(u), u - v \rangle \leq \liminf_{j \rightarrow \infty} \langle A(u_{n_j}), u_{n_j} - v \rangle = \langle f, u - v \rangle \quad \forall v \in \bigcup_n V_n.$$

We want to show that this holds for all $v \in V$. Take $v \in V$ and $\epsilon > 0$, then $\exists w^\epsilon \in \bigcup_n V_n$, such that $\|v - w^\epsilon\| < \epsilon$. Therefore,

$$\langle A(u) - f, u - v \rangle = \langle A(u) - f, u - w^\epsilon \rangle + \langle A(u) - f, w^\epsilon - v \rangle \leq (\|A(u)\|_{V^*} + \|f\|_{V^*})\epsilon.$$

By the arbitrariness of ϵ , it follows that $\langle A(u) - f, u - v \rangle \leq 0 \quad \forall v \in V$, this implies that $\forall w \in V$, taking $v = u - w$ yields $\langle A(u) - f, w \rangle \leq 0 \quad \forall w \in V$, thus we have our desired result

$$A(u) = f.$$

□

2.1.6 More about Monotone and Pseudomonotone Operators

When considering a problem, it is often easier to prove pseudomonotonicity as an implication of other properties. It is therefore necessary to understand which properties imply pseudomonotonicity. Pseudomonotone operators play an essential role in proving the existence of weak solutions to some boundary value problems. Its significance will be more apparent when we examine the approach to proving the existence of weak solutions in the next chapter.

In this section, we look at some more properties and results involving monotone and pseudomonotone operators $A : V \rightarrow V^*$, on reflexive Banach space V .

Lemma 2.1.16. *Let V be a reflexive Banach space.*

(i) *The sum of pseudomonotone operators remain pseudomonotone, i.e. A_1 and A_2 pseudomonotone implies $u \mapsto A_1(u) + A_2(u)$ is pseudomonotone.*

(ii) *A shift of a pseudomonotone operator remains pseudomonotone, i.e. A pseudomonotone implies $u \mapsto A(u + w)$ is pseudomonotone, for any $w \in V$.*

Proof: Note that the boundedness of $u \mapsto A_1(u) + A_2(u)$, as well as $A(\cdot + w)$ is straightforward to show. We will prove requirement (2.13) of pseudomonotonicity.

(i) Let A_1 and A_2 be pseudomonotone and assume that $u_n \rightharpoonup u$ and $\limsup_{n \rightarrow \infty} \langle (A_1 + A_2)(u_n), u_n - u \rangle \leq 0$. We first need to show

$$\limsup_{n \rightarrow \infty} \langle A_1(u_n), u_n - u \rangle \leq 0 \text{ and } \limsup_{n \rightarrow \infty} \langle A_2(u_n), u_n - u \rangle \leq 0. \quad (2.19)$$

By contradiction, assume that $\limsup_{n \rightarrow \infty} \langle A_1(u_n), u_n - u \rangle = \epsilon > 0$, then there exists a subsequence, such that $\lim_{k \rightarrow \infty} \langle A_1(u_{n_k}), u_{n_k} - u \rangle = \limsup_{n \rightarrow \infty} \langle A_1(u_n), u_n - u \rangle = \epsilon > 0$. Consider

$$\begin{aligned} 0 &\geq \limsup_{n \rightarrow \infty} (\langle A_1(u_n), u_n - u \rangle + \langle A_2(u_n), u_n - u \rangle) \\ &\geq \limsup_{k \rightarrow \infty} (\langle A_1(u_{n_k}), u_{n_k} - u \rangle + \langle A_2(u_{n_k}), u_{n_k} - u \rangle) \\ &\geq \liminf_{k \rightarrow \infty} \langle A_1(u_{n_k}), u_{n_k} - u \rangle + \limsup_{k \rightarrow \infty} \langle A_2(u_{n_k}), u_{n_k} - u \rangle. \end{aligned}$$

It follows that $\limsup_{k \rightarrow \infty} \langle A_2(u_{n_k}), u_{n_k} - u \rangle \leq -\epsilon < 0$. We also have

$$u_{n_k} \rightharpoonup u \text{ and } \limsup_{k \rightarrow \infty} \langle A_2(u_{n_k}), u_{n_k} - u \rangle \leq 0,$$

and hence, $\langle A_2(u), u - v \rangle \leq \liminf_{k \rightarrow \infty} \langle A_2(u_{n_k}), u_{n_k} - v \rangle \quad \forall v \in V$. Let $v = u$ then

$$0 \leq \liminf_{k \rightarrow \infty} \langle A_2(u_{n_k}), u_{n_k} - u \rangle \leq \limsup_{k \rightarrow \infty} \langle A_2(u_{n_k}), u_{n_k} - u \rangle \leq -\epsilon < 0.$$

Thus, a contradiction is reached. It follows then that (2.19) holds.

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle (A_1 + A_2)(u_n), u_n - v \rangle &\geq \liminf_{n \rightarrow \infty} \langle A_1(u_n), u_n - v \rangle + \liminf_{n \rightarrow \infty} \langle A_2(u_n), u_n - v \rangle \\ &\geq \langle A_1(u), u - v \rangle + \langle A_2(u), u - v \rangle \\ &= \langle (A_1 + A_2)(u), u - v \rangle. \end{aligned}$$

(ii) Let $u_n \rightharpoonup u$ and $\limsup_{n \rightarrow \infty} \langle A(u_n + w), u_n - u \rangle \leq 0$. We then have $u_n + w \rightharpoonup u + w$ and $\limsup_{n \rightarrow \infty} \langle A(u_n + w), (u_n + w) - (u + w) \rangle \leq 0$. Since A is assumed pseudomonotone we have:

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle A(u_n + w), u_n - v \rangle &= \liminf_{n \rightarrow \infty} \langle A(u_n + w), (u_n + w) - (v + w) \rangle \\ &\geq \langle A(u + w), (u + w) - (v + w) \rangle \\ &= \langle A(u + w), u - v \rangle. \end{aligned}$$

Therefore, $A(\cdot + w)$ is pseudomonotone. \square

Lemma 2.1.17. *Let V be a reflexive Banach space and $A : V \rightarrow V^*$ a strongly continuous operator, then A is pseudomonotone.*

Proof: We first show that A is bounded. Assume there is some bounded set $B \subset V$, where $A(B)$ is unbounded in V^* . Then, choose a sequence $\{A(u_n)\}_{n \in \mathbb{N}} \subset A(B)$ such that $\|A(u_n)\|_{V^*} \geq n$. Note that $\{u_n\}_n \subset B$ is bounded in V , so by the Eberlein-Šmulian Theorem, it follows that there exists a subsequence $\{u_{n_k}\}_k$ such that $u_{n_k} \rightharpoonup u$ in V . The strong continuity then yields $A(u_{n_k}) \rightarrow A(u)$ in V^* , the sequence is therefore bounded. However, we also have $\|A(u_{n_k})\|_{V^*} \geq n_k$, this contradicts $\{A(u_{n_k})\}_k$ being bounded. Hence, A is bounded. Assume $u_n \rightharpoonup u$, then the strong continuity yields $A(u_n) \rightarrow A(u)$, which implies

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = 0,$$

but we also have

$$\langle A(u), u - v \rangle = \lim_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle = \liminf_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle.$$

Thus, A is pseudomonotone. \square

Lemma 2.1.18. *Let V be a reflexive Banach space, then any bounded demicontinuous operator $A : V \rightarrow V^*$ satisfying*

$$u_n \rightharpoonup u \text{ and } \limsup_{n \rightarrow \infty} \langle A(u_n) - A(u), u_n - u \rangle \leq 0 \implies u_n \rightarrow u \quad (2.20)$$

is pseudomonotone.

Proof: Assume that $u_n \rightharpoonup u$ and $\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0$. It can be shown that

$$\limsup_{n \rightarrow \infty} \langle A(u_n) - A(u), u_n - u \rangle = \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle - \lim_{n \rightarrow \infty} \langle A(u), u_n - u \rangle.$$

Since $u_n \rightharpoonup u$, we have $\lim_{n \rightarrow \infty} \langle A(u), u_n - u \rangle = 0$ and therefore, $\limsup_{n \rightarrow \infty} \langle A(u_n) - A(u), u_n - u \rangle = \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0$. By (2.20), it follows that $u_n \rightarrow u$ and the demicontinuity of A yields $A(u_n) \rightarrow A(u)$. It then follows immediately that $\liminf_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle = \lim_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle = \langle A(u), u - v \rangle$ for any $v \in V$. \square

Lemma 2.1.19. *Let V be a reflexive Banach space. Any radially continuous monotone operator $A : V \rightarrow V^*$, satisfies (2.13). In particular, bounded radially continuous monotone operators are pseudomonotone.*

Proof: Consider $u_n \rightharpoonup u$ and $\limsup \langle A(u_n), u_n - u \rangle \leq 0$. We assume that A is monotone, thus, $\langle A(u_n), u_n - u \rangle \geq \langle A(u), u_n - u \rangle \rightarrow 0$, this yields $\liminf \langle A(u_n), u_n - u \rangle \geq 0$. Since $\liminf \langle A(u_n), u_n - u \rangle \leq \limsup \langle A(u_n), u_n - u \rangle \leq 0$, together we have

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = 0. \quad (2.21)$$

Take $u_\epsilon = (1 - \epsilon)u + \epsilon v$, where $\epsilon > 0$, from the assumed monotonicity of A ,

$$0 \leq \langle A(u_n) - A(u_\epsilon), u_n - u_\epsilon \rangle = \langle A(u_n) - A(u_\epsilon), \epsilon(u - v) + (u_n - u) \rangle. \quad (2.22)$$

Then, from $\langle A(u_n) - A(u_\epsilon), \epsilon(u - v) \rangle + \langle A(u_n) - A(u_\epsilon), u_n - u \rangle \geq 0$, it follows that

$$\epsilon \langle A(u_n), u - v \rangle \geq \langle A(u_\epsilon), u_n - u \rangle - \langle A(u_n), u_n - u \rangle + \epsilon \langle A(u_\epsilon), u - v \rangle. \quad (2.23)$$

Fix $\epsilon > 0$ and take the limit as $n \rightarrow \infty$. By (2.21) we have

$$\epsilon \liminf_{n \rightarrow \infty} \langle A(u_n), u - v \rangle \geq \epsilon \langle A(u_\epsilon), u - v \rangle. \quad (2.24)$$

Dividing by ϵ gives $\liminf_{n \rightarrow \infty} \langle A(u_n), u - v \rangle \geq \langle A(u_\epsilon), u - v \rangle = \langle A(u + \epsilon(v - u)), u - v \rangle$. Letting $\epsilon \rightarrow 0$, together with the assumed radial continuity results in

$$\liminf_{n \rightarrow \infty} \langle A(u_n), u - v \rangle \geq \lim_{\epsilon \rightarrow 0^+} \langle A(u + \epsilon(v - u)), u - v \rangle = \langle A(u), u - v \rangle. \quad (2.25)$$

Hence, (2.25) and (2.21) yields

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle \\ = \lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle + \liminf_{n \rightarrow \infty} \langle A(u_n), u - v \rangle \geq \langle A(u), u - v \rangle. \end{aligned} \quad (2.26)$$

If we further assume that A is bounded, then not only does this prove (2.13), but it also shows that A is pseudomonotone. \square

Remark 11. Furthermore, using similarly structured proofs (with Galerkin Approximations) and some of the properties mentioned previously, additional results showing the existence of solutions to $A : V \rightarrow V^*$ can be proved. The following is an example of such a result, which can be applied to quasilinear partial differential equations with lower order terms, which do not satisfy a monotonicity condition (see [63] for treatment of this result):

Let V be a reflexive Banach space. Suppose operator $A = A_1 + A_2 : V \rightarrow V^$ is coercive, A_1 radially continuous and monotone and A_2 strongly continuous. Then A is surjective.*

Observe that if A_1 is a bounded radially continuous monotone operator, then it is also pseudomonotone (from Lemma 2.1.19). By Lemma 2.1.17, we can see that if A_2 is strongly continuous, then it is also a pseudomonotone operator. The sum of pseudomonotone operators is pseudomonotone. Moreover, $A = A_1 + A_2$ is assumed coercive, hence, as a consequence of Theorem 2.1.15 (*Brézis*) we have that A is surjective. Note, that if $A_2 = 0$ then we obtain Theorem 2.1.11 (*Browder, Minty*).

2.1.7 Two Notions of Pseudomonotone Operators

In this section, we briefly examine the connection between two distinct notions of pseudomonotonicity. We will refer to the first type as K -pseudomonotonicity. This definition was proposed by S.Karamardian in 1976 (see [45]), its applications are often used in the treatment of optimization problems. The second type of pseudomonotonicity will be called B -pseudomonotonicity, it serves as a generalization of the notion of pseudomonotonicity introduced by H.Brézis (see Definition 2.1.12).

Note that for this section we refer to our usual definition of pseudomonotonicity (as in Definition 2.1.12), again simply as 'pseudomonotone', but without $A : K \rightarrow V^*$ required to be bounded (as in (2.12)).

Let K be a non-empty, closed and convex subset of a Banach space V .

The notion of K -pseudomonotonicity, credited to Karamardian is defined as follows:

Definition 2.1.20. An operator $A : K \rightarrow V^*$ is K -pseudomonotone if

$$\langle A(v), u - v \rangle \geq 0 \implies \langle A(u), u - v \rangle \geq 0 \quad (2.27)$$

for each $u, v \in K$.

The following is a weaker form of hemicontinuity to that which we defined previously, we will call this K -hemicontinuity.

Definition 2.1.21. An operator $A : K \rightarrow V^*$ is K -hemicontinuous if

$$t \mapsto \langle A(u + t(v - u)), v - u \rangle \quad (2.28)$$

is continuous at 0^+ for all $u, v \in K$.

Remark 12. The definition of hemicontinuity presented previously (in Section 2.1.2) implies K -hemicontinuity.

Proposition 2.1.22. Let K be a non-empty, closed and convex subset of Banach space V . A monotone and K -hemicontinuous operator $A : K \rightarrow V^*$ is pseudomonotone.

We now consider a more general notion of pseudomonotonicity than that introduced previously.

Definition 2.1.23. An operator $A : K \rightarrow V^*$ is B -pseudomonotone if for each $\{u_n\}_{n \in \mathbb{N}} \subset K$ and for all $u, v \in K$, the assumptions

$$u_n \rightarrow u \quad (2.29)$$

and

$$\langle A(u_n), (1 - t)u + tv - u_n \rangle \geq 0, \quad \forall t \in [0, 1], n \in \mathbb{N} \quad (2.30)$$

implies that

$$\langle A(u), v - u \rangle \geq 0. \quad (2.31)$$

Proposition 2.1.24. Let K be a non-empty, closed and convex subset of Banach space V . Every pseudomonotone operator $A : K \rightarrow V^*$ is also B -pseudomonotone.

Proof: Assume that $A : K \rightarrow V^*$ is pseudomonotone and that conditions (2.29) and (2.30) hold. Consider (2.30) with $t = 0$ and $t = 1$, this yields

$$\langle A(u_n), u - u_n \rangle \geq 0 \quad \text{and} \quad \langle A(u_n), v - u_n \rangle \geq 0 \quad \forall n \in \mathbb{N},$$

hence we have

$$\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle \leq 0.$$

As a result, $\langle A(u), u - v \rangle \leq \liminf_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle \leq 0$ for all $v \in K$. In conclusion, we have the desired result $\langle A(u), v - u \rangle \geq 0$.

□

The main result, which we seek to prove, establishes a connection between K -pseudomonotonicity and B -pseudomonotonicity. Before proving this result; we first prove a characterization of B -pseudomonotonicity which will be used. In order to do this, we define set-valued operators $A_1, A_2 : K \rightarrow 2^K$ by

$$A_1(v) := \{u \in K : \langle A(u), v - u \rangle \geq 0\} \quad \text{and} \quad A_2(v) := \{u \in K : \langle A(v), v - u \rangle \geq 0\}.$$

The notation 'wcl' will denote the weak closure of a set, and ' $[u, v]$ ', the line segment joining u and v . The result we seek to prove is that operator A is B -pseudomonotone if

$$\text{wcl} \left(\bigcap_{z \in [u, v]} A_1(z) \right) \cap [u, v] = \left(\bigcap_{z \in [u, v]} A_1(z) \right) \cap [u, v] \quad (2.32)$$

for all $u, v \in K$. Note the following argument:

Assume (2.32) is true. Moreover, assume (2.30) holds for $u, v \in K$, and let $u_n \rightharpoonup u$ in K . We want to show (2.31).

The assumption (2.30) implies $\langle A(u_n), z - u_n \rangle \geq 0$, for every $z \in [u, v]$. Consequently,

$$\{u_n\}_{n \in \mathbb{N}} \subset \bigcap_{z \in [u, v]} A_1(z).$$

Since $u_n \rightharpoonup u$ in K , it follows that

$$u \in \text{wcl} \left(\bigcap_{z \in [u, v]} A_1(z) \right) \cap [u, v] = \left(\bigcap_{z \in [u, v]} A_1(z) \right) \cap [u, v].$$

It is then clear that (2.31) is satisfied. One of the inclusions of (2.32) is trivial, so all that needs to be shown is that

$$\text{wcl} \left(\bigcap_{z \in [u, v]} A_1(z) \right) \cap [u, v] \subset \left(\bigcap_{z \in [u, v]} A_1(z) \right) \cap [u, v] \quad (2.33)$$

for all $u, v \in K$. To prove (2.33), the following will be proved:

$$\begin{aligned} \text{wcl} \left(\bigcap_{z \in [u, v]} A_1(z) \right) \cap [u, v] &\subset \text{wcl} \left(\bigcap_{z \in [u, v]} A_2(z) \right) \cap [u, v] \\ &\subset \left(\bigcap_{z \in [u, v]} A_2(z) \right) \cap [u, v] = \left(\bigcap_{z \in [u, v]} A_1(z) \right) \cap [u, v]. \end{aligned} \quad (2.34)$$

Theorem 2.1.25. *Let K be a non-empty, closed and convex subset of a Banach space V . If the operator $A : K \rightarrow V^*$ is K -pseudomonotone and K -hemicontinuous, then it is also B -pseudomonotone.*

Proof: In order to prove (2.32), it is enough to prove (2.34). The K -pseudomonotonicity of A yields

$$A_1(z) \subset A_2(z) \quad \text{for all } z \in K, \quad (2.35)$$

this shows that the first inclusion of (2.34) is valid.

The equality in (2.34) is shown as follows: First note that since $A : K \rightarrow V^*$ is K -pseudomonotone and K -hemicontinuous, $A : [u, v] \rightarrow V^*$ is also K -pseudomonotone and K -hemicontinuous. Also, $[u, v]$ is a non-empty, closed and convex subset of V . Take $\bar{u} \in \left(\bigcap_{z \in [u, v]} A_2(z) \right) \cap [u, v]$, then $\langle A(z), z - \bar{u} \rangle \geq 0$ for all $z \in [u, v]$. We want to show that $\langle A(\bar{u}), z - \bar{u} \rangle \geq 0$, for all $z \in [u, v]$. Consider $\bar{u} + t(z - \bar{u}) \in [u, v]$, where $t \in [0, 1]$. Using the K -pseudomonotonicity yields

$$\langle A(\bar{u} + t(z - \bar{u})), z - \bar{u} \rangle \geq 0.$$

Then, the assumed K -hemicontinuity gives

$$\langle A(\bar{u} + t(z - \bar{u})), z - \bar{u} \rangle \rightarrow \langle A(\bar{u}), z - \bar{u} \rangle \quad \text{as } t \rightarrow 0^+.$$

Hence, $\langle A(\bar{u}), z - \bar{u} \rangle \geq 0$, therefore

$$\bar{u} \in \left(\bigcap_{z \in [u, v]} A_1(z) \right) \cap [u, v].$$

Thus, we obtain the desired result,

$$\left(\bigcap_{z \in [u, v]} A_2(z) \right) \cap [u, v] \subset \left(\bigcap_{z \in [u, v]} A_1(z) \right) \cap [u, v].$$

The other inclusion follows easily. Hence,

$$\left(\bigcap_{z \in [u, v]} A_2(z) \right) \cap [u, v] = \left(\bigcap_{z \in [u, v]} A_1(z) \right) \cap [u, v].$$

Finally, we show the second inclusion. Take

$$\bar{u} \in \text{wcl} \left(\bigcap_{z \in [u, v]} A_2(z) \right) \cap [u, v], \quad (2.36)$$

then there exists a net $(u_i)_{i \in \mathbb{I}}$ (with directed set \mathbb{I}), such that $u_i \rightarrow \bar{u}$, with

$$\{u_i\}_{i \in \mathbb{I}} \subset \left(\bigcap_{z \in [u, v]} A_2(z) \right) \cap [u, v] \quad \forall i \in \mathbb{I}.$$

Thus, $\langle A(z), z - u_i \rangle \geq 0$ for all $z \in [u, v]$, $i \in \mathbb{I}$. Since $u_i \rightarrow \bar{u}$, it follows that $\langle A(z), z - \bar{u} \rangle \geq 0$ for all $z \in [u, v]$, thus we have

$$\bar{u} \in \left(\bigcap_{z \in [u, v]} A_2(z) \right) \cap [u, v].$$

□

Theorem 2.1.25 thus confirms an association between two distinct notions of pseudomonotonicity, namely, K -pseudomonotonicity and B -pseudomonotonicity.

Chapter 3

Applications to Variational Problems

In this chapter, we will start by describing the weak formulation and how it relates to the classical notion of a solution to a second order partial differential equation. Then we will use the abstract theory of pseudomonotone operators on boundary value problems to prove the existence of a weak solution. More specifically, we will focus on quasilinear second order partial differential equations of the form

$$\begin{cases} -\operatorname{div}(a(x, u, \nabla u)) + c(x, u, \nabla u) = g & \text{in } \Omega, \\ u|_{\Gamma} = u_D & \text{on } \Gamma_D, \\ \nu \cdot a(x, u, \nabla u) + b(x, u) = h & \text{on } \Gamma_N. \end{cases} \quad (3.1)$$

Above, $\Omega \subset \mathbb{R}^N$ is assumed to be a bounded, connected, Lipschitz domain. Moreover, we assume that mappings

$$a_i, c : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}, \quad b : \Gamma \times \mathbb{R} \rightarrow \mathbb{R} \text{ are Carathéodory functions for } i \in \{1, \dots, N\}. \quad (3.2)$$

By definition of Carathéodory functions, the functions are measurable in x and continuous almost everywhere with respect to $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$. We will take $g \in L^{p^*}(\Omega)$, $h \in L^{p^*}(\Gamma)$, where $p \in (1, +\infty)$. Assume the following growth conditions on mappings a, b and c ,

$$|a(x, s, \xi)| \leq \gamma_1(x) + C|s|^{\frac{(p^*-\epsilon)}{p^*}} + C|\xi|^{p-1} \quad \text{for some } \gamma_1 \in L^{p^*}(\Omega), \quad (3.3)$$

$$|b(x, s)| \leq \gamma_2(x) + C|s|^{p^*-\epsilon-1} \quad \text{for some } \gamma_2 \in L^{p^*}(\Gamma), \quad (3.4)$$

$$|c(x, s, \xi)| \leq \gamma_3(x) + C|s|^{p^*-\epsilon-1} + C|\xi|^{\frac{p}{p^*}} \quad \text{for some } \gamma_3 \in L^{p^*}(\Omega). \quad (3.5)$$

Lastly, we assume that $a(x, s, \xi)$ satisfies a monotonicity condition:

$$(a(x, s, \xi) - a(x, s, \bar{\xi})) \cdot (\xi - \bar{\xi}) \geq 0 \quad \forall a.e. \ x \in \Omega, \forall s \in \mathbb{R}, \forall \xi, \bar{\xi} \in \mathbb{R}^N. \quad (3.6)$$

3.1 Boundary Value Problems

As shown in (3.1), we seek to prove the existence of a weak solution to

$$-\operatorname{div}(a(x, u, \nabla u)) + c(x, u, \nabla u) = g \quad \text{in } \Omega. \quad (3.7)$$

It is more common to consider only a Dirichlet boundary condition, but we will instead consider a combination of boundary conditions (Dirichlet and Newton Boundary

conditions), thereby covering a more diverse range of boundary value problems. In doing so, we demonstrate how effective a tool pseudomonotone operators are in examining quasilinear equations of this type. We divide the boundary Γ into two disjoint open parts Γ_D and Γ_N . As shown in (3.1), the mixed boundary conditions are:

$$u|_{\Gamma} = u_D \quad \text{on } \Gamma_D, \quad (3.8)$$

$$\nu \cdot a(x, u, \nabla u) + b(x, u) = h \quad \text{on } \Gamma_N. \quad (3.9)$$

The classical solution $u \in C^2(\overline{\Omega})$ is not usually easy to attain, we therefore seek to describe a more general notion of a solution. We require this general solution, called a 'weak solution', to satisfy, first and foremost, the following conditions:

Condition 1: Any classical solution is also a weak solution.

Condition 2: If the data is assumed smooth and the weak solution belongs to $C^2(\overline{\Omega})$, then it is the classical solution.

3.1.1 Weak Formulation

In order to define the concept of a weak solution, we first need to describe the weak formulation. We will apply the procedure of deriving the weak formulation to equation (3.1). The weak formulation is obtained by applying the following steps:

- Multiply (3.7) by a test function. We will call this test function v .
- Integrate over the domain Ω .
- Use Green's formula (*Let $z = a(x, u, \nabla u)$. Refer to the appendix*).
- Substitute the Newton boundary condition into the boundary integral term in Green's formula. Since we consider a test function v , we have that $v|_D = 0$, so the integral over Γ_D vanishes.

The steps described above, applied to our quasilinear pde with mixed boundary value conditions, results in the following equation:

$$\begin{aligned} \int_{\Omega} (-\operatorname{div}(a(x, u, \nabla u)) + c(x, u, \nabla u))v dx & \quad (3.10) \\ & = \int_{\Omega} a(x, u, \nabla u) \cdot \nabla v + c(x, u, \nabla u)v dx + \int_{\Gamma} (b(x, u) - h)v dS. \end{aligned}$$

Note that from (3.7) we also have $\int_{\Omega} (-\operatorname{div}(a(x, u, \nabla u)) + c(x, u, \nabla u))v dx = \int_{\Omega} g v dx$. We, therefore, get the following important equation, which we refer to frequently in what follows:

$$\int_{\Omega} a(x, u, \nabla u) \cdot \nabla v + c(x, u, \nabla u)v dx + \int_{\Gamma_N} b(x, u)v dS = \int_{\Omega} g v dx + \int_{\Gamma_N} h v dS. \quad (3.11)$$

Remark 13. Sobolev spaces, named after the Russian mathematician, Sergei Sobolev, are sufficiently rich in derivatives (in the weak sense) and additional structure. It is a suitable setting for application in partial differential equations, as their solutions are naturally found here.

We are now in a position to describe the notion of a weak solution.

Definition 3.1.1. We regard $u \in W^{1,p}(\Omega)$ as a *weak solution* to the mixed boundary value problem (3.7), (3.8) and (3.9), if $u|_{\Gamma_D} = u_D$, and the integral (3.11) is satisfied for any test function $v \in W^{1,p}(\Omega)$, with $v|_{\Gamma_D} = 0$.

Condition 1 follows as a result of how the integral (3.11) was obtained. We will prove Condition 2 now, but first we will introduce the space of test functions:

$$V := \{v \in W^{1,p}(\Omega) : v|_{\Gamma_D} = 0\}. \quad (3.12)$$

An important observation is the restriction of v on Γ_D in the space of test functions V , i.e., $v|_{\Gamma_D} = 0$. In Definition 3.1.1, $u|_{\Gamma_D} = u_D$ is a separate requirement.

We must ensure that this setting does not compromise Condition 2.

Proposition 3.1.2. (*verification of Condition 2*) Let $a \in C^1(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N; \mathbb{R}^N)$, $b \in C(\overline{\Gamma_N} \times \mathbb{R})$, $c \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$, $g \in C(\overline{\Omega})$ and $h \in C(\Gamma_N)$. Then, it follows that the weak solution $u \in C^2(\overline{\Omega})$ described above is also the classical solution.

Proof: Consider (3.11), then

$$\begin{aligned} & \int_{\Omega} a(x, u, \nabla u) \cdot \nabla v + c(x, u, \nabla u)v dx - \int_{\Omega} (-\operatorname{div}(a(x, u, \nabla u)) + c(x, u, \nabla u))v dx \\ &= - \int_{\Gamma_N} b(x, u)v dS + \int_{\Gamma_N} hvdS, \end{aligned}$$

therefore,

$$\int_{\Omega} \operatorname{div}(a(x, u, \nabla u))v + a(x, u, \nabla u) \cdot \nabla v dx = \int_{\Gamma_N} (h - b(x, u))v dS$$

$$\text{Add the following to the LHS: } \int_{\Omega} (-\operatorname{div}(a(x, u, \nabla u)) + c(x, u, \nabla u))v dx - \int_{\Omega} gv dx$$

$$\begin{aligned} & \int_{\Omega} \operatorname{div}(a(x, u, \nabla u))v + a(x, u, \nabla u) \cdot \nabla v dx + \int_{\Omega} (-\operatorname{div}(a(x, u, \nabla u)) + c(x, u, \nabla u))v dx \\ & - \int_{\Omega} gv dx = \int_{\Gamma_N} (h - b(x, u))v dS. \end{aligned}$$

Green's Formula then yields:

$$\begin{aligned} & \int_{\Gamma_N} (a(x, u, \nabla u) \cdot \nu)v dS + \int_{\Omega} (-\operatorname{div}(a(x, u, \nabla u)) + c(x, u, \nabla u) - g)v dx \\ &= \int_{\Gamma_N} (h - b(x, u))v dS, \end{aligned}$$

hence,

$$\begin{aligned} & \int_{\Gamma_N} (a(x, u, \nabla u) \cdot \nu)v dS + \int_{\Omega} (-\operatorname{div}(a(x, u, \nabla u)) + c(x, u, \nabla u) - g)v dx \\ & - \int_{\Gamma_N} (h - b(x, u))v dS = 0. \end{aligned}$$

This leads to

$$\int_{\Omega} (\operatorname{div}(a(x, u, \nabla u)) - c(x, u, \nabla u) + g) v dx + \int_{\Gamma_N} (h - b(x, u) - \nu \cdot a(x, u, \nabla u)) v dS = 0. \quad (3.13)$$

Since $v|_{\Gamma} = 0$ by definition of V , it follows that the boundary integral in (3.13) vanishes. As v is taken arbitrarily from V , it follows that (3.7) holds a.e. $x \in \Omega$. Since $\operatorname{div}(a(x, u, \nabla u)) - c(x, u, \nabla u) + g$ is continuous, it is easy to show that it even holds everywhere.¹ Taking $v \in V$ in (3.13), we can also show that the boundary condition (3.9) holds. Note that the first boundary condition holds by our definition of a weak solution. \square

From Chapter 1, we have the continuous embedding, $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$, and the compact embedding, $W^{1,p}(\Omega) \Subset L^{p^*-\epsilon}(\Omega)$, where $\epsilon > 0$. From Theorem 1.5.11 (trace operator), we have that there exists a continuous mapping $T : W^{1,p}(\Omega) \rightarrow L^{p^{\#}}(\Gamma)$, and a compact mapping from $W^{1,p}(\Omega)$ into $L^{p^{\#}-\epsilon}(\Gamma)$. We will use these results after applying the Nemytskii Theorem.

Remark 14. Refer to the appendix for the notation p', p^* and $p^{\#}$.

The growth conditions allow the application of the Nemytskii Theorem (see appendix, Theorem 7.6.1). The continuity results which derive from applying the Nemytskii Theorem to mappings a , b and c are first stated, then the continuity of these mappings are proved.

$$\eta_a : W^{1,p}(\Omega) \times L^p(\Omega; \mathbb{R}^N) \rightarrow L^{p'}(\Omega; \mathbb{R}^N) \quad (\text{weak} \times \text{norm, norm})\text{-continuous,} \quad (3.14)$$

$$\eta_b \circ T : W^{1,p}(\Omega) \rightarrow L^{p^{\#}}(\Gamma) \quad (\text{weak} \times \text{norm})\text{-continuous,} \quad (3.15)$$

$$\eta_c : W^{1,p}(\Omega) \times L^p(\Omega; \mathbb{R}^N) \rightarrow L^{p^{*'}}(\Omega) \quad (\text{weak} \times \text{norm, norm})\text{-continuous.} \quad (3.16)$$

Let us first look at the continuity result (3.14) in some more detail. Recall that $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is assumed to be a Carathéodory function. The Nemytskii mappings η_a will map functions $u : \Omega \rightarrow \mathbb{R}$ and $\nabla u : \Omega \rightarrow \mathbb{R}^N$ to a function $\eta_a(u, \nabla u) : \Omega \rightarrow \mathbb{R}^N$ defined by

$$(\eta_a(u, \nabla u))(x) := a(x, u, \nabla u).$$

Then, using the Nemytskii Theorem, we find that η_a is bounded and continuous, mapping $L^{p^*-\epsilon}(\Omega) \times L^p(\Omega; \mathbb{R}^N) \rightarrow L^{p'}(\Omega; \mathbb{R}^N)$.

We show why $\eta_a : W^{1,p}(\Omega) \times L^p(\Omega; \mathbb{R}^N) \rightarrow L^{p'}(\Omega; \mathbb{R}^N)$ is (weak \times norm, norm)-continuous as follows:

Proof:

Take convergent sequences $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\varphi_n \rightarrow \varphi$ in $L^p(\Omega; \mathbb{R}^N)$. First consider $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, from Theorem 1.5.13 (Rellich Kondrachov Theorem), it follows that

¹ $C_0^\infty(\Omega)$ is dense in $W_0^{k,p}(\Omega)$, so we can use the du Bois-Reymond lemma (check appendix). Noting that, in this case, $\operatorname{div}(a(x, u, \nabla u)) - c(x, u, \nabla u) + g$ is taken as being continuous (therefore, also locally integrable) and taking $v \in C_0^\infty(\Omega)$ allows for an application of du Bois-Reymond lemma. This gives $\operatorname{div}(a(x, u, \nabla u)) - c(x, u, \nabla u) + g = 0$ almost everywhere. Since $\operatorname{div}(a(x, u, \nabla u)) - c(x, u, \nabla u) + g$ is continuous, it can be shown that $\operatorname{div}(a(x, u, \nabla u)) - c(x, u, \nabla u) + g = 0$ everywhere.

$u_n \rightarrow u$ in $L^{p^*-\epsilon}(\Omega)$. From Proposition 1.4.11, we have the existence of a pointwise convergent subsequence, and a function $h \in L^{p^*-\epsilon}(\Omega)$, where $h(x) \geq 0$. Further, we also have that $|u_{n_k}(x)| \leq h(x)$ a.e. $x \in \Omega$ (clearly also $|u(x)| \leq h(x)$ a.e. $x \in \Omega$). Similarly, consider $\varphi_{n_k} \rightarrow \varphi$ in $L^p(\Omega; \mathbb{R}^N)$. Again, we use Proposition 1.4.11, this yields a pointwise convergent subsequence, $\varphi_{n_{k_l}}(x) \rightarrow \varphi(x)$ a.e. $x \in \Omega$, and there exists $h_0 \in L^p(\Omega; \mathbb{R}^N)$ where $h_0(x) \geq 0$. Moreover, we also have $|\varphi_{n_{k_l}}(x)| \leq h_0(x)$ a.e. $x \in \Omega$ (clearly also $|\varphi(x)| \leq h_0(x)$ a.e. $x \in \Omega$). Consider also the subsequence $\{u_{n_{k_l}}\}_l$.

For simplicity of notation, these subsequences will be denoted by $\{\varphi_{n_k}\}_k$ and $\{u_{n_k}\}_k$.

Using the continuity of Carathéodory function $a(x, s, \xi)$, it follows that

$$a(x, u_{n_k}(x), \varphi_{n_k}(x)) \rightarrow a(x, u(x), \varphi(x)) \quad \text{a.e. } x \in \Omega.$$

We want to show that

$$\int_{\Omega} |\eta_a(u_{n_k}, \varphi_{n_k})(x) - \eta_a(u, \varphi)(x)|^{p'} dx \rightarrow 0. \quad (3.17)$$

Using Lemma 1.2.10, it follows that if we can prove (3.17) for the subsequence then it holds for the whole sequence. Using growth condition (3.3):

$$\begin{aligned} |\eta_a(u_{n_k}, \varphi_{n_k})(x) - \eta_a(u, \varphi)(x)|^{p'} &\leq C(|\eta_a(u_{n_k}, \varphi_{n_k})(x)|^{p'} + |\eta_a(u, \varphi)(x)|^{p'}) \\ &= C(|a(x, u_{n_k}(x), \varphi_{n_k}(x))|^{p'} + |a(x, u(x), \varphi(x))|^{p'}) \\ &\leq C(2\gamma_1(x)^{p'} + |u_{n_k}(x)|^{p^*-\epsilon} + |\varphi_{n_k}(x)|^p + |u(x)|^{p^*-\epsilon} + |\varphi(x)|^p) \\ &\leq C(2\gamma_1(x)^{p'} + 2h(x)^{p^*-\epsilon} + 2h_0(x)^p) \end{aligned}$$

for some constant $C > 0$ (where C is absorbed and chosen accordingly throughout the working). The result (3.17), follows after applying Theorem 1.3.10 (Lebesgue Dominated Convergence Theorem). Hence, we have

$$\eta_a : W^{1,p}(\Omega) \times L^p(\Omega; \mathbb{R}^N) \rightarrow L^{p'}(\Omega; \mathbb{R}^N) \text{ is (weak} \times \text{norm,norm)-continuous.} \quad (3.18)$$

□

Next, we want to show that $\eta_b \circ T : W^{1,p}(\Omega) \rightarrow L^{p^{\#'}}(\Gamma)$ is (weak, norm)-continuous.

Proof:

Take $u_n \rightarrow u$ in $W^{1,p}(\Omega)$, we want to show that $(\eta_b \circ T)(u_n) \rightarrow (\eta_b \circ T)(u)$. Since T is a compact operator (see Theorem 1.5.11), it follows that $T(u_n) \rightarrow T(u)$ in $L^{p^{\#'}-\epsilon}(\Gamma)$. By Proposition 1.4.11, it follows that there exists a subsequence $T(u_{n_k})(x) \rightarrow T(u)(x)$ a.e. $x \in \Gamma$, and a function $w \in L^{p^{\#'}-\epsilon}(\Gamma)$, where $w(x) \geq 0$ and $|T(u_{n_k})(x)| \leq w(x)$ a.e. $x \in \Gamma$ (clearly also $|T(u)(x)| \leq w(x)$ a.e. $x \in \Gamma$).

Since b is also a Carathéodory function, we have

$$b(x, T(u_{n_k})(x)) \rightarrow b(x, T(u)(x)) \quad \text{a.e. } x \in \Gamma.$$

We want to show that

$$\int_{\Gamma} |\eta_b \circ T(u_{n_k})(x) - \eta_b \circ T(u)(x)|^{p^{\#'}} dx \rightarrow 0. \quad (3.19)$$

Using Lemma 1.2.10, it follows that if we can prove (3.19) for the subsequence, then it holds for the whole sequence. Using growth condition (3.4):

$$\begin{aligned}
|\eta_b \circ T(u_{n_k})(x) - \eta_b \circ T(u)(x)|^{p^{\#'}} &\leq D(|\eta_b \circ T(u_{n_k})(x)|^{p^{\#'}} + |\eta_b \circ T(u)(x)|^{p^{\#'}}) \\
&= D(|b(x, T(u_{n_k})(x))|^{p^{\#'}} + |b(x, T(u)(x))|^{p^{\#'}}) \\
&\leq D(2\gamma_2(x)^{p^{\#'}} + |T(u_{n_k})(x)|^{(p^{\#}-\epsilon-1)p^{\#'}} + |T(u)(x)|^{(p^{\#}-\epsilon-1)p^{\#'}}) \\
&\leq D(2\gamma_2(x)^{p^{\#'}} + 2w(x)^{(p^{\#}-\epsilon-1)p^{\#'}})
\end{aligned}$$

for some constant $D > 0$. Note that since $(p^{\#}-\epsilon-1)\frac{p^{\#}}{p^{\#}-1} < p^{\#}-\epsilon$, it can be shown that $w(x)^{(p^{\#}-\epsilon-1)p^{\#'}} \in L^1(\Gamma)$. Then the result (3.19) follows by Theorem 1.3.10 (Lebesgue Dominated Theorem). Hence, we have

$$\eta_b \circ T : W^{1,p}(\Omega) \rightarrow L^{p^{\#'}}(\Gamma) \text{ is (weak, norm)-continuous.} \quad (3.20)$$

□

Remark 15. Using similar arguments, we can show that

$$\eta_c : W^{1,p}(\Omega) \times L^p(\Omega; \mathbb{R}^N) \rightarrow L^{p^{\#'}}(\Omega) \text{ is (weak } \times \text{ norm, norm)-continuous.} \quad (3.21)$$

The terms on the right-hand side of (3.11) being summable relies on the following observations. Given $v \in W^{1,p}(\Omega)$, using the continuous embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and the trace operator, which maps $W^{1,p}(\Omega)$ into $L^{p^{\#}}(\Omega)$, together with Hölder's Inequality implies $gv \in L^1(\Omega)$ and $hv|_{\Gamma} \in L^1(\Gamma)$.

For the Dirichlet boundary condition on Γ_D , we will assume

$$\exists w \in W^{1,p}(\Omega) \text{ such that } u_D = w|_{\Gamma}. \quad (3.22)$$

Consider $V := \{v \in W^{1,p}(\Omega) : v|_{\Gamma_D} = 0\}$, equipped with the Sobolev space norm and define the operators $A : W^{1,p}(\Omega) \rightarrow V^*$ and $f \in V^*$ by

$$\langle A(u), v \rangle := \text{left-hand side of (3.11)}, \quad (3.23)$$

$$\langle f, v \rangle := \text{right-hand side of (3.11)}. \quad (3.24)$$

Then, define operator $A_1 : V \rightarrow V^*$ by

$$A_1(u) := A(u + w). \quad (3.25)$$

Now, we check that $f \in V^*$.

$$\begin{aligned}
\|f\|_{V^*} &:= \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \{|\langle f, v \rangle|\} \\
&= \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \left\{ \left| \int_{\Omega} gv dx + \int_{\Gamma_N} hvdS \right| \right\} \\
&\leq \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \left\{ \int_{\Omega} |gv| dx + \int_{\Gamma_N} |hv| dS \right\} \\
&\leq \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \left\{ \|g\|_{L^{p^*}(\Omega)} \|v\|_{L^{p^*}(\Omega)} + \|h\|_{L^{p^{\#'}}(\Gamma_N)} \|v\|_{L^{p^{\#}}(\Gamma_N)} \right\} \quad (\text{by Hölder's Inequality}) \\
&\leq \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \left\{ \|g\|_{L^{p^*}(\Omega)} C \|v\|_{W^{1,p}(\Omega)} + \|h\|_{L^{p^{\#'}}(\Gamma_N)} D \|v\|_{W^{1,p}(\Omega)} \right\} \\
&\leq C \|g\|_{L^{p^*}(\Omega)} + D \|h\|_{L^{p^{\#'}}(\Gamma_N)},
\end{aligned}$$

where constant C is the norm of the embedding operator $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ and constant D is the norm of the trace operator $W^{1,p}(\Omega) \rightarrow L^{p^\#}(\Gamma_N)$. Note that f is clearly linear. Thus, we have shown that $f \in V^*$.

Similarly, we can check that $A(u) \in V^*$.

$$\begin{aligned}
\|A(u)\|_{V^*} &:= \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \{ |\langle A(u), v \rangle| \} \\
&\leq \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \left\{ \int_{\Omega} |a(x, u, \nabla u) \cdot \nabla v| dx + \int_{\Omega} |c(x, u, \nabla u)v| dx + \int_{\Gamma_N} |b(x, u)v| dS \right\} \\
&\leq \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \left\{ \|a(x, u, \nabla u)\|_{L^{p'}(\Omega; \mathbb{R}^N)} \|\nabla v\|_{L^p(\Omega; \mathbb{R}^N)} + \|c(x, u, \nabla u)\|_{L^{p^*}(\Omega)} \|v\|_{L^{p^*}(\Omega)} \right. \\
&\quad \left. + \|b(x, u)\|_{L^{p^\#}(\Gamma_N)} \|v\|_{L^{p^\#}(\Gamma_N)} \right\} \\
&\leq \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \left\{ \|a(x, u, \nabla u)\|_{L^{p'}(\Omega; \mathbb{R}^N)} \|v\|_{W^{1,p}(\Omega)} + \|c(x, u, \nabla u)\|_{L^{p^*}(\Omega)} C \|v\|_{W^{1,p}(\Omega)} \right. \\
&\quad \left. + \|b(x, u)\|_{L^{p^\#}(\Gamma_N)} D \|v\|_{W^{1,p}(\Gamma_N)} \right\} \\
&\leq \|a(x, u, \nabla u)\|_{L^{p'}(\Omega; \mathbb{R}^N)} + C \|c(x, u, \nabla u)\|_{L^{p^*}(\Omega)} + D \|b(x, u)\|_{L^{p^\#}(\Gamma_N)}.
\end{aligned}$$

It is clear that $A : W^{1,p}(\Omega) \rightarrow V^*$ is well-defined ($A(u)$ is clearly a linear operator).

Proposition 3.1.3. *There exists a solution $u_1 \in V$ to (2.16) for A_1 if and only if $u = u_1 + w \in W^{1,p}(\Omega)$ is the weak solution to our second order boundary value problem (3.1), where 'weak solution' is defined as in Definition 3.1.1.*

Proof:

(\implies) Assume $u_1 \in V$ is a solution to (2.16) for A_1 . Then, $f = A_1(u_1) = A_1(u - w) = A(u)$, so by (3.23), (3.24) and Definition 3.1.1, it is clear that u is a weak solution.

(\impliedby) Assume that $u = u_1 + w$ is the weak solution to the boundary value problem, then, $\langle A(u_1 + w), v \rangle = \langle f, v \rangle$ for all $v \in V$, thus, $A_1(u_1) = A(u_1 + w) = f$. \square

3.1.2 Existence of Weak Solutions

The existence of a weak solution is obtained as an application of Theorem 2.1.15 and Proposition 3.1.3. In order to achieve this, we need to prove that $A_1 : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ satisfies the necessary conditions. To this end, we will show that $A_1 : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ inherits the required properties of coercivity and pseudomonotonicity from A . It is apparent, that these results would then also hold for $A_1 : V \rightarrow V^*$.

In order to prove the pseudomonotonicity of A_1 , we have to show that A_1 is bounded.

Lemma 3.1.4. *Given that (3.2)-(3.5) hold, it follows that $A : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ is a bounded operator.*

Proof:

The idea is to show that A takes bounded sets in $W^{1,p}(\Omega)$ to bounded sets in $W^{1,p}(\Omega)^*$. Consider $\{u \in W^{1,p}(\Omega) : \|u\|_{W^{1,p}(\Omega)} \leq r\}$ for some $r > 0$ in $W^{1,p}(\Omega)$. We will show that

$A(\{u \in W^{1,p}(\Omega) : \|u\|_{W^{1,p}(\Omega)} \leq r\})$ is bounded in $W^{1,p}(\Omega)^*$.

$$\begin{aligned}
& \sup_{\|u\|_{W^{1,p}(\Omega)} \leq r} \|A(u)\|_{W^{1,p}(\Omega)^*} = \sup_{\|u\|_{W^{1,p}(\Omega)} \leq r} \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \{|\langle A(u), v \rangle|\} \\
&= \sup_{\|u\|_{W^{1,p}(\Omega)} \leq r} \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \left\{ \left| \int_{\Omega} a(x, u, \nabla u) \cdot \nabla v + \int_{\Omega} c(x, u, \nabla u) v dx + \int_{\Gamma_N} b(x, u) v dS \right| \right\} \\
&\leq \sup_{\|u\|_{W^{1,p}(\Omega)} \leq r} \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \left\{ \int_{\Omega} |a(x, u, \nabla u) \cdot \nabla v| dx + \int_{\Omega} |c(x, u, \nabla u) v| dx + \int_{\Gamma_N} |b(x, u) v| dS \right\} \\
&\leq \sup_{\|u\|_{W^{1,p}(\Omega)} \leq r} \sup_{\|v\|_{W^{1,p}(\Omega)} \leq 1} \left\{ \|a(x, u, \nabla u)\|_{L^{p'}(\Omega; \mathbb{R}^N)} \|\nabla v\|_{L^p(\Omega; \mathbb{R}^N)} + \|c(x, u, \nabla u)\|_{L^{p^*}(\Omega)} \|v\|_{L^p(\Omega)} \right. \\
&+ \left. \|b(x, u)\|_{L^{p^{\#}'}(\Gamma_N)} \|v\|_{L^{p^{\#}}(\Gamma_N)} \right\} \\
&\leq \sup_{\|u\|_{W^{1,p}(\Omega)} \leq r} \left\{ \|a(x, u, \nabla u)\|_{L^{p'}(\Omega; \mathbb{R}^N)} + C \|c(x, u, \nabla u)\|_{L^{p^*}(\Omega)} + D \|b(x, u)\|_{L^{p^{\#}'}(\Gamma_N)} \right\} \leq M.
\end{aligned}$$

Again, C is from the continuous embedding $W^{1,p}(\Omega) \subset L^p(\Omega)$, D is from the trace operator $T : W^{1,p}(\Omega) \rightarrow L^{p^{\#}}(\Gamma)$ and $M > 0$ is a constant depending on r . The last inequality above is obtained from the growth conditions. \square

Lemma 3.1.5. $A_1 : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ is a bounded operator.

Proof: We have already shown that A is bounded. Take bounded set $B \subset W^{1,p}(\Omega)$ (so for $b \in B$, it follows that $\|b\| \leq M$, where $M > 0$). It would suffice to show that $B + w$ is bounded. But, it is straightforward to show that a translation of a bounded set is bounded. \square

Lemma 3.1.6. (The Coercivity of A) Let the following hold:

$$\exists \epsilon_1, \epsilon_2 > 0, k_1 \in L^1(\Omega) \text{ such that } a(x, s, \xi) \cdot \xi + c(x, s, \xi) s \geq \epsilon_1 |\xi|^p + \epsilon_2 |s|^q - k_1(x), \quad (3.26)$$

$$\exists \beta < +\infty \text{ and } \exists k_2 \in L^1(\Gamma) \text{ such that } b(x, s) s \geq -\beta |s|^{q_1} - k_2(x) \quad (3.27)$$

for some $1 < q_1 < q \leq p$. Then $A : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ is coercive.

Proof: Using Theorem 1.5.14, yields

$$\begin{aligned}
\|u\|_{W^{1,p}(\Omega)}^q &\leq 2^{q-1} C_p^q (\|\nabla u\|_{L^p(\Omega; \mathbb{R}^N)}^q + \|u\|_{L^q(\Omega)}^q) \\
&\leq C_{p,q} (1 + \|\nabla u\|_{L^p(\Omega; \mathbb{R}^N)}^q + \|u\|_{L^q(\Omega)}^q).
\end{aligned} \quad (3.28)$$

Young's inequality states that

$$\forall \epsilon > 0 \text{ and } 1 < m < +\infty \text{ we have } ab \leq \epsilon a^m + C_\epsilon b^{m'},$$

where constant C_ϵ is taken as in (7.2). Using this inequality yields

$$|u(x)| |1| \leq \epsilon |u(x)|^m + C_\epsilon |1|^{m'} = \epsilon |u(x)|^m + C_\epsilon.$$

Let $m := \frac{q}{q_1} > 1$ then $|u(x)| \leq \epsilon |u(x)|^{\frac{q}{q_1}} + C_\epsilon$, therefore,

$$|u(x)|^{q_1} \leq \left(\epsilon |u(x)|^{\frac{q}{q_1}} + C_\epsilon \right)^{q_1} \leq 2^{q_1-1} \epsilon^{q_1} |u(x)|^q + C_\epsilon^{q_1}.$$

Using the boundedness of trace operator $T : W^{1,p}(\Omega) \rightarrow L^q(\Gamma)$, implies $\|u\|_{L^q(\Gamma)} \leq G\|u\|_{W^{1,p}(\Omega)}$, therefore,

$$\|u\|_{L^q(\Gamma)}^q \leq G^q \|u\|_{W^{1,p}(\Omega)}^q$$

hence,

$$2^{q_1-1} \epsilon^{q_1} \int_{\Gamma} |u|^q dS \leq 2^{q_1-1} \epsilon^{q_1} G^q \|u\|_{W^{1,p}(\Omega)}^q,$$

where G is the constant from the trace operator. Putting these inequalities together yields

$$\|u\|_{L^{q_1}(\Gamma)}^{q_1} = \int_{\Gamma} |u|^{q_1} dS \leq \int_{\Gamma} 2^{q_1-1} \epsilon^{q_1} |u|^q + C_{\epsilon}^{q_1} dS \leq 2^{q_1-1} \epsilon^{q_1} G^q \|u\|_{W^{1,p}(\Omega)}^q + C_{\epsilon}^{q_1} \text{meas}_{n-1}(\Gamma), \quad (3.29)$$

where $\epsilon > 0$ is chosen arbitrarily small.

For the next estimate, use (3.26) and (3.27):

$$\begin{aligned} \langle A(u), u \rangle &= \int_{\Omega} a(x, u, \nabla u) \cdot \nabla u + c(x, u, \nabla u) u dx + \int_{\Gamma} b(x, u) u dS \\ &\geq \int_{\Omega} (\epsilon_1 |\nabla u|^p + \epsilon_2 |u|^q - k_1) dx - \int_{\Gamma} (\beta |u|^{q_1} + k_2) dS. \end{aligned} \quad (3.30)$$

From (3.28), we have

$$\int_{L^p(\Omega; \mathbb{R}^N)} |\nabla u|^p dx + \int_{L^q(\Omega)} |u|^q dx \geq \left(\frac{\|u\|_{W^{1,p}(\Omega)}^q}{C_{p,q}} - 1 \right),$$

therefore,

$$\int_{\Omega} (\epsilon_1 |\nabla u|^p + \epsilon_2 |u|^q - k_1) dx \geq \min(\epsilon_1, \epsilon_2) \left(\frac{\|u\|_{W^{1,p}(\Omega)}^q}{C_{p,q}} - 1 \right) - \|k_1\|_{L^1(\Omega)}.$$

From (3.29), we have

$$\int_{\Gamma} (\beta |u|^{q_1} + k_2) dS \leq \beta 2^{q_1-1} \epsilon^{q_1} G^q \|u\|_{W^{1,p}(\Omega)}^q + \beta C_{\epsilon}^{q_1} \text{meas}_{n-1}(\Gamma) + \|k_2\|_{L^1(\Gamma)}.$$

Putting these inequalities into (3.30) yields the desired inequality,

$$\begin{aligned} \langle A(u), u \rangle &\geq \min(\epsilon_1, \epsilon_2) \left(\frac{\|u\|_{W^{1,p}(\Omega)}^q}{C_{p,q}} - 1 \right) - \|k_1\|_{L^1(\Omega)} - \beta 2^{q_1-1} \epsilon^{q_1} G^q \|u\|_{W^{1,p}(\Omega)}^q \\ &\quad - \beta C_{\epsilon}^{q_1} \text{meas}_{n-1}(\Gamma) - \|k_2\|_{L^1(\Gamma)}. \end{aligned} \quad (3.31)$$

Choose ϵ small enough, so that we have $\epsilon^{q_1} < \frac{\min(\epsilon_1, \epsilon_2)}{C_{p,q} \beta 2^{q_1-1} G^q}$, and note that $q > 1$. Then,

$\lim_{\|u\|_{W^{1,p}(\Omega)} \rightarrow \infty} \langle A(u), u \rangle = +\infty$. This shows the coercivity of A . \square

Lemma 3.1.7. $A_1 : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ is coercive.

Proof: Fix $w \in W^{1,p}(\Omega)$. If we take $c > 1$ and $R := \frac{c\|w\|_{W^{1,p}(\Omega)}}{c-1}$, then for all $u \in W^{1,p}(\Omega)$ with $\|u\|_{W^{1,p}(\Omega)} \geq R$ it holds that

$$\|u\|_{W^{1,p}(\Omega)} \leq c\|u+w\|_{W^{1,p}(\Omega)}. \quad (3.32)$$

Consider $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ taken as in Definition 2.1.14, so, φ is unbounded ($\lim_{t \rightarrow \infty} \varphi(t) = +\infty$) and $A(u) \geq \varphi(\|u\|_{W^{1,p}(\Omega)})\|u\|_{W^{1,p}(\Omega)}$. Define the mapping $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$\psi(r) := \begin{cases} c \inf_{v \in W^{1,p}(\Omega), \|v\|=r} \varphi(\|v+w\|_{W^{1,p}(\Omega)}) & \text{for } r \geq R, \\ 0 & \text{for } r < R. \end{cases} \quad (3.33)$$

Note that $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is well-defined:

This follows by observing that $\inf_{v \in W^{1,p}(\Omega), \|v\|=r} \varphi(\|v+w\|_{W^{1,p}(\Omega)}) \geq 0$ (the infimum exists by the boundedness of φ from below).

We also have that $\psi(r) \rightarrow +\infty$ as $r \rightarrow \infty$. To show this, assume to the contrary that ψ is bounded. By properties of the infimum, for each $n \in \mathbb{N}$ (chosen large enough) with $n \geq R$, there exists $u_n \in W^{1,p}(\Omega)$ such that $\|u_n\|_{W^{1,p}(\Omega)} = n$ and

$$\varphi(\|u_n+w\|_{W^{1,p}(\Omega)}) - 1 \leq \inf_{v \in W^{1,p}(\Omega), \|v\|=n} \varphi(\|v+w\|_{W^{1,p}(\Omega)}) = \psi(n) = \psi(\|u_n\|_{W^{1,p}(\Omega)}) \quad (3.34)$$

(this follows since we can choose a sequence converging to the infimum). So, if ψ is bounded by some constant $D > 0$, then we would have

$$\varphi(\|u_n+w\|_{W^{1,p}(\Omega)}) \leq \psi(\|u_n\|_{W^{1,p}(\Omega)}) + 1 \leq D + 1 =: M. \quad (3.35)$$

Clearly, if $n \rightarrow \infty$ then $\|u_n\|_{W^{1,p}(\Omega)} \rightarrow \infty$, and using (3.32) yields $\|u_n+w\|_{W^{1,p}(\Omega)} \rightarrow \infty$, but then $\limsup_{n \rightarrow \infty} \varphi(\|u_n+w\|_{W^{1,p}(\Omega)}) \leq M$ contradicts φ being unbounded. Thus, we have shown that $\psi(r) \rightarrow +\infty$ as $r \rightarrow \infty$.

It follows also that for all $u \in W^{1,p}(\Omega)$ with $\|u\|_{W^{1,p}(\Omega)} \geq R$ that

$$\begin{aligned} A_1(u) &= A(u+w) \geq \varphi(\|u+w\|_{W^{1,p}(\Omega)})\|u+w\|_{W^{1,p}(\Omega)} \\ &\geq \frac{1}{c} \varphi(\|u+w\|_{W^{1,p}(\Omega)})\|u\|_{W^{1,p}(\Omega)} \\ &\geq \psi(\|u\|_{W^{1,p}(\Omega)})\|u\|_{W^{1,p}(\Omega)}, \end{aligned}$$

and for all $u \in W^{1,p}(\Omega)$ with $\|u\|_{W^{1,p}(\Omega)} < R$ we have

$$A_1(u) = A(u+w) \geq \varphi(\|u+w\|_{W^{1,p}(\Omega)})\|u+w\|_{W^{1,p}(\Omega)} \geq 0 = \psi(\|u\|_{W^{1,p}(\Omega)})\|u\|_{W^{1,p}(\Omega)}.$$

Hence, A_1 is coercive. \square

We now seek to prove the pseudomonotonicity of A and by implication the pseudomonotonicity of A_1 (we have already shown that A and A_1 are bounded, thus only (2.13) needs to be proved).

Lemma 3.1.8. *(Proof of (2.13)) Assume (3.2)-(3.6) hold. Additionally, assume the following:*

$$(a(x, s, \xi) - a(x, s, \bar{\xi})) \cdot (\xi - \bar{\xi}) = 0 \implies \xi = \bar{\xi}, \quad (3.36)$$

$$\forall \xi_0 \in \mathbb{R}^N : \lim_{|\xi| \rightarrow \infty} \frac{a(x, s, \xi) \cdot (\xi - \xi_0)}{|\xi|} = +\infty \quad \text{uniformly for } s \text{ bounded}, \quad (3.37)$$

$$\exists \gamma \in L^{p^*'+\epsilon}(\Omega) \exists C \in \mathbb{R} : |c(x, s, \xi)| \leq \gamma(x) + C|s|^{p^*-\epsilon-1} + C|\xi|^{\frac{p-\epsilon}{p^*'}}. \quad (3.38)$$

Then $A : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ satisfies (2.13).

Proof: From the growth condition on c (as assumed in (3.38)), it follows from the Nemytskii Theorem (see appendix for description) that for some $\epsilon > 0$,

$$\eta_c : W^{1,p}(\Omega) \times L^p(\Omega; \mathbb{R}^N) \rightarrow L^{p^*'+\epsilon}(\Omega) \quad (3.39)$$

is (weak \times norm, norm)-continuous. Assume that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and

$$\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0. \quad (3.40)$$

We want to show $\liminf_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle \geq \langle A(u), u - v \rangle$ for all $v \in W^{1,p}(\Omega)$. Consider a notational adjustment which will assist in highlighting the difference between the higher and lower-order terms: define $B(w, u) \in W^{1,p}(\Omega)^*$ by

$$\langle B(w, u), v \rangle := \int_{\Omega} a(x, w, \nabla u) \cdot \nabla v + c(x, w, \nabla w) v dx + \int_{\Gamma_N} b(x, w) v dS, \quad (3.41)$$

where $u, w \in W^{1,p}(\Omega)$ (this implies that $A(u) = B(u, u)$). Consider the line segment between u and v , so, $u_t := (1-t)u + tv$, $t \in [0, 1]$. We then get the following result from the monotonicity of the principal part assumed in (3.6):

$$\langle B(u_n, u_n) - B(u_n, u_t), u_n - u_t \rangle = \int_{\Omega} (a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u_t)) \cdot \nabla (u_n - u_t) dx \geq 0. \quad (3.42)$$

Using (3.42), we obtain

$$\langle B(u_n, u_n), u_n - ((1-t)u + tv) \rangle \geq \langle B(u_n, u_t), u_n - ((1-t)u + tv) \rangle,$$

therefore,

$$\langle A(u_n), u_n - u \rangle + t \langle A(u_n), u - v \rangle \geq \langle B(u_n, u_t), u_n - u \rangle + t \langle B(u_n, u_t), u - v \rangle. \quad (3.43)$$

For now, assume the following results hold (a proof will be provided later):

$$\lim_{n \rightarrow \infty} \langle B(u_n, v), u_n - u \rangle = 0 \quad (3.44)$$

$$B(u_n, v) \rightharpoonup B(u, v) \quad \text{in } W^{1,p}(\Omega)^*. \quad (3.45)$$

Pass to the limit in (3.43), using claims (3.44) and (3.45):

$$\begin{aligned} t \liminf_{n \rightarrow \infty} \langle A(u_n), u - v \rangle &\geq \liminf_{n \rightarrow \infty} (-\langle A(u_n), u_n - u \rangle) + \liminf_{n \rightarrow \infty} \langle B(u_n, u_t), u_n - u \rangle \\ &\quad + t \liminf_{n \rightarrow \infty} \langle B(u_n, u_t), u - v \rangle \\ &= -\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle + t \langle B(u, u_t), u - v \rangle \\ &\geq t \langle B(u, u_t), u - v \rangle. \end{aligned}$$

Now, dividing by $t > 0$ yields $\liminf_{n \rightarrow \infty} \langle A(u_n), u - v \rangle \geq \langle B(u, u_t), u - v \rangle$. Let $t \rightarrow 0$, then $u_t \rightarrow u$ in $W^{1,p}(\Omega)$ implies $B(u, u_t) \rightarrow B(u, u)$ in $W^{1,p}(\Omega)^*$. This follows from growth condition (3.3), together with the Nemytskii Mappings Theorem:

Consider the composition $a \circ u : (x, \xi) \mapsto a(x, u(x), \xi)$, so $\eta_{a \circ u} : L^p(\Omega; \mathbb{R}^N) \rightarrow L^{p'}(\Omega; \mathbb{R}^N)$ is (norm, norm)-continuous. Let $\xi := \nabla u_t$ then $(\eta_{a \circ u}(\nabla u_t))(x) = a(x, u(x), \nabla u_t(x))$. By the (norm, norm)-continuity it follows that if $u_t \rightarrow u$ in $W^{1,p}(\Omega)$, then

$$a(x, u(x), \nabla u_t(x)) \rightarrow a(x, u(x), \nabla u(x)) \quad \text{in } L^{p'}(\Omega; \mathbb{R}^N).$$

Using Hölder's Inequality it follows that

$$\begin{aligned} & \|B(u, u_t) - B(u, u)\|_{(W^{1,p}(\Omega))^*} \\ & \leq \sup_{\substack{v \in W^{1,p}(\Omega) \\ \|v\|_{W^{1,p}(\Omega)} \leq 1}} \int_{\Omega} |a(x, u(x), \nabla u_t(x)) - a(x, u(x), \nabla u(x))| |\nabla v(x)| dx \\ & \leq \sup_{\substack{v \in W^{1,p}(\Omega) \\ \|v\|_{W^{1,p}(\Omega)} \leq 1}} \|a(x, u(x), \nabla u_t(x)) - a(x, u(x), \nabla u(x))\|_{L^{p'}(\Omega; \mathbb{R}^N)} \|\nabla v(x)\|_{L^p(\Omega; \mathbb{R}^N)} \\ & \leq \|a(x, u(x), \nabla u_t(x)) - a(x, u(x), \nabla u(x))\|_{L^{p'}(\Omega; \mathbb{R}^N)} \rightarrow 0. \end{aligned}$$

This shows that $u_t \rightarrow u$ in $W^{1,p}(\Omega) \implies B(u, u_t) \rightarrow B(u, u) = A(u)$ in $W^{1,p}(\Omega)^*$. Therefore, $\liminf_{n \rightarrow \infty} \langle A(u_n), u - v \rangle \geq \langle B(u, u), u - v \rangle = \langle A(u), u - v \rangle$.

Now, use the monotonicity of the principal part assumed in (3.6), this yields $\langle B(u_n, u_n) - B(u_n, u), u_n - u \rangle \geq 0$, then (3.44) with $v = u$ implies that

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle A(u_n), u_n - v \rangle & \geq \liminf_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle + \liminf_{n \rightarrow \infty} \langle A(u_n), u - v \rangle \\ & \geq \lim_{n \rightarrow \infty} \langle B(u_n, u), u_n - u \rangle + \liminf_{n \rightarrow \infty} \langle B(u_n, u_n) - B(u_n, u), u_n - u \rangle \\ & \quad + \liminf_{n \rightarrow \infty} \langle A(u_n), u - v \rangle \geq \langle A(u), u - v \rangle. \end{aligned} \tag{3.46}$$

Hence, we have shown that if our assumptions (3.44) and (3.45) hold, then the proof of (2.13) is complete.

We will now show that (3.44) and (3.45) hold. Since we assume that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega) \Subset L^{p^*-\epsilon}(\Omega)$, we have $u_n \rightarrow u$ in $L^{p^*-\epsilon}(\Omega)$. We also have $u_n|_{\Gamma} \rightarrow u|_{\Gamma}$ in $L^{p^{\#}-\epsilon}(\Gamma)$ (since the trace operator $T : W^{1,p}(\Omega) \rightarrow L^{p^{\#}-\epsilon}(\Gamma)$ is a compact operator, therefore, there exists a subsequence such that $T(u_{n_k}) \rightarrow T(u)$ in $L^{p^{\#}-\epsilon}(\Gamma)$). Lemma 1.2.10 can be used to show that even the whole sequence $\{T(u_n)\}_{n \in \mathbb{N}}$ converges in $L^{p^{\#}-\epsilon}(\Gamma)$). By the continuity of Nemytskii mappings induced by $a(\cdot, \cdot, \nabla v)$ and b , we obtain $a(x, u_n, \nabla v) \rightarrow a(x, u, \nabla v)$ in $L^{p'}(\Omega; \mathbb{R}^N)$, as well as $b(x, u_n) \rightarrow b(x, u)$ in $L^{p^{\#}}(\Gamma)$. Also, $\nabla(u_n - u) \rightharpoonup 0$ in $L^p(\Omega; \mathbb{R}^N)$ and $(u_n - u)|_{\Gamma} \rightharpoonup 0$ in $L^{p^{\#}}(\Gamma_N)$ (follows from the (weak, weak)-continuity of the trace operator T).

The previous observations are used in the following working: Add and subtract

$a(x, u, \nabla v) \cdot \nabla(u_n - u)$ from the integrand of $\int_{\Omega} a(x, u_n, \nabla v) \cdot \nabla(u_n - u) dx$, then

$$\begin{aligned} & \int_{\Omega} a(x, u_n, \nabla v) \cdot \nabla(u_n - u) dx \\ &= \int_{\Omega} (a(x, u_n, \nabla v) \cdot \nabla(u_n - u) + a(x, u, \nabla v) \cdot \nabla(u_n - u) - a(x, u, \nabla v) \cdot \nabla(u_n - u)) dx \\ &= \int_{\Omega} (a(x, u_n, \nabla v) - a(x, u, \nabla v)) \cdot \nabla(u_n - u) dx + \int_{\Omega} a(x, u, \nabla v) \cdot \nabla(u_n - u) dx. \end{aligned}$$

Consider the first term, then Hölder's Inequality yields

$$\begin{aligned} & \left| \int_{\Omega} (a(x, u_n, \nabla v) - a(x, u, \nabla v)) \cdot \nabla(u_n - u) dx \right| \\ & \leq \|a(x, u_n, \nabla v) - a(x, u, \nabla v)\|_{L^{p'}(\Omega; \mathbb{R}^N)} \|\nabla(u_n - u)\|_{L^p(\Omega; \mathbb{R}^N)}, \end{aligned}$$

where $\|a(x, u_n, \nabla v) - a(x, u, \nabla v)\|_{L^{p'}(\Omega; \mathbb{R}^N)} \rightarrow 0$ and $\nabla(u_n - u)$ is bounded in $L^p(\Omega; \mathbb{R}^N)$.

Consider the second term, then $\int_{\Omega} a(x, u, \nabla v) \cdot \nabla(u_n - u) dx \rightarrow 0$, since $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\Omega; \mathbb{R}^N)$.

Using similar arguments, it can be shown that $\int_{\Gamma_N} b(x, u_n)(u_n - u) dS \rightarrow 0$.

Together, these results yield

$$\int_{\Omega} a(x, u_n, \nabla v) \cdot \nabla(u_n - u) dx + \int_{\Gamma_N} b(x, u_n)(u_n - u) dS \rightarrow 0. \quad (3.47)$$

Furthermore, it can be shown that for any $z \in W^{1,p}(\Omega)$, we have

$$\int_{\Omega} a(x, u_n, \nabla v) \cdot \nabla z dx + \int_{\Gamma_N} b(x, u_n) z dS \rightarrow \int_{\Omega} a(x, u, \nabla v) \cdot \nabla z dx + \int_{\Gamma_N} b(x, u) z dS. \quad (3.48)$$

Since $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\nabla u_n \rightharpoonup \nabla u$ in $L^p(\Omega; \mathbb{R}^N)$, using the continuity and boundedness of the Nemytskii mapping in (3.39), it can be shown that $\{c(x, u_n, \nabla u_n)\}_{n \in \mathbb{N}}$ is bounded in $L^{p^* + \epsilon}(\Omega)$. For ϵ taken small enough, it follows that $(p^* - \epsilon)' = \frac{p^* - \epsilon}{p^* - \epsilon - 1} < p^* - \epsilon$, so using Hölder's Inequality yields

$$\left| \int_{\Omega} c(x, u_n, \nabla u_n)(u_n - u) dx \right| \leq \left(\int_{\Omega} |u_n - u|^{p^* - \epsilon} dx \right)^{\frac{1}{p^* - \epsilon}} \left(\int_{\Omega} |c(x, u_n, \nabla u_n)|^{(p^* - \epsilon)'} dx \right)^{\frac{1}{(p^* - \epsilon)'}},$$

then, since $u_n \rightarrow u$ in $L^{p^* - \epsilon}(\Omega)$, we get

$$\int_{\Omega} c(x, u_n, \nabla u_n)(u_n - u) dx \rightarrow 0. \quad (3.49)$$

Thus, (3.49) together with (3.47) yields (3.44).

We now seek to prove (3.45). To do this, we will first show that $\nabla u_n(x) \rightarrow \nabla u(x)$ a.e. $x \in \Omega$. First consider

$$\mathcal{B}_n(x) := (a(x, u_n(x), \nabla u_n(x)) - a(x, u_n(x), \nabla u(x))) \cdot \nabla(u_n(x) - u(x)). \quad (3.50)$$

From (3.6), it follows that

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} \int_{\Omega} \mathcal{B}_n(x) dx = \limsup_{n \rightarrow \infty} \langle B(u_n, u_n) - B(u_n, u), u_n - u \rangle \\ &\leq \limsup_{n \rightarrow \infty} \langle B(u_n, u_n), u_n - u \rangle - \liminf_{n \rightarrow \infty} \langle B(u_n, u), u_n - u \rangle \\ &= \limsup_{n \rightarrow \infty} \langle B(u_n, u_n), u_n - u \rangle - \lim_{n \rightarrow \infty} \langle B(u_n, u), u_n - u \rangle \leq 0. \end{aligned}$$

This clearly follows from our assumption that $\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0$ and (3.44). We then have

$$0 \leq \liminf_{n \rightarrow \infty} \int_{\Omega} \mathcal{B}_n(x) dx \leq \limsup_{n \rightarrow \infty} \int_{\Omega} \mathcal{B}_n(x) dx \leq 0.$$

Hence, $\lim_{n \rightarrow \infty} \int_{\Omega} \mathcal{B}_n(x) dx = 0$. Using Theorem 1.3.14 (Markov's Inequality), for $\epsilon > 0$,

$$\text{meas}(\{x \in \Omega : |\mathcal{B}_n(x)| \geq \epsilon\}) = \text{meas}(\{x \in \Omega : \mathcal{B}_n(x) \geq \epsilon\}) \leq \frac{1}{\epsilon} \int_{\Omega} \mathcal{B}_n(x) dx.$$

Taking the limit as $n \rightarrow \infty$, it follows that $\mathcal{B}_n \rightarrow 0$ in measure. It follows then from Proposition 1.4.10, that we can choose a subsequence such that

$$\mathcal{B}_n(x) \rightarrow 0 \quad \text{for a.e. } x \in \Omega. \quad (3.51)$$

Since we have that $u_n \rightarrow u$ in $L^{p^*-\epsilon}(\Omega) \subset L^1(\Omega)$, it follows again from Proposition 1.4.10 that we can choose a subsequence, such that

$$u_n(x) \rightarrow u(x) \quad \text{for a.e. } x \in \Omega. \quad (3.52)$$

Choose $x \in \Omega$ such that the following is satisfied (this can be done since these properties hold up to a set of measure zero):

Both (3.51) and (3.52) hold. Also, $\nabla u(x), \nabla u_n(x)$ and $\gamma(x)$ (as in (3.14)) are finite, and $a(x, \cdot, \cdot)$ is continuous (using the assumption that c is a Carathéodory function). Now assume that the sequence $\{\nabla u_n(x)\}_n$ is unbounded, hence there exists a subsequence such that $\nabla u_n(x) \rightarrow \infty$. Using the assumed coercivity from (3.37), with $\xi_0 = \nabla u(x)$, results in

$$\limsup_{n \rightarrow \infty} (a(x, u_n(x), \nabla u_n(x)) - a(x, u_n(x), \xi_0)) \cdot (\nabla u_n(x) - \xi_0) = +\infty.$$

This contradicts (3.51). Therefore, we have a subsequence and a $\xi \in \mathbb{R}^N$ such that $\nabla u_n(x) \rightarrow \xi$ in \mathbb{R}^N . Using (3.51), (3.52) and the continuity of $a(x, \cdot, \cdot)$, then passing to the limit in (3.50), yields

$$(a(x, u(x), \xi) - a(x, u(x), \nabla u(x))) \cdot (\xi - \nabla u(x)) = 0. \quad (3.53)$$

By (3.36), it follows that $\xi = \nabla u(x)$. Then, even the whole sequence $\{\nabla u_n(x)\}_n$ converges to $\nabla u(x)$. We have therefore shown that $\nabla u_n(x) \rightarrow \nabla u(x)$ a.e. $x \in \Omega$.

Since c is a Carathéodory function, it follows that

$$c(x, u_n(x), \nabla u_n(x)) \rightarrow c(x, u(x), \nabla u(x)) \quad \text{a.e. } x \text{ in } \Omega.$$

Also, since $c(x, u_n(x), \nabla u_n(x)) \in L^{p^{*'}+\epsilon}(\Omega)$, with $p^{*'}+\epsilon > 1$, it follows that $c(x, u_n(x), \nabla u_n(x))$ is uniformly integrable. Hence by Theorem 1.4.13 (Vitali's Convergence Theorem) it follows that

$$\lim_{n \rightarrow \infty} \int_E c(x, u_n(x), \nabla u_n(x)) dx = \int_E c(x, u(x), \nabla u(x)) dx, \quad (3.54)$$

for $E \subset \Omega$. As mentioned previously, $\{c(x, u_n(x), \nabla u_n(x))\}_{n \in \mathbb{N}}$ is bounded in $L^{p^{*'}+\epsilon}(\Omega)$. Therefore, there exists a subsequence such that $c(x, u_n(x), \nabla u_n(x)) \rightharpoonup \phi$ in $L^{p^{*'}+\epsilon}(\Omega)$. This yields

$$\lim_{n \rightarrow \infty} \int_E c(x, u_n(x), \nabla u_n(x)) dx = \int_E \phi dx,$$

but then,

$$\int_E c(x, u(x), \nabla u(x)) dx = \int_E \phi dx.$$

If we consider open balls $E := B(x, r)$, then using Theorem 1.3.12 (Lebesgue's Differentiation Theorem), as $r \rightarrow 0$, we can show that $c(x, u(x), \nabla u(x)) = \phi(x)$ a.e. $x \in \Omega$. It follows then that $c(x, u_n(x), \nabla u_n(x)) \rightharpoonup c(x, u(x), \nabla u(x))$ in $L^{p^{*'}+\epsilon}(\Omega)$. Using Lemma 1.2.10, it follows that the whole sequence $\{c(x, u_n(x), \nabla u_n(x))\}_n$ converges weakly to $c(x, u(x), \nabla u(x))$ in $L^{p^{*'}+\epsilon}(\Omega)$. From Theorem 1.5.12, we have that $W^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$, since $c(x, u_n(x), \nabla u_n(x)) \rightharpoonup c(x, u_n(x), \nabla u_n(x))$ in $L^{p^{*'}}(\Omega)$, it follows that

$$\int_{\Omega} c(x, u_n(x), \nabla u_n(x)) z dx \rightarrow \int_{\Omega} c(x, u(x), \nabla u(x)) z dx \quad (3.55)$$

for $z \in W^{1,p}(\Omega)$. From (3.55) and (3.48), we obtain (3.45). We have thus shown that (2.13) is satisfied. \square

Lemma 3.1.4 with Lemma 3.1.8 proves the pseudomonotonicity of $A : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$.

Lemma 3.1.9. $A_1 : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ is pseudomonotone.

Proof: This follows simply from Lemma 2.1.16. \square

Remark 16. Suppose $c(x, s, \xi)$ is independent of ξ , it is then no longer necessary for the assumptions of (3.36)-(3.38). The proof of pseudomonotonicity is simplified in this case. Consider $c(x, s, \xi) = \tilde{c}(x, s)$, since we assume $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $c(x, s, \xi)$ satisfies the growth condition (3.5), using the Neymetskii Theorem, yields the result that $\eta_{\tilde{c}} : W^{1,p}(\Omega) \rightarrow L^{p^{*'}}(\Omega)$ is (weak, norm)-continuous. Consequently, $\tilde{c}(x, u_n) \rightarrow \tilde{c}(x, u)$ in $L^{p^{*'}}(\Omega)$. We observe also that $u_n - u \rightharpoonup 0$ in $L^{p^*}(\Omega)$ (since $W^{1,p}(\Omega)$ is continuously embedded in $L^{p^*}(\Omega)$), this gives $\int_{\Omega} \tilde{c}(x, u_n)(u_n - u) dx \rightarrow 0$. This, together with (3.47), yields

$$\begin{aligned} & \langle B(u_n, v), u_n - u \rangle \\ &= \int_{\Omega} (a(x, u_n, \nabla v) \cdot \nabla(u_n - u) + \tilde{c}(x, u_n)(u_n - u)) dx + \int_{\Gamma_N} b(x, u_n)(u_n - u) dS \longrightarrow 0. \end{aligned} \quad (3.56)$$

This proves (3.44). It can also be shown that $\int_{\Omega} \bar{c}(x, u_n)z dx \rightarrow \int_{\Omega} \bar{c}(x, u)z dx$ for $z \in W^{1,p}(\Omega)$, which, together with (3.48), yields (3.45). Thus, the pseudomonotonicity is shown.

The requirements of Theorem 2.1.15 are satisfied. Applying Theorem 2.1.15 in the previous chapter, together with our previous results in this chapter, will allow us to show the existence of a weak solution to our second order boundary value problem (3.1).

The following result is credited to French mathematicians Jean Leray and Jacques-Louis Lions (see [43]).

Theorem 3.1.10. (*Leray-Lions(1965)*) *Let (3.2)-(3.6), (3.22), (3.26), (3.27) and (3.36)-(3.38) hold, then our second order boundary value problem (3.1) has a weak solution.*

Proof: From Lemmas 3.1.5, 3.1.7 and 3.1.9 it follows that $A_1 : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ is both pseudomonotone and coercive. These are exactly the requirements necessary for the application of Theorem 2.1.15. Using Theorem 2.1.15, allows us to prove that $A_1 : V \rightarrow V^*$ is surjective (so for $f \in V^*$, there is at least one solution to the equation $A_1(u) = f$). Then from Proposition 3.1.3, we can conclude that the boundary value problem (3.1) (with (3.8) and (3.9)) has a weak solution. \square

Remark 17. We might want to consider applying this approach to higher order equations. It would seem natural to first consider a generalization to equations involving $2k$ -order derivatives where $k \geq 2$. We would then have k -boundary conditions. A full treatment of this case can be found in [58].

3.1.3 Application of Anisotropic Operator

In the following section, we examine the problem of finding a weak solution to some nonlinear elliptic equation, by considering an operator from an anisotropic Sobolev space to its dual. We seek to prove the existence of a weak solution to the following boundary value problem

$$\begin{cases} -\operatorname{div}(a(x, u, \nabla u)) + c(x, u) = g & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \quad (3.57)$$

where we have open, bounded domain $\Omega \subset \mathbb{R}^N$, $0 < \epsilon < 1$ and $g \in L^{1+\epsilon}(\Omega)$. Let $\vec{p} = (p_1, \dots, p_N)$ and $\vec{p}' = (p'_1, \dots, p'_N)$, where $p_i > 1$.

Consider the following anisotropic Sobolev space:

$$W^{1, \vec{p}, \epsilon}(\Omega) := \left\{ u \in L^{1+\frac{1}{\epsilon}}(\Omega) : \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), \quad i \in \{1, \dots, N\} \right\}, \quad (3.58)$$

which is endowed with the norm

$$\|u\|_{1, \vec{p}, \epsilon} := \|u\|_{L^{1+\frac{1}{\epsilon}}(\Omega)} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}. \quad (3.59)$$

Then, let $W_0^{1, \vec{p}, \epsilon}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{1, \vec{p}, \epsilon}(\Omega)}$, which is endowed with the norm

$$\|u\| := \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}. \quad (3.60)$$

Theorem 1.5.23 states the equivalence of these norms. This equivalence, will be used in later arguments.

Define the nonlinear operator A on $V := W_0^{1, \vec{p}, \epsilon}(\Omega)$ into its dual (which we denote as $W^{-1, \vec{p}', \epsilon}(\Omega)$), by

$$A(u) := -\operatorname{div}(a(x, u, \nabla u)),$$

again we take $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ and $c : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ as Carathéodory functions. Assume that a and c satisfy the following conditions:

$$|a_i(x, s, \xi)| \leq \beta \left(k(x) + |s|^{p_0} + \sum_{j=1}^N |\xi_j|^{p_j} \right)^{1 - \frac{1}{p_i}}, \quad (3.61)$$

for a.e. $x \in \Omega$, $\forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, for $i \in \{1, \dots, N\}$. Where $\beta > 0$ is a real constant and $k \in L^1(\Omega)$ is a non-negative function.

$$\sum_{i=1}^N (a_i(x, s, \xi) - a_i(x, s, \xi^*)) (\xi_i - \xi_i^*) > 0, \quad (3.62)$$

for a.e. $x \in \Omega$, and every $\xi, \xi^* \in \mathbb{R}^N$, where $\xi \neq \xi^*$.

$$a(x, s, \xi) \xi \geq \alpha \sum_{i=1}^N |\xi_i|^{p_i}, \quad (3.63)$$

for a.e. $x \in \Omega$ and every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $\alpha > 0$ is a real constant.

Assume $c : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies

$$\sup_{|u| \leq s} |c(x, u)| \leq h_s(x), \quad (3.64)$$

for a.e. $x \in \Omega$, all $s > 0$ and some function $h_s \in L^{\frac{1}{1-\epsilon}}(\Omega)$, where $0 < \epsilon < 1$. Additionally, we assume that

$$c(x, u)u \geq 0 \quad (3.65)$$

for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$. Note that there is no assumed growth condition on c .

Theorem 3.1.11. *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, with smooth boundary. Assume that (3.61)-(3.65) are satisfied. Then, for any $g \in L^{1+\epsilon}(\Omega)$, where $0 < \epsilon < 1$, there exists at least one non-trivial weak solution $u \in W_0^{1, \vec{p}, \epsilon}(\Omega)$, to the problem (3.57), i.e.,*

$$\int_{\Omega} a(x, u, \nabla u) \cdot \nabla v dx + \int_{\Omega} c(x, u) v dx = \int_{\Omega} g v dx \quad \forall v \in W_0^{1, \vec{p}, \epsilon}(\Omega) \cap L^{\infty}(\Omega). \quad (3.66)$$

Proof: Consider

$$\langle A(u), v \rangle := \int_{\Omega} a(x, u, \nabla u) \cdot \nabla v dx \quad \forall u, v \in W^{1, \vec{p}, \epsilon}(\Omega). \quad (3.67)$$

We check that A is well-defined,

$$\begin{aligned}
\sum_{i=1}^N \int_{\Omega} \left| a_i(x, u, \nabla u) \frac{\partial v}{\partial x_i} \right| dx &\leq \beta \sum_{i=1}^N \int_{\Omega} \left(k(x) + |u(x)|^{1+\frac{1}{\epsilon}} + \sum_{j=1}^N \left| \frac{\partial u}{\partial x_j} \right|^{p_j} \right)^{\frac{1}{p'_i}} \left| \frac{\partial v}{\partial x_i} \right| dx \\
&\leq \beta \sum_{i=1}^N \left(\int_{\Omega} (k(x) + |u(x)|^{1+\frac{1}{\epsilon}} + \sum_{j=1}^N \left| \frac{\partial u}{\partial x_j} \right|^{p_j}) dx \right)^{\frac{1}{p'_i}} \left(\int_{\Omega} \left| \frac{\partial v}{\partial x_i} \right|^{p_i} dx \right)^{\frac{1}{p_i}} \\
&= \beta \sum_{i=1}^N \left(\|k\|_{L^1(\Omega)} + \|u\|_{L^{1+\frac{1}{\epsilon}}(\Omega)}^{1+\frac{1}{\epsilon}} + \sum_{j=1}^N \left\| \frac{\partial u}{\partial x_j} \right\|_{L^{p_j}(\Omega)}^{p_j} \right)^{\frac{1}{p'_i}} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \\
&\leq \beta \sum_{i=1}^N \left(\|k\|_{L^1(\Omega)} + C_1 \|u\|^{1+\frac{1}{\epsilon}} + \sum_{j=1}^N \|u\|^{p_j} \right)^{\frac{1}{p'_i}} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}(\Omega)},
\end{aligned}$$

where $C_1 > 0$ is some constant. Note that

$$\sum_{j=1}^N \|u\|^{p_j} \leq \begin{cases} N \|u\|^{p_{\min}} & \text{if } \|u\| < 1, \\ N \|u\|^{p_{\max}} & \text{if } \|u\| \geq 1. \end{cases}$$

Therefore, it can be shown that

$$\begin{aligned}
\sum_{i=1}^N \int_{\Omega} \left| a_i(x, u, \nabla u) \frac{\partial v}{\partial x_i} \right| dx \\
\leq \beta \sum_{i=1}^N \left(\|k\|_{L^1(\Omega)} + C_1 \|u\|^{1+\frac{1}{\epsilon}} + N \|u\|^{\gamma} \right)^{\frac{1}{p'_i}} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \leq D_1 \left(D_2 + \|u\|^{\delta} \right) \|v\|,
\end{aligned}$$

where γ is p_{\min} or p_{\max} . The last inequality, can be shown to hold for some $\delta > 0$ and some constants $D_1, D_2 > 0$. It follows, then, that

$$|\langle A(u), v \rangle| \leq D_1 (D_2 + \|u\|^{\delta}) \|v\| \quad \forall u, v \in W_0^{1, \vec{p}, \epsilon}(\Omega).$$

It is now clear that $A(u)$ is a linear bounded functional, and so A is well-defined, where $A : W_0^{1, \vec{p}, \epsilon}(\Omega) \rightarrow W^{-1, \vec{p}, \epsilon}(\Omega)$. Moreover, we have

$$\frac{|\langle A(u), v \rangle|}{\|v\|} \leq D_1 (D_2 + \|u\|^{\delta}),$$

therefore,

$$\begin{aligned}
\|A(u)\|_{W^{-1, \vec{p}, \epsilon}(\Omega)} &= \sup_{v \in W_0^{1, \vec{p}, \epsilon}(\Omega), v \neq 0} \left\{ \frac{|\langle A(u), v \rangle|}{\|v\|} \right\} \\
&\leq D_1 (D_2 + \|u\|^{\delta}).
\end{aligned}$$

It is, thus, apparent that A is a bounded operator. Since there is no assumed growth condition on c , we therefore first consider an approximate problem. Set for every $k \in \mathbb{N}$, the truncation mapping $T_k : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$T_k(z) := \begin{cases} z & \text{if } |z| \leq k, \\ k \frac{z}{|z|} & \text{if } |z| > k. \end{cases}$$

Consider $c_k(x, u) := T_k c(x, u)$ and the operator $S_k : W_0^{1, \vec{p}, \epsilon}(\Omega) \rightarrow W^{-1, \vec{p}, \epsilon}(\Omega)$ which maps $u \mapsto S_k(u)$, where the linear functional $S_k(u) : W_0^{1, \vec{p}, \epsilon}(\Omega) \rightarrow \mathbb{R}$ is defined as $\langle S_k(u), v \rangle := \int_{\Omega} c_k(x, u) v dx$. Using Hölder's Inequality, yields

$$\begin{aligned} |\langle S_k(u), v \rangle| &\leq \int_{\Omega} |c_k(x, u) v| dx \\ &\leq \|c_k\|_{L^{(1+\frac{1}{\epsilon})'}(\Omega)} \|v\|_{L^{1+\frac{1}{\epsilon}}(\Omega)} \leq kC \|v\| \quad \text{where } v \in L^{1+\frac{1}{\epsilon}}(\Omega), \end{aligned}$$

for some constant $C > 0$. Clearly then, S_k is well-defined. Furthermore,

$$\|S_k(u)\|_{W^{-1, \vec{p}, \epsilon}(\Omega)} \leq kC. \quad (3.68)$$

Step 1: Consider the approximate problem:

$$\begin{cases} -\operatorname{div}(a(x, u, \nabla u)) + c_k(x, u) = g & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \quad (3.69)$$

Let us prove that there exists $u_k \in W_0^{1, \vec{p}, \epsilon}(\Omega)$, such that $\langle (A + S_k)(u_k), v \rangle = \langle g, v \rangle$ for all $v \in W_0^{1, \vec{p}, \epsilon}(\Omega)$, that is,

$$\int_{\Omega} a(x, u_k, \nabla u_k) \cdot \nabla v dx + \int_{\Omega} c_k(x, u_k) v dx = \int_{\Omega} g v dx \quad \forall v \in W_0^{1, \vec{p}, \epsilon}(\Omega). \quad (3.70)$$

To this end, we will show that $A + S_k$ is bounded, coercive and pseudomonotone.

Claim 1: $A + S_k$ is bounded.

$$\begin{aligned} \|(A + S_k)(u)\|_{W^{-1, \vec{p}, \epsilon}(\Omega)} &\leq \|A(u)\|_{W^{-1, \vec{p}, \epsilon}(\Omega)} + \|S_k(u)\|_{W^{-1, \vec{p}, \epsilon}(\Omega)} \\ &\leq D_1(D_2 + \|u\|^\delta) + kC. \end{aligned}$$

Therefore, $A + S_k$ is bounded.

Claim 2: $A + S_k$ is coercive.

We will use that $c_k(x, u)u \geq 0$ a.e. $x \in \Omega$, $\forall u \in \mathbb{R}$ from (3.65). We want to show coercivity, by proving that

$$\frac{\langle A(u) + S_k(u), u \rangle}{\|u\|} \rightarrow \infty \quad \text{as } \|u\| \rightarrow \infty.$$

Consider

$$\begin{aligned} \langle A(u) + S_k(u), u \rangle &= \sum_{i=1}^N \int_{\Omega} a_i(x, u, \nabla u) \cdot \frac{\partial u}{\partial x_i} dx + \int_{\Omega} c_k(x, u) u dx \\ &\geq \int_{\Omega} a(x, u, \nabla u) \cdot \nabla u dx \\ &\geq \alpha \int_{\Omega} \sum_{i=1}^N \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx = \alpha \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i}. \end{aligned}$$

Let

$$I_1 := \left\{ i \in \{1, \dots, N\} : \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \leq 1 \right\} \text{ and } I_2 := \left\{ i \in \{1, \dots, N\} : \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} > 1 \right\},$$

then,

$$\begin{aligned} \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i} &= \sum_{i \in I_1} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i} + \sum_{i \in I_2} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_i} \\ &\geq \sum_{i \in I_1} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_{\max}} + \sum_{i \in I_2} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_{\min}} \\ &= \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_{\min}} - \sum_{i \in I_1} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_{\min}} + \sum_{i \in I_1} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_{\max}} \\ &\geq N \sum_{i=1}^N \frac{1}{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)}^{p_{\min}} - N \geq N \left(\sum_{i=1}^N \frac{1}{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \right)^{p_{\min}} - N = \frac{\|u\|^{p_{\min}}}{N^{p_{\min}-1}} - N. \end{aligned}$$

Since $p_{\min} > 1$, it follows that

$$\frac{\langle A(u) + S_k(u), u \rangle}{\|u\|} \rightarrow \infty \quad \text{when } \|u\| \rightarrow \infty.$$

Claim 3: $A + S_k$ is pseudomonotone.

Assume that $u_n \rightharpoonup u$ in $W_0^{1, \vec{p}, \epsilon}(\Omega)$ and $\limsup_{n \rightarrow \infty} \langle A(u_n) + S_k(u_n), u_n - u \rangle \leq 0$.

From Theorem 1.5.24 for anisotropic Sobolev spaces, it follows that $W_0^{1, \vec{p}, \epsilon}(\Omega) \Subset L^q(\Omega)$ where $1 \leq q \leq p^\#$. Therefore, there exists a subsequence such that

$$u_n(x) \rightarrow u(x) \quad \text{a.e. } x \in \Omega,$$

then, by the continuity of the Carathéodory function c , it follows that

$$c_k(x, u_n) \rightarrow c_k(x, u) \quad \text{a.e. } x \in \Omega.$$

By the Theorem 1.3.10 (Lebesgue Dominated Convergence Theorem), it follows that

$$\int_{\Omega} c_k(x, u_n) z dx \rightarrow \int_{\Omega} c_k(x, u) z dx, \quad \text{for } z \in W_0^{1, \vec{p}, \epsilon}(\Omega),$$

since $|c_k(x, u_n)z| \leq k|z|$. For the remainder of the proof of pseudomonotonicity, it can be shown that we have the necessary conditions that would allow us to follow a similarly structured proof of pseudomonotonicity presented previously in Lemma 3.1.8 and Remark 16.

Putting these results together with Theorem 2.1.15, yields the desired result, therefore there exists $u_k \in W_0^{1, \vec{p}, \epsilon}(\Omega)$, such that (3.70) holds.

Step 2: Define an increasing continuous mapping $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}$, such that

$$\zeta(\|u\|) := \frac{\|u\|^{p_{\min}-1}}{N^{p_{\min}-1}} - \frac{N}{\|u\|},$$

then,

$$\zeta(\|u_k\|)\|u_k\| \leq \langle A(u_k) + S_k(u_k), u_k \rangle = \langle g, u_k \rangle \leq \|g\|_{W^{-1, \vec{p}, \epsilon}(\Omega)} \|u_k\|,$$

thus, $\|u_k\| \leq \zeta^{-1}(\|g\|_{W^{-1, \vec{p}, \epsilon}(\Omega)}) < \infty$, which implies that $\{u_k\}_{k \in \mathbb{N}}$ is bounded independently of k . Using (3.65) and (3.70), we can show that

$$\begin{aligned} 0 &\leq \int_{\Omega} c_k(x, u_k) u_k dx \leq \|g\|_{W^{-1, \vec{p}, \epsilon}(\Omega)} \|u_k\| - \langle A(u_k), u_k \rangle \\ &\leq (\|g\|_{W^{-1, \vec{p}, \epsilon}(\Omega)} + \|A(u_k)\|_{W^{-1, \vec{p}, \epsilon}(\Omega)}) \|u_k\|. \end{aligned}$$

Consequently, there exists a constant $B > 0$ such that

$$\|u_k\| \leq B, \quad (3.71)$$

and

$$\int_{\Omega} c_k(x, u_k) u_k dx \leq B. \quad (3.72)$$

Furthermore, since A is a bounded operator as we have previously proved, we have that

$$\|A(u_k)\|_{W^{-1, \vec{p}, \epsilon}(\Omega)} \leq B' \quad (3.73)$$

for some constant $B' > 0$, which is independent of k .

Step 3: From Theorem 1.5.19, we have that $W_0^{1, \vec{p}, \epsilon}(\Omega)$ is a reflexive Banach space. It follows from Theorem 1.1.11 (Eberlein-Šmulian Theorem), that

$$u_k \rightharpoonup u \quad \text{in } W_0^{1, \vec{p}, \epsilon}(\Omega), \quad (3.74)$$

and

$$A(u_k) \rightharpoonup \psi \quad \text{in } W^{-1, \vec{p}, \epsilon}(\Omega). \quad (3.75)$$

Hence, we can choose a subsequence (denoted again by u_k) such that

$$u_k(x) \rightarrow u(x) \quad \text{a.e. } x \in \Omega \quad \text{and} \quad c_k(x, u_k) \rightarrow c(x, u) \quad \text{a.e. } x \in \Omega. \quad (3.76)$$

Let $\delta > 0$, then $|c_k(x, t)|\delta \leq |c_k(x, t)t|$ where $|t| \geq \delta$, this yields

$$|c_k(x, u_k)| \leq \sup_{|t| \leq \delta} |c_k(x, t)| + \delta^{-1} |c_k(x, u_k) u_k| \leq h_{\delta}(x) + \delta^{-1} |c_k(x, u_k) u_k|.$$

Consequently,

$$\int_E |c_k(x, u_k)| dx \leq \int_E h_{\delta}(x) + \delta^{-1} B,$$

where $E \subset \Omega$ is measurable and B is a constant as in (3.72). Since, from (3.64) we have $h_{\delta} \in L^{\frac{1}{1-\epsilon}}(\Omega)$, where $0 < \epsilon < 1$, it follows that h_{δ} is uniformly integrable. So, for any $\epsilon_1 > 0$, we can choose $\text{meas}(E)$ small enough so that

$$\int_E h_{\delta}(x) dx < \frac{1}{2} \epsilon_1,$$

then, taking $\delta = \frac{2B}{\epsilon_1}$ yields

$$\int_E |c_k(x, u_k)| dx \leq \epsilon_1.$$

Therefore, c_k is uniformly integrable. All the conditions necessary to use Theorem 1.4.13 (Vitali's Convergence Theorem) are satisfied. Thus,

$$c_k(x, u_k) \rightarrow c(x, u) \quad \text{in } L^1(\Omega).$$

Passing to the limit yields

$$\langle \psi, v \rangle + \int_{\Omega} c(x, u) v dx = \langle g, v \rangle \quad (3.77)$$

for all $v \in W_0^{1, \vec{p}, \epsilon}(\Omega) \cap L^\infty(\Omega)$.

What is left to show is that $A(u) = \psi$. Take $v = T_k(u)$ in (3.77). Observe that $\frac{\partial T_k(u(x))}{\partial x_i} \rightarrow \frac{\partial u(x)}{\partial x_i}$ a.e. $x \in \Omega$ (T_k is a Lipschitz function, consequently, $T_k(u) \in W_0^{1, \vec{p}, \epsilon}(\Omega)$

and using the chain rule it follows that $\frac{\partial T_k}{\partial x_i} = \frac{\partial T_k}{\partial u} \frac{\partial u}{\partial x_i}$, see page 296 of [51]) and also

$\left| \frac{\partial T_k(u(x))}{\partial x_i} \right| \leq \left| \frac{\partial u(x)}{\partial x_i} \right|$ for all $k \in \mathbb{N}$. Therefore, by Corollary 1.3.11 (Dominated Convergence in L^p) it follows that for each $i \in \{1, \dots, N\}$, we have $\left\| \frac{\partial T_k u}{\partial x_i} - \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}(\Omega)} \rightarrow 0$.

Therefore,

$$T_k(u) \rightarrow u \quad \text{in } W_0^{1, \vec{p}, \epsilon}(\Omega),$$

and because $\psi - g \in W^{-1, \vec{p}, \epsilon}(\Omega)$, we have that

$$\langle \psi - g, T_k u \rangle \rightarrow \langle \psi - g, u \rangle.$$

Since

$$|c(x, u) T_k u| \leq |c(x, u)| |u| \in L^1(\Omega)$$

and

$$c(x, u) T_k u(x) \rightarrow c(x, u) u(x) \quad \text{a.e. } x \in \Omega.$$

Using Theorem 1.3.10 (Lebesgue Dominated Convergence Theorem) implies that

$$c(x, u) T_k u \rightarrow c(x, u) u \quad \text{in } L^1(\Omega).$$

Therefore,

$$\langle \psi, u \rangle + \int_{\Omega} c(x, u) u dx = \langle g, u \rangle.$$

Now, take $v = u_k$ in (3.70). From (3.76) and (3.65), it follows that there exists a constant $m > 0$ such that $g(x)u_k(x) - c_k(x, u_k(x))u_k(x) \leq mg(x)$ a.e. $x \in \Omega$. Together with Theorem 1.3.9 (Reverse Fatou's Lemma), this yields

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle A(u_k), u_k \rangle &= \limsup_{k \rightarrow \infty} \int_{\Omega} a(x, u_k, \nabla u_k) \cdot \nabla u_k dx \\ &= \limsup_{k \rightarrow \infty} \int_{\Omega} (g(x)u_k(x) - c_k(x, u_k(x))u_k(x)) dx \\ &\leq \int_{\Omega} \limsup_{k \rightarrow \infty} (g(x)u_k(x) - c_k(x, u_k(x))u_k(x)) dx \\ &\leq \int_{\Omega} \limsup_{k \rightarrow \infty} g(x)u_k(x) + \limsup_{k \rightarrow \infty} (-c_k(x, u_k(x))u_k(x)) dx \\ &= \int_{\Omega} (g(x)u(x) - c(x, u(x))u(x)) dx = \langle g, u \rangle - \int_{\Omega} c(x, u(x))u(x) dx. \end{aligned}$$

Hence, we have

$$\limsup_{k \rightarrow \infty} \langle A(u_k), u_k \rangle \leq \langle \psi, u \rangle. \quad (3.78)$$

Furthermore, we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle A(u_k), u_k - u \rangle &= \limsup_{k \rightarrow \infty} (\langle A(u_k), u_k \rangle - \langle A(u_k), u \rangle) \\ &\leq \limsup_{k \rightarrow \infty} \langle A(u_k), u_k \rangle - \langle \psi, u \rangle \leq 0. \end{aligned}$$

The pseudomonotonicity of A yields

$$\begin{aligned} \langle A(u), u - v \rangle &\leq \liminf_{k \rightarrow \infty} \langle A(u_k), u_k - v \rangle \\ &\leq \limsup_{k \rightarrow \infty} (\langle A(u_k), u_k \rangle - \langle A(u_k), v \rangle) \\ &\leq \limsup_{k \rightarrow \infty} \langle A(u_k), u_k \rangle - \langle \psi, v \rangle \leq \langle \psi, u - v \rangle, \end{aligned}$$

this yields $A(u) = \psi$, so we have the desired result

$$\langle A(u), v \rangle + \int_{\Omega} c(x, u)v dx = \langle g, v \rangle \quad (3.79)$$

for all $v \in W_0^{1, \vec{p}, \epsilon}(\Omega) \cap L^\infty(\Omega)$.

□

The proof of Theorem 3.1.11 confirms the existence of a weak solution to boundary value problem (3.57).

Chapter 4

The Characterization of Pseudomonotone Operators in Divergence Form

In what follows, we will present a result by Boccardo and Dacorogna (1984) (see [4]). In doing so, a connection between pseudomonotone operators and the principal part of a second order partial differential equation (again, the principal part is a mapping $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$) will be established.

Assume that $A : V \rightarrow V^*$ is pseudomonotone, as described in Definition 2.1.12. We restrict ourselves to operators in divergence form over $V := W^{1,p}(\Omega)$, with $p > 1$, consider $\Omega \subset \mathbb{R}^N$ as a bounded open set, and

$$A(u) := -\operatorname{div} a(x, u(x), \nabla u(x)), \quad (4.1)$$

where $\langle A(u), v \rangle := \int_{\Omega} a(x, u, \nabla u) \cdot \nabla v dx$ for $u, v \in W^{1,p}(\Omega)$. Let $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a Carathéodory function satisfying

$$|a(x, s, \xi)| \leq k(x) + \beta(|s|^{p-1} + |\xi|^{p-1}), \quad (4.2)$$

for a.e. $x \in \Omega$, for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and for some $k \in L^{p'}(\Omega)$ and $\beta \geq 0$.

The theorem central to this chapter is stated as follows:

Theorem 4.0.12. *Suppose that the conditions mentioned above are satisfied for operator A and mapping a . If A is pseudomonotone on $W^{1,p}(\Omega)$, then a is monotone with respect to the last variable, i.e.,*

$$[a(x, s, \xi) - a(x, s, \eta), \xi - \eta] \geq 0, \quad (4.3)$$

for almost every $x \in \Omega$, for every $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$ and where $[\cdot, \cdot]$ denotes the scalar product on \mathbb{R}^N .

Proof of Theorem 4.0.12:

To prove this theorem, we require two lemmas.

Lemma 4.0.13. *Let $I \subset \mathbb{R}^N$ be a convex, bounded open set with Lipschitz boundary ∂I . Let $\{u_n\}_n$ and u be such that*

$$u_n \rightharpoonup^* u \quad \text{in } W^{1,\infty}(I), \quad (4.4)$$

$$\|\nabla u_n\|_{L^\infty(I)} \leq \sigma, \quad (4.5)$$

where $\sigma > 0$ is fixed (we denote the weak* convergence by the symbol ' \rightharpoonup^* '). Then, there exists a sequence $\{v_n\}_n$ such that

$$v_n \rightharpoonup^* u \quad \text{in } W^{1,\infty}(I), \quad (4.6)$$

$$v_n = u \quad \text{on } \partial I, \quad (4.7)$$

$$\|\nabla v_n\|_{L^\infty(I)} \leq 2\sigma, \quad (4.8)$$

$$\lim_{n \rightarrow \infty} \left| \int_I [a(x, v_n, \nabla v_n), \nabla(v_n - z)] - [a(x, u_n, \nabla u_n), \nabla(u_n - z)] dx \right| = 0 \quad (4.9)$$

for every $z \in W^{1,\infty}(I)$.

This lemma proves that the sequence $\{u_n\}_n$ may be altered slightly, so as to fix its value on the boundary, without significantly changing the value of $a(x, u_n, \nabla u_n)$.

Proof:

Since $u_n \rightharpoonup^* u$ in $W^{1,\infty}(I)$, using Theorem 1.5.16 and the Arzelá-Ascoli Theorem (Theorem 1.4.2) it can be shown that $u_n \rightarrow u$ in $L^\infty(I)$. Now, let

$$\delta_n := \frac{1}{\sigma} \|u_n - u\|_{L^\infty(I)} \quad (4.10)$$

and

$$I_n := \{x \in I : \text{dist}(x, \partial I) > \delta_n\}. \quad (4.11)$$

Define a sequence $\{\bar{v}_n\}_n$ as follows:

$$\bar{v}_n(x) := \begin{cases} u_n(x) & \text{if } x \in \bar{I}_n, \\ u(x) & \text{if } x \in \partial I. \end{cases} \quad (4.12)$$

We first demonstrate that u_n and u are Lipschitz continuous on I , with Lipschitz constant σ (it is then clear that u_n and u are Lipschitz continuous on the compact subset $\bar{I}_n \cup \partial I$).

Consider the proof for the case involving u . Since $u_n \rightharpoonup^* u$ in $W^{1,\infty}(I)$, it follows that $\nabla u_n \rightharpoonup^* \nabla u$ in $L^\infty(I; \mathbb{R}^N)$, therefore $\|\nabla u\|_{L^\infty(I)} \leq \liminf_{n \rightarrow \infty} \|\nabla u_n\|_{L^\infty(I)} \leq \sigma$. The gradient ∇u exists almost everywhere, it is also Lebesgue integrable and

$$u(x) = u(y) + \int_0^1 \nabla u(tx + (1-t)y) \cdot (x-y) dt,$$

since an element of a Sobolev space is absolutely continuous on almost every line segment.

Now, use the following procedure: Consider $\psi(t) := u(tx + (1-t)y)$ for $t \in [0, 1]$. Then, $\psi(1) - \psi(0) = \int_0^1 \psi'(t)dt = \int_0^1 (\nabla u(tx + (1-t)y)) \cdot (x-y)dt$, therefore,

$$\begin{aligned} |u(x) - u(y)| &\leq \int_0^1 |\nabla u(tx + (1-t)y)| |x-y| dt \\ &\leq \|\nabla u\|_{L^\infty(I)} |x-y| \\ &\leq \sigma |x-y| \quad \text{a.e on } I. \end{aligned}$$

Then, extend u to all of I . Thus, u is Lipschitz continuous on I with Lipschitz constant σ . Similarly, the result ($|u_n(x) - u_n(y)| \leq \sigma|x-y|$) for u_n can be shown, by using (4.5) instead of lower semi-continuity in the proof.

We now deduce that \bar{v}_n is locally Lipschitz on $\bar{I}_n \cup \partial I$, with Lipschitz constant 2σ . Take $x \in \bar{I}_n$ and $y \in \partial I$, choose $\{\tau\} = [x, y] \cap \partial I_n$, then,

$$\begin{aligned} |\bar{v}_n(x) - \bar{v}_n(y)| &\leq |\bar{v}_n(x) - \bar{v}_n(\tau)| + |\bar{v}_n(\tau) - \bar{v}_n(y)| \\ &\leq \sigma|x-\tau| + |u_n(\tau) - u(y)| \\ &\leq \sigma|x-\tau| + \|u_n - u\|_{L^\infty(I)} + |u(\tau) - u(y)| \leq 2\sigma|x-y|. \end{aligned} \tag{4.13}$$

The last inequality is obtained by observing that $|y-x| = |x-\tau| + |\tau-y|$, hence $\sigma|x-\tau| + |u(\tau) - u(y)| \leq \sigma|x-\tau| + \sigma|\tau-y| = \sigma|x-y|$. Also, $\|u_n - u\|_{L^\infty(I)} = \sigma\delta_n \leq \sigma|x-y|$.

Lemma 1.4.3 (Mac Shane's lemma) yields a sequence $\{v_n\}_n$, an extension of $\{\bar{v}_n\}_n$ to \bar{I} which satisfies (4.6)-(4.8). Note that (4.7) is obtained from our definition of \bar{v}_n , and inequality (4.8) follows from Mac Shane's lemma. Since v_n is Lipschitz continuous, it is also almost everywhere differentiable by Theorem 1.4.4 (Rademacher's Theorem).

Consequently, $\left| \frac{\partial v_n(x)}{\partial x_k} \right| = \lim_{h \rightarrow 0} \left| \frac{v_n(x+h \cdot e_k) - v_n(x)}{h} \right| \leq 2\sigma$, for $1 \leq k \leq N$, hence $\|\nabla v_n\|_{L^\infty(I)} \leq 2\sigma$.

We now show how to obtain (4.6), take $\phi \in L^1(I)$ then

$$\begin{aligned} 0 &\leq \left| \int_I v_n \phi dx - \int_I u \phi dx \right| \\ &= \left| \int_{I_n} u_n \phi dx + \int_{I \setminus I_n} v_n \phi dx - \int_I u \phi dx \right| \\ &= \left| \int_I u_n \phi dx - \int_{I \setminus I_n} u_n \phi dx + \int_{I \setminus I_n} v_n \phi dx - \int_I u \phi dx \right| \\ &\leq \left| \int_I u_n \phi dx - \int_{I \setminus I_n} u_n \phi dx + k \int_{I \setminus I_n} \phi dx - \int_I u \phi dx \right| \\ &\leq \left| \int_I u_n \phi dx - \int_I u \phi dx \right| + k \int_{I \setminus I_n} |\phi| dx + \int_{I \setminus I_n} |u_n \phi| dx. \end{aligned}$$

We used the fact that v_n is Lipschitz continuous on compact interval \bar{I} , hence, $|v_n(x)| \leq k$, for some constant $k > 0$. Since, $u_n \rightharpoonup^* u$ in $W^{1,\infty}(I)$, it follows that $\|u_n\|_{L^\infty(I)} \leq C$, for some constant $C > 0$. This gives $\int_{I \setminus I_n} |u_n \phi| dx = \int_I \chi_{I \setminus I_n} |u_n \phi| dx \leq C \int_I \chi_{I \setminus I_n} |\phi| dx$ (χ denotes the characteristic function). Since $\chi_{I \setminus I_n}(x) |\phi(x)| \rightarrow 0$ and $|\chi_{I \setminus I_n}(x)| |\phi(x)| \leq |\phi(x)|$,

the Dominated Convergence Theorem implies

$$\lim_{n \rightarrow \infty} \int_{I \setminus I_n} |u_n \phi| dx \leq C \lim_{n \rightarrow \infty} \int_I \chi_{I \setminus I_n} |\phi| dx = 0.$$

Similarly, using the Dominated Convergence Theorem gives $k \lim_{n \rightarrow \infty} \int_I \chi_{I \setminus I_n} |\phi| dx = 0$ (note that $|\chi_{I \setminus I_n}(x)| |\phi(x)| \leq |\phi(x)|$). Then, since $u_n \rightharpoonup^* u$ in $W^{1,\infty}(I)$, by definition we have $\int_I u_n \phi dx \rightarrow \int_I u \phi dx$. Consequently, $\left| \int_I u_n \phi dx - \int_I u \phi dx \right| \rightarrow 0$. Putting all of the above together yields

$$0 \leq \lim_{n \rightarrow \infty} \left| \int_I v_n \phi dx - \int_I u \phi dx \right| \leq 0.$$

Using the Squeeze Theorem, it follows that $v_n \rightharpoonup^* u$ in $L^\infty(I)$. Similarly, it can be shown that $\nabla v_n \rightharpoonup^* \nabla u$ in $L^\infty(I; \mathbb{R}^N)$. Hence, the desired result is obtained.

What remains is to prove (4.9). We will denote the right-hand side of (4.9) by ' b_n '. Using the definition of v_n , yields

$$\begin{aligned} b_n &= \left| \int_{I \setminus I_n} ([a(x, v_n, \nabla v_n), \nabla(v_n - z)] - [a(x, u_n, \nabla u_n), \nabla(u_n - z)]) dx \right| \quad (4.14) \\ &\leq \int_{I \setminus I_n} |a(x, v_n, \nabla v_n)| |\nabla(v_n - z)| dx + \int_{I \setminus I_n} |a(x, u_n, \nabla u_n)| |\nabla(u_n - z)| dx. \end{aligned}$$

From (4.2), (4.6) - (4.8) and Hölder's Inequality we obtain

$$b_n \leq D (\text{meas}(I \setminus I_n))^{\frac{1}{p}}, \quad (4.15)$$

where $D > 0$. Letting $n \rightarrow \infty$, gives the desired result. \square

Lemma 4.0.14. *Suppose A is as in Theorem 4.0.12, and let A be pseudomonotone on $W_0^{1,p}(\Omega)$. Then, for every convex, bounded open set $O \subset \Omega$, and for every sequence $\{u_n\}_n$ satisfying*

$$u_n \rightharpoonup^* u \quad \text{in } W_0^{1,\infty}(O), \quad (4.16)$$

$$\limsup_{n \rightarrow \infty} \int_O [a(x, u_n, \nabla u_n), \nabla(u_n - u)] dx \leq 0, \quad (4.17)$$

we obtain,

$$\begin{aligned} &\liminf_{n \rightarrow \infty} \int_O [a(x, u_n, \nabla u_n), \nabla(u_n - w)] dx \\ &\geq \int_O [a(x, u, \nabla u), \nabla(u - w)] dx \quad \text{for every } w \in W^{1,\infty}(O). \end{aligned} \quad (4.18)$$

Proof: Assume that (4.16) and (4.17) hold. Observe that $u_n \rightharpoonup^* u$ in $W_0^{1,\infty}(O)$ implies $u_n \rightharpoonup^* u$ in $W^{1,\infty}(O)$. Since weak* convergence implies boundedness, it follows that $\|\nabla u_n\|_{L^\infty(O)} \leq \|u_n\|_{W^{1,\infty}(O)} \leq \alpha$, for some $\alpha > 0$. Hence, conditions (4.4) and (4.5) are satisfied. Then, Lemma 4.0.13 implies the existence of a sequence $\{v_n\}_n$, such that

$$v_n \rightharpoonup^* u \quad \text{in } W^{1,\infty}(O), \quad (4.19)$$

$$v_n = u \quad \text{on } \partial O, \quad (4.20)$$

and

$$\left| \int_O ([a(x, v_n, \nabla v_n), \nabla(v_n - w)] - [a(x, u_n, \nabla u_n), \nabla(u_n - w)]) dx \right| \rightarrow 0 \quad \text{for } w \in W^{1,\infty}(O). \quad (4.21)$$

We extend u to \bar{u} by taking $\bar{u} = 0$ on $\Omega \setminus O$, then $\bar{u} \in W_0^{1,\infty}(\Omega)$. Similarly, since $v_n = u$ on ∂O , we define $\bar{v}_n = v_n$ on O and $\bar{v}_n = \bar{u}$ on $\Omega \setminus O$. Then,

$$\bar{v}_n \rightharpoonup^* \bar{u} \quad \text{in } W_0^{1,\infty}(\Omega), \quad (4.22)$$

where (4.22) is obtained by noting that for any $\phi \in L^1(\Omega)$, $\int_\Omega (\bar{v}_n - \bar{u})\phi dx \rightarrow 0$ and similarly, $\int_\Omega (\nabla \bar{v}_n - \nabla \bar{u}) \cdot \psi dx \rightarrow 0$ for any $\psi \in L^1(\Omega; \mathbb{R}^N)$. It must now be shown that

$$\limsup_{n \rightarrow \infty} \int_\Omega [a(x, \bar{v}_n, \nabla \bar{v}_n), \nabla(\bar{v}_n - \bar{u})] dx \leq 0, \quad (4.23)$$

which is required to use the pseudomonotonicity of A . Consider

$$\begin{aligned} \zeta_n &:= \int_\Omega [a(x, \bar{v}_n, \nabla \bar{v}_n), \nabla(\bar{v}_n - \bar{u})] dx \\ &= \int_O [a(x, v_n, \nabla v_n), \nabla(v_n - u)] dx \\ &= \int_O [a(x, v_n, \nabla v_n), \nabla(v_n - u)] dx - \int_O [a(x, u_n, \nabla u_n), \nabla(u_n - u)] dx \\ &\quad + \int_O [a(x, u_n, \nabla u_n), \nabla(u_n - u)] dx. \end{aligned} \quad (4.24)$$

Then,

$$\begin{aligned} &\limsup_{n \rightarrow \infty} \int_\Omega [a(x, \bar{v}_n, \nabla \bar{v}_n), \nabla(\bar{v}_n - \bar{u})] dx \\ &\leq \limsup_{n \rightarrow \infty} \left(\int_O [a(x, v_n, \nabla v_n), \nabla(v_n - u)] dx - \int_O [a(x, u_n, \nabla u_n), \nabla(u_n - u)] dx \right) \\ &\quad + \limsup_{n \rightarrow \infty} \int_O [a(x, u_n, \nabla u_n), \nabla(u_n - u)] dx \\ &\leq 0 + \limsup_{n \rightarrow \infty} \int_O [a(x, u_n, \nabla u_n), \nabla(u_n - u)] dx \leq 0. \end{aligned}$$

Both (4.17) and (4.21) are used in the last two inequalities. We can now use the pseudomonotonicity of A . Since $\bar{v}_n \rightharpoonup^* \bar{u}$ in $W_0^{1,\infty}(\Omega)$, we have $\bar{v}_n \rightharpoonup^* \bar{u}$ in $W_0^{1,p}(\Omega)$. It follows that

$$\liminf_{n \rightarrow \infty} \int_\Omega [a(x, \bar{v}_n, \nabla \bar{v}_n), \nabla(\bar{v}_n - z)] dx \geq \int_\Omega [a(x, \bar{u}, \nabla \bar{u}), \nabla(\bar{u} - z)] dx, \quad (4.25)$$

for every $z \in W_0^{1,p}(\Omega)$. We can choose z in such a way that $z \in W_0^{1,p}(\Omega)$ and $z = w$ on O , where $w \in W^{1,\infty}(O)$. Now, use (4.25) and the fact that $\bar{v}_n = \bar{u}$ on $\Omega \setminus O$, this yields

$$\liminf_{n \rightarrow \infty} \int_O [a(x, v_n, \nabla v_n), \nabla(v_n - w)] dx \geq \int_O [a(x, u, \nabla u), \nabla(u - w)] dx \quad (4.26)$$

for every $w \in W^{1,\infty}(O)$. Combining (4.21) and (4.26) we obtain (4.18), i.e.,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \int_O [a(x, u_n, \nabla u_n), \nabla(u_n - w)] dx &= \liminf_{n \rightarrow \infty} \int_O [a(x, u_n, \nabla v_n), \nabla(v_n - w)] dx \\ &\geq \int_O [a(x, u, \nabla u), \nabla(u - w)] dx \end{aligned}$$

for every $w \in W^{1,\infty}(O)$. \square

We now have the necessary preliminary results required to prove the main result, Theorem 4.0.12. We are required to show that (4.3) holds for every Lebesgue point $x \in \Omega$. Since a is a locally summable function, it follows that almost every point is a Lebesgue point and therefore we would have shown that (4.3) holds for almost all $x \in \Omega$.

To prove by contradiction assume that $x_0 \in \Omega$ is a Lebesgue point of a and let $s \in \mathbb{R}$, $\xi, \eta \in \mathbb{R}^N$ ($\xi \neq \eta$) be such that

$$[a(x_0, s, \xi) - a(x_0, s, \eta), \xi - \eta] < 0. \quad (4.27)$$

Step 1: Assuming that x_0 is a Lebesgue point, using (4.27) and Cauchy-Schwarz inequality, it is easy to show that

$$\lim_{R \rightarrow 0} \frac{1}{\text{meas}(H_R(x_0))} \int_{H_R(x_0)} [a(x, s, \xi) - a(x, s, \eta), \xi - \eta] dx < 0, \quad (4.28)$$

where we denote the hypercube of \mathbb{R}^N , centred at x_0 and with side of length R , by ' $H_R(x_0)$ '. From (4.28), it follows that for small enough R , we have that $H_R(x_0)$ and

$$\frac{1}{\text{meas}(H_R(x_0))} \int_{H_R(x_0)} [a(x, s, \xi) - a(x, s, \eta), \xi - \eta] dx < 0,$$

furthermore,

$$\int_{H_R(x_0)} [a(x, s, \xi) - a(x, s, \eta), \xi - \eta] dx < 0. \quad (4.29)$$

If we take R small enough, and consider

$$u(x) := \left[\frac{\xi + \eta}{2}, x - x_0 \right] + s \quad \text{for } x \in \Omega, \quad (4.30)$$

then,

$$\int_{H_R(x_0)} [a(x, u(x), \xi) - a(x, u(x), \eta), \xi - \eta] dx < 0. \quad (4.31)$$

For notational purposes, we will denote $H_R(x_0)$ by O .

Step 2: Construct a sequence $\{u_n\}_n$ (where u is defined as in (4.30)) such that

$$u_n \rightharpoonup^* u \quad \text{in } W^{1,\infty}(O) \quad (4.32)$$

and

$$\nabla u_n(x) = \begin{cases} \xi & \text{if } x \in O_1^n, \\ \eta & \text{if } x \in O_2^n, \end{cases} \quad (4.33)$$

where $O_1^n, O_2^n \subset O$, $O_1^n \cap O_2^n = \emptyset$ and

$$\text{meas}(O_1^n) = \text{meas}(O_2^n) = \frac{1}{2} \text{meas}(O). \quad (4.34)$$

Considering the characteristic functions on O_1^n and O_2^n , we also have

$$\chi_{O_1^n} \rightharpoonup^* \frac{1}{2} \quad \text{and} \quad \chi_{O_2^n} \rightharpoonup^* \frac{1}{2} \quad \text{in } L^\infty(O). \quad (4.35)$$

Now, we seek to give motivation to the partitioning of O , the construction of the sequence $\{u_n\}_n$ and associated claims mentioned above.

Consider, first, a change of coordinates: If the vectors ξ and η are fixed, then $\xi - \eta$ is a fixed vector, and we can build a base $(\xi - \eta, \zeta^{(2)}, \zeta^{(3)}, \dots, \zeta^{(N)})$ of \mathbb{R}^N . In this base $\xi - \eta = (1, 0, \dots, 0)$, hence,

$$[\xi - \eta, x] = (\xi_1 - \eta_1)x_1, \quad (4.36)$$

where $x = (x_1, \dots, x_N)$ and $(\xi - \eta) = (\xi_1 - \eta_1, \dots, \xi_N - \eta_N)$.

We, then, write $O := H_R(x) = \prod_{i=1}^N (\alpha_i, \beta_i) = (\alpha_1, \beta_1) \times \prod_{i=2}^N (\alpha_i, \beta_i)$. Now, we seek to partition the hypercube. We divide (α_1, β_1) into intervals of length $(\beta_1 - \alpha_1)2^{-n}$, where $n \in \mathbb{N}$. Again, divide each into two equal parts of equal length $(\beta_1 - \alpha_1)2^{-(n+1)}$. Then, the union of the first (respectively the second) subintervals is denoted by I_n (respectively J_n). Thus,

$$O_1^n := I_n \times \prod_{i=2}^N (\alpha_i, \beta_i) \quad (4.37)$$

and

$$O_2^n := J_n \times \prod_{i=2}^N (\alpha_i, \beta_i). \quad (4.38)$$

It is clear that (4.34) holds.

The argument to show $\chi_{O_1^n} \rightharpoonup^* \frac{1}{2}$ in (4.35) is set out as follows (a similar scheme may be employed to prove $\chi_{O_2^n} \rightharpoonup^* \frac{1}{2}$ in $L^\infty(O)$). We have a bounded sequence $\{\chi_{O_1^n}\}_n$ in $L^\infty(O)$, specifically $\|\chi_{O_1^n}\|_{L^\infty(O)} \leq 1$ for all $n \in \mathbb{N}$.

To verify that $\chi_{O_1^n} \rightharpoonup^* \frac{1}{2}$, it suffices to check that

$$\int_O g \chi_{O_1^n} dx \rightarrow \frac{1}{2} \int_O g dx, \quad (4.39)$$

for all g in some set $B \subset L^1(O)$, with norm-dense linear span. Once we have (4.39) for $g \in B$, we have it for all linear combinations of elements of B . Then, given any

$h \in L^1(O)$, we can find a g in the span of B , such that for all $\epsilon > 0$, it follows that $\|h - g\|_{L^1(O)} \leq \epsilon$. It can then be shown that

$$\int_O h \chi_{O_1^n} dx \rightarrow \frac{1}{2} \int_O h dx, \quad (4.40)$$

since $\left| \int_O h \chi_{O_1^n} dx - \frac{1}{2} \int_O h dx \right| \leq 2\epsilon + \left| \int_O g \chi_{O_1^n} dx - \frac{1}{2} \int_O g dx \right|$. We will consider the set of characteristic functions of dyadic cubes in O as our choice for B .

First, we write our hypercube as $O = \prod_{i=1}^N (\alpha_i, \beta_i) = \prod_{i=1}^N (\alpha_i, \alpha_i + R)$. Consider then for each $k \in \mathbb{N}$, the cubes

$$C(k, m) := \prod_{i=1}^N (\alpha_i + m_i 2^{-k} R, \alpha_i + (m_i + 1) 2^{-k} R), \quad (4.41)$$

where $m = (m_1, \dots, m_N)$, with $m_i \in \mathbb{Z}$ and $0 \leq m_i < 2^k$, for $i \in \{1, \dots, N\}$. Then, the system $\mathcal{C}(k) := \{C(k, m) : 0 \leq m_i < 2^k\}$ is a partition of O into dyadic cubes of side-length $2^{-k}R$. Looking at the system

$$\mathcal{C} := \bigcup_{k \in \mathbb{N}} \mathcal{C}(k), \quad (4.42)$$

it can be shown that the linear span of the set $\{\chi_C : C \in \mathcal{C}\}$ is norm-dense in $L^1(O)$. To show this, note that $C_c(O)$ is dense in $L^1(O)$. Then, take $f \in L^1(O)$ and $\epsilon > 0$. There exists a $g \in C_c(O)$, such that $\|f - g\|_{L^1(O)} < \frac{\epsilon}{2}$. We know that g is uniformly continuous, so there is a $\delta > 0$ such that

$$|x - y| < \delta \implies |g(x) - g(y)| < \frac{\epsilon}{2 \cdot \text{meas}(O)}.$$

Choose dyadic cubes $\{C_l\}_l$ so small that their diameter is smaller than δ , and then set $h(x) := \sum_l g(x_l) \chi_{C_l}$, where x_l is the centre of each cube and $\{C_l\}_l$ is a partition of O into small dyadic cubes (we can take partition $\{C_l\}_l = \mathcal{C}(k)$, for sufficiently large k). Therefore,

$$\|f - h\|_{L^1(O)} \leq \|f - g\|_{L^1(O)} + \|g - h\|_{L^1(O)} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus, we can take the set $\{\chi_C : C \in \mathcal{C}\}$ as our desired set $B \subset L^1(O)$, with norm-dense linear span. Take any $C_j \in B$ (consider C_j taken from partition $\{C_j\}_j = \mathcal{C}(k)$, for some $k \in \mathbb{N}$) and let $g := \chi_{C_j}$, then,

$$\int_O g \chi_{O_1^n} dx = \int_O \chi_{C_j \cap O_1^n} dx = \text{meas}(C_j \cap O_1^n) = \frac{1}{2} \text{meas}(C_j) = \frac{1}{2} \int_O \chi_{C_j} dx = \frac{1}{2} \int_O g dx,$$

for $n > k$ taken large enough. The same arguments can be made to show that $\chi_{O_2^n} \xrightarrow{*} \frac{1}{2}$. This is enough to conclude that (4.35) is valid.

Define $\varphi_n : (\alpha_1, \beta_1) \rightarrow \mathbb{R}$ as follows:

$$\varphi_n(\alpha_1) := 0,$$

$$\frac{d\varphi_n(z)}{dz} := \begin{cases} \frac{1}{2} & \text{if } z \in I_n, \\ -\frac{1}{2} & \text{if } z \in J_n, \end{cases}$$

then, also, $\varphi_n(\beta_1) = 0$. It is straightforward to show that $\varphi_n \rightarrow 0$ uniformly on (α_1, β_1) .¹ Now, let

$$u_n(x) := \varphi_n([\xi - \eta, x]) + u(x). \quad (4.43)$$

Check that (4.32) and (4.33) are satisfied: Not only is $\{\varphi_n\}_n$ uniformly convergent, but, since for each $n \in \mathbb{N}$, the functions φ_n are bounded, we have that the sequence is uniformly bounded. Consequently, $\sup_{x \in O} |\varphi_n(x)| \leq C$, for some constant $C \geq 0$.

We want to show $u_n \rightharpoonup^* u$ in $W^{1,\infty}(O)$. Therefore, we are required to first show that for any $\psi \in L^1(O)$,

$$\int_O (u_n - u)\psi dx \rightarrow 0,$$

which is

$$\int_O \varphi_n([\xi - \eta, x])\psi(x) dx \rightarrow 0.$$

Since $\varphi_n \rightarrow 0$ uniformly, it follows that $(u_n - u)(x)\psi(x) \rightarrow 0$ pointwise. We now use Lebesgue's Dominated Convergence Theorem. We can dominate $(u_n(x) - u(x))\psi(x)$, since $\{\varphi_n\}_n$ is uniformly bounded and $\psi(x) \in L^1(O)$. Therefore, $|(u_n(x) - u(x))\psi(x)| \leq C\psi(x)$. Then, by the Dominated Convergence Theorem,

$$\lim_{n \rightarrow \infty} \int_O (u_n - u)\psi dx = \int_O \lim_{n \rightarrow \infty} (u_n - u)\psi dx = 0.$$

The result $u_n \rightharpoonup^* u$ in $L^\infty(O)$ follows. Similarly, it can be shown that $\nabla u_n \rightharpoonup^* \nabla u$ in $L^\infty(O; \mathbb{R}^N)$.

We now check (4.33). If, say, $x \in O_1^n = I_n \times \prod_{i=2}^N (\alpha_i, \beta_i)$, then $x_1 \in I_n$. However, from the change of coordinates defined previously, we have that this is exactly $[\xi - \eta, x] \in I_n$. It follows then that

$$\begin{aligned} \nabla u_n(x) &= (\xi - \eta) \frac{d\varphi_n([\xi - \eta, x])}{dz} + \nabla u(x) \\ &= (\xi - \eta) \frac{d\varphi_n([\xi - \eta, x])}{dz} + \frac{\xi + \eta}{2}. \end{aligned}$$

Hence, $\nabla u_n(x) = \xi$ if $x \in O_1^n$ and $\nabla u_n(x) = \eta$ if $x \in O_2^n$. Therefore, (4.33) holds.

Step 3: If (4.27) implies that A is not pseudomonotone, then we obtain a contradiction.

First, note that

$$\begin{aligned} &\int_O [a(x, u_n(x), \nabla u_n(x)), \nabla(u_n(x) - u(x))] dx \\ &= \frac{1}{2} \int_{O_1^n} [a(x, u_n(x), \xi), \xi - \eta] dx - \frac{1}{2} \int_{O_2^n} [a(x, u_n(x), \eta), \xi - \eta] dx. \end{aligned}$$

It can be shown, using (4.31), (4.35) as well as the Lebesgue Dominated Convergence Theorem (to show that $a(x, u_n, \xi) \rightarrow a(x, u, \xi)$ in $L^1(\Omega)$) that

$$\limsup_{n \rightarrow \infty} \int_O [a(x, u_n, \nabla u_n), \nabla(u_n - u)] dx < 0. \quad (4.44)$$

¹by noting that φ is absolutely continuous and using the second fundamental theorem of calculus, it can be shown that $\sup_z |\varphi_n(z)| \leq \frac{K}{2^n} \rightarrow 0$, for some constant $K > 0$.

Then, after using Lemma 4.0.14 with $w = u$, we can conclude that

$$\liminf_{n \rightarrow \infty} \int_O [a(x, u_n, \nabla u_n), \nabla(u_n - u)] dx \geq 0, \quad (4.45)$$

thus, contradicting (4.44) and hence, showing that A is not pseudomonotone. Theorem 4.0.12 is thus proved by contradiction. \square

The proof of Theorem 4.0.12 confirms a direct association between a pseudomonotone operator in divergence form and the principal part of a second order partial differential equation.

Chapter 5

Elliptic Variational Inequalities

In real life, there are various phenomena that are described mathematically by some form of a variational inequality (VI). The concept of equilibrium, central to numerous applied disciplines is often formulated mathematically as a VI. They are also fundamental to the framework of the studies of elastoplastic materials and the so called obstacle problem.

The theory of VIs was introduced in 1967 by Lions and Stampacchia (see [44]) as a tool for the study of partial differential equations with applications mainly in mechanics. The VIs were initially studied in infinite-dimensional spaces, then later in finite-dimensional spaces by Smith and Dafermos (see [62] and [24]) for the study of traffic network.

In this thesis, we are interested in VIs in the infinite-dimensional setting. For the theory and the application of the VI in finite-dimensional setting, we mention works in connection with traffic network equilibrium, financial equilibrium, environmental network, transportation equilibrium, etc.

The purpose of this chapter is to show how the property of pseudomonotonicity plays an important role in the establishment of the well posedness of some variational inequalities. Let us first define the concept of a variational inequality.

Definition 5.0.15. Let $(V, \|\cdot\|)$ be a normed space with dual V^* . Let $A : V \rightarrow V^*$ and K be a non-empty closed convex subset of V .

- A variational problem of the type: Find $u \in K$ such that

$$\langle A(u), v - u \rangle \geq 0 \text{ for every } v \in K \quad (5.1)$$

is called a *variational inequality of the first kind*.

- A variational problem of the type: Find $u \in V$ such that

$$\langle A(u), v - u \rangle + j(v) - j(u) \geq 0 \text{ for every } v \in V, \quad (5.2)$$

where $j : V \rightarrow \bar{\mathbb{R}}$ is a proper functional, is called a *variational inequality of the second kind*.

Remark 18. Clearly (5.2) is reduced to (5.1) whenever $j = I_k$, the indicator function of the set K defined by

$$I_k(v) := \begin{cases} 0 & \text{if } v \in K, \\ \infty & \text{otherwise.} \end{cases}$$

In the Hilbert space setting $(H, [\cdot, \cdot])$, (5.1) and (5.2) often appear respectively in the forms

$$a(u, v - u) \geq [f, v - u] \quad \text{for every } v \in K \quad (5.3)$$

and

$$a(u, v - u) + j(v) - j(u) \geq [f, v - u] \quad \text{for every } v \in V, \quad (5.4)$$

where $a : H \times H \rightarrow \mathbb{R}$ is a bilinear form and $f \in V$.

We will need the notion of projection onto a closed convex subset of a Hilbert space.

Theorem 5.0.16. *Let K be a non-empty closed convex subset of H . Then for every $x \in H$, there exists a unique $y \in K$ such that*

$$\|x - y\| = \inf_{z \in K} \|x - z\|, \quad (5.5)$$

the unique element y is called the projection of x onto K and is denoted by $P_K(x)$. This is characterized by the variational inequality

$$[x - y, x - z] \leq 0 \quad \text{for all } z \in K. \quad (5.6)$$

Furthermore, the new mapping $P_K : H \rightarrow K$, defined by $P_K(x) := y$ is non-expansive, i.e.,

$$\|P_K(u) - P_K(v)\| \leq \|u - v\| \quad \text{for all } u, v \in H. \quad (5.7)$$

We know that the dual of \mathbb{R}^N can be shown to be isometrically isomorphic to \mathbb{R}^N . We can then state that there exists a linear surjective isometry $\zeta : (\mathbb{R}^N)^* \rightarrow \mathbb{R}^N$, which identifies $(\mathbb{R}^N)^*$ and \mathbb{R}^N . The existence of the mapping follows from Theorem 1.2.15 (Riesz-representation theorem).

The first theorem on the subject of variational inequalities is now presented.

Theorem 5.0.17. *Suppose $K \subset \mathbb{R}^N$ is compact and convex. Let*

$$f : K \rightarrow (\mathbb{R}^N)^*$$

be continuous. Then there is a $u \in K$ such that

$$\langle f(u), v - u \rangle \geq 0 \quad \text{for all } v \in K. \quad (5.8)$$

Proof: From the above and Riesz-representation theorem, it is clear that what we are looking for is a $u \in K$ such that $\langle f(u), v - u \rangle = [\zeta(f(u)), v - u] \geq 0$ for all $v \in K$. Equivalently, $u \in K$ such that $[u, v - u] \geq [u - \zeta(f(u)), v - u]$ for all $v \in K$. Consider the continuous mapping

$$P_K(I - \zeta(f)) : K \rightarrow K,$$

where $I(u) = u$ is the identity mapping. It follows from Corollary 2.1.5, that there exists a fixed point $u \in K$ such that $u = P_K(I - \zeta(f))(u)$. Using the characterization of the projection given in Theorem 5.0.16, it follows that

$$[u, v - u] \geq [u - \zeta(f(u)), v - u] \quad \text{for } v \in K.$$

□

In the following section, it is assumed that V is a reflexive Banach space with dual V^* . Take $K \subset V$ as a closed convex set. We now seek to establish a connection between monotonicity and the existence of a solution to a variational inequality.

Definition 5.0.18. The mapping $A : K \rightarrow V^*$ is *continuous on finite-dimensional subspaces* if, for any finite-dimensional subspace $M \subset V$, the restriction of A to $K \cap M$ is weakly continuous.

The following lemma by Minty (see [55]) will play a crucial role in the proof of our main theorem involving monotone operators.

Lemma 5.0.19 (Minty 1962). *Let K be a closed convex subset of V and $A : K \rightarrow V^*$ a monotone mapping, which is continuous on finite-dimensional subspaces. Then, $u \in K$ satisfies*

$$\langle A(u), v - u \rangle \geq 0 \text{ for all } v \in K \quad (5.9)$$

if and only if it satisfies

$$\langle A(v), v - u \rangle \geq 0 \text{ for all } v \in K. \quad (5.10)$$

Proof: (\implies) We first show that (5.9) implies (5.10). Using the monotonicity of A we have that

$$0 \leq \langle A(v) - A(u), v - u \rangle = \langle A(v), v - u \rangle - \langle A(u), v - u \rangle \text{ for } u, v \in K,$$

together with (5.9) this yields

$$u \in K : 0 \leq \langle A(u), v - u \rangle \leq \langle A(v), v - u \rangle \text{ for all } v \in K.$$

Thus, the first implication is proved.

(\impliedby) We now show that (5.10) implies (5.9). Take $w \in K$ and $0 \leq t \leq 1$. Consider $v = u + t(w - u)$, which is in K due to the convexity of K . By (5.10), for $t > 0$ we have

$$\langle A(u + t(w - u)), t(w - u) \rangle \geq 0,$$

dividing by $\frac{1}{t}$ yields

$$\langle A(u + t(w - u)), w - u \rangle \geq 0 \text{ for all } w \in K.$$

Consider the sequence $u + t_n(w - u)$, where $t_n := \frac{1}{n}$. It is straightforward to show that $u + t_n(w - u) = (1 - t_n)u + t_n w \rightarrow u$ in the norm topology. Since A is weakly continuous on the intersection of K , with the subspace spanned by u and w , we obtain

$$A((1 - t_n)u + t_n w) \rightharpoonup A(u).$$

It, therefore, follows that $\langle A(u), w - u \rangle \geq 0$ for any $w \in K$. \square

Theorem 5.0.20. *Let K be a closed bounded convex subset of V and let $A : K \rightarrow V^*$ be a monotone mapping, which is continuous on finite-dimensional subspaces. Then, there exists $u \in K$ such that*

$$\langle A(u), v - u \rangle \geq 0 \text{ for all } v \in K. \quad (5.11)$$

Also, the solution set is closed and convex and if A is assumed strictly monotone, then the solution u to (5.11) is unique.

Proof: Take $M \subset V$ to be a finite-dimensional subspace with dimension $N < \infty$. Without loss of generality we can assume that $0 \in K$. Now consider the injection mapping

$$j : M \rightarrow V$$

and the mapping

$$j^* : V^* \rightarrow M^*.$$

These mappings are such that

$$\langle f, j(u) \rangle_{V^*, V} = \langle j^* f, u \rangle_{M^*, M} \quad \text{whenever } u \in M, f \in V^*.$$

Note that both of j and j^* chosen in this way are continuous. Set $K_M := K \cap M$. Consider the variational inequality (VI_M): Find $u_M \in K_M$ such that

$$\langle j^* A j(u_M), v - u_M \rangle_{M^*, M} \geq 0 \quad \text{for all } v \in K_M. \quad (5.12)$$

Since $u_M, v \in M$, upon examining the left-hand side of (5.12), it is clear that $j(v - u) = v - u$ and

$$\langle A(u_M), v - u_M \rangle_{V^*, V} = \langle A(j(u_M)), j(v - u_M) \rangle_{V^*, V} = \langle j^* A j(u_M), v - u_M \rangle_{M^*, M}.$$

M is a closed subspace of V (since M is a finite-dimensional subspace), therefore $(M, \|\cdot\|)$ is a Banach space. Now K_M is a closed convex bounded subset of $(M, \|\cdot\|)$. Hence, by Kakutani's Theorem, K_M is weakly compact in $(M, \|\cdot\|)$. Further, we have that since M is finite-dimensional, it is also linearly homeomorphic to \mathbb{R}^N , and endowing it with a scalar product allows us to show that it is even isometrically isomorphic to \mathbb{R}^N . This is enough to satisfy the conditions necessary to use Theorem 5.0.17, which yields the existence of an element $u_M \in K_M$ which satisfies (5.12), then

$$\langle A(u_M), v - u_M \rangle_{V^*, V} \geq 0 \quad \text{for all } v \in K_M$$

By Lemma 5.0.19, we have

$$\langle A(v), v - u_M \rangle_{V^*, V} \geq 0 \quad \text{for all } v \in K_M.$$

We now define the set

$$S(v) := \{u \in K : \langle A(v), v - u \rangle \geq 0\}. \quad (5.13)$$

From the previous working, we have that $S(v) \neq \emptyset$ for all $v \in K_M$. It is straightforward to show that $S(v)$ is weakly closed for each $v \in K$. Since K is weakly closed, it follows that $S(v)$ is a closed subset of K , where K is endowed with the weak topology. Furthermore, K is weakly compact by Kakutani's Theorem. Moreover, $\{S(v) : v \in K\}$ is a family of weakly closed subsets of the weakly compact set K . Now, note that the variational inequality (5.11) has a solution if and only if $\bigcap_{v \in K} S(v) \neq \emptyset$. To this end, we will prove that $\{S(v) : v \in K\}$ has the finite intersection property. Take $\{v_1, \dots, v_m\} \subset K$. We claim that

$$S(v_1) \cap S(v_2) \cap \dots \cap S(v_m) \neq \emptyset. \quad (5.14)$$

Take M to be the finite-dimensional subspace of V spanned by $\{v_1, \dots, v_m\}$, while defining $K_M := K \cap M$ as before. By the same argument as before, there exists an element $u_M \in K_M$ such that

$$\langle A(v), v - u_M \rangle_{V^*, V} \geq 0 \quad \text{for all } v \in K_M.$$

In particular, we consider

$$\langle A(v_i), v_i - u_M \rangle_{V^*, V} \geq 0 \quad \text{for } i \in \{1, \dots, m\},$$

this yields $u_M \in S(v_i)$ for $i \in \{1, \dots, m\}$. As a consequence, for any finite collection $\{v_1, \dots, v_m\} \subset K$, (5.14) holds. Therefore, there exists an element

$$u \in \bigcap_{v \in K} S(v),$$

this gives

$$u \in K : \langle A(v), v - u \rangle \geq 0 \quad \text{for all } v \in K.$$

Again using Lemma 5.0.19 yields

$$u \in K : \langle A(u), v - u \rangle \geq 0 \quad \text{for all } v \in K.$$

This solution is not necessarily unique and it can be shown that the set of solutions is a closed convex subset of K as follows.

We first show that given $u_1, u_2 \in U := \{u \in K : \langle A(u), v - u \rangle \geq 0, \forall v \in K\}$, it follows that $tu_1 + (1-t)u_2 \in U$, where $t \in [0, 1]$. Since we have $u_1, u_2 \in U$, it can be shown, using Lemma 5.0.19, that $\langle A(v), v - (tu_1 + (1-t)u_2) \rangle \geq 0$ for all $v \in K$. Again using Lemma 5.0.19 yields

$$\langle A(tu_1 + (1-t)u_2), v - (tu_1 + (1-t)u_2) \rangle \geq 0 \quad \text{for all } v \in K.$$

Hence, the solution set U is convex.

We now show that U is also closed. We know that $U \subset \bar{U}$, so we will prove the other inclusion. Take $u \in \bar{U}$, then there exists a sequence $\{u_n\}_n \subset U$ such that $u_n \rightarrow u$ in K . Therefore, $v - u_n \rightarrow v - u$ in K , for any $v \in K$. We have $\langle A(u_n), v - u_n \rangle \geq 0$, and so by Lemma 5.0.19 it follows that $\langle A(v), v - u_n \rangle \geq 0$. But since $v - u_n \rightarrow v - u$, we have $\langle A(v), v - u \rangle \geq 0$. Using Lemma 5.0.19 again, yields $\langle A(u), v - u \rangle \geq 0$. Hence, $u \in U$ and it follows that $U = \bar{U}$. \square

Corollary 5.0.21. *Let H be a Hilbert space and $K \subset H$ a closed bounded convex set (non-empty). Suppose that $f : K \rightarrow K$ is non-expansive (has Lipschitz constant ≤ 1). Then f has a non-empty closed convex subset of fixed points.*

Proof: Since H is isometrically isomorphic to H^* , instead of considering this isometric isomorphism $\zeta : H \rightarrow H^*$ in composition with f , we note that there is no reason to distinguish between these objects so we take $H^* = H$ (where we associate the bilinear pairing $\langle \cdot, \cdot \rangle$ with the inner-product $[\cdot, \cdot]$ as in Theorem 1.2.15 (Riesz-Frechet Representation Theorem)). As a consequence, we consider the mapping $f : K \rightarrow K \subset H = H^*$. It is given that f is continuous, so as a result we have that $I - f$ is continuous. What is left to prove is that $I - f$ is a monotone operator.

This can be achieved using the Cauchy-Schwarz inequality:

$$\begin{aligned} [f(u) - f(v), u - v] &\leq [f(u) - f(v), f(u) - f(v)]^{\frac{1}{2}} [u - v, u - v]^{\frac{1}{2}} \\ &= \|f(u) - f(v)\| \|u - v\| \\ &\leq \alpha [u - v, u - v] \end{aligned}$$

for all $u, v \in K$. Since $\alpha \in [0, 1]$, it follows that $[(I - f)(u) - (I - f)(v), u - v] \geq 0$ for all $u, v \in K$. We can now apply Theorem 5.0.20. The result follows. \square

The following results also hold for monotone operators:

Theorem 5.0.22. *Let K be a closed convex subset of V and let $A : K \rightarrow V^*$ be a monotone mapping, which is continuous on finite-dimensional subspaces. Then, there exists $u \in K$ such that*

$$\langle A(u), v - u \rangle \geq 0 \quad \text{for any } v \in K$$

if and only if there exists a $R > 0$, such that we have at least one solution, $u_R \in K_R$, of the variational inequality

$$\langle A(u_R), v - u_R \rangle \geq 0 \quad \text{for any } v \in K_R,$$

where $K_R := K \cap \{v : \|v\| \leq R\}$ and $\|u_R\| < R$.

Corollary 5.0.23. *Take K as a closed convex set (non-empty). Let $A : K \rightarrow V^*$ be monotone, coercive and continuous on finite-dimensional subspaces. Then, it follows that there exists $u \in K$ such that*

$$\langle A(u), v - u \rangle \geq 0 \quad \text{for any } v \in K.$$

Definition 5.0.24. Let V be a Banach space, $K \subset V$ a closed convex set, $A : K \rightarrow V^*$ an operator and $f \in V^*$. Then, the *variational inequality* is the following problem: Find $u \in K$ satisfying

$$\langle A(u), v - u \rangle \geq \langle f, v - u \rangle \quad \text{for all } v \in K. \quad (5.15)$$

Below we show how a pseudomonotone operator can be used in an existence proof involving a variational inequality. Before doing so, we note the following.

Remark 19. It can be shown that the following definition of pseudomonotonicity is equivalent to that of Definition 2.1.12 presented previously. Consider operator $A : K \rightarrow V^*$, then A is pseudomonotone if

$$A \text{ is bounded,} \quad (5.16)$$

and

$$u_n \rightharpoonup u \text{ in } V, \quad u_n \in K \quad \text{and} \quad \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0 \quad (5.17)$$

implies

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = 0 \quad \text{and} \quad A(u_n) \rightharpoonup A(u) \text{ in } V^*. \quad (5.18)$$

Theorem 5.0.25. *Take V as a reflexive, separable, Banach space and $K \subset V$ a closed, convex, bounded subset. Assume that $A : K \rightarrow V^*$ is pseudomonotone. Then, for all $f \in V^*$ there exists $u \in K$ which satisfies (5.15), i.e.,*

$$\langle A(u), v - u \rangle \geq \langle f, v - u \rangle \quad \text{for all } v \in K.$$

Proof: Take a sequence $(e_k)_k$ of linearly independent vectors in V , such that setting

$$V_n := \text{span}\{e_1, \dots, e_n\},$$

yields $V = \overline{\bigcup_n V_n}$. Now take $K_n := V_n \cap K$. It is simple to show that $K_n \subset V_n$ is a closed, convex, bounded subset of V . It can then also be shown that $K = \overline{\bigcup_n K_n}$. Our first goal is to show the existence of solutions $u_n \in K_n$ of the variational inequalities for all n , i.e.,

$$\langle A(u_n), v - u_n \rangle \geq \langle f, v - u_n \rangle \quad \text{for all } v \in K_n. \quad (5.19)$$

Now, endow V_n with an inner product $[\cdot, \cdot]$. It can be shown that for any n -dimensional real inner product space, there exists a linear map $f : V_n \rightarrow \mathbb{R}^n$ which is an isomorphism of inner product spaces (it follows also that V_n is then isometrically isomorphic to \mathbb{R}^n). The equivalence of the new inner-product induced norm and the original norm stems from the fact that we are in a finite-dimensional subspace.

Consider a linear functional $g \in V^*$ and its continuous linear restriction to the finite-dimensional subspace V_n . Since V_n is a Hilbert space, we can use the Riesz-representation theorem to show that there exists a continuous linear operator $B : V^* \rightarrow V_n$ such that $\langle g, w \rangle = [B(g), w]$ for all $w \in V_n$. This allows (5.19) to be restated as

$$[B(A(u_n)), v - u_n] \geq [B(f), v - u_n] \quad \text{for all } v \in K_n,$$

which is equivalent to

$$[u_n, v - u_n] \geq [u_n + B(f) - B(A(u_n)), v - u_n] \quad \text{for all } v \in K_n. \quad (5.20)$$

This is a familiar characterization of a projection, which we encountered in Theorem 5.0.16. Consider the projection $P_n : V_n \rightarrow K_n$. It follows, then, that

$$u_n = P_n(u_n + B(f) - B(A(u_n))). \quad (5.21)$$

Now, define an operator $\psi_n : K_n \rightarrow K_n$:

$$\psi_n(v) := P_n(v + B(f) - B(A(v))) \quad v \in K_n. \quad (5.22)$$

At this point, we seek to use Brouwer's fixed point theorem, therefore, we must ensure the necessary requirements are met.

First, we check the continuity of ψ_n . Note that the weak and strong topologies coincide with respect to K_n . Assume that $v_k \rightarrow v$ in K_n . Using Lemma 2.1.13, it follows that A is demicontinuous and hence $A(v_k) \rightharpoonup A(v)$ in V^* , further, we have that $B(A(v_k)) \rightarrow B(A(v))$ in K_n . Together with the continuity of the P_n (shown in Theorem 5.0.16) it yields

$$P_n(v_k + B(f) - B(A(v_k))) \rightarrow P_n(v + B(f) - B(A(v))) \quad \text{as } k \rightarrow \infty.$$

The Brouwer's fixed point theorem then implies that ψ_n has a fixed point. This shows the existence of a solution for (5.21).

We now examine the sequence $(u_n)_n$ of solutions to (5.21). Seeing as K is bounded in reflexive space V , by Theorem 1.1.11 (Eberlein-Šmulian Theorem), there is a convergent subsequence of $\{u_n\}_n$ (which for notational simplicity will again be denoted as $\{u_n\}_n$), then,

$$u_n \rightharpoonup u \quad \text{in } V. \quad (5.23)$$

Since K is convex, K is closed in both the weak and strong topology, hence, $u \in K$.

In order to use the pseudomonotonicity of A , we need to first show

$$\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0. \quad (5.24)$$

As previously noted, $\bigcup_n K_n$ is dense in K , it follows that for some arbitrary ϵ there exists a $u_0 \in \bigcup_n K_n$ such that

$$\|u - u_0\|_V \leq \epsilon. \quad (5.25)$$

Since $u_0 \in \bigcup_n K_n$, it follows that $u_0 \in K_n$, for some n taken large enough. Then, using (5.19), gives

$$\langle A(u_n), u_n - u_0 \rangle \leq \langle f, u_n - u_0 \rangle.$$

Using (5.25), and noting that $\|A(u_n)\|_{V^*}$ is bounded, yields

$$\langle A(u_n), u_n - u \rangle = \langle A(u_n), u_n - u_0 \rangle + \langle A(u_n), u_0 - u \rangle \leq \langle f, u_n - u_0 \rangle + C\epsilon,$$

for some constant $C \geq 0$. This leads to

$$\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq \langle f, u_n - u_0 \rangle + C\epsilon \leq \|f\|_{V^*} \|u - u_0\| + C\epsilon \leq C_1\epsilon.$$

Thus, by the arbitrariness of ϵ , we obtain (5.24).

The pseudomonotonicity of A now gives

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = 0 \quad \text{and} \quad A(u_n) \rightharpoonup A(u) \quad \text{in } V^*. \quad (5.26)$$

Now, for any $v \in \bigcup_n K_n$, we have $v \in K_n$ for n large enough. Using (5.19) we obtain that

$$\langle A(u_n), v - u \rangle + \langle A(u_n), u - u_n \rangle = \langle A(u_n), v - u_n \rangle \geq \langle f, v - u_n \rangle.$$

Hence, passing to the limit, using once again Remark 19, we get that

$$\langle A(u), v - u \rangle \geq \langle f, v - u \rangle \quad \text{for any } v \in \bigcup_n K_n. \quad (5.27)$$

Since $\bigcup_n K_n$ is dense in K , for any arbitrary $v \in K$, there exists a sequence $\{v_m\}_m \subset \bigcup_n K_n$, such that $v_m \rightarrow v$ in K . Then, using (5.27) gives the desired solution of the variational inequality (5.15). \square

We now consider a result, which relaxes the requirement that $K \subset V$ is bounded.

Theorem 5.0.26. *Let V be a reflexive, separable Banach space and $K \subset V$ a closed, convex subset. Assume that $A : K \rightarrow V^*$ is pseudomonotone and coercive in the following way: there exists $v_0 \in K$ such that*

$$\lim_{\|v\| \rightarrow \infty} \frac{\langle A(v), v - v_0 \rangle}{\|v\|_V} \rightarrow \infty. \quad (5.28)$$

Then, for any $f \in V^$ there exists a solution $u \in K$ of (5.15).*

Proof: Consider subsets $B_R := \{v \in V : \|v\| \leq R\}$ and $K_R := K \cap B_R$. It is simple to show that K_R is closed, convex and bounded. It follows, by Theorem 5.0.25, that there exists a $u_R \in K_R$ such that

$$\langle A(u_R), v - u_R \rangle \geq \langle f, v - u_R \rangle \quad \text{for any } v \in K_R. \quad (5.29)$$

Claim: $\exists M > 0$ such that $\|u_R\|_V \leq M$ for R large enough.

In order to prove by contradiction, assume that there exists a sequence $\{R_n\}_n$ of real numbers such that $R_n \rightarrow \infty$ and $\|u_{R_n}\|_V \rightarrow \infty$. Let n be large enough such that $\|v_0\| \leq R_n$. Taking $v = v_0$ in (5.29) of R_n we get

$$\langle A(u_{R_n}), v_0 - u_{R_n} \rangle \geq \langle f, v_0 - u_{R_n} \rangle \geq -\|f\|_{V^*} \|v_0 - u_{R_n}\|_V.$$

Hence,

$$\frac{\langle A(u_{R_n}), u_{R_n} - v_0 \rangle}{\|u_{R_n}\|_V} \leq \|f\|_{V^*} \frac{\|v_0\|_V + \|u_{R_n}\|_V}{\|u_{R_n}\|_V} \leq 2\|f\|_{V^*}$$

for n taken large enough. This contradicts the coercivity condition (5.28).

So, by the boundedness of $\{u_R\}_R$ it follows by the Eberlein-Šmulian Theorem that there exists a weakly convergent subsequence $\{u_{R_k}\}_k$ and $u \in V$ such that

$$u_{R_k} \rightharpoonup u \quad \text{in } V. \quad (5.30)$$

Using Theorem 1.2.5, and noting that $u_{R_k} \in K$, where K is assumed closed and convex, it follows that $u \in K$. For all $v \in K$, we can find k large enough such that $v \in K_{R_k}$. Hence, v can be used in (5.29) for R_k , that is,

$$\langle A(u_{R_k}), v - u_{R_k} \rangle \geq \langle f, v - u_{R_k} \rangle. \quad (5.31)$$

In particular, taking $v = u$ we get

$$\langle A(u_{R_k}), u_{R_k} - u \rangle \leq \langle f, u_{R_k} - u \rangle.$$

Hence, $\limsup_{k \rightarrow \infty} \langle A(u_{R_k}), u_{R_k} - u \rangle \leq \lim_{k \rightarrow \infty} \langle f, u_{R_k} - u \rangle = 0$ implies (since A is pseudomonotone) from Remark 19 that

$$\lim_{k \rightarrow \infty} \langle A(u_{R_k}), u_{R_k} - u \rangle = 0 \quad \text{and} \quad A(u_{R_k}) \rightharpoonup A(u) \quad \text{in } V^*. \quad (5.32)$$

From (5.31), we get $\langle A(u_{R_k}), v - u \rangle + \langle A(u_{R_k}), u - u_{R_k} \rangle \geq \langle f, v - u_{R_k} \rangle$. Hence, by passing to the limit as $k \rightarrow \infty$, we obtain using (5.32) that $\langle A(u), v - u \rangle \geq \langle f, v - u \rangle$. \square

If we assume that $A : K \rightarrow V^*$ is strictly monotone, then the solution of (5.15) is unique: Assume that u_1 and u_2 are solutions. Taking $v = u_2$ and $v = u_1$ yields

$$\langle A(u_1), u_2 - u_1 \rangle \geq \langle f, u_2 - u_1 \rangle \text{ and } \langle A(u_2), u_1 - u_2 \rangle \geq \langle f, u_1 - u_2 \rangle,$$

together this yields $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \leq 0$ and it follows, then, that $u_1 = u_2$.

We now present an example of how pseudomonotone operators can be used in an existence proof of a variational inequality.

Example: Assume $\Omega \subset \mathbb{R}^N$ is bounded. Divide Γ as follows,

$$\Gamma_0 := \left\{ x \in \Gamma : \frac{\partial}{\partial \xi_j} a_i(x, s, \xi) \nu_i \nu_j = 0, \quad \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N \right\},$$

$$\Gamma_1 := \left\{ x \in \Gamma_0 : \frac{\partial}{\partial s} a_i(x, s, \xi) \nu_i = 0, \quad \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N \right\},$$

$$\Gamma_2 = \Gamma_0 \setminus \Gamma_1 \quad \text{and} \quad \Gamma_3 = \Gamma \setminus \Gamma_0,$$

where $\bar{\nu} = (\nu_1, \dots, \nu_N)$ is the outward normal to Γ . Suppose that $\bar{W}^{1,p}$ is the closure of $\{u \in C(\bar{\Omega}) : u = 0 \text{ on } \Gamma_3\}$ in the norm of $W^{1,p}(\Omega)$. Let

$$K := \{v \in W^{1,p}(\Omega) : v \leq \psi \text{ a.e. } x \in \Omega\},$$

where $\psi \in C^2(\bar{\Omega})$, $\psi = 0$ on Γ_3 is a given function. Consider the *Keldys-Fichera obstacle problem*:

$$\left\{ \begin{array}{ll} -\frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + c(x, u, \nabla u) \leq g(x) & \text{where } x \in \Omega, \\ u \leq \psi & \text{a.e. } x \in \Omega, \\ \left(-\frac{\partial}{\partial x_i} a_i(x, u, \nabla u) + c(x, u, \nabla u) - g \right) (u - \psi) = 0 & \text{where } x \in \Omega, \\ u = 0 & x \in \Gamma_2 \cup \Gamma_3, \end{array} \right. \quad (5.33)$$

where $\frac{\partial}{\partial x_j} a_i(x, s, \xi) \xi_i \xi_j > 0$, for a.e. $x \in \Omega$, $\forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

A function $u \in \bar{W}^{1,p} \cap K$ is a weak solution of (5.33), if for any $v \in K$, the variational inequality

$$\int_{\Omega} \left(a_i(x, u, \nabla u) \frac{\partial}{\partial x_i} (v - u) + c(x, u, \nabla u) (v - u) \right) dx \geq \int_{\Omega} g(x) (v - u) dx + \int_{\Gamma_0} a_i(x, 0, 0) \nu_i (v - u) ds \quad (5.34)$$

holds.

Suppose conditions (3.2), (3.3), (3.5) and (3.6) are satisfied and $a_i(x, 0, 0) \in W^{\frac{1}{p}, p'}(\Gamma)$, then problem (5.33) has a weak solution.

See [68] for an extended review of this obstacle problem.

Remark 20. A more recent paper (see [36]), investigates a connection between variational inequalities and the notion of K -pseudomonotonicity introduced by Karamardian (as we presented in Definition 2.1.20), where in particular, existence results are studied.

Chapter 6

Conclusion and Discussion

6.1 Brief Review of Topics Presented

In the earlier chapters, we provided the abstract theory, which served as a foundation for the applications to variational problems presented in later chapters. The two main theorems presented in Chapter 2 are Theorem 2.1.11 (Browder, Minty, 1963) and Theorem 2.1.15 (Brézis, 1968). The importance of Theorem 2.1.15 becomes apparent when dealing with problems where there is no monotonicity condition on the operator or perturbation thereof.

In the section which followed, we examined some properties of monotone and pseudomonotone operators, concluding with a review of two commonly used definitions of pseudomonotonicity, one of which was proposed by S. Karamardian and the other, a generalization of the notion of pseudomonotonicity introduced by H. Brézis.

In Chapter 3, we applied the abstract theory to a second order boundary value problem (with mixed boundary conditions). We started by motivating the concept of a weak formulation. Then, by transforming the problem into a functional equation, $A(u) = f$, satisfying the necessary requirements of pseudomonotonicity and coercivity, using the abstract theory presented previously, we proved the existence of a weak solution.

An example of a problem, which is suitable for this type of approach, would be the case where the principal part of a quasilinear elliptic equation is monotone, but the lower order terms only have strongly continuous perturbations. Thus, we would seek to prove the existence of a weak solution to

$$A(u) + B(u) = f \quad \text{in } V^*,$$

where $A : V \rightarrow V^*$ is monotone and hemicontinuous and $B : V \rightarrow V^*$ is strongly continuous. Hence, by Remark 11, we have that operator $A + B$ is pseudomonotone. The following boundary value problem is a standard example:

$$\begin{cases} \operatorname{div}(|\nabla u|^{p-2}\nabla u) + c(u) = g & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \quad (6.1)$$

where $\langle A(u), v \rangle := \int_{\Omega} |\nabla u|^{p-2}\nabla u \cdot \nabla v dx$, $\langle B(u), v \rangle := \int_{\Omega} c(u)v dx$ and $\langle g, v \rangle := \int_{\Omega} gv dx$.

We concluded Chapter 3 with an examination of a second order boundary value problem governed by an anisotropic operator, and proved the existence of a weak solution to this problem using certain ideas presented previously in Chapter 3.

Furthermore, we sought to gain deeper insight into how pseudomonotone operators relate to the terms of second order partial differential equations, which are used to define the pseudomonotone operator. To this end, we established a result (Theorem 4.0.12) that exhibits a connection between the monotonicity of the principal part of a second order partial differential equation and the pseudomonotone operator. We discussed the differential operator

$$A(u) := \operatorname{div}(a(x, u(x), \nabla u(x))),$$

where the operator was defined as $\langle A(u), v \rangle := \int_{\Omega} a(x, u, \nabla u) \cdot \nabla v dx$ for all $u, v \in W^{1,p}(\Omega)$.

In Chapter 5, we briefly introduced elliptic variational inequalities, mentioning also, the significant role this subject plays in diverse areas of applied science. We placed emphasis on presenting existence results, which use monotone (see Theorem 5.0.20, Theorem 5.0.22 and Corollary 5.0.23) or pseudomonotone operators (see Theorem 5.0.25 and Theorem 5.0.26). We also proved Lemma 5.0.19 (by Minty(1962)), which is used in the proof of Theorem 5.0.20.

6.2 Further Applications of Monotone and Pseudomonotone Operators

This section is concerned with other applications involving monotone and pseudomonotone operators not previously mentioned.

6.2.1 Nonlinear Stationary Problems

Application in the Calculus of Variations

The connection between monotone operators and the calculus of variations can be considered by briefly examining a typical problem. We previously considered the problem of finding a solution to the equation

$$A(u) = f \quad u \in V.$$

We can further explore the case where there exists a functional $g : V \rightarrow \mathbb{R}$, where $g'(u) = A(u) - f$ on V . That is, the equation

$$g'(u) = 0 \quad u \in V, \tag{6.2}$$

along with the minimum problem,

$$\operatorname{argmin}(g(u)) \quad u \in V. \tag{6.3}$$

Among other results, we can show that the solutions of (6.3) are also the solutions of (6.2). This type of investigation leads to numerous applications in diverse areas of applied science (see [17] for a paper by Browder(1970) on pseudomonotone operators and the calculus of variations).

6.2.2 Nonlinear Non-Stationary Problems

Maximal Monotone Operators

Within the theory of monotone operators, an important concept is that of maximal monotone operators. Browder contributed extensively to the development of this topic, with his existence results being particularly noteworthy (see [15], [11] and [16]). Among others, this theory can be applied to evolution problems (of first and second order) and variational inequalities (elliptic and parabolic).

The following is one of the initial ideas presented by Minty (see [55]), in 1962, on the theory of monotone operators:

Consider a Hilbert space H , with real scalars and with inner product $[\cdot, \cdot]$. Minty showed that a monotone operator

$$A : D(A) \subset V \rightarrow V \quad (6.4)$$

on a real Hilbert space is maximal monotone if and only if

$$R(A + I) = V.$$

Specifically, we have that each continuous monotone operator $A : V \rightarrow V$, on a Hilbert space V is maximal monotone and $A + I : V \rightarrow V$ is a homeomorphism, so that

$$A(u) + u = f \quad u \in V \quad (6.5)$$

has a unique solution u for each $f \in V$.

Pseudomonotone Perturbations of Maximal Monotone Mappings

Let V be a reflexive Banach space. Assume $K \subset V$ is a non-empty closed, convex bounded subset of V . If we assume that $A : K \rightarrow V^*$ is maximal monotone and $B : K \rightarrow V^*$ is pseudomonotone, then there is a Theorem (Browder 1968), which states that for $f \in V^*$, there exists a solution to

$$A(u) + B(u) = f. \quad (6.6)$$

A more detailed treatment of these topics can be found in [67].

Pseudomonotone Operators and First-Order Evolution Problems

The treatment of the following initial-value problem provides a prime example of an application of pseudomonotone operators with regard to evolution problems:

Let V be a separable reflexive Banach space that is embedded continuously and densely into a Hilbert space H (we have $V \subset H \simeq H^* \subset V^*$).

Consider the Cauchy-problem:

$$\begin{cases} \frac{du}{dt} + A(t, u(t)) = f(t) & \text{for a.e. } t \in I, \\ u(0) = u_0 & \text{(Initial Condition) ,} \end{cases} \quad (6.7)$$

where $A : I \times V \rightarrow V^*$, with $I := [0, T]$, is a Carathéodory function. In [58], the problem is first examined for $A : V \rightarrow V^*$ (using the Rothe Method), where A is assumed

pseudomonotone, and then later generalized to the case where $A : I \times V \rightarrow V^*$, as described above.

For a full description of the methods used to examine this type of problem, consult [58], [61] and [67] (for applications to quasilinear parabolic equations see [58]).

6.3 Discussion of Current Research

As an example of a topic, which has more recently been explored and, which has potential for significant future research, we briefly introduce the idea of using Young measures in proving the existence of weak solutions to quasilinear elliptic boundary value problems. The advantage of using this method is that it allows us to prove existence results for problems, which do not satisfy the requirements needed to use the previously defined methods. This is achieved by introducing a weaker notion of monotonicity, called *quasimonotonicity*. *Young measures* play a pivotal role in the following method.

Definition 6.3.1. A *Young measure* on $\Omega \times \mathbb{R}^n$ is a positive measure λ on $\Omega \times \mathbb{R}^n$, such that for any Borel set $A \subset \Omega$, it follows that $\lambda(A \times \mathbb{R}^n) = \mu(A)$.

Definition 6.3.2. Suppose that $u : \Omega \rightarrow \mathbb{R}^n$ is measurable, the *Young Measure* ν associated with u is the unique Young measure carried by the graph of u . Thus, it is the image of μ by the mapping $x \mapsto (x, u(x))$, .i.e., for Borel subsets $A \subset \Omega$ and $B \subset \mathbb{R}^n$, it follows that $\nu(A \times B) = \mu(A \cap u^{-1}(B))$. Also, for any positive measurable $\phi : \Omega \times \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ (or ν -integrable), we have

$$\int_{\Omega \times \mathbb{R}^n} \phi d\nu = \int_{\Omega} \phi(x, u(x)) \mu(dx).$$

Definition 6.3.3. The *disintegration* of a Young measure λ , is a family, $(\lambda_x)_x$ of probabilities on \mathbb{R}^n such that

$$\int_{\Omega \times \mathbb{R}^n} \phi d\lambda = \int_{\Omega} \left(\int_{\mathbb{R}^n} \phi(x, y) \lambda_x(dy) \right) \mu(dx),$$

for all positive measurable functions $\phi : \Omega \times \mathbb{R}^n \rightarrow \bar{\mathbb{R}}$ (or λ -integrable).

Remark 21. Note that when ν is associated to u , we obtain $\nu_x = \delta_{u(x)}$ (where $\delta_{u(x)}$ denotes the Dirac mass at $u(x)$). For a full treatment and description of the case where we consider *gradient Young measures*, refer to [41].

Denote by $\mathcal{M}^{m \times n}$, the real vector space of $m \times n$ matrices equipped with inner-product $[M, N] := \sum_{i \in m} \sum_{j \in n} a_{ij} b_{ij}$.

Definition 6.3.4. A mapping $h : \mathcal{M}^{m \times n} \rightarrow \mathcal{M}^{m \times n}$ is said to be strictly *p-quasimonotone*, if

$$\int_{\mathcal{M}^{m \times n}} [h(\tau) - h(\bar{\tau}), (\tau - \bar{\tau})] d\nu(\tau) > 0 \quad (6.8)$$

for all homogenous $W^{1,p}$ gradient Young measures ν , with center of mass $\bar{\tau} = \langle \nu, id \rangle$, which are not a single Dirac mass.

Consider the Dirichlet problem of the type

$$\begin{cases} -\operatorname{div}(a(x, u(x), \nabla u(x))) = g & \text{on } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \quad (6.9)$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain, $u : \Omega \rightarrow \mathbb{R}^m$ and $g \in W^{-1,p'}(\Omega)$ for $p \in (1, \infty)$.

Consider the following conditions:

- $a : \Omega \times \mathbb{R}^m \times \mathcal{M}^{m \times n} \rightarrow \mathcal{M}^{m \times n}$ is a Carathéodory function.
- Growth and coercivity conditions,

$$|a(x, s, \xi)| \leq \gamma_1(x) + C(|s|^q + |\xi|^{p-1})$$

and

$$[a(x, s, \xi), \xi] \geq -\gamma_2(x) - \gamma_3(x)|s|^\alpha + D|\xi|^p,$$

where $C \geq 0$, $D > 0$, $\gamma_1 \in L^{p'}(\Omega)$, $\gamma_2 \in L^1(\Omega)$, $\gamma_3 \in L^{\left(\frac{p}{\alpha}\right)'(\Omega)}$, $0 < \alpha < p$ and $0 < q \leq n \frac{p-1}{n-p}$.

Let one of the following conditions hold for mapping a :

- For all $x \in \Omega$ and $s \in \mathbb{R}^n$, the mapping $\xi \mapsto a(x, s, \xi)$ is C^1 and mapping a is monotone as in (3.6).
- There exists a mapping $f : \Omega \times \mathbb{R}^m \times \mathcal{M}^{m \times n} \rightarrow \mathbb{R}$ such that $a(x, s, \xi) = \frac{\partial f}{\partial \xi}(x, s, \xi)$ and $\xi \mapsto f(x, s, \xi)$ is both convex and C^1 .
- a is strictly monotone.
- a is strictly p -quasimonotone in ξ .

This leads to the following existence result.

Theorem 6.3.5. *If $a : \Omega \times \mathbb{R}^m \times \mathcal{M}^{m \times n} \rightarrow \mathcal{M}^{m \times n}$ satisfies the conditions mentioned previously, then the boundary value problem (6.9) has a weak solution $u \in W_0^{1,p}(\Omega)$ for any $g \in W^{-1,p}(\Omega)$.*

Remark 22. A new type of monotonicity called p -quasimonotonicity is introduced in Definition 6.3.4. It is more appropriate to consider, as new research directions:

- A comparison study between p -quasimonotonicity and the various notions of monotonicity and pseudomonotonicity described in previous chapters.
- The study of p -quasimonotonicity in connection with variational inequalities.

6.4 End Note

In this thesis, we explored some significant ideas on the subject of pseudomonotone and monotone operators. By extending our study from the abstract theory, presenting some of the fundamental existence results of Browder, Minty and Brézis, it became apparent that pseudomonotone and monotone operators have a wide scope of application in various areas of mathematics and applied science. The relevance of these operators in the areas of partial differential equations, variational inequalities, mechanics and others, make them indispensable tools for current research in mathematical sciences.

Chapter 7

Appendix

7.1 Notation

The conjugate exponent of $p \in [1, +\infty]$ is defined as $p' := \frac{p}{p-1}$, the exponent in the continuous embedding $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ is denoted by p^* , where

$$p^* := \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ \text{an arbitrarily large real} & \text{if } p = N, \\ +\infty & \text{if } p > N, \end{cases}$$

and the exponent in the trace operator $u \mapsto u|_{\Gamma} : W^{1,p}(\Omega) \rightarrow L^{p^{\#}}(\Gamma)$ is denoted by $p^{\#}$, where

$$p^{\#} := \begin{cases} \frac{Np-p}{N-p} & \text{if } p < N, \\ \text{an arbitrarily large real} & \text{if } p = N, \\ +\infty & \text{if } p > N. \end{cases}$$

7.2 Convolution and Mollification

Suppose that $\Omega \subset \mathbb{R}^N$ is open, $\epsilon > 0$ and $\Omega_{\epsilon} := \{x \in \Omega : \text{dist}(x, \Gamma) > \epsilon\}$. The standard mollifier $\eta \in C^{\infty}(\mathbb{R}^N)$ is defined as

$$\eta(x) := \begin{cases} Ce \left(\frac{1}{|x|^2 - 1} \right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where the constant $C > 0$ is chosen so that $\int_{\mathbb{R}^N} \eta dx = 1$. For each $\epsilon > 0$, let

$$\eta_{\epsilon}(x) := \frac{1}{\epsilon^n} \eta\left(\frac{x}{\epsilon}\right).$$

The functions η_{ϵ} are C^{∞} . It also holds that

$$\int_{\mathbb{R}^N} \eta_{\epsilon} dx = 1, \quad \text{spt}(\eta_{\epsilon}) \subset B(0, \epsilon).$$

We now look at what it means to 'mollify' a function.

Definition 7.2.1. If $f : \Omega \rightarrow \mathbb{R}$ is locally integrable, then we define its *mollification* as

$$f^\epsilon := \eta_\epsilon * f \text{ in } \Omega_\epsilon,$$

which is the *convolution*,

$$f^\epsilon(x) = \int_{\Omega} \eta_\epsilon(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta_\epsilon(y)f(x-y)dy$$

for $x \in \Omega_\epsilon$.

Some properties of mollifiers:

- $f^\epsilon \in C^\infty(\Omega_\epsilon)$.
- $f^\epsilon \rightarrow f$ almost everywhere as $\epsilon \rightarrow 0$.
- If $f \in C(\Omega)$ then $f^\epsilon \rightarrow f$ uniformly on compact subsets of U .
- If $1 \leq p < \infty$ and $f \in L^p_{loc}(\Omega)$ then $f^\epsilon \rightarrow f$ in $L^p_{loc}(\Omega)$.

7.3 Young's Inequality

Young's inequality states that if a and b are non-negative real numbers and p and q are positive real numbers so that $\frac{1}{p} + \frac{1}{q} = 1$, then we have that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \quad (7.1)$$

The adapted Young's Inequality states that given any $\epsilon > 0$, it follows that

$$ab \leq \epsilon a^p + C_\epsilon b^{p'}, \text{ where } C_\epsilon := \frac{\epsilon^{\frac{1}{p-1}} p^{\frac{1}{p'}}}{p-1}. \quad (7.2)$$

7.4 Hölder's Inequality

Suppose that (X, ξ, λ) is a measure space and take $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then for all measurable functions f, g we have

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q}. \quad (7.3)$$

7.5 Multidimensional Integration by Parts

Suppose that $\Omega \subset \mathbb{R}^N$ is open and bounded with piecewise smooth boundary Γ . Take v and z as being continuously differentiable functions, then integration by parts yields

$$\int_{\Omega} \frac{\partial v}{\partial x_i} z dx = \int_{\Gamma} v z \nu_i dS - \int_{\Omega} v \frac{\partial z}{\partial x_i} dx \text{ for all } i \in \{1, \dots, N\},$$

where we consider ν as the outward unit surface normal to Γ . Summing over $i \in \{1, \dots, N\}$ and considering $v \in W^{1,p}(\Omega)$ and $z \in W^{1,p'}(\Omega; \mathbb{R}^N)$ yields *Green's formula*,

$$\int_{\Omega} \nabla v \cdot z dx = \int_{\Gamma} v(z \cdot \nu) dS - \int_{\Omega} v(\nabla \cdot z) dx. \quad (7.4)$$

7.6 Nemytskii Mappings

Consider the integers k, n_0, n_1, \dots, n_k . A mapping $a : \Omega \times \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k} \rightarrow \mathbb{R}^{n_0}$ is a Carathéodory mapping if $a(\cdot, r_1, \dots, r_j) : \Omega \rightarrow \mathbb{R}^{n_0}$ is measurable for all $(r_1, \dots, r_k) \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k}$ and $a(x, \cdot) : \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k} \rightarrow \mathbb{R}^{n_0}$ is continuous for almost every $x \in \Omega$. Then the Nemytskii mappings, which we denote by η_a , map functions $u_i : \Omega \rightarrow \mathbb{R}^{n_i}$, $i \in \{1, \dots, k\}$ to a function $\eta_a(u_1, \dots, u_k) : \Omega \rightarrow \mathbb{R}^{n_0}$, which we define as

$$(\eta_a(u_1, \dots, u_k))(x) = a(x, u_1(x), \dots, u_k(x)).$$

Theorem 7.6.1. (*Nemytskii Mappings on L^p Spaces*)

If $a : \Omega \times \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_k} \rightarrow \mathbb{R}^{n_0}$ is a Carathéodory mapping and the functions $u_i : \Omega \rightarrow \mathbb{R}^{n_i}$, $i \in \{1, \dots, k\}$ are measurable, then $\eta_a(u_1, \dots, u_k)$ is measurable. Furthermore, if a satisfies the growth condition

$$|a(x, r_1, \dots, r_k)| \leq \gamma(x) + C \sum_{i=1}^k |r_i|^{p_i/p_0} \quad \text{for some } \gamma \in L^{p_0}(\Omega), \quad (7.5)$$

with $1 \leq p_i < +\infty$, $1 \leq p_0 < +\infty$, then η_a is a bounded continuous mapping $L^{p_1}(\Omega; \mathbb{R}^{n_1}) \times \dots \times L^{p_k}(\Omega; \mathbb{R}^{n_k}) \rightarrow L^{p_0}(\Omega; \mathbb{R}^{n_0})$.

7.7 The du-Bois-Reymond Lemma

Consider locally integrable function f defined on an open set $\Omega \subset \mathbb{R}^N$. If

$$\int_{\Omega} f(x)\psi(x)dx = 0 \quad (7.6)$$

for all $\psi \in C_0^\infty(\Omega)$, then $f(x) = 0$ almost everywhere for $x \in \Omega$.

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