SOME STRUCTURAL THEOREMS FOR INELASTIC SOLIDS:

AN INTERNAL VARIABLE APPROACH

by

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Declaration

I hereby declare that this thesis has not been presented for any other degree at any other university. Appendices A and B contain parts of the thesis that have been published or are to be published with the permission of the University.

Signature removed 1st September, 1976.

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Summary

The theory of inelastic solids involving thermodynamic potential functions with internal variables is reviewed.

Use is made of the condition for stable thermodynamic equilibrium in order to obtain dual minimum principles for the equilibrium state of a solid inelastic body. This leads to dual forms of the incremental (or rate) theorems and their respective extended forms. The extended static incremental theorem is applied to a pin-jointed truss and an algorithm suggested for solution of the ensuing programming problem. Numerical examples are given.

A class of bounding theorems is also studied from the point of view of the potential functions. Bounds on the work and complementary work are obtained and properties of the bounding functions examined. Finally, the bound on a functional, which has been used to obtain general work and displacement bounds for dynamically loaded structures, is discussed.

Contents

List of important symbols

U	Internal energy
S	Entropy
T	Temperature
F	Free (Helmholtz) energy
H	Gibbs energy
u	Internal energy per unit volume
s	Entropy per unit volume
f	Free energy per unit volume
h	Gibbs energy per unit volume
u_i	Displacement of a point in a strained body
ε_{ij}	Cartesian infinitesmal strain tensor
σ_{ij}	Cartesian stress tensor
χ_α	Internal variables
X_α	Internal forces
e, p	Superscripts denoting elastic and plastic (inelastic) behaviour of thermodynamic quantities
D	Energy dissipation rate per unit volume
F_i	Imposed body force
P_i	Imposed surface traction
u_i	Imposed surface displacement
W^p, $\overline{W}^p$	Functions used in kinematic incremental theorems
U^p, $\tilde{U}^p$, $\overline{U}^p$	Functionals used in kinematic incremental theorems
Ω^c, $\overline{\Omega}^c$	Functions used in static incremental theorems
V^c, $\tilde{V}^c$, $\overline{V}^c$	Functionals used in static incremental theorems
δ_i	Extension of a bar
δ_i^p	Inelastic extension of a bar
N_i	Axial force in a bar
W^c, $\overline{W}^c$	Functions used in the application of the extended static incremental theorem to a truss
W	Work functional
$\tilde{W}$	Work bounding function
Ω	Complementary work functional
$\tilde{\Omega}$	Complementary work bounding function
W^s	Functional introduced by Ponter
w	Bound on W^s

Chapter 1. Thermodynamics and its application to strained solid continua

1.1 Introduction

In recent years various authors have shown that the framework of irreversible thermodynamics involving internal variables provides a unified approach to the study of strained inelastic solids. Suitable equations of state and kinetic or rate equations have been used by Kestin (1968, 1973), Rice (1970, 1971), Kestin and Rice (1970) and Martin (1975e) to obtain the well-known constitutive equations for metals exhibiting linear and non-linear creep and time-independent plasticity, the last involving either hardening or perfect plasticity.

These successes with the constitutive equations suggest that this formalism may be exploited as a basis for the demonstration of structural theorems. The reasons for such an attempt are slightly different for the two classes of structural theorems which we will consider. The rate theorems of time-independent plasticity have a form which is reminiscent of certain minimum principles for the equilibrium state of a statically loaded body. We expect that this similarity and the physical basis for the rate theorems would be made explicit by the application of thermodynamic principles. The bounding theorems of time-independent and time-dependent plasticity have hitherto been proved for particular materials or classes of materials. That this can be done suggests that the different materials have some property or properties in common on which the theorems depend. Thermodynamics provides a general framework involving potential functions for the description of matter. We expect that the microscopic and macroscopic properties of individual materials would be expressed in general and succinct properties of the potential functions involved. The intention therefore is to illuminate and generalise certain structural theorems by proving them from the properties of the thermodynamic potential functions.

Section 1.2 is a brief review of the basic concepts of thermodynamics and is taken from Callen (1960) and the review of Martin (1975d). In section 1.3, using the ideas from section 1.2, we define state variables and discuss equations of state and kinetic equations for certain classes of inelastic materials. Sufficient conditions for stability are given. This section is taken largely from Kestin and Rice (1970) and Martin (1975e). In section 1.4 we apply the conditions for thermodynamic equilibrium to loaded isothermal bodies. The connection between the equilibrium state and dual minimum principles is discussed.

Chapter 2 is concerned with the kinematic and static rate theorems. These theorems give the response of a time-independent inelastic structure to imposed load and displacement rates. The starting point for both theorems is the condition of thermodynamic equilibrium and extended forms of each are obtained. Chapter 3 deals with the application of the extended static rate theorem to a pin-jointed space truss, and numerical examples are given.

Chapters 4 and 5 discuss bounding properties of the constitutive equations which are used in connection with deformation and bounding theories of plasticity. Chapter 4 is concerned with minimum work and maximum complementary work functions and paths, while Chapter 5 discusses the bound on a functional introduced by Ponter (1970).

We will limit discussion to small, isothermal deformations in all cases.

1.2 Review of classical thermodynamics

1.2.1 The first law

The internal energy of a system can be altered only by adding heat to it or by performing work on it or by altering the amount of matter present. For our purposes the last possibility may be ignored. Thus if heat dQ is added to and work dW is performed on the system, the change in the internal energy U is, by the conservation of energy,

$$dU = dQ + dW. \tag{1}$$

Equation 1 expresses the *first law of thermodynamics*.

1.2.2 State variables and quasi-static processes

The state of a thermodynamic system in equilibrium may be completely specified by its internal energy U and a finite number (n) of other independent extensive quantities x_i, $i = 1,2,\dots,n$. For systems of interest to us the $\{x_i\}$ will be displacement variables. In passing from one equilibrium state to a neighbouring equilibrium state, it is assumed that the states traversed by the system are describable in terms of $\{U,x_i\}$. Physically, this means that the system traverses a series of equilibrium states or undergoes a *quasi-static process*.

1.2.3 The entropy maximum principle

The basic problem in classical thermodynamics is to determine the equilibrium state of a system which has certain known conditions (constraints) imposed on it. Of all the states satisfying the constraints, the *constrained equilibrium state* is characterised by a maximum principle. The quantity which is thus maximised is the entropy S, a potential function of the state variables $\{U,x_i\}$.

1.2.4 Equations of state, reversible heat and work, the second law

Entropy is an extensive quantity, is continuously differentiable and is a monotonically increasing function of the internal energy. This implies that the so-called *fundamental equation* $S = S(U,x_i)$ can be solved to give $U = U(S,x_i)$, an alternative form of the fundamental equation. U is continuously differentiable and is a monotonic increasing function of S. The differential of U is

$$dU = \frac{\partial U}{\partial S}\, dS + \frac{\partial U}{\partial x_i}\, dx_i . \tag{2}$$

By definition, the thermodynamic temperature T is

$$T = \frac{\partial U}{\partial S} = T(S,x_i), \tag{3}$$

and the (intensive) forces X_i conjugate to the extensive variables x_i are

$$X_i = \frac{\partial U}{\partial x_i} = X_i(S,x_j). \tag{4}$$

Equations 3 and 4 are the *equations of state* of the system. If they are known, the fundamental equation can be inferred to within an arbitrary constant.

Equation 2 may be written

$$dU = TdS + X_i dx_i \tag{5}$$

$$= dQ^0 + dW^0 , \tag{6}$$

where

$$dQ^0 = TdS, \tag{7}$$

$$dW^0 = X_i dx_i , \tag{8}$$

are termed the reversible heat and reversible work respectively. Equation 7 is referred to as the *first part of the second law.*

Considering equations 5 and 6 as giving the same small change in U as equation 1 we obtain

$$dS = \frac{dQ}{T} + \frac{dW - dW^0}{T} . \tag{9}$$

It is seen that the change in the entropy of the system may be split into two parts. One part is due to a quantity dQ of heat entering the system, the other is due to a difference between the work performed on the system and the reversible work. The latter is referred to as the entropy production within the system $d\xi$.

The *second part of the second law* states that

$$d\xi = \frac{dW - dW^0}{T} \geq 0. \tag{10}$$

1.2.5 The internal energy minimum principle

An isolated system is one for which the internal energy U is fixed. The equilibrium state of such a system maximises $S(U,x_i)$ subject to U fixed and whatever constraints are imposed on the x_i's. The properties of the entropy function are sufficient to ensure that this equilibrium state also minimises $U(S,x_i)$ subject to S fixed and the same set of constraints on the x_i's.

Now the minimum of $U(S,x_i)$ with S fixed but with no other constraints satisfies

$$\frac{\partial U}{\partial x_i} = X_i = 0. \tag{11}$$

Thus the forces X_i conjugate to unconstrained extensive parameters x_i are zero in the equilibrium state.

1.2.6 The Legendre transformations

When we desire to express the fundamental equation and equations of state in terms of some or all of the intensive variables as independent variables we make use of the appropriate Legendre transformation. If we require $T = \partial U/\partial S$ and $\{x_i\}$ as independent variables we form

$$F = U(S,x_i) - TS. \tag{12}$$

The differential of F is

$$dF = TdS + X_i dx_i - (TdS + SdT) \tag{13}$$

$$= - SdT + X_i dx_i. \tag{14}$$

We see that $F = F(T,x_i)$ and

$$\frac{\partial F}{\partial T} = - S, \tag{15}$$

$$\frac{\partial F}{\partial x_i} = X_i, \tag{16}$$

are equations of state. $F(T,x_i)$ is the *Helmholtz potential* or *free energy* of the system.

Similarly we might require $T = \partial U/\partial S$ and $\{X_i = \partial U/\partial x_i\}$ as independent variables. We form

$$H = X_i x_i - F(T,x_i), \tag{17}$$

of which the differential is

$$dH = SdT + x_i dX_i . \tag{18}$$

We see that $H = H(T,X_i)$ and

$$\frac{\partial H}{\partial T} = S, \tag{19}$$

$$\frac{\partial H}{\partial X_i} = x_i, \tag{20}$$

are equations of state. $H(T,X_i)$ is the *Gibbs energy* of the system.

1.2.7 The free energy minimum principle

Consider an isolated composite system consisting of a subsystem whose fundamental equation is $U = U(S,x_i)$, in diathermal contact with a heat source whose fundamental equation is $U_h = T^0 S_h$. The internal energy U_c of the composite system is

$$U_c = U(S,x_i) + T^0 S_h . \tag{21}$$

A necessary condition for U_c to be a minimum at constant total entropy is

$$\delta U_c = 0, \tag{22}$$

subject to

$$\delta S_c = \delta S + \delta S_h = 0. \tag{23}$$

Thus the unconstrained equilibrium state satisfies

$$\delta U_c = \delta U(S,x_i) - T^0 \delta S \tag{24}$$

$$= \delta(U - T^0 S) = 0. \tag{25}$$

Recalling the definition of the free energy (equation 12) we see that

$$\delta U_c = 0 \iff \delta F(T,x_i) = 0 \tag{26}$$

subject to $T = T^0$, where F is the free energy of the subsystem. It is readily shown that $\delta F = 0$ at constant T is a minimum. We conclude that if the temperature of a system is known, then the unconstrained equilibrium state minimises the free energy at constant temperature. This result is the *free energy minimum principle.*

1.3 Constitutive equations for inelastic materials

1.3.1 State variables and equations of state

Consider a small element in a body of an inelastic material which is subjected to small deformations. Following Kestin and Rice (1970), the thermodynamic state of the element is assumed to be described by the temperature T, the macroscopic strain tensor ε_{ij}, and the internal variables χ_α, $\alpha = 1,2,\ldots,n$.

We identify points within the body by Cartesian coordinates (x_1,x_2,x_3). If $u_i(x_j)$ are the small displacements of a point in the element from some initial configuration, the strain tensor ε_{ij} is defined by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) . \tag{27}$$

The internal variables χ_α characterise the microstructural re-arrangements within the element and are in general necessary to complete the description of its state. They are assumed to be *measurable* but not directly *controllable.*

The fundamental equation of the element is assumed to be known and of the form

$$f = f(T,\varepsilon_{ij},\chi_\alpha), \tag{28}$$

where f is the free energy per unit volume.

The equations of state are

$$s = -\frac{\partial f}{\partial T}, \tag{29}$$

$$\sigma_{ij} = \frac{\partial f}{\partial \varepsilon_{ij}}, \tag{30}$$

$$X_\alpha = -\frac{\partial f}{\partial \chi_\alpha}, \tag{31}$$

where s is the entropy per unit volume, σ_{ij} is the Cartesian stress tensor and $-X_\alpha$ are the "internal forces" conjugate to the internal variables. Hence the differential of f is

$$df = -s dT + \sigma_{ij} d\varepsilon_{ij} - X_\alpha d\chi_\alpha . = \frac{\partial f}{\partial T} dT + \frac{\partial f}{\partial \varepsilon_{ij}} d\varepsilon_{ij} + \frac{\partial f}{\partial \chi_\alpha} d\chi_\alpha \tag{32}$$

The Gibbs energy per unit volume is

$$h = \sigma_{ij}\varepsilon_{ij} - f(T,\varepsilon_{ij},\chi_\alpha). \tag{33}$$

It follows that $h = h(T,\sigma_{ij},\chi_\alpha)$ and

$$s = \frac{\partial h}{\partial T}, \tag{34}$$

$$\varepsilon_{ij} = \frac{\partial h}{\partial \sigma_{ij}}, \tag{35}$$

$$X_\alpha = \frac{\partial h}{\partial \chi_\alpha}, \tag{36}$$

are equations of state. Hence the differential of h is

$$dh = s dT + \varepsilon_{ij} d\sigma_{ij} + X_\alpha d\chi_\alpha . \tag{37}$$

For the special case where

$$\frac{\partial \varepsilon_{ij}}{\partial \sigma_{k\ell}} = \frac{\partial^2 h}{\partial \sigma_{ij}\partial \sigma_{k\ell}},$$

is independent of χ_α and

$$\frac{\partial \varepsilon_{ij}}{\partial \chi_\alpha} = \frac{\partial^2 h}{\partial \sigma_{ij}\partial \chi_\alpha},$$

is constant, the strains, entropy per unit volume and free energy per unit volume may be split into elastic and plastic parts. To see this we consider

$$\varepsilon_{ij}(\sigma'_{k\ell},\chi'_\alpha) = \int_0^{\sigma'_{k\ell}} \frac{\partial\varepsilon_{ij}}{\partial\sigma_{k\ell}} d\sigma_{k\ell} + \int_0^{\chi'_\alpha} \frac{\partial\varepsilon_{ij}}{\partial\chi_\alpha} d\chi_\alpha \quad . \tag{38}$$

$$= \varepsilon^e_{ij}(\sigma'_{k\ell}) + \varepsilon^p_{ij}(\chi'_\alpha) \quad , \tag{39}$$

where we define

$$\varepsilon^e_{ij}(\sigma'_{k\ell}) = \int_0^{\sigma'_{k\ell}} \frac{\partial\varepsilon_{ij}}{\partial\sigma_{k\ell}} d\sigma_{k\ell} \; , \tag{40}$$

$$\varepsilon^p_{ij}(\chi'_\alpha) = \int_0^{\chi'_\alpha} \frac{\partial\varepsilon_{ij}}{\partial\chi_\alpha} d\chi_\alpha = \frac{\partial\varepsilon^p_{ij}}{\partial\chi_\alpha}\chi'_\alpha \quad . \tag{41}$$

Equations 40 and 41 give the elastic and plastic strains respectively.

Now if ε_{ij}, χ_α are independent state variables we can make use of the division $\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}(\chi_\alpha)$ and use ε^e_{ij}, χ_α as independent state variables. In this case equation 32 becomes

$$df = -sdT + \sigma_{ij}d\varepsilon^e_{ij} + \left(\sigma_{ij}\frac{\partial\varepsilon^p_{ij}}{\partial\chi_\alpha} - X_\alpha\right)d\chi_\alpha \; . \tag{42}$$

We will let state functions denoted by the superscript e be those associated with thermoelastic behaviour with $\chi_\alpha = 0$. Thus

$$df^e(T,\varepsilon^e_{ij}) = -s^e dT + \sigma_{ij}d\varepsilon^e_{ij} \quad , \tag{43}$$

with

$$s^e = -\frac{\partial f^e}{\partial T} \; , \tag{44a}$$

$$\sigma_{ij} = \frac{\partial f^e}{\partial\varepsilon^e_{ij}} \; . \tag{44b}$$

Subtracting equation 43 from equation 42 we obtain

$$df - df^e = df^p = -(s - s^e)dT + \left(\sigma_{ij}\frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} - X_\alpha\right)d\chi_\alpha , \tag{45}$$

and conclude that $f^p = f - f^e$ is independent of ε^e_{ij}. Thus we may put

$$f(T,\varepsilon^e_{ij} + \varepsilon^p_{ij}(\chi_\alpha),\chi_\alpha) = f^e(T,\varepsilon^e_{ij}) + f^p(T,\chi_\alpha). \tag{46}$$

With equations 44,

$$s^p = s - s^e = -\frac{\partial f^p}{\partial T} , \tag{47a}$$

$$\sigma_{ij}\frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} - X_\alpha = \frac{\partial f^p}{\partial \chi_\alpha} , \tag{47b}$$

are the equations of state.

Using equation 33, the Gibbs energy per unit volume in this case is

$$h(T,\sigma_{ij},\chi_\alpha) = h^e(T,\sigma_{ij}) + \sigma_{ij}\varepsilon^p_{ij}(\chi_\alpha) - f^p(T,\chi_\alpha), \tag{48}$$

where

$$h^e(T,\sigma_{ij}) = \sigma_{ij}\varepsilon^e_{ij} - f^e(T,\varepsilon^e_{ij}). \tag{49}$$

The equations of state are

$$\varepsilon_{ij} = \frac{\partial h^e}{\partial \sigma_{ij}} + \varepsilon^p_{ij}(\chi_\alpha), \tag{50a}$$

$$X_\alpha = \sigma_{ij}\frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} - \frac{\partial f^p}{\partial \chi_\alpha} . \tag{50b}$$

The splitting of ε_{ij}, s and f into elastic and inelastic parts was demonstrated by Kestin and Rice (1970) for the case $(\partial\varepsilon_{ij}/\partial\sigma_{k\ell}) = \text{constant}$.

The mechanical work per unit volume done on the element during a small quasi-static change in strain $d\varepsilon_{ij}$ is

$$dW = \sigma_{ij}d\varepsilon_{ij} . \tag{51}$$

The reversible work per unit volume is, on comparing equations 8 and 32,

$$dW^0 = \sigma_{ij} d\varepsilon_{ij} - X_\alpha d\chi_\alpha \ . \tag{52}$$

Referring to equation 10, the entropy production per unit volume is

$$d\xi = \frac{X_\alpha d\chi_\alpha}{T} \ , \tag{53a}$$

of which the rate form is

$$\dot{\xi} = \frac{X_\alpha \dot{\chi}_\alpha}{T} \ , \tag{53b}$$

which is required to be non-negative by the second part of the second law.

1.3.2 The kinetic equations

In order to complete the description of the material we require phenomenological rate or kinetic equations giving the rates of change of the internal variables. We will assume them to be of the form

$$\dot{\chi}_\alpha = P_\alpha(X_\beta) \ , \tag{54a}$$

or

$$X_\alpha = P_\alpha^{-1}(\dot{\chi}_\beta) \ , \tag{54b}$$

which must always be satisfied along with the equations of state (equation 31).

The entropy production rate

$$\frac{D(\dot{\chi}_\alpha)}{T} = \frac{X_\alpha \dot{\chi}_\alpha}{T} \ , \tag{55}$$

is required to be non-negative.

We will now examine three examples of classes of kinetic equations used for time-independent and time-dependent plasticity in metals.

(i) Linear and non-linear creep

$$\dot{\chi}_\alpha = \phi^n \frac{\partial\phi}{\partial X_\alpha}, \tag{56}$$

where $\phi(X_\alpha)$ is homogenous and of degree one in X_α and n is an odd positive integer. The linear case occurs for $n = 1$. From equation 55

$$D = X_\alpha \dot{\chi}_\alpha$$

$$= \phi^{n+1}(X_\alpha). \tag{57}$$

Since $\dot{\chi}_\alpha$ is homogenous and of degree n in X_α it follows from equation 57 that D is homogenous and of degree $(n + 1)/n$ in $\dot{\chi}_\alpha$. Also, by applying Euler's theorem

$$\frac{\partial D}{\partial \dot{\chi}_\alpha} = X_\alpha + \frac{\partial X_\alpha}{\partial \dot{\chi}_\beta} \dot{\chi}_\beta \tag{58a}$$

$$= \frac{n+1}{n} X_\alpha, \tag{58b}$$

or

$$X_\alpha = \frac{1}{p+1} \frac{\partial D}{\partial \dot{\chi}_\alpha}, \tag{58c}$$

where $p = \frac{1}{n}$. Equation 58c is the inverse of equation 56.

(ii) Viscoplasticity

$$\dot{\chi}_\alpha = \{\phi(X_\alpha) - \phi_0\}^n \frac{\partial\phi}{\partial X_\alpha} \qquad \text{if} \quad \phi(X_\alpha) > \phi_0$$

$$= 0 \qquad \text{if} \quad \phi(X_\alpha) \le \phi_0, \tag{59}$$

where $\phi(X_\alpha)$ is homogenous and of degree one in X_α, n is an odd positive integer and ϕ_0 is a constant.

For this material we have a range of values of X_α for which $\dot{\chi}_\alpha = 0$. This set is bounded by $\{X_\alpha : \phi(X_\alpha) = \phi_0\}$ which we term the *yield surface*.

From the equation

$$\frac{\partial D}{\partial \dot{x}_\alpha} = X_\alpha + \frac{\partial X_\alpha}{\partial \dot{x}_\beta}\dot{x}_\beta ,$$

it is seen that $\partial D/\partial \dot{x}_\alpha$ is discontinuous (multivalued) at $\dot{x}_\alpha = 0$.

(iii) Time-independent plasticity

$$\dot{x}_\alpha = \lambda \frac{\partial \phi}{\partial X_\alpha} , \tag{60a}$$

where $\phi(X_\alpha)$ is a continuously differentiable single-valued *limit function* for which $\phi(X_\alpha = 0) < 0$. Internal forces such that $\phi(X_\alpha) > 0$ are not obtainable and

$$\begin{aligned} \lambda &\geq 0 \quad \text{if} \quad \phi = 0 \quad \text{and} \quad \dot{\phi} = 0 , \\ \lambda &= 0 \quad \text{if} \quad \phi < 0, \quad \text{or} \quad \phi = 0 \quad \text{and} \quad \dot{\phi} < 0. \end{aligned} \tag{60b}$$

The set $\{X_\alpha : \phi(X_\alpha) = 0\}$ is termed the *limit surface*. Equations 60 are obtained in the limit as n tends to infinity in equation 56 (Martin, 1975e).

More generally, limit surfaces with corners are described by the kinetic equations

$$\dot{x}_\alpha = \sum_{i=1}^{m} \lambda^{(i)} \frac{\partial \phi^{(i)}}{\partial X_\alpha} ,$$

where $\phi^{(i)}(X\ )$ is a continuously differentiable, single valued limit function such that $\phi^{(i)}(X_\alpha = 0) < 0$. $\phi^{(i)}(X_\alpha) > 0$ is not obtainable and

$$\lambda^{(i)} \geq 0 \quad \text{if} \quad \phi^{(i)} = 0 \quad \text{and} \quad \dot{\phi}^{(i)} = 0 ,$$

$$\lambda^{(i)} = 0 \quad \text{if} \quad \phi^{(i)} < 0, \quad \text{or} \quad \phi^{(i)} = 0 \quad \text{and} \quad \dot{\phi}^{(i)} < 0.$$

We will restrict ourselves to single limit functions when discussing time-independent plasticity. In all cases, however, the extension to more than one limit function is readily achieved.

From the definition of D (equation 55) or by letting $n \to \infty$ we see that D is homogenous and of degree one in $\dot{x}_\alpha$ and so

$$X_\alpha = \frac{\partial D}{\partial \dot{\chi}_\alpha} . \tag{61}$$

As in viscoplastic behaviour, $\partial D/\partial\dot{\chi}_\alpha$ is discontinuous at $\dot{\chi}_\alpha = 0$. All values of X_α obtained from equation 61 with $\dot{\chi}_\alpha \neq 0$ lie on the limit surface $\phi(X_\alpha) = 0$.

$X_\alpha = -\frac{\partial f}{\partial \chi_\alpha}$

Since the equation of state (equation 31) implies

$$\dot{X}_\alpha = -\left\{\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha}\,\dot{\varepsilon}_{ij} + \frac{\partial^2 f}{\partial\chi_\alpha\partial\chi_\beta}\,\dot{\chi}_\beta\right\} , \tag{62}$$

equation 60a and its conditions (equation 60b) provide sufficient information to determine λ uniquely for particular values of $\phi(X_\alpha)$ and $\dot{\varepsilon}_{ij}$.

Let us assume that $\phi(X_\alpha) = 0$ in which case $\lambda = 0$ if $(\partial\phi/\partial X_\alpha)\dot{X}_\alpha < 0$ and $\lambda \geq 0$ if $(\partial\phi/\partial X_\alpha)\dot{X}_\alpha = 0$. As we will see in section 1.3.3, $f(T,\varepsilon_{ij},\chi_\alpha)$ is convex at constant T, so $\partial^2 f/\partial\chi_\alpha\partial\chi_\beta$ is positive definite. Thus equation 62 may be used to show that

$$\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha}\,\frac{\partial\phi}{\partial X_\alpha}\,\dot{\varepsilon}_{ij} > 0 \iff \frac{\partial\phi}{\partial X_\alpha}\,\dot{X}_\alpha < 0 , \tag{63a}$$

$$\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha}\,\frac{\partial\phi}{\partial \chi_\alpha}\,\dot{\varepsilon}_{ij} \leq 0 \iff \frac{\partial\phi}{\partial X_\alpha}\,\dot{X}_\alpha = 0 . \tag{63b}$$

Using equations 60a and 62 we may solve

$$\frac{\partial\phi}{\partial X_\alpha}\,\dot{X}_\alpha = 0 ,$$

to give

$$\lambda = -\frac{\dfrac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha}\,\dfrac{\partial\phi}{\partial X_\alpha}\,\dot{\varepsilon}_{ij}}{\dfrac{\partial^2 f}{\partial\chi_\alpha\partial\chi_\beta}\,\dfrac{\partial\phi}{\partial X_\alpha}\,\dfrac{\partial\phi}{\partial X_\beta}} , \tag{64a}$$

which is the value it takes if $\phi(X_\alpha) = 0$ and

$$\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha}\,\frac{\partial\phi}{\partial X_\alpha}\,\dot{\varepsilon}_{ij} \leq 0 . \tag{64b}$$

(i.e. yielding)

For all other cases

$$\lambda = 0 . \tag{64c}$$

We note that $\lambda = \lambda(\phi, \dot{\varepsilon}_{ij})$ may be obtained by minimising

$$P(\lambda) = \lambda \frac{\partial^2 f}{\partial \varepsilon_{ij} \partial \chi_\alpha} \frac{\partial \phi}{\partial X_\alpha} \dot{\varepsilon}_{ij} + \frac{1}{2}\lambda^2 \frac{\partial^2 f}{\partial \chi_\alpha \partial \chi_\beta} \frac{\partial \phi}{\partial X_\alpha} \frac{\partial \phi}{\partial X_\beta} , \tag{65}$$

with respect to variations in λ subject to $\lambda \geq 0$ if $\phi(X_\alpha) = 0$, $\lambda = 0$ if $\phi(X_\alpha) < 0$.

To see this we set $dP/d\lambda = 0$ and solve for λ. will get eq. (64a) If

$$\frac{\partial^2 f}{\partial \varepsilon_{ij} \partial \chi_\alpha} \frac{\partial \phi}{\partial X_\alpha} \dot{\varepsilon}_{ij} \leq 0 ,$$

the least value of P is given by equation 64a. If, however,

$$\frac{\partial^2 f}{\partial \varepsilon_{ij} \partial \chi_\alpha} \frac{\partial \phi}{\partial X_\alpha} \dot{\varepsilon}_{ij} > 0 ,$$

the least value of P subject to $\lambda \geq 0$ is given by $\lambda = 0$.

Alternatively, we may use equation 36 to obtain

$$\dot{X}_\alpha = \frac{\partial^2 h}{\partial \sigma_{ij} \partial \chi_\alpha} \dot{\sigma}_{ij} + \frac{\partial^2 h}{\partial \chi_\alpha \partial \chi_\beta} \dot{\chi}_\beta , \tag{66}$$

and, using equations 60a and 60b, obtain λ uniquely for particular values of $\phi(X_\alpha)$ and $\dot{\sigma}_{ij}$. Let us assume that $\phi(X_\alpha) = 0$ in which case $\lambda = 0$ if $(\partial\phi/\partial X_\alpha)\dot{X}_\alpha < 0$ and $\lambda \geq 0$ if $(\partial\phi/\partial X_\alpha)\dot{X}_\alpha = 0$. As we will see in section 1.3.3 $h(T, \sigma_{ij}, \chi_\alpha)$ is concave with T, σ_{ij} constant so $\partial^2 h/\partial\chi_\alpha \partial\chi_\beta$ is negative definite. Thus equation 66 may be used to show that

$$\frac{\partial^2 h}{\partial \sigma_{ij} \partial \chi_\alpha} \frac{\partial \phi}{\partial X_\alpha} \dot{\sigma}_{ij} < 0 \iff \frac{\partial \phi}{\partial X_\alpha} \dot{X}_\alpha < 0 , \tag{67a}$$

$$\frac{\partial^2 h}{\partial \sigma_{ij} \partial \chi_\alpha} \frac{\partial \phi}{\partial X_\alpha} \dot{\sigma}_{ij} \geq 0 \iff \frac{\partial \phi}{\partial X_\alpha} \dot{X}_\alpha = 0 . \tag{67b}$$

Using equations 60a and 66 we may solve

$$\frac{\partial \phi}{\partial X_\alpha} \dot{X}_\alpha = 0 ,$$

to give

$$\lambda = - \frac{\dfrac{\partial^2 h}{\partial \sigma_{ij} \partial \chi_\alpha} \dfrac{\partial \phi}{\partial X_\alpha} \dot{\sigma}_{ij}}{\dfrac{\partial^2 h}{\partial \chi_\alpha \partial \chi_\beta} \dfrac{\partial \phi}{\partial X_\alpha} \dfrac{\partial \phi}{\partial X_\beta}} , \qquad (68a)$$

which is the value it takes if $\phi(X_\alpha) = 0$ and

$$\frac{\partial^2 h}{\partial \sigma_{ij} \partial \chi_\alpha} \frac{\partial \phi}{\partial X_\alpha} \dot{\sigma}_{ij} \geq 0 . \qquad (68b)$$

For all other cases

$$\lambda = 0 . \qquad (68c)$$

We note that $\lambda = \lambda(\phi, \dot{\sigma}_{ij})$ may be obtained by maximising

$$Q(\lambda) = \lambda \frac{\partial^2 h}{\partial \sigma_{ij} \partial \chi_\alpha} \frac{\partial \phi}{\partial X_\alpha} \dot{\sigma}_{ij} + \frac{1}{2}\lambda^2 \frac{\partial^2 h}{\partial \chi_\alpha \partial \chi_\beta} \frac{\partial \phi}{\partial X_\alpha} \frac{\partial \phi}{\partial X_\beta} . \qquad (69)$$

with respect to variations in λ subject to $\lambda \geq 0$ if $\phi(X_\alpha) = 0$, $\lambda = 0$ if $\phi(X_\alpha) < 0$.

To see this we set $dQ/d\lambda = 0$ and solve for λ. If

$$\frac{\partial^2 h}{\partial \sigma_{ij} \partial \chi_\alpha} \frac{\partial \phi}{\partial X_\alpha} \dot{\sigma}_{ij} \geq 0 ,$$

the maximum value of Q is given by equation 68a satisfying the constraint $\lambda \geq 0$. If, however,

$$\frac{\partial^2 h}{\partial \sigma_{ij} \partial \chi_\alpha} \frac{\partial \phi}{\partial X_\alpha} \dot{\sigma}_{ij} < 0 ,$$

the maximum value of Q subject to $\lambda \geq 0$ is given by $\lambda = 0$.

Equations 68 are the familiar equations of kinematic hardening plasticity giving the rates of change (or increments) of internal variables in terms of rates of change (or increments) of stress. The limit surface is $\phi(X_\alpha(\sigma_{ij},\chi_\alpha)) = 0$, which may be interpreted as giving the *current yield surface* in stress space.

1.3.3 Sufficient conditions for stability

In order to ensure global and material stability in the thermodynamic sense and in the sense of Drucker (1951) we assume for a particular temperature T^0 that $f(T^0,\varepsilon_{ij},\chi_\alpha)$ and $D(\dot{\chi}_\alpha)$ are convex functions of their respective auguments. The convexity of D is expressed by

$$D(\dot{\chi}'_\alpha) - D(\dot{\chi}''_\alpha) \geq \left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\dot{\chi}''_\alpha} (\dot{\chi}'_\alpha - \dot{\chi}''_\alpha) \ . \tag{70}$$

Materials which are completely or partially time-independent (e.g. visco-plasticity) have a discontinuity in $X_\alpha(\dot{\chi}_\beta)$ and $\partial D/\partial\dot{\chi}_\alpha$ at $\dot{\chi}_\alpha = 0$. We see from equation 58a that

$$\{X_\alpha : \dot{\chi}_\alpha = 0\} \equiv \{\frac{\partial D}{\partial \dot{\chi}_\alpha} : \dot{\chi}_\alpha = 0\} \ . \tag{71}$$

Thus we may evaluate equation 70 at $\dot{\chi}''_\alpha = 0$ to obtain

$$D(\dot{\chi}'_\alpha) - X^*_\alpha \dot{\chi}'_\alpha \geq 0 \ , \tag{72}$$

where X^*_α is any point inside or on the *limit surface* $\phi(X_\alpha) = 0$ (time-independent plasticity) or the *yield surface* $\phi(X_\alpha) = \phi_0$ (viscoplasticity). Equation 72 may be written

$$(X_\alpha - X^*_\alpha)\dot{\chi}_\alpha \geq 0 \ , \tag{73}$$

where X_α and $\dot{\chi}_\alpha$ are related through the kinetic equations. For time-independent plasticity, inequality 73 expresses the maximum plastic work principle. Martin (1975b) has shown that the Drucker stability condition is implied by the convexity of $f(T^0,\varepsilon_{ij},\chi_\alpha)$ and inequality 73.

We now show that the convexity of $f(T^0,\varepsilon_{ij},\chi_\alpha)$ implies the convexity at constant χ_α and the concavity at constant σ_{ij} of $h(T^0,\sigma_{ij},\chi_\alpha)$, and the convexity of $f^e(T^0,\varepsilon^e_{ij})$ and $f^p(T^0,\chi_\alpha)$.

Ignoring temperature, the convexity of f is expressed by

$$f(\varepsilon'_{ij},\chi'_\alpha) - f(\varepsilon''_{ij},\chi''_\alpha) \geq (\varepsilon'_{ij} - \varepsilon''_{ij}) \left.\frac{\partial f}{\partial \varepsilon_{ij}}\right|_{\varepsilon''_{ij},\chi''_\alpha}$$

$$+ (\chi'_\alpha - \chi''_\alpha) \left.\frac{\partial f}{\partial \chi_\alpha}\right|_{\varepsilon''_{ij},\chi''_\alpha} \tag{74}$$

$$= (\varepsilon'_{ij} - \varepsilon''_{ij})\sigma''_{ij} - (\chi'_\alpha - \chi''_\alpha)X''_\alpha \quad , \tag{75}$$

where

$$\sigma''_{ij} = \left.\frac{\partial f}{\partial \varepsilon_{ij}}\right|_{\varepsilon''_{ij},\chi''_\alpha} \quad ,$$

$$X''_\alpha = -\left.\frac{\partial f}{\partial \chi_\alpha}\right|_{\varepsilon''_{ij},\chi''_\alpha} \quad ,$$

are the equations of state, (equations 30 and 31). Thus inequality 74 may be written as

$$\sigma''_{ij}\varepsilon''_{ij} - f(\varepsilon''_{ij},\chi''_\alpha) - \{\sigma'_{ij}\varepsilon'_{ij} - f(\varepsilon'_{ij},\chi'_\alpha)\}$$

$$\geq (\sigma''_{ij} - \sigma'_{ij})\varepsilon'_{ij} - (\chi'_\alpha - \chi''_\alpha)X''_\alpha \quad , \tag{76}$$

where $\sigma'_{ij} = \sigma_{ij}(\varepsilon'_{ij},\chi'_\alpha)$. Noting equations 33, 35 and 36, inequality 76 may be written

$$h(\sigma''_{ij},\chi''_\alpha) - h(\sigma'_{ij},\chi'_\alpha) \geq (\sigma''_{ij} - \sigma'_{ij}) \left.\frac{\partial h}{\partial \sigma_{ij}}\right|_{\sigma'_{ij},\chi''_\alpha}$$

$$- (\chi'_\alpha - \chi''_\alpha) \left.\frac{\partial h}{\partial \chi_\alpha}\right|_{\sigma''_{ij},\chi''_\alpha} \quad . \tag{77}$$

Putting $\chi'_\alpha = \chi''_\alpha$, it is seen that inequality 77 expresses the convexity at constant χ_α of $h(T^0,\sigma_{ij},\chi_\alpha)$. Putting $\sigma'_{ij} = \sigma''_{ij}$ inequality 77 becomes

$$h(\sigma'_{ij},\chi'_\alpha) - h(\sigma'_{ij},\chi''_\alpha) \leq (\chi'_\alpha - \chi''_\alpha) \left.\frac{\partial h}{\partial \chi_\alpha}\right|_{\sigma'_{ij},\chi''_\alpha} \quad , \tag{78}$$

which expresses the concavity at constant σ_{ij} of $h(T^0,\sigma_{ij},\chi_\alpha)$.

We now consider materials for which $\varepsilon_{ij} = \varepsilon^e_{ij} + \varepsilon^p_{ij}(\chi_\alpha)$, $f = f(T,\varepsilon^e_{ij}) + f^p(T,\chi_\alpha)$. Inequalities 74 and 75 may be written

$$f^e(\varepsilon^{e\prime}_{ij}) - f^e(\varepsilon^{e\prime\prime}_{ij}) + f^p(\chi'_\alpha) - f^p(\chi''_\alpha) \geq (\varepsilon^{e\prime}_{ij} - \varepsilon^{e\prime\prime}_{ij})\sigma''_{ij} + (\chi'_\alpha - \chi''_\alpha)(\sigma''_{ij}\frac{\partial\varepsilon^p_{ij}}{\partial\chi_\alpha} - X_\alpha), \quad (79)$$

since $\partial\varepsilon^p_{ij}/\partial\chi_\alpha$ is an array of constants. Putting $\chi'_\alpha = \chi''_\alpha$ in this inequality yields an inequality expressing the convexity of $f^e(\varepsilon^e_{ij})$, while putting $\varepsilon^{e\prime}_{ij} = \varepsilon^{e\prime\prime}_{ij}$ yields an inequality expressing the convexity of $f^p(\chi_\alpha)$.

1.3.4 Continuous thermodynamic systems

All of the previous section refers to a single element of the material. A body of finite volume V may be considered as consisting of a discrete set of such elements. In the limit as the volume of each element tends to zero we assume the principle of local state as in Kestin and Rice (1970). The thermodynamic state of a point x_i in the body is described by the temperature $T(x_i)$, strains $\varepsilon_{ij}(x_i)$ and the internal variables $\chi_\alpha(x_i)$ at that point. These state variables are field quantities over V. If x_i are Cartesian coordinates and $u_i(x_j)$ is the small, continuously differentiable displacement field from some initial relaxed configuration, the strain tensor field $\varepsilon_{ij}(x_i)$ is defined by equation 27. Unlike other field quantities the internal variable field $\chi_\alpha(x_i)$ does not satisfy any continuity conditions. The fundamental equation is the free energy per unit volume $f(T(x_i),\varepsilon_{ij}(x_i),\chi_\alpha(x_i))$, which we assume is known. The equations of state are field equations and are as expressed in equations 29, 30 and 31. They give the entropy per unit volume s, stresses σ_{ij} and internal forces X_α as functions of the state variables at each point in the body. The free energy F of the body is

$$F = \int_V f dV . \quad (80)$$

We define the Gibbs energy per unit volume h at each point by equation 33. The Gibbs energy H of the body is

$$H = \int_V h dV . \tag{81}$$

Equations 80 and 81 are the forms of the fundamental equation we will require since we deal with isothermal deformations.

The second part of the second law is assumed to be satisfied locally, i.e. that $\dot{\xi} \geq 0$ everywhere. Since temperature gradients are assumed to vanish throughout the body, the entropy production rate per unit volume at each point is

$$\dot{\xi}(x_i) = \frac{D(\dot{\chi}_\alpha(x_i))}{T} . \tag{82}$$

All the equations in section 1.3.2 go over to the continuous case unchanged, and the sufficient conditions for stability are assumed to be satisfied at each point.

1.4 Equilibrium of a loaded body

Consider a body of volume V and surface S consisting of a inelastic material covered by the assumptions in section 1.3, subject to small, isothermal deformations. The displacement of a point in the body is characterised by $u_i(x_j)$ where (x_1,x_2,x_3) are the Cartesian coordinates of the point. Following the assumption of small displacements the strain-displacement relations take the form

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) . \tag{83}$$

The body is subjected to constant conservative body forces $\hat{F}_i$ on V, constant conservative tractions $\hat{P}_i$ on part of the surface S_p and constant displacements $\hat{u}_i$ on the remainder of the surface S_u.

1.4.1 The free energy minimum principle

We now examine the consequences of the free energy minimum principle when applied to such a loaded body.

The free energy of the composite system consisting of the body at temperature T^0 and the applied forces is

$$F = \int_V f(T^0, \varepsilon_{ij}, \chi_\alpha) dV - \int_V \hat{F}_i u_i dV - \int_{S_p} \hat{P}_i u_i dS \quad . \tag{84}$$

At any instant, the equilibrium state is that which minimises F with respect to variations in unconstrained variables. The constraints on the system are

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{over} \quad V, \tag{85a}$$

$$u_i = \hat{u}_i \quad \text{over} \quad S_u \;, \tag{85b}$$

$$\chi_\alpha = \int_0^t \dot{\chi}_\alpha(\tau) d\tau \quad \text{over} \quad V, \tag{85c}$$

where the history $\dot{\chi}_\alpha(\tau)$, $0 \leq \tau \leq t$, is determined by the history $X_\alpha(\tau)$, $0 \leq \tau \leq t$, through the kinetic equation

$$\dot{\chi}_\alpha = P_\alpha(X_\beta) \; .$$

It follows that the only unconstrained variations are $\delta\varepsilon_{ij}$ and δu_i satisfying the incremental form of equation 85a and $\delta u_i = 0$ over S_u. The variation in F due to such variations is, to first order,

$$\delta F = \int_V \sigma_{ij} \delta\varepsilon_{ij} dV - \int_V \hat{F}_i \delta u_i dV - \int_{S_p} \hat{P}_i \delta u_i dS \; , \tag{86}$$

where we use equation 30. Substituting for $\delta\varepsilon_{ij}$ from the incremental form of equation 85a and using $\sigma_{ij} = \sigma_{ji}$,

$$\delta F = \int_V \left(\frac{\partial}{\partial x_j}(\sigma_{ij}\delta u_i) - \frac{\partial}{\partial x_j}(\sigma_{ij})\delta u_i\right) dV - \int_V \hat{F}_i \delta u_i dV - \int_{S_p} \hat{P}_i \delta u_i dS. \tag{87}$$

On using Gauss' theorem and the constraint $\delta u_i = 0$ over S_u,

$$\delta F = \int_V -\left(\frac{\partial\sigma_{ij}}{\partial x_j} + \hat{F}_i\right)\delta u_i dV + \int_{S_p} (\sigma_{ij}\nu_j - \hat{P}_i)\delta u_i dS \quad , \tag{88}$$

where ν_j is the unit outward normal over S.

Thus $\delta F = 0$ for arbitrary δu_i if and only if

$$\frac{\partial\sigma_{ij}}{\partial x_j} + \hat{F}_i = 0 \text{ over } V \text{ ,} \tag{89a}$$

$$\sigma_{ij}\nu_j = \hat{P}_i \text{ over } S_p \tag{89b}$$

The convexity of $f(T^0, \varepsilon_{ij}, \chi_\alpha)$ implies that this stationary point is the minimum of F. Equations 89 are the equilibrium equations for the body.

It is seen that the minimum free energy principle is a generalisation of the kinematic minimum principle of Colonetti (1918, 1950).

1.4.2 The Gibbs energy minimum principle

It is sometimes useful to use the stresses σ_{ij} and internal variables χ_α as the independent state variables. We will therefore obtain a dual minimum principle in terms of these quantities for the equilibrium state of the same loaded body. Equation 86 is

$$\delta F = \int_V \delta f dV - \int_V \delta(\hat{F}_i u_i) dV - \int_{S_p} \delta(\hat{P}_i u_i) dS, \tag{90}$$

which on using Gauss' theorem may be written as

$$\delta F = \int_V \{\delta f - \delta(\sigma^s_{ij}\varepsilon_{ij})\} dV + \int_{S_u} \delta(\sigma^s_{ij}\nu_j\hat{u}_i) dS, \tag{91}$$

where

$$\frac{\partial\sigma^s_{ij}}{\partial x_j} + \hat{F}_i = 0 \text{ over } V \tag{92a}$$

$$\sigma^s_{ij}\nu_j = \hat{P}_i \quad \text{over} \quad S_p \quad . \tag{92b}$$

A field σ^s_{ij} satisfying equations 92 is termed *statically admissible*. It is seen that variations $\delta\sigma^s_{ij}$ must satisfy

$$\frac{\partial(\delta\sigma^s_{ij})}{\partial x_j} = 0 \quad \text{over} \quad V \ , \tag{93a}$$

$$\delta\sigma^s_{ij}\nu_j = 0 \quad \text{over} \quad S_p \ , \tag{93b}$$

We have seen that the condition $\delta F = 0$ implies that the stress field

$$\sigma_{ij} = \frac{\partial f}{\partial \varepsilon_{ij}} \ ,$$

is statically admissible (equations 89). Thus if $\delta F = 0$ we may certainly replace σ^s_{ij} by $\partial f/\partial\varepsilon_{ij}$ in equation 91, which may then be written

$$\delta F = \int_V \left\{\frac{\partial f}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij} - \left(\frac{\partial f}{\partial\varepsilon_{ij}}\delta\varepsilon_{ij} + \varepsilon_{ij}\delta\sigma^s_{ij}\right)\right\} dV - \int_V \delta\sigma^s_{ij}\nu_j\hat{u}_i dS \tag{94a}$$

$$= \int_V -\varepsilon_{ij}\delta\sigma^s_{ij} dV + \int_{S_u} \delta\sigma^s_{ij}\nu_j\hat{u}_i dS$$

$$= -\left[\int_V \frac{\partial h(T^\circ,\sigma_{ij},\chi_\alpha)}{\partial\sigma_{ij}}\delta\sigma^s_{ij} dV - \int_{S_u} \delta\sigma^s_{ij}\nu_j\hat{u}_i dS\right] \ , \tag{94b}$$

on noting equation 35. We remark that in equation 94b, $\sigma_{ij} = \partial f/\partial\varepsilon_{ij}$ is the statically admissible stress field which is the solution to the problem. Defining the Gibbs energy of the system by

$$H = \int_V h(T^\circ,\sigma_{ij},\chi_\alpha) dV - \int_{S_u} \sigma_{ij}\nu_j\hat{u}_i dS \ , \tag{95}$$

we see that, at constant χ_α

$$\left[\begin{array}{l} \qquad \delta F = 0 \\ \text{subject to } \varepsilon_{ij} \text{ and } u_i \\ \text{satisfying equations 85a} \\ \text{and 85b} \end{array}\right] \Leftrightarrow \left[\begin{array}{l} \qquad \delta H = 0 \\ \text{subject to } \sigma_{ij} \\ \text{satisfying equations 92} \end{array}\right] . \tag{96}$$

Due to the convexity at constant χ_α of $h(T^0, \sigma_{ij}, \chi_\alpha)$ as discussed in section 1.3.3, the stationary point of H is the minimum of H. This result is the *Gibbs energy minimum principle.*

We will now show that if H is stationary subject to the conditions stated above then the strain field $\varepsilon_{ij} = \partial h/\partial\sigma_{ij}$ and the displacement field u_i satisfy the strain-displacement relations (equations 85a and 85b). Consider variations in H due to variations $\delta\sigma^s_{ij}$ satisfying equations 93.

$$\delta H = \int_V \varepsilon_{ij}\delta\sigma^s_{ij}dV - \int_{S_u} \delta\sigma^s_{ij}\nu_j\hat{u}_i dS \ , \qquad (97)$$

where we use equation 35. Let u_i be a displacement field over V. Equation 97 may be written

$$\delta H = \int_V \varepsilon_{ij}\delta\sigma^s_{ij}dV - \int_S \delta\sigma^s_{ij}\nu_j\hat{u}_i dS$$

$$- \int_{S_u} \delta\sigma^s_{ij}\nu_j\hat{u}_i dS + \int_{S_u} \delta\sigma^s_{ij}\nu_j u_i dS \ , \qquad (98)$$

where we use equation 93b. Using Gauss' theorem, equation 93a and $\delta\sigma^s_{ij} = \delta\sigma^s_{ji}$, equation 98 becomes

$$\delta H = \int_V \left[\varepsilon_{ij} - \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]\delta\sigma^s_{ij}dV + \int (u_i - \hat{u}_i)\delta\sigma^s_{ij}\nu_j dS \ . \qquad (99)$$

Thus $\delta H = 0$ for arbitrary $\delta\sigma^s_{ij}$ if and only if

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{over } V \qquad (100a)$$

$$u_i = \hat{u}_i \quad \text{over } S_u \qquad (100b)$$

It is seen that the Gibbs energy minimum principle is a generalisation of the static minimum principle of Colonetti (1918, 1950).

The relation between the strain-displacement equations, the equilibrium equations and the minimum principles may be expressed by

$$\begin{bmatrix} \delta F = 0 \quad \text{subject to} \\ \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ over } V \\ u_i = \hat{u}_i \text{ over } S_u \\ \chi_\alpha = \int_0^t \dot{\chi}_\alpha d\tau \\ \quad = \hat{\chi}_\alpha \text{ over } V \end{bmatrix} \iff \begin{bmatrix} \delta H = 0 \quad \text{subject to} \\ \frac{\partial \sigma_{ij}}{\partial x_j} + \hat{F}_i = 0 \text{ over } V \\ \sigma_{ij}\nu_j = \hat{P}_i \text{ over } S_p \\ \chi_\alpha = \hat{\chi}_\alpha \text{ over } V \end{bmatrix}$$

$$\iff \begin{bmatrix} \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \text{ over } V \\ u_i = \hat{u}_i \text{ over } S_u \\ \frac{\partial \sigma_{ij}}{\partial x_j} + \hat{F}_i = 0 \text{ over } V \\ \sigma_{ij}\nu_j = \hat{P}_i \text{ over } S_p \\ \chi_\alpha = \hat{\chi}_\alpha \text{ over } V \end{bmatrix} \tag{101}$$

The last set of equations comprise the full set of field equations for the body; if u_i, ε_{ij} and χ_α are the unknown independent fields over V, this set and the knowledge of the history of $\dot{\chi}_\alpha$ enable us to calculate the state of the body.

Chapter 2. The incremental theorems for time-independent plasticity

2.1 Introduction

The minimum principles for the rate or incremental problem in time independent plasticity have been the subject of study over a number of years. The principles were first established in a weak form by Prager (1942, 1946), and extended to smooth yield surfaces by Hodge and Prager (1948). The conventional form for smooth yield surfaces was finally given by Greenberg (1949a, 1949b). Koiter (1953) further generalised the principles to cover singular yield surfaces. Further discussions of the conventional form of the minimum principles have been given by Hill (1956), Drucker (1958), Koiter (1960) and Hodge (1968).

In this conventional form the rate (or incremental) problem is considered as a boundary value problem in which traction or displacement rates are specified on the surface S of a body of volume V. The stress rates $\dot{\sigma}_{ij}$ are required to satisfy the rate form of the equilibrium equations, and the strain rates $\dot{\varepsilon}_{ij}$ and the displacement rates $\dot{u}_i$ are required to satisfy the strain rate, displacement rate relations. The constitutive equations are given in terms of stress rates $\dot{\sigma}_{ij}$ and total strain rates $\dot{\varepsilon}_{ij}$. These equations depend on the previous stress or strain history, and take a different form depending upon whether an element of material is elastic or plastic and unloading or plastic and loading.

An alternative approach has been presented more recently by the Italian school. This approach is based on the work of Colonetti (1918, 1950) who considered elastic bodies subjected to loading and imposed inelastic strains. The solution is given as the superposition of two elastic problems, one involving loading and no inelastic strains, and the other no loading and imposed inelastic strains. A rate (or incremental) form of approach can also be given.

Ceradini (1966) and Maier (1969) in the static and kinematic cases respectively considered what additional requirements must be imposed if the inelastic strain rates, the elastic strain rates and the stress rates must satisfy the plastic constitutive relations. This resulted in two new minimum principles of a quadratic programming form: quadratic functions of the plastic strain rates must be minimised subject to linear inequality constraints. Ceradini's theorem was derived directly from the conventional form, while Maier used quadratic programming arguments to establish the kinematic form. It was shown by Martin (1975c) that Maier's theorem follows from the conventional form of the kinematic theorem if use is made of a further property of the constitutive relation in the form of an inequality concerning an arbitrary division of strain rate into elastic and plastic parts. This result was further generalised by Martin (1975a) who gave directly a quadratic programming form of the kinematic minimum principle in which total strain rates and plastic strain rates are variables and the principle of superposition is not used.

Recently, attention has also been given to internal variable theories of plasticity which have a sound thermodynamic basis. This work suggests the problem of basing the minimum principles of the rate problem on the appropriate thermodynamic minimum principle for statically loaded bodies undergoing isothermal deformation. This does not appear to have been considered in previous work where, for example, the formal relation between the classical potential and complementary energy theorems of elasticity and the rate theorems of plasticity have not been formally explored.

It is the intention of this chapter to study this relation. We begin by considering the kinematic theorems in section 2.2. The free energy minimum principle is applied to two adjacent states of loading to give an equilibrium condition which is a form of Colonetti's kinematic principle in incremental form. Inclusion of the properties of the limit function yields the conventional kinematic minimum principle in terms of the incremental strains alone and finally the extended minimum principle in terms of incremental strains and internal variables. In section 2.3, using stresses and internal variables as independent parameters of state, we apply the Gibbs energy minimum principle to two adjacent states of loading to obtain an incremental form of Colonetti's static principle. Inclusion of the properties of the limit function yields the conventional static minimum principle in terms of incremental stresses and internal variables. Finally, in section 2.4 we note that the incremental theorems may be reduced to rate theorems.

2.2 The kinematic incremental theorems

Consider a body of volume V and surface S, consisting of a time-independent plastic material and subject to small isothermal deformations. The body is subjected to time-dependent body faces $\hat{F}_i(t)$ over V, tractions $\hat{P}_i(t)$ on part of the surface S_p and displacements $\hat{u}_i(t)$ on the remainder of the surface S_u. Omitting reference to temperature, the state of the body at time t is described by the strain and internal variable fields $\varepsilon_{ij}(t)$ and $\chi_\alpha(t)$ over V. The displacement field over V is $u_i(t)$ and satisfies

$$u_i(t) = \hat{u}_i(t) \quad \text{over} \quad S_u \ . \tag{1a}$$

The strain-displacement relations are

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{over} \quad V \ . \tag{1b}$$

Evidently we may apply the free energy minimum principle at an instant $t = t_o$. As we have seen in section 1.4, this enables us to calculate the state of the body provided the constrained internal variable field is known. We will assume that this is the case and that $\chi_\alpha(t_o) = \chi_\alpha^*$ over V. The strain field $\varepsilon_{ij}(t_o) = \varepsilon_{ij}^*$ and the displacement field $u_i(t_o) = u_i^*$ satisfy equations 1 and, from equations 101 of chapter 1, the stress field

$$\sigma_{ij}^* = \left.\frac{\partial f}{\partial \varepsilon_{ij}}\right|_{\varepsilon_{ij}^*, \chi_\alpha^*}$$

satisfies

$$\frac{\partial \sigma_{ij}^*}{\partial x_j} + \hat{F}_i(t_o) = 0 \quad \text{over} \quad V \ , \tag{2a}$$

$$\sigma_{ij}^* \nu_j = \hat{P}_i(t_o) \quad \text{over} \quad S_p \ , \tag{2b}$$

where ν_j is the unit outward normal over S. The internal forces

$$X_\alpha^* = -\left.\frac{\partial f}{\partial \chi_\alpha}\right|_{\varepsilon_{ij}^*, \chi_\alpha^*} ,$$

must satisfy the limit condition $\phi(X^*_\alpha) \leq 0$.

We now consider the (new) system at time $t = t_o + \Delta t$ where Δt is small. The imposed loads and displacements have been increased by $\Delta\hat{F}_i = \dot{\hat{F}}_i \Delta t$, $\Delta\hat{P}_i = \dot{\hat{P}}_i \Delta t$, $\Delta\hat{u}_i = \dot{\hat{u}}_i \Delta t$. Let the increase in the displacement field be Δu_i. We represent the state of the new system by $\varepsilon^*_{ij} + \Delta\varepsilon_{ij}$ and $\chi^*_\alpha + \Delta\chi_\alpha$. The dependent variables become $\sigma^*_{ij} + \Delta\sigma_{ij}$ and $X^*_\alpha + \Delta X_\alpha$.

Clearly we may apply the free energy minimum principle to this new system and assert that

$$F = \int_V f(\varepsilon^*_{ij} + \Delta\varepsilon_{ij}, \chi^*_\alpha + \Delta\chi_\alpha)dV - \int_V (\hat{F}_i + \Delta\hat{F}_i)(u^*_i + \Delta u_i)dV$$

$$- \int_{S_p} (\hat{P}_i + \Delta\hat{P}_i)(u^*_i + \Delta u_i)dS \ , \qquad (3)$$

must be a minimum subject to the constraints

$$\varepsilon^*_{ij} + \Delta\varepsilon_{ij} = \frac{1}{2}\left\{\frac{\partial}{\partial x_j}(u^*_i + \Delta u_i) + \frac{\partial}{\partial x_i}(u^*_j + \Delta u_j)\right\} \quad \text{over } V \ , \qquad (4a)$$

$$u^*_i + \Delta u_i = \hat{u}_i + \Delta\hat{u}_i \quad \text{over } S_u \ , \qquad (4b)$$

$$\chi^*_\alpha + \Delta\chi_\alpha \text{ is fixed over } V \ . \qquad (4c)$$

Since equations 1 are satisfied by ε^*_{ij} and u^*_i and since χ^*_α is fixed, equations 4 imply

$$\Delta\varepsilon_{ij} = \frac{1}{2}\left\{\frac{\partial \Delta u_i}{\partial x_j} + \frac{\partial \Delta u_j}{\partial x_i}\right\} \quad \text{over } V \ , \qquad (5a)$$

$$\Delta u_i = \Delta\hat{u}_i \quad \text{over } V \ , \qquad (5b)$$

$$\Delta\chi_\alpha \text{ is fixed over } V \ . \qquad (5c)$$

Incremental fields Δu_i and $\Delta\varepsilon_{ij}$ satisfying equations 5a and 5b will be termed *kinematically admissible*.

We note that to second order we may put

$$f(\varepsilon^*_{ij} + \Delta\varepsilon_{ij}, \chi^*_\alpha + \Delta\chi_\alpha) = f(\varepsilon^*_{ij}, \chi^*_\alpha) + \frac{\partial f}{\partial \varepsilon_{ij}} \Delta\varepsilon_{ij} + \frac{\partial f}{\partial \chi_\alpha} \Delta\chi_\alpha$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial \varepsilon_{ij} \partial \varepsilon_{k\ell}} \Delta\varepsilon_{ij} \Delta\varepsilon_{k\ell} + \frac{\partial^2 f}{\partial \varepsilon_{ij} \partial \chi_\alpha} \Delta\varepsilon_{ij} \Delta\chi_\alpha$$

$$+ \frac{1}{2} \frac{\partial^2 f}{\partial \chi_\alpha \partial \chi_\beta} \Delta\chi_\alpha \Delta\chi_\beta \quad . \tag{6}$$

All derivatives are evaluated at $\varepsilon^*_{ij}, \chi^*_\alpha$.

Since variations in $\varepsilon^*_{ij} + \Delta\varepsilon_{ij}$, $u^*_i + \Delta u_i$ may be treated as variations in $\Delta\varepsilon_{ij}$, Δu_i and denoted by $\delta\varepsilon_{ij}$, δu_i we see that the corresponding variation in F is

$$\delta F = \left[\int_V \sigma^*_{ij} \delta\varepsilon_{ij} dV - \int_V \hat{F}_i \delta u_i dV - \int_{S_p} \hat{P}_i \delta u_i dS \right]$$

$$+ \left[\int_V \left(\frac{\partial^2 f}{\partial \varepsilon_{ij} \partial \varepsilon_{k\ell}} \Delta\varepsilon_{ij} + \frac{\partial^2 f}{\partial \varepsilon_{k\ell} \partial \chi_\alpha} \Delta\chi_\alpha \right) \delta\varepsilon_{k\ell} dV \right.$$

$$\left. - \int_V \Delta\hat{F}_i \delta u_i dV - \int_{S_p} \Delta\hat{P}_i \delta u_i dS \right] \quad . \tag{7}$$

We require that $\delta F = 0$ for arbitrary variations $\delta\varepsilon_{ij}$, δu_i such that $\Delta\varepsilon_{ij} + \delta\varepsilon_{ij}$, $\Delta u_i + \delta u_i$ are kinematically admissible. But we know that the first set of terms within square brackets vanishes for such variations since it gives the first variation of the free energy of the system at time $t = t_o$. (See equations 86 - 89 of chapter 1). That the sum of the remaining terms vanishes implies that

$$U^P = \int_V \left(\frac{1}{2} \frac{\partial^2 f}{\partial \varepsilon_{ij} \partial \varepsilon_{k\ell}} \Delta\varepsilon_{ij} \Delta\varepsilon_{k\ell} + \frac{\partial^2 f}{\partial \varepsilon_{ij} \partial \chi_\alpha} \Delta\varepsilon_{ij} \Delta\chi_\alpha \right) dV$$

$$- \int_V \Delta\hat{F}_i \Delta u_i dV - \int_{S_p} \Delta\hat{P}_i \Delta u_i dS \quad , \tag{8}$$

is stationary with respect to variations in the kinematically admissible $\Delta\varepsilon_{ij}$, Δu_i with $\Delta\chi_\alpha$ held constant. Due to the convexity of $f(\varepsilon_{ij},\chi_\alpha)$, U^p takes its least value when it is stationary. Since

$$\Delta\sigma_{ij} = \left.\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}}\right|_{\varepsilon^*_{ij},\chi^*_\alpha} \Delta\varepsilon_{k\ell} + \left.\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha}\right|_{\varepsilon^*_{ij},\chi^*_\alpha} \Delta\chi_\alpha \tag{9}$$

it is readily established that U^p is stationary if and only if

$$\begin{aligned} \frac{\partial\Delta\sigma_{ij}}{\partial x_j} + \Delta\hat{F}_i &= 0 \quad \text{over } V , \\ \Delta\sigma_{ij}\nu_j &= \Delta\hat{P}_i \quad \text{over } S_p . \end{aligned} \tag{10}$$

These equations are the incremental equilibrium equations. Equation 8 is the incremental form of the kinematic minimum principle obtained by Colonetti.

In order to complete the solution of the incremental problem we need to ascertain $\Delta\chi_\alpha$. Returning to equations 60a and 64 of chapter 1 we see that in their incremental form they give $\Delta\chi_\alpha$ as a function of $\phi(X^*_\alpha)$ and $\Delta\varepsilon_{ij}$. Substituting for $\Delta\chi_\alpha$ from the incremental form of these equations into the second set of terms within square brackets in equation 7 we see that

$$\int_V \left\{ \frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}} \Delta\varepsilon_{k\ell} - \left[\frac{\left(\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha} \frac{\partial\phi}{\partial X_\alpha}\right)\left(\frac{\partial^2 f}{\partial\varepsilon_{k\ell}\partial\chi_\beta} \frac{\partial\phi}{\partial X_\beta} \Delta\varepsilon_{k\ell}\right)}{\left(\frac{\partial^2 f}{\partial\chi_\alpha\partial\chi_\beta} \frac{\partial\phi}{\partial X_\alpha} \frac{\partial\phi}{\partial X_\beta}\right)} \right] \right\} \delta\varepsilon_{ij} dV$$

$$- \int_V \Delta\hat{F}_i \delta u_i dV - \int_{S_p} \Delta\hat{P}_i \delta u_i dS = 0 , \tag{11}$$

where the term in square brackets is included if and only if

$$\phi(X^*_\alpha) = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha} \frac{\partial\phi}{\partial X_\alpha} \Delta\varepsilon_{ij} < 0 . \tag{12}$$

We may readily establish that equations 11 and 12 imply that the incremental solution is given by the least value of

$$\hat{U}^P = \int_V W^P(\Delta\varepsilon_{ij})dV - \int_V \Delta\hat{F}_i \Delta u_i dV - \int_{S_p} \Delta\hat{P}_i \Delta u_i dS \tag{13}$$

with respect the kinematically admissible $\Delta\varepsilon_{ij}$, Δu_i where

$$W^P = \frac{1}{2}\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}}\Delta\varepsilon_{ij}\Delta\varepsilon_{k\ell} - \frac{1}{2}\frac{\left(\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha}\frac{\partial\phi}{\partial X_\alpha}\Delta\varepsilon_{ij}\right)^2}{\left(\frac{\partial^2 f}{\partial\chi_\alpha\partial\chi_\beta}\frac{\partial\phi}{\partial X_\alpha}\frac{\partial\phi}{\partial X_\beta}\right)}, \tag{14a}$$

when $\phi(X^*_\alpha) = 0$ and $\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha}\frac{\partial\phi}{\partial X_\alpha}\Delta\varepsilon_{ij} < 0$,

and

$$W^P = \frac{1}{2}\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}}\Delta\varepsilon_{ij}\Delta\varepsilon_{k\ell}, \tag{14b}$$

otherwise.

Alternatively, using equations 64 of chapter 1 and equation 9

$$W^P = \frac{1}{2}\Delta\sigma_{ij}\Delta\varepsilon_{ij}, \tag{15}$$

where $\Delta\sigma_{ij}$ is the stress increment associated with $\Delta\varepsilon_{ij}$ through the constitutive relations.

This result is the classical incremental theorem given in the form of Greenberg (1949a, 1949b).

We now make use of the minimum principle given in equation 65 of chapter 1 in order to obtain $\Delta\chi_\alpha(\phi,\Delta\varepsilon_{ij})$. It is seen that

$$W^P = \frac{1}{2}\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}}\Delta\varepsilon_{ij}\Delta\varepsilon_{k\ell} + \min.\{P(\lambda)\} \tag{16}$$

subject to $\lambda \geq 0$ if $\phi(X^*_\alpha) = 0$, $\lambda = 0$ if $\phi(X^*_\alpha) < 0$.

Alternatively

$$W^p \le \overline{W}^p = \frac{1}{2}\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}}\Delta\varepsilon_{ij}\Delta\varepsilon_{k\ell} + \dot{P}(\lambda)$$

$$= \frac{1}{2}\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}}\Delta\varepsilon_{ij}\Delta\varepsilon_{k\ell} + \frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha}\frac{\partial\phi}{\partial X_\alpha}\Delta\varepsilon_{ij}\lambda$$

$$+ \frac{1}{2}\frac{\partial^2 f}{\partial\chi_\alpha\partial\chi_\beta}\frac{\partial\phi}{\partial X_\alpha}\frac{\partial\phi}{\partial X_\beta}\lambda^2 \tag{17}$$

subject to $\lambda \ge 0$ if $\phi(X^*_\alpha) = 0$ and $\lambda = 0$ if $\phi(X^*_\alpha) < 0$. $W^p = \overline{W}^p$ when λ takes its correct value for given $\phi(X^*_\alpha)$ and $\Delta\varepsilon_{ij}$.

We define

$$\overline{U}^p = \int_V \overline{W}^p(\Delta\varepsilon_{ij},\lambda)dV - \int_V \Delta\hat{F}_i\Delta u_i dV - \int_{S_p} \Delta\hat{P}_i\Delta u_i dS \, . \tag{18}$$

For any given field $\Delta\varepsilon_{ij}(x_i)$ we see from equation 16 that

$$\overline{U}^p \ge \tilde{U}^p \quad , \tag{19}$$

subject to $\lambda \ge 0$ if $\phi(X^*_\alpha) = 0$ and $\lambda = 0$ if $\phi(X^*_\alpha) < 0$. $\overline{U}^p = \tilde{U}^p$ if we minimise $\overline{W}^p$ with respect to λ subject to these constraints in a pointwise fashion throughout the body. We may then further assert that the incremental solution is given by the least value of $\overline{U}^p$ with respect to kinematically admissible fields Δu_i, $\Delta\varepsilon_{ij}$ and with respect to λ subject to $\lambda \ge 0$ if $\phi(X^*_\alpha) = 0$ and $\lambda = 0$ if $\phi(X^*_\alpha) < 0$. This is the extended minimum principle given by Martin (1975a).

2.3 The static incremental theorems

Again, consider the problem of a body of time-independent plastic material of volume V and surface S, subject to small isothermal deformations with time-dependent body forces $\hat{F}_i(t)$ over V, tractions $\hat{P}_i(t)$ over S_p, and displacements $\hat{u}_i(t)$ over S_u. Let the displacement field over V be $u_i(t)$. We use the stress and internal variable fields $\sigma_{ij}(t)$ and $\chi_\alpha(t)$

over V to describe the state of the body at time t.

Evidently we may apply the Gibbs energy minimum principle at $t = t_o$. As we have seen in section 1.4, this allows us to calculate the state of the body provided the constrained internal variable field is known. We will assume that this is the case and that $\chi_\alpha(t_o) = \chi_\alpha^*$ over V. From equations 101 of chapter 1, the stress field $\sigma_{ij}(t_o) = \sigma_{ij}^*$ satisfies

$$\frac{\partial \sigma_{ij}^*}{\partial x_j} + \hat{F}_i(t_o) = 0 \quad \text{over} \quad V \; , \tag{20a}$$

$$\sigma_{ij}^* \nu_j = \hat{P}_i(t_o) \quad \text{over} \quad S_p \; . \tag{20b}$$

The strain field

$$\varepsilon_{ij}^* = \left.\frac{\partial h}{\partial \sigma_{ij}}\right|_{\sigma_{ij}^*, \chi_\alpha^*}$$

and the displacement field $u_i(t_o) = u_i^*$ satisfy

$$\varepsilon_{ij}^* = \frac{1}{2}\left(\frac{\partial u_i^*}{\partial x_j} + \frac{\partial u_j^*}{\partial x_i}\right) \quad \text{over} \quad V \; ,$$

$$u_i^* = \hat{u}_i(t_o) \quad \text{over} \quad S_u \; . \tag{21}$$

The internal forces

$$X_\alpha^* = \left.\frac{\partial h}{\partial \chi_\alpha}\right|_{\sigma_{ij}^*, \chi_\alpha^*}$$

must satisfy the limit condition $\phi(X_\alpha^*) \leq 0$.

We now consider the (new) system at time $t = t_o + \Delta t$ where Δt is small. The imposed loads and displacements have been increased by small amounts $\Delta\hat{F}_i = \dot{\hat{F}}_i \Delta t$, $\Delta\hat{P}_i = \dot{\hat{P}}_i \Delta t$, $\Delta\hat{u}_i = \dot{\hat{u}}_i \Delta t$. Let the increase in the displacement field over V be Δu_i. We represent the state of the new system by $\sigma_{ij}^* + \Delta\sigma_{ij}$ and $\chi_\alpha^* + \Delta\chi_\alpha$. The dependent variables become $\varepsilon_{ij}^* + \Delta\varepsilon_{ij}$, $X_\alpha^* + \Delta X_\alpha$.

Clearly we may apply the Gibbs energy minimum principle to this new system and assert that

$$H = \int_V h(\sigma^*_{ij} + \Delta\sigma_{ij}, \chi^*_\alpha + \Delta\chi_\alpha)dV - \int_{S_u} (\sigma^*_{ij} + \Delta\sigma_{ij})\nu_j(\hat{u}_i + \Delta\hat{u}_i)dS \ , \tag{22}$$

must be a minimum subject to the constraints

$$\frac{\partial}{\partial x_j}(\sigma^*_{ij} + \Delta\sigma_{ij}) + \hat{F}_i(t_o) + \Delta\hat{F}_i = 0 \quad \text{over} \quad V \ , \tag{23a}$$

$$(\sigma^*_{ij} + \Delta\sigma_{ij})\nu_j = \hat{P}_i(t_o) + \Delta\hat{P}_i \quad \text{over} \quad S_p \tag{23b}$$

$$\chi^*_\alpha + \Delta\chi_\alpha \quad \text{is fixed over} \quad V \ . \tag{23c}$$

Noting equations 20a and 20b and that χ^*_α is fixed, equations 23 imply

$$\frac{\partial\Delta\sigma_{ij}}{\partial x_j} + \Delta\hat{F}_i = 0 \quad \text{over} \quad V \ , \tag{24a}$$

$$\Delta\sigma_{ij}\nu_j = \Delta\hat{P}_i \quad \text{over} \quad S_p \ , \tag{24b}$$

$$\Delta\chi_\alpha \quad \text{is fixed over} \quad V \ . \tag{24c}$$

An incremental stress field satisfying equations 24a and 24b will be termed *statically admissible.*

We note that to second order we may put

$$\begin{aligned} h(\sigma^*_{ij} + \Delta\sigma_{ij}, \chi^*_\alpha + \Delta\chi_\alpha) &= h(\sigma^*_{ij}, \chi^*_\alpha) + \frac{\partial h}{\partial\sigma_{ij}}\Delta\sigma_{ij} + \frac{\partial h}{\partial\chi_\alpha}\Delta\chi_\alpha \\ &+ \frac{1}{2}\frac{\partial^2 h}{\partial\sigma_{ij}\partial\sigma_{k\ell}}\Delta\sigma_{ij}\Delta\sigma_{k\ell} + \frac{\partial^2 h}{\partial\sigma_{ij}\partial\chi_\alpha}\Delta\sigma_{ij}\Delta\chi_\alpha \\ &+ \frac{1}{2}\frac{\partial^2 h}{\partial\chi_\alpha\partial\chi_\beta}\Delta\chi_\alpha\Delta\chi_\beta \ . \end{aligned} \tag{25}$$

All derivatives are evaluated at σ^*_{ij}, χ^*_α.

Since variations in $\sigma^*_{ij} + \Delta\sigma_{ij}$ may be treated as variations in $\Delta\sigma_{ij}$ and denoted by $\delta\sigma_{ij}$, the corresponding variation in H is

$$\delta H = \left[\int_V \varepsilon^*_{ij}\delta\sigma_{ij}dV - \int_{S_u} \hat{u}_i\delta\sigma_{ij}\nu_j dS\right]$$

$$+ \left[\int_V \left(\frac{\partial^2 h}{\partial\sigma_{ij}\partial\sigma_{k\ell}}\Delta\sigma_{ij} + \frac{\partial^2 h}{\partial\sigma_{k\ell}\partial\chi_\alpha}\Delta\chi_\alpha\right)\delta\sigma_{k\ell}dV\right.$$

$$\left. - \int_{S_u} \Delta\hat{u}_i\delta\sigma_{ij}\nu_j dS\right] . \tag{26}$$

We require that $\delta H = 0$ for arbitrary variations $\delta\sigma_{ij}$ such that $\Delta\sigma_{ij} + \delta\sigma_{ij}$ is statically admissible. But we know that the first set of terms within square brackets vanishes for such variations since they give the first variation in the Gibbs energy of the system at time $t = t_o$. (See equations 97 - 100 of chapter 1). That the sum of the remaining terms vanishes implies that

$$V^c = \int_V \left(\frac{1}{2}\frac{\partial^2 h}{\partial\sigma_{ij}\partial\sigma_{k\ell}}\Delta\sigma_{ij}\Delta\sigma_{k\ell} + \frac{\partial^2 h}{\partial\sigma_{ij}\partial\chi_\alpha}\Delta\sigma_{ij}\Delta\chi_\alpha\right)dV$$

$$- \int_{S_u} \Delta\hat{u}_i\Delta\sigma_{ij}\nu_j dS , \tag{27}$$

is stationary for variations in the statically admissible $\Delta\sigma_{ij}$ with $\Delta\chi_\alpha$ held constant. Due to the convexity at constant χ_α of $h(\sigma_{ij}, \chi_\alpha)$, (see section 1.3.3), V^c takes its least value under these conditions. Since

$$\Delta\varepsilon_{ij} = \left.\frac{\partial^2 h}{\partial\sigma_{ij}\partial\sigma_{k\ell}}\right|_{\sigma^*_{ij},\chi^*_\alpha}\Delta\sigma_{k\ell} + \left.\frac{\partial^2 h}{\partial\sigma_{ij}\partial\chi_\alpha}\right|_{\sigma^*_{ij},\chi^*_\alpha}\Delta\chi_\alpha , \tag{28}$$

it may also be established that V^c is stationary if and only if

$$\Delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial\Delta u_i}{\partial x_j} + \frac{\partial\Delta u_j}{\partial u_i}\right) \text{ over } V , \tag{29a}$$

$$\Delta u_i = \Delta \hat{u}_i \quad \text{over} \quad S_u \ . \tag{29b}$$

These equations are the incremental strain-displacement relations. Equation 27 is the incremental form of the static minimum principle obtained by Colonetti.

In order to complete the solution of the incremental problem we need to ascertain $\Delta\chi_\alpha$. Returning to equations 60a and 68 of chapter 1, we see that in their incremental form, they give $\Delta\chi_\alpha$ as a function of $\phi(X^*_\alpha)$ and $\Delta\sigma_{ij}$. Substituting for $\Delta\chi_\alpha$ from the incremental form of these equations into the second set of terms within square brackets in equation 26 we see that

$$\int_V \left\{ \frac{\partial^2 h}{\partial\sigma_{ij}\partial\sigma_{k\ell}} \Delta\sigma_{ij} - \left[\frac{\left(\dfrac{\partial^2 h}{\partial\sigma_{k\ell}\partial\chi_\alpha} \dfrac{\partial\phi}{\partial X_\alpha}\right)\left(\dfrac{\partial^2 h}{\partial\sigma_{ij}\partial\chi_\beta} \dfrac{\partial\phi}{\partial X_\beta} \Delta\sigma_{ij}\right)}{\left(\dfrac{\partial^2 h}{\partial\chi_\alpha\partial\chi_\beta} \dfrac{\partial\phi}{\partial X_\alpha} \dfrac{\partial\phi}{\partial X_\beta}\right)} \right] \right\} \delta\sigma_{k\ell} dV$$

$$- \int_{S_u} \Delta\hat{u}_i \delta\sigma_{ij} \nu_j dS = 0 , \tag{30}$$

where the term in square brackets is included if and only if

$$\phi(X^*_\alpha) = 0 \quad \text{and} \quad \frac{\partial^2 h}{\partial\sigma_{ij}\partial\chi_\alpha} \frac{\partial\phi}{\partial X_\alpha} \Delta\sigma_{ij} > 0 \ . \tag{31}$$

We may readily establish that equations 30 and 31 imply that the incremental solution is given by the least value of

$$\tilde{V}^c = \int_V \Omega^c(\Delta\sigma_{ij}) dV - \int_{S_u} \Delta\hat{u}_i \Delta\sigma_{ij} \nu_j dS , \tag{32}$$

with respect variations in the statically admissible field $\Delta\sigma_{ij}$, where

$$\Omega^c = \frac{1}{2} \frac{\partial^2 h}{\partial\sigma_{ij}\partial\sigma_{k\ell}} \Delta\sigma_{ij}\Delta\sigma_{k\ell} - \frac{1}{2} \frac{\left(\dfrac{\partial^2 h}{\partial\sigma_{ij}\partial\chi_\alpha} \dfrac{\partial\phi}{\partial X_\alpha} \Delta\sigma_{ij}\right)^2}{\left(\dfrac{\partial^2 h}{\partial\chi_\alpha\partial\chi_\beta} \dfrac{\partial\phi}{\partial X_\alpha} \dfrac{\partial\phi}{\partial X_\beta}\right)} \tag{33a}$$

when $\phi(X^*_\alpha) = 0$ and

$$\frac{\partial^2 h}{\partial\sigma_{ij}\partial\chi_\alpha}\ \frac{\partial\phi}{\partial X_\alpha}\ \Delta\sigma_{ij}\quad > \quad 0\ ,$$

and

$$\Omega^c \quad = \quad \frac{1}{2}\,\frac{\partial^2 h}{\partial\sigma_{ij}\partial\sigma_{k\ell}}\ \Delta\sigma_{ij}\Delta\sigma_{k\ell}\ , \tag{33b}$$

otherwise.

Alternatively, using equations 68 of chapter 1 and equation 28

$$\Omega^c \quad = \quad \frac{1}{2}\,\Delta\bar{\sigma}_{ij}\Delta\varepsilon_{ij}\ , \tag{34}$$

where $\Delta\varepsilon_{ij}$ is the strain increment associated with $\Delta\sigma_{ij}$ through the constitutive relations.

This result is the classical static incremental theorem given in the form of Greenberg (1949a, 1949b).

Returning to equation 69 of chapter 1, it is seen that for a particular $\Delta\sigma_{ij}$, the value of λ which satisfies the limit condition maximises

$$\bar{\Omega}^c(\Delta\sigma_{ij},\lambda) \quad = \quad \frac{1}{2}\,\frac{\partial^2 h}{\partial\sigma_{ij}\partial\sigma_{k\ell}}\ \Delta\sigma_{ij}\Delta\sigma_{k\ell}\ +\ \frac{\partial^2 h}{\partial\sigma_{ij}\partial\chi_\alpha}\ \frac{\partial\phi}{\partial X_\alpha}\ \Delta\sigma_{ij}\lambda$$

$$+\ \frac{1}{2}\,\frac{\partial^2 h}{\partial\chi_\alpha\partial\chi_\beta}\ \frac{\partial\phi}{\partial X_\alpha}\ \frac{\partial\phi}{\partial X_\beta}\ \lambda^2\ , \tag{35}$$

subject to $\lambda \geq 0$ if $\phi(X^*_\alpha) = 0$, $\lambda = 0$ if $\phi(X^*_\alpha) < 0$. It follows that for any particular field $\Delta\sigma'_{ij}(x_i)$, the field $\lambda'(x_i)$ which satisfies the limit condition at each point maximises

$$\bar{V}^c(\Delta\sigma'_{ij},\lambda) \quad = \quad \int_V \bar{\Omega}^c(\Delta\sigma'_{ij},\lambda)\,dV\ -\ \int_{S_u} \Delta\hat{u}_i\Delta\sigma'_{ij}\nu_j\,dS\ , \tag{36}$$

in a pointwise fashion throughout the body subject to $\lambda \geq 0$ if $\phi(X^*_\alpha) = 0$, $\lambda < 0$ if $\phi(X^*_\alpha) < 0$. Alternatively, let $\lambda'(x_i)$ be associated with a particular field $\Delta\sigma'_{ij}(x_i)$ through equations 68 of chapter 1. Then

$$\bar{V}^c(\Delta\sigma'_{ij},\lambda) \quad \leq \quad \bar{V}^c(\Delta\sigma'_{ij},\lambda')\ , \tag{37}$$

for all λ subject to $\lambda \geq 0$ if $\phi(X^*_\alpha) = 0$, $\lambda = 0$ if $\phi(X^*_\alpha) < 0$.

Returning to equation 27, it is seen that for a particular field $\Delta\chi'_\alpha(x_i) = \lambda'(x_i)\{\partial\phi/\partial X_\alpha\}$, the field $\Delta\sigma'_{ij}(x_i)$ which satisfies the incremental strain-displacement relations (equations 29) minimises $\overline{V}^c(\Delta\sigma_{ij},\lambda)$ with respect to statically admissible fields $\Delta\sigma_{ij}$. Alternatively

$$\overline{V}^c(\Delta\sigma'_{ij},\lambda') \leq \overline{V}^c(\Delta\sigma_{ij},\lambda') , \tag{38}$$

for all statically admissible $\Delta\sigma_{ij}$.

Thus if the fields $\Delta\sigma'_{ij}(x_i)$, $\Delta\chi'_\alpha(x_i) = \lambda'(x_i)\{\partial\phi/\partial X_\alpha\}$ where $\partial\phi/\partial X_\alpha$ is evaluated at X^*_α, are the solution to the static incremental problem, then from inequalities 37 and 38 we see that

$$\overline{V}^c(\Delta\sigma'_{ij},\lambda) \leq \overline{V}^c(\Delta\sigma'_{ij},\lambda') \leq \overline{V}^c(\Delta\sigma_{ij},\lambda') , \tag{39}$$

for all statically admissible fields $\Delta\sigma_{ij}$ and for all fields λ satisfying $\lambda \geq 0$ if $\phi(X^*_\alpha) = 0$, $\lambda = 0$ if $\phi(X^*_\alpha) < 0$.

The solution to the statically formulated incremental problem thus satisfies a minimum-maximum principle; $\overline{V}^c$ is minimised with respect to the statically admissible stress field $\Delta\sigma_{ij}$ and maximised in a pointwise fashion with respect to the partially constrained internal variable field $\Delta\chi_\alpha$.

2.4 The rate theorems

All the incremental theorems discussed in this chapter may be reduced to rate theorems. The functionals which we have obtained are all homogenous and of degree two, and so may be divided by $(\Delta t)^2$. In the limit as $\Delta t \to 0$ we recover rate theorems in terms of $(\dot{\varepsilon}_{ij}, \dot{\chi}_\alpha)$ or $(\dot{\sigma}_{ij}, \dot{\chi}_\alpha)$.

Let the body be subjected to load rates $\dot{\hat{F}}_i$ over V, $\dot{\hat{P}}_i$ over S_p and displacement rates $\dot{\hat{u}}_i$ over S_u. It is assumed that the state of the body is known.

The extended kinematic rate theorem states that the response of the body, i.e. the strain rate and internal variable rate fields $\dot{\varepsilon}_{ij}(x_i)$, $\dot{\chi}_\alpha(x_i) = \lambda(x_i)\{\partial\phi/\partial X_\alpha\}$, are those that minimise see eq. (17) & eq. (6)

$$\bar{U}^P = \int_V \left\{\frac{1}{2}\frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\varepsilon_{k\ell}}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{k\ell} + \frac{\partial^2 f}{\partial\varepsilon_{ij}\partial\chi_\alpha}\frac{\partial\phi}{\partial X_\alpha}\dot{\varepsilon}_{ij}\lambda + \frac{1}{2}\frac{\partial^2 f}{\partial\chi_\alpha\partial\chi_\beta}\frac{\partial\phi}{\partial X_\alpha}\frac{\partial\phi}{\partial X_\beta}\lambda^2\right\}dV$$

$$- \int_V \dot{\hat{F}}_i\dot{u}_i dV - \int_{S_p}\dot{\hat{P}}_i\dot{u}_i dS \ , \qquad (40)$$

with respect to the kinematically admissible fields $\dot{u}_i$, $\dot{\varepsilon}_{ij}$ and with respect to λ subject to $\lambda \geq 0$ if $\phi(X_\alpha) = 0$, $\lambda = 0$ if $\phi(X_\alpha) < 0$.

Alternatively the extended static rate theorem states that the response of the body, i.e. the stress rate and internal variable rate fields $\dot{\sigma}_{ij}(x_i)$, $\dot{\chi}_\alpha(x_i) = \lambda(x_i)\{\partial\phi/\partial X_\alpha\}$ are those for which

$$\bar{V}^c = \int_V \left\{\frac{1}{2}\frac{\partial^2 h}{\partial\sigma_{ij}\partial\sigma_{k\ell}}\dot{\sigma}_{ij}\dot{\sigma}_{k\ell} + \frac{\partial^2 h}{\partial\sigma_{ij}\partial\chi_\alpha}\frac{\partial\phi}{\partial X_\alpha}\dot{\sigma}_{ij}\lambda + \frac{1}{2}\frac{\partial^2 h}{\partial\chi_\alpha\partial\chi_\beta}\frac{\partial\phi}{\partial X_\alpha}\frac{\partial\phi}{\partial X_\beta}\lambda^2\right\}dV$$

$$- \int_{S_u}\dot{\hat{u}}_i\dot{\sigma}_{ij}\nu_j dS \ , \qquad (41)$$

is minimised with respect to the statically admissible field $\dot{\sigma}_{ij}$ and maximised with respect to λ subject to $\lambda \geq 0$ if $\phi(X_\alpha) = 0$, $\lambda = 0$ if $\phi(X_\alpha) < 0$.

Chapter 3. An algorithm for the application of the extended static incremental theorem to a pin-jointed truss

3.1 Introduction

When applied to a structure whose state can be represented by a finite number of variables, the rate (or incremental) theorems of chapter 2 become programming problems. Martin and Reddy (1976) have applied the kinematic incremental theorem to a general pin-jointed truss and have suggested an algorithm for the solution of the ensuing programming problem.

In section 3.2 we formulate the extended static incremental theorem for a general pin-jointed truss, and in section 3.3 apply an algorithm similar to that of Martin and Reddy (1976) for the solution of the minimum-maximum problem. In section 3.4, in order to illustrate the algorithm, two numerical examples are discussed.

Examples of the use of the conventional static rate theorems for such discretised structures may be found in the work of Sayegh and Rubinstein (1972) and Hodge (1973).

3.2 Application to a pin-jointed truss

To illustrate the use of the extended static incremental theorem, consider an assembly of pin-jointed, inelastic, time-independent bars. External loads are applied only at the nodes, and since we are considering small deformations, the deformations (extensions) of the bars are related linearly to the nodal displacements which are defined in a Cartesian co-ordinate system. It will be assumed that certain of the nodal displacements are constrained to be zero over the entire loading program. The remaining (unknown) displacements are ordered and are represented by a column vector $\{u\}$ of say n elements. The specified external loads can be represented by a similarly ordered column vector $\{P\}$.

Let there be m bars, the i-th bar having uniform crossectional area A_i and length ℓ_i. Let its total and plastic extensions be δ_i and

δ_i^P respectively. We will assume that the state of the i-th bar can be represented by δ_i and δ_i^P and that the free energy of the i-th bar is

$$f_i(\delta_i,\ \delta_i^P) = \frac{1}{2}\frac{A_iE_i}{\ell_i}(\delta_i - \delta_i^P)^2 + \frac{1}{2}\frac{A_iE_i^P}{\ell_i}(\delta_i^P)^2 , \tag{1}$$

where E_i and E_i^P are constants having the dimensions of energy per unit volume and summation over repeated indices is not implied.

Thus the equations of state are

$$N_i = \frac{\partial f_i}{\partial \delta_i}$$

$$= \frac{A_iE_i}{\ell_i}(\delta_i - \delta_i^P) , \tag{2}$$

and

$$N_i^P = -\frac{\partial f_i}{\partial \delta_i^P}$$

$$= \frac{A_iE_i}{\ell_i}(\delta_i - \delta_i^P) - \frac{A_iE_i^P}{\ell_i}(\delta_i^P) . \tag{3}$$

Note that if E_i is the elastic modulus then the forces conjugate to the extensions are the axial forces. N_i^P is the internal force in the bar.

We see that the Gibbs energy of the i-th bar is

$$h_i(N_i,\ \delta_i^P) = \frac{1}{2}\frac{\ell_i}{A_iE_i}(N_i)^2 + N_i\delta_i^P - \frac{1}{2}\frac{A_iE_i^P}{\ell_i}(\delta_i^P)^2 . \tag{4}$$

The equations of state obtainable from this fundamental equation are

$$\delta_i = \frac{\partial h_i}{\partial N_i}$$

$$= \frac{\ell_i}{A_iE_i}N_i + \delta_i^P , \tag{5}$$

and

$$N_i^p = \frac{\partial h}{\partial \delta_i^p}$$

$$= N_i - \frac{A_i E_i^p}{\ell_i} \delta_i^p \quad . \tag{6}$$

In their incremental form, the general kinetic equations for time-independent plasticity (equations 60 of chapter 1) are

$$\Delta \chi_\alpha = \lambda \frac{\partial \phi}{\partial X_\alpha} \tag{7a}$$

where

$$\begin{aligned} &\lambda \geq 0 \text{ if } \phi = 0 \text{ and } \Delta\phi = 0 \\ &\lambda = 0 \text{ if } \phi < 0 \text{ or } \phi = 0 \text{ and } \Delta\phi < 0 , \end{aligned} \tag{7b}$$

and $\phi > 0$ is not attainable.

In this example the limit surface for the i-th bar consists of two points in the one-dimensional N_i^p - space. We will assume them to be $+N_i^{op}$ and $-N_i^{op}$ respectively. The limit function for the i-th bar $\phi^{(i)}(N_i^p)$ is

$$\phi^{(i)} = |N_i^p| - N_i^{op} ,$$

and so the gradient of $\phi^{(i)}$ is

$$\frac{\partial \phi^{(i)}}{\partial N_i^p} = \begin{cases} +1 \text{ if } N_i^p > 0 \\ -1 \text{ if } N_i^p < 0 . \end{cases}$$

In obtaining an expression for $\Delta\sigma_i^p$ corresponding to equations 7 we dispense with $\partial\phi^{(i)}/\partial N_i^p$ and replace the non-negativity constraint on λ by a constraint on the sign of $\Delta\sigma_i^p$. Thus

$$
\begin{aligned}
\Delta\delta_i^p &\geq 0 \quad \text{if} \quad N_i^p = +N_i^{op} \quad \text{and} \quad \Delta N_i^p = 0 \\
\Delta\delta_i^p &= 0 \begin{cases} \text{if} \quad N_i^p = +N_i^{op} \quad \text{and} \quad \Delta N_i^p < 0 \\ \text{if} \quad -N_i^{op} < N_i^p < +N_i^{op} \\ \text{if} \quad N_i^p = -N_i^{op} \quad \text{and} \quad \Delta N_i^p > 0 \end{cases} \\
\Delta\delta_i^p &\leq 0 \quad \text{if} \quad N_i^p = -N_i^{op} \quad \text{and} \quad \Delta N_i^p = 0 \,,
\end{aligned}
\tag{8}
$$

and $|N_i^p| > N_i^{op}$ is not attainable. The incremental form of equation 6 may be used to show that

$$
\Delta\delta_i^p = \Delta N_i \frac{\ell_i}{A_i E_i^p} \begin{cases} \text{if} \quad N_i^p = +N_i^{op} \quad \text{and} \quad \Delta N_i > 0 \\ \text{if} \quad N_i^p = -N_i^{op} \quad \text{and} \quad \Delta N_i < 0 \,, \end{cases}
\tag{9}
$$

$$\Delta\delta_i^p = 0 \quad \text{otherwise.}$$

Equations 2, 3 and 9 are the familiar equations of a bar whose elastic response is linear and whose inelastic response exhibits linear kinematic hardening. //

The free energy of the truss and conservative loads is

$$
F = \sum_{i=1}^{m} \left[\frac{1}{2} \frac{A_i E_i}{\ell_i} (\delta_i - \delta_i^p)^2 + \frac{1}{2} \frac{A_i E_i^p}{\ell_i} (\delta_i^p)^2 \right] - \{u\}^T \{P\} \,.
\tag{10}
$$

We may express equation 10 in matrix form by introducing the diagonal $m \times m$ elastic and plastic stiffness matrices $[S]$ and $[S^p]$. The diagonal elements of $[S]$ and $[S^p]$ are $A_i E_i / \ell_i$ and $A_i E_i^p / \ell_i$ respectively. We also introduce the column vectors $\{\delta\}$ and $\{\delta^p\}$ comprised of the ordered extensions δ_i and plastic extensions δ_i^p, the ordering being the same as in forming $[S]$ and $[S^p]$. Equation 10 becomes

$$
F = (\{\delta\} - \{\delta^p\})^T [S] (\{\delta\} - \{\delta^p\}) + \{\delta^p\}^T [S^p] \{\delta\} - \{u\}^T \{P\}.
\tag{11}
$$

By the free energy minimum principle, F must, at any instant be a minimum subject to $\{\delta^p\}$ fixed and $\{\delta\}$ and $\{u\}$ satisfying the strain-displacement relations

$$
\{\delta\} = [B]\{u\},
\tag{12}
$$

where $[B]$ is the deformation matrix. Equation 12 gives the bar extensions in terms of nodal displacements as we assumed above.

It is readily shown that F is a minimum under these circumstances if and only if

$$\{P\} = [B]^T\{N\} \quad , \tag{13}$$

which are the equilibrium equations for the system. In its incremental form equation 13 defines the relation between statically admissible load increments and stress increments.

Now, if at any instant the state $(\{N\}, \{\delta^P\})$, of the truss is known we may calculate its response $(\{\Delta N\}, \{\Delta\delta^P\})$, to a given load increment $\{\{\Delta^P\}$, by using the extended static incremental theorem of chapter 2.

Using equation 4 it is seen that $\overline{V}^c$ (defined in equations 35 and 36 of chapter 2) is

$$\overline{V}^c(\{\Delta N\}, \{\Delta\delta^P\}) = \sum_{i=1}^{m} \left[\frac{1}{2} \frac{\ell_i}{A_i E_i} (\Delta N_i)^2 + \Delta N_i \Delta\delta_i^P - \frac{1}{2} \frac{A_i E_i^P}{\ell_i} (\Delta\delta_i^P)^2 \right] . \tag{14}$$

The constraints on $\Delta\chi_\alpha$ in the extended static incremental theorem of chapter 2 are weaker than in equations 7 and are as follows

$$\Delta\chi_\alpha = \lambda \frac{\partial\phi}{\partial X_\alpha} , \tag{15a}$$

where

$$\begin{aligned} \lambda &\geq 0 \text{ if } \phi = 0 \\ \lambda &= 0 \text{ if } \phi < 0 . \end{aligned} \tag{15b}$$

These become, in this example

$$\Delta\delta_i^P \geq 0 \text{ if } N_i^P = +N_i^{oP} , \tag{16a}$$

$$\Delta\delta_i^P = 0 \text{ if } -N_i^{oP} < N_i^P < +N_i^{oP} , \tag{16b}$$

$$\Delta\delta_i^P \leq 0 \text{ if } N_i^P = -N_i^{oP} . \tag{16c}$$

We seek $\{\Delta N\}^*$, $\{\Delta\delta^P\}^*$ such that

$$\overline{V}^c(\{\Delta N\}^*, \{\Delta\delta^P\}) \le \overline{V}^c(\{\Delta N\}^*, \{\Delta\delta^P\}^*) \le \overline{V}^c(\{\Delta N\}, \{\Delta\delta^P\}^*) \quad , \tag{17}$$

for all $\{\Delta N\}$ subject to

$$\{\Delta P\} = [B]^T\{\Delta N\} \quad ,$$

and for all $\{\Delta\delta^P\}$ satisfying equations 16.

We include the constraint that $\{\Delta P\}$ and $\{\Delta N\}$ are statically admissible by introducing a Lagrange multipliers μ_i represented by the column vector $\{\mu\}$. For fixed $\{\Delta\delta^P\}^*$, the requirement that $\overline{V}^c(\{\Delta N\}, \{\Delta\delta^P\}^*)$ should be a minimum subject to

$$\{\Delta P\} = [B]^T\{\Delta N\} \quad ,$$

is expressed by requiring that for fixed $\{\Delta\delta^P\}^*$,

$$W^c(\{\Delta N\}, \{\Delta\delta^P\}^*, \{\mu\}) = \overline{V}^c(\{\Delta N\}, \{\Delta\delta^P\}^*)$$

$$+ \{\mu\}^T(\{\Delta P\} - [B]^T\{\Delta N\}) \quad , \tag{18}$$

should be stationary.

Since $\{\mu\}^T(\{\Delta P\} - [B]^T\{\Delta N\})$ is independent of $\{\Delta\delta^P\}$, from equation 17 we may write

$$W^c(\{\Delta N\}^*, \{\Delta\delta^P\}, \{\mu\}^*) \le W^c(\{\Delta N\}^*, \{\Delta\delta^P\}^*, \{\mu\}^*) \quad , \tag{19}$$

where $(\{\Delta N\}^*, \{\Delta\delta^P\}^*, \{\mu\}^*)$ is the solution to the problem and $\Delta\delta_i^P$ satisfies the constraints in equations 16.

Combining equations 18 and 19, the solution we seek is that for which

$$W^c = \overline{V}^c(\{\Delta N\}, \{\Delta\delta^P\}) + \{\mu\}^T(\{\Delta P\} - [B]^T\{\Delta N\}) \quad , \tag{20}$$

is stationary with respect to variations in $\{\Delta N\}$, $\{\mu\}$ and a maximum with respect to variations in $\{\Delta\delta^P\}$ satisfying equations 16.

We may put the expression for $\overline{V}^c(\{\Delta N\}, \{\Delta\delta^P\})$, (equation 14), in matrix form by defining the m x m diagonal matrix $[C]$ with terms on the diagonal given by ℓ_i/A_iE_i, the ordering being the same as in $\{\delta\}$. We note that $[C] = [S]^{-1}$. Remembering that $[S^P]$ is the m x m diagonal matrix whose diagonal elements are $A_iE_i^P/\ell_i$,

$$\overline{V}^c = \frac{1}{2}\{\Delta N\}^T[C]\{\Delta N\} + \{\Delta N\}^T\{\Delta\delta^P\} - \frac{1}{2}\{\Delta\delta^P\}^T[S^P]\{\Delta\delta^P\} . \qquad (21)$$

Equation 20 may be simplified if we introduce the combined column vector $\{\Delta N : \mu : \Delta\delta^P\}$ made up of the m elements each of $\{\Delta N\}$ and $\{\Delta\delta^P\}$ and the n elements of $\{\mu\}$, and the combined column vector $\{0 : \Delta P : 0\}$ made of 2m zero elements and the n elements of $\{\Delta P\}$. Equations 20 and 21 may therefore be written

$$\overline{W}^c = \frac{1}{2}\{\Delta N : \mu : \Delta\delta^P\}^T[A]\{\Delta N : \mu : \Delta\delta^P\} + \{\Delta N : \mu : \Delta\delta^P\}^T\{0 : \Delta P : 0\} , \qquad (22a)$$

where $[A]$ is the symmetric (2m + n) x (2m + n) symmetric matrix given by

$$[A] = \left[\begin{array}{c|c|c} [C] & -[B] & [I] \\ \hline -[B]^T & [0] & [0] \\ \hline [I] & [0] & -[S^P] \end{array}\right] , \qquad (22b)$$

in which $[I]$ is the identity matrix.

We may interpret the Lagrange multipliers $\{\mu\}$ by noting if the first order variation in $\overline{W}^c$ is zero we have

$$[A]\{\Delta N : \mu : \Delta\delta^P\} = -\{0 : \Delta P : 0\} , \qquad (23)$$

of which one set of equations is

$$[C]\{\Delta N\} - [B]\{\mu\} + \{\Delta\delta^P\} = 0 ,$$

or

$$\{\Delta\delta\} = [B]\{\mu\} . \qquad (24)$$

Comparing equation 24 with equation 12 it is evident that when $\overline{W}^c$ is stationary

$$\{\mu\} = \{\Delta u\} , \tag{25}$$

giving the nodal displacement increments.

3.3 An algorithm for the extended static incremental theorem

In applying this theorem we make use of an algorithm similar to that used by Martin and Reddy (1976) for the extended kinematic incremental theorem.

Let us consider a generic incremental problem in which $\{N\}$, $\{\delta^P\}$ and $\{\Delta P\}$ are known. We consider

$$\overline{W}^c = \frac{1}{2}\{\Delta N : \mu : \Delta\delta^P\}^T[A]\{\Delta N : \mu : \Delta\delta^P\} + \{\Delta N : \mu : \Delta\delta^P\}^T\{0 : \Delta P : 0\}, \tag{26}$$

and seek $\{\Delta N\}^*$, $\{\mu\}^*$, $\{\Delta\delta^P\}^*$ such that $\overline{W}^c$ is stationary with respect to variations in $\{\Delta N\}$, $\{\mu\}$ and a maximum with respect to variations in $\Delta\delta^P_i$ such that

$$\Delta\delta^P_i \geq 0 \quad \text{if} \quad N^P_i = +N^{op}_i \tag{27a}$$

$$\Delta\delta^P_i = 0 \quad \text{if} \quad -N^{op}_i < N^P_i < +N^{op}_i \tag{27b}$$

$$\Delta\delta^P_i \leq 0 \quad \text{if} \quad N^P_i = -N^{op}_i , \tag{27c}$$

in which

$$N^P_i = N_i - \frac{A_i E^P_i}{\ell_i}\delta^P_i . \tag{28}$$

As we have seen, the unconstrained stationary point of $\overline{W}^c$ satisfies

$$[A]\{\Delta N : \mu : \Delta\delta^P\} = -\{0 : \Delta P : 0\} . \tag{29}$$

A priori knowledge of which of the $\Delta\delta_i^p$'s were constrained at zero would enable us to delete rows and columns of equation 29 and solve the remaining equations to obtain the solution. The following algorithm enables one to make a guess as to which $\Delta\delta_i^p$'s are constrained at zero in terms of the solution for the previous guess.

Referring to equation 29:-

(i) We identify those bars satisfying $|N_i^p| < N_i^{op}$ to which we assign $\Delta\delta_i^p = 0$.

(ii) We assign $\Delta\delta_i^p \neq 0$ in all remaining bars, i.e. for which $|N_i^p| = N_i^{op}$.

(iii) We eliminate rows and columns of equation 29 corresponding to the elements of $\{\Delta\delta^p\}$ which have been set equal to zero.

(iv) Solve the remaining equations for $\{\Delta N\}$, $\{\mu\}$ and the supposed non-zero elements of $\{\Delta\delta^p\}$. This is a trial solution.

(v) To check the trial solution we consider each bar for which $|N_i^p| = N_i^{op}$. From equation 9, the trial solution is correct if

$$\left.\begin{array}{l} \Delta N_i < 0 \text{ if } N_i^p = +N_i^{op} \\ \Delta N_i > 0 \text{ if } N_i^p = -N_i^{op} \end{array}\right\} \text{ if } \Delta\delta_i^p \text{ was assumed to be zero}$$

$$\left.\begin{array}{l} \Delta\delta_i^p \geq 0 \text{ if } N_i^p = +N_i^{op} \\ \Delta\delta_i^p \leq 0 \text{ if } N_i^p = -N_i^{op} \end{array}\right\} \text{ if } \Delta\delta_i^p \text{ was assumed to be non-zero}$$

(vi) If these checks are satisfied the solution has been found. If not, for bars satisfying $|N_i^p| = N_i^{op}$, we modify the choice of which bars have $\Delta\delta_i^p \neq 0$ and return to step (iii). If $\Delta\delta_i^p$ was assumed to be zero and ΔN_i has an incorrect sign it is now assumed to be non-zero. If $\Delta\delta_i^p$ was assumed to be non-zero and has an incorrect sign, it is now assumed to be zero.

3.4 Numerical examples

3.4.1 Three bar truss

To illustrate the use of the algorithm we consider the three bar pin-jointed plane truss in figure 1, subject to load components (P, Q).

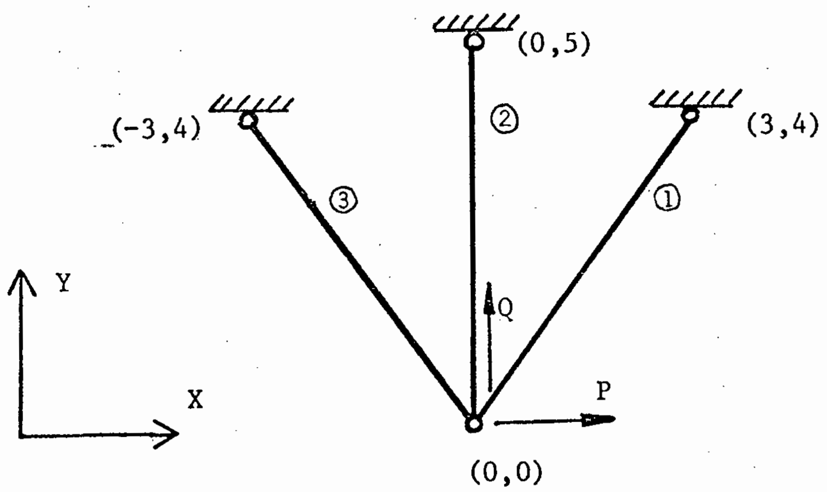

Figure 1. Three bar plane truss

Cartesian coordinates are given for each node. All bars have

$$\ell_i = 5 \text{ metres}$$

$$\frac{\ell_i}{A_i E_i} = 0{,}25 \times 10^{-4} \quad \text{m/kN}$$

$$\frac{A_i E_i^p}{\ell_i} = 1 \times 10^4 \quad \text{kN/m}$$

$$N_i^{op} = 200 \text{ kN}.$$

The numbers in circles next to the bars give their ordering. The node (0,0) is the only unconstrained node and its displacements are (u,v). The relations between bar extensions and nodal displacements are

$$\delta_1 = -(3/5)u - (4/5)v$$

$$\delta_2 = -u \qquad (30)$$

$$\delta_3 = (3/5)u - (4/5)v .$$

Hence

$$[B] = \begin{bmatrix} -0,6 & -0,8 \\ -1,0 & 0 \\ 0,6 & -0,8 \end{bmatrix} . \qquad (31)$$

The matrices $[C]$ and $[S^P]$ are

$$[C] = 0,25 \times 10^{-4}[I]$$

$$[S^P] = 10^4[I] \qquad (32)$$

where $[I]$ is the identity matrix.

From the expression for $[A]$ (equation 22b),

$$[A] = \left[\begin{array}{ccc|cc|ccc} 0,25 \times 10^{-4} & 0 & 0 & 0,6 & 0,8 & 1 & 0 & 0 \\ 0 & 0,25 \times 10^{-4} & 0 & 1,0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0,25 \times 10^{-4} & -0,6 & 0,8 & 0 & 0 & 0 \\ \hline 0,6 & 1,0 & -0,6 & 0 & 0 & 0 & 0 & 0 \\ 0,8 & 0 & 0,8 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & -10^4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -10^4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -10^4 \end{array}\right] .$$

(33)

Consider the proportional loading program shown in figure 2. Units (m. or kN.) are omitted.

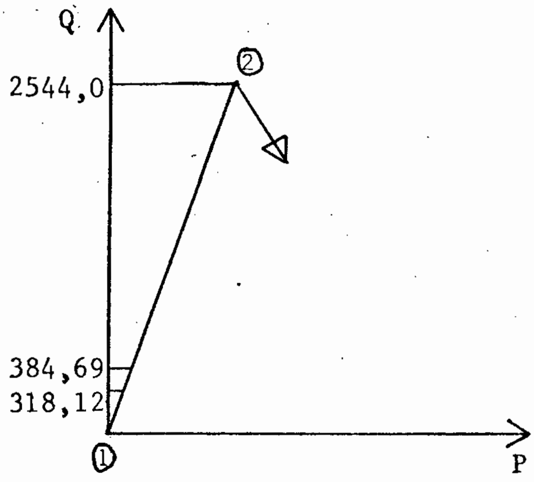

Figure 2. Loading program

From point (1) to point (2), $P = \frac{1}{3} Q$. At point (2) we consider the beginning of a load path for which

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} +2 \\ -3 \end{bmatrix} . \tag{34}$$

We begin the process at $(P,Q) = (0,0)$ and consider an increment

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} . \tag{35}$$

Initially $N_i^P = 0$ for all bars so we set $\{\Delta\delta^P\} = 0$, and delete the last three rows and columns of $[A]$ in the set of equations

$$[A]\{\Delta N : \mu : \Delta\delta^P\} = \{0\ 0\ 0\ -1\ -3\ 0\ 0\ 0\}^T . \tag{36}$$

We note that from equation 25

$$\{\mu\} = \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} .$$

The solution of this set of equations is

$$\begin{bmatrix} \Delta N_1 \\ \Delta N_2 \\ \Delta N_3 \\ \Delta u \\ \Delta v \end{bmatrix} = \begin{bmatrix} -1,8860 \\ -1,3158 \\ -0,21929 \\ 0,34722 \times 10^{-4} \\ 0.32895 \times 10^{-4} \end{bmatrix} . \quad (37)$$

This solution is valid for any increment magnitude provided all the bars remain elastic. The longest possible such increment is obtained by multiplying equations 35 to 37 by (200/1,886) = 106,04. This load increment causes bar (1) to yield, and at this point

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 106,04 \\ 318,12 \end{bmatrix} \quad \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} -200,00 \\ -139,53 \\ -\ 23,256 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = 10^{-3} \begin{bmatrix} 3,6822 \\ 3,4884 \end{bmatrix} \quad \begin{bmatrix} \delta_1^P \\ \delta_2^P \\ \delta_3^P \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} . \quad (38)$$

Now we continue to load proportionally. We guess that for the next increment $\Delta\delta_1^P < 0$, $\Delta N_1 < 0$ with all other bars continuing to behave elastically. Thus we solve equation 36 with the last two rows and columns of $[A]$ deleted.

The solution is

$$\begin{bmatrix} \Delta N_1 \\ \Delta N_2 \\ \Delta N_3 \\ \Delta u \\ \Delta v \\ \Delta\delta_1^P \end{bmatrix} = \begin{bmatrix} -1,0047 \\ -2,7259 \\ 0,66199 \\ 1,1844 \times 10^{-4} \\ 0,68746 \times 10^{-4} \\ -1,0047 \times 10^{-4} \end{bmatrix} . \quad (39)$$

The sign of $\Delta\delta_1^P$ agrees with our guess so no iteration is necessary. The solution is valid for any increment magnitude provided no further bars become plastic. Considering equations 38 and 39, the next bar to do so will be bar (2). This occurs for an increment given by multiplying equations

35, 36 and 39 by $(200 - 139{,}53)/2{,}7259 = 22{,}183$.

After such an increment, adding the results to equation 38 we have

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 128{,}23 \\ 384{,}69 \end{bmatrix} \qquad \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} -222{,}29 \\ -200{,}00 \\ -8{,}5714 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = 10^{-3} \begin{bmatrix} 6{,}3095 \\ 5{,}0000 \end{bmatrix} \qquad \begin{bmatrix} \delta_1^P \\ \delta_2^P \\ \delta_3^P \end{bmatrix} = 10^{-3} \begin{bmatrix} 2{,}2286 \\ 0 \\ 0 \end{bmatrix}. \tag{40}$$

Continuing the load path we guess that bars (1) and (2) deform plastically, while (3) remains elastic. Thus we delete the last row and column of $[A]$ and solve equation 36.

The solution is

$$\begin{bmatrix} \Delta N_1 \\ \Delta N_2 \\ \Delta N_3 \\ \Delta u \\ \Delta v \\ \Delta \delta_1^P \\ \Delta \delta_2^P \end{bmatrix} = \begin{bmatrix} -1{,}9326 \\ -1{,}2411 \\ -2{,}6596 \\ 1{,}9577 \times 10^{-4} \\ 1{,}5514 \times 10^{-4} \\ -1{,}9326 \times 10^{-4} \\ -1{,}2411 \times 10^{-4} \end{bmatrix}, \tag{41}$$

and we see that the signs of ΔN_1, ΔN_2, $\Delta\delta_1^P$, $\Delta\delta_2^P$ agree with our guess so no iteration is necessary. The largest possible increment with bar (3) remaining elastic is $(200 - 8{,}5714)/1{,}2411 = 719{,}77$.

After such an increment, adding the results to equation 40 we have

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 848{,}00 \\ 2544{,}0 \end{bmatrix} \qquad \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} -1613{,}3 \\ -1093{,}3 \\ -200{,}00 \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = 10^{-3} \begin{bmatrix} 147{,}22 \\ 116{,}67 \end{bmatrix} \qquad \begin{bmatrix} \delta_1^P \\ \delta_2^P \\ \delta_3^P \end{bmatrix} = 10^{-3} \begin{bmatrix} -141{,}33 \\ -89{,}333 \\ 0 \end{bmatrix}. \tag{42}$$

We now change the direction of the load increment as in figure 2 and put

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix} = \begin{bmatrix} +2 \\ -3 \end{bmatrix} .$$

It is now not clear which of the bars have $\Delta\delta_i^p \neq 0$. As an initial guess, we assume that $\Delta\delta_i^p \neq 0$ for all bars, i.e. $\Delta N_i < 0$, $\Delta\delta_i^p < 0$. The equations we have to solve are

$$[A]\{\Delta N \; : \; \mu \; \vdots \; \Delta\delta^p\} = \{0 \; 0 \; 0 \; -2 \; +3 \; 0 \; 0 \; 0\}^T \quad . \tag{43}$$

The solution is

$$\begin{bmatrix} \Delta N_1 \\ \Delta N_2 \\ \Delta N_3 \\ \Delta u \\ \Delta v \\ \Delta\delta_1^p \\ \Delta\delta_2^p \\ \Delta\delta_3^p \end{bmatrix} = \begin{bmatrix} -0,30702 \\ 0,65789 \\ 1,3597 \\ 1,7361 \times 10^{-4} \\ -0,8224 \times 10^{-4} \\ -0,30702 \times 10^{-4} \\ 0,65789 \times 10^{-4} \\ 1,3597 \times 10^{-4} \end{bmatrix} . \tag{44}$$

We see that the signs of ΔN_2 and ΔN_3 disagree with our assumption and so we put $\Delta\delta_2^p = \Delta\delta_3^p = 0$. Solving equation 43 with the last two rows and columns of $[A]$ deleted gives

$$\begin{bmatrix} \Delta N_1 \\ \Delta N_2 \\ \Delta N_3 \\ \Delta u \\ \Delta v \\ \Delta\delta_1^p \end{bmatrix} = \begin{bmatrix} -0,16351 \\ 0,42835 \\ 1,5031 \\ 0,48352 \times 10^{-4} \\ 0,10709 \times 10^{-4} \\ -0,16351 \times 10^{-4} \end{bmatrix} \tag{45}$$

The signs of ΔN_1 and $\Delta\delta_1^p$ agree with our second assumption and so equation 45 is the solution.

3.4.2 Six bar truss

A computer program has been written which uses the extended static extremum principle to analyse an arbitrary space truss subject to arbitrary load histories consisting of straight line segments in load space.

The descriptive flow chart for the program is shown in figure 4. The program was used to analyse the six bar pin-jointed truss shown in figure 3.

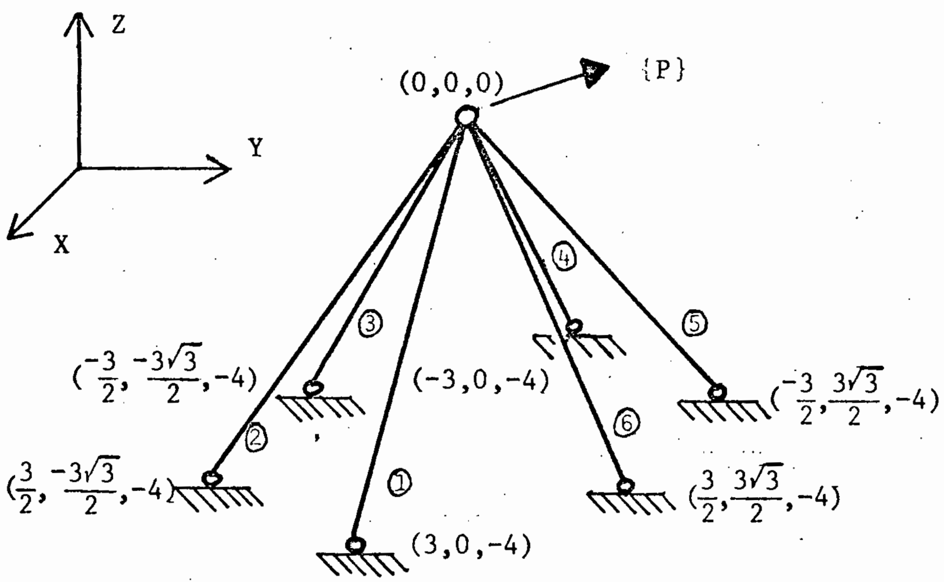

Figure 3. Six bar space truss

Cartesian coordinates are given for each node. All bars have

$$\ell_i = 5 \text{ metres}$$

$$\frac{\ell_i}{A_i E_i} = 0{,}25 \times 10^{-4} \quad \text{m/kN}$$

$$\frac{A_i E_i^p}{\ell_i} = 1 \times 10^4 \quad \text{kN/m}$$

$$N_i^{op} = 500 \quad \text{kN.}$$

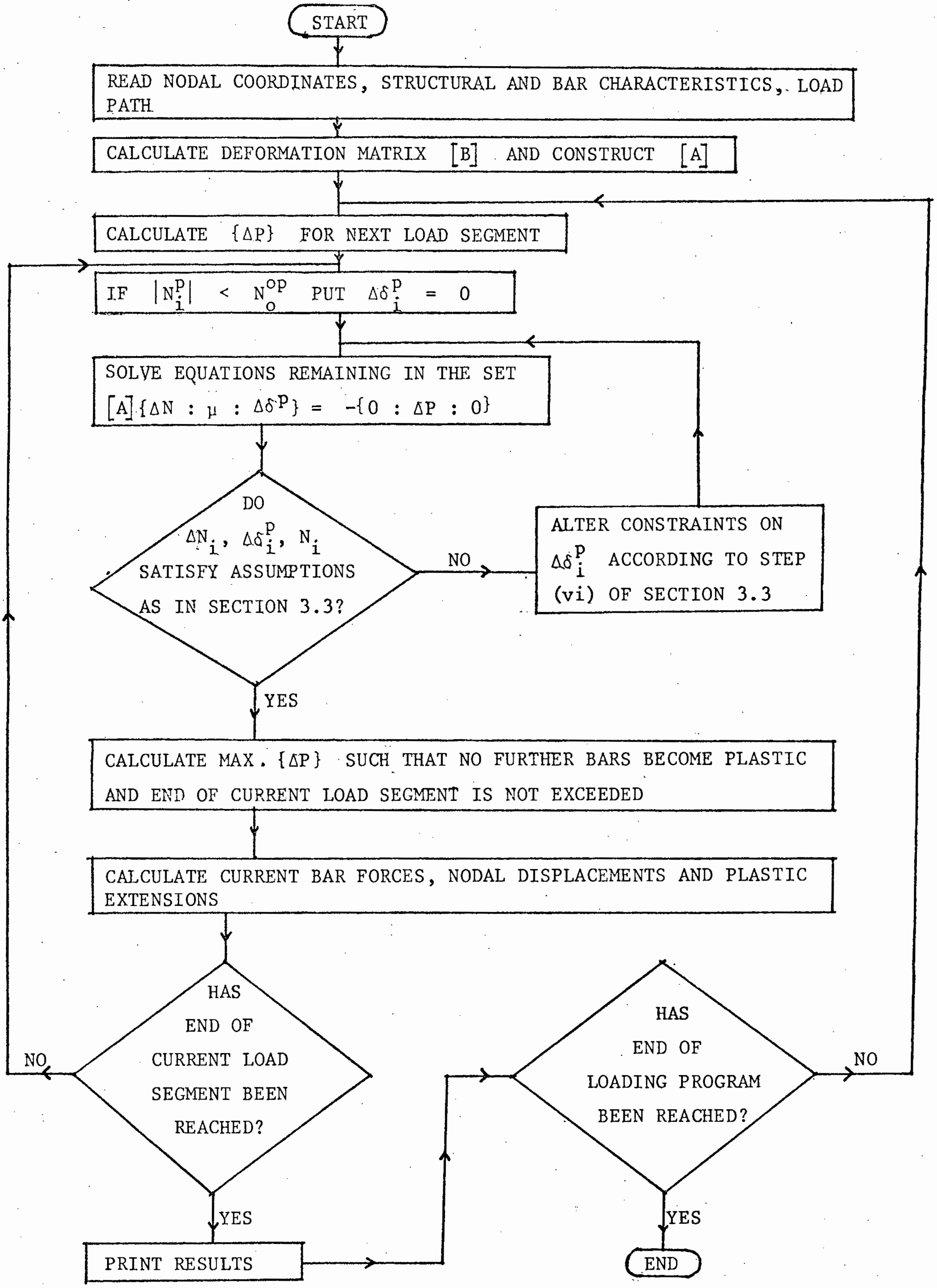

Figure 4. Flow chart for the application of extended static incremental theorem

The numbers in circles next to the bars give their ordering. The node (0, 0, 0) is the only unconstrained node and has displacements (u, v, w). It is subjected to a load vector {P}, the components of which are in terms of the same Cartesian coordinate system used to define the nodes.

We will consider the loading program shown in figure 5, which begins at {P} = (0, 0, 0) kN, and is restricted to the Px - Pz plane.

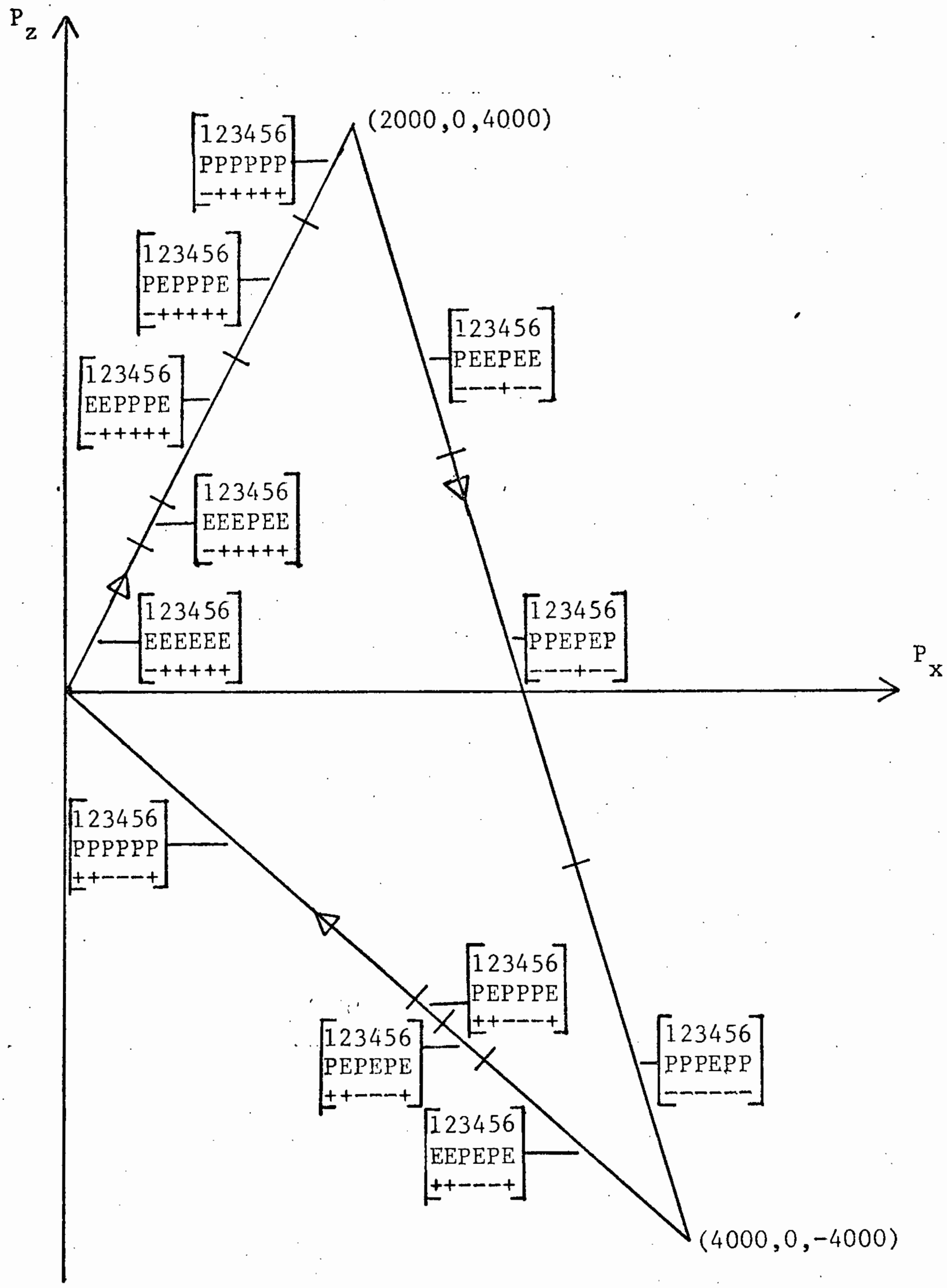

Figure 5. Loading program

Figure 5 also shows the behaviour of the truss at each stage of the loading program. Each segment of the load history is divided into regions within which each element of the truss remains either elastic or plastic. For each such region, figure 5 shows the bar number, whether it is deforming elastically or plastically, and the sign of ΔN_i. Note in the loading segment from (2000, 0, 4000) to (4000, 0, -4000), that unloading occurs in the fourth member. Units (m. or kN.) are omitted.

Initially the state of the truss is $\{N\} = 0$, $\{\delta^P\} = 0$. At

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 2000 \\ 0 \\ 4000 \end{bmatrix}, \qquad (46a)$$

the state of the truss is

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{bmatrix} = \begin{bmatrix} -677{,}78 \\ 544{,}44 \\ 1388{,}9 \\ 1811{,}1 \\ 1388{,}9 \\ 544{,}44 \end{bmatrix}, \qquad \begin{bmatrix} \delta_1^P \\ \delta_2^P \\ \delta_3^P \\ \delta_4^P \\ \delta_5^P \\ \delta_6^P \end{bmatrix} = 10^{-1} \begin{bmatrix} -0{,}17778 \\ 0{,}044444 \\ 0{,}88889 \\ 1{,}3111 \\ 0{,}88889 \\ 0{,}044444 \end{bmatrix}. \qquad (46b)$$

The nodal displacements are

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = 10^{-1} \begin{bmatrix} 1{,}7593 \\ 0 \\ 0{,}88542 \end{bmatrix}. \qquad (46c)$$

At

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 4000 \\ 0 \\ -4000 \end{bmatrix}, \qquad (47a)$$

the state of the truss is

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{bmatrix} = \begin{bmatrix} -2964,2 \\ -1944,4 \\ 95,062 \\ 1663,0 \\ 95,062 \\ -1944,4 \end{bmatrix} , \begin{bmatrix} \delta_1^p \\ \delta_2^p \\ \delta_3^p \\ \delta_4^p \\ \delta_5^p \\ \delta_6^p \end{bmatrix} = 10^{-1} \begin{bmatrix} -2,4642 \\ -1,4444 \\ 0,59506 \\ 1,4777 \\ 0,59506 \\ -1,4444 \end{bmatrix} . \tag{47b}$$

The nodal displacements are

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = 10^{-1} \begin{bmatrix} 4,2490 \\ 0 \\ -0,81983 \end{bmatrix} . \tag{47c}$$

Finally at

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} , \tag{48a}$$

the state of the truss is

$$\begin{bmatrix} N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \end{bmatrix} = \begin{bmatrix} -133,33 \\ 133,33 \\ -133,33 \\ 133,33 \\ -133,33 \\ 133,33 \end{bmatrix} , \begin{bmatrix} \delta_1^p \\ \delta_2^p \\ \delta_3^p \\ \delta_4^p \\ \delta_5^p \\ \delta_6^p \end{bmatrix} = 10^{-1} \begin{bmatrix} -0,63333 \\ -0,36667 \\ 0,36667 \\ 0,63333 \\ 0,36667 \\ -0,36667 \end{bmatrix} . \tag{48b}$$

The nodal displacements are

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = 10^{-1} \begin{bmatrix} 1,1111 \\ 0 \\ 0 \end{bmatrix} . \tag{48c}$$

Conclusion

We have demonstrated how the extended static incremental theorem when applied to a truss becomes a programming problem. An algorithm similar to that of Martin and Reddy (1976) has been shown to be applicable for its solution. Convergence was found to be rapid but remains unproven. The size and conditioning of the matrix $[A]$ makes the method less efficient than the displacement method used by Martin and Reddy (1976).

Chapter 4. Extremal paths and functions derived from thermodynamic potential functions

4.1 Introduction

The development of bounding theorems in plasticity (see, for example, Martin (1966a, 1966b), Hodge (1966), Maier (1969a, 1969b)) introduced the problem of determining bounds on work and complementary work for deformation along strain and stress paths where only the initial and terminal values are known. The bounding problem can be precisely defined as follows, for both time-independent and time-dependent plasticity.

Consider a homogeneously strained element of material of unit volume. Small, isothermal deformations are considered, and the strain and conjugate stress tensors are denoted by ε_{ij} and σ_{ij} respectively. The element is subjected to some (unspecified) strain history $\varepsilon_{ij}(t)$, $0 \leq t \leq T$, subject to an unstrained (and unstressed) initial state $\varepsilon_{ij}(0) = 0$ and a given terminal strain $\varepsilon_{ij}(T) = \bar{\varepsilon}_{ij}$. The associated stress history is $\sigma_{ij}(t)$, with $\sigma_{ij}(0) = 0$. The work done in deforming the material element in the period $0 \leq t \leq T$ is

$$W(\bar{\varepsilon}_{ij}) = \int_0^T \sigma_{ij}(t)\, \dot{\varepsilon}_{ij}(t)dt \; . \tag{1}$$

We seek a *work bounding function* $\tilde{W}(\bar{\varepsilon}_{ij})$ such that

$$\tilde{W}(\bar{\varepsilon}_{ij}) \leq W(\bar{\varepsilon}_{ij}) \tag{2}$$

for any choice of $\bar{\varepsilon}_{ij} = \varepsilon_{ij}(T)$ and any choice of $\varepsilon_{ij}(t)$.

Similarly, we may impose a stress path $\sigma_{ij}(t)$, with $\sigma_{ij}(0) = 0$, $\sigma_{ij}(T) = \bar{\sigma}_{ij}$, with an associated strain path $\varepsilon_{ij}(t)$. The complementary work done in the period $0 \leq t \leq T$ is

$$\Omega(\bar{\sigma}_{ij}) = \int_0^T \varepsilon_{ij}(t)\, \dot{\sigma}_{ij}(t)dt \; . \tag{3}$$

We seek a *complementary work bounding function* $\tilde{\Omega}(\bar{\sigma}_{ij})$ such that

$$\tilde{\Omega}(\bar{\sigma}_{ij}) \geq \Omega(\bar{\sigma}_{ij}) \tag{4}$$

for any choice of $\bar{\sigma}_{ij} = \sigma_{ij}(T)$ and any choice of $\sigma_{ij}(t)$.

Work bounding functions for several models of materials obeying specific constitutive equations have been derived (for example, Martin (1966a, 1966b), Hodge (1966), Martin and Ponter (1966), Maier (1969a), Soechting and Lance (1969). The problem has also been discussed in a general context by Ponter (1968, 1969) and Ponter and Martin (1972). In this latter approach the concepts of a minimum work path and a maximum complementary work path for given terminal strain and stress respectively were introduced. We then define the *minimum work function* $\bar{W}(\bar{\varepsilon}_{ij})$ as the work done along the minimum work path, so that

$$\bar{W}(\bar{\varepsilon}_{ij}) = \min.\left\{W = \int_0^T \sigma_{ij}\dot{\varepsilon}_{ij}dt : \varepsilon_{ij}(0) = 0, \ \varepsilon_{ij}(T) = \bar{\varepsilon}_{ij}\right\} \tag{5}$$

Similarly the maximum complementary work function $\bar{\Omega}(\bar{\sigma}_{ij})$ is the work done along the maximum complementary work path, so that

$$\bar{\Omega}(\bar{\sigma}_{ij}) = \max.\left\{\Omega = \int_0^T \varepsilon_{ij}\dot{\sigma}_{ij}dt : \sigma_{ij}(0) = 0, \ \sigma_{ij}(T) = \bar{\sigma}_{ij}\right\} \tag{6}$$

Evidently

$$\bar{W}(\bar{\varepsilon}_{ij}) \leq W(\bar{\varepsilon}_{ij}), \quad \bar{\Omega}(\bar{\sigma}_{ij}) \geq \Omega(\bar{\sigma}_{ij}) , \tag{7}$$

and the minimum work and maximum complementary work functions are work and complementary work bounding functions respectively; indeed, they are the optimal choices for the bounding functions.

On the assumption that the material is stable in the sense of Drucker (1951) (see Ponter (1968, 1969) and Ponter and Martin (1972)), several interesting properties of $\bar{W}$ and $\bar{\Omega}$ were found. The functions are both convex, and are potential functions in the sense that the derivative of $\bar{W}$ with respect to strain gives the terminal stress for the minimum work path, and the derivative of $\bar{\Omega}$ with respect to stress gives the terminal strain for the maximum complementary work path. Further, the minimum work path maps a path in stress space which is the maximum complementary work path for that terminal stress. These results indicate that $\overline{W}$ and $\overline{\Omega}$ can be considered as the strain energy and complementary energy of a hypothetical, stable elastic material which bears a special relation to the plastic material.

In this chapter we present an alternative and more general approach to the determination of bounding functions, based on the internal variable model of plasticity. In section 4.2 a lower bound $\tilde{W}$ on the work W to a particular strain $\overline{\varepsilon}_{ij}$ is obtained and its potential nature demonstrated. In section 4.3 an upper bound $\tilde{\Omega}$ on the complementary work Ω to a particular stress $\overline{\sigma}_{ij}$ is obtained and its potential nature demonstrated. The duality of $\tilde{W}$ and $\tilde{\Omega}$ is also discussed. In section 4.4 the convexity of $\tilde{W}$ is proved and in section 4.5 the conditions that $\tilde{W}$ and $\tilde{\Omega}$ are physically attainable are given. Finally in section 4.6 a generalised Maxwell model for non-linear creep in metals is considered as an example.

In all cases the essential property we will require is the convexity of f and D.

4.2 The work bounding function

We assume that at time $t = 0$ the material element is undeformed such that $\varepsilon_{ij}(0) = 0$, $\chi_\alpha(0) = 0$. Consider isothermal quasi-static deformation denoted by the strain path $\varepsilon_{ij}(t)$ over the time period $0 \leq t \leq T$. The work done by external agencies as the element is deformed along this path is

$$W = \int_0^T \sigma_{ij}(t)\dot{\varepsilon}_{ij}(t)dt \quad . \tag{8}$$

We introduce a terminal strain constraint $\varepsilon_{ij}(T) = \overline{\varepsilon}_{ij}$, and seek a lower bound on W. From equations 32 and 55 of chapter 1 we see that

$$W = f(\overline{\varepsilon}_{ij}, \chi_\alpha(T)) + \int_0^T X_\alpha(t)\dot{\chi}_\alpha(t)dt \tag{9a}$$

$$= f(\overline{\varepsilon}_{ij}, \chi_\alpha(T)) + \int_0^T D(\dot{\chi}_\alpha)dt \ . \tag{9b}$$

It is clear that the functional W depends on $\overline{\varepsilon}_{ij}$ and the history $\chi_\alpha(t)$, so we put

$$W = W(\overline{\varepsilon}_{ij}, \chi_\alpha(t)) . \tag{10}$$

We choose to bound W in two steps. First we adopt an arbitrary terminal value $\chi_\alpha(T) = \hat{\chi}_\alpha$ and seek the internal variable history $\chi_\alpha(t)$ which gives the least value of W subject to $\chi_\alpha(T) = \hat{\chi}_\alpha$. We define

$$\hat{W}(\overline{\varepsilon}_{ij}, \hat{\chi}_\alpha) = \min.\left\{ W(\overline{\varepsilon}_{ij}, \chi_\alpha(t)) : \chi_\alpha(T) = \hat{\chi}_\alpha \right\} . \tag{11}$$

Secondly we seek $\overline{\chi}_\alpha$ such that $\hat{W}(\overline{\varepsilon}_{ij}, \hat{\chi}_\alpha)$ is a minimum and define

$$\tilde{W}(\overline{\varepsilon}_{ij}) = \min.\left\{ \hat{W}(\overline{\varepsilon}_{ij}, \hat{\chi}_\alpha) \right\} . \tag{12}$$

We are then assured that

$$\tilde{W}(\overline{\varepsilon}_{ij}) \leq W(\overline{\varepsilon}_{ij}, \chi_\alpha(t)) . \tag{13}$$

Consider two internal variable paths $\chi'_\alpha(t)$ and $\chi''_\alpha(t)$ satisfying

$$\begin{aligned} \chi'_\alpha(0) &= \chi''_\alpha(0) = 0 \\ \chi'_\alpha(T) &= \chi''_\alpha(T) = \hat{\chi}_\alpha . \end{aligned} \tag{14}$$

$f(\overline{\varepsilon}_{ij}, \hat{\chi}_\alpha)$ is not affected by such variations and so

$$\Delta W = W(\overline{\varepsilon}_{ij}, \chi'_\alpha(t)) - W(\overline{\varepsilon}_{ij}, \chi''_\alpha(t)) \tag{15}$$

$$= \int_0^T \left\{ D(\dot{\chi}'_\alpha(t)) - D(\dot{\chi}''_\alpha(t)) \right\} dt . \tag{16}$$

Now the convexity of $D(\dot{\chi}_\alpha)$ is expressed by

$$D(\dot{\chi}'_\alpha) - D(\dot{\chi}''_\alpha) \leq \left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\dot{\chi}'_\alpha} (\dot{\chi}'_\alpha - \dot{\chi}''_\alpha) . \tag{17}$$

If $\dot{\chi}'_\alpha = 0$ and $\partial D/\partial \dot{\chi}_\alpha$ is discontinuous (multivalued) at $\dot{\chi}_\alpha = 0$, equation 17 is valid when $\partial D/\partial \dot{\chi}_\alpha$ takes any value associated with $\dot{\chi}_\alpha = 0$.

Thus

$$\Delta W \leq \int_0^T \left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\dot{\chi}'_\alpha(t)} (\dot{\chi}'_\alpha(t) - \dot{\chi}''_\alpha(t))dt \,. \tag{18}$$

Integrating by parts and using equations 14,

$$\Delta W \leq -\int_0^T \frac{d}{dt}\left\{\left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\dot{\chi}'_\alpha(t)}\right\} (\chi'_\alpha(t) - \chi''_\alpha(t))dt \,. \tag{19}$$

Now let $\chi'_\alpha(t)$ be the path for which W is a minimum subject to equations 14. We require for all other paths $(\chi''_\alpha(t))$ that $\Delta W \leq 0$. From equation 19 a sufficient condition that this is so is

$$\frac{d}{dt}\left\{\left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\dot{\chi}'_\alpha(t)}\right\} = 0 \,, \tag{20}$$

which is satisfied for all materials if

$$\dot{\chi}'_\alpha(t) = \frac{\hat{\chi}_\alpha}{T} \,, \tag{21}$$

a constant.

If $\partial D/\partial\dot{\chi}_\alpha$ is discontinuous at $\dot{\chi}_\alpha = 0$ and we are considering $\hat{\chi}_\alpha = 0$, the positive-definiteness of $D(\dot{\chi}_\alpha)$ ensures that the least path is one for which $\dot{\chi}_\alpha = 0$.

Thus

$$\begin{aligned} \hat{W}(\overline{\varepsilon}_{ij}, \hat{\chi}_\alpha) &= f(\overline{\varepsilon}_{ij}, \hat{\chi}_\alpha) + \int_0^T D\left(\frac{\hat{\chi}_\alpha}{T}\right)dt \\ &= f(\overline{\varepsilon}_{ij}, \hat{\chi}_\alpha) + TD\left(\frac{\hat{\chi}_\alpha}{T}\right) \,. \end{aligned} \tag{22}$$

We now seek $\overline{\chi}_\alpha$ such that

$$\begin{aligned} \tilde{W}(\overline{\varepsilon}_{ij}) &= \hat{W}(\overline{\varepsilon}_{ij}, \overline{\chi}_\alpha) \\ &\leq \hat{W}(\overline{\varepsilon}_{ij}, \hat{\chi}_\alpha) \,. \end{aligned} \tag{23}$$

Consider two end points $\hat{\chi}'_\alpha$ and $\hat{\chi}''_\alpha$.

$$\Delta\hat{W} = \hat{W}(\bar{\varepsilon}_{ij}, \hat{\chi}'_\alpha) - \hat{W}(\bar{\varepsilon}_{ij}, \hat{\chi}''_\alpha)$$

$$= f(\bar{\varepsilon}_{ij}, \hat{\chi}'_\alpha) - f(\bar{\varepsilon}_{ij}, \hat{\chi}''_\alpha) + T\left\{D\left(\frac{\hat{\chi}'_\alpha}{T}\right) - D\left(\frac{\hat{\chi}''_\alpha}{T}\right)\right\} . \tag{24}$$

First we assume

$$\left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\hat{\chi}'_\alpha}{T}} ,$$

is well defined. From the convexity of $f(\varepsilon_{ij}, \chi_\alpha)$ and $D(\dot{\chi}_\alpha)$,

$$\Delta\hat{W} \leq \left.\frac{\partial f}{\partial \chi_\alpha}\right|_{\bar{\varepsilon}_{ij},\, \hat{\chi}'_\alpha} (\hat{\chi}'_\alpha - \hat{\chi}''_\alpha) + T \left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\hat{\chi}'_\alpha}{T}} \left(\frac{\hat{\chi}'_\alpha}{T} - \frac{\hat{\chi}''_\alpha}{T}\right) \tag{25}$$

$$= \left(\left.\frac{\partial f}{\partial \chi_\alpha}\right|_{\bar{\varepsilon}_{ij},\, \hat{\chi}'_\alpha} + \left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\hat{\chi}'_\alpha}{T}}\right) (\hat{\chi}'_\alpha - \hat{\chi}''_\alpha) . \tag{26}$$

Now if $\hat{\chi}'_\alpha = \bar{\chi}_\alpha$, the end point for which $\hat{W}$ is a minimum, we require for all other points $(\hat{\chi}''_\alpha)$ that $\Delta\hat{W} \leq 0$. A sufficient condition that this is so is

$$\left.\frac{\partial f}{\partial \chi_\alpha}\right|_{\bar{\varepsilon}_{ij},\, \bar{\chi}_\alpha} + \left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\bar{\chi}_\alpha}{T}} = 0 . \tag{27}$$

Second, if $\hat{\chi}'_\alpha = 0$ and $\partial D/\partial\dot{\chi}_\alpha$ is discontinuous at $\dot{\chi}_\alpha = 0$ we consider equation 24 with $\hat{\chi}'_\alpha = 0$. We have

$$\Delta\hat{W} = f(\bar{\varepsilon}_{ij}, 0) - f(\bar{\varepsilon}_{ij}, \hat{\chi}''_\alpha) - TD\left(\frac{\hat{\chi}''_\alpha}{T}\right) . \tag{28}$$

From the convexity of $f(\varepsilon_{ij}, \chi_\alpha)$,

$$\Delta\hat{W} \leq -\left.\frac{\partial f}{\partial \chi_\alpha}\right|_{\bar{\varepsilon}_{ij},\, 0} \hat{\chi}''_\alpha - TD\left(\frac{\hat{\chi}''_\alpha}{T}\right)$$

$$= -T\left\{D\left(\frac{\hat{\chi}''_\alpha}{T}\right) - X_\alpha(\bar{\varepsilon}_{ij}, 0)\,\frac{\hat{\chi}''_\alpha}{T}\right\} , \tag{29}$$

where we use the definition of X_α. Returning to inequality 72 of chapter 1, we see that

$$D\left(\frac{\hat{\chi}''_\alpha}{T}\right) - X_\alpha(\bar{\varepsilon}_{ij}, 0)\,\frac{\hat{\chi}''_\alpha}{T} \geq 0 \quad , \tag{30}$$

for all $\hat{\chi}''_\alpha$ if $X_\alpha(\bar{\varepsilon}_{ij}, 0)$ lies inside or on the limit surface (time-independent plasticity) or the yield surface (viscoplasticity).

Thus if $(\partial D/\partial\dot{\chi}_\alpha)$ is discontinuous at $\dot{\chi}_\alpha = 0$ we choose $\bar{\chi}_\alpha = 0$ if $X_\alpha(\bar{\varepsilon}_{ij}, 0)$ lies inside or on the limit or yield surface. In all other cases equation 27 applies.

We may now define

$$\tilde{W}(\bar{\varepsilon}_{ij}) = f(\bar{\varepsilon}_{ij}, \bar{\chi}_\alpha) + TD\left(\frac{\bar{\chi}_\alpha}{T}\right) \quad , \tag{31}$$

and we are assured that provided $\bar{\chi}_\alpha$ is chosen according to equation 27 (or, where applicable, equation 30)

$$\tilde{W}(\bar{\varepsilon}_{ij}) \leq W = \int_0^T \sigma_{ij}(t)\dot{\varepsilon}_{ij}(t)dt \quad , \tag{32}$$

with $\bar{\varepsilon}_{ij} = \varepsilon_{ij}(T)$ fixed.

On considering variations in $\bar{\varepsilon}_{ij}$ and using equations 31 and 27, we note that

$$\delta\hat{W} = \left.\frac{\delta f}{\delta\varepsilon_{ij}}\right|_{\bar{\varepsilon}_{ij}, \bar{\chi}_\alpha} \delta\bar{\varepsilon}_{ij} + \left(\left.\frac{\delta f}{\delta\chi_\alpha}\right|_{\bar{\varepsilon}_{ij}, \bar{\chi}_\alpha} + \frac{\delta D(\bar{\chi}_\alpha/T)}{\delta(\bar{\chi}_\alpha/T)}\right) \frac{\delta\bar{\chi}_\alpha}{\delta\bar{\varepsilon}_{ij}}\,\delta\bar{\varepsilon}_{ij} \tag{33}$$

$$= \left.\frac{\delta f}{\delta\varepsilon_{ij}}\right|_{\bar{\varepsilon}_{ij}, \bar{\chi}_\alpha} \delta\bar{\varepsilon}_{ij}$$

$$= \bar{\sigma}_{ij}\,\delta\bar{\varepsilon}_{ij} \quad . \tag{34}$$

The stress $\bar{\sigma}_{ij}$ is the stress associated with the state $(\bar{\varepsilon}_{ij}, \bar{\chi}_\alpha)$. This result is valid when $\bar{\chi}_\alpha = 0$ and $\delta D/\delta\dot{\chi}$ is discontinuous at $\dot{\chi}_\alpha = 0$, since in this case

$$\frac{\partial \bar{\chi}_\alpha}{\partial \bar{\varepsilon}_{ij}} = 0 . \tag{35}$$

Equation 34 is sufficient to establish that $\tilde{W}(\bar{\varepsilon}_{ij})$ is a potential function relating $\bar{\sigma}_{ij}$ and $\bar{\varepsilon}_{ij}$ where

$$\bar{\sigma}_{ij} = \frac{\partial \tilde{W}}{\partial \bar{\varepsilon}_{ij}} . \tag{36}$$

4.3 The complementary work bounding function

Consider an element of material subject to a stress path $\sigma_{ij}(t)$ with $\sigma_{ij}(0) = \varepsilon_{ij}(0) = \chi_\alpha(0) = 0$ and a terminal stress constraint $\sigma_{ij}(T) = \bar{\sigma}_{ij}$. The complementary work done along this path is

$$\Omega = \int_0^T \varepsilon_{ij}\dot{\sigma}_{ij}dt . \tag{37}$$

We seek an upper bound $\tilde{\Omega}$ on Ω.

In order to compute Ω in the internal variable framework we recall equations 37 and 55 of chapter 1. We see that for the imposed terminal constraint $\sigma_{ij}(T) = \bar{\sigma}_{ij}$,

$$\Omega = h(\bar{\sigma}_{ij}, \chi_\alpha(T)) - \int_0^T X_\alpha(t)\dot{\chi}_\alpha(t)dt \tag{38a}$$

$$= h(\bar{\sigma}_{ij}, \chi_\alpha(T)) - \int_0^T D(\dot{\chi}_\alpha)dt . \tag{38b}$$

Thus the functional Ω depends on the terminal stress $\bar{\sigma}_{ij}$ and the internal variable history $\chi_\alpha(t)$. We put

$$\Omega = \Omega(\bar{\sigma}_{ij}, \chi_\alpha(t)) . \tag{39}$$

As in the previous section for W, we bound Ω in two steps. First, for an arbitrary terminal value $\chi_\alpha(T) = \hat{\chi}_\alpha$, we seek the history $\chi_\alpha(t)$ which gives the maximum value of Ω subject to $\chi_\alpha(T) = \hat{\chi}_\alpha$. We define

$$\hat{\Omega}(\bar{\sigma}_{ij}, \hat{\chi}_\alpha) = \max.\left\{\Omega(\bar{\sigma}_{ij}, \chi_\alpha(t)) : \chi_\alpha(T) = \hat{\chi}_\alpha\right\}. \tag{40}$$

Secondly we seek $\bar{\chi}_\alpha$ such that $\hat{\Omega}(\bar{\sigma}_{ij}, \hat{\chi}_\alpha)$ is a maximum and define

$$\tilde{\Omega}(\bar{\sigma}_{ij}) = \max.\left\{\hat{\Omega}(\bar{\sigma}_{ij}, \hat{\chi}_\alpha)\right\} . \tag{41}$$

We are then assured that

$$\tilde{\Omega}(\bar{\sigma}_{ij}) \geq \Omega(\bar{\sigma}_{ij}, \chi_\alpha(t)) . \tag{42}$$

The first part of the bounding problem is identical to the problem posed and solved in equations 14 to 21. Hence we define

$$\hat{\Omega} = h(\bar{\sigma}_{ij}, \hat{\chi}_\alpha) - TD\left(\frac{\hat{\chi}_\alpha}{T}\right) . \tag{43}$$

We now seek $\bar{\chi}_\alpha$ such that

$$\begin{aligned} \tilde{\Omega}(\bar{\sigma}_{ij}) &= \hat{\Omega}(\bar{\sigma}_{ij}, \bar{\chi}_\alpha) \\ &\geq \Omega(\bar{\sigma}_{ij}, \hat{\chi}_\alpha) . \end{aligned} \tag{44}$$

Consider two end points $\hat{\chi}'_\alpha$ and $\hat{\chi}''_\alpha$.

$$\Delta\hat{\Omega} = \hat{\Omega}(\bar{\sigma}_{ij}, \hat{\chi}'_\alpha) - \hat{\Omega}(\bar{\sigma}_{ij}, \hat{\chi}''_\alpha) \tag{45}$$

$$= h(\bar{\sigma}_{ij}, \hat{\chi}'_\alpha) - h(\bar{\sigma}_{ij}, \hat{\chi}''_\alpha) - T\left\{D\left(\frac{\hat{\chi}'_\alpha}{T}\right) - D\left(\frac{\hat{\chi}''_\alpha}{T}\right)\right\} . \tag{46}$$

First we assume

$$\left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\hat{\chi}'_\alpha}{T}}$$

is well defined. As discussed in section 1.3.3, the convexity of $f(\varepsilon_{ij}, \chi_\alpha)$ implies the concavity at constant σ_{ij} of $h(\sigma_{ij}, \chi_\alpha)$. Using this property and the convexity of D,

$$\Delta\hat{\Omega} \geq \frac{\partial h}{\partial \chi_\alpha}\bigg|_{\bar{\sigma}_{ij},\,\hat{\chi}'_\alpha} (\hat{\chi}'_\alpha - \hat{\chi}''_\alpha) - T\,\frac{\partial D}{\partial \dot{\chi}_\alpha}\bigg|_{\frac{\hat{\chi}'_\alpha}{T}} \left(\frac{\hat{\chi}'_\alpha}{T} - \frac{\hat{\chi}''_\alpha}{T}\right) \tag{47}$$

$$= \left\{\frac{\partial h}{\partial \chi_\alpha}\bigg|_{\bar{\sigma}_{ij},\,\hat{\chi}'_\alpha} - \frac{\partial D}{\partial \dot{\chi}_\alpha}\bigg|_{\frac{\hat{\chi}'_\alpha}{T}}\right\} (\hat{\chi}'_\alpha - \hat{\chi}''_\alpha) \quad . \tag{48}$$

Now, if $\hat{\chi}'_\alpha = \bar{\chi}_\alpha$, the end point for which $\hat{\Omega}$ is a maximum, we require for all other points $(\hat{\chi}''_\alpha)$ that $\Delta\tilde{\Omega} \geq 0$. A sufficient condition that this is so is

$$\frac{\partial h}{\partial \chi_\alpha}\bigg|_{\bar{\sigma}_{ij},\,\bar{\chi}_\alpha} - \frac{\partial D}{\partial \dot{\chi}_\alpha}\bigg|_{\frac{\bar{\chi}_\alpha}{T}} = 0 \ . \tag{49}$$

Second, if $\hat{\chi}'_\alpha = 0$ and $(\partial D/\partial \dot{\chi}_\alpha)$ is discontinuous at $\dot{\chi}_\alpha = 0$, we consider equation 47 with $\hat{\chi}'_\alpha = 0$. The argument is similar to that given in equations 35 to 37 and will not be repeated. We again choose $\bar{\chi}_\alpha = 0$ if

$$D\left(\frac{\hat{\chi}''_\alpha}{T}\right) - X_\alpha(\bar{\sigma}_{ij}, 0)\,\frac{\hat{\chi}''_\alpha}{T} \geq 0 \ , \tag{50}$$

for all $\hat{\chi}''_\alpha$, i.e. if $X_\alpha(\bar{\sigma}_{ij}, 0)$ lies inside as on the yield or limit surface.

Thus we may now define

$$\tilde{\Omega}(\bar{\sigma}_{ij}) = h(\bar{\sigma}_{ij}, \bar{\chi}_\alpha) - TD\left(\frac{\bar{\chi}_\alpha}{T}\right) \ , \tag{51}$$

and provided $\bar{\chi}_\alpha$ is chosen according to equation 49 (or where applicable, equation 50)

$$\tilde{\Omega}(\bar{\sigma}_{ij}) \geq \Omega = \int_0^T \varepsilon_{ij}(t)\dot{\sigma}_{ij}(t)\,dt \ , \tag{52}$$

with $\sigma_{ij}(T) = \bar{\sigma}_{ij}$ fixed.

Considering variations in $\bar{\sigma}_{ij}$ and using equations 51 and 49,

$$\delta\tilde{\Omega} = \left.\frac{\partial h}{\partial \sigma_{ij}}\right|_{\bar{\sigma}_{ij},\, \bar{\chi}_\alpha} \delta\bar{\sigma}_{ij} + \left\{\left.\frac{\partial h}{\partial \chi_\alpha}\right|_{\bar{\sigma}_{ij},\, \bar{\chi}_\alpha} - \frac{\partial D(\dot{\bar{\chi}}_\alpha/T)}{\partial(\dot{\bar{\chi}}_\alpha/T)}\right\} \frac{\partial \bar{\chi}_\alpha}{\partial \bar{\sigma}_{ij}} \delta\bar{\sigma}_{ij} \tag{53}$$

$$= \left.\frac{\partial h}{\partial \sigma_{ij}}\right|_{\bar{\sigma}_{ij},\, \bar{\chi}_\alpha} \delta\bar{\sigma}_{ij}$$

$$= \bar{\varepsilon}_{ij}\delta\bar{\sigma}_{ij} \ . \tag{54}$$

The strain $\bar{\varepsilon}_{ij}$ is the strain associated with the state $\bar{\sigma}_{ij}, \bar{\chi}_\alpha$. Equation 54 is valid when $\bar{\chi}_\alpha = 0$ and $\partial D/\partial\dot{\chi}_\alpha$ is discontinuous at $\dot{\chi}_\alpha = 0$ since in this case

$$\frac{\partial \bar{\chi}_\alpha}{\partial \bar{\sigma}_{ij}} = 0 \ . \tag{55}$$

Equation 54 is sufficient to establish that $\tilde{\Omega}(\bar{\sigma}_{ij})$ is a potential function relating $\bar{\sigma}_{ij}$ and $\bar{\varepsilon}_{ij}$ where

$$\bar{\varepsilon}_{ij} = \frac{\partial\tilde{\Omega}}{\partial\bar{\sigma}_{ij}} \ . \tag{56}$$

It is readily seen that $\tilde{\Omega}$ and $\tilde{W}$ are dual functions: for any choice of $\bar{\varepsilon}_{ij}$ we may calculate $\bar{\chi}_\alpha$ from equation 27 and $\bar{\sigma}_{ij}$ from equation 36; if we then use this value of $\bar{\sigma}_{ij}$ in equation 49 we determine precisely the same $\bar{\chi}_\alpha$, and equation 56 gives the original $\bar{\varepsilon}_{ij}$. Further, from equations 31 and 51 and equation 33 of chapter 1,

$$\tilde{W}(\bar{\varepsilon}_{ij}) + \tilde{\Omega}(\bar{\sigma}_{ij}) = \bar{\sigma}_{ij}\bar{\varepsilon}_{ij} \ . \tag{57}$$

4.4 Convexity of $\tilde{W}$ and $\tilde{\Omega}$

Convexity of f and D is sufficient to show that $\tilde{W}(\bar{\varepsilon}_{ij})$ and $\tilde{\Omega}(\bar{\sigma}_{ij})$ are convex functions of their respective arguments.

Consider two states $\bar{\varepsilon}'_{ij}$ and $\bar{\varepsilon}''_{ij}$ and the corresponding terminal internal variables $\bar{\chi}'_\alpha$ and $\bar{\chi}''_\alpha$. From the convexity of $f(\varepsilon_{ij}, \chi_\alpha)$

$$f(\overline{\varepsilon}'_{ij}, \overline{\chi}'_\alpha) - f(\overline{\varepsilon}''_{ij}, \overline{\chi}''_\alpha) \geq (\overline{\varepsilon}'_{ij} - \overline{\varepsilon}''_{ij}) \left.\frac{\partial f}{\partial \varepsilon_{ij}}\right|_{\overline{\varepsilon}''_{ij}, \overline{\chi}''_\alpha} + (\overline{\chi}'_\alpha - \overline{\chi}''_\alpha) \left.\frac{\partial f}{\partial \chi_\alpha}\right|_{\overline{\varepsilon}''_{ij}, \overline{\chi}''_\alpha} . \tag{58}$$

Similarly, if $D(\dot{\chi}_\alpha)$ is convex and

$$\left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\overline{\chi}''_\alpha}{T}}$$

is well defined,

$$D\left(\frac{\overline{\chi}'_\alpha}{T}\right) - D\left(\frac{\overline{\chi}''_\alpha}{T}\right) \geq \left(\frac{\overline{\chi}'_\alpha}{T} - \frac{\overline{\chi}''_\alpha}{T}\right) \left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\overline{\chi}''_\alpha}{T}} . \tag{59}$$

Multiplying inequality 59 by T and adding the result to inequality 58 we see that

$$\left\{f(\overline{\varepsilon}'_{ij}, \overline{\chi}'_\alpha) + TD\left(\frac{\overline{\chi}'_\alpha}{T}\right)\right\} - \left\{f(\overline{\varepsilon}''_{ij}, \overline{\chi}''_\alpha) + TD\left(\frac{\overline{\chi}''_\alpha}{T}\right)\right\} \geq$$

$$(\overline{\varepsilon}'_{ij} - \overline{\varepsilon}''_{ij}) \left.\frac{\partial f}{\partial \varepsilon_{ij}}\right|_{\overline{\varepsilon}''_{ij}, \overline{\chi}''_\alpha} + (\overline{\chi}'_\alpha - \overline{\chi}''_\alpha) \left[\left.\frac{\partial f}{\partial \chi_\alpha}\right|_{\overline{\varepsilon}''_{ij}, \overline{\chi}''_\alpha} + \left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\overline{\chi}''_\alpha}{T}}\right] . \tag{60}$$

However, the last term vanishes as a result of equation 27. Using equations 31 and 36, inequality 60 becomes

$$\tilde{W}(\overline{\varepsilon}'_{ij}) - \tilde{W}(\overline{\varepsilon}''_{ij}) \geq (\overline{\varepsilon}'_{ij} - \overline{\varepsilon}''_{ij}) \left.\frac{\partial \tilde{W}}{\partial \overline{\varepsilon}_{ij}}\right|_{\overline{\varepsilon}''_{ij}} , \tag{61}$$

which establishes that $\tilde{W}(\overline{\varepsilon}_{ij})$ is convex.

If $\overline{\chi}''_\alpha = 0$ and $\partial D/\partial \dot{\chi}_\alpha$ is discontinuous at $\dot{\chi}_\alpha = 0$, we add the positive-definite quantity $T\{D(\overline{\chi}'_\alpha/T)\}$ to both sides of inequality 58 to obtain

$$f(\bar{\varepsilon}'_{ij}, \bar{\chi}'_\alpha) + TD\left(\frac{\bar{\chi}'_\alpha}{T}\right) - f(\bar{\varepsilon}_{ij}, 0) \geq$$

$$(\bar{\varepsilon}'_{ij} - \bar{\varepsilon}''_{ij}) \left.\frac{\partial f}{\partial \varepsilon_{ij}}\right|_{\bar{\varepsilon}''_{ij}, 0} + T\left[\frac{\bar{\chi}'_\alpha}{T} \left.\frac{\partial f}{\partial \chi_\alpha}\right|_{\bar{\varepsilon}''_{ij}, 0} + D\left(\frac{\bar{\chi}'_\alpha}{T}\right)\right] . \tag{62}$$

In this case the term in square brackets is non-negative for arbitrary $\bar{\chi}'_\alpha$ due to the convexity of D as expressed in inequality 72 of chapter 1. We may thus delete it to obtain equation 61.

It may be noted in passing that, from equation 32

$$W(\bar{\varepsilon}'_{ij}) \geq \tilde{W}(\bar{\varepsilon}'_{ij}) , \tag{63}$$

where $W(\bar{\varepsilon}'_{ij})$ is the work done along an arbitrary strain path $\varepsilon_{ij}(t)$ with $\varepsilon_{ij}(0) = 0$, $\varepsilon_{ij}(T) = \bar{\varepsilon}_{ij}$. Combining inequalities 61 and 63

$$W(\bar{\varepsilon}'_{ij}) - \tilde{W}(\bar{\varepsilon}''_{ij}) \geq (\bar{\varepsilon}'_{ij} - \bar{\varepsilon}''_{ij}) \left.\frac{\partial \tilde{W}}{\partial \bar{\varepsilon}_{ij}}\right|_{\bar{\varepsilon}''_{ij}} , \tag{64a}$$

or, using equation 57

$$W(\bar{\varepsilon}'_{ij}) + \Omega(\bar{\sigma}''_{ij}) \geq \bar{\sigma}''_{ij}\bar{\varepsilon}'_{ij} . \tag{64b}$$

This result was established by Martin (1966) for time-independent materials and by Ponter (1969) for time-dependent materials.

4.5 Realisability of paths

The internal variable history associated with the work bounding function,

$$\dot{\chi}_\alpha(t) = \frac{\bar{\chi}_\alpha}{T} , \tag{65}$$

does not in general correspond to a history which can actually be achieved by the imposition of a strain path $\varepsilon_{ij}(t)$. It follows $\tilde{W}(\bar{\varepsilon}_{ij})$ is less than the least work required to deform the material element through a strain history $\varepsilon_{ij}(t)$ with $\varepsilon_{ij}(0) = 0$, $\varepsilon_{ij}(T) = \bar{\varepsilon}_{ij}$. Alternatively $\tilde{W}(\bar{\varepsilon}_{ij})$ is less than the work along the minimum work path. This does not affect the utilisation of $\tilde{W}(\bar{\varepsilon}_{ij})$ in the static and dynamic bounding theorems, although it may lead to loss in accuracy.

This result follows on noting that

$$-\dot{X}_\alpha = \frac{\partial^2 f}{\partial \varepsilon_{ij} \partial \chi_\alpha} \dot{\varepsilon}_{ij} + \frac{\partial^2 f}{\partial \chi_\alpha \partial \chi_\beta} \dot{\chi}_\beta \quad . \tag{66}$$

Now, a minimum path satisfies

$$\dot{\chi}_\alpha(t) = \frac{\bar{\chi}_\alpha}{T} \ ,$$

which implies, from the kinetic equation

$$X_\alpha = X_\alpha(\dot{\chi}_\beta) \ , \tag{67}$$

that

$$\dot{X}_\alpha = 0 \ . \tag{68}$$

Thus equation 66 becomes

$$\frac{\partial^2 f}{\partial \varepsilon_{ij} \partial \chi_\alpha} \dot{\varepsilon}_{ij}(t) = -\frac{\partial^2 f}{\partial \chi_\alpha \partial \chi_\beta} \frac{\bar{\chi}_\beta}{T} \quad . \tag{69}$$

In the general case, equation 69 cannot be solved to give $\dot{\varepsilon}_{ij}(t)$.

In the exceptional case where these equations can be inverted, however, a strain path can be found which provides the required internal variable history. In this case the work bounding function and the work along the minimum work path coincide. A majority of the specific models for which minimum work paths have been computed fall into this exceptional case; in these specific models the plastic strains and the internal variables coincide. For any model in which equation 69 can be solved for $\dot{\varepsilon}_{ij}(t)$, the present approach will provide the minimum work path and the work bounding function will provide the minimum realisable work.

For the class of materials discussed in sections 1.3.2 involving linear and non-linear creep and time-independent plasticity, the kinetic equations can be expressed in the form

$$X_\alpha = \frac{1}{p + 1} \frac{\partial D}{\partial \dot{\chi}_\alpha} \quad , \tag{70}$$

where p is the reciprocal of an odd positive integer, and D is homogenous and of degree $p + 1$ in $\dot{\chi}_\alpha$.

In the cases where a realisable path is implied by $\dot{\chi}_\alpha(t)$ constant, the internal force history $X_\alpha(t)$ can also be calculated. The internal forces are constant over the interval $0 < t < T$. Since $X_\alpha(t \leq 0) = 0$, and the terminal value $X_\alpha(T) = \overline{X}_\alpha$ is given independently by

$$\overline{X}_\alpha = \left.\frac{\partial h}{\partial \chi_\alpha}\right|_{\overline{\sigma}_{ij},\ \overline{\chi}_\alpha} \quad , \tag{71}$$

it follows that the function $X_\alpha(t)$, $0 \leq t \leq T$ exhibits discontinuities at $t = 0$ and $t = T$. From equation 70

$$\left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\overline{\chi}_\alpha}{T}} = (p + 1) X_\alpha(t) \ . \tag{72}$$

Substituting equations 71 and 72 into equation 49,

$$X_\alpha(t) = \frac{1}{p + 1} \overline{X}_\alpha, \quad 0 < t < T \ . \tag{73}$$

This result was given by Ponter (1969) for Maxwell models of creep.

4.6 Example

It is instructive to rederive the result given by Ponter (1969) for the generalised Maxwell model for non-linear creep of metals. Consider isothermal, incompressible deformation of an element of unit volume. Let e_{ij} be the strain deviator and s_{ij} the stress deviator, replacing ε_{ij} and σ_{ij} respectively. The constitutive equations under consideration are conventionally given in the form

$$e_{ij} = e^e_{ij} + e^p_{ij} \qquad (74a)$$

$$\dot{e}^e_{ij} = \dot{s}_{ij}/2G \qquad (74b)$$

$$\left(\frac{\dot{e}^e_{ij}}{\dot{e}_o}\right) = \phi^n \frac{\partial\phi}{\partial(s_{ij}/s_o)} , \qquad (74c)$$

where e^e_{ij}, e^p_{ij} are respectively the elastic and inelastic components of strain, G is the shear modulus, $\dot{e}_o$, s_o are constants with the dimensions of strain rate and stress respectively, and ϕ is homogeneous and of degree one in the components of (s_{ij}/s_o). Noting that

$$D = s_{ij}\dot{e}^p_{ij} = s_o\dot{e}_o\phi^{n+1} , \qquad (75)$$

we see that D is homogeneous and of degree $(n + 1)/n$ in the components of $\dot{e}_{ij}$. This implies that $n = 1/p$, (see equation 70).

Omitting reference to temperature, we may identify e^p_{ij} as the internal variable, and put

$$f(e_{ij}, e^p_{ij}) = G(e_{ij} - e^p_{ij})(e_{ij} - e^p_{ij}) . \qquad (76)$$

It is seen that

$$\frac{\partial f}{\partial e_{ij}} = 2G(e_{ij} - e^p_{ij}) = 2G(e^e_{ij}) = s_{ij} , \qquad (77a)$$

$$\frac{\partial f}{\partial e^p_{ij}} = -2G(e_{ij} - e^p_{ij}) = -s_{ij} . \qquad (77b)$$

Consequently, the forces conjugate to e^p_{ij} (equivalent to X_α) are the s_{ij}. Similarly, we see that

$$h(s_{ij}, e^p_{ij}) = \frac{1}{4G} s_{ij}s_{ij} + s_{ij}e^p_{ij} . \qquad (78)$$

Then

$$\frac{\partial h}{\partial s_{ij}} = \frac{s_{ij}}{2G} + e^p_{ij} = e_{ij}, \quad \frac{\partial h}{\partial e^p_{ij}} = s_{ij} . \qquad (79)$$

Using equation 73, it may be seen immediately that $\hat{s}_{ij} = s_{ij}(t)$, $0 \leq t \leq T$, is given in terms of a terminal stress $\bar{s}_{ij}$ by means of the relation

$$\hat{s}_{ij} = \frac{1}{p+1}\,\bar{s}_{ij} = \frac{n}{n+1}\,\bar{s}_{ij}\,. \tag{80}$$

From equation 74c, integrating over the interval $[0,T]$,

$$\bar{e}^{p}_{ij} = \dot{e}_o T\phi^{n}\left(\frac{\hat{s}_{ij}}{s_o}\right)\left.\frac{\partial\phi}{\partial\left(\frac{s_{ij}}{s_o}\right)}\right|_{\frac{\hat{s}_{ij}}{s_o}}\,. \tag{81}$$

Thus, from equation 78, using the requirement that ϕ is homogeneous and of degree one,

$$h(\bar{s}_{ij},\bar{e}^{p}_{ij}) = \frac{1}{4G}\,\bar{s}_{ij}\bar{s}_{ij} + \left(\frac{n+1}{n}\right) s_o\left(\frac{\hat{s}_{ij}}{s_o}\right)\bar{e}^{p}_{ij}$$

$$= \frac{1}{4G}\,\bar{s}_{ij}\bar{s}_{ij} + \left(\frac{n+1}{n}\right) s_o\dot{e}_o T\phi^{(n+1)}\left(\frac{\hat{s}_{ij}}{s_o}\right). \tag{82}$$

Further,

$$TD(e^{p}_{ij}/T) = \frac{1}{T^{p}}\,D(\bar{e}^{p}_{ij}) = s_o\dot{e}_o T\phi^{(n+1)}\left(\frac{\hat{s}_{ij}}{s_o}\right)\,. \tag{83}$$

Finally, referring to equation 51, equations 82 and 83 give

$$\aleph(\bar{s}_{ij}) = \frac{1}{4G}\,\bar{s}_{ij}\bar{s}_{ij} + \frac{1}{n}\,s_o\dot{e}_o T\phi^{(n+1)}\left\{\frac{n}{(n+1)}\,\frac{\bar{s}_{ij}}{s_o}\right\}. \tag{84}$$

It follows then that

$$\bar{e}_{ij} = \frac{1}{2G}\,\bar{s}_{ij} + \left(\frac{n+1}{n}\right)\dot{e}_o T\phi^{n}\left\{\frac{n}{(n+1)}\,\frac{\bar{s}_{ij}}{s_o}\right\}\left.\frac{\partial\phi}{\partial\left(\frac{s_{ij}}{s_o}\right)}\right|_{\frac{n}{(n+1)}\frac{\bar{s}_{ij}}{s_o}}\,. \tag{85}$$

This equation cannot be inverted without a precise definition of the function ϕ. With this further information, $\tilde{W}$ may be computed from the relation

$$\tilde{W}(\bar{e}_{ij}) = \bar{s}_{ij}\bar{e}_{ij} - \tilde{\Omega}(\bar{s}_{ij}) . \tag{86}$$

Chapter 5. Further extremal properties of the constitutive equations

5.1 Introduction

Ponter (1970, 1972, 1974) has derived displacement and work bounds on elastic-plastic bodies subject to quasi-static dynamic loading. Subsequently, Ponter (1975a) included inertial effects and obtained general bounding theorems, which include as special cases the previous results, and the results of Martin (1965, 1966a) for the maximum displacement at a point on an elastic-plastic body subjected to an impulse.

In the theory a functional W^s was defined as follows. Let $\sigma^*_{ij}(t)$, $0 \leq t \leq T$, be any prescribed stress history and let $\sigma_{ij}(t)$, $0 \leq t \leq T$, be an independent stress history with its associated plastic strain rate history $\dot{\varepsilon}^p_{ij}(t)$ such that $\varepsilon^p_{ij}(0) = 0$. Then

$$W^s(\sigma_{ij}, \sigma^*_{ij}) = \int_0^T (\sigma_{ij}(t) - \sigma^*_{ij}(t))\dot{\varepsilon}^p_{ij}(t)dt \,. \tag{1}$$

Central to the theory is the assumption that for any prescribed stress history $\sigma^*_{ij}(t)$, there exists $w(\sigma^*_{ij})$ such that

$$w(\sigma^*_{ij}) \leq W^s(\sigma_{ij}, \sigma^*_{ij}) \quad , \tag{2}$$

for all stress histories $\sigma_{ij}(t)$.

Ponter (1975b) has obtained expressions for $w(\sigma^*_{ij})$ for the following classes of materials and histories $\sigma^*_{ij}(t)$.

(i) For materials satisfying the Drucker stability condition for instantaneous changes and for which maximum complementary work paths (m-paths) exist, with $\sigma_{ij}(t)$ constant,

$$w(\sigma^*_{ij}) = -\max.\left\{\int_0^{\sigma^*_{ij}} \varepsilon^p_{ij}d\sigma_{ij} : \varepsilon^p_{ij}(\sigma_{ij} = 0) = 0\right\} \,. \tag{3}$$

(ii) For perfectly plastic materials with $\sigma^*_{ij}(t)$ remaining in or on the limit surface,

$$w(\sigma^*_{ij}) = 0 \,. \tag{4}$$

(iii) For materials exhibiting isotropic hardening whose current yield surface is $\phi(\sigma_{ij})$,

$$w(\sigma^*_{ij}) = -\max.\left\{\int_0^{\sigma^*_{ij}(t_o)} \varepsilon^p_{ij} d\sigma_{ij} : \varepsilon^p_{ij}(\sigma_{ij} = 0) = 0\right\} . \tag{5}$$

where t_o is the instant during $0 \leq t \leq T$ when $\phi(\sigma^*_{ij}(t))$ achieves its maximum value.

(iv) For materials exhibiting linear kinematic hardening whose yield surface at a state of plastic strain ε^p_{ij} is $\phi(\sigma_{ij} - C_{ijk\ell}\varepsilon^p_{k\ell})$,

$$w(\sigma^*_{ij}) \geq -\frac{1}{2} C_{ijk\ell}\varepsilon^{p*}_{ij}\varepsilon^{p*}_{k\ell} , \tag{6a}$$

provided that some ε^{p*}_{ij} exists such that

$$\phi(\sigma^*_{ij}(t) - C_{ijk\ell}\varepsilon^{p*}_{k\ell}) \leq 0 \text{ for } 0 \leq t \leq T . \tag{6b}$$

(v) For materials exhibiting stationary state creep where the plastic strain rates $\dot{\varepsilon}^p_{ij}$ are given by

$$\dot{\varepsilon}^p_{ij} = \frac{k}{n+1} \frac{\partial}{\partial\sigma_{ij}} \phi^{n+1}(\sigma_{ij}) , \tag{7a}$$

where $\phi(\sigma_{ij})$ is homogeneous and of degree one in σ_{ij},

$$w(\sigma^*_{ij}) = -\int_0^T \frac{k}{n}\left\{\frac{n}{n+1} \phi(\sigma^*_{ij}(t))\right\}^{n+1} dt . \tag{7b}$$

(vi) For materials exhibiting viscoplastic behaviour for which the plastic strain rates $\dot{\varepsilon}^p_{ij}$ are given by

$$\begin{aligned} \dot{\varepsilon}^p_{ij} &= \frac{k}{n+1} \frac{\partial}{\partial\sigma_{ij}}\left\{\phi(\sigma_{ij}) - \sigma_o\right\}^{n+1} && \text{if } \phi > \sigma_o \\ &= 0 && \text{if } \phi \leq \sigma_o , \end{aligned} \tag{8}$$

where $\phi(\sigma_{ij})$ is homogeneous and of degree one in σ_{ij},

$$w(\sigma^*_{ij}) = \int_0^T P(\sigma^*_{ij}(t))dt , \tag{9a}$$

where

$$P = \begin{cases} \frac{k}{n}\left(\frac{n}{n+1}\right)^{n+1} \{\phi(\sigma^*_{ij}) - \sigma_o\}^{n+1} & \text{if } \phi(\sigma^*_{ij}) > \sigma_o \\ 0 & \text{if } \phi(\sigma^*_{ij}) \leq \sigma_o \, . \end{cases} \tag{9b}$$

However, sufficient conditions for the existence of a lower bound are not established and the observation is made that despite the superficial similarity between W^s and the Drucker stability condition, the two concepts appear distinct.

In this chapter we examine the functional W^s in the light of the thermodynamic potential functions involving internal variables. We are thus enabled in section 5.2 to derive a set of sufficient conditions for the existence of a lower bound on W^s and in section 5.3 discuss two examples in which they are satisfied. The first is mainly to illustrate the application of the sufficient conditions and the second deals with the case $\sigma^*_{ij}(t) = \sigma^*_{ij}$ a constant. In section 5.4 a theorem is proved in which the lower bound on W^s for the case $\sigma^*_{ij}(t) = \sigma^*_{ij}$ is compared with the upper bound over all stress histories of the functional

$$\Omega_p = \int_0^T \varepsilon^p_{ij}\dot{\sigma}_{ij}dt \, , \tag{10}$$

subject to $\varepsilon^p_{ij}(0) = \sigma_{ij}(0) = 0$ and $\sigma_{ij}(T) = \sigma^*_{ij}$. In section 5.5 sufficient conditions for the lower bound on W^s are found for the case $f^p(\chi_\alpha) = 0$. Apart from being a special case, the assumptions made are weaker than in section 5.2. This enables the lower bound on W^s to be found for the class of materials exhibiting non-linear creep. Finally, in section 5.6, for the case of time-independent plasticity with $\dot{\sigma}^*_{ij} \neq 0$, which does not satisfy the sufficient conditions of section 5.2, a lower bound on W^s is obtained using a simpler and more general technique than Ponter in cases (ii) and (iv) above.

In all cases, the crucial property we make use of is the convexity of $f^p(\chi_\alpha)$ and $D(\dot{\chi}_\alpha)$.

5.2 Sufficient conditions for the minimum of W^s

We begin by obtaining an expression for W^s in terms of the thermodynamic potential functions. We consider materials whose strains and free energy may be split into elastic and inelastic parts as in equations 38 to 50 of chapter 1. We will assume that deformations are imposed isothermally and so reference to temperature is omitted.

Multiplying equation 47b of chapter 1 with $dT = 0$ by $\dot{\chi}_\alpha$, and integrating with respect to time between $t = 0$ and $t = T$ gives

$$\int_0^T \sigma_{ij}\dot{\varepsilon}^p_{ij}dt = f^p(\chi_\alpha)\Big]_{\chi_\alpha(0)}^{\chi_\alpha(T)} + \int_0^T D(\dot{\chi}_\alpha)dt . \tag{11}$$

Let us assume that the stress $\sigma^*_{ij}(t)$, $0 \leq t \leq T$, is given and that $\chi_\alpha(0) = 0$. Using equations 1 and 11,

$$W^s = \int_0^T (\sigma_{ij} - \sigma^*_{ij})\dot{\varepsilon}^p_{ij}dt$$

$$= f^p(\chi_\alpha(T)) + \int_0^T \left[D(\dot{\chi}_\alpha) - \sigma^*_{ij}\frac{\partial\varepsilon^p_{ij}}{\partial\chi_\alpha}\dot{\chi}_\alpha\right]dt . \tag{12}$$

Note that this functional depends on the stress history $\sigma^*_{ij}(t)$ and the internal variable history $\chi_\alpha(t)$. Thus

$$W^s = W^s(\sigma^*_{ij}(t), \chi_\alpha(t)) . \tag{13}$$

We will attempt to find a history $\chi'_\alpha(t)$ such that

$$W^s(\sigma^*_{ij}(t), \chi'_\alpha(t)) \leq W^s(\sigma^*_{ij}(t), \chi_\alpha(t)) , \tag{14}$$

for arbitrary continuous $\chi_\alpha(t)$ subject to $\chi_\alpha(0) = 0$. Then

$$w(\sigma^*_{ij}(t)) = W^s(\sigma^*_{ij}(t), \chi'_\alpha(t)) , \tag{15}$$

is the lower bound on W^s. We term $w(\sigma^*_{ij})$ the *lower bound* on W^s in order to distinguish between the minimum of $W^s(\sigma^*_{ij}(t), \chi_\alpha(t))$ over all *continuous* histories $\chi_\alpha(t)$ subject to $\chi_\alpha(0) = 0$ and the minimum of $W^s(\sigma^*_{ij}(t), \chi_\alpha(t))$

over all *physically attainable* histories $\chi_\alpha(t)$ subject to $\chi_\alpha(0) = 0$. The latter would be referred to as the *minimum* of W^s.

The procedure in obtaining the extremal history $\chi'_\alpha(t)$ will be as follows:-

(i) Put $\chi_\alpha(T) = \hat{\chi}_\alpha$ and seek a history $\chi_\alpha(t)$ which gives the least value of W^s for this particular value of $\chi_\alpha(T)$. Define

$$\overline{W}^s(\sigma^*_{ij}(t), \hat{\chi}_\alpha) = \min. \{W^s(\sigma^*_{ij}(t), \chi_\alpha(t)) : \chi_\alpha(T) = \hat{\chi}_\alpha\} . \tag{16}$$

(ii) Choose the value of $\hat{\chi}_\alpha$ which minimises $\overline{W}^s(\sigma^*_{ij}(t), \hat{\chi}_\alpha)$. Let it be $\overline{\chi}_\alpha$. We see that

$$\begin{aligned} w(\sigma^*_{ij}(t)) &= \min. \{\overline{W}^s(\sigma^*_{ij}(t), \hat{\chi}_\alpha)\} \\ &= \overline{W}^s(\sigma^*_{ij}(t), \overline{\chi}_\alpha) . \end{aligned} \tag{17}$$

Consider two paths $\chi'_\alpha(t)$ and $\chi''_\alpha(t)$ satisfying $\chi'_\alpha(0) = \chi''_\alpha(0) = 0$, $\chi'_\alpha(T) = \chi''_\alpha(T) = \hat{\chi}_\alpha$. Note that $f^P(\hat{\chi}_\alpha)$ is unaffected by such variations and so, using equation 12,

$$\begin{aligned} \Delta W^s &= W^s(\sigma^*_{ij}(t), \chi'_\alpha(t)) - W^s(\sigma^*_{ij}(t), \chi''_\alpha(t)) \\ &= \int_0^T \left\{ D(\dot{\chi}'_\alpha) - D(\dot{\chi}''_\alpha) - \sigma^*_{ij} \frac{\partial \varepsilon^P_{ij}}{\partial \chi_\alpha} (\dot{\chi}'_\alpha - \dot{\chi}''_\alpha) \right\} dt . \end{aligned} \tag{18}$$

Using the convexity of $D(\dot{\chi}_\alpha)$

$$\Delta W^s \leq \int_0^T \left\{ \left. \frac{\partial D}{\partial \dot{\chi}_\alpha} \right|_{\dot{\chi}'_\alpha(t)} (\dot{\chi}'_\alpha - \dot{\chi}''_\alpha) - \sigma^*_{ij} \frac{\partial \varepsilon^P_{ij}}{\partial \chi_\alpha} (\dot{\chi}'_\alpha - \dot{\chi}''_\alpha) \right\} dt \tag{19}$$

$$= - \int_0^T \left\{ \frac{d}{dt} \left(\left. \frac{\partial D}{\partial \dot{\chi}_\alpha} \right|_{\chi'_\alpha(t)} \right) - \sigma^*_{ij} \frac{\partial \varepsilon^P_{ij}}{\partial \chi_\alpha} \right\} (\chi'_\alpha - \chi''_\alpha) dt , \tag{20}$$

on integration by parts. ($\dot{\sigma}^*_{ij}$ is assumed to exist). Now let $\chi'_\alpha(t)$ be a minimum path. We require for all other paths $(\chi''_\alpha(t))$ that $\Delta W^s \leq 0$. A sufficient condition that this is so is

$$\frac{d}{dt}\left\{\frac{\partial D}{\partial \dot{\chi}_\alpha}\bigg|_{\dot{\chi}'_\alpha(t)}\right\} = \dot{\sigma}^*_{ij}\,\frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha}\,, \tag{21}$$

or that

$$\frac{\partial D}{\partial \dot{\chi}_\alpha}\bigg|_{\dot{\chi}'_\alpha(t)} - \frac{\partial D}{\partial \dot{\chi}_\alpha}\bigg|_{\dot{\chi}'_\alpha(0)} = (\sigma^*_{ij}(t) - \sigma^*_{ij}(0))\,\frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha}\,. \tag{22}$$

We will now assume for any choice of $\chi_\alpha(T) = \hat{\chi}_\alpha$ that there exists a history $\chi_\alpha(t)$ satisfying equation 22 and the end conditions $\chi_\alpha(0) = 0$, $\chi_\alpha(T) = \hat{\chi}_\alpha$. A consequence of this assumption is that we may associate a value of -

$$\frac{\partial D}{\partial \dot{\chi}_\alpha}\bigg|_{\dot{\chi}_\alpha(0)}$$

with every value of $\hat{\chi}_\alpha$, i.e. there exists a function

$$\frac{\partial D}{\partial \dot{\chi}_\alpha}\bigg|_{\dot{\chi}_\alpha(0)} = h_\alpha(\hat{\chi}_\beta)\,. \tag{23}$$

Thus we may put

$$\overline{W}^s(\sigma^*_{ij}(t),\, \hat{\chi}_\alpha) = \min.\,\{W^s(\sigma^*_{ij}(t),\, \chi_\alpha(t)) : \chi_\alpha(T) = \hat{\chi}_\alpha\}$$

$$= f^p(\hat{\chi}_\alpha) + \int_0^T \left[D(\dot{\chi}_\alpha) - \sigma^*_{ij}\,\frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha}\,\dot{\chi}_\alpha\right] dt\,, \tag{24}$$

where $\dot{\chi}_\alpha$ satisfies equation 22.

We now proceed to the second part of the bounding problem. Consider two different end conditions $\overline{\chi}'_\alpha$ and $\overline{\chi}''_\alpha$. Put

$$\Delta\overline{W}^s = \overline{W}^s(\sigma^*_{ij}(t),\, \overline{\chi}'_\alpha) - \overline{W}^s(\sigma^*_{ij}(t),\, \overline{\chi}''_\alpha)\,. \tag{25}$$

Let the extreme paths to $\overline{\chi}'_\alpha$ and $\overline{\chi}''_\alpha$ (i.e. which satisfy equation 22) be $\chi'_\alpha(t)$ and $\chi''_\alpha(t)$ respectively. Thus

$$\int_0^T \dot{\chi}'_\alpha dt = \overline{\chi}'_\alpha \quad \text{and} \quad \int_0^T \dot{\chi}''_\alpha dt = \overline{\chi}''_\alpha\,. \tag{26}$$

Using equation 24,

$$\Delta\overline{W}^s = f^p(\overline{\chi}'_\alpha) - f^p(\overline{\chi}''_\alpha) + \int_0^T \left\{ D(\dot{\chi}'_\alpha) - D(\dot{\chi}''_\alpha) + \sigma^*_{ij} \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} (\dot{\chi}'_\alpha - \dot{\chi}''_\alpha) \right\} dt \; . \tag{27}$$

But using the convexity of $f^p(\chi_\alpha)$ and $D(\dot{\chi}_\alpha)$,

$$\Delta\overline{W}^s \leq \frac{\partial f^p}{\partial \overline{\chi}'_\alpha} (\overline{\chi}'_\alpha - \overline{\chi}''_\alpha) + \int_0^T \left(\left. \frac{\partial D}{\partial \dot{\chi}_\alpha} \right|_{\dot{\chi}'_\alpha(t)} - \sigma^*_{ij}(t) \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} \right) (\dot{\chi}'_\alpha - \dot{\chi}''_\alpha) dt \tag{28}$$

$$= \frac{\partial f^p}{\partial \overline{\chi}'_\alpha} (\overline{\chi}'_\alpha - \overline{\chi}''_\alpha) + \left(\left. \frac{\partial D}{\partial \dot{\chi}_\alpha} \right|_{\dot{\chi}'_\alpha(0)} - \sigma^*_{ij}(0) \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} \right) \int_0^T (\dot{\chi}'_\alpha - \dot{\chi}''_\alpha) dt, \tag{29}$$

since $\dot{\chi}'_\alpha(t)$ satisfies equation 22. Thus

$$\Delta\overline{W}^s \leq \left(\frac{\partial f^p}{\partial \overline{\chi}'_\alpha} + \left. \frac{\partial D}{\partial \dot{\chi}_\alpha} \right|_{\dot{\chi}'_\alpha(0)} - \sigma^*_{ij}(0) \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} \right) (\overline{\chi}'_\alpha - \overline{\chi}''_\alpha) \; , \tag{30}$$

where we use equations 26.

Now let $\overline{\chi}'_\alpha$ be the end point for which $\overline{W}^s$ is a minimum. We require for all other points $(\overline{\chi}''_\alpha)$ that $\Delta\overline{W}^s \leq 0$. A sufficient condition that this is so is

$$\frac{\partial f^p}{\partial \overline{\chi}'_\alpha} + \left. \frac{\partial D}{\partial \dot{\chi}_\alpha} \right|_{\dot{\chi}'_\alpha(0)} = \sigma^*_{ij}(0) \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} \; , \tag{31}$$

where

$$\left. \frac{\partial D}{\partial \dot{\chi}_\alpha} \right|_{\dot{\chi}'_\alpha(0)}$$

is a function of $\overline{\chi}'_\alpha$ (equation 23).

In order to calculate $w(\sigma^*_{ij}(t))$ we note that

$$D(\dot{\chi}_\alpha) = X_\alpha \dot{\chi}_\alpha$$

and so

$$X_\alpha = \frac{\partial D}{\partial \dot{\chi}_\alpha} - \frac{\partial X_\alpha}{\partial \dot{\chi}_\beta} \dot{\chi}_\beta \; . \tag{32}$$

Thus for a path $\chi_\alpha(t)$ satisfying equation 22 and terminating at a point $\bar{\chi}_\alpha$ satisfying equation 31,

$$\begin{aligned} W^s &= w(\sigma^*_{ij}(t)) \\ &= f^P(\bar{\chi}_\alpha) + \int_0^T \left\{ X_\alpha - \sigma^*_{ij} \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} \right\} \dot{\chi}_\alpha dt \\ &= f^P(\bar{\chi}_\alpha) + \int_0^T \left(\frac{\partial D}{\partial \dot{\chi}_\alpha} - \sigma^*_{ij} \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} \right) \dot{\chi}_\alpha dt - \int_0^T \frac{\partial X_\alpha}{\partial \dot{\chi}_\beta} \dot{\chi}_\alpha \dot{\chi}_\beta dt . \end{aligned} \tag{33}$$

Making use of equations 22 and 31, equation 33 becomes

$$w(\sigma^*_{ij}(t)) = f^P(\bar{\chi}_\alpha) - \frac{\partial f^P}{\partial \bar{\chi}_\alpha} \bar{\chi}_\alpha - \int_0^T \frac{\partial X_\alpha}{\partial \dot{\chi}_\beta} \dot{\chi}_\alpha \dot{\chi}_\beta dt . \tag{34}$$

We are then assured that

$$w(\sigma^*_{ij}(t)) \leq W^s(\sigma^*_{ij}(t), \chi_\alpha(t)) , \tag{35}$$

for all continuous paths $\chi_\alpha(t)$ such that $\chi_\alpha(0) = 0$.

5.3 Examples satisfying the sufficient conditions

The above derivation of $w(\sigma^*_{ij}(t))$ relies on the assumption that for any choice of $\hat{\chi}_\alpha$ there exists a history $\chi_\alpha(t)$ satisfying equation 22 and the end conditions $\chi_\alpha(0) = 0$, $\chi_\alpha(T) = \hat{\chi}_\alpha$. We will examine two examples where this assumption is valid.

5.3.1 Example 1. Time-dependent Maxwell material

In order to illustrate the method, consider the material whose (isothermal) fundamental equation is

$$f(\varepsilon_{ij}, \varepsilon^p_{ij}) = \frac{1}{2} C_{ijk\ell}(\varepsilon_{ij} - \varepsilon^p_{ij})(\varepsilon_{k\ell} - \varepsilon^p_{k\ell}) + \frac{1}{2} K^P \varepsilon^p_{ij} \varepsilon^p_{ij} , \tag{36}$$

in which the plastic strains (ε^p_{ij}) are the internal variables. Thus the equations of state are

$$q_{ij} = \frac{\partial f}{\partial \varepsilon_{ij}}$$

$$= C_{ijk\ell}(\varepsilon_{k\ell} - \varepsilon^p_{k\ell}) , \quad (37)$$

$$s_{ij} = -\frac{\partial f}{\partial \varepsilon^p_{ij}}$$

$$= \sigma_{ij} - K^p \varepsilon^p_{ij} , \quad (38)$$

where s_{ij} are the internal forces.

Let the kinetic equations be linear;

$$\frac{\dot{\varepsilon}^p_{ij}}{\dot{\varepsilon}_o} = \frac{s_{ij}}{s_o} = \frac{\sigma_{ij} - K^p \varepsilon^p_{ij}}{s_o} , \quad (39)$$

where $\dot{\varepsilon}_o$ and s_o are constants having the dimensions of strain rate and stress respectively. Thus

$$D(\dot{\varepsilon}^p_{ij}) = \frac{s_o}{\dot{\varepsilon}_o} \dot{\varepsilon}^p_{ij} \dot{\varepsilon}^p_{ij} , \quad (40)$$

and

$$\dot{\varepsilon}^p_{ij} = \frac{\dot{\varepsilon}_o}{2s_o} \frac{\partial D}{\partial \dot{\varepsilon}^p_{ij}} . \quad (41)$$

We will assume a specific form for the prescribed stress history $\sigma^*_{ij}(t)$. Let it be

$$\sigma^*_{ij}(t) = \sigma^o_{ij} \frac{t}{T} , \quad 0 \leq t \leq T . \quad (42)$$

The first requirement for a minimum path is that the history of $\partial D/\partial \dot{\varepsilon}^p_{ij}$ is given by equation 22 viz.

$$\left.\frac{\partial D}{\partial \dot{\varepsilon}^p_{ij}}\right|_{\dot{\varepsilon}^p_{ij}(t)} = \sigma^o_{ij} \frac{t}{T} + \left.\frac{\partial D}{\partial \dot{\varepsilon}^p_{ij}}\right|_{\dot{\varepsilon}^p_{ij}(0)} . \quad (43)$$

Substituting into equation 41 and integrating gives

$$\varepsilon^p_{ij}(T) = \int_0^T \dot{\varepsilon}^p_{ij}\,dt$$

$$= \int_0^T \frac{\dot{\varepsilon}_o}{2s_o}\left\{\overset{o}{\sigma}_{ij}\frac{t}{T} + \left.\frac{\partial D}{\partial \dot{\varepsilon}^p_{ij}}\right|_{\dot{\varepsilon}^p_{ij}(0)}\right\}dt$$

$$= \frac{T\dot{\varepsilon}_o}{2s_o}\left\{\frac{\overset{o}{\sigma}_{ij}}{2} + \left.\frac{\partial D}{\partial \dot{\varepsilon}^p_{ij}}\right|_{\dot{\varepsilon}^p_{ij}(0)}\right\}. \tag{44}$$

Thus if we require that $\varepsilon^p_{ij}(T) = \hat{\varepsilon}^p_{ij}$, we choose $\dot{\varepsilon}^p_{ij}(0)$ such that

$$\left.\frac{\partial D}{\partial \dot{\varepsilon}^p_{ij}}\right|_{\dot{\varepsilon}^p_{ij}(0)} = \frac{2s_o}{T\dot{\varepsilon}_o}\hat{\varepsilon}^p_{ij} - \frac{\overset{o}{\sigma}_{ij}}{2}, \tag{45}$$

which is the function we defined in equation 23.

The second requirement for a minimum path is that $\varepsilon^p_{ij}(T) = \bar{\varepsilon}^p_{ij}$ satisfies equation 31. Thus

$$K^p\bar{\varepsilon}^p_{ij} + \frac{2s_o}{\dot{\varepsilon}_o}\frac{\bar{\varepsilon}^p_{ij}}{T} - \frac{\overset{o}{\sigma}_{ij}}{2} = 0, \tag{46}$$

or

$$\bar{\varepsilon}^p_{ij} = \frac{\overset{o}{\sigma}_{ij}}{2}\,\frac{\dot{\varepsilon}_o T}{K^p\dot{\varepsilon}_o T + 2s_o}. \tag{47}$$

Therefore, using equations 41, 43, 45 and 47, an extreme path $\varepsilon^p_{ij}(t)$ is given by

$$\varepsilon^p_{ij} = \frac{\dot{\varepsilon}_o\overset{o}{\sigma}_{ij}}{4s_o}\left\{\frac{t}{T} - \frac{tK^p\dot{\varepsilon}_o T}{K^p\dot{\varepsilon}_o T + 2s_o}\right\}. \tag{48}$$

In evaluating $w(\dot{\sigma}^*_{ij}(t))$ from equation 34 we note that the expression equivalent to $\partial X_\alpha/\partial \dot{\chi}_\beta$ in this example is, from equation 39,

$$\frac{\partial s_{ij}}{\partial \dot{\varepsilon}^p_{k\ell}} = \frac{s_o}{\dot{\varepsilon}_o}\delta_{ik}\delta_{j\ell}. \tag{49}$$

Substitution from equations 48 and 49 into equation 34 yields

$$w(q^o_{ij}, \frac{t}{T}) = -\frac{q^o_{ij} q^o_{ij}}{8} \left(\frac{\dot{\varepsilon}_o T}{\dot{\varepsilon}_o K^P T + 2s_o}\right)^2 K^P$$

$$- \frac{\dot{\varepsilon}_o \sigma^o_{ij} \sigma^o_{ij}}{16 s_o} \int_0^T \left\{\frac{4t^2}{T} - \frac{2tK^P\dot{\varepsilon}_o}{K^P\dot{\varepsilon}_o T + 2s_o} + \left(\frac{K^P\dot{\varepsilon}_o T}{K^P\dot{\varepsilon}_o T + 2s_o}\right)^2\right\} dt$$

$$= -\frac{\dot{\varepsilon}_o 2\sigma^o_{ij}\sigma^o_{ij} T^2}{8} \left[\frac{K^P}{(\dot{\varepsilon}_o K^P T - 2s_o)^2} + \frac{1}{2s_o\dot{\varepsilon}_o}\left\{\frac{4}{3T} - \frac{K^P\dot{\varepsilon}_o}{K^P\dot{\varepsilon}_o T + 2s_o} + T\left(\frac{K^P\dot{\varepsilon}_o}{K^P\dot{\varepsilon}_o T + 2s_o}\right)^2\right\}\right]$$

(50)

5.3.2 Example 2. $\sigma^*_{ij}(t) = \sigma^*_{ij}$, a constant, $0 \leq t \leq T$

If $\dot{\sigma}^*_{ij} = 0$ for all t such that $0 \leq t \leq T$, then equation 22 becomes

$$\frac{\partial D}{\partial \dot{\chi}_\alpha} = \text{constant}\ , \tag{51}$$

which is satisfied for all materials included in chapter 1 if $\dot{\chi}_\alpha(t)$ is constant. Our assumption is therefore valid and it follows that equation 23 becomes

$$\left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\dot{\chi}_\alpha(0)} = \left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\chi_\alpha}{T}} . \tag{52}$$

The criterion for the choice of $\overline{\chi}_\alpha$ (equation 31) is

$$\frac{\partial f^P}{\partial \overline{\chi}_\alpha} + \left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\overline{\chi}_\alpha}{T}} = \sigma^*_{ij} \frac{\partial \varepsilon^P_{ij}}{\partial \chi_\alpha} . \tag{53}$$

If $\partial D/\partial \dot{\chi}_\alpha$ is discontinuous at $\dot{\chi}_\alpha = 0$. that is if there exists a limit surface $\phi(X_\alpha) = 0$ (time-independent plasticity) or a yield surface $\phi(X_\alpha) = \phi_o$ (viscoplasticity) (see 1.3.2), we solve equation 53 by noting that

(i) If $\phi\left(\sigma^*_{ij} \frac{\partial \varepsilon^P_{ij}}{\partial \chi_\alpha}\right) \leq \begin{Bmatrix} 0 \\ \phi_o \end{Bmatrix}$,

then equation 53 is satisfied by $\overline{\chi}_\alpha = 0$.

(ii) If $\phi\left(\sigma^*_{ij}\,\frac{\partial \varepsilon^p_{ij}}{\partial \overline{\chi}_\alpha}\right) > \begin{Bmatrix} 0 \\ \phi_o \end{Bmatrix}$,

then equation 53 is satisfied (if at all) by some non-zero $\overline{\chi}_\alpha$, and thus

$$\left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\overline{\chi}_\alpha}{T}}$$

is well defined.

Thus if $\sigma^*_{ij}(t) = \sigma^*_{ij}$, then from equation 34,

$$\begin{aligned} W^s &\leq w(\sigma^*_{ij}) \\ &= f^p(\overline{\chi}_\alpha) - \frac{\partial f^p}{\partial \overline{\chi}_\alpha}\overline{\chi}_\alpha - \int_0^T \left.\frac{\partial X_\alpha}{\partial \dot{\chi}_\beta}\right|_{\frac{\overline{\chi}_\beta}{T}} \frac{\overline{\chi}_\beta}{T}\,\frac{\overline{\chi}_\alpha}{T}\,dt \\ &= f^p(\overline{\chi}_\alpha) - \frac{\partial f^p}{\partial \overline{\chi}_\alpha}\overline{\chi}_\alpha - \frac{1}{T}\left.\frac{\partial X_\alpha}{\partial \dot{\chi}_\beta}\right|_{\frac{\overline{\chi}_\beta}{T}} \overline{\chi}_\alpha\overline{\chi}_\beta\,. \end{aligned} \tag{54}$$

Alternatively, evaluating equation 12 for a path $\chi_\alpha = \overline{\chi}_\alpha \frac{t}{T}$,

$$w(\sigma^*_{ij}) = f^p(\overline{\chi}_\alpha) - \sigma^*_{ij}\varepsilon^p_{ij}(\overline{\chi}_\alpha) + TD\left(\frac{\overline{\chi}_\alpha}{T}\right). \tag{55}$$

We note that for time-independent plasticity

$$\begin{aligned} \frac{\partial X_\alpha}{\partial \dot{\chi}_\beta}\dot{\chi}_\beta &= \frac{\partial^2 D}{\partial \dot{\chi}_\alpha \partial \dot{\chi}_\beta}\dot{\chi}_\beta \\ &= 0\,, \end{aligned} \tag{56}$$

since D is homogeneous and of degree one in $\dot{\chi}_\alpha$. Thus in this case

$$w(\sigma^*_{ij}) = f^p(\overline{\chi}_\alpha) - \frac{\partial f^p}{\partial \overline{\chi}_\alpha}\overline{\chi}_\alpha\,. \tag{57}$$

Further, for perfect plasticity $f^p(\chi_\alpha) = 0$ and so equation 57 becomes

$$w(\sigma^*_{ij}) = 0\,. \tag{58}$$

We note that for perfect plasticity, equation 53 can only be satisfied if

$$\phi\left(\sigma^*_{ij}\, \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha}\right) \leq 0,$$

i.e. if σ^*_{ij} lies in or on the limit surface in stress space.

5.4 A theorem for the lower bound on W^S for the case $\sigma^*_{ij}(t) = $ constant

In this section we relate the lower bound on W^S for the case $\sigma^*_{ij}(t) = \sigma^*_{ij}$ a constant, to the upper bound on the complementary work to σ^*_{ij}. We introduce the functional

$$\Omega_p = \int_0^T \varepsilon^p_{ij}(t)\dot{\sigma}_{ij}(t)dt , \tag{59}$$

where ε^p_{ij} and σ_{ij} are related through the constitutive equations. An upper bound $\tilde{\Omega}_p(\sigma^*_{ij})$ is then sought such that

$$\tilde{\Omega}_p(\sigma^*_{ij}) \geq \Omega_p , \tag{60}$$

for all stress histories $\sigma_{ij}(t)$ subject to $\varepsilon^p_{ij}(0) = \sigma_{ij}(0) = 0$, $\sigma_{ij}(T) = \sigma^*_{ij}$. It will be seen that $\tilde{\Omega}_p(\sigma^*_{ij})$ is closely related to $\tilde{\Omega}(\sigma^*_{ij})$, the upper bound on the complementary work to σ^*_{ij} which was obtained in chapter 4. We then prove the following theorem.

If $\sigma^*_{ij}(t) = \sigma^*_{ij}$ a constant, $0 \leq t \leq T$, then

$$-\tilde{\Omega}_p(\sigma^*_{ij}) = w(\sigma^*_{ij}) \leq W^S(\sigma^*_{ij}(t), \chi_\alpha(t)) , \tag{61}$$

for arbitrary continuous $\chi_\alpha(t)$ subject to $\chi_\alpha(0) = 0$.

In order to examine equation 59 we recall equations 39 and 50a of chapter 1. We see that

$$\int_0^{\sigma^*_{ij}} \varepsilon^p_{ij} d\sigma_{ij} = \int_0^T (\varepsilon_{ij} - \varepsilon^e_{ij})\dot{\sigma}_{ij}dt , \tag{62}$$

and note that

$$\int_0^T \varepsilon^e_{ij}\dot{\sigma}_{ij}dt = h^e(\sigma^*_{ij}) \,, \tag{63}$$

where $\sigma_{ij}(0) = 0$, $\sigma_{ij}(T) = \sigma^*_{ij}$, is path independent.

Given the convexity of $f(\varepsilon_{ij}, \chi_\alpha)$ and $D(\dot{\chi}_\alpha)$, then it may be shown (equations 43 to 52 of chapter 4) that sufficient conditions for the functional

$$\Omega = \int_0^T \varepsilon_{ij}\dot{\sigma}_{ij}dt$$

$$= h(\sigma^*_{ij}, \chi_\alpha) - \int_0^T D(\dot{\chi}_\alpha)dt \,, \tag{64}$$

where $\sigma_{ij}(0) = \chi_\alpha(0) = 0$, $\sigma_{ij}(T) = \sigma^*_{ij}$, to be a maximum are that the history $\chi_\alpha(t)$ satisfies

$$\dot{\chi}_\alpha = \frac{\overline{\chi}_\alpha}{T} \text{ constant,} \tag{65}$$

$$\frac{\partial h}{\partial \overline{\chi}_\alpha} = \sigma^*_{ij}\frac{\partial \varepsilon^P_{ij}}{\partial \chi_\alpha} - \frac{\partial f^P}{\partial \overline{\chi}_\alpha} = \left.\frac{\partial D}{\partial \dot{\chi}_\alpha}\right|_{\frac{\overline{\chi}_\alpha}{T}} \,, \tag{66}$$

if $\partial D/\partial\dot{\chi}_\alpha$ is continuous.

If $\partial D/\partial\dot{\chi}_\alpha$ is discontinuous at $\dot{\chi}_\alpha = 0$, then we choose $\overline{\chi}_\alpha = 0$ if

$$X_\alpha(\sigma^*_{ij}, \chi_\alpha = 0) = \sigma^*_{ij}\frac{\partial \varepsilon^P_{ij}}{\partial \chi_\alpha}$$

lies inside or on the limit surface (time-independent plasticity) or the yield surface (viscoplasticity). If $\sigma^*_{ij}(\partial\varepsilon^P_{ij}/\partial\chi_\alpha)$ does not lie in or on such a surface then equations 65 and 66 are sufficient conditions for the maximum.

Thus, from equation 64

$$\Omega \leq \tilde{\Omega}(\sigma^*_{ij})$$

$$= h(\sigma^*_{ij}, \overline{\chi}_\alpha) - TD\left(\frac{\overline{\chi}_\alpha}{T}\right) . \tag{67}$$

Using equation 48 of chapter 1 we see that

$$\tilde{\Omega}(\sigma^*_{ij}) = \overline{h^e}(\sigma^*_{ij}) + \sigma^*_{ij}\varepsilon^p_{ij}(\overline{\chi}_\alpha) - f^p(\overline{\chi}_\alpha) - TD\left(\frac{\overline{\chi}_\alpha}{T}\right) . \tag{68}$$

So in view of equations 62 and 63

$$\tilde{\Omega}_p(\sigma^*_{ij}) = \sigma^*_{ij}\varepsilon^p_{ij}(\overline{\chi}_\alpha) - f^p(\overline{\chi}_\alpha) - TD\left(\frac{\overline{\chi}_\alpha}{T}\right) , \tag{69}$$

where $\overline{\chi}_\alpha$ satisfies equation 66 with its proviso where a yield or limit surface exists.

Now comparing equations 53 and 66 with their respective provisos where a yield or limit surface exists we see that they provide the same criterion for $\overline{\chi}_\alpha$ in terms of σ^*_{ij}. Comparing equations 55 and 69 we conclude that if $\sigma^*_{ij}(t) = \sigma^*_{ij}$ a constant for $0 \leq t \leq T$, then

$$-\tilde{\Omega}_p(\sigma^*_{ij}) = w(\sigma^*_{ij}) \leq W^s(\sigma^*_{ij}, \chi_\alpha(t)) , \tag{70}$$

for arbitrary continuous $\chi_\alpha(t)$ subject to $\chi_\alpha(0) = 0$.

This result has been obtained by Ponter (1975b), (see equation 3), using the Drucker stability condition for instantaneous changes and the existence of maximum complementary work paths.

5.5 A lower bound on W^s for the case $f^p = 0$

We will now derive conditions for the extreme path and lower bound on W^s for the special case $f^p(\chi_\alpha) = 0$. Although this case is included in the treatment of the general case it is useful to treat it separately since it will not prove necessary to assume the existence of a history $\chi_\alpha(t)$ satisfying equation 22 for arbitrary $\chi_\alpha(T)$.

Referring to equation 12

$$W^s(\sigma^*_{ij}(t), \chi_\alpha(t)) = \int_0^T \left\{D(\dot{\chi}_\alpha) - \sigma^*_{ij}\frac{\partial\varepsilon^p_{ij}}{\partial\chi_\alpha}\dot{\chi}_\alpha\right\}dt . \tag{71}$$

Considering two independent histories $\chi'_\alpha(t)$ and $\chi''_\alpha(t)$,

$$\Delta W^s = W^s(\sigma^*_{ij}(t), \chi'_\alpha(t)) - W^s(\sigma^*_{ij}(t), \chi''_\alpha(t))$$

$$= \int_0^T \left\{ D(\dot{\chi}'_\alpha) - D(\dot{\chi}''_\alpha) - \sigma^*_{ij} \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} (\dot{\chi}'_\alpha - \dot{\chi}''_\alpha) \right\} dt \; . \qquad (72)$$

Using the convexity of $D(\dot{\chi}_\alpha)$

$$\Delta W^s \leq \int_0^T \left(\left. \frac{\partial D}{\partial \dot{\chi}_\alpha} \right|_{\dot{\chi}'_\alpha(t)} - \sigma^*_{ij} \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha} \right) (\dot{\chi}'_\alpha - \dot{\chi}''_\alpha) dt \; . \qquad (73)$$

Now let $\chi'_\alpha(t)$ be the extremal path. We require for all other paths $(\chi''_\alpha(t))$ that $\Delta W^s \leq 0$. A sufficient condition that this is so is

$$\left. \frac{\partial D}{\partial \dot{\chi}_\alpha} \right|_{\dot{\chi}'_\alpha(t)} = \sigma^*_{ij}(t) \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha}, \quad 0 \leq t \leq T \; . \qquad (74)$$

It is seen from equation 34 that in this case

$$w(\sigma^*_{ij}) = - \int_0^T \frac{\partial \chi_\alpha}{\partial \dot{\chi}_\beta} \dot{\chi}_\alpha \dot{\chi}_\beta dt \; , \qquad (75)$$

where $\dot{\chi}_\alpha$ satisfies equation 74.

5.5.1 Example. Non-linear creep for metals

The usual example for non-linear creep for metals has $f^p(\chi_\alpha) = 0$ and the kinetic equations in the form

$$\frac{\dot{\varepsilon}^p_{ij}}{\dot{\varepsilon}_o} = \phi^n \frac{\partial \phi}{\partial(\sigma_{ij}/\sigma_o)} \; , \qquad (76)$$

where $\dot{\varepsilon}_o$ and σ_o are constants having the dimensions of strain rate and stress respectively, n is taken to be an odd positive integer and $\phi(\sigma_{ij}/\sigma_o)$ is homogeneous and of degree one in (σ_{ij}/σ_o).

We have examined such a form of the kinetic equations in section 1.3.2. From equation 51 of chapter 1 we see that

$$\frac{\partial D}{\partial \dot{\varepsilon}^p_{ij}} = \frac{n+1}{n} \sigma_{ij} \; , \qquad (77)$$

since q_{ij} are the forces conjugate to ε^p_{ij}.

Thus if $\sigma^*_{ij}(t)$ is the prescribed stress history, equation 74 for the extreme path for W^S is

$$\frac{n+1}{n}\sigma_{ij}(t) = \sigma^*_{ij}(t) . \tag{78}$$

To evaluate $w(\sigma^*_{ij}(t))$ from equation 75 we note that since (σ_{ij}/σ_o) is homogeneous and of degree $\frac{1}{n}$ in $(\dot{\varepsilon}^p_{ij}/\dot{\varepsilon}_o)$,

$$\begin{aligned}\frac{\partial\sigma_{ij}}{\partial\dot{\varepsilon}^p_{k\ell}}\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{k\ell} &= \dot{\varepsilon}_o\sigma_o\frac{\partial(\sigma_{ij}/\sigma_o)}{\partial(\dot{\varepsilon}^p_{k\ell}/\dot{\varepsilon}_o)}\frac{\dot{\varepsilon}^p_{ij}}{\dot{\varepsilon}_o}\frac{\dot{\varepsilon}^p_{k\ell}}{\dot{\varepsilon}_o}\\ &= \frac{\dot{\varepsilon}_o}{n}\sigma_{ij}\phi^n\frac{\partial\phi}{\partial(\sigma_{ij}/\sigma_o)} ,\end{aligned} \tag{79}$$

which is the counterpart in this example of the term

$$\frac{\partial X_\alpha}{\partial\dot{\chi}_\beta}\dot{\chi}_\alpha\dot{\chi}_\beta .$$

If a minimum path (equation 78) is followed the right hand side of equation 79 becomes

$$\frac{\dot{\varepsilon}_o}{n+1}\sigma^*_{ij}(t)\phi^n\left.\frac{\partial\phi}{\partial(\sigma_{ij}/\sigma_o)}\right|_{\frac{n}{n+1}\frac{\sigma^*_{ij}(t)}{\sigma_o}}$$

Integration gives

$$w(\sigma^*_{ij}(t)) = -\frac{\dot{\varepsilon}_o\sigma_o}{n}\left(\frac{n}{n+1}\right)^{n+1}\int_0^T\phi^{n+1}\left(\frac{\sigma^*_{ij}(t)}{\sigma_o}\right)dt , \tag{80}$$

which is the result obtained by Ponter (1975b), (see equation 7b).

5.6 A lower bound on W^S for time-independent plasticity

We now consider those materials for which D is homogeneous and of degree one in $\dot{\chi}_\alpha$. Let the limit surface be described by $\phi(X_\alpha) = 0$. For a general stress history $\sigma^*_{ij}(t)$, the assumptions made in section 5.2

do not apply due to the restriction on the range of values for $\partial D/\partial\dot{\chi}_\alpha$. In this section we will show that W^s is bounded if $\sigma^*_{ij}(t)$ satisfies certain conditions.

We define $X^*_\alpha(t)$ by

$$X^*_\alpha = \sigma^*_{ij}(t)\frac{\partial\varepsilon^p_{ij}}{\partial\chi_\alpha} - \frac{\partial f^p}{\partial\chi^*_\alpha} , \tag{81}$$

where χ^*_α is a constant, and we assume that for the particular history $\sigma^*_{ij}(t)$ there exists some χ^*_α such that

$$\phi(X^*_\alpha(t)) \leq 0, \quad 0 \leq t \leq T . \tag{82}$$

This assumption is satisfied if $\sigma^*_{ij}(t)$ remains in or on the limit surface for perfect plasticity or remains in or on some current yield surface for kinematic hardening plasticity.

From equation 12 we have that

$$W^s(\sigma^*_{ij}(t), \chi_\alpha(t)) = f^p(\chi_\alpha(T)) + \int_0^T \left(X_\alpha - \sigma^*_{ij}\frac{\partial\varepsilon^p_{ij}}{\partial\chi_\alpha}\right)\dot{\chi}_\alpha dt$$

$$= f^p(\chi_\alpha(T)) + \int_0^T (X_\alpha - X^*_\alpha)\dot{\chi}_\alpha dt - \int_0^T \frac{\partial f^p}{\partial\chi^*_\alpha}\dot{\chi}_\alpha dt , \tag{83}$$

where we make use of equation 81. Equation 83 becomes

$$W^s(\sigma^*_{ij}(t), \chi_\alpha(t)) = f^p(\chi_\alpha(T)) - \frac{\partial f^p}{\partial\chi^*_\alpha}\chi_\alpha(T) + \int_0^T (X_\alpha - X^*_\alpha)\dot{\chi}_\alpha dt \tag{84}$$

$$= f^p(\chi_\alpha(T)) - f^p(\chi^*_\alpha) - \frac{\partial f^p}{\partial\chi^*_\alpha}(\chi_\alpha(T) - \chi^*_\alpha) + \int_0^T (X - X^*_\alpha)\dot{\chi}_\alpha dt$$

$$+ f^p(\chi^*_\alpha) - \frac{\partial f^p}{\partial\chi^*_\alpha}\chi^*_\alpha . \tag{85}$$

Due to the convexity of $f^p(\chi_\alpha)$ and $D(\dot{\chi}_\alpha)$ (as expressed in inequality 73 of chapter 1) the first line of equation 85 is non-negative. Thus we may write

$$W^s(\sigma^*_{ij}(t), \chi_\alpha(t)) \geq f^p(\chi^*_\alpha) - \frac{\partial f^p}{\partial \chi^*_\alpha} \chi^*_\alpha \quad , \tag{86}$$

where χ^*_α is defined in equations 81 and 82.

For the special case of linear kinematic hardening

$$f^p = \frac{1}{2} \frac{\partial^2 f^p}{\partial \chi_\alpha \partial \chi_\beta} \chi_\alpha \chi_\beta \quad , \tag{87}$$

so equation 86 becomes

$$W^s(\sigma^*_{ij}(t), \chi_\alpha(t)) \geq -\frac{1}{2} \frac{\partial^2 f^p}{\partial \chi_\alpha \partial \chi_\beta} \chi^*_\alpha \chi^*_\beta \quad . \tag{88}$$

For perfect plasticity $f^p(\chi_\alpha) = 0$ and equation 82 is satisfied only if

$$\phi\left(\sigma^*_{ij}(t) \frac{\partial \varepsilon^p_{ij}}{\partial \chi_\alpha}\right) \leq 0, \quad 0 \leq t \leq T \; .$$

In this case

$$W^s(\sigma^*_{ij}(t), \chi_\alpha(t)) \geq 0 \; . \tag{89}$$

The results for linear kinematic hardening and perfect plasticity were obtained by Ponter (1975b), (see equations 4 and 6).

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Appendix A

A note on the derivation of the kinematic rate theorem of plasticity from the free energy minimum principle.

A NOTE ON THE DERIVATION OF THE KINEMATIC RATE THEOREM OF PLASTICITY FROM THE FREE ENERGY MINIMUM PRINCIPLE*

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Abstract:

Various forms of the rate or incremental minimum principles of classical plasticity have been given, being generally derived in an ad hoc manner from the equations governing the problem. In this note the kinematic theorems are considered from the point of view of the free energy minimum principle for statically loaded bodies: it is shown that various minimum principles may be derived from a single variational form.

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1. Introduction

The minimum principles for the rate or incremental problem in time independent plasticity have been the subject of study over a number of years. The principles were first established in a weak form by Prager [1], [2], and extended to smooth yield surfaces by Hodge and Prager [3]. The conventional form for smooth yield surfaces was finally given by Greenberg [4], [5]. Koiter [6] further generalised the principles to cover singular yield surfaces. Further discussions of the conventional form of the minimum principles have been given by Hill [7], Drucker [8], Koiter [9] and Hodge [10].

In this conventional form the rate (or incremental) problem is considered as a boundary value problem in which traction or displacement rates are specified on the surface S of a body of volume V. The stress rates $\dot{\sigma}_{ij}$ are required to satisfy the rate form of the equilibrium equations, and the strain rates $\dot{\epsilon}_{ij}$ and the displacement rates $\dot{u}_i$ are required to satisfy the strain rate, displacement rate relations. The constitution equations are given in terms of stress rates $\dot{\sigma}_{ij}$ and total strain rates $\dot{\epsilon}_{ij}$. These equations depend on the previous stress or strain history, and take a different form depending upon whether an element of material is elastic or plastic and unloading or plastic and loading.

An alternative approach has been presented more recently by the Italian school. This approach is based on the work of Colonetti [11], [12] who considered elastic bodies subjected to loading and imposed inelastic strains. The solution is given as the superposition of two elastic problems, one involving loading and no inelastic strains, and the other no loading and imposed inelastic strains. A rate (or incremental) form of approach can also be given.

Ceradini [13] and Maier [14] in the static and kinematic cases respectively considered what additional requirements must be imposed if the inelastic strain rates, the elastic strain rates and the stress rates must satisfy the plastic constitutive relations. This resulted in two new minimum principles of a quadratic programming form: quadratic functions of the plastic strain rates must be minimised subject to linear inequality constraints. Ceradini's theorem was derived directly from the conventional form, while Maier used quadratic programming arguments to establish the kinematic form. It was shown by Martin [15] that Maier's theorem follows from the conventional form of the kinematic theorem if use is made of a further property of the constitutive relation in the form of an inequality concerning an arbitrary division of strain rate into elastic and plastic parts. This result was further generalised by Martin [16] who gave directly a quadratic programming form of the kinematic minimum principle in which total strain rates and plastic strain rates are variables and the principle of superposition is not used.

Recently, attention has also been given to internal variable theories of plasticity which have a sound thermodynamic basis. This work suggests the problem of basing the minimum principles of the rate problem on the appropriate thermodynamic minimum principle for statically loaded bodies undergoing isothermal deformation. This does not appear to have been considered in previous work where, for example, the formal relation between the classical potential and complementary energy theorems of elasticity and the rate theorems of plasticity have not been formally explored.

It is our intention in this paper to study this relation. We shall limit ourselves to the kinematic theorems, and the starting point will be the internal variable description of a time independent plastic material under isothermal conditions and the Helmholtz free energy minimum principle for a body subjected to conservative loads. We shall demonstrate how the application of the free energy minimum principle to two adjacent states of loading gives an equilibrium condition which may in turn be interpreted as a form of Colonetti's principle in incremental form, as the conventional kinematic minimum principle in terms of total strains, and the extended minimum principle in terms of total strains and internal variables.

This formulation was attempted by Martin [17] for a truss problem. However, the formulation was not completely correct. The present paper rectifies the argument and generalises it to the continuum case.

2. Constitutive Relations

We consider a time-independent plastic material subject to isothermal small deformations. The internal variable model of Kestin and Rice [18] and Rice [19] is adopted. Since the temperature T remains constant it will not be referred to in the following discussion. The remaining thermodynamic state variables are the macroscopic strain ϵ_{ij} and the internal variables χ_α $(\alpha = 1, \ldots, n)$. The fundamental equation is taken in the form

$$f = f(\epsilon_{ij}, \chi_\alpha) \tag{1}$$

where f is the Helmholtz free energy per unit volume.

Small changes in f as a result of changes in ϵ_{ij} and χ_α are given by

$$df = \sigma_{ij}\, d\epsilon_{ij} - X_\alpha d\chi_\alpha, \tag{2}$$

where σ_{ij} is the stress tensor and X_α are the internal forces conjugate to the internal variables. Comparison of equations (1) and (2) gives the equations of state

$$\sigma_{ij} = \frac{\partial f}{\partial \epsilon_{ij}}, \quad X_\alpha = -\frac{\partial f}{\partial \chi_\alpha}. \tag{3}$$

To complete the description of the mechanical behaviour of the material a kinetic relation between the internal variables and the internal forces must be introduced.

For time-independent plasticity we introduce a relation of the form

$$\dot{\chi}_\alpha = \lambda \frac{\partial \phi}{\partial X_\alpha} \tag{4}$$

where $\phi(X_\alpha)$ is a continuously differentiable yield function. Internal forces such that $\phi(X_\alpha) > 0$ are not admitted, and

$$\begin{aligned} \lambda &= 0 \quad \text{if} \quad \phi < 0 \\ &\qquad \text{or} \quad \phi = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial X_\alpha}\dot{X}_\alpha < 0, \\ \lambda &\geq 0 \quad \text{if} \quad \phi = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial X_\alpha}\dot{X}_\alpha = 0. \end{aligned} \tag{5}$$

perfectly plastic

More general yield functions may be characterised by a number of yield functions, $\phi_1, \phi_2, \ldots, \phi_m$, with

$$\dot{\chi}_\alpha = \lambda_1 \frac{\partial \phi_1}{\partial X_\alpha} + \lambda_2 \frac{\partial \phi_2}{\partial X_\alpha} + \ldots + \lambda_m \frac{\partial \phi_m}{\partial X_\alpha}. \tag{6}$$

Each λ_k, ϕ_k is subjected to equation (5). However, we shall carry out our argument in terms of a single yield function, since a generalisation to the kinetic equation to equation (6) is readily effected.

Conventional time-independent plastic materials are subjected to stability restrictions, usually in the sense of Drucker [20]. It is sufficient for our purposes (see, for example, Martin [21]) that we assume that $f(\epsilon_{ij}, \chi_\alpha)$ is a homogeneous quadratic convex function, and that $\phi(X_\alpha)$ is convex and such that $\phi(X_\alpha = 0) < 0$. These assumptions ensure Drucker stability, and provide a model in which the elastic behaviour is linear.

We shall carry out our argument in <u>incremental</u> rather than rate terms. It is convenient, therefore, to rephrase equations (4) and (5) in terms of an infinitesimal increment in the internal variables, $\Delta\chi_\alpha$;

$$\Delta\chi_\alpha = \Lambda \frac{\partial\phi}{\partial X_\alpha} , \tag{7}$$

where $\Lambda = 0$ if $\phi < 0$

or $\phi = 0$ and $\dfrac{\partial\phi}{\partial X_\alpha} \Delta X_\alpha < 0$,

$\Lambda \geq 0$ if $\phi = 0$ and $\dfrac{\partial\phi}{\partial X_\alpha} \Delta X_\alpha = 0$,

The multiplier Λ may be determined explicitly in terms of the change in strain $\Delta\epsilon_{ij}$. From equations (3) and (7)

$$\Delta X_\alpha = - \left\{ \frac{\partial^2 f}{\partial\chi_\alpha \, \partial\epsilon_{ij}} \Delta\epsilon_{ij} + \frac{\partial^2 f}{\partial\chi_\alpha \, \partial\chi_\beta} \Delta\chi_\beta \right\}$$

$$= - \left\{ \frac{\partial^2 f}{\partial\chi_\alpha \, \partial\epsilon_{ij}} \Delta\epsilon_{ij} + \Lambda \frac{\partial^2 f}{\partial\chi_\alpha \, \partial\chi_\beta} \frac{\partial\phi}{\partial X_\beta} \right\} . \tag{8}$$

Thus the condition $\Delta X_\alpha (\partial\phi/\partial X_\alpha) = 0$ becomes

$$\frac{\partial\phi}{\partial X_\alpha} \left\{ \frac{\partial^2 f}{\partial X_\alpha \, \partial\epsilon_{ij}} \Delta\epsilon_{ij} + \Lambda \frac{\partial^2 f}{\partial X_\alpha \, \partial X_\beta} \frac{\partial\phi}{\partial X_\beta} \right\} = 0 \tag{9}$$

Solving for Λ, bearing in mind that $\Lambda \geq 0$, we find

$$\Lambda = -\frac{\dfrac{\partial^2 f}{\partial\epsilon_{ij}\,\partial X_\alpha}\dfrac{\partial\phi}{\partial X_\alpha}\Delta\epsilon_{ij}}{\dfrac{\partial^2 f}{\partial X_\alpha\,\partial X_\beta}\dfrac{\partial\phi}{\partial X_\alpha}\dfrac{\partial\phi}{\partial X_\beta}} \quad \text{if} \quad \frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial X_\alpha}\frac{\partial\phi}{\partial X_\alpha}\Delta\epsilon_{ij} \leq 0, \tag{10a}$$

$$\Lambda = 0 \quad \text{if} \quad \frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial X_\alpha}\frac{\partial\phi}{\partial X_\alpha}\Delta\epsilon_{ij} \geq 0. \tag{10b}$$

This follows from our assumption that f is convex, in which case

$$\frac{\partial^2 f}{\partial X_\alpha\,\partial X_\beta}\frac{\partial\phi}{\partial X_\alpha}\frac{\partial\phi}{\partial X_\beta} > 0. \tag{11}$$

We may also observe that equations (10) can be obtained from the condition that

$$H(\Lambda) = \Lambda\frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial X_\alpha}\frac{\partial\phi}{\partial X_\alpha}\Delta\epsilon_{ij} + \tfrac{1}{2}\Lambda^2\frac{\partial^2 f}{\partial X_\alpha\,\partial X_\beta}\frac{\partial\phi}{\partial X_\alpha}\frac{\partial\phi}{\partial X_\beta} \tag{12a}$$

ahould be a minimum with respect to Λ, subject to

$$\Lambda \geq 0. \tag{12b}$$

This follows directly by setting $\partial H/\partial\Lambda = 0$, and solving for Λ. If

$$\frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial X_\alpha}\frac{\partial\phi}{\partial X_\alpha}\Delta\epsilon_{ij} \leq 0 \tag{13}$$

the least value of H is given by equation (10a), satisfying the constant $\Lambda \geq 0$. If, however,

$$\frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial X_\alpha}\frac{\partial\phi}{\partial X_\alpha}\Delta\epsilon_{ij} \geq 0, \tag{14}$$

the least value of H, subject to $\Lambda \geq 0$, is given by $\Lambda = 0$.

3. Equilibrium of a Loaded Body

Consider a body of volume V and surface S, consisting of a material described by the constitutive equations given in the previous section, and subject to small isothermal deformations. The displacement of a point in the body is characterised by $u_i(x_1, x_2, x_3)$, where $x_i (i = 1, 2, 3)$ are the coordinates of the point in a Cartesian coordinate system. Following the assumption of small displacements, the strain-displacement relations take the form

$$\epsilon_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \qquad (15)$$

Let us assume that at time t the body is subjected to conservative body forces $\hat{F}_i$ on V, conservative traction $\hat{P}_i$ on part of the surface S_p and displacements $\hat{u}_i$ on the remainder of the surface S_u. Since the history of loading has not been specified, we do not have sufficient information to determine the strains $\epsilon^*_{ij}(x_i) = \epsilon_{ij}(x_i, t)$, the displacements $u^*_i(x_i) = u_i(x_i, t)$, the internal variables $\chi^*_\alpha(x_i) = \chi_\alpha(x_i, t)$, the stresses $\sigma^*_{ij}(x_i) = \sigma_{ij}(x_i, t)$ and the internal forces $X^*_\alpha(x_i) = X_\alpha(x_i, t)$ throughout the body. The body is in a state of constrained equilibrium, with $\phi(X^*_\alpha) \leq 0$ at each point.

However, we can recover certain information about this constrained equilibrium state. Suppose that we regard the χ^*_α as fixed, and consider variations $\Delta\epsilon_{ij}$, Δu_i in the strain and displacement fields while ignoring the constraints of the kinetic equation (7). The body must be in unconstrained equilibrium with respect to such variations.

To formalise this result we consider the Helmholtz free energy of the body and the conservative loads; this is

$$U = \int_V f(\epsilon^*_{ij}, \chi^*_\alpha)\,dV - \int_V \hat{F}_i\, u^*_i\, dV - \int_{S_p} \hat{P}_i\, u^*_i\, dS. \qquad (16)$$

Consider variations in U with respect to variations $\Delta\epsilon_{ij}$ in ϵ^*_{ij}, Δu_i in u_i, subject to equation (15), with χ^*_α held fixed. Then

$$\Delta U = \int_V \sigma^*_{ij}\, \Delta\epsilon_{ij}\, dV - \int_V \hat{F}_i\, \Delta u_i\, dV - \int_{S_p} \hat{P}_i\, \Delta u_i\, dS, \qquad (17)$$

where $$\sigma^*_{ij} = \left.\frac{\partial f}{\partial \epsilon_{ij}}\right|_{\epsilon^*_{ij},\, \chi^*_\alpha}. \qquad (18)$$

Using Gauss' Theorem, we show that $\Delta U = 0$ if and only if

$$\frac{\partial \sigma^*_{ij}}{\partial x_j} + \hat{F}_i = 0 \quad \text{on } V, \tag{19a}$$

$$\sigma^*_{ij}\,\nu_j = \hat{P}_i \quad \text{on } S_p, \tag{19b}$$

where ν_j is the unit outward normal vector on S.

These equations are the equilibrium relations for small deformations, and must indeed be satisfied for any loading. Hence U is stationary, and may readily be shown to take its least value with respect to unconstrained variations in the strains and displacements with the internal variables held constant. The result was obtained by Colonetti [11], [12].

4. <u>The Kinematic Incremental Theorems</u>

Let us now suppose that the loads on the body are changed infinitesimally, so that at time $t + \Delta t$ we have body forces $\hat{F}_i + \Delta F_i$ on V, tractions $\hat{P}_i + \Delta P_i$ on S_p and displacements $\hat{u}_i + \Delta u_i$ on S_u. Let us further suppose that the solution is given by strains $\epsilon^*_{ij} + \Delta\epsilon_{ij}$, displacements $u^*_i + \Delta u_i$, internal variables $\chi^*_\alpha + \Delta\chi_\alpha$, stresses $\sigma^*_{ij} + \Delta\sigma_{ij}$ and internal forces $X^*_\alpha + \Delta X_\alpha$.

Evidently, we may apply the result given in the previous section to this new loading state and assert that

$$\tilde{U} = \int f(\epsilon^*_{ij} + \Delta\epsilon_{ij}, \chi^*_\alpha + \Delta\chi_\alpha)\, dV - \int (\hat{F}_i + \Delta F_i)(u^*_i + \Delta u_i) dV$$
$$- \int_{S_p} (\hat{P}_i + \Delta P_i)(u^*_i + \Delta u_i) dS \tag{20}$$

must be stationary with respect to unconstrained variations in the strain and displacement fields, subject to equation (15) in incremental form, with the internal variables held constant. We further note that, to second order, we may put

$$f(\epsilon^*_{ij} + \Delta\epsilon_{ij}, \chi^*_\alpha + \Delta\chi_\alpha) = f(\epsilon^*_{ij}, \chi^*_\alpha) + \frac{\partial f}{\partial \epsilon_{ij}}\Delta\epsilon_{ij} + \frac{\partial f}{\partial \chi_\alpha}\Delta\chi_\alpha$$
$$+ \tfrac{1}{2}\frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\epsilon_{kl}}\Delta\epsilon_{ij}\,\Delta\epsilon_{kl} + \frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\chi_\alpha}\Delta\epsilon_{ij}\,\Delta\chi_\alpha$$
$$+ \tfrac{1}{2}\frac{\partial^2 f}{\partial\chi_\alpha\,\partial\chi_\beta}\Delta\chi_\alpha\,\Delta\chi_\beta. \tag{21}$$

All derivatives are evaluated at ϵ^*_{ij}, χ^*_α.

Since variations in $\epsilon^*_{ij} + \Delta\epsilon_{ij}$, $u^*_i + \Delta u_i$ may be treated as variations in $\Delta\epsilon_{ij}$, Δu_i, and denoted by $\delta\epsilon_{ij}$, δu_i, we see that

$$\delta\mho = \left\{ \int_V \sigma^*_{ij}\, \delta\epsilon_{ij}\, dV - \int_V \hat{F}_i\, \delta u_i - \int_{S_p} \hat{P}_i\, \delta u_i \right\}$$

$$+ \left\{ \int_V \frac{\partial^2 f}{\partial\epsilon_{ij}\, \partial\epsilon_{kl}} \Delta\epsilon_{ij}\, \delta\epsilon_{kl}\, dV + \int_V \frac{\partial^2 f}{\partial\epsilon_{ij}\, \epsilon\chi_\alpha} \Delta\chi_\alpha\, \delta\epsilon_{ij}\, dV \right.$$

$$\left. - \int_V \Delta F_i\, \delta u_i\, dV - \int_{S_p} \Delta P_i\, \delta u_i\, dS \right\} = 0 \qquad (22)$$

The first set of terms within parentheses vanishes as a result of the equilibrium requirements at time t i.e. as a result of the stationarity of the functional given in equation (17). That variations of the second set of terms vanish implies that

$$V = \int_V \tfrac{1}{2} \frac{\partial^2 f}{\partial\epsilon_{ij}\, \partial\epsilon_{kl}} \Delta\epsilon_{ij}\, \Delta\epsilon_{kl}\, dV + \int_V \frac{\partial^2 f}{\partial\epsilon_{ij}\, \partial\chi_\alpha} \Delta\epsilon_{ij}\, \Delta\chi_\alpha\, dV - \int_V \Delta F_i\, \Delta u_i\, dV$$

$$- \int_{S_p} \Delta P_i\, \Delta u_i\, ds \qquad (23)$$

is stationary with respect to variations in $\Delta\epsilon_{ij}$, Δu_i with $\Delta\chi_\alpha$ held constant. We may readily establish that V takes its least value when it is stationary. Since

$$\Delta\sigma_{ij} = \left.\frac{\partial^2 f}{\partial\epsilon_{ij}\, \partial\epsilon_{kl}}\right|_{\epsilon^*_{ij},\, \chi^*_\alpha} \Delta\epsilon_{kl} + \left.\frac{\partial^2 f}{\partial\epsilon_{ij}\, \partial\chi_\alpha}\right|_{\epsilon_{ij},\, \chi_\alpha} \Delta\chi_\alpha, \qquad (24)$$

it may further be established that V is stationary if and only if

$$\begin{aligned} \frac{\partial\Delta\sigma_{ij}}{\partial x_j} + \Delta F_i &= 0 \quad \text{on } V, \\ \Delta\sigma_{ij}\, \nu_j &= \Delta P_i \quad \text{on } S_p \end{aligned} \qquad (25)$$

These equations are the incremental equilibrium relations.

The changes in the internal variables $\Delta\chi_\alpha$, of course, are not known <u>a priori</u>, and hence equations (25) do not provide sufficient information to solve the incremental problem. However, we may observe from equations (7) and (10) that the changes in the internal variables $\Delta\chi_\alpha$ may be regarded as functions of the changes in the strains $\Delta\epsilon_{ij}$. From the second term in parentheses in equation (22), and equation (10), we see that

$$\int_V \left\{ \frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\epsilon_{kl}} \Delta\epsilon_{kl} - \left[\frac{\left(\dfrac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\chi_\alpha} \dfrac{\partial\phi}{\partial\chi_\alpha} \right) \left(\dfrac{\partial^2 f}{\partial\epsilon_{kl}\,\partial\chi_\beta} \dfrac{\partial\phi}{\partial\chi_\beta} \Delta\epsilon_{kl} \right)}{\dfrac{\partial^2 f}{\partial\chi_\alpha\,\partial\chi_\beta} \dfrac{\partial\phi}{\partial\chi_\alpha} \dfrac{\partial\phi}{\partial\chi_\beta}} \right] \right\} \delta\epsilon_{ij}\, dV$$

$$- \int \Delta F_i\, \delta u_i\, dV - \int \Delta P_i\, \delta u_i\, dV = 0, \qquad (26)$$

where the term in square brackets is included only if

$$\phi = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\chi_\alpha} \frac{\partial\phi}{\partial\chi_\alpha} \Delta\epsilon_{ij} < 0. \qquad (27)$$

We may readily establish that equations (26) and (27) imply that the incremental solution is given by the least value of

$$\tilde{V} = \int_V W^o(\Delta\epsilon_{ij})dV - \int_V \Delta F_i\, \Delta u_i\, dV - \int_{S_p} \Delta P_i\, \Delta u_i\, dS \qquad (28)$$

with respect to changes in $\Delta\epsilon_{ij}$, Δu_i where

$$W^o = \tfrac{1}{2} \frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\epsilon_{kl}} \Delta\epsilon_{ij}\, \Delta\epsilon_{kl} - \tfrac{1}{2} \frac{\left(\dfrac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\chi_\alpha} \dfrac{\partial\phi}{\partial\chi_\alpha} \Delta\epsilon_{ij} \right)^2}{\left(\dfrac{\partial^2 f}{\partial\chi_\alpha\,\partial\chi_\beta} \dfrac{\partial\phi}{\partial\chi_\alpha} \dfrac{\partial\phi}{\partial\chi_\beta} \right)} \qquad (29a)$$

when $\phi = 0$ and $\dfrac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\chi_\alpha} \dfrac{\partial\phi}{\partial\chi_\alpha} \Delta\epsilon_{ij} < 0$

and

$$W^o = \tfrac{1}{2} \frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\epsilon_{kl}} \Delta\epsilon_{ij}\, \Delta\epsilon_{kl}$$

when $\phi = 0$ and $\dfrac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\chi_\alpha} \dfrac{\partial\phi}{\partial\chi_\alpha} \Delta\epsilon_{ij} \geq 0$, (29b)

or $\phi < 0$.

Alternatively, using equations (24) and (10),

$$W^o = \tfrac{1}{2}\,\Delta\sigma_{ij}\,\Delta\epsilon_{ij}, \tag{30}$$

where $\Delta\sigma_{ij}$ is the stress increment associated with the strain increment $\Delta\epsilon_{ij}$ through the constitutive relations.

This result is the classical kinematic incremental theorem, given in the form of Greenberg [4], [5].

If we now make use of equations (12), we may note that

$$W^o = \tfrac{1}{2}\,\frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\epsilon_{kl}}\,\Delta\epsilon_{ij}\,\Delta\epsilon_{kl} + \min\,\{H(\Lambda)\} \tag{31}$$

subject to $\Lambda \geq 0$. Alternatively

$$\begin{aligned} W^o \leq \overline{W}^o &= \tfrac{1}{2}\,\frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\epsilon_{kl}}\,\Delta\epsilon_{ij}\,\Delta\epsilon_{kl} + H(\Lambda) \\ &= \tfrac{1}{2}\,\frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial\epsilon_{kl}}\,\Delta\epsilon_{ij}\,\Delta\epsilon_{kl} + \Lambda\,\frac{\partial^2 f}{\partial\epsilon_{ij}\,\partial X_\alpha}\,\frac{\partial\phi}{\partial X_\alpha}\,\Delta\epsilon_{ij} \\ &\quad + \frac{\Lambda^2}{2}\,\frac{\partial^2 f}{\partial X_\alpha\,\partial X_\beta}\,\frac{\partial\phi}{\partial X_\alpha}\,\frac{\partial\phi}{\partial X_\beta} \end{aligned} \tag{32}$$

subject to $\Lambda \geq 0$. $\overline{W}^o = W^o$ when Λ takes its correct value for given $\Delta\epsilon_{ij}$.

It follows then that for any given field $\Delta\epsilon_{ij}(x_i)$

$$\overline{V} \geq \tilde{V}, \tag{33}$$

where

$$\overline{V} = \int_V \overline{W}^o(\Delta\epsilon_{ij}, \Lambda)\,dV - \int_V \Delta F_i\,\Delta u_i\,dV - \int_{S_p} \Delta P_i\,\Delta u_i\,dS \tag{34}$$

and $\tilde{V}$ is given by equation (28). $\overline{V} = \tilde{V}$ if we minimise $\overline{W}^o$ with respect to Λ in a pointwise fashion throughout the body. We may then further assert that the incremental solution is given by the least value of $\overline{V}$ with respect to the fields $\Delta\epsilon_{ij}$, Δu_i subject to equation (15) and with respect to Λ subject to $\Lambda \leq 0$. This is the extended minimum principle given by Martin [16].

5. Conclusions

The incremental theorems discussed above may be reduced to rate theorems. All terms in the functionals which are minimised are homogeneous and of degree two in the increments of strain and internal variables, and hence may be divided by $(\Delta t)^2$. In the limit as $\Delta t \to 0$, we recover the rate minimum principles in terms of $\dot{\varepsilon}_{ij}$, $\dot{\chi}_{\alpha}$.

It has been demonstrated, therefore, that the minimum principles for the rate or incremental problem in the classical and extended form may be derived directly from the minimum principle given by Colonetti [12], which in turn is derived from an application of the Helmholtz free energy minimum principle to the constrained equilibrium state achieved by time-independent plastic materials undergoing isothermal deformation.

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Appendix B

Work bounding functions for plastic materials.

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Work Bounding Functions for Plastic Materials[1]

The minimum work and maximum complementary work potentials for both time-independent and time-dependent plasticity are reconsidered from the viewpoint of internal variable theories. It is shown that the minimum work and maximum complementary work can be bounded in a simple and direct manner. The bounds provide the minimum work and maximum complementary work under certain limitations.

Introduction

The development of bounding theorems in plasticity (see, for example, Martin [1, 2],[2] Hodge [3], Maier [4, 5]) introduced the problem of determining bounds on work and complementary work for deformation along strain and stress paths where only the initial and terminal values are known. The bounding problem can be precisely defined as follows, for both time-independent and time-dependent plasticity.

Consider a homogeneously strained element of material of unit volume. Small, isothermal deformations are considered, and the strain and conjugate stress tensors are denoted by ϵ_{ij} and σ_{ij}, respectively. The element is subjected to some (unspecified) strain history $\epsilon_{ij}(t)$, $0 \le t \le T$, subject to an unstrained (and unstressed) initial state $\epsilon_{ij}(0) = 0$ and a given terminal strain $\epsilon_{ij}(T) = \bar{\epsilon}_{ij}$. The associated stress history is $\sigma_{ij}(t)$, with $\sigma_{ij}(0) = 0$. The work done in deforming the material element in the period $0 \le t \le T$ is

$$W(\bar{\epsilon}_{ij}) = \int_0^T \sigma_{ij}(t)\dot{\epsilon}_{ij}(t)dt. \tag{1}$$

We seek a *work bounding function* $\tilde{W}(\bar{\epsilon}_{ij})$ such that

$$\tilde{W}(\bar{\epsilon}_{ij}) \le W(\bar{\epsilon}_{ij}) \tag{2}$$

for any choice of $\bar{\epsilon}_{ij} = \epsilon_{ij}(T)$ and any choice of $\epsilon_{ij}(t)$.

Similarly, we may impose a stress path $\sigma_{ij}(t)$, with $\sigma_{ij}(0) = 0$, $\sigma_{ij}(T) = \bar{\sigma}_{ij}$, with an associated strain path $\epsilon_{ij}(t)$. The complementary work done in the period $0 \le t \le T$ is

$$\Omega(\bar{\sigma}_{ij}) = \int_0^T \epsilon_{ij}(t)\dot{\sigma}_{ij}(t)dt. \tag{3}$$

We seek a *complementary work bounding function* $\tilde{\Omega}(\bar{\sigma}_{ij})$ such that

$$\tilde{\Omega}(\bar{\sigma}_{ij}) \ge \Omega(\bar{\sigma}_{ij}) \tag{4}$$

for any choice of $\bar{\sigma}_{ij} = \sigma_{ij}(T)$ and any choice of $\sigma_{ij}(t)$.

Work bounding functions for several models of materials obeying specific constitutive equations have been derived (for example, Martin [1, 2], Hodge [3], Martin and Ponter [6], Maier [4], Soechting and Lance [7]). The problem has also been discussed in a general context by Ponter [8, 9] and Ponter and Martin [10]. In the latter approach the concepts of a minimum work path and a maximum complementary work path for given terminal strain and stress, respectively, were introduced. We then define the *minimum work function* $\bar{W}(\bar{\epsilon}_{ij})$ as the work done along the minimum work path, so that

$$\bar{W}(\bar{\epsilon}_{ij}) = \min\left\{W = \int_0^T \sigma_{ij}\dot{\epsilon}_{ij}dt : \epsilon_{ij}(0) = 0, \epsilon_{ij}(T) = \bar{\epsilon}_{ij}\right\} \tag{5}$$

Similarly the maximum complementary work function $\bar{\Omega}(\bar{\sigma}_{ij})$ is the work done along the maximum complementary work path, so that

$$\bar{\Omega}(\bar{\sigma}_{ij}) = \max\left\{\Omega = \int_0^T \epsilon_{ij}\dot{\sigma}_{ij}dt : \sigma_{ij}(0) = 0, \sigma_{ij}(T) = \bar{\sigma}_{ij}\right\} \tag{6}$$

Evidently

$$\bar{W}(\bar{\epsilon}_{ij}) \le W(\bar{\epsilon}_{ij}), \qquad \bar{\Omega}(\bar{\sigma}_{ij}) \ge \Omega(\bar{\sigma}_{ij}), \tag{7}$$

and the minimum work and maximum complementary work functions are work and complementary work bounding functions, respectively; indeed, they are the optimal choices for the bounding functions.

On the assumption that the material is stable in the sense of Drucker [11] (see [8–10] for details), several interesting properties of $\bar{W}$ and $\bar{\Omega}$ were found. The functions are both convex, and are potential functions in the sense that the derivative of $\bar{W}$ with respect to strain gives the terminal stress for the minimum work path, and the derivative of $\bar{\Omega}$ with respect to stress gives the terminal strain for the maximum complementary work path. Further, the minimum work

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[2] Numbers in brackets designate References at end of paper.

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path maps a path in stress space which is the maximum complementary work path for that terminal stress. These results indicate that $\bar{W}$ and $\bar{\Omega}$ can be considered as the strain energy and complementary energy of a hypothetical, stable elastic material which bears a special relation to the plastic material.

In this paper we present an alternative approach to the determination of bounding functions, based on an internal variable model of plasticity. We shall derive bounding functions for a fairly broad class of materials, and show that these bounding functions also possess the properties outlined in the previous paragraph. Under certain circumstances the bounding functions are the minimum work and maximum complementary work functions.

Internal Variable Model

The model of the constitutive equations is based on the form given by Kestin and Rice [12] and Rice [13]. Consider an element of material of unit volume subjected to a homogeneous macroscopic strain ϵ_{ij} and a uniform temperature θ. Changes in density are neglected. The work done by external agencies during a small change in strain $d\epsilon_{ij}$ is

$$dW = \sigma_{ij} d\epsilon_{ij}, \tag{8}$$

where σ_{ij} is the stress tensor.

The description of the material behavior is derived from a thermodynamic fundamental equation

$$f = f(\theta, \epsilon_{ij}, \xi_\alpha), \tag{9}$$

where f is the Helmholtz free energy per unit volume and $\xi_\alpha(\alpha = 1, \ldots\ldots, n)$ are internal variables. The equations of state are

$$s = -\frac{\partial f}{\partial \theta}, \qquad \sigma_{ij} = \frac{\partial f}{\partial \epsilon_{ij}}, \qquad X_\alpha = -\frac{\partial f}{\partial \xi_\alpha}, \tag{10}$$

where s is the entropy per unit volume and X_α are the internal forces conjugate to the internal variables. The reversible work occurring as a result of small changes in ϵ_{ij}, ξ_α under isothermal conditions is

$$dW^0 = df = \sigma_{ij} d\epsilon_{ij} - X_\alpha d\xi_\alpha \tag{11}$$

The entropy production rate

$$\dot{\zeta} = \frac{\dot{W} - \dot{W}^0}{\theta} = \frac{X_\alpha \dot{\xi}_\alpha}{\theta} \tag{12}$$

is required to be nonnegative.

The internal variable model is completed by the addition of kinetic equations governing the rate of change of ξ_α. We shall suppose that these equations take the form

$$X_\alpha = \frac{1}{(p+1)} \frac{\partial D}{\partial \dot{\xi}_\alpha}, \tag{13}$$

where $D = D(\dot{\xi}_\alpha)$ is homogeneous and of degree $(p + 1)$ in the components of $\dot{\xi}_\alpha$. The index p will be taken to be the reciprocal of an odd integer, so that $1 \geq p \geq 0$.

Equations (10) and (13) describe a fairly wide class of materials, including linear viscoelasticity ($p = 1$), nonlinear creep ($0 < p < 1$) and time-independent plasticity ($p = 0$). It should be noted that in the limiting case $p = 0$ the discontinuous nature of time-independent plasticity is apparent in the observation that $\partial D/\partial \dot{\xi}_\alpha$ is discontinuous at $\dot{\xi} = 0$. The values of $X_\alpha = \partial D/\partial \dot{\xi}_\alpha$ (which is homogeneous and of degree zero in the components of $\dot{\xi}_\alpha$) for $\dot{\xi}_\alpha \neq 0$ form a hypersurface in the X_α space which may be interpreted as a limit surface. When $\dot{\xi}_\alpha = 0$, X_α may take any value which lies within or on this limit surface. The conventional *yield surface* of time-independent plasticity is obtained by mapping this limit surface into the stress space for given values of ξ_α (see equations (37) for a direct relation between σ_{ij}, X_α, ξ_α).

Since $D(\dot{\xi}_\alpha)$ is homogeneous and of degree $(p + 1)$ in $\dot{\xi}_\alpha$, we see that

$$X_\alpha \dot{\xi}_\alpha = \frac{1}{(p+1)} \frac{\partial D}{\partial \dot{\xi}_\alpha} \dot{\xi}_\alpha = D(\dot{\xi}_\alpha). \tag{14}$$

The thermodynamic restriction on the entropy production rate (equation (12)) requires therefore that D should be nonnegative, vanishing if and only if $\dot{\xi}_\alpha = 0$. We shall further assume that both f and D are convex functions of their respective arguments. These are sufficient conditions to ensure stability in the thermodynamic sense (Martin [14]) and in the sense of Drucker [11]. While it cannot be shown that the conditions are necessary for stability in the sense of Drucker, it is apparent that convexity of f and D is not unduly restrictive in the sense that it limits the model to a greater extent than Drucker's postulates.

The Work Bounding Function

We assume that at time $t = 0$ the material element is undeformed, such that $\epsilon_{ij}(0) = 0$, $\xi_\alpha(0) = 0$. Consider isothermal quasi-static deformation denoted by the strain path $\epsilon_{ij}(t)$ over the time period $0 \leq t \leq T$. The work done by external agencies as the element is deformed along this path is

$$W = \int_0^T \sigma_{ij}(t) \dot{\epsilon}_{ij}(t) dt \tag{15}$$

We introduce a terminal strain constraint $\epsilon_{ij}(T) = \bar{\epsilon}_{ij}$, and seek a lower bound on W. From equations (8) and (11) we see that

$$dW = dW^0 + X_\alpha d\xi_\alpha \tag{16a}$$

and hence

$$W = f\{\bar{\epsilon}_{ij}, \xi_\alpha(T)\} + \int_0^T X_\alpha(t) \dot{\xi}_\alpha(t) dt. \tag{16b}$$

The internal variable history $\xi_\alpha(t)$ and the terminal value $\xi_\alpha(T)$ are not known.

We choose to bound W in two steps. First, we adopt an arbitrary terminal value $\hat{\xi}_\alpha = \xi_\alpha(T)$, and seek the internal variable history $\xi_\alpha(t)$, subject to $\xi_\alpha(0) = 0$, $\xi_\alpha(T) = \hat{\xi}_\alpha$, which gives the least value of

$$I = \int_0^T X_\alpha(t) \dot{\xi}_\alpha(t) dt. \tag{17}$$

We then determine the appropriate value of $\hat{\xi}_\alpha$, say $\bar{\xi}_\alpha$, which minimises

$$\hat{W} = f\{\bar{\epsilon}_{ij}, \hat{\xi}_\alpha\} + I_{\min}\{\hat{\xi}_\alpha\}. \tag{18}$$

Consider two internal variable paths $\xi_\alpha^{(1)}(t)$, $\xi_\alpha^{(2)}(t)$, both subject to $\xi_\alpha(0) = 0$, $\xi_\alpha(T) = \hat{\xi}_\alpha$. From the convexity of D, at any instant t such that $0 \leq t \leq T$ we may write

$$\begin{aligned} D(\dot{\xi}_\alpha^{(1)}) - D(\dot{\xi}_\alpha^{(2)}) &\geq (\dot{\xi}_\alpha^{(1)} - \dot{\xi}_\alpha^{(2)}) \left.\frac{\partial D}{\partial \dot{\xi}_\alpha}\right|_{\dot{\xi}_\alpha^{(2)}} \\ &= (p+1)(\dot{\xi}_\alpha^{(1)} - \dot{\xi}_\alpha^{(2)}) X_\alpha^{(2)}. \end{aligned} \tag{19}$$

Integrating over the interval $[0, T]$,

$$\int_0^T D(\dot{\xi}_\alpha^{(1)}) dt - \int_0^T D(\dot{\xi}_\alpha^{(2)}) dt \geq (p+1) \int_0^T (\dot{\xi}_\alpha^{(1)} - \dot{\xi}_\alpha^{(2)}) X_\alpha^{(2)} dt. \tag{20a}$$

If we choose $\dot{\xi}_\alpha^{(2)}(t)$ in such a way that $X_\alpha^{(2)}(t)$ is *constant*, it is evident that

$$\int_0^T (\dot{\xi}_\alpha^{(1)} - \dot{\xi}_\alpha^{(2)}) X_\alpha^{(2)} dt = 0, \tag{20b}$$

and

$$\int_0^T D(\dot{\xi}_\alpha^{(2)}) dt \leq \int_0^T D(\dot{\xi}_\alpha^{(1)}) dt. \tag{20c}$$

Further, from equation (13), a sufficient condition that $X_\alpha^{(2)}$ is constant is that

$$\dot{\xi}_\alpha^{(2)} = \frac{\xi_\alpha(T)}{T} = \frac{\hat{\xi}_\alpha}{T} \tag{21}$$

and is constant. This holds also for the case $p = 0$.

Consequently a *sufficient* condition that I should be a minimum is that we choose the path given by equation (21), in which case, using the homogeneity of D,

$$I_{\min} = \int_0^T X_\alpha^{(2)}\dot{\xi}_\alpha^{(2)}dt = \left(\frac{1}{p+1}\right)\int_0^T \frac{\partial D}{\partial \dot{\xi}_\alpha}\bigg|_{(\hat{\xi}_\alpha/T)} \left(\frac{\hat{\xi}_\alpha}{T}\right) dt$$

$$= TD\left\{\frac{\hat{\xi}_\alpha}{T}\right\}$$

$$= \frac{1}{T^p} D\{\hat{\xi}_\alpha\}. \tag{22}$$

In the case where $p \neq 0$, the argument may be extended to show that X_α and $\dot{\xi}_\alpha$ constant are also necessary for I to be a minimum; however, we do not need to make use of the result.

After substituting equation (22) into equation (18), we now consider

$$\hat{W} = f(\bar{\epsilon}_{ij}, \hat{\xi}_\alpha) + \frac{1}{T^p} D\{\hat{\xi}_\alpha\}. \tag{23}$$

We now seek the value of $\hat{\xi}_\alpha$, say $\bar{\xi}_\alpha$, for which $\hat{W}$ has its least value. For $p \neq 0$, we proceed by variational methods. For arbitrary variations in $\hat{\xi}_\alpha$,

$$\delta\hat{W} = \frac{\partial f}{\partial \xi_\alpha}\bigg|_{\bar{\epsilon}_{ij},\hat{\xi}_\alpha} \delta\hat{\xi}_\alpha + \frac{1}{T^p}\frac{\partial D(\xi_\alpha)}{\partial \xi_\alpha}\bigg|_{\hat{\xi}_\alpha} \delta\hat{\xi}_\alpha$$

$$= \left\{\frac{\partial f}{\partial \xi_\alpha}\bigg|_{\bar{\epsilon}_{ij},\hat{\xi}_\alpha} + \frac{1}{T^p}\frac{\partial D(\xi_\alpha)}{\partial \xi_\alpha}\bigg|_{\hat{\xi}_\alpha}\right\} \delta\hat{\xi}_\alpha \tag{24}$$

Thus $\hat{W}$ is stationary when the $\xi_\alpha(T) = \hat{\xi}_\alpha = \bar{\xi}_\alpha$, where $\bar{\xi}_\alpha$ is given by the solution of the equation

$$-\frac{\partial f}{\partial \xi_\alpha}\bigg|_{\bar{\epsilon}_{ij},\bar{\xi}_\alpha} = \frac{1}{T^p}\frac{\partial D(\xi_\alpha)}{\partial \xi_\alpha}\bigg|_{\bar{\xi}_\alpha}. \tag{25}$$

Convexity of both f and D is sufficient condition that $\hat{W}$ will have a global minimum at the stationary value.

We may now define

$$\bar{W}(\bar{\epsilon}_{ij}) = f(\bar{\epsilon}_{ij}, \bar{\xi}) + \frac{1}{T^p} D(\bar{\xi}_\alpha), \tag{26}$$

and we are assured that provided $\bar{\xi}_\alpha$ is chosen according to equation (25),

$$\bar{W}(\bar{\epsilon}_{ij}) \leq W = \int_0^T \sigma_{ij}(t)\dot{\epsilon}_{ij}(t)dt, \tag{27}$$

with $\bar{\epsilon}_{ij} = \epsilon_{ij}(T)$ fixed.

On considering variations in $\bar{\epsilon}_{ij}$, and using equation (25), we note that

$$\delta\bar{W} = \frac{\partial f}{\partial \epsilon_{ij}}\bigg|_{\bar{\epsilon}_{ij},\bar{\xi}_\alpha} \delta\bar{\epsilon}_{ij} + \left\{\frac{\partial f}{\partial \xi_\alpha}\bigg|_{\bar{\epsilon}_{ij},\bar{\xi}_\alpha} + \frac{1}{T^p}\frac{\partial D(\xi_\alpha)}{\partial \xi_\alpha}\bigg|_{\bar{\xi}_\alpha}\right\}\frac{\partial \bar{\xi}_\alpha}{\partial \bar{\epsilon}_{ij}}\delta\bar{\epsilon}_{ij}$$

$$= \frac{\partial f}{\partial \epsilon_{ij}}\bigg|_{\bar{\epsilon}_{ij},\bar{\xi}_\alpha} = \bar{\sigma}_{ij}\delta\bar{\epsilon}_{ij}, \tag{28}$$

The stress $\bar{\sigma}_{ij}$ is the stress associated with the state $\bar{\epsilon}_{ij}$, $\bar{\xi}_\alpha$. Equation (28) is sufficient to establish that $\bar{W}(\bar{\epsilon}_{ij})$ is a potential function relating $\bar{\sigma}_{ij}$ and $\bar{\epsilon}_{ij}$, where

$$\bar{\sigma}_{ij} = \frac{\partial \bar{W}}{\partial \bar{\epsilon}_{ij}} \tag{29}$$

In the limiting case $p = 0$, $\hat{W}$ (equation (23)) exhibits a discontinuity when $\hat{\xi}_\alpha = 0$. We must recognize that for certain choices of $\bar{\epsilon}_{ij}$ the least value of $\hat{W}$ can occur for $\hat{\xi}_\alpha = 0$. Consider

$$\hat{W}(\bar{\epsilon}_{ij}, \hat{\xi}_\alpha = 0) = f(\bar{\epsilon}_{ij}, \hat{\xi}_\alpha = 0) \tag{30a}$$

and

$$\hat{W}(\bar{\epsilon}_{ij}, \hat{\xi}_\alpha = \delta\hat{\xi}_\alpha) = f(\bar{\epsilon}_{ij}, \delta\hat{\xi}_\alpha) + D(\delta\hat{\xi}_\alpha). \tag{30b}$$

From the convexity of D (see equation (19), with $\hat{\xi}_\alpha^{(1)} = \delta\hat{\xi}_\alpha$, $\hat{\xi}_\alpha^{(2)} = 0$) we note that

$$D(\delta\hat{\xi}_\alpha) \geq \frac{\partial D}{\partial \xi_\alpha}\delta\hat{\xi}_\alpha, \tag{31}$$

where $\partial D/\partial \xi_\alpha$ may be evaluated for *any* nonzero value of ξ_α. Hence, from equations (30) and (31),

$$\delta\hat{W} = \hat{W}(\bar{\epsilon}_{ij}, \delta\hat{\xi}_\alpha) - \hat{W}(\bar{\epsilon}_{ij}, 0)$$

$$\geq \left(\frac{\partial f}{\partial \xi_\alpha}\bigg|_{\bar{\epsilon}_{ij},\hat{\xi}_\alpha=0} + \frac{\partial D}{\partial \xi_\alpha}\right)\delta\hat{\xi}_\alpha \tag{32}$$

Putting $\partial D/\partial \xi_\alpha = X_\alpha$ and $(\partial f/\partial \xi_\alpha)\bar{\epsilon}_{ij}$, $\hat{\xi}_\alpha = 0 = -\bar{X}_\alpha$, we see that

$$\delta\hat{W} \geq 0 \tag{33a}$$

if

$$(X_\alpha - \bar{X}_\alpha)\delta\xi_\alpha \geq 0. \tag{33b}$$

Inequality (33b) will be satisfied if the internal forces $\bar{X}_\alpha$ associated with $\bar{\epsilon}_{ij}$, $\hat{\xi}_\alpha = 0$ are such that they lie within or on the hypersurface in X_α space defined by the values of $\partial D/\partial \xi_\alpha$. In this case (33b) is the maximum plastic work inequality. This condition cannot be precisely defined without a specific form for D.

Thus, for the case $p = 0$, we choose $\bar{\xi}_\alpha = 0$ if inequality (33b) is satisfied. If not, equation (25) holds with $p = 0$. With proper recognition of the manner in which $\bar{\xi}_\alpha$ changes with $\bar{\epsilon}_{ij}$, it can also be shown that equation (29) holds for $p = 0$.

The Complementary Work Bounding Function

Consider an element of material subject to a stress path $\sigma_{ij}(t)$, with $\sigma_{ij}(0) = \epsilon_{ij}(0) = \xi_\alpha(0) = 0$ and a terminal stress constraint $\sigma_{ij}(T) = \bar{\sigma}_{ij}$. The complementary work done along this path is

$$\Omega = \int_0^T \epsilon_{ij}\dot{\sigma}_{ij}dt. \tag{34}$$

We seek an upper bound $\tilde{\Omega}$ on Ω.

In order to compute $\tilde{\Omega}$ in the internal variable framework we introduce the Gibb's function

$$h = \sigma_{ij}\epsilon_{ij} - f(\theta, \epsilon_{ij}, \xi_\alpha). \tag{35}$$

For small changes in the thermodynamic variables

$$dh = (\sigma_{ij}d\epsilon_{ij} + \epsilon_{ij}d\sigma_{ij}) + sd\theta - \sigma_{ij}d\epsilon_{ij} + X_\alpha d\xi_\alpha$$

$$= sd\theta + \epsilon_{ij}d\sigma_{ij} + X_\alpha d\xi_\alpha. \tag{36}$$

This is a sufficient condition to establish that

$$h = h(\theta, \sigma_{ij}, \xi_\alpha)$$

and that

$$s = \frac{\partial h}{\partial \theta}, \qquad \epsilon_{ij} = \frac{\partial h}{\partial \sigma_{ij}}, \qquad X_\alpha = \frac{\partial h}{\partial \xi_\alpha} \tag{37}$$

Further, on comparing equations (34) and (36) for isothermal deformation,

$$d\Omega = \epsilon_{ij}d\sigma_{ij} = dh - X_\alpha d\xi_\alpha \tag{38}$$

For the imposed terminal constraint $\sigma_{ij}(T) = \bar{\sigma}_{ij}$, therefore,

$$\Omega = \int_0^T \epsilon_{ij}\dot{\sigma}_{ij}dt = h\{\bar{\sigma}_{ij}, \xi_\alpha(T)\} - \int_0^T X_\alpha(t)\dot{\xi}_\alpha(t)dt$$

$$= h\{\bar{\sigma}_{ij}, \xi_\alpha(T)\} - I\{\xi_\alpha(T)\}. \tag{39}$$

In order to bound Ω, we first seek $\xi_\alpha(t)$ which minimizes I for an arbitrarily chosen $\xi_\alpha(T) = \hat{\xi}_\alpha$. This is identical to the problem posed in equation (17) *et seq.*, and hence we define

$$\hat{\Omega} = h\{\bar{\sigma}_{ij}\hat{\xi}_\alpha\} - \frac{1}{T^p}D(\hat{\xi}_\alpha). \tag{40}$$

We now seek the largest value of $\hat{\Omega}$. For arbitrary variations in $\hat{\xi}_\alpha$, when $p \neq 0$.

$$\delta\hat{\Omega} = \frac{\partial h}{\partial \xi_\alpha}\bigg|_{\bar{\sigma}_{ij},\hat{\xi}_\alpha} \delta\hat{\xi}_\alpha - \frac{1}{T^p}\frac{\partial D(\xi_\alpha)}{\partial \xi_\alpha}\bigg|_{\hat{\xi}_\alpha} \delta\hat{\xi}_\alpha$$

$$= \left\{\frac{\partial h}{\partial \xi_\alpha}\bigg|_{\bar{\sigma}_{ij},\hat{\xi}_\alpha} - \frac{1}{T^p}\frac{\partial D(\xi_\alpha)}{\partial \xi_\alpha}\bigg|_{\hat{\xi}_\alpha}\right\}\delta\hat{\xi}_\alpha \tag{41}$$

Thus $\hat{\Omega}$ is stationary when $\hat{\xi}_\alpha = \bar{\xi}_\alpha$, where $\bar{\xi}_\alpha$ is given by the solution of the equation

$$\left.\frac{\partial h}{\partial \xi_\alpha}\right|_{\bar{\sigma}_{ij},\bar{\xi}_\alpha} = \frac{1}{T^p}\left.\frac{\partial D(\xi_\alpha)}{\partial \xi_\alpha}\right|_{\bar{\xi}_\alpha}. \tag{42}$$

Convexity of both h and D is a sufficient condition that $\hat{\Omega}$ will have a global maximum at the stationary value; convexity of h follows from convexity of f.

Thus we may now define

$$\bar{\Omega}(\bar{\sigma}_{ij}) = h(\bar{\sigma}_{ij}, \bar{\xi}_\alpha) - \frac{1}{T^p} D(\bar{\xi}_\alpha) \tag{43}$$

and, provided that $\bar{\xi}_\alpha$ is chosen according to equation (42),

$$\bar{\Omega}(\bar{\sigma}_{ij}) \geq \Omega = \int_0^T \epsilon_{ij}(t)\dot{\sigma}_{ij}(t)dt \tag{44}$$

with $\sigma_{ij}(T) = \bar{\sigma}_{ij}$ fixed.

Considering variations in $\bar{\sigma}_{ij}$, and using equation (25), we note that

$$\delta\bar{\Omega} = \left.\frac{\partial h}{\partial \sigma_{ij}}\right|_{\bar{\sigma}_{ij},\bar{\xi}_\alpha} \delta\bar{\sigma}_{ij} + \left\{\left.\frac{\partial h}{\partial \xi_\alpha}\right|_{\bar{\sigma}_{ij},\bar{\xi}_\alpha} - \frac{1}{T^p}\left.\frac{\partial D(\xi_\alpha)}{\partial \xi_\alpha}\right|_{\bar{\xi}_\alpha}\right\}\frac{\partial \bar{\xi}_\alpha}{\partial \bar{\sigma}_{ij}}\delta\bar{\sigma}_{ij}$$
$$= \left.\frac{\partial h}{\partial \sigma_{ij}}\right|_{\bar{\sigma}_{ij},\bar{\xi}_\alpha} = \bar{\epsilon}_{ij}\delta\bar{\sigma}_{ij}. \tag{45}$$

The strain $\bar{\epsilon}_{ij}$ is the strain associated with the state $\bar{\sigma}_{ij}$, $\bar{\xi}_\alpha$. Equation (45) is sufficient to establish that $\bar{\Omega}(\bar{\sigma}_{ij})$ is a potential function relating $\bar{\sigma}_{ij}$ and $\bar{\epsilon}_{ij}$, where

$$\bar{\epsilon}_{ij} = \frac{\partial\bar{\Omega}}{\partial\bar{\sigma}_{ij}}. \tag{46}$$

It is readily seen that $\bar{\Omega}$ and $\bar{W}$ are dual functions: for any choice of $\bar{\epsilon}_{ij}$ we may calculate $\bar{\xi}_\alpha$ from equation (25) and $\bar{\sigma}_{ij}$ from equation (29); if we then use this value of $\bar{\sigma}_{ij}$ in equation (42) we determine precisely the same $\bar{\xi}_\alpha$, and equation (46) gives the original $\bar{\epsilon}_{ij}$. Further, from equations (26), (43), and (35) we see that

$$\bar{W}(\bar{\epsilon}_{ij}) + \bar{\Omega}(\bar{\sigma}_{ij}) = \bar{\sigma}_{ij}\bar{\epsilon}_{ij}. \tag{47}$$

In the limiting case $p = 0$, $\hat{\Omega}$ also exhibits a discontinuity when $\hat{\xi}_\alpha = 0$. It must again be recognised that for certain choices of $\bar{\sigma}_{ij}$ the greatest value of $\hat{\Omega}$ may occur for $\bar{\xi}_\alpha = 0$. The argument closely parallels that given in equation (30) *et seq.*, and will not be repeated in detail. We again choose $\bar{\xi}_\alpha = 0$ if

$$(X_\alpha - \bar{X}_\alpha)\delta\xi_\alpha \geq 0 \tag{48a}$$

where

$$X_\alpha = \frac{\partial D(\xi_\alpha)}{\partial \xi_\alpha} \tag{48b}$$

is derived from any nonzero choice of ξ_α, and

$$\bar{X}_\alpha = \left.\frac{\partial h}{\partial \xi_\alpha}\right|_{\bar{\sigma}_{ij},\xi_\alpha=0}. \tag{48c}$$

This condition will be satisfied when $\bar{X}_\alpha$ lies within or on the limit surface in X_α space. If inequality (48a) does not hold for arbitrary $\delta\xi_\alpha$, $\bar{\xi}_\alpha$ is chosen according to equation (42). Equation (46) also holds for $p = 0$, with proper consideration given to the manner in which $\bar{\xi}_\alpha$ changes with $\bar{\sigma}_{ij}$.

Convexity of $\bar{W}$ and $\bar{\Omega}$

Convexity of f and D is sufficient to show that $\bar{W}(\bar{\epsilon}_{ij})$ and $\bar{\Omega}(\bar{\sigma}_{ij})$ are convex functions of their respective arguments. It will suffice to demonstrate this for the work bounding function.

Consider two terminal states $\bar{\epsilon}'_{ij}$ and $\bar{\epsilon}''_{ij}$, and the corresponding terminal internal variables $\bar{\xi}'_\alpha, \bar{\xi}''_\alpha$. If $f(\epsilon_{ij}, \xi_\alpha)$ is convex, we may write

$$f(\bar{\epsilon}'_{ij}, \bar{\xi}'_\alpha) - f(\bar{\epsilon}''_{ij}, \bar{\xi}''_\alpha) \geq (\bar{\epsilon}'_{ij} - \bar{\epsilon}''_{ij})\left.\frac{\partial f}{\partial \epsilon_{ij}}\right|_{\bar{\epsilon}''_{ij},\bar{\xi}''_\alpha}$$
$$+ (\bar{\xi}'_\alpha - \bar{\xi}''_\alpha)\left.\frac{\partial f}{\partial \xi_\alpha}\right|_{\bar{\epsilon}''_{ij},\bar{\xi}''_\alpha}. \tag{49}$$

Similarly, if $D(\xi_\alpha)$ is convex,

$$D(\bar{\xi}'_\alpha) - D(\bar{\xi}''_\alpha) \geq (\bar{\xi}'_\alpha - \bar{\xi}''_\alpha)\left.\frac{\partial D(\xi_\alpha)}{\partial \xi_\alpha}\right|_{\bar{\xi}''_\alpha} \tag{50}$$

Multiplying inequality (50) by $1/T^p$, and adding inequalities (49) and (50), we see that

$$\left\{f(\bar{\epsilon}'_{ij}, \bar{\xi}'_\alpha) + \frac{1}{T^p}D(\bar{\xi}'_\alpha)\right\} - \left\{f(\bar{\epsilon}''_{ij}, \bar{\xi}''_\alpha) + \frac{1}{T^p}D(\bar{\xi}''_\alpha)\right\}$$
$$\geq (\bar{\epsilon}'_{ij} - \bar{\epsilon}''_{ij})\left.\frac{\partial f}{\partial \epsilon_{ij}}\right|_{\bar{\epsilon}''_{ij},\bar{\xi}''_\alpha}$$
$$+ (\bar{\xi}'_\alpha - \bar{\xi}'')\left[\left.\frac{\partial f}{\partial \xi_\alpha}\right|_{\bar{\epsilon}''_{ij},\bar{\xi}''_\alpha} + \frac{1}{T^p}\left.\frac{\partial D}{\partial \xi_\alpha}\right|_{\bar{\xi}''_\alpha}\right]. \tag{51}$$

However, the last term vanishes in view of equation (25) or is nonnegative in view of equation (33b). Using equations (26) and (29), inequality (51) then becomes

$$\bar{W}(\bar{\epsilon}'_{ij}) - \bar{W}(\bar{\epsilon}''_{ij}) \geq (\bar{\epsilon}'_{ij} - \bar{\epsilon}''_{ij})\left.\frac{\partial \bar{W}}{\partial \bar{\epsilon}_{ij}}\right|_{\bar{\epsilon}''_{ij}}. \tag{52}$$

which establishes that $\bar{W}(\bar{\epsilon}_{ij})$ is convex.

It may be noted in passing that, from equation (27),

$$W(\bar{\epsilon}'_{ij}) \geq \bar{W}(\bar{\epsilon}'_{ij}), \tag{53}$$

where $W(\bar{\epsilon}'_{ij})$ is the work done along an arbitrary strain path $\epsilon_{ij}(t)$ with $\epsilon_{ij}(0) = 0$, $\epsilon_{ij}(T) = \bar{\epsilon}'_{ij}$. Combining inequalities (52) and (53),

$$W(\bar{\epsilon}'_{ij}) - \bar{W}(\bar{\epsilon}''_{ij}) \geq (\bar{\epsilon}'_{ij} - \bar{\epsilon}''_{ij})\left.\frac{\partial \bar{W}}{\partial \bar{\epsilon}_{ij}}\right|_{\bar{\epsilon}''_{ij}}, \tag{54a}$$

or using equation (47),

$$W(\bar{\epsilon}'_{ij}) + \bar{\Omega}(\bar{\sigma}''_{ij}) \geq \bar{\sigma}''_{ij}\bar{\epsilon}'_{ij}. \tag{54b}$$

This result was established by Martin [1] for time-independent materials, and by Ponter [9] for time dependent materials.

Realizability of Paths

The internal variable history associated with the work bounding function.

$$\dot{\xi}_\alpha(t) = \frac{\bar{\xi}_\alpha}{T}, \tag{55}$$

does not in general correspond to a history which can in fact be achieved by the imposition of a strain path $\epsilon_{ij}(t)$. It follows that $\bar{W}(\bar{\epsilon}_{ij})$ is less than the least work required to deform the material element through a strain history $\epsilon_{ij}(t)$ with $\epsilon_{ij}(0) = 0$, $\epsilon_{ij}(T) = \bar{\epsilon}_{ij}$, or alternatively, $\bar{W}(\bar{\epsilon}_{ij})$ is less than the work along the minimum work path. This does not affect the utilisation of $\bar{W}(\bar{\epsilon}_{ij})$ in the static and dynamic bounding theorems, although it may lead to loss in accuracy.

This result follows on noting that, from equation (10),

$$\dot{X}_\alpha = \frac{\partial^2 f}{\partial\epsilon_{ij}\partial\xi_\alpha}\dot{\epsilon}_{ij} + \frac{\partial^2 f}{\partial\xi_\alpha\partial\xi_\beta}\dot{\xi}_\beta. \tag{56}$$

When $\dot{X}_\alpha(t) = 0$, following equations (20),

$$\frac{\partial^2 f}{\partial\epsilon_{ij}\partial\xi_\alpha}\dot{\epsilon}_{ij}(t) = -\frac{\partial^2 f}{\partial\xi_\alpha\partial\xi_\beta}\dot{\xi}_\beta(t). \tag{57}$$

For given $\dot{\xi}_\alpha(t) = \bar{\xi}_\alpha/T$, these equations cannot in the general case be inverted to give $\dot{\epsilon}_{ij}(t)$.

In the exceptional case where these equations can be inverted, however, a strain path can be found which provides the required internal variable history. In this case the work bounding function and the work along the minimum work path coincide. A majority of the specific models for which minimum work paths have been computed fall into this exceptional case; in these specific models the plastic strains and the internal variables coincide. For any model in which equation (57) can be solved for $\epsilon_{ij}(t)$ the present approach will provide the minimum work path and the working bounding function will provide the minimum realizable work.

In the cases where a realizable path is implied by $\dot{\xi}_\alpha(t)$ constant, the internal force history can also be calculated. The internal forces are also constant over the interval $0 < t < T$. Since $X_\alpha(t \leq 0) = 0$, and the terminal value $\bar{X}_\alpha = X_\alpha(T)$ is given independently by

$$\bar{X}_\alpha = \frac{\partial h}{\partial \xi_\alpha}\bigg|_{\bar{\sigma}_{ij}, \bar{\xi}_\alpha}, \tag{58}$$

it follows that the function $X_\alpha(t)$, $0 \leq t \leq T$ exhibits discontinuities at $t = 0$ and $t = T$. Considering equations (13) and (42), we may put

$$\frac{1}{T^p}\frac{\partial D(\xi_\alpha)}{\partial \xi_\alpha}\bigg|_{\dot{\xi}_\alpha} = \frac{\partial D(\xi_\alpha/T)}{\partial(\xi_\alpha/T)}\bigg|_{\xi_\alpha/T} = (p+1)X_\alpha(t). \tag{59}$$

Thus substituting equations (58) and (59) into (42)

$$X_\alpha(t) = \frac{1}{p+1}\bar{X}_\alpha(t), \qquad 0 < t < T. \tag{60}$$

This result was given by Ponter [9] for Maxwell models of creep.

Example

It is instructive to rederive the result given by Ponter [9] for the generalized Maxwell model for nonlinear creep of metals. Consider isothermal, incompressible deformation of an element of unit volume. Let e_{ij} be the strain deviator and s_{ij} the stress deviator, replacing ϵ_{ij} and σ_{ij}, respectively. The constitutive equations under consideration are conventionally given in the form

$$e_{ij} = e_{ij}^e + e_{ij}^p \tag{61a}$$

$$e_{ij}^e = s_{ij}/2G \tag{61b}$$

$$\left(\frac{\dot{e}_{ij}^e}{\dot{e}_0}\right) = \phi^n \frac{\partial\phi}{\partial(s_{ij}/s_0)} \tag{61c}$$

where e_{ij}^e, e_{ij}^p are, respectively, the elastic and inelastic components of strain, G is the shear modulus, $\dot{e}_0$, s_0 are constants with the dimensions of strain rate and stress, respectively, and ϕ is homogeneous and of degree one in the components of (s_{ij}/s_0). Noting that

$$D = s_{ij}\dot{e}_{ij}^p = s_0\dot{e}_0\phi^{n+1}, \tag{62}$$

we may fairly readily establish that D is homogeneous and of degree $(n+1)/n$ in the components of $\dot{e}_{ij}$. This implies that $n = 1/p$.

Omitting reference to temperature θ, we may identify e_{ij}^p as the internal variable, and put

$$f(e_{ij}, e_{ij}^p) = 4G(e_{ij} - e_{ij}^p)(e_{ij} - e_{ij}^p). \tag{63}$$

It is seen that

$$\frac{\partial f}{\partial e_{ij}} = 2G(e_{ij} - e_{ij}^p) = 2G(e_{ij}^e) = s_{ij}, \tag{64a}$$

$$\frac{\partial f}{\partial e_{ij}^p} = -2G(e_{ij} - e_{ij}^p) = -s_{ij} \tag{64b}$$

Consequently, the forces conjugate to e_{ij}^p (equivalent to X_α) are the s_{ij}. Similarly, we see that

$$h(s_{ij}, e_{ij}^p) = \frac{1}{4G}s_{ij}s_{ij} + s_{ij}e_{ij}^p. \tag{65}$$

Then

$$\frac{\partial h}{\partial s_{ij}} = \frac{s_{ij}}{2G} + e_{ij}^p = e_{ij}, \quad \frac{\partial h}{\partial e_{ij}^p} = s_{ij}. \tag{66}$$

Using equation (60), it may be seen immediately that $\hat{s}_{ij} = s_{ij}(t)$, $0 < t < T$, is given in terms of a terminal stress $\bar{s}_{ij}$ by means of the relation

$$\hat{s}_{ij} = \frac{1}{p+1}\bar{s}_{ij} = \frac{n}{n+1}\bar{s}_{ij}. \tag{67}$$

From equation (61c), integrating over the interval $[0, T]$,

$$\bar{e}_{ij}^p = \dot{e}_0 T\phi^n\left(\frac{\hat{s}_{ij}}{s_0}\right)\frac{\partial\phi}{\partial\left(\frac{s_{ij}}{s_0}\right)}\Bigg|_{\hat{s}_{ij}/s_0} \tag{68}$$

Thus, from equation (65), using the requirements that ϕ is homogeneous and of degree one,

$$h(\bar{s}_{ij}, \bar{e}_{ij}^p) = \frac{1}{4G}\bar{s}_{ij}\bar{s}_{ij} + \left(\frac{n+1}{n}\right)s_0\left(\frac{\hat{s}_{ij}}{s_0}\right)\bar{e}_{ij}^p$$

$$= \frac{1}{4G}\bar{s}_{ij}\bar{s}_{ij} + \left(\frac{n+1}{n}\right)s_0\dot{e}_0T\phi^{(n+1)}\left(\frac{\hat{s}_{ij}}{s_0}\right). \tag{69}$$

Further,

$$\frac{1}{T^p}D(\bar{e}_{ij}^p) = s_0\dot{e}_0T\phi^{(n+1)}\left(\frac{\hat{s}_{ij}}{s_0}\right). \tag{70}$$

Finally, referring to equation (43), equations (69) and (70) give

$$\Omega(\bar{s}_{ij}) = \frac{1}{4G}\bar{s}_{ij}\bar{s}_{ij} + \frac{1}{n}s_0\dot{e}_0T\phi^{(n+1)}\left\{\frac{n}{(n+1)}\frac{\bar{s}_{ij}}{s_0}\right\}. \tag{71}$$

It follows then that

$$\bar{e}_{ij} = \frac{1}{2G}\bar{s}_{ij} + \left(\frac{n+1}{n}\right)\dot{e}_0T\phi^n\left\{\frac{n}{(n+1)}\frac{\bar{s}_{ij}}{s_0}\right\}\frac{\partial\phi}{\partial\left(\frac{s_{ij}}{s_0}\right)}\Bigg|_{n/(n+1)\bar{s}_{ij}/s_0} \tag{72}$$

This equation cannot be inverted without a precise definition of the function ϕ. With this further information, $\bar{W}$ may be computed from the relation

$$W(\bar{e}_{ij}) = \bar{s}_{ij}\bar{e}_{ij} - \Omega(\bar{s}_{ij}). \tag{73}$$

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