

Nonlinear Vibration of Beams and Plates Resting on Elastic Foundations Having Nonlinear Stiffness Properties



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Thesis Presented for the Degree of
DOCTOR OF PHILOSOPHY

in the Department of Civil Engineering
UNIVERSITY OF CAPE TOWN

August 2023

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Declaration

I hereby declare that the work described in this dissertation is, except for quotations and citations which have been duly acknowledged, my own work and has not been submitted in whole or in part as an exercise for a degree at this University or any other educational establishment.

Edwige Feulefack Songong

August 2023

Abstract

Beam elements are used to model components in which one dimension (the length) is significantly greater than the other two dimensions. On the other hand, a plate is a structural element characterised by its thickness being very small compared with the other two dimensions. Beams and plates find civil and mechanical engineering applications and are typically used to construct buildings, bridges, floors, pavements, and runways. Investigating the vibration of beams and plates is an ongoing topic, and researchers are becoming more interested in it. Because the structure foundation is the most important part of the building process, engineers and designers dedicate more attention to that part of the structure. This contributes to the desire to build strong, safe, and economic structures for a sustainable environment.

Beams and plates on elastic foundations (BPoEFs) also find many applications in civil engineering. They are frequently used to design structural members of buildings, railroads, airports, highways, and railway tracks. For example, when studying railway track behaviour, a beam on an elastic foundation (BoEF) theory is often employed, and the study can be extended to track dynamics, noise, and vibration. Winkler (1867) was the first to introduce the topic of BPoEFs and developed it to analyse railroad tracks. Since then, many other researchers have extended the concept, giving rise to other foundation types.

The foundation can be linear or nonlinear, depending on the purpose for which the structure will be used. For example, to model the interaction between the beam or plate and an elastic foundation, the entire beam or plate and the foundation are modelled without modelling the foundation itself. Then, the modelling can become less complex depending on the foundation type and property (linear or nonlinear). Therefore, linear analysis is appropriate for easier cases, while nonlinear analysis is left for other complex cases. Nevertheless, it is important to note that linear analysis is limited to simple structures and is only valid for design. As a result, nonlinear analysis is recommended because it can be used for design and realisation purposes.

Beams and plates resting on linear elastic foundations (BPoLEFs) have been extensively studied, whereas beams and plates resting on nonlinear elastic foundations (BPoNEFs) have been neglected and constitute the interest of the current thesis. A beam simply supported at both ends and a rectangular plate also simply supported are considered. Linear and nonlinear foundations are considered to capture the beam and plate behaviour and response. The foundation is assumed to be an assembly of discrete linear or nonlinear springs or connectors. First, the beam on a linear foundation is studied as the basis and then extended to the beam on a nonlinear foundation. Three cases of nonlinearities are discussed: quadratic, cubic, and the combination of quadratic and cubic. The straightforward expansion method (SEM) is used to find solutions to the governing nonlinear differential equation of the beam. Finally, the natural frequencies and the corresponding mode shapes of the system beam-foundation are derived. We note that the overall thesis only focuses on free vibration analyses.

A rectangular plate on an elastic foundation (PoEF) is studied to extend the research on beams. Here also, linear and nonlinear foundations are considered. The same analyses as those of the beam are carried out on the plate to cover all structure types. Both studies on BoEFs and PoEFs revealed that the natural frequencies of the system increase with the increase of the nonlinear stiffness parameter of the foundation. Interestingly, the same increase in the nonlinear stiffness of the foundation tends to decrease the beam and the plate temporal vibration amplitude. Comparing linear to nonlinear results reveals that neglecting the nonlinearity of the foundation produces differences which are more significant at lower modes. This confirms the fact that, if the model of a beam or a plate resting on a nonlinear elastic foundation is simplified to a linear one, the results obtained from the linear analysis might not accurately represent the real system.

Finite Element Analysis (FEA) in Abaqus is used for the numerical analysis. Here also, the natural frequencies and corresponding mode shapes of the beam and the plate with foundation are derived. Extensive parametric studies conducted on the beam and plate vibration reveal that the foundation's nonlinear stiffness is indispensable because it lowers the beam and plate vibration amplitude. Having validated the results from Abaqus, they are used as a benchmark to validate the analytical results. Satisfactory agreement is found between the results of the two methods, with details of each method shown. The results from this thesis show that the nonlinear foundation can accurately be used to control the vibrations of the beam and plate model. The findings further reveal that adding nonlinear quadratic or cubic or mixed parity stiffness properties in the foundation when studying BPoEFs is beneficial and should be adapted for design and realisation purposes.

Acknowledgements

Firstly, I would like to thank God for his guidance, blessings, and protection throughout this PhD journey.

My sincere appreciation goes to my supervisor at the University of Cape Town, Prof. Alphose Zingoni, for his direction and all the support (academically and financially) during this research work. His guidance, constructive feedback, and criticism have improved me academically and enhanced my career as a researcher. Today I am very proud of myself and everything that I have achieved.

I express my deepest gratitude to the following financial support and materials received for this research: The Organization for Women in Science for the Developing World (OWSD) Fellowships, The Swedish International Development Corporation Agency (SIDA), The International Students' Scholarship and the Civil Engineering P/G Scholar from the University of Cape Town. This PhD research programme was made possible with these funds.

My deepest gratitude goes to my lovely fiancé L. Njouakoua for his attention, patience, understanding and encouragement throughout my PhD study. I would also like to express my appreciation to my parents, V. Jiata and J. P. Takala, brothers and sisters Ghislain, Mireille, Cyrille and Wilfried, my uncle Tsoupa Thomas, Mr and Mrs Tchonko for their constant calls to check up on me and prayers that have given me the courage to go through these years.

Special thanks to people within the University of Cape Town community for their various contributions towards the successful completion of this thesis. You made my stay at the university enjoyable. This includes C. Kaluba, K. Mudenda, Dr Matongo, T. Matose, Dr Enoma, C. Mpofu, R. Geswindt, A. Dalwai, I. Ncube, A. Courie, B. Nyoni and A. Ibrahim.

I would also like to thank my friends and mentors for their moral support, prayers and encouragement during this PhD journey. These go to Mr and Mrs Ambassa, Merlin, Mr and Mrs Dika, Dr Fotsa, Axel, Mrs Kounga, Christian, Franck, and others. Finally, I extend my thanks to Javad Mohseni for his guidance through modelling in Abaqus.

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List of Abbreviations

AGS	Approximate Global Size
AVM	Averaging Method
BoEF	Beam on an Elastic Foundation
BoLEF	Beam on a Linear Elastic Foundation
BoNEF	Beam on a Nonlinear Elastic Foundation
BoNCEF	Beam on a Nonlinear Cubic Elastic Foundation
BoNQEF	Beam on a Nonlinear Quadratic Elastic Foundation
BPoEF	Beam and Plate on an Elastic Foundation
BPoLEF	Beam and Plate on a Linear Elastic Foundation
BPoNEF	Beam and Plate on a Nonlinear Elastic Foundation
DOF	Degrees of Freedom
DQM	Differential Quadrature Method
EBBT	Euler-Bernoulli Beam Theory
FDM	Finite Difference Method
FEA	Finite Element Analysis
FEM	Finite Element Method
HP	Hamilton's Principle
HBM	Harmonic Balance Method
LM	Lindstedt's Method
L-PM	Lindstedt-Poincaré Method
MFM	Mesh-free method
MDOF	Multi-Degrees Of Freedom
MTSM	Multiple Time Scales Method
NDE	Nonlinear Differential Equation
NNMs	Nonlinear Normal Modes
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation

PEM	Parameter Expansion Method
PoEF	Plate on an Elastic Foundation
PoLEF	Plate on a Linear Elastic Foundation
PoNEF	Plate on a Nonlinear Elastic Foundation
SDOF	Single-Degree of Freedom
SEM	Straightforward Expansion Method
TBT	Timoshenko Beam Theory
TDOF	Two-Degree of Freedom
VIM	Variational Iteration Method

List of Symbols

α_2	Nonlinear parameter due to quadratic stiffness properties
β_3	Nonlinear parameter due to cubic stiffness properties
ν	Poisson ratio
ω_l, ω_n	Linear natural frequencies
$\Omega_q, \Omega_c, \Omega_{qc}$	Nonlinear natural frequencies due to quadratic, cubic, and quadratic-cubic stiffness properties
ρ	Density of a steel beam
ε	Small perturbation parameter
a	Length of the plate
b	Width of the plate
D	Flexural rigidity of the plate
E	Young's modulus
h	Thickness of the plate
I	Second moment of inertia
k_2	Nonlinear quadratic stiffness parameter
k_3	Nonlinear cubic stiffness parameter
k_b	Stiffness parameter of the beam
k_l	Linear stiffness parameter

L	Length of the beam
l	width of the beam
m, n	Mode number
r_q, r_c	Nonlinear relative ratio to quadratic and cubic stiffness properties
S	Cross-sectional area of the beam
t	Time parameter
U	Transversal displacement of the beam
W	Transversal displacement of the plate
x	Horizontal coordinate

Chapter 1

General Introduction

This chapter introduces the research and outlines the background and rationale for the study. The chapter subsequently presents the research problem, the research aims and objectives, its limitations, and finally provides a chapter-by-chapter overview of the thesis.

1.1 Background to the Study

Beams and plates on elastic foundations (BPoEFs) are civil and mechanical engineering topics. They have been of interest in the past decades, and linear cases have been extensively studied. However, beams and plates resting on nonlinear foundations have been neglected and are open to further investigations. BPoEFs are also essential topics in vibration analysis, and an understanding of the behaviour of the beam or plate under dynamic actions is of interest. Moreover, determining the system's natural frequencies and mode shapes thoroughly explains the structural behaviour. By and large, the dynamic analysis of structures has been a field of interest to many researchers. In that regard, structures are designed to resist more deformation caused by external forces/loads. In addition, the growth of knowledge in structural engineering has led to the possibility of producing accurate designs for more extensive and taller structures with fewer materials and less effort. This has been very promising, and the applicability of new advancements allows for creating more stable structures that can bear various dynamic loads.

Emile Winkler (Winkler, 1867) was the first to formulate a problem of a beam on a deformable foundation. Therefore, the model of a beam, resting on an elastic foundation having a linear force-deflection relationship, is known as a beam on the Winkler foundation.

When such a model is excited by different types of loadings, the principle of superposition can be employed to obtain the entire solution of the system. Since its inception, Winkler's BPoEF technique has had many uses and extensions. For example, Biot (1937) extended the solution of Winkler's model to the case where the foundation is a half-space, and the applied force is sinusoidal. That resulted in foundation stiffness that was effectively a function of the applied load's spatial frequency. His research would eventually find considerable application in surface layer wrinkling studies, where the surface layer's thickness is small compared to the underlying material. Also, Hetényi (1971) provided solutions for many BPoEF geometries and loading scenarios, including applications to cylindrical pressure vessels, torsion, and buckling. This source remains a complete and seminal work detailing the various results of Winkler's foundation forecasts. The same solution is also applicable to the bending of non-shallow spherical shells (Zingoni, 2018a).

Because of the limitations of Winkler's foundation, many other types of foundations have been developed to improve the foundation design for tall buildings and structures for sustainable infrastructures (Filonenko-Borodich, 1940; Pasternak, 1954; Kerr, 1964). For example, Eisenberger and Clastornik (1987) and Shirima and Giger (1992) investigated a beam resting on a two-parameter elastic foundation. Eisenberger and Clastornik (1987) found that using the cubic shape functions of a beam element and adding the foundation contribution as element foundation stiffness matrices produce somewhat better outcomes and require less computing time. As for Shirima and Giger (1992), the effects of the second elastic foundation parameter which is introduced to consider the coupling effect of the linear elastic springs should not be overlooked, even though the first elastic foundation parameter appears to always influence the results of interest significantly. Nevertheless, including the second parameter could enhance simulation accuracy, but such a method has the downside of making the determination parameters complex.

There exist many methods to solve problems involving BPoEFs. For linear cases, the method of separation of variables (Myint-U and Debnath, 2007; Dawkins, 2018) and the principle of superposition can be used to solve the governing equation. These procedures yield acceptable results. In nonlinear cases, however, such methods are no longer valid, and other analytical or approximation methods are used. Examples include the multiple scale method (MSM), the Lindstedt method (LM), the iteration method, the straightforward expansion method (SEM), and the parameter expansion method (PEM), etc. (Nayfeh, 1973; Nayfeh and Mook, 1995; Luongo, 1996). Because of the problem complexity of some nonlinear cases, only numerical simulations can solve the equation of motion. Examples

are the Finite Element Methods (FEM) (Zienkiewicz et al., 1977; Mourelatos and Parsons, 1987; Rao, 2004; Reddy, 2005; Logan, 2011), the Finite Element Analysis (FEA) (Pao, 1986; Cook et al., 2001), and the Mesh-free method (MFM) (Lucy, 1977; Belytschko et al., 1994; Sunitha et al., 2008; Lan, 2020). Though the same analytical procedure can be applied when studying a beam or a plate on a nonlinear elastic foundation, in using the FEA, the analysis is different due to the dimensions not being the same. For this reason, further steps in addition to the beam case are considered to model the plate (3D modelling) on an elastic foundation.

1.2 Notion of Nonlinearity

The term nonlinearity refers to a situation where the independent and dependent variables do not have a linear relationship. Moreover, any input changes do not linearly change the output. By and large, nonlinearity can be divided into three types: geometric (large deformations, large strains); material (plasticity, creep, viscoelasticity or viscoplasticity); and boundary condition (contact) (Plumbridge et al., 2003; Nastran, 2004). Individually or combined, these nonlinear factors may occur. Geometric nonlinearity occurs when the relation displacement-strain is nonlinear. A practical example is a large deflection of a cantilever beam (Banerjee et al., 2008).

Regarding material nonlinearity, three types can be distinguished: nonlinear elasticity; nonlinear plasticity; and rubber (the structure loads and unloads through different paths) (Poorjamshidian et al., 2014; Wagg and Neild, 2014). There exist other categories and subcategories of nonlinear material behaviour such as nonlinear viscoplasticity and nonlinear hypoplasticity. A nonlinear viscoplastic material refers to a nonlinear material that shows both viscous and plastic properties. When subjected to very large deformation which is no longer linear, the material does not recover its initial shape, and the relation describing the material response is nonlinear. An example of a nonlinear viscoplastic material is a polymer (Perzyna, 1966). On the other hand, hypoplastic material models have been shown to be a good choice for describing the realistic behaviour of soils, which is both nonlinear and inelastic (Fellin, 2002). We should highlight that nonlinear behaviour in soils can be associated with large shear strains or increased material damping ratio (Ji et al., 2021).

Finally, boundary nonlinearity plays a role in the vibration response of structures, and its incorporation in any computational model helps produce accurate results (Mao et al., 2017; Kanda and Sugiura, 2018).

The analysis of nonlinear problems is inherently more complex than that of linear problems. As a result, the principle of superposition (which stipulates that on a linear structure, the resultant deflection, stress, or tension in a system is the arithmetic sum of the effects of each force separately applied) no longer holds. Overall, linearity is a useful feature but mainly applies to simple situations. However, nonlinearity is needed for real-life modelling because it can accurately represent complex situations.

Regarding the types of foundations, linear and nonlinear cases can be differentiated. For the analysis of the beam or plate resting on a linear elastic foundation, the displacement is assumed to be infinitesimally small, and the material is linearly elastic. Moreover, the displacement response is a linear function of the applied load vector, and no second elastic foundation parameter is required. Though the complete linear analysis of the system provides accurate results, the linear approximation is only convenient for design purposes. Furthermore, for both design and realisation purposes, linear systems and analyses may fail to represent the real structure adequately. Therefore, nonlinear systems and analyses are used, for instance, where Hooke's law is no longer valid. In this case, the stiffness of the foundation is now dependent on the displacement. As a result, the system can be purely nonlinear owing to the system's inherent properties. Sometimes solutions are difficult to obtain for the governing nonlinear equations. As such, this thesis considers nonlinearity in the vibration analysis of beams and plates, and the goal is to study beams and plates having nonlinear stiffness properties in the foundation.

1.3 Stiffness of Elastic Foundations

Subgrade materials are commonly characterised by their stiffness. The subgrade must be capable of withstanding the load transmitted from the pavement to the structure. Therefore, the more resistant a subgrade is, the more load it can withstand before reaching a critical deformation value. Subgrade performance generally depends on load-bearing capacity which can be affected by the type of soil. Soils are inherently nonlinear, and the nonlinear stiffness parameters for soil are related to soil composition and current state. The value of the stiffness/modulus of subgrade (as represented in the literature) ranges from 0.1 MPa/m for weak support to 270 MPa/m or higher for strong/hard support (Bowles, 1988; Christopher et al., 2010; Costa et al., 2010). In the context of real infrastructures such as roads, runways, or railways, the character of nonlinearity can also be discussed. For example, in road dynamics,

the pavement is often modelled as a beam or plate resting on an elastic or viscoelastic foundation, while the vehicle is viewed as a moving load acting on the road.

Studies have been conducted recently mostly focusing on the nonlinear and viscoplastic properties of pavement materials as compared to linear materials. In these studies, the road is modelled as an infinite beam resting on an elastic foundation because a road is regarded as a structure with an infinite length (Cebon, 1993; Chen, 2002; Li et al., 2013). For example, Li et al. (2013) modelled a nonlinear vehicle-road coupled system as a seven-DOF system vehicle moving on a simply supported rectangular thin plate on a nonlinear viscoelastic foundation. It was found that road pavements have nonlinear properties which must be considered when describing the mechanical properties of roads. It was then concluded that considering nonlinearity is beneficial and necessary to model a road pavement structure accurately, whereas linear analysis may result in unsafe design.

On the other hand, track-ground behaviour can be differentiated when considering linear and nonlinear soil behaviour. Costa et al. (2010) studied the influence of the nonlinear behaviour of the soil on the response of railway tracks. It was shown that an equivalent linear model could be used to approximate the nonlinear soil behaviour, but in the passage of trains at high-speed on the railway, the method cannot represent the change of the soil properties. The study also revealed that nonlinearity has a very significant influence when considered and as a result, the critical speed is lowered when the nonlinear model is used to simulate the response or behaviour of railways. Because the support structure of a railway track is highly nonlinear under dynamic loading, it is imperative to use a nonlinear relation to accurately approximate the behaviour of a railway track instead of using a linear one (Dahlberg, 2002).

1.4 Motivation for the Choice of Foundation/Model

Studying beams and plates resting on elastic foundations may enhance structural understanding and promote the need to build efficiently using fewer materials and less effort. Furthermore, it may help designers and practitioners in designing structures that resist more deformation when subjected to external excitation. However, not enough information is available on these models, especially when the beam or plate rests on a nonlinear elastic foundation. Because the governing equations of such models are nonlinear and complex in finding solutions, many designers fall back on linear cases, which sometimes fail to give an accurate representation of the real structure. To avoid such a situation, Jorge et al. (2015) have proven the importance and benefit of incorporating nonlinearity into the foundation of a

beam or a plate. This motivates the need to explore beams on nonlinear elastic foundations in this thesis. The research will further investigate the same study on plates resting on nonlinear elastic foundations to cover 2D and 3D modelling and understand the structural response.

1.5 Research Problem

Beams and plates are essential in structural construction, but not enough studies have addressed them, mainly when they rest on nonlinear elastic foundations. Numerous studies have investigated beams and plates supported on linear elastic foundations, and the solutions to the governing equations could easily be found. However, beams and plates on nonlinear foundation cases have been neglected and constitute the motivation of the current research.

1.6 Research Aim and Objectives

This research aims to integrate nonlinearities in developing and validating beam and plate models and enhance their resistance when subjected to loads or initial displacements. Stiffness nonlinearity will be used to achieve that goal, and its importance when incorporated into the foundation of structures will be discussed with particular attention to beams and plates. Therefore, the present work will achieve on the following objectives:

1. Develop approximate solutions for the nonlinear vibration of beams and plates on elastic foundations with nonlinear stiffness properties.
2. Compare the results with those of beams and plates supported on linear elastic foundations.
3. Evaluate the relative differences incurred by ignoring the nonlinearity of the foundation stiffness.

1.7 Limitations of Study

This thesis aims to study BPoEFs and emphasises on nonlinear cases. Specific methods and analyses have been selected to cover the scope of the study. Therefore, analytical studies will use the straightforward expansion method (SEM) to investigate the free vibration (no

external excitation is considered in the entire thesis) of beams and plates on nonlinear elastic foundations. Moreover, numerical studies will use the FEA in Abaqus. Finally, the work will only be theoretical, and no physical or experimental test will be performed in the laboratory.

1.8 Thesis Structure

This dissertation is organized into seven chapters. The current chapter gives a brief background to the study and highlights the research problem, aim, and objectives. The rest of the thesis contains the following chapters:

Chapter 2 provides a detailed literature review of beams and plates on elastic foundations and the concept of nonlinearity. It also discusses the approaches employed in the investigations and identifies key research findings and gaps in the open literature. The chapter further highlights the research significance and the methodology employed.

Chapter 3 covers beams on elastic foundations using linear theory. This chapter serves as the basis of this research. First, the system's equation of motion is derived, and the solution is found using analytical and numerical methods. Finally, the results of the beam resting on the linear elastic foundation are compared to those of the simple beam.

Chapter 4 addresses the nonlinear vibration of beams on elastic foundations. First, the equations of motion for free vibration of beams with quadratic and cubic stiffness nonlinearities of the foundation are derived and solved using an expansion method. Thereafter, numerical analysis in Abaqus is carried out, and the comparison between the two results is performed for validation.

Chapter 5 focuses on a parametric study of beams resting on nonlinear elastic foundations, based on the results obtained in Chapter 4. The effects of quadratic and cubic stiffness nonlinearities on the beam's vibration amplitude and natural frequencies are explored.

Chapter 6 presents the study of a plate on a linear and nonlinear elastic foundation. The same analyses as those of the beam are performed. The results obtained from nonlinear analyses are compared to those from linear analyses. Moreover, the results from the plate are compared to those of the beam to ensure a complete understanding of the structural behaviour.

Chapter 7 concludes the thesis by providing a summary of significant findings of the research and suggestions for future studies.

Chapter 2

Literature Review

2.1 Introduction

This chapter reviews the available work in the literature concerning beams and plates on elastic foundations (BPoEFs). Some background regarding nonlinearity and vibration analysis will be provided. Moreover, methods for deriving and solving nonlinear problems are also addressed.

2.2 Some Definitions and Concepts

Vibration is a mechanical phenomenon whereby oscillations occur about an equilibrium point (Zingoni, 2018b), with a reversible energy exchange between kinetic and potential energies. Therefore, any vibrating system must have a component that stores kinetic energy (mass or inertial element) and potential energy (spring or elastic element). The latter can also be a spring-like component that holds the potential energy and converts it into kinetic energy in the form of motion. Without those components, a system cannot be classified as a vibrating system. Some examples of vibration phenomena are: a motion felt by passengers in an automobile travelling over a bumpy road; the action of tree branches; heartbeat; the motion of a guitar string; swaying of tall buildings due to strong wind or earthquake; or the movement of an aeroplane in turbulence (Wagg and Neild, 2009). Rigid or deformable systems such as engineering devices and machines subjected to dynamic loads exhibit vibration phenomena. Since most engineering structures carry a dynamic load in their service life, it is important

to study them in order to predict the responses of systems and avoid serious damage (Rao, 2009).

Vibrations can be desirable or undesirable, depending on where they appear and their characteristics. Undesirable vibrations waste energy in the system, reduce efficiency, and produce noise and unwanted sound (Crawford and Crawford, 1992). These imply discomfort in the system, and effects such as fatigue, reduction of life, damage, or failure of structures. Some examples are the vibration motion of electric motors and engines. In such cases, eliminating vibrations may save maintenance costs or even human lives. On the other hand, examples of desirable vibrations include the reed in a woodwind instrument or harmonica, the vibration of a mobile phone or the cone of a loudspeaker (Takács et al., 2012). When vibrations are beneficial, they may be used to improve the quality and efficiency of a process.

Mechanical vibrations can be classified as free or forced vibration depending on whether an outside force is absent or present, respectively (Zingoni, 2018b). Free vibration is when the system is left to vibrate independently after initial disturbance. Moreover, disregarding loss of energy due to damping, the system's energy is constant and equal to the initial energy. Consequently, the system is excited by an initial condition such as displacement, velocity, or acceleration. A simple pendulum is a typical example. As for forced vibrations, they appear when an external force is acting on the system, inducing vibrations (Crawford and Crawford, 1992; Takács et al., 2012; Zingoni, 2018b). Therefore, external energy is supplied to the system through an applied force or an imposed displacement excitation—also, the system's energy changes over time.

Vibrations can be linear or nonlinear, depending on the system's essential components. When all essential members in a vibratory system (spring, mass, damper) behave linearly, the system is linear. As a result, the principle of superposition holds. However, the system is nonlinear when one or some essential components act nonlinearly. In this case, the principle of superposition does not hold, and the system's response needs further analysis (Singiresu et al., 1995). Nonlinear systems are often difficult (or impossible) to model, and their behaviour concerning a given variable (for instance, time) is complicated to predict. Therefore, where applicable, one should approximate a nonlinear system as a linear one to simplify its modelling.

Systems can be classified according to discrete and continuous types of mathematical models based on their DOF (Zingoni, 2018b). The degree of freedom (DOF) is the number of independent coordinates required to describe the motion of a system entirely. Let n be the DOF of a given system; $n = 1$ corresponds to a Single-Degree of Freedom (SDOF)

system, whereas $n > 1$ is for a Multi-degrees of Freedom (MDOF) system. Discrete or lumped parameter systems have a finite number of DOF. They are usually modelled by interconnected masses, dampers, and springs. Continuous or distributed systems are those with an infinite number of DOF. Here, the mass, damping, and stiffness vary as continuous functions of the generalised coordinates. A mathematical variable can be discrete (countably infinite/finite).

On the other hand, a variable can also be continuous, taking on infinitely uncountable values. Hence, modelling a system as discrete or continuous depends on the analysis purpose and the expected accuracy of the results. When symmetry exists in such systems, special methods can be used to study vibration phenomena and simplify the calculations (Zingoni, 2018b).

2.2.1 Background on Nonlinear Systems

Most structures or mechanical systems behave nonlinearly under certain conditions. For example, nonlinearity in a differential equation can be due to stiffness or damping. It appears as a product of two variables, a power of the same variable, a function of a square or a product of variables. Studies have shown that the presence of nonlinear stiffness on a structure can have both benefits and inconvenient effects (Esmailzadeh et al., 2018).

Three crucial factors can significantly affect the dynamic performance of a system: nonlinear stiffness, damping and excitation force/amplitude. Many fundamental problems of mechanics, including those of vibrations, have long been recognised as nonlinear. Although the commonly adopted linear treatments are satisfactory for most purposes, they are not always adequate. Theoretically, impossible phenomena in linear systems often occur in nonlinear systems. Zhang and Huang (2010) summarised three critical parts to justify a vibration problem. The first one is the vibration source related to the loading applied to the system. The second is the vibration path referring to the components where vibrations are generated, for example, the damper, the stiffness or the mass. The last one is the vibration response and its effects on the receiver.

An ordinary differential equation (ODE) mathematically describes a nonlinear dynamical system with the following form (Esmailzadeh et al., 2018):

$$\ddot{x} + f(x, \dot{x}, t) = 0, \quad (2.1)$$

where x is the displacement, $\dot{x}$ is the derivative of the displacement x over time t , and $f(x, \dot{x}, t)$ stands for a nonlinear function. Since the basic components of a vibratory system are mass, damper, and spring, the nonlinearity in the governing differential equation may be introduced through any of them. This means the nonlinear function of Eq. (2.1) can be due to the stiffness, the damping or the mass. It can even be both of the first two components, as expressed in the following equation:

$$f(x, \dot{x}, t) = k(x)x + c(x)\dot{x}. \quad (2.2)$$

Here the function $f(x, \dot{x}, t)$ is nonlinear due to the terms $k(x)$ and $c(x)$, standing for the stiffness and the damping, respectively. In the case of a system with many DOFs, Eq. (2.1) can be generalized to an equation with the given DOFs.

2.2.2 The Importance of Nonlinear Analysis in Modelling

The concept of nonlinearity is found everywhere because all systems behave nonlinearly to some extent, and the introduction of nonlinearity in some systems may improve their performance. Moreover, it can be used to improve mechanical designs, especially in the area of vibration control. Although linear modelling provides good approximations in many cases and thus good results, it fails in some instances due to the misrepresentation of the real situation. Thus, nonlinear modelling has been of concern to engineers and researchers. The main reason nonlinear analysis must be considered is that it makes the model of the system more reliable. In addition, it reveals specific necessary behaviour of a system that cannot be predicted using linear analysis. It also allows for obtaining accurate values required to simulate the real-world behaviour of systems. Therefore, the objective of the engineer or practitioner is achieved more efficiently than in a linear system.

Furthermore, if the linear analysis/modelling fails, nonlinearity might affect the structure's response. In this case, only the nonlinear analysis or modelling can be used to describe the motion fully. For example, Zhao et al. (2018) explored the nonlinear vibration of an elastic string with large amplitude and curvature. The authors showed that the results obtained from the nonlinear analysis were more accurate than those from the linear analysis.

The approach to dealing with nonlinear problems is not unique. Owing to this, once detected that a problem is nonlinear, one can experiment with several techniques before

solving it. For example, a fixed beam at both ends with a thickness h is considered and represented in the following.

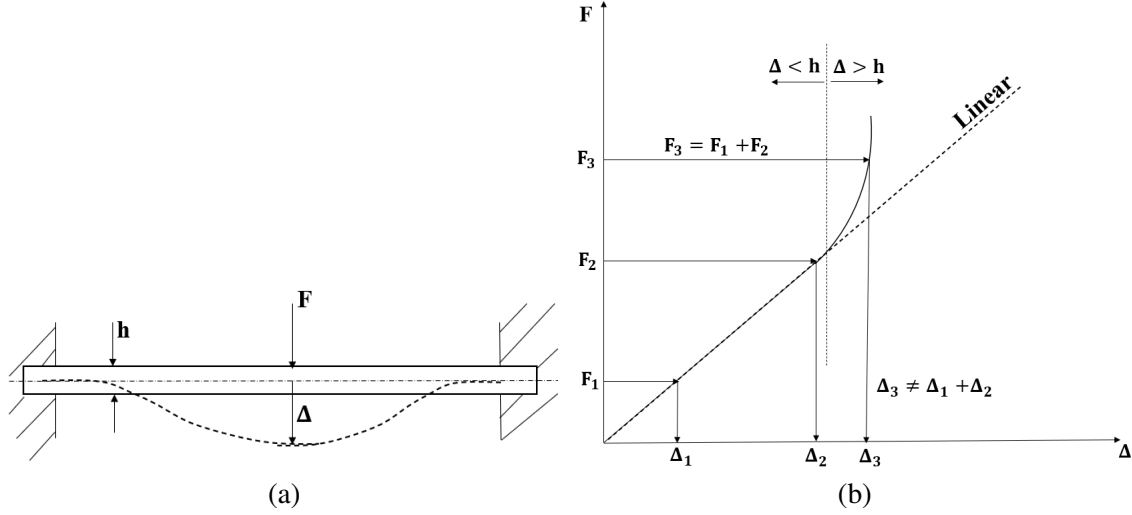


Figure 2.1: (a) A fixed beam in both ends with a deflected shape and (b) the plot of the load versus the deflection.

Figure 2.1a presents the beam deflection Δ when a load F is applied at its middle. Figure 2.1b shows a plot of the applied load as a function of the deflection. In Figure 2.1b, we see that when the deflection is less than the thickness of the beam ($\Delta < h$), it is the linear range. In this case, the superposition principle holds, and the following equation holds: $F_3 = F_1 + F_2 \Rightarrow \Delta_3 = \Delta_1 + \Delta_2$. On the contrary, when the deflection is greater than the thickness of the beam ($\Delta > h$) due to the increase of the applied loads, the superposition principle is no longer valid. Therefore, the equation is transformed into $F_3 = F_1 + F_2 \Rightarrow \Delta_3 \neq \Delta_1 + \Delta_2$. This case is the nonlinear range, meaning that the stiffness of the beam is nonlinear; more precisely, a hardening stiffness. This example is similar to that of Tomlinson and Worden (2000), where the authors identified nonlinear characteristics in a nonlinear structure and proposed the harmonic balance method (HBM) to solve the nonlinear equation.

The most common nonlinearities encountered in systems are due to polynomial stiffness and damping. These nonlinearities are usually amplitude, frequency, and velocity-dependent (Mahmoodi et al., 2004). Therefore, the major problem is their amplitude dependence when dealing with nonlinear elements, making a traditional linear modal analysis a complex task. This condition generates stiffness/damping coefficients which vary continuously with the level of the response. Hence, using nonlinear elements in dynamical systems influences their performance to a considerable extent. The same finding appears when those nonlinear elements are added to the system.

It is important to highlight that classical Coulomb friction produces a nonlinear response which is not a polynomial stiffness or damping. For example, rubber is a super-elastic material with a nonlinear response. Fractional derivative models have been used to characterise and describe the behaviour of viscoelastic materials (such as polymers and rubbers) with a small number of parameters. The method uses the framework of a standard spring-dashpot mechanical model. Solving dynamics problems in vibration systems with rubber requires knowledge of the mechanical properties of the rubber. Rubber springs have been used in railway systems. For example, Shi and Wu (2016) and Zhang and Zhu (2017) used a fractional derivative model of a nonlinear rubber spring for the dynamic analysis of a vehicle-track coupled system. They studied the influence of the nonlinear properties of the rubber on the dynamic interaction between the track and the railway vehicle. The following subsection presents an extensive survey of the causes of nonlinearity in structures.

2.3 Causes of Nonlinearity in a Structure

In many functional structures, nonlinearity is primarily concentrated in a single component. The causes of nonlinearity arise from one factor or a combination of many factors in the system. For instance, there is stiffness nonlinearity or damping nonlinearity. In structural dynamics, nonlinearity can appear in a system due to the material, large deformations, large displacements, inertia, dry friction effects, and boundary conditions. In this thesis, particular interest is in the nonlinear vibration of systems with nonlinearities due to stiffness.

In nonlinear problems, nonlinear effects are classified using the following three main categories: material, geometry, and boundary condition (Plumbridge et al., 2003; Nastran, 2004). The first two types are considered when dealing with large deformations, and the third is mainly related to contact. Structures exhibiting a combination of those three primary sources of nonlinearity may exist. The algorithms to solve the resulting nonlinear differential equations will generally be set up to handle nonlinear effects from various sources. In what follows, brief details of those causes of nonlinearity are elaborated.

2.3.1 Geometric Nonlinearity

Geometric nonlinearity is the first and simplest one, arising from the system's potential energy and in the form of pure kinematics. A typical example is a nonlinear curvature in a cantilever beam where the load direction of an object changes. It occurs due to the modification of

the geometry of the system and involves problems with large displacement (Kelly, 2012). An example where geometric nonlinearity is encountered is a simple pendulum system, as shown in Figure 2.2.

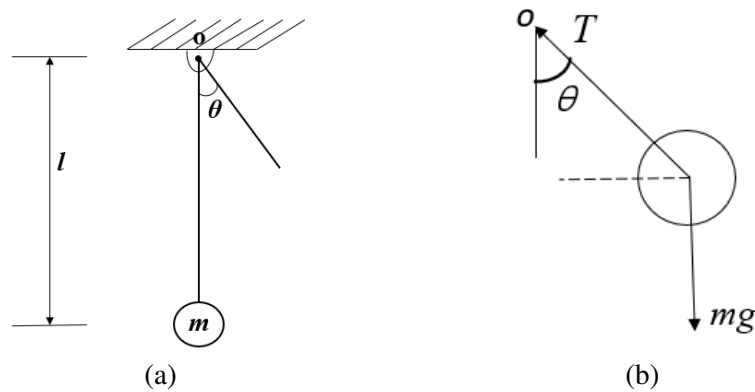


Figure 2.2: (a) Simple pendulum and (b) free body diagram.

For example, Wagg and Neild (2014) investigated the pendulum in Figure 2.2. They highlighted that the pendulum has approximately linear behaviour for a minimal displacement angle θ , but an inherent nonlinear behaviour when the angular displacement becomes larger. When the oscillation amplitude increases, the linear equation's validity ends. Then, the best description of the oscillating system is obtained by solving a nonlinear differential equation of motion. The derivation of the differential equation describing the movement of the mass of the pendulum presented in Figure 2.2 is obtained by summing all the moments concerning the fixed axis of rotation O . That gives the following equation of motion:

$$ml^2\ddot{\theta} + mgl \sin \theta = 0 \Rightarrow \ddot{\theta} + \frac{g}{l} \sin \theta = 0, \quad (2.3)$$

where l , θ , g are the length of the pendulum, the angle, and the acceleration due to gravity. This equation could also be obtained by deriving the system's kinetic and potential energies. Eq. (2.3) is a nonlinear equation due to the presence of $\sin \theta$ in the equation. To obtain an acceptable solution, a two-terms approximation for $\sin \theta$ near $\theta = 0$ is used. This gives:

$$\sin \theta \simeq \theta - \frac{1}{6}\theta^3 + \frac{1}{120}\theta^5 - \dots. \quad (2.4)$$

By only retaining the first two terms of this approximation, Eq. (2.3) becomes:

$$ml^2\ddot{\theta} + mgl \left(\theta - \frac{\theta^3}{6} \right) = 0 \Leftrightarrow \ddot{\theta} + \frac{g}{l} \left(\theta - \frac{\theta^3}{6} \right) = 0, \quad (2.5)$$

which is also a nonlinear equation. Here the nonlinearity appears because of θ^3 , which is a geometric nonlinearity. Due to that term, it is hard to find an exact analytical solution to Eq. (2.5). However, some approximate analytical methods exist to find a very close solution to the expected one.

One approach is that, for small oscillations, $\sin \theta \simeq \theta$ and Eq. (2.5) reduces into:

$$\ddot{\theta} + \frac{g}{l}\theta = 0. \quad (2.6)$$

Eq. (2.6) is a linear equation satisfied by the simple pendulum with a solution of the form:

$$\theta(t) = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right) \Rightarrow \theta(t) = \theta_0 \cos(\omega_0 t), \quad (2.7)$$

where θ_0 is the amplitude of the pendulum and ω_0 is the natural frequency of oscillations. When the oscillations are not small, one must deal with the nonlinear equation directly. Beléndez et al. (2007) proposed an exact solution of the nonlinear equation of the simple pendulum using the Jacobi elliptic function $sn(u; m)$ as given by Eq. (2.8).

$$\theta(t) = 2 \arcsin \left\{ \sin\left(\frac{\theta_0}{2}\right) sn \left[\Gamma\left(\sin^2\left(\frac{\theta_0}{2}\right)\right) - \omega_0 t; \sin^2\left(\frac{\theta_0}{2}\right) \right] \right\}, \quad (2.8)$$

where $\Gamma()$ stands for the complete elliptical integral of the first kind (Abramowitz and Stegun, 1965). The plots presented in Figure 2.3 correspond to the expressions of the angular displacement $\theta(t)$ as a function of $\omega_0 t$ given by Eqs. (2.7) and (2.8). Here, the oscillation amplitudes have been considered as $\theta_0 = 0.1\pi$ and $\theta_0 = 2\pi$, respectively for each subfigure.

In Figure 2.3a, there is a shift between the two curves, each corresponding to the linear and nonlinear solutions with the same maximum and minimum amplitudes. On the other hand, Figure 2.3b shows that the amplitude of the nonlinear solution has considerably reduced and that of the linear solution has increased as compared to the plots in Figure 2.3a. This shows that the initial angle affects the variation of the pendulum amplitude of vibration. Therefore, by neglecting nonlinear terms, the results obtained may not correctly describe the system's behaviour. Also, note that the shape of the curve can be different each time a new value is assigned to ω_0 .

Geometric effects related to vibration can result from the excessive deformation of the structure's stiffness. Due to this, the geometry of the design will vary accordingly and will imply a systematic change in the geometric stiffness of the structure (Cai et al., 2017).

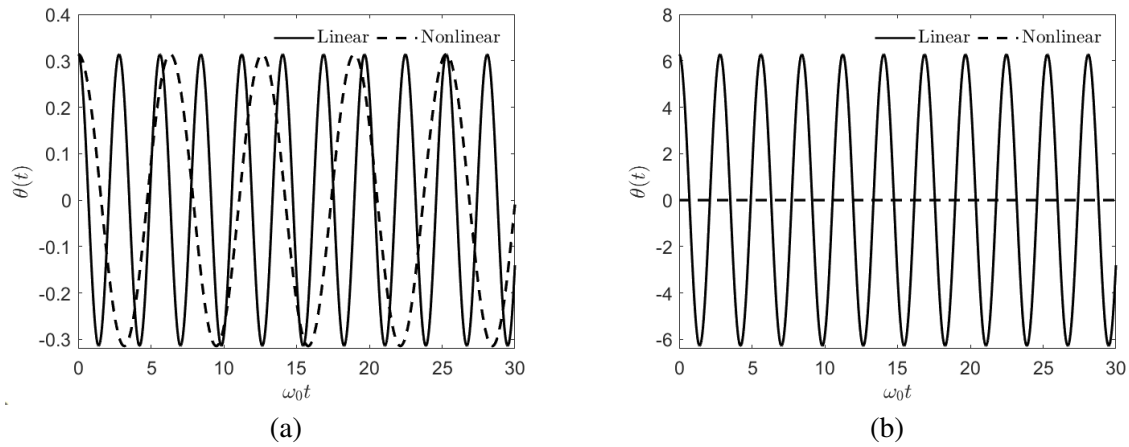


Figure 2.3: Comparison between linear (dashed line) and nonlinear (continuous line) plots of the angular displacement of a simple pendulum: (a) $\theta_0 = 0.1\pi$ and (b) $\theta_0 = 2\pi$.

Geometric effects can be sudden and unexpected, and the characterisation of geometric nonlinearity is complex. Therefore, they are considered to avoid any numerical simulation failing to predict the structure's real behaviour. Some applications of geometric nonlinearity are slender structures in aerospace, civil and mechanical engineering, tensile structures such as cables and inflatable membranes, and metal and plastic forming (Cook et al., 2001).

2.3.2 Material Nonlinearity

Material nonlinearity is due to nonlinearity in the material behaviour when the relationship between stress and strain or force-displacement is essentially nonlinear regardless of if the material is still elastic. In this case, the linear elastic theory may fail when material properties are essentially unchanged by loading. Thus, the nonlinear elastoplastic theory must be used to describe the motion. For example, Wagg and Neild (2014) listed different materials together with their causes of nonlinearity. The first type is the nonlinear elasticity, characterised by the fact that the stress-strain relationship is nonlinear, as shown in Figure 2.4a. The loading and unloading paths followed for material nonlinearity are the same. This means that after loading the material, it returns to its initial position following the unloading part without any permanent deformation.

Another example of material nonlinearity is nonlinear plasticity. That is illustrated in Figure 2.4b. A load applied to a plastic material deforms permanently due to the load action. Also, it should be noted that all materials fail when a certain large deformation is

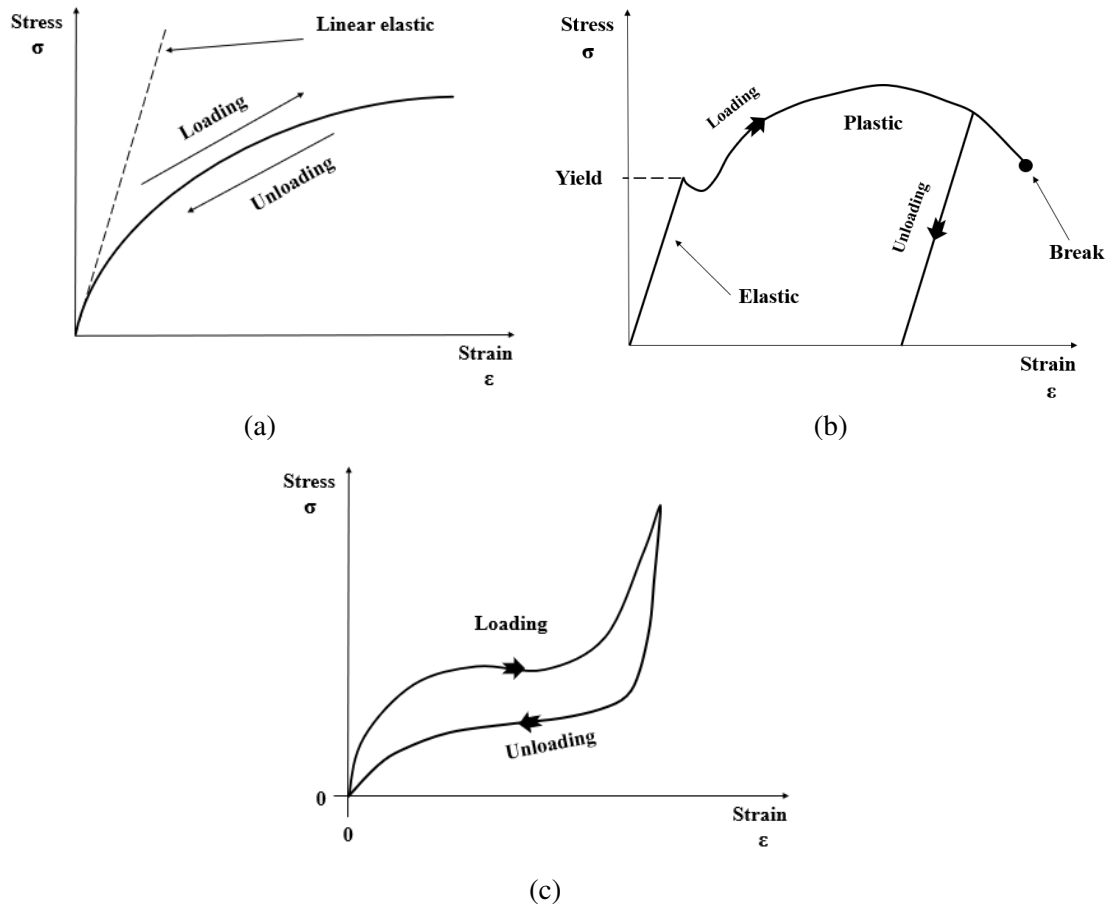


Figure 2.4: Representation of material nonlinearity: (a) elastic, (b) plastic, and (c) rubber.

reached. A real example of applying a material fulfilling these conditions is low-carbon steel. The last example is when the structure follows different paths through loading and unloading. An example of application is the rubber-like material displayed in Figure 2.4c. In this last case, the overall shape is a hysteresis curve as is the case for an elastoplastic material. Nevertheless, not only rubber has a strong hysteresis; other materials such as metals exhibit hysteresis when exposed to high-magnitude load cycles. In this case, the Bauschinger effect (Moss and Basic, 2013) is produced, that is a decrease in the yield strength of the material in compression because of prior deformation in tension.

Although material nonlinearities mainly occur in civil engineering, they deal with nonlinear materials such as concrete and steel. Thus, incorporating nonlinear properties of materials allows for accurate calculation of the stress distribution and makes it possible to verify when certain load-bearing capacity aspects of the material such as plastic collapse. Also, the damage or fatigue of the material due to excessive vibration amplitudes can destroy

the structure. For beams, the nonlinear relationship between the stress and the strain causes material nonlinearity (Poorjamshidian et al., 2014). In this case, considering the nonlinear effects is extremely important during the beam's analysis because fatigue on the beam's material and overall failure of the beam can arise when the vibration amplitude is too high.

2.3.3 Boundary Condition Nonlinearity

Boundary condition nonlinearity refers to an abrupt change in the boundary conditions of the structure during its analysis. Boundary conditions are easily applied to continuous systems such as beams, plates, and shells. For example, Watanabe (1978) studied a beam fixed at one end and the other, supported by a spring with clearance and hysteresis loop characteristics and nonlinear boundary conditions. Because nonlinearity was due to the boundary condition, the problem was treated as a nonlinear vibration problem. The results obtained from the study of the system were then compared to those from an SDOF system. The comparison revealed similarity between the two results. That similarity of the results confirms that sometimes, it is possible to reduce the study of a continuous system to that of an SDOF system. To illustrate this scenario, Figure 2.5 shows a cantilever beam with an applied load at the free end.

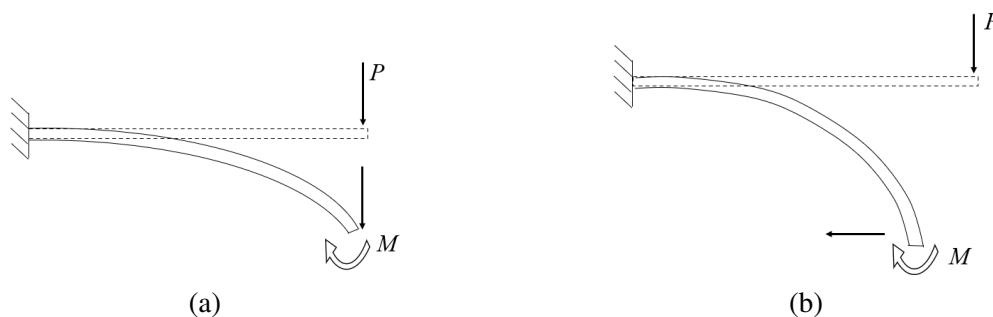


Figure 2.5: Deformation of the cantilever beam due to an applied load: (a) linear case and (b) nonlinear case.

In Figure 2.5, M and P stand respectively for the applied torque and load. The trajectory followed by the cantilever beam in the linear case is directly proportional to the initial position of the beam. This is why the arrow indicating the load P is pointing vertically downward as depicted in Figure 2.5a. However, that is not the same in the nonlinear case. The application of the same amount of loading at the free end of the cantilever beam induces the large curvature of the beam and gives rise to the nonlinear effect. In this specific case, the arrow materialising the direction of the load P is now horizontally as shown in Figure

2.5b. This change in geometry as the structure deforms is considered in setting up the strain-displacement and equilibrium equations.

Kanda and Sugiura (2018) considered quadratic boundary conditions nonlinearity. The multiple time scales method (MTSM) was used to investigate the cumulative nonlinear guided waves caused by internal resonance. The results showed that nonlinear effects on finite amplitude of two modes satisfy a phase-matching condition for internal resonance. Wang et al. (2021) explored the forced vibration response of a non-uniform beam with nonlinear boundary conditions. They analysed the influence of boundary nonlinearity and excitation amplitude on the vibration response of non-uniform beams. They found that adjusting the nonlinear spring property can control the beam's vibration. The proposed approach could analyse vibration problems of non-uniform beams with nonlinear boundary conditions and subject to any load.

Although taking boundary condition nonlinearity into account could complicate the determination of mode shapes of the structure, it is crucial in determining the vibration response of structures, producing more accurate results.

2.4 Stiffness: Definition, Applications, and Importance in Vibration Analysis

2.4.1 Definition of Stiffness

Stiffness is the ability or extent to which an object or elastic body resists deformation in response to applied loads/forces. It is also the ability of a structure or mechanical system to sustain external loads without being excessively deformed. Thus, it is one of the most critical design criteria for systems and mechanical components. For Takács et al. (2012), it is a spring-like property that describes the level of resisting force resulting when a body changes in length. The stiffness can be linear, implying that the restoring force is proportional to the displacement. On the other hand, it can be nonlinear. In this case, the restoring force is not proportional to the displacement, and the connection is nonlinear.

2.4.2 Stiffness Nonlinearity in Vibration

Stiffness nonlinearity appears when a spring no longer obeys Hooke's law and is mainly linked to the geometry of the spring's structure. It can take many forms, such as a softening or hardening stiffness, a bilinear stiffness, clearance, or saturation. Stiffness nonlinearity in a system or structure can be disadvantageous or advantageous. The former case refers to instances where the effects produced are contrary to the expectations and can lead to damage. This explains why engineers sometimes try to eliminate (stiffness) nonlinearity from structural design. For the latter case, introducing stiffness nonlinearity into a system can improve its performance and help vibration isolation.

For example, Brennan (2017) presented the possibilities of considering nonlinear stiffness for vibration neutralisers, nonlinear vibration isolators, and nonlinear energy harvesters. The author further indicated that desirable effects are produced when the system implements stiffness nonlinearity carefully. In contrast, the wrong implementation could have undesirable consequences. Thus, the big challenge for an engineer is to use nonlinear stiffness advantages to minimise unwanted dynamic effects such as large jumps in the vibration. Since these jumps can occur at specific frequencies, they can be dangerous and cause damage to the structure. Moreover, since nonlinear stiffness is present in real-world applications, it is imperative to consider it for the dynamic analysis of structures and mechanical systems (Rivin, 2010).

Because of many case studies on stiffness nonlinearities in engineering structures, it is possible to summarise that the development of satisfactory results when linear analysis techniques fail is a very active study area in today's structural dynamics society. Therefore, considering the nonlinear regime during the design process shows a need to develop strategies for understanding the nonlinear response of structures.

2.4.3 Applications and Importance of Stiffness in Vibration Analysis

The stiffness effect plays a vital role in the performance of mechanical systems. As an example, variable stiffness can be used for vibration control. The deflection and the dynamic response of a structure to dynamic loads are both dependent on their stiffness. That is because the loads carried by the elements of a structure made up of multiple structural elements are proportional to the relative stiffness of each component (Heyman and Jacques, 1998). Thus,

an element with significant stiffness attracts more load, allowing a structure to withstand deformations easily.

To reduce vibrations, the amplitude response of the structure must be as small as possible. Increasing the stiffness of the design is one of the possibilities. Turner (1956) showed that increasing the stiffness of a structure reduces the amplitude of the external excitation and thus results in a vibration reduction. However, increasing the stiffness increases the frequency of vibrations. An example of this latter case is a spring-mass system with a natural frequency given by $\omega = \sqrt{\frac{k}{m}}$, where k is the stiffness of the spring, and m is the mass attached to the spring. Then, increasing k will increase ω . Turner (1956) performed analyses and applied them to structures such as plates, trusses, and beams. The findings revealed that stiffness plays an important role when considering structures' vibration analysis. Furthermore, it also contributes to determining the structure's vibration modes.

Incorporating stiffness nonlinearity in structures is not a new topic but needs to be explored in vibration analysis. That will improve structural performance and enhance the structural design. The effect of stiffness nonlinearity on the mechanical systems' performance is due to the influence of deformations on the dynamic or vibration stability of the system. The applications of stiffness are broad and differ from the area of study. In this thesis, only its application in vibration is of interest. Although considering stiffness in the system is very important, one should pay attention to its assigned value to avoid some damage. For example, according to Rivin (2010), excessive deformations of some system components may be caused by an inadequate stiffness value. That can result in dynamic instability and increase the vibration amplitude of the external excitation, respectively, for self-excited and forced structures, thus, implying the shortening of life or destruction of the structure.

The resonance frequencies for nonlinear systems may increase or decrease with the increase of the vibration amplitude. The same phenomenon is observed in the vibration of systems where the natural frequency or the stiffness increases for hardening nonlinearity and decreases for softening nonlinearity with increasing excitation amplitude. As a result, changing the stiffness of a structure may have both benefits and inconvenient effects on the vibrations of the design. In addition, the vibration response of the resonance and other vibration processes can be reduced if the variation of the stiffness follows a specified control law. Examples of this latter case are random vibration and pulse vibrations (Rivin, 2010; Brennan, 2017). Hence, modifying the system's stiffness can reduce the complexity of a problem to be solved.

2.5 Types of Nonlinear Stiffness

The most common types of nonlinearities encountered in dynamic testing are polynomial stiffness and damping, clearances, impacts, friction, and saturation effects. As one would expect, these nonlinearities are usually amplitude, velocity, and frequency-dependent. Still, it is usual to simplify and idealise these to incorporate into the analysis, simulation and prediction capabilities.

Considering an SDOF system oscillator with nonlinear damping and stiffness terms, the equation of motion is given by (Tomlinson and Worden, 2000):

$$m\ddot{x} + f_d(\dot{x}) + f_e(x) = y(t), \quad (2.9)$$

where $m\ddot{x}$, $f_d(\dot{x})$, $f_e(x)$, and $y(t)$ respectively stand for the inertia, the damping, the elastic and the external excitation terms. Two sorts of nonlinearity are generally distinguished: localised and non-localised. The nonlinearities are localised when incorporating elements such as actuators, bearings, linkages, or elastomeric in the structure as stiffness or damping. In other cases, they act as non-localised nonlinearities. The following paragraphs present some types of nonlinearities.

2.5.1 Cubic Nonlinearity/Cubic Stiffness

The cubic stiffness nonlinearity can be characterised by the presence of a cubic term in the expression of the restoring force. It is the most investigated form of nonlinearity. The following figure displays an example of a system with nonlinear stiffness.

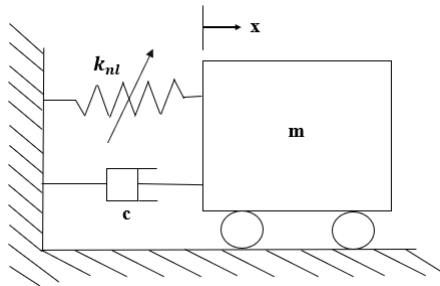


Figure 2.6: System with nonlinear stiffness k_{nl} .

Figure 2.6 presents a mass m attached to a nonlinear spring with a stiffness coefficient k_{nl} , where the subscript stands for nonlinear and a damper with a damping coefficient c on

a fixed support. The displacement of the mass is horizontal and given by x . The system in Figure 2.6 is considered nonlinear due to the presence of the nonlinear stiffness coefficient k_{nl} . Here, two different equations are considered: one without and the other with a quadratic term.

First case: Equation without a quadratic term

This example was presented in work done by Burgess (1988). For this case, the equation of motion is given by the following:

$$\begin{aligned} m\ddot{x} + c\dot{x} + k_1x + k_3x^3 &= f(t) \\ \Rightarrow \ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2x + \gamma\omega_n^2x^3 &= F(t). \end{aligned} \quad (2.10)$$

Eq. (2.10) is the forced Duffing equation with $\xi = \frac{c}{2m\omega_n}$, $\omega_n = \sqrt{\frac{k_1}{m}}$, $\gamma = \frac{k_3}{k_1}$, and $F(t) = \frac{f(t)}{m}$. It can be seen that the restoring force is given by $f_r(x) = k_1x + k_3x^3$, combining a linear term k_1x and a nonlinear term k_3x^3 . This means that, for $k_3 > 0$, at high level of excitation, the restoring force will be greater than that expected from the linear term alone.

Depending on the value of γ , the stiffness can be a softening ($\gamma < 0$) or a hardening ($\gamma > 0$) cubic nonlinearity. In the former case, the effective stiffness decreases as the level of excitation increases. This complements the idea of Nayfeh (1973) who showed that stiffness nonlinearity could produce softening or hardening effects in the dynamics of mechanical oscillators. Examples of systems having hardening behaviour are clamped plates and beams. One should note that the restoring force changes its sign at a certain distance from equilibrium for softening cubic systems and begins to drive the system to infinity. Consistently, systems with such characteristics have higher-order polynomial terms in the stiffness with positive coefficients that dominate at high levels and restore stability (Nayfeh, 1973).

Second case: Equation with a quadratic term

Here, the nonlinear equation combines both quadratic and cubic stiffnesses:

$$\begin{aligned} m\ddot{x} + c\dot{x} + k_1x + k_2x^2 + k_3x^3 &= f(t) \\ \Rightarrow \ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2x + \alpha\omega_n^2x^2 + \gamma\omega_n^2x^3 &= F(t) \end{aligned} \quad (2.11)$$

where $\alpha = \frac{k_2}{k_1}$ and $\gamma = \frac{k_3}{k_1}$ and $F(t) = \frac{f(t)}{m}$. If damping and excitation force are not considered, Eq. (2.11) is reduced to:

$$\ddot{x} + \omega_n^2x + \alpha\omega_n^2x^2 + \gamma\omega_n^2x^3 = 0. \quad (2.12)$$

Eq. (2.12) is a nonlinear differential equation with both quadratic and cubic nonlinearities, also called a mixed-parity equation. An equation similar to Eq. (2.12) will be solved in other chapters while studying BPoEFs having nonlinear quadratic and cubic stiffness properties. Figure 2.7 displays the plots of the restoring force $f_r(x)$ for the cases above presented.

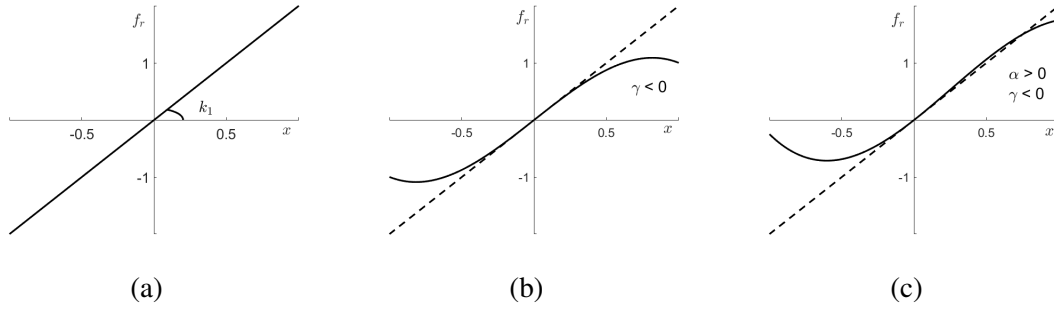


Figure 2.7: Plots of the restoring force: (a) linear, (b) nonlinear cubic, and (c) nonlinear quadratic-cubic.

In Figure 2.7a, the plot of a linear restoring force is shown whereas in Figures 2.7b and 2.7c those of a nonlinear restoring force are shown. These plots were obtained for $k_1 = 2$, $k_2 = 0.75$, and $k_3 = 1$. Therefore, for different values of these stiffnesses, the plots in Figure 2.7 could be modified.

2.5.2 Backlash

A backlash is a clearance or lost motion in a mechanism caused by gaps between parts of a system. Because nearly all mechanical systems are susceptible to backlash, system performance may significantly deteriorate and may even become unstable if backlash is ignored throughout the control design process. This type of nonlinearity is particular because the input moves while the output does not. Figure 2.8 presents an example of a system with backlash.

The elements constituting the figure are a mass m ; two springs with a stiffness coefficient k ; and two dampers with a damping coefficient c . The mass can move horizontally, and x describes its displacement, and the gap between parts is x_0 . For this type of nonlinearity, the equation of motion of the SDOF system presented in Figure 2.8 is given by

$$\begin{aligned} \ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2(x - x_0) &= f(t), & |x| > x_0 \\ \ddot{x} &= f(t), & |x| < x_0 \end{aligned} \quad (2.13)$$

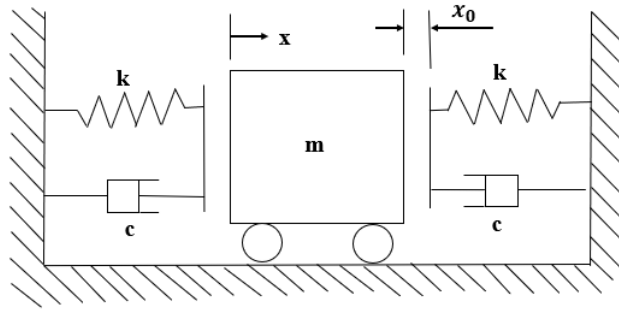


Figure 2.8: System with backlash

This equation is nonlinear, with the nonlinearity due to the stiffness because the displacement is not linear and takes values according to its sign. Figure 2.9 shows a plot example of a restoring force of a system with backlash.

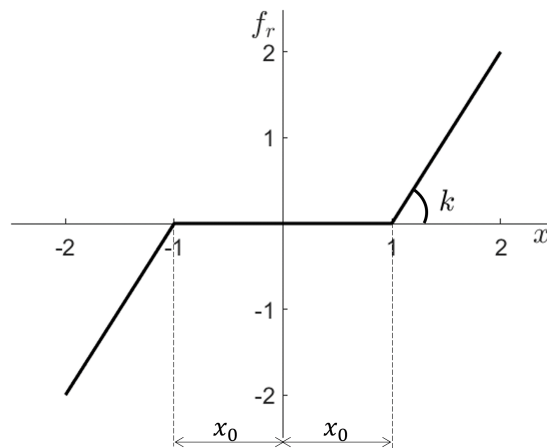


Figure 2.9: Plot showing a relationship between displacement x and restoring force f_r for backlash.

Nonlinearity must first be identified depending on the application and the response amplitude. Then, a suitable test is performed to classify it as backlash. For example, Burgess (1988) stated that in a system with backlash, decreasing the input force level or increasing the gap x_0 implies the increase of a nonlinear effect. That means one can easily vary the nonlinear effect while analysing a system with backlash. Moreover, if the load is removed in a system with backlash, it does not return to its starting position but needs a restoring force. Unfortunately, studying systems with backlash is infrequent and requires further investigations.

Backlash is mainly considered to have negative characteristics because it can give rise to uncertainty and inaccuracy in applications requiring precise positioning. However, it can be tolerated in some cases. However, the system's inaccuracy can allow unwanted motion and free play (unrestricted movement/activity), implying vibration noise and reducing the system's performance. Consequently, if the backlash is unwanted, it can be minimised using precision manufactured components or eliminated by preloading (Solon, 2010).

On the other hand, backlash created by clearances is beneficial because it can provide enough allowance for misalignment or clearance to pass contaminants. Then, suppose some backlash is tolerated in the design of a structure. In that case, the additional clearance can accommodate the changes and prevent both binding and the problems which can cause expansion and contraction. That means tolerating a small amount of backlash can be cost-effective by not sourcing more expensive and higher precision components.

2.5.3 Piecewise linear stiffness

The restoring force of a piecewise linear stiffness system can be given by the following (Tomlinson and Worden, 2000):

$$f_r(x) = \begin{cases} k_2x + (k_1 - k_2)a, & x > a \\ k_1x, & |x| < a \\ k_2x - (k_1 - k_2)a, & x < -a \end{cases} \quad (2.14)$$

Thompson et al. (1983) used the modelling of structures as piecewise-linear systems and investigated the harmonic and subharmonic responses using numerical integrations. The authors found that chaotic motions may be observed in impact oscillators when one stiffness on the system becomes infinity. However, it must be highlighted that the investigation of systems having the restoring force given by Eq. (2.14) is also rare and, therefore, needs more studies. An example plot of a system with a piecewise restoring force as given in Eq. (2.14) is shown in Figure 2.10.

Two kinds of nonlinear elastic force exist: the piecewise linear force and the material nonlinear force. Zhang and Huang (2010) studied the effects of nonlinear stiffness parameters on a vibration transfer path of a 5-DOF system. First, the authors discussed two cases of nonlinear elastic forces. Then, they compared the stiffness between the two forces regarding the sensitivity of the system. The finding showed that the sensitivity concerning the equivalent

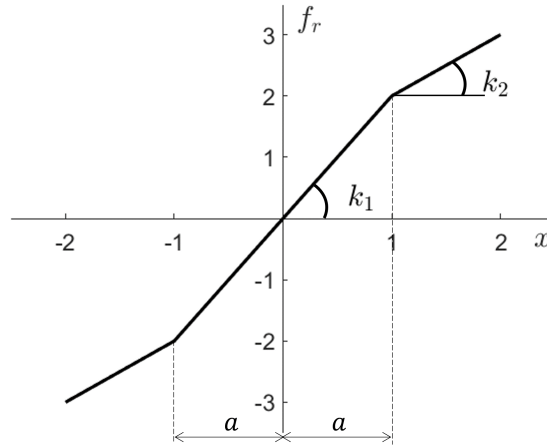


Figure 2.10: Plot showing a relationship between displacement x and restoring force f_r for piecewise linear stiffness.

linear stiffness of the piecewise linear stiffness is more extensive than that of the nonlinear material stiffness. The authors also found the results beneficial in efficiently decreasing noise and vibration transfer in vibration transfer path systems.

2.5.4 Bilinear Stiffness

Bilinear stiffness is one type of piecewise linear stiffness. It has discontinuities in the stiffness coefficients, meaning that the restoring forces are not smooth functions of the state variables. Because bilinear stiffness in the system causes an abrupt change in its stiffness, it is crucial to understand the dynamic response and vibration transmission of structures incorporating bilinear stiffness. That will avoid any misbehaviour of the system. Figure 2.11 shows an example of an SDOF system comprising bilinear stiffness.

Figure 2.11 has the same elements as in Figure 2.9, but there exist differences between the two systems. For example, Figure 2.11 does not have a gap like in Figure 2.9, and the stiffness coefficients of its springs are different.

Generally, a bilinear spring has two distinct stiffness values, k_1 and k_2 joined at the knee of the stiffness curve, denoted by the letter s . Then, depending on the sign of the displacement, the stiffness will turn to take one of the two different values k_1 and k_2 (Burgess, 1988). Shi et al. (2019) explored the dynamics and vibration transmission behaviour of dynamical systems with bilinear stiffness and bilinear damping elements. They used the harmonic

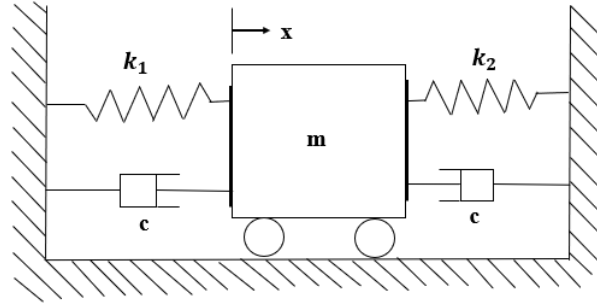


Figure 2.11: System with bilinear stiffness

balance method (HBM) and compared the results with those from numerical integration. It was shown that, when the bilinear stiffness is present in the system, the force transmission increases at high excitation frequencies for the single and coupled oscillator. However, the force transmission increases for the single and the coupled oscillator at less time-averaged input power and high time-averaged power transmission through the interface.

Studies exist on vibration analysis of systems incorporating bilinear stiffness. For example, Shaw and Holmes (1983) obtained the harmonic response of an SDOF oscillator with bilinear stiffness. They found that the system with bilinear stiffness can exhibit harmonic, subharmonic and chaotic motions, and a period-doubling bifurcation. Froio et al. (2018b) studied the dynamic response of beams on bilinear elastic foundations under an oscillating and concentrated moving load. They numerically investigated the effects of the foundation's stiffness and damping on the critical velocity of the moving load using the finite element method (FEM). Research on vibrating systems incorporating bilinear stiffness is also limited and therefore needs further studies and clarifications.

The motion of an SDOF system presented in Figure 2.9 is described by the following equations (Burgess, 1988):

$$\begin{aligned}\ddot{x} + 2\xi\omega_n\dot{x} + \omega_1^2x &= f(t), & x < 0 \\ \ddot{x} + 2\xi\omega_n\dot{x} + \omega_2^2x &= f(t), & x \geq 0\end{aligned}\quad (2.15)$$

where $\omega_1 = \sqrt{\frac{k_1}{m}}$ and $\omega_2 = \sqrt{\frac{k_2}{m}}$. The equivalent linear equation of Eq. (2.15) is given by

$$\ddot{x} + 2\xi\omega_n\dot{x} + \left(\frac{k_1 + k_2}{2m}\right)x = f(t) \quad (2.16)$$

For a system incorporating bilinear stiffness, the natural frequency considers the two stiffnesses corresponding to the positive and negative displacement. For example, considering the natural frequency $\frac{k_1+k_2}{2m}$, one can say that the two stiffnesses k_1 and k_2 will dominate the time history. This condition is particular and makes bilinear stiffness very important. The next section deals with methods to derive governing equations of models. Figure 2.12 shows a plot of a restoring force with bilinear stiffness.

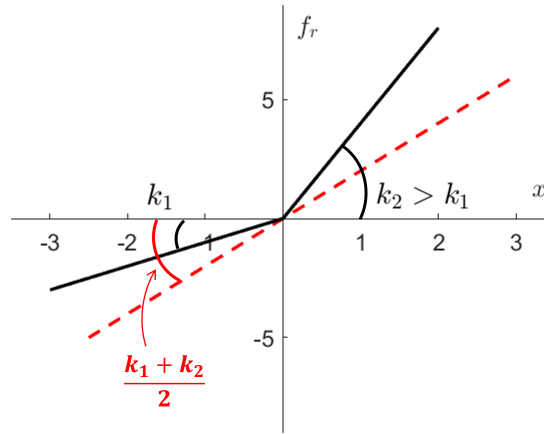


Figure 2.12: Plot showing a relationship between displacement x and restoring force f_r for bilinear stiffness.

2.6 Vibration Modelling of Systems

Modal analysis is a methodology for examining the dynamic behaviour of systems, structures, and buildings. The parameters' frequencies, damping values, and vibration modes describe the mechanical characteristics of structural systems. Therefore, the system's vibration interpretation can be facilitated based on measurements by knowing these parameters. Besides, modifying the structural system and formulating the requirements for retrofitting take these parameters into account (Czichos, 2013). Vibration modelling presents mathematical formulations of problems. This section highlights the methods of formulating nonlinear problems with their advantages and disadvantages.

2.6.1 Newton's Second Law of Motion

Regarding Newton's second law, the equations of motion of a vibrating system are derived using the following equation(s) (Wu, 2013):

$$\sum F_x = m\ddot{x} \quad \text{and/or} \quad \sum M_x = J_x\ddot{\theta}_x, \quad (2.17)$$

where F_x denotes the active and reactive forces and M_x denotes the bending or torsional moments in the x -direction and about the x -axis; m is the mass of the system, $\ddot{x}$ is the acceleration in the x -direction, while J_x and $\ddot{\theta}_x$ are the mass moment of inertia and the angular acceleration about the x -axis, respectively, where the over-dot represents the differentiation with respect to time t . The procedure for deriving the equation of motion of a system using Newton's second law requires first establishing the number of DOF and formulating all terms in only those variables. A system having MDOF has many masses. In that case, Eq. (2.17) should take a generalized form.

2.6.1.1 Advantages of Newton's Second Law

Newton's second law only considers the DOF, which are the variables fully describing the system's motion; the other variables remain in terms of the DOF. This process reduces the number of equations to solve and thus simplifies the problem. Moreover, using this method, one can easily find the relation between the motion and the force, and can calculate the acceleration. Therefore, the velocity and position can be determined knowing the force acting on the system. This helps the engineers map a given object's trajectories and position at a given time.

2.6.1.2 Disadvantages of Newton's Second Law

By applying Newton's second law, if the resulting equations of motion are nonlinear, neglecting nonlinear terms could affect the solution by giving an inadequate description of the motion. As a result, these solutions will be non-valid to the problem. Additionally, the method cannot be applied to problems where the mass varies according to the position but can be applied to a system where the mass varies according to time. In this specific case, one must apply the product rule to the rate of change of momentum. Newton's second law is inaccurate for problems with more than one variable. That is because it can be challenging to

write down all the forces when they point in various directions taking into account all the variables (Baruh, 2000).

2.6.1.3 Application of Newton's Second Law to a Simple Spring-Mass System

Many nonlinear dynamical systems in engineering can be described as an SDOF, the most straightforward representation of a dynamic system. Thus, given an MDOF system, one can write the resulting equations of motion in the form of an equation of an SDOF system. Then, the problem to solve becomes straightforward.

Consider an SDOF spring-mass system with a mass (m) and a stiffness (k). In addition, a damper (c) is also needed if the system is damped. The following figure presents an example of an undamped and unforced SDOF system.

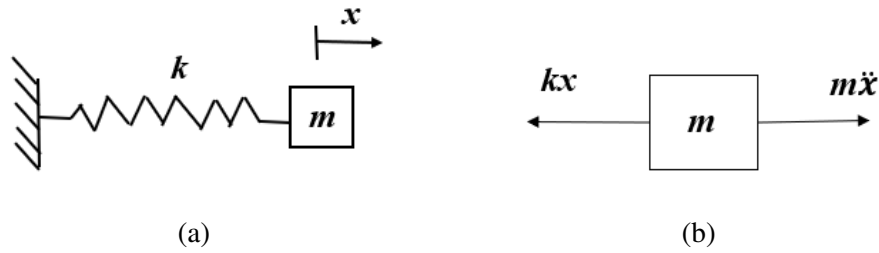


Figure 2.13: (a) Simple spring-mass system and (b) Free body diagram.

Using Newton's second law $\sum F_x = m\ddot{x}$, the equation of motion of the system presented in Figure 2.13 is given by:

$$-kx = m\ddot{x} \Rightarrow m\ddot{x} + kx = 0, \quad (2.18)$$

where m , k , and x are the mass, linear stiffness, and spring displacement. Eq. (2.18) is the linear equation of the simple spring-mass linear system. Nonetheless, when the spring extension becomes so great, or the response is not directly proportional to the action that produces it, Eq. (2.18) is no longer valid for the system. In this case, the spring exhibits a characteristic such that, for instance, the restoring force increases more rapidly than the displacement. So, the equation of motion becomes:

$$m\ddot{x} + k_n l x = 0, \quad (2.19)$$

where k_{nl} represents the nonlinear stiffness equivalent to the linear stiffness of the SDOF system in Figure 2.13. Here, the stiffness coefficient depends on the geometry and material properties of the system. These properties introduce nonlinearity into the system. For example, Eq. (2.19) can be written as:

$$m\ddot{x} + k_l x \pm k_3 x^3 = 0, \quad (2.20)$$

where x^3 represents the cubic nonlinearity. The system is nonlinear because the restoring force $k_l x \pm k_3 x^3$ is a combination of linear ($k_l x$) and nonlinear ($\pm k_3 x^3$) parts. One could also include a term with x^2 representing the quadratic nonlinearity. The equation of motion will become nonlinear with both quadratic and cubic nonlinearities. Suppose the sign in front of the nonlinearity is positive. In that case, it is a hardening characteristic, whereas the negative sign refers to the softening aspect.

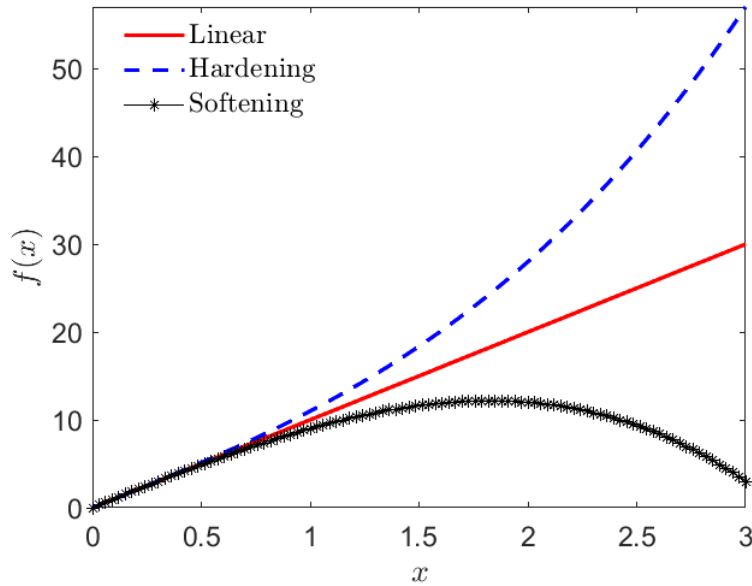


Figure 2.14: Plot of linear and nonlinear cubic curves of a simple spring-mass system.

Plotting on the same figure, the force-displacement characteristic for a linear and a nonlinear spring-mass system with the equation of motion in Eq. (2.20) gives Figure 2.14. The figure shows that the nonlinear curve bends upward for a hardening spring or downward for softening spring. In contrast, the linear curve is just a straight line. The curves were obtained in MATLAB for arbitrary dimensionless constants $k_l = 10$ and $k_3 = 1$.

2.6.2 The Energy Method

Based on the energy method, the equations of motion of a vibrating system are determined using the following equation (Tomlinson and Worden, 2000):

$$\frac{d}{dt}(T + V) = \dot{W}, \quad (2.21)$$

where T and V are the vibrating system's total kinetic and potential energies, respectively. On the other hand, $\dot{W}$ is the system's full power (that is, the work done by the external loads per unit of time). Applying this method requires determining the number of DOF, as in the case of Newton's second law, and then deriving the respective kinetic energy, potential energy, and power. Notwithstanding the linearity of a system, one can apply the energy method to every conservative system. A conservative system is where the sum of the kinetic and potential energies is a constant, and the force can be derived from the potential energy. That is not the case for non-conservative systems.

2.6.2.1 Advantages of the Energy Method

The energy method appears to be the most useful and powerful for analysing statically indeterminate structures and continuous systems with known mode shapes. The latter case is done by evaluating the kinetic and potential energies of the system. Moreover, the energy method can be applied when Newton's second law fails to derive the mathematical model of conservative systems.

2.6.2.2 Disadvantages of the Energy Method

The energy method does not define the velocity direction because it appears in the kinetic energy expression under the squared form. Also, it does not define the spontaneity of the process of deriving the equation of motion when stresses and displacements are not linear functions of the applied loads because in this case, the principle of superposition is no more applicable.

2.6.2.3 Application of the Energy Method to a Two-DOF System

Let us consider a Two-DOF (TDOF) system with the physical representation given in Figure 2.15, where the position of the mass m is fully defined by the elongation s and the angle θ .

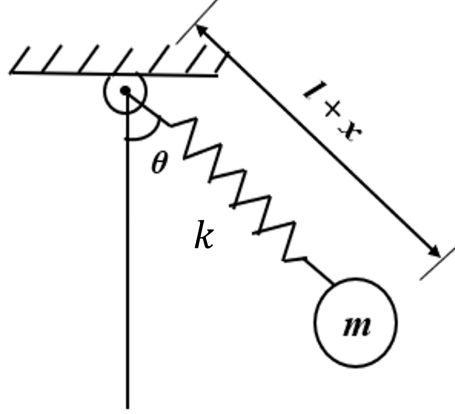


Figure 2.15: Oscillation of a swinging spring.

For both small angular oscillations and deformation of the spring, the system presented in Figure 2.15 is linear. The equation of motion of the TDOF displayed in Figure 2.15 is subsequently derived by applying the energy method.

The kinetic energy is given by:

$$T = \frac{1}{2}m[\dot{x}^2 + (l+x)^2\dot{\theta}^2], \quad (2.22)$$

whereas the potential energy is given by:

$$V = \frac{1}{2}kx^2 - mg(l+x)\cos\theta. \quad (2.23)$$

At equilibrium, it follows that:

$$kx_0 = mg \Rightarrow x_0 = \frac{mg}{k}. \quad (2.24)$$

Using the energy method implies:

$$\begin{aligned} \frac{d}{dt}(T+V) = \dot{W} \Rightarrow m\ddot{x} + m(l+x)\dot{\theta}^2 + m(l+x)^2\dot{\theta}\ddot{\theta} + kx\dot{x} - mg\dot{x}\cos\theta + \\ mg(l+x)\dot{\theta}\sin\theta = \dot{W}. \end{aligned} \quad (2.25)$$

By assuming that the mechanical energy is conserved, Eq. (2.25) is transformed into:

$$\ddot{x} + \frac{k}{m}x - (l+x)\dot{\theta}^2 - g \cos \theta = 0 \quad (2.26a)$$

and

$$\ddot{\theta} + \frac{g}{(l+x)} \sin \theta + \frac{2}{l+x} \dot{x}\dot{\theta} = 0 \quad (2.26b)$$

- Assuming that x and θ are small, Eqs. (2.26a) and (2.26b) give:

$$\ddot{x} + \omega_n^2 x = 0 \quad (2.27a)$$

$$\ddot{\theta} + \Omega_n^2 \theta = 0, \quad (2.27b)$$

where $\omega_n = \frac{k}{m}$ and $\Omega_n = \frac{g}{l+x_0}$. Eqs. (2.26a) and (2.26b) are linear equations of a simple spring-mass system and a simple pendulum respectively. Therefore, the solutions for Eq. (2.27a) and (2.27b) will be:

$$x(t) = x_0 \cos(\omega_n t + \varphi) \quad (2.28a)$$

$$\theta(t) = \theta_0 \cos(\Omega_n t + \psi) \quad (2.28b)$$

However, in general, these solutions do not predict the real behaviour of the system. So, one should choose the initial conditions wisely, which is not easy.

- Assuming x and θ are not small, Eqs. (2.26a) and (2.26b) are nonlinear differential equations. Then, their solutions can be found using nonlinear analysis or by first assuming the solution for each of the two generalised coordinates x and θ are replaced into the equations of motion to obtain the respective amplitudes and frequencies.

2.6.3 The Principle of Virtual Work

In 1717, Johann Bernoulli stated that the principle of virtual work is essentially a statement of the static or dynamic equilibrium of a mechanical engineering system (Capecchi, 2012). Denoted by δr , a virtual displacement is an infinitesimal and imaginary displacement without time passage. It also obeys the rules of differential calculus and takes place instantaneously; that is, no time is necessary to materialize it (Hashemi Kachapi and Ganji, 2014), meaning that $\delta t = 0$.

An applied system with N particles in a three-dimensional space is considered with the following Cartesian coordinates: $(x_1, y_1, z_1; x_2, y_2, z_2; \dots, x_N, y_N, z_N)$. The system is also subjected to k holonomic constraints (constraints in which the generalised coordinates are related by a constraint equation) $\psi_j(x_1, y_1, z_1; x_2, y_2, z_2; \dots, x_N, y_N, z_N, t) = 0$, $j = 1, 2, \dots, k$. Then, the condition for virtual displacements $\delta x_1, \delta y_1, \delta z_1$, etc. to be compatible with the constraints of the system is that the constraint equations must still be satisfied. This means that:

$$\psi_j(x_1 + \delta x_1, y_1 + \delta y_1, z_1 + \delta z_1, \dots, z_n + \delta z_n, t) = 0. \quad (2.29)$$

During the virtual displacement, the time t remains constant. By using Taylor's series around the original position to expand Eq. (2.29), and by neglecting the higher-order terms in $\delta x_1, \delta y_1, \delta z_1$ etc., Eq. (2.29) holds:

$$\psi_j(x_1, y_1, z_1, \dots, z_n, t) + \sum_{i=1}^N \left(\frac{\partial \psi_j}{\partial x_i} \delta x_i + \frac{\partial \psi_j}{\partial y_i} \delta y_i + \frac{\partial \psi_j}{\partial z_i} \delta z_i \right) = 0. \quad (2.30)$$

This implies:

$$\sum_{i=1}^N \left(\frac{\partial \psi_j}{\partial x_i} \delta x_i + \frac{\partial \psi_j}{\partial y_i} \delta y_i + \frac{\partial \psi_j}{\partial z_i} \delta z_i \right) = 0, \quad (2.31)$$

since the first term of Eq. (2.30) is zero. Note that there should be compatibility between the constraints and the condition for the virtual displacements.

Let $\bar{\mathbf{F}}_i$ be the force acting on particle i , which is subject to a virtual displacement, $\delta \bar{\mathbf{r}}_i$. Then, the virtual work of the system is

$$\delta W = \sum_{i=1}^N \bar{\mathbf{F}}_i \cdot \delta \bar{\mathbf{r}}_i. \quad (2.32)$$

Eq. (2.32) represents the virtual work performed by the resultant force vector $\bar{\mathbf{F}}_i$ over the virtual displacement vector $\delta \bar{\mathbf{r}}_i$ of particle i . When the system is in balance, the resultant force acting on each particle is zero. Hence, under balanced conditions, it follows that

$$\bar{\mathbf{F}}_i + \bar{\mathbf{R}}_i = 0, \quad (2.33)$$

where $\bar{\mathbf{R}}_i$ is the reaction force. Therefore, there is no virtual work done by all the forces moving through an arbitrary virtual displacement that is compatible with the constraints,

meaning that virtual work is zero. Hence,

$$\sum_{i=1}^N (\bar{\mathbf{F}}_i + \bar{\mathbf{R}}_i) \cdot \delta \bar{\mathbf{r}}_i = \sum_{i=1}^N \bar{\mathbf{F}}_i \cdot \delta \bar{\mathbf{r}}_i + \sum_{i=1}^N \bar{\mathbf{R}}_i \cdot \delta \bar{\mathbf{r}}_i. \quad (2.34)$$

In Eq. (2.34), the work done by the constraint forces is $\sum_{i=1}^N \bar{\mathbf{R}}_i \cdots \delta \bar{\mathbf{r}}_i$. During a virtual displacement, no work is done by many constraint forces since either they are equal or perpendicular to the displacement. Also, the opposite reaction forces cancel the work done.

2.6.3.1 Advantages of the Principle of Virtual Work

The equilibrium position of a system with applied known loads is easy to determine by applying virtual work. Moreover, the analysis of the system is done in whole and without reactive forces and works for internal frictional forces during any virtual displacement. As a result, the analysis of the system is simplified because the forces keeping the system from moving can be neglected.

2.6.3.2 Disadvantages of the Principle of Virtual Work

The principle of virtual work neglects the internal frictional forces, and the solution becomes inconsistent (Capecchi, 2012). Besides, the method is only powerful for static systems but not for dynamic systems because it does not consider inertial forces. On the other hand, the principle of virtual work only takes forces but considers torques less critical because these latter effects are actions applied at particular points/locations.

2.6.4 D'Alembert's Principle

D'Alembert's principle is an extension of the principle of virtual work to a form of dynamics by considering the inertia forces and the systems to be in dynamic equilibrium (Baruh, 2000). It states that the resultant of the external forces and the kinetic reaction acting on a body equals zero. As in the case of virtual work, let a system of N particles. It is assumed that a typical mass particle m_i in a system of particles ($i = 1, 2, \dots, N$) is acted upon by the applied force $\bar{\mathbf{F}}_i$ and the constraint force $\bar{\mathbf{R}}_i$.

Then the equation of motion using d'Alembert's principle can be written as:

$$\bar{\mathbf{F}}_i + \bar{\mathbf{R}}_i - m_i \ddot{\mathbf{r}}_i = 0, \quad i = 1, 2, \dots, N, \quad (2.35)$$

where $m_i \ddot{\mathbf{r}}_i$ is the inertia force. Eq. (2.35) means that the sum of all external and inertial forces acting on each system's particle is zero. This is known as d'Alembert's principle. By extending the principle of virtual work to a dynamic equilibrium state, the following equation can be written:

$$\delta W = \sum_{i=1}^N (\bar{\mathbf{F}}_i + \bar{\mathbf{R}}_i - m_i \ddot{\mathbf{r}}_i)_i \delta \bar{\mathbf{r}}_i = 0. \quad (2.36)$$

Eq. (2.36) combines the virtual work, the principle of statics and d'Alembert's principle and is often referred to as the generalised principle of d'Alembert (or the Lagrange version of d'Alembert's principle) (Hashemi Kachapi and Ganji, 2014). If the system has workless constraints and $\bar{\mathbf{r}}_i$ is chosen to be the reversible virtual displacements consistent with the constraints, it happens that:

$$\delta W = \sum_{i=1}^N (\bar{\mathbf{F}}_i - m_i \ddot{\mathbf{r}}_i)_i \cdots \delta \bar{\mathbf{r}}_i = 0. \quad (2.37)$$

In Eq. (2.37), the sum of the applied force $\bar{\mathbf{F}}_i$ and the inertia force, $m_i \ddot{\mathbf{r}}_i$ or $(\bar{\mathbf{F}}_i - m_i \ddot{\mathbf{r}}_i)$ is called the effective force acting on the particle m_i .

2.6.4.1 Advantages of d'Alembert's Principle

D'Alembert's principle is more practical and economical in problems with constraints. It considers moments not just at the centre of mass, but at any point, implying more straightforward calculations. By choosing an appropriate point of application of the moment, any force from the moment equation can be eliminated. It is a more general method that transforms a problem in dynamics into a problem in statics. It is also more suitable for non-holonomic constraints (cannot be written as an algebraic equation) systems.

2.6.4.2 Disadvantages of d'Alembert's Principle

D'Alembert's principle cannot be applied for irreversible displacements, for example, sliding friction. Moreover, the calculation of accelerations and derivatives of the angular momentum involves much algebra and transforms the problem into a tedious one. Furthermore,

d'Alembert's principle is not integral. In this case, the best method is that of Lagrange's equations (Harrison and Nettleton, 1997).

2.6.5 The Lagrange Equations

Lagrange's equations of motion for a dynamic system are derived using an application of d'Alembert's principle and the principle of virtual work. Considering a dynamic system of N particles as before, one can use d'Alembert's principle to derive the equation of motion (Baruh, 2000; Hashemi Kachapi and Ganji, 2014). In subsection 2.5.4, the following equation was obtained:

$$\sum_{j=1}^N \left(\vec{\mathbf{F}}_j - m_j \ddot{\vec{\mathbf{r}}}_j \right) \delta \vec{\mathbf{F}}_j = 0, \quad (2.38)$$

where $\vec{\mathbf{F}}_j$ is the exciting force on the i th particle of mass m_j . For an n DOF system, choosing $q_1, q_2, \dots, q_n$ as n generalized coordinates implies

$$\vec{\mathbf{r}}_j = \vec{\mathbf{r}}_j(q_1, q_2, \dots, q_n, t), \quad (2.39)$$

which is the transformation equation between the vector coordinates of the N particles and the n generalized coordinates. Based on Eq. (2.39), the velocities of the particles are derived as.

$$\dot{\vec{\mathbf{r}}}_j = \sum_{i=1}^n \frac{\partial \vec{\mathbf{r}}_j}{\partial q_i} \dot{q}_i + \frac{\partial \vec{\mathbf{r}}_j}{\partial t}. \quad (2.40)$$

On the other hand, the virtual displacement $\delta \vec{\mathbf{r}}_j$ in terms of the n generalized coordinates is:

$$\delta \vec{\mathbf{r}}_j = \sum_{i=1}^n \frac{\partial \vec{\mathbf{r}}_j}{\partial q_i} \delta q_i. \quad (2.41)$$

Then Eq. (2.38) becomes:

$$\sum_{j=1}^N \sum_{i=1}^n \left(\bar{\mathbf{F}}_j \frac{\partial \vec{\mathbf{r}}_j}{\partial q_i} - m_j \ddot{\vec{\mathbf{r}}}_j \frac{\partial \vec{\mathbf{r}}_j}{\partial q_i} \right) \delta q_i = 0, \quad (2.42a)$$

$$\text{where } \sum_{j=1}^N \bar{\mathbf{F}}_j \frac{\partial \vec{\mathbf{r}}_j}{\partial q_i} = Q_i. \quad (2.42b)$$

Eq. (2.42b) represents the generalised forces in the direction of the i th generalised coordinate and the second term is related to the kinetic energy of the system, the latter given by:

$$T = \frac{1}{2} \sum_{j=1}^N m_j \dot{\mathbf{r}}_j^2. \quad (2.43)$$

It follows that:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) = \sum_{j=1}^N m_j \ddot{\mathbf{r}}_j \frac{\partial \dot{\mathbf{r}}_j}{\partial \dot{q}_i} + \sum_{j=1}^N m_j \dot{\mathbf{r}}_j \frac{d}{dt} \left(\frac{\partial \dot{\mathbf{r}}_j}{\partial \dot{q}_i} \right). \quad (2.44)$$

Also,

$$\frac{\partial T}{\partial q_i} = \sum_{j=1}^N m_j \ddot{\mathbf{r}}_j \frac{\partial \dot{\mathbf{r}}_j}{\partial q_i}. \quad (2.45)$$

Taking into account that $\frac{\partial \ddot{\mathbf{r}}_j}{\partial \dot{q}_i} = \frac{\partial \ddot{\mathbf{r}}_j}{\partial q_i}$ and substituting Eq. (2.45) into Eq. (2.44) gives:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) = \sum_{j=1}^N m_j \ddot{\mathbf{r}}_j \frac{\partial \dot{\mathbf{r}}_j}{\partial q_i} + \frac{\partial T}{\partial q_i}. \quad (2.46)$$

By replacing Eq. (2.42b) and Eq. (2.46) into Eq. (2.42a), it follows that:

$$\sum_{i=1}^n \left[\frac{\partial T}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) + Q_i \right] \delta q_i = 0. \quad (2.47)$$

Since the generalized coordinates q_i are independents, Eq. (2.47) holds if the terms under the square bracket are equal to zero. That is:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} - Q_i = 0. \quad (2.48)$$

For a conservative system, the following equation is to be used:

$$Q_i = -\frac{\partial V}{\partial q_i}. \quad (2.49)$$

By putting Eq. (2.49) into Eq. (2.48), the equation of motion of a vibrating system is given by:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = 0, \quad (i = 1, 2, \dots, n) \quad (2.50)$$

In the case of a non-conservative system, one should use

$$Q_i = -\frac{\partial V}{\partial q_i} + Q'_i, \quad (2.51)$$

transforming Eq. (2.48) into:

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q'_i. \quad (2.52)$$

Note that the definitions of T and V are the same as those for Eq. (2.21). To use Lagrange's method, the first step is to determine the system's number of DOF and formulate all the energy terms in only those variables. The next step is to derive the equations of motion by using either Eq. (2.50) or Eq. (2.52) for a conservative or a non-conservative system, respectively. The method can also be used when Newton's second law is not applicable.

2.6.5.1 Advantages of Lagrange's Method

Lagrange's method provides a straightforward way to solve complicated problems (linear and nonlinear) as it can be applied to any coordinate system. There is no need to project forces (as is the case of Newton's second law) or velocities in the axes because the derivation of the equation of motion takes generalised coordinates (Baruh, 2000). Also, it can be used for both conservative and non-conservative systems and is most potent for continuous systems.

2.6.5.2 Disadvantages of Lagrange's Method

Nonlinear terms are sometimes neglected during the analysis, which produces an inconsistent solution to the problem. The method loses its validity when dealing with systems with non-conservative forces such as friction because of no potential energy. In this case, non-conservative forces may be incorporated into the equation by hand.

2.6.6 The Hamilton Principle

According to Hamilton's principle (HP), the equations of motion of a vibrating system and the associated boundary conditions are obtained from (Wu, 2013):

$$\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc}) dt = 0, \quad (2.53)$$

where δT and δV are the variations of T and V . Here, T and V are the same as those for Eqs. (2.21) and (2.50) if the system is conservative and for Eqs. (2.21) and (2.52) if the system is nonconservative. δW_{nc} is the virtual work done by the non-conservative loads, while t_1 and t_2 are instants of time t at which the varied and true paths for the deformations of the vibrating system intersect so that the variations of the associated parameters q_i ($i = 1; 2; \dots$) are equal to zero (that is, $dq_i = 0$ at $t = t_1$ and $t = t_2$).

2.6.6.1 Advantages of the Hamilton's Principle

The equations of motion derived from the HP are more straightforward to compute than those from Lagrange's equations. The principle does not need any linearisation of the problem, and the obtained approximate solution has high accuracy and can be used for frequency calculation. It is very suitable for a system with known time intervals.

2.6.6.2 Disadvantages of the Hamilton's Principle

The HP is restricted to conservative systems. Moreover, it does not fulfil the primary objective of a dynamical system to study the system's evolution with time. Therefore, assuming that the time in the beginning and the end of the time interval is known is not appropriate.

2.7 Continuous Systems with Nonlinear Stiffness

Natural frequencies for geometrically nonlinear systems vary according to vibration amplitude, whereas they are amplitude-invariant quantities in linear systems. Generally, the equation of motion of a one-dimensional nonlinear continuous damped system is assumed to be (Nayfeh and Mook, 1995):

$$\ddot{y}(x, t) + c\dot{y}(x, t) + L[y(x, t)] + N[\ddot{y}, \dot{y}, y] = 0, \quad (2.54)$$

where c is a damping factor, L and N are the linear and nonlinear operators on y , respectively. The solution of this equation requires initial and boundary conditions, the latter depending on the system's boundaries: hinged, clamped, fixed. However, if the system is linear, the system has an infinite number n of natural frequencies ω_n , and the equation of motion is reduced to:

$$\ddot{y}(x, t) + c\dot{y}(x, t) + L[y(x, t)] = 0. \quad (2.55)$$

The following subsections present some systems with nonlinear stiffness. The derivation of their equations of motion will be provided, and solutions will be found where possible.

2.7.1 Stretched String with a Concentrated Mass

It is assumed that the vibrating string is flexible and does not require any force to bend. Also, the force necessary to bend a string is characterised using stiffness. Young's modulus, which measures the stiffness of solid material, also determines the sufficient force to stretch the string. Gomez et al. (2007) qualitatively explored the stiffness effect on the wave velocity along a string. The authors showed that the effect of stiffness is small for a longer string. The authors also recalled that stiffness causes the frequency spectrum of a string not to be a harmonic series, reducing the degree of resonance along the string.

The system to be studied in this section is represented in Figure 2.16. It shows a pretensioned string at its initial position and the resulting deflection when a load P is applied to the mass.

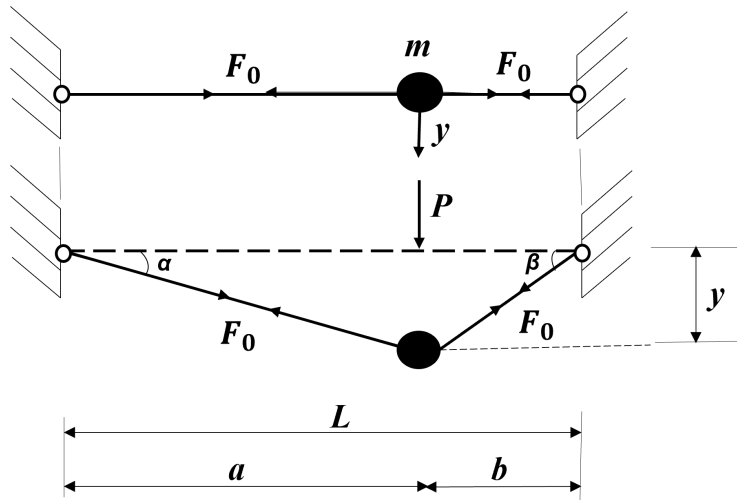


Figure 2.16: Vibration of a weighted string.

To derive the equation of motion of Figure 2.16, first, the load P is expressed as:

$$\begin{aligned}
 P &= F_0 \sin \alpha + F_0 \sin \beta = F_0 \left(\frac{y}{a} + \frac{y}{b} \right) \\
 &= F_0 \left(\frac{y}{a} + \frac{y}{b} \right) = F_0 \frac{a+b}{ab} y,
 \end{aligned} \tag{2.56}$$

where F_0 is the initial tension, y is the deflection and $k = F_0 \frac{a+b}{ab}$ is the linear stiffness. Eq. (2.56) is only valid for a small deformation. If the deflection is very large, the stiffness is no more linear, and there is another term added to the expression of k given by:

$$(AE - F_0) \left(\frac{a^3 + b^3}{2a^3b^3} \right), \quad (2.57)$$

where A is the cross-sectional area, and E is the elastic modulus of the string. Therefore, the restoring force of the nonlinear string becomes:

$$F_r = F_0 \frac{a+b}{ab} y + (AE - F_0) \left(\frac{a^3 + b^3}{2a^3b^3} \right) y^3. \quad (2.58)$$

The following approximately gives the equation of motion of the string:

$$m\ddot{y} + F_0 \left(\frac{L}{ab} \right) y + (AE - F_0) \left(\frac{a^3 + b^3}{2a^3b^3} \right) y^3 = 0. \quad (2.59)$$

Denoting the unstretched length of the half string by L_0 , for $a = b$, the initial tension and the restoring force become respectively:

$$F_0 = AE \left(\frac{a - L_0}{L_0} \right) \quad \text{and} \quad F_r \simeq AE \left[2 \left(\frac{a}{L_0} - 1 \right) \left(\frac{y}{a} \right) + \left(2 - \frac{a}{L_0} \right) \left(\frac{y}{a} \right)^3 \right]. \quad (2.60)$$

To draw the comparison between the linear stretched string to that of the nonlinear, three cases are distinguished: $\frac{a}{L_0} < 2$, $\frac{a}{L_0} = 2$, and $\frac{a}{L_0} > 2$. Supposing that $\frac{a}{L_0} = 1.5$ and $\frac{a}{L_0} = 2.5$ respectively for the cases $\frac{a}{L_0} < 2$ and $\frac{a}{L_0} > 2$, the following equations are obtained:

$$F_r = AE \left[\frac{y}{a} + 0.5 \left(\frac{y}{a} \right)^3 \right] \quad (2.61a)$$

$$F_r = AE \left[2 \frac{y}{a} \right] \quad (2.61b)$$

$$F_r = AE \left[3 \frac{y}{a} - 0.5 \left(\frac{y}{a} \right)^3 \right], \quad (2.61c)$$

where Eqs. (2.61a) and (2.61c) are nonlinear whereas Eq. (2.61b) is linear. Plotting F_r/AE versus y/a in MATLAB for these three equations, Figure 2.17 is obtained.

From Figure 2.17, it is easy to differentiate a linear restoring force plot from the nonlinear one (hardening and softening). For example, Dalir (2019) obtained the mode shapes and natural frequencies of a beam and affirmed that when the coefficient of the cubic

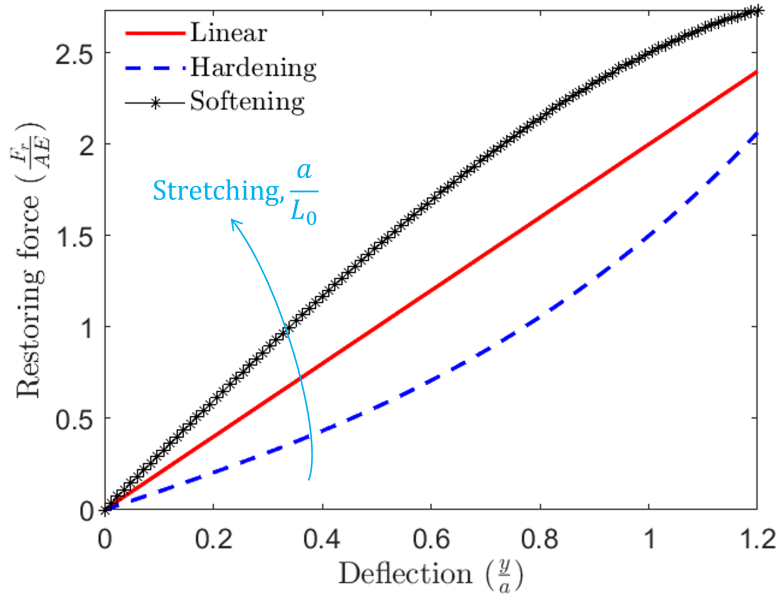


Figure 2.17: Plot of stiffness curves for a stretched string.

term is positive, increasing the amplitude of motion increases the driving forces. If it is negative, increasing the motion amplitude will decrease those forces. Thus, the natural frequency increases with the motion amplitude for nonlinear hardening systems. On the other hand, the natural frequency is inverted to the motion amplitude for nonlinear softening systems.

2.7.2 Transverse Vibration of Beams

Researchers are becoming more concerned with analysing mechanical structures such as beams, plates, and shells because of their broad application in industry. Beams have attracted much attention compared to other structural members due to their significant applications in civil, mechanical, aerospace, and marine engineering. For example, many civil structures such as bridges and space frames can be modelled as beams. The equation of motion of the beam is easy to derive, and the solutions are also easy to obtain. However, specific methods to find exact mathematical or analytical solutions to the vibration of beams with geometric nonlinearity are scarce in the literature. Therefore, recourse is made toward approximate and numerical solutions.

2.7.2.1 Equation of Motion of Simple Beams

A beam is a structure with one of its dimensions longer along the longitudinal axis than the dimensions along with other directions. It is also an elongated and straight rigid structural element that withstands applied loadings by bending or flexure as primary deformation (Zingoni et al., 2000). Beams are vital and are generally used in civil engineering structures for the construction of buildings and to bear loads. In those cases, they can be made of steel or metal, concrete or wood. In addition, they also serve as elements of the framework and support floors and walls in crossing structures such as bridges and viaducts (Billington et al., 2020). In this latter case, they are subjected to various stresses, which can be constant, variable or mobile. Therefore, the study of vibrations of beams is an active topic of interest. Moreover, when subjected to loads, their study allows the understanding of the behaviour of structures. The equation of motion of a simple beam in transverse vibration is derived using the system displayed in Figure 2.18.

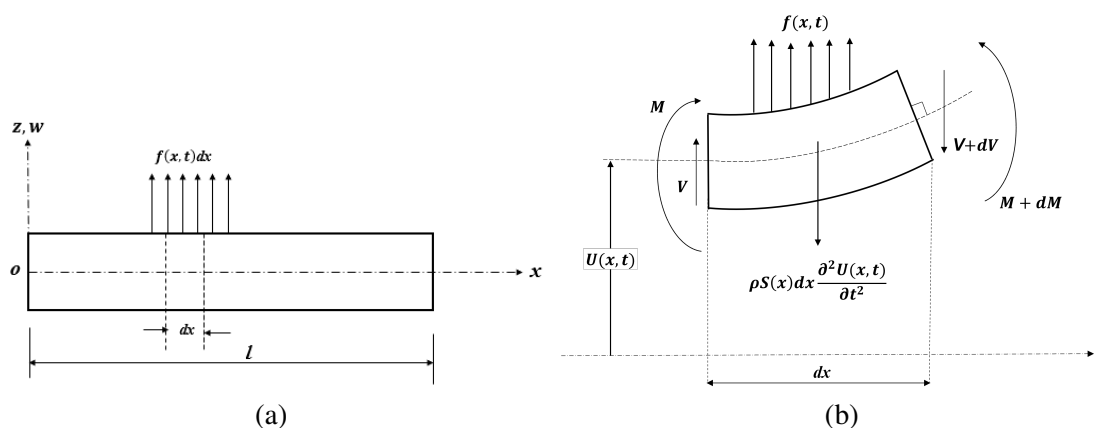


Figure 2.18: Transverse vibration of a thin beam (Rao, 2007): (a) an element of a beam with a transversal applied load and (b) a free-body diagram.

In Figure 2.18, the force $f(x, t)$ is the external transverse force per unit length of the beam, V is the shear force, and M is the bending moment. Rao (2007) studied the transverse vibration of beams like the one presented in Figure 2.18. The equation of motion was derived according to the Rayleigh (Rayleigh, 1896), Euler-Bernoulli (Timoshenko, 1983), and the Timoshenko (Timoshenko, 1921) beam theories. In the Rayleigh theory, only the rotary inertia effect is considered. In contrast, the Euler-Bernoulli beam theory (EBBT) neglects both the impact of the rotary inertia and the shear deformation of thin beams. It is important to note that one of the disadvantages of the Euler-Bernoulli beam theory is that because it does not account for the effects of shear, it does not always predict deflections. Therefore,

when the frequencies increase and tend to infinity, the prediction of phase velocities becomes unrealistic. As a result, the Timoshenko beam theory (TBT) has been proposed to produce finite phase velocities by considering both the effect of the rotary inertia and the shear deformation of beams in the derivation of the governing equation. For TBT, during bending or after deformation, the beam plane sections initially normal to the neutral axis of the beam may remain plane during bending but no longer normal to that axis (become curved) (Hutchinson, 2001). Rao (2007) found the exact solutions to relatively simple cases of continuous systems problems associated with free and forced vibrations. The accurate solutions allowed a valuable understanding of the system's behaviour by directly providing the natural frequencies and corresponding mode shapes. However, for a system extended to a two or three-dimensional problem, the exact solutions were challenging to find, and only approximate solutions were relied on.

For a transverse vibration of a thin beam like that presented in Figure 2.18, the inertial force is given by (Rao, 2007):

$$\rho S(x)dx \frac{\partial w}{\partial t^2}(x,t), \quad (2.62)$$

where $S(x)$ is the cross-sectional area of the beam at the abscissa x . On the one hand, the forces along the z direction are given by:

$$V - (V + dV) + f(x,t)dx = \rho S(x)dx \frac{\partial w}{\partial t^2}(x,t), \quad (2.63)$$

and the moments are given by:

$$(M + dM) - M - (V + dV)dx + f(x,t)dx \frac{dx}{2} = 0. \quad (2.64)$$

On the other hand, the differentiation of V and M concerning x are as follows:

$$dV = \frac{\partial V}{\partial x}dx \text{ and } dM = \frac{\partial M}{\partial x}dx. \quad (2.65)$$

Therefore, Eqs. (2.63) and (2.64) are transformed into:

$$-\frac{\partial V}{\partial x}dx + f(x,t)dx = \rho S(x)dx \frac{\partial w}{\partial t^2} \quad (2.66a)$$

$$\frac{\partial M}{\partial x}dx - Vdx + \frac{\partial V}{\partial x}dx dx + f(x,t)dx \frac{dx}{2} = 0 \Rightarrow \frac{\partial M}{\partial x} - V = 0. \quad (2.66b)$$

From Eqs. (2.66a) and (2.66b), the equation of the transverse vibration of simple beams is derived as:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho S \frac{\partial^2 w}{\partial t^2} = f(x, t). \quad (2.67)$$

For free vibration, the solution of Eq. (2.67) is obtained as

$$U(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right) \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] \quad \text{with} \quad \omega_n = \left(\frac{n\pi}{l}\right)^2 \sqrt{\frac{EI}{\rho S}}, \quad (2.68)$$

where A_n and B_n can be found knowing the system's initial conditions. Thus, the exact solution of the beam in Figure 2.18 is straightforward because the resulting equation of motion is linear. However, if the boundary conditions are assumed to be nonlinear, or by modifying the beam's material to that of a nonlinear one, the resulting equation of motion becomes nonlinear. In this case, Young's modulus E or the area moment of inertia of the cross-section of the beam I or both will no more be constant but vary as a function of square or any other power of x .

2.7.2.2 Equation of Motion of a Beam on an Elastic Foundation

Beams on elastic foundations (BoEFs), on the other hand, are models of interest for problems in various fields, such as railway engineering, aerospace, and physics (Wang et al., 2005). They have attracted a great deal of attention from engineers and practitioners due to their several applications in engineering in general and in particular civil and mechanical engineering (Hetényi, 1971; Kerr, 1984; Thambiratnam and Zhuge, 1996; Wang et al., 2005; Barber, 2011). BoEFs are used in engineering design such as automobile frames, bridge decks, pavements, pipelines, runways, and railroad tracks (Winkler, 1867; Hetényi and Hetbenyi, 1946; Froio and Rizzi, 2016; Froio et al., 2018a; Younesian et al., 2019). Because the foundation can be linear or nonlinear, the reaction produced by the foundation when the load is applied to the structure entirely depends on the foundation components. Therefore, the load, amplitude, and frequency type will help any engineer or designer design the foundation accordingly. We may also note that the axisymmetric bending of non-shallow shells of revolution in the vicinity of the shell edge may be approximated by an equation of the beam-on-elastic-foundation type (Zingoni, 1997).

In the research conducted concerning BoEFs, different foundation models were used such as Winkler, Pasternak, Vlasov, Filonenko-Borodich (Kerr, 1964). Thus, modelling the interaction between the beam and the foundation in each study is essential for analysing the

beam vibration. Among many applications of this type of structure, one can cite vibrating machines on elastic foundations, buildings, bridges, and railroad tracks (Hayir, 2010).

The dynamic response of structures resting on elastic foundations is commonly applied in engineering, and BoEFs have applications in structural mechanics. Some examples are roadways, rails, runways analysis, and design (Huang and Thambiratnam, 2002). Wang et al. (2005) presented a review of the critical areas of development, modelling of soil media and various analytical and numerical approaches in analysing structures on elastic foundations. To model a BoEF, the analysis only considers a tiny element of the vibrating BoEF (Rao, 2007). For example, a free-body diagram of such an element is displayed in Figure 2.19.

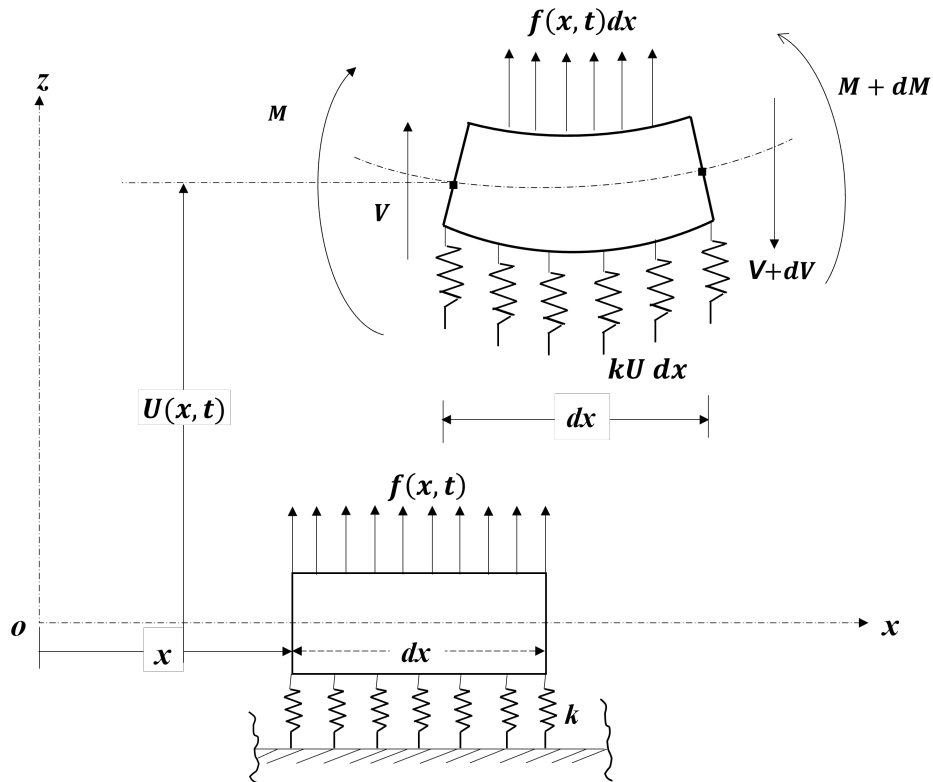


Figure 2.19: An element of a beam on an elastic foundation.

The equation of motion of an element of a beam resting on an elastic foundation is written by modifying Eq. (2.67) as follows:

$$EI \frac{\partial^4 U}{\partial x^4} + \rho S \frac{\partial^2 U}{\partial t^2} + kU = f(x, t), \quad (2.69)$$

where k is the spring stiffness of the foundation. To solve Eq. (2.69) for free vibration, $U(x, t)$ is assumed to be:

$$U(x, t) = \sum_{n=1}^{\infty} X_n(x) [A_n \cos(\omega_n t) + B_n \sin(\omega_n t)]. \quad (2.70)$$

Next, Eq. (2.70) is replaced into Eq. (2.69) and it gives:

$$\frac{d^4 X_n}{dx^4} - \left(\frac{\rho S}{EI} \omega_n^2 - \frac{k}{EI} \right) X_n = 0. \quad (2.71)$$

By setting:

$$\alpha_n^4 = \left(\frac{\rho S}{EI} \omega_n^2 - \frac{k}{EI} \right), \quad (2.72)$$

Eq. (2.71) becomes:

$$\frac{d^4 X_n}{dx^4} - \alpha_n^4 X_n = 0, \quad (2.73)$$

with the solution given by:

$$X_n(x) = C_1 \cos \alpha_n x + C_2 \cosh \alpha_n x + C_3 \sin \alpha_n x + C_4 \sinh \alpha_n x, \quad (2.74)$$

where C_1, C_2, C_3 , and C_4 are constants to be determined using the boundary conditions of the beam. From Eq. (2.72), the natural frequencies are deduced as:

$$\omega_n = \sqrt{\frac{EI}{\rho S} \left[\alpha_n^4 + \frac{k}{EI} \right]}. \quad (2.75)$$

The spatial solution given by Eq. (2.74) is valid only if the elastic foundation is linear. When the beam rests on a nonlinear elastic foundation, nonlinear terms are added into Eq. (2.69) according to the type of nonlinearity considered, for example, quadratic, cubic, and boundary conditions, transforming the expression of α_n and, thus, ω_n into a complex one.

According to Akour (2010), there exist different categories to classify BoEFs, among which the most important three are given as follows.

- The first category is the simplest representation and involves linear beams on linear elastic foundations (BoLEFs) (Winkler, 1867). Here, the foundation is made with linear elastic elements, and the modelling of the structure can be done quickly using linear analysis.
- The second includes linear beams on nonlinear elastic foundations (Beaufait and Hoadley, 1980; Mo et al., 2009; Alimoradzadeh et al., 2019). This case concerns foundations made with springs or other materials having nonlinear properties, whereas the beam has

linear properties. Typical examples of this type of material are soils since they are highly nonlinear.

- The third category involves nonlinear BoNEFs (Mahmoodi et al., 2004). For this kind of structure, the nonlinearities of the beams can be due to the material or the geometry of the beam itself. It should also be noted that nonlinear beams can experience relatively large deflections (Akour, 2010). Nevertheless, it is worth highlighting that studying models involving both a nonlinear beam and a nonlinear elastic foundation is time-consuming and requires extensive calculations. Consequently, those models are rare and not of interest in this thesis. Some examples of work related to the first and second categories are given in the subsequent paragraphs.

2.7.2.3 Some available Work on Beams on Elastic Foundations

BoLEFs have been extensively explored. For example, De Rosa and Lippiello (2021) used an analytical approach to analyse the vibrations of cracked Timoshenko beams resting on Winkler elastic foundations. It was found that the natural frequencies are highly dependent on the beam's length and boundary conditions, and they increased with the increase of the foundation parameter. More study models of BoLEFs can be found in (Lee, 1994; Pan and Atluri, 1995; Lee, 1998; Mo et al., 2009).

Concerning a linearly varying foundation, Froio and Rizzi (2016, 2017) presented beams lying on a Winkler's type foundation and analytically examined a specific linear variation of the foundation's stiffness coefficient. In addition, they derived the explicit closed-form analytical solution of the beam's static deflection in terms of polynomial functions. They also conducted parametric studies to accurately investigate the influence of the beam-foundation mechanical parameters on the resulting elastic bending structural response.

Problems involving BoNEFs are often challenging to solve due to the intermediate supports added to the beam. However, these added supports allow the structure to resist more deformations when subjected to dynamic loads than beams with linear or without foundations. When confronted with those problems, the most convenient way is to treat the issues nonlinearly. That is because linearising them can lead to significant relative differences during the modelling of the system. Moreover, it can present a wrong interpretation of the real behaviour of the structure. Therefore, the interest is turned towards structures with foundations having nonlinear stiffness characteristics because they generally find wide applications in many areas of engineering such as mechanical and civil engineering, particularly in foundation

analysis and design (Santee and Gonçalves, 2006). Some examples of BoNEFs are columns and pipes supported by soils, vibrating machines on elastic foundations network of beams in the construction of floor systems for ships, buildings and bridges, submerged floating tunnels, buried pipelines, and railroads tracks (Pellicano and Mastroddi, 1997; Akour, 2010; Younesian et al., 2012).

BoNEFs have also attracted some attention. For instance, Nayfeh (1995), Pellicano and Mastroddi (1997), Qaisi (1998), and Santee and Gonçalves (2006) all studied the cases of BoNEFs with cubic stiffness nonlinearity. The differences between these works were related to external excitation and the method used during the analysis. However, the results obtained from these studies were similar. Similar work was presented by Mahmoodi et al. (2004). Models for constructing nonlinear natural frequencies and mode shapes of a one-dimensional continuous system with cubic nonlinearities were investigated. The authors studied two cases: a hinged-hinged beam resting on a nonlinear viscous foundation, and a hinged-hinged geometrically nonlinear beam resting on a nonlinear viscous foundation. The mode shapes and natural frequencies of the system were derived. It was found that the presence of only nonlinear viscous damping in the system cannot transform the natural frequencies to be nonlinear. However, the presence of a stiffness nonlinearity made it possible.

Nayfeh and Mook (1995) studied a simply supported Euler-Bernoulli BoNEF having quadratic and cubic nonlinearities. The resulting temporal equation of this type of model is known as a mixed-parity equation (Elías-Zúñiga, 2012). Nayfeh and Mook (1995) obtained natural frequencies and the corresponding mode shapes of the system using two approaches: the first approach consisted of using the MTSM. The second was to use the Galerkin method to discretise the equation and proceed with the MTSM and the invariant manifold approach in order to solve it. Some years later, Qaisi (2003) obtained the solutions for the free vibration of the same system studied by Nayfeh and Mook (1995). Assuming the time dependence to be harmonic, the problem was solved as a nonlinear eigenvalue problem. The results revealed that the cubic nonlinearity significantly affects both the lower frequencies and the first mode shape for a first-order approximation. In contrast, the second and third nonlinear modes remain unchanged, and the vibration frequencies increased due to nonlinearity, despite the constant fundamental frequency.

On the other hand, Younesian et al. (2012) also studied the oscillations of BoNEFs with cubic stiffness nonlinearity. The authors obtained analytical solutions to the governing equation of the system and the natural frequencies for the response of the beam using the variational iteration method (VIM). The relative differences were significant when comparing

the results to the equivalent linear system, confirming that the linear analysis could not handle the nonlinear system. The results presented by Younesian et al. (2012) are applicable to railway track systems with strong nonlinearity.

Some available researches involving BoEFs having nonlinear stiffness properties consider an external excitation. For example, Mamandi et al. (2012) used the nonlinear normal modes (NNMs) to analyse a slender, simply supported beam subjected to an axial load and resting on a nonlinear elastic foundation. The frequencies and the corresponding mode shapes were extracted using perturbation methods. Also, a parametric study evaluating some model parameters concerning the nonlinear response of the beam was executed. The results showed that the nonlinear frequencies increased and implied higher vibration amplitude when the nonlinear stiffness coefficient increased. Nevertheless, it is important to highlight that in some cases, increasing the nonlinear stiffness coefficient may decrease the vibration amplitude of the system.

Ghannadiasl and Khodapanah (2018) explored the dynamic analysis of the Euler-Bernoulli cracked BoEF subjected to the concentrated load. It was found that the foundation's stiffness affects the beam vibration characteristic. Froio et al. (2018a,b) also studied the dynamic response of beams on bilinear elastic foundations under an oscillating and concentrated moving load in one article and derived the complete analytical solution of the steady-state response of a two-dimensional uniform infinite Euler-Bernoulli beam supported on a Pasternak elastic foundation and subjected to a constant moving load in the other. In addition, they numerically investigated the effects of the stiffness of the foundation and damping on the critical velocity of the moving load using the FEM approach. The modified multi-level residue HBM was used by Rahman et al. (2019) to solve the nonlinear vibration problem of a BoNEF and subjected to an applied load. It was found that the excitation parameters influence the beam's nonlinear vibration.

The nonlinear elastic foundation model with linear plus cubic stiffness has been widely recognised as one of the most reliable and convenient modelling for the dynamic analysis of railway tracks. As an application, Jorge et al. (2015) investigated the dynamic response of BoNEFs subjected to moving loads. The velocity of the load was constant throughout the study. The authors found that the presence of nonlinearity in the foundation lowered the amplitude of the beam's displacement. Besides, as the nonlinear part of the foundation increased, the displacement decreased. Apart from that, high displacements resulted from the increased applied load. Therefore, it could be possible to reduce the amplitude of vibration of the beam due to the applied load by increasing the number of springs of the

foundation. However, this can considerably increase the computational time and requires more processors.

Froio et al. (2016, 2017) conducted similar research to that of Jorge et al. (2015). In both articles, the authors investigated the dynamic response of a simply supported Euler-Bernoulli beam on a nonlinear cubic elastic foundation. The beam was subjected to a harmonic-varying moving load with a constant velocity. The authors examined the critical velocity dependency on the harmonic load's mean value. It was proved that the critical velocity value and number both depend on the characteristic parameters of the moving load. Numerical analysis was performed using FEM to characterise the beam's critical velocity–frequency curves for a zero and a nonzero mean value of the harmonic moving load amplitude. It was found that the foundation's subgrade reaction depends on the beam's displacements according to a nonlinear cubic constitutive law. The authors highlighted how the harmonic moving load's frequency and velocity influence the beam-foundation system's response. Practical implications emerged from their results, especially in lowering the speed ranges for trains to meet structural requirements or reduce passenger discomfort. Comparable work to that by Froio et al. (2016, 2017) was done by Ibrayev (2019), where the response of beams resting on linear and nonlinear elastic foundations and subjected to a harmonic moving load was explored. The frequencies and the mode shapes were extracted using numerical simulations. It was found that increasing the stiffness and the shear parameter implies an increase in the frequencies of the system. In the nonlinear case studied, the frequencies of vibration of the beam increased due to nonlinear terms in the equation of motion.

Recently, Alimoradzadeh et al. (2021) studied the free vibration and thermal buckling of a beam resting on a nonlinear elastic foundation. A year later, Alimoradzadeh and Akbas (2022) investigated the nonlinear dynamic responses of a functionally graded composite beam resting on the nonlinear viscoelastic foundation subjected to moving mass with the temperature rising. Euler-Bernoulli beam theory and the HP were used to derive the equation of motion. A parametric study highlighted some parameters' effects on the dynamic response of the beam.

It is beyond the scope of this thesis to present an exhaustive literature review. For more information on BoNEFs, reference can be made to Pellicano and Mastroddi (1997); Chien and Chen (2005); Santee and Gonçalves (2006); Malekzadeh and Setoodeh (2007); Esfahani et al. (2013); Yaghoobi and Torabi (2013); Soares and Gonçalves (2018); Mahmoudpour et al. (2018); Mohamed et al. (2018).

2.7.3 Transverse Vibration of Plates

A plate is a three-dimensional solid having one of its dimensions smaller than the two others. That smallest dimension is the thickness, separating the top layer from the bottom layer. There exist two types of plates: thin and thick plates. The former refers to the case where a vertical straight line normal to the middle plane of the plate remains normal even after the deformation of the plate. In this case, the thickness of the plate is minimal compared to the other two dimensions (length and width). As for a thick plate, a vertical straight line passing through the middle of the plate and perpendicular to the middle plane of the plate changes after the plate's deformation (Shwetha and Subrahmanya, 2018). The vibration of plates is important in studying practical systems such as bridges, decks, hydraulic structures, pavements of highways and airport runways, aeroplanes, and machine parts (Hetényi and Hetbenyi, 1946; Younesian et al., 2019). All displacements on the elastic surface define a transversal vibration of a plate. Each point, if not restrained, can experience is animated by a vertical movement to the equilibrium position of the curvature surface.

2.7.3.1 Equation of Motion of a Simple Rectangular Plate

A schematic characterisation of a simple plate is given in Figure 2.20 inspired by Soni et al. (2019).

Figure 2.20 exhibits a rectangular plate on which are represented the moment and the shear forces resultants. M_x and M_y stand for the bending moments along the x and y axes, respectively, while M_{xy} or M_{yx} is the twisting moment. On the other hand, V_x , V_y and V_{xy} or V_{yx} are the shear forces along the x and y directions and twisting shear, respectively. An external force $f(x, y, t)$ is also applied on the plate. In what follows, each quantity will be defined accordingly. To establish the upcoming equation of motion of a simple plate, the following references are used: Timoshenko et al. (1959), Clough (1993), Nayfeh and Mook (1995), Zingoni et al. (2000), Rao (2007), and Logan (2011).

From Hooke's law, one can write:

$$\sigma = E\varepsilon \text{ implying } \begin{cases} \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \\ \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \end{cases} \quad (2.76)$$

From Eq. (2.76), σ is the stress with σ_x and σ_y its components along the x and y directions respectively; ε is the strain with ε_x and ε_y its components along the x and y directions

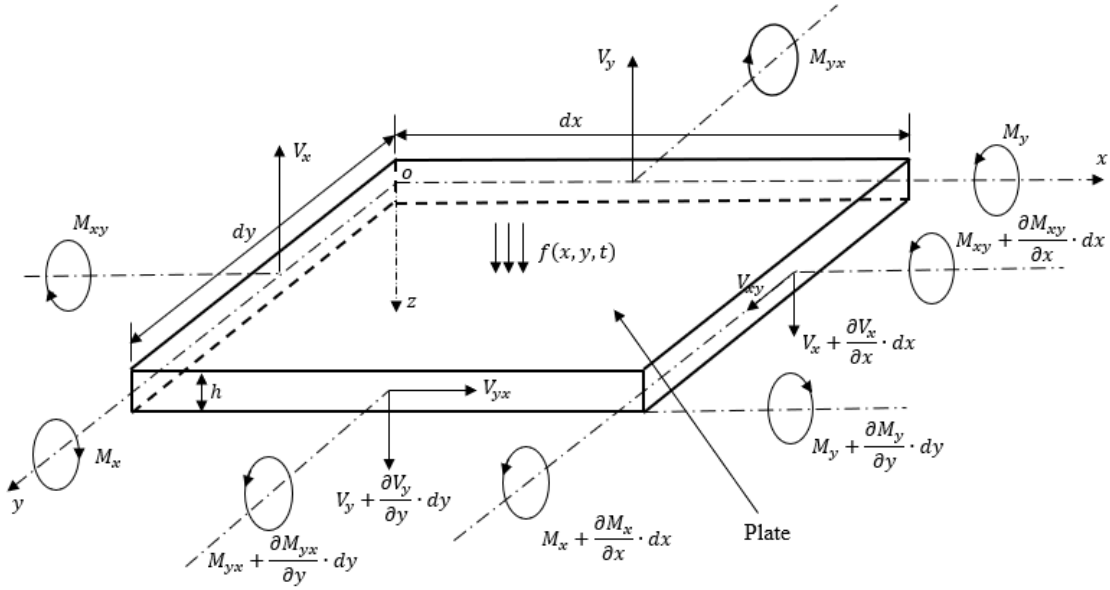


Figure 2.20: An element of the plate with bending moments and shear forces represented.

respectively; E is Young's modulus and ν is Poisson's ratio. Note that $\sigma = \epsilon E$ is only valid for uni-axial conditions. Eq. (2.76) allows deducing the following expressions for the stress:

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y); \quad \sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \quad (2.77)$$

On the other hand, the strain can be expressed as a function of the curvature as:

$$\epsilon_x = \frac{z}{C_x}; \quad \epsilon_y = \frac{z}{C_y} \quad (2.78)$$

where $\frac{1}{C_x}$ and $\frac{1}{C_y}$ stand respectively for the curvatures along the x and y directions. By replacing Eq. (2.78) into Eq. (2.77), it happens that:

$$\sigma_x = \frac{zE}{1-\nu^2} \left(\frac{1}{C_x} + \nu \frac{1}{C_y} \right); \quad \sigma_y = \frac{zE}{1-\nu^2} \left(\frac{1}{C_y} + \nu \frac{1}{C_x} \right). \quad (2.79)$$

Knowing that the moment is equal to the force times the distance, the following equation holds:

$$dM = z dF \quad (2.80)$$

By integrating both sides of Eq. (2.80), the moment acting over the cross-sectional area S of the plate is obtained as:

$$M_x = \int_S z dF \quad \text{and} \quad M_y = \int_S z dF \quad (2.81)$$

Considering that the stress is the force per unit area, the following equations hold:

$$M_x = \int_S z(\sigma_x dS), \quad \text{with } dS = dydz \quad \text{and} \quad M_y = \int_S z(\sigma_y dS), \quad \text{with } dS = dx dz, \quad (2.82)$$

with dS an element of the area. By replacing σ_x and σ_y by their expressions given in Eq. (2.79), Eq. (2.82) becomes:

$$M_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{z^2 E}{1 - \nu^2} \left(\frac{1}{C_x} + \nu \frac{1}{C_y} \right) dx dz, \quad \text{and} \quad M_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{z^2 E}{1 - \nu^2} \left(\frac{1}{C_y} + \nu \frac{1}{C_x} \right) dx dz, \quad (2.83)$$

where h is the plate's thickness. For the sake of simplification, the following is set:

$$D = \frac{E}{1 - \nu^2} \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2 dz = \frac{E h^3}{12(1 - \nu^2)}. \quad (2.84)$$

D is known as the plate's flexural rigidity. For beams, $D = EI$. Also, the curvature of the beam is expressed as the displacement of the beam as given by the following equations:

$$\frac{1}{C_x} = -\frac{\partial^2 W}{\partial x^2} \quad \text{and} \quad \frac{1}{C_y} = -\frac{\partial^2 W}{\partial y^2} \quad (2.85)$$

Taking Eqs. (2.83) to (2.85) into account, the expressions of the moments become:

$$M_x = -D \left(\frac{\partial^2 W}{\partial x^2} + \nu \frac{\partial^2 W}{\partial y^2} \right) \quad \text{and} \quad M_y = -D \left(\frac{\partial^2 W}{\partial y^2} + \nu \frac{\partial^2 W}{\partial x^2} \right) \quad (2.86)$$

Considering the left face of the plate in Figure 2.17, V_x , M_x , and M_{xy} remain as is. At the right face of the plate, these quantities are transformed into:

$$V_x + \frac{\partial V_x}{\partial x} dx, \quad M_x + \frac{\partial M_x}{\partial x} dx, \quad \text{and} \quad M_{xy} + \frac{\partial M_{xy}}{\partial x} dx. \quad (2.87)$$

On the other hand, the back and the front faces of the plate in Figure 2.17, respectively give:

$$V_y, \quad M_y, \quad \text{and} \quad M_{yx}, \quad \text{and} \quad (2.88)$$

$$V_y + \frac{\partial V_y}{\partial y} dy, M_y + \frac{\partial M_y}{\partial y} dy, \text{ and } M_{yx} + \frac{\partial M_{yx}}{\partial x} dy \quad (2.89)$$

Therefore, applying the equilibrium of forces, the following equation is obtained:

$$\left(V_x + \frac{\partial V_x}{\partial x} dx \right) dy - V_x dy + \left(V_y + \frac{\partial V_y}{\partial y} dy \right) dx - V_y dx = 0, \quad (2.90)$$

$$\text{implying } \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0. \quad (2.91)$$

Applying the sum of the forces equal to zero along the x -direction gives:

$$-\left(M_y + \frac{\partial M_y}{\partial y} dy \right) dx + M_y dy + V_y dx dy + \left(M_{xy} + \frac{\partial M_{xy}}{\partial x} dx \right) dy - M_{xy} dy = 0. \quad (2.92)$$

Eq. (2.92) can also be simplified into:

$$\frac{\partial M_{xy}}{\partial x} - \frac{\partial M_y}{\partial y} + V_y = 0. \quad (2.93)$$

Similarly, applying the sum of the forces equal to zero along the y -direction, it gives:

$$\left(M_x + \frac{\partial M_x}{\partial x} dx \right) dy - M_x dy - V_x dx dy + \left(M_{yx} + \frac{\partial M_{yx}}{\partial y} dy \right) dx - M_{yx} dx = 0. \quad (2.94)$$

Eq. (2.94) can also be rewritten as:

$$\frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x} - V_x = 0. \quad (2.95)$$

To proceed, V_y and V_x are extracted from Eqs. (2.93) and (2.95). This gives respectively:

$$V_y = \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} \text{ and } V_x = \frac{\partial M_{yx}}{\partial y} + \frac{\partial M_x}{\partial x}. \quad (2.96)$$

By replacing Eq. (2.96) into Eq. (2.91) and because $M_{xy} = -M_{yx}$, it gives:

$$\frac{\partial^2 M_x}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = 0. \quad (2.97)$$

Before proceeding, the relations between bending moments and shear forces are recalled to be:

$$M_{xy} = - \int_{-\frac{h}{2}}^{\frac{h}{2}} V_{xy} z dz \text{ and } M_{yx} = - \int_{-\frac{h}{2}}^{\frac{h}{2}} V_{yx} z dz. \quad (2.98)$$

That implies

$$M_{xy} = -M_{yx} = D(1 - \nu) \frac{\partial^2 w}{\partial x \partial y}. \quad (2.99)$$

When Eqs. (2.86) and (2.99) are put into Eq. (2.97) the governing equation of a simple rectangular thin plate is derived as

$$D \left(\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) = 0. \quad (2.100)$$

Eq. (2.100) which is often referred to as the *biharmonic equation of plate bending* (Timoshenko et al., 1959; Zingoni et al., 2000), is valid only for the static case study. Because time is considered, the study case is dynamic. Therefore, inertial forces must be considered, transforming Eq. (2.100) into:

$$\rho h \frac{\partial^2 W}{\partial t^2} + D \left(\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) = 0. \quad (2.101)$$

Eq. (2.101) is the general governing equation of motion of a simple plate in movement where $W = W(x, y, t)$ represents the transversal displacement, or deflection of the plate.

2.7.3.2 Equation of a Rectangular Plate on an Elastic Foundation

As with beams, there are problems involving plates on elastic foundations (PoEFs). Some applications include foundation slabs of heavy machines and buildings and reinforced concrete pavements of highways and airport runways. By assuming the reaction force of the foundation to be proportional to the deflection, Eq. (2.101) is rewritten as:

$$\rho h \frac{\partial^2 W}{\partial t^2} + D \left(\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) + kW = 0, \quad (2.102)$$

where k represents the stiffness of the elastic foundation. Eq. (2.102) is the governing equation of a plate resting on a linear elastic foundation (PoLEF). The solution of Eq. (2.102) is assumed to be:

$$W(x, y, t) = W(x, y) \exp^{i\omega t}. \quad (2.103)$$

By replacing Eq. (2.103) into Eq. (2.102), it gives:

$$\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} - \frac{1}{D}(\rho h \omega^2 - k)W = 0, \quad (2.104)$$

which can be written as:

$$\nabla^4 W - \lambda^4 W = 0, \text{ with } \lambda^4 = \frac{1}{D}(\rho h \omega^2 - k). \quad (2.105)$$

For a simply supported plate, the natural frequencies can be deduced from Eq. (2.105) as:

$$\omega_{mn} = \left\{ \frac{D\pi^2}{4\rho h} \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right]^2 + \frac{k}{4\pi^2 \rho h} \right\}^{\frac{1}{2}}, \quad (2.106a)$$

where a and b are the length and width of the plate, respectively, and the plate displacement is given by:

$$W(x, y, t) = W_{mn}(x, y) [A_{mn} \cos(\omega_{mn} t) + B_{mn} \sin(\omega_{mn} t)], \quad (2.106b)$$

$$\text{with } W_{mn}(x, y) = \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right). \quad (2.106c)$$

A_{mn} and B_{mn} are amplitude coefficients which can be determined using the initial conditions of the problem, and m and n are integers representing the mode's number.

The solution of Eq. (2.102) given by Eq. (2.106b) is only valid for a linear elastic foundation system. However, the term characterising nonlinearity is added to Eq. (2.102) for a nonlinear elastic foundation. Consequently, the equation becomes nonlinear and is given by:

$$\rho h \frac{\partial^2 W}{\partial t^2} + D \left(\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) + k_{nl} W = f(x, y, t), \quad (2.107)$$

where k_{nl} is the nonlinear stiffness coefficient of the foundation. This last equation will be solved later when studying plates on nonlinear elastic foundations (PoNEFs).

2.7.3.3 Some available Work on Plates on Elastic Foundations

Due to various applications of PoEFs, authors have been interested in studying systems formed by such models. For example, the effect of elastic foundations on the mode shapes instability and vibration problems of simply supported rectangular plates was explored by Raju and Rao (1990). The results issued from their work could help understand the dynamic stability of PoEFs. Idowu et al. (2017) achieved the vibrations of a damped orthotropic simply supported rectangular plate resting on the Winkler foundation and subjected to dynamic loads. It was shown that damping affects the vibration of solid structures significantly as there is a substantial dependence between the damping ratio and the deflection profile.

On the other hand, Yang and Shen (2001) investigated the vibration analysis of an initially stressed functionally graded rectangular thin PoEF with clamped boundary conditions. They showed that the dynamic responses and vibration characteristics of the plate are influenced by foundation stiffness, shape and duration of the impulsive load, and initial membrane stress. Some years later, the differential quadrature method (DQM) was used by Malekzadeh and Karami (2004) to study the free vibration analysis of homogeneous linearly varying thick PoEF. They showed the foundation's effect on the nonlinear static and dynamic responses of the plate. Moreover, it was concluded that DQM is an alternative numerical tool for structural analysis.

Chien and Chen (2005, 2006), and Chen et al. (2009) studied the nonlinear vibration of laminated plates on a nonlinear elastic foundation. The Galerkin method was used to derive the equation of motion, which was solved numerically using the Runge–Kutta method. It was found that the initial stresses, foundation stiffness, modulus ratio, and vibration amplitude all affect the plate's frequency responses. The free Vibrations of thin rectangular plates with mixed or fully clamped boundary conditions on Winkler foundations were studied by Motaghian et al. (2012). The governing differential equations for plates were solved to find closed-form solutions.

Furthermore, a new mathematical approach was developed by Motaghian et al. (2012) to determine the exact analytical solution of the free vibration of the plate. The results were presented through parametric and verification studies. The proposed mathematical method could successfully obtain exact solutions to free vibration problems of plates on partial elastic foundations with various boundary conditions. Similar work to that by Chien and Chen (2005, 2006) was conducted by Gao et al. (2017) on the stability and dynamic characteristics of a composite plate resting on a Winkler-Pasternak elastic foundation. The fourth-order Runge-Kutta method (Cortell, 1993) was used to solve the equation of motion. Some parameters, such as damping ratios and temperature variations, affected the dynamic behaviour of the plate.

Static analysis was applied by Duc et al. (2017) to conduct the free vibration of nanocomposites rectangular plates reinforced by single-walled carbon nanotubes and resting on two-parameter elastic foundations. HP and first-order shear deformation were used to derive the governing equation, solved using the Navier solution. The natural frequencies and mode shapes of the plate were obtained. It was found that the mode's number and geometrical parameters strongly affect the system's natural frequencies. Recently, Ghorbanpour et al. (2018) presented the dynamic performance of a porous rectangular plate resting on the Pasternak

foundation and subjected to a dynamic transverse load. They also used Hamilton's principle and the energy method to derive the governing equation of the model, which was solved using the DQM. The results were considered useful for designing aircraft, automobiles, pneumatic conveying, material handling and control systems.

Sayyad and Ghugal (2018) used a trigonometric plate theory to analyse simply supported plates resting on Winkler elastic foundations. The principle of virtual work was used to derive the equation of motion, and the solution was obtained using double trigonometric series. Some years later, Esmaeilzadeh et al. (2020) carried out the transient responses of axially moving nanoplate using a relaxation technique combined with the Newmark method. For more research about PoEFs, readers are referred to Voyiadjis and Kattan (1986); El-Zafrany et al. (1995); Wang et al. (2005); Civalek (2007); Younesian et al. (2011); Gao et al. (2018).

2.8 Vibration Analysis of Nonlinear Systems

The vibration analysis of a system determines the response to specified dynamic excitations. Its process includes mathematical modelling, derivation of the governing differential equations of motion, finding the solutions arising for the equations of motion, and providing the interpretation of the response results (Rao, 2007).

Vibration analysis is also the determination of oscillation frequencies concerning time, the relative position of the various components of the physical system during its motion, and the amplitude of the oscillatory motion of these components (Zingoni, 2018b). The solution to nonlinear problems tends to be more complex and computationally intensive than linear problems. Moreover, because the superposition principle is invalid for nonlinear systems, often one can only obtain approximate solutions. Another alternative is using iterative solutions, tedious calculations, and extensive mathematical studies to get the solutions.

Given a dynamic system, the mathematical approach to solving the resulting differential equation of motion depends on the nature of the system and the analyst's convenience. Generally, it is required to use less computational, easy and simplified methods. However, for predicting the dynamic behaviour of structures and mechanical devices, engineers rely on mechanical models of the system under study, such as theoretical (with a set of equations, ODEs, and partial differential equations (PDEs)) or numerical (FEM, DQM, FDM). There exist, in general, no methods capable of yielding an exact solution of an arbitrarily selected nonlinear differential equation established using any of the formulations in section 2.5 or

others not mentioned here. Thus, for certain classes of nonlinear differential equations often encountered, the only available methods are those of various approximation procedures.

Mead (1987) presented some steps to follow in order to analyse a structure. The first is to convert the physical system into a simplified mathematical model (use the concept of an equivalent system). The second is to determine the differential equation of motion of the system (use either the Newtonian approach, d'Alembert's principle, the energy method, the Lagrange equations, the HP, or any other available method). The third is to solve the equation of motion to obtain the response. The last step is the interpretation of the result for the physical system. Following those steps, a complete structure analysis is guaranteed. Some analytical methods for approximating solutions to nonlinear differential equations are presented in what follows below.

2.8.1 Perturbation Methods

Some essential methods for solving nonlinear differential equations are perturbation methods. The use of these methods was first limited to some calculations but the vital contributions of Poincaré (1892) and other mathematicians have broadened the applicability of the approaches to include a more general field of nonlinear mechanics. The methods apply to differential equations in which a relatively small dimensionless parameter is associated with nonlinear terms. Applying perturbation methods requires first developing the desired quantities in powers of the small parameter (by applying the Taylor expansion method). Then, multiply the coefficients, which are functions of the independent variable. Next, split the equation in terms of the power of the small parameter. Last, determine the coefficients of the developments one by one, usually by solving a sequence of linear equations. By proceeding this way, a severe difficulty may often be encountered in the form of the so-called secular terms. Those terms grow indefinitely when the time tends to infinity and can thus destroy the convergence of the series solution. For more details about secular terms, the reader is referred to Rao (1990). Frequently, the first term in the series is the linear response corresponding to the response to the system if all nonlinear terms were removed.

Perturbation methods are extremely reliable and applicable, especially when the coefficient of a nonlinear term is of a smaller order than other terms. The accuracy of an approximate solution obtained by perturbation methods is testified by comparing it to the numerical solution, which is considered as the reference one. In what follows, different perturbation techniques are outlined, such as the straightforward expansion method (SEM),

the Lindstedt method (LM), the parameter expansion method (PEM), the MTSM and the averaging method (AVM) (Nayfeh, 1973).

2.8.1.1 The Straightforward Expansion Method (SEM)

The SEM only solves nonlinear differential equations in which a small parameter is present. The method also considers the expansion of the response, which is valid for small but finite amplitude motions. Many authors (Nayfeh and Mook, 1995; He, 1999a; Nayfeh, 2011; Amore and Fernández, 2018; Moreno-Pulido et al., 2021) have applied the method to various types of nonlinear problems. For example, Amore and Fernández (2018) used the method to solve Duffing's equation given by:

$$\ddot{x} + \omega_0^2 x + \gamma_2 x^2 + \gamma_3 x^3 = 0. \quad (2.108)$$

By using a small parameter ε , the solution of Eq. (2.108) is expressed as:

$$x(t, \varepsilon) = \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \varepsilon^3 x_3(t) + \dots. \quad (2.109)$$

Replacing Eq. (2.109) into Eq. (2.108) and setting the coefficients like powers of ε equal to zero, the following linear differential equations are obtained:

$$\text{for } \varepsilon : \quad \ddot{x}_1 + \omega_0^2 x_1 = 0 \quad (2.110a)$$

$$\text{for } \varepsilon^2 : \quad \ddot{x}_2 + \omega_0^2 x_2 = -\gamma_2 x_1^2 \quad (2.110b)$$

$$\text{for } \varepsilon^3 : \quad \ddot{x}_3 + \omega_0^2 x_3 = -2\gamma_2 x_1 x_2 - \gamma_3 x_1^3 = 0 \quad (2.110c)$$

The solution of Eq. (2.110a) is found as:

$$x_1(t) = A_1 \cos(\omega_0 t + \varphi_1) \quad (2.111)$$

where A_1 and φ_1 are respectively the amplitude and the phase angle, which can be determined using the initial conditions of the system. Implementing $x_1(t)$ into Eq. (2.110b) and performing some mathematical calculations, the solution x_2 is derived as:

$$x_2(t) = A_2 \cos(\omega_0 t + \varphi_2) - \frac{\gamma_2 A_1^2}{6\omega_0^2} \left[3 - \cos(2\omega_0 t + 2\varphi_1) \right] \quad (2.112)$$

For Eq. (2.110c), x_1 and x_2 are replaced by their expressions and some mathematical calculations are developed. This gives the solution for x_3 . The same procedure is applied to find the solutions for the order of $\varepsilon > 3$. Therefore, the solution of Eq. (2.108) is given by:

$$x(t, \varepsilon) = \varepsilon A_1 \cos(\omega_0 t + \varphi_1) + \varepsilon^2 \left\{ A_2 \cos(\omega_0 t + \varphi_2) - \frac{\gamma_2 A_1^2}{6 \omega_0^2} \left[3 - \cos(2\omega_0 t + 2\varphi_1) \right] \right\} \quad (2.113)$$

This solution (Eq. (2.113)) goes up to order three only and can be extended if the solution to the higher order is of interest. Amore and Fernández (2018) based their research on the idea of Nayfeh and Mook (1995), according to which there are two alternative ways to use the initial conditions. The first requires substituting the solution Eq. (2.109) into the initial conditions and equating coefficients of like powers of ε . The second alternative can ignore the initial conditions and the homogeneous solution in the x_n for $n \geq 2$, until the last step. The latter approach is the most frequently used because it requires less algebra than the first. Furthermore, the solution of the equation involves only the steady state, which does not depend on initial conditions. Nonetheless, both approaches are equivalent and give similar results.

While solving equations using the SEM, from the third-order approximation, there is a presence of secular terms with which the solution grows without bound as t tends to infinity. Hence, that solution does not correct the first two solutions. That means that the solution as t increases, is not uniformly valid.

2.8.1.2 The Lindstedt Method (LM)

Also called the Lindstedt-Poincaré method (L-PM)(Poincaré, 1892), the LM is the modification of the SEM. It allows the determination of the periodic or almost-periodic orbits of the nonlinear system (limit cycles) (Luongo, 1996). Although the L-PM cannot provide solutions that evolve aperiodically on a slow time scale, it gives a procedure to build asymptotic approximations of periodic solutions.

Let the quasi-harmonic system with the following equation of motion (Mickens, 2010):

$$\ddot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}), \quad (2.114)$$

where ε is a small parameter, and $f(x, \dot{x})$ is a nonlinear analytic function, with the nonlinearity due to the position or the velocity. The nonlinear term $\varepsilon f(x, \dot{x})$ affects not only the amplitude but also the period of the system. The essence of the LM is to produce a periodic solution of a given nonlinear differential equation of every order of approximation because the period of oscillation is affected by the nonlinear term. To solve Eq. (2.108) using the LM, the solution should be assumed in the form:

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + \varepsilon^3 x_3(t) + \dots \quad (2.115)$$

with the period $\frac{2\pi}{\omega}$, where:

$$\omega = \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \varepsilon^3 \omega_3 + \dots \quad (2.116)$$

This method assumes that the angular frequency and the solution vary as a function of the amplitude A_0 , representing the amplitude of the solution without perturbation. The LM also eliminates the secular terms in each approximation step by requiring the solution to be periodic in each step.

Only weakly nonlinear systems apply the L-PM. That method had been revised and called the modified L-PM (Nayfeh, 1973). Unlike the L-PM where the small parameter ε in Eq. (2.114) must be small for the method to be used, using the modified L-PM, ε may not necessarily be small. In applying the latter method, one can transform the equation of the strongly nonlinear system into a weakly nonlinear one and thus use the standard L-PM to solve it. Moreover, the modified L-PM can be applied to a wide range of nonlinear problems and converges to the exact solution more quickly than the standard L-PM (Amore and Fernández, 2018).

2.8.1.3 The Parameter Expansion Method (PEM)

He (2006) was the first to propose the PEM. The idea of the method is to consider the expansion representing the response to be a function of multiple independent variables (Mickens, 2010). The technique is an extension of the LM and can be used to construct an approximate general solution to a system of ODEs.

Moreover, the PEM can be used to obtain the system's behaviour in the neighbourhood of a limit cycle or an equilibrium point (Sedighi et al., 2012). It has advantages over the L-PM, as it can treat damped systems conveniently. The PEM does not need to have a time

transformation like the L-PM. That is because its basic constitution is to expand the solution and some parameters in the equation, and it is one of the easiest approaches to nonlinear oscillators (Hashemi Kachapi and Ganji, 2014).

The process of using the PEM is the following:

1. Write the variable and all coefficients of the equation to be solved into an expansion of the power series of a chosen parameter, for example, p ;
2. Substitute the expression of each coefficient in the equation and gather the terms following the power of p ;
3. Equate the terms with the identical powers of p ;
4. Solve a sequence of linear equations for which the solution of the equation at a given stage depends on the expressions of the variables obtained in the previous step.

As an example, in solving Eq. (2.108) using the PEM, the procedure will be the following. First, each coefficient of the equation is written in the power of the parameter p as:

$$x = x_0 + px_1 + p^2x_2 + \dots \quad (2.117a)$$

$$1 = \omega_0^2 + p\omega_1 + p^2\omega_2 + \dots \quad (2.117b)$$

$$\gamma_2 = p\alpha_1 + p^2\alpha_2 + \dots \quad (2.117c)$$

$$\gamma_3 = p\beta_1 + p^2\beta_2 + \dots \quad (2.117d)$$

Then, Eqs. (2.117a) to (2.117d) are replaced into Eq. (2.108). That gives:

$$\begin{aligned} (\ddot{x}_0 + p\ddot{x}_1 + p^2\ddot{x}_2 + \dots) + (\omega_0^2 + p\omega_1 + p^2\omega_2 + \dots)(x_0 + px_1 + p^2x_2 + \dots) + \\ (p\alpha_1 + p^2\alpha_2 + \dots)(x_0 + px_1 + p^2x_2 + \dots)^2 + \\ (p\beta_1 + p^2\beta_2 + \dots)(x_0 + px_1 + p^2x_2 + \dots) = 0. \end{aligned} \quad (2.118)$$

Next, by developing Eq. (2.118), linear equations are obtained and solved successively, as in the case of the previous methods mentioned. Finally, combining all the intermediary solutions will form the overall solution of Eq. (2.108).

2.8.1.4 The Multiple Time Scales Method (MTSM)

Applying the MTSM requires writing the original time at which the system evolves in different time scales by using a small parameter as $T_n = \varepsilon^n t$; $n = 1, 2, \dots$. These different time scales can be multiple independent variables or scales instead of a single variable

(Amore and Fernández, 2018). The MTSM is a more general technique that does not require the solution to depend periodically on the slow time variables. Its idea is to describe the evolution of the solution over long time scales of the order of a chosen system parameter. That is by introducing an additional low time variable as a function of that parameter and time, or using various time or space variables scaled differently and considered independent. Thereafter, a solution is found in terms of the new variables (Meirovitch, 2010) without forgetting that the number of independent time scales needed depends on how the expansion is carried out. For example, to solve Eq. (2.108) using the MTSM, first, the equation is written as:

$$\ddot{x} + \omega_0^2 x + \varepsilon \gamma_2 x^2 + \varepsilon \gamma_3 x^3 = 0. \quad (2.119)$$

Next, the solution is supposed to be:

$$x(t, \varepsilon) = \varepsilon x_1(T_0, T_1, T_2, \dots) + \varepsilon^2 x_2(T_0, T_1, T_2, \dots) + \dots + \varepsilon^n x_n(T_0, T_1, T_2, \dots), \quad (2.120)$$

where $T_0 = t$ is a fast time scale characterising motions occurring at the linear natural frequencies of the system. In contrast, $T_1 = \varepsilon t$ and $T_2 = \varepsilon^2 t$ are slow scales indicating the shift in the system frequencies due to the quadratic and cubic nonlinearities, respectively (Nayfeh and Mook, 1995).

The MTSM is robust and works well but is mathematically more difficult, and the dependent variable depends explicitly on the multiple scales inherent in the system (He et al., 2022). Thus, the precise scales and the dependent variable are expanded in the perturbation series. Furthermore, the secular terms are eliminated from the governing equations obtained in different orders. For Badeti et al. (2018), the main characteristic of this method is that it systematically separates the problem of local dynamics in time/space from the problem of slow larger-scale variations. Thus, the solution to a problem using the MTSM agrees with that obtained using the L-PM. Still, the former can solve damping systems more efficiently than the latter, even though it is more mathematically difficult.

2.8.1.5 The Averaging Method (AVM)

The solutions based on the AVM mostly start with the variation of parameters to transform the dependent variable. The AVM is more sophisticated for initial value problems and can be applied to systems with a dynamic of the following (Wagg and Neild, 2014):

$$\ddot{x} + \omega_n^2 x = \varepsilon N(x, \dot{x}), \quad (2.121)$$

where $N(x, \dot{x})$ is a nonlinear function. Equation (2.121) is sometimes referred to as the standard or normal form. For vibration analysis, understanding how external excitation affects the system is crucial. However, this force is often omitted from examples of averaging. This technique has some advantages over the PEM. For example, it has easy computations, and the principle of applying the method is elementary.

The AVM applies not only to the study of periodic oscillations, but also to the study of transient oscillations, where both the amplitude and phase of oscillations change slowly with time (Mickens, 2010). Its main idea is not to solve Eq. (2.121) but to find another more straightforward system in which solutions are close to the solutions of Eq. (2.121) for a sufficiently long-time interval. The simplification that can be achieved using the AVM is to eliminate the independent variable t from the considered equations; that is, to reduce the practical order of the system by one.

All the above-stated perturbation methods have shortcomings. Because they require a small parameter in the differential equation of motion, their application is restricted to weakly nonlinear systems only. If these methods must be applied to strongly nonlinear systems, they must be modified. A good choice of the small parameter may give good results close to the exact solution. However, that choice is not easy and challenging, resulting in severe harmful effects for an unsuitable choice of the small parameter. Furthermore, the small parameter value should be in the interval $0 < \text{parameter} < 1$, which restricts the range of the solution. Therefore, in strongly nonlinear systems, one can use other approximations or numerical methods to check the result from the perturbation methods.

Perturbation methods were the first analytical and approximate methods to find approximate analytical solutions to nonlinear differential equations (NDEs). However, several methods have been introduced and developed recently to obtain approximate solutions for NDEs due to their complexity and the difficulty of solving them through traditional perturbation techniques. For example, the VIM (He, 1999b), the max-min approach (He, 2008), the iteration method (Hoang et al., 2017), and the HBM (Beléndez et al., 2008; Molla et al., 2016; Hosen et al., 2020). The following subsections present other approximation methods.

2.8.2 The Harmonic Balance Method (HBM)

The principle of HBM is used to determine analytical approximations to the periodic solutions of differential equations using a truncated Fourier series representation. For Grolet and Thouverez (2012) the method offers the ability to treat the case of a highly nonlinear system

with, for instance, friction or geometric nonlinearities. The most important parameter of this method is the number of selected harmonics. That number is not typically known ahead of time in a situation requiring convergence studies to ensure accurate solution estimation. If that number is high, the solution procedure can quickly become complex and challenging. Nevertheless, in many cases, it turns out that the solution rapidly converges in terms of harmonics, which leads to systems with more practical dimensions.

The first part of the HBM assumes a harmonic (cosine or sine waves) solution in terms of unknown coefficients and an angular frequency, determined using initial conditions for the problem. Once that assumed solution is substituted into the governing equation, the second part of the process balances the coefficients of the harmonic terms. The last step is to solve the algebraic equations. By doing so, the user can calculate the approximate response of the steady state. Then, to balance all the harmonic terms, the trial solution for the displacement must be extended to include a summation of all the relevant harmonics and subharmonics. In general, the trial solution for a nonlinear system with a primary response at frequency ω_r would take the form (Mickens, 1996; Haddad and Chellaboina, 2011):

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_r t) + b_n \sin(n\omega_r t)), \quad n = 1, 2, 3, \dots \quad (2.122)$$

where a_n and b_n are coefficients to be determined using the initial conditions. For a weakly nonlinear system, the first harmonic frequency in this expression, ω_r , corresponds to the forcing frequency if the system is forced, or to a natural frequency, if the system is unforced. The HBM is approximate because higher harmonics sometimes are neglected during the development (to simplify the problem). It applies a trial solution with fewer terms (often limited to just at frequency ω_r). It balances just the terms present in the trial solution while ignoring any higher frequency terms generated by the nonlinearity.

2.8.3 Variational Approach for Nonlinear Oscillators

One can describe a physical problem using two primary ways: differential equations with initial or boundary conditions and variational principles. Both ways are different because the differential equations model requires well-built local differentiability (smoothness) of the physical field. On the other hand, the variational principle requires weaker local smoothness or only local integrability. The latter has many advantages: simple and compact in form

and comprehensive. Moreover, it provides all necessary information for characterising the problem under consideration (He, 2007).

Variational methods are popular tools for nonlinear analysis. Compared with other approximate analytical methods, they provide physical insight into the nature of the problem's solution. Moreover, the obtained solutions are the best among all the possible trial functions. The VIM was first proposed by He (2001). It is a reliable method for solving nonlinear differential equations using successive iterations (Bayat et al., 2012). The main idea behind iteration methods is to rewrite the original nonlinear differential equation into another simple one formed by a sequence of linear equations. This method will require the user to solve the sequence of linear equations. Thus, the only occurring problem is reformulating the original nonlinear differential equation to construct a viable iteration scheme. The following lines describe the principle to use to simplify the problem and solve it directly (Mickens, 2010):

1- Assume that the differential equation to be solved is given by:

$$F(\ddot{x}, x) = 0, \quad x(0) = A, \quad \dot{x}(0) = 0. \quad (2.123)$$

Moreover, assume that it can be rewritten in the form:

$$\ddot{x} + f(\ddot{x}, x) = 0. \quad (2.124)$$

2- Add $\Omega^2 x$ to both sides to obtain:

$$\ddot{x} + \Omega^2 x = \Omega^2 x - f(\ddot{x}, x) \equiv G(\ddot{x}, x), \quad (2.125)$$

where Ω^2 is an unknown constant.

3- Formulate the iteration scheme in the following way

$$\ddot{x}_{k+1} + \Omega_k^2 x_{k+1} = G(x_k, \ddot{x}_k); \quad k = 0, 1, 2, \dots \quad (2.126)$$

with $x_0(t) = A \cos(\Omega_0 t)$, such that x_{k+1} satisfies the initial conditions $x_{k+1}(0) = A$, and $\dot{x}_{k+1}(0) = 0$.

4- At each stage of the iteration, Ω_k is determined by the requirement that secular terms should not occur in the full solution of $x_{k+1}(t)$.

5- This procedure gives a sequence of solutions: $x_0(t)$, $x_1(t)$, $\dots$, and all solutions are easy to calculate. The only difficulty may be the algebraic intensity required to complete the calculations.

Having explicit and exact solutions to nonlinear problems is very difficult and even limited in physics and engineering. That is why nonlinear problems are always solved using approximate solutions or qualitative theory. The approximation methods proposed in this section for solving nonlinear systems work well for weakly nonlinear systems. Still, those methods can be used with accurate results for strongly nonlinear systems. However, one should note that perturbation methods can be used for systems with a basic exact nonlinear solution exists. Each method has advantages and disadvantages for solving differential equations of nonlinear systems. The choice of a technique is of the user's convenience.

2.8.4 Finite Element Method (FEM)

Finite element methods are numerical methods for approximating the solutions to mathematical problems that are usually formulated to precisely state an idea of some aspect of physical reality. The method is characterised by a variational formulation, a discretisation strategy, one or more solution algorithms and post-processing procedures (Smith et al., 2013). Moreover, it is a numerical formulation that divides the continuous structure into several discrete elements. Furthermore, it requires the displacement at the meeting points of the elements to be compatible and that the internal forces at these points be in equilibrium. The method is also based on fully nonlinear analysis and is a standard technique for nonlinear structural analysis, structural dynamics, and fluid flow (Huang, 2007; Logan, 2011). The procedure of applying FEM to solve real engineering problems is called Finite Element Analysis (FEA).

FEA involves the mathematical modelling of a physical system to predict its structural behaviour. The method requires the geometry, the material properties, boundary conditions, etc. (Logan, 2011; Sapsis et al., 2012). The FEA is a perfect method for analysing problems over complex domains. Because of that, it is used when the desired precision varies over the entire domain when the solution is deficient in smoothness, or when the domain changes (as during a state reaction with a moving boundary). One of the advantages of the FEA is its ability to generalise a structural problem and solve it by subdividing a large system into a smaller one or simpler parts called finite elements (Zienkiewicz et al., 1977; Pao, 1986).

One of the engineering simulation programs that use FEA is Abaqus. The program can solve problems ranging from relatively simple linear analyses to the most challenging nonlinear simulations. In addition to standard material models, Abaqus has a comprehensive list of material models for simulating engineering materials such as rock, soils, metal, rubber, polymers, and composites (Khennane, 2013). It is also the most accessible FEM software at the University of Cape Town. For these reasons, Abaqus will be adopted as the numerical tool for the FEM analyses.

Another method that uses FEA is ANSYS. Tiwari and Kuppa (2014) used that method to solve a problem of a beam on a two-parameter elastic foundation under static and dynamic loads. On the other hand, Frydrysek et al. (2013) used ANSYS to solve a problem of a beam on a Winkler's foundation. The mechanical contact was simulated using spring elements between the beam as a 2D element (BEAM54) and the rigid ground. It was found that the number of elements over the beam's length influences the results. However, comparing the results to theoretical ones showed agreement between the two methods.

There are three distinct steps in Abaqus analysis: preprocessing, simulation, and post-processing. The flowchart in Figure 2.21 gives details regarding each stage.

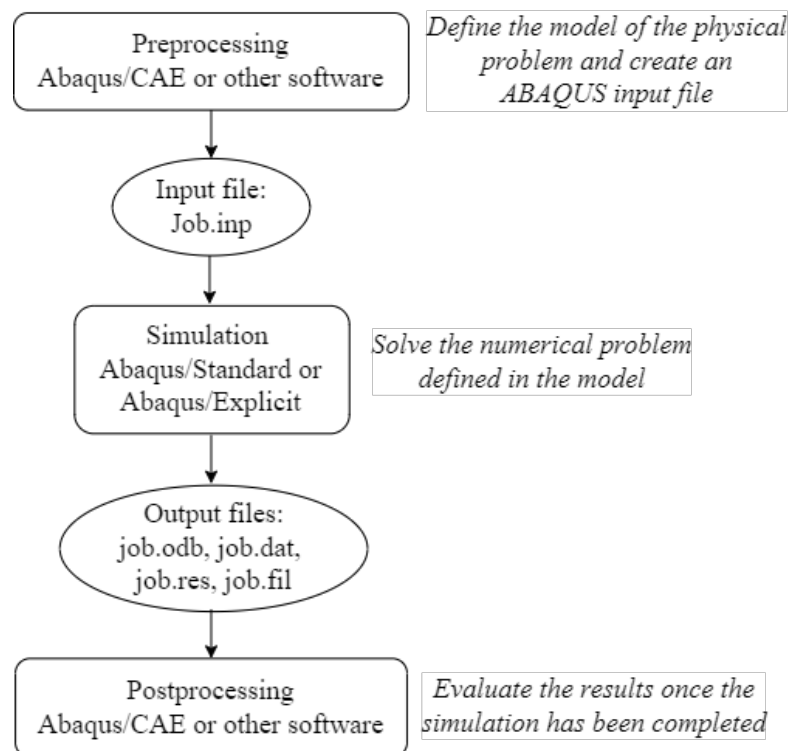


Figure 2.21: Stages of a complete simulation in Abaqus (SIMULIA, 2008).

FEA in Abaqus can also be used to calculate the natural frequencies and mode shapes for any linear system. While applying the method, the user only needs to define the parameters and the properties of the structure to model. Moreover, the user must create the steps to extract the natural frequencies and apply loads. In the end, the number of natural frequencies extracted depends on the number entered by the user.

Some shortcomings of using Abaqus are as follows: the method relies on mesh or elements connected through nodes, and sometimes the mesh construction can be very costly and time-consuming. Moreover, the entire mesh must be refined from the start when the desired accuracy is not achieved, thus making all previous works a waste. For problems with large deformation creating distortions, the accuracy of the results can be questioned.

2.8.5 Other Numerical Methods for solving Problems of BPoEFs

There exist other numerical methods for solving soil structure interaction problems such as BPoEFs.

- **Mesh-free method (MFM):** developed by Lucy (1977) uses a set of scattered nodes in the domain to represent the problem domain. The method does not require meshing the domain either while formulating the problem or solving it. The MFM has excellent accuracy and rapid convergence. Element-free Galerkin method is one of the mesh-free methods developed by Belytschko et al. (1994) to alleviate mesh bias. The method uses the moving least square interpolation to construct the trial and test functions for the weak form of the variational principle and the weight functions (Sunitha et al., 2008; Lan, 2020).

- **Finite element displacement method:** Mourelatos and Parsons (1987) used the finite element displacement method to study BoEF and formulated the problem as a three-dimensional one. The foundation combined the effects of the Filonecko-Borodich and Pasternak models. The practical application of this type of model of BoEF is found in the area of rotor dynamics.

Regarding all the numerical methods listed above, FEA in Abaqus has been applied for the solution of models represented by BPoEFs presented in this thesis. This method has been selected because it has many advantages as compared to others: it is very easy and standard to apply boundary conditions in Abaqus while other methods may require special knowledge. Moreover, there is a possibility to model the nonlinear foundation by defining the data of the nonlinear restoring force of the foundation. Once the mesh has been refined, the accuracy of

the results cannot be contested. Furthermore, the method solves the problem by subdividing a large system into a smaller one or simpler parts called finite elements so that the resulting simple equations, assembled together form a large system of equations that models the entire problem (Logan, 2011; Sapsis et al., 2012).

2.9 Summary of Findings of Literature Review

Reading through the available work, some gaps have been identified. The following points give details.

- The study of the effect of elastic foundations on the mode shapes instability and vibration problems of simply supported beams and rectangular plates has attracted much attention. However, most available works are related to static and simple linear cases. Those researches were also limited to a one-dimensional system with limited information about two-dimensional and three-dimensional ones. Apart from that, the support flexibility of the system was sometimes neglected to simplify the analysis. The differential equations of simplified systems were derived, and the analytical solutions of those equations were also easy to get using available methods. Because the obtained results sometimes failed to give the system's real representation, studies have been conducted using nonlinear analyses.

- In comparison, continuous systems with stiffness nonlinearity are the least investigated due to the complexity of the equations of motion. Due to this, only a few studies exist on some nonlinear continuous systems having geometric properties or cubic stiffness nonlinearity. For these systems, numerical or approximate solutions using perturbation methods are obtained because of the complexity of solving the resulting nonlinear differential equations using exact analytical methods. Sometimes, the equations of motion are transformed into linear differential equations where possible, which means that nonlinear terms are simply neglected. As a result, significant relative differences occur in the modelling, and the results are inaccurate for the given nonlinear system.

- The dynamic response of beams and plates resting on nonlinear elastic foundations (BPoNEFs) having quadratic and cubic stiffness properties has attracted less attention, although necessary in understanding the dynamic behaviour of structures such as bridges, roadways, and runways. Many other types of nonlinearities of the foundations are being investigated. Some involve cubic nonlinearity, but further studies are still required. Also, no research has been done on the vibration of beams and plates resting on nonlinear quadratic

elastic foundations. This means that the quadratic nonlinearity of the foundation is quite rare. At the same time, there is limited research on BPoEFs having quadratic and cubic stiffness properties. Therefore, there is a need to investigate the effects of a foundation having quadratic and cubic stiffness nonlinearities on the beam and plate vibration response and natural frequencies. This type of system is used in railway engineering to model railroad and railway tracks (Jorge et al., 2015; Froio et al., 2016; Younesian et al., 2019).

- It is important to mention that in the cases of roads, tracks, and runways, the beam may not be considered as simply supported structures because they have infinite lengths. They can be seen as finite structures without support at their boundaries. Therefore, in order to represent roads, tracks, and runways, a reasonable length of the beam may be chosen, or a finite structure with simple supports in addition to the elastic foundation may be considered to allow comparison. Note that, finite structures with supports at the boundaries are more relevant to ground beams and ground slabs in a building.

- Although exact analytical methods can yield complete analytical solutions, the solutions obtained are usually implicit. Therefore, they can only solve problems under limited simple conditions. Moreover, available solutions are not easy to implement by engineers and practitioners due to their inconsistency. Accordingly, for complex problems such as BPoNEFs having quadratic and cubic stiffness properties, it is better to use approximate methods such as the SEM to solve the resulting nonlinear differential equations. Unfortunately, research using that particular method has not been reported for solving problems of BPoEFs. However, it is also a powerful analytical method.

- On the other hand, the Galerkin procedure simplifies or reduces the nonlinear PDEs of BPoEFs into nonlinear ODEs. Then, numerical methods such as the DQM, the FDM, the mix-max approach, the Runge Kutta method, and the modal superposition method have been extensively used to solve the resulting ODEs. In contrast, FEA in Abaqus, one of the available finite element solvers has been neglected because advanced knowledge of the modelling technique is sometimes needed to analyse a complex nonlinear problem. As a result, FEA in Abaqus is still needed as a benchmark for analytical results.

2.10 Statement of Research

The importance of nonlinear stiffness in analysing and designing structural systems and machines must be addressed for safe design and construction. The main objective of this

research is to respond to the need to improve structural performance and investigate the effects of the nonlinear quadratic and nonlinear cubic stiffness of the foundation on the vibration response of structural and mechanical systems compared to that of structural systems with linear foundation stiffness. The study will also provide new development for predicting the performance of nonlinear structural systems by analysing beams and plates resting on nonlinear elastic foundations. The length of the beam or the plate will be chosen within a range limit so that for structures such as roads, tracks, and runways, the formulation presented in this thesis can be adapted or modified to cover the specific case at hand. The investigation will be performed using classical beam theory through analytical and numerical analyses in order to:

- obtain closed-form solutions for the equations of motion of the models of interest;
- determine differences in behaviour (particularly natural frequencies and mode shapes) between beams and plates on *nonlinear* elastic foundations and beams and plates on *linear* elastic foundations;
- better understand the vibration response of beams and plates on elastic foundations when the elastic-stiffness law of the foundation is specifically:
 - i. linear;
 - ii. nonlinear quadratic;
 - iii. nonlinear cubic;
 - iv. nonlinear quadratic-cubic.

2.11 Justification of the Research and Applicability of its Outcomes

Beams and plates resting on elastic foundations find many civil and mechanical engineering applications. Unfortunately, knowledge of their structural behaviour is scarce, especially when the foundation has nonlinear quadratic stiffness properties. Moreover, the applications recorded are mostly centred on linear cases study. Consequently, this research is needed since nonlinearity due to stiffness is an essential factor that can bring a suitable modification to the response of beams and plates when subjected to dynamic loads. It can also enhance structural vibration response. The role of quadratic and cubic stiffness nonlinearities in vibration analysis will be shown when considered during the analysis and design of structures such as beams and plates on nonlinear elastic foundations. The SEM is an accurate analytical

approximate method to solve such problems. This research aims to close this gap by better understanding the structural behaviour of beams and plates resting on nonlinear quadratic and nonlinear cubic elastic foundations and providing design advice.

2.12 Research Questions

The main problem of this research is understanding the influence of nonlinear foundation stiffness on the vibration response of beams and plates resting on elastic foundations. That will provide helpful information for engineers and designers. The following research questions need to be addressed:

- How does stiffness nonlinearity in the elastic foundation affect the vibration behaviour of beams and plates?
- What are the relative differences incurred by ignoring the nonlinearity in the foundation stiffness?
- How can such nonlinear vibrations of beams and plates be controlled?

2.13 Research Methodology

Nonlinear systems are often difficult (or impossible) to accurately model, and their behaviour concerning a given variable (for instance, time) is complicated to predict. Therefore, one should approximate a nonlinear system as a linear one where applicable in order to simplify its modelling.

The methodologies employed will be analytical solution procedures, approximate and exact, and numerical methods where needed. Next, both analytical and numerical results will be compared to validate assumptions and predictions made. The research will follow the steps below:

1. The vibration of beams resting on nonlinear elastic foundations is first studied and then extended to that of plates resting on nonlinear elastic foundations later in the thesis.
2. Approximations will be introduced in the exact governing differential equations of motions to obtain approximate analytical solutions. These solutions will be plotted to

appreciate the mode shapes of the beam and the plate, as well as the variation of the natural frequencies of the system.

3. Accurate FEM modelling of the problems (beams and plates on nonlinear elastic foundations) will be carried out using the commercial software Abaqus to obtain precise numerical solutions for natural frequencies and mode shapes. The accurate numerical solutions will be used to validate and benchmark the accuracy of the approximate analytical solutions.

4. Once validated, the results of the nonlinear elastic foundations will be compared to those of the linear elastic foundations to identify the relative differences of ignoring the nonlinearities of the foundations.

5. Parametric studies focusing on the variation of the stiffness of the foundation will be carried out to enable general conclusions and design recommendations to be made.

Chapter 3

Linear Vibration of Beams

3.1 Introduction

The present chapter focuses on studying the linear vibration of beams. The chapter investigates the transverse vibrations of a simple beam and the beam on an elastic foundation (BoEF) having linear stiffness properties. Analytical studies using the separation of variables and numerical studies using the Finite Element Analysis (FEA) in Abaqus will be presented. Particular attention is devoted to analysing the effect of the foundation having linear stiffness properties on the response of a beam structure.

This chapter is structured as follows: Section 3.2 concentrates on the study of vibrations of a simple beam. This section will present results from analytical studies. In Section 3.3, numerical studies using FEA in Abaqus are conducted, and the results are also presented. Sections 3.4 and 3.5 focus on analytical and numerical studies of a BoEF having linear stiffness properties. Both analytical and numerical studies are based on free vibration only. Section 3.6, explores the effects of increasing some parameters on the beam's natural frequencies and vibration amplitude. Section 3.7, concludes the chapter.

3.2 Analytical Study of the Transverse Vibrations of a Simple Slender Beam

In this section, the transversal vibration of a simple beam is studied. The analysis is performed on a simple beam so that when the foundation is added, the modification per the beam's

amplitude of vibration and natural frequencies can be assessed. Therefore, it is clear that the frequency of vibration of the beam may increase in the presence of the elastic foundation. As for the amplitude, it is governed by the initial conditions applied to the system, and it may not be straightforward to comment on its variation due to the stiffness of the elastic foundation. However, the development in the subsequent sections will help us highlight the main trends. To begin with, the system's equation of motion is derived. Then, the solution is found using an analytical method. Finally, the natural frequencies and mode shapes of the beam are determined.

3.2.1 Derivation of the Equation of Motion of a Simple Slender Beam

The system to be studied in this section is that of a beam displayed in Figure 3.1.

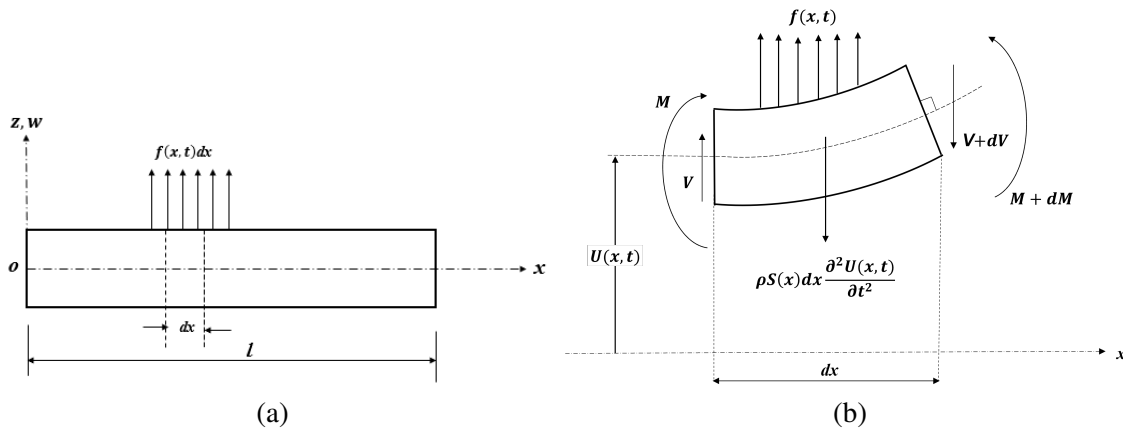


Figure 3.1: Transverse vibration of a slender beam.

Figure 3.1 adapted from Rao (2007)) shows two sub-figures: (a) is a beam on which is applied a transversal load on an element and (b) is a free-body diagram of an element of the beam. The force $f(x, t)$ is the external transverse force per unit length of the beam, V is the shear force, M is the bending moment, and $U(x, t)$ is the transversal displacement at any distance x along the length of the beam. In what follows, the governing equation of the system in Figure 3.1 is derived and solved using the procedure provided in the books by Nayfeh and Nayfeh (1994) and Rao (2007).

For transverse vibration of a slender beam such as that presented in Figure 3.1, the inertial force is derived from the kinetic energy (Nayfeh and Pai, 2008) and given by:

$$\rho S(x)dx \frac{\partial^2 U(x,t)}{\partial t^2}, \quad (3.1)$$

where the transversal displacement $U(x,t)$ varies along the z -direction. On the other hand, the forces along the z -direction satisfy the following equation:

$$V - (V + dV) + f(x,t)dx = \rho S(x)dx \frac{\partial^2 U}{\partial t^2}(x,t), \quad (3.2)$$

whereas the moments satisfy the following:

$$(M + dM) - M - (V + dV)dx + f(x,t)dx \frac{dx}{2} = 0. \quad (3.3)$$

Knowing that dV and dM can be written as:

$$dV = \frac{\partial V}{\partial x}dx \text{ and } dM = \frac{\partial M}{\partial x}dx. \quad (3.4)$$

Eqs. (3.2) and (3.3) are transformed into:

$$-\frac{\partial V}{\partial x}dx + f(x,t)dx = \rho S(x)dx \frac{\partial^2 w}{\partial t^2} \quad (3.5a)$$

$$\frac{\partial M}{\partial x}dx - Vdx + \frac{\partial V}{\partial x}dx dx + f(x,t)dx \frac{dx}{2} = 0 \Rightarrow \frac{\partial M}{\partial x} - V = 0. \quad (3.5b)$$

Also, the moment-curvature relationship is recalled as (Timoshenko and Gere, 2009; Caglar et al., 2015) that

$$M = EI \frac{\partial^2 w}{\partial x^2} \quad (3.5c)$$

Then, from Eqs. (3.5a), (3.5b), and (3.5c) and using the Euler-Bernoulli beam theory as recommended by Rao (2007), the equation of the transverse vibration of beams is obtained as follows:

$$EI \frac{\partial^4 U(x,t)}{\partial x^4} + \rho S \frac{\partial^2 U(x,t)}{\partial t^2} = f(x,t). \quad (3.6)$$

3.2.2 Solution of the Governing Equation using the Separation of Variables

For free vibration, the separation of variables (Dawkins, 2018; Myint-U and Debnath, 2007) is used to solve Eq. (3.6). Also known as the Fourier method, the separation of variables is one of the more widely used methods for solving ordinary and partial differential equations. Algebra allows rewriting an equation so that each variable occurs on a different side of the equation. Then, using the separation of variables, $U(x, t)$ is written as a product of two functions: one depending on time t and the other depending on space x . That is:

$$U(x, t) = X(x)T(t). \quad (3.7)$$

In Eq. (3.7), $X(x)$ is a shape function representing the ratio of the displacement at any distance x taken along the length of the beam. By replacing Eq. (3.7) into Eq. (3.6) with the external excitation being equal to zero, it happens that:

$$EIT(t)\frac{\partial^4 X(x)}{\partial x^4} + \rho SX(x)\frac{\partial^2 T(t)}{\partial t^2} = 0. \quad (3.8)$$

This implies:

$$\frac{1}{X(x)}EI\frac{\partial^4 X(x)}{\partial x^4} = -\frac{1}{T(t)}\rho S\frac{\partial^2 T(t)}{\partial t^2}. \quad (3.9)$$

Eq. (3.9) holds if both sides of the equation are equal to the same constant. If the constant is called ω^2 , the following equation holds:

$$\frac{1}{X(x)}EI\frac{\partial^4 X(x)}{\partial x^4} = -\frac{1}{T(t)}\rho S\frac{\partial^2 T(t)}{\partial t^2} = \omega^2, \quad (3.10)$$

where ω^2 is called the separation constant. Therefore, Eq. (3.10) can be split into two different equations and the partial derivatives transformed to right derivatives since each equation depends on one variable only, says:

$$\frac{d^2 T(t)}{dt^2} + \omega^2 T(t) = 0, \quad (3.11a)$$

$$\frac{EI}{\rho S}\frac{d^4 X(x)}{dx^4} - \omega^2 X(x) = 0. \quad (3.11b)$$

Equation (3.11a) is a homogeneous linear ordinary differential equation to the second-order (Gundersen, 1988). Consequently, its solution is known from mathematical calculations and

is equal to:

$$T(t) = a \cos \omega t + b \sin \omega t, \quad (3.12)$$

where a and b are constants to be determined using the initial conditions of the system.

On the other hand, Eq. (3.11b) is a homogeneous fourth-order linear ordinary differential equation. It needs further calculations for the solution to be obtained. The equation can be rewritten as:

$$\frac{d^4 X(x)}{dx^4} - \frac{\rho S}{EI} \omega^2 X(x) = 0. \quad (3.13)$$

To simplify Eq. (3.13) a new parameter β^4 is set so that $\beta^4 = \frac{\rho S}{EI} \omega^2$. Therefore, Eq. (3.13) is transformed into:

$$\frac{d^4 X(x)}{dx^4} - \beta^4 X(x) = 0 \quad (3.14)$$

Let $X(x) = e^{\mu x}$ be a solution of Eq. (3.14) for some μ , a constant to be determined. Then, the characteristic equation of Eq. (3.14) is derived to be:

$$\mu^4 - \beta^4 = 0, \text{ implying that } \mu = \begin{cases} \pm \beta \\ \pm i\beta \end{cases} \quad (3.15)$$

From the characteristic equation, the general solution of (3.14) is found to be (Barber, 2010)

$$X(x) = A_1 e^{\beta x} + A_2 e^{-\beta x} + A_3 e^{i\beta x} + A_4 e^{-i\beta x}. \quad (3.16)$$

By rearranging this expression of $X(x)$, it can also be put in the form:

$$X(x) = A'_1 (\cos \beta x + \cosh \beta x) + A'_2 (\cos \beta x - \cosh \beta x) + A'_3 (\sin \beta x + \sinh \beta x) + A'_4 (\sin \beta x - \sinh \beta x), \quad (3.17)$$

where $A'_i (i = 1, \dots, 4)$ are integration constants to be determined using the boundary conditions of the beam. That means that any shape $X(x)$ may be selected, which satisfies the geometric boundary conditions of the beam. At this stage, one can combine Eqs. (3.12) and (3.17) to obtain the expression of $U(x, t)$, the solution of Eq. (3.6) for free vibration. However, the complete solution for a particular structure requires expressions for the displacement, slope, moment, and shear at the supports, which must be substituted in $U(x, t)$. Therefore, the solution depends on the type of supports and thus the boundary conditions (Lai et al., 2008).

To proceed, a simply supported beam at both ends is considered. The boundary conditions impose no displacement and no moment at both ends (Zingoni, 2018b). The equations explaining that are the following:

$$\begin{cases} U(0,t) = U(L,t) = 0 & \Rightarrow X(0) = X(L) = 0 \\ \text{and} \\ \frac{\partial^2 U(0,t)}{\partial x^2} = \frac{\partial^2 U(L,t)}{\partial x^2} = 0 & \Rightarrow \frac{d^2 X(0)}{dx^2} = \frac{d^2 X(L)}{dx^2} = 0. \end{cases} \quad (3.18)$$

Then, by applying the above boundary conditions, it is found that

$$A'_1 = A'_2 = 0, \quad (3.19a)$$

$$\begin{cases} A'_3(\sin \beta L + \sinh \beta L) + A'_4(\sin \beta L - \sinh \beta L) = 0 \\ A'_3(-\sin \beta L + \sinh \beta L) - A'_4(\sin \beta L + \sinh \beta L) = 0. \end{cases} \quad (3.19b)$$

The first sub-equation of Eq. (3.19b) implies:

$$A'_3 = \frac{-\sin \beta L + \sinh \beta L}{\sin \beta L + \sinh \beta L} A'_4. \quad (3.20a)$$

On the other hand, for Eq. (3.19b) to hold, its determinant must be equal to zero. That means:

$$\begin{vmatrix} \sin \beta L + \sinh \beta L & \sin \beta L - \sinh \beta L \\ -\sin \beta L + \sinh \beta L & -\sin \beta L - \sinh \beta L \end{vmatrix} = 0 \quad (3.20b)$$

$$\begin{aligned} \Rightarrow & -\sin^2 \beta L - 2 \sin \beta L \sinh \beta L - \sinh^2 \beta L + \sin^2 \beta L + \sinh^2 \beta L - 2 \sin \beta L \sinh \beta L = 0 \\ \Rightarrow & \sin \beta L \sinh \beta L = 0. \end{aligned} \quad (3.21)$$

Eq. (3.21) implies that $\beta = 0$ is one root of the equation. This also implies that one trivial solution for Eq. (3.14) is $X = 0$. Since $U(x,t) = X(x)T(t)$, it is noticed that $U(x,t) = 0$ is also one trivial solution of Eq. (3.6) without a second member. But the fact is that this solution does not help in the sense that, for the displacement equal to 0, there is no motion. Hence, other roots of Eq. (3.21) which will produce non-zero solutions of Eq. (3.14) and thus non-zero displacement are of interest.

The other roots are obtained by considering $\beta \neq 0$. However, from the trigonometric hyperbolic functions (Mitsuishi and Yang, 1999), $\sinh \beta L$ exists and is different from zero.

Then, Eq. (3.21) is satisfied for $\sin \beta L = 0$, implying $A'_3 = A'_4$. Hence, the n solutions of equation $\sin \beta L = 0$ are given by:

$$\beta_n = \frac{\pi n}{L}, \text{ where } n \in \mathbb{N}^* \text{ is the mode's number.} \quad (3.22)$$

Thus, the eigenfunctions $X_n(x)$, solutions of Eq. (3.14) can be written as:

$$X_n(x) = A'_n \sin\left(\frac{n\pi}{L}x\right). \quad (3.23)$$

By combining Eqs. (3.12) and (3.23), the solution to Eq. (3.6) for free vibration is deduced as:

$$U(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \left[A_n \cos(\omega_n t) + B_n \sin(\omega_n t) \right] \text{ with } \omega_n = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho S}}, \quad (3.24)$$

where $A_n = A'_n a$ and $B_n = A'_n b$ can be found knowing the initial conditions of the system. It is seen that the exact solution of the beam in Figure 3.1 is straightforward because the resulting equation of motion is linear. Table 3.1 shows the values of a steel beam to be considered in calculations.

Table 3.1: Properties of the steel beam for the analysis.

Item	Notation	Value
Young's modulus	E	200 GPa
Length	L	2 m
width	l	0.05 m
height	h	0.03 m
Poisson's ratio	ν	0.3
mass density	ρ	8050 kg/m ²

The values presented in Table 3.1 are the mechanical and physical properties of the steel beam to be used for our analysis. The values of Young's modulus E , Poisson's ratio ν , and density ρ were chosen in the standard range of values, whereas the beam's dimensions were chosen for illustration purposes and as an example of application. Note that the length of 2 m has been used because ground beams in a building can vary from 2 m being the shortest to 20 m being the longest. The values in Table 3.1 will be used in all the studies in this thesis involving a steel beam unless otherwise specified. Then, the second moment of inertia I and

the cross-sectional area S can be respectively calculated as:

$$I = \frac{1}{12}l \cdot h^3 = 1.125 \times 10^{-7} \text{ m}^4 \text{ and } S = l \cdot h = 15 \times 10^{-4} \text{ m}^2.$$

Taking those values for I and S into account, the natural frequencies ω_n of the simple slender beam are found as:

$$\begin{aligned} \omega_n &= \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{EI}{\rho S}} \\ &= 106.5 \cdot n^2 \text{ rad/s}, \quad n \in \mathbb{N}^* \end{aligned} \quad (3.25)$$

Hence, for each mode n , we have a new natural frequency. Based on Eq. (3.25), the natural frequencies increase with the mode's number because other parameters in the equation are all constant. Next, the first initial condition is imposed so that at $t = 0$, the displacement is equal to 1, meaning that $U(x, t = 0) = 1$, implying $A_n = 1$. On the other hand, the second initial condition is such that the derivative of the displacement is equal to ω_n , meaning that $\dot{U}_n(x, t = 0) = \omega_n$. This gives $B_n = 1$. Those values for A_n and B_n are to be used for every mode. Then, the transversal displacements of the beam for different modes n by varying x along the beam length are obtained for a fixed time $t = 1$ s. Figure 3.2 shows the results.

Figure 3.2 shows the results obtained from the plot of $U(x, t)$ as a function of x for four values of the parameter n . For all the cases, the normalised amplitude is equal to 1. That is because of the presence of cosine and sine functions in the expression of the displacement $U(x, t)$ (see Eq. (3.24)). Therefore, the actual spatial amplitude depends on how the system behaves initially in motion. In what follows, a numerical study using FEA in Abaqus is performed.

It is important to state that analytical methods have some advantages over numerical ones. First, an analytical method can be faster, and the solution is often considered to be exact. Second, through mathematical expressions, the method allows viewing what variables are important and how their relationships to other variables could affect the results or solutions, meaning that there is transparency. Finally, it is clear through mathematical expressions to see the effect of the inputs on the outputs. Nevertheless, we must sometimes have recourse to a numerical method because it can do more steps beyond what mathematical expressions would. This also serves as a benchmark in order to validate the results obtained through an analytical method. The following section focuses on a numerical study using FEA.

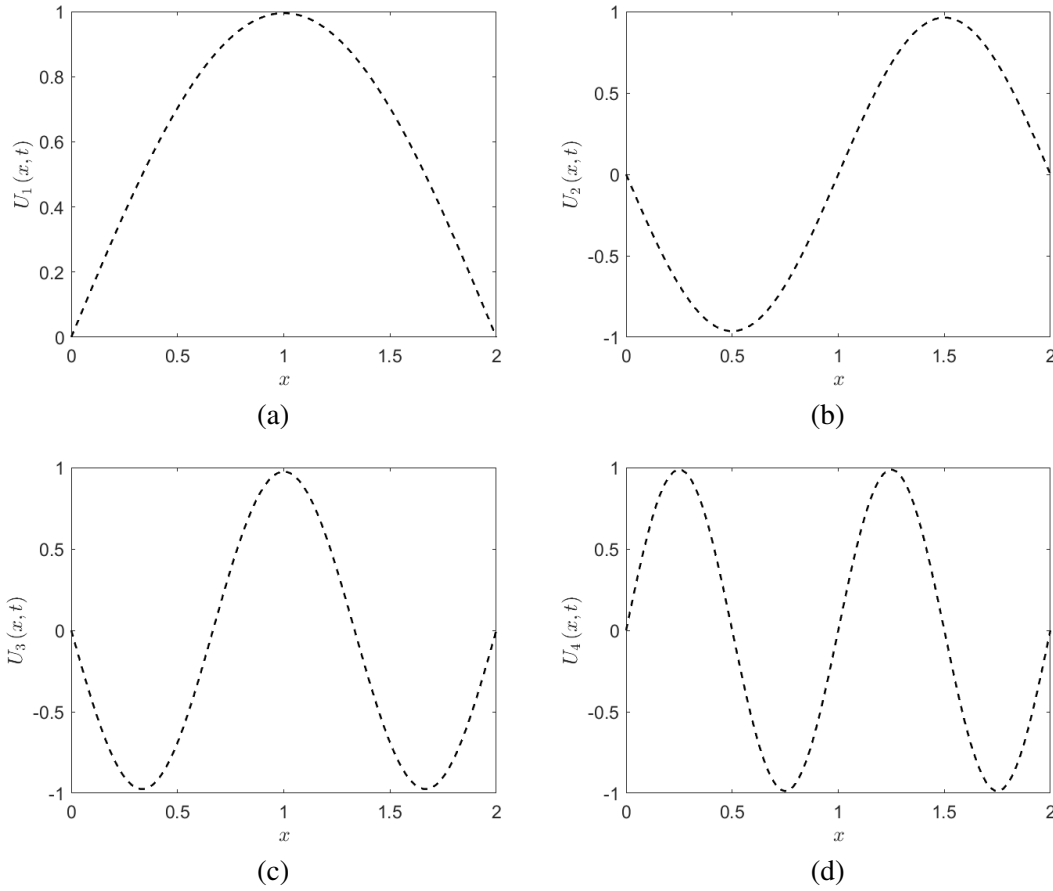


Figure 3.2: Spatial responses of the simple slender beam: (a) to (d) are modes 1 to 4.

3.3 Numerical Study of the Simple 2D Beam Element using FEA in Abaqus

3.3.1 Theoretical Formulation of the Finite Element Model of the Dynamic Behaviour of the Simple Slender Beam

The beam to be investigated in this section is a simple 2D beam element of length L having the properties revealed in Table 3.1. In an attempt to give a brief theoretical formulation of the finite element model, the beam is considered as displayed in Figure 3.3.

Figure 3.3 shows a two-node beam element of length L . Each beam node has two degrees of freedom (DOFs) because displacement and slope continuities over the beam are required. These give in total four DOFs: two vertical displacements (w_1 at node one and w_2

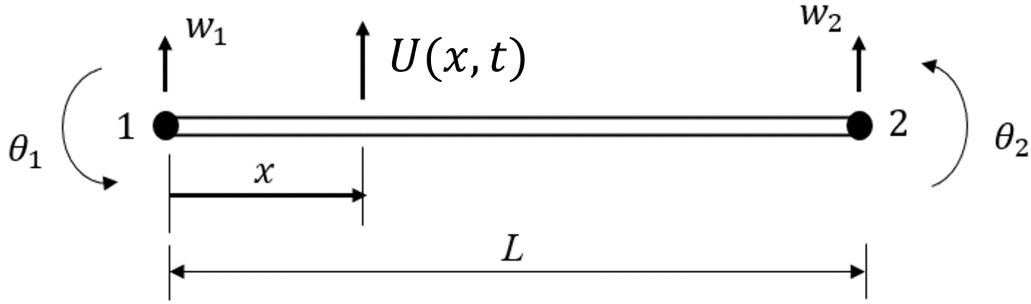


Figure 3.3: A two-node beam element of length L (Zingoni, 2018b).

at node two) and two rotations (θ_1 at node one and θ_2 at node two). The overall displacement of the beam $U(x, t)$ is formed with both spatial and temporal parts. Then, the next thing is to derive the shape functions by knowing the representation of the beam element and the number of DOFs as above-shown in Figure 3.3.

The polynomial function which is used to represent the spatial part of the beam displacement is a cubic polynomial defined as (Cook et al., 2001; Hutton, 2004; Zingoni, 2018b):

$$U = U(x, t) = a_0 + a_1x + a_2x^2 + a_3x^3, \quad (3.26)$$

where a_0 , a_1 , a_2 , and a_3 are polynomial coefficients to be determined. Taking the slope of Eq. (3.26), it holds

$$\frac{dU(x, t)}{dx} = a_1 + 2a_2x + 3a_3x^2. \quad (3.27)$$

Then, nodal boundary conditions (Cook et al., 2001; Hutton, 2004; Zingoni, 2018b) are applied to Eqs. (3.26) and (3.27) such that the following equations hold:

$$\text{at } x = 0, w = w_1 \text{ and } \frac{dw}{dx} = \theta_1; \quad (3.28a)$$

$$\text{at } x = L, w = w_2 \text{ and } \frac{dw}{dx} = \theta_2. \quad (3.28b)$$

Taking Eqs. (3.28) into account in Eqs. (3.26) and (3.27), the coefficients a_0 , a_1 , a_2 , and a_3 are deduced in terms of the nodal DOFs as:

$$a_0 = w_1, \text{ and } a_1 = \theta_1, \quad (3.29a)$$

$$w_2 = a_0 + a_1L + a_2L^2 + a_3L^3, \text{ and } \theta_2 = a_1 + 2a_2L + 3a_3L^2. \quad (3.29b)$$

Hence, solving for a_2 and a_3 as a function of w_1 , w_2 , θ_1 , and θ_2 knowing a_0 and a_1 gives:

$$a_2 = \frac{3}{L^2}(w_2 - w_1) - \frac{1}{L}(2\theta_1 + \theta_2) \quad \text{and} \quad a_3 = -\frac{2}{L^3}(w_2 - w_1) + \frac{1}{L^2}(\theta_1 + \theta_2). \quad (3.30)$$

To proceed, the expressions for a_0 , a_1 , a_2 , and a_3 are substituted back into the expression of $U(x, t)$ in Eq. (3.26), and the equation becomes:

$$\begin{aligned} U(x, t) &= w_1 + \theta_1 x + \left[\frac{3}{L^2}(w_2 - w_1) - \frac{1}{L}(2\theta_1 + \theta_2) \right] x^2 - \left[\frac{2}{L^3}(w_2 - w_1) - \frac{1}{L^2}(2\theta_1 + \theta_2) \right] x^3 \\ &= \left(1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3 \right) w_1 + \left(x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3 \right) \theta_1 + \\ &\quad \left(\frac{3}{L^2}x^2 - \frac{2}{L^3}x^3 \right) w_2 + \left(-\frac{1}{L}x^2 + \frac{1}{L^2}x^3 \right) \theta_2 \\ &= N_1(x)w_1 + N_2(x)\theta_1 + N_3(x)w_2 + N_4(x)\theta_2, \end{aligned} \quad (3.31)$$

$$\text{with } N_1(x) = 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3; \quad N_2(x) = x - \frac{2}{L}x^2 + \frac{1}{L^2}x^3; \quad (3.32)$$

$$N_3(x) = \frac{3}{L^2}x^2 - \frac{2}{L^3}x^3; \quad \text{and} \quad N_4(x) = -\frac{1}{L}x^2 + \frac{1}{L^2}x^3.$$

The expressions in Eq. (3.32) stand for the shape functions for a simple 2D beam element of length L . Figure 3.4 shows the plots of the above shape functions obtained in MATLAB. The parameter x along the horizontal axis varies along the beam length L

Considering the expressions of the shape functions as given in Eq. (3.32), $U(x, t)$ can be rewritten on a matrix form as follows:

$$U(x, t) = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = [\mathbf{N}] \{ \mathbf{d} \}, \quad (3.33)$$

where $[\mathbf{N}]$ is the shape function matrix and $\{ \mathbf{d} \}$ is the nodal degree of freedom vector.

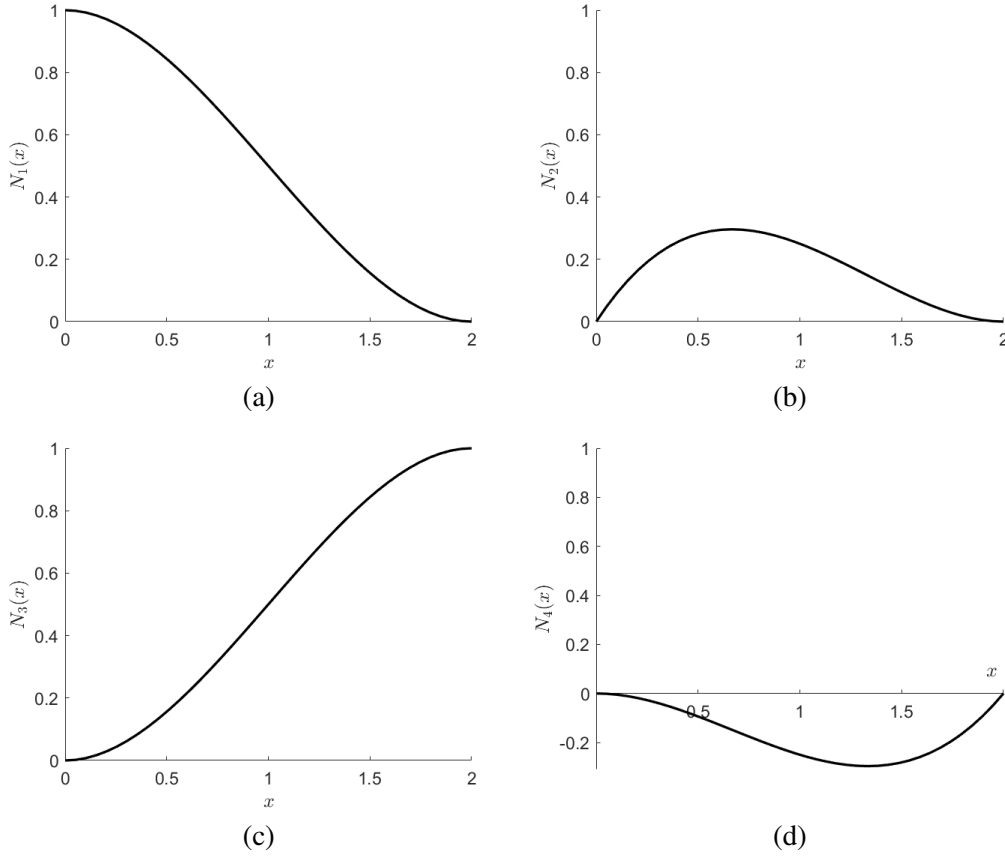


Figure 3.4: Plots of the shape functions: (a) $N_1(x)$, (b) $N_2(x)$, (c) $N_3(x)$, (d) $N_4(x)$.

In Subsection 3.2.1, the equation of motion of the simple beam element was derived and given by:

$$\rho S \frac{\partial^2 U(x,t)}{\partial t^2} + EI \frac{\partial^4 U(x,t)}{\partial x^4} = f(x,t). \quad (3.34)$$

Since the displacement $w = U(x,t)$ can be written in terms of the nodal DOF and the shape functions as:

$$U(x,t) = N_1(x)w_1(t) + N_2(x)\theta_1(t) + N_3(x)w_2(t) + N_4(x)\theta_2(t), \quad (3.35)$$

with $N_i(x)$, ($i = 1, \dots, 4$) given in Eq. (3.32), the application of the Galerkin method to Eq. (3.34) for a finite beam element of length L results in the following residual equations (Hutton, 2004; Rao, 2004):

$$\int_0^L N_i(x) \left(\rho S \frac{\partial^2 U(x,t)}{\partial t^2} + EI \frac{\partial^4 U(x,t)}{\partial x^4} \right) dx = \int_0^L N_i(x) f(x,t) dx, \quad i = 1, \dots, 4. \quad (3.36)$$

All the parameters of Eq. (3.36) have already been defined in the previous sections and remain unchanged. Since ρ and S are both constants, it follows that:

$$\rho S \int_0^L N_i (N_1 \ddot{w}_1 + N_2 \ddot{\theta}_1 + N_3 \ddot{w}_2 + N_4 \ddot{\theta}_2) dx = \rho S \int_0^L N_i [\mathbf{N}] \begin{Bmatrix} \ddot{w}_1 \\ \ddot{\theta}_1 \\ \ddot{w}_2 \\ \ddot{\theta}_2 \end{Bmatrix} dx. \quad (3.37)$$

$$\text{implying } \rho S \int_0^L [\mathbf{N}]^T [\mathbf{N}] \begin{Bmatrix} \ddot{w}_1 \\ \ddot{\theta}_1 \\ \ddot{w}_2 \\ \ddot{\theta}_2 \end{Bmatrix} dx = [\mathbf{M}] \{\ddot{\mathbf{d}}\}, \quad (3.38)$$

where $[\mathbf{N}]^T$ is the transpose of the shape function matrix $[\mathbf{N}]$, $[\mathbf{M}] = \rho S \int_0^L [\mathbf{N}]^T [\mathbf{N}] dx$ is the consistent mass matrix for a 2D beam, and $\{\ddot{\mathbf{d}}\}$ is the second derivative concerning time of the nodal degree of freedom vector $\{\mathbf{d}\}$. This allows us to write the elements of the consistent mass matrix as:

$$[\mathbf{M}] = \rho S \int_0^L [\mathbf{N}]^T [\mathbf{N}] dx = \rho S \int_0^L \begin{bmatrix} N_1 N_1 & N_1 N_2 & N_1 N_3 & N_1 N_4 \\ N_2 N_1 & N_2 N_2 & N_2 N_3 & N_2 N_4 \\ N_3 N_1 & N_3 N_2 & N_3 N_3 & N_3 N_4 \\ N_4 N_1 & N_4 N_2 & N_4 N_3 & N_4 N_4 \end{bmatrix} dx \quad (3.39)$$

After calculating all the products $N_i N_j$, ($i, j = 1, \dots, 4$) and replacing them into Eq. (3.39), the elements of the mass matrix $[\mathbf{M}]$ are given by the following numerical values:

$$[\mathbf{M}] = \frac{\rho S L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \quad (3.40)$$

On the other hand, the second integral term of Eq. (3.36) is given by:

$$\int_0^L N_i(x) EI \frac{\partial^4 U(x,t)}{\partial x^4} dx = \int_0^L N_i(x) \frac{d^2}{dx^2} \left(EI \frac{d^2 U(x,t)}{dx^2} \right) dx, \quad i = 1, \dots, 4. \quad (3.41)$$

By performing twice the integral by parts over Eq. (3.41) it happens that:

$$\int_0^L \frac{d^2 N_i(x)}{dx^2} EI \frac{d^2 U(x,t)}{dx^2} dx = \int_0^L N_i(x) EI \frac{d^4 U(x,t)}{dx^4} dx - \left[N_i(x) \frac{d}{dx} \left(EI \frac{d^2 U(x,t)}{dx^2} \right) \right]_0^L + \left[\frac{dN_i(x)}{dx} EI \frac{d^2 U(x,t)}{dx^2} \right]_0^L, \quad i = 1, \dots, 4. \quad (3.42)$$

We know that the governing equation for a beam flexure is expressed as follows (Cook et al., 2001):

$$\frac{d^2}{dx^2} \left(EI \frac{d^2 U(x,t)}{dx^2} \right) = q(x,t), \quad (3.43)$$

where $q(x,t)$ is a distributed load force per unit length of the beam, the first term on the right-hand side of Eq. (3.42) becomes:

$$\int_0^L N_i(x) EI \frac{d^4 U(x,t)}{dx^4} dx = \int_0^L N_i(x) q(x,t) dx, \quad i = 1, \dots, 4. \quad (3.44)$$

The second and third terms of the right-hand side of Eq. (3.42) can respectively be written as:

$$- \left[N_i(x) EI \frac{d^3 U(x,t)}{dx^3} \right]_0^L = - \left[N_i(x) V(x) \right]_0^L \quad (3.45a)$$

and

$$\left[\frac{dN_i(x)}{dx} EI \frac{d^2 U(x,t)}{dx^2} \right]_0^L = \left[\frac{dN_i(x)}{dx} M(x) \right]_0^L \quad (3.45b)$$

where $V(x)$ and $M(x)$ are respectively the shear force and bending moment at element nodes of the beam. Therefore, Eq. (3.42) is transformed into:

$$\int_0^L \frac{d^2 N_i(x)}{dx^2} EI \frac{d^2 U(x,t)}{dx^2} dx = \int_0^L N_i(x) q(x,t) dx - \left[N_i(x) V(x) \right]_0^L + \left[\frac{dN_i(x)}{dx} M(x) \right]_0^L, \quad \text{where } i = 1, \dots, 4. \quad (3.46)$$

Eq. (3.46) can be written on the matrix form as $[\mathbf{K}] \{\mathbf{d}\} = \{\mathbf{f}\}$, where $[\mathbf{K}]$ is the stiffness matrix with its elements defined as:

$$K_{ij} = \int_0^L \frac{d^2 N_i(x)}{dx^2} EI \frac{d^2 U(x,t)}{dx^2} dx = EI \int_0^L \frac{d^2 N_i(x)}{dx^2} \frac{d^2 U(x,t)}{dx^2} dx, \quad i, j = 1, \dots, 4, \quad (3.47)$$

and $\{\mathbf{f}\}$ is the force vector with the following expression:

$$f_i(x, t) = \int_0^L N_i(x) q(x, t) dx - \left[N_i(x) V(x) \right]_0^L + \left[\frac{dN_i(x)}{dx} M(x) \right]_0^L, \quad i = 1, \dots, 4. \quad (3.48)$$

Because for small displacements the curvature $\mathcal{C}$ of the beam is approximately equal to the second derivative of the displacement with respect to x , the following equation holds:

$$\mathcal{C}(x) \simeq \frac{d^2 U(x, t)}{dx^2} = L(Nd) = \frac{d^2(Nd)}{dx^2} B d, \quad (3.49)$$

where $L = \frac{d^2}{dx^2}$ is an operator and $\mathbf{B} = L(N) = \frac{d^2 N}{dx^2}$ is a matrix equals to the second derivative of the shape function. The expressions for B_1 , B_2 , B_3 , and B_4 are deduced as follows:

$$B_1(x) = \frac{d^2 N_1(x)}{dx^2} = -\frac{6}{L^2} + \frac{12}{L^3} x; \quad B_2(x) = \frac{d^2 N_2(x)}{dx^2} = -\frac{4}{L} + \frac{6}{L^2} x; \quad (3.50)$$

$$B_3(x) = \frac{d^2 N_3(x)}{dx^2} = \frac{6}{L^2} - \frac{12}{L^3} x; \quad \text{and} \quad B_4(x) = \frac{d^2 N_4(x)}{dx^2} = -\frac{2}{L} x + \frac{6}{L^2} x.$$

Therefore,

$$B(x) = \begin{bmatrix} B_1 & B_2 & B_3 & B_4 \end{bmatrix}. \quad (3.51)$$

Next, the element stiffness matrix follows as

$$\begin{aligned} \mathbf{K} &= \int_0^L \mathbf{B}^T (EI) \mathbf{B} dx = EI \int_0^L \mathbf{B}^T \mathbf{B} dx \quad \text{since } EI \text{ is constant} \\ &= EI \int_0^L \begin{bmatrix} B_1 B_1 & B_1 B_2 & B_1 B_3 & B_1 B_4 \\ B_2 B_1 & B_2 B_2 & B_2 B_3 & B_2 B_4 \\ B_3 B_1 & B_3 B_2 & B_3 B_3 & B_3 B_4 \\ B_4 B_1 & B_4 B_2 & B_4 B_3 & B_4 B_4 \end{bmatrix} dx \end{aligned} \quad (3.52)$$

where EI is taken to be constant with the values of E and I given in Subsection 3.2.1. Hence, the elements of the matrix $[\mathbf{K}]$ are calculated and given by:

$$[\mathbf{K}] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \quad (3.53)$$

By combining the mass and stiffness matrices as well as the force vector, the finite element equation of motion for a simple 2D beam element is obtained as:

$$[\mathbf{M}] \{\ddot{\mathbf{U}}\} + [\mathbf{K}] \{\mathbf{U}\} = \int_0^L [\mathbf{N}]^T f(x,t) + \begin{Bmatrix} -V_1(x) \\ -M_1(x) \\ -V_2(x) \\ -M_2(x) \end{Bmatrix} \quad (3.54)$$

where $V_1(x)$, $V_2(x)$ and $M_1(x)$, $M_2(x)$, are respectively shear forces and bending moments at nodes 1 and 2. Since the beam is simply supported at both ends, the bending moments at nodes 1 and 2 are zero. Therefore, the last term of Eq. (3.54) vanishes. Further, no force is acting on the beam. Consequently, the right-hand side of Eq. (3.54) is equal to zero, and the equation which remains is given by:

$$[\mathbf{M}] \{\ddot{\mathbf{U}}\} + [\mathbf{K}] \{\mathbf{U}\} = [0] \quad (3.55)$$

This last equation (Eq. (3.55)) is the general form of the eigensystem encountered in structural engineering using the FEM (Bathe, 2006). In the next sections, it will be solved numerically.

3.3.2 Description of the Numerical Procedure

To obtain numerical solutions corresponding to the analytical results presented in Figure 3.2, the FEA is used to analyse a simply supported beam with the properties given in Table 3.1 via Abaqus/Standard (Documentation Abaqus, 2019). The beam is chosen to be a 2D deformable element. In what follows, flowcharts used for our analysis in Abaqus are presented. Each figure represents the procedure adopted for each module. Recourse was also made to Documentation Abaqus (2019), and SIMULIA (2017a,b).

In the *Part module*, the parts of a given model are created, managed, and edited. The geometry of the part is also defined and the space (2D or 3D) specified. For example, Figure 3.5 shows the flowchart followed to create a 2D planar, deformable beam and input five as the approximate size. Then, a beam of length $L = 2m$ with two nodes using connected lines was created. Once everything had been input correctly, the model was saved.

The *Property module* is where the material properties of the model are defined, and the beam section is created. Figure 3.6 presents a flowchart of the *Property module*. The density value, Young's modulus, and Poisson's ratio were taken as input from Table 3.1. Then, a

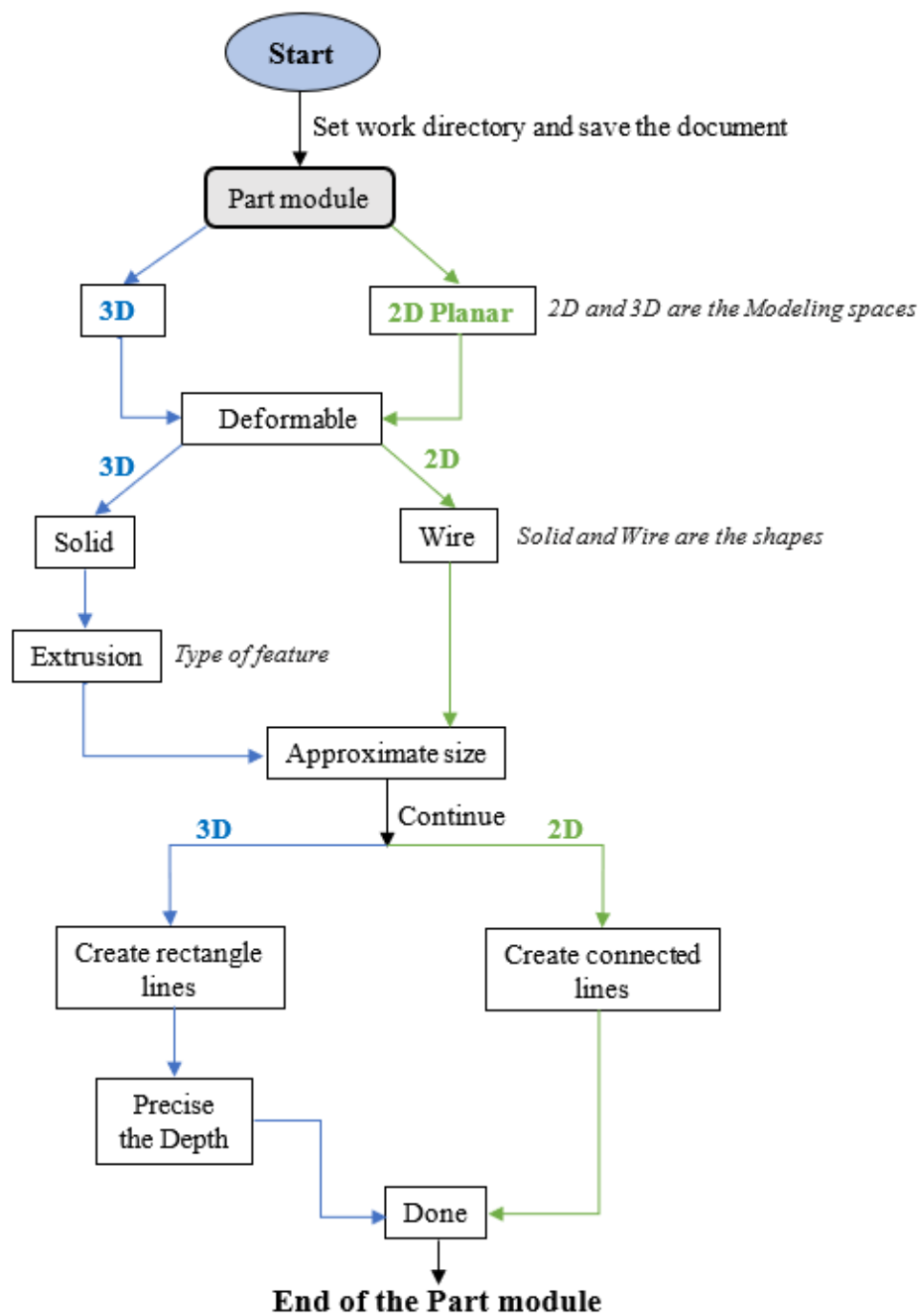


Figure 3.5: Flowchart for the Part module.

rectangular section of $0.05 \text{ m} \times 0.03 \text{ m}$ was created and assigned to the beam. Finally, this module ended by assigning an orientation to the beam as $(0.0, 0.0, -1.0)$, which was normal to the plane where the motion occurred.

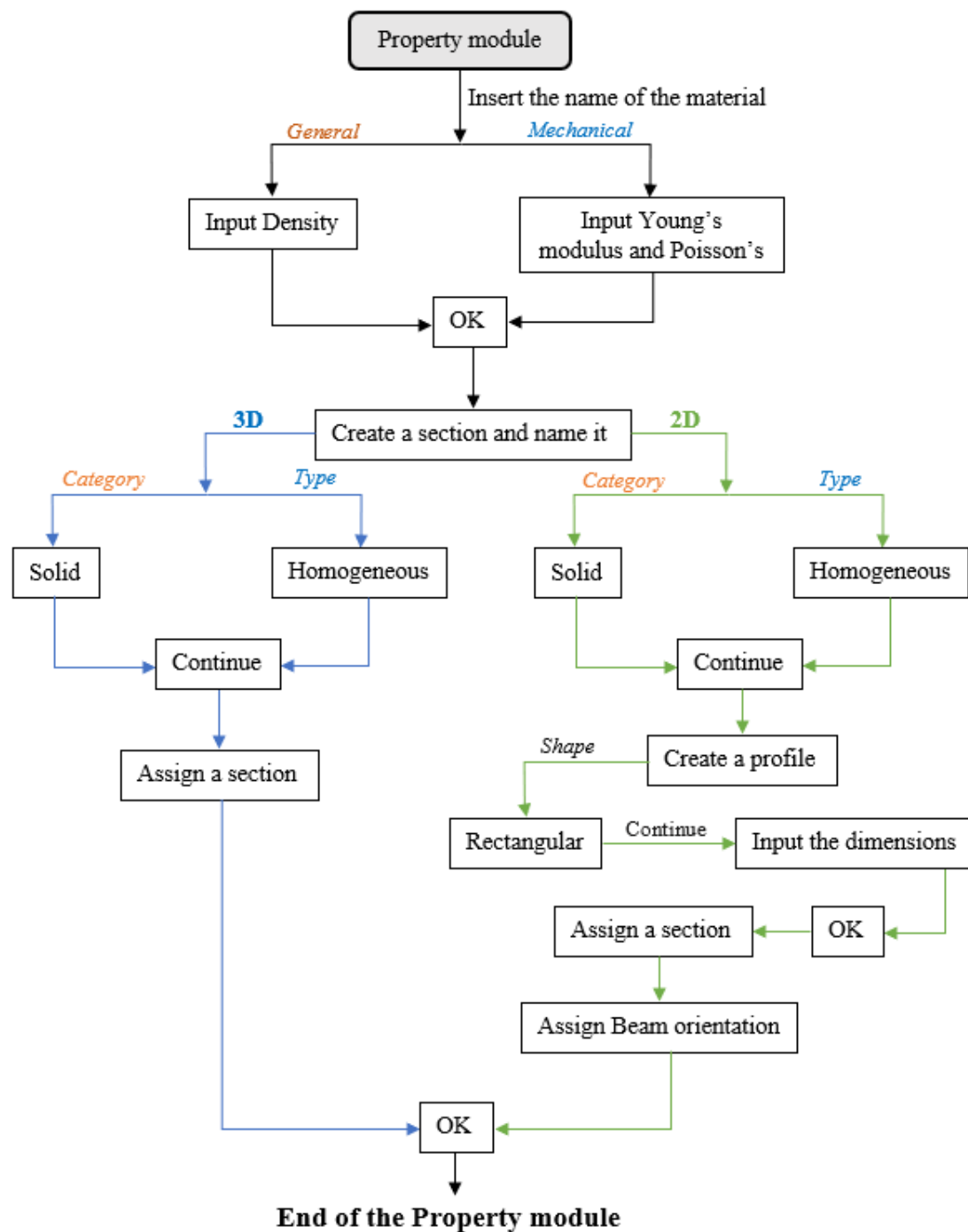


Figure 3.6: Flowchart for the Property module.

In the *Assembly module* instances are created in the parts from the model, and, even if the model only has one part, instances must still be created. This means that, if more than one part is created for the model, the *Assembly module* is used to combine them with only

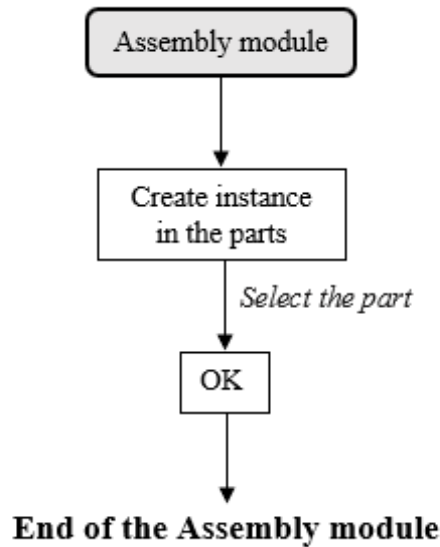


Figure 3.7: Flowchart for the Assembly module.

one part. Then, this new part is chosen to continue the analysis depending on the type of structure to model. As far as this thesis is concerned, only one part was created, and the flowchart in Figure 3.7 describes the procedure followed to assemble the part. Since there was no interaction between the beam and another element, the model was saved and directly moved to the *Step module*.

In the *Step module*, one or a sequence of analysis steps is defined and used to define the loads and boundary conditions of the model. It is also the module where a given number of natural frequencies are requested, and the output variables are specified. Thus, one can create more than one step and specify the output for each step created. The flowchart in Figure 3.8 gives details of the scheme followed. Because this chapter and the entire thesis concentrate on free vibration, only one step was created for solving the eigenfrequency problem using the linear perturbation procedure. Then, the Lanczos method was used to extract the eigenvalues and ten natural frequencies were requested. This method was chosen because it has general capabilities compared to other methods. For instance, it is a very efficient method for calculating eigenvalues and the corresponding mode shapes (Paige, 1972, 1980; SIMULIA, 2017b). After finishing with the *step module*, the *Load module* was concentrated on.

In the *Load module*, loads and boundary conditions are applied to the part. Figure 3.9 gives details about the module. Since this thesis only deals with free vibration, this module was restricted to applying boundary conditions to the 2D beam. The load part of the *Load*

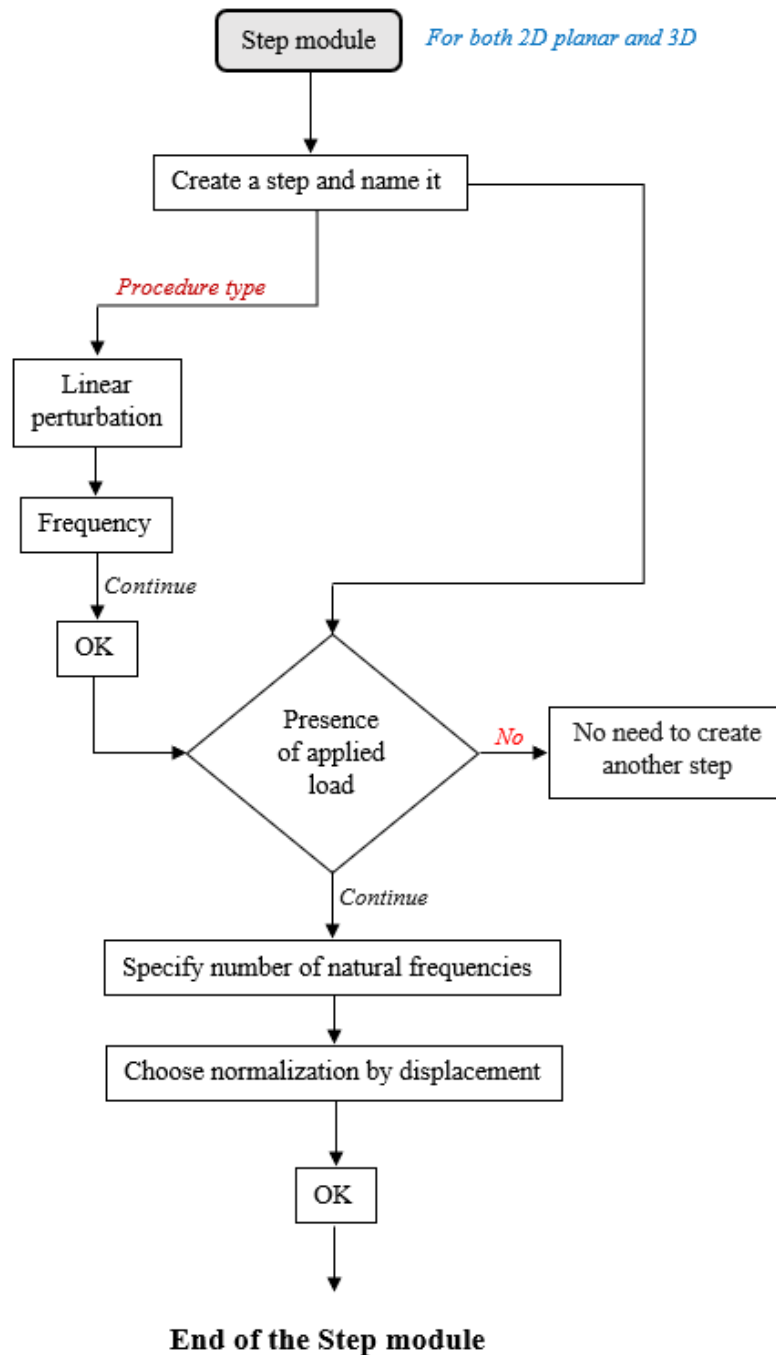


Figure 3.8: Flowchart for the Step module.

module is shown here just to draw attention to other features of the *Load module*. Given that the current study concerns a simply supported beam, the boundary conditions were applied at the initial step, and *Displacement/Rotation* was chosen as the boundary type. The left end was considered a pinned support; therefore, U_1 and U_2 were ticked. In contrast, the right

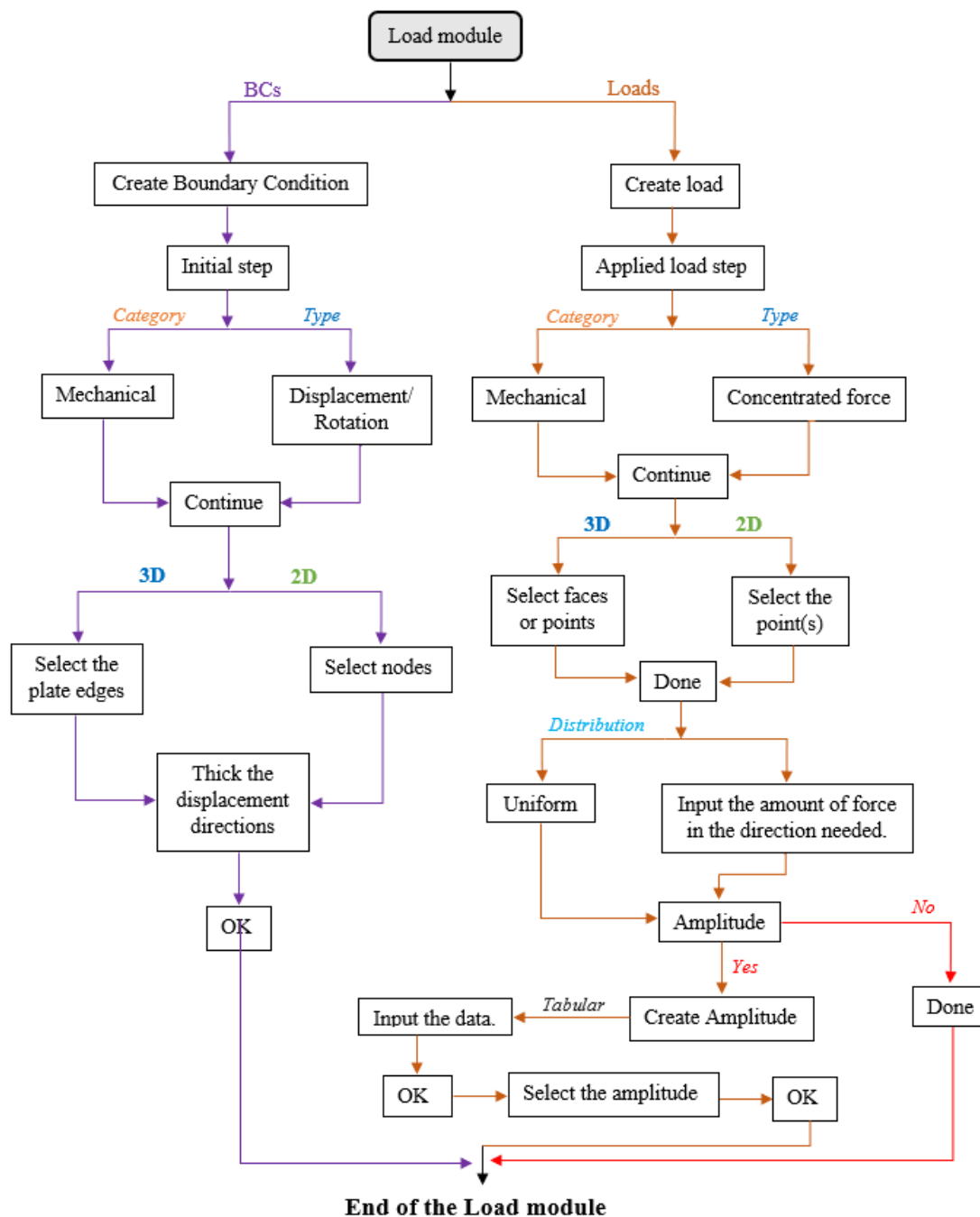


Figure 3.9: Flowchart for the Load module.

end was considered on roller support, so only U_2 was ticked. As for the initial displacement, there was no possibility of defining it in Abaqus. Nevertheless, it was assumed that there was no displacement at the initial state of the beam.

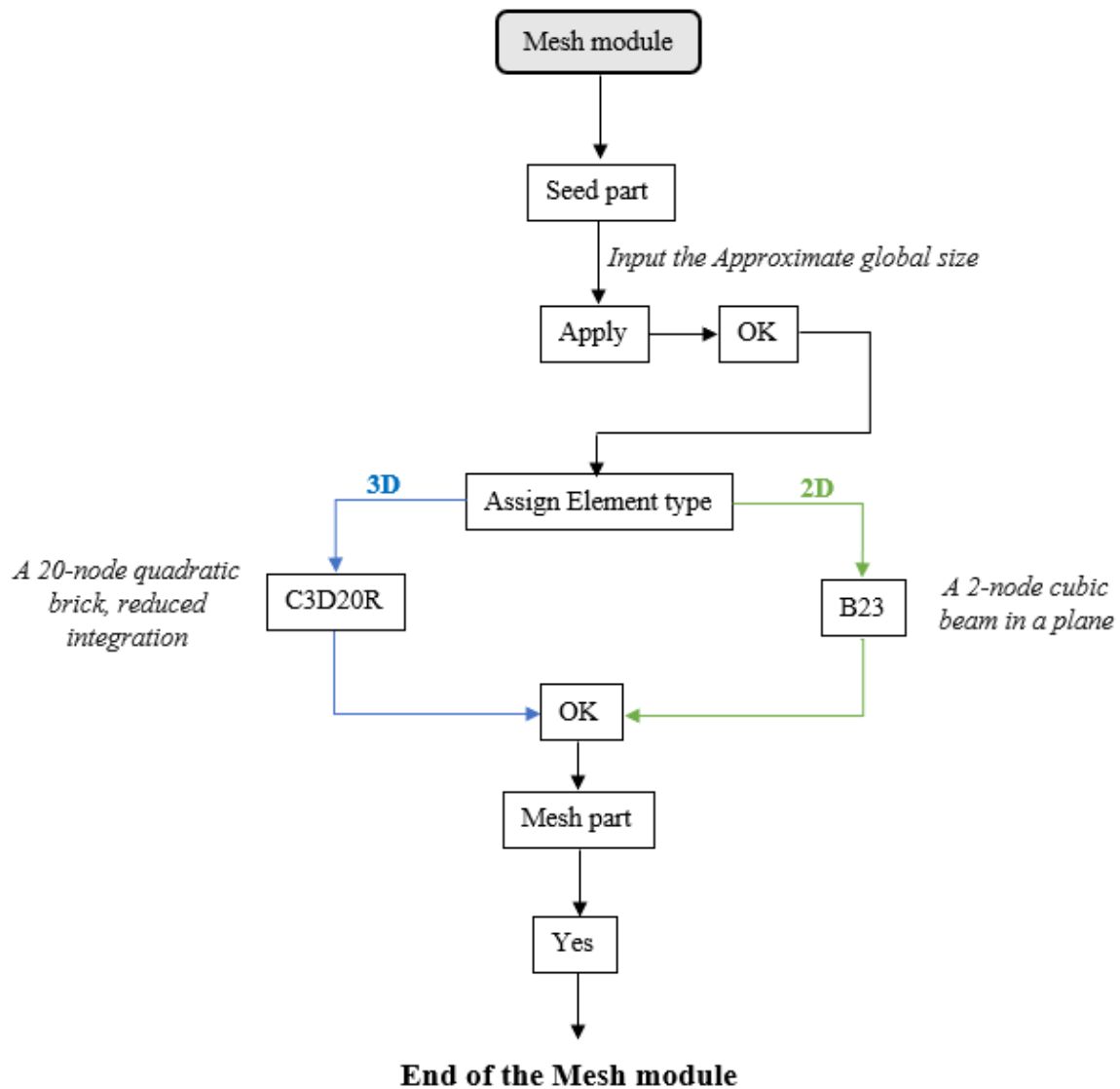


Figure 3.10: Flowchart for the Mesh module.

In Figure 3.10, a flowchart of the *Mesh module* is displayed, which contains tools that allow users to generate meshes on parts and assemblies created for a model. During the meshing of a part, the Element type directly related to the model dimension space must also be assigned. For the current study, the part was seeded with 0.003 as an approximate global size. Then, *B23* was chosen as an element type, a 2-node beam in plane with cubic interpolation. This element type is chosen because it does not account for shear deformation, and therefore, it is well-suitable to the modelling in Abaqus for the Euler-Bernoulli beam theory. The figure below reveals the output of the tasks performed in the Abaqus modules.

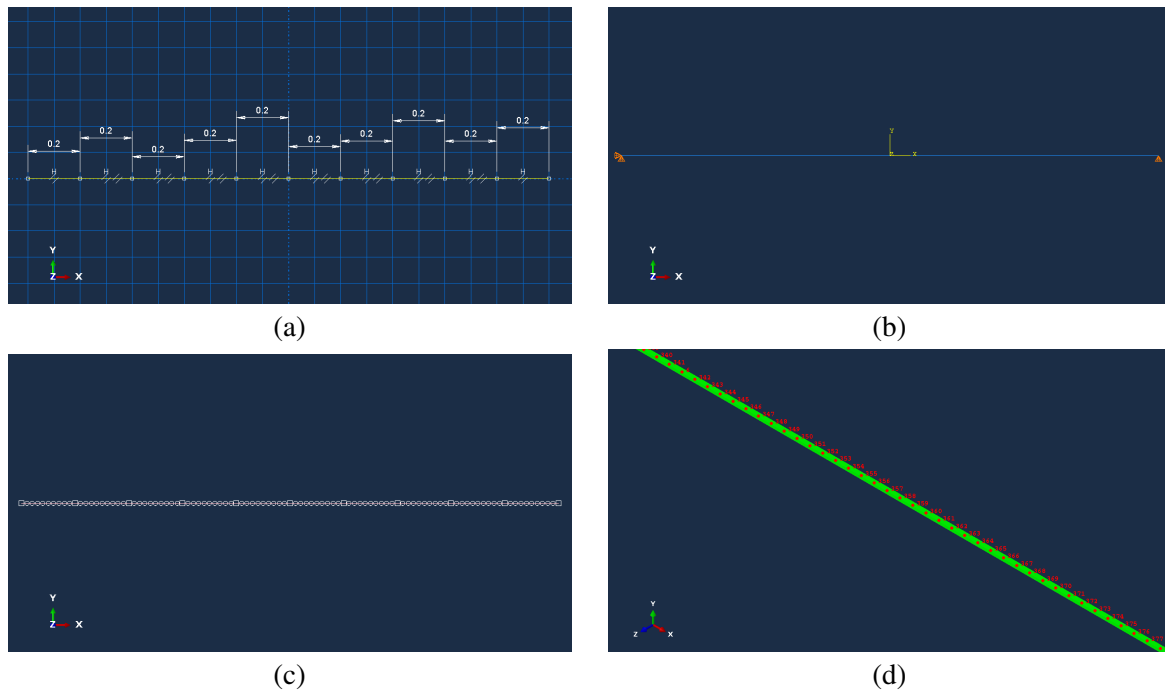


Figure 3.11: Tasks performed in the modules in Abaqus: (a) Creation of the part; (b) Boundary conditions applied; (c) Part seeded; and (d) Mesh generated.

Figure 3.11 shows a 2-node beam (Figure 3.11a), the boundary conditions applied (Figure 3.11b), and the mesh generated after seeding the part. The nodes generated on the mesh can be seen as materialised by the red dots (see Figure 3.11d). After meshing the part, 670 elements were generated.

Once all the steps required to define a model had been undertaken, attention was given to the *Job module* shown in Figure 3.12. That module allowed creating a job for the model, submitting and monitoring its evolution. It is important to highlight that when the analysis is completed without errors, it is also crucial to check the message file to see if there is anything to be highlighted. Moreover, check the model's settings and parameters to ensure that everything has been entered correctly wherever needed. It is equally advised to run several analyses for each case to confirm consistency with the results. Sometimes the analysis can abort. In this case, one must verify that the parameters were entered and defined correctly. The current analysis was completed successfully, and different files were printed, including the *.dat* file. This is a text file in which the analysis input file processor generates information about the model definition. In Abaqus, it is a tabular output of results (Giner et al., 2009). The *.dat* file also contains the values of the natural frequencies requested while entering the input in the *Step module*.

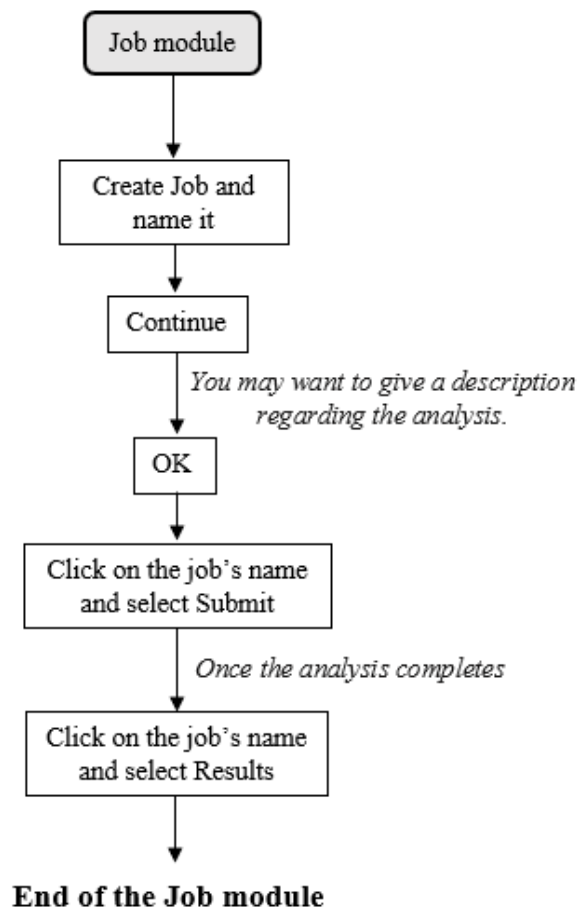


Figure 3.12: Flowchart for the Job module.

The last task in Abaqus is to go through the *Visualisation module*, the one used to visualise the output or results of the analysis as figures. The section below gives details of the outputs.

3.3.3 Results of the Numerical Simulation of a Simple Slender Beam using FEA in Abaqus

In the *Visualization module*, one selects the variable to display in the viewport and decides on the shape, deformed or undeformed. For the present case, the deformed shape of the beam displacement was chosen. That appears as mode shapes the viewport. Table 3.2 displays the natural frequencies obtained during the analysis in Abaqus and those calculated manually from the analytical study.

Table 3.2: Values of the natural frequencies for the first ten modes of vibration of a simple slender beam simply supported at both ends.

Mode n	ω_n (rad/s)		Differences (%)
	Analytical	Abaqus	
1	106.5	106.51	0.0094
2	426.0	426.04	0.0094
3	958.6	958.58	0.0021
4	1704.1	1704.1	0.00
5	2662.7	2662.7	0.00
6	3834.3	3834.3	0.00
7	5219.0	5219.0	0.00
8	6816.6	6816.6	0.00
9	8627.3	8627.3	0.00
10	10651	10651	0.00

Table 3.2 gives the values of the natural frequencies recorded for the first ten vibration modes for analytical calculations and numerical analyses in Abaqus. It can be seen that the differences between the two methods are tiny at fundamental modes and become zero as the mode's number increases. The resulting small percentage of differences justifies the fact that there is an agreement between the analytical and numerical results. Moreover, this confirms that using the *B23* beam element in Abaqus perfectly represents the same assumptions made during analytical calculations. Then, the numerical results are regarded as the benchmark of the solution. Having the frequencies, the mode shapes can be visualised using the *Visualisation module*. Figure 3.13 shows the beam at mode 0, while Figures 3.14 and 3.15 show the outputs for the first four modes of vibration.

In Figure 3.13, it is seen that the beam is at rest at the initial state ($t = 0$). Therefore, there is no displacement, and the legend shows only 0 as the value of the displacement U along the direction $U2$ (the y -direction). Although the initial displacement could not be defined as an input, the current output tells that Abaqus understands it and takes it into account.

Figure 3.14 reveals the analysis results as mode shapes. All information printed in this figure describes each mode shape. The deformation scale factor is the same for all the modes, meaning that the beam deforms similarly at each mode. Moreover, the eigenvalues and natural frequencies are given in cycles/time, increasing as the mode's number increases. The appreciation of the mode shapes is further interpreted by plotting the beam's displacement as a function of the beam's length. That is shown in Figure 3.15.

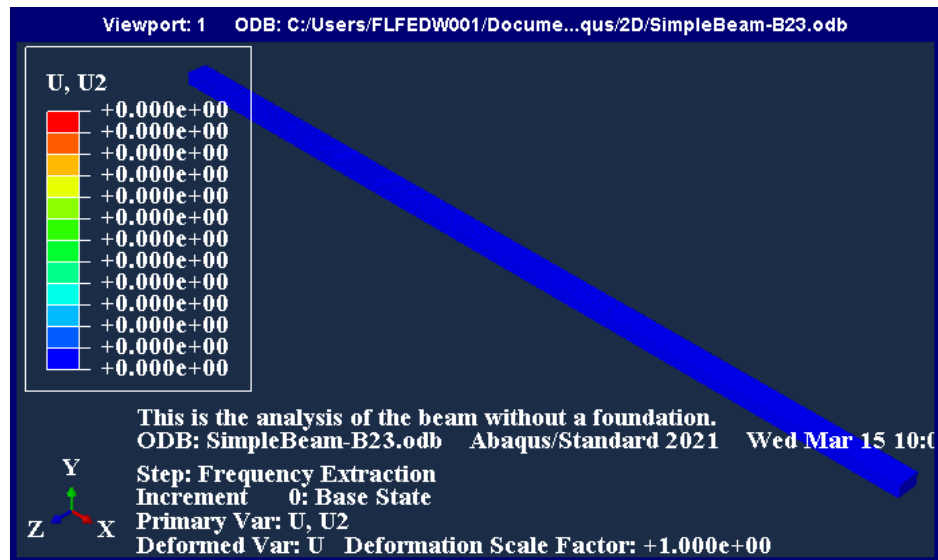


Figure 3.13: Base state of the simple beam.

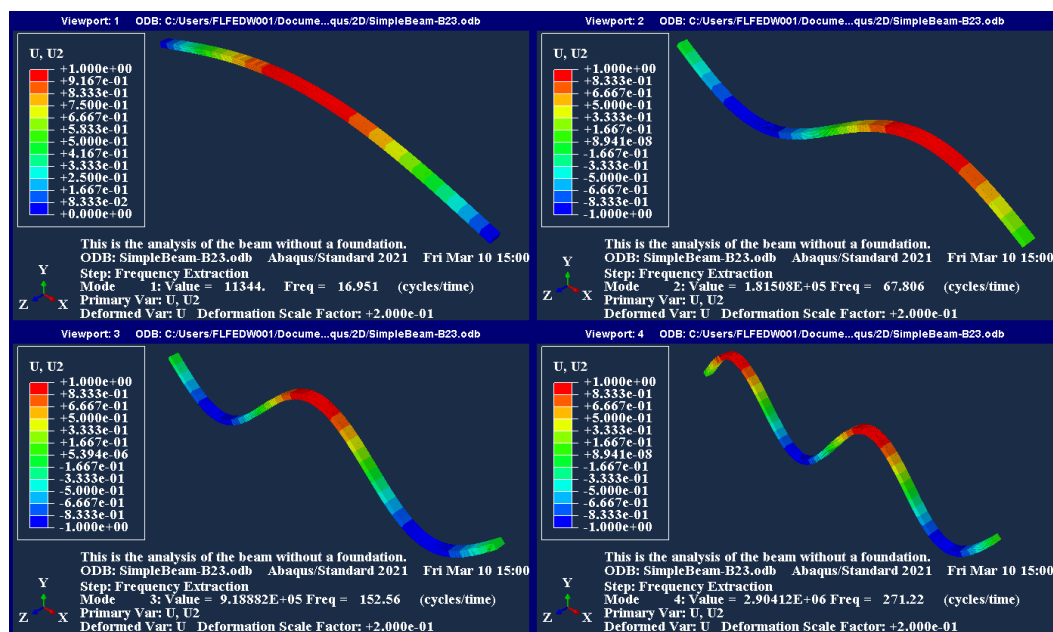


Figure 3.14: Mode shapes of the simple beam obtained numerically in Abaqus.

In Figure 3.15, the beam's minimum and maximum spatial vibration amplitude for the first four modes are pictured. Interestingly, by comparing Figure 3.15 to that shown in Fig 3.2 (obtained analytically), it is noticed that although there exist slight differences between the frequencies issued from the two methods (presented in Table 3.2), the shapes, maximum, and minimum amplitudes for the corresponding modes look alike. This confirms an agreement between the analytical and the numerical methods. In addition, the solutions from Abaqus

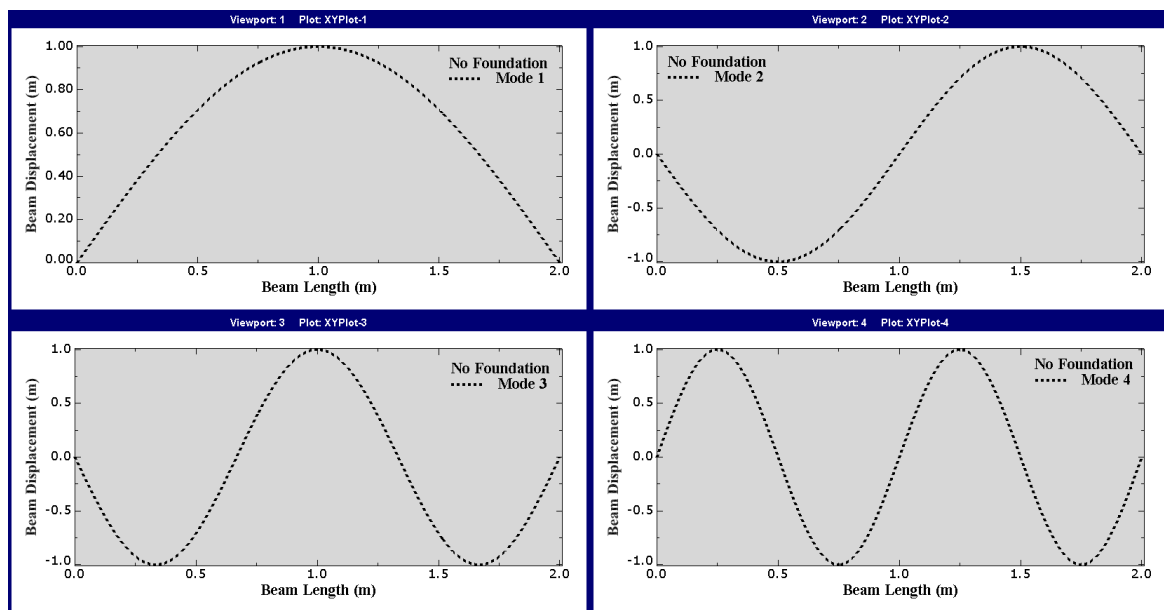


Figure 3.15: Spatial displacement of the simple beam obtained numerically in Abaqus.

taken as a benchmark allow validation of the analytical results obtained using the separation of variables.

To track the temporal evolution of the beam, the beam's amplitude is plotted as a function of time. The fixed position at which these plots were obtained is the centre of the beam, meaning at $x = \frac{L}{2}$. Figure 3.16 displays the outputs obtained from MATLAB. The initial conditions as already defined in 3.2.2, are given by 1 and ω_n , respectively, for the displacement and derivative of the displacement at $t = 0$. Moreover, the analytical and numerical curves were plotted on the same graph to compare results from the two methods.

Figure 3.16 shows the motion of one point on the beam with respect to time. It is seen that there is no difference between analytical and numerical plots; all the curves mix. At each mode of vibration, both curves start simultaneously, and the maximum and minimum amplitudes are reached simultaneously as time increases. It is clear that the periods are not the same at each mode of vibration because the natural frequencies increase as the mode's number increases. Therefore, for a higher mode, the oscillation period will be small compared to that at the fundamental mode. The fact that there is no difference between analytical and numerical curves confirms the agreement between the two methods. We highlight here that the response at even modes is nonzero because we did not consider the spatial part of the displacement solution while plotting the curves. The response of the beam will definitely tend to zero at even mode if that is considered.

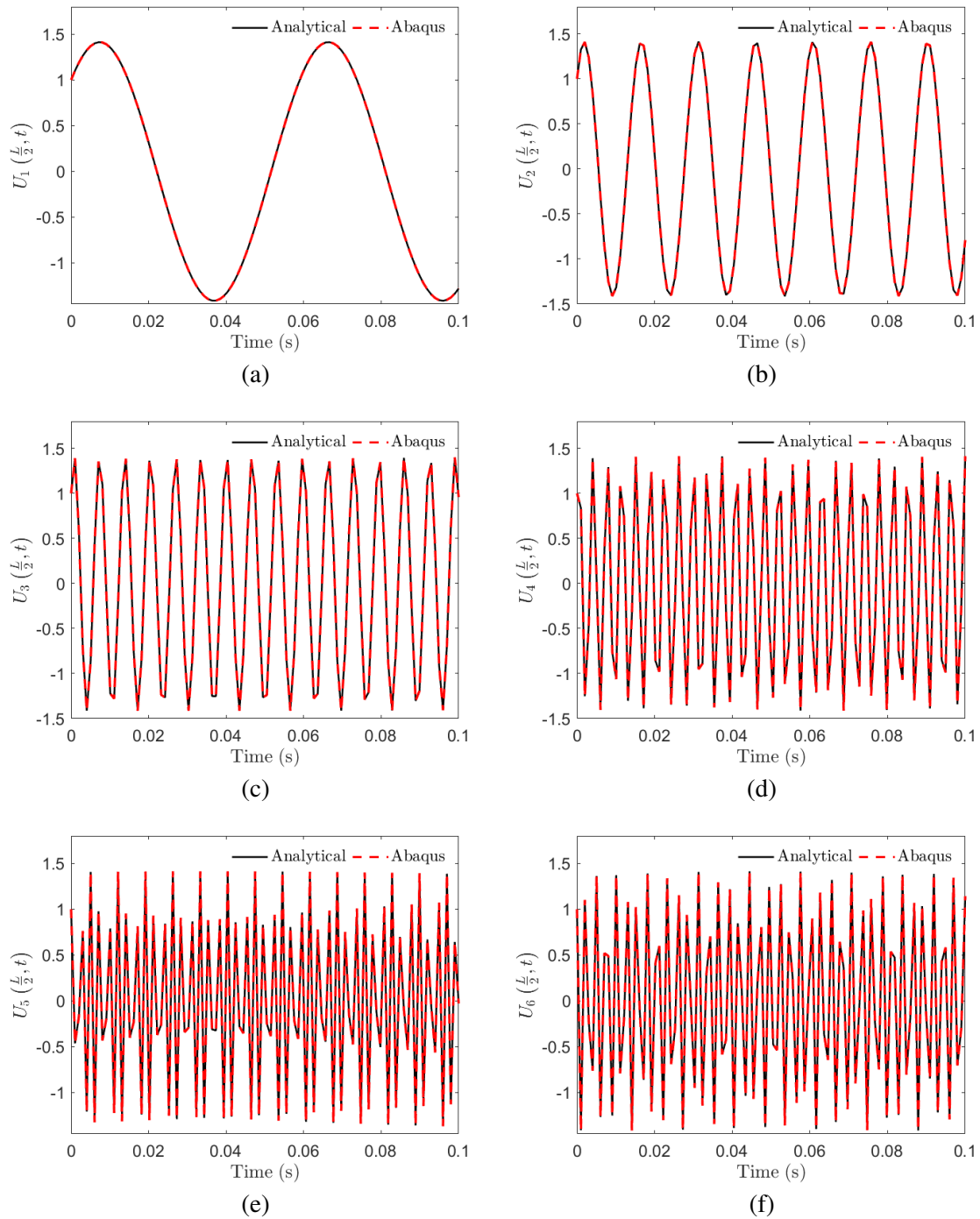


Figure 3.16: Temporal displacement at the centre of the beam: (a) to (f) are modes 1 to 6.

The following section studies the vibration of a 2D beam element on an elastic foundation having linear stiffness properties. The same procedure will be followed as in the case of a simple beam. Moreover, analytical and numerical analyses will be performed. In addition,

the influence of the foundation's linear elastic coefficient on the beam's vibration will be highlighted. Furthermore, the results issued from the two methods will be compared.

3.4 Vibration of Beams on Elastic Foundations having Linear Stiffness Properties

The support foundation is considered elastic if the reaction force offered by the continuous support is a function of the displacement of the members, and a beam resting on elastic supports is said to be a BoEF. According to Dillard et al. (2018), Emil Winkler (1867) was the first to formulate a problem of an elastic beam on a deformable foundation. He initially proposed that model in application to sleepers and rails on earth supports upon which they rest (Dillard et al., 2018). Therefore, the model of a BoEF with a linear force-deflection relationship is known as the beam on the Winkler Foundation (Winkler, 1867). It is important to point out that although Winkler's model is advantageous, it has some limitations. Firstly, there is no interaction between the springs. Secondly, using that approach, the continuous characteristic of the foundations is not perfectly represented, and all the predictions are considered at one point. Lastly, it is not helpful in the predictions of ground movement, which may arise due to earthquakes (Hayir, 2010).

Examples of application of BoEFs with linear stiffness properties include but are not limited to the concrete footing on earth foundations, railroad tracks, and long steel pipes resting on a series of elastic springs (Jones, 1997). In railroad track applications, the member subjected to load can be supported on continuous foundations or by sleepers (also called railroad ties), meaning that the reactions due to external loading are distributed along the length of the member. As for slab tracks, they have continuous support.

3.4.1 Analytical Study of a Beam on an Elastic Foundation using the Separation of Variables

In this subsection, the equation of motion of a beam resting on elastic linear springs is first derived. Then, the resulting equation of motion is solved using the separation of variables. The figure below displays the system to be studied.

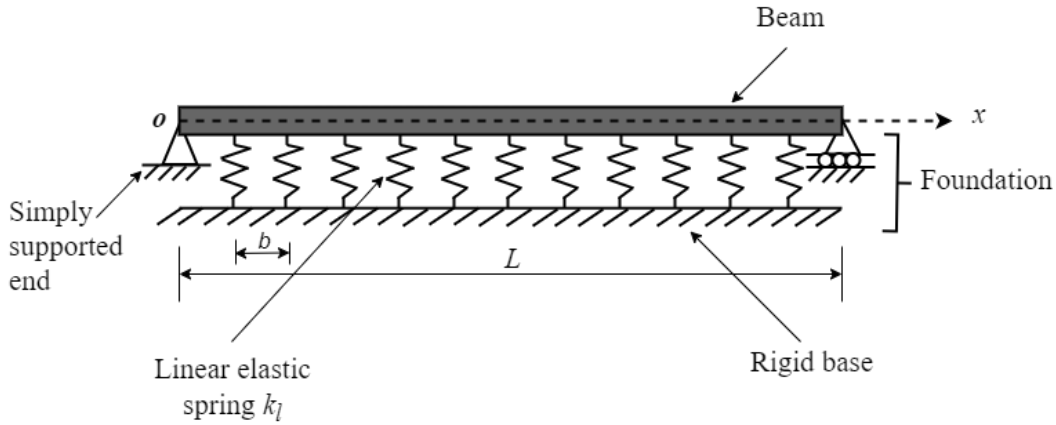


Figure 3.17: Beam on an elastic foundation having linear elastic springs.

Figure 3.17 adopted from Hetényi (1971) presents a beam of length L with the linear elastic foundation made of spring supports equally spaced at a distance b and having each a stiffness coefficient per unit length k_l . The motion of the beam is described by a transversal movement $U(x, t)$. In the real situation, the distribution of forces acting on the beam by the foundation is a nonlinear function of the beam length coordinate. Nevertheless, as the beam length decreases or the beam becomes stiffer and stiffer, the distribution tends to behave linearly. In this case, solving problems of BoEFs as those of linear problems produces relevant results.

With a focus on establishing the governing equation of the system in Figure 3.17, an element of the beam is considered where to apply the shear, moment, restoring, and excitation forces. Then, to the procedure followed in the case of the simple beam in Section 3.2, is added the force generated by the foundation. The following figure, inspired by Limkatanyu et al. (2013) gives a brief representation of an element of the beam with forces applied.

Figure 3.18 shows an element of the beam with the applied forces such as the shear forces $V(x)$ and $V(x + dx)$, the moments $M(x)$ and $M(x + dx)$, the excitation force $f(x, t)$ and the reaction force generated by the springs of the foundation $k_l U(x, t)$. One can notice that, by considering the beam subjected to any vertical downward force $f(x, t)$ as shown in Figure 3.18, there is a reaction force $k_l U(x, t)$ produced by each spring to counteract the downward force. The dashed line represents the reference or neutral axis of the beam. Following the procedure used in Subsection 3.2.1, all the forces along the z -direction are summed, and this allows writing:

$$-V(x + dx) + V(x) = f(x, t) - k_l U(x, t) \Rightarrow \frac{dV(x)}{dx} = f(x, t) - k_l U(x, t). \quad (3.56)$$

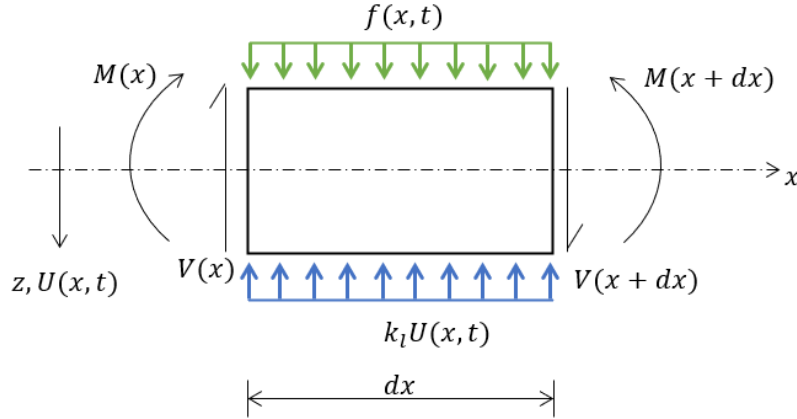


Figure 3.18: An element of the beam and the forces applied.

Knowing the relationship between the shear force and the moment, the following equation holds:

$$\frac{dM(x)}{dx} = V(x). \quad (3.57)$$

Considering Eq. (3.56) into Eq. (3.57) gives:

$$\frac{d^2M(x)}{dx^2} = f(x, t) - k_l U(x, t). \quad (3.58)$$

Also, the relation between the moment and the curvature is recalled as being:

$$M(x) = EI \frac{d^2U(x, t)}{dx^2}. \quad (3.59)$$

By replacing $M(x)$ (Eq. (3.59)) into Eq. (3.58) it holds:

$$EI \frac{d^4U(x, t)}{dx^4} + k_l U(x, t) = f(x, t). \quad (3.60)$$

Eq. (3.60) is valid only in static analysis. Since time is considered, the inertia force given by $\rho S \frac{\partial^2 U(x, t)}{\partial t^2}$ must be added. Therefore, Eq. (3.60) is transformed into:

$$\rho S \frac{\partial^2 U(x, t)}{\partial t^2} + EI \frac{\partial^4 U(x, t)}{\partial x^4} + k_l U(x, t) = f(x, t), \quad (3.61)$$

where the first term stands for the inertia force, and the second and third terms both represent the elastic force. The last term denotes the external excitation acting on the system to induce vibrations. Since the beam rests on continuous elastic supports, the solution to the problem is

found using Eq. (3.61) in which the value of k_l is given by:

$$k_l = \frac{k_b}{b}, \text{ where } k_b \text{ stands for the beam stiffness defined as } k_b = \frac{12EI}{L^3}, \quad (3.62)$$

This relation of the linear stiffness of the foundation (see Eq. (3.62)) is considered here based on the range found in the literature (Barber, 2011; Frydrysek et al., 2013). Nevertheless, it should be noted that the stiffness value of the foundation depends on the properties of the soil. It is clear that as k_l tends to infinity, the foundation gets stiffer. We recall that only free vibration is considered in this thesis, so, Eq. (3.61) is rewritten as:

$$\rho S \frac{\partial^2 U(x,t)}{\partial t^2} + EI \frac{\partial^4 U(x,t)}{\partial x^4} + k_l U(x,t) = 0 \quad (3.63)$$

To proceed, the transversal displacement $U(x,t)$ is written using the separation of variables as given by the following equation:

$$U(x,t) = Q(x)\tau(t). \quad (3.64)$$

Then, Eq. (3.57) becomes:

$$Q(x) \frac{\partial^2 \tau(t)}{\partial t^2} + \frac{EI}{\rho S} \tau(t) \frac{\partial^4 Q(x)}{\partial x^4} + \frac{k_l}{\rho S} Q(x) \tau(t) = 0. \quad (3.65)$$

This also implies:

$$Q(x) \frac{\partial^2 \tau(t)}{\partial t^2} + \left[\frac{EI}{\rho S} \frac{\partial^4 Q(x)}{\partial x^4} + \frac{k_l}{\rho S} Q(x) \right] \tau(t) = 0, \quad (3.66)$$

which can also be rewritten as:

$$\begin{aligned} \frac{1}{\tau(t)} \frac{\partial^2 \tau(t)}{\partial t^2} + \frac{1}{Q(x)} \left[\frac{EI}{\rho S} \frac{\partial^4 Q(x)}{\partial x^4} + \frac{k_l}{\rho S} Q(x) \right] &= 0 \\ \Rightarrow \underbrace{\frac{1}{Q(x)} \left[\frac{EI}{\rho S} \frac{\partial^4 Q(x)}{\partial x^4} + \frac{k_l}{\rho S} Q(x) \right]}_{g(x)} &= - \underbrace{\frac{1}{\tau(t)} \frac{\partial^2 \tau(t)}{\partial t^2}}_{h(t)}. \end{aligned} \quad (3.67)$$

Having $g(x) = h(t)$, implies $g \equiv \text{constant} = \gamma^2$, where γ will be determined shortly. Hence,

$$\frac{1}{Q(x)} \left[\frac{EI}{\rho S} \frac{\partial^4 Q(x)}{\partial x^4} + \frac{k_l}{\rho S} Q(x) \right] = \gamma^2 = -\frac{1}{\tau(t)} \frac{\partial^2 \tau(t)}{\partial t^2} \quad (3.68)$$

From Eq. (3.68), the partial derivatives are now right derivatives and given by:

$$\frac{d^2 \tau(t)}{dt^2} + \gamma^2 \tau(t) = 0 \quad (3.69a)$$

$$\frac{d^4 Q(x)}{dx^4} - \left[\frac{\rho S}{EI} \gamma^2 - \frac{k_l}{EI} \right] Q(x) = 0 \quad (3.69b)$$

Because Eq. (3.69a) is a second-order differential equation with no second member, its solution is known and deduced as:

$$\tau(t) = A \cos(\gamma t) + B \sin(\gamma t), \quad (3.70)$$

where A and B are constants to be determined using the initial conditions of the system. Now it remains solving the second equation given by:

$$\frac{d^4 Q(x)}{dx^4} - \left[\frac{\rho S}{EI} \gamma^2 - \frac{k_l}{EI} \right] Q(x) = 0. \quad (3.71)$$

To simplify the Eq. (3.71), a new parameter λ^4 is set such that:

$$\frac{1}{EI} (\rho S \gamma^2 - k_l) = \lambda^4, \quad (3.72)$$

Therefore, Eq. (3.71) is reduced to:

$$\frac{d^4 Q(x)}{dx^4} - \lambda^4 Q(x) = 0 \quad (3.73)$$

The characteristic equation issue from Eq. (3.73) is:

$$\Omega^4 - \lambda^4 = 0 \Rightarrow \Omega = \begin{cases} \pm \lambda \\ \pm i\lambda \end{cases} \quad (3.74)$$

Then, the solution of Eq. (3.73) is found to be:

$$Q(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x} + C_3 e^{i\lambda x} + C_4 e^{-i\lambda x}. \quad (3.75)$$

However, $Q(x)$ can also be written as:

$$Q(x) = D_1(\cos \lambda x + \cosh \lambda x) + D_2(\cos \lambda x - \cosh \lambda x) + D_3(\sin \lambda x + \sinh \lambda x) + D_4(\sin \lambda x - \sinh \lambda x), \quad (3.76)$$

where the coefficients $D_i (i = 1, \dots, 4)$ can be determined using boundary conditions of the beam. It is supposed that the beam is simply supported at both ends $x = 0$ and $x = L$. That implies that the boundary conditions are known. Then, applying the boundary conditions for the displacement, results in:

Then, applying the boundary conditions results in the following:

$$D_1 = D_2 = 0, \quad (3.77a)$$

$$\sin \lambda L \sinh \lambda L = 0. \quad (3.77b)$$

Undoubtedly, $\lambda = 0$ is a trivial solution of Eq. (3.77b) but is not needed here because nonzero displacements are of interest. So, only nonzero roots ($\lambda \neq 0$) are considered. However, knowing that for $\lambda \neq 0$, $\sinh \lambda L$ does not vanish, Eq. (3.77b) is satisfied if $\sin \lambda L = 0$ which will require $D_3 = D_4$. Hence, the m solutions of the equation $\sin \lambda L = 0$ are given by:

$$\lambda_m = \frac{\pi m}{L}, \quad m \in \mathbb{N}^*. \quad (3.78)$$

Therefore, the solution $Q_m(x)$ of Eq. (3.73) is deduced to be:

$$Q_m(x) = D_m \sin \left(\frac{m\pi}{L} x \right). \quad (3.79)$$

From Eq. (3.72), γ is found as follows:

$$(\rho S \gamma^2 - k_l) = EI \lambda^4 \Rightarrow \gamma^2 = \frac{1}{\rho S} (k_l + EI \lambda^4). \quad (3.80)$$

Then, replacing λ_m by its expression given in Eq. (3.78), the natural frequencies ω_l of the beam resting on a linear elastic foundation are expressed as follows:

$$\omega_l^2 = \frac{1}{\rho S} \left[k_l + EI \left(\frac{m\pi}{L} \right)^4 \right], \quad \text{implying} \quad \omega_l = \sqrt{\frac{1}{\rho} \left[\frac{k_l}{S} + \frac{EI}{S} \left(\frac{m\pi}{L} \right)^4 \right]}. \quad (3.81)$$

Thus, by combining Eqs. (3.70) and (3.79), the transversal displacement of the beam resting on a linear elastic foundation is obtained as:

$$\begin{aligned} U(x,t) &= \sum_{m=0}^{\infty} \left[A \cos(\omega_l t) + B \sin(\omega_l t) \right] D_m \sin\left(\frac{m\pi}{L}x\right) \\ &= \sum_{m=0}^{\infty} \left[A_m \cos(\omega_l t) + B_m \sin(\omega_l t) \right] \sin\left(\frac{m\pi}{L}x\right), \end{aligned} \quad (3.82)$$

where A_m and B_m are respectively given by $A \cdot D_m$ and $B \cdot D_m$.

Eq. (3.82) is the solution of the governing equation of a linear beam resting on a linear elastic foundation. Then, having the constants A_m , B_m , the natural frequencies and the time, the solutions can be obtained as plots. In the following subsection, the results using MATLAB of Eq. (3.82) are presented.

3.4.2 Analytical Results of a Beam on a Linear Elastic Foundation (BoLEF)

All the assumptions in this thesis represent an idealisation that closely approximates the real situation. The amplitudes A_m and B_m at each mode will be considered constant and equal to 1 as in Subsection 3.2.1. This is to avoid giving a greater value that will impose an initial amplitude to the beam's displacement, and only consider the effects of the foundation on the beam. On the other hand, the distance between two consecutive springs will be $b = 0.2$ m. To appreciate the analytical results, MATLAB was used to plot the displacement $U(x,t)$ given in Eq. (3.82) by varying the coordinate x along the beam length. The output produces spatial amplitudes of vibration of the beam. These are shown in Figure 3.19.

Figure 3.19 exhibits the plots of the first four modes of vibration of the linear BoLEF, obtained from MATLAB for $t = 1$. The maximum amplitude is one because it is a sine function. Comparing Figure 3.19 to Figure 3.2, it can be seen that the curves look the same, with the max and min amplitudes reached at the same time value. That was expected because the stiffness of a vibrating system does not affect the shape or amplitude of the sine/cosine curve, but the frequencies. That means that by focusing on Figure 3.19 it can be seen that the effects of the linear elastic foundation on the beam's response are not noticeable because the behaviour of the beam remains unchanged. The following subsection presents the numerical results recorded after completing the analysis in Abaqus.

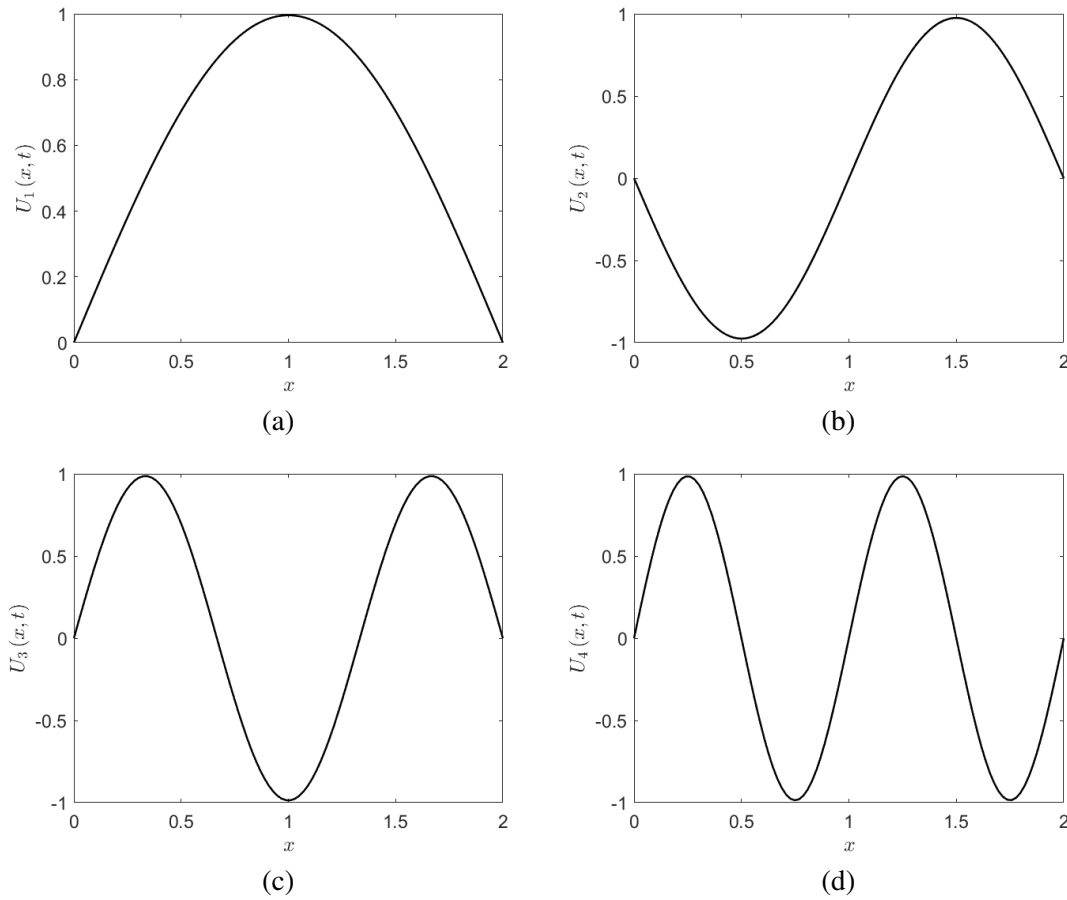


Figure 3.19: First four mode shapes of the simply supported beam resting on a linear elastic foundation: (a) to (d) are modes 1 to 4.

3.5 Numerical Analysis of a BoLEF using FEA in Abaqus

Due to the considerable time, energy, and effort involved in finding solutions to possible problems apropos of BoEFs using analytical methods, numerical methods or analyses are often applied in order to benchmark the solution. In this chapter, FEA through Abaqus/Standard is used to analyse the beam presented in Figure 3.17, validate, and benchmark the accuracy of the analytical solutions. During the analysis in Abaqus, a part was considered a 2D planar beam element. Then, the beam was divided into 10 elements separated by nodes. The number of DOFs was equal to two at each beam extremity as for a simple beam. Moreover, the step was taken as a linear perturbation to request the natural frequencies.

3.5.1 Theoretical Formulation of the Finite Element Model of the Dynamic Behaviour of the BoLEF

When building a model theoretically, one constructs the structure and adds the elastic foundations as several springs on a rigid base. In Abaqus, it is not the case, and the elastic foundations are a simple way of including the stiffness effects of supports without modelling the details of the supports. The foundation is assigned by specifying its stiffness per unit length of the beam. The theoretical formulation of the finite element model of a BoLEF is derived using the following figure.

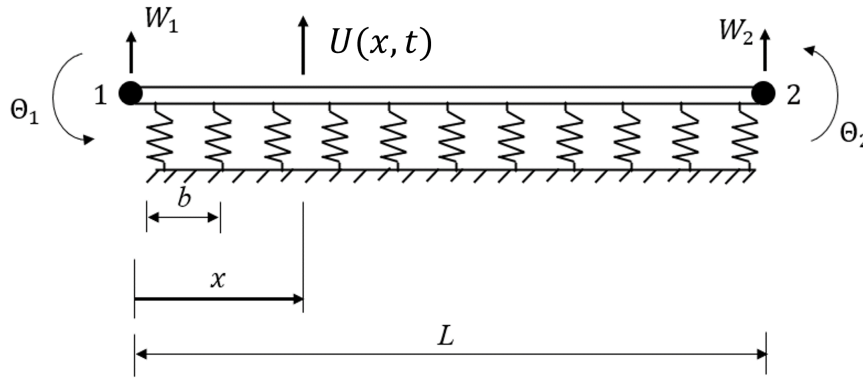


Figure 3.20: A two-node beam element of length L resting on an elastic foundation.

Figure 3.20 adopted from (Tiwari and Kuppaa, 2014) shows a 2-nodded BoEF. The elastic foundation can be seen here as a system of mutually independent springs. In Abaqus, the *Interaction module* is used to create a linear or nonlinear elastic foundation and assign it to the beam. However, the equation for the FEM formulation of the model must first be derived.

The procedure starts by finding the shape functions. These are obtained by following the same procedure as Section 3.3 from Eq. (3.26) to Eq. (3.32). Cook et al. (2001) and Tiwari and Kuppaa (2014) stated that the shape functions of a beam resting on a linear elastic foundation remain identical to those of a simple beam. Consequently, the shape functions given in Eq. (3.32) are the same to be used in this section.

The formulation to be presented is that of Eq. (3.63) recalled here as:

$$\rho S \frac{\partial^2 U(x, t)}{\partial t^2} + EI \frac{\partial^4 U(x, t)}{\partial x^4} + k_l U(x, t) = 0. \quad (3.83)$$

Here also, the nodal displacements are assumed to be time-dependent. By applying the Galerkin method to Eq. (3.83), the residual equation is obtained as follows:

$$\int_0^L N_i(x) \left(\rho S \frac{\partial^2 U(x,t)}{\partial t^2} + EI \frac{\partial^4 U(x,t)}{\partial x^4} + k_l U(x,t) \right) dx = 0, \quad i = 1, \dots, 4 \quad (3.84)$$

The integral of the first term of Eq. (3.84) allows deriving the mass matrix. Since the springs used here are massless, the foundation does not impact the inertial force. Bearing that in mind, the mass matrix of a 2D BoEF is the same as a simple 2D beam. That is the previously calculated one given in Eq. (3.40).

On the contrary, the same cannot be considered for the stiffness matrix. This is because there are now two elastic forces: one produced by the beam $\left(EI \frac{\partial^4 U(x,t)}{\partial x^4} \right)$ and the other produced by the foundation $(k_l U(x,t))$. Therefore, the stiffness matrix given in Eq. (3.53) must be modified by adding that of the foundation. To begin with, the following equation holds:

$$\int_0^L N_i(x) k_l U(x,t) dx = k_l \int_0^L N_i(x) [\mathbf{N}] \{\mathbf{D}\} dx, \quad i = 1, \dots, 4 \quad (3.85)$$

By varying i four equations are obtained, which, after combining them, transform the right-hand side of Eq. (3.85) into:

$$\mathbf{K}_f = k_l \int_0^L [\mathbf{N}]^T [\mathbf{N}] \{\mathbf{D}\} dx, \quad (3.86)$$

where $[\mathbf{N}]$ and $[\mathbf{N}]^T$ are respectively the shape function matrix and its transpose; $\{\mathbf{D}\}$ is the nodal DOF vector defined as:

$$\{\mathbf{D}\} = \begin{Bmatrix} W_1 \\ \Theta_1 \\ W_2 \\ \Theta_2 \end{Bmatrix}; \quad (3.87)$$

while $\mathbf{K}_f$ is the element stiffness matrix of the foundation given by

$$\mathbf{K}_f = k_l \int_0^L \begin{bmatrix} N_1 N_1 & N_1 N_2 & N_1 N_3 & N_1 N_4 \\ N_2 N_1 & N_2 N_2 & N_2 N_3 & N_2 N_4 \\ N_3 N_1 & N_3 N_2 & N_3 N_3 & N_3 N_4 \\ N_4 N_1 & N_4 N_2 & N_4 N_3 & N_4 N_4 \end{bmatrix} dx \quad (3.88)$$

By calculating all the $N_{ij} = N_i N_j$ we obtain the elements of the matrix $[\mathbf{K}_f]$. Then, the stiffness matrix of the linear elastic foundation is determined as follows

$$[\mathbf{K}_f] = k_l \begin{bmatrix} \frac{13}{35}L & \frac{11}{210}L^2 & \frac{9}{70}L & -\frac{13}{420}L^2 \\ \frac{11}{210}L^2 & \frac{1}{105}L^3 & \frac{13}{420}L^2 & -\frac{1}{140}L^3 \\ \frac{9}{70}L & \frac{13}{420}L^2 & \frac{13}{35}L & -\frac{11}{210}L^2 \\ -\frac{13}{420}L^2 & -\frac{1}{140}L^3 & -\frac{11}{210}L^2 & \frac{1}{105}L^3 \end{bmatrix} \quad (3.89)$$

By combining Eq. (3.53) and Eq. (3.89) the stiffness matrix of the BoLEF $[\mathbf{K}_{\text{BoEF}}]$ is obtained as:

$$[\mathbf{K}_{\text{BoEF}}] = [\mathbf{k}_b] + [\mathbf{K}_f]$$

$$= \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} + k_l \begin{bmatrix} \frac{13}{35}L & \frac{11}{210}L^2 & \frac{9}{70}L & -\frac{13}{420}L^2 \\ \frac{11}{210}L^2 & \frac{1}{105}L^3 & \frac{13}{420}L^2 & -\frac{1}{140}L^3 \\ \frac{9}{70}L & \frac{13}{420}L^2 & \frac{13}{35}L & -\frac{11}{210}L^2 \\ -\frac{13}{420}L^2 & -\frac{1}{140}L^3 & -\frac{11}{210}L^2 & \frac{1}{105}L^3 \end{bmatrix} \quad (3.90)$$

To proceed, the relation between the foundation stiffness and the beam foundation given in Eq. (3.62) is used. Then, by replacing k_l by its expression, Eq. (3.90) becomes:

$$[\mathbf{K}_{\text{BoEF}}] = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} + \frac{EI}{L^3} \begin{bmatrix} \frac{13}{35} & \frac{11}{35}L & \frac{54}{35} & -\frac{13}{35}L \\ \frac{11}{35}L & \frac{4}{35}L^2 & \frac{13}{35}L & -\frac{3}{35}L^2 \\ \frac{54}{35} & \frac{13}{35}L & \frac{13}{35} & -\frac{11}{35}L \\ -\frac{13}{35}L & -\frac{3}{35}L^2 & -\frac{11}{35}L & \frac{4}{35}L^2 \end{bmatrix}$$

$$= \frac{EI}{35L^3} \begin{bmatrix} 433 & 221L & -366 & 197L \\ 221L & 144L^2 & -197L & 67L^2 \\ -366 & -197L & 433 & -221L \\ 197L & 67L^2 & -221L & 144L^2 \end{bmatrix} \quad (3.91)$$

To analyse a 2D BoLEF in Abaqus, several options exist to model the foundation. The first is to use spring elements by selecting *Assembly module* → *Special* → *Springs/Dashpots*. The second option is to use linear connectors with elasticity (spring's constant). As part of this chapter and the rest of the thesis, the second option will be used, which is depicted in Figure 3.21. It shows the flowchart of the *Interaction module* which details the procedure followed to define the linear elastic foundation in Abaqus.

To model a BoLEF in Abaqus, first, a part was created with connected lines distant by 0.2 m. The points of connection of those lines were seen as nodes at which to attach elastic elements to create the foundation. So, to create a beam of length 2 m, ten lines were created, generating 11 nodes. Then, the properties were defined, and a section was created and assigned, followed by the beam orientation. Finally, the part was assembled. In the *interaction module*, eleven wires were created at the ten nodes initiated in the part, as shown in Figure 3.22a. Furthermore, linear connectors were built and assigned to the wires, as shown in Figure 3.22b. When the inputs were all defined, the analysis was launched. The section thereafter presents the results.

3.5.2 Numerical Results from Studying the BoLEF

The analysis was completed satisfactorily, allowing us to obtain the natural frequencies for the first ten vibration modes. These were printed on the *.dat* file and are now shown in the second column of Table 3.3.

Table 3.3 displays the values of the natural frequencies ω_l for the first ten modes of vibration of the BoLEF from analytical calculations and numerical analysis in Abaqus. These values were obtained for analytical calculations by varying the mode number m , the other parameters being constant. For the numerical analysis, the natural frequencies were requested at the Linear perturbation step. It is vital highlighting that the stiffness coefficient k_l was

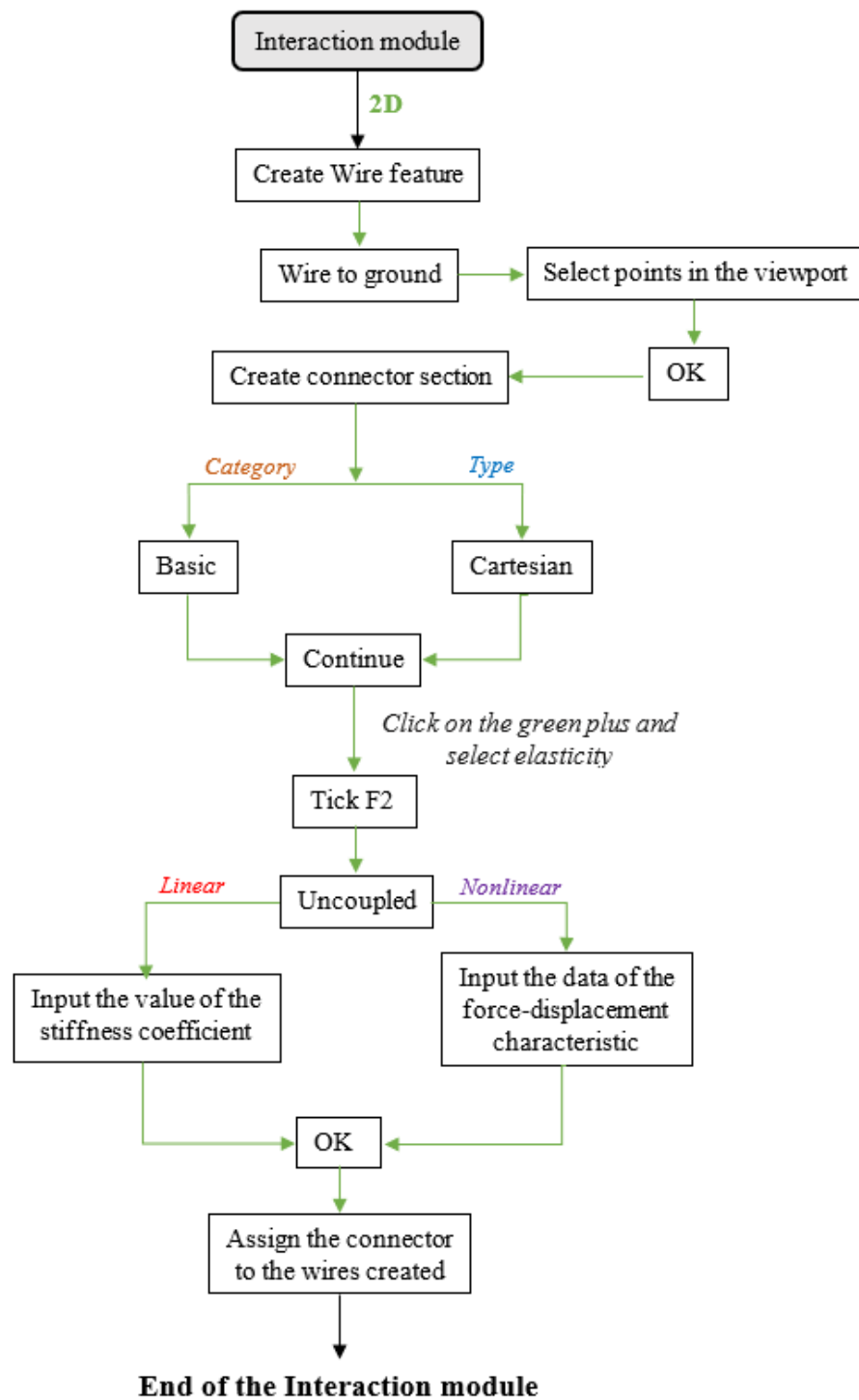


Figure 3.21: Flowchart for the Interaction module.

equal to the beam's stiffness over the distance between two successive springs. Considering the linear stiffness of the foundation in relation to that of the beam's stiffness has been

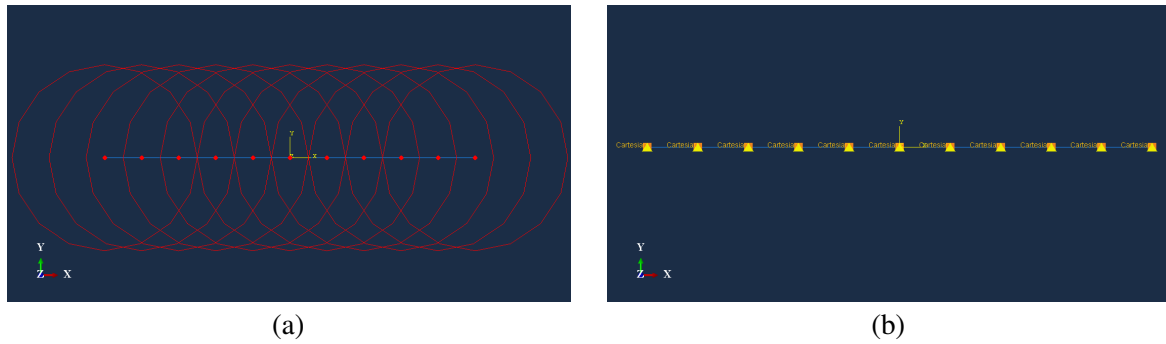


Figure 3.22: Assign a linear elastic foundation to the 2D beam element: (a) wires created and (b) connectors created and assigned to the wires.

Table 3.3: Natural frequencies for the first ten modes of vibration of a simply supported BoLEF.

Mode m	ω_l (rad/s)		Differences (%)
	Analytical	Abaqus	
1	285.17	285.16	0.004
2	501.5	500.88	0.004
3	994.42	994.25	0.001
4	1724.6	1724.6	0.00
5	2675.8	2675.8	0.00
6	3843.4	3843.4	0.00
7	5225.7	5225.7	0.00
8	6821.7	6821.7	0.00
9	8631.3	8631.3	0.00
10	10654	10651	0.00

adapted by many authors and it is a way to see the variation of the natural frequencies when the beam's stiffness increases. Moreover, the foundation stiffness value depends on the soil properties. Knowing the values for E , I , and L as shown in Table 3.1 and replacing them into Eq. (3.62), the linear stiffness coefficient k_l of the foundation was found as $k_l = 1.69 \times 10^5$ N/m². This value is in the range of those reported in the literature. The tiny differences resulting in the comparison between the natural frequencies obtained from the two methods confirm that there is an agreement.

Comparing Table 3.3 to Table 3.2 reveals that the presence of the foundation increases the natural frequencies. Thus, the expectation is confirmed that adding the elastic foundation to the beam may not affect the spatial amplitude of vibration of the beam, but instead increase the frequencies due to the stiffness coefficient of the foundation added to the formula used

to compute the frequencies. Then, by increasing stiffness, the vibration frequencies will inevitably increase because the natural frequency ω_l is proportional to $\sqrt{k_l}$.

For more understanding and a good interpretation of the numerical results, the mode shapes are displayed, and Figure 3.22 reveals the outcome. In addition, the corresponding plots representing the beam's displacement obtained as a function of the beam's length are disclosed in 3.23.

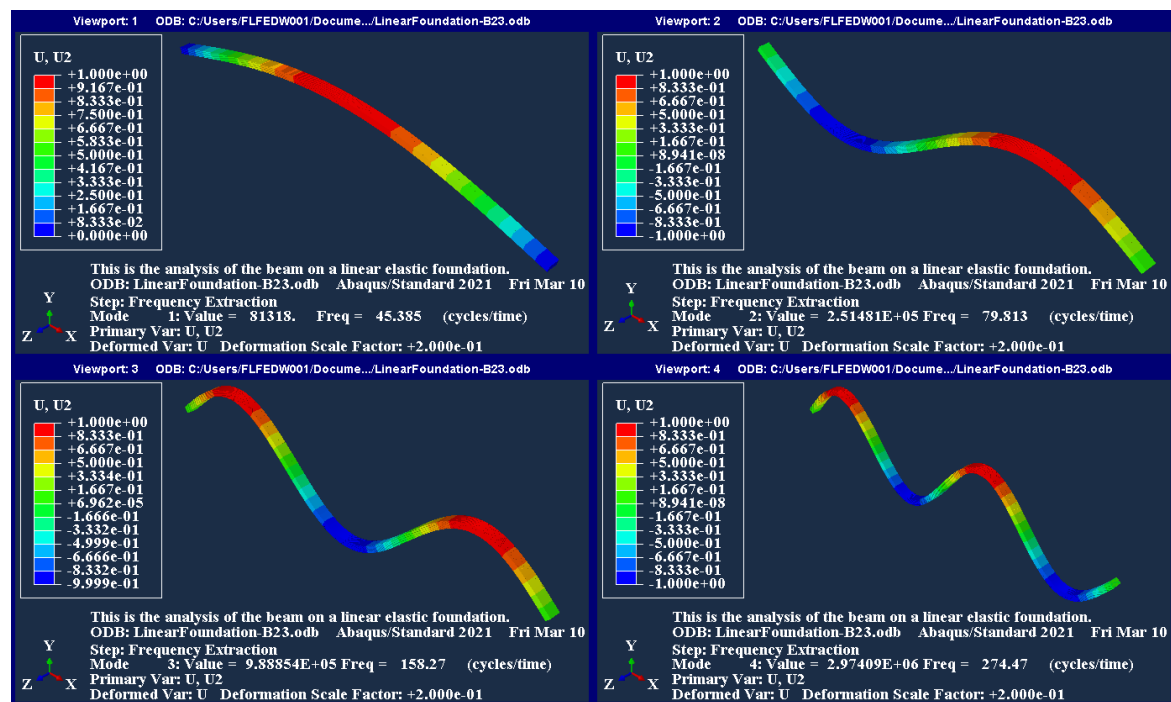


Figure 3.23: Mode shapes of the beam on a linear elastic foundation obtained in Abaqus.

In Figure 3.23 the first four mode shapes of the beam are shown. The information printed on each sub-figure gives a clear description of the curve. Comparing Figure 3.23 to Figure 3.13, there is no difference because the beam shape, the legend's values and the deformation scale factors are all the same. The natural frequencies have increased due to the stiffness coefficient of the foundation.

Similarly, in Figure 3.24, the same findings regarding Figure 3.23. Moreover, comparing Figure 3.24 to Figure 3.19, it can be seen that although some differences were recorded between the analytical and numerical natural frequencies, for all the four sub-figures, the maximum and minimum spatial amplitudes remain the same. This is because, in Abaqus, the output was requested as displacement normalised. In the figure below, the temporal plots are revealed.

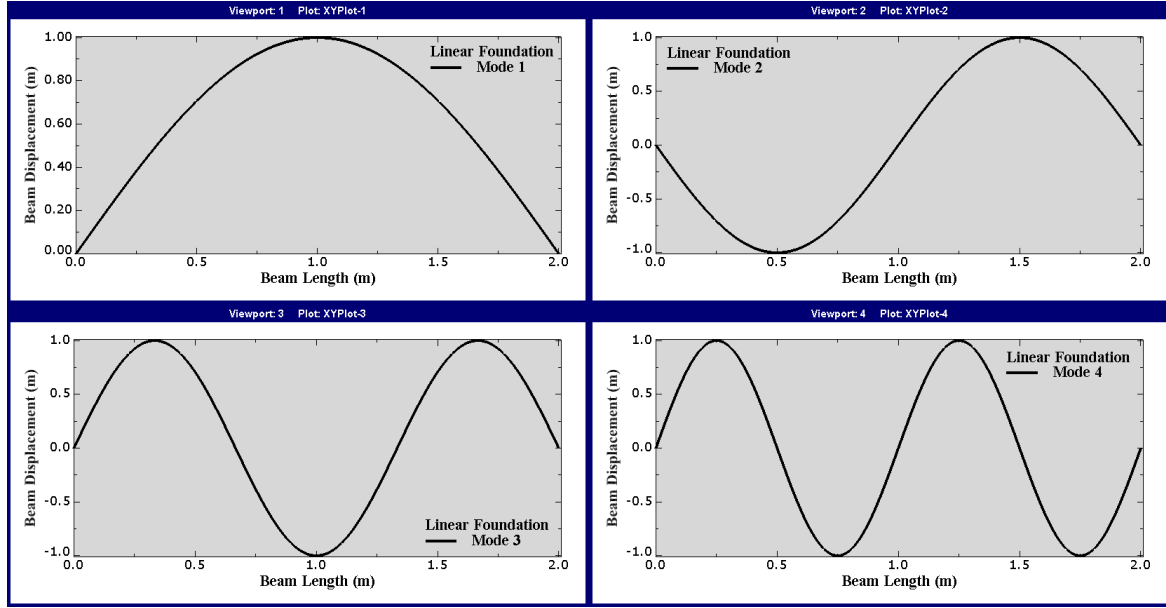


Figure 3.24: Displacement of the beam on a linear elastic foundation obtained in Abaqus.

Figure 3.25 exhibits the plots of the temporal amplitude of the BoLEF. Six modes are presented, and one can see that there is no difference between analytical and numerical plots. This confirms that the analytical results are a match to the numerical ones.

To talk about a realistic case, we consider a BoEF modelled as a rail track structure. In (Raymond, 1985; Selig and Li, 1994; Burrow et al., 2009), the track foundation modulus u for a single rail is defined as:

$$u = \left(\frac{k^4}{64EI} \right)^{\frac{4}{3}}, \quad (3.92)$$

where k is the track stiffness, E is the rail Young's modulus of elasticity, and I is the moment of inertia of the rail. It should be noted that the stiffness k includes the effect of EI whereas u represents only the remainder of the superstructure (fastener and ties) and substructure (ballast, subballast, and subgrade) (Selig and Li, 1994).

The track performance is determined by the value of its track modulus. Therefore, for $u < 14$ MPa, the track is considered to be poor or weak; for $14 \text{ MPa} < u < 28 \text{ MPa}$ the track is average; and for $u \geq 28 \text{ MPa}$ the track is considered good. From Eq. (3.92), the stiffness of the foundation is deduced as:

$$k = (64EIu^3)^{\frac{1}{4}} \quad (3.93)$$

To ensure a good track performance under traffic loading, Selig and Li (1994) suggest that the track modulus must be $u = 28 \text{ MPa}$ for a single rail. Then, for the rail type UIC 60, the value

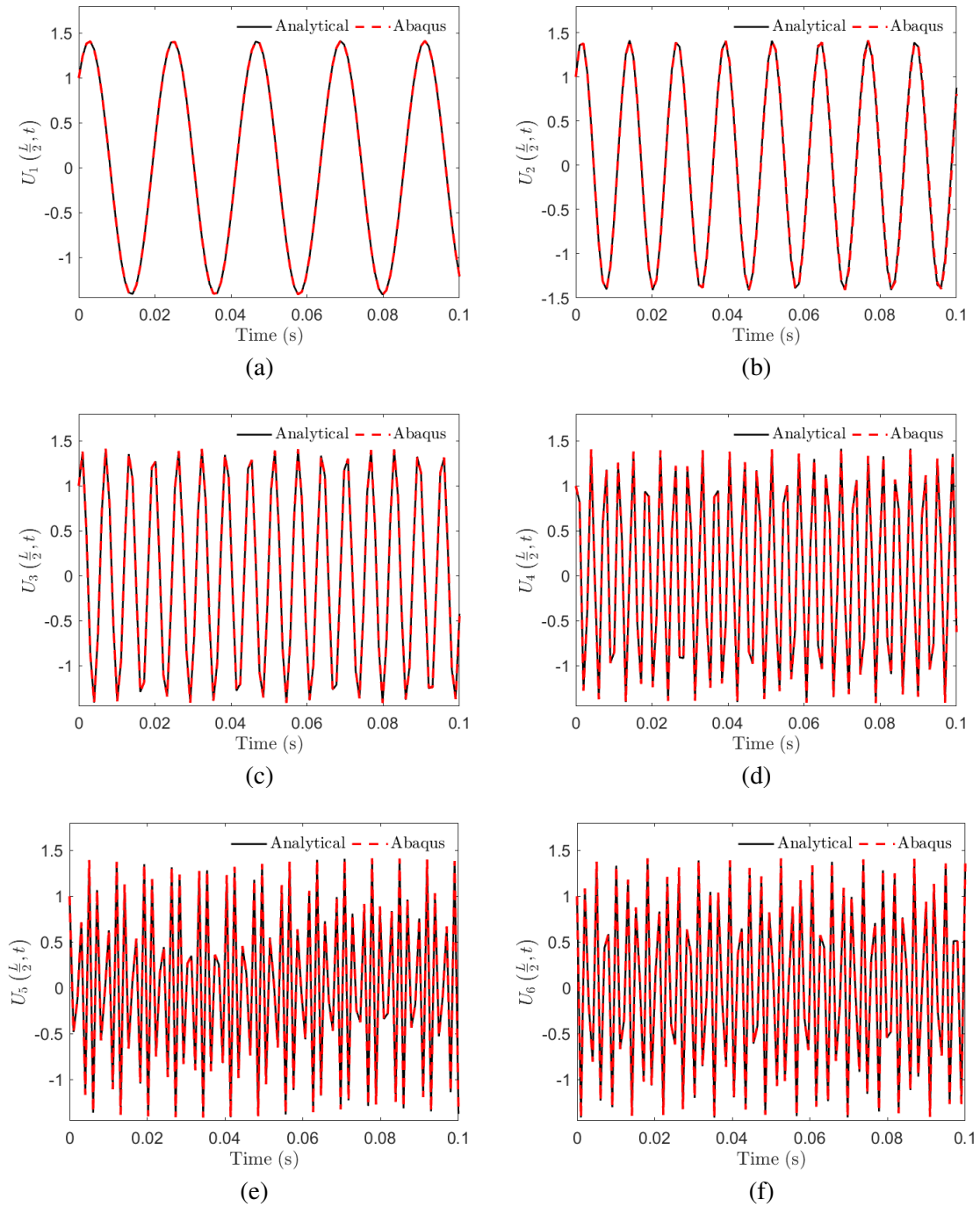


Figure 3.25: Temporal displacement at the centre of the BoLEF: (a) to (f) are modes 1 to 6.

of the foundation is found to be $k = 5.5 \times 10^7 \text{ N/m}^3$. The table below shows the parameters' values for this rail type.

Table 3.4: Properties of the rail type UIC 60 (Burrow et al., 2009).

Item	Notation	Value
Young's modulus	E	210 GPa
Length	L	20 m
Cross-section	S	$76.7 \times 10^{-4} \text{ m}^4$
Poisson's ratio	ν	0.35
Second moment of inertia	I	$3038.3 \times 10^{-8} \text{ m}^4$
Density	ρ	7850 kg/m^3

We note that the standard length of the rail type UIC 60 is between 12 m and 25 m. So for the present calculations, the length is considered as $L = 20 \text{ m}$.

To further consider other cases, many studies (Biot, 1937; Vallabhan and Das, 1991; Daloglu and Vallabhan, 2000) have been carried out to determine the coefficient of subgrade reaction or subgrade modulus k as given by:

$$k = \frac{0.95E_s}{1 - \nu_s^2} \left[\frac{B^4 E_s}{(1 - \nu_s^2)EI} \right]^{0.108}, \quad (3.94)$$

where E_s is the soil modulus of elasticity, ν_s is the soil Poisson's ratio, B the beam's width, and EI the beam's flexural rigidity. Therefore, by considering different types of soils and knowing their properties and the beam's dimensions, the subgrade modulus can be found using Eq. 3.94. The table below shows some types of soils and their properties (Bowles, 1988; Gaylord et al., 1992).

Table 3.5: Modulus of elasticity, Poisson's ratio and modulus of subgrade for different types of clay soils.

Type of soil	Modulus of elasticity E_s (MPa)	Poisson's ratio ν_s	Modulus k (N/m ³)
Soft	5 - 25 (10)	0.43	1.26×10^7
Medium	15 - 50 (30)	0.3	3.77×10^7
Hard	50 - 100 (60)	0.35	8.47×10^7
Sandy	25 - 250 (120)	0.2 - 0.3 (0.2)	1.65×10^8

Table 3.5 puts in view the mechanical properties of types of clay soils E_s and ν_s and their subgrade modulus k . Note that the values in parentheses are the ones used for calculations. To obtain the values for k , the beam had the properties given in Table 3.1 except the length which was changed to 20 m. Table 3.6 reveals the natural frequencies calculated using Eq. (3.81) and the values in Tables 3.4 and 3.5.

Table 3.6: Natural frequencies in rad/s for the rail type UIC 60 and those for different types of clay soils.

Mode m	UIC 60 Rail	Clay soil			
		Soft	Medium	Hard	Sandy
1	1351.7	1022.3	1767.7	2648.1	3699.0
2	1352.0	1022.3	1767.7	2648.1	3699.0
3	1353.6	1022.4	1767.7	2648.1	3699.0
4	1357.7	1022.5	1767.8	2648.1	3699.0
5	1366.5	1022.7	1767.9	2648.2	3699.1
6	1382.2	1023.0	1768.1	2648.3	3699.2
7	1407.8	1023.6	1768.5	2648.6	3699.3
8	1446.1	1024.6	1769.0	2648.9	3699.6
9	1500.1	1025.9	1769.8	2649.5	3700.0
10	1572.3	1027.8	1770.9	2650.2	3700.5

Table 3.6 shows that the natural frequencies of the rail UIC 60 modelled as a BoLEF have drastically increased as compared to those of the beam without foundation given in Table 3.2. In this case, the contribution of the foundation is much higher due to the stiffness of the subgrade. On the other hand, the natural frequencies are also higher for each case of soil type and increase with the subgrade modulus k . Regarding the values in Table 3.6, it can be confirmed that as the mode's number increases, the increase in the natural frequencies is not too significant.

The results obtained in Subsection 3.5.2 are those for free vibration of the BoEF with a constant length. However, what happens if the stiffness of the foundation is increased? Also, does the variation of the beam length influence the beam's amplitude of vibration? These assumptions are considered in the following section to see the variation of the beam's vibration amplitude and frequencies of the system beam-foundation with the increase of the beam stiffness on the one hand and the beam length on the other. Both analytical and numerical studies will be performed, and the natural frequencies will be calculated.

3.6 Influence of some Parameters in the Vibration of BoLEFs

This section is more about a parametric study of the free vibration of a BoEF having linear stiffness properties. The influence of the linear stiffness coefficient of the foundation on the beam temporal amplitude of vibration is investigated. Moreover, the beam length's effect on

the beam's vibration response is also explored. The natural frequencies for each value of the parameters will be calculated for analytical and numerical studies. Finally, the differences between analytical and numerical results will be highlighted.

3.6.1 Influence of the Stiffness of the Foundation

Because the foundation's stiffness is the parameter that helps resist the beam's deformation when the beam vibrates, or when a load is applied to the beam, therefore, its presence is significant in discussing beams or plates on elastic foundations. In the current subsection, different stiffness values of the foundation will be assigned, and their effects on the beam's deformation or vibration amplitude will be investigated. The corresponding natural frequencies obtained from analytical and numerical methods for each stiffness coefficient are presented in Table 3.7.

Table 3.7, shows different relations between the foundation's stiffness coefficient k_l and the beam stiffness k_b . The corresponding values of the natural frequencies ω_l of the BoLEF obtained for different modes m from both analytical and numerical analyses and the differences between the two methods are revealed. In this table, the values corresponding to $k_l = 0k_b$ and $k_l = 0.5k_b$ are not reported because these were already given in Tables 3.2 and 3.3 for the study of a simple beam and for the study of the BoLEF, respectively. Then, examining Table 3.7 carefully, it is observed that the natural frequencies increase when both the spring's stiffness coefficient (read from left to right) and the mode's number (read from top to bottom) increase. The differences observed here are insignificant. Therefore, an agreement is confirmed between the two methods. Hence, the results from Abaqus are a benchmark for the analytical results.

To proceed, the values reported in Table 3.7 are used to plot the natural frequencies ω_l for the first ten vibration modes of the BoLEF. The result is given in Figure 3.26.

Figure 3.26 shows the plots of the natural frequencies ω_l issued from the analytical calculations (see Figure 3.26a) and the logarithm plot of these natural frequencies (see Figure 3.26b). From each sub-figure, the natural frequencies increase, as both the mode's number and the stiffness coefficient increase, as displayed in Table 3.7. Moreover, the gap between two consecutive plots becomes large as the ratio $\frac{k_l}{k_b}$ increases. Nevertheless, when that ratio tends to infinity, the foundation becomes infinitely rigid, and the beam cannot vibrate. This case is not of interest in this thesis.

Table 3.7: Natural frequencies for the first ten modes of vibration of a BoLEF obtained for different stiffness coefficients.

Mode	$k_l = 0.25k_b$			$k_l = k_b$			$k_l = 2k_b$			$k_l = 4k_b$			$k_l = 10k_b$		
	ω_l (rad/s)		Diff	ω_l (rad/s)		Diff	ω_l (rad/s)		Diff	ω_l (rad/s)		Diff	ω_l (rad/s)		Diff
m	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)
1	121.81	121.81	0	159.12	159.12	0	198.23	198.23	0	259.32	259.31	0.004	388.71	388.68	0.008
2	430.12	430.12	0	442.13	442.13	0	457.7	457.66	0.009	487.25	487.25	0	566.8	566.77	0.005
3	960.4	960.4	0	965.85	965.85	0	973.05	973.05	0	987.31	987.31	0	1028.9	1028.9	0
4	1705.2	1705.2	0	1708.2	1708.2	0	1712.3	1712.3	0	1720.5	1720.5	0	1744.7	1744.7	0
5	2663.4	2663.4	0	2665.4	2665.4	0	2668.0	2668.0	0	2673.2	2673.2	0	2668.8	2668.8	0
6	3834.8	3834.8	0	3836.2	3836.2	0	3838.0	3838.0	0	3841.6	3841.6	0	3852.5	3852.5	0
7	5219.3	5219.3	0	5220.3	5220.3	0	5221.6	5221.6	0	5224.3	5224.3	0	5232.3	5232.3	0
8	6816.9	6816.9	0	6817.6	6817.6	0	6818.6	6818.6	0	6820.7	6820.7	0	6826.8	6826.8	0
9	8627.5	8627.5	0	8628.1	8628.1	0	8628.9	8628.9	0	8630.5	8630.5	0	8635.3	8635.3	0
10	10651	10651	0	10651	10651	0	10652	10651	0.009	10654	10651	0.03	10657	10651	0.06

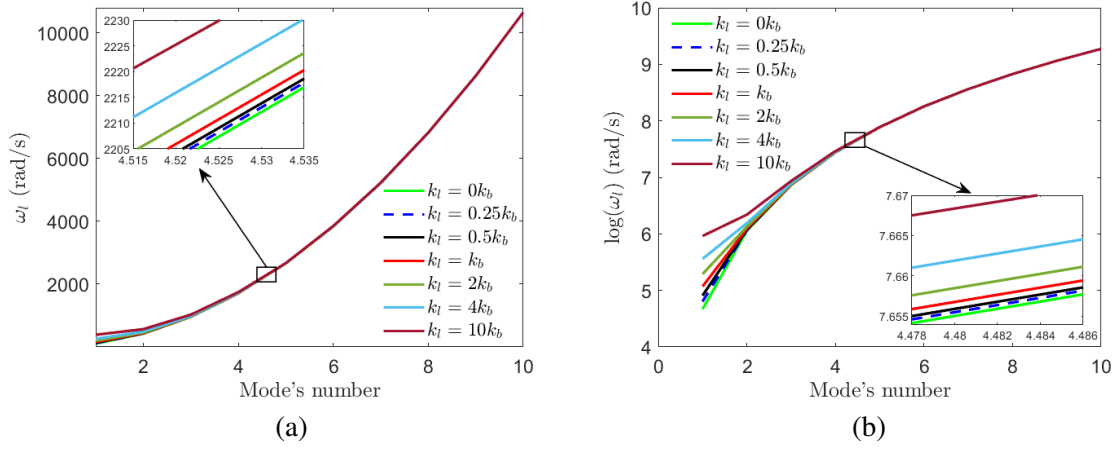


Figure 3.26: (a) Natural frequency ω_l of the BoLEF and (b) Logarithm plot of the natural frequency ω_l versus the mode's number for different relations between the stiffness coefficient of the beam k_b and that of the foundation k_l

For further analysis, the zoom on a portion of each subfigure highlights that the frequencies obtained from analytical and numerical methods vary similarly. Hence, Figure 3.26 helps in pointing out that the natural frequencies of BoLEFs depend considerably on the evolution of the spring stiffness coefficient.

In Figure 3.27, are displayed the temporal amplitudes of the beam at modes 1, 2, 3, and 4. These plots were obtained using MATLAB with the initial conditions given as $U(\frac{L}{2}, 0) = 1$ and $\dot{U}(\frac{L}{2}, 0) = 0$, and the frequencies used in the programme were calculated analytically. It can be observed that for the four cases presented here, the gaps between the curves corresponding to each stiffness relation decrease as the mode's number increases: at the fundamental mode, all the curves can easily be distinguished (see Figure 3.27a) whereas, at mode 4, all the curves are mingled (see Figure 3.27d). The maximum and minimum amplitudes remain similar to those obtained in previous temporal plots. The above results reveal that increasing the linear foundation's stiffness does not modify the beam's amplitude of vibration, only the natural frequencies of the system formed by the beam and the foundation increase. However, there might be some cases where increasing the linear stiffness coefficient inevitably affects the beam's vibration amplitude.

3.6.2 Influence of the Beam's Length

The length of the beam is significant in designing a structure. This is because it determines the total measure of a member along its longitudinal axis. In this subsection, the interest is

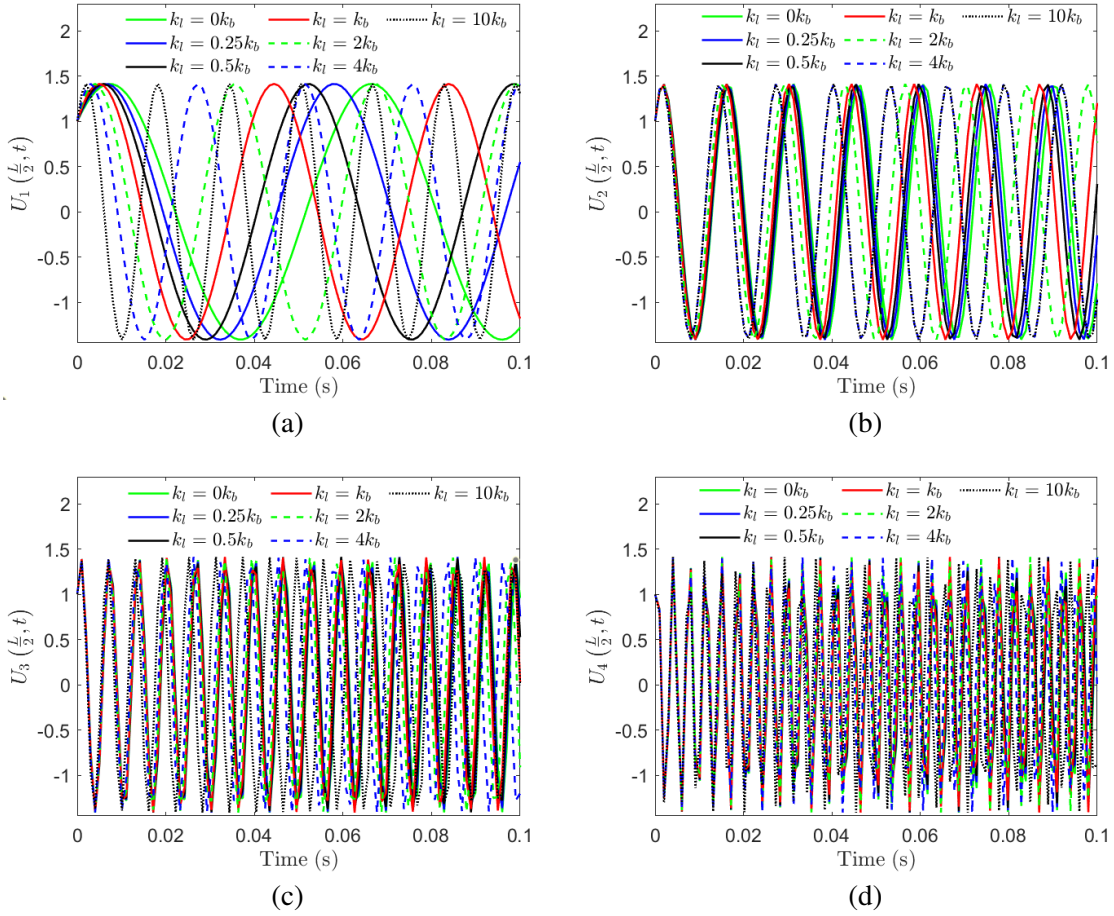


Figure 3.27: Temporal displacement at the centre of the BoLEF obtained for different stiffness coefficients: (a) to (d) are modes 1 to 4.

turned to the length's effects on the beam's amplitude when its value increases. Table 3.8 displays the natural frequencies calculated for different lengths of the beam, say $L=2$ m, $L=4$ m, $L=10$ m, and $L=20$ m for the stiffness coefficients $k_l = 0.5k_b$.

In Table 3.8, the natural frequencies decrease as the beam's length increases (read from left to right) and increase as the mode's number increases (read from top to bottom). The same remark concerning the length of the beam was made by Asiri (2018), where the linear and nonlinear free vibration of beams were investigated. Also, as the beam's length increases, the values of the frequencies become very small. This can be the case for a road structure. For all the cases presented in Table 3.8, the differences between the analytical and numerical frequencies are very small. Figure 3.28 shows the plots of the first four mode shapes of the transversal displacement of the BoLEF obtained for different beam lengths.

Table 3.8: Natural frequencies of the BoLEF obtained for $k_l = 0.5k_b$ by varying the beam length L .

Mode	$L=2$ m			$L=4$ m			$L=10$ m			$L=20$ m		
	ω_l (rad/s)		Diff	ω_l (rad/s)		Diff	ω_l (rad/s)		Diff	ω_l (rad/s)		Diff
m	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)
1	285.17	285.16	0.004	265.9	265.9	0	264.57	264.95	0.14	167.3	167.2	0.06
2	501.5	501.48	0.004	285.17	285.16	0.004	265.08	266.16	0.4	167.36	167.25	0.06
3	994.42	994.41	0.001	356.9	356.9	0	267.3	269.77	0.9	167.58	167.47	0.07
4	1724.6	1724.6	0	501.5	501.5	0	273.17	278.67	1.9	168.17	168.06	0.07
5	2675.8	2675.8	0	716.32	716.31	0.0014	285.17	285.92	0.3	169.4	169.3	0.06
6	3843.4	3843.4	0	994.42	994.41	0.001	305.78	311.34	1.8	171.6	171.53	0.04
7	5225.7	5225.7	0	1331.3	1331.3	0	337	337.16	0.05	175.3	175.15	0.09
8	6821.7	6821.7	0	1724.6	1724.6	0	379.9	383.17	0.9	180.7	180.55	0.08
9	8631.3	8631.3	0	2173	2173	0	434.8	436.24	0.3	188.2	188.13	0.04
10	10654	10651	0.028	2675.8	2675.8	0	501.48	502.49	0.2	198.33	198.23	0.05

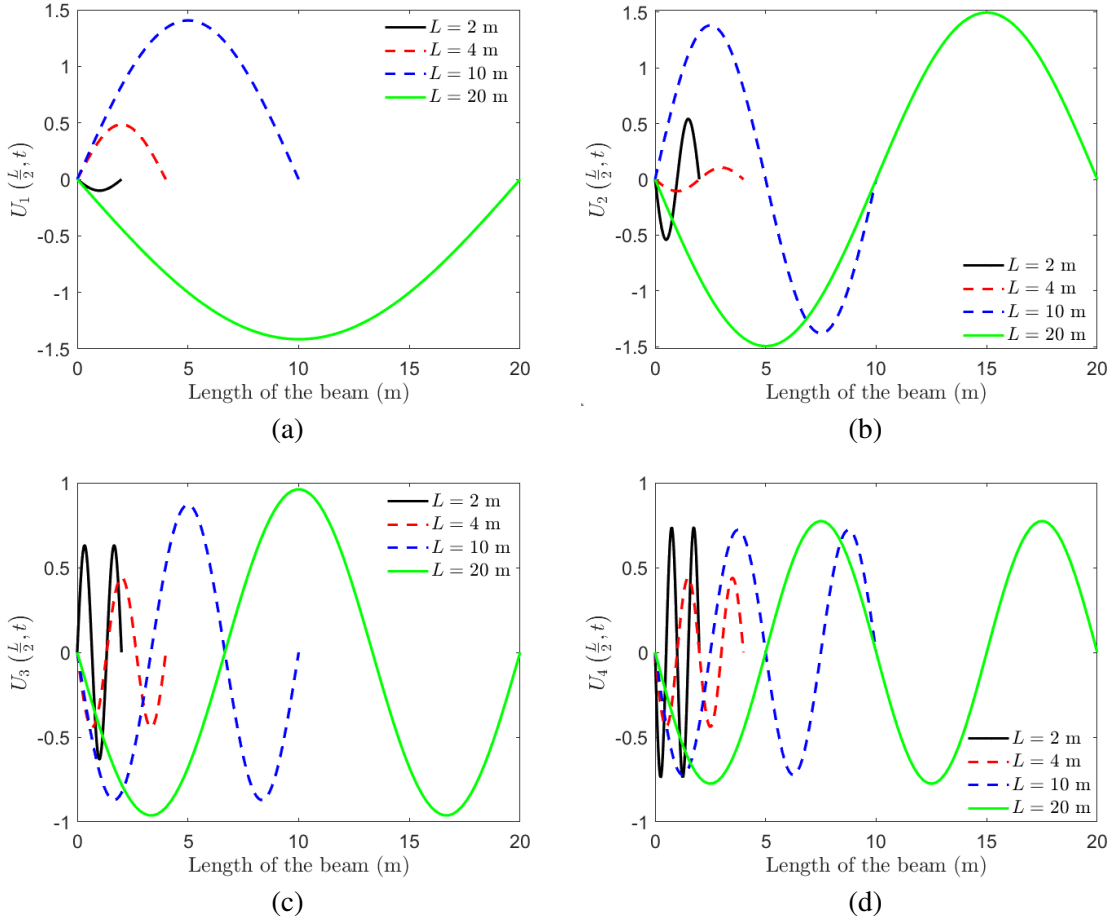


Figure 3.28: Spatial displacement of the BoLEF obtained for different beam lengths: (a) to (d) are modes 1 to 4.

In Figure 3.28 the curves for all four cases are plotted on the same sub-figure for each mode to appreciate the evolution of the beam spatial amplitude for each specific beam's length. The beam's spatial amplitude varies from one mode to another for each beam's length. As far as the current findings are concerned, the decrease in the natural frequencies is considerably related to the increase in the beam length. In contrast, a good interpretation cannot be given regarding the beam spatial amplitude of vibration for the increase of the beam's length because the curves are all mixed at each of the four modes presented. We note that all the curves were plotted at $t = 1$ s.

Figure 3.29 shows the temporal amplitudes of the BoLEF obtained from different beam lengths. All the curves have similar maximum and minimum amplitudes. Again, a proper relation between the beam length and vibration amplitude cannot be stated. It should be highlighted that for structures such as roads modelled as beams, the length is infinite and for

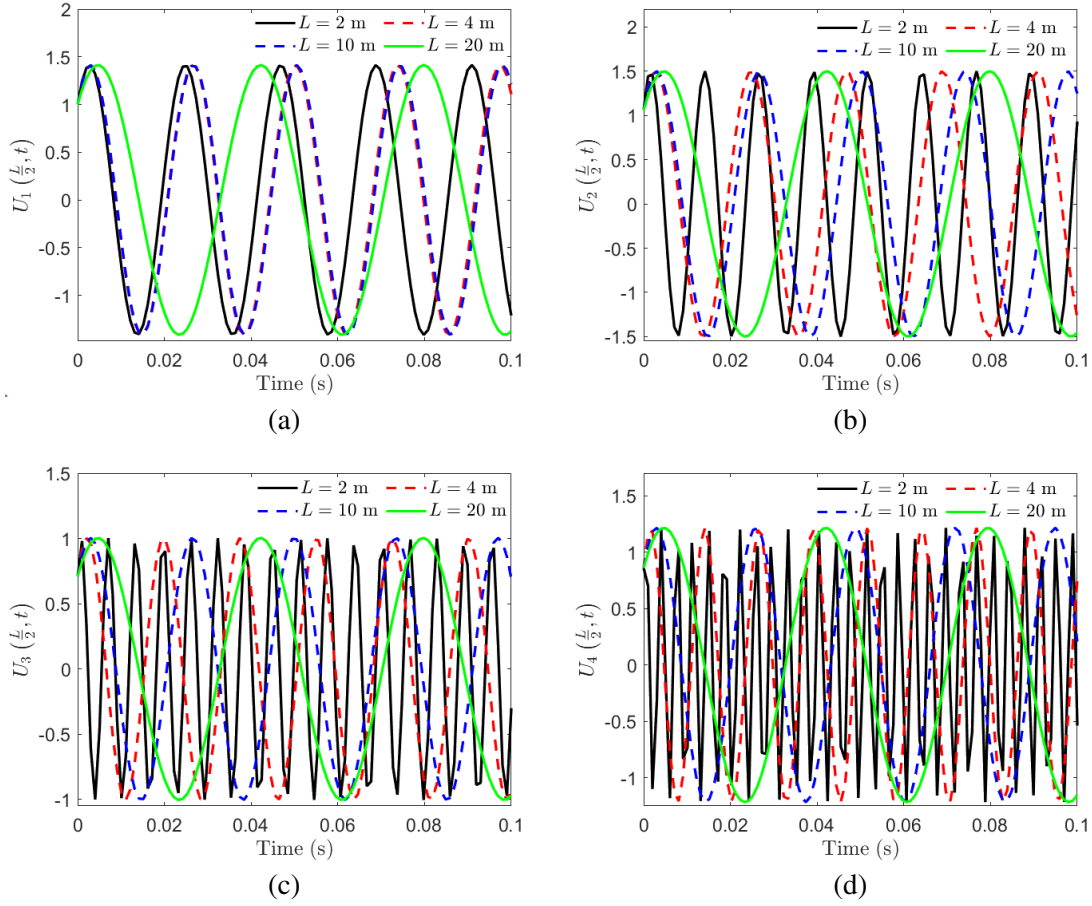


Figure 3.29: Temporal amplitudes of the BoLEF obtained for different beam lengths: (a) to (d) are modes 1 to 4.

calculation purposes, a reasonable finite length must be used. The length of 2 m is chosen in this thesis as a lower limit for illustration purposes, and to apply the concept presented in this thesis. Therefore, other beam lengths were considered to extend the study. We note that to reduce the vibrations, it is advisable to stiffer or put dampers or absorbers under the beam.

3.7 Summary and Conclusions

Chapter 3 of this thesis studied the vibrations of a simple beam simply supported at both ends and that of a beam simply supported at both ends resting on a linear elastic foundation, both beams made with steel. In addition, an analytical study using the method of separation of variables and a numerical study using FEA in Abaqus were carried out. From the above findings, the following conclusions can be drawn.

1. Theoretical formulation of the finite element model of the dynamic behaviour of the simple beam was given to know the dynamic formulation needed to be solved. To reflect the Euler-Bernoulli beam theory used for analytical calculations, *B23* beam element was used in Abaqus, since it does not account for shear deformation. The results for the simple beam using the separation of variables were compared to those obtained in Abaqus. The natural frequencies obtained from the two methods were presented, and the tiny differences confirmed an agreement between the two methods.

2. The same study on a BoEF having linear stiffness properties was conducted. There was also a good degree of accuracy between the analytical and numerical results of the natural frequencies because the differences calculated were almost negligible for all the cases presented. A practical case of a BoEF modelled as a UIC 60 rail and a BoEF having different types of clay soil properties revealed significant values of the natural frequencies which increased slightly with the mode's number.

3. Two parametric studies were conducted to appreciate the beam's vibration amplitude evolution when some parameters vary. For the first study, the foundation stiffness was supposed to have different values in relation to the beam stiffness, and for the second study, different beam lengths were considered. For the former case, it was found that increasing the stiffness of the foundation results in increasing the frequencies of the system beam-foundation. Moreover, as the mode's number increased, there was no difference between the plot corresponding to each stiffness relation because all the curves were mingled.

4. As for the latter case, it was found that increasing the beam length implied a decrease in the natural frequencies of the system beam-foundation. By comparing the curves for each case it was not possible to tell whether the amplitude of vibration of the beam was increasing or decreasing regarding the increase of the beam length because all the curves were mixed and changing from one mode to another. For an infinitely long beam, it is advised to use a reasonable finite length for illustration and calculation purposes.

5. In this chapter's work, the differences between analytical and numerical results were almost negligible. Hence, the numerical results from FEA in Abaqus are an excellent benchmark for the solution. Consequently, the analytical results are accurate and valid.

6. The present study results are a basis and will serve as references for the next chapter on the nonlinear vibration of beams. These will help in highlighting the relative differences of neglecting the nonlinearities of the foundation, which is the primary objective of this research.

Chapter 4

Nonlinear Vibration of Beams on Elastic Foundations

4.1 Introduction

Nonlinear dynamics is the branch of physics that studies systems governed by nonlinear equations rather than linear ones. It is also a research field based on nonlinear ordinary or partial differential equations. Nonlinear dynamical systems constitute most of the dynamic systems encountered daily. That is why mathematicians, physicists, and engineers are interested in them (De Canete et al., 2011). Solving nonlinear equations using analytical methods has always been a challenging problem. However, many numerical methods exist to find the solution to nonlinear equations quickly. The choice of the method depends on its convergence, computational costs, and the time spent formulating the solutions.

Many researchers have been using the Finite Element Method (FEM) to explore beams and plates' dynamic responses due to their relatively rapid convergence. This chapter will use connectors to model the beam's nonlinear foundation numerically using Finite Element Analysis (FEA) in Abaqus. Moreover, the influence of the nonlinearities of the foundation on the beam's dynamics will be highlighted. The straightforward expansion method (SEM) will be used to find an approximate solution to the nonlinear differential equation, and the results will be compared to those obtained numerically from Abaqus. Also, the present results will be compared to those obtained in the last chapter on the linear vibration of beams.

This chapter is structured as follows: Section 4.2 will present a general equation of a system with nonlinearity. Section 4.3 will focus on an analytical study of a steel beam on an

elastic foundation (BoEF), with three cases of nonlinearity of the foundation added: nonlinear quadratic, nonlinear cubic, and a combination of nonlinear quadratic and nonlinear cubic. Section 4.4 will deal with numerical analysis in Abaqus of the three nonlinear analytical cases. In Section 4.5, a comparison of the present results to linear results will be performed. Lastly, in Section 4.6, conclusions are drawn from the findings.

4.2 Continuous Systems with Nonlinear Stiffness

Linear systems can be obtained by making some approximations in the dynamics of nonlinear systems, if possible. In some cases, efficient results can be found after linearising a nonlinear system, but these results could be inconsistent and the solution erroneous in other cases. One must often rely on the nonlinear system directly to obtain acceptable results.

A vibrating system can be seen as a process in which input signals are transformed into other signals called outputs. A continuous system is a one that the state variables change continuously over time or other variables, and an output signal is generated in response to the input signal (Gajic, 2003). It has infinite DOF, and the displacement depends on two variables: displacement and time. Continuous systems are encountered in many branches of engineering, such as civil, mechanical, aerospace, and architectural. In addition, designing many structural devices requires an accurate prediction of their dynamic performance and vibration characteristics (Shabana and Ling, 1997; Rao, 2007; Leissa and Qatu, 2011). Of course, linear, and nonlinear continuous systems also exist. The difference between the two systems is that the natural frequencies of nonlinear continuous systems vary according to the vibration amplitude. On the other hand, in continuous linear systems, they are invariant quantities (Nayfeh and Nayfeh, 1994; Maccari, 1999; Shmaliy, 2007; Leissa and Qatu, 2011).

The type of nonlinearity to be focused on in this thesis is that of the foundations. Three cases of expressions of the restoring force produced by the springs of the foundations will be considered: linear and nonlinear quadratic, linear and nonlinear cubic, and linear and nonlinear quadratic and nonlinear cubic, meaning that the following expressions for the restoring force will be used (Nayfeh and Mook, 1995; Tomlinson and Worden, 2000; Tellı and Kopmaz, 2006)

$$F_q(x) = k_l x + k_2 x^2; \quad (4.1a)$$

$$F_c(x) = k_l x + k_3 x^3; \quad (4.1b)$$

$$F_{qc}(x) = k_l x + k_2 x^2 + k_3 x^3; \quad (4.1c)$$

where $F_q(x)$, $F_c(x)$, $F_{qc}(x)$ respectively stand for the quadratic, cubic, and quadratic-cubic restoring forces; k_l , k_2 , and k_3 are the linear, nonlinear quadratic, and nonlinear cubic stiffness coefficients, respectively. From Eqs. (4.1a), (4.1b), and (4.1c) one can notice that the overall stiffness of the spring changes with the displacement x , whereas the coefficients k_l , k_2 , and k_3 remain constant. Plotting Eqs. 4.1a, 4.1b, and 4.1c in MATLAB, the results shown in Figure 4.1 are obtained. The plots of all three restoring forces are shown as a function of the displacement for arbitrary values of the stiffness coefficients.

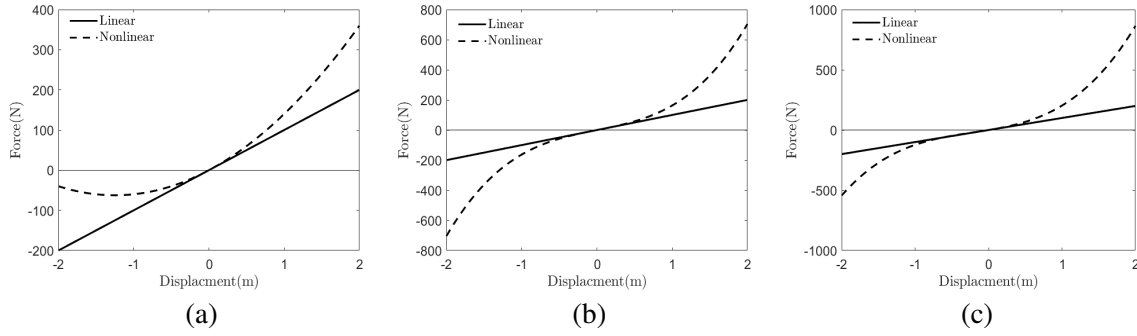


Figure 4.1: Behaviour of stiffness elements: (a) linear and quadratic, (b) linear and cubic, and (c) linear and quadratic-cubic.

The shape of each curve allows differentiating one from another. Although Figures 4.1b and 4.1c do not differ greatly from one another, the shape of Figure 4.1c considerably depends on the relative contribution of 4.1a and 4.1b. Throughout this chapter, those expressions of the restoring forces will be considered, and the values of the nonlinear stiffness coefficients will be defined to produce a foundation model that is suitable for minor beam displacements compared to the linear elastic foundation while also being stiffer for large displacements (Nguyen and Duhamel, 2008). The following section details the study of the beam on a nonlinear elastic foundation having each of the above-defined nonlinearities.

4.3 Analytical Study of a BoEF having Nonlinear Stiffness Properties

When researching or studying beams and plates resting on nonlinear elastic foundations (BPoEFs), attention is restricted to planar and non-rotational motions. The equations of motion are derived from inertial effects, elastic effects, and nonlinear effects (of the foundation). This section focuses on the vibration of a BoEF with nonlinear stiffness properties. Compared

to the linear case investigated in Chapter 3, the stiffness coefficients of the foundation in the present chapter will be nonlinear. The system to be analysed in this chapter was adapted from Hetényi (1971) and is illustrated in Figure 4.2.

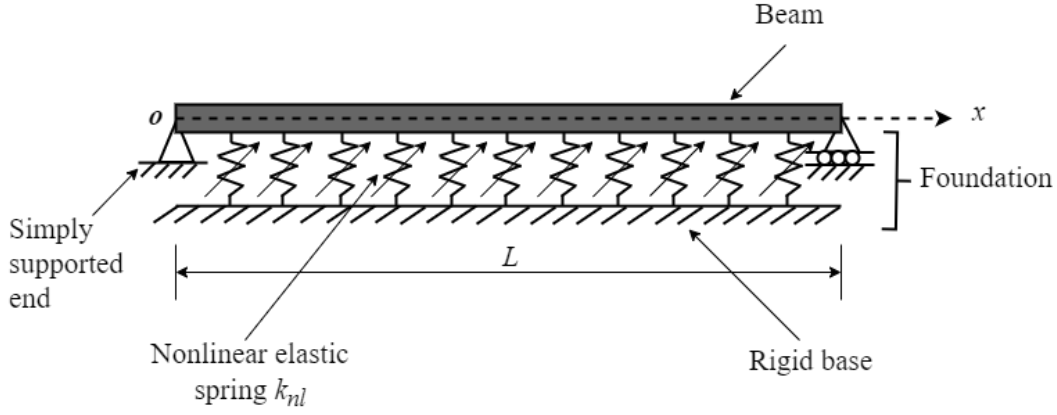


Figure 4.2: BoEF foundation having nonlinear springs (Hetényi, 1971).

Figure 4.2, shows a beam supported with nonlinear elastic springs having each a stiffness coefficient per unit length $k_{nl} = k_{nl}(U)$ where U is the beam's transversal displacement; x is the direction taken along the longitudinal axis of the beam, and L is the beam's length. One extremity of each spring is fixed to the beam, whereas the other relies on rigid support regarded as ground. The distance between two consecutive springs is given by b . Those springs are implemented in the model as discrete elastic supports. The theoretical formulation and solution procedure presented in this section are based on the treatment given by Nayfeh and Nayfeh (1994) and Rao (2007). Although the procedures followed in those books were specific to linear systems, they have been adapted to the present nonlinear system.

The following assumptions will be considered during the beam analysis in this chapter:

- The beam is simply supported and symmetric with respect to the vertical plane.
- The beam material is linearly elastic, homogeneous, and continuous.
- The beam plane section remains plane before and after deformation.
- No external force is applied to the beam, except an imposed initial displacement.

Considering the system presented in Figure 4.2, the Hamilton principle is used to derive the governing equation of the system. The kinetic energy is given by (Nayfeh and Pai, 2008):

$$E_c = \frac{1}{2} \int_0^L \rho S \left(\frac{\partial U(x,t)}{\partial t} \right)^2 dx, \quad (4.2)$$

where E_c , ρ , S , L , and $U(x, t)$ stand for the kinetic energy, the beam's density, cross-sectional area, length, and transversal displacement, respectively. Also, the potential energy is given by (Nayfeh and Pai, 2008):

$$E_p = \frac{1}{2} \int_0^L EI \left(\frac{\partial^2 U(x, t)}{\partial x^2} \right)^2 + \frac{1}{2} \int_0^L k_{nl} U(x, t)^2 dx, \quad (4.3)$$

where E_p , E , and I are the potential energy, Young's modulus, and the inertial moment of the beam, respectively. k_{nl} is the coefficient of the foundation stiffness per unit length.

By using the Hamilton principle and the Lagrangian of a system (Hamilton, 1834; Wells, 1967), the following equation holds:

$$\delta \int_{t_1}^{t_2} (E_c - E_p) dt = 0. \quad (4.4)$$

Then, substituting Eqs. (4.2) and (4.3) into (4.4), the governing equation of the system presented in Figure 4.2 is the following:

$$\rho S \frac{\partial^2 U(x, t)}{\partial t^2} + EI \frac{\partial^4 U(x, t)}{\partial x^4} + k_{nl} U(x, t) = 0, \quad (4.5)$$

where the first term is the inertial force per unit length, and the second and third terms both represent the elastic force per unit length.

4.3.1 Case of a BoEF having Linear and Nonlinear Quadratic Stiffness Properties

For a system having linear and nonlinear quadratic stiffness coefficients, the expression of the stiffness coefficient k_{nl} in Eq. (4.5) is given by (Nayfeh and Mook, 1995; Tomlinson and Worden, 2000) :

$$k_{nl} = k_l + k_2 U(x, t), \quad (4.6)$$

where $U(x, t)$ is the transverse deflection of the beam; k_l and k_2 are respectively the linear and the nonlinear quadratic stiffness coefficients per unit length. Then, replacing Eq. (4.6) into Eq. (4.5) it gives:

$$\rho S \frac{\partial^2 U(x, t)}{\partial t^2} + EI \frac{\partial^4 U(x, t)}{\partial x^4} + k_l U(x, t) + k_2 U(x, t)^2 = 0. \quad (4.7)$$

EI in Eq. (4.7) is the beam's flexural rigidity, which is constant along the beam's length, while ρS is the beam mass per unit length. In the current section, the aim is to solve Eq. (4.7). First, the separation of variables (Myint-U and Debnath, 2007; Dawkins, 2018) is used to write the displacement of $U(x, t)$ as a product of spatial and temporal parts as:

$$U(x, t) = X(x)T(t), \quad (4.8)$$

where $X(x)$ depends only on the coordinate x and $T(t)$ depends only on time t . By replacing Eq. (4.8) into Eq. (4.7), it holds:

$$EI \frac{d^4 X(x)}{dx^4} T(t) + \frac{d^2 T(t)}{dt^2} X(x) + k_l X(x) T(t) + k_2 X(x)^2 T(t)^2 = 0. \quad (4.9)$$

Note that Eq. (4.9) is now written with the right derivatives because the functions to be derived depend only on one variable

Next, it is supposed that the beam is simply supported at both ends. These beam ends are chosen because the solution of the spatial part is known from the literature (Nayfeh and Mook, 1995; Rao, 2007; Nayfeh and Pai, 2008). Therefore, the focus is only on the procedure to find the solution of the equation satisfied by the temporal part. That allows writing the solution of the spatial part as:

$$X_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right), \quad (4.10)$$

where n stands for an integer representing the mode's number, A_n is the amplitude of vibration of the spatial displacement. For the upcoming equations, the solutions approach will follow those from Nayfeh and Mook (1995), Rao (2007), and Nayfeh and Pai (2008). Also, it must be highlighted that the solution in Eq. (4.10) has been chosen because it automatically satisfies the simply supported boundary conditions for the displacement and bending moment of a simply supported beam.

In order to derive the modal equation satisfied by Eq. (4.9), that is the equation with the only parameter being the time, Eq. (4.10) is substituted into Eq. (4.9) and it gives:

$$\sum_{n=0}^{\infty} \left\{ \left[EI \left(\frac{n\pi}{L} \right)^4 + k_l \right] \sin\left(\frac{n\pi}{L}x\right) T_n(t) + \rho S \sin\left(\frac{n\pi}{L}x\right) \frac{d^2 T_n(t)}{dt^2} + \frac{1}{2} k_2 A_n T_n(t)^2 - \frac{1}{2} k_2 A_n \cos\left(\frac{2n\pi}{L}x\right) T_n(t)^2 \right\} = 0. \quad (4.11)$$

Furthermore, the orthogonality of trigonometric functions is applied to Eq. (4.11). This is done by first multiplying the whole equation by $\sin\left(\frac{m\pi}{L}x\right)$, where m is an integer. Then, the obtained expression is integrated by parts over the entire length of the beam (from 0 to L). That is:

$$\sum_{n=0}^{\infty} \left\{ \left[EI \left(\frac{n\pi}{L} \right)^4 + k_l \right] T_n(t) \int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx + \frac{1}{2} k_2 A_n T_n(t)^2 \int_0^L \sin\left(\frac{m\pi}{L}x\right) dx + \rho S \frac{d^2 T_n(t)}{dt^2} \int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx - \frac{1}{2} k_2 A_n T_n(t)^2 \int_0^L \cos\left(\frac{2n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx \right\} = 0. \quad (4.12)$$

The following equations are recalled from the integral calculus of trigonometric functions:

$$\int_0^L \sin\left(\frac{m\pi}{L}x\right) dx = \frac{(1 - (-1)^m)L}{m\pi} \quad (4.13a)$$

$$\int_0^L \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \begin{cases} \frac{L}{2} & \text{if } n = m \neq 0 \\ 0 & \text{if } n \neq m \end{cases} \quad (4.13b)$$

$$\int_0^L \cos\left(\frac{2n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}x\right) dx = \frac{(1 - (-1)^m)mL}{\pi(m^2 - 4n^2)}. \quad (4.13c)$$

Using Eqs. (4.13) into Eq. (4.12) transforms it into:

$$\frac{d^2 T_n(t)}{dt^2} + \left[\frac{EI}{\rho S} \left(\frac{n\pi}{L} \right)^4 + \frac{k_l}{\rho S} \right] T_n(t) + \frac{1 - (-1)^m}{m\pi} \left[\frac{4n^2}{4n^2 - m^2} \right] \frac{k_2 A_n}{\rho S} T_n(t)^2 = 0. \quad (4.14)$$

This last equation is also called a modal equation (Nayfeh and Mook, 1995) regarding the mode's number n . To simplify it, there are two possibilities: either suppose $n = m$ and discard all other terms, or just assign a fixed value to m . For this thesis, the second possibility is chosen because if $n \neq m$, the integral given in Eq. (4.13b) is equal to 0, and this will imply that the term in square brackets in Eq. (4.12) vanishes. By doing so, the linear part of the equation will also vanish, and as a result, the derivation of the natural frequencies of the system will not make any sense. Moreover, by considering $m = n$, the last term in Eq. (4.14) will vanish for even modes, and the equation will resume to that of a linear system.

Since the effects of the quadratic nonlinearity are of interest herein, the term characterising nonlinearity must remain in the equation so that its influence on the system's natural frequencies can be investigated. Therefore, all the factors carrying k_2 cannot be equal to zero.

Keeping all these in mind, a fixed value to m is assigned, says $m = 1$ so that Eq. (4.14) is modified into:

$$\frac{d^2 T_n(t)}{dt^2} + \left[\frac{EI}{\rho S} \left(\frac{n\pi}{L} \right)^4 + \frac{k_l}{\rho S} \right] T_n(t) + 4 \left[\frac{2n^2}{(4n^2 - 1)\pi} \right] \frac{k_2 A_n}{\rho S} T_n(t)^2 = 0. \quad (4.15)$$

Next, the following parameters are set to simplify Eq. (4.15):

$$\omega_l^2 = \frac{EI}{\rho S} \left(\frac{n\pi}{L} \right)^4 + \frac{k_l}{\rho S}; \quad \alpha_2 = \left[\frac{8n^2}{(4n^2 - 1)\pi} \right] \frac{k_2 A_n}{\rho S}. \quad (4.16)$$

Therefore, Eq. (4.15) is reduced to:

$$\ddot{T}_n + \omega_l^2 T_n + \alpha_2 T_n^2 = 0, \quad (4.17)$$

where $\ddot{T}_n$ is the second derivative of T_n with respect to the time t , ω_l is a linear parameter depending on the stiffness coefficient k_l , and represents the linear natural frequency as obtained in 3, and α_2 is the nonlinear parameter depending on the nonlinear stiffness coefficient k_2 . Then, the nonlinearity represented here by the parameter α_2 will intervene in the calculations of the nonlinear natural frequencies of the system and the beam's vibration amplitude.

Meanwhile Eq. (4.17) is a nonlinear differential equation to the second order, it cannot be transformed into a linear equation due to the term $\alpha_2 T_n^2$. Hence, its solution should be obtained using any analytical methods available in the literature dealing with nonlinear equations or using a numerical method. In the present chapter, the SEM (Nayfeh and Mook, 1995; He, 2004) is used to solve Eq. (4.17). The procedure of the method has already been explained in Chapter 2. To apply the method, first, a new time τ is set such that:

$$\tau = \Omega_q t, \quad (4.18)$$

where Ω_q is the nonlinear natural frequency of the entire system which will be determined shortly. Then, replacing t in Eq. (4.17), a modified equation is obtained:

$$\Omega_q^2 \ddot{T}_n + \omega_l^2 T_n + \alpha_2 T_n^2 = 0, \quad (4.19)$$

where $\ddot{T}_n$ stands for the second derivative of T with respect to the new time τ . Then, the solution of Eq. (4.19) is assumed to be:

$$T_n(t) = \varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3 + \cdots + \varepsilon^n T_n, \text{ with } \varepsilon \ll 1. \quad (4.20)$$

Also, the nonlinear natural frequency Ω_q is expressed using the small parameter:

$$\Omega_q = \omega_l + \varepsilon \Omega_1 + \varepsilon^2 \Omega_2 + \varepsilon^3 \Omega_3 + \cdots + \varepsilon^n \Omega_n, \text{ with } \varepsilon \ll 1. \quad (4.21)$$

By replacing Eqs. (4.20) and (4.21) into Eq. (4.19), it gives:

$$\begin{aligned} & \varepsilon \omega_l^2 (\ddot{T}_1 + T_1) + \varepsilon^2 (\omega_l^2 \ddot{T}_2 + \omega_l^2 T_2 + \alpha_2 T_1^2 + 2\omega_l \Omega_1 \ddot{T}_1) + \\ & \varepsilon^3 \left[\omega_l^2 \ddot{T}_3 + \omega_l^2 T_3 + 2\omega_l \Omega_1 \ddot{T}_2 + (\Omega_1 + 2\omega_l \Omega_2) \ddot{T}_1 + 2\alpha_2 T_1 T_2 \right] + \cdots = 0 \end{aligned} \quad (4.22)$$

Since Eq. (4.22) is grouped in terms of powers of ε , solving it implies solving the terms of each power of ε by equating them to zero. That results in solving the following successive equations where the solution of an equation at a given order greater than 1 of ε requires the solution from the previous equations. These are:

$$\text{for order } \varepsilon^1 : \quad \ddot{T}_1 + T_1 = 0 \quad (4.23a)$$

$$\text{for order } \varepsilon^2 : \quad \omega_l^2 (\ddot{T}_2 + T_2) = -2\omega_l \Omega_1 \ddot{T}_1 - \alpha_2 T_1^2 \quad (4.23b)$$

$$\text{for order } \varepsilon^3 : \quad \omega_l^2 (\ddot{T}_3 + T_3) = -2\omega_l \Omega_1 \ddot{T}_2 - (\Omega_1 + 2\omega_l \Omega_2) \ddot{T}_1 - 2\alpha_2 T_1 T_2 \quad (4.23c)$$

Eq. (4.23a) is a homogeneous linear ordinary differential equation to the second order (Gundersen, 1988). Therefore, its solution is known and given by:

$$T_1(\tau) = B_1 \cos(\tau + \varphi_1), \quad (4.24)$$

where B_1 is the amplitude of vibration, and its value will later be specified. The substitution of Eq. (4.24) into Eq. (4.23b), implies:

$$\omega_l^2 (\ddot{T}_2 + T_2) = -2\omega_l \Omega_1 B_1 \cos(\tau + \varphi_1) - \frac{\alpha_2 B_1^2}{2} - \frac{\alpha_2 B_1^2}{2} \cos(2\tau + 2\varphi_1). \quad (4.25)$$

The solution of Eq. (4.25) has two parts: one for the homogeneous equation and the other for the particular equation. The homogeneous solution is given by:

$$T_2(\tau) = B_2 \cos(\tau + \varphi_2). \quad (4.26)$$

Before proceeding to find the particular solution, one should note that due to the presence of the secular term in Eq. (4.25) given by $-2\omega_l \Omega_1 B_1 \cos(\tau + \varphi_1)$ the solution will be unbounded.

That term must be eliminated to allow bounded solutions for the system. This results in solving:

$$-2\omega_l\Omega_1B_1\cos(\tau+\varphi_1)=0 \quad (4.27)$$

The only reasonable solution for Eq. (4.27) to hold is to have $\Omega_1 = 0$. The particular solution which matches the right-hand side of Eq. (4.25) is supposed to be:

$$T_{2P}(\tau) = \lambda + \gamma\cos(2\tau + 2\varphi_1) \quad (4.28)$$

Next, putting Eq. (4.28) into Eq. (4.25) λ and γ are found to be:

$$\lambda = -\frac{\alpha_2}{2\omega_l^2}B_1^2 \quad \text{and} \quad \gamma = \frac{\alpha_2}{6\omega_l^2}B_1^2. \quad (4.29)$$

Consequently, Eq. (4.28) is transformed into:

$$T_{2P}(\tau) = -\frac{\alpha_2}{6\omega_l^2}B_1^2[3 - \cos(2\tau + 2\varphi_1)]. \quad (4.30)$$

Hence, the solution of Eq. (4.23b) is given by:

$$T_2(\tau) = B_2\cos(\tau + \varphi_2) - \frac{\alpha_2}{6\omega_l^2}B_0^2[3 - \cos(2\tau + 2\varphi_1)]. \quad (4.31)$$

Now, it remains solving Eq. (4.23c). Considering that $\Omega_1 = 0$, the equation becomes:

$$\begin{aligned} \omega_l^2(\ddot{T}_3 + T_3) = & \left[2\omega_l\Omega_2B_1 + \frac{\alpha_2^2}{\omega_l^2}B_1^3 - \frac{\alpha_2^2}{6\omega_l^2}B_1^3 \right] \cos(\tau + \varphi_1) - \\ & \frac{\alpha_2^2}{6\omega_l^2}B_1^3 \cos(3\tau + 3\varphi_1) - \alpha_2B_1B_2\cos(2\tau + \varphi_1 + \varphi_2) - \alpha_2B_1B_2\cos(\varphi_1 - \varphi_2). \end{aligned} \quad (4.32)$$

By using the same procedure as to find the solution of Eq. (4.23b), the solution of the homogeneous equation of Eq. (4.32) is deduced as:

$$T_{3H}(\tau) = B_3\cos(\tau + \varphi_3) \quad (4.33)$$

In order also to suppress the secular terms from Eq. (4.32) which is that with $\cos(\tau + \varphi_1)$, and allow the bounded response of the system, the following equation is solved:

$$2\omega_l\Omega_2B_1 + \frac{\alpha_2^2}{\omega_l^2}B_1^3 - \frac{\alpha_2^2}{6\omega_l^2}B_1^3 = 0 \quad (4.34)$$

Then Ω_2 is obtained as follows:

$$\Omega_2 = -\frac{\alpha_2^2}{3\omega_l^3} B_1^2 \quad (4.35)$$

On the other hand, the particular solution of Eq. (4.32) is supposed to have the same form as its right-hand side. So, solving it step by step and bearing in mind that the first term on the right-hand side is equal to zero, three particular equations are formed. By assuming three intermediary solutions, the overall solution of Eq. (4.32) is supposed to be:

$$T_{3P}(\tau) = a_1 \cos(2\tau + \varphi_1 + \varphi_2) + a_2 \cos(\varphi_1 - \varphi_2) + a_3 \cos(3\tau + 3\varphi_1), \quad (4.36)$$

with a_1 , a_2 , and a_3 , the coefficients or amplitudes of each of the three intermediary particular equations to be determined. Replacing Eq. (4.36) and its second derivative with respect to τ into Eq. (4.32) gives:

$$\begin{aligned} & -4a_1 \cos(2\tau + \varphi_1 + \varphi_2) - 9a_3 \cos(3\tau + 3\varphi_1) + \\ & a_1 \cos(2\tau + \varphi_1 + \varphi_2) + a_2 \cos(\varphi_1 - \varphi_2) + a_3 \cos(3\tau + 3\varphi_1) = \\ & -\frac{\alpha_2^2}{6\omega_l^4} B_1^3 \cos(3\tau + 3\varphi_1) - \frac{\alpha_2}{\omega_l^2} B_1 B_2 \cos(2\tau + \varphi_1 + \varphi_2) - \frac{\alpha_2}{\omega_l^2} B_1 B_2 \cos(\varphi_1 - \varphi_2). \end{aligned} \quad (4.37)$$

It is worth highlighting that the coefficients $a_i (i = 1, \dots, 3)$ are obtained by equating the terms of the same argument of cosine. By doing so, these coefficients are deduced as:

$$a_1 = -\frac{\alpha_2}{3\omega_l^2} B_1 B_2; \quad a_2 = \frac{\alpha_2}{\omega_l^2} B_1 B_2; \quad a_3 = -\frac{\alpha_2^2}{48\omega_l^4} B_1^3. \quad (4.38)$$

By replacing $a_i (i = 1, \dots, 3)$ into Eq. (4.36), the particular equation has the following solution:

$$\begin{aligned} T_{3P}(\tau) = & -\frac{\alpha_2}{3\omega_l^2} B_1 B_2 \cos(2\tau + \varphi_1 + \varphi_2) + \frac{\alpha_2}{\omega_l^2} B_1 B_2 \cos(\varphi_1 - \varphi_2) - \\ & \frac{\alpha_2^2}{48\omega_l^4} B_1^3 \cos(3\tau + 3\varphi_1) \end{aligned} \quad (4.39)$$

Combining Eqs. (4.33) and (4.39), the solution of Eq. (4.23c) is obtained as:

$$\begin{aligned} T_3(\tau) = & B_3 \cos(\tau + \varphi_3) - \frac{\alpha_2}{3\omega_l^2} B_1 B_2 \cos(2\tau + \varphi_1 + \varphi_2) + \frac{\alpha_2}{\omega_l^2} B_1 B_2 \cos(\varphi_1 - \varphi_2) - \\ & \frac{\alpha_2^2}{48\omega_l^4} B_1^3 \cos(3\tau + 3\varphi_1) \end{aligned} \quad (4.40)$$

Hence, the overall solution for Eq. (4.19) to order three of ε is deduced to be:

$$\begin{aligned} T_n(t) = & \varepsilon B_1 \cos(\Omega_q t + \varphi_1) + \varepsilon^2 \left\{ B_2 \cos(\Omega_q t + \varphi_2) - \frac{\alpha_2}{6\omega_l^2} B_1^2 [3 - \cos(2\Omega_q t + 2\varphi_1)] \right\} \\ & + \varepsilon^3 \left\{ B_3 \cos(\Omega_q t + \varphi_3) - \frac{\alpha_2}{3\omega_l^2} B_1 B_2 \cos(2\Omega_q t + \varphi_1 + \varphi_2) + \frac{\alpha_2}{\omega_l^2} B_1 B_2 \cos(\varphi_1 - \varphi_2) \right. \\ & \left. - \frac{\alpha_2^2}{48\omega_l^4} B_1^3 \cos(3\Omega_q t + 3\varphi_1) \right\}. \end{aligned} \quad (4.41)$$

In Eq. (4.41) the terms containing powers of $\varepsilon > 3$ are neglected because these do not bring more modification to the solution (He, 2003; Mohapatra, 2020). Moreover, the solutions at greater orders of $\varepsilon > 3$ are minimal compared to those at or close to the fundamental order and become minor as the order of the parameter ε increases. Since ε is very small (tends to zero), its powers tend faster to zero. Therefore, subsequent terms of the solution tend to zero faster than previous terms. For instance, if $\varepsilon = 10^{-5}$, then we have $\varepsilon^2 = 10^{-10}$ and $\varepsilon^3 = 10^{-15}$. Therefore, the second and third terms in the right-hand side of Eq. (4.41) become very small as compared to the first term. For this reason, they are neglected.

It follows that the nonlinear natural frequency using Eq. (4.21) is given by:

$$\Omega_q = \omega_l + \varepsilon(0) + \varepsilon^2 \left(\frac{\alpha_2^2}{3\omega_l^3} B_1^2 \right) + \dots + \varepsilon^n \Omega_n, \text{ with } \varepsilon \ll 1, \quad (4.42)$$

It is worth highlighting that the positive value of Ω_2 was kept to work with positive values of the natural frequencies and also to obtain reasonable results of the natural frequencies. We note that if ε tends to zero, the nonlinear natural frequency Ω_q given in Eq. (4.42) tends to $\Omega_q = \omega_l$, which is the linear natural frequency. From Eq. (4.42), Ω_q is deduced as:

$$\Omega_q = \omega_l \left[1 + \frac{\alpha_2^2}{3\omega_l^4} \varepsilon^2 B_1^2 \right], \quad (4.43)$$

By replacing $X_n(t)$ and $T_n(t)$ into Eq. (4.8), the overall solution of the governing equation of the beam resting on a nonlinear quadratic elastic foundation (Eq. (4.7)) is found to be:

$$\begin{aligned} U_n(x, t) = & A_n \sin\left(\frac{n\pi}{L}x\right) \left[\varepsilon B_1 \cos(\Omega_q t + \varphi_1) + \varepsilon^2 \left\{ -\frac{\alpha_2}{6\omega_l^2} B_1^2 [3 - \cos(2\Omega_q t + 2\varphi_1)] \right. \right. \\ & \left. \left. + B_2 \cos(\Omega_q t + \varphi_2) \right\} + \varepsilon^3 \left\{ B_3 \cos(\Omega_q t + \varphi_3) - \frac{\alpha_2}{3\omega_l^2} B_1 B_2 \cos(2\Omega_q t + \varphi_1 + \varphi_2) \right. \right. \\ & \left. \left. + \frac{\alpha_2}{\omega_l^2} B_1 B_2 \cos(\varphi_1 - \varphi_2) - \frac{\alpha_2^2}{48\omega_l^4} B_1^3 \cos(3\Omega_q t + 3\varphi_1) \right\} \right]. \end{aligned} \quad (4.44)$$

The above equation determines the beam's transversal displacement $U_n(x, t)$ on a nonlinear quadratic elastic foundation. In what follows, the solution $U_n(x, t)$ is limited to the third-order approximation. Then, to determine the amplitudes B_1 , B_2 , and B_3 , the following initial conditions are considered:

$$T_n(0) = 0.35, \quad \dot{T}_n(0) = 0 \quad (4.45)$$

These conditions are chosen so that the results from analytical calculations can be closer enough to the results from numerical analysis using FEM. The first condition gives:

$$\begin{aligned} T_n(0) = 0.35 \Rightarrow \varepsilon B_1 \cos(\varphi_1) + \varepsilon^2 \left\{ B_2 \cos(\varphi_2) - \frac{\alpha_2}{6\omega_l^2} B_1^2 [3 - \cos(2\varphi_1)] \right\} + \\ \varepsilon^3 \left\{ B_3 \cos(\varphi_3) - \frac{\alpha_2}{3\omega_l^2} B_1 B_2 \cos(\varphi_1 + \varphi_2) + \frac{\alpha_2}{\omega_l^2} B_1 B_2 \cos(\varphi_1 - \varphi_2) - \right. \\ \left. \frac{\alpha_2^2}{48\omega_l^4} B_1^3 \cos(3\varphi_1) \right\} = 0.35. \end{aligned} \quad (4.46)$$

Taking the first derivative of $T_n(t)$ and equating it to zero gives:

$$\begin{aligned} \dot{T}_n(0) = 0 \Rightarrow -\varepsilon B_1 \Omega_q \sin(\varphi_1) + \varepsilon^2 \left\{ -\Omega_q B_2 \sin(\varphi_2) - \frac{\alpha_2 \Omega_q}{3\omega_l^2} B_1^2 \sin(2\varphi_1) \right\} + \\ \varepsilon^3 \left\{ -B_3 \Omega_q \sin(\varphi_3) + \frac{\alpha_2 \Omega_q}{3\omega_l^2} B_1 B_2 \sin(\varphi_1 + \varphi_2) + \frac{3\alpha_2^2 \Omega_q}{48\omega_l^4} B_1^3 \sin(3\varphi_1) \right\} = 0 \end{aligned} \quad (4.47)$$

From Eq. (4.47), each term of powers of ε are equated to 0. This implies:

$$B_1 \Omega_q \sin(\varphi_1) = 0, \quad (4.48a)$$

$$-\Omega_q B_2 \sin(\varphi_2) - \frac{\alpha_2 \Omega_q}{3\omega_l^2} B_1^2 \sin(2\varphi_1) = 0, \text{ and} \quad (4.48b)$$

$$-\Omega_q \left[B_3 \sin(\varphi_3) - \frac{\alpha_2}{3\omega_l^2} B_1 B_2 \sin(\varphi_1 + \varphi_2) - \frac{3\alpha_2^2}{48\omega_l^4} B_1^3 \sin(3\varphi_1) \right] = 0. \quad (4.48c)$$

Since B_1 and Ω_q are both nonzero parameters, Eq. (4.48a) implies:

$$\sin(\varphi_1) = 0 \Rightarrow \varphi_1 = 0 + \Omega_1 \pi, \quad \Omega_1 \in \mathbb{Z}. \quad (4.49)$$

By implementing Eq. (4.49) into Eq. (4.48b), it follows that:

$$\sin(\varphi_2) = 0 \Rightarrow \varphi_2 = 0 + \Omega_2 \pi, \quad \Omega_2 \in \mathbb{Z}. \quad (4.50)$$

Replacing φ_1 and φ_2 into Eq. (4.48c), it results in:

$$\sin(\varphi_3) = 0 \Rightarrow \varphi_3 = 0 + \Omega_3 \pi, \Omega_3 \in \mathbb{Z}. \quad (4.51)$$

Hence $\varphi_1 = \varphi_2 = \varphi_3 = 0$, transforming Eq. (4.46) into:

$$\varepsilon B_1 + \varepsilon^2 \left\{ B_2 - \frac{\alpha_2}{3\omega_l^2} B_1^2 \right\} + \varepsilon^3 \left\{ B_3 - \frac{\alpha_2}{3\omega_l^2} B_1 B_2 + \frac{\alpha_2}{\omega_l^2} B_1 B_2 - \frac{3\alpha_2^2}{48\omega_l^4} B_1^3 \right\} = 0.35 \quad (4.52)$$

By setting the terms of the same power of ε together, it holds that:

$$B_1 = 35 \text{ for } \varepsilon = 0.01 \quad (4.53a)$$

$$\varepsilon^2 \left\{ B_2 - \frac{\alpha_2}{3\omega_l^2} B_1^2 \right\} = 0, \text{ and} \quad (4.53b)$$

$$\varepsilon^3 \left\{ B_3 + \frac{2\alpha_2}{3\omega_l^2} B_1 B_2 - \frac{\alpha_2^2}{16\omega_l^4} B_1^3 \right\} = 0. \quad (4.53c)$$

Solving Eqs. (4.53b) and (4.53c), respectively gives:

$$B_2 = \frac{\alpha_2}{3\omega_l^2} B_1^2, \text{ and } B_3 = -\frac{23\alpha_2^2}{144\omega_l^2} B_1^3. \quad (4.54)$$

Thus, the expression of $T_n(t)$ is found in terms of B_1 , ω_l , and α_2 as:

$$\begin{aligned} T_n(t) = & \varepsilon B_1 \cos(\Omega_q t) + \frac{\alpha_2}{6\omega_l^2} \varepsilon^2 B_1^2 \left[2\cos(\Omega_q t) - 3 + \cos(2\Omega_q t) \right] - \\ & \frac{\alpha_2^2}{3\omega_l^4} \varepsilon^3 B_1^3 \left\{ \frac{23}{48} \cos(\Omega_q t) + \frac{1}{3} [\cos(2\Omega_q t) - 3] + \frac{1}{16} \cos(3\Omega_q t) \right\} \end{aligned} \quad (4.55)$$

Hence, the final solution of Eq. (4.7) is:

$$\begin{aligned} U_n(x, t) = & A_n \sin\left(\frac{n\pi}{L}x\right) \left\{ \varepsilon B_1 \cos(\Omega_q t) + \frac{\alpha_2 \varepsilon^2}{6\omega_l^2} B_1^2 \left[2\cos(\Omega_q t) - 3 + \cos(2\Omega_q t) \right] \right. \\ & \left. - \frac{\alpha_2^2}{3\omega_l^4} \varepsilon^3 B_1^3 \left[\frac{23}{48} \cos(\Omega_q t) + \frac{1}{3} (\cos(2\Omega_q t) - 3) + \frac{1}{16} \cos(3\Omega_q t) \right] \right\} \end{aligned} \quad (4.56)$$

Throughout this chapter and the rest of the thesis, only free vibration will be considered, meaning that no external excitation will be considered. Also, the same parameters provided in Table 3.1 of chapter 3 will be used for both analytical and numerical calculations. Moreover,

the values of I , the second moment of inertia, and S , the cross-section will remain constant. The same parameters are considered to allow consistency when comparing the results from the linear analysis to those from the nonlinear analysis, which will result from the present study. The distance b between two consecutive springs will be 0.2 m.

To obtain the vibration of the BoEF with quadratic nonlinearity in the foundation, the parameters of the system have to be chosen wisely. The value of the amplitude A_n can be chosen as equal to 1 so that it does not influence the results as time evolves. The same value was used by some authors (Hasheminejad and Rafsanjani, 2011; Mamandi et al., 2012; Kumar et al., 2015) This is also in accord with the statement in many studies in the literature. As for ε , it is chosen in the interval $0 < \varepsilon < 1$ because the perturbation/approximate solutions are acceptable only for small values of ε (He, 2003; Mohapatra, 2020).

Relying on the fact that the foundation must not be too stiff, the question is what stiffness value can be used? Knowing how the vibration response of the beam varies with the foundation stiffness is useful, as this allows us to predict the vibration behaviour of the beam for a given foundation stiffness. However, the vibration response of the beam cannot be modified by changing the foundation stiffness, as this depends on fixed soil properties unless the soil properties can be changed through soil-improvement measures. The relevant thing to remember is that if the foundation is too stiff, instead of the problem being the vibration of a beam on an elastic foundation, it becomes the vibration of the elastic foundation. Therefore, the central deflection of the simply supported beam is first considered, without an elastic foundation. This has a value of δ_0 under a known uniformly distributed static loading w (per unit length) and is given by:

$$\delta_0 = \frac{5wL^4}{384EI}. \quad (4.57)$$

The foundation stiffness k depends on the foundation's properties. As an example, the stiffness of a track on a subgrade depends on the properties of the soil and the value can be changed based on soil improvement. Based on this, the net effect of k_1 and k_2 must be such that the net deflection δ_k of the BoEF has a value ranging from $0.5\delta_0$ to $1.0\delta_0$. That is:

$$0.5\delta_0 < \delta_k < 1.0\delta_0 \text{ under the same loading } w. \quad (4.58)$$

Then, the elastic foundation can be considered not to be too stiff. At the same time, writing the expression in Eq. (4.58) as a function of the beam stiffness gives:

$$\delta_0 = \frac{5wL}{32k_b}, \text{ where } k_b = \frac{12EI}{L^3} \text{ is the beam stiffness.} \quad (4.59)$$

It follows that the foundation's stiffness is considered such that the stiffness of the spring has the same numerical value as that of the beam, meaning $k = k_b$. Then, following the idea by Frydrysek et al. (2013) the linear and nonlinear stiffnesses of the foundation are respectively equal to:

$$k_l = \frac{k_b}{b} \text{ for the linear stiffness.} \quad (4.60a)$$

$$\text{and } k_2 = \frac{k_b}{b^2} \text{ for the nonlinear quadratic stiffness.} \quad (4.60b)$$

where b is the distance separating two consecutive springs forming the foundation. By performing the numerical application of k_b using the values given in Table 4.2 is found as:

$$k_b = 33750 \text{ N/m.} \quad (4.61)$$

This value of k_b is the same used in the last chapter and will be used for this chapter and the rest of the thesis. Hence, replacing k_b into Eqs. (4.60a) and (4.60b) the stiffnesses of the foundation are deduced as:

$$k_l = 1.69 \times 10^5 \text{ N/m}^2 \text{ and } k_2 = 8.44 \times 10^5 \text{ N/m}^3 \quad (4.62)$$

These values of k_l the linear stiffness coefficient and k_2 the nonlinear quadratic stiffness coefficient of the foundation will be used to calculate the nonlinear natural frequencies Ω_q of the whole system. The table below displays the values of the parameters to be used for analytical calculations.

Table 4.1: Values of parameters used as input in MATLAB for the beam on a nonlinear quadratic elastic foundation.

(a)			(b)		
n	ω_l (rad/s)	α_2 (N/kg·m ²)	n	ω_l (rad/s)	α_2 (N/kg·m ²)
1	285.17	1.2×10^5	6	3843.4	8.96×10^4
2	501.5	9.5×10^4	7	5225.7	8.95×10^4
3	994.42	9.2×10^4	8	6821.7	8.93×10^4
4	1724.6	9.04×10^4	9	8631.3	8.93×10^4
5	2675.8	8.99×10^4	10	10651	8.92×10^4

Other parameters	
k_l	$1.69 \times 10^5 \text{ N/m}^2$
k_2	$8.44 \times 10^5 \text{ N/m}^3$
A_n	1
ε	0.01
B_1	35

In Table 4.1 two sub-tables are presented: the first one (Table 4.1a) reads the linear natural frequencies ω_l and another parameter α_2 (carrying nonlinearity) which both depend

on the mode's number n . The second one (see Table 4.1b) reveals other parameters which are constant. These values were used as input in MATLAB. Then, the spatial displacement $U_n(x, t)$ is plotted for the first four modes of vibration of the beam as displayed in Figure 4.3.

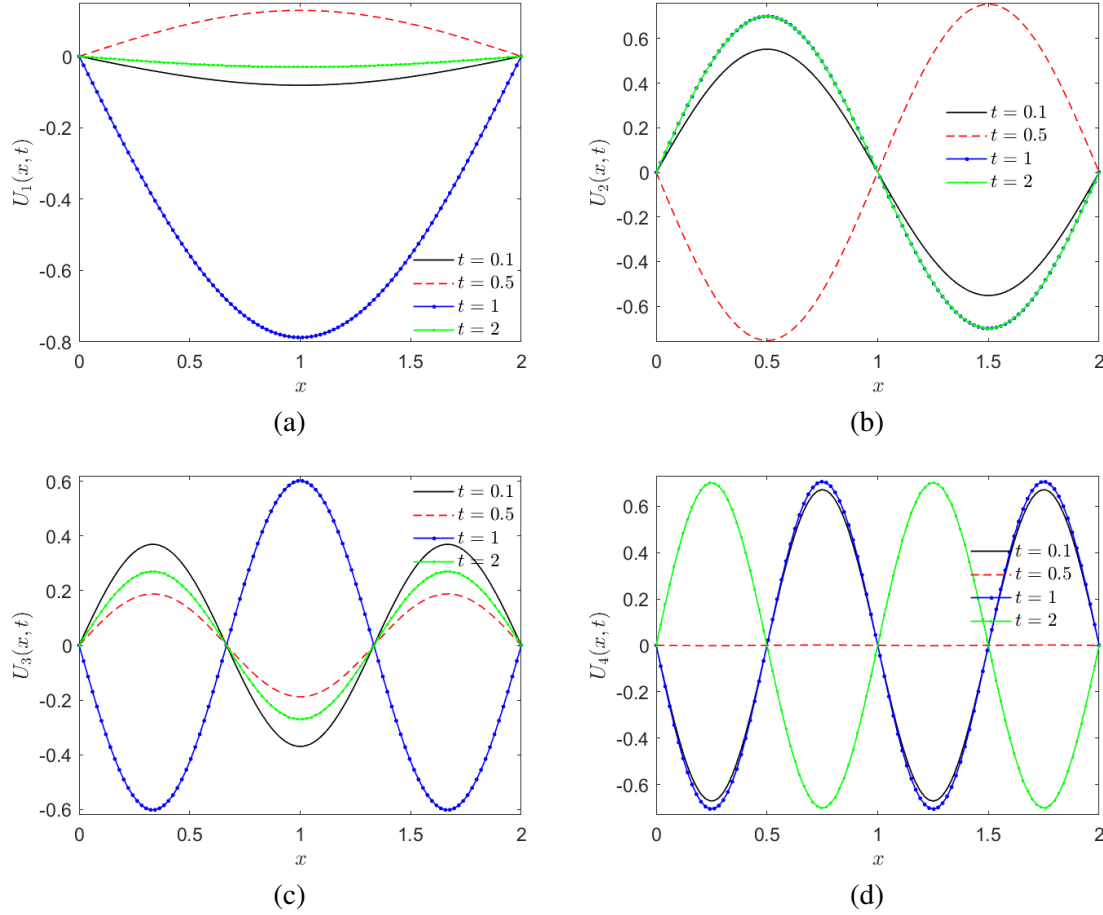


Figure 4.3: Vibration modes of the spatial displacement of the simply supported beam with a nonlinear quadratic foundation: (a) to (d) are modes 1 to 4.

Figure 4.3 reveals MATLAB outputs of the beam's spatial displacement by varying the coordinate x along the beam's length, while other parameters remained as in Table 4.1. Plotting $U_n(x, t)$ as a function of x is like plotting a graph for which the amplitude depends on the mode's number because the term in braces (see Eq. (4.56)) is constant with respect to x . That means the displacement plot is obtained at a given time, and the resulting vibration amplitude is just that of a spatial amplitude. Several plots are shown here at different times to highlight the impact of the term in braces given in Eq. (4.56). The natural frequencies corresponding to the plots in Figure 4.3 are given in the following table.

Table 4.2: Natural frequencies of the beam on a nonlinear quadratic elastic foundation corresponding to the analytical plots presented in Figure 4.3.

Mode's number n	1	2	3	4
Ω_q (rad/s)	299.2	508.6	998.1	1726.8

The natural frequencies presented in Table 4.2 increase with the mode's number n . Although the terms in brace given in Eq. (4.56) seem constant with respect to x , plotting the curves at each mode for different time values shows different amplitudes. For three of all the cases plotted, it is noticed that the curves obtained at $t = 0.5$ s have the minimum amplitude (see Figures 4.3a, 4.3c, and 4.3d), whereas at $t = 1$ s, the amplitudes are maximum. Unfortunately, it is not possible to define a specific relation between amplitude and time. Nevertheless, the outcome of the plots reveals that the terms in braces cannot simply be neglected because this could lead to erroneous results. Moreover, plotting the curves for different time values exposes that the beam's amplitude may decrease or increase for some modes for a given time, which is a good outcome.

In order to obtain the temporal amplitude of vibration of the beam resting on a nonlinear quadratic elastic foundation, the solution $U_n(x, t)$ was plotted at a fixed spatial position $x = \frac{L}{2}$, the centre of the beam by varying the time t . The outcome is shown in Figure 4.4.

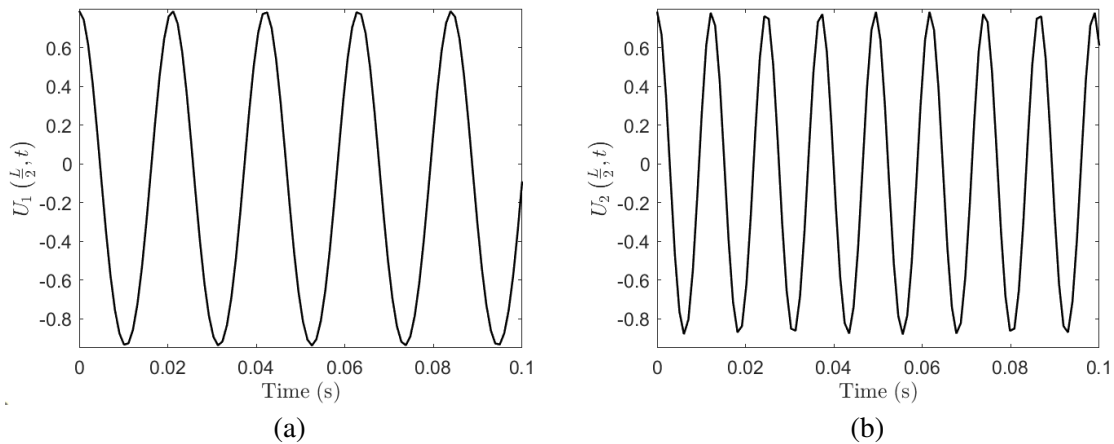


Figure 4.4 sets out the temporal plots of the beam displacement at its midspan. These plots were obtained in MATLAB using the parameters defined in the previous paragraphs and varying the time t from 0 to 0.1 s. The timestep is chosen very small to enable the visualisation of the oscillations. the maximum and minimum amplitudes are similar for all four modes.

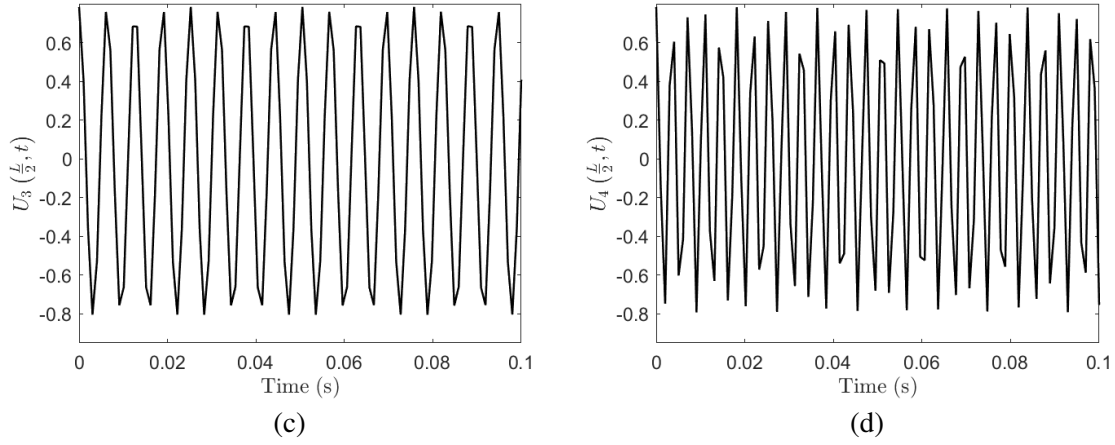


Figure 4.4: Temporal amplitude of vibration of the simply supported beam on a nonlinear quadratic elastic foundation obtained at beam midspan ($x = \frac{L}{2}$): (a) to (d) are modes 1 to 4.

4.3.2 Study of the BoEF having Linear and Nonlinear Cubic Stiffness Properties

For the BoEF having linear and nonlinear cubic stiffness properties, the expression of the stiffness coefficient k_{nl} is given by (Tomlinson and Worden, 2000):

$$k_{nl} = k_l + k_3 U(x, t)^2, \quad (4.63)$$

with k_l the linear and k_3 the nonlinear cubic stiffness coefficients per unit length and $U(x, t)$ is considered as the transversal displacement of the beam. Therefore, Eq. (4.7) becomes:

$$\rho S \frac{\partial^2 U(x, t)}{\partial t^2} + EI \frac{\partial^4 U(x, t)}{\partial x^4} + k_l U(x, t) + k_3 U(x, t)^3 = 0. \quad (4.64)$$

For this second case, $U(x, t)$ is written on the following form:

$$U(x, t) = Q(x)\Gamma(t), \quad (4.65)$$

where $Q(x)$ and $\Gamma(t)$ are the spatial and temporal parts of the transversal displacement, respectively. When Eq. (4.65) is replaced into Eq. (4.64), and it follows that:

$$EI \frac{\partial^4 Q(x)}{\partial x^4} \Gamma(t) + \frac{\partial^2 \Gamma(t)}{\partial t^2} Q(x) + k_l Q(x) \Gamma(t) + k_3 Q(x)^3 \Gamma(t)^3 = 0. \quad (4.66)$$

Following the procedure in Nayfeh and Mook (1995), Rao (2007), and Nayfeh and Pai (2008), the spatial solution of a simply supported beam is given by:

$$Q_m(x) = B_m \sin\left(\frac{m\pi}{L}x\right), \quad (4.67)$$

where B_m is the amplitude of vibration and m is an integer representing the mode's number. When putting $Q_m(x)$ into Eq. (4.66), since the only variable is $\Gamma(t)$, the partial derivatives are transformed into ordinary derivatives. It follows that:

$$\sum_{m=0}^{\infty} \left\{ \left[EI \left(\frac{m\pi}{L} \right)^4 + k_l \right] \sin\left(\frac{m\pi}{L}x\right) \Gamma_m(t) + \rho S \sin\left(\frac{m\pi}{L}x\right) \frac{d^2 \Gamma_m(t)}{dt^2} + \frac{1}{4} k_3 B_m^2 \left[3 \sin\left(\frac{m\pi}{L}x\right) - \sin\left(\frac{3m\pi}{L}x\right) \right] \Gamma_m(t)^3 \right\} = 0. \quad (4.68)$$

Next, the orthogonality of sine functions is applied to Eq. (4.68). This gives:

$$\begin{aligned} \sum_{m=0}^{\infty} \left\{ \left[EI \left(\frac{m\pi}{L} \right)^4 + k_l \right] \Gamma_m(t) \int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx + \right. \\ \rho S \frac{d^2 \Gamma_m(t)}{dt^2} \int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx + \\ \frac{3}{4} k_3 B_m^2 \Gamma_m(t)^3 \int_0^L \sin\left(\frac{m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx - \\ \left. \frac{1}{4} k_3 B_m^2 \Gamma_m(t)^3 \int_0^L \sin\left(\frac{3m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx \right\} = 0. \end{aligned} \quad (4.69)$$

The following equation holds from the integral of trigonometric functions:

$$\int_0^L \sin\left(\frac{3m\pi}{L}x\right) \sin\left(\frac{n\pi}{L}x\right) dx = 0 \quad \text{for both } m = n \text{ and } m \neq n. \quad (4.70)$$

Only the special case for $m = n$ is considered, while other cases are discarded so that, some terms in Eq. (4.69) will not vanish. Therefore, the outcomes of the integrals given in Eqs. (4.13a) and (4.70) are substituted into Eq. (4.69) and the following equation is obtained:

$$\frac{d^2 \Gamma_m(t)}{dt^2} + \left[\frac{EI}{\rho S} \left(\frac{m\pi}{L} \right)^4 + \frac{k_l}{\rho S} \right] \Gamma_m(t) + \frac{3}{4} \frac{k_3}{\rho S} B_m^2 \Gamma_m(t)^3 = 0. \quad (4.71)$$

This last equation is called a modal equation. To simplify it, the parameters below are set:

$$\omega_l^2 = \frac{EI}{\rho S} \left(\frac{m\pi}{L} \right)^4 + \frac{k_l}{\rho S}; \quad \text{and} \quad \beta_3 = \frac{3}{4} \frac{k_3 B_m^2}{\rho S}. \quad (4.72)$$

Then, Eq. (4.71) is transformed into:

$$\ddot{\Gamma}_m(t) + \omega_l^2 \Gamma_m(t) + \beta_3 \Gamma_m(t)^3 = 0. \quad (4.73)$$

Eq. (4.73) is a second-order nonlinear cubic differential equation and cannot be transformed into a linear equation due to the presence of a term $\beta_3 \Gamma_m(t)^3$. It is also an undamped Duffing's equation (Korsch et al., 2007; Moser, 1963; Thompson and Stewart, 2002) named after Georg Duffing (1918). Apart from that, Eq. (4.73) describes the vibrations of a mass attached to a nonlinear spring having the restoring force of $\omega_l \Gamma_m + \beta_3 \Gamma_m^3$. ω_l is the same parameter as defined in Subsection 4.3.1. Thus, the cubic nonlinearity expressed here with the parameter β_3 will mainly contribute to the expressions of the nonlinear natural frequencies. Nevertheless, it is worth mentioning that when ω_l remains positive and $\beta_3 < 0$, the spring is identified as a softening spring, whereas for $\beta_3 > 0$, the spring is a hardening spring (Soares and Gonçalves, 2018). The solution of Eq. (4.73) is found using the SEM.

To begin with, the same time τ is set with the new nonlinear natural frequency such that:

$$\tau = \Omega_c t, \quad (4.74)$$

where Ω_c for this second case of study is the nonlinear natural frequency due to the cubic nonlinearity that is assumed to be:

$$\Omega_c = \omega_l + \varepsilon \Omega_1 + \varepsilon^2 \Omega_2 + \varepsilon^3 \Omega_3 + \cdots + \varepsilon^n \Omega_m, \text{ with } \varepsilon \ll 1. \quad (4.75)$$

Then, Eq. (4.73) is transformed into:

$$\Omega_c^2 \ddot{\Gamma}_m(\tau) + \omega_l^2 \Gamma_m(\tau) + \beta_3 \Gamma_m(\tau)^3 = 0. \quad (4.76)$$

The double dot above Γ_m stands for the derivative with respect to the new time τ . Because the method to be used is a perturbation method, the parameter ε to Eq. (4.76) in front of the nonlinear term and the solution of the equation is assumed to be:

$$\Gamma_m(\tau) = \Gamma_0 + \varepsilon \Gamma_1 + \varepsilon^2 \Gamma_2 + \cdots + \varepsilon^m \Gamma_m, \text{ with } \varepsilon \ll 1. \quad (4.77)$$

By substituting $\Gamma_m(\tau)$ and Ω_c into Eq. (4.76) the following equation is obtained:

$$\begin{aligned} & \omega_l^2(\ddot{\Gamma}_0 + \Gamma_0) + \varepsilon(\omega_l^2\ddot{\Gamma}_1 + \omega_l^2\Gamma_1 + 2\omega_l\Omega_1\ddot{\Gamma}_0 + \beta_3\Gamma_0^3) + \\ & \varepsilon^2[\omega_l^2\ddot{\Gamma}_2 + \omega_l^2\Gamma_2 + 2\omega_l\Omega_1\ddot{\Gamma}_1 + (\Omega_1^2 + 2\omega_l\Omega_2)\ddot{\Gamma}_0 + 3\beta_3\Gamma_0^2\Gamma_1] + \dots = 0 \end{aligned} \quad (4.78)$$

Since Eq. (4.78) is grouped in terms of powers of ε , solving it implies solving the following equations:

$$\text{Order } \varepsilon^0: \quad \ddot{\Gamma}_0 + \Gamma_0 = 0 \quad (4.79a)$$

$$\text{Order } \varepsilon^1: \quad \omega_l^2(\ddot{\Gamma}_1 + \Gamma_1) = -2\omega_l\Omega_1\ddot{\Gamma}_0 - \beta_3\Gamma_0^3 \quad (4.79b)$$

$$\text{Order } \varepsilon^2: \quad \omega_l^2(\ddot{\Gamma}_2 + \Gamma_2) = -2\omega_l\Omega_1\ddot{\Gamma}_1 - (\omega_l\Omega_2 + \Omega_1^2)\ddot{\Gamma}_0 - 3\beta_3\Gamma_0^2\Gamma_1 \quad (4.79c)$$

Eq. (4.79a) is a homogeneous second-order differential equation. For that reason, its solution can be in the form:

$$\Gamma_0(\tau) = C_0 \cos(\tau + \phi_0), \quad (4.80)$$

Putting Eq. (4.80) back into Eq. (4.79b), the following is obtained:

$$\omega_l^2(\ddot{\Gamma}_1 + \Gamma_1) = \left[2\omega_l\Omega_1C_0 - \frac{3}{4}\beta_3C_0^3\right]\cos(\tau + \phi_0) - \frac{1}{4}\beta_3C_0^3\cos(3\tau + 3\phi_0) \quad (4.81)$$

The solution of Eq. (4.81) has two parts: homogeneous and particular equations. The homogeneous equation has the same form as that of Eq. (4.79a). That implies:

$$\Gamma_{1H}(\tau) = C_1 \cos(\tau + \phi_1). \quad (4.82)$$

To obtain the solution of the particular equation satisfying Eq. (4.81), the secular term carried by $\cos(\tau + \phi_0)$ is first eliminated by setting:

$$2\omega_l\Omega_1C_0 - \frac{3}{4}\beta_3C_0^3 = 0 \quad (4.83)$$

Solving Eq. (4.83) with Ω_1 as variable, Ω_1 is found to be:

$$\Omega_1 = \frac{3}{8\omega_l}\beta_3C_0^2 = 0 \quad (4.84)$$

It follows that the solution of the particular equation from Eq. (4.81) is given by:

$$\Gamma_{1p}(\tau) = \frac{\beta_3}{32\omega_l^2} C_0^3 \cos(3\tau + 3\phi_0). \quad (4.85)$$

Combining Eqs. (4.82) and (4.85), the solution of Eq. (4.79b) is found as follows:

$$\Gamma_1(\tau) = C_1 \cos(\tau + \phi_1) + \frac{\beta_3}{32\omega_l^2} C_0^3 \cos(3\tau + 3\phi_0). \quad (4.86)$$

Therefore, stopping at this order, the system's nonlinear natural frequency is written as:

$$\Omega_c = \omega_l \left[1 + \frac{3}{8\omega_l^2} \varepsilon \beta_3 C_0^2 \right], \quad (4.87)$$

This expression of the nonlinear natural frequencies Ω_c was also found by Pellicano and Vakakis (2001) and Mamandi et al. (2012) using nonlinear normal modes through an energy-based method and multiple time scales method (MTSM), respectively. Then, the overall solution of Eq. (4.76) is deduced to be:

$$\Gamma_m(t) = C_0 \cos(\Omega_c t + \phi_0) + \varepsilon \left[C_1 \cos(\Omega_c t + \phi_1) + \frac{\beta_3}{32\omega_l^2} C_0^3 \cos(3\Omega_c t + 3\phi_0) \right]. \quad (4.88)$$

Next, replacing $Q_m(x)$ and $\Gamma_m(t)$ into Eq. (4.65) where m is the mode's number, it holds that:

$$U_m(x, t) = B \sin\left(\frac{m\pi}{L}x\right) \left\{ C_0 \cos(\Omega_c t + \phi_0) + \varepsilon \left[C_1 \cos(\Omega_c t + \phi_1) + \frac{\beta_3}{32\omega_l^2} C_0^3 \cos(3\Omega_c t + 3\phi_0) \right] \right\}. \quad (4.89)$$

Furthermore, $V_m(t)$ (Eq. (4.89)) is written in a simplified form, by determining the C_1 as a function of C_0 , ω_l , and β_3 . To do this, the following initial conditions are set:

$$\Gamma_m(0) = 0.8, \text{ and} \quad (4.90a)$$

$$\dot{\Gamma}_m(0) = 0 \quad (4.90b)$$

Eq. (4.90b) implies:

$$\phi_0 = \phi_1 = 0. \quad (4.91)$$

Substituting Eq. (4.91) into the equation given by $\Gamma_m(0) = 0.8$, the following is obtained:

$$C_0 + \varepsilon \left[C_1 + \frac{1}{32\omega_l^2} \beta_3 C_0^3 \right] = 1 \quad (4.92)$$

Solving Eq. (4.92) with the variables C_1 and C_2 gives

$$C_0 = 0.8, \text{ and } C_1 = -\frac{\beta_3}{32\omega_l^2} C_0^3 \quad (4.93)$$

Thus, the expression of $\Gamma(t)$ is found to be:

$$\Gamma_m(t) = C_0 \cos(\Omega_c t) + \varepsilon \left[-\frac{\beta_3}{32\omega_l^2} C_0^3 \cos(\Omega_c t) + \frac{1}{32\omega_l^2} \beta_3 C_0^3 \cos(3\Omega_c t) \right]. \quad (4.94)$$

Hence, the solution of Eq. (4.66) till the first-order is deduced as:

$$U_m(x, t) = B_m \sin\left(\frac{m\pi}{L}x\right) \left\{ C_0 \cos(\Omega_c t) + \varepsilon \left[-\frac{\beta_3}{32\omega_l^2} C_0^3 \cos(\Omega_c t) + \frac{\beta_3}{32\omega_l^2} C_0^3 \cos(3\Omega_c t) \right] \right\}. \quad (4.95)$$

The vibration of the BoEF with cubic nonlinearity in the foundation is obtained by knowing the system's parameters. The linear stiffness coefficient equals $k_l = 1.68 \times 10^5$ N/m² (the same value as in Subsection 4.3.1); $B_m = 1$ following the same explanation as for the value of A_n ; $C_0 = 0.2$, $\varepsilon = 0.01$ playing the same role as in the last case studied; and t having the same values as well. As for the nonlinear cubic stiffness coefficient, its value is assigned so that:

$$k_3 = \frac{3k_b}{b^3} = \frac{3k_l}{b^2}, \quad (4.96)$$

Putting the value of k_l the linear stiffness coefficient into Eq. (4.96) and knowing that $b = 0.2$ m, k_3 is obtained:

$$k_3 = 1.3 \times 10^7 \text{ N/m}^4. \quad (4.97)$$

What follows is to plot the displacement $U_m(x, t)$ for the first four modes of vibration ($m = 1, \dots, 4$) by varying the coordinate x along the beam length. This is shown in Figure 4.5.

Figure 4.5 shows the plots obtained from MATLAB of the first four modes of vibration of a simply supported beam resting on a foundation having nonlinear cubic stiffness properties. The shapes of the plots at each mode look different to those in Figure 4.3. The

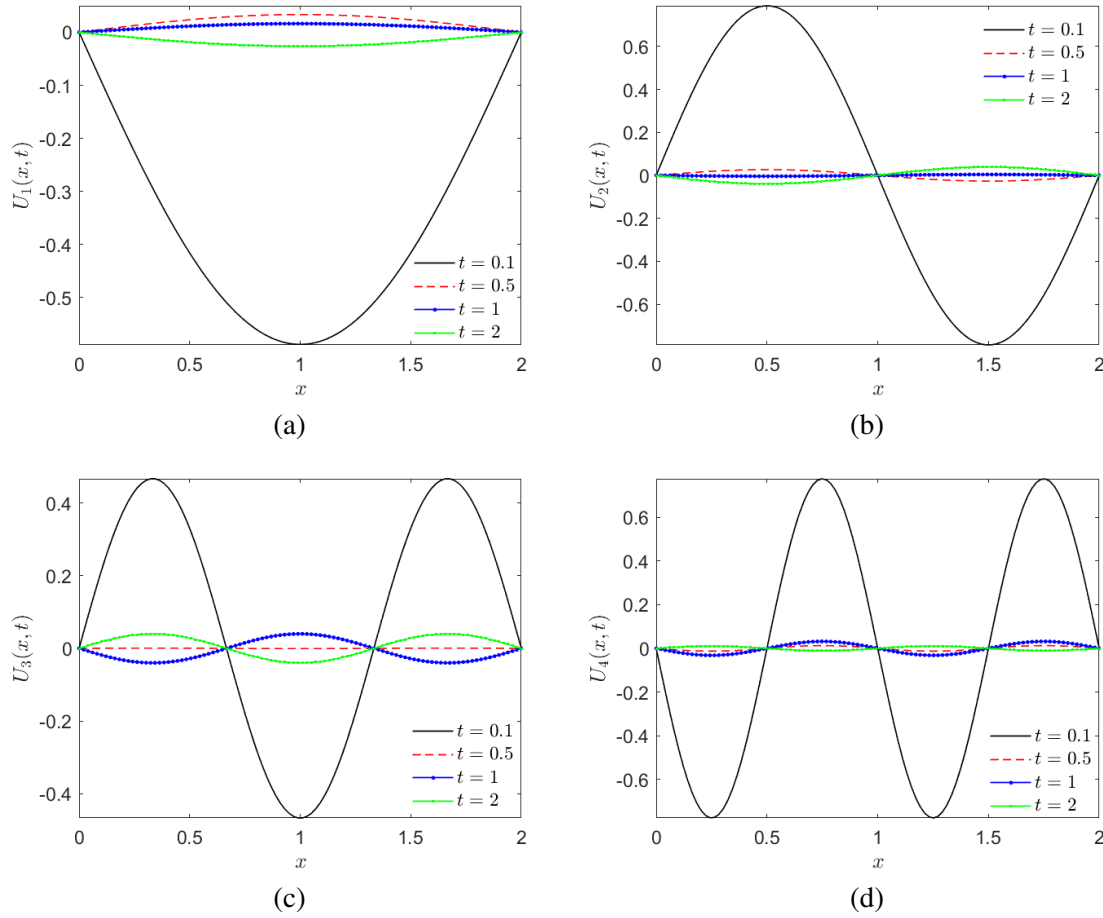


Figure 4.5: Vibration modes of the spatial displacement of the simply supported beam on a nonlinear cubic elastic foundation: (a) to (d) are modes 1 to 4.

spatial amplitudes of vibration of the beam for $t = 0.1$ s are the highest for all four modes. Nevertheless, the amplitudes at other times have considerably reduced. Therefore, replacing quadratic nonlinearity with cubic nonlinearity brings changes to the beam's spatial vibration amplitude at given times. It is worth highlighting that plots with different times values show the effects of the temporal part of the displacement solution. As a result, assigning different values to the time decreases the beam's spatial amplitude at some modes. The corresponding natural frequencies to these modes are given in Table 4.3.

In Table 4.3 the natural frequencies have slightly decreased compared to those obtained for the case with quadratic nonlinearity in the foundation. This is because the nonlinear parameter β_3 does not depend on the mode's number as was the case for the BoEF having nonlinear quadratic stiffness properties. To proceed, the behaviour of the beam as time

Table 4.3: Natural frequencies of the beam on a nonlinear cubic elastic foundation corresponding to the plots presented in Figure 4.5.

Mode's number m	1	2	3	4
Ω_c (rad/s)	290.2	504.3	995.9	1725.4

evolves is explored. Since the solution of the governing equation of the system has already been obtained, the plots are produced by varying the time.

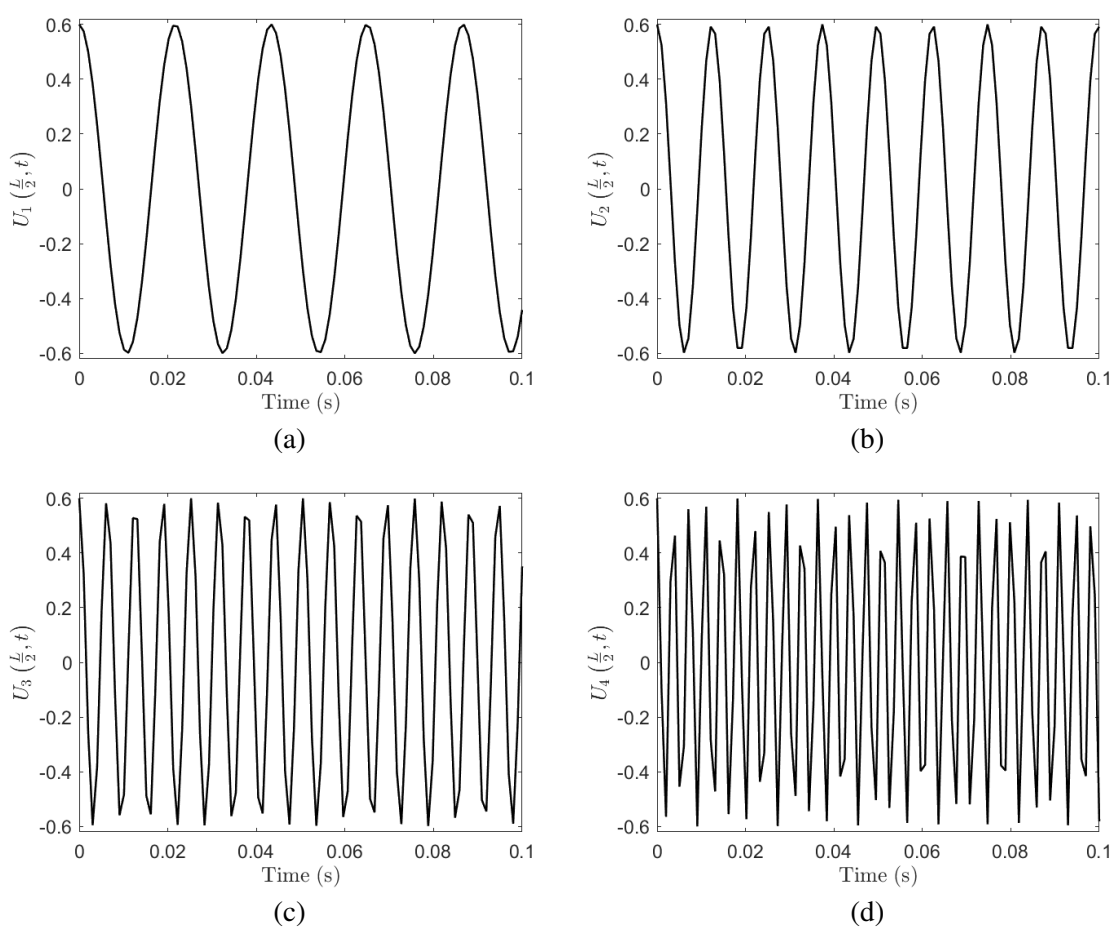


Figure 4.6: Temporal amplitude of vibration of the simply supported beam on a nonlinear cubic elastic foundation at midspan ($x = \frac{L}{2}$): (a) to (d) are modes 1 to 4.

Figure 4.6 shows that the plots have similar amplitudes. This observation is the same as that in Figure 4.4. Comparing Figure 4.6 to Figure 4.4, the paths followed by the beam for both cases look similar in each mode. Note that the amplitude of the restoring force produced by the beam having a nonlinear cubic elastic foundation is smaller than that of the beam with

quadratic nonlinearity when the spatial coordinate x varies within the interval $[0, 1]$. The following study case combines quadratic and cubic nonlinearities in the foundation.

4.3.3 Case of a BoEF having Linear and Nonlinear Quadratic and Cubic Stiffness Properties

This third case of the study considers the following expression of k_{nl}

$$k_{nl} = k_l + k_2 U(x, t) + k_3 U(x, t)^2, \quad (4.98)$$

where $U(x, t)$ represents the transversal displacement of the beam. We note that the two cases studied above are just particular cases of this third one. Then, taking the nonlinear stiffness relation given in Eq. (4.98), the equation of motion Eq. (4.5) is transformed into:

$$\rho S \frac{\partial^2 U(x, t)}{\partial t^2} + EI \frac{\partial^4 U(x, t)}{\partial x^4} + k_l U(x, t) + k_2 U(x, t)^2 + k_3 U(x, t)^3 = 0, \quad (4.99)$$

where the beam's transversal displacement $U(x, t)$ is assumed to have the following expression:

$$U(x, t) = P(x) \Theta(t) \quad (4.100)$$

In this subsection, the same boundary conditions to the beam are considered so that the solution to Eq. (4.99) is given by:

$$P_i(x) = C_i \sin \left(\frac{i\pi}{L} x \right), \quad (4.101)$$

where i is an integer representing the mode's number. Then, reference is made to the work by Rao (2007), Nayfeh and Mook (1995), and Nayfeh and Pai (2008), to find the solution of Eq. (4.99). First, Eqs. (4.100) and (4.101) are replaced into Eq. (4.99). Then, after applying orthogonality and simplifying the resulting equation, it holds:

$$\frac{d^2 \Theta_i(t)}{dt^2} + \left[\frac{EI}{\rho S} \left(\frac{i\pi}{L} \right)^4 + \frac{k_l}{\rho S} \right] \Theta_i(t) + \frac{1 - (-1)^j}{j\pi} \left[\frac{8i^2}{4i^2 - j^2} \right] \frac{k_2 C_i}{\rho S} \Theta_i(t)^2 + \frac{3}{4} \frac{k_3 C_i^2}{\rho S} \Theta_i(t)^3 = 0. \quad (4.102)$$

Eq. (4.102) is further simplified by setting:

$$\omega_l^2 = \frac{EI}{\rho S} \left(\frac{i\pi}{L} \right)^4 + \frac{k_l}{\rho S}; \quad \gamma_2 = \left[\frac{8(1 - (-1)^i)}{3i\pi} \right] \frac{k_2 C_i}{\rho S}; \quad \text{and} \quad \gamma_3 = \frac{3}{4} \frac{k_3 C_i^2}{\rho S}. \quad (4.103)$$

In Eq. (4.103), ω_l is the linear natural frequency, whereas γ_2 and γ_3 are the parameters carrying the nonlinear quadratic and nonlinear cubic parts of Eq. (4.98), respectively. So, Eq. (4.99) is transformed into:

$$\ddot{\Theta}_i(t) + \omega_l^2 \Theta_i(t) + \gamma_2 \Theta_i(t)^2 + \gamma_3 \Theta_i(t)^3 = 0. \quad (4.104)$$

Eq. (4.104) is a mixed parity equation and cannot be transformed into a linear equation. It is expected that the nonlinearity presented in this third case of study through the parameters γ_2 and γ_3 will considerably affect the nonlinear natural frequencies of the system and the beam's vibration amplitude.

In order to solve Eq. (4.104) using the SEM, the time τ is newly set as:

$$\tau = \Omega_{qc} t, \quad (4.105)$$

where Ω_{qc} is the nonlinear natural frequency of the system with the subscript qc standing for quadratic-cubic. Then, Eq. (4.104) is transformed into:

$$\Omega_{qc}^2 \ddot{\Theta}_i + \omega_l^2 \Theta_i + \gamma_2 \Theta_i^2 + \gamma_3 \Theta_i^3 = 0, \quad (4.106)$$

where $\ddot{\Theta}_i$ represents the second derivative of Θ_i with respect to the new time τ . As in the two cases presented above, Ω_{qc} is defined as follows:

$$\Omega_{qc} = \omega_l + \varepsilon \Omega_1 + \varepsilon^2 \Omega_2 + \varepsilon^3 \Omega_3 + \cdots + \varepsilon^i \Omega_i, \quad \varepsilon \ll 1, \quad (4.107)$$

where ε is a small parameter. Also, the solution of Eq. (4.106) is supposed to be:

$$\Theta_i(t) = \varepsilon \Theta_1 + \varepsilon^2 \Theta_2 + \varepsilon^3 \Theta_3 + \cdots + \varepsilon^i \Theta_i. \quad (4.108)$$

When Eqs. (4.107) and (4.108) are put into Eq. (4.99), it results in:

$$\begin{aligned} & \varepsilon \omega_l^2 (\ddot{\Theta}_1 + \Theta_1) + \varepsilon^2 (\omega_l^2 \ddot{\Theta}_2 + \omega_l^2 \Theta_2 + 2\omega_l \Omega_1 \ddot{\Theta}_1 + \gamma_2 \Theta_1^2) + \\ & \varepsilon^3 \left[\omega_l^2 (\ddot{\Theta}_3 + \Theta_3) + 2\omega_l \Omega_1 \ddot{\Theta}_2 + (\Omega_1^2 + 2\omega_l \Omega_1) \ddot{\Theta}_1 + 2\gamma_2 \Theta_1 \Theta_2 + \gamma_3 \tau_1^3 \right] + \cdots = 0 \end{aligned} \quad (4.109)$$

The solution of Eq. (4.109) implies solving the following equations:

$$\text{Order } \varepsilon^1 : \quad \omega_l^2 (\ddot{\Theta}_1 + \Theta_1) = 0 \quad (4.110a)$$

$$\text{Order } \varepsilon^2 : \quad \omega_l^2 (\ddot{\Theta}_2 + \Theta_2) = -2\omega_l \Omega_1 \ddot{\Theta}_1 - \gamma_2 \Theta_1^2 \quad (4.110b)$$

$$\text{Order } \varepsilon^3 : \quad \omega_l^2 (\ddot{\Theta}_3 + \Theta_3) = -2\omega_l \Omega_1 \ddot{\Theta}_2 - (\Omega_1^2 + 2\omega_l \Omega_1) \ddot{\Theta}_1 - 2\gamma_2 \Theta_1 \Theta_2 - \gamma_3 \Theta_1^3 \quad (4.110c)$$

Therefore, the solution of Eq. (4.110a) is given by:

$$\Theta_1(\tau) = D_1 \cos(\tau + \psi_1). \quad (4.111)$$

Replacing Eq. (4.111) into Eq. (4.110b), it results that:

$$\omega_l^2 (\ddot{\Theta}_2 + \Theta_2) = 2\omega_l \Omega_1 D_1 \cos(\tau + \psi_1) - \frac{\gamma_2 D_1^2}{2} [1 + \cos(2\tau + 2\psi_1)]. \quad (4.112)$$

The homogeneous equation of Eq. (4.112) has the following solution:

$$\Theta_{2H} = D_2 \cos(\tau + \psi_2). \quad (4.113)$$

Eliminating the secular term from Eq. (4.112) means solving:

$$2\omega_l \Omega_1 D_1 \cos(\tau + \psi_1) = 0. \quad (4.114)$$

The only reasonable solution for Eq. (4.114) to hold is for $\Omega_1 = 0$. Next, the solution of the particular equation satisfied by Eq. (4.112) is assumed, considering that the first term on the right-hand side is equal to zero to be formed by a constant plus a cosine function as:

$$\Theta_{2P}(\tau) = \mu + \eta \cos(2\tau + 2\psi_1) \quad (4.115)$$

By replacing Eq. (4.115) into Eq. (4.112), it happens that μ and η are given by:

$$\mu = -\frac{\gamma_2}{2\omega_l^2} D_1^2 \quad \text{and} \quad \eta = \frac{\gamma_2}{6\omega_l^2} D_1^2. \quad (4.116)$$

Taking Eq. (4.116) into account, the solution of Eq. (4.110b) is found to be:

$$\Theta_2(\tau) = D_2 \cos(\tau + \psi_2) - \frac{\gamma_2}{6\omega_l^2} D_1^2 [3 - \cos(2\tau + 2\psi_1)]. \quad (4.117)$$

Now, the solution for Eq. (4.110c) is determined by replacing Θ_1 and Θ_2 into the right-hand side of the equation. Knowing that $\Omega_1 = 0$, Eq. (4.110c) is modified as:

$$\begin{aligned} \omega_l^2(\ddot{\Theta}_3 + \Theta_3) = & \left[2\omega_l\Omega_2D_1 + \frac{5\gamma_2^2}{6\omega_l^2}D_1^3 - \frac{3\gamma_3}{4}D_1^3 \right] \cos(\tau + \psi_1) - \gamma_2D_1D_2 \cos(\psi_1 - \psi_2) - \\ & - \gamma_2D_1D_2 \cos(2\tau + \psi_1 + \psi_2) - \left(\frac{\gamma_2^2}{6\omega_l^2} + \frac{\gamma_3}{4} \right) D_1^3 \cos(3\tau + 3\psi_1). \end{aligned} \quad (4.118)$$

By proceeding the same way as to find the solution of Eq. (4.110b), the following is deduced:

$$\Theta_{3H}(\tau) = D_3 \cos(\tau + \psi_3). \quad (4.119)$$

As far as the particular equation is concerned, the secular terms are eliminated by solving the equation below:

$$\left[2\omega_l\Omega_2D_1 + \frac{5\gamma_2^2}{6\omega_l^2}D_1^3 - \frac{3\gamma_3}{4}D_1^3 \right] \cos(\tau + \psi_1) = 0. \quad (4.120)$$

Solving Eq. (4.120) with variable Ω_2 gives:

$$\Omega_2 = \frac{9\gamma_3\omega_l^2 - 10\gamma_2^2}{24\omega_l^3} D_1^2. \quad (4.121)$$

By repeating the procedure followed previously, the solution of the particular equation derived from Eq. (4.118) is found in an account that the term with $\cos(\tau + \psi_1)$ vanishes as follows:

$$\begin{aligned} \Theta_{3P}(\tau) = & \frac{\gamma_2}{3\omega_l^2} D_1D_2 \cos(2\tau + \psi_1 + \psi_2) - \frac{\gamma_2}{\omega_l^2} D_1D_2 \cos(\psi_1 - \psi_2) + \\ & \frac{2\gamma_2^2 + 3\gamma_3\omega_l^2}{96\omega_l^4} D_1^3 \cos(3\tau + 3\psi_1). \end{aligned} \quad (4.122)$$

As a result, the solution of Eq. (4.110c) is deduced to be:

$$\begin{aligned} \Theta_3(\tau) = & D_3 \cos(\tau + \psi_3) + \frac{\gamma_2}{3\omega_l^2} D_1D_2 \cos(2\tau + \psi_1 + \psi_2) + \\ & \frac{\gamma_2}{\omega_l^2} D_1D_2 \cos(\psi_1 - \psi_2) + \frac{2\gamma_2^2 + 3\gamma_3\omega_l^2}{96\omega_l^4} D_1^3 \cos(3\tau + 3\psi_1). \end{aligned} \quad (4.123)$$

Thus, the overall solution for Eq. (4.106) up to order 3 is found to be:

$$\begin{aligned} \Theta(t) = \varepsilon D_1 \cos(\Omega_{qc}t + \psi_1) + \varepsilon^2 \left\{ D_2 \cos(\Omega_{qc}t + \psi_2) - \frac{\gamma_2}{6\omega_l^2} D_1^2 \left[3 - \cos(2\tau + 2\psi_1) \right] \right\} + \\ \varepsilon^3 \left\{ D_3 \cos(\tau + \psi_3) + \frac{\gamma_2}{3\omega_l^2} D_1 D_2 \cos(2\tau + \psi_1 + \psi_2) - \frac{\gamma_2}{\omega_l^2} D_1 D_2 \cos(\psi_1 - \psi_2) + \right. \\ \left. \frac{2\gamma_2^2 + 3\gamma_3\omega_l^2}{96\omega_l^4} D_1^3 \cos(3\tau + 3\psi_1) \right\}. \quad (4.124) \end{aligned}$$

Knowing Ω_1 and Ω_2 , the nonlinear natural frequencies can be written to the second order combining both nonlinear quadratic and nonlinear cubic stiffness coefficients as:

$$\Omega_{qc} = \omega_l + \frac{9\gamma_3\omega_l^2 - 10\gamma_2^2}{24\omega_l^3} \varepsilon^2 D_1^2, \quad (4.125)$$

It is important to stress that Nayfeh et al. (1980) obtained this same expression of the nonlinear natural frequency using the MTSM. This, therefore, gives accountability to the present work.

Then, by replacing the expressions for $P_i(t)$ and $\Theta(t)$ into Eq. (4.100), the transversal displacement of the BoEF having nonlinear quadratic and nonlinear cubic stiffness properties is found as:

$$\begin{aligned} U_i(x, t) = C_i \sin\left(\frac{i\pi}{L}x\right) \left[\varepsilon D_1 \cos(\Omega_{qc}t + \psi_1) + \varepsilon^2 \left\{ D_2 \cos(\Omega_{qc}t + \psi_2) - \right. \right. \\ \left. \frac{\gamma_2}{6\omega_l^2} D_1^2 \left[3 - \cos(2\tau + 2\psi_1) \right] \right\} + \varepsilon^3 \left\{ \frac{\gamma_2}{3\omega_l^2} D_1 D_2 \cos(2\tau + \psi_1 + \psi_2) + \right. \\ \left. D_3 \cos(\tau + \psi_3) - \frac{\gamma_2}{\omega_l^2} D_1 D_2 \cos(\psi_1 - \psi_2) + \frac{2\gamma_2^2 + 3\gamma_3\omega_l^2}{96\omega_l^4} D_1^3 \cos(3\tau + 3\psi_1) \right\} \right]. \quad (4.126) \end{aligned}$$

Finally, the solution is considered up to the third order of ε , and the following initial conditions are assumed to simplify Eq. (4.126):

$$\Theta_i(0) = 0.3 \quad \text{and} \quad \dot{\Theta}_i(0) = 0. \quad (4.127)$$

The above initial conditions enable finding

$$\psi_1 = \psi_2 = \psi_3 = 0, \quad (4.128)$$

and writing

$$\varepsilon D_1 + \varepsilon^2 \left\{ D_2 - \frac{\gamma_2}{3\omega_l^2} D_1^2 \right\} + \varepsilon^3 \left\{ \frac{\gamma_2}{3\omega_l^2} D_1 D_2 + D_3 - \frac{\gamma_2}{\omega_l^2} D_1 D_2 + \frac{2\gamma_2^2 + 3\gamma_3 \omega_l^2}{96\omega_l^4} D_1^3 \right\} = 0.3. \quad (4.129)$$

For Eq. (4.129) to hold, the following expressions for D_1 , D_2 , and D_3 will be considered:

$$D_1 = 30 \text{ since } \varepsilon = 0.01, \quad (4.130a)$$

$$D_2 = \frac{\gamma_2}{3\omega_l^2} D_1^2, \quad (4.130b)$$

$$D_3 = \left(\frac{-29\gamma_2^2}{144\omega_l^4} + \frac{\gamma_3}{32\omega_l^2} \right) D_1^3. \quad (4.130c)$$

Thus, the expression of $\Theta_i(t)$ is found in terms of D_1 , ω_l , γ_2 , and γ_3 as:

$$\begin{aligned} \Theta_i(t) = & \varepsilon D_1 \cos(\Omega_{qc}t) + \frac{\gamma_2}{6\omega_l^2} \varepsilon^2 D_1^2 \left[2\cos(\Omega_{qc}t) - 3 + \cos(2\Omega_{qc}t) \right] + \\ & \varepsilon^3 D_1^3 \left[\left(\frac{-29\gamma_2^2}{144\omega_l^4} + \frac{\gamma_3}{32\omega_l^2} \right) \cos(\Omega_{qc}t) + \frac{\gamma_2^2}{9\omega_l^2} \cos(2\Omega_{qc}t) - \frac{\gamma_2^2}{3\omega_l^4} + \frac{2\gamma_2^2 + 3\gamma_3 \omega_l^2}{96\omega_l^4} \cos(3\Omega_{qc}t) \right]. \end{aligned} \quad (4.131)$$

This allows for deducing the overall solution of Eq.(4.99) to be:

$$\begin{aligned} U_i(x,t) = & C_i \sin\left(\frac{i\pi}{L}x\right) \left\{ \varepsilon D_1 \cos(\Omega_{qc}t) + \frac{\gamma_2}{6\omega_l^2} \varepsilon^2 D_1^2 \left[2\cos(\Omega_{qc}t) - 3 + \cos(2\Omega_{qc}t) \right] + \right. \\ & \left. \varepsilon^3 D_1^3 \left[\left(\frac{-29\gamma_2^2}{144\omega_l^4} + \frac{\gamma_3}{32\omega_l^2} \right) \cos(\Omega_{qc}t) + \frac{\gamma_2^2}{9\omega_l^2} \cos(2\Omega_{qc}t) - \frac{\gamma_2^2}{3\omega_l^4} + \frac{2\gamma_2^2 + 3\gamma_3 \omega_l^2}{96\omega_l^4} \cos(3\Omega_{qc}t) \right] \right\}. \end{aligned} \quad (4.132)$$

For the BoEF having both quadratic and cubic nonlinearities in the foundation, the displacement $W_i(x,t)$ is plotted versus the beam's length for different time values as in the previous two cases. Figure 4.7 displays the curves obtained from MATLAB.

Figure 4.7 also shows the plots of the beam's spatial displacement for the first four vibration modes. The shape of these curves has not changed as compared to Figures 4.3 and 4.5. However, looking at the maximum amplitude, it can be observed that it has considerably reduced for the current case compared to those in Figures 4.3 and 4.5. That means combining

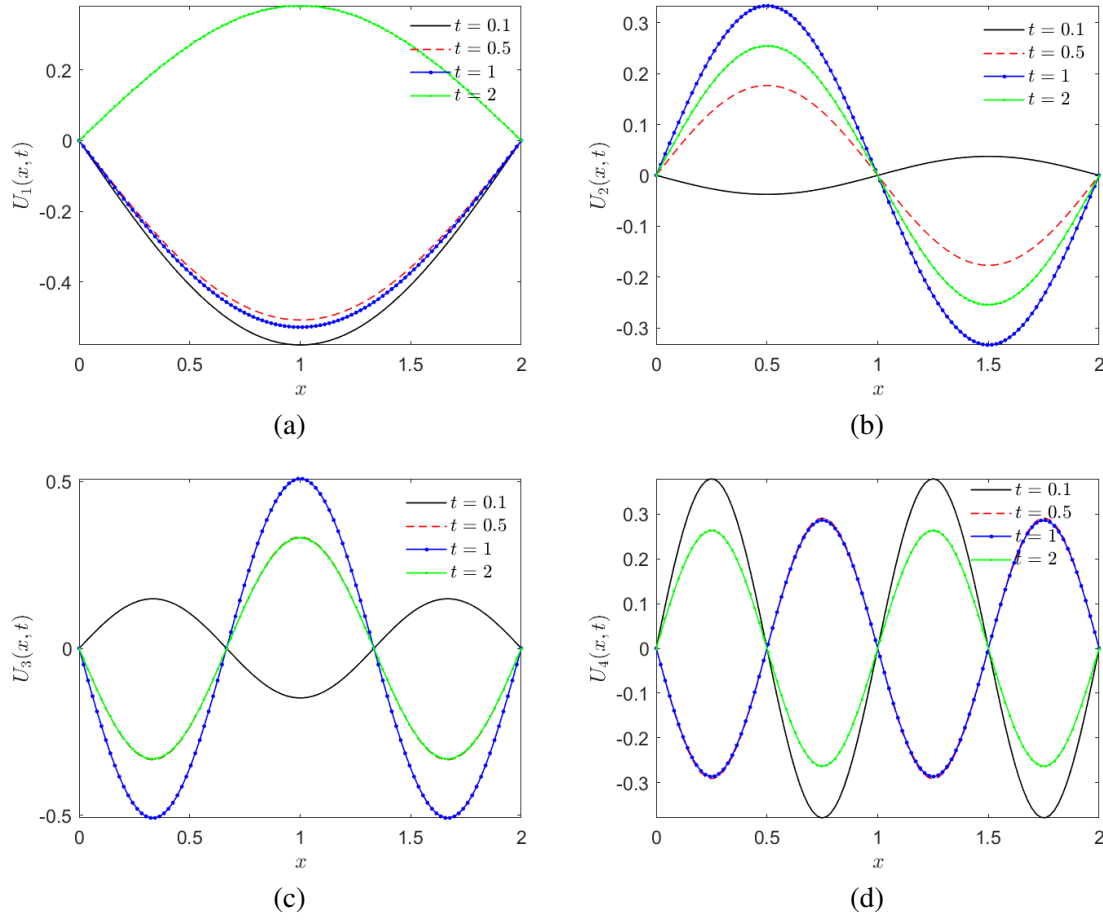


Figure 4.7: Vibration modes of the spatial displacement of the simply supported BoEF having nonlinear quadratic and cubic stiffness properties: (a) to (d) are modes 1 to 4.

both nonlinear quadratic and nonlinear cubic effects in the expression of the restoring force is a good advantage. Table 4.4 gives the natural frequencies corresponding to Fig. 4.7.

Table 4.4: Natural frequencies of the beam on a nonlinear quadratic-cubic elastic foundation corresponding to the plots presented in Figure 4.7.

Mode's number i	1	2	3	4
Ω_{qc} (rad/s)	301.1	511.01	999.3	1727.4

Table 4.4 shows the natural frequencies of the BoEF with mixed nonlinearity in the foundation. It is seen that these frequencies are greater than those in Tables 4.2 and 4.3. As a result, the conclusion is that when both nonlinearities are combined in one system, each contributes to the natural frequencies. It is important to clarify that the increase of the natural

frequencies is dramatic due to the specific value of the nonlinear stiffness parameters chosen for each case. Therefore, for a small value of the nonlinear stiffness parameter, the increase will be minimal. Figure 4.8 shows the temporal plots.

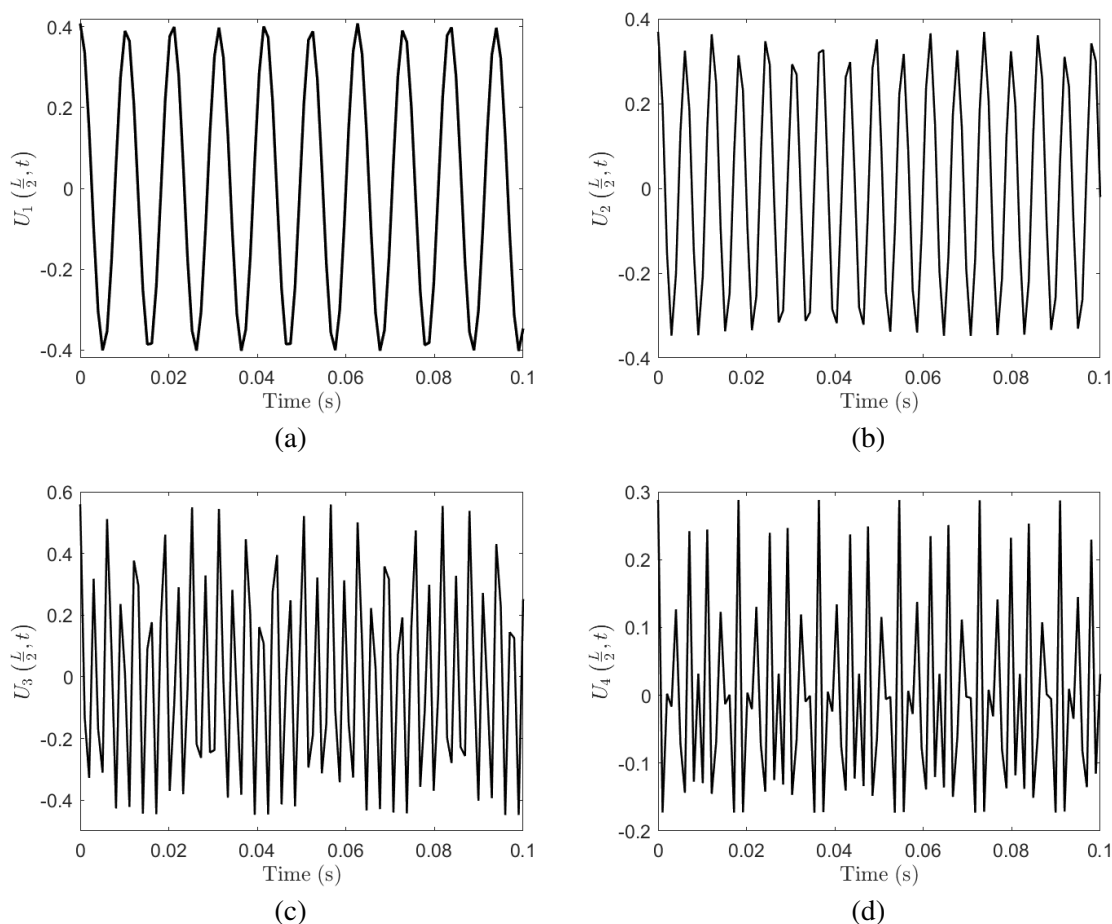


Figure 4.8: Temporal amplitude of the simply supported BoEF having nonlinear quadratic and nonlinear cubic stiffness properties: (a) to (d) are modes 1 to 4.

Figure 4.8 exhibits the temporal plots of the beam resting on a nonlinear quadratic-cubic elastic foundation. A prediction was made regarding the behaviour of the beam which may decrease when both nonlinearities are combined, and with the actual results, that prediction is validated. In that regard, the temporal amplitude has reduced as compared to those in Figures 4.4 and 4.6. In addition, the comment is that, although the quadratic nonlinearity added to the foundation has almost similar effects to those of the cubic nonlinearity, their combination is more beneficial. As far as the three cases studied are concerned, the nonlinearity of the foundation has a huge impact on the natural frequencies of the entire system. That impact is also acknowledged in the beam's vibration amplitude to some extent.

By studying BoEFs having the three nonlinearities of the foundation above presented, it was realised that the expressions of the natural frequencies for each case were not the same but increased. Moreover, from the plots obtained, it was proved that nonlinearity due to the stiffness in the foundation lowers the beam's spatial amplitude of vibration for a specific time value and decreases the beam's temporal amplitude at midspan. These findings accord with the work by Jorge et al. (2015). In order to confirm the analytical results produced in the previous sections, the next section will present results obtained using FEA in Abaqus/Standard.

4.4 Numerical Results

4.4.1 Description of the Procedure Followed in Abaqus

Determining the parameters of the foundation in engineering practice is vital for accurate foundation modelling. For example, suppose the foundation is considered as soil. In that case, one may pay attention that due to the mechanical properties of soils, assuming a linear behaviour may fail in representing the foundation model accurately. In this thesis, only the response of the beam is of concern, not the foundation. Therefore, interest is not in modelling the foundation itself, but in highlighting its nonlinear effects on the beam's vibration amplitude and comparing them to those of a beam with linear foundation.

There exist many options to implement a foundation in Abaqus. The choice should be made to adapt the correct configuration for the foundation to behave correctly. The procedure is complex to realize nonlinear foundations in Abaqus and can only be done using connectors because they can be linear or nonlinear. In addition, a force-displacement characteristic of the spring is needed as input (Reinås et al., 2011). The analysis starts with creating a wire feature for the nodes selected. For each wire, one end is connected to the beam and the other to the ground (Documentation Abaqus, 2019; SIMULIA, 2017a,b). After this, the connectors' behaviours are defined by selecting Nonlinear and specifying the data collected by plotting a force-displacement or moment-rotation characteristic. The data can be obtained from a plot using other software. In the present thesis, MATLAB is used. The analysis is performed in Abaqus by moving from one module to another until the analysis is completed. The schemes presented in Chapter 3 are the same to be adopted here. However, for the *Interaction module*, only the nonlinear part was followed. The flowchart in Figure 4.9 shows what needs to be done in the *interaction module*.

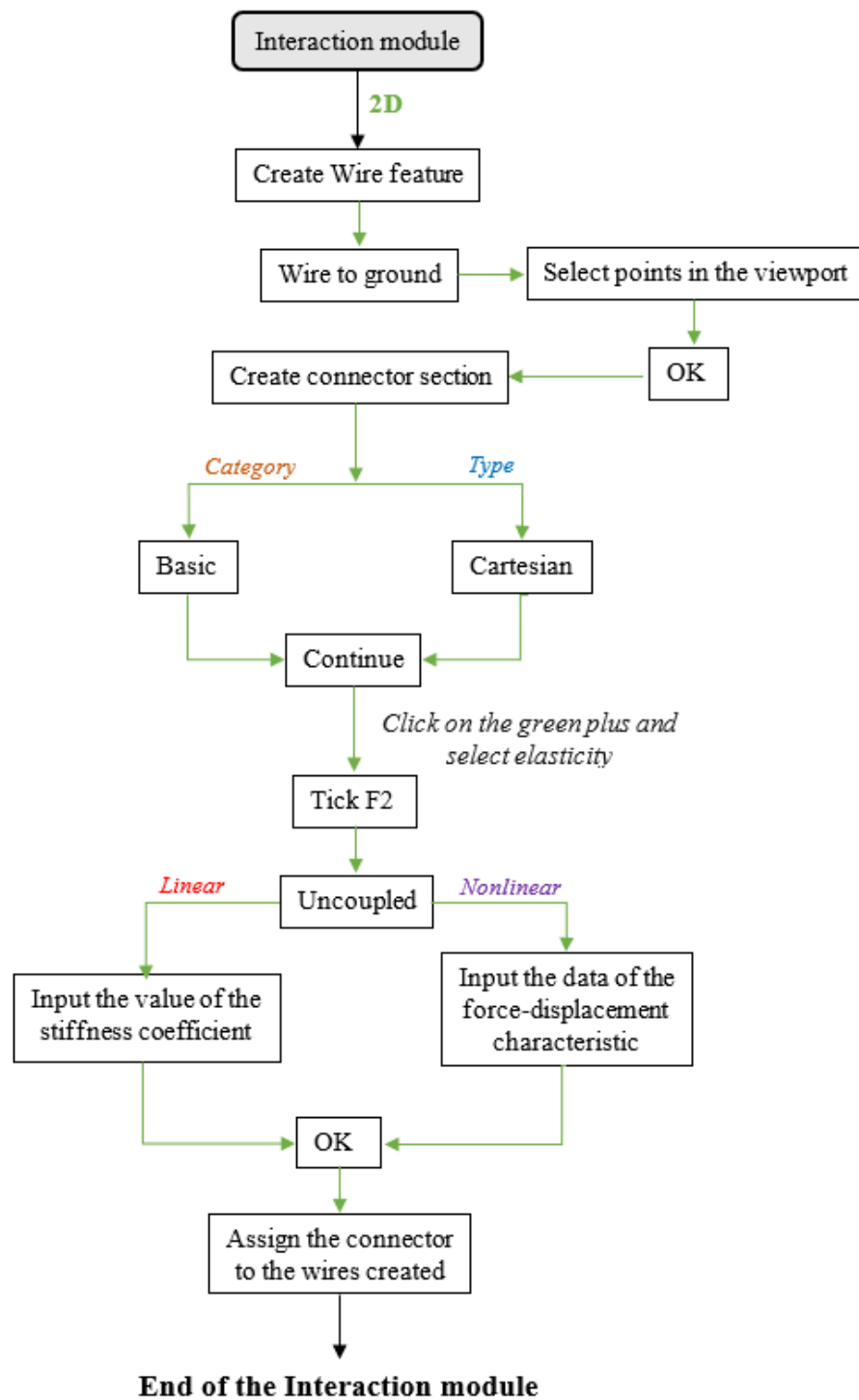


Figure 4.9: Flawchart for the Interaction module.

The current chapter uses the *Interaction module* to define the nonlinear elastic foundation and assign it to the beam. Figure 4.9 shows the scheme to be followed for all the cases

presented in this chapter dealing with nonlinear analysis in Abaqus. There will only be differences regarding the data of the restoring forces, which will change according to the type of nonlinearity. It must also be highlighted that Cartesian was chosen as a connector type because this work uses Cartesian coordinates. Moreover, $F2$ was ticked for the direction of the spring because the springs should be vertical and along the y -axis. That is justified because the interest lies in studying the transversal vibration of the beam, which is along the y -direction. The figure below displays the outcome of the tasks performed in the Abaqus modules.

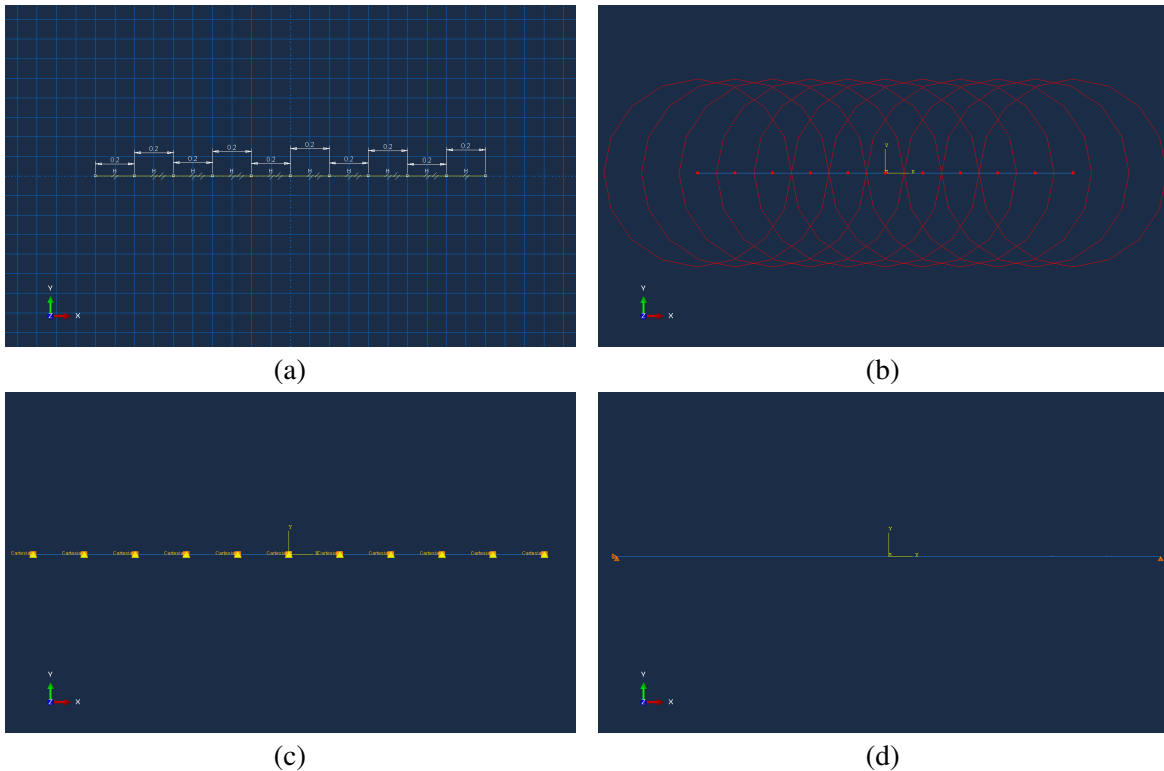


Figure 4.10: Tasks performed in the modules in Abaqus: (a) creation of the part, (b) wire created, (c) connectors assigned to the wires, and (d) boundary conditions.

Figure 4.10 shows the images representing the outcome of the tasks performed in the modules in Abaqus. The part was created by joining different lines of length 0.2 m each (see Figure 4.10a), the wires created at each of the joints at which to insert the nonlinear connectors (see Figure 4.10b), the connectors assigned to the wires at each joint (see Figure 4.10c), and the boundary conditions assigned to the beam (see Figure 4.10d). After applying the boundary conditions, we moved to the *Mesh module*.

After seeding the part with a value of 0.003, we assigned the element type *B23*, a 2-node beam in plane with cubic interpolation, which is equivalent to the Euler-Bernoulli beam. We then proceeded and meshed the part; the mesh generated 670 elements on the part created. Finally, in the *Job module* we created a job and submitted it for the beam analysis on the nonlinear elastic foundation. After that, we present the results obtained from the analysis in Abaqus for each of the three nonlinear cases presented in Subsections 4.3.1 to 4.3.3.

4.4.2 Numerical Results of a Beam on a Nonlinear Quadratic Elastic Foundation

To solve the equation of motion of a BoEF having nonlinear quadratic stiffness properties using FEA in Abaqus, the theoretical formulation of the equation is first written. The equation satisfied by the beam and the foundation was given in Subsection 4.3.1 by Eq. (4.9). In Chapter 3, the displacement as a function of the shape functions and nodal DOF was expressed as:

$$U(x, t) = N_1(x)w_1(t) + N_2(x)\theta_1(t) + N_3(x)w_2(t) + N_4(x)\theta_2(t). \quad (4.133)$$

To proceed, the Galerkin method (Fletcher, 1984) is applied to Eq. (4.9) and it gives:

$$\int_0^L N_i(x) \left(\rho S \frac{\partial^2 U(x, t)}{\partial t^2} + EI \frac{\partial^4 U(x, t)}{\partial x^4} + k_l U(x, t) + k_2 U(x, t)^2 \right) dx = \int_0^L 0 dx, \quad (4.134)$$

where $i = 1, \dots, 4$. In Eq. (4.134), the first, second, and third terms of the left-hand side allow for finding the mass, beam stiffness, and linear stiffness foundation matrices, respectively. Then, to find the nonlinear stiffness matrix of the foundation, the following is written (Fletcher, 1984):

$$\begin{aligned} \int_0^L N_i(x) k_2 U(x, t)^2 dx &= \int_0^L N_i k_2 \left\{ N_j^2 d_j^2 + 2N_1 N_2 w_1 \theta_1 + 2N_1 N_3 w_1 w_2 + \right. \\ &\quad \left. 2N_1 N_4 w_1 \theta_2 + 2N_2 N_3 \theta_1 w_2 + 2N_2 N_4 \theta_1 \theta_2 + 2N_3 N_4 w_2 \theta_2 \right\} dx \\ &= \int_0^L N_i k_2 \left\{ N_j^2 d_j^2 + 2N_i N_j d_i d_j \right\} dx, \end{aligned} \quad (4.135)$$

where $i, j = 1, \dots, 4$, N_j^2 and d_j^2 are the square of the shape function matrix elements $[\mathbf{N}]$ and the nodal degree of freedom vector elements $\{\mathbf{d}\}$, respectively. It should be highlighted

that the term $2N_iN_jd_id_j$ is valid only for $i \neq j$. Therefore:

$$\begin{aligned} \int_0^L N_i(x) \left(EI \frac{\partial^4 U(x,t)}{\partial x^4} + k_l U(x,t) + k_2 U(x,t)^2 \right) dx = EI \int_0^L \mathbf{B} \mathbf{B}^T \mathbf{B} dx + \\ k_l \int_0^L \mathbf{N}^T \mathbf{N} dx + k_2 \int_0^L \mathbf{N}^T N_i^2 d_i^2 dx + 2k_2 \int_0^L \underbrace{N_i N_j d_i d_j}_{i \neq j} dx = \mathbf{K}_b + \mathbf{K}_l + \mathbf{K}_q, \end{aligned} \quad (4.136)$$

where $\mathbf{B}$ is the matrix formed by the second derivative of the shape function matrix, and $\mathbf{N}^T$ is the transpose of the shape function matrix $\mathbf{N}$. From the right-hand side of Eq. (4.136), $\mathbf{K}_b$ represents the beam element stiffness matrix; $\mathbf{K}_l$ is the linear element stiffness matrix; and $\mathbf{K}_q$ is the nonlinear quadratic element stiffness matrix respectively defined as:

$$\mathbf{K}_b = EI \int_0^L \mathbf{B}^T \mathbf{B} dx \quad (4.137a)$$

$$\mathbf{K}_l = k_l \int_0^L \mathbf{N}^T \mathbf{N} dx \quad (4.137b)$$

$$\mathbf{K}_q = k_2 \int_0^L \mathbf{N}^T \mathbf{N}^2 d^2 dx + 2k_2 \int_0^L \underbrace{N_i N_j d_i d_j}_{i \neq j} dx. \quad (4.137c)$$

Now the stiffness of the entire system beam-foundation is defined $[\mathbf{K}_{\text{BNQEF}}]$ as follows:

$$[\mathbf{K}_{\text{BNQEF}}] = [\mathbf{K}_b] + [\mathbf{K}_l] + [\mathbf{K}_q], \quad (4.138)$$

with the subscript of $[\mathbf{K}_{\text{BNQEF}}]$ standing for the beam on the nonlinear quadratic elastic foundation. One can notice the presence of the new matrix $[\mathbf{K}_q]$ due to the quadratic nonlinearity. Having the detailed expression of the matrix of the entire nonlinear system $\mathbf{K}_b$, $\mathbf{K}_l$, $\mathbf{K}_q$, and hence $[\mathbf{K}_{\text{BNQEF}}]$ can be calculated. This is not of interest here. Instead, the focus is on the numerical analysis in Abaqus.

Figure 4.11 exhibits a plot of the restoring force with quadratic nonlinearity whose data were used during the analysis in Abaqus. The figure was obtained by plotting the function $k_l x + k_2 x^2$ with the stiffness coefficients $k_l = 1.69 \times 10^5 \text{ N/m}^2$ and $k_2 = 8.44 \times 10^5 \text{ N/m}^3$. A rapidly increasing curve is observed because the nonlinear quadratic term is present in the expression of the restoring force. Table 4.5 records the natural frequencies obtained from analytical and numerical simulations and the differences between the two.

In Table 4.5 the natural frequencies are recorded for the first ten vibration modes from analytical calculations and numerical analysis in Abaqus of a BoEF having linear

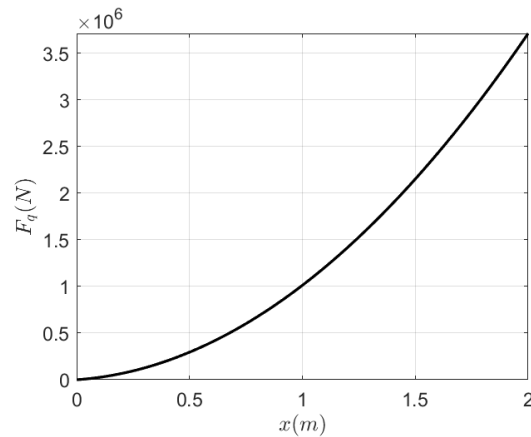


Figure 4.11: Behaviour of the nonlinear quadratic spring.

Table 4.5: Natural frequencies for the first ten modes of vibration of a simply supported beam resting on a linear and nonlinear quadratic elastic foundation.

Mode's number n	Ω_q (rad/s)		Differences (%)
	Analytical	Abaqus	
1	299.2	297.3	0.64
2	508.6	508.5	0.02
3	998.1	997.96	0.014
4	1726.8	1726.6	0.012
5	2675.9	2677.2	0.011
6	3843.5	3744.4	0.003
7	5226.3	5226.3	0.00
8	6821.7	6822.2	0.01
9	8631.5	8631.7	0.002
10	10654	10651	0.028

and nonlinear quadratic stiffness properties. The results from Abaqus serve as a reference. Although the perturbation analysis carried out in Abaqus is linear, we note that the nonlinear part of the modelling intervenes when defining the nonlinear properties of the foundation while using the *Interaction module*. The differences shown in Table 4.5 are very small and confirm the agreement between the two methods. The mode shapes can now be visualised using the *Visualization module* in Abaqus as shown in Figures 4.12 and 4.13.

Figure 4.12 shows the beam response along the y -direction for modes 1 to 4. The maximum amplitude of the mode shapes remains equal to 1 for all four mode shapes (see legend). That would have been the same for any other modes because, in Abaqus, the eigenvectors are either mass normalised or displacement normalised. More importantly, the

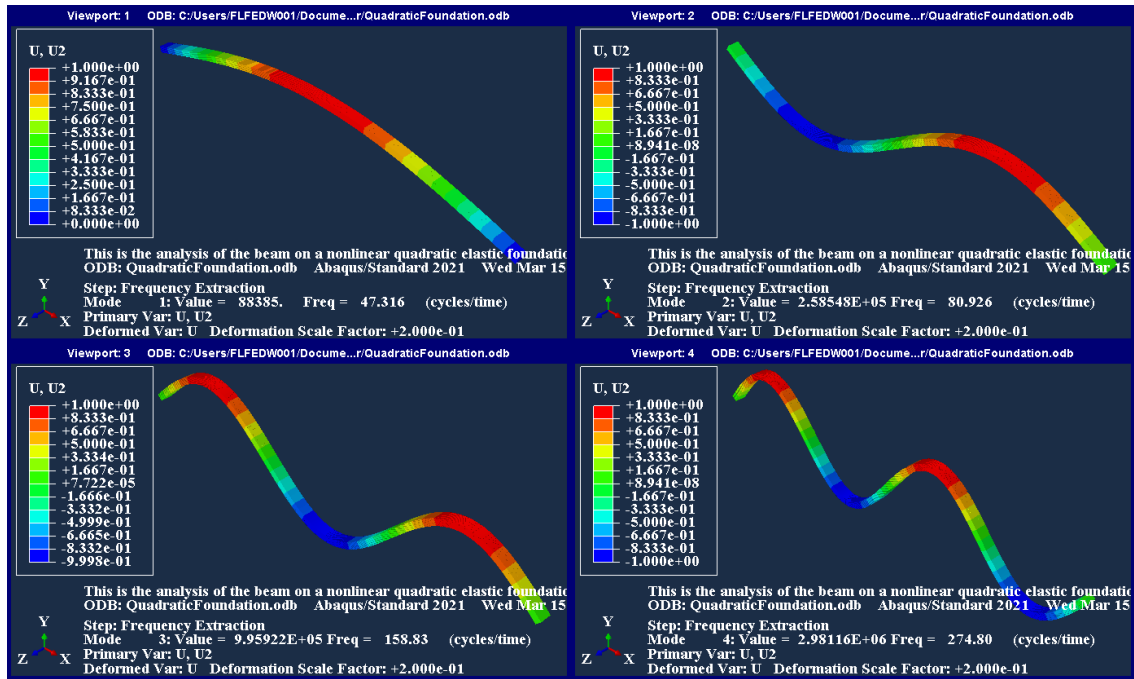


Figure 4.12: Mode shapes of the beam with a nonlinear quadratic stiffness foundation.

normalisation choice does not affect the values of the natural frequencies, but it only scales the amplitudes of mode shapes up or down. So, in the context of this thesis, displacement normalised is chosen because the beam transversal displacement is the focus, the reason why the maximum displacement amplitude is equal to 1.

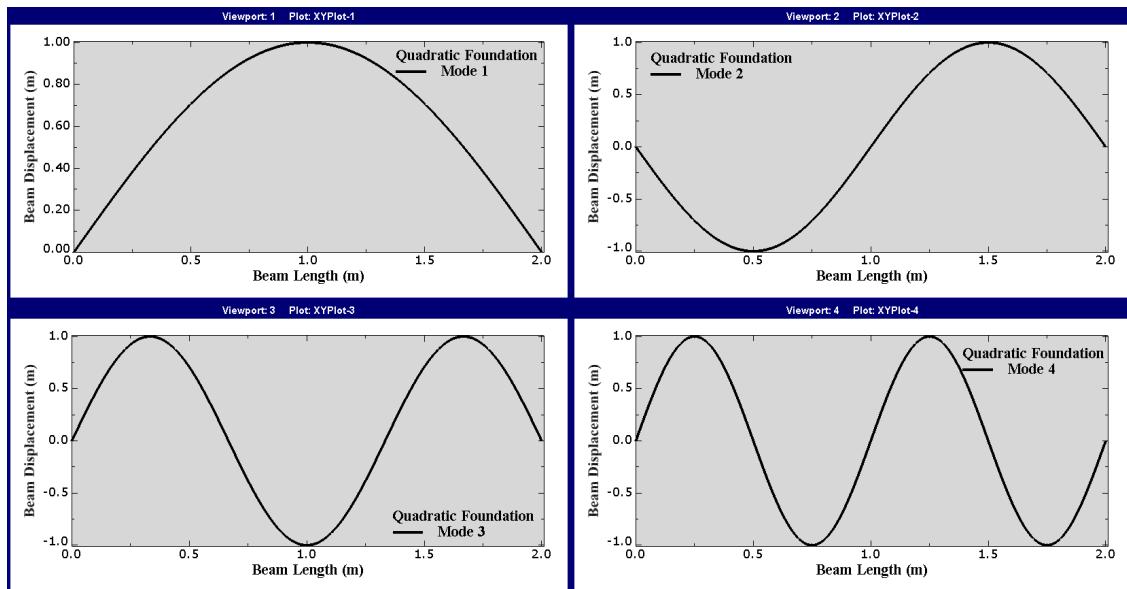


Figure 4.13: Displacement of the beam with a nonlinear quadratic stiffness foundation.

On the other hand, Figure 4.13 displays the beam's displacement corresponding to the above mode shapes. The curves were obtained by creating a path in the *Visualization module* in Abaqus. By referring to the plots obtained in Figure 4.3, the normalisation of the displacements obtained from analytical calculations could produce the same amplitudes as those from numerical analyses by choosing a proper time. Of course, that is not the interest here, but only to highlight that the temporal part of the solution has a significant contribution.

In order to address the beam behaviour as time goes on, the beam vibration is studied at the beam's centre. The parameters involved in this case remain the same as those given in Subsection 4.3.1.

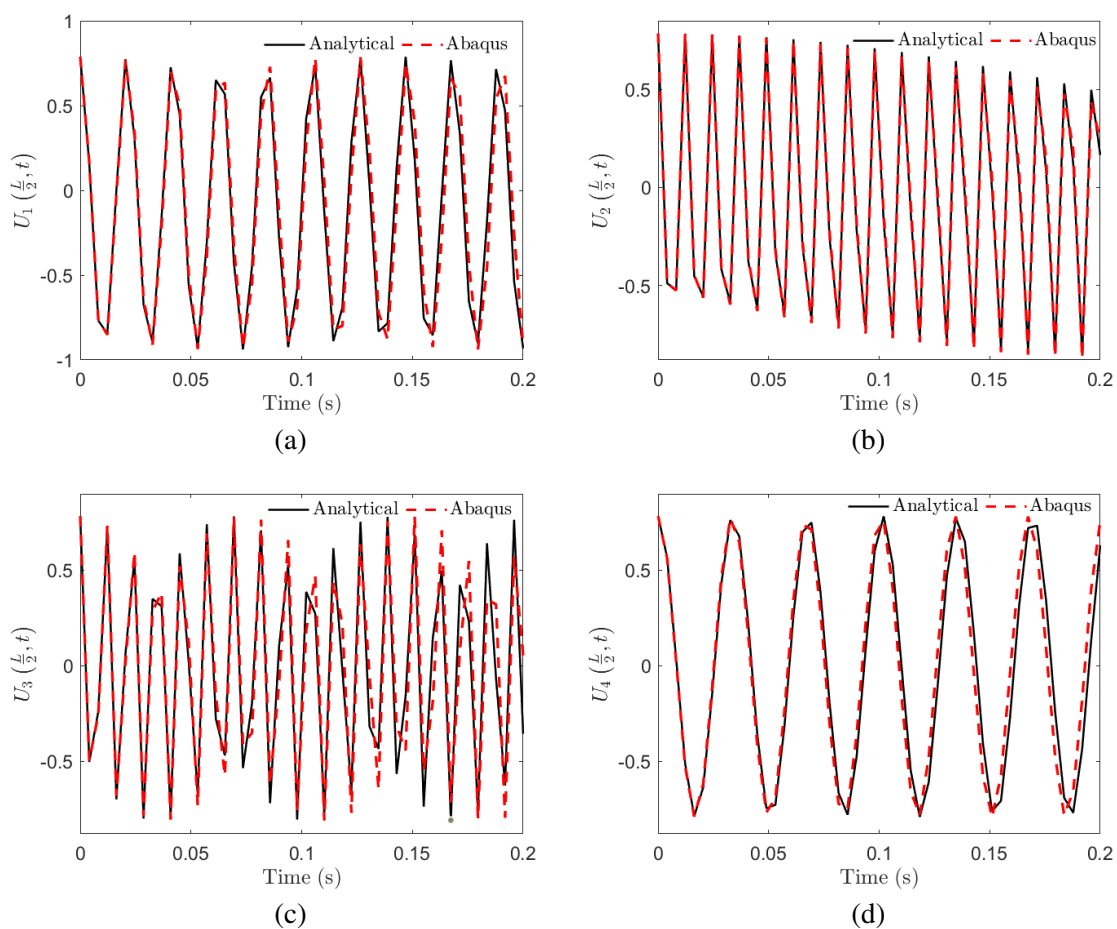


Figure 4.14 compares analytical and numerical temporal amplitudes of the beam on a nonlinear quadratic elastic foundation. It can be seen that the response at the middle of the beam is nonzero in modes 2, 4, and 6. This is because we only plotted the temporal part of the displacement solution in order to have the same interpretation for all the modes. However, it is clear that by considering the spatial part of the solution, the response at even modes

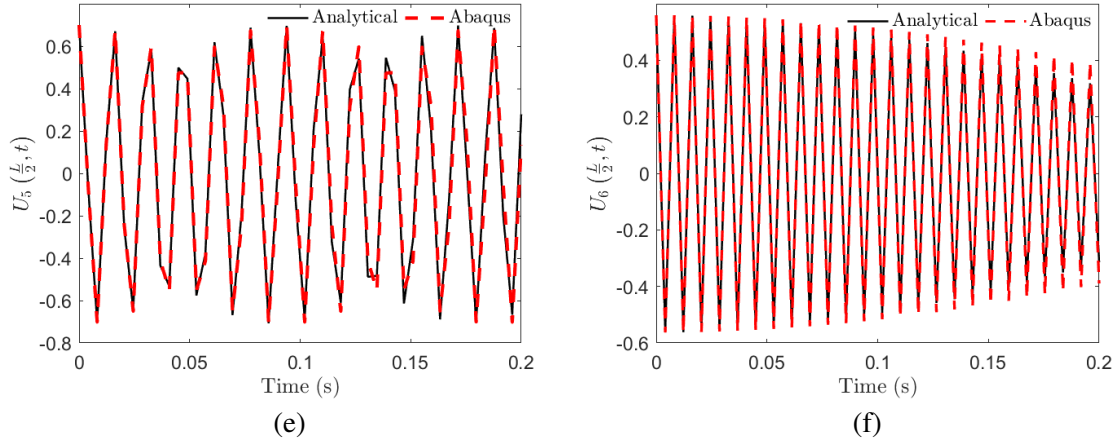


Figure 4.14: Temporal displacement at the centre of the beam on a nonlinear quadratic elastic foundation: (a) to (f) are modes 1 to 6.

will tend to be zero. The same comments apply to the temporal plots corresponding to the cases of the beam on a nonlinear cubic and nonlinear quadratic-cubic elastic foundation to be presented in the upcoming sections. It can be attested from Figure 4.14 that the paths followed by the curves at each mode are the same with the same maximum amplitudes. That confirms the agreement between the analytical and the numerical methods. Therefore, the numerical results are a benchmark for the analytical results.

4.4.3 Results of a BoEF having a Nonlinear Cubic Stiffness Foundation

Finding the solution of the equation of motion of the beam on the nonlinear cubic elastic foundation using FEA is similar to that of the beam on a nonlinear quadratic elastic foundation. By applying the Galerkin method to Eq. (4.64), the term with cubic nonlinearity becomes:

$$\int_0^L N_i(x) k_3 U(x, t)^3 dx = \int_0^L N_i(x) k_3 \left[N_1(x) w_1(t) + N_2(x) \theta_1(t) + N_3(x) w_2(t) + N_4(x) \theta_2(t) \right]^3 dx = \int_0^L N_i k_3 \left\{ N_j^3 d_j^3 + 3 \underbrace{N_j^2 N_k d_j^2 d_k}_{j \neq k} + 6 \underbrace{N_i N_j N_k d_i d_j d_k}_{i \neq j \neq k} \right\} dx, \quad (4.139)$$

where $i, j, k = 1, \dots, 4$; N_j^2 and N_j^3 , d_j^2 and d_j^3 are the square and cubic of the elements of the shape function matrix $[\mathbf{N}]$ and those of the nodal degree of freedom vector $\{\mathbf{d}\}$, respectively.

Therefore:

$$\int_0^L N_i(x) \left(EI \frac{\partial^4 U(x,t)}{\partial x^4} + k_l U(x,t) + k_2 U(x,t)^2 + k_3 U(x,t)^3 \right) dx = EI \int_0^L \mathbf{B}^T \mathbf{B} dx + k_l \int_0^L \mathbf{N}^T \mathbf{N} dx + k_3 \int_0^L \mathbf{N}^T \left\{ N_j^3 d_j^3 + 3 \underbrace{N_i^2 N_j d_i^2 d_j}_{i \neq j} + 6 \underbrace{N_i N_j N_k d_i d_j d_k}_{i \neq j \neq k} \right\} dx = \mathbf{K}_b + \mathbf{K}_l + \mathbf{K}_c. \quad (4.140)$$

From the right-hand side of Eq. (4.140), $\mathbf{K}_b$ represents the beam element stiffness matrix, $\mathbf{K}_l$ is the linear element stiffness matrix, and $\mathbf{K}_c$ is the nonlinear cubic element stiffness matrix, respectively defined as

$$\mathbf{K}_b = EI \int_0^L \mathbf{B}^T \mathbf{B} dx, \quad \mathbf{K}_l = k_l \int_0^L \mathbf{N}^T \mathbf{N} dx, \quad (4.141a)$$

$$\mathbf{K}_c = k_3 \int_0^L \mathbf{N}^T N_j^3 d_j^3 dx + 3k_3 \int_0^L \mathbf{N}^T \underbrace{N_i^2 N_j d_i^2 d_j}_{i \neq j} dx + 6k_3 \int_0^L \mathbf{N}^T \underbrace{N_i N_j N_k d_i d_j d_k}_{i \neq j \neq k} dx. \quad (4.141b)$$

For this second case, the stiffness matrix carrying the nonlinearity is more complicated to find than in the previous case. Once more, determining all the matrix elements is not of interest here. The restoring force to be considered here has the plot in Figure 4.15. The curve was obtained by plotting $k_l x + k_3 x^3$ versus x with $k_l = 1.69 \times 10^5 \text{ N/m}^2$ and $k_3 = 1.3 \times 10^7 \text{ N/m}^4$.

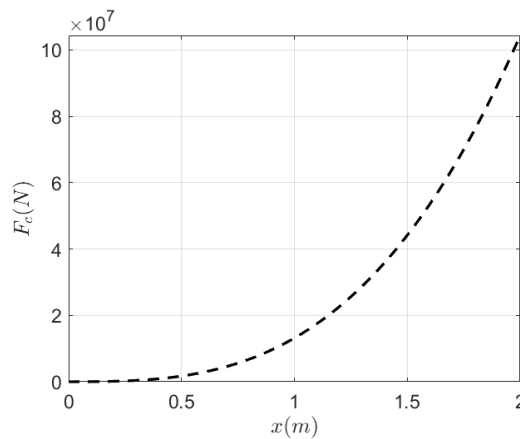


Figure 4.15: Behaviour of the nonlinear cubic spring.

Figure 4.15 shows a plot of a cubic function. It is a curve that increases as the coordinate x increases. However, for x lower than 1, precisely in the interval $[0, 0.5]$, the curve seems not to increase. On the contrary, after $x = 0.5$, the increase is much more significant. That is because, for those values of x smaller than 1, raising them to the power 3 gives smaller values than the initial value itself. The data recorded were used in Abaqus to define the nonlinear foundation having cubic stiffness properties. The table below contains the natural frequencies recorded.

Table 4.6: Natural frequencies for the first ten modes of vibration of the simply supported beam resting on a linear and nonlinear cubic elastic foundation.

Mode's number m	Ω_c (rad/s)		
	Analytical	Abaqus	Differences (%)
1	290.2	288.8	0.45
2	504.3	503.6	0.14
3	995.9	995.5	0.04
4	1725.4	1725.2	0.012
5	2676.4	2676.2	0.007
6	3843.8	3843.7	0.003
7	5225.9	5225.9	0.00
8	6821.9	6821.9	0.00
9	8631.5	8631.4	0.001
10	10654	10651	0.028

Table 4.6 shows the frequencies recorded for the first ten vibration modes from analytical calculations and numerical analysis in Abaqus of the BoEF having linear and nonlinear cubic stiffness properties. Comparing this table to Table 4.5, there is no significant gap regarding the percentage of differences produced, but the frequencies recorded for the present case have slightly decreased.

Figure 4.16 exhibits the beam's first four mode shapes of vibration. The legend unveils that the response's maximum amplitude is still one for the same reason highlighted in Subsection 4.4.2. Therefore, the same comments apply here.

On the other hand, Figure 4.17 shows the plots of the spatial displacement of the beam obtained in Abaqus. The shape of each plot remains unchanged, and the amplitude of vibration of the beam is also the same. As these results in Figure 4.17 do not give more insight into the comparison between analytical and numerical results, the temporal plots are produced in that regard.

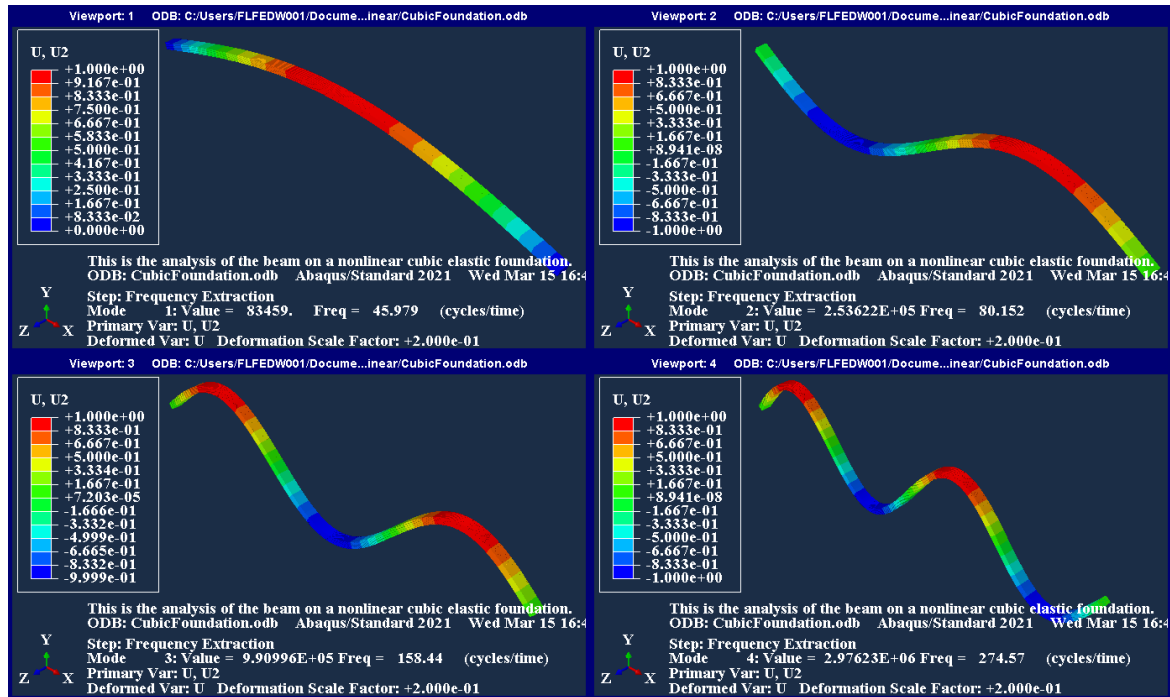


Figure 4.16: Mode shapes of the beam with a nonlinear cubic stiffness foundation.

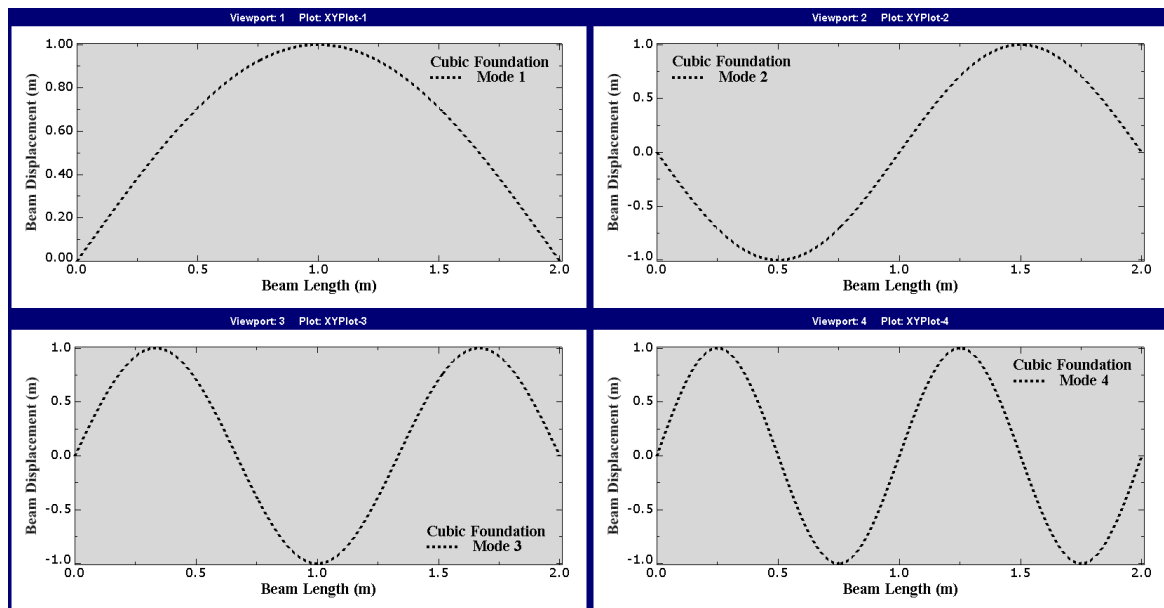


Figure 4.17: Displacement of the beam with a nonlinear cubic stiffness foundation.

In Figure 4.18, are shown the temporal plots corresponding to the displacement of the beam resting on a nonlinear cubic foundation. Again, the analytical plots mingle with

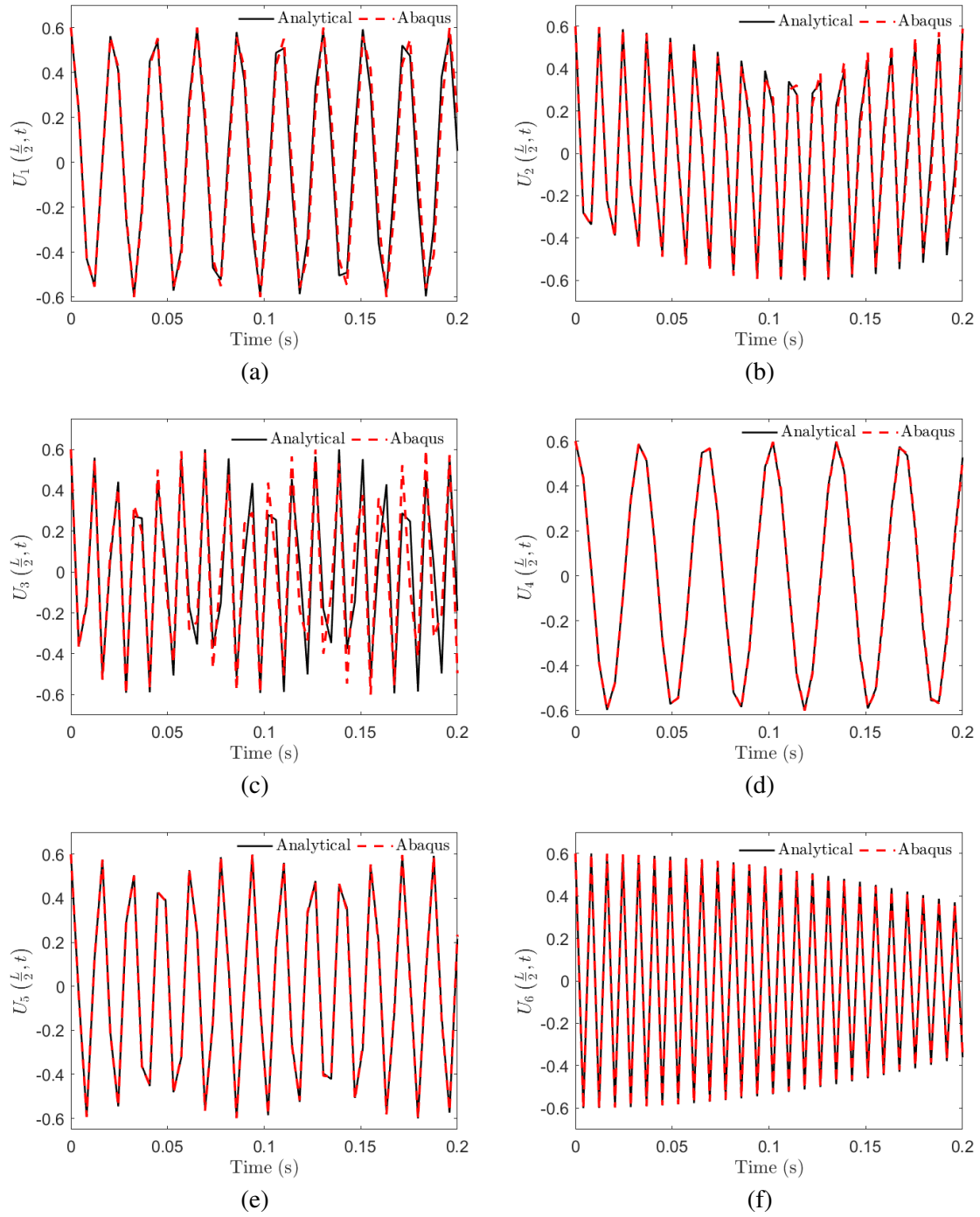


Figure 4.18: Temporal displacement at the centre of the beam on a nonlinear cubic elastic foundation: (a) to (f) are modes 1 to 6.

numerical plots and have the same amplitudes at each mode. Once more, the agreement between the two methods is pronounced here.

4.4.4 Numerical results of a BoEF having Quadratic and Cubic Stiffness Nonlinearities of the Foundation

In the current subsection, analysis is performed in Abaqus for a BoEF having both nonlinear quadratic and nonlinear cubic stiffness properties of the foundation. That aims to see to which extent the beam's natural frequencies and mode shapes can be modified due to this combination of nonlinearities. The restoring force to be considered for this third case is presented in Figure 4.19.

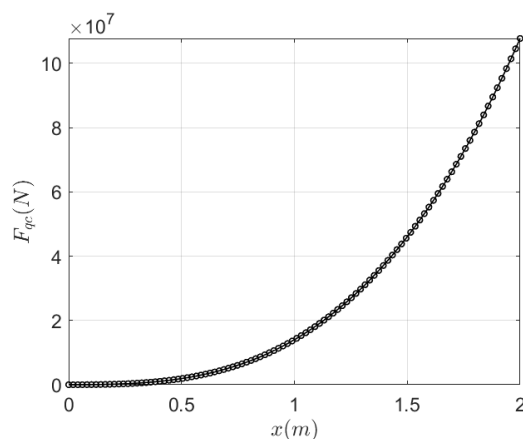


Figure 4.19: Behaviour of the nonlinear quadratic-cubic spring.

Figure 4.19 shows the plot of the restoring force combining the linear and nonlinear quadratic and nonlinear cubic terms. This curve was obtained by plotting the expression $k_l x + k_2 x^2 + k_3 x^3$, with the stiffness coefficients $k_l = 1.69 \times 10^5 \text{ N/m}^2$, $k_2 = 8.44 \times 10^5 \text{ N/m}^3$, and $k_3 = 1.3 \times 10^7 \text{ N/m}^4$. It is also an increased curve which looks more like the one presented in Subsection 4.4.3. This is justified since the cubic power of the displacement x is very small for minimal values, influencing the entire function. Comparing all restoring force plots, Figure 4.19 has the maximum amplitude. The data recorded from plotting the present curve allowed for appropriately assigning nonlinear quadratic and nonlinear cubic stiffness properties to the connectors during the analysis in Abaqus/Standard. Table 4.7 reads the values of the nonlinear natural frequencies obtained.

In Table 4.7, it can be seen that the frequencies have increased from those presented in Tables 4.6 and 4.5. Then, considering a soil foundation having nonlinear quadratic and cubic stiffness properties to the beam foundation significantly affects the natural frequencies at which the beam vibrates. The following figures show the mode shapes obtained.

Table 4.7: Natural frequencies for the first ten modes of vibration of a simply supported beam resting on a linear and nonlinear quadratic-cubic elastic foundation.

Mode's number i	Ω_{qc} (rad/s)		Differences (%)
	Analytical	Abaqus	
1	301.1	300.9	0.07
2	511.01	510.6	0.08
3	999.3	999.03	0.03
4	1727.4	1727.2	0.12
5	2677.6	2677.6	0.00
6	3844.7	3844.6	0.003
7	5226.6	5226.5	0.002
8	6822.6	6822.4	0.003
9	8631.9	8631.8	0.001
10	10655	10651	0.04

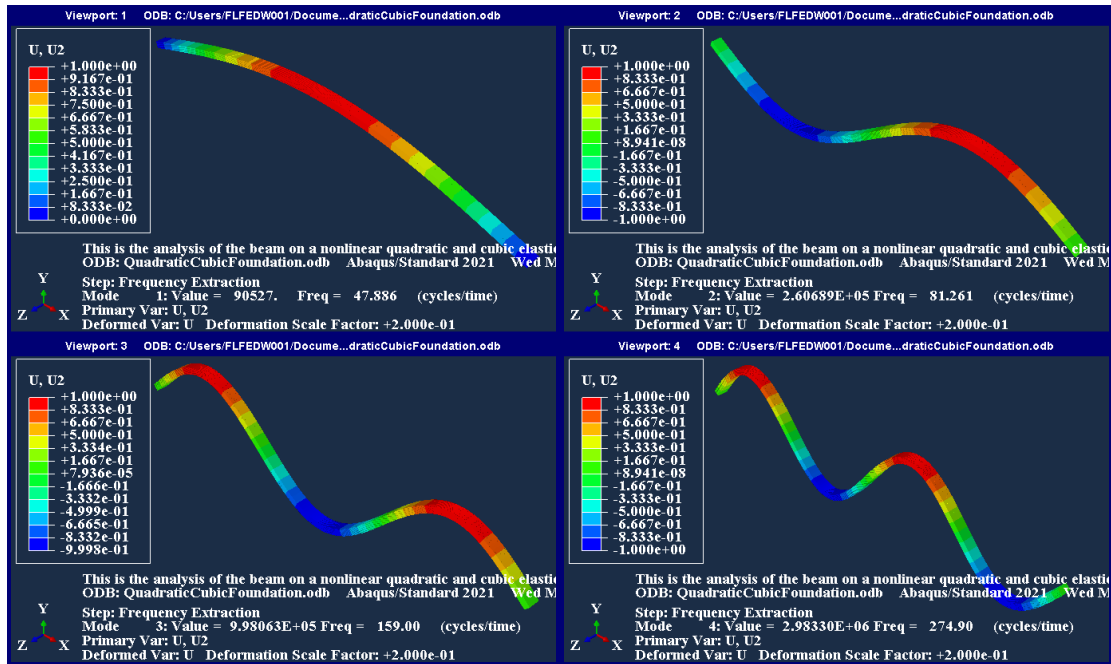


Figure 4.20: Mode shapes of the beam with a nonlinear quadratic-cubic stiffness foundation.

Given Figure 4.20, the same findings can be observed as those in Figures 4.12 and 4.16 regarding the beam shapes and amplitudes. Only the eigenvalues and natural frequencies have increased. As far as Figure 4.21 is concerned, the amplitudes look the same as those previously presented in Figures 4.13 and 4.17. Consequently, the same comments also apply here. Figure 4.22 compares temporal plots issued from analytical calculations and numerical analyses of the BoEF having mixed parity stiffness properties.

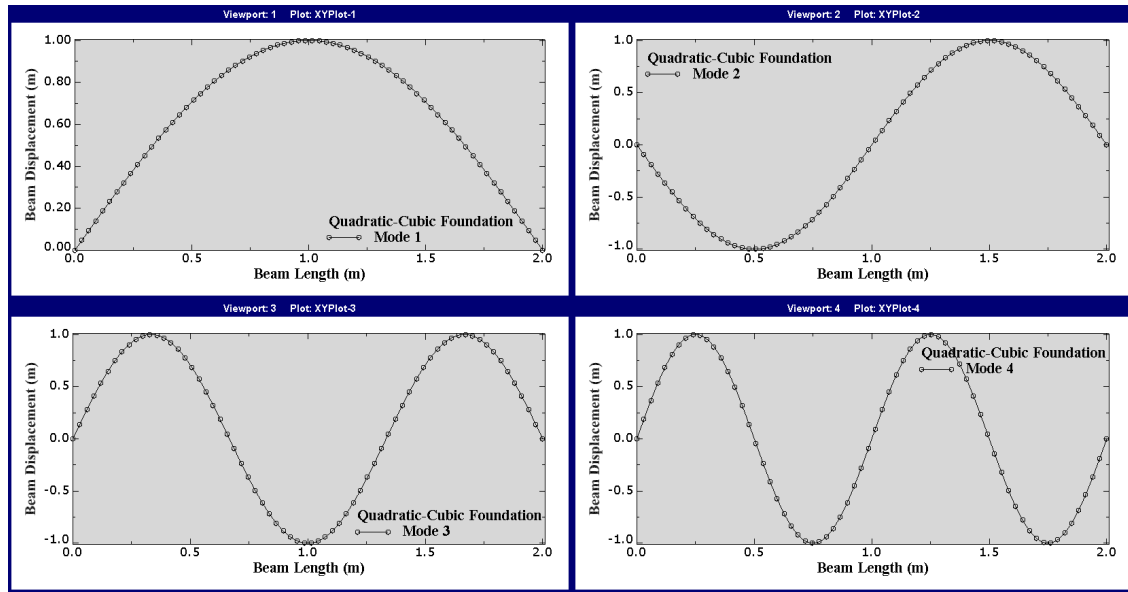
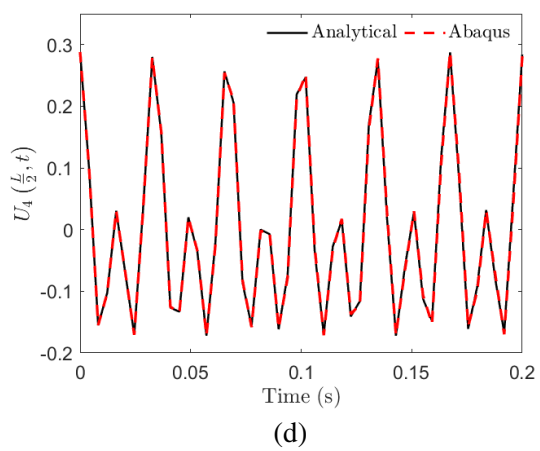
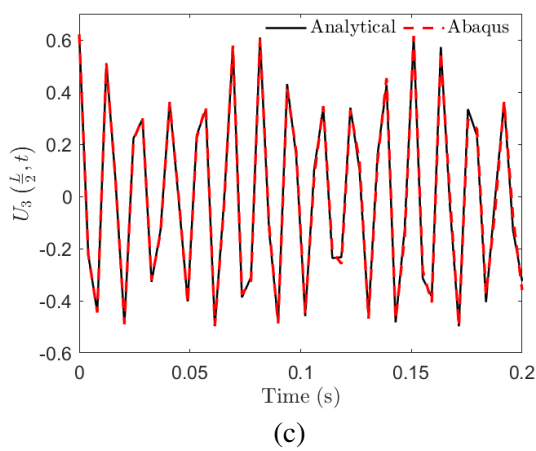
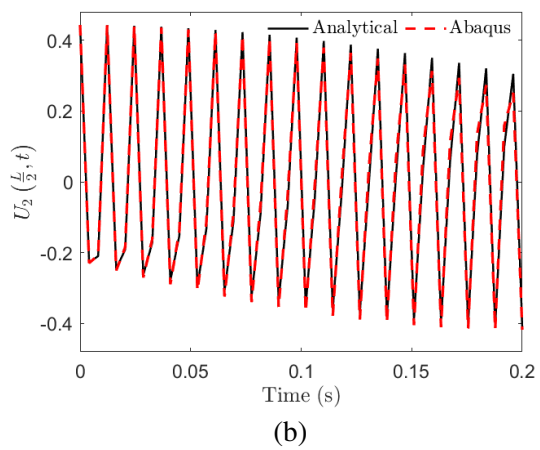
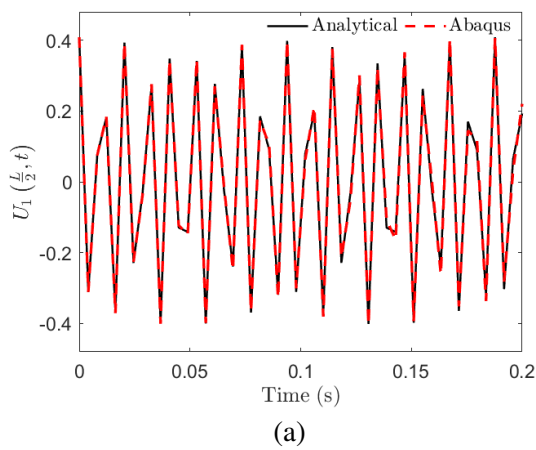


Figure 4.21: Displacement of the beam with a nonlinear quadratic-cubic stiffness foundation.



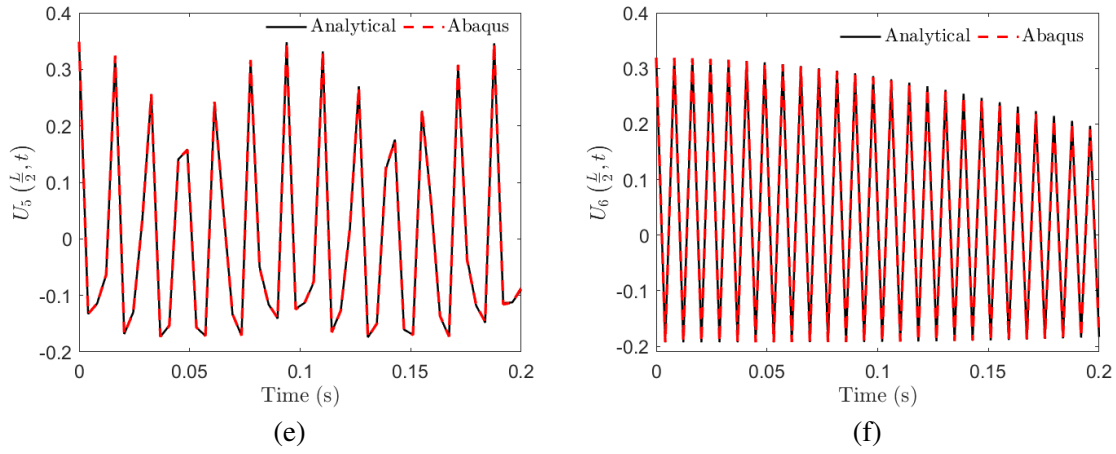


Figure 4.22: Temporal displacement at the centre of the beam on a nonlinear quadratic-cubic elastic foundation: (a) to (f) are modes 1 to 6.

In Figure 4.22 it can be observed that the curves follow the same paths with the same maximum amplitudes, except those at mode six, which look different in terms of shapes. The current findings confirm that the agreement between the two methods is also confirmed. Since the three nonlinear types of nonlinear problems investigated here produce similar findings, the FEA results in Abaqus serve as an excellent benchmark for the analytical results.

4.5 Comparison of the Present Results to those obtained from the Linear Analysis

In this section, the results of the present study (nonlinear analysis) are compared to those obtained from the linear analysis in Chapter 3. The justification for conducting the nonlinear study after the linear study is that the nonlinear study is regarded as an extension of the complete analysis. In this case, the analysis is performed beyond the linear analysis assumptions. Then, the effect of the nonlinearity introduced can easily be explained. Moreover, the results from both analyses can be compared and the differences in neglecting the nonlinearity highlighted, which is the primary aim of this thesis and the purpose of the current chapter. The tables below display the natural frequencies calculated from both analyses.

Tables 4.8, 4.9, and 4.10 contain the natural frequencies computed analytically and numerically for the first ten modes of vibration. ω_l , Ω_q , Ω_c , and Ω_{qc} respectively stand for the natural frequency of the beam with linear, nonlinear quadratic, nonlinear cubic, and

Table 4.8: Comparison between linear and nonlinear quadratic natural frequencies.

Mode	Analytical			Abaqus		
	ω_l (rad/s)	Ω_q (rad/s)	Diff (%)	ω_l (rad/s)	Ω_q (rad/s)	Diff (%)
1	285.17	299.2	4.7	285.12	297.3	4.1
2	501.5	508.6	1.4	501.48	508.5	1.4
3	994.42	998.1	0.4	994.41	997.96	0.4
4	1724.6	1726.8	0.13	1724.6	1726.6	0.12
5	2675.8	2677.5	0.06	2675.8	2677.2	0.05
6	3843.4	3844.5	0.03	3843.4	3844.4	0.026
7	5225.7	5226.3	0.01	5225.7	5226.3	0.011
8	6821.7	6821.5	0.03	6821.7	6822.2	0.007
9	8631.3	8631.5	0.02	8631.3	8631.7	0.005
10	10654	10654	0.00	10651	10651	0.00

Table 4.9: Comparison between linear and nonlinear cubic natural frequencies.

Mode	Analytical			Abaqus		
	ω_l (rad/s)	Ω_c (rad/s)	Diff (%)	ω_l (rad/s)	Ω_c (rad/s)	Diff (%)
1	285.17	290.2	1.73	285.12	288.9	1.3
2	501.5	504.3	0.6	501.48	503.6	0.4
3	994.42	995.9	0.15	994.41	995.5	0.11
4	1724.6	1725.4	0.05	1724.6	1725.2	0.03
5	2675.8	2676.4	0.02	2675.8	2676.2	0.01
6	3843.4	3843.8	0.01	3843.4	3843.7	0.008
7	5225.7	5225.9	0.004	5225.7	5225.9	0.004
8	6821.7	6821.9	0.003	6821.7	6821.9	0.003
9	8631.3	8631.5	0.002	8631.3	8631.4	0.001
10	10654	10654	0.00	10651	10651	0.00

nonlinear quadratic-cubic stiffness coefficients of the foundation. The remark here is that the nonlinear natural frequencies are more significant for analytical and numerical analyses than those from the linear analysis. The results are per the findings in work by Ibrayev (2019). The distinction between the nonlinear natural frequencies is minimal, and their comparison to the linear natural frequencies produces almost the same differences. Nevertheless, for the BoEF having quadratic and cubic stiffness nonlinearities, the differences computed from the natural frequencies are higher than those of the beam supported on other nonlinear foundations.

Turning attention to the lowest natural frequencies suitable for design checks, especially the fundamental mode, it is seen that differences are high compared to other modes. One of the reasons is that the linear natural frequency at mode 1 is very small as compared to other

Table 4.10: Comparison between linear and nonlinear quadratic-cubic natural frequencies.

Mode	Analytical			Abaqus		
	ω_l (rad/s)	Ω_{qc} (rad/s)	Diff (%)	ω_l (rad/s)	Ω_{qc} (rad/s)	Diff (%)
1	285.17	301.1	5.3	285.12	300.9	5.2
2	501.5	511.01	1.9	501.48	510.6	1.8
3	994.42	999.3	0.5	994.41	999.03	0.5
4	1724.6	1727.4	0.2	1724.6	1727.2	0.2
5	2675.8	2677.6	0.07	2675.8	2677.6	0.07
6	3843.4	3844.7	0.03	3843.4	3844.6	0.03
7	5225.7	5226.6	0.02	5225.7	5226.5	0.02
8	6821.7	6822.6	0.01	6821.7	6822.4	0.01
9	8631.3	8631.9	0.007	8631.3	8631.8	0.006
10	10654	10655	0.009	10651	10651	0.00

modes and since its square root is inversely proportional to the nonlinear natural frequency at that same mode, it results that its effect is not as significant as in other modes. In addition, the nonlinear stiffness parameter α_2 decreases as the mode number increases which has the same effect in the nonlinear natural frequencies. The results in Tables 4.8, 4.9, and 4.10 show that, by neglecting the nonlinearity of the foundation, the structure may not be as vital to support all the excitation acting on it. On the other hand, assuming the structure to be linear, a good representation of the real situation may not be obtained. Consequently, a nonlinear problem should be solved as is for the accurate design of sustainable structures. In what follows, nonlinear temporal plots are compared to linear plots.

Figure 4.23 displays the plots of the beam's displacement versus time, representing the comparison between the temporal amplitudes of vibration of the BLEF to those of the beam resting on a nonlinear elastic foundation. Those plots in MATLAB were obtained after analytical calculations, and the results were produced to clearly show the responses of the beam when nonlinear stiffness coefficients are added to the foundation. From these outputs, it can be seen that the curve of the BLEF at each of the modes presented has the maximum amplitude of vibration. On the other hand, there are close amplitudes curves for both the beam with quadratic nonlinearity and the one with cubic nonlinearity.

In contrast, the foundation of the beam with quadratic and cubic stiffness nonlinearities has minimal amplitude. Then, it is confirmed that the presence of nonlinearity in the foundation influences the natural frequencies of the entire system and can reduce the amplitude of vibration of the beam. Hence, when the foundation is assumed to be linear, the differences materialised here with the gaps between the linear and nonlinear curves are neglected.

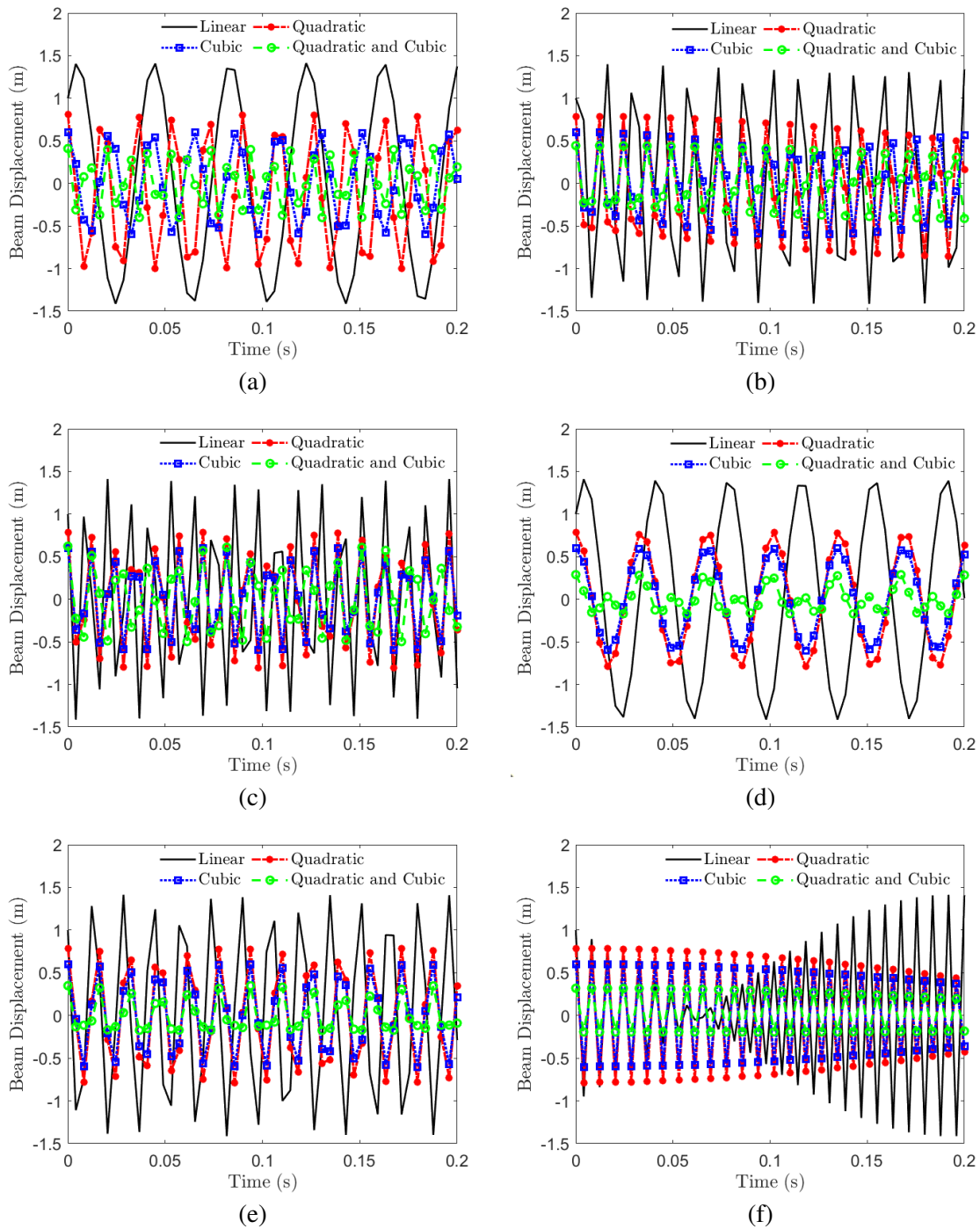


Figure 4.23: Comparison between the amplitudes of vibration of a BoEF obtained from analytical calculations: (a) to (f) are modes 1 to 6.

These could impact the representation of the real structure. The following figure shows the corresponding plots in Abaqus.

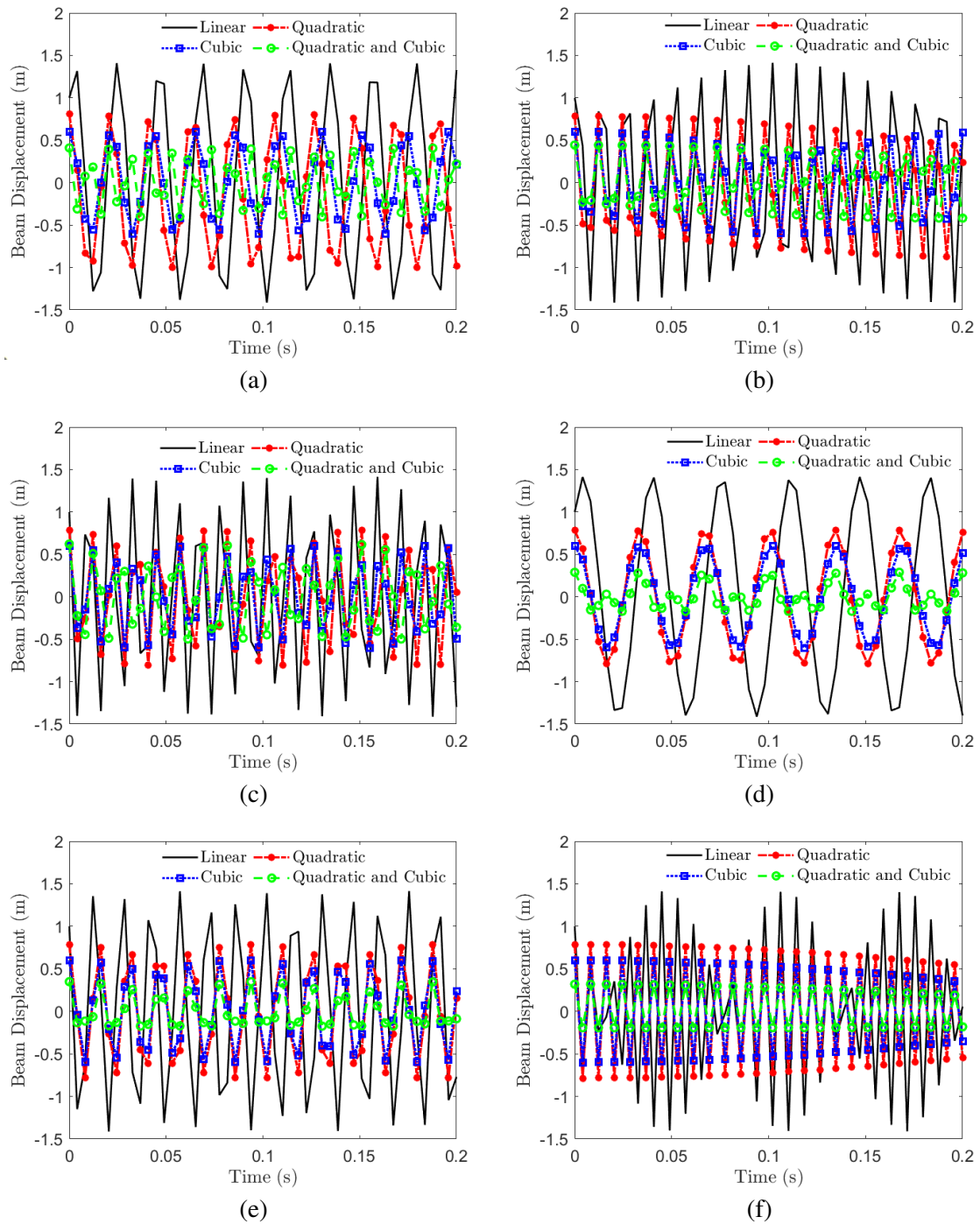


Figure 4.24: Comparison between linear and nonlinear temporal amplitudes of vibration of the BoEF obtained using the natural frequencies computed in Abaqus: (a) to (f) are modes 1 to 6.

Figure 4.24 also compares the linear and nonlinear plots of the beam's displacement versus Time. These plots were obtained in MATLAB using the natural frequencies recorded

in Abaqus/Standard and considering the other parameters from analytical calculations. It can also be observed that the BoEF with both quadratic and cubic stiffness nonlinearities has a minimum amplitude of the curves compared to others.

From both analytical and numerical results (Figures 4.23 and 4.24), the linear curves have the maximum amplitudes of vibration at each mode. Moreover, the gaps between the linear and nonlinear plots justify the differences highlighted in Tables 4.8, 4.9, and 4.10 for each case. Looking at the figures in the three cases presented above, the temporal amplitude is mostly influenced by the presence of nonlinearity in the equation of motion. The current findings are in accord with other works available in the literature. For example, Jorge et al. (2015) also highlighted the decrease of the displacement amplitude in the presence of nonlinearity. In addition, the fact that the amplitude of vibration of the beam decreases when the nonlinearity increases, confirms that the nonlinear foundation is stronger to withstand the deformation which may arise due to any load applied to the beam.

To extend the present study, numerical modelling was performed on a cantilever BoEF in Abaqus. Both linear and nonlinear analyses were conducted. In Appendix A, the natural frequencies and temporal plots are presented. The outcome reveals that the natural frequencies increase with nonlinearity. Moreover, it is observed that the vibration amplitude for the cantilever BoLEF is more significant than that of the BoNEF. Furthermore, the cantilever BoEF having nonlinear quadratic and nonlinear cubic stiffness properties has the smallest vibration amplitude for all the modes presented. This also shows that having a strong foundation is beneficial for controlling the beam's vibration amplitude.

Furthermore, it is interesting to perform some analyses for a simply supported beam resting on a linear and nonlinear cubic elastic foundation with a negative contribution from the cubic term. The table below shows the natural frequencies calculated from analytical and numerical methods as well as the differences which arise.

In Table 4.11, the results from both analytical and numerical analyses are similar. It is seen that the natural frequencies for the BoEF having nonlinear stiffness properties with a negative contribution from the cubic term Ω_c are smaller than those of the BoEF having linear stiffness properties ω_l . Therefore, for the softening foundation, the nonlinear natural frequencies of the BoEF will always be smaller than those of the beam with another foundation.

Table 4.11: Natural frequencies for the BoNEF having linear and a negative contribution from the cubic term.

Mode	Analytical			Abaqus		
	ω_l (rad/s)	Ω_c (rad/s)	Diff (%)	ω_l (rad/s)	Ω_c (rad/s)	Diff (%)
1	285.17	280.2	1.77	285.12	281.4	1.3
2	501.5	498.6	0.6	501.48	499.3	0.44
3	994.42	993.0	0.14	994.41	993.3	0.11
4	1724.6	1723.7	0.05	1724.6	1723.9	0.04
5	2675.8	2675.3	0.02	2675.8	2675.4	0.01
6	3843.4	3843.1	0.008	3843.4	3843.2	0.005
7	5225.7	5225.4	0.006	5225.7	5225.4	0.006
8	6821.7	6821.5	0.003	6821.7	6821.6	0.001
9	8631.3	8631.1	0.002	8631.3	8631.2	0.001
10	10654	10654	0.00	10651	10651	0.00

4.6 Summary and Conclusions

A simply supported BoEF having nonlinear stiffness properties is studied in this chapter. Three types of nonlinear stiffnesses of the foundation were investigated: quadratic, cubic, and a combination of quadratic and cubic. Analytical analysis using the straightforward expansion method was conducted to find the approximate solutions to the equations of motion issued from those three cases. The solutions of the equations were therefore plotted using MATLAB. On the other hand, Finite Element analyses developed in Abaqus/Standard were used to perform numerical analyses and benchmark the accuracy of the approximate solutions. These analytical and numerical analyses were carried out to capture the behaviour of the beam resting on the nonlinear foundation without modelling the foundation itself. Therefore, the following conclusions were drawn.

1. Nonlinearity was added to the linear equation, and the solution to the nonlinear differential equation was found. The nonlinear natural frequencies were also calculated from analytical and numerical methods. Minimal differences were found between the natural frequencies calculated from the analytical method and those recorded during the analysis in Abaqus. Good similarities were noticed by comparing the plots of the beam's displacements versus the beam's length for each method. Moreover, the maximum and minimum amplitudes remained unchanged for the two methods.

2. It was also found that the natural frequencies obtained analytically from the three nonlinear cases studied significantly increased compared to those obtained from the linear

analysis in Chapter 3. The modelling in Abaqus considered nonlinearity in the foundation only and linear perturbation was used to request the natural frequencies. Similar results to the analytical ones were obtained.

3. Considering a soil foundation having cubic stiffness properties implied a slight decrease in the natural frequencies. By considering a negative contribution of the cubic term, the nonlinear natural frequencies were smaller than those from the beam on a linear elastic foundation. On the other hand, comparing the study of a beam resting on a nonlinear cubic elastic foundation to that of a BoEF having both nonlinear quadratic and nonlinear cubic stiffness properties showed that the vibration frequencies increased, and the displacement amplitude of the beam reduced. Nonetheless, conducting the three nonlinear analyses showed that the increase of the nonlinear stiffness of the foundation modulus increases the beam's natural frequencies. On the contrary, the same increase of the nonlinear stiffness coefficient may imply a significant decrease in the amplitude of vibration of the beam.

4. Regarding the plots of the beam spatial displacement at different times at some modes revealed that the temporal part of the solution cannot just be neglected because at different times, the beam exhibits different amplitudes. The nonlinear foundation in some cases revealed a slight decrease in the spatial vibration amplitude of the beam but caused a considerable influence on the natural frequencies at which the structure vibrated.

5. Using *B23* as an element type in Abaqus was suitable to compare the numerical analysis to the Euler-Bernoulli beam theory used analytically. As a result, comparing results from the two methods gave minimal differences in the natural frequencies and the same paths to the temporal plots. These, therefore, are valid proof that the results from Abaqus are an excellent benchmark for analytical results. Hence, the present analytical results from SEM are accurate and can be validated.

6. Considering a foundation that will reflect the quadratic and cubic stiffness nonlinearities in one system produced beneficial effects as it considerably reduced the beam's vibration amplitude. The extension of the numerical analysis to the case of a cantilever beam on an elastic foundation revealed the same observation as the simply supported beam with the same foundation. These observations could be regarded as a means of vibration control of beams.

7. The current chapter enabled understanding and drawing attention to the fact that nonlinearity is critical when designing the foundation of a structure because the more vital the foundation is, the more robust the structure can resist deformation. Moreover, the linear analysis is only suitable for simple cases of beams supported on linear foundations.

Chapter 5

Parametric Study

5.1 Introduction

A parametric study on a beam resting on an elastic foundation having linear, nonlinear quadratic and cubic stiffness properties is conducted in the current chapter. The purpose of the chapter is to highlight the effects of the foundation's stiffness parameters on the beam's amplitude of vibration. Then, it will be established how the beam's natural frequencies and mode shapes vary with the linear and nonlinear stiffness parameters (as appropriate). As in the previous chapters, both analytical and numerical studies will be conducted. The analytical research will use the straightforward expansion method, whereas the numerical analysis will concentrate on FEA in Abaqus/Standard. The chapter will consider a 2D beam resting on a linear and nonlinear quadratic elastic foundation on the one hand and a linear and nonlinear cubic elastic foundation on the other. Some of the results presented in this chapter have been recently reported (Songong and Zingoni, 2022).

The present chapter extends the chapter on the nonlinear vibration of beams (Chapter 4). Two cases of nonlinearities will be regarded: quadratic and cubic. First, different values are assigned to the linear stiffness coefficient, then the nonlinear quadratic stiffness coefficient is varied, and for all the cases, the corresponding natural frequencies are calculated. Second, the beam displacement is plotted versus time. Furthermore, the same study is performed by varying the nonlinear cubic stiffness parameter. Finally, from the two nonlinear investigations, it will be defined which parameter value of the foundation best controls the beam vibration response. We note that the value of the beam's stiffness k_b will remain as $k_b = 33750$ N/m as given in the previous chapters.

5.2 Effects of the Nonlinear Quadratic Stiffness Coefficient of the Elastic Foundation

In this section, the linear stiffness coefficient is fixed, and different values are assigned to the nonlinear quadratic stiffness coefficient. This will facilitate a parametric study on a beam resting on a nonlinear quadratic elastic foundation. In addition, analytical and numerical results of the natural frequencies will be presented.

5.2.1 Analytical Calculations of the Natural Frequencies of the BoEF having Nonlinear Quadratic Stiffness Properties

A parametric analytical study is conducted in this subsection to picture the evolution of the natural frequencies when the nonlinear stiffness coefficient increases. It is recalled that in Chapter 4, the equation of motion of the beam resting on a nonlinear quadratic elastic foundation was solved. That same equation is the one to be considered in this subsection and given by:

$$\rho S \frac{\partial^2 U(x,t)}{\partial t^2} + EI \frac{\partial^4 U(x,t)}{\partial x^4} + k_l U(x,t) + k_2 U(x,t)^2 = 0, \quad (5.1)$$

where $U(x,t)$ is the transverse displacement of the beam and $k_2 U(x,t)^2$ is the nonlinear quadratic term. The solution to Eq. (5.1) was found in Chapter 4 to be:

$$U_n(x,t) = A_n \sin\left(\frac{n\pi}{L}x\right) \left\{ \varepsilon B_1 \cos(\Omega_q t) + \frac{\alpha_2 \varepsilon^2}{6\omega_l^2} B_1^2 \left[2 \cos(\Omega_q t) - 3 + \cos(2\Omega_q t) \right] - \frac{\alpha_2^2}{3\omega_l^4} \varepsilon^3 B_1^3 \left[\frac{23}{48} \cos(\Omega_q t) + \frac{1}{3} \left(\cos(2\Omega_q t) - 3 \right) + \frac{1}{16} \cos(3\Omega_q t) \right] \right\}, \quad (5.2)$$

where n is the mode's number. The expression of the natural frequencies Ω_q due to the quadratic nonlinearity was also given by:

$$\Omega_q = \omega_l \left[1 + \frac{\alpha_2^2}{3\omega_l^4} \varepsilon^2 B_1^2 \right]. \quad (5.3)$$

In Eqs. (5.2) and (5.3), ω_l is the linear natural frequency and α_2 is the coefficient carrying nonlinearity. Both parameters are recalled being:

$$\omega_l^2 = \frac{EI}{\rho S} \left(\frac{n\pi}{L} \right)^4 + \frac{k_l}{\rho S} \text{ implying } \omega_l = \sqrt{\frac{EI}{\rho S} \left(\frac{n\pi}{L} \right)^4 + \frac{k_l}{\rho S}}, \quad (5.4a)$$

$$\text{and } \alpha_2 = \left[\frac{8n^2}{(4n^2 - 1)\pi} \right] \frac{k_2 A_n}{\rho S}. \quad (5.4b)$$

To proceed, a fix relationship is defined between the linear stiffness coefficient k_l and the beam stiffness k_b by mean of the linear relation $\mu = \frac{k_l}{k_b}$. Then, by giving different fixed values to μ , the nonlinear stiffness coefficient k_2 is varied concerning the linear stiffness coefficient by mean of the ratio:

$$r_q = \frac{k_2}{k_l}. \quad (5.5)$$

Eq. (5.5) is defined to record the amount of nonlinearity added to the foundation easily. Moreover, it will help determine how the natural frequencies are modified concerning that ratio. Then, for each fixed value of μ , different values to r_q will be assigned, and the corresponding nonlinear natural frequencies Ω_q will be computed.

5.2.1.1 Case I: $\mu = \frac{k_l}{k_b} = 0.25$

For this first case, values to the ratio r are assigned starting from 0.25. Then, a new value of r_q is obtained by adding 0.25 to the previous one. This process stops when the value $r_q = 2$ is reached, especially when the nonlinear stiffness coefficient k_2 is twice the linear stiffness coefficient k_l . Table 5.1 displays the nonlinear natural frequencies computed.

The natural frequencies presented in Table 5.1 are those obtained for $\mu = 0.25$. The values of ω_l , α_2 , ε , and B_1 were taken from Chapter 4, and these helped for computing the nonlinear natural frequencies Ω_q given in Eq. (5.3) for different modes. It is realised from the results that the natural frequencies increase rapidly with the mode's number, but slightly with the nonlinear ratio r_q . Moreover, the natural frequencies computed when nonlinearity is present (for $r_q \neq 0$) are similar to those obtained in the absence of nonlinearity (for $r_q = 0$) when $r_q < 1$. Furthermore, as the beam stiffness is greater than the linear and nonlinear stiffness coefficients, the effect of the foundation is almost negligible.

Table 5.1: Nonlinear natural frequencies Ω_q (rad/s) of the beam on a linear and nonlinear quadratic elastic foundation obtained for $k_l = 0.25k_b$.

n	$r_q = 0$	$r_q = 0.25$	$r_q = 0.5$	$r_q = 0.75$	$r_q = 1$	$r_q = 1.25$	$r_q = 1.5$	$r_q = 1.75$	$r_q = 2$
1	121.81	121.84	121.92	122.06	122.25	122.51	122.81	123.18	123.6
2	430.12	430.12	430.13	430.13	430.15	430.16	430.18	430.21	430.24
3	960.4	960.4	960.4	960.4	960.41	960.41	960.41	960.41	960.41
4	1705.2	1705.2	1705.2	1705.2	1705.2	1705.2	1705.2	1705.2	1705.2
5	2663.4	2663.4	2663.4	2663.4	2663.4	2663.4	2663.4	2663.4	2663.4
6	3834.8	3834.8	3834.8	3834.8	3834.8	3834.8	3834.8	3834.8	3834.8
7	5219.3	5219.3	5219.3	5219.3	5219.3	5219.3	5219.3	5219.3	5219.3
8	6816.9	6816.9	6816.9	6816.9	6816.9	6816.9	6816.9	6816.9	6816.9
9	8627.5	8627.5	8627.5	8627.5	8627.5	8627.5	8627.5	8627.5	8627.5
10	10651	10651	10651	10651	10651	10651	10651	10651	10651

5.2.1.2 Case 2: $\mu = \frac{k_l}{k_b} = 0.5$

Here, the same procedure is followed as in case 1. However, the relation between the linear stiffness coefficient and that of the beam is given by $k_l = 0.5k_b$. In Table 5.2, the natural frequencies are revealed.

Table 5.2: Nonlinear natural frequencies Ω_q (rad/s) of the beam on a linear and nonlinear quadratic elastic foundation obtained for the case $k_l = 0.5k_b$.

n	$r_q = 0$	$r_q = 0.25$	$r_q = 0.5$	$r_q = 0.75$	$r_q = 1$	$r_q = 1.25$	$r_q = 1.5$	$r_q = 1.75$	$r_q = 2$
1	135.4	135.48	135.72	136.13	136.69	137.43	137.93	138.38	139.1
2	434.16	434.17	434.19	434.23	434.28	434.34	434.42	434.51	434.62
3	962.22	962.22	962.22	962.23	962.23	962.24	962.24	962.25	962.26
4	1706.2	1706.2	1706.2	1706.2	1706.2	1706.2	1706.2	1706.2	1706.2
5	2664	2664.1	2664.1	2664.1	2664.1	2664.1	2664.1	2664.1	2664.1
6	3835.2	3835.3	3835.3	3835.3	3835.3	3835.3	3835.3	3835.3	3835.3
7	5219.6	5219.6	5219.6	5219.6	5219.6	5219.6	5219.6	5219.6	5219.6
8	6817.1	6817.1	6817.1	6817.1	6817.1	6817.1	6817.1	6817.1	6817.1
9	8627.7	8627.7	8627.7	8627.7	8627.7	8627.7	8627.7	8627.7	8627.7
10	10651	10651	10651	10651	10651	10651	10651	10651	10651

In Table 5.2, observations similar to those in Table 5.1 are made. Additionally, there is a slight increase in the natural frequencies, though the remark is only at modes 1, 2, and 3. The values at other modes remain equal to those without nonlinearity (when $r_q = 0$).

On the one hand, based on the natural frequencies already computed and shown in Tables 5.1 and 5.2, it is realised that as the coefficient μ increases, the natural frequencies increase as well. However, this increase is not particularly significant based on the amount of nonlinearity added. These two sets of values are provided to show the evolution of the

nonlinear natural frequencies when the ratio r_q increases and to highlight that the amount of nonlinearity added dictates the variation of the nonlinear natural frequencies. To extend the study, other values were assigned to μ from 0 to 2, and the corresponding nonlinear natural frequencies were calculated. The aim was to push the understanding and picture the variation of the natural frequencies when both μ and thus the nonlinear ratio r_q increase. In Appendix B, the complete results are set out from Tables B.1 to B.6.

On the other hand, in the two cases above-presented and those in Appendix B, the natural frequencies computed by considering the presence of the quadratic nonlinearity in the elastic foundation increase with the increase of the parameter μ and the ratio r_q . Then, the value of r_q determines the amount of nonlinearity added to the foundation. As a result, for $r_q < 1$, the nonlinear quadratic parameter is smaller than the linear parameter. Hence, the effects of the nonlinearity are not pictured, and the natural frequencies computed are 90% of the linear stiffness coefficient k_l . Conversely, for $r_q > 1$ the natural frequencies are more of the nonlinear stiffness coefficient k_2 . It is worth mentioning that the increase is small here due to the different values assigned to the ratio r_q , which vary by adding 0.25 to its previous value. Therefore, for all values of $r_q > 2$, the increase of the nonlinear natural frequencies will be more significant.

5.2.2 Analytical Plots Corresponding to the Natural Frequencies of the BoEF having Nonlinear Quadratic Stiffness Properties

In this subsection, the results issued by plotting the displacement $U_n(x, t)$ as a function of the time t are presented. The figures to be put in view here were obtained in MATLAB. These will reveal the temporal plots of the beam displacement at the beam midspan corresponding to the above natural frequencies while varying the time t . The plots will be shown using natural frequencies obtained at modes 1 to 4 only. Other modes were not of interest because, for those specific cases, there are no significant differences from the natural frequencies calculated for all the instances. The time t will be kept between the interval 0 s to 0.1 s.

Figure 5.1 shows the plots of $U_1(x, t)$ when $x = \frac{L}{2}$. The curves presented in each subfigure look similar. Then, to analyse the results, zoom plots were produced as shown by the small rectangle and the arrow in blue. This helps to identify the curve corresponding to each ratio r_q . Moreover, it is observed that as the value of the linear ratio $\mu = \frac{k_l}{k_b}$ increases, the maximum amplitude slightly increases as well. Similarly, as the nonlinear stiffness ratio

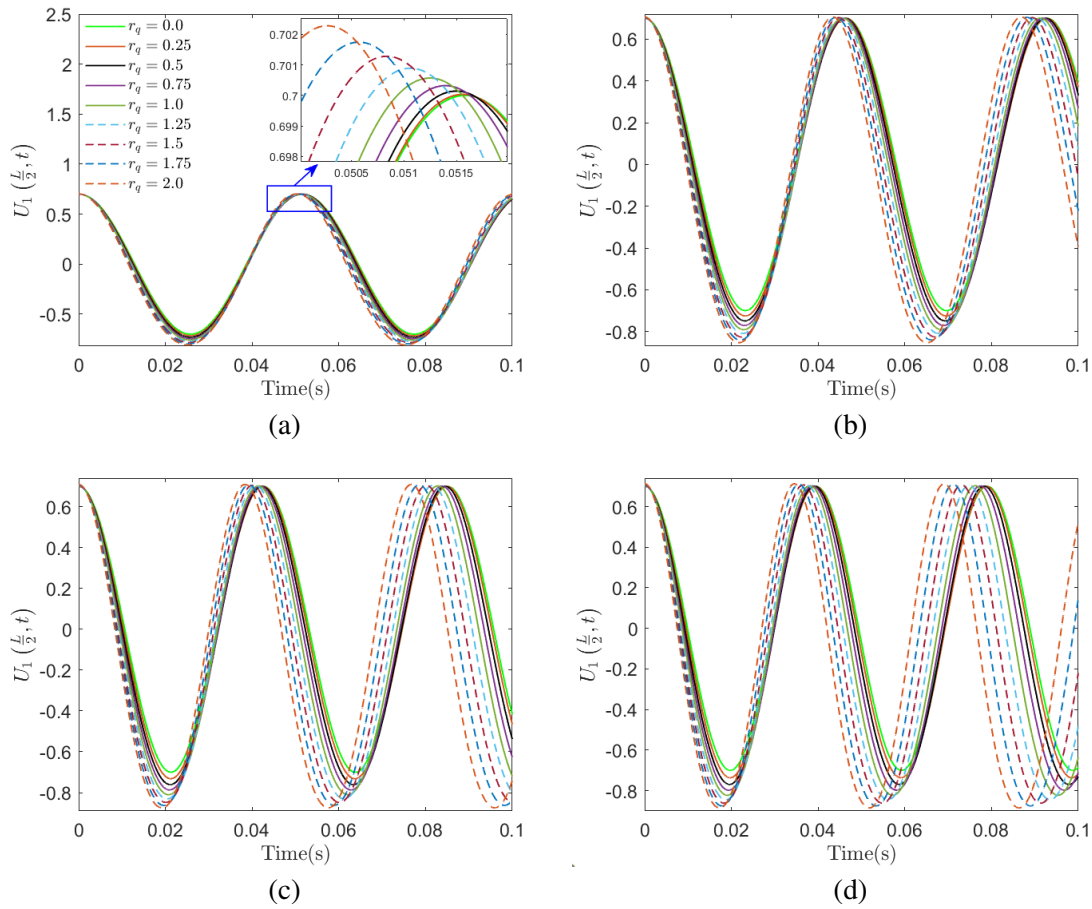
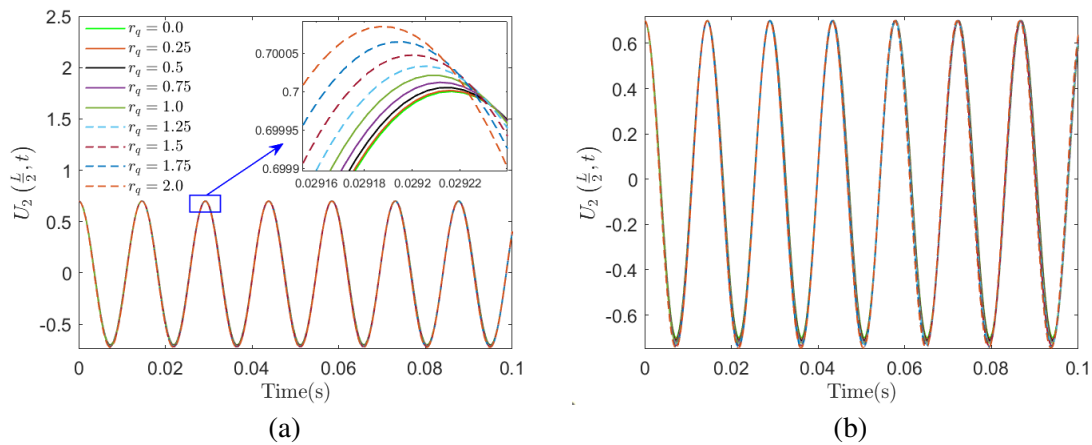


Figure 5.1: Displacement at the centre of the beam on a nonlinear quadratic elastic foundation obtained at mode 1: (a) $k_l = 0.25 \times k_b$, (b) $k_l = 0.5 \times k_b$, (c) $k_l = 0.75 \times k_b$, and (d) $k_l = k_b$.

r_q increases, the gap between two consecutive curves in the same subfigure becomes more significant. This implies an increase in the vibration amplitude of the beam.



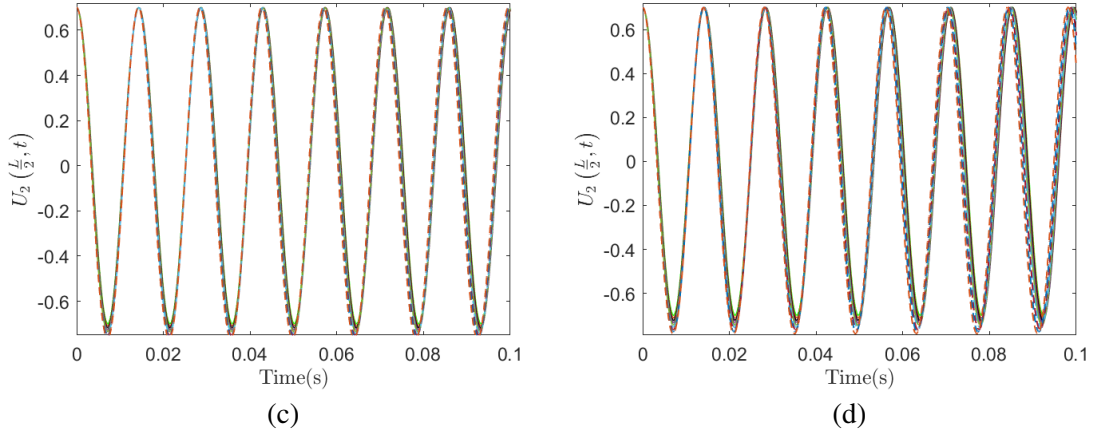
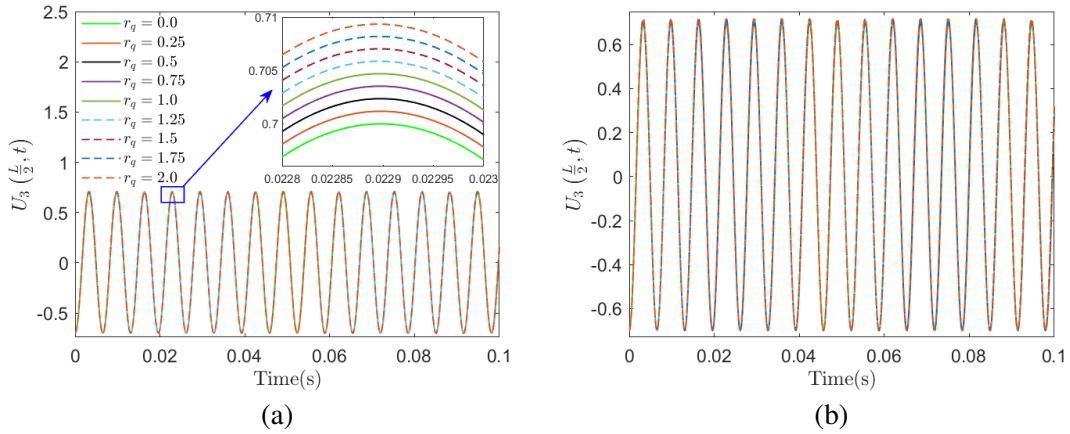


Figure 5.2: Displacement at the centre of the beam on a nonlinear quadratic elastic foundation obtained at mode 2: (a) $k_l = 0.25 \times k_b$, (b) $k_l = 0.5 \times k_b$, (c) $k_l = 0.75 \times k_b$, and (d) $k_l = k_b$.

Figure 5.2 shows that the beam's amplitude of vibration increases slightly due to the increase of the ratio r_q . This can be apprehended on the zoom plot. On each subfigure, all the curves do not have their maximum simultaneously because they cross themselves at some point. As a result, the curve with the maximum amplitude turns to the minimum amplitude at the lower end. Furthermore, the number of curvatures between the time interval has increased, unlike that at mode one.



As for mode 3, many cycles are formed compared to the plots in Figures 5.1 and 5.2. Then, focusing on the zoom plots and comparing amplitudes, there is an increase in the maximum amplitude as the ratio r_q increases. These details can only be viewed on the zoom plot since all the main plots in each subfigure mix with each other.

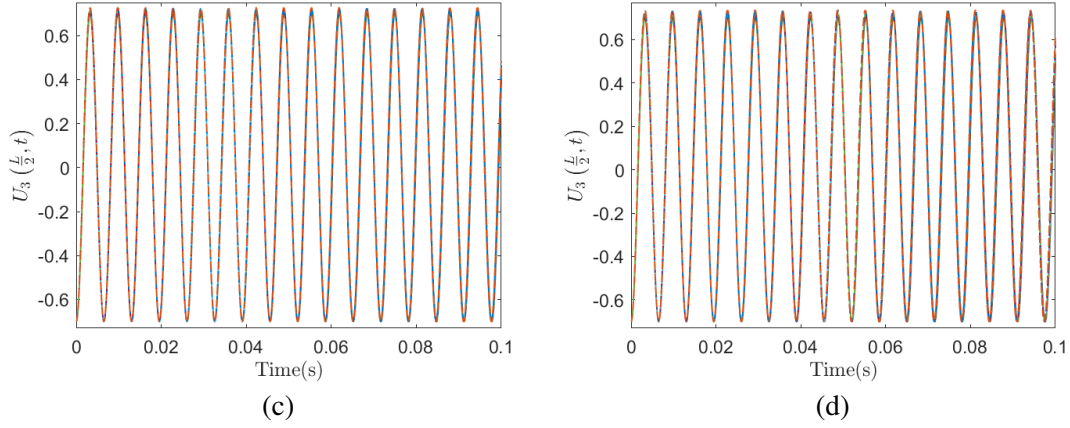


Figure 5.3: Displacement at the centre of the beam on a nonlinear quadratic elastic foundation obtained at mode 3: (a) $k_l = 0.25 \times k_b$, (b) $k_l = 0.5 \times k_b$, (c) $k_l = 0.75 \times k_b$, and (d) $k_l = k_b$.

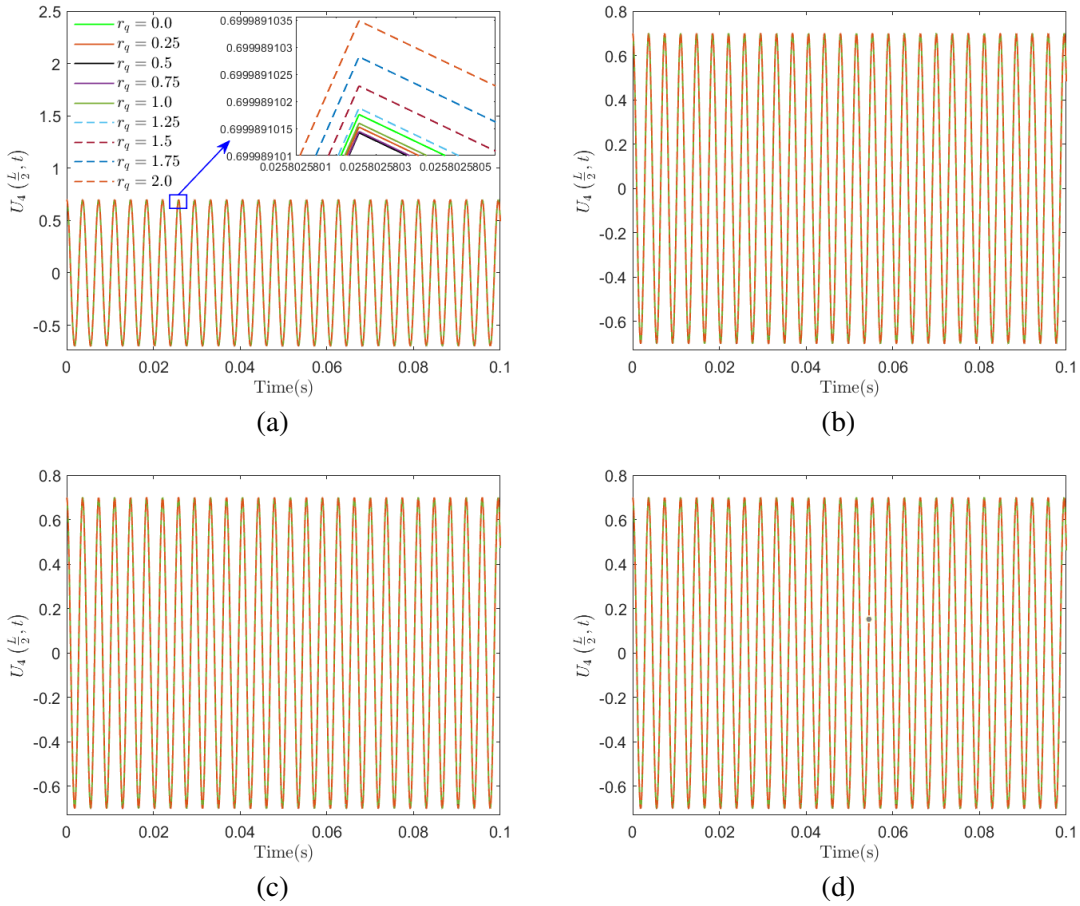


Figure 5.4: Displacement at the centre of the beam on a nonlinear quadratic elastic foundation obtained at mode 4: (a) $k_l = 0.25 \times k_b$, (b) $k_l = 0.5 \times k_b$, (c) $k_l = 0.75 \times k_b$, and (d) $k_l = k_b$.

Moving to mode four, the findings are the same as those in Figure 5.2. By looking at all the plot cases from Figures 5.1 to 5.4, an increase is remarked in the temporal amplitude of vibration as the relation between the linear stiffness coefficient k_l and the beam stiffness increases. Moreover, the maximum vibration amplitude increases as the nonlinear ratio r_q increases. As a result, by assuming the system is linear, one neglects some essential features, and in the end, the representation of the actual or real situation is inadequate. This is clarified by plotting the temporal displacement for k_l varying from $1.25k_b$ to $2k_b$. The results are shown in Subsection B.1.2 in Appendix B.

We should highlight that the amplitude for free vibration is studied to see the effects of the nonlinear stiffness of the foundation when the beam is not excited so that for forced vibration, the effects of the foundation in response to the applied external excitation are clearly captured. For example, Songong and Zingoni (2022) investigated the forced vibration of a beam on a nonlinear cubic elastic foundation and found that the presence of nonlinearity in the foundation significantly decreases the beam's vibration amplitude. The following subsection compares the analytical results to those from Abaqus.

5.2.3 Comparison of Analytical Results to those from FEA in Abaqus

To obtain numerical results, some analyses are performed in Abaqus. The same procedure is followed as in Chapter 4. For each case, the data recorded from the force-displacement characteristic curve obtained from MATLAB are used as input into Abaqus to define the springs' nonlinear properties of the foundation. In this subsection, results from numerical analyses using FEA in Abaqus are compared to those from analytical calculations presented in Subsection 5.2.1. Moreover, the differences between the two methods will also be identified, and the range of those differences will help validate or invalidate the analytical results.

5.2.3.1 Case 1: Comparison of the Natural Frequencies corresponding to $\frac{k_l}{k_b} = 0.25$

Here, the frequencies from analytical and numerical analyses are compared when the ratio between the linear stiffness of the foundation and beam stiffness is $\mu = 0.25$. The values are given in Table 5.3.

Table 5.3: Comparison between the natural frequencies Ω_q obtained analytically to those obtained in Abaqus for the beam on a nonlinear quadratic elastic foundation: Case 1 $k_l = 0.25k_b$.

	Ω_q (rad/s)		Diff (%)	Ω_q (rad/s)		Diff (%)	Ω_q (rad/s)		Diff (%)
Mode	Anal	Abaq		Anal	Abaq		Anal	Abaq	
n	$r_q = 0$			$r_q = 0.25$			$r_q = 0.5$		
1	121.81	121.81	0	121.84	121.88	0.03	121.92	121.96	0.03
2	430.12	430.12	0	430.12	430.14	0.005	430.13	430.16	0.007
3	960.4	960.4	0	960.4	960.41	0.001	960.4	960.42	0.002
4	1705.2	1705.2	0	1705.2	1705.2	0	1705.2	1705.2	0
5	2663.4	2663.4	0	2663.4	2663.4	0	2663.4	2663.4	0
6	3834.8	3834.8	0	3834.8	3834.8	0	3834.8	3834.8	0
7	5219.3	5219.3	0	5219.3	5219.3	0	5219.3	5219.3	0
8	6816.9	6816.9	0	6816.9	6816.9	0	6816.9	6816.9	0
9	8627.5	8627.5	0	8627.5	8627.5	0	8627.5	8627.5	0
10	10651	10651	0	10651	10651	0	10651	10651	0
n	$r_q = 0.75$			$r_q = 1.0$			$r_q = 1.25$		
1	122.06	122.03	0.02	122.25	122.1	0.1	122.51	122.17	0.3
2	430.13	430.18	0.01	430.15	430.2	0.01	430.16	430.22	0.01
3	960.4	960.42	0.002	960.41	960.44	0.003	960.41	960.45	0.004
4	1705.2	1705.2	0	1705.2	1705.2	0	1705.2	1705.2	0
5	2663.4	2663.4	0	2663.4	2663.4	0	2663.4	2663.4	0
6	3834.8	3834.8	0	3834.8	3834.8	0	3834.8	3834.8	0
7	5219.3	5219.3	0	5219.3	5219.3	0	5219.3	5219.3	0
8	6816.9	6816.9	0	6816.9	6816.9	0	6816.9	6816.9	0
9	8627.5	8627.5	0	8627.5	8627.5	0	8627.5	8627.5	0
10	10651	10651	0	10651	10651	0	10651	10651	0
n	$r_q = 1.5$			$r_q = 1.75$			$r_q = 2.0$		
1	122.81	122.25	0.5	123.18	122.32	0.7	123.6	122.39	0.9
2	430.18	430.24	0.02	430.21	430.26	0.01	430.24	430.28	0.01
3	960.41	960.46	0.005	960.41	960.46	0.005	960.41	960.48	0.007
4	1705.2	1705.2	0	1705.2	1705.2	0	1705.2	1705.2	0
5	2663.4	2663.4	0	2663.4	2663.4	0	2663.4	2663.4	0
6	3834.8	3834.8	0	3834.8	3834.8	0	3834.8	3834.8	0
7	5219.3	5219.3	0	5219.3	5219.3	0	5219.3	5219.3	0
8	6816.9	6816.9	0	6816.9	6816.9	0	6816.9	6816.9	0
9	8627.5	8627.5	0	8627.5	8627.5	0	8627.5	8627.5	0
10	10651	10651	0	10651	10651	0	10651	10651	0

5.2.3.2 Case 2: Comparison of the Natural Frequencies corresponding to $\frac{k_l}{k_b} = 0.5$ Table 5.4: Comparison between the natural frequencies Ω_q obtained analytically to those obtained in Abaqus for the beam on a nonlinear quadratic elastic foundation: Case 2 $k_l = 0.5k_b$.

	Ω_q (rad/s)		Diff (%)	Ω_q (rad/s)		Diff (%)	Ω_q (rad/s)		Diff (%)
Mode	Anal	Abaq		Anal	Abaq		Anal	Abaq	
n	$r_q = 0$			$r_q = 0.25$			$r_q = 0.5$		
1	135.4	135.39	0.007	135.48	135.53	0.04	135.72	135.66	0.04
2	434.16	434.16	0	434.17	434.2	0.007	434.19	434.24	0.01
3	962.22	962.22	0	962.22	962.24	0.002	962.22	962.26	0.004
4	1706.2	1706.2	0	1706.2	1706.2	0	1706.2	1706.2	0
5	2664.0	2664.1	0.004	2664.1	2664.1	0	2664.1	2664.1	0
6	3835.2	3835.3	0.003	3835.3	3835.3	0	3835.3	3835.3	0
7	5219.6	5219.6	0	5219.6	5219.6	0	5219.6	5219.6	0
8	6817.1	6817.1	0	6817.1	6817.1	0	6817.1	6817.1	0
9	8627.7	8627.7	0	8627.7	8627.7	0	8627.7	8627.7	0
10	10651	10651	0	10651	10651	0	10651	10651	0
n	$r_q = 0.75$			$r_q = 1.0$			$r_q = 1.25$		
1	136.13	135.79	0.23	136.69	135.92	0.6	137.43	136.04	1.02
2	434.23	434.28	0.01	434.28	434.32	0.01	434.34	434.36	0.005
3	962.23	962.28	0.005	962.23	962.29	0.006	962.24	962.31	0.007
4	1706.2	1706.2	0	1706.2	1706.2	0	1706.2	1706.2	0
5	2664.1	2664.1	0	2664.1	2664.1	0	2664.1	2664.1	0
6	3835.3	3835.3	0	3835.3	3835.3	0	3835.3	3835.3	0
7	5219.6	5219.6	0	5219.6	5219.6	0	5219.6	5219.6	0
8	6817.1	6817.1	0	6817.1	6817.1	0	6817.1	6817.1	0
9	8627.7	8627.7	0	8627.7	8627.7	0	8627.7	8627.7	0
10	10651	10651	0	10651	10651	0	10651	10651	0
n	$r_q = 1.5$			$r_q = 1.75$			$r_q = 2.0$		
1	137.93	136.17	1.3	138.38	136.3	1.5	139.1	135.43	2.7
2	434.42	434.4	0.005	434.51	434.44	0.01	434.62	434.49	0.03
3	962.24	962.33	0.001	962.25	962.35	0.01	962.26	962.37	0.01
4	1706.2	1706.3	0.006	1706.2	1706.3	0.006	1706.2	1706.3	0.006
5	2664.1	2664.1	0	2664.1	2664.1	0	2664.1	2664.1	0
6	3835.3	3835.3	0	3835.3	3835.3	0	3835.3	3835.3	0
7	5219.6	5219.6	0	5219.6	5219.6	0	5219.6	5219.6	0
8	6817.1	6817.1	0	6817.1	6817.1	0	6817.1	6817.1	0
9	8627.7	8627.7	0	8627.7	8627.7	0	8627.7	8627.7	0
10	10651	10651	0	10651	10651	0	10651	10651	0

In Tables 5.3 and 5.4, the natural frequencies corresponding to both analytical calculations and numerical analysis in Abaqus are displayed. The percentage of differences is provided to see whether there is an agreement between the two methods. As can be noticed in those tables, the differences recorded are tiny. As a result, computing the mean corresponding to the differences calculated for each case of the ratio r_q , will give a percentage less than 10%. The complete results of the natural frequencies for the other six cases computed up to the value of $\mu = \frac{k_l}{k_b} = 2$ are given in Appendix B. Because of the findings mentioned above, the FEA results in Abaqus are an excellent benchmark for the analytical results. Therefore, the analytical results can be confidently validated. The effects of the cubic stiffness coefficient in the elastic foundation are investigated in the following section.

5.3 Effects of the Nonlinear Cubic Stiffness Coefficient of the Elastic Foundation

The study to be carried out in this section describes that of the beam resting on a nonlinear cubic elastic foundation. This study is similar to that in Section 5.2. The only difference lies in changing the quadratic nonlinearity to the cubic nonlinearity. Then, different values are assigned to the cubic stiffness coefficient of the foundation while the linear stiffness coefficient remains fixed. This will help in performing a parametric study on a beam resting on a nonlinear cubic elastic foundation. This section will conduct analytical and numerical studies, and the results will be presented.

5.3.1 Analytical Calculations of the Nonlinear Natural Frequencies

This subsection concentrates on finding the natural frequencies of the BoEF having nonlinear cubic stiffness properties. In Chapter 4 the governing equation satisfied by such model was derived as:

$$\rho S \frac{\partial^2 U(x,t)}{\partial t^2} + EI \frac{\partial^4 U(x,t)}{\partial x^4} + k_l U(x,t) + k_3 U(x,t)^3 = f(x,t). \quad (5.6)$$

with $U(x,t)$ standing for the transversal displacement and $k_3 U(x,t)^3$ the nonlinear cubic term. The straightforward expansion method is used to solve Eq. (5.6). Then, the solution to

the first order of the small parameter ε for free vibration of the beam was found to be:

$$U_n(x, t) = B_n \sin\left(\frac{n\pi}{L}x\right) \left\{ C_0 \cos(\Omega_c t) + \varepsilon \left[-\frac{\beta_3}{32\omega_l^2} C_0^3 \cos(\Omega_c t) + \frac{1}{32\omega_l^2} \beta_3 C_0^3 \cos(3\Omega_c t) \right] \right\} \quad (5.7)$$

where n is the mode's number, and the expression of the nonlinear natural frequency Ω_c was as follows:

$$\Omega_c = \omega_l \left[1 + \frac{3}{8\omega_l^2} \varepsilon \beta_3 C_0^2 \right]. \quad (5.8)$$

It is recalled that in Eqs. (5.7) and (5.8), ω_l is the linear natural frequency, whereas β_3 is the parameter carrying the nonlinear cubic term in the equation. Both parameters are respectively given by:

$$\omega_l^2 = \frac{EI}{\rho S} \left(\frac{n\pi}{L} \right)^4 + \frac{k_l}{\rho S} \text{ implying } \omega_l = \sqrt{\frac{EI}{\rho S} \left(\frac{n\pi}{L} \right)^4 + \frac{k_l}{\rho S}}, \quad (5.9a)$$

$$\text{and } \beta_3 = \frac{3}{4} \frac{k_3 B_n^2}{\rho S}. \quad (5.9b)$$

The values of all the parameters involved in this section will be taken from Chapter 4, unless otherwise specified. The same relationship between the linear stiffness coefficient k_l and the beam stiffness coefficient k_b is considered, as was the case in Section 5.2. Then, by giving the same fixed values as before, the nonlinear stiffness coefficient k_3 is assigned values with respect to the linear stiffness coefficient using the following ratio as was the case with the quadratic nonlinearity. That is:

$$r_c = \frac{k_3}{k_l}. \quad (5.10)$$

It can be noticed that the expression of r_c considered previously remains unchanged due to the same denominator k_l . Then, by assigning different values to r_c , the corresponding nonlinear natural frequencies Ω_c are computed. In what follows, present the results on a case-by-case basis are presented.

5.3.1.1 Case 1: $\frac{k_l}{k_b} = 0.25$

Here, the linear stiffness coefficient is $k_l = 0.25 \times k_b$. Then, the ratio r_c varies, ranging from 0.25 to 2 with the step of 0.25. Table 5.5 displays the natural frequencies computed.

Table 5.5: Nonlinear natural frequencies Ω_c of the beam on a linear and nonlinear cubic elastic foundation obtained for $k_l = 0.25k_b$.

m	$r_c = 0$	$r_c = 0.25$	$r_c = 0.5$	$r_c = 0.75$	$r_c = 1$	$r_c = 1.25$	$r_c = 1.5$	$r_c = 1.75$	$r_c = 2$
1	121.81	121.82	121.83	121.84	121.85	121.86	121.87	121.88	121.89
2	430.12	430.12	430.12	430.13	430.13	430.13	430.13	430.14	430.14
3	960.4	960.4	960.41	960.41	960.41	960.41	960.41	960.41	960.41
4	1705.2	1705.2	1705.2	1705.2	1705.2	1705.2	1705.2	1705.2	1705.2
5	2663.4	2663.4	2663.4	2663.4	2663.4	2663.4	2663.4	2663.4	2663.4
6	3834.8	3834.8	3834.8	3834.8	3834.8	3834.8	3834.8	3834.8	3834.8
7	5219.3	5219.3	5219.3	5219.3	5219.3	5219.3	5219.3	5219.3	5219.3
8	6816.9	6816.9	6816.9	6816.9	6816.9	6816.9	6816.9	6816.9	6816.9
9	8627.5	8627.5	8627.5	8627.5	8627.5	8627.5	8627.5	8627.5	8627.5
10	10651	10651	10651	10651	10651	10651	10651	10651	10651

The natural frequencies shown in Table 5.5 are not too different from those in Table 5.1. As such, those in the current table vary slightly compared to those in Table 5.1. This can be explained by the difference in the initial conditions chosen for both cases. Additionally, the choice of the amplitude C_0 of the temporal solution of the governing equation with cubic nonlinearity also plays a part.

5.3.1.2 Case 2: $\frac{k_l}{k_b} = 0.5$

Table 5.6: Nonlinear natural frequencies Ω_c of the beam on a linear and nonlinear cubic elastic foundation obtained for $k_l = 0.5k_b$.

m	$r_c = 0$	$r_c = 0.25$	$r_c = 0.5$	$r_c = 0.75$	$r_c = 1$	$r_c = 1.25$	$r_c = 1.5$	$r_c = 1.75$	$r_c = 2$
1	135.4	135.41	135.43	135.45	136.46	135.48	135.5	135.52	135.54
2	434.16	434.17	434.17	434.18	434.18	434.18	434.19	434.2	434.2
3	962.22	962.22	962.23	962.23	962.23	962.23	962.24	962.24	962.24
4	1706.2	1706.2	1706.2	1706.2	1706.2	1706.2	1706.2	1706.2	1706.2
5	2664	2664.1	2664.1	2664.1	2664.1	2664	2664	2664	2664.1
6	3835.2	3835.3	3835.3	3835.3	3835.3	3835.2	3835.2	3835.2	3835.3
7	5219.6	5219.6	5219.6	5219.6	5219.6	5219.6	5219.6	5219.6	5219.6
8	6817.1	6817.1	6817.1	6817.1	6817.1	6817.1	6817.1	6817.1	6817.1
9	8627.7	8627.7	8627.7	8627.7	8627.7	8627.7	8627.7	8627.7	8627.7
10	10651	10651	10651	10651	10651	10651	10651	10651	10651

When dealing with the second case, the same variation of the natural frequencies as in case 1 is observed. However, the natural frequencies obtained at the first three modes are mostly the ones with some increase, while those at subsequent modes remain invariable. A

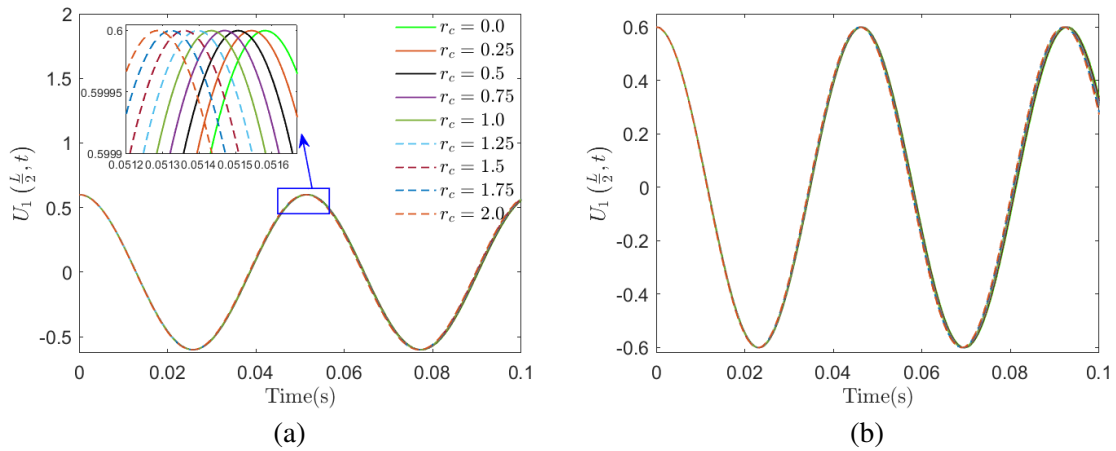
complete list of all the cases undertaken can be viewed in Appendix B. Details are presented in Tables B.13 to B.18.

Given Tables 5.5 and 5.6 as well as those in Appendix B, the comment is that the natural frequencies of the beam resting on a nonlinear cubic elastic foundation vary to a slight extent when the ratio r_c between the nonlinear cubic stiffness coefficient and the linear stiffness coefficient increases.

It is important to note that although the ratio r_c was kept the same for both cases investigated in Sections 5.2 and 5.3, the results are not the same. It is observed that the results in the latter section are slightly lower than those obtained in the former section for the first three modes but are the same for modes 4 to 10. Nonetheless, the slight increase in the nonlinear natural frequencies is suspected to be due to the small nonlinearity added in each new case. Therefore, the conclusion is that for more significant values of the nonlinear ratio r_c such as $r_c > 2$, the natural frequencies will considerably increase based on the type of nonlinearity.

5.3.2 Analytical Results Corresponding to the Study of the Beam on a Nonlinear Cubic Elastic Foundation

This subsection presents the results obtained by plotting the displacement $U_n(x, t)$ by varying the time t on a specific interval. MATLAB is used to obtain the graphs at the beam midspan. The interest in these graphs helps in understanding how the amplitude of vibration of the beam is modified when the nonlinear cubic stiffness coefficient increases.



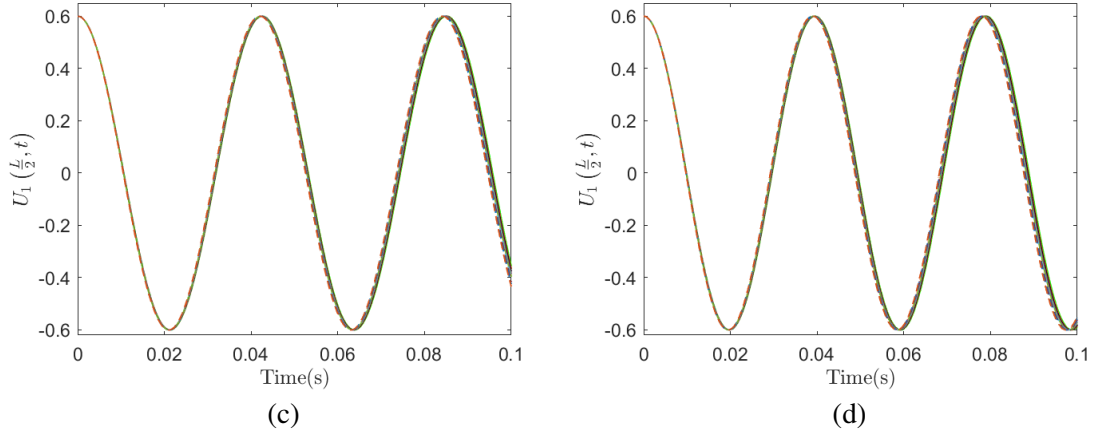


Figure 5.5: Displacement at the centre of the beam on a nonlinear cubic elastic foundation for obtained at mode 1: (a) $k_l = 0.25 \times k_b$, (b) $k_l = 0.5 \times k_b$, (c) $k_l = 0.75 \times k_b$, and (d) $k_l = k_b$.

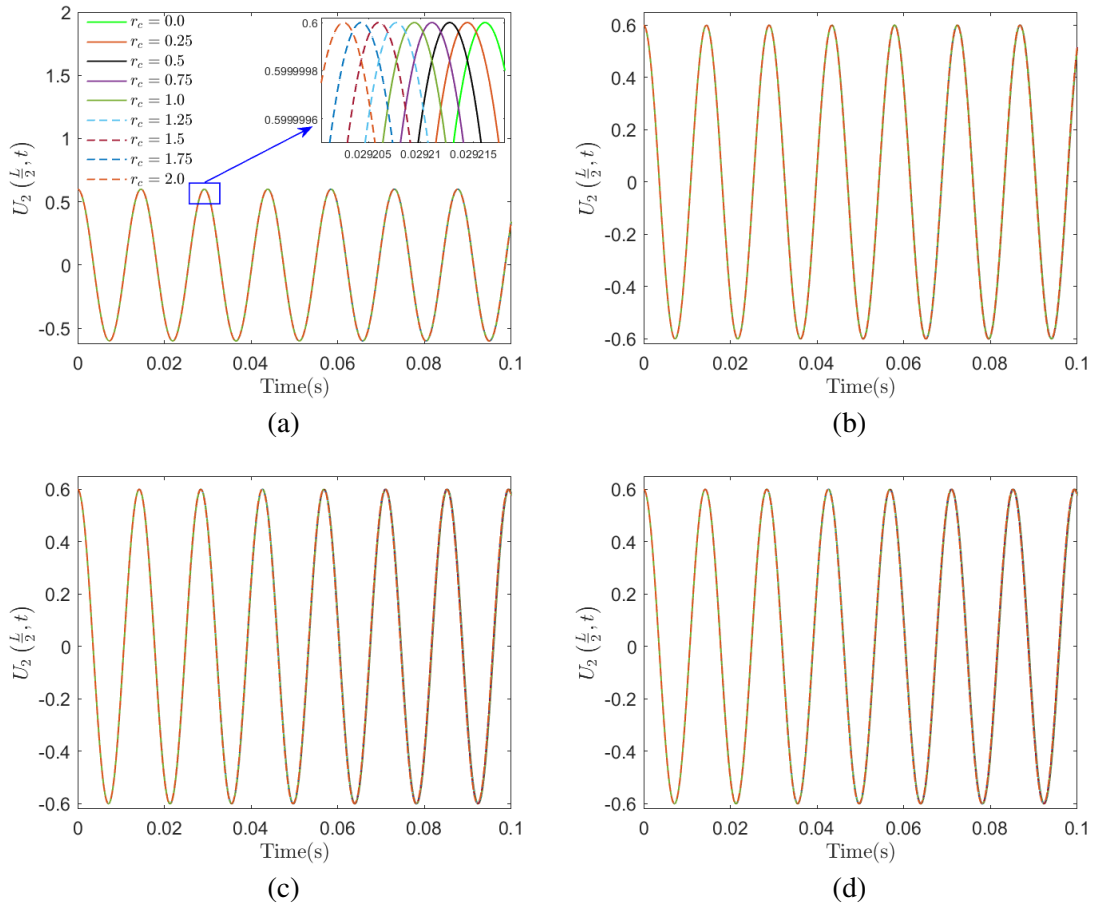


Figure 5.6: Displacement at the centre of the beam on a nonlinear cubic elastic foundation obtained at mode 2: (a) $k_l = 0.25 \times k_b$, (b) $k_l = 0.5 \times k_b$, (c) $k_l = 0.75 \times k_b$, and (d) $k_l = k_b$.

The curves in Figure 5.5 were obtained by plotting the displacement $U_n(x, t)$ at beam midspan (for $x = \frac{L}{2}$) while varying the time from 0s to 0.1s. The results show that all the curves (see each subfigure) obtain their maximum at the same value of the displacement $U_1(\frac{L}{2}, t)$ close to 0.6. However, the corresponding values on the abscissas are not the same for each curve. More precisely, the abscissa decreases as the ratio r_c increases (see the abscissa on the zoomed plots). The same observation can be made as the value of the linear stiffness coefficient k_l increases with respect to the beam stiffness coefficient k_b (see Figures 5.5a to 5.5d).

Concerning the plots at mode 2 (see Figure 5.6), the remarks highlighted in Subsection 5.2.2 are also valid here. Focusing on each subfigure, all the curves have the same maximum amplitude but different times. Moreover, as r_c increases, the time for that maximum amplitude decreases.

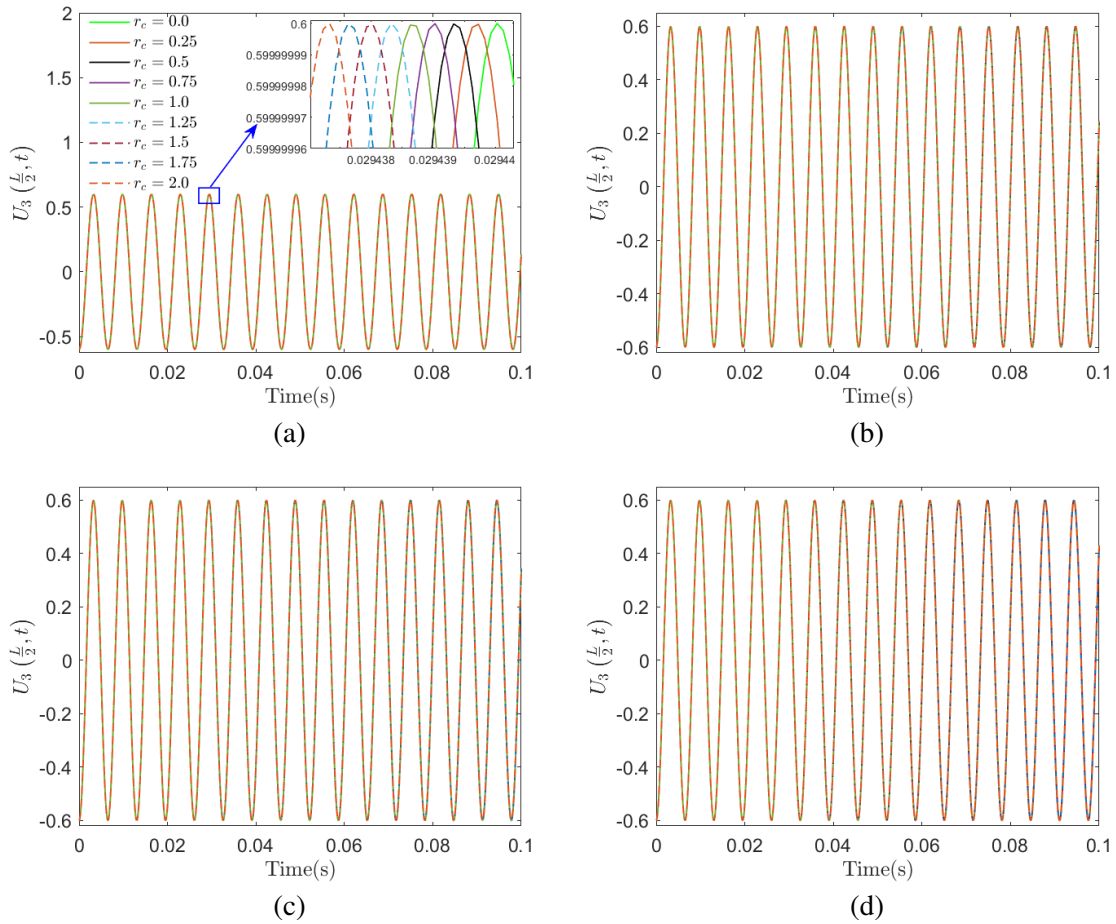


Figure 5.7: Displacement at the centre of the beam on a nonlinear cubic elastic foundation obtained at mode 3: (a) $k_l = 0.25 \times k_b$, (b) $k_l = 0.5 \times k_b$, (c) $k_l = 0.75 \times k_b$, and (d) $k_l = k_b$.

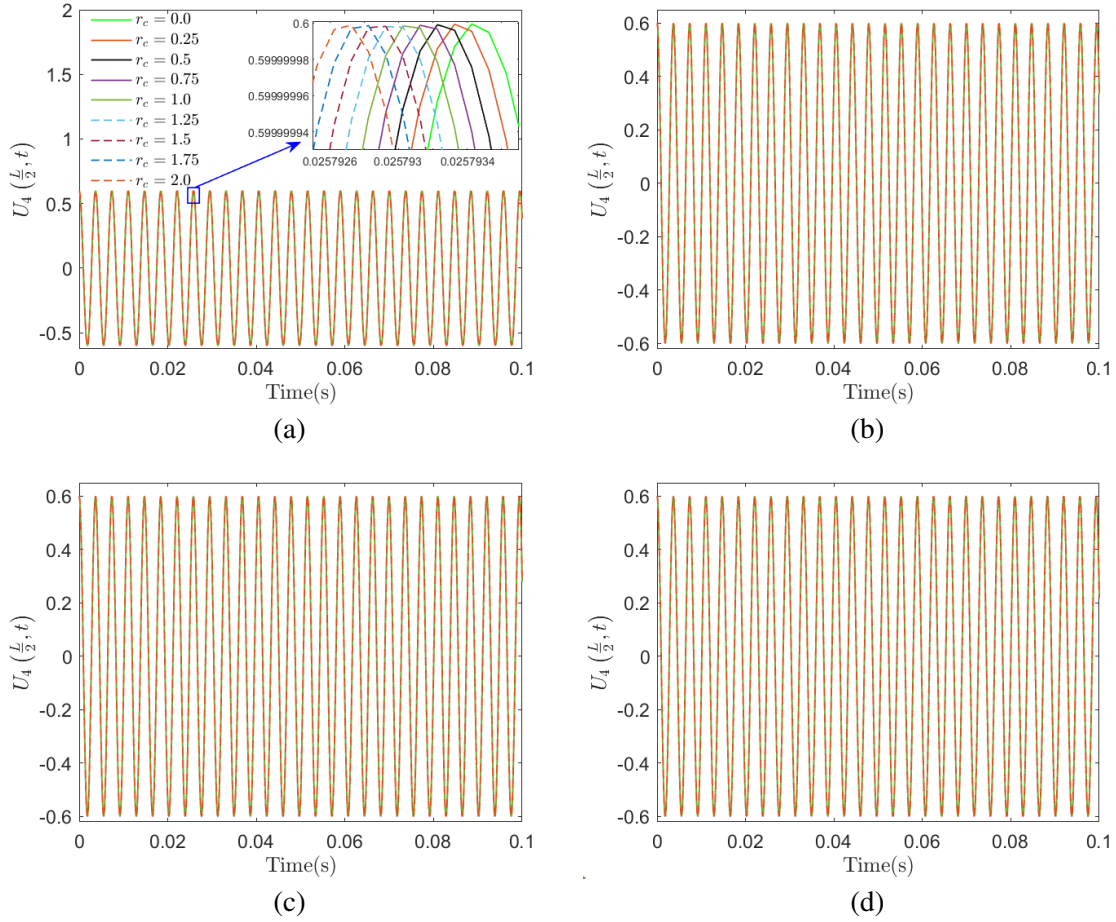


Figure 5.8: Displacement at the centre of the beam on a nonlinear quadratic elastic foundation obtained at mode 4: (a) $k_l = 0.25 \times k_b$, (b) $k_l = 0.5 \times k_b$, (c) $k_l = 0.75 \times k_b$, and (d) $k_l = k_b$.

As far as modes 3 and 4 are concerned, many cycles are observed compared to the plots in Figures 5.5 and 5.6. However, the remarks concerning Figures 5.7 and 5.8 are similar to those of the two previous cases (see Figures 5.5 and 5.6). Other plots related to k_l varying from $1.25k_b$ to $2k_b$ are shown in Subsection B.2.3 in Appendix B. In Figures 5.5 to 5.8, it is not obvious to talk about the amplitude as the mode's number increases because they all obtain their maximum at the same value on the y-axis. Only the number of cycles formed increases as the mode's number increases. Moreover, all the curves mix, and it is difficult to tell what is happening at each mode. Since the differences at each vibration mode are very minimal, the curves can be considered more or less the same from a practical point of view, and smoother curves could be obtained by increasing the time step. Also, by focusing on the zoom plots, the gap between two consecutive curves is the same in all the modes.

5.3.3 Comparison of Analytical Results to those obtained using FEA in Abaqus of the BoEF having Nonlinear Cubic Stiffness Properties

This subsection compares nonlinear natural frequencies of the beam on a nonlinear cubic elastic foundation computed using FEA in Abaqus to those obtained from analytical calculations and presented in Subsection 5.3.1. The procedure to be followed here to obtain the natural frequencies in Abaqus has already been mentioned in the last paragraph in Subsection 5.2.3 and is similar to the present case. Only the nonlinearity is different. The differences from the comparison will also be calculated for all the cases to approve the analytical results. In what follows, the findings are presented.

5.3.3.1 Case 1: $\frac{k_l}{k_b} = 0.25$

Table 5.7: Comparison between the natural frequencies Ω_c obtained analytically to those obtained in Abaqus for the beam on a nonlinear cubic elastic foundation: Case 1 $k_l = 0.25k_b$.

	Ω_c (rad/s)			Diff	Ω_c (rad/s)			Diff	Ω_c (rad/s)			Diff
Mode	Anal	Abaq	(%)		Anal	Abaq	(%)		Anal	Abaq	(%)	
n	$r_c = 0$				$r_c = 0.25$				$r_c = 0.5$			
1	121.81	121.81	0		121.82	121.81	0.008		121.83	121.81	0.02	
2	430.12	430.12	0		430.12	430.12	0		430.12	430.12	0	
3	960.4	960.4	0		960.4	960.4	0		960.41	960.4	0.001	
4	1705.2	1705.2	0		1705.2	1705.2	0		1705.2	1705.2	0	
5	2663.4	2663.4	0		2663.4	2663.4	0		2663.4	2663.4	0	
6	3834.8	3834.8	0		3834.8	3834.8	0		3834.8	3834.8	0	
7	5219.3	5219.3	0		5219.3	5219.3	0		5219.3	5219.3	0	
8	6816.9	6816.9	0		6816.9	6816.9	0		6816.9	6816.9	0	
9	8627.5	8627.5	0		8627.5	8627.5	0		8627.5	8627.5	0	
10	10651	10651	0		10651	10651	0		10651	10651	0	
n	$r_c = 0.75$				$r_c = 1.0$				$r_c = 1.25$			
1	121.84	121.82	0.02		121.85	121.82	0.02		121.86	121.82	0.03	
2	430.13	430.12	0.002		430.13	430.12	0.002		430.13	430.12	0.002	
3	960.41	960.4	0.001		960.41	960.4	0.001		960.41	960.41	0	
4	1705.2	1705.2	0		1705.2	1705.2	0		1705.2	1705.2	0	
5	2663.4	2663.4	0		2663.4	2663.4	0		2663.4	2663.4	0	
6	3834.8	3834.8	0		3834.8	3834.8	0		3834.8	3834.8	0	
7	5219.3	5219.3	0		5219.3	5219.3	0		5219.3	5219.3	0	
8	6816.9	6816.9	0		6816.9	6816.9	0		6816.9	6816.9	0	
9	8627.5	8627.5	0		8627.5	8627.5	0		8627.5	8627.5	0	

Continued on next page

Table 5.7 – continued from previous page

n	$r_c = 1.5$			$r_c = 1.75$			$r_c = 2.0$		
10	10651	10651	0	10651	10651	0	10651	10651	0
n	$r_c = 1.5$			$r_c = 1.75$			$r_c = 2.0$		
1	121.87	121.82	0.04	121.88	121.82	0.05	121.89	121.82	0.06
2	430.13	430.12	0.002	430.14	430.12	0.005	430.14	430.12	0.005
3	960.41	960.41	0	960.41	960.41	0	960.41	960.41	0
4	1705.2	1705.2	0	1705.2	1705.2	0	1705.2	1705.2	0
5	2663.4	2663.4	0	2663.4	2663.4	0	2663.4	2663.4	0
6	3834.8	3834.8	0	3834.8	3834.8	0	3834.8	3834.8	0
7	5219.3	5219.3	0	5219.3	5219.3	0	5219.3	5219.3	0
8	6816.9	6816.9	0	6816.9	6816.9	0	6816.9	6816.9	0
9	8627.5	8627.5	0	8627.5	8627.5	0	8627.5	8627.5	0
10	10651	10651	0	10651	10651	0	10651	10651	0

5.3.3.2 Case 2: $\frac{k_l}{k_b} = 0.5$

Table 5.8: Comparison between the natural frequencies Ω_c obtained analytically to those obtained in Abaqus for the beam on a nonlinear cubic elastic foundation: Case 2 $k_l = 0.5k_b$.

	Ω_c (rad/s)		Diff (%)	Ω_c (rad/s)		Diff (%)	Ω_c (rad/s)		Diff (%)
Mode	Anal	Abaq		Anal	Abaq		Anal	Abaq	
n	$r_c = 0$			$r_c = 0.25$			$r_c = 0.5$		
1	135.4	135.39	0.007	135.41	135.4	0.007	135.43	135.4	0.02
2	434.16	434.16	0	434.17	434.16	0.002	434.17	434.16	0.002
3	962.22	962.22	0	962.22	962.22	0	962.23	962.22	0.001
4	1706.2	1706.2	0	1706.2	1706.2	0	1706.2	1706.2	0
5	2664	2664	0	2664.1	2664	0.004	2664.1	2664	0.004
6	3835.2	3835.2	0	3835.3	3835.2	0.003	3835.3	3835.2	0.003
7	5219.6	5219.6	0	5219.6	5219.6	0	5219.6	5219.6	0
8	6817.1	6817.1	0	6817.1	6817.1	0	6817.1	6817.1	0
9	8627.7	8627.7	0	8627.7	8627.7	0	8627.7	8627.7	0
10	10651	10651	0	10651	10651	0	10651	10651	0
n	$r_c = 0.75$			$r_c = 1.0$			$r_c = 1.25$		
1	135.45	135.4	0.04	136.46	135.41	0.8	135.48	135.41	0.05
2	434.18	434.16	0.005	434.18	434.16	0.005	434.18	434.16	0.005
3	962.23	962.22	0.001	962.23	962.22	0.001	962.23	962.22	0.001
4	1706.2	1706.2	0	1706.2	1706.2	0	1706.2	1706.2	0
5	2664.1	2664	0.004	2664.1	2664	0.004	2664	2664	0

Continued on next page

Table 5.8 – continued from previous page

n	$r_c = 0.75$			$r_c = 1.0$			$r_c = 1.25$		
6	3835.3	3835.2	0.003	3835.3	3835.2	0.003	3835.2	3835.2	0
7	5219.6	5219.6	0	5219.6	5219.6	0	5219.6	5219.6	0
8	6817.1	6817.1	0	6817.1	6817.1	0	6817.1	6817.1	0
9	8627.7	8627.7	0	8627.7	8627.7	0	8627.7	8627.7	0
10	10651	10651	0	10651	10651	0	10651	10651	0
n	$r_c = 1.5$			$r_c = 1.75$			$r_c = 2.0$		
1	135.5	135.41	0.07	135.52	135.41	0.08	135.54	135.42	0.09
2	434.19	434.17	0.005	434.2	434.17	0.007	434.2	434.17	0.007
3	962.24	962.22	0.002	962.24	962.22	0.002	962.24	962.22	0.002
4	1706.2	1706.2	0	1706.2	1706.2	0	1706.2	1706.2	0
5	2664	2664	0	2664	2664	0	2664.1	2664.1	0
6	3835.2	3835.2	0	3835.2	3835.2	0	3835.3	3835.3	0
7	5219.6	5219.6	0	5219.6	5219.6	0	5219.6	5219.6	0
8	6817.1	6817.1	0	6817.1	6817.1	0	6817.1	6817.1	0
9	8627.7	8627.7	0	8627.7	8627.7	0	8627.7	8627.7	0
10	10651	10651	0	10651	10651	0	10651	10651	0

The comparison between analytical and numerical results of the natural frequencies of the beam on a nonlinear cubic elastic foundation showed in Tables 5.7, 5.8 and those in Appendix B produces minor differences. That confirms that the results from Abaqus are an excellent benchmark for the solution. Thus, the analytical results can be validated. The differences between analytical and numerical analyses could be because the former approximates the beam as a line and thus ignores the section change, while the FEA does not. However, it can also be seen that the finite element model does not accurately approximate analytical solutions for higher modes. The relative differences in each table explain that.

5.4 Effects of some Parameters on the Nonlinear Natural Frequencies of the BoNEF

This section explores the effects of the beam's vibration amplitude and that of the relative nonlinear stiffness ratio. Two cases of nonlinearities of the beam's foundation are investigated, the nonlinear quadratic and the nonlinear cubic elastic foundations.

5.4.1 Effect of the Beam's Vibration Amplitude

In this subsection, the effect of the beam's vibration amplitude on the ratio between the nonlinear natural frequencies Ω_q and Ω_c to the linear natural frequency ω_l is studied. The aim is to appreciate the variation of the nonlinear natural frequencies when the beam's amplitude increases because due to the change of stiffness, the natural frequency may be amplitude-dependent. The same numerical values of the nonlinear stiffness coefficients k_2 and k_3 are assigned. Figure 5.9 shows the plots of each case.

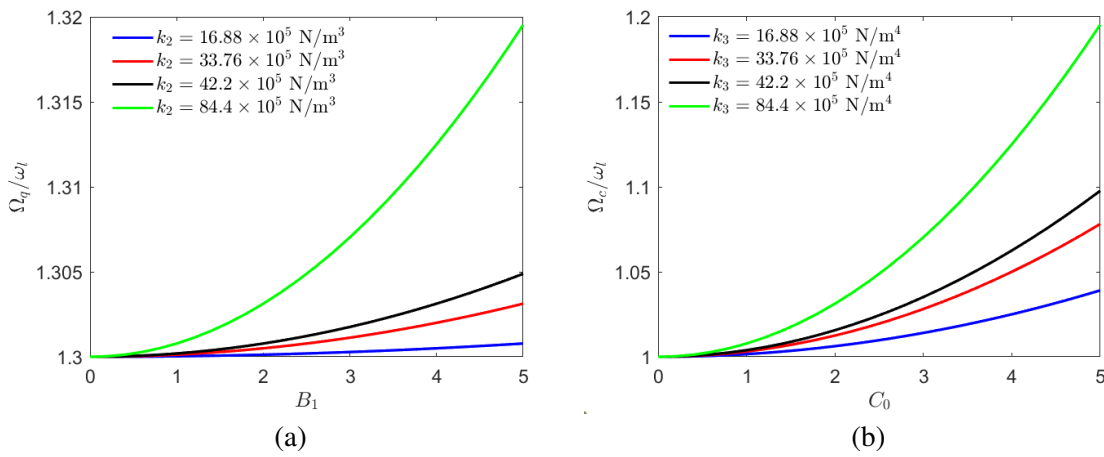


Figure 5.9: Effect of beam's vibration amplitude on natural frequency ratio at mode 2: (a) nonlinear quadratic elastic foundation and (b) nonlinear cubic elastic foundation.

It is seen from Figure 5.9 that the ratio of natural frequencies at mode two increases with the increase of the nonlinear stiffness ratio and with the vibration amplitude for both types of nonlinearities. This proves that the nonlinear natural frequency directly relates to the foundation's nonlinear stiffness. In addition, the nonlinear natural frequencies increase as the amplitudes B_1 and C_0 increase. This observation is the same as that reported by Kanani et al. (2014).

5.4.2 Effect of the Nonlinear Stiffness Ratio

This section analyses the variation of the nonlinear natural frequencies Ω_q and Ω_c as the nonlinear stiffness parameters of the foundation k_2 and k_3 increase, respectively. On the one hand, to observe the effects of k_2 on the nonlinear natural frequency Ω_q , an analytical parametric study is undertaken on Eqs. (5.3) and (5.4). To achieve this at the second mode

($n=2$), for example, the value of k_2 is selected, and α_2 is evaluated from the second of Eq. (5.4) while keeping A and ρS constant. Then, the value of k_l is selected, and ω_l is evaluated from the first of Eq. (5.4), keeping EI and ρS constant. Therefore, knowing α_2 and ω_l at mode 1, an evaluation of Ω_q is performed from Eq. (5.3), taking fixed values of ε and B_1 .

On the other hand, the effects of k_3 on the nonlinear natural frequency Ω_c are observed by performing a parametric study on Eqs. (5.8) and (5.9). This is done by selecting the value of k_3 and evaluating β_3 from the second of Eq. (5.9) while keeping B and ρS constant. Next, the value of k_l is selected, and ω_l is evaluated from the first of Eq. 5.9 at a given mode n , keeping EI and ρS constant. Hence, the values of β_3 and ω_l allow evaluating Ω_c from Eq. (5.8) at that mode, taking fixed values of ε and C_0 . The figure below displays the output in MATLAB.

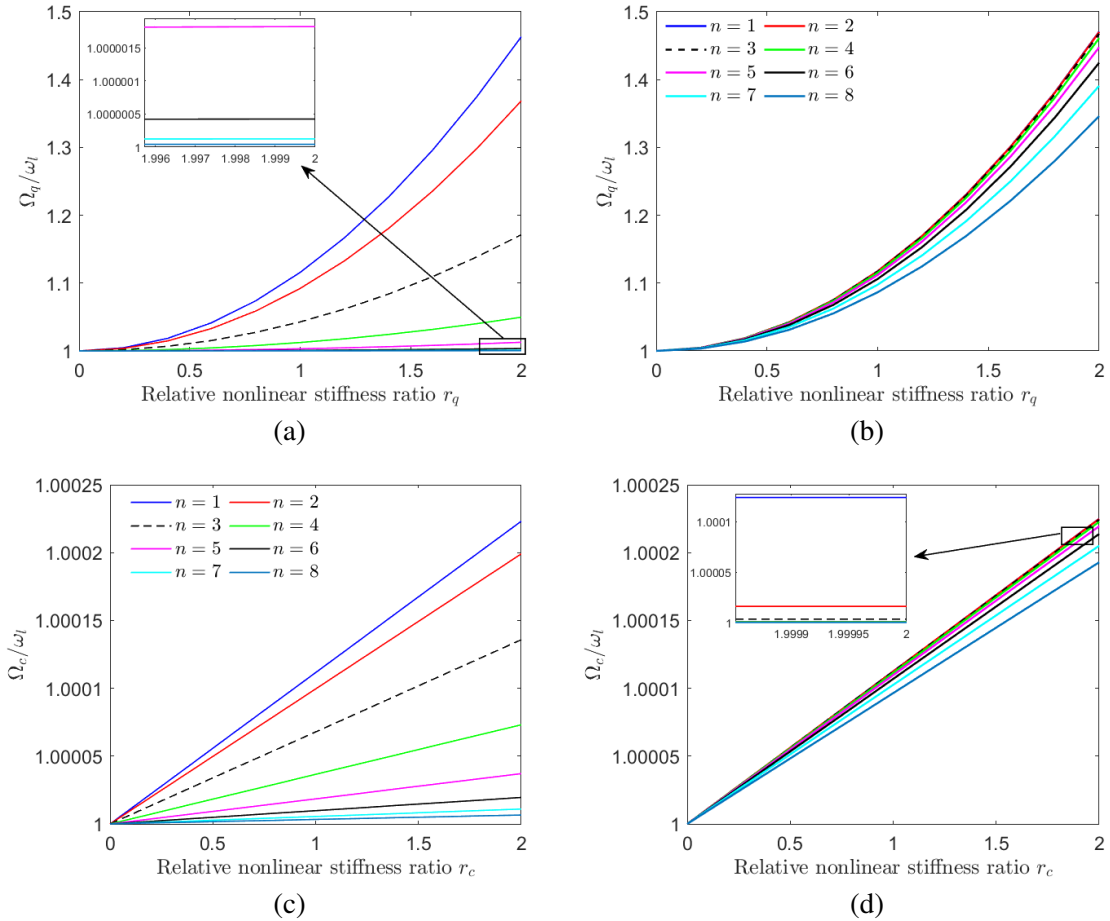


Figure 5.10: Plots of the dimensionless ratio of the natural frequency of the BoNEF: (a) and (c) for $k_l = 100 \times k_b$; (b) and (d) for $k_l = 10000 \times k_b$.

Figure 5.10 shows the plots of the first eight modes of the nonlinear natural frequency Ω_q (for the beam on a nonlinear quadratic elastic foundation) versus the relative nonlinear stiffness ratio r_q (see Figures 5.10a and 5.10b) and those of the nonlinear natural frequency Ω_c (for the beam on a nonlinear cubic elastic foundation) versus the relative nonlinear stiffness ratio r_c (see Figures 5.10c and 5.10d). For all modes presented, the natural frequencies increase as the nonlinear relative ratio increases. The plots in Figure 5.10 confirm that for fixed and variable linear stiffness k_l of the foundation, the nonlinear natural frequencies Ω_q and Ω_c will always increase with the relative nonlinear stiffness ratio r_q and r_c , respectively. Therefore, the nonlinear stiffness of the foundation plays a significant role in determining the nonlinear natural frequencies.

5.5 Summary and Conclusions

This chapter conducted a parametric study of the beam resting on a nonlinear elastic foundation. Two specific cases of nonlinear foundations were dealt with: the first involved quadratic nonlinearity, and the other concerned cubic nonlinearity. From the findings arise the conclusions below:

1. Although the same value was assigned to the ratio between each nonlinear stiffness and the linear stiffness of the foundation, the natural frequencies for both studies were similar. Also, the natural frequencies, considering the nonlinear cubic stiffness coefficient, were slightly smaller than those calculated with the nonlinear quadratic stiffness.
2. By defining a relation between the nonlinear stiffness and that of the linear stiffness by the mean of the parameter r_q or r_c , the amount of nonlinearity to the foundation was quantified, and the variation of the natural frequencies with nonlinearity was pictured. It was found that, for lower values of the ratio r_q or r_c (r_q or $r_c < 1$), the natural frequencies were more of the linear stiffness k_l . In contrast, the natural frequencies increased for r_q or $r_c > 1$ and represented more of the nonlinearity. Though the increase was minimal (for example at $n = 1$, it gave 0.26 for $r_q = 1.25$, 0.56 for $r_q = 1.5$, 0.93 for $r_q = 1.75$, and 1.35 for $r_q = 2$) and only noticeable in the first three modes, it was concluded that nonlinearity could not be overlooked. Similar increased values were obtained for values of r_c greater than 1.
3. The natural frequencies increased slightly, sensibly 0.3% with the nonlinearity, meaning for r_q or r_c varying from 1.25 to 2. Those obtained for all the modes greater than 3 were practically identical to those without nonlinearity. This was due to the small variation

of the nonlinear stiffness in relation to the linear stiffness for each new case. However, for cases with r_q or $r_c > 2$ (for example $r_q = 3$ or $r_c = 3$) it is confirmed that the increase of the nonlinear natural frequencies would be more significant and around 5%.

4. The displacement plots as a function of time revealed that, as the value of the ratio r_q increases, the period of vibration between two consecutive curves increases for the beam with quadratic nonlinearity in the foundation. In contrast, when the ratio r_c increased, the gap between two successive curves was the same for the beam with cubic nonlinearity in the foundation.

5. The extended parametric study was necessary to understand the effects of some system parameters on the nonlinear natural frequencies of the BoNEF. It was found that increasing the beam's vibration amplitude increases the nonlinear natural frequencies. Therefore, as the effective stiffness of the beam-foundation system increases, even if the vibration amplitude remains invariable, the natural frequencies will always increase. The designer can control the beam's amplitude by making the foundation strong enough to withstand the vibrations, by applying a control system, or by even implementing vibration absorbers to the beam's structure.

6. Comparing the natural frequencies obtained analytically to those computed from Abaqus showed tiny percentage of differences, with the mean of the relative differences for each case of the nonlinear stiffness ratios of the foundation r_q and r_c sensibly equal to 0.01%, almost negligible. This enabled us to consider the latter results as the benchmark and validate the analytical results.

7. Conducting this parametric study helped further to understand the purpose of nonlinearity to a great extent. Based on that study, the relative nonlinear ratio of the foundation stiffness was found to play a vital role in the variation of the nonlinear natural frequencies. Furthermore, the findings clearly show that quadratic nonlinearity influences the beam's vibration response by decreasing it, and the system's natural frequencies by increasing them more than cubic nonlinearity.

Chapter 6

Linear and Nonlinear Vibration of Plates on Elastic Foundations

6.1 Introduction

Plates are used in civil engineering to construct buildings, airports, and bridges. While a beam has a single bending moment resistance, a plate can resist bending on two axes. In addition to that, a plate has a twisting moment. The present chapter studies the free vibration of a plate on an elastic foundation (PoEF) having linear and quadratic and cubic stiffness nonlinearities. When studying the movement of a simple plate, it is easy to find the plate displacement knowing the boundary conditions of the plate and the external excitation acting on the system. Since the current study deals with free vibration only, an initial displacement will be imposed on the plate, and the plate response as time evolves will be captured.

This chapter is structured as follows: Section 6.2 focuses on a plate on a linear elastic foundation (PoLEF) and presents analytical results. The analytical study of plates resting on three types of nonlinear elastic foundations is conducted in Section 6.3. Section 6.4 carries out numerical studies using Finite Element Analysis (FEA) in Abaqus. Section 6.5 will compare analytical and numerical results, whereas Section 6.6 will compare linear results to nonlinear ones. In Section 6.7 the present analytical results are compared to the beam study results. The last section will conclude the chapter.

6.2 Study of a Plate on a Linear Elastic Foundation

The vibration of plates is essential in studying practical systems such as bridges, decks, hydraulic structures, pavements of highways and airport runways, aeroplanes, and machine parts (Elzein, 2013). This section concentrates on the analytical study of a plate on a linear elastic foundation (PoLEF). The model considered in this chapter is adapted from work by Hetényi (1971) and shown in Figure 6.1.

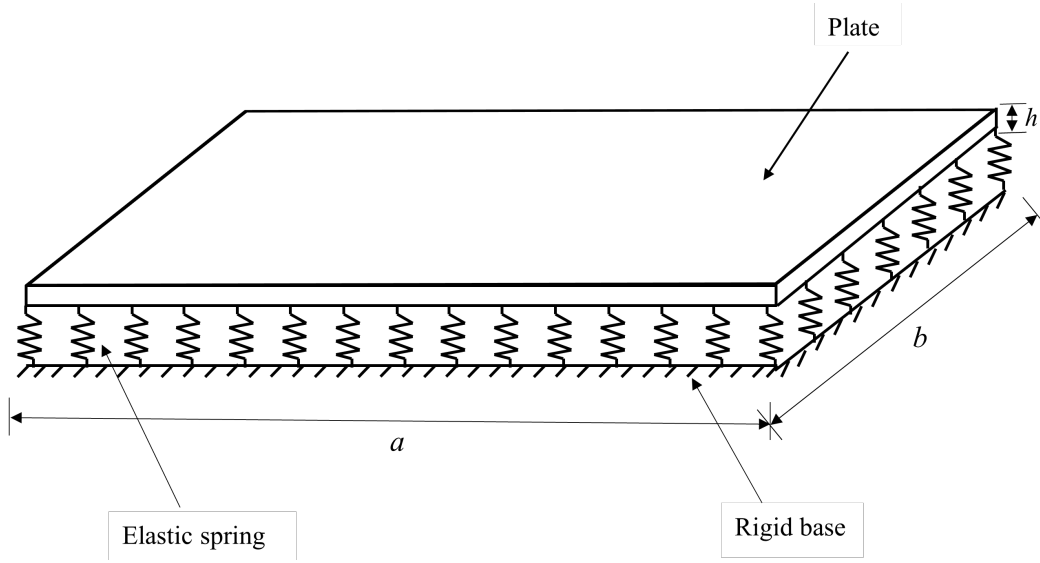


Figure 6.1: Plate resting on an elastic foundation.

Figure 6.1 shows a plate resting on elastic springs, which form the foundation of the plate. The elastic springs of the foundation have linear stiffness properties. The thickness of the plate is h , while a and b are the length and width of the plate. Then, the distance separating two consecutive springs is assumed to be the same throughout this chapter. The plate and the foundation are on top of the rigid base. The equation of motion of the system depicted in Figure 6.1 is as follows (Raju and Rao, 1990):

$$\rho h \frac{\partial^2 W}{\partial t^2} + D \left(\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) + k_l W = 0, \quad (6.1)$$

where k_l stands for the linear stiffness coefficient of the elastic foundation and $W = W(x, y, t) = W_h(x, y, t) + W_p(x, y, t)$ is the transversal displacement of the plate, with $W_h(x, y, t)$ the solution of the homogeneous equation and $W_p(x, y, t)$ the solution of the particular equation. The

parameter D is known as the plate's flexural rigidity, and its expression is given as follows.

$$D = \frac{Eh^3}{12(1 - \nu^2)} \quad (6.2)$$

For beams, $D = EI$. The first and third terms of Eq. (6.1) relate to the inertial effects and the foundation resistance respectively, while the second term is the well-known biharmonic formulation for the static equilibrium of a rectangular plate experiencing small elastic deformations and supported only at the edges (Timoshenko et al., 1959; Zingoni et al., 2000).

6.2.1 Solution of the Equation of Motion

Using the separation of variables, the solution to Eq. (6.1) is assumed to be (Dawkins, 2018; Myint-U and Debnath, 2007):

$$W_H(x, y, t) = \phi(x, y)T_h(t), \quad (6.3)$$

where $\phi(x, t)$ represents the spatial part of the solution, whereas $T_h(t)$ represents the temporal part. By putting Eq. (6.3) into Eq. (6.1), it gives:

$$\rho h \phi \frac{\partial^2 T_h}{\partial t^2} + D T_h \left(\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right) + k_l \phi T_h = 0 \quad (6.4)$$

Eq. (6.4) can also be written as (Raju and Rao, 1990):

$$\phi \frac{\partial^2 T_h}{\partial t^2} + \left[\frac{D}{\rho h} \left(\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right) + \frac{k_l}{\rho h} \phi \right] T_h = 0 \quad (6.5)$$

which may be re-arranged to give:

$$\frac{1}{\phi} \left[\frac{D}{\rho h} \left(\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right) + \frac{k_l}{\rho h} \phi \right] = -\frac{1}{T_h} \frac{\partial^2 T_h}{\partial t^2}. \quad (6.6)$$

Eq. (6.6) holds if, and only if, both sides are equal to the same constant. If that constant is denoted μ_{mn}^2 , we may write:

$$\frac{1}{\phi} \left[\frac{D}{\rho h} \left(\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right) + \frac{k_l}{\rho h} \phi \right] = -\frac{1}{T_h} \frac{\partial^2 T_h}{\partial t^2} = \mu_{mn}^2. \quad (6.7)$$

Therefore, Eq. (6.7) can be split into two differential equations: one with the variable T and the other with the variable ϕ . These are respectively given by:

$$\frac{d^2 T_h}{dt^2} + \mu_{mn}^2 T_h = 0, \quad (6.8a)$$

and

$$\left[\frac{D}{\rho h} \left(\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right) + \frac{k_l}{\rho h} \phi \right] - \mu_{mn}^2 \phi = 0. \quad (6.8b)$$

In Eq. (6.8a) the right derivatives are used because the only variable here is T , which depends only on one parameter t . The second equation (Eq. (6.8b)) remains with partial derivatives because the variable ϕ depends on two parameters x and y .

On the other hand, it can be noticed that Eq. (6.8a) is a linear differential equation to the second-order in T ; its solution is known from the mathematical developments and equals to:

$$T_h(t) = A_{mn} \cos(\mu_{mn} t) + B_{mn} \sin(\mu_{mn} t), \quad (6.9)$$

where A_{mn} and B_{mn} are amplitude coefficients which will be determined later using the initial conditions of the problem, and μ_{mn} will be used to find the natural frequencies of the system.

As for Eq. (6.8b), it is a linear partial differential equation, but to the fourth-order with two parameters x and y . The equation to be solved is defined as follows:

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} - \frac{1}{D} (\rho h \mu_{mn}^2 - k_l) \phi = 0. \quad (6.10)$$

To proceed, Eq. (6.10) is transformed by assuming the solution of the equation to be (El-Zafrany et al., 1995):

$$\phi(x, y) = X(x)Y(y), \quad (6.11)$$

By replacing Eq. (6.11) into Eq. (6.10) and rearranging it, it holds:

$$\underbrace{\frac{1}{X(x)} \frac{\partial^4 X(x)}{\partial x^4}}_{a(x)} = \underbrace{-\frac{2}{X(x)Y(y)} \frac{\partial^4 X(x)Y(y)}{\partial x^2 \partial y^2} - \frac{1}{Y(y)} \frac{\partial^4 Y(y)}{\partial y^4} + \frac{1}{D} (\rho h \mu_{mn}^2 - k_l)}_{b(x,y)}. \quad (6.12)$$

Eq. (6.12) holds only if $a(x) = b(x, y) = \text{constant}$. Calling that constant λ^4 it follows that:

$$\frac{1}{X(x)} \frac{\partial^4 X(x)}{\partial x^4} = -\frac{2}{X(x)Y(y)} \frac{\partial^4 X(x)Y(y)}{\partial x^2 \partial y^2} - \frac{1}{Y(y)} \frac{\partial^4 Y(y)}{\partial y^4} + \frac{1}{D} (\rho h \mu_{mn}^2 - k_l) = \lambda^4. \quad (6.13)$$

Therefore, Eq. (6.12) can be transformed into the following two equations (El-Zafrany et al., 1995):

$$\frac{\partial^4 X(x)}{\partial x^4} - \lambda^4 X(x) = 0, \quad (6.14a)$$

$$-\frac{2}{X(x)Y(y)} \frac{\partial^4 X(x)Y(y)}{\partial x^2 \partial y^2} - \frac{1}{Y(y)} \frac{\partial^4 Y(y)}{\partial y^4} + \frac{1}{D}(\rho h \mu_{mn}^2 - k_l) - \lambda^4 = 0. \quad (6.14b)$$

Since Eq. (6.14a) depends only on one parameter x , the partial derivatives are transformed into right derivatives, giving:

$$\frac{d^4 X(x)}{dx^4} - \lambda^4 X(x) = 0, \quad (6.15)$$

It follows that the characteristic equation satisfying Eq. (6.15) is:

$$r^4 - \lambda^4 = 0, \text{ meaning that } r = \pm\lambda, \pm i\lambda, \quad (6.16)$$

where i is the complex number. Then, $X(x)$ is deduced as:

$$X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + c_3 e^{i\lambda x} + c_4 e^{-i\lambda x}, \quad (6.17)$$

where $c_i (i = 1, \dots, 4)$ are integration constants.

However, the relations between the exponential function, cosine, sine, and hyperbolic cosine and sine functions allow to rewrite $X(x)$ as:

$$X(x) = C_1(\cos \lambda x + \cosh \lambda x) + C_2(\cos \lambda x - \cosh \lambda x) + C_3(\sin \lambda x + \sinh \lambda x) + C_4(\sin \lambda x - \sinh \lambda x), \quad (6.18)$$

where $C_j (j = 1, 2, \dots, 4)$ can be determined using the boundary conditions of the plate.

The model considered is that of a simply supported plate at all its edges so that the deflection and bending moments at all edges are equal to zero. The boundary conditions satisfying that type of plate are defined as follows (Zingoni et al., 2000):

$$\left\{ \begin{array}{l} (W_h)_{x=0} = (U_h)_{x=a} = 0, \quad (W_h)_{y=0} = (U_h)_{y=b} = 0, \\ \left(\frac{\partial^2 W_h}{\partial x^2} + \nu \frac{\partial^2 W_h}{\partial y^2} \right)_{x=0} = 0 \quad \text{and} \quad \left(\frac{\partial^2 W_h}{\partial x^2} + \nu \frac{\partial^2 W_h}{\partial y^2} \right)_{x=a} = 0, \\ \left(\frac{\partial^2 W_h}{\partial y^2} + \nu \frac{\partial^2 W_h}{\partial x^2} \right)_{y=0} = 0 \quad \text{and} \quad \left(\frac{\partial^2 W_h}{\partial y^2} + \nu \frac{\partial^2 W_h}{\partial x^2} \right)_{y=b} = 0. \end{array} \right. \quad (6.19)$$

In Eq. (6.19), ν stands for Poisson's ratio. These boundary conditions are applied using the expression of $X(x)$ given in Eq. (6.18). Then,

$$(W_h)_{x=0} = 0 \Rightarrow (X)_{x=0} = 0 \Rightarrow 2C_1 = 0 \Rightarrow C_1 = 0 \quad (6.20)$$

On the other hand:

$$\left(\frac{\partial^2 W_h}{\partial x^2} + \nu \frac{\partial^2 W_h}{\partial y^2} \right)_{x=0} = 0 \Rightarrow \left(Y(y) \frac{\partial^2 X(x)}{\partial x^2} + \nu X(x) \frac{\partial^2 Y(y)}{\partial y^2} \right)_{x=0} = 0. \quad (6.21)$$

Knowing the second derivative of $X(x)$ with respect to x , we can write:

$$\left(\frac{\partial^2 X(x)}{\partial x^2} \right)_{x=0} = 0 \Rightarrow -2\lambda^2 C_2 = 0 \Rightarrow C_2 = 0. \quad (6.22)$$

Since $C_1 = C_2 = 0$, the other boundary conditions (at the coordinate $x = a$) are used to find C_3 and C_4 . This allows solving the following system:

$$\begin{cases} C_3(\sin \lambda a + \sinh \lambda a) + C_4(\sin \lambda a - \sinh \lambda a) = 0 \\ C_3(-\sin \lambda a + \sinh \lambda a) + C_4(-\sin \lambda a - \sinh \lambda a) = 0 \end{cases} \quad (6.23)$$

It should be highlighted that the interest here is in non-trivial solutions. These are the solution with C_3 and C_4 all different from zero. Then, the system of equations Eq. (6.23) has non-trivial solutions if its determinant is equal to zero, meaning that:

$$\det \begin{vmatrix} \sin \lambda a + \sinh \lambda a & \sin \lambda a - \sinh \lambda a \\ -\sin \lambda a + \sinh \lambda a & -\sin \lambda a - \sinh \lambda a \end{vmatrix} = 0. \quad (6.24)$$

That implies:

$$\begin{aligned} -(\sin \lambda a + \sinh \lambda a)^2 + (\sin \lambda a - \sinh \lambda a)^2 &= 0 \\ \Rightarrow -4 \sin \lambda a \sinh \lambda a &= 0. \end{aligned} \quad (6.25)$$

Because $\sinh \lambda a$ always exist and is different from zero, Eq. (6.25) holds only if:

$$\sin \lambda a = 0 \Rightarrow \lambda a = m\pi \Rightarrow \lambda = \frac{m\pi}{a}, \quad m \in \mathbb{N}^* \quad (6.26)$$

By replacing $\sin \lambda a$ into one of the equations in Eq. (6.23), it gives:

$$C_3 = C_4 \quad (6.27)$$

Hence $X(x)$ is found to be $2C_3 \sin \lambda x$, which can be written as (Leissa, 1969)

$$X(x) = C_m \sin \lambda x = C_m \sin \left(\frac{m\pi}{a} x \right). \quad (6.28)$$

Since both $X(x)$ and $Y(y)$ play a symmetric role, the procedure to derive the expression of $Y(y)$ is the same as that to find $X(x)$. Therefore, the expression for $Y(y)$ is deduced as:

$$Y(y) = E_n \sin \lambda y = E_n \sin \left(\frac{n\pi}{b} y \right), \quad n \in \mathbb{N}^* \quad (6.29)$$

Putting together Eqs. (6.28) and (6.29), the spatial solution $\phi(x, y)$ is given by:

$$\phi(x, y) = C_m E_n \sin \left(\frac{m\pi}{a} x \right) \sin \left(\frac{n\pi}{b} y \right). \quad (6.30)$$

The expression of $\phi(x, y)$ is used to determine the expression of the natural frequency μ_{mn} . To do so, $\phi(x, y)$ (given in Eq. (6.30)) and its derivatives are replaced into Eq. (6.8b). The equation is as follows:

$$\frac{D}{\rho h} \left(\frac{\partial^4 \phi(x, y)}{\partial x^4} + 2 \frac{\partial^4 \phi(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi(x, y)}{\partial y^4} \right) + \frac{k_l}{\rho h} \phi(x, y) - \mu_{mn}^2 \phi(x, y) = 0. \quad (6.31)$$

By replacing the respective derivatives of $\phi(x, y)$ into Eq. (6.31) it holds:

$$\frac{D}{\rho h} \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2 \phi(x, y) + \frac{k_l}{\rho h} \phi(x, y) - \mu_{mn}^2 \phi(x, y) = 0. \quad (6.32)$$

By simplifying $\phi(x, y)$ from each term of Eq. (6.32), it happens that μ_{mn} is deduced as:

$$\mu_{mn}^2 = \frac{D}{\rho h} \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2 + \frac{k_l}{\rho h}. \quad (6.33)$$

Considering that $\mu = 2\pi\omega_{mn}$, the natural frequencies of a rectangular PoLEF can be calculated according to the following equation:

$$\omega_{mn} = \sqrt{\left\{ \frac{D\pi^2}{4\rho h} \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right]^2 + \frac{k_l}{4\pi^2 \rho h} \right\}}. \quad (6.34)$$

By combining Eqs. (6.9) and (6.30), the expression of the displacement $U(x, y, t)$, solution to the governing equation of motion of the plate on a linear elastic foundation (Eq. (6.1)) is deduced as:

$$W(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} C_m E_n \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) [A_{mn} \cos(\omega_{mn}t) + B_{mn} \sin(\omega_{mn}t)]. \quad (6.35)$$

For the sake of simplification, Eq. (6.35) can also be rewritten as:

$$W(x, y, t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} W_0 \sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right) [A_{mn} \cos(\omega_{mn}t) + B_{mn} \sin(\omega_{mn}t)], \quad (6.36)$$

where $W_0 = C_m E_n$ stands for the spatial amplitude of vibration of the plate.

The solution of Eq. (6.1) given by Eq. (6.36) is only valid for a PoLEF. If the plate is excited, the expression of the force is replaced in Eq. (6.1), and the resulting equation is solved for both homogeneous and particular equations. However, forced vibration is not of interest in this thesis. On the other hand, if the elastic foundation is nonlinear, the term indicating the type of nonlinearity is added to Eq. (6.1). A plate with a nonlinear elastic foundation will be investigated later in this chapter.

In this chapter, the plate used for any calculations will have the properties in the table below.

Table 6.1: Properties of the steel plate for analytical and numerical studies of the PoEF.

Item	Notation	Value
Young's modulus	E	200 GPa
Length	a	2 m
Width	b	0.5 m
Thickness	h	0.04 m
Poisson's ratio	ν	0.3
Mass density	ρ	8050 kg/m ³

These values are the same as those used in the beam case. Therefore, given the current study on a plate, how will the natural frequencies change from 2D modelling to 3D modelling? Using Young's modulus, Poisson's ratio, and the plate thickness, the plate flexural rigidity is found as:

$$D = \frac{2 \times 10^{11} \times 0.04^3}{12(1 - 0.3^2)} = 1.1722 \times 10^6 \text{ Pa} \cdot \text{m}^3. \quad (6.37)$$

6.2.2 Natural Frequencies and Displacement of the Plate at Midspan

Knowing the model's parameters as shown in Table 6.1, the flexural rigidity $D = 1.722 \times 10^6$ Pa·m³, and the linear stiffness coefficient of the foundation $k_l = 1.69 \times 10^5$ N/m², the natural frequencies at which the plate vibrates are calculated using Eq. (6.34). These are given in Table 6.2.

Table 6.2: Natural frequencies of the plate without and with a linear elastic foundation obtained from analytical calculations.

Mode's number	ω_{nm} (rad/s)	
	No foundation	Linear foundation
1	402.79	402.9
2	473.87	474.0
3	592.33	592.4
4	758.19	758.23
5	971.43	971.5
6	1232.1	1232.1
7	1540.1	1540.1
8	1611.2	1611.2
9	1729.6	1729.6
10	1895.5	1895.5

In Table 6.2 the natural frequencies obtained for the plate without foundation and those for a plate on a linear elastic foundation are displayed. The last section defined all the parameters used to obtain those values. As for the linear stiffness coefficient, this was taken as $k_l = 1.69 \times 10^5$ N/m². It is clear that if the foundation is calibrated with higher soil stiffness, the natural frequencies will increase. One should note that the mode number for a plate does not vary like that of the beam.

Because the plate has two parameters m and n that determine the mode's number, one parameter was fixed while the other was given different values to calculate the natural frequencies of the plate. Therefore, the natural frequencies in Table 6.2 were obtained by fixing $n = 1$ and varying m , then $n = 2$ and varying m . The same procedure will be followed to compute the natural frequencies of all other plate systems in this chapter. Figure 6.2 exhibits the temporal displacement of the plate supported by a linear elastic foundation.

Figure 6.2 shows the temporal plots of the plate displacement at midspan. It can be seen that the maximum amplitude of vibration is close to 1 for all the six modes presented. To obtain these plots, only the displacement's temporal was used part since the spatial part's

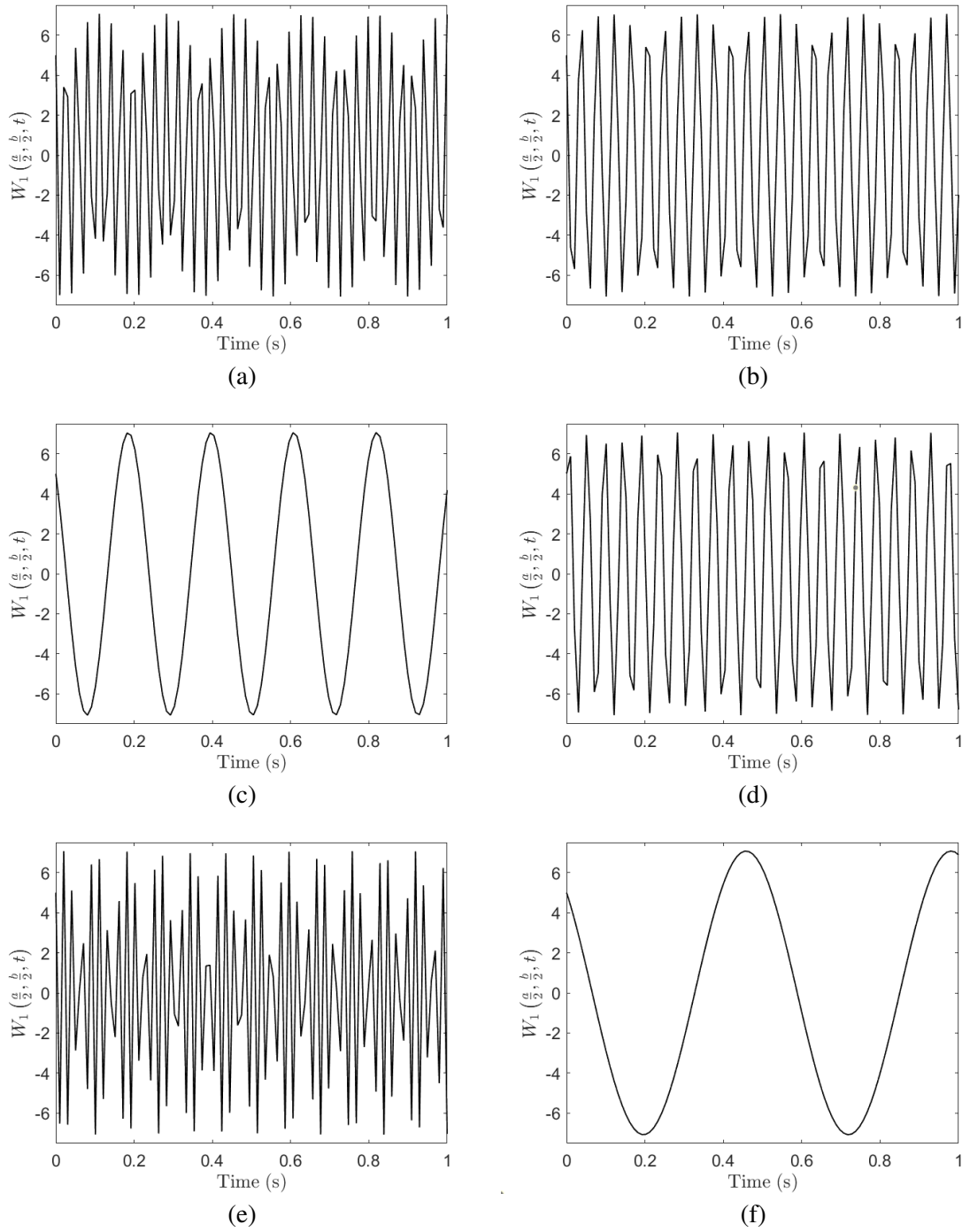


Figure 6.2: Temporal displacement at the centre of the PoLEF.

amplitude oscillates between -1 and 1. The next section investigates the vibrations of the plate supported on a nonlinear elastic foundation. Three types of nonlinearities will be explored.

6.3 Analytical Study of a Plate on a Nonlinear Elastic Foundation

The plate to be studied in the current section has the same design as presented in 6.3. However, the difference depends on the type of foundation. The former had a linear elastic foundation, whereas the present example has nonlinear stiffness properties. The linear equation given in Eq. (6.1) is now transformed into:

$$\rho h \frac{\partial^2 W}{\partial t^2} + D \left(\frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4} \right) + k_{nl} W = 0, \quad (6.38)$$

k_{nl} is the nonlinear stiffness coefficient. Three types of nonlinearities are investigated: the first is quadratic, the second is cubic, and the third combines quadratic and cubic nonlinearities.

6.3.1 Foundation having Quadratic Stiffness Properties

For a plate on a nonlinear quadratic elastic foundation, the nonlinearity is given by (Nayfeh and Mook, 1995; Tomlinson and Worden, 2000):

$$k_{nl} = k_l + k_2 W_q, \quad (6.39)$$

so that Eq. (6.38) is modified into:

$$\rho h \frac{\partial^2 W_q}{\partial t^2} + D \left(\frac{\partial^4 W_q}{\partial x^4} + 2 \frac{\partial^4 W_q}{\partial x^2 \partial y^2} + \frac{\partial^4 W_q}{\partial y^4} \right) + k_l W_q + k_2 W_q^2 = 0. \quad (6.40)$$

Eq. (6.40) is a nonlinear differential equation because of $k_2 W_q^2$. Consequently, its solution is not as straightforward as that of the linear case studied in Section 6.3. W_q , k_l , and k_2 are, respectively, the plate transversal displacement, the linear and nonlinear quadratic stiffness coefficients.

Here also, the solution of Eq. (6.40) has both spatial and temporal parts. That is assumed to be (Clough, 1993):

$$W_q(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} W_{ij}(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varphi_{ij}(x, y) \tau_{qij}(t), \quad i, j \in \mathbb{N}^*. \quad (6.41)$$

Next, the plate is supposed simply supported. This means that the spatial solution of Eq. (6.40) is known and given by (Leissa, 1969):

$$\varphi_{ij}(x, y) = X_i \sin\left(\frac{i\pi}{a}x\right) Y_j \sin\left(\frac{j\pi}{b}y\right), \quad i, j \in \mathbb{N}^*, \quad (6.42)$$

where X_i and Y_i are spatial amplitudes which can be determined using the boundary conditions of the problem.

By putting the derivatives of $\varphi(x, y)$ and the square expressions of $W(x, y, t)$ into Eq. (6.40), the partial derivatives with respect to time become the right derivatives and it holds:

$$\begin{aligned} \rho h X_i Y_j \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{j\pi}{b}y\right) \frac{d^2 \tau_q(t)}{dt^2} + D \left\{ \left[\left(\frac{i\pi}{a}\right)^2 + \left(\frac{j\pi}{b}\right)^2 \right]^2 + k_l \right\} \tau_q(t) \times \\ X_i Y_j \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{j\pi}{b}y\right) + k_2 X_i^2 Y_j^2 \sin^2\left(\frac{i\pi}{a}x\right) \sin^2\left(\frac{j\pi}{b}y\right) \tau_q(t)^2 = 0. \end{aligned} \quad (6.43)$$

Then, by applying orthogonality to Eq. (6.43). the whole equation is multiplied by $\sin\left(\frac{m\pi}{a}x\right) \sin\left(\frac{n\pi}{b}y\right)$ and is integrated over the plate's ends as follows:

$$\begin{aligned} \frac{d^2 \tau_q(t)}{dt^2} \rho h X_i Y_j \int_0^a \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) dx \int_0^b \sin\left(\frac{j\pi}{b}y\right) \sin\left(\frac{n\pi}{b}y\right) dy + \\ \left\{ D \left[\left(\frac{i\pi}{a}\right)^2 + \left(\frac{j\pi}{b}\right)^2 \right]^2 + k_l \right\} X_i Y_j \tau_q(t) \int_0^a \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) dx \times \\ \int_0^b \sin\left(\frac{j\pi}{b}y\right) \sin\left(\frac{n\pi}{b}y\right) dy + \\ \tau_q(t)^2 k_2 X_i^2 Y_j^2 \int_0^a \sin^2\left(\frac{i\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) dx \int_0^b \sin^2\left(\frac{j\pi}{b}y\right) \sin\left(\frac{n\pi}{b}y\right) dy = 0. \end{aligned} \quad (6.44)$$

Then, the integrals are calculated such that:

$$\int_0^a \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) dx = \begin{cases} \frac{a}{2} & \text{if } i = m \neq 0 \\ 0 & \text{if } i \neq m \end{cases} \quad (6.45a)$$

$$\int_0^b \sin\left(\frac{j\pi}{b}y\right) \sin\left(\frac{n\pi}{b}y\right) dy = \begin{cases} \frac{b}{2} & \text{if } j = n \neq 0 \\ 0 & \text{if } j \neq n \end{cases} \quad (6.45b)$$

$$\int_0^a \sin^2 \left(\frac{i\pi}{a} x \right) \sin \left(\frac{m\pi}{a} x \right) dx = \frac{1}{2} \int_0^a \left[1 - \cos \left(\frac{2i\pi}{a} x \right) \right] \sin \left(\frac{m\pi}{a} x \right) dx$$

$$= \frac{a}{m\pi} (1 - (-1)^m) + \frac{a}{4\pi} \left[\frac{(-1)^{2i-m}}{2i-m} - \frac{(-1)^{2i+m}}{2i+m} \right] \quad (6.45c)$$

$$\int_0^b \sin^2 \left(\frac{j\pi}{b} y \right) \sin \left(\frac{n\pi}{b} y \right) dy = \frac{1}{2} \int_0^b \left[1 - \cos \left(\frac{2j\pi}{b} y \right) \right] \sin \left(\frac{n\pi}{b} y \right) dy$$

$$= \frac{b}{n\pi} (1 - (-1)^n) + \frac{b}{4\pi} \left[\frac{(-1)^{2j-n}}{2j-n} - \frac{(-1)^{2j+n}}{2j+n} \right] \quad (6.45d)$$

When Eqs. (6.45a) to (6.45d) are put into Eq. (6.44) it happens that:

$$\frac{ab}{4} \rho h X_i Y_j \frac{d^2 \tau(t)}{dt^2} + \frac{ab}{4} \left\{ D \left[\left(\frac{i\pi}{a} \right)^2 + \left(\frac{j\pi}{b} \right)^2 \right]^2 + k_l \right\} X_i Y_j \tau(t) +$$

$$\tau(t)^2 k_2 X_i^2 Y_j^2 \left\{ \frac{a}{m\pi} (1 - (-1)^m) + \frac{a}{4\pi} \left[\frac{(-1)^{2i-m}}{2i-m} - \frac{(-1)^{2i+m}}{2i+m} \right] \right\} \times$$

$$\left\{ \frac{b}{n\pi} (1 - (-1)^n) + \frac{b}{4\pi} \left[\frac{(-1)^{2j-n}}{2j-n} - \frac{(-1)^{2j+n}}{2j+n} \right] \right\} = 0. \quad (6.46)$$

By assuming $i = m$ and $j = n$, Eq. (6.46) is transformed into:

$$\frac{ab}{4} \rho h \frac{d^2 \tau_q(t)}{dt^2} + \frac{ab}{4} \left\{ \frac{D}{\rho h} \left[\left(\frac{i\pi}{a} \right)^2 + \left(\frac{j\pi}{b} \right)^2 \right]^2 + \frac{k_l}{\rho h} \right\} \tau_q(t) +$$

$$\frac{4X_i Y_j}{ab \rho h} k_2 \tau_q(t)^2 \left\{ \frac{a}{i\pi} (1 - (-1)^i) + \frac{a}{4\pi} \left[\frac{(-1)^i}{i} - \frac{(-1)^{3i}}{3i} \right] \right\} \times$$

$$\left\{ \frac{b}{j\pi} (1 - (-1)^j) + \frac{b}{4\pi} \left[\frac{(-1)^j}{j} - \frac{(-1)^{3j}}{3j} \right] \right\} = 0. \quad (6.47)$$

For the sake of simplification, the following parameters are set:

$$\mu_{ij}^2 = \frac{D}{\rho h} \left[\left(\frac{i\pi}{a} \right)^2 + \left(\frac{j\pi}{b} \right)^2 \right]^2 + \frac{k_l}{\rho h} \Rightarrow \mu_{ij} = \sqrt{\frac{1}{\rho h} \left\{ k_l + D \left[\left(\frac{i\pi}{a} \right)^2 + \left(\frac{j\pi}{b} \right)^2 \right]^2 \right\}} \quad (6.48a)$$

$$\alpha_{ij} = \frac{4X_i Y_j}{\rho h \pi} k_2 \left\{ \frac{1}{i} (1 - (-1)^i) + \frac{1}{4} \left[\frac{(-1)^i}{i} - \frac{(-1)^{3i}}{3i} \right] \right\} \times \left\{ \frac{1}{j} (1 - (-1)^j) + \frac{1}{4} \left[\frac{(-1)^j}{j} - \frac{(-1)^{3j}}{3j} \right] \right\}, \quad (6.48b)$$

The flexural rigidity D keeps its value given in Eq. (6.2). If $i = m$ and $j = n$, the expression of μ_{ij} is similar to Eq. (6.33). As for α_{ij} , it will be calculated later. Therefore, with these parameters, Eq. (6.47) is reduced to:

$$\ddot{\tau}_q + \mu_{ij}^2 \tau_q(t) + \alpha_{ij} \tau_q(t)^2 = 0, \quad (6.49)$$

where $\ddot{\tau}_q$ stands for the second derivative of τ_q concerning the time. Now, it remains to find the solution of Eq. (6.49), which is a nonlinear ordinary differential equation to the second-order in τ_q . The straightforward expansion method (He, 2004; Nayfeh and Mook, 1995) is used, as described in the previous chapters. Eq. (6.49), it is the same equation as Eq. (4.17) solved in Chapter 4 using the same method. Only the names of the variables differ, and the subscripts of the parameters are now two (i and j). Due to this, there is no need to solve Eq. (6.49) again, but we adopt the same solution of Eq. (4.17) knowing that the present study is that of a plate in comparison to that of Chapter 4 which was on a beam.

Thus, the solution of Eq. (6.49) is found to be:

$$\begin{aligned} \tau_{qij}(t) = & \varepsilon A_1 \cos(\Omega_q t) + \frac{\alpha_{ij}}{6\mu_{ij}^2} \varepsilon^2 A_1^2 \left[2\cos(\Omega_q t) - 3 + \cos(2\Omega_q t) \right] - \\ & \frac{\alpha_{ij}^2}{3\mu_{ij}^4} \varepsilon^3 A_1^3 \left\{ \frac{23}{48} \cos(\Omega_q t) + \frac{1}{3} [\cos(2\Omega_q t) - 3] + \frac{1}{16} \cos(3\Omega_q t) \right\} \end{aligned} \quad (6.50)$$

where ε is a small parameter ($\varepsilon \ll 1$) equal to 0.01, A_1 is the amplitude of vibration of the plate, which value can be obtained by applying the initial conditions. Hence, the overall solution to Eq. (6.40) is deduced as:

$$\begin{aligned} W_q(x, y, t) = & \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{j\pi}{b}y\right) \left\{ \varepsilon A_1 \cos(\Omega_q t) + \frac{\alpha_{ij}\varepsilon^2}{6\mu_{ij}^2} A_1^2 \left[2\cos(\Omega_q t) - 3 + \right. \right. \\ & \left. \left. \cos(2\Omega_q t) \right] - \frac{\alpha_{ij}^2\varepsilon^3}{3\mu_{ij}^4} A_1^3 \left[\frac{23}{48} \cos(\Omega_q t) + \frac{1}{3} (\cos(2\Omega_q t) - 3) + \frac{1}{16} \cos(3\Omega_q t) \right] \right\}. \end{aligned} \quad (6.51)$$

It should be highlighted that Ω_q where the subscript q stands for quadratic, is the nonlinear natural frequency of the entire nonlinear plate system given by:

$$\Omega_q = \omega_{ij} \left[1 + \frac{\alpha_{ij}^2}{3\omega_{ij}^4} \varepsilon^2 A_1^2 \right] = \sqrt{\left\{ \frac{D\pi^2}{4\rho h} \left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2 \right]^2 + \frac{k}{4\pi^2\rho h} \right\} \left[1 + \frac{\alpha_{ij}^2}{3\omega_{ij}^4} \varepsilon^2 A_1^2 \right]}. \quad (6.52)$$

Eq. (6.51) is the solution of the equation governing the free vibration of a PoEF having linear and nonlinear quadratic stiffness properties. The difference between this solution and that obtained in the beam analysis under the same elastic foundation properties relies upon the spatial part of the solution. Here, there is another term with the coordinate y , which is $\sin\left(\frac{j\pi}{b}y\right)$. Consequently, plate problems are three-dimensional due to two spatial variables x and y , and one temporal variable t . The upcoming subsection offers the study of a plate on a nonlinear cubic elastic foundation.

6.3.2 Foundation having Cubic Stiffness Properties

This second case has the following nonlinearity (Tomlinson and Worden, 2000):

$$k_{nl} = k_l + k_3 W_c^2, \quad (6.53)$$

So, Eq. (6.38) is modified into:

$$\rho h \frac{\partial^2 W_c}{\partial t^2} + D \left(\frac{\partial^4 W_c}{\partial x^4} + 2 \frac{\partial^4 W_c}{\partial x^2 \partial y^2} + \frac{\partial^4 W_c}{\partial y^4} \right) + k_l W_c + k_3 W_c^3 = 0. \quad (6.54)$$

where W_c , k_l , and k_3 are respectively the transversal displacement of the plate due to the nonlinear cubic elastic foundation, the linear, and nonlinear cubic stiffness coefficients. Eq. (6.54) is a nonlinear differential equation due to the term $k_3 W_c^3$. The solution is assumed to have the following expression:

$$W_c(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} W_{ij}(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varphi_{ij}(x, y) \tau_{cij}(t), \quad i, j \in \mathbb{N}^*, \quad (6.55)$$

where $\varphi_{ij}(x, y)$ has the same expression as given in Eq (6.42). Supposing that the plate is simply supported, the spatial part $\varphi(x, y)$ is known. Then replacing the derivatives of $\varphi(x, y)$ and the cubic of $W_c(x, y, t)$ into Eq. (6.54), the equation below is obtained:

$$\begin{aligned} \rho h X_i Y_j \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{j\pi}{b}y\right) \frac{d^2 \tau_c(t)}{dt^2} + D \left[\left(\frac{i\pi}{a}\right)^2 + \left(\frac{j\pi}{b}\right)^2 \right]^2 \sin\left(\frac{i\pi}{a}x\right) \times \\ \sin\left(\frac{j\pi}{b}y\right) X_i Y_j \tau_c(t) + k_l X_i Y_j \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{j\pi}{b}y\right) \tau(t) + \\ k_3 X_i^3 Y_j^3 \sin^3\left(\frac{i\pi}{a}x\right) \sin^3\left(\frac{j\pi}{b}y\right) \tau_c(t)^3 = 0. \end{aligned} \quad (6.56)$$

After applying orthogonality to Eq. (6.56), the following integrals of trigonometric functions need to be found to simplify the resulting equation:

$$\int_0^a \sin^3\left(\frac{i\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) dx = -\frac{3}{4} \int_0^a \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{m\pi}{a}x\right) dx = \begin{cases} -\frac{3a}{8} & \text{if } i = m \neq 0 \\ 0 & \text{if } i \neq m \end{cases} \quad (6.57a)$$

$$\int_0^b \sin^3\left(\frac{j\pi}{b}y\right) \sin\left(\frac{n\pi}{b}y\right) dy = -\frac{3}{4} \int_0^b \sin\left(\frac{j\pi}{b}y\right) \sin\left(\frac{n\pi}{b}y\right) dy = \begin{cases} -\frac{3b}{8} & \text{if } j = n \neq 0 \\ 0 & \text{if } j \neq n \end{cases} \quad (6.57b)$$

Replacing Eqs. (6.57a) and (6.57b) into Eq. (6.56) it reduces to:

$$\rho h \frac{d^2 \tau_c(t)}{dt^2} + \left\{ \frac{D}{\rho h} \left[\left(\frac{i\pi}{a} \right)^2 + \left(\frac{j\pi}{b} \right)^2 \right]^2 + \frac{k_l}{\rho h} \right\} \tau_c(t) + \frac{9}{16\rho h} k_3 X_i^2 Y_j^2 \tau_c(t)^3 = 0. \quad (6.58)$$

In Eq. (6.58), the parameter μ_{ij} set before in Subsection 6.3.1 can be identified from the second term. As for the third term, a new parameter β_{ij} is set so that:

$$\beta_{ij} = \frac{9}{16\rho h} k_3 X_i^2 Y_j^2. \quad (6.59)$$

Therefore, Eq. (6.58) is simplified as:

$$\ddot{\tau}_c + \mu_{ij}^2 \tau_c(t) + \beta_{ij} \tau_c(t)^3 = 0, \quad (6.60)$$

where $\ddot{\tau}$ stands for the second derivative of τ concerning the time. As for the parameter β_{ij} , it will be calculated later. Now, it remains to find the solution of Eq. (6.60), which is a nonlinear ODE to the second-order in τ with cubic nonlinearity. Moreover, it is the same equation as Eq. (4.73) solved in Chapter 4 using the same method. So, the same solution of Eq. (4.73) is adopted, and the solution of Eq. (6.60) is deduced as being:

$$\tau_{cij}(t) = B_1 \cos(\Omega_c t) + \varepsilon \left[-\frac{\beta_{ij}}{32\mu_{ij}^2} B_1^3 \cos(\Omega_c t) + \frac{\beta_{ij}}{32\mu_{ij}^2} B_1^3 \cos(3\Omega_c t) \right], \quad (6.61)$$

where B_1 is the plate's amplitude of vibration, which will be obtained shortly by applying the initial conditions. Hence, putting (6.55) and (6.61) together, the overall solution of Eq.

(6.54) is found as:

$$W_c(x, y, t) = X_i \sin\left(\frac{i\pi}{a}x\right) Y_j \sin\left(\frac{j\pi}{b}y\right) \left\{ B_1 \cos(\Omega_c t) - \frac{\beta_{ij}\varepsilon}{32\mu_{ij}^2} B_1^3 \left[\cos(\Omega_c t) - \cos(3\Omega_c t) \right] \right\}. \quad (6.62)$$

Ω_c where the subscript c stands for cubic, is the nonlinear natural frequency of the entire system with cubic nonlinearity of the foundation given by:

$$\Omega_c = \omega_{ij} \left[1 + \frac{3\beta_{ij}}{8\omega_{ij}^2} \varepsilon B_1^2 \right] = \sqrt{\left\{ \frac{D\pi^2}{4\rho h} \left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2 \right]^2 + \frac{k}{4\pi^2\rho h} \right\} \left[1 + \frac{3\beta_{ij}}{8\omega_{ij}^2} \varepsilon B_1^2 \right]}. \quad (6.63)$$

Eq. (6.62) is the solution of the equation governing the free vibration of a plate resting on a linear and nonlinear cubic elastic foundation. The formula for nonlinear natural frequencies given in Eq. (6.63) has the parameter β_{ij} which includes the cubic stiffness coefficient k_3 . The subsequent subsection investigates a plate supported on an elastic foundation with quadratic and cubic stiffness nonlinearities.

6.3.3 Foundation having Quadratic and Cubic Stiffness Properties

The nonlinearity of the foundation to be considered in this subsection combines those investigated in the former subsections. That is given by:

$$k_{nl} = k_l + k_2 W_{qc} + k_3 W_{qc}^2, \quad (6.64)$$

so that in the absence of external excitation, Eq. (6.38) is modified into:

$$\rho h \frac{\partial^2 W_{qc}}{\partial t^2} + D \left(\frac{\partial^4 W_{qc}}{\partial x^4} + 2 \frac{\partial^4 W_{qc}}{\partial x^2 \partial y^2} + \frac{\partial^4 W_{qc}}{\partial y^4} \right) + k_l W_{qc} + k_2 W_{qc}^2 + k_3 W_{qc}^3 = 0. \quad (6.65)$$

Eq. (6.65) is a nonlinear differential equation due to the presence of the terms $k_2 W_{qc}^2$ and $k_3 W_{qc}^3$. Consequently, its solution is not trivial. Thanks to the SEM, the approximate solution can be found. Using the separation of variables, the general solution of Eq. (6.65) is assumed as:

$$W_{qc}(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} W_{ij}(x, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \phi_{ij}(x, y) \tau_{qcij}(t), \quad i, j \in \mathbb{N}^*. \quad (6.66)$$

Replacing Eq. (6.66) into Eq. (6.65) gives:

$$\begin{aligned} \rho h X_i Y_j \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{j\pi}{a}y\right) \frac{d^2 \tau_{qc}(t)}{dt^2} + D \left[\left(\frac{i\pi}{a}\right)^2 + \left(\frac{j\pi}{b}\right)^2 \right]^2 X_i Y_j \tau_{qc}(t) \times \\ \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{j\pi}{a}y\right) + k_l X_i Y_j \sin\left(\frac{i\pi}{a}x\right) \sin\left(\frac{j\pi}{a}y\right) \tau_{qc}(t) + k_2 X_i^2 Y_j^2 \tau(t)^2 \times \\ \sin^2\left(\frac{i\pi}{a}x\right) \sin^2\left(\frac{j\pi}{a}y\right) + k_3 X_i^3 Y_j^3 \sin^3\left(\frac{i\pi}{a}x\right) \sin^3\left(\frac{j\pi}{a}y\right) \tau_{qc}(t)^3 = 0. \end{aligned} \quad (6.67)$$

By following the same procedure like in Subsections 6.3.1 and 6.3.2, the following equation is obtained:

$$\begin{aligned} \frac{ab}{4} \rho h \frac{d^2 \tau_{qc}(t)}{dt^2} + \frac{ab}{4} \left\{ \frac{D}{\rho h} \left[\left(\frac{i\pi}{a}\right)^2 + \left(\frac{j\pi}{b}\right)^2 \right]^2 + \frac{k_l}{\rho h} \right\} \tau_{qc}(t) + \\ \frac{4X_i Y_j}{ab \rho h} k_2 \tau_{qc}(t)^2 \left\{ \frac{a}{i\pi} (1 - (-1)^i) + \frac{a}{4\pi} \left[\frac{(-1)^i}{i} - \frac{(-1)^{3i}}{3i} \right] \right\} \times \left\{ \frac{b}{j\pi} (1 - (-1)^j) + \right. \\ \left. \frac{b}{4\pi} \left[\frac{(-1)^j}{j} - \frac{(-1)^{3j}}{3j} \right] \right\} + \frac{9}{64 \rho h} k_3 X_i^2 Y_j^2 \tau_{qc}(t)^3 = 0. \end{aligned} \quad (6.68)$$

Moreover, simplifying Eq. (6.68) gives:

$$\ddot{\tau}_{qc} + \mu_{ij}^2 \tau_{qc}(t) + \alpha_{ij} \tau_{qc}(t)^2 + \beta_{ij} \tau_{qc}(t)^3 = 0, \quad (6.69)$$

To solve Eq. (6.69), the SEM is also used. Interestingly, Eq. (6.69) is similar to Eq. (4.104) solved in Chapter 4 using the same method. Therefore, the solution is adapted here according to the parameters in Eq. (6.69) as:

$$\begin{aligned} \tau_{qcij}(t) = \varepsilon C_1 \cos(\Omega_{qc} t) + \frac{\alpha_{ij}}{6\mu_{ij}^2} \varepsilon^2 C_1^2 \left[2 \cos(\Omega_{qc} t) - 3 + \cos(2\Omega_{qc} t) \right] + \\ \varepsilon^3 C_1^3 \left[\left(\frac{-29\alpha_{ij}^2}{144\mu_{ij}^4} + \frac{\beta_{ij}}{32\mu_{ij}^2} \right) \cos(\Omega_{qc} t) + \frac{\alpha_{ij}^2}{9\mu_{ij}^4} \cos(2\Omega_{qc} t) - \right. \\ \left. \frac{\alpha_{ij}^2}{3\mu_{ij}^4} + \frac{2\alpha_{ij}^2 + 3\beta_{ij}\mu_{ij}^2}{96\mu_{ij}^4} \cos(3\Omega_{qc} t) \right], \end{aligned} \quad (6.70)$$

where C_1 is the amplitude of vibration of the plate to be determined by using the initial conditions. By combining Eqs. (6.42) and (6.70), the complete solution of Eq. (6.65) governing the free vibration of a PoEF having linear and quadratic and nonlinear cubic

stiffness properties is:

$$W_{qc}(x, y, t) = X_i \sin\left(\frac{i\pi}{a}x\right) Y_j \sin\left(\frac{j\pi}{b}y\right) \left\{ \varepsilon C_1 \cos(\Omega_{qc}t) + \frac{\alpha_{ij}}{6\mu_{ij}^2} \varepsilon^2 C_1^2 \left[-3 + 2 \cos(\Omega_{qc}t) + \cos(2\Omega_{qc}t) \right] + \varepsilon^3 C_1^3 \left[\left(\frac{-29\alpha_{ij}^2}{144\mu_{ij}^4} + \frac{\beta_{ij}}{32\mu_{ij}^2} \right) \cos(\Omega_{qc}t) + \frac{\alpha_{ij}^2}{9\mu_{ij}^4} \cos(2\Omega_{qc}t) - \frac{\alpha_{ij}^2}{3\mu_{ij}^4} + \frac{2\alpha_{ij}^2 + 3\beta_{ij}\mu_{ij}^2}{96\mu_{ij}^4} \cos(3\Omega_{qc}t) \right] \right\}, \quad (6.71)$$

where Ω_{qc} is the nonlinear natural frequency of the system with mixed parity nonlinearity given by:

$$\Omega_{qc} = \omega_{ij} \left[1 + \frac{9\beta_{ij}\omega_{ij}^2 - 10\alpha_{ij}^2}{24\omega_{ij}^4} \varepsilon^2 C_1^2 \right] \\ = \sqrt{\left\{ \frac{D\pi^2}{4\rho h} \left[\left(\frac{i}{a}\right)^2 + \left(\frac{j}{b}\right)^2 \right]^2 + \frac{k_l}{4\pi^2\rho h} \right\}} \left[1 + \frac{9\beta_{ij}\omega_{ij}^2 - 10\alpha_{ij}^2}{24\omega_{ij}^4} \varepsilon^2 C_1^2 \right]. \quad (6.72)$$

In the three nonlinear cases studied above, it could be highlighted that the temporal equation for a plate resting on a nonlinear elastic foundation is like the beam with the same foundation properties. Only the plate's spatial or overall solution (spatial and temporal combined) is modified due to another coordinate y . Moreover, the parameters that define the temporal part of the solution are distinct for the plate and the beam. That modification can also be seen in the formulas of the natural frequency. The following section will provide analytical results.

6.3.4 Results of Analytical Studies

This section reports the natural frequencies issued from the analytical solution of the plate resting on a nonlinear elastic foundation. Moreover, the corresponding plots to the above-mentioned nonlinear cases will be revealed. First, a programme is written in MATLAB to calculate the respective natural frequencies and then these are used to perform the plots. In the table below, the parameters' values used are recorded.

Table 6.3: Parameters' values used as input in MATLAB to calculate the natural frequencies analytically.

Parameters and their values					
k_1	$1.69 \times 10^5 \text{ N/m}^2$	ϵ	0.01	X_1	2
k_2	$8.44 \times 10^5 \text{ N/m}^3$	A_1	100	Y_1	2
k_3	$1.3 \times 10^7 \text{ N/m}^4$	B_1	0.5	$\rho \cdot h$	322 kg/m^2
β_{ij}	$3.6 \times 10^5 \text{ N/kg} \cdot \text{m}^2$	C_1	30	D	$1.17 \times 10^6 \text{ Pa} \cdot \text{m}^3$

On the one hand, Table 6.3 puts in view the parameters with their respective constant values, which were used as input in MATLAB. Those values remain constant throughout the rest of the chapter unless otherwise specified.

Table 6.4: Values of α_{ij} and the nonlinear natural frequencies of the PoEF calculated analytically.

Mode	$\alpha_{ij} \text{ (N/kg} \cdot \text{m}^2)$	$\Omega_q \text{ (rad/s)}$	$\Omega_c \text{ (rad/s)}$	$\Omega_{qc} \text{ (rad/s)}$
1	224280	413.1	419.9	432.1
2	10194	474.3	488.4	509.1
3	74758	601.4	604.0	619.2
4	5097.2	758.3	767.3	780.2
5	44855	972.2	978.5	988.5
6	3398.1	1232.2	1237.6	1245.6
7	32039	1540.2	1544.5	1550.9
8	463.4	1611.2	1615.4	1621.5
9	3398.1	1729.6	1733.6	1739.3
10	231.7	1895.5	1899.1	1904.3

On the other hand, Table 6.4 reveals the parameter α_{ij} due to nonlinear quadratic stiffness and the plate's nonlinear natural frequencies calculated analytically. The natural frequencies for the first ten modes are presented. It is noted that the natural frequencies in the presence of nonlinearity increase as the nonlinearity changes from quadratic to the mixed parity passing by cubic.

Figure 6.3 depicts the plots of the temporal transversal displacement of the plate $W(x, y, t)$ obtained from analytical calculations in MATLAB. The curves corresponding to each of the three nonlinear foundation types are shown. The plots reveal that the plate with the quadratic-cubic elastic foundation has a minimal amplitude than the plate with other nonlinear foundations. That confirms that the structure can withstand more deformation when the foundation is robust.

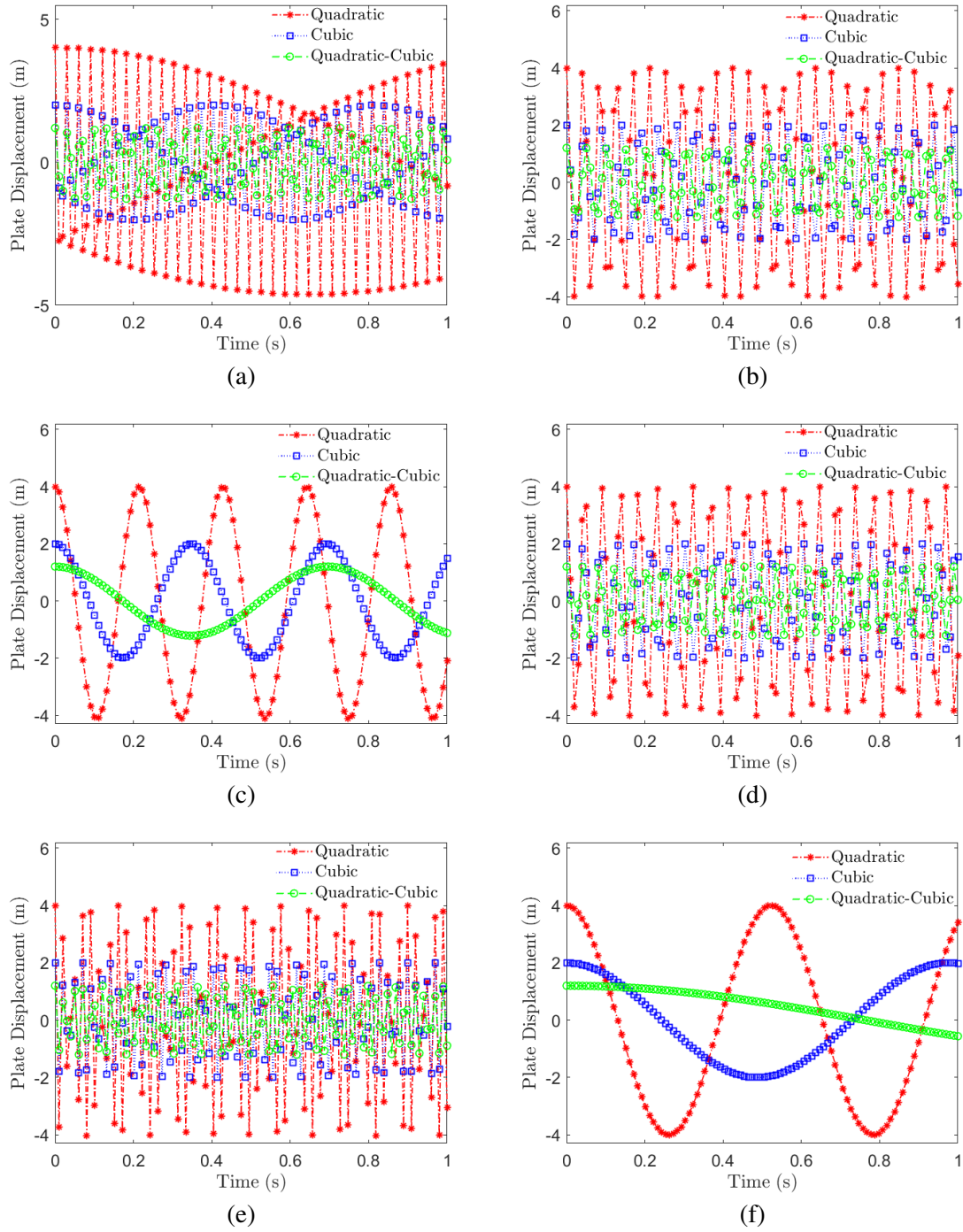


Figure 6.3: Analytical temporal displacement at the centre of the PoNEF: (a) to (f) are modes 1 to 6.

6.3.5 Linear and Nonlinear Natural Frequencies of the Plate on an Elastic Foundation with the same stiffness values

The table below shows the natural frequencies of the PoEF having linear, nonlinear quadratic and nonlinear cubic stiffness properties. These values are obtained by considering the same numerical value for the linear and nonlinear stiffness parameters equal to 8.45×10^6 .

Table 6.5: Natural frequencies in (rad/s) of the PoEF for $k_1 = k_2 = k_3$.

Mode's number	No Foundation	Linear	Nonlinear		
	ω_0	ω_l	Ω_q	Ω_c	Ω_{qc}
1	402.79	403.6	424.3	414.6	438.5
2	473.87	474.6	475.9	483.9	504.1
3	592.3	592.9	598.7	600.4	624.5
4	758.18	758.6	758.7	764.5	793.9
5	971.43	971.8	974.7	976.3	998.2
6	1232.1	1232.3	1232.3	1235.9	1254.0
7	1540.1	1540.3	1540.7	1543.2	1557.5
8	1611.2	1611.4	1611.4	1614.1	1628.0
9	1729.6	1729.8	1729.8	1732.4	1745.3
10	1895.5	1895.6	1895.6	1898.0	1909.8

Table 6.5 reveals that by using the same numerical value for the linear and nonlinear stiffness parameters of the foundation, the nonlinear natural frequencies of the PoEF will always be greater than the linear ones.

In conducting a study using one method exclusively, one may find it challenging to validate the results with 100% accuracy. For this reason, numerical studies must be conducted to validate the above analytical results. These are achieved using FEA in Abaqus.

6.4 Numerical Modelling

The procedure followed in the plate analysis is slightly different from that in the previous chapters, which concentrated on studying and modelling a 2D beam. Since the present study analyses a 3D plate, additional requirements are considered for the modelling. The general equation of finite element for the transverse deflection of a thin plate is:

$$[\mathbf{M}] \{\ddot{\mathbf{q}}\} + [\mathbf{K}] \{\mathbf{q}\} = 0, \quad (6.73)$$

where $[\mathbf{M}]$ is the mass matrix, $[\mathbf{K}]$ is the stiffness matrix, $\{\ddot{\mathbf{q}}\}$ is the nodal acceleration vector, and $\{\mathbf{q}\}$ is the nodal displacement vector. Note that the matrix $[\mathbf{K}]$ has different values depending on the type of foundation. For a nonlinear foundation, the stiffness matrix is more complex, and Eq. (6.73) is modified accordingly and becomes:

$$[\mathbf{M}] \{\ddot{\mathbf{q}}\} + [\mathbf{K}_{\text{nonl}}] \{\mathbf{q}\} = 0, \quad (6.74)$$

with $[\mathbf{K}_{\text{nonl}}]$ standing for the nonlinear stiffness matrix of the entire system. The entries of the matrices $[\mathbf{M}]$, $\{\mathbf{q}\}$, and $[\mathbf{K}_{\text{nonl}}]$ are not of interest in this thesis and will not be determined. Therefore, the modelling in Abaqus (Documentation Abaqus, 2019; SIMULIA, 2017a,b) is the next step.

6.4.1 Tasks required to perform the Numerical Analysis

To obtain the results in Abaqus, the software is first launched, and one proceeds as described in the tasks below (Documentation Abaqus, 2019; SIMULIA, 2017a,b):

Task 1: Create a 3D Deformable solid and input five as approximate global size. Then, in the *Part module*, draw a rectangle in the sketcher with a length of 2 m and width of 0.5 m and click done. In the pop-up menu that appears, input 0.04 for extrusion. Moreover, create a surface and select the bottom face of the plate perpendicular to the z -direction. That surface will serve when defining the elastic foundation. At the end of this module, a 3D solid part is created. Figures 6.4a and 6.4b depict the outcome in the viewport. Next, we move to the *Property module*.

Task 2: In the *Property module*, the material is defined, and the properties of a steel beam are assigned: the density, Young's modulus, and Poisson's coefficient. Then, a solid homogeneous section is created and assigned to the part created.

Task 3: In the *Assembly module*, Instances is double-clicked on, and the part created before is selected. Then, independent (mesh on part) is clicked on, and the module is completed by clicking on OK.

Task 4: Since the modelling is that of a plate on an elastic foundation, the most crucial step is the one at which to define the elastic foundation. That is done in the *Interaction module* as follows: create an interaction at the initial step, choose Elastic foundation and

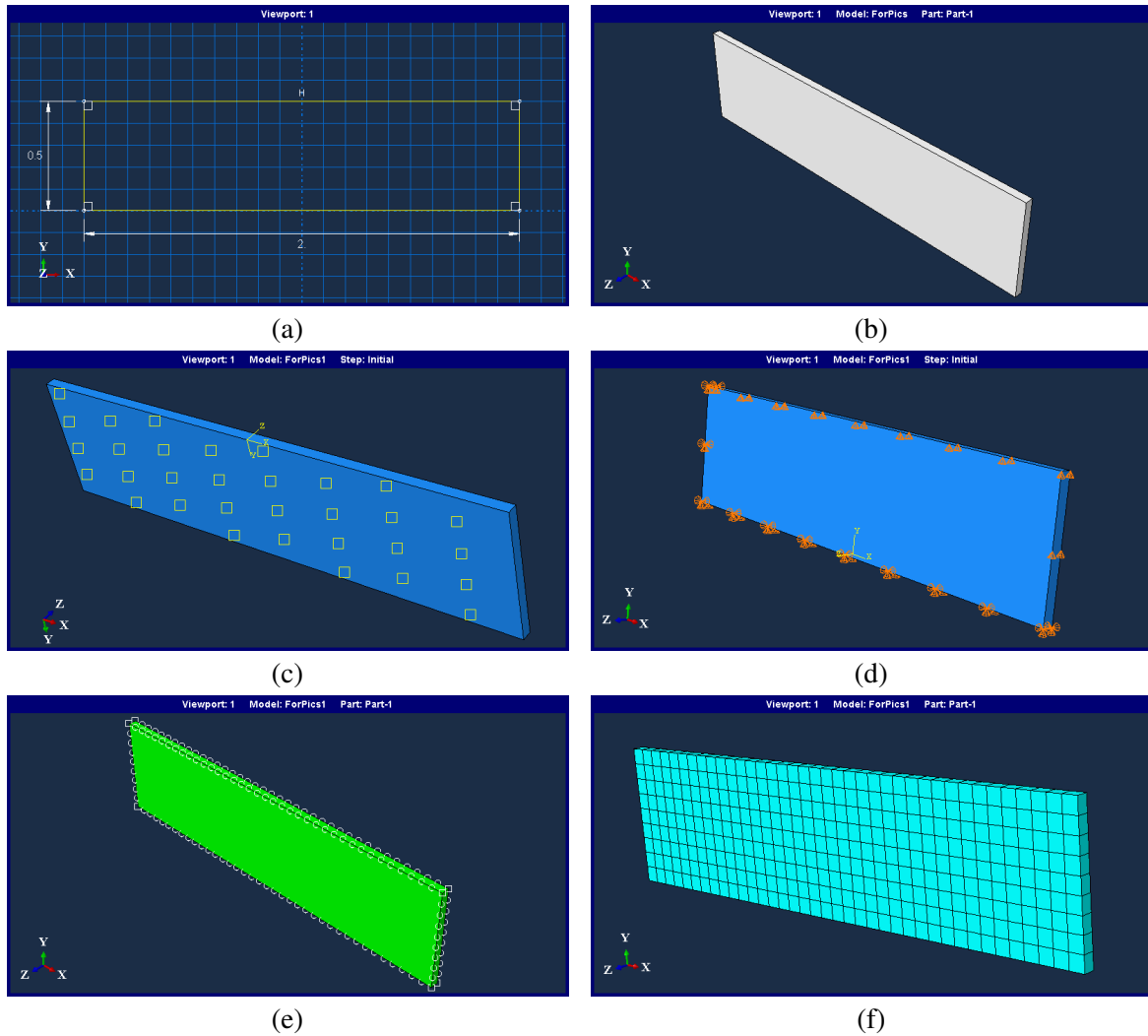


Figure 6.4: Different views of the plate: (a) section sketch of the plate; (b) the part created; (c) plate with the elastic foundation; (d) plate with simply supported boundary conditions; (e) plate with seeds; and (f) plate meshed.

click Continue. Next, in the viewport, select the surface created in **Task 1** and enter the value of the linear stiffness coefficient. We now move to the *Step module*.

Figure 6.5 details how to define the foundation of the 3D plate in Abaqus using the *Interaction module*. At the end of the *Interaction module*, the foundation is assigned to the plate. However, this is only for the plate supported on a linear elastic foundation. Concerning the nonlinear case, it will be described later. 6.4c portrays the plate with the linear foundation. The next module is the *Step module*.

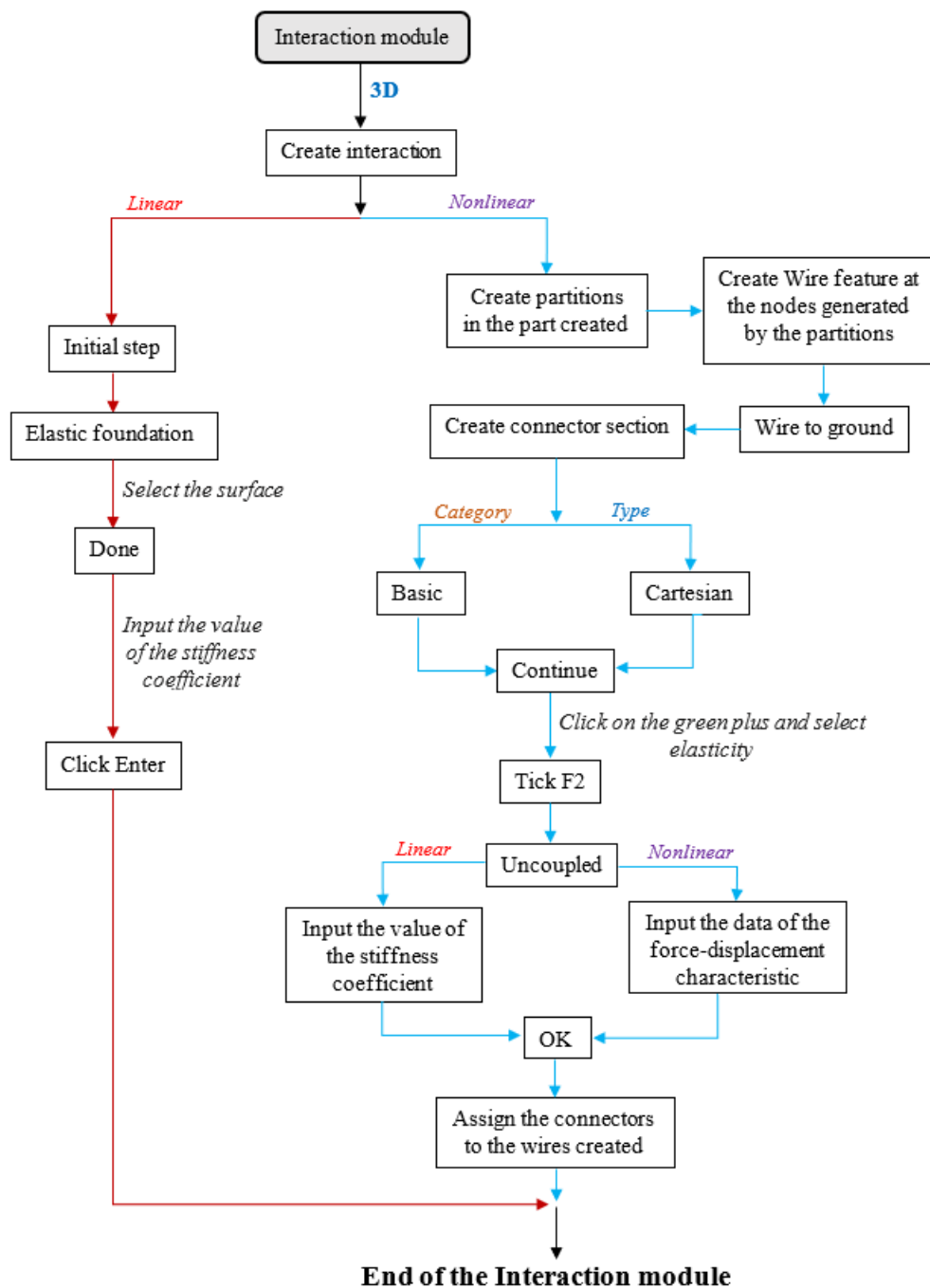


Figure 6.5: Flowchart of the Interaction module: procedure to add the foundation to a 3D model in Abaqus.

Task 5: In the *Step module*, ten natural frequencies are requested, and the output is required to be displacement normalised. Then, the simply supported boundary conditions are created to the plate in the *Load module*. This follows by clicking on create boundary

condition and ticking $U1$, and $U2$ for the left and bottom edges of the plate and only $U2$ for the right and top edges of the plate. The outcome is shown in Figure 6.4d.

Task 6: In the *Mesh module*, the part is seeded with the size of 0.053 (see Figure 6.4e), and C3D8R is used as element types (standard 3D stress, reduced integration) which is an 8-node linear brick, reduced integration with hourglass control. Here, the thick plate element is regarded as a thin plate. Figure 6.4f shows the resulting mesh with 342 elements.

Task 7: After meshing the part, the *Job module* was used to create a job and submit it. Finally, the job completed, and the results could be visualised using the *Visualization module*. The results obtained are presented in the subsequent subsection.

6.4.2 Numerical Results for Linear Elastic Foundation

This subsection presents the results from analysing the free vibration of a PoLEF in Abaqus. The table below contains the natural frequencies recorded from the analysis.

Table 6.6: Natural frequencies of the plate recorded from the linear analysis in Abaqus.

Mode's number	ω_{nm} (rad/s)	
	No foundation	Linear foundation
1	401.09	401.72
2	479.99	480.51
3	593.83	594.25
4	733.49	733.82
5	895.36	895.62
6	1229.7	1230.1
7	1451.7	1451.8
8	1509.7	1509.8
9	1623.4	1623.6
10	1823.9	1824.1

In Table 6.6 the first ten modes of linear natural frequencies are displayed. The natural frequencies of the plate with a linear elastic foundation are slightly similar to those without a foundation. However, it is clear that increasing the linear stiffness coefficient k_l will expand the linear natural frequencies. Therefore, the first six mode shapes of the plate taken from the viewport in Abaqus are given below.

In Figure 6.6 the mode shapes of the beam resting on a linear elastic foundation are displayed. On each sub-figure, one can count the number of modes. The maximum

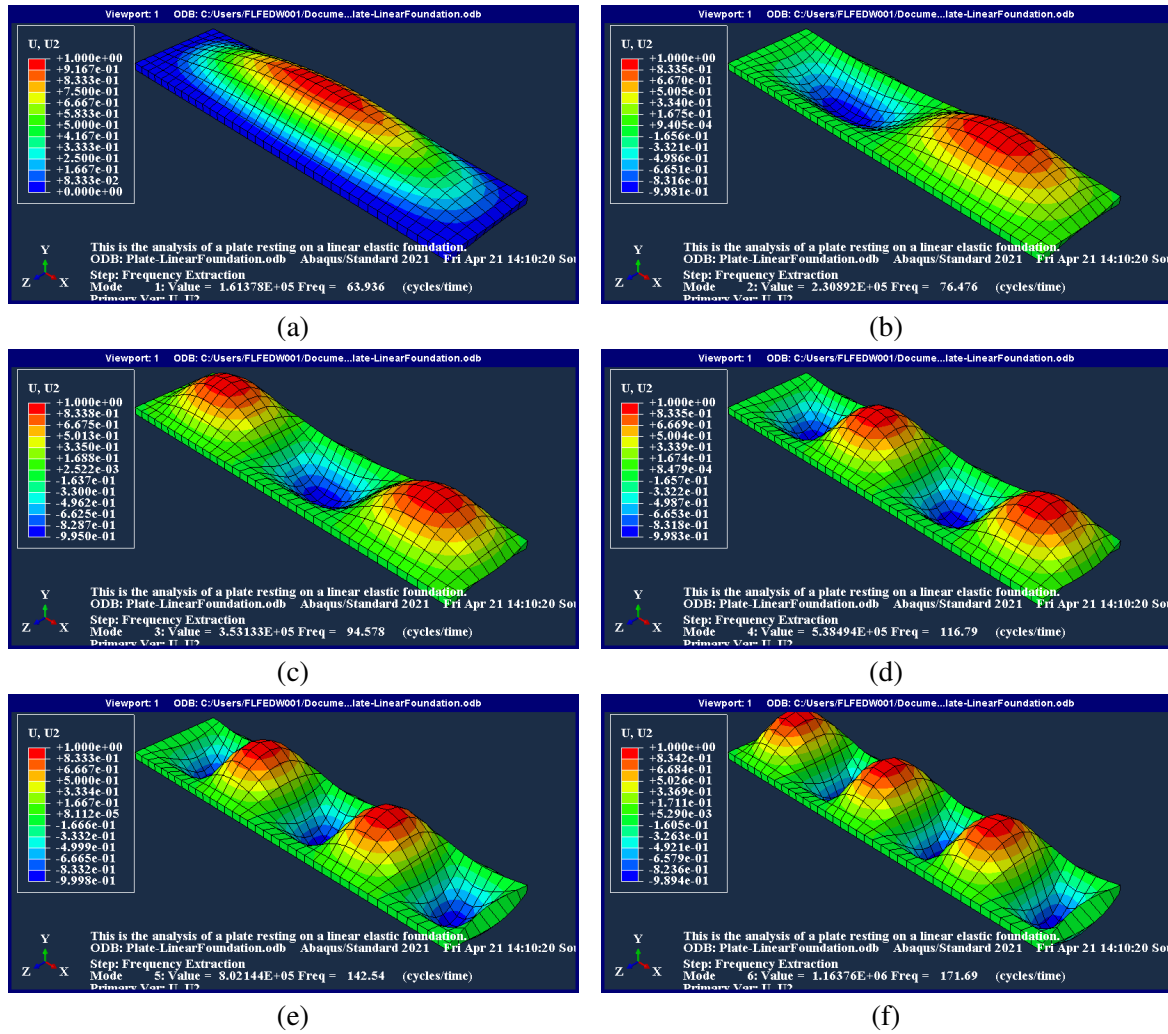


Figure 6.6: Mode shapes of the plate on a linear elastic foundation from Abaqus: (a) to (f) are modes 1 to 6.

displacement at all modes shows 1.00 because the results were requested as displacement normalised. The subsection below deals with nonlinear analyses of the PoEF in Abaqus.

6.4.3 Numerical Results for Nonlinear Elastic Foundation

In the current subsection, the main interest is in the numerical analysis of a PoEF having nonlinear stiffness properties. As was earlier mentioned, the nonlinear foundation is not defined as the linear foundation in Abaqus. Therefore, the tasks described in Subsection 6.4.1 are identical for linear and nonlinear modelling of a PoEF, except **Task 4**. That is because in Abaqus, it is not possible to assign the nonlinear foundation directly as the linear foundation.

To define the nonlinear foundation and assign it to the plate (3D model) in Abaqus, first, **Task 1** is performed. Still, in the *Part module*, partitions are created on the surface created and these partitions generate new nodes (see Figure 6.7a). Then, **Task 2** and **Task 3** are performed (see Subsection 6.4.1) and finally, **Task 4** is focused on to complete the *Interaction module*.

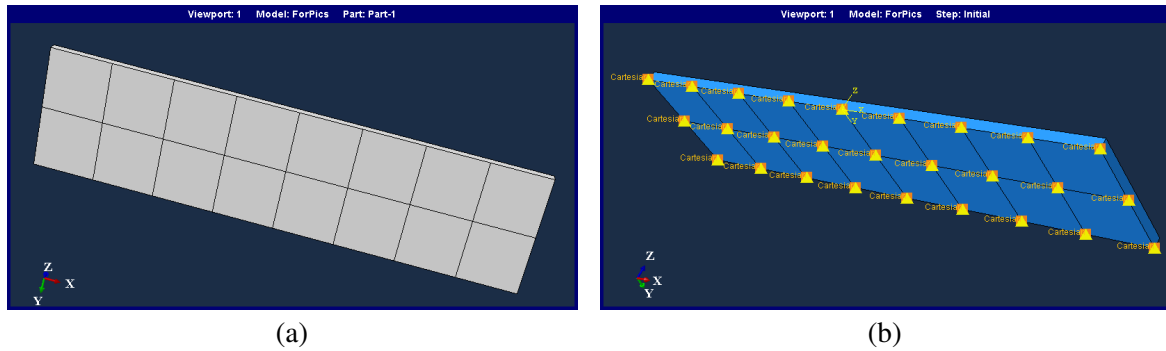


Figure 6.7: (a) Partition of the plate and (b) Nonlinear connectors.

In the *Interaction module*, wires are created with one end connected to the bottom surface of the plate and the other to the ground at the nodes generated when creating the partitions in **Task 1**. Then, elastic connectors are created, and nonlinear properties are assigned to them. This procedure of creating and assigning connectors has already been described in 4, precisely in Subsection 4.4.1. Moreover, in Figure 6.5 details are also given regarding what needs to be done to define a nonlinear foundation to model a 3D plate in Abaqus. Since three types of nonlinear foundations of the plate are investigated in this chapter, each analysis in Abaqus considered the nonlinear type of the foundation accordingly. In Figure 6.7b, one can see the plate with connectors representing the nonlinear foundation.

As far as other tasks are concerned, they remain the same as those described in Subsection 6.4.1. Nevertheless, before concluding on the mesh size, several analyses were performed, and the relevant details are revealed shortly.

Table 6.7 shows the mesh's approximate global sizes (AGS), the number of elements generated, and the fundamental natural frequencies of the plate on a nonlinear quadratic elastic foundation. It can be observed that the frequencies increase with the mesh size. At the same time, the number of elements decreases with the increase in the mesh size. Finally, it is highlighted that the same mesh size was used to analyse the plate resting on the cubic or nonlinear quadratic-cubic elastic foundation.

Table 6.7: Number of elements used to capture optimum mesh for the PoNEF.

AGS	Number of elements	Ω_{mn} (rad/s) at mode 1
0.1	120	517.26
0.09	144	484.76
0.07	232	435.27
0.06	264	426.40
0.058	288	419.62
0.055	360	409.32
0.054	370	410.11
0.053	380	411.11
0.052	400	406.21
0.051	400	406.21
0.050	400	406.21

Based on the values in Table 6.7, it is confirmed that the area of convergence of the solution is when the AGS is between 0.05 and 0.055. Then, the AGS of 0.053 is concluded to be the best fit, referring to the fundamental natural frequency from analytical results. More importantly, 0.053 generates 380 elements, unlike the linear analysis that gave 342 for the same seed number. That is because the partitions added the number of nodes and, thus, increased the number of elements.

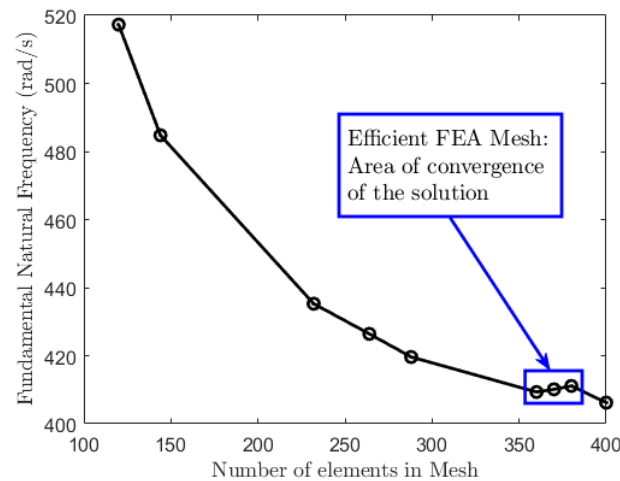


Figure 6.8: Graph of Mesh Convergence for the finite element model of the PoEF in Abaqus.

In Figure 6.8, the plot of the graph of Mesh convergence in Abaqus is displayed. This enables specifying which mesh size to use for the acceptable solution. The small rectangle in blue delimits the domain. Since the mesh suitable for the model is now decided, the analysis

can be submitted, and the output of the displacement modes visualised in the viewport. The nonlinear natural frequencies requested for the first ten vibration modes are presented in Table 6.8.

Table 6.8: Nonlinear natural frequencies of the PoNEF in (rad/s) recorded in Abaqus.

Mode's number	Ω_q	Ω_c	Ω_{qc}
1	411.1	418.9	430.2
2	483.5	489.9	499.4
3	600.3	606.1	624.5
4	739.3	749.7	765.2
5	913.3	916.5	951.4
6	1234.0	1235.7	1239.5
7	1476.7	1485.9	1496.7
8	1574.3	1584.7	1598.3
9	1661.1	1676.5	1701.9
10	1874.8	1895.5	1910.0

Regarding the values shown in Table 6.8, the first ten natural frequencies for the plate with quadratic (Ω_q), cubic (Ω_c), and quadratic-cubic (Ω_{qc}) nonlinear elastic foundations can be identified. It is remarked that, as the type of nonlinearity changes, the nonlinear natural frequencies also vary. The plate with the mixed foundation has more significant values than the plate with other types of nonlinear foundations. Therefore, based on this outcome and the one in Subsection 6.3.4, it is confirmed that, as the nonlinear stiffness coefficient of the foundation increases, the nonlinear natural frequencies increase accordingly, which was expected. The mode shapes from the nonlinear analyses are as shown in Figure 6.9.

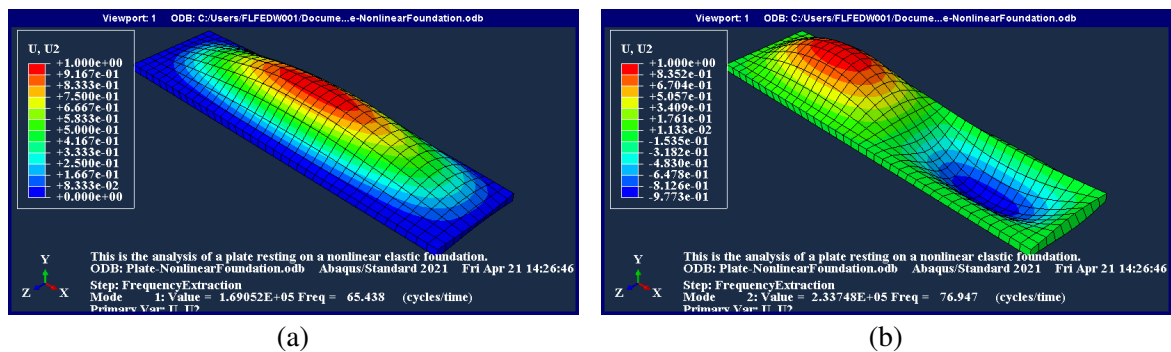


Figure 6.9 shows the mode shapes of the plate resting on a nonlinear elastic foundation. The plate shapes and the maximum displacement are identical to those of the PoLEF. Therefore, looking at the natural frequencies given in Tables 6.6 and 6.8 the differences between

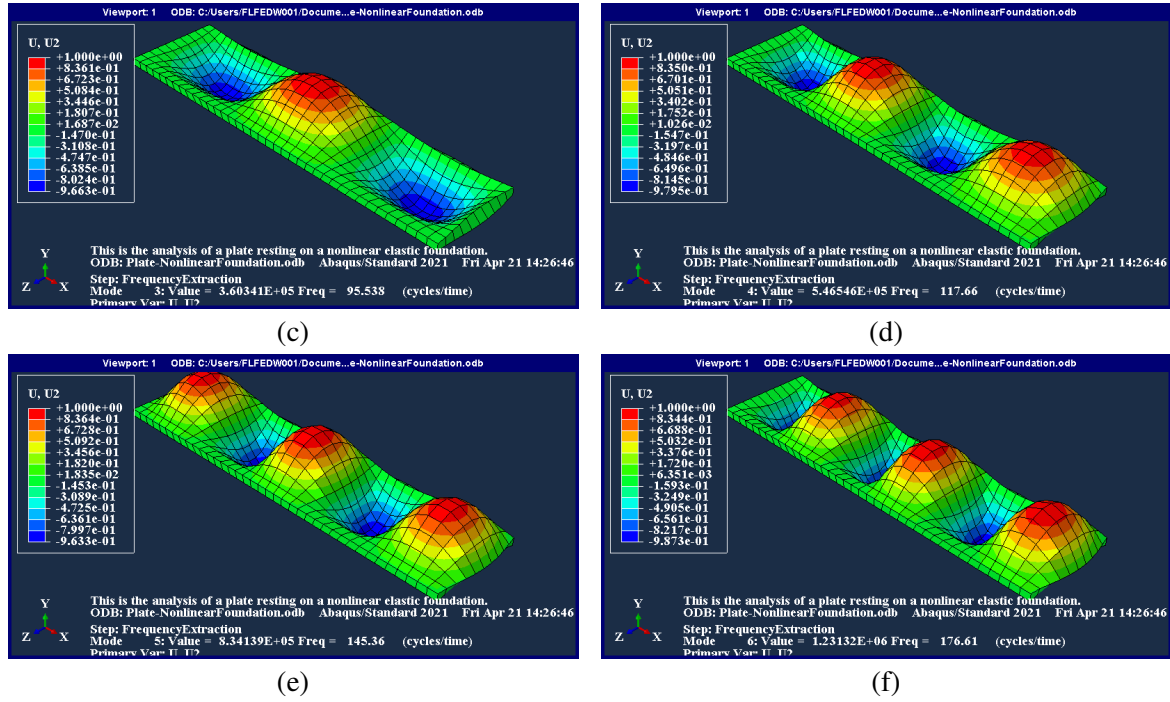
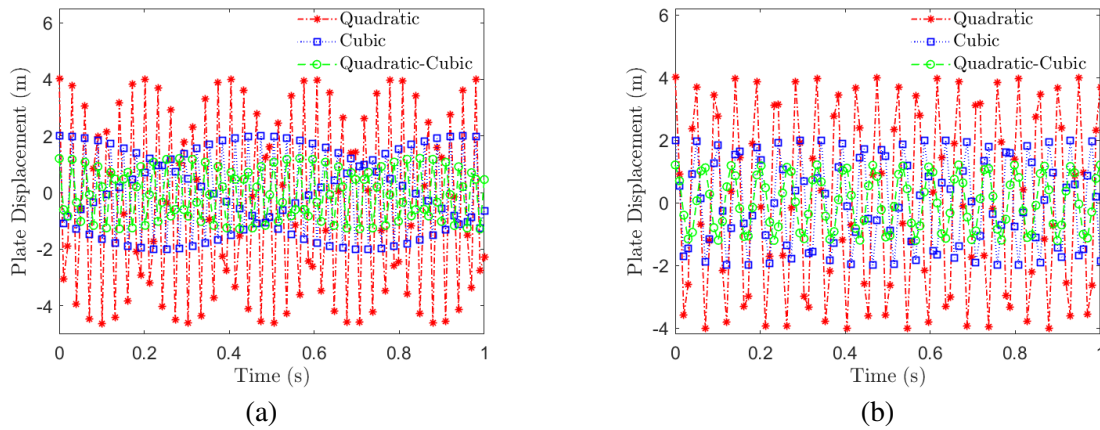


Figure 6.9: Mode shapes of the plate resting on a nonlinear elastic foundation in Abaqus: (a) to (f) are modes 1 to 6.

the plate on a linear and that on a nonlinear elastic foundation are easily noted. The temporal plots of the plate displacement are shown in Figure 6.10.



From the curves given in Figure 6.10, it is noted that the plate supported by a nonlinear quadratic-cubic elastic foundation has the smallest vibration amplitude for all the modes presented. The same observation was made during the analytical study. That confirms that

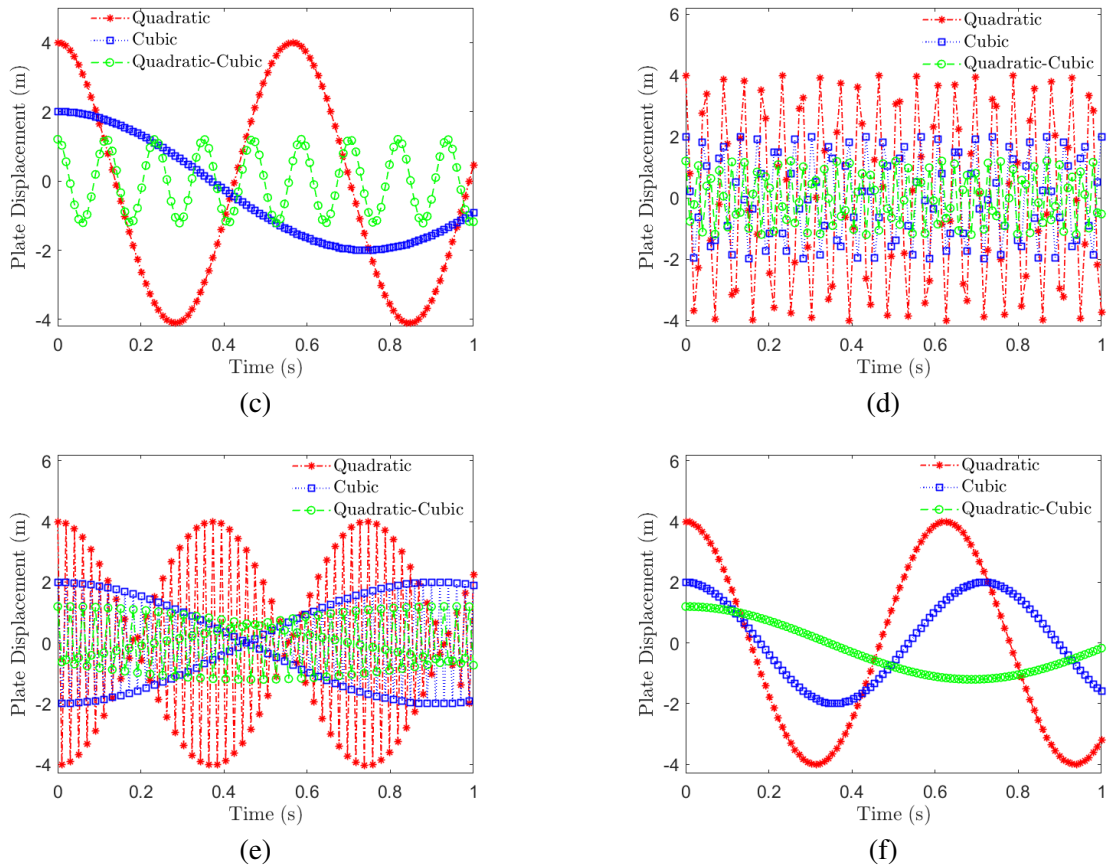


Figure 6.10: Temporal displacement plots of the PoNEF obtained using the natural frequencies computed in Abaqus: (a) to (f) are modes 1 to 6.

a plate with a strong foundation can withstand more deformation than a plate with a weak foundation. The following section compares analytical and numerical results.

6.5 Comparison of Analytical and Numerical Results

The present section compares analytical results to those obtained from Abaqus of the PoEF. The aim is to validate the analytical results. To achieve that, the natural frequencies from analytical calculations are compared to those obtained from the numerical analysis in Abaqus. The table below displays the differences calculated.

Table 6.9 shows the differences calculated by comparing the natural frequencies issued from analytical and numerical analyses. These values were obtained by using Tables 6.4 and 6.8. It can be seen that the differences generated are tiny and less than 10%. This confirms

Table 6.9: Differences in percentage (%) recorded while comparing analytical results to those from Abaqus.

Mode's number	$Diff_{Linear}$	$Diff_{Quadratic}$	$Diff_{Cubic}$	$Diff_{quadratic-cubic}$
1	0.3	0.5	0.2	0.4
2	1.4	1.9	0.3	1.9
3	0.3	0.2	0.3	0.8
4	3.3	2.6	2.3	1.9
5	8.5	6.4	6.8	3.9
6	0.2	0.1	0.2	0.5
7	6.1	4.3	3.9	3.6
8	6.8	2.3	1.9	1.4
9	6.5	4.1	3.4	2.2
10	3.9	1.1	0.2	0.3

the agreement between analytical and numerical results. Thus, the results from Abaqus are an excellent benchmark for analytical results. Figure 6.11 shows the plots of the natural frequencies obtained from analytical and numerical analyses.

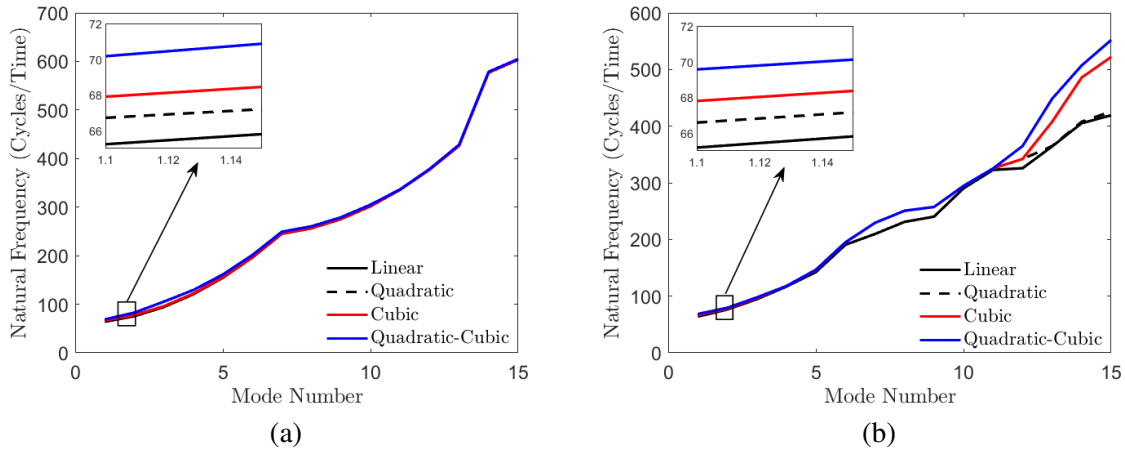


Figure 6.11: Plots of the natural frequencies in Cycles/Time of the PoEF: (a) Analytical and (b) Abaqus.

Figure 6.11 reveals that the natural frequencies increase as the mode number increases, which was expected. However, the zoom plot shows that the natural frequencies of the plate resting on a nonlinear quadratic-cubic elastic foundation are slightly above other natural frequencies. Moreover, the increase of the natural frequencies is similar for both analyses. Finally, the comparison between analytical and numerical results is extended by plotting the plate's displacement as time evolves. These are shown in Figure 6.12.

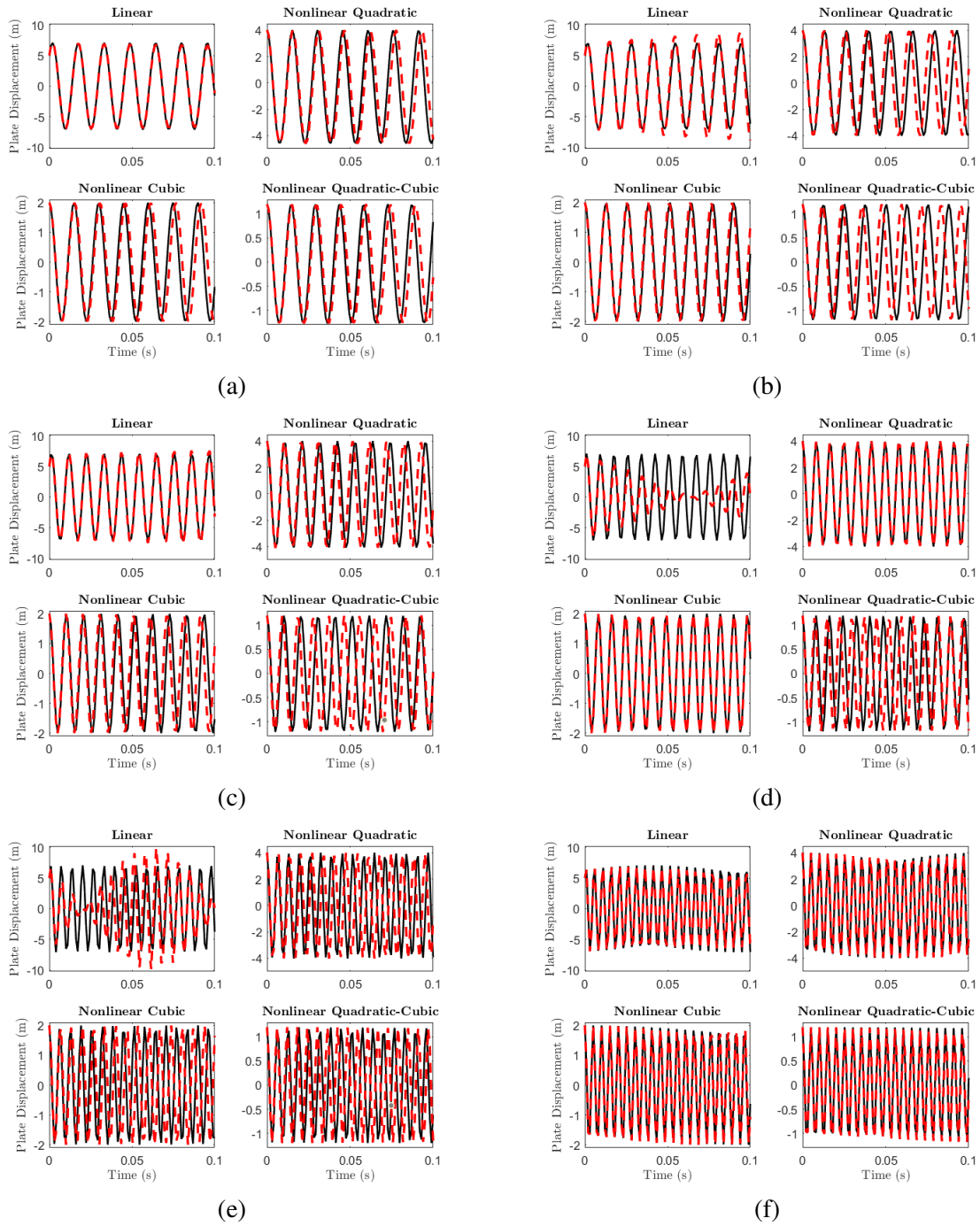


Figure 6.12: Comparison between analytical (in black) and numerical (in red) temporal displacement plots of the PoEF: (a) to (f) are modes 1 to 6.

Figure 6.12 indicates that the path followed by the curves corresponding to the analytical and numerical analyses are similar to a great extent. Moreover, the maximum and minimum

vibration amplitudes are reached to the same displacement value, though at slightly different times. Here also, the agreement between the two methods is confirmed.

6.6 Comparison of Linear and Nonlinear Results

This section compares linear and nonlinear analyses. The aim is to highlight the relative differences of neglecting nonlinearity when modelling plates on elastic foundations. That is achieved by comparing the natural frequencies recorded from linear and nonlinear analyses. Moreover, the amplitudes of the plate as time evolves are also compared to picture the effects of nonlinearity on the plate's behaviour. The following table record the natural frequencies of the plate with the three types of nonlinear foundations investigated in this chapter compared to the linear ones. The relative differences are also provided to emphasise the comparison.

Table 6.10: Comparison between linear and nonlinear natural frequencies of the PoEF.

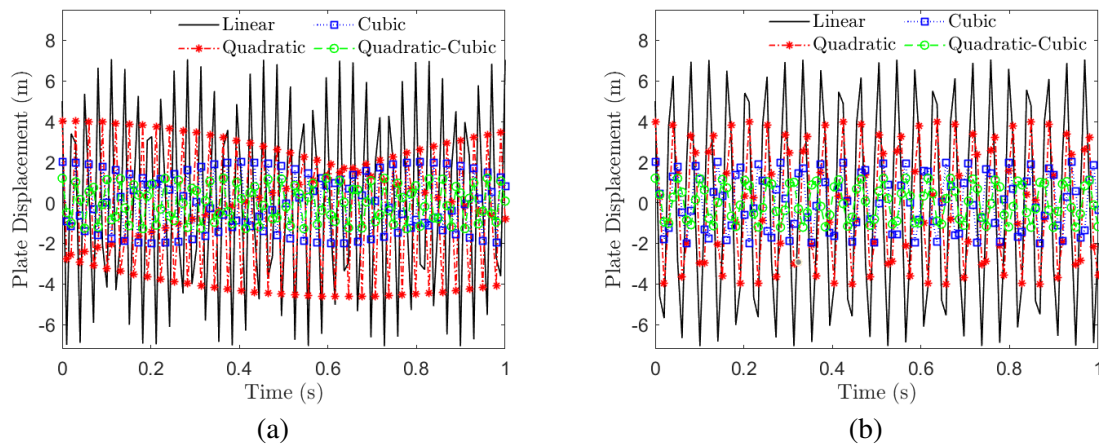
Linear and Quadratic						
Mode	Analytical			Abaqus		
	ω_l (rad/s)	Ω_q (rad/s)	Diff (%)	ω_l (rad/s)	Ω_q (rad/s)	Diff (%)
1	402.9	413.1	2.5	401.72	411.1	2.3
2	474.0	474.3	0.06	480.51	483.5	0.6
3	592.4	601.4	1.5	594.25	600.3	1.01
4	758.23	758.3	0.01	733.82	739.3	0.74
5	971.5	972.2	0.07	895.62	913.3	1.94
6	1232.1	1232.2	0.01	1230.1	1234	0.32
7	1540.1	1540.2	0.01	1451.8	1476.7	1.69
8	1611.2	1611.2	0	1509.8	1574.3	4.1
9	1729.6	1729.6	0	1623.6	1661.1	2.3
10	1895.5	1895.5	0	1824.1	1874.8	2.7
Linear and Cubic						
Mode	ω_l (rad/s)	Ω_c (rad/s)	Diff (%)	ω_l (rad/s)	Ω_c (rad/s)	Diff (%)
1	402.9	419.9	4.05	401.72	418.9	4.1
2	474.0	488.4	2.9	480.51	489.9	1.92
3	592.4	604.0	1.9	594.25	606.1	1.96
4	758.23	767.3	1.2	733.82	749.7	2.1
5	971.5	978.5	0.7	895.62	916.5	2.3
6	1232.1	1237.6	0.4	1230.1	1235.7	0.45
7	1540.1	1544.5	0.3	1451.8	1485.9	2.3
8	1611.2	1615.4	0.3	1509.8	1584.7	4.7
9	1729.6	1733.6	0.2	1623.6	1676.5	3.2

Continued on next page

Table 6.10 – continued from previous page

Linear and Cubic						
Mode	ω_l (rad/s)	Ω_c (rad/s)	Diff (%)	ω_l (rad/s)	Ω_c (rad/s)	Diff (%)
10	1895.5	1899.1	0.2	1824.1	1895.5	3.8
Linear and Quadratic-Cubic						
Mode	ω_l (rad/s)	Ω_{qc} (rad/s)	Diff (%)	ω_l (rad/s)	Ω_{qc} (rad/s)	Diff (%)
1	402.9	432.1	6.8	401.72	430.2	6.6
2	474.0	509.1	6.9	480.51	499.4	3.8
3	592.4	619.2	4.3	594.25	624.5	4.8
4	758.23	780.2	2.8	733.82	765.2	4.1
5	971.5	988.5	1.7	895.62	951.4	5.9
6	1232.1	1245.6	1.1	1230.1	1239.5	0.8
7	1540.1	1550.9	0.7	1451.8	1496.7	3
8	1611.2	1621.5	0.6	1509.8	1598.6	5.6
9	1729.6	1739.3	0.6	1623.6	1701.9	4.6
10	1895.5	1904.3	0.5	1824.1	1910	5.5

In Table 6.10, the relative differences calculated when comparing linear to nonlinear natural frequencies increase following the type of nonlinear foundation of the plate. For example, the foundation with quadratic and nonlinear cubic stiffness properties has the highest differences. This once more confirms that, as the nonlinear parameter of the foundation increases, the nonlinear natural frequencies also increase. The temporal plots of the displacement are shown below.



Figures 6.13 and 6.14 show the curves to compare linear and nonlinear analyses. It is seen that the PoLEF has the maximum amplitude for both analytical and numerical analyses. In contrast, the plate supported on a mixed parity nonlinear elastic foundation has the smallest amplitude. The good thing is that all the nonlinear cases have smaller amplitudes than the

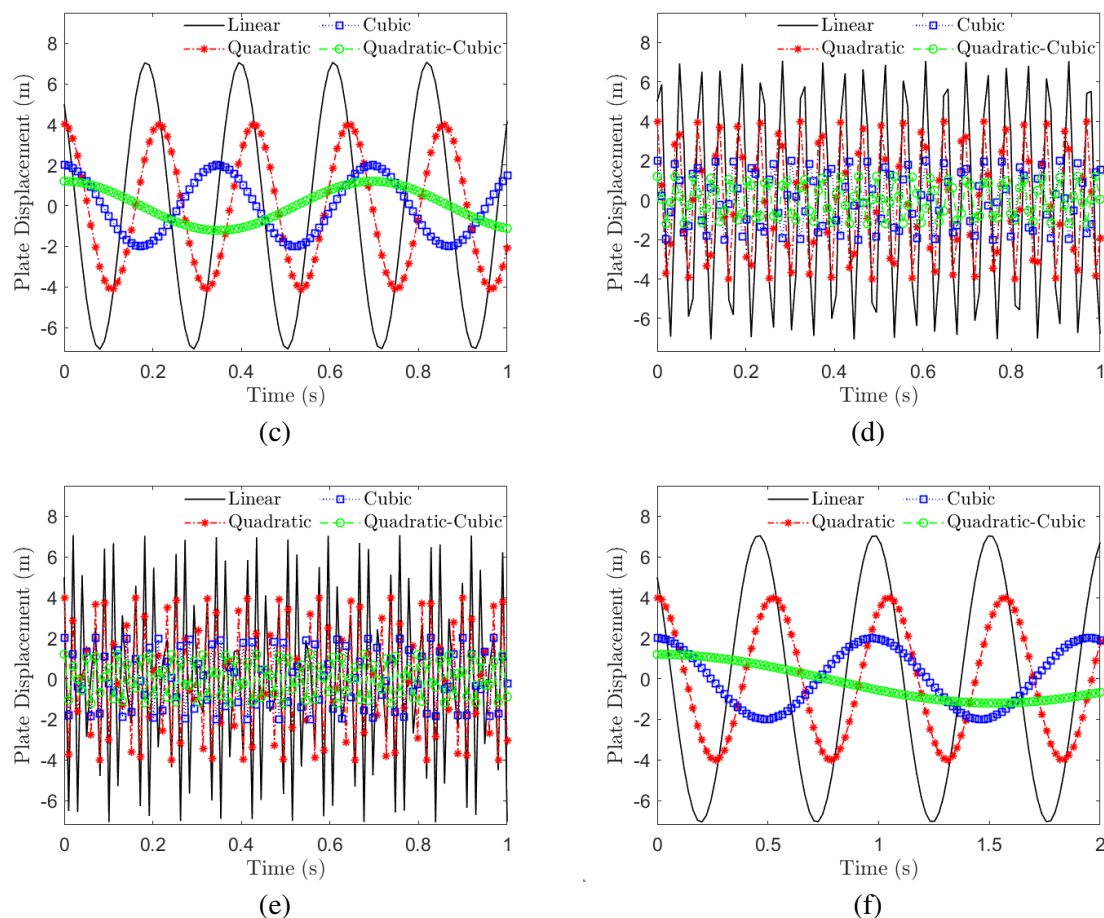
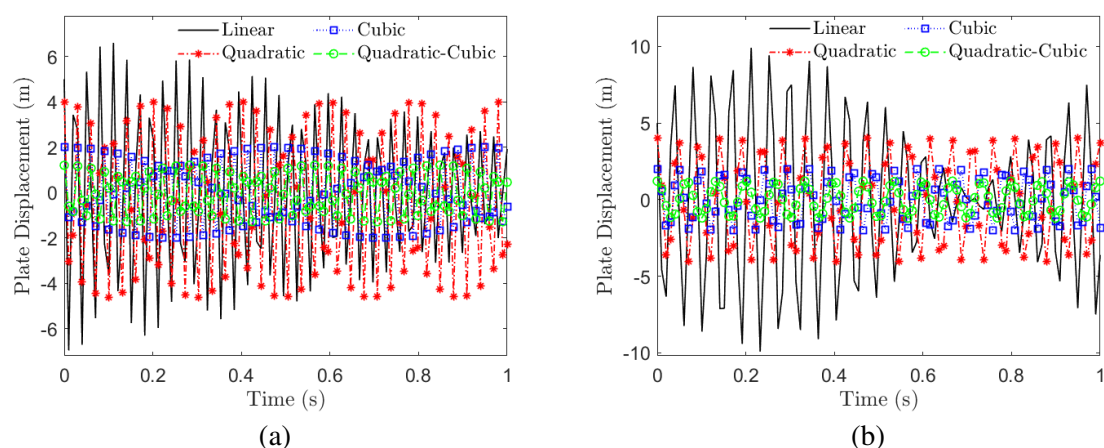


Figure 6.13: Comparison between linear and nonlinear temporal plots of the plate displacement obtained analytically from MATLAB: (a) to (f) are modes 1 to 6.



linear case. Therefore, the presence of nonlinearity in the plate's foundation is beneficial and

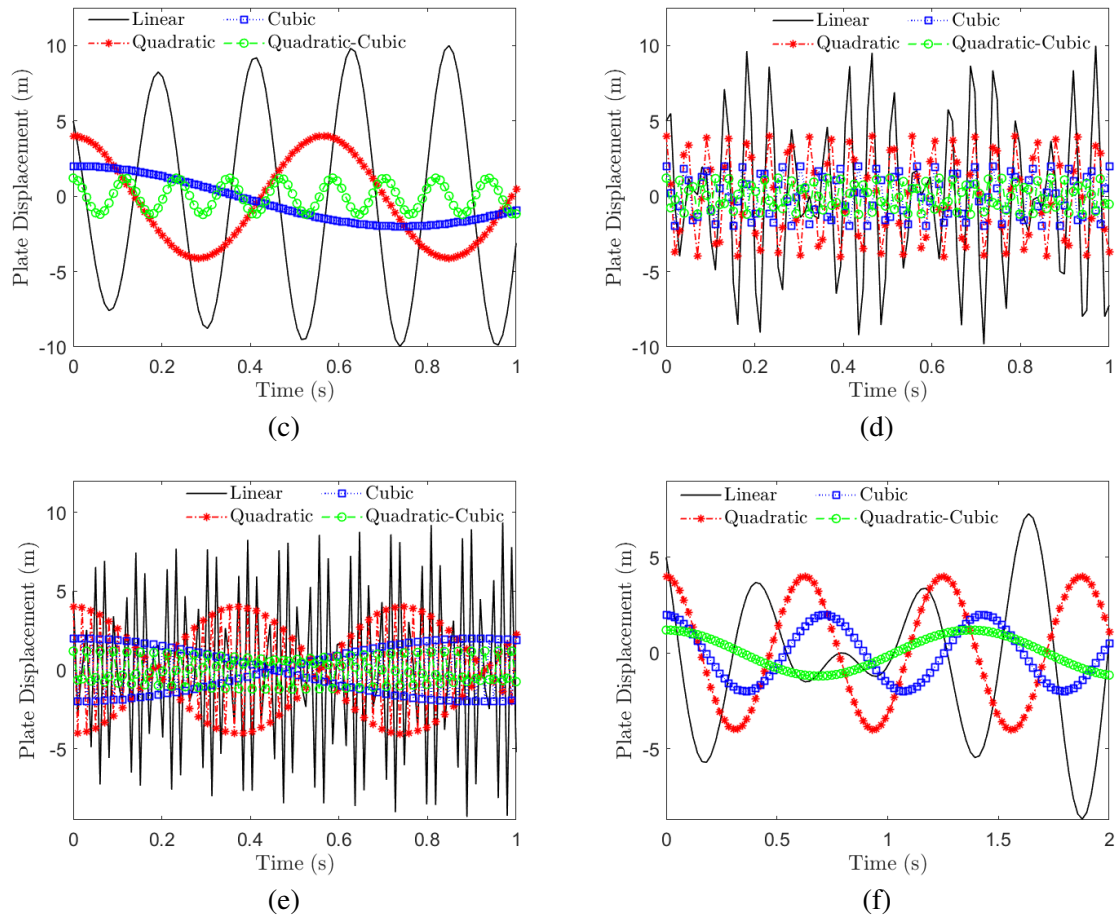


Figure 6.14: Comparison between linear and nonlinear temporal plots of the plate displacement obtained using the natural frequencies from FEA in Abaqus: (a) to (f) are modes 1 to 6.

lowers the plate's amplitude of vibration. Thus, depending on the foundation stiffness, the plate's vibration amplitude can be smaller or larger.

6.7 Modelling of the PoEF using Another Element Type

6.7.1 Modelling of the PoEF using C3D20R

In this subsection, the results from other types of modelling are presented. First of all, the modelling using C3D20R element type which is a general purpose quadratic brick element, with reduced integration (2x2x2 integration points) performed. The table below shows the natural frequencies recorded.

Table 6.11: Natural frequencies of the PoEF in (rad/s) recorded from C3D20R modelling in Abaqus.

Mode's number	Linear	Nonlinear		
	ω_l	Ω_q	Ω_c	Ω_{qc}
1	3787.7	3788.3	3789.2	3790.6
2	4141.2	4141.8	4142.6	4143.8
3	4764.2	4764.7	4765.4	4766.6
4	5676.9	5677	5677.1	5677.2
5	6885.4	6885.7	6886.1	6886.7
6	8381.2	8381.4	8381.8	8382.3
7	9660.1	9660.1	9660.1	9660.1
8	10150	10150	10150	10150
9	11576	11576	11576	11576
10	11921	11921	11921	11921

In Table 6.11 we can see that the natural frequencies are highly significant for the modelling using C3D20R element type as compared to the one with C3D8R (see Table 6.8). Surprisingly, these natural frequencies in Table 6.11 do not match the analytical results given in Table 6.4. Therefore, the C3D20R element type can be used to model PoEFs but cannot be used as a benchmark for the analytical results presented in this thesis.

6.7.2 Modelling of the Plate using Four-node Shell Element S4R

Modelling PoEFs using shell elements follows the same procedure as using solid elements only the base feature (shape and type) changes to shell planar. The modelling was performed using the same AGS of 0.053 and this generated 400 elements. Moreover, the same boundary conditions applied to the solid shape were also applied to the shell shape. The table below displays the natural frequencies recorded after the analysis was completed.

The natural frequencies of the PoEF modelled in Abaqus using shell elements are presented in Table 6.12. It can be seen that the results are higher than those obtained for the PoEF modelled using a solid shape. Moreover, the present results are not close to those obtained from analytical calculations. Therefore, they cannot be used as a benchmark for the analytical results.

In view of the results presented in Tables 6.11 and 6.12, it is clear that although a PoEF can be modelled using different types of elements, some of the elements might not be suitable as a benchmark because of a large gap resulting from the comparison. Therefore, the element

Table 6.12: Natural frequencies in (rad/s) recorded from the analysis of a PoEF using shell elements in Abaqus.

Mode's number	Linear	Nonlinear		
	ω_l	Ω_q	Ω_c	Ω_{qc}
1	2519.4	2521.1	2523.7	2527.7
2	2927.7	2929.1	2931.4	2934.8
3	3622.0	3623.2	3625.0	3627.8
4	4610.5	4611.5	4612.9	4615.1
5	5896.3	5897.0	5898.1	5899.6
6	7480.1	7480.6	7481.5	7482.9
7	9362.4	9362.9	9363.6	9364.7
8	9660.5	9660.5	9660.5	9660.5
9	9680.1	9680.1	9680.1	9680.1
10	10047	10047	10047	10047

type C3D8R considered here as a thin plate is a benchmark for the analytical results because the differences resulting from the comparison between the two methods are tiny.

6.8 Comparison of Plate versus Beam Response

Now that analytical and numerical results of beams and plates resting on elastic foundations have been obtained, the structural response can be fully understood. This section aims to grasp how the beam structural response is modified to that of the plate.

First, a beam is a long structural member with minimal lateral and transverse dimensions and is characterised by bending, whereas a plate is thin in one axis relative to the other two axes. The supports and loads are in one direction for the beam, whereas they are in two directions for the plate. Then, looking at the equation of motion of the PoEF, there are some similarities to that of the beam supported on the foundation having the same properties. For example, the effects produced on the beam, or the plate supported on an elastic foundation generate inertial, elastic, and restoring (created by the foundation) forces.

Despite similarities in the governing equations of the BEF and the PoEF, differences exist concerning the spatial solution of both equations. The beam has one spatial variable, x , whereas the plate has two, x and y . Consequently, two parameters, m and n define the mode's number of the plate, whereas, for the beam, it is only one parameter, either m or n . Although the temporal equation is the same for the beam and the plate, the plate's solution

Table 6.13: Comparison between the expressions of the natural frequencies of the BoEF and those of the PoEF.

Foundation Type	Beam	Plate
Linear	$\omega_m = \sqrt{\frac{1}{\rho} \left[\frac{k_l}{S} + \frac{EI}{S} \left(\frac{m\pi}{L} \right)^4 \right]}$	$\Omega_{mn} = \sqrt{\left\{ \frac{D}{\rho h} \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right]^2 + \frac{k_l}{\rho h} \right\}}$
Linear and Quadratic	$\omega_q = \omega_m \left[1 + \frac{\alpha_m^2}{3\omega_m^4} \varepsilon^2 A_1^2 \right]$	$\Omega_q = \omega_{mn} \left[1 + \frac{\alpha_{mn}^2}{3\omega_{mn}^4} \varepsilon^2 A_1^2 \right]$
Linear and Cubic	$\omega_c = \omega_m \left[1 + \frac{3}{8\omega_m^2} \varepsilon \beta_3 B_1^2 \right]$	$\Omega_c = \omega_{mn} \left[1 + \frac{3\beta_{mn}}{8\omega_{mn}^2} \varepsilon B_1^2 \right]$
Linear and Quadratic-Cubic	$\omega_{qc} = \omega_m \left[\frac{9\beta_3 \omega_m^2 - 10\alpha_m^2}{24\omega_m^4} \varepsilon^2 C_1^2 \right]$	$\Omega_{qc} = \omega_{mn} \left[1 + \frac{9\beta_{mn} \omega_{mn}^2 - 10\alpha_{mn}^2}{24\omega_{mn}^4} \varepsilon^2 C_1^2 \right]$

is slightly different from that of the beam. Table 6.13 shows the formulas to calculate the natural frequencies for the plate and those of the beam with different foundation properties. It is acknowledged that the expressions are almost similar for each foundation type, but the parameters define each expression differently.

Analytical calculations are used to find the natural frequencies of the plate and compare them to those of the beam by varying the parameters m and n from 1 to 10. Moreover, the same values are assigned to the constant parameters A_1 , B_1 , and C_1 to perform the comparison. Furthermore, the same boundary conditions were applied to the beam and the plate. Table 6.14 displays the parameters' values.

Table 6.14: Values of the parameters to be used for comparing the natural frequencies of the BoEF and those of the PoEF.

Parameters and their values					
k_l	$1.69 \times 10^5 \text{ N/m}^2$	$\rho \cdot S$	12.075 kg/m	$a = L$	2 m
k_2	$8.44 \times 10^5 \text{ N/m}^3$	$\rho \cdot h$	322 kg/m^2	b	0.5 m
k_3	$1.3 \times 10^7 \text{ N/m}^4$	β_3	$8.07 \times 10^5 \text{ N/kg} \cdot \text{m}^2$	A_1	100
D	$1.17 \times 10^6 \text{ Pa} \cdot \text{m}^3$	β_{ij}	$3.6 \times 10^5 \text{ N/kg} \cdot \text{m}^2$	B_1	0.5
EI	$2.25 \times 10^4 \text{ Pa} \cdot \text{m}^4$	ε	0.01	C_1	30

In Table 6.15 the first ten natural frequencies of the beam resting on an elastic foundation are compared to those of the plate on the same foundation type. These values were obtained

Table 6.15: Comparison between the natural frequencies in (rad/s) of the BoEF and those of the PoEF.

Mode's number	Linear		Quadratic		Cubic		Quadratic-Cubic	
	Beam	Plate	Beam	Plate	Beam	Plate	Beam	Plate
1	285.17	402.9	335.8	413.1	316.4	419.8	372.7	432.1
2	501.5	474.0	507.4	474.3	519.3	488.4	553.8	509.1
3	994.42	592.4	995.1	601.4	1003.4	604.0	1021.1	619.2
4	1724.6	758.23	1724.7	758.3	1729.7	767.3	1740.0	780.2
5	2675.8	971.5	2675.9	972.2	2679.2	978.5	2685.8	988.5
6	3843.4	1232.1	3843.5	1232.2	3845.8	1237.6	3850.4	1245.6
7	5225.7	1540.1	5225.7	1540.2	5227.4	1544.5	5230.7	1550.9
8	6821.7	1611.2	6821.7	1611.2	6823.0	1615.4	6825.6	1621.5
9	8631.3	1729.6	8631.3	1729.6	8632.3	1733.6	8634.4	1739.3
10	10654	1895.4	10654	1895.5	10655	1899.1	10657	1904.3

by considering the natural frequencies of the plate calculated in the previous sections of this chapter and using the same parameters as given in Table 6.14 to calculate those of the beam with the formulas in Table 6.13. Four foundation types are presented: linear, nonlinear quadratic, nonlinear cubic, and nonlinear quadratic-cubic. It is observed that the plate has significant values of the natural frequencies only at the fundamental mode for all the cases, while for $n > 1$ they are lower as compared to those of the beam. Therefore, the plate differs from the beam in the dimensions, the mode's number, the solution of the equation of motion, as well as in the natural frequencies as confirmed in Table 6.15. It is important to stress that in case the natural frequencies of the plate are calculated by considering the mode's number $m = n$, the values in Table 6.15 will be more significant for the plate.

6.9 Summary and Conclusions

Extensive studies on the vibration of plates resting on linear elastic foundations have been carried out. Because nonlinear cases have been neglected, and since nonlinearity is important in vibration analysis, this chapter has dealt with the linear and nonlinear study of vibrations of a PoEF. First, the linear case was conducted, followed by the nonlinear. The primary goal was to highlight the nonlinear effects of the foundation on the plate's vibration response and underline the relative differences of neglecting the nonlinearity of the foundation. Then, both analytical studies using the SEM and numerical analysis using FEA in Abaqus were performed. The findings have enabled us to draw the conclusions below.

1. The straightforward expansion method was used to conduct analytical studies. First, the solutions to the governing equations of the plate resting on different types of foundations were derived. For example, the solution to the linear case study was uncomplicated, whereas the nonlinear solution necessitated tedious calculations. Since the resulting temporal equation of the PoNEF was similar to that of the beam with the same foundation properties, the solution of the beam case was adapted to fit that of the plate. Ultimately, the natural frequencies for both linear and nonlinear systems were calculated.

2. The plate's natural frequencies without foundation were slightly minimal compared to those of the plate resting on a linear elastic foundation. As the foundation properties changed from linear to nonlinear, the natural frequencies also varied and increased as the foundation's nonlinear stiffness parameter increased. It was found that for an elastic foundation having higher soil stiffness, the natural frequencies will be more significant.

3. To validate analytical results, numerical analysis using FEA in Abaqus was performed. However, it was realised that it is not straightforward to model a 3D plate on a nonlinear elastic foundation compared to a plate with a linear foundation. Abaqus already has an integrative feature to create the linear elastic foundation, which makes modelling a PLEF effortless. On the other hand, modelling a plate on a nonlinear elastic foundation necessitated advanced knowledge of FEA in Abaqus. Ultimately, one possible way was to first create partitions on the plate's surface to support the foundation. Then, create wires to ground with the nodes generated after creating the partition. Finally, create nonlinear connectors having properties of the foundation nonlinearity and assign them to the wires. These steps enabled us to perform analyses for the three types of nonlinearities of the foundation investigated. The analyses were completed successfully. The nonlinear natural frequencies recorded from each nonlinear analysis were significant than the linear ones.

4. Comparing analytical to numerical results proved an agreement between the two methods with minimal differences, which is proof that the results from Abaqus are an excellent benchmark for analytical results. On the other hand, comparing linear to nonlinear results revealed relative differences which increased as the nonlinear stiffness parameter of the foundation increased.

5. The plate's foundation having both quadratic and cubic nonlinearities was found to have more significant natural frequencies. Consequently, the vibration amplitude of the plate supported on such a foundation was minimal compared to that of the plate resting on other nonlinear foundations. This proved that when the foundation is stronger, the system plate-foundation is equipped to withstand external excitations which may arise on the plate.

6. Other element types were used in Abaqus to study PoEFs, namely C3D20R and S4R which are general purpose quadratic brick element, with reduced integration (2x2x2 integration points) and four-node shell element, respectively. It was found that for both analyses, the natural frequencies of the PoEFs were more significant as compared to those obtained using C3D8R which is a general purpose linear brick element, with reduced integration (1x1x1 integration point). Therefore, the results from C3D20R and S4R could not be used as a benchmark because comparing these to analytical results could produce large discrepancies.

7. Comparing a plate (3D model) to a beam (2D model) revealed that the solution of the equation of motion is not the same in both cases. The natural frequencies of the PoEF were lower as compared to those of the beam except at mode 1. However, it was noticed that if the plate's mode number takes values like that of the beam, the natural frequencies of the plate will be far greater than those of the beam having the same foundation properties.

8. The current results on the plate and those of the beam in previous chapters allow for an understanding of the structural response. Above all, the findings of 2D and 3D study models confirm that incorporating nonlinearity in the foundation of beams and plates is beneficial because it may help lower the structural amplitude of vibration when excitation arises.

Chapter 7

Conclusions and Recommendations

7.1 Introduction

This chapter brings together the key findings and conclusions from the preceding chapters. It also makes recommendations for future research.

7.2 Summary of Research

The primary goal of this research was to respond to the need to improve structural performance and investigate the effects of the nonlinear stiffness of the foundation on the vibration response of structural and mechanical systems compared to that of structural systems with linear foundation stiffness. The study also provided new development for predicting the performance of nonlinear structural systems by analysing beams and plates resting on nonlinear elastic foundations (BPoEFs). The investigation was performed using classical beam theory through analytical and numerical analyses. This was done in order to:

- obtain closed-form solutions for the equations of motion of the models of interest;
- evaluate the effect of nonlinearity of foundation stiffness on the natural frequencies of beams and plates, by comparing with the response of beams and plates on linear elastic foundations;
- understand the vibration response of beams and plates on elastic foundations when the stiffness law of the elastic foundation is as follows:

- i. linear;
- ii. nonlinear quadratic;
- iii. nonlinear cubic;
- iv. nonlinear quadratic-cubic.

The above objectives were met throughout this thesis. The findings and conclusions of the various studies are discussed in the following subsections.

7.2.1 Free Vibration of Beams on Elastic Foundations

The vibration of beams resting on linear elastic foundations was presented in Chapter 3. The system's equation of motion was derived, and the solution was quickly found using the separation of variables. A comparison between analyses of a simple beam and a beam supported on a linear elastic foundation was made.

In Chapter 4, the vibration of beams resting on elastic foundations having three different nonlinearities was investigated: quadratic nonlinear foundation, cubic nonlinear foundation, and quadratic-cubic nonlinear foundation. The straightforward expansion method was used to solve the governing equation of each case. The natural frequencies and mode shapes were obtained, and the effects of the nonlinear stiffness coefficient of the foundation on the vibration response were investigated. It was found that when the nonlinear stiffness parameter of the foundation increases, the natural frequencies increase also. For further investigations, the study of a BoNEF with a negative contribution from the cubic term was conducted. The results of the natural frequencies revealed slightly lower values for the nonlinear case as compared to the linear case. Overall, it was also found that the presence of nonlinearity in the foundation was of benefit because it contributed to decreasing the beam's vibration amplitude.

The parametric study was proposed in Chapter 5, where two cases of nonlinear foundations of the beam were investigated: the first with nonlinear quadratic and the second with nonlinear cubic stiffness properties. The effects of the foundation stiffness nonlinearities on the natural frequencies and the beam's vibration response were explored. It was found that as the nonlinear stiffness parameter increases, the natural frequencies of the system beam-foundation also increase. In contrast, increasing the nonlinear stiffness coefficient decreased the beam's vibration amplitude for the foundation having quadratic stiffness nonlinearity. However, the amplitude remained the same for the one with cubic stiffness nonlinearity. Comparing analytical and FEA results provided excellent agreement, which proved that

the results from Abaqus are an excellent benchmark for the analytical results. Therefore, analytical results could be validated. Finally, the parametric study revealed that quadratic nonlinearity affects the system and its natural frequencies more than cubic nonlinearity.

The increasing errors calculated between linear and nonlinear natural frequencies highlighted that one may fail to represent the real situation by neglecting nonlinearity. A new parameter, the relative nonlinear ratio, was introduced. It was seen that the natural frequencies increase with that ratio. The amplitude of vibration of the system with nonlinearity was small compared to the system without nonlinearity. The results obtained in Abaqus were compared with the analytical solution. An agreement was found between the methods.

The numerical analysis of the beam resting on an elastic foundation in Abaqus was straightforward for both linear and nonlinear cases. On that account, this thesis designed new flowcharts to simplify the analysis. For the BoNEF, one only needed the force-displacement characteristic of the restoring force corresponding to each nonlinear case. The natural frequencies and the corresponding mode shapes were obtained from linear and nonlinear analyses. Finally, the differences in neglecting the nonlinearity of the foundation were highlighted in the two analyses.

7.2.2 Free Vibration of Plates on Elastic Foundations

Linear and nonlinear vibration of plates resting on elastic foundations were presented in 6 as an extension of the cases of beams. Both analytical and numerical analyses were carried out. The solutions to the nonlinear differential equation of motion were also found using the straightforward expansion method by adapting the solution from the beam case. The linear case was easier, whereas the nonlinear case required further development. Nevertheless, the plate's temporal equation was similar to that of the beam, and the solution to the equation also looked similar. Only the parameter values were different. Comparing spatial solutions of the beam and the plate revealed a difference in the plate solution because of another spatial coordinate y .

On the other hand, two integers, m and n defined the plate's mode, unlike the beam, which was only either m or n . surprisingly, comparing the beam's natural frequencies and those of the plate showed that the plate's natural frequencies are only greater at mode 1 and lower than those of the beam when the mode number increases. However, it was realised that if the same values are given to m and n at each mode, the natural frequencies of a PoEF will be far greater than those of a BoEF.

In addition to analytical studies, numerical analysis using FEA in Abaqus was carried out on the 3D plate resting on the elastic foundation. FEA of the plate on a linear elastic foundation was easier because, in Abaqus, there is already a possibility to choose an elastic foundation and enter the linear stiffness coefficient of the foundation. On the contrary, the analysis of the plate resting on a nonlinear elastic foundation was more complex, and there was no direct procedure to assign a nonlinear foundation to the 3D plate. After many attempts, it was realised that one possibility was first to create partitions on the surface on which to define the foundation. These partitions generated new nodes, which were later used. Then, connectors were created at those nodes and assigned nonlinear properties to them. In the end, the overall nonlinear connectors formed the nonlinear foundation. More importantly, it was noted that the choice of parameters is very important in producing acceptable results. Therefore, to facilitate the modelling in Abaqus of the 3D PNEF, new procedures were described, and these detailed the tasks to be followed.

Analytical results were compared to FEA results from Abaqus. Excellent agreement was found using C3D8R element type and the analytical results could be validated. On the other hand, the modelling using C3D20R and S4R element types produced large gaps between analytical and numerical results and could not be used as a benchmark. On the whole, comparing linear and nonlinear results provided increasing errors with the increase of the nonlinear stiffness coefficient of the foundation. It was also found that the nonlinear parameter of the foundation is beneficial in decreasing the amplitude of vibration of the plate. Therefore, it is concluded that nonlinearity should not be neglected.

7.2.3 Contribution of the Work

Studying beams and plates resting on elastic foundations having both quadratic and cubic nonlinear stiffness properties showed that both nonlinear stiffness parameters of the soil foundation influence the beam's or plate's vibration amplitude and natural frequencies. It is clear that building the foundation with greater stiffness properties can be a meaningful way of vibration control of beam and plate structures. As a result, this thesis serves as a tool to direct engineers and practitioners in designing stronger and more sustainable structures for a better environment. Moreover, the findings show that the goal of this thesis has been achieved.

The study on BPoEFs revealed that natural frequencies do not vary according to the vibration amplitude for a continuous system resting on a linear elastic foundation. However,

for a continuous system resting on a nonlinear elastic foundation, given that the expression to calculate the nonlinear natural frequencies contains the vibration amplitude, the nonlinear natural frequencies will increase if that amplitude increases.

Investigating quadratic stiffness nonlinearity on the one hand, and cubic on the other, was important to highlight the effects of each foundation stiffness on the system's natural frequencies and the beam's and plate's vibration amplitudes. Therefore, details were given for each case, and the findings clearly showed that the impacts of quadratic stiffness nonlinearity are more significant than cubic.

7.3 Recommendations for Future Research

The results and limitations of the present study point toward topics to be addressed in the future. The following paragraphs give recommendations for further research.

1. First of all, the beam considered in this thesis was simply supported at both ends. Other boundary conditions could be explored to see how different boundary conditions combine with different forms of nonlinearities. Moreover, the analytical models were limited to simply supported beams and plates resting on linear and nonlinear elastic foundations. To cover other structure types, the work could be extended to investigate other models, such as cables and shells supported on the same foundations. Finally, this thesis focused on using the straightforward expansion method and FEA in Abaqus. Other methods and algorithms could be adopted for those types of models.

2. The linear and nonlinear analyses of the beam or the plate in this study were limited to free vibrations, and only initial displacements were imposed. However, because structures will continue to be exposed to external excitation, the study could be extended by exploring the forced vibration of beams or plates resting on nonlinear elastic foundations.

3. Apart from the nonlinearities considered in this thesis, the influence of other types of nonlinearities of the foundation on the vibration response of beams or plates could be investigated. Other nonlinearities that could be explored are bilinear stiffness, piecewise linear stiffness, backlash, and boundary condition nonlinearities.

4. The analytical results presented in this thesis could be compared to real measurements for practical use and validation.

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Appendix A

Natural Frequencies and Displacements of the Cantilever Beam on a Nonlinear Elastic Foundation

A.1 Comparison between Linear and Nonlinear Analyses of the Fixed end-free Beam on an Elastic Foundation

The natural frequencies for both linear and nonlinear analyses in Abaqus are given in the tables below.

Tables A.1, A.2, and A.2 show the natural frequencies for the cantilever beam on an elastic foundation (BoEF) recorded from Abaqus. They reveal that the natural frequencies increase with nonlinearity. The first six modes of the temporal plots of the displacement of a cantilever BoEF obtained using the natural frequencies above-displayed are given in Figure A.1.

Table A.1: Comparison between linear and nonlinear quadratic natural frequencies for the cantilever beam.

Mode m	ω_l (rad/s) Linear	Ω_q (rad/s) Nonlinear Quadratic	Diff (%)
1	285.78	298.97	4.41
2	380.96	392.71	2.99
3	729.24	735.39	0.84
4	1338.6	1342	0.25
5	2177.8	2179.9	0.1
6	3236.3	3237.7	0.04
7	4510.5	4511.6	0.02
8	5999.3	6000.2	0.01
9	7702.1	7702.8	0.009
10	9619.5	9620.2	0.007

Table A.2: Comparison between linear and nonlinear cubic natural frequencies for the cantilever beam.

Mode m	ω_l (rad/s) Linear	Ω_c (rad/s) Nonlinear Cubic	Diff (%)
1	285.78	289.85	1.40
2	380.96	384.56	0.94
3	729.24	731.11	0.26
4	1338.6	1339.6	0.07
5	2177.8	2178.4	0.03
6	3236.3	3236.7	0.01
7	4510.5	4510.9	0.01
8	5999.3	5999.6	0.01
9	7702.1	7702.3	0.002
10	9619.5	9619.7	0.002

Table A.2: Comparison between linear and non-linear quadratic-cubic natural frequencies for the cantilever beam.

Mode m	ω_l (rad/s) Linear	Ω_{qc} (rad/s) Nonlinear Quadratic-cubic	Diff (%)
1	285.78	302.85	5.64
2	380.96	396.21	3.85
3	729.24	737.25	1.09
4	1338.6	1343	0.33
5	2177.8	2180.5	0.12
6	3236.3	3238.1	0.06
7	4510.5	4511.9	0.03
8	5999.3	6000.4	0.018
9	7702.1	7703	0.012
10	9619.5	9620.4	0.009

Figure A.1 shows that the vibration amplitude for the cantilever BoEF having nonlinear quadratic and cubic stiffness properties has the smallest amplitude of vibration. The outcome highlights the benefit of having a strong foundation.

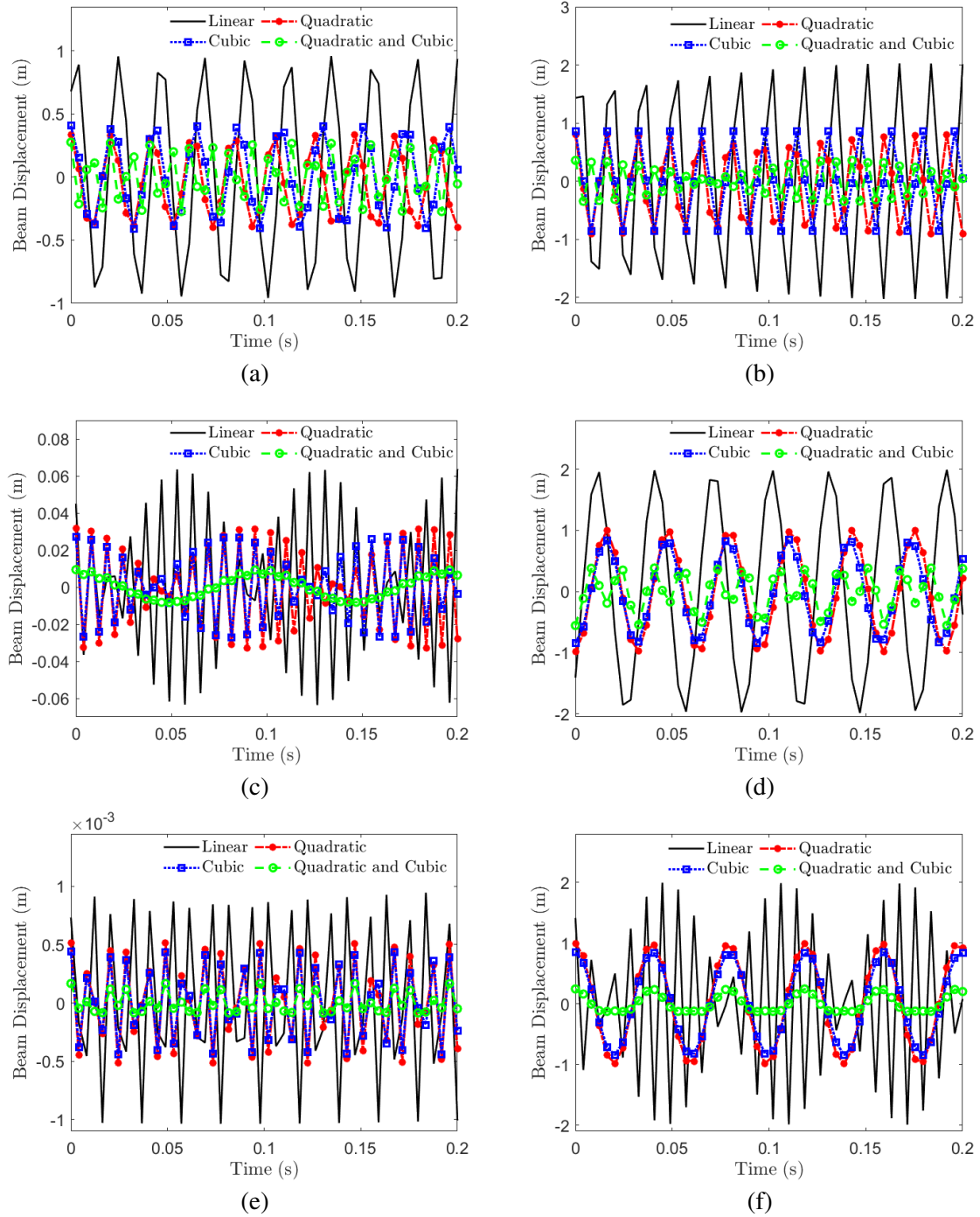


Figure A.1: Comparison between the vibration amplitudes of a cantilever BoEF obtained using the natural frequencies recorded from Abaqus: (a) to (f) are modes 1 to 6.

Appendix B

Natural Frequencies and Temporal Plots of Beams on Nonlinear Quadratic and Nonlinear Cubic Elastic Foundations

B.1 Foundation with Quadratic Nonlinearity

B.1.1 Natural Frequencies of the BoNQEF

Tables B.1 to B.6 displays the nonlinear natural frequencies calculated analytically by varying the relative nonlinear stiffness ratio $r_q = \frac{k_2}{k_1}$ with different values of the linear stiffness coefficient k_1 taken as a factor of the beam's stiffness coefficient k_b .

Table B.1: Nonlinear natural frequencies Ω_q (rad/s) of the beam on a linear and nonlinear quadratic elastic foundation obtained for the case $k_1 = 0.75k_b$.

[illegible]

Table B.2: Nonlinear natural frequencies Ω_q (rad/s) of the beam on a linear and nonlinear quadratic elastic foundation obtained for the case $k_1 = k_b$.

[illegible]

Table B.3: Nonlinear natural frequencies Ω_q (rad/s) of the beam on a linear and nonlinear quadratic elastic foundation obtained for the case $k_1 = 1.25k_b$.

[illegible]

Table B.4: Nonlinear natural frequencies Ω_q (rad/s) of the beam on a linear and nonlinear quadratic elastic foundation obtained for the case $k_1 = 1.5k_p$.

[illegible]

Table B.5: Nonlinear natural frequencies Ω_q (rad/s) of the beam on a linear and nonlinear quadratic elastic foundation obtained for the case $k_1 = 1.75k_p$.

[illegible]

Table B.6: Nonlinear natural frequencies Ω_q (rad/s) of the beam on a linear and nonlinear quadratic elastic foundation obtained for the case $k_1 = 2k_p$.

[illegible]

B.1.2 Temporal Plots related to the Natural Frequencies at Modes 1 to 4 for other Values of the Linear Stiffness parameter for the BoNQEF

Figures B.1 to B.4 exhibit the temporal plots of the beam on an elastic foundation having nonlinear quadratic stiffness properties. These plots are to complete those presented in Subsection 5.2.2.

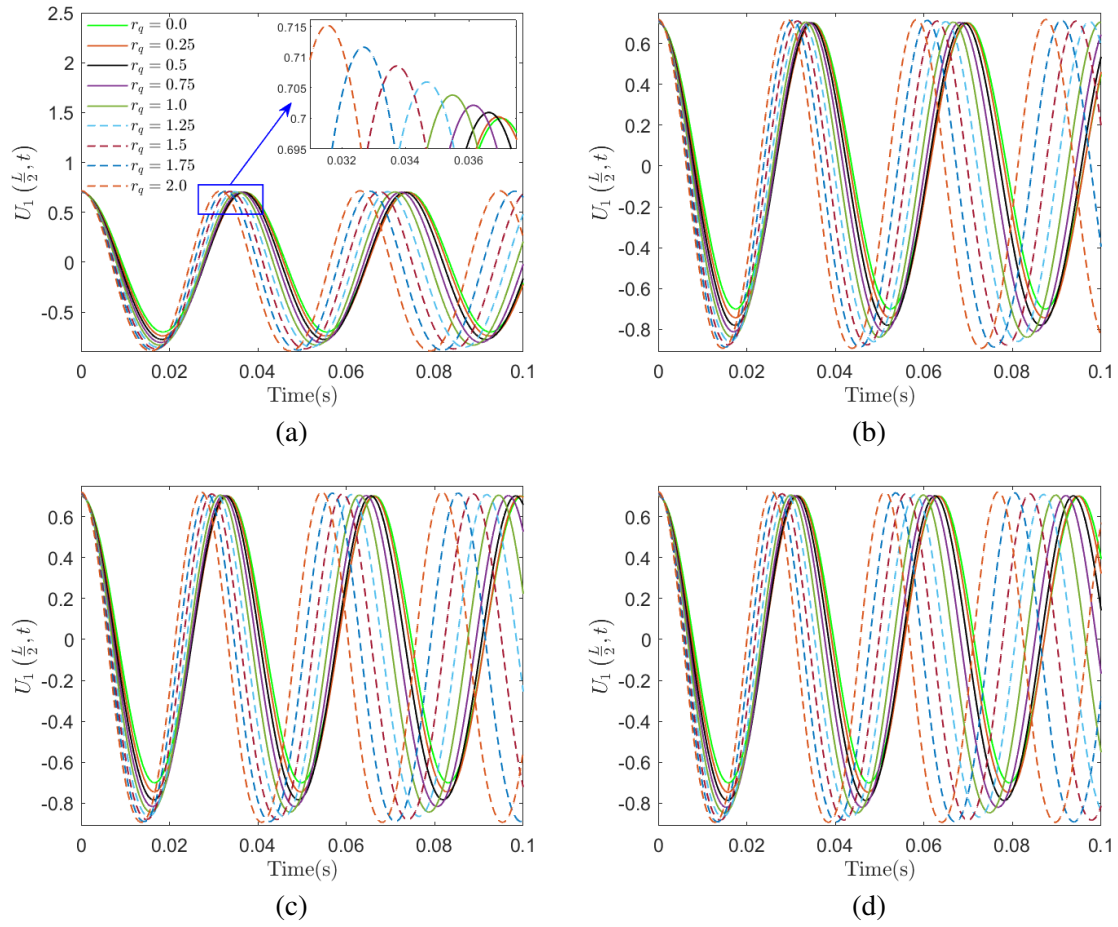


Figure B.1: Temporal displacement at the centre of the beam on a nonlinear quadratic elastic foundation obtained at mode 1: (a) $k_1 = 1.25 \times k_b$, (b) $k_1 = 1.5 \times k_b$, (c) $k_1 = 1.75 \times k_b$, and (d) $k_1 = 2 \times k_b$.

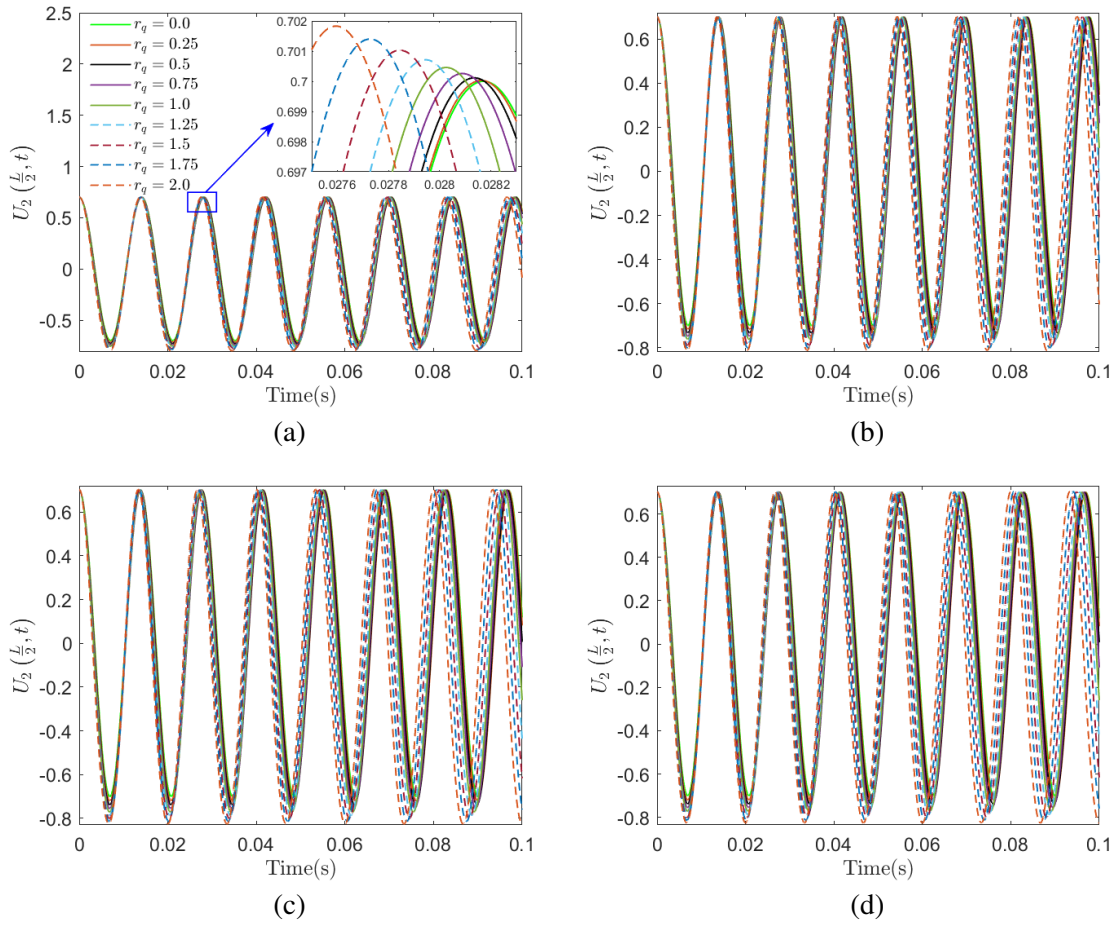
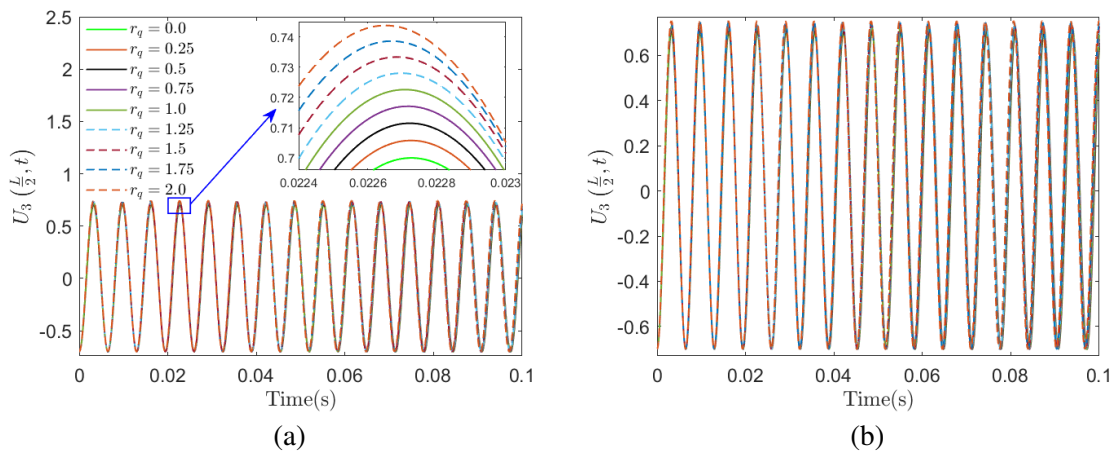


Figure B.2: Temporal displacement at the centre of the beam on a nonlinear quadratic elastic foundation obtained at mode 2: (a) $k_1 = 1.25 \times k_b$, (b) $k_1 = 1.5 \times k_b$, (c) $k_1 = 1.75 \times k_b$, and (d) $k_1 = 2 \times k_b$.



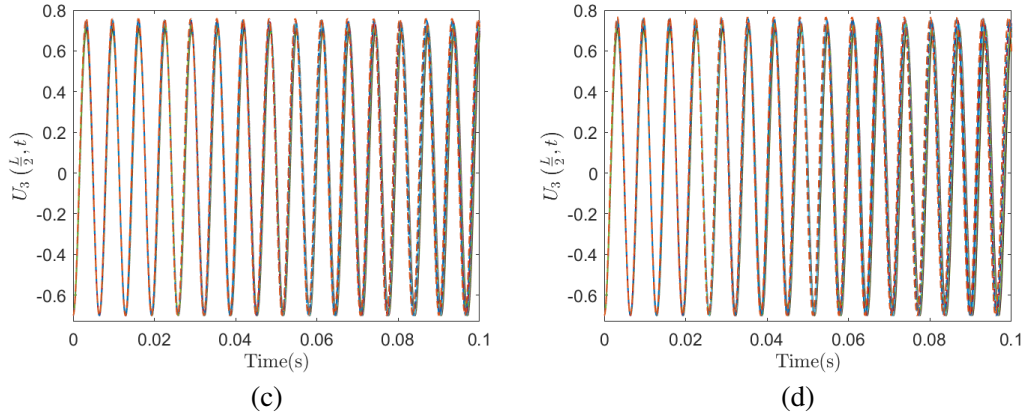


Figure B.3: Temporal displacement at the centre of the beam on a nonlinear quadratic elastic foundation obtained at mode 3: (a) $k_1 = 1.25 \times k_b$, (b) $k_1 = 1.5 \times k_b$, (c) $k_1 = 1.75 \times k_b$, and (d) $k_1 = 2 \times k_b$.

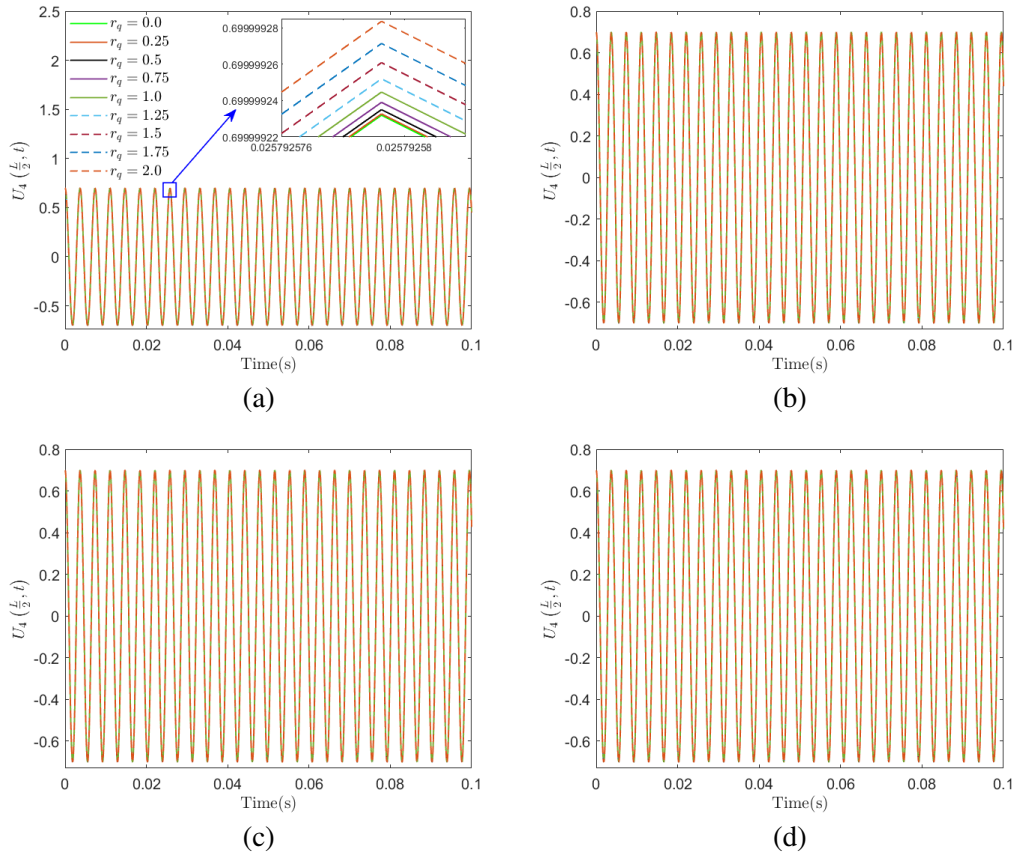


Figure B.4: Temporal displacement at the centre of the beam on a nonlinear quadratic elastic foundation obtained at mode 4: (a) $k_1 = 1.25 \times k_b$, (b) $k_1 = 1.5 \times k_b$, (c) $k_1 = 1.75 \times k_b$, and (d) $k_1 = 2 \times k_b$.

B.1.3 Comparison between Analytical and Numerical Natural Frequencies of the BoNQEF

Tables B.7 to B.12 show the comparison between nonlinear natural frequencies calculated analytically to those computed in Abaqus. These were obtained by varying the relative nonlinear stiffness ratio $r_q = \frac{k_2}{k_1}$ with different values of the linear stiffness coefficient k_1 taken as a factor of the beam's stiffness coefficient k_b . It should be highlighted that Tables B.7 to B.12 complete those in Subsection 5.2.3.

Table B.7: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNQE: Case 3 $k_1 = 0.75k_p$.

	Ω_q (rad/s)		Diff	Ω_q (rad/s)		Diff	Ω_q (rad/s)		Diff
Mode	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)
n	$r_q = 0$			$r_q = 0.25$			$r_q = 0.5$		
1	147.73	147.73	0	147.88	147.91	0.02	148.3	148.09	0.1
2	438.17	438.17	0	438.18	438.23	0.01	438.23	438.29	0.01
3	964.04	964.04	0	964.04	964.06	0.002	964.04	964.09	0.005
4	1707.2	1707.2	0	1707.2	1707.2	0	1707.2	1707.3	0.006
5	2664.7	2664.7	0	2664.7	2664.7	0	2664.7	2664.7	0
6	3835.7	3835.7	0	3835.7	3835.7	0	3835.7	3835.7	0
7	5220.0	5220.0	0	5220	5220.0	0	5220.0	5220.0	0
8	6817.4	6817.4	0	6817.4	6817.4	0	6817.4	6817.4	0
9	8627.9	8627.9	0	8627.9	8627.9	0	8627.9	8627.9	0
10	10651	10651	0	10651	10651	0	10651	10651	0
n	$r_q = 0.75$			$r_q = 1.0$			$r_q = 1.25$		
1	149	148.27	0.5	149.99	148.45	1.04	150.8	148.63	1.5
2	438.31	438.35	0.009	438.42	438.41	0.002	438.56	438.47	0.02
3	964.05	964.12	0.007	964.06	964.14	0.008	964.07	964.17	0.01
4	1707.2	1707.3	0.006	1707.2	1707.3	0.006	1707.2	1707.3	0.006
5	2664.7	2664.7	0	2664.7	2664.7	0	2664.7	2664.7	0
6	3835.7	3835.7	0	3835.7	3835.7	0	3835.7	3835.7	0
7	5220.0	5220.0	0	5220.0	5220.0	0	5220.0	5220.0	0
8	6817.4	6817.4	0	6817.4	6817.4	0	6817.4	6817.4	0
9	8627.9	8627.9	0	8627.9	8627.9	0	8627.9	8627.9	0
10	10651	10651	0	10651	10651	0	10651	10651	0
n	$r_q = 1.5$			$r_q = 1.75$			$r_q = 2.0$		
1	151.55	148.81	1.8	152.16	148.98	2.1	153.53	149.46	2.7
2	438.73	438.53	0.05	438.94	438.59	0.08	439.17	438.65	0.1
3	964.08	964.2	0.01	964.1	964.23	0.01	964.12	964.25	0.01
4	1707.2	1707.3	0.006	1707.2	1707.3	0.006	1707.2	1707.3	0.006
5	2664.7	2664.8	0.004	2664.7	2664.8	0.004	2664.7	2664.8	0.004
6	3835.7	3835.7	0	3835.7	3835.7	0	3835.7	3835.8	0.003
7	5220.0	5220.0	0	5220.0	5220.0	0	5220.0	5220.0	0
8	6817.4	6817.4	0	6817.4	6817.4	0	6817.4	6817.4	0
9	8627.9	8627.9	0	8627.9	8627.9	0	8627.9	8627.9	0
10	10651	10651	0	10651	10651	0	10651	10651	0

Table B.8: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNQEF: Case 4 $k_1 = k_p$.

	Ω_q (rad/s)			Diff	Ω_q (rad/s)			Diff	Ω_q (rad/s)			Diff
Mode	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)
n	$r_q = 0$			$r_q = 0.25$			$r_q = 0.5$					
1	159.12	159.12	0	159.32	159.34	0.01	159.92	159.56	0.2			
2	442.13	442.13	0	442.16	442.21	0.01	442.24	442.29	0.01			
3	965.85	965.85	0	965.85	965.88	0.003	965.86	965.92	0.006			
4	1708.2	1708.2	0	1708.2	1708.3	0.006	1708.2	1708.3	0.006			
5	2665.4	2665.4	0	2665.4	2665.4	0	2665.4	2665.4	0			
6	3836.2	3836.2	0	3836.2	3836.2	0	3836.2	3836.2	0			
7	5220.3	5220.3	0	5220.3	5220.3	0	5220.3	5220.3	0			
8	6817.6	6817.6	0	6817.6	6817.6	0	6817.6	6817.6	0			
9	8628.1	8628.1	0	8628.1	8628.1	0	8628.1	8628.1	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			
n	$r_q = 0.75$			$r_q = 1.0$			$r_q = 1.25$					
1	160.48	159.78	0.4	161.53	160.0	0.9	162.33	160.23	1.3			
2	442.38	442.37	0.002	442.57	442.45	0.03	442.82	442.53	0.07			
3	965.87	965.95	0.008	965.9	965.99	0.01	965.91	966.03	0.01			
4	1708.2	1708.3	0.006	1708.3	1708.3	0	1708.3	1708.3	0			
5	2665.4	2665.4	0	2665.4	2665.4	0	2665.4	2665.4	0			
6	3836.2	3836.2	0	3836.2	3836.2	0	3836.2	3836.2	0			
7	5220.3	5220.3	0	5220.3	5220.3	0	5220.3	5220.3	0			
8	6817.6	6817.6	0	6817.6	6817.6	0	6817.6	6817.6	0			
9	8628.1	8628.1	0	8628.1	8628.1	0	8628.1	8628.1	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			
n	$r_q = 1.5$			$r_q = 1.75$			$r_q = 2.0$					
1	162.84	160.45	1.5	163.4	160.66	1.7	165.3	160.88	2.75			
2	443.11	442.61	0.1	443.47	442.69	0.2	443.88	442.77	0.3			
3	965.93	966.06	0.01	965.96	966.1	0.01	966	966.14	0.01			
4	1708.3	1708.4	0.006	1708.3	1708.4	0.006	1708.3	1708.4	0.006			
5	2665.4	2665.4	0	2665.4	2665.4	0	2665.4	2665.5	0.0034			
6	3836.2	3836.2	0	3836.2	3836.2	0	3836.2	3836.2	0			
7	5220.3	5220.3	0	5220.3	5220.3	0	5220.3	5220.3	0			
8	6817.6	6817.6	0	6817.6	6817.7	0.001	6817.6	6817.7	0.001			
9	8628.1	8628.1	0	8628.1	8628.1	0	8628.1	8628.1	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			

Table B.9: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNQEF: Case 5 $k_1 = 1.25k_b$.

	Ω_q (rad/s)			Diff	Ω_q (rad/s)			Diff	Ω_q (rad/s)			Diff
Mode	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)
n	$r_q = 0$			$r_q = 0.25$			$r_q = 0.5$			$r_q = 0.5$		
1	169.74	169.74	0	170.0	170.0	0	170.78	170.26	0.3			
2	446.07	446.07	0	446.11	446.17	0.01	446.23	446.27	0.01			
3	967.65	967.65	0	967.66	967.7	0.004	967.67	967.74	0.007			
4	1709.3	1709.3	0	1709.3	1709.3	0	1709.3	1709.3	0			
5	2666.0	2666.0	0	2666.0	2666.0	0	2666.0	2666.0	0			
6	3836.6	3836.6	0	3836.6	3836.6	0	3836.6	3836.6	0			
7	5220.6	5220.6	0	5220.6	5220.6	0	5220.6	5220.6	0			
8	6817.9	6817.9	0	6817.9	6817.9	0	6817.9	6817.9	0			
9	8628.3	8628.3	0	8628.3	8628.3	0	8628.3	8628.3	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			
n	$r_q = 0.75$			$r_q = 1.0$			$r_q = 1.25$			$r_q = 1.25$		
1	171.2	170.52	0.4	171.84	170.78	0.6	172.87	171.04	1.1			
2	446.44	446.36	0.02	446.73	446.46	0.06	447.1	446.56	0.1			
3	967.69	967.79	0.01	967.71	967.83	0.01	967.75	967.88	0.01			
4	1709.3	1709.3	0	1709.3	1709.4	0.006	1709.3	1709.4	0.006			
5	2666.0	2666.1	0.004	2666.0	2666.1	0.004	2666	2666.1	0.004			
6	3836.6	3836.6	0	3836.6	3836.7	0.003	3836.6	3836.7	0.003			
7	5220.6	5220.7	0.002	5220.6	5220.7	0.002	5220.6	5220.7	0.002			
8	6817.9	6817.9	0	6817.9	6817.9	0	6817.9	6817.9	0			
9	8628.3	8628.3	0	8628.3	8628.3	0	8628.3	8628.3	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			
n	$r_q = 1.5$			$r_q = 1.75$			$r_q = 2.0$			$r_q = 2.0$		
1	173.5	171.3	1.3	174.94	171.55	1.98	176.1	171.81	2.5			
2	447.6	446.66	0.2	448.1	446.76	0.3	448.72	446.86	0.4			
3	967.8	967.93	0.01	967.84	967.97	0.01	967.89	968.02	0.01			
4	1709.3	1709.4	0.006	1709.3	1709.4	0.006	1709.3	1709.5	0.01			
5	2666.0	2666.1	0.004	2666.0	2666.1	0.004	2666.0	2666.1	0.004			
6	3836.6	3836.7	0.003	3836.6	3836.7	0.003	3836.6	3836.7	0.003			
7	5220.6	5220.7	0.002	5220.6	5220.7	0.002	5220.6	5220.7	0.002			
8	6817.9	6817.9	0	6817.9	6817.9	0	6817.9	6817.9	0			
9	8628.3	8628.3	0	8628.3	8628.3	0	8628.3	8628.3	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			

Table B.10: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNQE: Case 6 $k_1 = 1.5k_b$.

	Ω_q (rad/s)			Diff	Ω_q (rad/s)			Diff	Ω_q (rad/s)			Diff
Mode	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)
n	$r_q = 0$			$r_q = 0.25$			$r_q = 0.5$					
1	179.74	179.74	0	179.88	180.03	0.08	180.3	180.33	0.02			
2	449.97	449.97	0	450.03	450.08	0.01	450.2	450.2	0			
3	969.46	969.46	0	969.46	969.51	0.005	969.48	969.56	0.008			
4	1710.3	1710.3	0	1710.3	1710.3	0	1710.3	1710.3	0			
5	2666.7	2666.7	0	2666.7	2666.7	0	2666.7	2666.7	0			
6	3837.1	3837.1	0	3837.1	3837.1	0	3837.1	3837.1	0			
7	5221.0	5221.0	0	5221.0	5221.0	0	5221.0	5221.0	0			
8	6818.1	6818.1	0	6818.1	6818.1	0	6818.1	6818.1	0			
9	8628.5	8628.5	0	8628.5	8628.5	0	8628.5	8628.5	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			
n	$r_q = 0.75$			$r_q = 1.0$			$r_q = 1.25$					
1	180.99	180.62	0.2	181.97	180.91	0.6	182.22	181.21	0.6			
2	450.5	450.32	0.04	450.9	450.44	0.1	451.42	450.55	0.2			
3	969.5	969.62	0.01	969.54	969.67	0.01	969.59	969.73	0.01			
4	1710.3	1710.4	0.006	1710.3	1710.4	0.005	1710.3	1710.4	0.006			
5	2666.7	2666.7	0	2666.7	2666.7	0	2666.7	2666.7	0			
6	3837.1	3837.1	0	3837.1	3837.1	0	3837.1	3837.1	0			
7	5221.0	5221.0	0	5221.0	5221.0	0	5221.0	5221.0	0			
8	6818.1	6818.2	0.001	6818.1	6818.2	0.001	6818.1	6818.2	0.001			
9	8628.5	8628.5	0	8628.5	8628.5	0	8628.5	8628.5	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			
n	$r_q = 1.5$			$r_q = 1.75$			$r_q = 2.0$					
1	182.75	181.5	0.7	184.2	181.79	1.3	186.5	182.08	2.4			
2	452.1	450.67	0.3	452.82	450.79	0.5	453.7	450.91	0.6			
3	969.65	969.78	0.01	969.72	969.84	0.01	969.8	969.89	0.01			
4	1710.3	1710.5	0.01	1710.3	1710.5	0.01	1710.3	1710.5	0.01			
5	2666.7	2666.8	0.004	2666.7	2666.8	0.004	2666.7	2666.8	0.004			
6	3837.1	3837.1	0	3837.1	3837.1	0	3837.1	3837.1	0			
7	5221.0	5221.0	0	5221.0	5221.0	0	5221.0	5221.0	0			
8	6818.1	6818.2	0.001	6818.1	6818.2	0.001	6818.1	6818.2	0.001			
9	8628.5	8628.5	0	8628.5	8628.5	0	8628.5	8628.5	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			

Table B.11: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNQE: Case 7 $k_1 = 1.75k_p$.

	Ω_q (rad/s)		Diff (%)	Ω_q (rad/s)		Diff (%)	Ω_q (rad/s)		Diff (%)
Mode	Anal	Abaq		Anal	Abaq		Anal	Abaq	
n	$r_q = 0$			$r_q = 0.25$			$r_q = 0.5$		
1	189.21	189.21	0	189.37	189.54	0.09	189.86	189.86	0
2	453.83	453.83	0	453.9	453.97	0.01	454.14	454.1	0.01
3	971.26	971.26	0	971.26	971.32	0.006	971.29	971.38	0.01
4	1711.3	1711.3	0	1711.3	1711.3	0	1711.3	1711.4	0.006
5	2667.3	2667.3	0	2667.3	2667.3	0	2667.3	2667.4	0.004
6	3837.5	3837.5	0	3837.5	3837.5	0	3837.5	3837.5	0
7	5221.3	5221.3	0	5221.3	5221.3	0	5221.3	5221.3	0
8	6818.4	6818.4	0	6818.4	6818.4	0	6818.4	6818.4	0
9	8628.7	8628.7	0	8628.7	8628.7	0	8628.7	8628.7	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
n	$r_q = 0.75$			$r_q = 1.0$			$r_q = 1.25$		
1	190.67	190.19	0.3	191.71	190.51	0.6	192.27	190.83	0.7
2	454.53	454.24	0.06	455.07	454.38	0.2	455.76	454.51	0.3
3	971.32	971.45	0.01	971.37	971.51	0.01	971.44	971.52	0.008
4	1711.3	1711.4	0.006	1711.3	1711.5	0.01	1711.3	1711.5	0.01
5	2667.3	2667.4	0.004	2667.3	2667.4	0.004	2667.3	2667.4	0.004
6	3837.5	3837.5	0	3837.5	3837.5	0	3837.5	3837.6	0.003
7	5221.3	5221.3	0	5221.3	5221.3	0	5221.3	5221.4	0.002
8	6818.4	6818.4	0	6818.4	6818.4	0	6818.4	6818.4	0
9	8628.7	8628.7	0	8628.7	8628.7	0	8628.7	8628.7	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
n	$r_q = 1.5$			$r_q = 1.75$			$r_q = 2.0$		
1	193.08	191.16	1.0	194.5	191.48	1.6	196.6	191.8	2.5
2	456.61	454.65	0.4	457.6	454.78	0.6	458.8	454.92	0.9
3	971.52	971.64	0.01	971.6	971.7	0.01	971.7	971.76	0.006
4	1711.4	1711.5	0.006	1711.4	1711.6	0.01	1711.4	1711.6	0.01
5	2667.3	2667.5	0.007	2667.3	2667.5	0.007	2667.3	2667.5	0.007
6	3837.5	3837.6	0.003	3837.5	3837.6	0.003	3837.5	3837.6	0.003
7	5221.3	5221.4	0.001	5221.3	5221.4	0.002	5221.3	5221.4	0.002
8	6818.4	6818.4	0	6818.4	6818.5	0.001	6818.4	6818.5	0.001
9	8628.7	8628.7	0	8628.7	8628.7	0	8628.7	8628.7	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01

Table B.12: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNQEF: Case 8 $k_1 = 2k_b$.

	Ω_q (rad/s)		Diff (%)	Ω_q (rad/s)		Diff (%)	Ω_q (rad/s)		Diff (%)
Mode	Anal	Abaq		Anal	Abaq		Anal	Abaq	
n	$r_q = 0$			$r_q = 0.25$			$r_q = 0.5$		
1	198.23	198.23	0	198.41	198.58	0.09	198.97	198.54	0.2
2	457.67	457.66	0.002	457.76	457.82	0.01	458.06	457.97	0.02
3	973.05	973.05	0	973.06	973.13	0.007	973.09	973.2	0.01
4	1712.3	1712.3	0	1712.3	1712.4	0.006	1712.3	1712.4	0.006
5	2668.0	2668.0	0	2668.0	2668.0	0	2668.0	2668.0	0
6	3838.0	3838.0	0	3838.0	3838.0	0	3838.0	3838.0	0
7	5221.6	5221.6	0	5221.6	5221.6	0	5221.6	5221.7	0.002
8	6818.6	6818.6	0	6818.6	6818.7	0.001	6818.6	6818.7	0.001
9	8628.9	8628.9	0	8628.9	8628.9	0	8628.9	8628.9	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
n	$r_q = 0.75$			$r_q = 1.0$			$r_q = 1.25$		
1	199.88	199.29	0.3	200.2	199.65	0.3	201.4	200	0.7
2	458.55	458.13	0.09	459.2	458.28	0.2	460.1	458.44	0.4
3	973.14	973.27	0.01	973.2	973.34	0.01	973.3	973.42	0.01
4	1712.3	1712.5	0.01	1712.4	1712.5	0.006	1712.4	1712.5	0.006
5	2668.0	2668.1	0.004	2668.0	2668.1	0.004	2668.0	2668.1	0.004
6	3838.0	3838.0	0	3838.0	3838.1	0.003	3838.0	3838.1	0.003
7	5221.6	5221.7	0.002	5221.6	5221.7	0.002	5221.6	5221.7	0.002
8	6818.6	6818.7	0.001	6818.6	6818.7	0.001	6818.6	6818.7	0.001
9	8628.9	8628.9	0	8628.9	8628.9	0	8628.9	8628.9	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
n	$r_q = 1.5$			$r_q = 1.75$			$r_q = 2.0$		
1	202.2	200.35	0.9	204.3	200.7	1.8	206.9	201.05	2.9
2	461.2	458.59	0.6	462.5	458.74	0.8	463.95	458.9	1.1
3	973.4	973.49	0.01	973.52	973.56	0.004	973.7	973.63	0.007
4	1712.4	1712.6	0.01	1712.4	1712.6	0.01	1712.4	1712.7	0.02
5	2668.0	2668.1	0.004	2668.0	2668.2	0.007	2668.0	2668.2	0.007
6	3838.0	3838.1	0.003	3838.0	3838.1	0.003	3838.0	3838.1	0.003
7	5221.6	5221.7	0.002	5221.6	5221.7	0.002	5221.6	5221.7	0.002
8	6818.6	6818.7	0.001	6818.6	6818.7	0.001	6818.6	6818.7	0.001
9	8628.9	8628.9	0	8628.9	8628.9	0	8628.9	8628.9	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01

[illegible]

cubic elastic foundation obtained for the case $k_1 = 1.25k_b$.

[illegible]

cubic elastic foundation obtained for the case $k_1 = 1.5k_b$.

[illegible]

cubic elastic foundation obtained for the case $k_1 = 1.75k_b$.

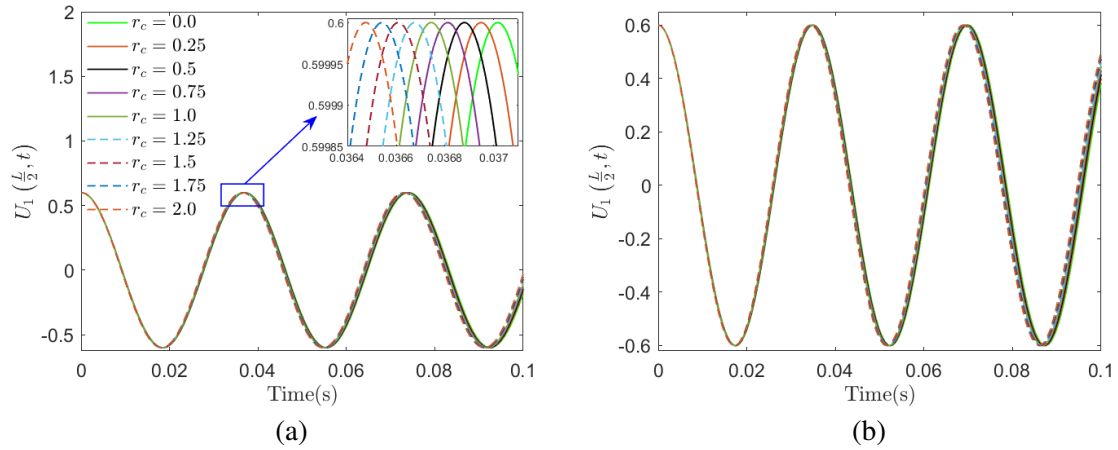
[illegible]

Table B.18: Nonlinear natural frequencies Ω_c (rad/s) of the beam on a linear and nonlinear cubic elastic foundation obtained for the case $k_1 = 2k_b$.

m	$r_c = 0$	$r_c = 0.25$	$r_c = 0.5$	$r_c = 0.75$	$r_c = 1.0$	$r_c = 1.25$	$r_c = 1.5$	$r_c = 1.75$	$r_c = 2.0$
1	198.23	198.28	198.32	198.37	198.42	198.47	198.52	198.56	198.61
2	457.67	457.69	457.7	457.73	457.75	457.77	457.8	457.81	457.83
3	973.05	973.06	973.07	973.08	973.09	973.1	973.11	973.12	973.13
4	1712.3	1712.3	1712.3	1712.3	1712.4	1712.4	1712.4	1712.4	1712.4
5	2668	2668	2668	2668	2668	2668	2668	2668	2668
6	3838	3838	3838	3838	3838	3838	3838	3838	3838
7	5221.6	5221.6	5221.6	5221.6	5221.6	5221.6	5221.6	5221.6	5221.6
8	6818.6	6818.6	6818.6	6818.6	6818.6	6818.6	6818.6	6818.6	6818.6
9	8628.9	8628.9	8628.9	8628.9	8628.9	8628.9	8628.9	8628.9	8628.9
10	10652	10652	10652	10652	10652	10652	10652	10652	10652

B.2.2 Temporal Plots related to the Natural Frequencies at Modes 1 to 4 for other Values of the Linear Stiffness parameter for the BoNCEF

Figures B.5 to B.8 exhibit the temporal displacement plots of the beam on an elastic foundation having nonlinear cubic stiffness properties. These plots were obtained in MATLAB and are meant to complete those presented in Subsection 5.3.2.



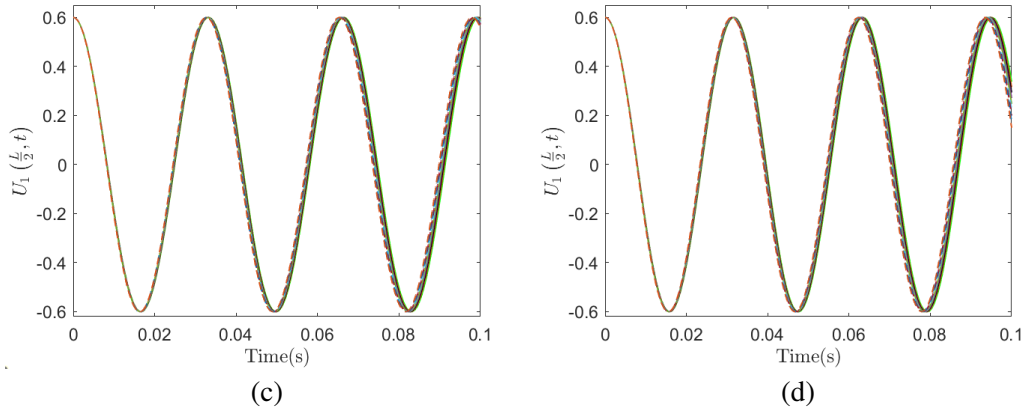


Figure B.5: Temporal displacement at the centre of the beam on a nonlinear cubic elastic foundation obtained at mode 1: (a) $k_1 = 1.25 \times k_b$, (b) $k_1 = 1.5 \times k_b$, (c) $k_1 = 1.75 \times k_b$, and (d) $k_1 = 2 \times k_b$

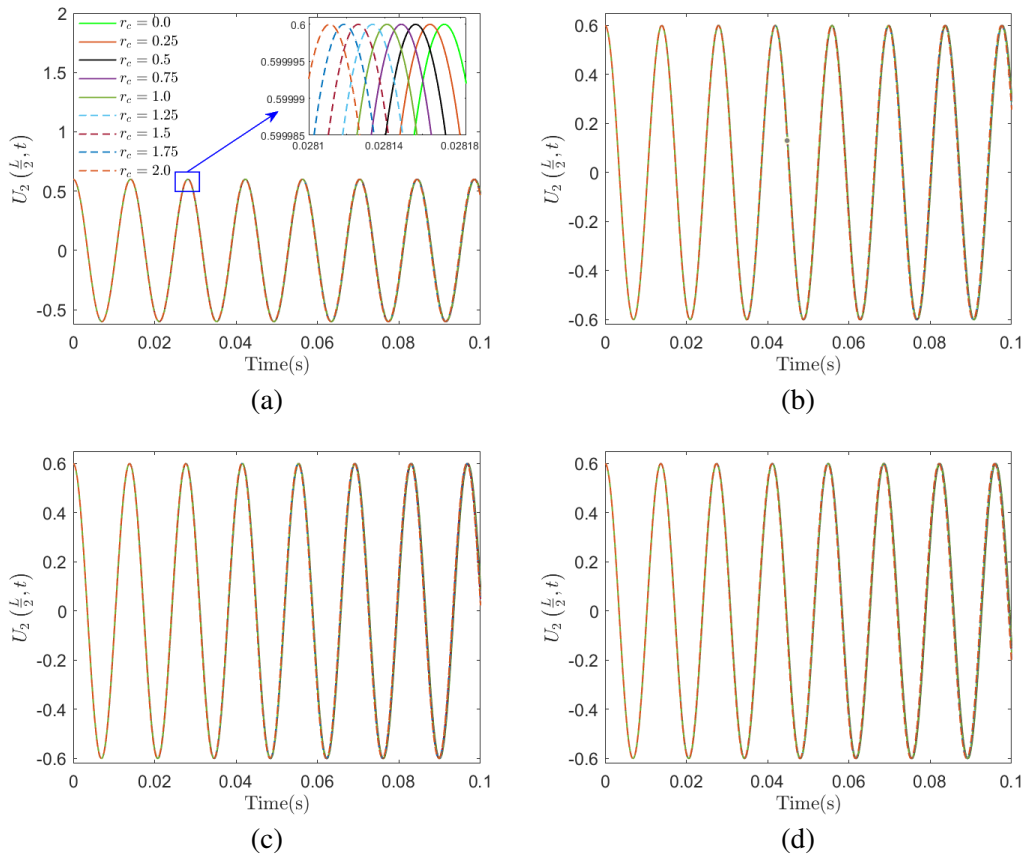


Figure B.6: Temporal displacement at the centre of the beam on a nonlinear cubic elastic foundation obtained at mode 2: (a) $k_1 = 1.25 \times k_b$, (b) $k_1 = 1.5 \times k_b$, (c) $k_1 = 1.75 \times k_b$, and (d) $k_1 = 2 \times k_b$

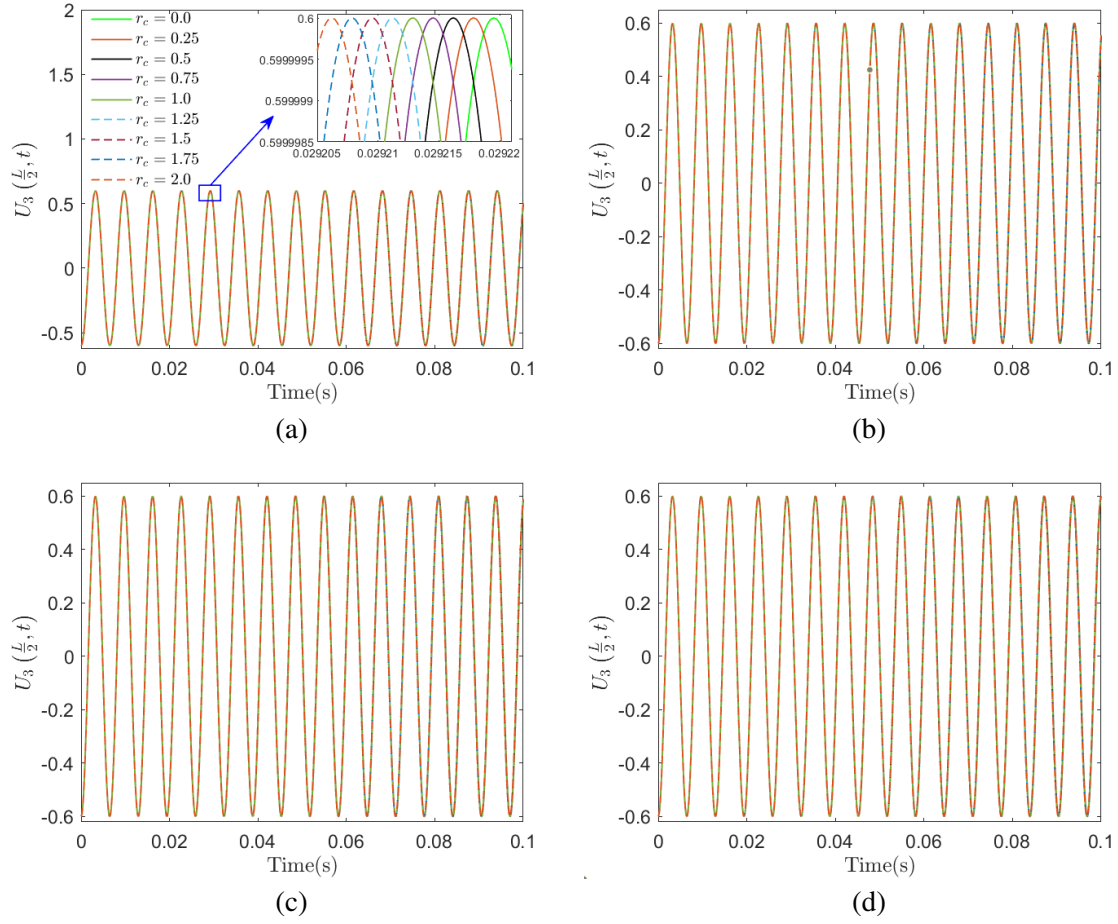
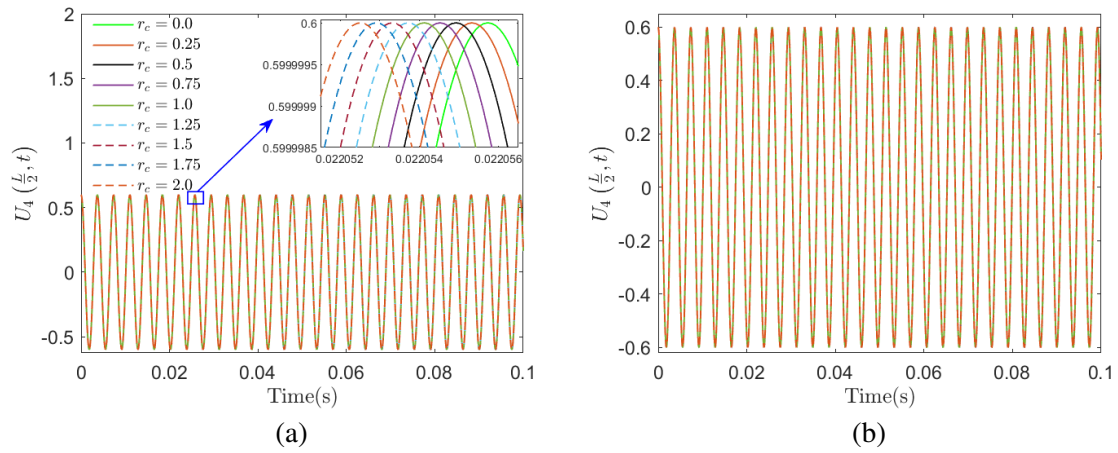


Figure B.7: Temporal displacement at the centre of the beam on a nonlinear cubic elastic foundation obtained at mode 3: (a) $k_1 = 1.25 \times k_b$, (b) $k_1 = 1.5 \times k_b$, (c) $k_1 = 1.75 \times k_b$, and (d) $k_1 = 2 \times k_b$



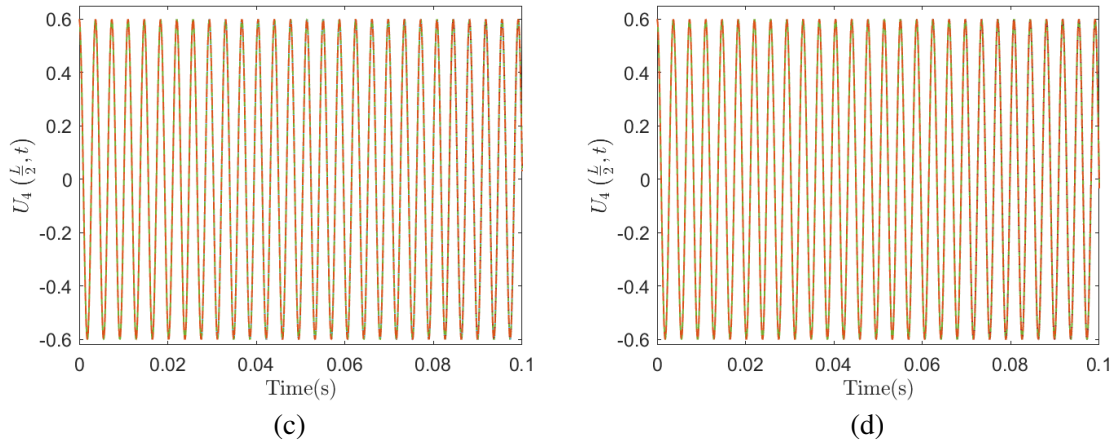


Figure B.8: Temporal displacement at the centre of the beam on a nonlinear cubic elastic foundation obtained at mode 4: (a) $k_1 = 1.25 \times k_b$, (b) $k_1 = 1.5 \times k_b$, (c) $k_1 = 1.75 \times k_b$, and (d) $k_1 = 2 \times k_b$

B.2.3 Comparison between Analytical and Numerical Natural Frequencies of the BoNCEF

Tables B.7 to B.12 show the comparison between nonlinear natural frequencies calculated analytically to those computed in Abaqus. These were obtained by varying the relative nonlinear stiffness ratio $r_c = \frac{k_3}{k_1}$ with different values of the linear stiffness coefficient k_1 taken as a factor of the beam's stiffness coefficient k_b .

Table B.19: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNCEF: Case 3 $k_1 = 0.75k_b$.

	Ω_c (rad/s)			Diff	Ω_c (rad/s)			Diff	Ω_c (rad/s)			Diff
Mode	Anal	Abaq	(%)		Anal	Abaq	(%)		Anal	Abaq	(%)	
m	$r_c = 0$				$r_c = 0.25$				$r_c = 0.5$			
1	147.73	147.73	0		147.75	147.74	0.007		147.78	147.74	0.03	
2	438.17	438.17	0		438.17	438.17	0		438.18	438.17	0.002	
3	964.04	964.04	0		964.04	964.04	0		964.04	964.04	0	
4	1707.2	1707.2	0		1707.2	1707.2	0		1707.2	1707.2	0	
5	2664.7	2664.7	0		2664.7	2664.7	0		2664.7	2664.7	0	
6	3835.7	3835.7	0		3835.7	3835.7	0		3835.7	3835.7	0	
7	5220	5220	0		5220	5220	0		5220	5220	0	
8	6817.4	6817.4	0		6817.4	6817.4	0		6817.4	6817.4	0	
9	8627.9	8627.9	0		8627.9	8627.9	0		8627.9	8627.9	0	
10	10651	10651	0		10651	10651	0		10651	10651	0	
m	$r_c = 0.75$				$r_c = 1.0$				$r_c = 1.25$			
1	147.8	147.75	0.03		147.83	147.75	0.05		147.85	147.75	0.07	
2	438.19	438.17	0.005		438.19	438.17	0.005		438.2	438.17	0.007	
3	964.04	964.04	0		964.05	964.04	0.001		964.05	964.04	0.001	
4	1707.2	1707.2	0		1707.2	1707.2	0		1707.2	1707.2	0	
5	2664.7	2664.7	0		2664.7	2664.7	0		2664.7	2664.7	0	
6	3835.7	3835.7	0		3835.7	3835.7	0		3835.7	3835.7	0	
7	5220	5220	0		5220	5220	0		5220	5220	0	
8	6817.4	6817.4	0		6817.4	6817.4	0		6817.4	6817.4	0	
9	8627.9	8627.9	0		8627.9	8627.9	0		8627.9	8627.9	0	
10	10651	10651	0		10651	10651	0		10651	10651	0	
m	$r_c = 1.5$				$r_c = 1.75$				$r_c = 2.0$			
1	147.88	147.76	0.08		147.9	147.76	0.09		147.93	147.76	0.1	
2	438.21	438.17	0.009		438.22	438.17	0.01		438.23	438.18	0.01	
3	964.06	964.04	0.002		964.06	964.04	0.002		964.06	964.04	0.002	
4	1707.2	1707.2	0		1707.2	1707.2	0		1707.2	1707.2	0	
5	2664.7	2664.7	0		2664.7	2664.7	0		2664.7	2664.7	0	
6	3835.7	3835.7	0		3835.7	3835.7	0		3835.7	3835.7	0	
7	5220	5220	0		5220	5220	0		5220	5220	0	
8	6817.4	6817.4	0		6817.4	6817.4	0		6817.4	6817.4	0	
9	8627.9	8627.9	0		8627.9	8627.9	0		8627.9	8627.9	0	
10	10651	10651	0		10651	10651	0		10651	10651	0	

Table B.20: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNCEF: Case 4 $k_1 = k_b$.

	Ω_c (rad/s)		Diff (%)	Ω_c (rad/s)		Diff (%)	Ω_c (rad/s)		Diff (%)
Mode	Anal	Abaq		Anal	Abaq		Anal	Abaq	
m	$r_c = 0$			$r_c = 0.25$			$r_c = 0.5$		
1	159.12	159.12	0	159.15	159.12	0.02	159.18	159.13	0.03
2	442.13	442.13	0	442.14	442.13	0.002	442.15	442.14	0.002
3	965.85	965.85	0	965.85	965.85	0	965.86	965.85	0.0012
4	1708.2	1708.2	0	1708.2	1708.2	0	1708.2	1708.2	0
5	2665.4	2665.4	0	2665.4	2665.4	0	2665.4	2665.4	0
6	3836.2	3836.2	0	3836.2	3836.2	0	3836.2	3836.2	0
7	5220.3	5220.3	0	5220.3	5220.3	0	5220.3	5220.3	0
8	6817.6	6817.6	0	6817.6	6817.6	0	6817.6	6817.6	0
9	8628.1	8628.1	0	8628.1	8628.1	0	8628.1	8628.1	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
m	$r_c = 0.75$			$r_c = 1.0$			$r_c = 1.25$		
1	159.2	159.13	0.04	159.24	159.13	0.07	159.27	159.14	0.08
2	442.17	442.14	0.007	442.17	442.14	0.007	442.19	442.14	0.01
3	965.86	965.85	0.0017	965.86	965.85	0.001	965.87	965.85	0.002
4	1708.3	1708.2	0.006	1708.3	1708.2	0.006	1708.3	1708.2	0.006
5	2665.4	2665.4	0	2665.4	2665.4	0	2665.4	2665.4	0
6	3836.2	3836.2	0	3836.2	3836.2	0	3836.2	3836.2	0
7	5220.3	5220.3	0	5220.3	5220.3	0	5220.3	5220.3	0
8	6817.6	6817.6	0	6817.6	6817.6	0	6817.6	6817.6	0
9	8628.1	8628.1	0	8628.1	8628.1	0	8628.1	8628.1	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
m	$r_c = 1.5$			$r_c = 1.75$			$r_c = 2.0$		
1	159.3	159.15	0.09	159.33	159.15	0.1	159.36	159.16	0.1
2	442.2	442.14	0.01	442.21	442.15	0.01	442.22	442.15	0.02
3	965.87	965.85	0.002	965.88	965.85	0.003	965.88	965.85	0.003
4	1708.3	1708.2	0.006	1708.3	1708.2	0.006	1708.3	1708.2	0.006
5	2665.4	2665.4	0	2665.4	2665.4	0	2665.4	2665.4	0
6	3836.2	3836.2	0	3836.2	3836.2	0	3836.2	3836.2	0
7	5220.3	5220.3	0	5220.3	5220.3	0	5220.3	5220.3	0
8	6817.6	6817.6	0	6817.6	6817.6	0	6817.6	6817.6	0
9	8628.1	8628.1	0	8628.1	8628.1	0	8628.1	8628.1	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01

Table B.21: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNCEF: Case 5 $k_1 = 1.25k_b$.

	Ω_c (rad/s)		Diff (%)	Ω_c (rad/s)		Diff (%)	Ω_c (rad/s)		Diff (%)
Mode	Anal	Abaq		Anal	Abaq		Anal	Abaq	
m	$r_c = 0$			$r_c = 0.25$			$r_c = 0.5$		
1	169.74	169.74	0	169.78	169.75	0.02	169.81	169.75	0.03
2	446.07	446.07	0	446.08	446.07	0.002	446.09	446.07	0.004
3	967.65	967.65	0	967.66	967.65	0.001	967.66	967.65	0.001
4	1709.3	1709.3	0	1709.3	1709.3	0	1709.3	1709.3	0
5	2666	2666	0	2666	2666	0	2666	2666	0
6	3836.6	3836.6	0	3836.6	3836.6	0	3836.6	3836.6	0
7	5220.6	5220.6	0	5220.6	5220.6	0	5220.6	5220.6	0
8	6817.9	6817.9	0	6817.9	6817.9	0	6817.9	6817.9	0
9	8628.3	8628.3	0	8628.3	8628.3	0	8628.3	8628.3	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
m	$r_c = 0.75$			$r_c = 1.0$			$r_c = 1.25$		
1	169.84	169.76	0.05	169.88	169.76	0.07	169.92	169.77	0.09
2	446.11	446.07	0.009	446.12	446.08	0.009	446.13	446.08	0.01
3	967.67	967.66	0.001	967.68	967.66	0.002	967.68	967.66	0.002
4	1709.3	1709.3	0	1709.3	1709.3	0	1709.3	1709.3	0
5	2666	2666	0	2666	2666	0	2666	2666	0
6	3836.6	3836.6	0	3836.6	3836.6	0	3836.6	3836.6	0
7	5220.6	5220.6	0	5220.6	5220.6	0	5220.6	5220.6	0
8	6817.9	6817.9	0	6817.9	6817.9	0	6817.9	6817.9	0
9	8628.3	8628.3	0	8628.3	8628.3	0	8628.3	8628.3	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
m	$r_c = 1.5$			$r_c = 1.75$			$r_c = 2.0$		
1	169.95	169.77	0.1	169.99	169.78	0.1	170.02	169.79	0.1
2	446.15	446.08	0.02	446.16	446.08	0.02	446.17	446.08	0.02
3	967.69	967.66	0.003	967.69	967.66	0.003	967.7	967.66	0.004
4	1709.3	1709.3	0	1709.3	1709.3	0	1709.3	1709.3	0
5	2666	2666	0	2666	2666	0	2666	2666	0
6	3836.6	3836.6	0	3836.6	3836.6	0	3836.6	3836.6	0
7	5220.6	5220.6	0	5220.6	5220.6	0	5220.6	5220.6	0
8	6817.9	6817.9	0	6817.9	6817.9	0	6817.9	6817.9	0
9	8628.3	8628.3	0	8628.3	8628.3	0	8628.3	8628.3	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01

Table B.22: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNCEF: Case 6 $k_1 = 1.5k_b$.

	Ω_c (rad/s)		Diff (%)	Ω_c (rad/s)		Diff (%)	Ω_c (rad/s)		Diff (%)
Mode	Anal	Abaq		Anal	Abaq		Anal	Abaq	
m	$r_c = 0$			$r_c = 0.25$			$r_c = 0.5$		
1	179.74	179.74	0	179.78	179.75	0.02	179.82	179.75	0.04
2	449.97	449.97	0	449.98	449.97	0.002	449.99	449.97	0.004
3	969.46	969.46	0	969.46	969.46	0	969.47	969.46	0.001
4	1710.3	1710.3	0	1710.3	1710.3	0	1710.3	1710.3	0
5	2666.7	2666.7	0	2666.7	2666.7	0	2666.7	2666.7	0
6	3837.1	3837.1	0	3837.1	3837.1	0	3837.1	3837.1	0
7	5221	5221	0	5221	5221	0	5221	5221	0
8	6818.1	6818.1	0	6818.1	6818.1	0	6818.1	6818.1	0
9	8628.5	8628.5	0	8628.5	8628.5	0	8628.5	8628.5	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
m	$r_c = 0.75$			$r_c = 1.0$			$r_c = 1.25$		
1	179.86	179.76	0.06	179.9	179.76	0.08	179.74	179.77	0.02
2	450.01	449.97	0.009	450.03	449.98	0.01	450.05	449.98	0.02
3	969.48	969.46	0.002	969.48	969.46	0.002	969.49	969.46	0.003
4	1710.3	1710.3	0	1710.3	1710.3	0	1710.3	1710.3	0
5	2666.7	2666.7	0	2666.7	2666.7	0	2666.7	2666.7	0
6	3837.1	3837.1	0	3837.1	3837.1	0	3837.1	3837.1	0
7	5221	5221	0	5221	5221	0	5221	5221	0
8	6818.1	6818.1	0	6818.1	6818.1	0	6818.1	6818.1	0
9	8628.5	8628.5	0	8628.5	8628.5	0	8628.5	8628.5	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
m	$r_c = 1.5$			$r_c = 1.75$			$r_c = 2.0$		
1	179.98	179.78	0.1	180.02	179.78	0.1	180.06	179.79	0.2
2	450.06	449.98	0.02	450.08	449.98	0.02	450.09	450.05	0.009
3	969.5	969.46	0.004	969.51	969.46	0.005	969.51	969.46	0.005
4	1710.3	1710.3	0	1710.3	1710.3	0	1710.3	1710.3	0
5	2666.7	2666.7	0	2666.7	2666.7	0	2666.7	2666.7	0
6	3837.1	3837.1	0	3837.1	3837.1	0	3837.1	3837.1	0
7	5221	5221	0	5221	5221	0	5221	5221	0
8	6818.1	6818.1	0	6818.1	6818.1	0	6818.1	6818.1	0
9	8628.5	8628.5	0	8628.5	8628.5	0	8628.5	8628.5	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01

Table B.23: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNCEF: Case 7 $k_1 = 1.75k_b$.

	Ω_c (rad/s)			Diff	Ω_c (rad/s)			Diff	Ω_c (rad/s)			Diff
Mode	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)	Anal	Abaq	(%)
m	$r_c = 0$			$r_c = 0.25$			$r_c = 0.5$					
1	189.21	189.21	0	189.25	189.22	0.02	189.3	189.89	0.3			
2	453.83	453.83	0	453.85	453.83	0.004	453.87	453.84	0.007			
3	971.26	971.26	0	971.26	971.26	0	971.27	971.26	0.001			
4	1711.3	1711.3	0	1711.3	1711.3	0	1711.3	1711.3	0			
5	2667.3	2667.3	0	2667.3	2667.3	0	2667.3	2667.3	0			
6	3837.5	3837.5	0	3837.5	3837.5	0	3837.5	3837.5	0			
7	5221.3	5221.3	0	5221.3	5221.3	0	5221.3	5221.3	0			
8	6818.4	6818.4	0	6818.4	6818.4	0	6818.4	6818.4	0			
9	8628.7	8628.7	0	8628.7	8628.7	0	8628.7	8628.7	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			
m	$r_c = 0.75$			$r_c = 1.0$			$r_c = 1.25$					
1	189.34	189.23	0.06	189.38	189.24	0.07	189.43	189.24	0.1			
2	453.89	453.84	0.01	453.91	453.84	0.01	453.92	453.85	0.02			
3	971.28	971.26	0.002	971.29	971.26	0.003	971.3	971.26	0.004			
4	1711.3	1711.3	0	1711.3	1711.3	0	1711.3	1711.3	0			
5	2667.3	2667.3	0	2667.3	2667.3	0	2667.3	2667.3	0			
6	3837.5	3837.5	0	3837.5	3837.5	0	3837.5	3837.5	0			
7	5221.3	5221.3	0	5221.3	5221.3	0	5221.3	5221.3	0			
8	6818.4	6818.4	0	6818.4	6818.4	0	6818.4	6818.4	0			
9	8628.7	8628.7	0	8628.7	8628.7	0	8628.7	8628.7	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			
m	$r_c = 1.5$			$r_c = 1.75$			$r_c = 2.0$					
1	189.47	189.25	0.1	189.52	189.25	0.1	189.56	189.26	0.2			
2	453.94	453.85	0.02	453.96	453.85	0.02	453.98	453.85	0.03			
3	971.31	971.26	0.005	971.32	971.26	0.006	971.32	971.27	0.005			
4	1711.3	1711.3	0	1711.3	1711.3	0	1711.3	1711.3	0			
5	2667.3	2667.3	0	2667.3	2667.3	0	2667.3	2667.3	0			
6	3837.5	3837.5	0	3837.5	3837.5	0	3837.5	3837.5	0			
7	5221.3	5221.3	0	5221.3	5221.3	0	5221.3	5221.3	0			
8	6818.4	6818.4	0	6818.4	6818.4	0	6818.4	6818.4	0			
9	8628.7	8628.7	0	8628.7	8628.7	0	8628.7	8628.7	0			
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01			

Table B.24: Comparison between the natural frequencies obtained analytically to those obtained in Abaqus for the BoNCEF: Case 8 $k_1 = 2k_b$.

	Ω_c (rad/s)		Diff (%)	Ω_c (rad/s)		Diff (%)	Ω_c (rad/s)		Diff (%)
Mode	Anal	Abaq		Anal	Abaq		Anal	Abaq	
m	$r_c = 0$			$r_c = 0.25$			$r_c = 0.5$		
1	198.23	198.23	0	198.28	198.23	0.03	198.32	198.25	0.04
2	457.67	457.66	0.002	457.69	457.67	0.004	457.7	457.67	0.007
3	973.05	973.05	0	973.06	973.05	0.001	973.07	973.06	0.001
4	1712.3	1712.3	0	1712.3	1712.3	0	1712.3	1712.3	0
5	2668	2668	0	2668	2668	0	2668	2668	0
6	3838	3838	0	3838	3838	0	3838	3838	0
7	5221.6	5221.6	0	5221.6	5221.6	0	5221.6	5221.6	0
8	6818.6	6818.6	0	6818.6	6818.6	0	6818.6	6818.6	0
9	8628.9	8628.9	0	8628.9	8628.9	0	8628.9	8628.9	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
m	$r_c = 0.75$			$r_c = 1.0$			$r_c = 1.25$		
1	198.37	198.25	0.06	198.42	198.25	0.09	198.47	198.26	0.1
2	457.73	457.67	0.01	457.75	457.68	0.02	457.77	457.68	0.02
3	973.08	973.06	0.002	973.09	973.06	0.003	973.1	973.06	0.004
4	1712.3	1712.3	0	1712.4	1712.3	0.006	1712.4	1712.3	0.006
5	2668	2668	0	2668	2668	0	2668	2668	0
6	3838	3838	0	3838	3838	0	3838	3838	0
7	5221.6	5221.6	0	5221.6	5221.6	0	5221.6	5221.6	0
8	6818.6	6818.6	0	6818.6	6818.6	0	6818.6	6818.6	0
9	8628.9	8628.9	0	8628.9	8628.9	0	8628.9	8628.9	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01
m	$r_c = 1.5$			$r_c = 1.75$			$r_c = 2.0$		
1	198.52	198.27	0.1	198.56	198.28	0.1	198.61	198.28	0.2
2	457.8	457.68	0.03	457.81	457.69	0.03	457.83	457.69	0.03
3	973.11	973.06	0.005	973.12	973.06	0.006	973.13	973.06	0.007
4	1712.4	1712.3	0.006	1712.4	1712.3	0.006	1712.4	1712.3	0.006
5	2668	2668	0	2668	2668	0	2668	2668	0
6	3838	3838	0	3838	3838	0	3838	3838	0
7	5221.6	5221.6	0	5221.6	5221.6	0	5221.6	5221.6	0
8	6818.6	6818.6	0	6818.6	6818.6	0	6818.6	6818.6	0
9	8628.9	8628.9	0	8628.9	8628.9	0	8628.9	8628.9	0
10	10652	10651	0.01	10652	10651	0.01	10652	10651	0.01