EQUILIBRIA IN OVERLAPPING GENERATIONS MODELS

by Heide C. U. Mackensen

Paper submitted
in partial fulfilment of the requirements for the degree of
Master of Commerce in Economics

School of Economics
University of Cape Town

1992

ACKNOWLEDGEMENTS

I wish to thank my supervisor, Prof. Haim Abraham, for his informed insight and invaluable assistance towards the completion of this thesis. Many thanks also to Anthony Leiman for proof-reading.

The financial assistance of the Institute for Research Development towards this research is hereby acknowledged. Opinions expressed in this work, or conclusions arrived at, are those of the author and are not to be attributed to the Institute for Research Development.

TABLE OF CONTENTS

ABSTRACT

Interest rates are fundamental in the explanation of equilibrium prices over time, because they provide the link between the present and the future. Capturing this dynamic feature, the overlapping generations model is particularly suitable to address the interest rate problem, as has been shown by Paul Samuelson, David Gale and Costas Azariadis.

This thesis reviews their contribution to the theory of interest: with his consumption-loan model, Samuelson sets the analytical framework for subsequent research. Furthermore, he demonstrates that the optimal interest rate is unstable, implying that a competitive economy may fail to approach the social optimum. The Samuelson and classical sets of assumptions are consolidated in the intertemporal exchange model of Gale. Its equilibrium nature, however, ignores the sequential adjustment of disequilibrium interest rates to their equilibrium values. Consequently it is difficult to comment on the direction of causality involved in the interest rate determination, unless a clearing house is introduced which simultaneously resolves the starting-up, continuity and causality problems.

Departing from the full certainty scenario, Azariadis analyses the existence and likelihood of self-fulfilling prophecies. It is shown that the implications of the economy's assumed Markovian structure are twofold: while facilitating the parametric treatment of the transition probabilities, it negates the question concerning the likelihood of sunspot equilibria. Within the specified framework it is impossible to explain how the economy *arrives* at such equilibria; it is only possible to identify the conditions that *maintain* (once they exist) these self-fulfilling prophecies.

INTRODUCTION

Interest rates provide the link between present and future prices, they are, therefore, fundamental when explaining the evolution of equilibrium prices. In addition, the inverse relationship between interest rates and consumption levels (given certain assumptions), implies that changing intertemporal prices translate to business cycles. With these two intertwined issues in mind, this paper sets out to critically review a selection of the major articles that investigate the determinants of interest rates and consumption levels over time.

Reviewing the economic literature on interest rates, it is clear that, at the turn of the century, Eugen von Böhm-Bawerk dominated the theory of capital and interest with his "agio" theory. Having taken a grand equilibrium approach, he separated capitalists and entrepreneurs into lenders and borrowers in an aggregate capital market where present goods would be exchanged for future ones. Supply and demand forces in this intertemporal market, Böhm-Bawerk maintained, establish a systematic premium on present goods vis-à-vis future goods. The particular causes that Böhm-Bawerk adduced for the positivity of the rate of interest include two psychological reasons: that agents typically expect to be better off in the future, and that they typically do not feel future wants as intensely as present ones. A third reason postulated stems from his belief that Nature's time-consuming methods are more productive, i.e., the sustainable yield of renewable resources is represented as a compound rate, establishing the technological superiority of present over future goods (Brems 1988, Samuelson 1967). The market rate of interest is thus determined, on one hand, by intertemporal consumer behaviour as based on preferences and expected incomes; and on the other hand, by producer behaviour as based on the intertemporal structure of roundabout methods of production.

Böhm-Bawerk's analysis was taken up by Irving Fisher and developed into the "impatience and opportunity" theory. By extending Walras's general equilibrium theory to include intertemporal choices and relationships, while simultaneously simplifying it by considering an aggregate commodity, Fisher succeeded in presenting a "definitive model of general equilibrium determination of interest rates" (Samuelson 1967, p.30), presenting insights into the problem of intertemporal allocation not offered by his predecessors.

The Fisher theory of interest is based on the notions of time preference and investment opportunities. Fisher advocates impatience as an explanation of interest and believes that, in a stationary equilibrium, consumers will require positive interest; and that only those technologies and investment opportunities affording a net rate of return at least equal to this pure time preference rate would be used (Tobin 1987).

Another milestone in the history of the theory of interest was reached by **Paul Samuelson** in 1958. With his famous consumption-loans model he was the first to capture the essential point that finite lived individuals exist in an infinitely lived economy, and thus to address the interest rate problem in a dynamic population model. In his exposition, he raises some fundamental questions, such as the multiplicity of stationary equilibrium rates and their stability properties. The interest rates that he derives in a competitive economy can be negative - in contrast to the classicals like Böhm-Bawerk and Fisher. This observation led **David Gale** (1973) to consolidate the opposing sets of assumptions in a single model; through which he could expose the qualitative differences between the two approaches to the interest rate problem. In particular, he considers their implications for the stationary and nonstationary behaviour of the model.

Departing from the full certainty scenario, **Costas Azariadis** (1981) investigates the emergence of business cycles caused merely by subjective beliefs. He argues that even in a world in which uncertainty is excluded from the structural components of the economy, agents may take actions which tend to bear out their expectations. Azariadis is mainly concerned with the likelihood of these self-fulfilling prophecies, but it will be argued that the assumed Markovian structure of this model prevents him from doing so.

This thesis is broadly structured as follows: chapter one introduces the general overlapping generations framework that is used to discuss the selected articles. Chapter two will briefly familiarise the reader with Samuelson's analysis of the interest rate problem, summarising the main questions raised. The third chapter is concerned with Gale's generalisation of Samuelson's framework. It is in two sections, the first delineating the main issues addressed by Gale, the second offering a critical review of his argument. Chapter four examines the impact of uncertainty on the model. In section one we outline Azariadis's model and elucidate the concept of extraneous uncertainty. The second part of chapter four highlights some of its shortcomings, inconsistencies and redundancies. The main findings are then summarised in the conclusion.

Chapter 1

THE GENERAL MODEL

The papers by Samuelson, Gale and Azariadis mentioned above will be discussed within the general framework of an overlapping generations model without bequests. We divide every agent's life into J distinct periods of equal length, where J takes on any positive integer value. The number of people that are born at the beginning of a period is denoted by N_1, and the number of people who die at the end of the period by N_J. There are thus N_j^t members of age j = 1, . . . , J alive in any time period t.

Each agent receives an exogenous endowment e_j in the *j*th period of life (j = 1, . . . , J). This endowment can be interpreted as labour endowment which enables the individual to produce output. Due to the assumptions that only **one** good is produced, and that all agents are identical and hence have access to the same technology, these endowments can be expressed as units of the good produced. Throughout the paper we will deal with endowments and keep production implicit, i.e. focus on pure exchange rather than productive models, and hence concentrate on the theory of interest rather than the theory of capital[1].

The number of goods consumed by an individual of age j in period t is given by c_j^t. For all generations the representative agent's preferences are given by a smooth monotone, concave utility function $V(c_1, \ldots, c_J)$. Hence any trade that occurs is reflected by the divergence of the lifetime consumption pattern $\mathbf{c} = \left(c_1^t, \ldots, c_J^{t+J-1}\right)$ from the endowment vector $e = (e_1, \ldots, e_J)$. Noting that both, the utility function and the endowment vector remain invariant over time, consumption levels may change over time in accordance with changing prices. The fact that we are dealing with a single good economy enables us to focus on intertemporal preferences that cause price changes over time, as reflected by the interest rate r_t, or equivalently, by the interest factor $\rho_t = 1 + r_t = p_t / p_{t+1}$. The product is assumed completely perishable and must therefore be consumed in the period of its production. Consequently, it can serve

[1] Historically, the theory of interest was developed in the context of relative prices, while the classics developed the theory of capital in the context of reproduction: whereas the former was designed to explain the exchange of wealth of the individuals with the interest rate as the price relating to the intertemporal exchange of wealth, the latter centered on the explanation of the growth of the wealth of nations (Meacci 1989).

neither as a store of value nor as a medium of exchange; in due course (section 3.2.3) other candidates for these purposes will be considered.

Having fixed the notation, general consumer behaviour can be defined formally. At the beginning of his life the **individual** maximises lifetime utility $V(c_1, \ldots, c_J)$ subject to his **budget constraint**; which requires simply that his total compounded lifetime consumption must equal total compounded lifetime endowment,

$$\sum_{j=1}^{J}\left[\prod_{k=j}^{J-1} \rho_{t+k-1}\left(c_j^{t+j-1} - e_j^{t+j-1}\right)\right] = 0 \qquad [1.A]$$

where, by convention, $\rho_{t+j-1} = 1$ if $j > J - 1$.

Alternatively, total compounded excess demands $\mathbf{z} = \mathbf{c} - \mathbf{e}$ must be zero (or, equivalently, total net savings $\mathbf{s} = \mathbf{e} - \mathbf{c}$ must be zero), i.e.,

$$\sum_{j=1}^{J}\left[\prod_{k=j}^{J-1} \rho_{t+k-1}\ z_j^{t+j-1}\right] = -\sum_{j=1}^{J}\left[\prod_{k=j}^{J-1} \rho_{t+k-1}\ s_j^{t+j-1}\right] = 0 \qquad [1.A']$$

for any given t.

The corresponding Lagrange function yields the first order conditions

$$V_j + \lambda \prod_{k=j}^{J-1} \rho_{t+k-1} = 0 \qquad \forall t = 1, \ldots, J \text{ (J equations)} \qquad [1.B']$$

where λ is the Lagrange multiplier, and V_j denotes the partial derivative of V with respect to c_j, i.e., V_j refers to the marginal utility of consumption at time $t + j + 1$ when the consumer is of age j. Eliminating the Lagrange multiplier, this set of J equations can now be reduced by one equation to read

$$\rho_{t+j-1} = \frac{V_j}{V_{j+1}} \qquad \forall t = 1, \ldots, J - 1 \text{ (J - 1 equations).} \qquad [1.B]$$

These equalities clearly state that the individual is in equilibrium if in every period of his life the marginal rate of subjective intertemporal substitution coincides with the corresponding intertemporal price. For parametric interest rates we can derive the demand functions $c_j^t(\rho_t, \ldots, \rho_{t+J-2})$ and hence the corresponding excess demand $z_j^t(\rho_t, \ldots, \rho_{t+J-2})$ or savings functions $s_j^t(\rho_t, \ldots, \rho_{t+J-2})$.

In addition to the budget constraint that each individual faces, the population as a whole is constrained by the **market clearing** condition which requires that in every time period t aggregate consumption equals aggregate endowment[2]. Consider a population that grows geometrically at a constant rate γ and suppose that at time t there are N_J^t members in their last period of life. Then the demographic structure implies that there will be γN_J^t members of age J - 1, or in general, $\gamma^{J-j} N_J^t$ people in their *j*th period of life. Remembering that the utility function is representative and remains invariant over time, the market clearing condition can be obtained by summing all excess demands of all population members that are alive in period t. That means

$$\sum_{j=1}^{J} (\gamma)^{J-j} N_J^t \; z_j^t(\rho_t, \ldots, \rho_{t+J-2}) = 0 \qquad \forall t. \qquad [1.C]$$

We can now define a **short run equilibrium** as an element of the sequence $\{c\}_t$ of consumption vectors that satisfies simultaneously the individual budget constraint [1.A] and the feasibility condition [1.C] for a given period t. A **stationary** (or long run) **equilibrium**, by contrast, is characterised by a constant sequence $\{c\}_t$ where the equilibrium consumption pattern and hence also the corresponding interest factor ρ are time independent, i.e., their values remain invariant over time.

Let us now turn to the papers by Samuelson, Gale, and Azariadis to consider their use of the above model, the alterations they made to it, and the results and inferences they finally obtained.

---

2 Recall that real investment is impossible because the good is perishable.

Chapter 2

SAMUELSON'S CONSUMPTION-LOAN MODEL

In his article "An Exact Consumption-Loan Model of Interest with or without Social Contrivance of Money" Samuelson (1958) considers successive generations of agents who enter the labour market, work for about forty-five years and then retire. Not having a social security system, these people want to save during their working years to provide for a retirement during which they will not earn any income. This desired saving pattern could easily be realised, if the good produced were durable. His question is how these agents will provide for their old age given that the good is perfectly perishable and thus cannot be carried over from one period to the next?

2.1) The Model

To formulate the problem, Samuelson makes some simplifying assumptions: all agents of all generations are identical except, of course, for their birth dates. In particular, a representative agent's life is divided into three periods - two productive ones during each of which the exogenously given labour endowment enables him to produce one unit of the good; followed by a non-productive retirement period without any endowment[3]. The agents' utility function $V(c_1, c_2, c_3)$ does not assume any systematic subjective time preference. In addition, the demographic structure is restricted to a population that grows geometrically at a constant rate γ, where a stationary population corresponds to a zero growth rate ($\gamma = 1$). Thus at any point in time the young and middle-aged outnumber the old by the fixed proportions $\gamma^2:1$ and $\gamma:1$, respectively.

Given these assumptions, what will be the intertemporal terms of trade in a perfect capital market under perfect certainty? In other words, which equilibrium levels of the interest factors ρ_t for any time t will clear the market for present and future consumption?

For this purpose we should clearly delineate the agents' behaviour and the constraints they face individually and collectively. Referring to the general framework of the

[3] Contrary to our general specifications outlined in the introduction, Samuelson considers the special case where $e_3 = 0$.

overlapping generations model outlined in chapter one, we can now easily derive the constraints that are relevant in Samuelson's setup. We simply substitute the number of generations that are alive at any point in time (J = 3) and the lifetime endowment vector of each individual **e** = (e_1, e_2, e_3) = (1, 1, 0) to obtain the **individual budget** constraint

$$\rho_t \rho_{t+1}(c_1^t - 1) + \rho_{t+1}(c_2^{t+1} - 1) + (c_3^{t+2} - 0) = 0 \qquad [2.A]$$

or, in terms of individual net savings,

$$\rho_t \rho_{t+1}\, s_1^t + \rho_{t+1}\, s_2^{t+1} + s_3^{t+2} = 0. \qquad [2.A']$$

Similarly, the **market clearing** condition now becomes

$$\gamma^2\, N\, s_1^t(\rho_t,\, \rho_{t+1}) + \gamma\; N\, s_2^t(\rho_{t-1},\, \rho_t) + N\, s_3^t(\rho_{t-2},\, \rho_{t-1}) = 0 \quad \forall t \qquad [2.C]$$

where N refers to the number of retired consumers in period t. The population is homogenous, therefore dividing by N one obtains this condition as

$$\gamma^2\, s_1^t(\rho_t,\, \rho_{t+1}) + \gamma\; s_2^t(\rho_{t-1},\, \rho_t) + s_3^t(\rho_{t-2},\, \rho_{t-1}) = 0 \qquad \forall t. \qquad [2.C']$$

These two equations, however, pose a very difficult problem as the market clearing condition for each period contains past, current and future interest factors, leaving us with more unknowns than equations. Extending the time period under consideration to any finite length will not resolve the problem, because this process adds as many unknowns as it adds equations. Thus, to define an equilibrium path of interest rates we have to determine **all** interest rates between the present and the infinitely far future. To sidestep this "planning-until-infinity" problem, Samuelson assumes a constant interest factor ρ for all periods. The obvious mathematical solution satisfying both the stationary budget constraint

$$\rho^2\, s_1 + \rho\, s_2 + s_3 = 0 \qquad [2.A^*]$$

and a stationary market clearing condition

$$N\left[\gamma^2\, s_1(\rho,\, \rho) + \gamma\; s_2(\rho,\, \rho) + s_3(\rho,\, \rho)\right] = 0 \qquad [2.C^*]$$

would be an interest factor equal to the biological growth rate ($\rho = \gamma$) and, correspondingly, an interest rate $r = \gamma - 1$. Thus a growing population would have a positive equilibrium interest rate, a decaying one a negative rate of interest.

Samuelson demonstrates that this outcome represents the "optimal" interest factor in both, the constant and the growing population, in the sense that it is a solution to the constrained utility maximisation problem. It does not matter, whether we formulate the optimality conjecture in terms of lifetime utility of the representative agent or in terms of the current utility of a cross-sectional family, since the fixed proportion $\gamma^2:\gamma:1$ of such a family's age distribution ensures that the one utility function is a monotonic transform of the other. Consequently, the two maximisation problems must yield the same results, implying that the intertemporal choice problem can be condensed to a one-period problem.

2.2) Common Sense Explanations

Samuelson looks for heuristic explanations of his result: every agent of each generation shares the same characteristics except, of course, for the birth date. Thus, any hypothetical transfer through time is essentially a trade with a member of another generation, which, in a stationary population ($\gamma = 1$) is a one-to-one physical transfer, implying a zero interest rate. In a growing population, where the age distribution is skewed in favour of the younger productive ages ($\gamma > 1$), total production increases along with the population size. This increased production allows the old to enjoy higher consumption levels compared to the stationary state, because there are more workers to support them, explaining the positive interest rate.

The above explanation is rendered not convincing by the premise that voluntary trade will only take place if mutually beneficial. Without altruism, social security or any store of value, the old have no claims on the young. Agents must therefore provide for their own retirement while being middle-aged by passing some consumption goods on to the young in return for consumption in the following period. This forces them to consume in excess of their endowment while young ($c_1 > 1$ or $s_1 < 0$) and to save in only one period of their life, namely when they are middle-aged ($s_2 > 0$). The resulting "hump saving" pattern, however, is incompatible with Samuelson's assumption of no systematic subjective time preference. At the interest factor $\rho = 1$, the latter assumption would prescribe equal consumption levels for each of the three periods

($c_j = {}^2/_3$ for j = 1, 2, 3), implying that the endeavour to save is spread over both productive periods. But the young are unable to save, because there are no suitable trading partners with whom they can enter an intertemporal trade agreement: neither the middle-aged nor the old will be able to repay at a future date, either because they are old and hence without any endowment, or because they do not live anymore. In this scenario, market forces will clearly not bring about the optimal interest rate ($\rho = \gamma = 1$). This inconsistency of the common sense explanation and the impeccable mathematical solution suggests that the equilibrium equations have multiple solutions. In particular, if we assume that the model starts at the beginning of biological time, the biological rate of interest will never emerge in the free market although it satisfies the optimality conjecture (Samuelson's "impossibility theorem").

To delineate the paradox more clearly Samuelson considers the two-generation case[4] where only the young produce, the old retire. Intergenerational trade cannot take place because no two potential trading partners of different generations both live for two consecutive time periods, implying that *quid pro quo*, or in fact any mutual exchange, is impossible. The interest factor is thus indeterminate, yet the corresponding mathematics allows for $\rho = \gamma$ as a solution. Even if we adopt a multilateral view of trade, for example in the stationary three-generation model, positive savings of the young at an interest factor equal to 1 is logically impossible. In the absence of any systematic time preference, the social optimum configuration ($\rho = \gamma$) can never be reached by the competitive market, or even be approached over time. Samuelson provides a numerical example to confirm this instability of the biological interest rate by deriving a negative value for the actual competitive market rate of interest that would reflect the cost that agents must bear to bribe other generations to provide for their retirement. He concludes that in the stationary population case a negative market interest rate will obtain rather than the biological zero interest rate that corresponds to the social optimum. And in the general case, where the population changes ($\gamma \neq 1$), the competitive market rate will be below the biological rate of interest ($\rho_m < \gamma$). Increasing the productive years relative to the retirement years will narrow down the differential as the optimal rate γ is approached from below. Note, that the market interest rates converge to the competitive equilibrium rate over time (and remain at that level forever after), implying that the latter ensures the stability of the system - which the biological rate of interest fails to do.

[4] By introducing an overlap between workers of different ages, the three-period model is essentially equivalent to a general n-period model, and thus very different from the two-period case.

According to Samuelson, legislating social security will enable society as a whole to attain the social optimum if, by law, the young are assured of their retirement subsistence, provided they support the currently old. Insofar as such a contract is binding for the yet unborn, the young will overcome their reluctance to give part of their endowment to the old who will never be able to return it. Consequently everybody, including the currently young will be better off, since they will be on the receiving end when old. The optimal unstable equilibrium has thus become stable by decree.

Alternatively, we could introduce fiat money as a medium of exchange and a store of value. Then money earned during the productive years would give the agents a claim on workers of subsequent time periods even though no real compensation is possible. Samuelson concludes with the conjecture that a constant total money stock might lead to the socially optimal interest factor γ, with prices falling at a rate $^1/_\gamma$. Thus money itself can serve as a social contract.

In concluding this chapter let us highlight the important contribution of Samuelson's paper. Not only does he point to some difficulties we do encounter when determining the equilibrium path of interest rates, but he also reveals a fundamental deficiency inherent in the free pricing system: assuming no systematic time preference a "hump saving" pattern will emerge, and the implied stationary equilibrium interest rate is below the optimal biological rate of interest. The instability of the latter thus indicates that the competitive economy approaches the suboptimal outcome[5], where individuals fail to attain the maximum level of utility that were possible under the biological interest rate. The simultaneous actions of self-interested individuals do not (necessarily) ensure the attaining of a social optimum and some kind of social collusion is necessary for this purpose.

These results have paved the way for further research in the determination of interest rates and the role of money as an optimal store of wealth. The following chapter focuses on the determination of interest rates following Gale's (1973) article "Pure Exchange Equilibrium of Dynamic Economic Models".

---

[5] Samuelson argues that the competitive outcome is situated on the Pareto-efficiency frontier but is not ethically optimal in terms of a social welfare function. Gale later refutes its Pareto optimality and Malinvaud (1985) demonstrates that "in the overlapping generations model, stationary competitive equilibria may exist that are not Pareto efficient" (p.311).

Chapter 3

GALE'S PURE EXCHANGE MODELS

Like Samuelson, Gale (1973) studies the competitive equilibria over time of a one good, pure exchange model within the overlapping generations framework, but attempts a more general analysis of the interest rate determination problem. In particular, he observes that the assumptions that led earlier writers (Böhm-Bawerk, Fisher) to postulate positive interest rates, are directly opposite to Samuelson's assumptions which give rise to negative rates of interest.

To clarify this dichotomy, let us briefly review the classical theory of interest. The general equilibrium solution of a pure intertemporal trade problem, where agents exchange the incomes of various periods of their lives, determines the interest rate in each period. In general, income is highly desired during the agents' youth due to the impatience to spend income and due to the opportunity to invest it. However, people are assumed to receive more income towards the later years of their lives, emphasising the scarcity of income during youth. This scenario induces individuals to shift part of their income stream towards the early years of their lives, explaining why interest rates should be positive.

Samuelson, by contrast, rules out both, systematic time preference and the opportunity to invest income. Moreover, he does not grant his agents any income in their old age. The two cases are thus based on opposing assumptions and should therefore be expected to display fundamentally different results.

Let us now consider Gale's intertemporal exchange model which encompasses both the worlds of the classics and of Samuelson. We shall see how it exposes the qualitative differences of the underlying assumptions and their implications for both stationary equilibria and for the nonsteady state behaviour in the two models.

3.1) THE MODEL

Gale uses the general structure of the overlapping generations model as delineated in the first chapter, but confines his analysis to the two-generation case (J = 2) where each agent lives for two periods only. As before constant population growth, only one perishable good and no store of value are assumed. It is also assumed that agents of all generations receive the same income stream over time as specified by the lifetime endowment vector **e** = (e_1, e_2). At the beginning of his life, the representative **individual** maximises lifetime utility V(**c**) = $V(c_1^t, c_2^{t+1})$ subject to the **budget constraint**

$$\rho_t(e_1 - c_1^t) + (e_2 - c_2^{t+1}) = 0 \qquad [3.A]$$

where savings earn interest ρ_t - 1. Corresponding to the set of optimality conditions [1.B] a single first order condition that equates the interest factor with the individual's marginal rate of intertemporal substitution, is now obtained:

$$\rho_t = \frac{V_1(c_1, c_2)}{V_2(c_1, c_2)}. \qquad [3.B]$$

Upon substitution, Gale is able to summarise the individual's behaviour by the so-called **offer curve**

$$V_1(c_1, c_2)[e_1 - c_1^t] + V_2(c_1, c_2)[e_2 - c_2^{t+1}] = 0. \qquad [3.D]$$

This curve depicts all consumption patterns at which the individual is in equilibrium. Each combination of c_1 and c_2 corresponds to a unique interest rate as implied by the first order condition [3.B]. The ability of all individuals to simultaneously attain a personal equilibrium depends on the total resources available to society as given by the **market clearing condition** corresponding to [1.C],

$$\gamma\ [e_1 - c_1^t] + [e_2 - c_2^t] = 0. \qquad [3.C]$$

Any **equilibrium programme** of the sequence $\{c\}_t$ must satisfy both, the individual budget constraint [3.A] and the market clearing condition [3.C], i.e., it must be both *competitive* and *feasible* - in Gale's terminology.

3.1.1) Stationary Equilibria

Parallel to Samuelson's treatment of the interest rate problem, Gale first considers stationary equilibria where both the consumption level **c** and hence the interest factor ρ are constant over time. The stationary equivalents of the individual and aggregate budget constraints [3.A] and [3.C] are given by

$$\rho\,[e_1 - c_1] + [e_2 - c_2] = 0 \qquad [3.A^*]$$

and

$$\gamma\,[e_1 - c_1] + [e_2 - c_2] = 0. \qquad [3.C^*]$$

The resulting equilibrium condition

$$(\rho - \gamma)\,[e_1 - c_1] = 0 \qquad [3.E]$$

thus requires either that the interest factor is equal to the population growth rate ($\rho = \gamma$), or that no trade occurs ($e_1 = c_1$). Consequently, a steady state equilibrium must be either a **golden rule** (or **optimal)** programme where the biological interest rate obtains, or a **balanced** programme in which there is no trade between generations and hence everybody consumes according to the given endowment pattern. Gale deduces that, without using the preference ordering "there are at most two possible steady state equilibria" (p.19). However, one needs to incorporate preferences to determine a unique consumption level in the golden rule case, as the condition $\rho = \gamma$ implies that the individual budget and the feasibility constraints coincide. Hence they are simultaneously satisfied for an infinite number of points on this line, allowing for *infinitely many* (not only two) steady state equilibria. This inconsistency can easily be resolved by introducing preferences[6] to determine the unique optimal stationary equilibrium $\mathbf{c}^*$.

---

[6] Gale in fact refers to revealed preferences in proving his theorem (p.19).

In his attempt to examine the sign of interest rates in general, Gale clearly distinguishes between the classical and the Samuelson cases:

1) In the **classical** model people are impatient in the sense that they want to consume in excess of their endowment when young, i.e., $c_1^* > e_1$. Equivalently, the interest factor associated with the no trade equilibrium, $\tilde{\rho}$, exceeds the population growth rate, i.e., $\tilde{\rho} > \gamma$. Thus a constant population size accords with positive interest rates[7]:

$$\tilde{\rho} = 1 + \tilde{r} > \gamma = 1 \qquad \Rightarrow \qquad \tilde{r} > 0.$$

2) The **Samuelson** case refers to the opposite scenario where agents save in their youth, i.e., $c_1^* < e_1$, or equivalently $\tilde{\rho} < \gamma$, implying negative subjective interest rates, $\tilde{r} < 0$. This is a slight generalisation of Samuelson's (1958) original model which involves the extreme value of $e_2 = 0$.

The distinction between these two cases is important when considering the **Pareto optimality** of stationary states. Gale claims in his theorem 3 (p.21) that the "no trade equilibrium is Pareto optimal in the classical case and not in the Samuelson case."[8] Let us examine the second part of the statement: if, as Samuelson suggests, the young consume $c_1^* < e_1$ instead of e_1, the utility they derive in this period is lessened, ($V(c_1^*, e_2) < V(e_1, e_2)$ with $c_2 = e_2$ constant), implying that they are worse off in their youth. Focusing on lifetime utility, however, $\mathbf{c}^*$ is preferred to $\mathbf{e}$ as indicated by a higher indifference curve (Fig.D). Hence, if agents realise c_2^* when old, the sacrifice during their youth is overcompensated in their old age.

3.1.2) Stability of Nonstationary Programmes

Let us now investigate the nonsteady behaviour of the models, in particular their stability. Starting from given initial conditions, will the economy approach a stationary equilibrium, and if so, which one?

---

[7] The interest factor $\tilde{\rho}$ and the corresponding interest rate $\tilde{r}$ should be interpreted as the *subjective* interest factor and interest rate, respectively, as determined by the slope of the indifference curve through **e**, since no interest rates can be determined if no trade occurs (Fig.D).

[8] Note that $\mathbf{c}^*$ is Pareto optimal in both cases.

Recall that the offer curve is a function of an individual's equilibrium consumption levels, c_1^t and c_2^{t+1}, over two *consecutive* time periods, while the market clearing constraint is a function of the consumption levels c_1^t and c_2^t of two different generations in the *same* time period. Assuming that we can solve the former for c_2^{t+1}, we can rearrange equations [3.D] and [3.C] to obtain

$$c_2^{t+1} = f\,[c_1^t] \qquad [3.D']$$

and

$$c_1^t = g\,[c_2^t] = e_1 + \tfrac{1}{\gamma}(e_2 - c_2^t). \qquad [3.C']$$

The resulting composite function

$$c_2^{t+1} = f\,[\,g\,[c_2^t]\,] \qquad [3.CD]$$

clearly gives successive values of consumption by the old over time and thus shows how the consumption levels of one generation depend on those of preceding generations. For any given initial value c_2^0 these equations completely determine the consumption patterns for *all* future generations[9], and in view of the individuals' budget constraints [3.A] also the corresponding interest rates. Using Fig.A we can easily trace out the time path that describes the evolution of equilibrium consumption programmes: starting with an initial value of c_2^0 the corresponding c_1^0 value is given by [3.C'], i.e., by a horizontal arrow towards the market clearing line. Proceeding in a vertical direction until the offer curve is reached (with the exact value given by [3.D']), the subsequent consumption level c_2^1 can be read off the graph.

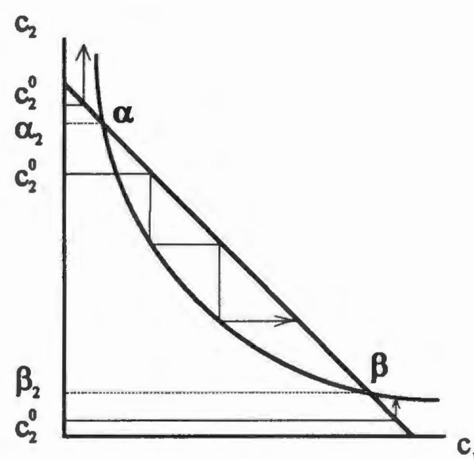


Fig. A

The model's behaviour is sensitive to the initial value of c_2^0. Denoting the two stationary equilibria at the points of intersection by α and β, we can observe that any initial value c_2^0 between α_2 and β_2 gives rise to a time path that moves away from α and towards β. For any starting point $c_2^0 < \beta_2$, the time path will also approach β; but if $c_2^0 > \alpha_2$ the resulting time path will explode, illustrating a kind of breakdown of the

[9] Note that Samuelson has remarked on this planning-until-infinity issue already.

economy. Consequently, α is unstable, β is locally stable. In the classical case $\alpha = \mathbf{e}$ and $\beta = \mathbf{c}^*$, implying that the balanced programme is unstable and the golden rule programme is locally stable. The opposite can be observed in the Samuelson model where $\alpha = \mathbf{c}^*$ and $\beta = \mathbf{e}$. Hence we get the paradoxical result that the Pareto preferred golden rule equilibrium is unstable while the Pareto suboptimal no trade equilibrium is stable.

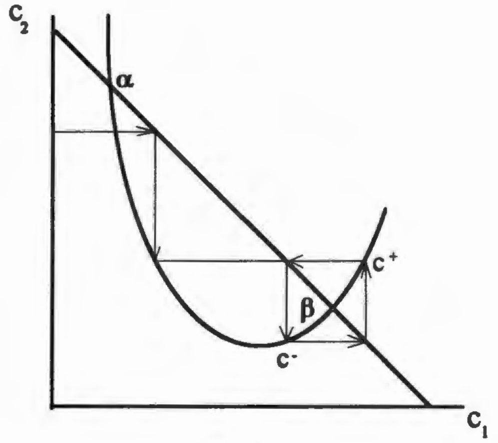


Fig.B

Note that the monotonic convergence to β requires that the offer curve is negatively sloped everywhere. Gale gives an example (using a quadratic utility function) to show the emergence of business cycles under perfect foresight. Such an oscillation between two lifetime consumption vectors $\mathbf{c}^+$ and $\mathbf{c}^-$ is illustrated in Fig.B. While this scenario requires that the offer curve is upward sloping at β, a positive slope will not guarantee a stable limit cycle. Loosely speaking, a fairly flat, yet upward sloping, offer curve will maintain the local stability of β; but convergence will be nonmonotonic. If the offer curve gets steeper at β, the equilibrium point becomes unstable and the time path exhibits a two-period cycle (Fig.B). Further increases in the steepness of the offer curve will produce cycles of increasing periodicity, and eventually an aperiodic time path (exhibiting chaos). The exact specifications that distinguish among stationary equilibria, periodic and aperiodic cycles, however, do not fall under the scope of this paper[10].

[10] The interested reader may refer to the literature on chaos theory, *inter alia* Butler (1990), Day (1982) and Rosser (1990).

3.2) COMMENTS

3.2.1) Systematic Time Preference

It is often held that the interest rate expresses the intensity of preference for present over future consumption, but this relationship can be more complex. To avoid any confusion, let us define systematic time preference and clarify the distinction between the classical and the Samuelson models. As used by Gale, the dichotomy between the two cases depends on the distribution of lifetime endowment *relative* to the individual's intertemporal preferences and not on systematic time preference *per sé*. According to Malinvaud's (1985) definition of impatience, however, the latter notion refers to systematic preference for the present over the future in the sense that at any point on the line $c_1 = c_2$ the indifference curve has a gradient whose absolute value is greater than one.

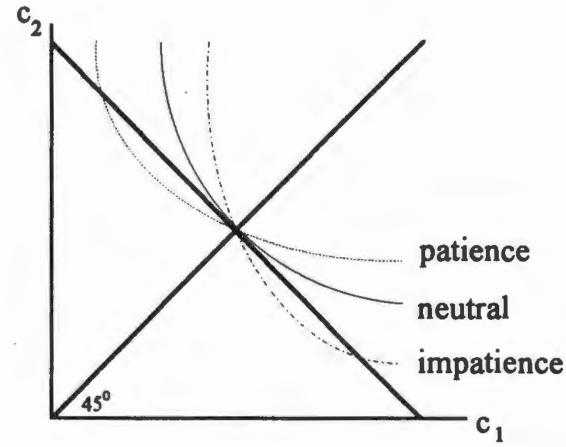


Fig.C

Preferences	η
impatient	> 1
neutral	= 1
patient	< 1

Following this definition all possibilities are illustrated in Fig.C and listed in the table, where η denotes the absolute value of the indifference curve's slope on the 45° line.

Given agent preferences and the population growth rate, whether the classical and Samuelson case arises depends on the position of the endowment vector relative to the desired lifetime consumption pattern: thus, depicted graphically (Fig.D), whenever the desired level of consumption c_1^* is approached from above (i.e. when e is situated somewhere on the bold section of the market clearing constraint) we are dealing with the Samuelson case - irrespective of any systematic time preference.

Although positive interest rates are usually associated with impatient preferences[11] we can show for example the Samuelson model exhibiting negative subjective interest rates $\tilde{r}$ ($\tilde{\rho} < 1$) compatible with systematic preference for present consumption á la Malinvaud as illustrated in Fig.D.

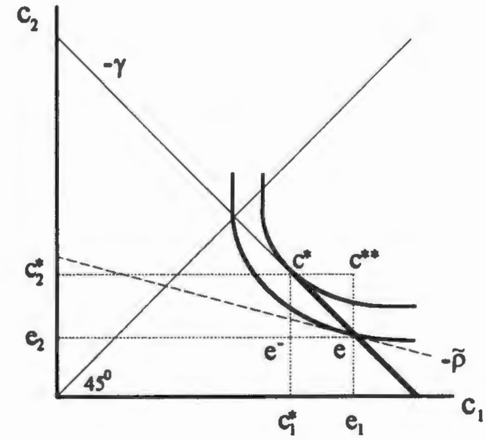


Fig.D

Thus Samuelson's statement that, "nothing is said about whether, subjectively, men systematically discount future consumptions or satisfactions" implying that he is "ignoring Böhm's second cause of systematic time preference" (p.469), while compatible with Malinvaud's definition, is contradicted if systematic time preference is defined in terms of the value $\tilde{\rho}$ takes on relative to γ. In the sequel we shall use the terms **impatient** and **patient** to indicate that $\eta > 1$ and $\eta < 1$, respectively, independently of the value taken by $\tilde{\rho}/\gamma$. The value of $\tilde{\rho}/\gamma$ merely serves to distinguish between the **classical** and the **Samuelson** case: if it exceeds 1 the classical case applies; if it is below 1 we are dealing with the Samuelson case.

3.2.2) Direction of Causality in the Interest Rate Determination

Each of the successive points on the offer curve (c_1^t, c_2^{t+1}), as traced out by the time path, corresponds to a different value of ρ_t except, of course, at the stationary equilibria, α and β, where the corresponding interest factors remain constant over time. This raises the question; *how* are these interest rates determined? Considering that the individuals' maximisation problem involves the choice of lifetime consumption patterns based on parametric interest rates[12] (implying that the causality runs from ρ_t to c_1^t and c_2^{t+1}), we are able to explain how successive consumption levels come about, but not how the interest rates are formed.

---

[11] For example I.Fisher (1961), quoted in Gale.

[12] The parametric treatment of interest rates together with changing rates implied by the time path points to a fundamental inconsistency: why should economic agents base their decisions on interest rates as if they would remain at the given values, whilst knowing that they will change over time?

Let us investigate whether the intertemporal system of equations [3.C'] and [3.D'] suggests any direction of causality. The different values are calculated in the sequence

$$c_2^t \xrightarrow{g} c_1^t \xrightarrow{f} c_2^{t+1} \xrightarrow{g} c_1^{t+1} \qquad \text{etc.} \qquad [3.F]$$

This clearly demonstrates that, given the consumption level of the old, consumption by the young in the same time period is fully determined by the market clearing constraint [3.C'], which is a function of the endowments (e_1 and e_2) and of the population growth rate (γ). Thus consumption by the young would be predetermined by the parent generation, independently of both their own preferences and of the prevailing interest factor. This result would be rather disturbing if, for example, the young were endowed with all the income available as is the case in Samuelson's original model where **e** = (1, 0). It should be pointed out, however, that the time path which solves the intertemporal system of equations, satisfies the optimality principle. This has the advantage that successive equilibrium values may be calculated in *any* sequence, including the one given by [3.F]. On the other hand, it does *not* entail any information on the direction of causality.

Thus, Gale's model is an equilibrium model in the sense that the equilibrium interest rates in each period (on which agents base their decisions) are implicitly assumed to satisfy the aggregate budget. The underlying dynamic process describing how the interest rate of a given period attains its equilibrium value, is, however, ignored. The model includes no specification (as for example a differential equation) that describes how relative prices - or equivalently, interest rates - change in response to unsatisfied excess demands and supplies.

Let us, for example, consider a separable utility function $V(c_1^t, c_2^{t+1})$ such that the marginal utility of consumption at one age is independent of the consumption level at another age. That means

$$V_1(c_1^t,\ c_2^{t+1}) = V_1(c_1^t)$$

and $$V_2(c_1^t,\ c_2^{t+1}) = V_2(c_2^{t+1}).$$

Suppose that, for a fixed level of $\bar{c}_1^t$ (predetermined by the parent generation) the agent spends his entire income, but that the values for $\bar{c}_1^t$, c_2^{t+1} and ρ_t satisfying his budget constraint do not maximise his utility. Assume, without loss of generality, that the agent's disequilibrium position is characterised by

$$\rho_t < \frac{V_1(\bar{c}_1^t)}{V_2(c_2^{t+1})}. \qquad [3.G]$$

Differentiating the individual budget constraint [3.A] we obtain

$$\frac{dc_2^{t+1}}{d\rho_t} = e_1 - \bar{c}_1^t$$

which takes on a positive or negative sign according to the definitions of the Samuelson and classical cases, respectively. Now consider a Samuelson agent's attempt to restore his personal equilibrium by raising the interest factor ρ_t. The consumption level c_2^{t+1} would rise accordingly. Due to diminishing marginal utility of consumption, this lowers the denominator, raising both sides of the inequality [3.G]. To ensure the convergence to an equilibrium interest rate we therefore need to impose the restriction that

$$\frac{d\rho}{dc_2} > \frac{d}{dc_2}\left(\frac{V_1}{V_2}\right).$$

Thus, in a sequential approach, the consumption levels c_1^t and c_2^{t+1} and the interest rate ρ_t will not necessarily satisfy simultaneously the individual budget, the optimality and the market clearing conditions. This further emphasises two points, firstly, that the **direction of causality** is not given by [3.F], and secondly, that Gale is concerned with the **existence** of equilibrium rates of interest, while ignoring the conditions for convergence from initial disequilibrium to an optimal level.

In the following section we will reconsider the time path followed by the economy and elucidate the role that a clearing house may have in interest rate determination.

3.2.3) Starting-up and Continuity of the Time Path

In view of the stability (in the Samuelson case) or Pareto optimality (in the classical case) of the no trade equilibrium, there is no tendency to move away from the balanced programme, implying that no such time path is initiated. Moreover, as Samuelson (1958) pointed out, no voluntary intergenerational trade can possibly occur in the two generation model, because two successive generations, A and B, share only one time period in which both are alive.

In the **Samuelson** case, intergenerational trade is impossible in the two period case, unless we design rules of the game that assure agents of their own retirement subsistence if they support the old generation during their own youth. Young individuals will be reluctant to provide for the currently old if they cannot simultaneously acquire a claim on the offspring's resources to prevent their lifetime consumption vector from dropping to $\mathbf{e}^-$ (Fig.D). A second, possibly insignificant disincentive for saving while young is the opportunity cost[13] involved in the adjustment process. Suppose the adjustment takes place in period t, where the agents from generation A are old, and those from generation B are young. In previous periods no trade took place, implying that each type-A agent's consumption level during his youth is fixed at $c_1^{t-1} = e_1$. Then any current savings of the B-agents that are transferred will shift the consumption vector of the old A-agents upwards along the vertical line $c_1 = e_1$. The first generation (B) that saves $e_1 - c_1^*$ to attain the desired lifetime consumption $\mathbf{c}^*$, will correspondingly shift the preceding generation (A) to point c^{**} on an even higher indifference curve (Fig.D). If, however, the adjustment process could be postponed by exactly one period then the B-agents would be able to attain the "superoptimal" point c^{**} and thereby secure the once-off utility gain for themselves rather than conveying it to their parents. Agents of subsequent generations will, of course, follow the same reasoning and, by symmetry will encounter no incentive to save in their first period of life. Clearly, *quid pro quo* does not hold and therefore the golden rule equilibrium will not emerge in the Samuelson case even though it is Pareto preferred[14] to the no trade equilibrium. Consequently, the population would indefinitely remain at the Pareto suboptimal no trade equilibrium.

Due to the fact that the no trade equilibrium is Pareto optimal in the **classical** case, this economy too will not depart from the socially suboptimal balanced programme,

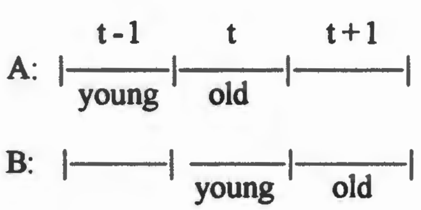
---

[13] In the classical case these are actual costs which have to be born by one or more generations, depending on the speed of the adjustment process. Thus nobody can be made better off without hurting someone else, implying that the no trade equilibrium is Pareto optimal.

[14] The higher indifference curve can be attained without reducing anyone's lifetime utility.

implying that in neither model will the economy spontaneously deviate from the no trade scenario; moreover, that even were a deviation initiated, no time path could reasonably be expected to continue. To obviate the problems one needs to ask under what conditions it is economically reasonable to start with an initial consumption level c^0 different from **e**. What adjustments can alleviate the starting-up problem and ensure the continuity of intergenerational trade over time?

Will it help to relax our assumptions somewhat by allowing agents to issue **IOU**'s? Suppose agents A are old, B are young in period t. In the classical model the young desire to spend in excess of their endowment e_1 and would thus aim to borrow $c_1^* - e_1$ from old agents A, issuing them IOU's over this amount. In the following period, t+1, however, agents A have died and are thus unable to cash in their IOU's from agents B. Similarly in the Samuelson model, where the young wish to save the amount $e_1 - c_1^*$ by lending it to the old agents A[15], the IOU's cannot be cashed in since agents A are dead in the next period and hence are unable to repay this amount. Thus in both cases *quid pro quo* is impossible, because either the creditors (in the classical model) or the debtors (in the Samuelson case) do not live in the period when repayment is due, hence no intergenerational trade will occur.



Let us now introduce a **central clearing house** into the model and reconsider the **classical case**. As before young agents B aim to borrow $c_1^* - e_1$ from old agents A, but for the same reasons outlined in the previous section these utility maximising individuals will not voluntarily give up the required portion of their endowment, $(^1/_\gamma)[e_2 - c_1^*]$, but will instead consume $c_2^t = e_2$. Even the presence of a clearing house does not alleviate the "starting-up" problem[16], since it does not own any endowments and is unable to circumvent the aggregate budget constraint [3.C]. It can, however, ensure continuity in the time path towards the golden rule equilibrium as soon as a once-off government intervention, in the form of income tax levied on the old, forces

---

[15] Note that the optimal bundle c_1^* in the Samuelson case does not generally coincide with c_1^* of the classical model as the distinction arises from different preferences relative to endowments.

[16] Recall that this problem arises due to the fact that the no trade equilibrium is Pareto optimal in the classical model.

them to forfeit a part ε of their endowment. This tax levy forces the currently old to a lifetime consumption bundle indicated by point A below the offer curve (Fig.E(i)). Simultaneously it allows the clearing house to lend $^{\varepsilon}/_{\gamma}$ to each young B-individual in return for an IOU. The realised consumption level of the latter, $c_1^t = e_1 + {}^{\varepsilon}/_{\gamma}$, will most probably fall short of the target level c_1^*, but nevertheless resembles a gain in their utility level compared to the no trade scenario in spite of repayment, $\rho_t\,{}^{\varepsilon}/_{\gamma}$, due in the next period, t+1, when B-agents become old and C-agents are born.

The clearing house holds the IOU's issued by agents B in their youth and is thus able to ensure that IOU's are being honoured - which is a prerequisite for future intergenerational trade: unless the clearing house receives repayments from B-agents, it is unable to extend any credit to the young agents C. The clearing house has a the second important function, namely, the determination of the interest factor. Due to the fact that the consumption good is perishable, the clearing house is unable to accumulate stock over time. Moreover, it has to lend out exactly the amount it has collected via tax levies (in period 1) or debt repayments (in all subsequent periods) to avoid any net welfare loss. Noting that the supply of credit available in each period is determined by a contract between the parent generation and the clearing house agreed upon in the previous period[17], how do they decide on the market clearing interest rate?

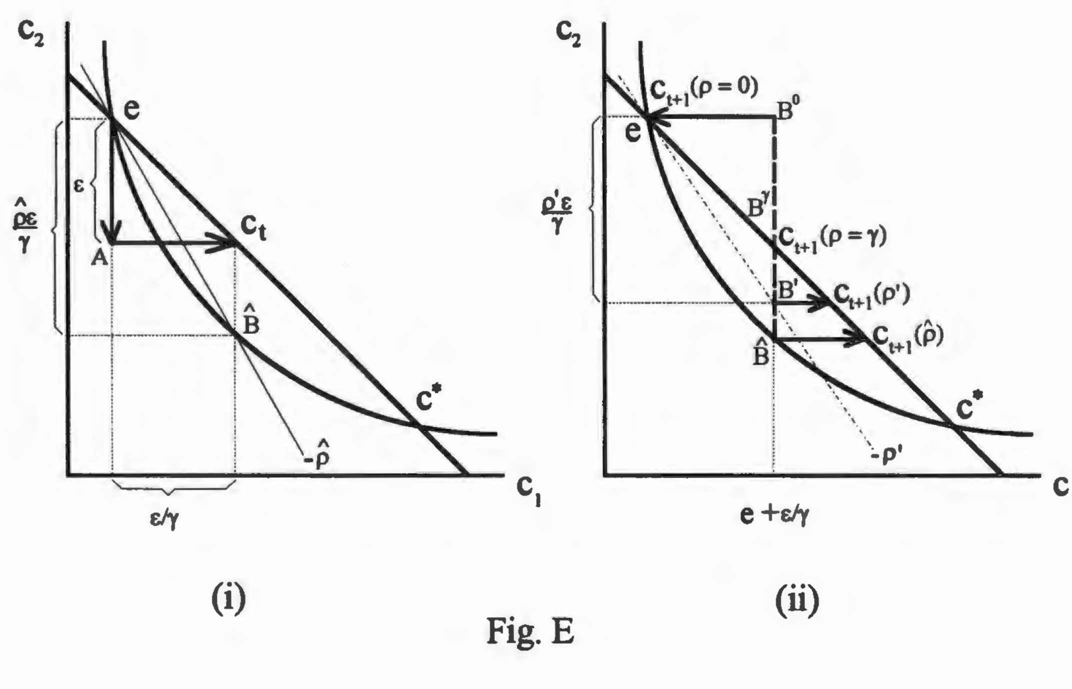


Fig. E

[17] This excludes period 1 in which the supply was fixed by decree.

Provided that the clearing house knows the agents' preferences ordering, it would never charge an interest factor above $\hat{\rho}$. From the offer curve in Fig.E it is clear that at any interest factor above $\hat{\rho}$ the young B-agents will borrow less than the available amount ε/γ; thus leaving the clearing house with an excess supply which, as it perishes, constitutes a waste for society . If, however, the clearing house extends its credit on more favourable terms (at ρ' say) the B-agents need only repay the smaller amount $\rho_t' \varepsilon/\gamma$ (instead of $\hat{\rho}_t \varepsilon/\gamma$) and their lifetime consumption bundle would accordingly shift upwards along the vertical line $c_1 = e_1 + \varepsilon/\gamma$, from $\hat{\mathbf{B}}$ to **B'**. Given the limited resources of the clearing house, the agents' excess demands prevalent at the lower interest factor ρ' can only be partly satisfied. Consequently, point **B'** involves a corner solution for the individual, but it is nevertheless preferred to the consumption vector $\hat{\mathbf{B}}$. The clearing house can thus increase the attained utility level of a generation by lowering the interest rate; but in doing so it will slow down the transition process to the social optimum considerably (or even reverse it). We therefore need to establish a lower bound for the interest factor. Let us start with the extreme case of $\rho_t = 0$, which enables B-agents to consume their total endowment e_2 when they are old. But their implied consumption pattern $\mathbf{B}^{\circ}$ indicates that the economy will move back to the no trade position **e** within one period (Fig.E(ii)).

The next alternative interest factor is one corresponding to the biological rate of population increase: given such a biological interest factor $\rho_t = \gamma$, we will observe the economy being locked into the position $\mathbf{B}^{\gamma}$ indefinitely. Thus, in order to navigate the economy towards the target outcome $\mathbf{c}^*$, the clearing house must consistently charge an interest factor exceeding the population growth rate. Taking the latter as a lower bound, the choice of the exact value of ρ within the established range $(\gamma, \hat{\rho}]$ is essentially a policy issue of the clearing house: if it aims to spread the potential benefits over all generations it should opt for a value close to the population growth rate, and an accordingly slow adjustment process. If its priority lies in a speedy transition to $\mathbf{c}^*$, however, it should settle for the relevant factor $\hat{\rho}$ prescribed by the offer curve. The resulting time path will zig-zag between the offer curve and the market clearing constraint as illustrated before, tracing the fastest route to $\mathbf{c}^*$ (for a given initial tax ε). If the objective entails an instantaneous adjustment to the social optimum, then the tax levy charged from the old must be large enough, i.e., $\varepsilon = e_2 - c_2^*$. In this extreme scenario the old agents of a single generation bear the entire adjustment costs, illustrating that the speed of transition involves costs in the form of redistributing prospective utility gains of currently alive generations towards greater actual gains for future generations.

Let us now investigate whether the **Samuelson model** gives rise to similar observations. Unless the agents have some guarantee of repayment in the future, they are unlikely to save during their youth, and hence the Pareto preferred golden rule programme will not be attained. In the presence of a central clearing house whose sole purpose is to provide such desired guarantee, young B-agents will deposit an amount δ in return for an IOU[18].

The lower bound of the interest factor is given by $\tilde{\rho}$: at this rate the young will not be induced to save; instead they will consume their entire endowment (Fig.F). As ρ rises, voluntary deposits δ will rise in accordance with the shape of the offer curve, as long as backward bending offer curves are excluded. Moreover, $\rho_t < \gamma$ to satisfy the aggregate budget constraint.

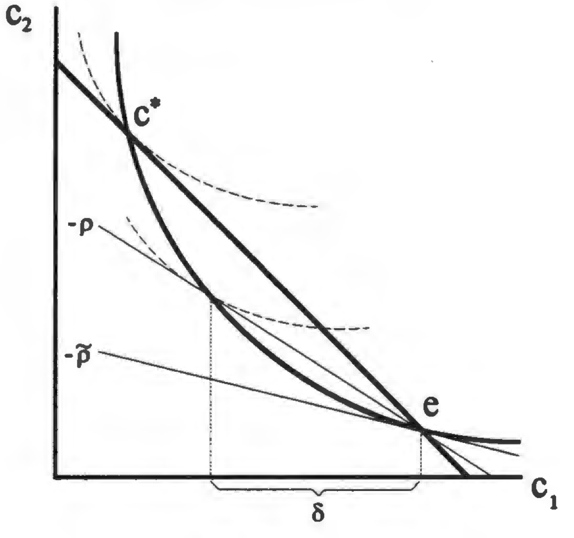


Fig. F

Having established the range of values ρ_t can take on, what guidelines should the clearing house follow in order to decide on interest charges? It can and should simply set $\rho_t = \gamma$ to facilitate the movement to $\mathbf{c}^*$. In contrast to the classical model, here is no conflict of interest between any generation currently alive and the society at large. A closely related observation is that, in the Samuelson model, the clearing house need not know the individuals' preferences in order to guide the economy towards the optimal equilibrium as it is in the agents' interests to attain the Pareto preferred golden rule programme instantaneously. Against this background, the time path towards **e**, as prescribed by the deterministic equations [3.C'] and [3.D'], will never materialise in reality, as agents will attain higher indifference curves by instantaneously jumping to $\mathbf{c}^*$ if the clearing house exists to ensure the continuity of intergenerational trade.

---

[18] Whether the clearing house distributes this amount among old A-agents or not is of secondary importance for the young.

3.2.4) Conclusion

Gale's results are similar to Samuelson's, but his analysis is extended to incorporate both, positive and negative interest rates. Moreover, he clarifies the role that systematic subjective time preference plays in this regard. His framework is of value in analysing the effects of any institutional or policy changes (such as the introduction of IOU's and a clearing house) on the resulting time path[19]. It must be emphasised, however, that Gale's model is an equilibrium model that ignores the sequential adjustment of disequilibrium interest rates to their equilibrium values. It is thus difficult to comment on the direction of causality inherent in the interest rate determination, unless a clearing house is introduced which simultaneously facilitates the continuity of intergenerational trade and resolves the causality problem.

[19] It might be interesting to analyse these problems in a game-theoretic framework, but such an approach is beyond the scope of this thesis.

Chapter 4:

AZARIADIS' SELF-FULFILLING PROPHECIES

So far we have assumed that agents operate under full certainty and therefore need not form expectations about future outcomes. Let us now introduce uncertainty into the model, but restrict the randomness to prices and hence output levels. To exclude the presence of intrinsic uncertainty, we assume that the structural elements of the economy, such as endowments, preferences and technology, remain in the realm of certainty. Instead of considering uncertainty inherent in the structure of the economy, we are now concerned with subjective beliefs. Suppose that a certain phenomenon - call it "sunspot" - is completely unrelated to economic activity. What happens if agents believe that these sunspots are relevant to economic activity, even though there is, objectively speaking, no causal relationship between the two? To answer this question we will consider Azariadis' (1981) article "Self-fulfilling Prophecies" which demonstrates that random prices arise merely because they are believed to be stochastic. Using the overlapping generations framework again, we will analyse his claim that "*extraneous uncertainty* is both possible and 'frequent' among rational expectations equilibria" (p.380).

4.1) THE MODEL

4.1.1) Perfect Foresight

Azariadis begins with a one good, two generations model under perfect foresight without population growth. In contrast to Gale (1973), however, he makes it explicit that the young use their labour endowment n, to produce y units of the perishable good via a constant returns to scale production function[20] ($n_1 = y_1 \leq e_1 = 1$). As in Samuelson's model, agents receive no endowment in their last period of life ($e_2 = 0$). Azariadis, however, goes even further and restricts all consumption to old age ($c_1 = 0$), thus clearly distinguishing between the agents who produce and those who consume. By implication nobody can consume even the smallest fraction of the own output,

[20] For simplicity let us use the identity function $y^t = n^t$. We thus remain essentially in the pure exchange model, while simultaneously allowing for the extension to the productive model.

forcing the young to trade all their produce for the unit of fiat money, m = 1, held exclusively by the old.

To accommodate the above specifications, agents' preferences are expressed in terms of current work and future consumption, as given by the utility function

$$V\left(n_1^t,\ c_2^{t+1}\right) = u\left(c_2^{t+1}\right) - g\left(n_1^t\right)$$

where u and g are smooth monotone, concave (hence well-behaved) functions, i.e.,

$$u',\ g' > 0 \qquad [4.i]$$

and $$u'',\ g'' \leq 0. \qquad [4.ii]$$

Analogous to the general constrained optimisation problem of chapter one, agents will maximise their utility subject to the budget[21]

$$\rho_t\, n_1^t = c_2^{t+1}. \qquad [4.A]$$

This lifetime budget constraint is effectively a combination of the liquidity constraints agents face in their two periods of life,

$$p_t\, n_1^t = m_d^t$$

and $$p_{t+1}\, c_2^{t+1} = m_s^{t+1}.$$

Since fiat money is the only store of value, the amount earned during the agent's youth is carried over to the following period; yielding his lifetime budget. Equilibrium in the money market, by contrast, requires that the young agents' demand for nominal money balances, m_d^t, equal the corresponding supply of them by the old in the same time period, m_s^t. Satisfying Walras's law, this equality implies that the product market clears simultaneously.

Returning to the individual's choice problem, we will observe equilibrium when the levels of labour input and future consumption satisfy the first order condition

$$g'\left(n_1^t\right) = \rho_t\, u'\left(c_2^{t+1}\right). \qquad [4.B]$$

---

[21] For the sake of consistency with previous chapters, I have substituted $p_t/p_{t+1} = \rho_t$. Current absolute prices p_t are then uniquely determined by the liquidity constraint of the old, i.e., $c_2 = m/p_t$.

Defining $$U(c) = c\,u'(c), \qquad G(n) = n\,g'(n) \qquad [4.iii]$$

this equation becomes

$$\underbrace{c_2^{t+1}\; n_1^t\, g'(n_1^t)}_{G(n_1^t)} \;=\; \rho_t\, n_1^t\; \underbrace{c_2^{t+1}\, u'(c_2^{t+1})}_{U(c_2^{t+1})}.$$

In view of the budget constraint, the production function and the market clearing condition (analogous to [1.C]),

$$y_1^t(\rho_t) = c_2^t(\rho_{t-1}), \qquad [4.C]$$

this yields the following law of motion[22]:

$$\begin{aligned} &y^{t+1} = 0 && \text{if } y^t = 0 \\ &U(y^{t+1}) = G(y^t) && \text{if } y^t > 0. \end{aligned} \qquad [4.D]$$

Given an initial output level, the functions U and G (which are essentially demand and supply functions) are instrumental in the transition of the economy from one state into the following one. We therefore state the most important properties below, with a fuller discussion of their characteristics left to Appendix 2. The assumptions

$$\lim_{y\to 0} G(y^t) = 0$$

and $$\lim_{y\to 1} G(y^t) = \infty \qquad [4.iv]$$

together with the monotonicity of g' imply that the function G is increasing, i.e.,

$$G'(y^t) > 0. \qquad [4.v]$$

The rate of change of U, by contrast, depends on the nature of the relationship between current leisure and future consumption. In particular,

---

[22] We can omit the subscripts that refer to the individuals' age, since only the old consume.

	$U'(y^t) < 0$	if they are **gross complements**,	[4.vi]
and	$U'(y^t) > 0$	if they are **gross substitutes**[23].	[4.vii]

A **stationary equilibrium** is then characterised by

$$U(y^{**}) = G(y^{**}) \qquad [4.D^*]$$

where the output remains at the constant level y^{**} for all time periods.

4.1.2) Extraneous Uncertainty

Let us now introduce extraneous uncertainty into the model: rational agents will form expectations based on past and current observations of prices and output levels. Denoting the information set available to agents in period t by Ω_t, they now maximise their expected lifetime utility function

$$E\left[V \mid \Omega_t\right] = E\left[u\left(c_2^{t+1}\right) \mid \Omega_t\right] - g\left(n_1^t\right)$$

subject to the expected budget constraint

$$\tilde{\rho}_t\, n_1^t = c_2^{t+1} \qquad [4.A']$$

where the tilde indicates randomness. The corresponding first order condition

$$E\left[\tilde{\rho}_t\, u'\left(c_2^{t+1}\right) \mid \Omega_t\right] = g'\left(n_1^t\right) \qquad [4.B']$$

illustrates that the probability distribution of $\tilde{\rho}_t$ is conditional on the information set Ω_t. In view of the expected budget constraint the probability distribution of $\tilde{y}^{t+1}$ (or $\tilde{c}_2^{t+1}$) at the individual level is derived from that of $\tilde{\rho}_t$.

---

[23] The signs can be established by differentiating U(c) = c u'(c) with respect to y:

$$U' = \frac{dU(c)}{dy} = \frac{du}{dc}\frac{dc}{dy} + c\frac{d^2u}{dc\,dy} = u'\frac{dc}{dy},$$

because the mixed derivative equals zero. Substitutability of leisure and consumption means that $\frac{dc}{dy} > 0$, implying that $U' > 0$; *mutatis mutandis* for gross complementarity implying $U' < 0$.

The distinctive characteristic of extrinsic uncertainty (as opposed to intrinsic uncertainty) lies in the independence of individual events from joint events in the aggregate. The implied absence of any covariance in the information set can be expressed by the equality of the expected value of a product of two terms with the product of their expected values conditional on the same information set, in particular

$$E\left[c_2^{t+1} u'(c_2^{t+1}) \mid \Omega_t\right] = E\left[c_2^{t+1} \mid \Omega_t\right] E\left[u'(c_2^{t+1}) \mid \Omega_t\right].$$

Interpreting this equality, it must be emphasised that the expectation of the marginal utility of future consumption, $E\left[u'(c_2^{t+1}) \mid \Omega_t\right]$, is formed at the individual level, but that the expected level of consumption in the following period, $E\left[c_2^{t+1} \mid \Omega_t\right]$, depends on the aggregation of all such beliefs (as well as the market clearing constraint) and is, therefore, **independent** of the former even though it is based on the same information set. We will now use the independence of expectations underlying our extrinsic uncertainty assumption, together with the definitions of U and G, the individual and aggregate budget constraints and the production function, to manipulate the first order condition [4.B'] and ultimately yield the modified law of motion [4.D'], analogous to [4.D]:

$$\begin{aligned}
& n_1^t E\left[c_2^{t+1} \mid \Omega_t\right] E\left[\tilde{\rho}_t u'(c_2^{t+1}) \mid \Omega_t\right] &&= E\left[c_2^{t+1} \mid \Omega_t\right] n_1^t g'(n_1^t) \\
\Rightarrow\ & n_1^t E\left[\tilde{\rho}_t c_2^{t+1} u'(c_2^{t+1}) \mid \Omega_t\right] &&= E\left[c_2^{t+1} \mid \Omega_t\right] n_1^t g'(n_1^t) \\
\Rightarrow\ & n_1^t E\left[\tilde{\rho}_t \mid \Omega_t\right] E\left[U(c_2^{t+1}) \mid \Omega_t\right] &&= E\left[c_2^{t+1} \mid \Omega_t\right] G(n_1^t) \\
\Rightarrow\ & E\left[U(y_2^{t+1}) \mid \Omega_t\right] &&= G(y_1^t)
\end{aligned} \qquad [4.D']$$

Azariadis confines the equilibrium output to solutions with the **Markov property** where $\Omega_t = y^t$ and output y^t attains one of the two discrete values of the set $\{y^1, y^2\}$. The corresponding stationary transition probabilities,

$$\begin{aligned}
q_1 &= P\,[y^{t+1} = y^1 \mid y^t = y^1] & 1 - q_1 &= P\,[y^{t+1} = y^2 \mid y^t = y^1] \\
q_2 &= P\,[y^{t+1} = y^2 \mid y^t = y^2] & 1 - q_2 &= P\,[y^{t+1} = y^1 \mid y^t = y^2],
\end{aligned} \qquad [4.E]$$

reflect the stochastic properties of any variable **believed** to influence economic activity.

A **self-fulfilling equilibrium** can now be defined as a set of numbers (q_1, q_2, y^1, y^2), with all elements lying in the interval [0,1], satisfying the law of motion [4.D'], i.e.,

$$q_1 U(y^1) + (1 - q_1) U(y^2) = G(y^1)$$
$$(1 - q_2) U(y^1) + q_2 U(y^2) = G(y^2). \quad [4.F]$$

4.1.3) Necessary Conditions

Azariadis derives two special assumptions that exclude extraneous uncertainty, namely:

1) consumption and leisure are gross substitutes

and/or 2) consumption and leisure are gross complements and $q_1 + q_2 \geq 1$.

Consequently a **necessary** but **not sufficient** condition[24] for sunspot equilibria is

gross complementarity with $q_1 + q_2 < 1$,

or equivalently[25], $U(y^2) \leq G(y^1) < G(y^2) \leq U(y^1)$ with $y^2 > y^1$. [C1]

Referring to the shaded areas in Fig.H(iv), this necessary condition can be illustrated in the y^1-y^2 plane by areas above (below) the 45° line, bounded above (below) by the line $U(y^1) = G(y^2)$ and below (above) by the line $U(y^2) = G(y^1)$. These areas are non-empty if, at the stationary equilibrium S, the curve

α: $U(y^1) = G(y^2)$

is steeper than curve β: $U(y^2) = G(y^1)$. [4.G]

For this purpose it is sufficient that the law of motion in equation [4.D] yields a **locally stable** stationary equilibrium in the perfect foresight case, i.e., that

$$|G'(y^{**})| < |U'(y^{**})|$$

or

$$\frac{G'(y^{**})}{U'(y^{**})} + 1 > 0 \quad [C2]$$

where y^{**} indicates a stationary equilibrium.

---

[24] For the remainder of this section we will assume gross complementarity.

[25] This equivalence is shown in Appendix 1.

4.1.4) Likelihood of Self-fulfilling Prophecies

Let us now review how Azariadis treats the question of the likelihood of sunspot equilibria. He claims that, "under the assumptions made . . . extraneous uncertainty is not only possible but 'probable' as well; for most configurations of the exogenous probabilities, q_1 and q_2 such that $q_1 + q_2 \leq 1$, there exists one stationary equilibrium and at least *two* other distinct equilibria such that $y^1 \neq y^2$ " (p.389). For this purpose he defines the function

$$T(q, y) = \frac{G(y) - q\,U(y)}{1-q} \qquad [4.H]$$

where $\lim_{y \to 1} G(y) = \infty, \qquad \lim_{y \to 0} U(y) = B \leq \infty.$ [4.viii]

Using equations [4.F] and [4.G] it is obvious that, for self-fulfilling prophecies to exist

$$\text{I: } U(y^1) = T(q_2, y^2)$$
and $$\text{II: } U(y^2) = T(q_1, y^1) \qquad [4.J]$$

must hold simultaneously. This means that, drawing curves I and II as in Azariadis' article (fig.3, p.390)[26] and treating the probabilities as parameters, enables us to easily locate any existing sunspot equilibria at the points of intersection. Based on this figure, Azariadis concludes that extraneous uncertainty will characterise at least two-thirds of equilibria in the region

$$R = \{q_1, q_2 \in [0, 1] \mid q_1 + q_2 < 1, q_1 \neq q_2\},$$

or equivalently, at least one-third of **all** equilibria in the region

$$R^+ = \{q_1, q_2 \in [0, 1] \mid q_1 \neq q_2\}.$$

[26] Fig.L portrays essentially the same information.

4.2) CRITIQUE

4.2.1) Necessary Conditions Revisited

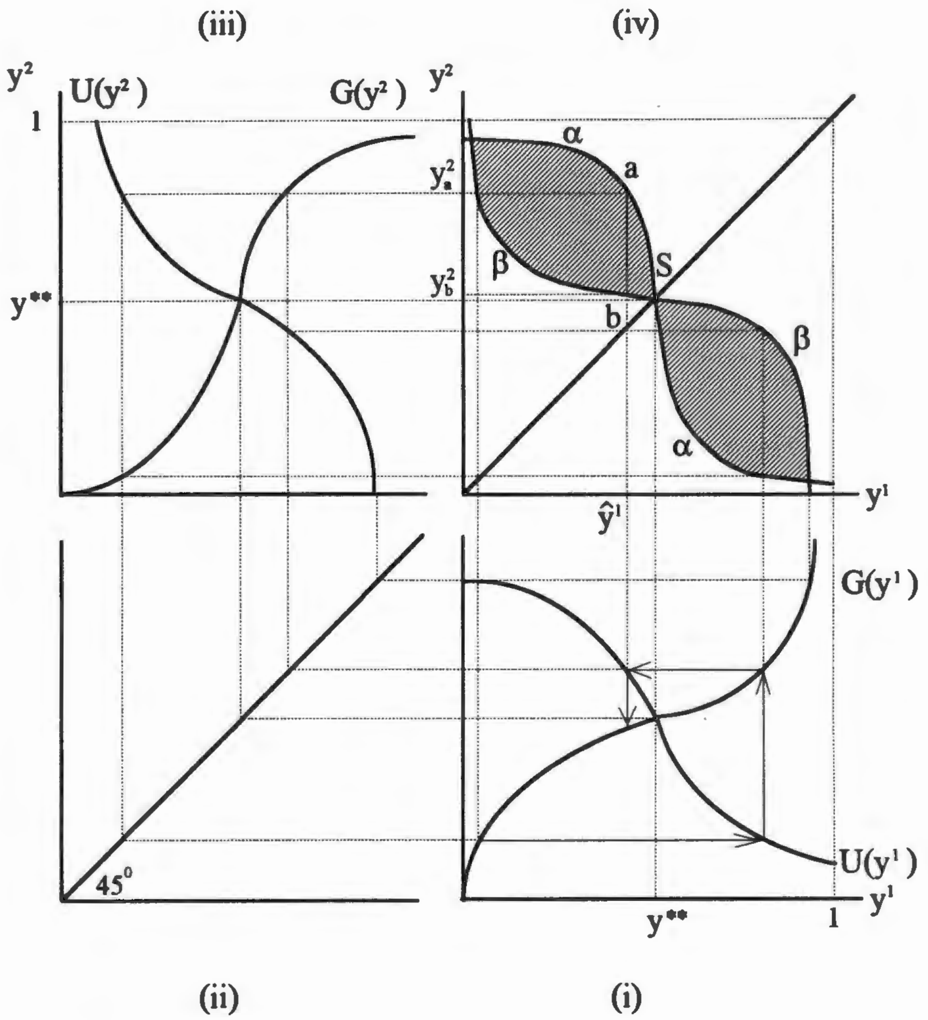


Fig.H

Referring to Fig.H, let us explain how the local stability condition [C2] translates to the relative steepness of the curves a and b: Quadrants (i) and (iii), which are symmetrical with respect to $y^1 = y^2$, show the curves $U(y^1)$, $G(y^1)$ and $U(y^2)$, $G(y^2)$, respectively[27]. Curve a can now be derived in quadrant (iv) by finding those values of y^1 and y^2 that satisfy $U(y^1) = G(y^2)$. Similarly, curve b shows the combinations of y^1, y^2 such that $U(y^2) = G(y^1)$ - the mirror image of a with respect to $y^1 = y^2$. It can now easily be observed that a stable stationary equilibrium (characterised by U being steeper

[27] For explanations of the shape of curves U and G refer to Appendix 2.

than G) corresponds to α cutting β from above. Consequently, there exists a non-empty set K, indicated by the shaded areas, that satisfy the necessary condition [C1]:

Consider, for example, points **a**: $U(\hat{y}^1) = G(y_a^2)$ and **b**: $U(y_b^2) = G(\hat{y}^1)$.

At any point below **a**, $y^2 \le y_a^2 \;\Rightarrow\; G(y^2) \le G(y_a^2) = U(\hat{y}^1)$.

At any point above **b**, $y^2 \ge y_b^2 \;\Rightarrow\; U(y^2) \le U(y_b^2) = G(\hat{y}^1)$.

Since $y^2 > \hat{y}^1$, the line **ab** satisfies $U(y^2) \le G(\hat{y}^1) < G(y^2) \le U(\hat{y}^1)$.

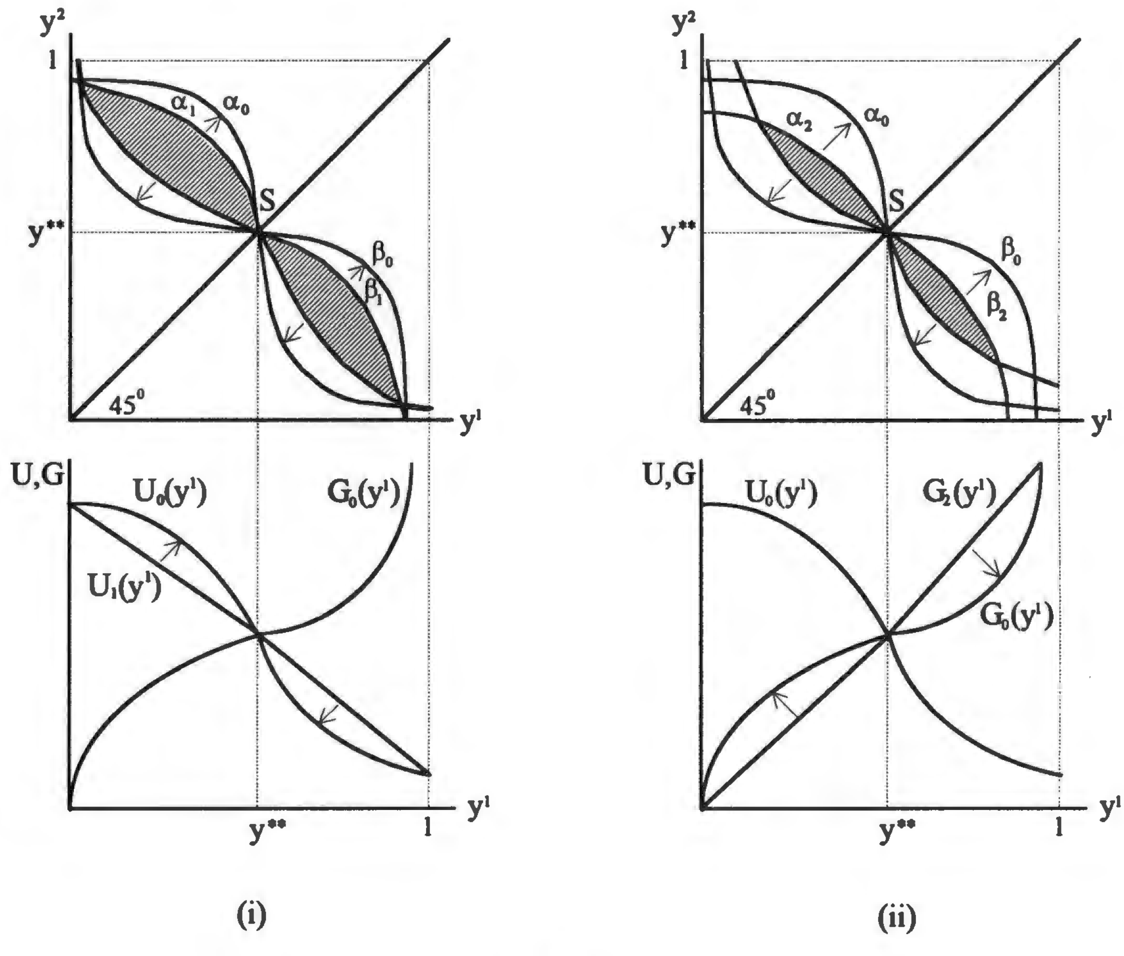
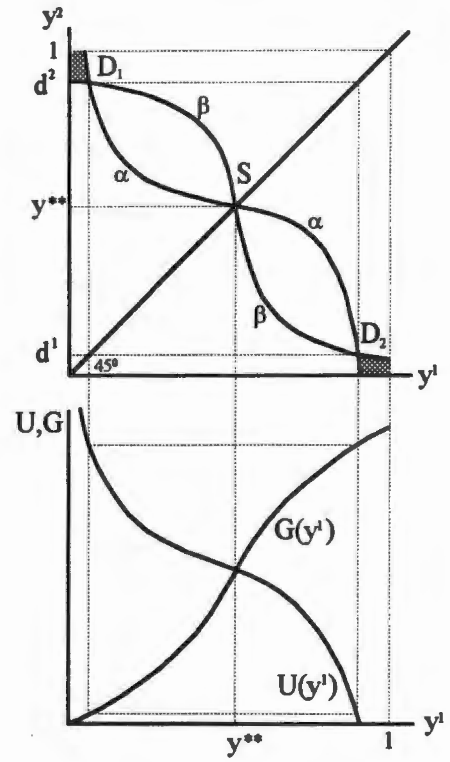


Fig.J

Thus condition [C2] evaluated at the stationary equilibrium is sufficient to ensure non-empty areas that satisfy the necessary condition [C1]. One can ask, is [C2] necessary? To answer this question let us first investigate the sensitivity of these areas to the shapes of G and U. Referring to Fig.J, it becomes clear that the shaded areas between α and β increase the more concave G and U for $0 \le y \le y^{**}$, and the more convex G and U for $y^{**} \le y < 1$ are; i.e., the steeper U and the flatter G become. By symmetry, a reversal of the curvatures of G and U will at least decrease the size of set K.

To investigate the impact on the size of area K, consider for example Fig.K[28]: the stationary equilibrium y^{**} is unstable, because the steeper curve $G(y^{**})$ relative to $U(y^{**})$ in panel (ii) translates to line β intersecting α from above at S in panel (i). Therefore, combinations of y^1 and y^2 in the vicinity of the stationary equilibrium S will never give rise to self-fulfilling prophecies. However, sunspot equilibria are still possible in the shaded areas in the north-western and south-eastern corners of panel (i). The corresponding set K will be non-empty if somewhere in the unit square curve α cuts β from above, for example D_1 and D_2. The points of intersection $D_1(d^1, d^2)$ and $D_2(d^2, d^1)$ are stable limit cycles in the perfect foresight case[29] characterised by

(i)



(ii)

Fig.K

$$|G'(d^i)| < |U'(d^j)| \qquad i, j = 1, 2, i \neq j.$$

Thus, instead of confining stability to the stationary equilibrium $S(y^{**}, y^{**})$, we can change this sufficient condition to become necessary and sufficient for [C1]; namely if there exists a pair of output levels $(y^1, y^2) \in (0, 1)$ such that,

$$|G'(y^1)| < |U'(y^2)|$$
$$|G'(y^2)| < |U'(y^1)|. \qquad [C2*]$$

To conclude this section, let us summarise the results developed so far. Azariadis has established a necessary condition for self-fulfilling prophecies [C1] and a condition sufficient for this necessary condition [C2]. We have suggested that the latter can easily be extended to become necessary and sufficient for [C1]. But, unless we show under which circumstances curves I and II have indeed points of intersection other

---

28 The shapes of U and G as illustrated violate some of the specifications we have assumed for the remaining analysis.

29 This scenario is very similar to Gale's example of cycling.

than the stationary equilibrium S[30], we are still lacking a sufficient condition that guarantees extrinsic uncertainty equilibria. Azariadis thus identifies cases that are **necessary** for sunspot equilibria and also gives an example of self-fulfilling prophecies (fig.3, p.390), but he does not prove the **existence** of such equilibria under the stated conditions.

Thus in contrast to his claim of having identified "a set of sufficient . . . conditions that guarantee the existence of replicating equilibria with extraneous uncertainty" (p.388), he has isolated only necessary conditions for extraneous uncertainty. This, however, does not ensure the existence of sunspot equilibria as such. It merely stipulates under which circumstances self-fulfilling prophecies are **possible**. This shortcoming will be further investigated and rectified in the following sections.

4.2.2) Sufficient Conditions Revisited

To identify a sufficient condition for self-fulfilling prophecies we need to verify that curves I and II indeed assume the shapes as illustrated. Given Azariadis' assumptions, we have established the following properties in Appendix 3:

(P1) Both curves are monotonically downward sloping in the y^1-y^2 plane, i.e., they lie in the shaded areas, cutting $S(y^{**}, y^{**})$.

(P2) Curve I (II) cuts the line $y^1 = 1$ ($y^2 = 1$) on the interval y^2 (y^1) $\in [0, y^{**}]$.

(P3) Curve I (II) cuts the y^2 (y^1) axis on the interval y^2 (y^1) $\in [y^{**}, 1]$.

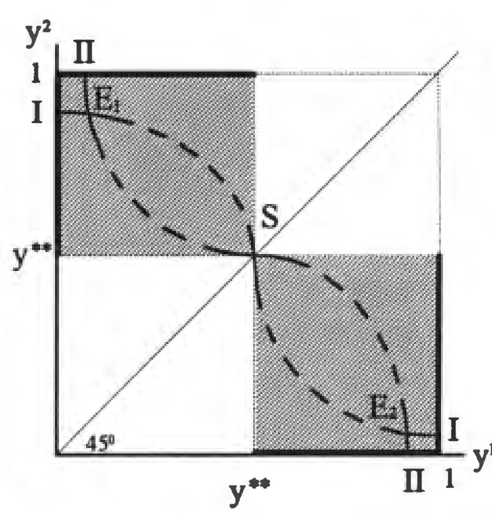


Fig.L

30 Recall that such points of intersection indicate the existence of sunspot equilibria, [4.J].

These properties, however, do not suffice to guarantee the existence of the two points of intersection E_1, E_2. We further need to impose the local stability condition on the stationary equilibrium in the extrinsic uncertainty case[31], i.e., that at S, curve I is steeper than curve II, which will then ensure the shape of the curves as illustrated in Fig.L. This condition implies that:

$$\left.\frac{dy^2}{dy^1}\right|_{I(y^{**})} < \left.\frac{dy^2}{dy^1}\right|_{II(y^{**})}$$

$$\Leftrightarrow^{32} \quad \frac{(1-q_2)\,U'(y^{**})}{G'(y^{**})-q_2\,U'(y^{**})} < \frac{G'(y^{**})-q_1\,U'(y^{**})}{(1-q_1)\,U'(y^{**})}$$

$$\Leftrightarrow^{33} \quad (1-q_1)(1-q_2)(U')^2 > (G'-q_1\,U')(G'-q_2\,U')$$

$$\Leftrightarrow \quad (G'-U')(q_1+q_2) < \frac{(G')^2-(U')^2}{U'}$$

$$\Leftrightarrow \quad (q_1+q_2) < \frac{G'}{U'}+1. \qquad [C3]$$

With [C3] we have thus established a **sufficient** condition for sunspot equilibria. But what is its underlying logic? To facilitate the interpretation recall that stability under full certainty is characterised by

$$|G'| < |U'|, \qquad [C2]$$

this means, supply must change at a lower rate than demand. Note that the introduction of extrinsic uncertainty subjects demand, but not supply, to randomness. To adjust for this presence of uncertainty, we therefore need to multiply the right hand side by the probability that the economy switches from one state to another. We thus obtain

$$|G'| < |U'|(1-(q_1+q_2))$$

[31] This condition is analogous to local stability in the perfect foresight case, where at S, α is steeper than β [C2].

[32] This slope is derived in Appendix 3.

[33] Keeping in mind that this equality is evaluated at y^{**}, we can omit this value.

where the sum of q_1 and q_2 reflects the probability that no transition takes place[34]. This inequality is equivalent to and therefore explains the meaning of [C3].

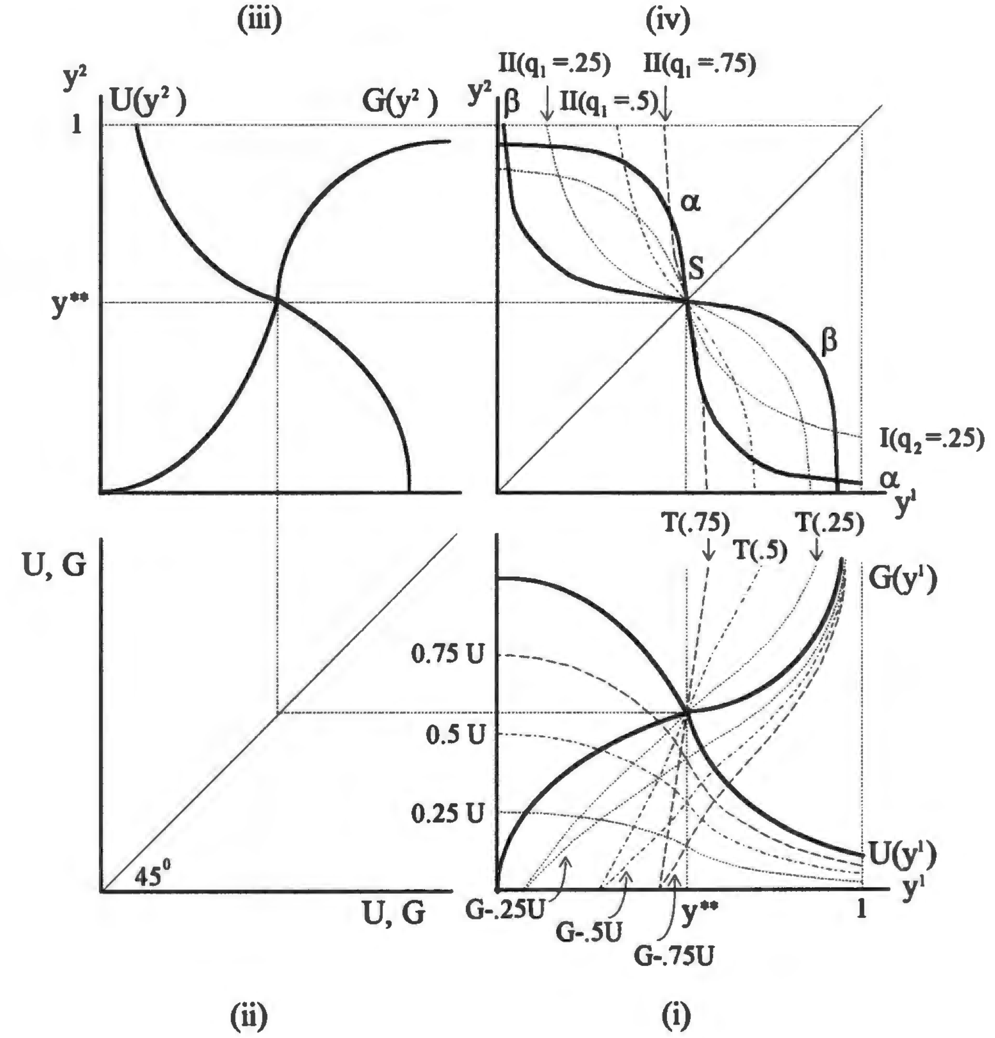


Fig.M

Let us now consider the graphical derivation of the curves I and II, using the same 4-quadrant graph as for the derivation of α and β (Fig.H). In order to derive curve

$$II: U(y^2) = T(q_1, y^1)$$

it is necessary to fix q_1. This enables us to draw $q_1 U(y^1)$, hence $G(y^1) - q_1 U(y^1)$,

[34] At this stage it becomes obvious why $q_1 + q_2 < 1$ is necessary for sunspot equilibria: a violation of this condition implies a nonpositive probability that the economy oscillates between two states of nature, making uncertainty equilibria impossible.

and ultimately

$$T(q_1, y^1) = \frac{G(y^1) - q_1 U(y^1)}{1 - q_1}.$$

Fig.M quadrant (i) shows the three curves $T(q_1 = \frac{1}{4}, y^1)$, $T(q_1 = \frac{1}{2}, y^1)$ and $T(q_1 = \frac{3}{4}, y^1)$. Using these and the curve $U(y^2)$ in quadrant (iii), we can derive three curves of type II, each corresponding to a specific probability q_1. Similarly, a set of curves type I can be derived.

It can now easily be observed that curve II (I) is bound by curve β (α) and the line $y^1 = y^{**}$ ($y^2 = y^{**}$, respectively):

$\lim_{q \to 0} T(q_1, y^1) = G(y^1)$ — i.e., curve II approaches β as q_1 falls to zero,

and

$\lim_{q \to 1} T(q_1, y^1) = \infty$ — i.e., curve II approaches the vertical line $y^1 = y^{**}$ as q_1 rises to one.

By symmetry, curve I approaches α as q_2 falls to zero, and it approaches the horizontal line $y_2 = y^{**}$ as q_2 rises to one.

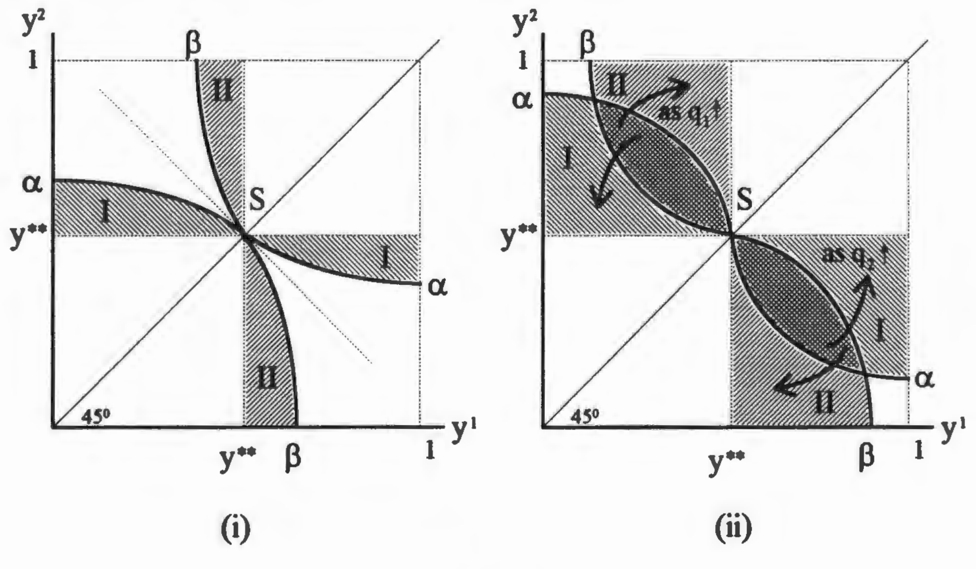


Fig.N

We can therefore draw the following conclusions:

1) If α is flatter than β, I and II cannot intersect each other except at S (Fig.N(i)). This implies that an unstable stationary equilibrium in the perfect foresight case prevents extrinsic uncertainty (unless the curves α and β take on different shapes as, for example, in Fig.K). This result is consistent with the statement stipulating that condition [C3] is sufficient for sunspot equilibria.

2) As long as α is steeper than β at S, there exist some probabilities such that I and II do intersect, as indicated by the overlapping shaded areas in Fig.N(ii). But Azariadis' restriction $q_1 + q_2 < 1$ is not sufficient to ensure this existence. With $q_1 < 1$ sufficiently close to 1, curve II may be steeper than $I(q_2 = 0)$ which is equal to the α curve; in this case no points of intersection and consequently no sunspot equilibria would exist. Thus, even in Azariadis' example, the set of probabilities that characterises extraneous uncertainty may be smaller than

$$R = \{q_1, q_2 \in [0, 1] \mid q_1 + q_2 < 1, q_1 \neq q_2\}.$$

Imposing the sufficient condition [C3], this set will be reduced to

$$R^- = \{q_1, q_2 \in [0, 1] \mid q_1 + q_2 < \frac{G'}{U'} + 1 < 1, q_1 \neq q_2\}$$

indicated by the shaded area in Fig.P, excluding the 45° ray. The combinations of (q_1, q_2) in this set give rise to sunspot equilibria unless $y^1 = y^2$. Thus according to Fig.L, at least two-thirds of the equilibria in this smaller region R^- are characterised by extraneous uncertainty.

Azariadis based his conclusion that "at least 'one-third' and less than 'one-half' of all equilibria" (p.390) are self-fulfilling prophecies on the set R instead of the set R^- which will be smaller than R to the extent that the value of $-G'/U'$ at S exceeds zero. The lower bound of the likelihood of sunspot equilibria may therefore be substantially smaller than one-third, depending on the value $1 + G'/U'$ takes on, i.e., depending on agents' preferences, in

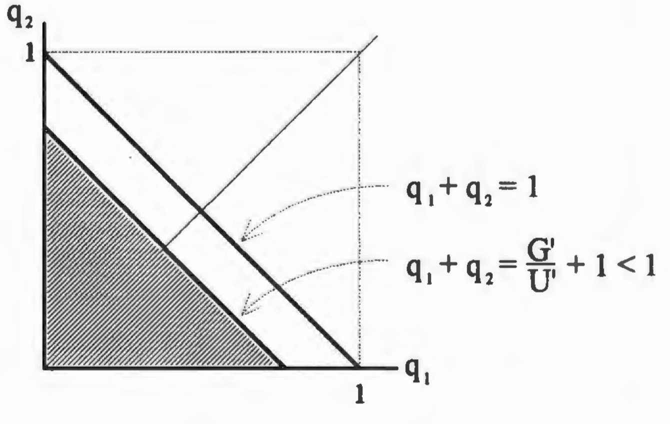


Fig.P

particular on the respective slopes of U and G.

Within Azariadis' framework we have thus identified the combinations of q_1 and q_2 that are **necessary** and **sufficient** for sunspot equilibria[35]. Our next task will be to investigate plausible combinations of q_1 and q_2 on *a priori* grounds, given that our assumptions hold.

4.2.3) Conceptual Inconsistencies

At this stage, I would like to emphasise that y^t is an element of the discrete set $\{y^1, y^2\}$, implying that Fig.H(iv) conveys some information about possible pairs of the two states y^1 and y^2, and the relative sizes of $U(y^i)$, $G(y^i)$ for i = 1, 2. It may, however, be misleading to introduce the local stability condition [C2] of the law of motion [4.D] into the same set of axes, as this would immediately suggest a time path as illustrated in Fig.H(i), where - in contrast to our restriction - y would take on different values over time.

To avoid any confusion we should clearly distinguish between the discrete values of y^1, y^2 and any **linear combinations** thereof. Suppose the latter can be expressed as the continuous variables

$$\begin{aligned} x^1 &= E\left[y^{t+1} \mid y^t = y^1\right] = q_1 y^1 + (1-q_1)y^2 \\ x^2 &= E\left[y^{t+1} \mid y^t = y^2\right] = (1-q_2)y^1 + q_2 y^2 \end{aligned} \qquad [4.K]$$

with $x^1, x^2 \in [y^1, y^2]$.

That means, x^i is the **expected value** of y^{t+1}, contingent on the present state $y^t = y^i$. This definition now allows us to rewrite the **law of motion** that incorporates extrinsic uncertainty [4.D'] as:

$$\begin{aligned} U(x^1) &= U\left[q_1 y^1 + (1-q_1)y^2\right] = G(y^1) \\ U(x^2) &= U\left[(1-q_2)y^1 + q_2 y^2\right] = G(y^2). \end{aligned} \qquad [4.D'']$$

[35] That means $q_1 + q_2 < 1$ is necessary, $q_1 + q_2 < G'/U' + 1$ is sufficient.

Moreover, any set (q_1, q_2, y^1, y^2) satisfying equation [4.D"] is a sunspot equilibrium. An interesting aside is that this definition only coincides with Azariadis' definition of a self-fulfilling equilibrium [4.F] if

$$U[q_1 y^1 + (1-q_1)y^2] = q_1 U(y^1) + (1-q_1)U(y^2)$$
$$U[(1-q_2)y^1 + q_2 y^2] = (1-q_2)U(y^1) + q_2 U(y^2).$$

We should therefore deduce that Azariadis has implicitly assumed linear U functions[36] - an assumption that allows him to construct the T(q, y) function designed to show the likelihood of self-fulfilling prophecies (as discussed a previously in section 4.1.4).

For the time being, we want to consider the more general case without restricting the U function to linearity. In particular, we want to investigate whether the **transformation** of Azariadis' analysis in terms of the two Markov states y^1, y^2 into a system of the expected values x^1, x^2 adds any new insights.

In view of [4.D"] and [4.K] we have four equations in the six variables y^i, x^i, q_i for i = 1, 2. Moreover, the assumptions on U and G (in particular their continuity and monotonicity) are sufficient to ensure that, for any given pair (y^1, y^2), we can solve for the unique set (x^1, x^2, q_1, q_2) that corresponds to a sunspot equilibrium. Picking an arbitrary point (y^1, y^2), the specific functional forms of U and G and the fact that self-fulfilling prophecies satisfy [4.D"], i.e., $U(x^i) = G(y^i)$ for i = 1, 2, enable us to solve for the corresponding expected values (x^1, x^2) unambiguously. This, in turn, allows us to solve for the equilibrium probabilities (q_1, q_2), because the definitions of x^i can be rewritten as functions in y and x values

$$q_1(x^1, y^1, y^2) = \frac{x^1 - y^2}{y^1 - y^2}$$

and $$q_2(x^2, y^1, y^2) = \frac{x^2 - y^1}{y^2 - y^1}. \qquad [4.K']$$

Schematically the procedure could be represented as follows:

$$\left[(y^1, y^2) \xleftrightarrow{[3.D']} (x^1, x^2) \right] \xrightarrow{[3.K]} (q_1, q_2). \qquad [4.L]$$

---

[36] Strictly speaking this is inconsistent with the nonlinearity of U as derived in Appendix 2, but a detailed discussion of possible ways to resolve these inconsistencies is beyond the scope of this paper.

The one-to-one correspondence ensured by equations [4.D"] thus allows the transformation from the y^1-y^2 space into the x^1-x^2 space, and vice versa. Consequently we can derive the curves

$$\alpha': \; U(x^1) = G(x^2)$$

and $$\beta': \; U(x^2) = G(x^1) \qquad [4.G']$$

with each of their points (x^1, x^2) corresponding to exactly one point (y^1, y^2) on the α and β curve, respectively. Although no probabilities feature in the latter curves explicitly, α and β can be easily obtained from the definition of sunspot equilibria by setting q_2 and q_1 equal to zero. Thus any point $\boldsymbol{\alpha}(y^1 = \alpha^1, y^2 = \alpha^2)$ on the α curve would typically[37] correspond to the point $\boldsymbol{\alpha}'(x^1 \neq \alpha^2, y^2 = \alpha^1)$ and the probabilities $0 \leq q_1 \leq 1$, $q_2 = 0$. Similarly, any point $\boldsymbol{\beta}(y^1 = \beta^1, y^2 = \beta^2)$ would typically correspond to the point $\boldsymbol{\beta}'(x^1 = \beta^2, y^2 \neq \beta^1)$ and the probabilities $q_1 = 0$, $0 \leq q_2 \leq 1$. The transition of the y^1-y^2 plane into the x^1-x^2 plane thus involves, loosely speaking, a "distorted reflection" through the stationary equilibrium point S, where the distortion involves an elongation or compression in any direction - horizontal for points on the α curve, vertical for points on the β curve (Fig.Q). Despite the distortion, the transformation preserves the ordering of the points in the sense that the images of α and β remain continuous and those of all points within the shaded areas in Fig.H(iv) are accordingly bounded by the α' and β' curves in the x^1-x^2 space. Thus, by construction, the α' and β' curves serve as an envelope to possible sunspot equilibria. Consequently, Azariadis' result concerning the necessary condition of self-fulfilling prophecies (namely that the equilibrium probabilities corresponding to the points in the shaded areas satisfy the condition $q_1 + q_2 < 1$) carries over when analysed in terms of the expected values x^1, x^2.

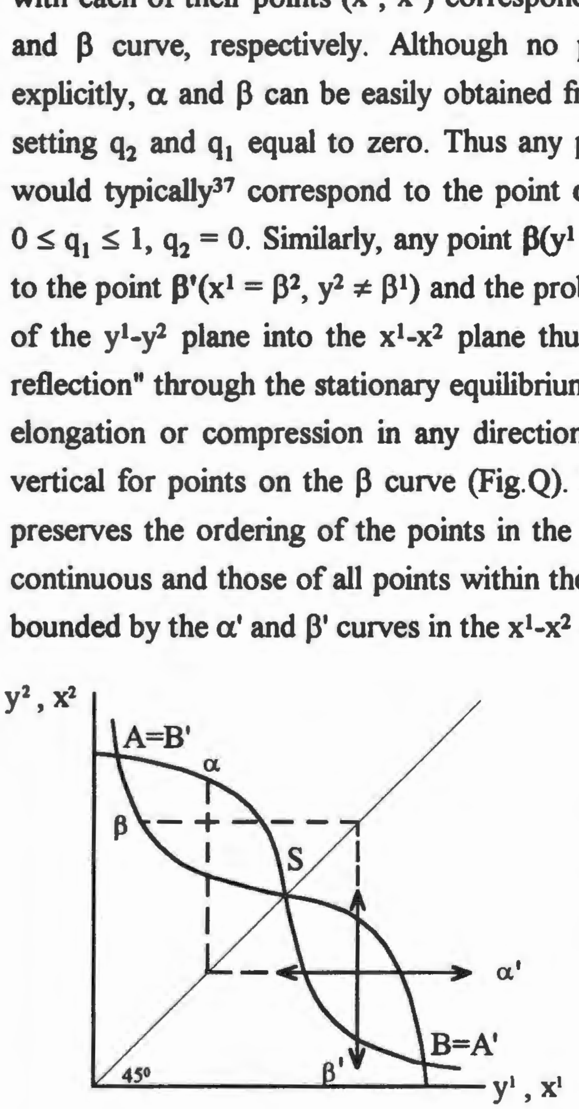


Fig.Q

[37] This case applies except at the points of intersection of α and β where $q_1 = q_2 = 0$, and $(y^1 = \alpha^1 = \beta^1, y^2 = \alpha^2 = \beta^2)$ would correspond to $(x^1 = \alpha^2 = \beta^2, x^2 = \alpha^1 = \beta^1)$. At these points the economy would oscillate between the two states y^1 and y^2 with certainty.

Apart from the fact that the analysis in terms of these expected values is conceptually superior to the discrete y^1-y^2 framework, we should enquire whether this alternative exposition generates new information. While the transformation does not produce new results, it changes the informational content of the diagrams. By definition every point in the x^1-x^2 space is based on, and therefore incorporates, the values of y^i, q_i $(i = 1, 2)$ that allow for extrinsic uncertainty equilibria. Moreover, its derivation clearly shows the correspondences among any three sets of pairs (y^1, y^2), (x^1, x^2) and (q_1, q_2) as summarised in [4.L]: because x^i is not monotonic in q_i, a given pair of (q_1, q_2) may have multiple solutions (y^1, y^2) that satisfy the conditions for sunspot equilibria[38], implying that the relationship between the sets $\mathbf{y} = \{(y^1, y^2) \mid y^i \in (0, 1)\}$ and $\mathbf{q} = \{(q_1, q_2) \mid q_i \in (0, 1), q_1 + q_2 < 1\}$ is one-to-many in contrast to the one-to-one correspondence between y and $\mathbf{x} = \{(x^1, x^2) \mid x^i \in (0, 1)\}$.

This asymmetry does not cause a problem for our analysis since it makes more economic sense to **identify possible outcomes** to which the transition probabilities apply than to **fix these probabilities parametrically** without specifying the states of nature[39]. Despite this, Azariadis has followed the latter approach by constructing the T(q, y) function. Given this conceptual difficulty and noting that the construction of the general T function[40] relies on the unduly restrictive linearity of the U function - which, as mentioned already, causes inconsistencies - one advantage of analysis in terms of expected outcomes become apparent: although it does not yield new results, it does not suffer from the above mentioned shortcomings.

4.2.4) The Likelihood of Sunspot Equilibria Revisited

Azariadis explicitly uses the rational expectations framework[41], hence his agents do not make systematic errors. Suppose now that a variable is deemed by public opinion to have a bearing on economic activity[42] such that $\{y^t\}$ follows the Markov process, with transition probabilities given by [4.E].

---

[38] Azariadis follows this approach.

[39] According to the definition of the transition probabilities [4.E], this approach is impossible.

[40] Despite its deficiences, the T function is an analytical construct usefull to delineate the relationship between the individuals' preference ordering (incorporated in the U and G functions) and the size of the set R^- more clearly.

[41] "extraneous uncertainty is both possible and 'frequent' among rational expectations equilibria", p.380.

[42] We will abstract from the question whether it is indeed rational for agents to do so, as rational expectations are merely taken to mean that agents do not make systematic errors.

Suppose $q_1 + q_2 \geq 1$. This rules out sunspot equilibria completely, because the implied probability that the economy oscillates, $1 - (q_1 + q_2)$, is nonpositive. Thus all such (q_1, q_2) values correspond to the stationary equilibrium value y^{**} unless the economy remains at disequilibrium. Although agents expect extrinsic uncertainty, in this scenario the equilibrium outcome is systematically y^{**} (with probability 1) - in contrast to their belief. Hence, their expectations will have to adjust to prevent a systematic expectational error: rational agents will either believe that the economy follows the stationary path with certainty, or the probabilities will be reevaluated such that $q_1 + q_2 < 1$. Thus, rational expectations is inconsistent with agents' belief in the existence of sunspot equilibria with $q_1 + q_2 \geq 1$.

A similar argument applies to any pair of transition probabilities that does not coincide with the unique pair of equilibrium probabilities corresponding to a given pair of output levels. For two possible states of nature $(\bar{y}^1, \bar{y}^2)$ that may occur, there is exactly one corresponding pair of probabilities $(\bar{q}_1, \bar{q}_2)$ that will generate extrinsic uncertainty equilibria. Taking the transition probabilities to represent agents' beliefs, how do we know that they believe in exactly that unique pair of probabilities that equilibrates the market? The answer lies in the assumed Markov property with stationary transition probabilities: the economy can only oscillate between the states $\bar{y}^1, \bar{y}^2$ at the specific probabilities $\bar{q}_1, \bar{q}_2$ if this pair resembles an equilibrium; if the probabilities were to deviate from these equilibrium values then $(\bar{y}^1, \bar{y}^2)$ could never be a sunspot equilibrium (because of the uniqueness of the corresponding probabilities). Hence output values cannot swing between $\bar{y}^1, \bar{y}^2$ at any stationary transition probabilities $(\hat{q}_1, \hat{q}_2) \neq (\bar{q}_1, \bar{q}_2)$, because $(\bar{y}^1, \bar{y}^2, \hat{q}_1, \hat{q}_2)$ reflects a disequilibrium position, contradicting the Markov property. Thus it is essentially the Markov property that ensures the existence of self-fulfilling prophecies, provided that the required conditions hold. Hence the question on the likelihood of sunspot equilibria is directly related to the probability that the required Markov structure will emerge: if it holds, extraneous uncertainty equilibria will arise with probability one.

An alternative approach would be to abandon the Markov property and reconsider the model's behaviour. If we assume, for example, that beliefs are exogenously given, it is highly unlikely that the believed probabilities match the unique pair of equilibrium probabilities. Consequently, we would end up with a scenario that would portray the other extreme, namely the virtual non-existence of sunspot equilibria. Another option would be to incorporate a learning process, where the question arises: during such an adjustment process, will an initial disequilibrium move to a sunspot equilibrium or to the stationary one? Thus, without any specification of the formation of beliefs, we are unable to comment on the likelihood of sunspot equilibria.

4.2.5) Conclusion

Azariadis' restriction of the equilibrium price set[43] to solutions with the Markov property is fundamental for his analysis: on the one hand it facilitates the parametric treatment of the transition probabilities because they are stationary by definition, on the other hand their stationarity makes it impossible to answer the question; how likely is it that extrinsic uncertainty equilibria will emerge? Agents believe in the values of q_1 and q_2 only if the economy followed exactly these transition probabilities in the past. But the stationary nature of these probabilities implies that the economy must be in equilibrium, as no disequilibrium would maintain their stationarity. Equilibrium is only obtained if the law of motion [4.D'] is satisfied. Comparing this line of argument with Azariadis' definition of a self-fulfilling equilibrium [4.F], we must deduce that an oscillation between two outcomes following the steady Markov probabilities is an extraneous uncertainty equilibrium already.

The assumed Markovian structure thus makes analysis of the likelihood of self-fulfilling prophecies redundant. Consequently, we must rephrase the question to be answered in this context, and investigate to what extent such beliefs are self-perpetuating. Azariadis' model can be used to identify the necessary and sufficient conditions that **maintain** sunspot equilibria, but it cannot explain how the economy **arrives** at such an equilibrium.

[43] In view of the liquidity constraint of the old, $c_2 = {}^{m}/p_t$, output values y can be interpreted as the commodity price of money.

CONCLUSION

This paper has briefly reviewed Samuelson's (1958) consumption-loan model to familiarise the reader with the widely used overlapping generations approach to the interest rate problem. Apart from setting the analytical framework, Samuelson's main contribution was found to lie in the demonstration that the optimal interest rate is unstable, implying that a competitive economy may fail to approach the social optimum.

Gale (1973) not only derived similar results, he refined and extended them in his intertemporal exchange model. This facilitated a clear distinction between the Samuelson and the classical assumptions, and their respective implications for the Pareto optimality and stability of the stationary states. It was emphasised that the equilibrium nature of his model ignored the sequential approach of disequilibrium rates of interest to their equilibrium levels. Thus, while it specified the characteristics of equilibrium interest rates, the model failed to explain **how** these values were attained so as to ensure the simultaneous equilibrium at the individual and aggregate levels. This shortcoming, as well as the problems concerning the launching and continuity of intergenerational trade, can be rectified by the introduction of a clearing house.

In the last chapter, we analyse the existence and likelihood of self-fulfilling prophecies as portrayed by Azariadis (1981). We note that his conclusions are largely based on an example, they cannot possibly claim general applicability. In our attempt to identify general conditions that guarantee the existence of sunspot equilbria (to complement the necessary ones put forward by Azariadis), the following two observations are made: firstly, the properties of demand and supply functions influence the stability of the economic system under both, perfect foresight and uncertainty. Secondly, stability in the full certainty case is a prerequisite for, and may translate, to stability in the stochastic setting, depending on the transition probabilities, if one adds the proviso that special cases as illustrated in Fig.K are excluded. It was also shown that for any two states of nature there exists a unique pair of transition probabilities that gives rise to an extrinsic uncertainty equilibrium. This one-to-many correspondence between consumption levels and probabilities, together with the assumed Markovian structure of the economy, makes it redundant to examine the likelihood of self-fulfilling prophecies in the given framework.

APPENDIX 1

To prove the equivalence of gross complementarity with $q_1 + q_2 < 1$, and the condition $U(y^2) \leq G(y^1) < G(y^2) \leq U(y^1)$ it can be shown that:

1) Given gross complementarity (i.e. $U' < 0$), $q_1, q_2 \in [0,1]$
   and the definition of sunspot equilibria

$$\Rightarrow \quad U(y^2) \leq G(y^1) < G(y^2) \leq U(y^1) \qquad \text{if } y^2 > y^1.$$

Let $y^2 > y^1 \quad \Rightarrow \quad U(y^2) < U(y^1) \quad \Rightarrow \quad U(y^2) - U(y^1) < 0$

$$[4.F] \quad \Rightarrow \quad \begin{cases} q_1 = \dfrac{U(y^2) - G(y^1)}{U(y^2) - U(y^1)} & 1 - q_1 = \dfrac{G(y^1) - U(y^1)}{U(y^2) - U(y^1)} \\[2ex] q_2 = \dfrac{G(y^2) - U(y^1)}{U(y^2) - U(y^1)} & 1 - q_2 = \dfrac{U(y^2) - G(y^2)}{U(y^2) - U(y^1)} \end{cases}$$

$q_1, q_2 \in [0,1] \Rightarrow \quad q_1, 1-q_1, q_2, 1-q_2 > 0.$

As their denominators are negative, so must be their numerators.

Thus
$$U(y^2) \leq G(y^1) \leq U(y^1)$$
$$U(y^2) \leq G(y^2) \leq U(y^1)$$

Since $G' > 0 \quad \Rightarrow \quad U(y^2) \leq G(y^1) < G(y^2) \leq U(y^1).$

2) Given $U(y^2) \leq G(y^1) < G(y^2) \leq U(y^1)$ if $y^2 > y^1$,
and the existence of sunspot equilibria

$$\Rightarrow \quad q^1 + q^2 < 1.$$

[4.F] $\Rightarrow$

$$q_1 + q_2 = \frac{U(y^2) - G(y^1)}{U(y^2) - U(y^1)} + \frac{G(y^2) - U(y^1)}{U(y^2) - U(y^1)}$$

$$= \frac{U(y^2) - U(y^1)}{U(y^2) - U(y^1)} + \frac{G(y^2) - G(y^1)}{U(y^2) - U(y^1)}$$

$$= \quad 1 \quad + \quad \Delta$$

The result, $\Delta < 0$, follows directly from $U(y^2) \leq G(y^1) < G(y^2) \leq U(y^1)$,

i.e., the numerator of Δ $\quad G(y^2) - G(y^1) > 0$
its denominator $\quad U(y^2) - U(y^1) < 0$.

APPENDIX 2

In line with our restrictions on U and G [4.i - vi, viii, ix] these curves exhibit the following properties:

$G(0) = 0$,	$\lim_{y\to 1} G(y) = \infty$,	$G'(y) > 0$	[4.iv, v]
$G''(y) > 0$	as $y \to 1$,	i.e., G is convex	
$G''(y)$ is of indeterminate sign as $y \to 0$		(in Fig.H, $G''(0) < 0$)	
$U(0) = B < \infty$			[4.viii]
$U(1) = M > 0$		(Appendix 3)	[4.ix]
$U'(y) < 0$			[4.vi]
$U''(y)$ is of indeterminate sign as $y \to 1$		(in Fig.H, $U''(1) > 0$).	
$U''(y) < 0$ as $y \to 0$ because		$U' = u' + yu'' < 0$	

$$\Leftrightarrow \quad u' < y\,|u''|,\quad u' > 0,\ u'' < 0 \qquad \text{[by 4.i]}$$

$$\Leftrightarrow \quad \begin{Bmatrix} 0 < u'(0) < 0\,|u''(0)| \\ 0 < u'(1) < |u''(1)| \\ u'(1) < u'(0) \end{Bmatrix}$$

$$\Rightarrow \quad 0 < u'(1) < u'(0) < 0\,|u''(0)| = 0,$$

contradicting $u'' < 0$

unless $\lim_{y\to 0} u''(y) = -\infty$

$$\text{Thus} \quad \lim_{y\to 0} U''(y) = \lim_{y\to 0} 2u''(y) + yu''' = \lim_{y\to 0} 2u''(y) + 0 = -\infty < 0$$

This also implies that $U(0) = B < \infty$ rather than $B \leq \infty$.

APPENDIX 3

Property 1

Given q_1, total differentiation of II: $(1-q_1)\, U(y^2) = G(y^1) - q_1\, U(y^1)$

yields $(1-q_1)\, U'(y^2)\, dy^2 = G'(y^1)\, dy^1 - q_1\, U'(y^1)\, dy^1$

$$\Rightarrow \qquad \left.\frac{dy^2}{dy^1}\right|_{II} = \frac{G'(y^1) - q_1\, U'(y^1)}{(1-q_1)\; U'(y^2)} < 0$$

because $G' > 0$, $U' < 0$ (from [4.v, vi]).

By symmetry, I is also downward sloping.

Property 2

It is sufficient to show that curve I does not cut the y^1 axis on the interval [0, 1]. Recall, that by [4.H, J] the equation for I

$$U(y^1) = T(q_2, y^2) = \frac{G(y^2) - q_2\, U(y^2)}{1-q_2}.$$

Let $M = \lim_{y\to 1} U(y)$.

Using [4.ii, iii], $M = \lim_{y\to 1} y\, u'(y) = \lim_{y\to 1} u'(y) > 0$ [4.ix]

Defining $\overline{y}^2$ as the y^2 value corresponding to $y^1 = 1$, it satisfies

$$T(q_2, \overline{y}^2) = \frac{G(\overline{y}^2) - q_2\, U(\overline{y}^2)}{1-q_2} = M$$

Now consider $\lim_{y \to 0} T(q_2, y^2) = \lim_{y \to 0} \dfrac{G(y^2) - q_2 U(y^2)}{1 - q_2}$

$$= \frac{0 - q_2 B}{1 - q_2} \qquad \text{because of [4.v,viii]}$$

$$= -\frac{q_2}{1 - q_2} B < 0$$

Since $T(q_2, y^2)$ is increasing in y^2 ($G' > 0$, $U' < 0$) and

$$T(q_2, \bar{y}^2) = M \geq -\frac{q_2}{1 - q_2} B = T(q_2, 0),$$

we can conclude that $\bar{y}^2 \geq 0$.

Since the curves I and II are monotonically downward sloping, (P2) is established.

<u>Property 3</u>

It is sufficient to show that curve I cuts the y^2 axis on the interval [0, 1].

$\lim_{y \to 1} U(y^1) = B \leq \infty$ from [4.viii]

The y^2 value corresponding to $y^1 = 0$, call it $\hat{y}^2$, satisfies

$$T(q_2, \hat{y}^2) = \frac{G(\hat{y}^2) - q_2 U(\hat{y}^2)}{1 - q_2} = B$$

Now consider $\lim_{y \to 1} T(q_2, y^2) = \lim_{y \to 1} \dfrac{G(y^2) - q_2 U(y^2)}{1 - q_2}$

$$= \frac{\infty - q_2 M}{1 - q_2} \qquad \text{because of [4.viii, ix]}$$

$$= \infty$$

Thus $B = T(q_2, \hat{y}^2) \leq \lim_{y \to 1} T(q_2, y^2) = \infty$

$\Rightarrow$ $\hat{y}^2 \leq 1$, establishing (P3).

BIBLIOGRAPHY

Arrow, K.J. and Hahn, F.H., General competitive analysis (Holden-Day and Oliver & Boyd, 1971).

Azariadis, C., "Self-fulfilling prophecies", Journal of Economic Theory, vol.25, 1981, pp.380-396.

Brems, H., "Time and interest: Böhm-Bawerk and Åkerman-Wicksell", History of Political Economy, vol.20(4), 1988, pp.565-581.

Butler, A., "A methodological approach to chaos: are economists missing the point?" Federal Reserve Bank of St. Louis. Review, vol.72(2), 1990, pp.36-48.

Cass, D. and Shell,K., "Do sunspots matter?" Journal of Political Economy, vol.91(2), 1983, pp.193-227.

Day, R.H., "Irregular growth cycles", American Economic Review, vol.72(3), 1982, pp.406-414.

Eatwell, J. et al eds., The New Palgrave. A Dictionary of Economics in four volumes (Macmillan Press, 1987).

Fellner, W. et al, Ten economic studies in the tradition of Irving Fisher (John Wiley & Sons, 1967).

Fischer, S., "Samuelson, Paul Anthony", The New Palgrave. A Dictionary of Economics, ed. Eatwell, J. et al, (Macmillan Press, 1987).

Fisher, F.M., "Stability and competitive equilibrium in two models of search and individual price adjustment", Journal of Economic Theory, vol.6(5), 1973, pp.446-470.

Fisher, I., The theory of interest (Augustus M. Kelley, 1961).

Gale, D., "Pure exchange equilibrium of dynamic economic models", Journal of Economic Theory, vol.6, 1973, pp.12-36.

Grandmont, J.M., "On endogenous competitive business cycles", Econometrica, vol.53(5), 1985, pp.995-1045.

Grandmont, J.M., Money and value. A reconsideration of classical and neoclassical monetary theories (Cambridge University Press, 1983).

Hahn, F., Money and inflation (Basil Blackwell, 1981).

Hennings, K.H., "Böhm-Bawerk, Eugen von", The New Palgrave. A Dictionary of Economics, ed. Eatwell, J. et al, (Macmillan Press, 1987).

Malinvaud, E., Lectures on microeconomic theory (Elsevier Science Publishers B.V., revised edition, 1985).

Markland, R.E, Topics in Management Science (John Wiley & Sons, second edition, 1983).

Meacci, F., "Irving Fisher and the classics on the notion of capital: upheaval and continuity in economic thought", History of Political Economy, vol.21(3), 1989, pp.409-424.

Render, B. and Stair, R.M.jr., Quantitative analysis for management (Allyn and Baron, second edition, 1985).

Rosser, J.B.jr., "Chaos theory and the New Keynesian Economics", The Manchester School, vol.58(3), 1990, pp.265-291.

Samuelson, P.A., "Irving Fisher and the theory of capital", in Ten economic studies in the tradition of Irving Fisher, Fellner, W. et al (John Wiley & Sons, 1967).

Samuelson, P.A., "An exact consumption-loan model of interest with or without the social contrivance of money", Journal of Political Economy, vol.66, 1958, pp.467-482.

Scarf, H., "Some examples of global instability of the competitive equilibrium", International Economic Review, vol.1(3), 1960, pp.157-172.

Spear, S.E., "Sufficient conditions for the existence of sunspot equilibria", Journal of Economic Theory, vol.34, 1984, pp.360-370.

Starrett, D.A., "On golden rules, the 'biological theory of interest,' and competitive inefficiency", Journal of Political Economy, vol.80(2), 1972, pp.276-291.

Tobin, J., "Fisher, Irving", The New Palgrave. A Dictionary of Economics, ed. Eatwell, J. et al, (Macmillan Press, 1987).