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FIXED POINTS FOR NONEXPANSIVE MAPPINGS
IN BANACH SPACES

by

P. JEGANATHAN

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of Dr. D.R. Smart for the degree of
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DEDICATION

I dedicate this thesis to my parents, who instilled in me an abiding respect for the dignity of work.

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LIST of SYMBOLS

We give a list of the symbols used, which may not be standard symbols to the reader, and a brief indication to their meaning.

To those symbols, for which we have given their definitions, we will give a reference to the page where it is defined.

Symbol	Page	Meaning
$B(x,r)$		closed ball of center x with radius r in the norm topology
$\mathcal{B}(A)$	53	$\cap \{B \supset A : B \text{ closed ball}\}$
$\partial(S)$		boundary of set S
$\delta(S)$	10	diameter of set S
$\text{cl}(S)$		closure of set S in the weak topology
\bar{S}		closure of set S in the norm topology
$\pi\text{-cl}(S)$	69	closure of set S in the topology of weak pointwise convergence
$\text{co}(S)$		convex hull of set S
$\overline{\text{co}}(S)$		norm closure of the convex hull of set S
$F(T)$	6	set of fixed points for a map T , in its domain
$N(F)$	69	$\{f : C \rightarrow C : f \text{ is nonexpansive and } fx = x \text{ for all } x \in F\}$
$r_x(A)$	10	$\sup \{d(x,y) : y \in A\}$
$r(H,K)$	22	$\inf \{r_x(H) : x \in K\}$
$r(H)$	22	$\inf \{r_x(H) : x \in X\}$
$\mathcal{C}(H,K)$	22	$\{x \in K : r_x(H) = r(H,K)\}$
$C(H)$	22	$\{x \in X : r_x(H) = r(H)\}$

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INTRODUCTION

Fixed point Theory is a branch of mathematics having a wide spectrum of applications in not only areas of mathematics, but also in many practical fields such as physics and economics. For example, in economics, the proofs of the existence of equilibrium for various economic systems are based on fixed point theorems.

Although a significant proportion of the theory lies in the branch of functional analysis, fixed point theory also resides in areas such as algebraic topology and degree theory in the sense that proofs of some fixed point theorems use arguments involving these areas. For example, there have been different approaches taken to prove the well known Brouwer fixed point theorem: a proof via degree theory is possible, but the most concise argument seems to be an algebraic topological one using homology functors.

As we are concerned with fixed point theorems for nonexpansive mappings, we shall confine ourselves to functional analysis, where we will be interacting mainly with related areas of mathematics such as General Topology, the theory of Locally Convex Spaces and Measure Theory.

Many results have been produced on the topic of fixed points for (non-linear) nonexpansive mappings in Banach spaces, as early as in 1965. We therefore find it justifiable to give a unified presentation of some of these results.

By a nonexpansive mapping, we mean a mapping which maps a metric space into itself such that it does not increase distances. The most obvious example of a nonexpansive mapping is any linear operator on a normed space which has norm less than or equal to one.

Some authors use the term contraction mapping instead of the term nonexpansive mapping. However, we shall use the term contraction mapping to imply that it is mapping in a metric space which strictly decreases distances.

Thus, clearly, contraction mappings are special cases of nonexpansive mappings.

As for applications, a significant publication by Browder ([Browder0]) appeared in 1965, where nonexpansive mappings were used to find periodic solutions for nonlinear equations of evolution.

Moreover, the classical Banach fixed point theorem for contraction mappings gives rise to applications in finding solutions to differential equations and integral equations; yielding both existence and uniqueness theorems for such equations in function spaces. For some other applications of Banach's fixed point theorem see, for instance, [Kreyszig] chapter 5 or [Smart] chapter 1.

Let us now give a brief outline of the direction we shall take.

Chapter 0 contains the necessary definitions and some standard theorems and examples.

Chapter 1 is focused on obtaining fixed points for single nonexpansive mappings.

Schauder's fixed point theorem tells us that any compact convex set in a normed space has the fixed point property. It is thus interesting to ask in what ways one can weaken the assumptions on the domain and/or space, and still obtain a fixed point. We show that a nonexpansive mapping $T: C \rightarrow C$ has a fixed point if:

- C is a bounded closed convex set, where C has a compact subset which is repeatedly approached by all orbits of the map (1.2.1);
- C is a bounded closed convex set with normal structure in a Banach space, where C has a weakly compact subset which is repeatedly approached by all orbits of the map (1.2.4);

- C is a nonempty bounded closed convex set with normal structure in a reflexive Banach space (1.3.2);
- C is a nonempty closed convex (not necessarily bounded) set with normal structure in a reflexive Banach space such that the map has a bounded orbit (1.3.5);
- C is nonempty, bounded closed and convex in a uniformly convex Banach space (1.4.3);
- C is nonempty, bounded closed and convex with asymptotic normal structure in a reflexive Banach space (1.5.3).

In addition, we also look at some results involving nonexpansive mappings in Hilbert spaces.

Following this survey, we consider the special class of metric spaces known as hyperconvex spaces. This class is familiar to categorical topologists, but not so familiar to analysts. However, we will prove some important results involving nonexpansive mappings in hyperconvex spaces.

The two sections following this give some conditions under which the set of fixed points of a nonexpansive mapping is a nonexpansive retract of the domain.

Finally, we give an example of a weakly compact convex set which lacks the fixed point property for nonexpansive mappings. As mentioned earlier, the results we have in our survey which assume weaker conditions than compactness were proven only by including additional assumptions. It is evident to us that when the authors proved these results the question on whether fixed points existed without such additional assumptions remained open. But considering this example which appeared in 1981, we find it justifiable that the additional assumptions should be included.

In chapter 2, we extend these ideas to commuting families of nonexpansive mappings. Markov and Kakutani have given a weak extension to Schauder's fixed point theorem by showing the existence of common fixed points for families of commuting continuous mappings which are affine. The assumption on the mappings to be affine makes this a weak extension to Schauder's fixed point theorem.

Indeed, a theorem of De Marr gives a proper extension to Schauder's fixed point theorem for nonexpansive mappings, by assuming the domain to be compact and convex.

On the other hand, by giving a counter example, we will show that assuming weak compactness instead of compactness may not yield a common fixed point for a family of nonexpansive mappings in a Banach space.

In addition to extending the ideas in chapter 1 to families of nonexpansive mappings, we will introduce a specific class of mappings called demicompact mappings, where we generalise the theorem of De Marr which we have mentioned above.

The following results are original and to the best of our knowledge they are not available in any published literature: 1.7.10, 1.7.11, 1.7.12, 1.7.14, 2.1.2, 2.1.6, 2.7.6 and 2.7.7.

CHAPTER 0

BASIC CONCEPTS

The purpose of this chapter is to equip the reader with the prerequisites in fixed point theory necessary to understand the remaining chapters.

Nearly all the definitions and results are quoted without any reference as they are obtainable from standard books on fixed point theory, such as [Smart].

To avoid pathology we assume throughout this thesis that the domains of functions are nonempty.

We will also assume that the reader is familiar with basic ideas from Functional Analysis, General Topology, the theory of Locally Convex Spaces and Measure Theory.

0.1 DEFINITION

Let X be a set and let S be a nonempty subset of X .

A map $f : S \rightarrow X$ is said to have a fixed point if there exists some x in S for which $f(x) = x$.

0.2 DEFINITIONS

Let X be a metric space and let $T : X \rightarrow X$.

Then T is said to be

- (i) a **nonexpansive mapping** if for every $x, y \in X$, $d(Tx, Ty) \leq d(x, y)$
- (ii) a **contraction mapping** if there exists some $k \in (0, 1)$ such that for every $x, y \in X$, $d(Tx, Ty) \leq k d(x, y)$.

Remarks:

It is clear that contraction mappings are special cases of nonexpansive mappings. Further, from the definitions of nonexpansive mapping it is clear that nonexpansive mappings are continuous (in fact, uniformly continuous).

0.3 DEFINITION

A set S in a topological space X is said to have the fixed point property if every continuous mapping which maps S into itself has a fixed point in S .

0.4 DEFINITION

A set S in a metric space X is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping which maps S into itself has a fixed point in S .

0.5 DEFINITION

Let (X, d) be a metric space and let $f : X \rightarrow X$ be a continuous mapping.

Suppose $\epsilon > 0$. If there exists x in X such that $d(f(x), x) < \epsilon$ then x is said to be an ϵ -fixed point for f .

It is clear that a fixed point will always be an ϵ -fixed point for any $\epsilon > 0$.

Alternatively, some authors use the term **almost fixed points** to mean ϵ -fixed points.

However, we shall be consistent by using the term ϵ -fixed points.

0.6 NOTATION

Let X be any set and let S be a nonempty subset of X . Suppose T maps S into X , i.e. $T : S \rightarrow X$. Then we denote the set of fixed points for T in S by $F(T)$.

The following is a trivial observation which is a property of a fixed point set. But it is worth noting it here as we will be using it without much reference.

0.7 PROPOSITION

Let X be a Hausdorff topological space and let $f : X \rightarrow X$ be a continuous mapping. Then $F(f)$ is closed.

The proof of this is trivial by using nets and uniqueness of limits.

However, a word of caution: If the map f is defined on a subset, say S of X , then $F(f)$ need not be closed in X (unless S is closed in X). Instead, it will be closed in the subspace formed by restricting the metric to S . For example, restrict the identity mapping to the open interval $(0,1)$ in \mathbb{R} , where \mathbb{R} has the usual metric. The set of fixed points is clearly $(0,1)$, which is closed in $(0,1)$, but not in \mathbb{R} .

This will not be of much concern to us because we will be working with mappings that have closed domain.

Let us state some well known results which we shall be using quite frequently.

These results can be found in standard books on fixed point theory such as [Smart] or [DG] (Dugundji, J and Granas, A), and so we omit the proofs.

The following is a classical result well known to analysts as Banach's fixed point theorem.

0.8 THEOREM

Let X be a nonempty complete metric space and let $T : X \rightarrow X$ be a contraction mapping. Then T has a unique fixed point.

The following result is known as Schauder's fixed point theorem.

0.9 THEOREM (Schauder)

Any nonempty compact convex set in a normed space has the fixed point property.

Note the strengths and the weakness of this result: it is demanded that the domain be a compact convex subset of a normed space. Under these hypotheses, however, any continuous mapping has a fixed point.

We shall now give a generalisation of Schauder's fixed point theorem known as Tychonov's fixed point theorem.

0.10 THEOREM (Tychonov)

Let X be a locally convex space and let C be a closed convex set in X . If $f: X \rightarrow X$ is a continuous mapping such that closure of $f(C)$ is compact then f has a fixed point.

If X is normed and if C is compact then, by continuity of f , $f(C)$ is compact and hence closed, since X has the norm topology. Thus $\overline{f(C)} = f(C)$ is compact. Hence Schauder's fixed point theorem is contained in Tychonov's fixed point theorem.

The following definition leads us to a useful proposition as we will be using it in some of our proofs in the following chapters.

0.11 DEFINITION [Kreyszig] 6.2-2

A norm $\|\cdot\|$ is said to be **strictly convex** if for all distinct x, y of norm 1, $\|x + y\| < 2$.

A normed space with such a norm is called a **strictly convex normed space** (or simply strictly convex space).

As an example, [Kreyszig] shows that any Hilbert space norm is strictly convex (lemma 6.2-4).

The proof is straightforward by the use of the parallelogram identity.

0.12 PROPOSITION

Let X be a strictly convex space. Let C be a convex set in X and let $T: C \rightarrow X$ be a nonexpansive mapping. Then the set, $F(T)$, of fixed points for T is convex.

The following theorem is known as the Schaefer's theorem.

0.13 THEOREM (Schaefer)

Let X be a normed space and let T be a continuous mapping of X into X which maps every bounded set into a compact set. Then either

- (i) the equation $x = \lambda Tx$ has a solution for $\lambda = 1$, or
- (ii) the set of all such solutions x , for $0 < \lambda < 1$, is unbounded.

So far we have discussed some elementary properties, such as closedness and convexity, of the set of fixed points for a continuous mapping. The following definitions describe a property for sets, in general, which is important for our purpose.

0.14 DEFINITIONS

Let X be a topological space and let S_1 and S_2 be subsets of X .

Then S_1 is said to be a retract S_2 if $S_1 \subset S_2$ and there exists a continuous mapping $r: S_2 \rightarrow S_1$ such that $r|_{S_1} = I$, where I denotes the identity mapping on S_2 .

The mapping r is said to be a retraction from S_2 onto S_1 .

If X is a metric space and r is a nonexpansive mapping then S_1 is said to be a nonexpansive retract of S_2 and r is said to be a nonexpansive retraction from S_2 onto S_1 .

We shall be using the following notations frequently.

0.15 DEFINITION [Baillon]

For a bounded subset A of a metric space X , define:

- (a) $r_x(A) = \sup \{d(x,y) : y \in A\}$ for $x \in X$
- (b) $\delta(A) = \sup \{r_x(A) : x \in A\} = \sup \{d(x,y) : x, y \in A\}$; the diameter of A .

CHAPTER 1

EXISTENCE OF FIXED POINTS FOR NONEXPANSIVE MAPPINGS

Our aim here is to investigate the existence of fixed points for nonexpansive mappings in some nontrivial cases and to discuss some properties of sets which are related to the fixed point property.

§ 1.1 is concerned with preliminaries.

§ 1.2 to § 1.6 are concerned with the existence of fixed points for nonexpansive mappings in either the general class of Banach spaces or in some specific classes of Banach spaces such as reflexive Banach spaces or uniformly convex Banach spaces.

In § 1.7 we introduce a special class of metric spaces called hyperconvex spaces, where we show that a bounded hyperconvex space has the fixed point property for nonexpansive mappings.

In § 1.8 and § 1.9 we discuss some cases in which the set of fixed points for a nonexpansive mapping is a nonexpansive retract.

In § 1.10 we will give an example of a set which does not have the fixed point property for nonexpansive mappings, which answers the question of whether or not a weakly compact convex set in a Banach space has the fixed point property for nonexpansive mappings.

§ 1.1 Preliminaries

Let us state and prove the following well known theorems.

1.1.1 THEOREM

Let X be a nonempty metric space and let $T: X \rightarrow X$ be a contraction mapping. Then for every $\epsilon > 0$ there will an ϵ -fixed point for T .

PROOF

Let $\epsilon > 0$.

Since T is a contraction mapping, there exists $0 < k < 1$ such that

$$d(Tx, Ty) \leq k d(x, y) \quad \forall x, y \in X.$$

Hence for any $x \in X$ we have that $d(T^{n+1}x, T^n x) \leq k^n d(Tx, x)$.

If $d(Tx, x) = 0$, for some $x \in X$, then x is a fixed point and hence x is an ϵ -fixed point.

So assume that $d(Tx, x) \neq 0$ for every $x \in X$.

Let $x \in X$.

Now $k^n d(Tx, x) < \epsilon$ for any $n > (\ln \epsilon - \ln d(Tx, x)) / \ln k$.

Hence $T^n x$ is an ϵ -fixed point, for any n satisfying this inequality. □

1.1.2 THEOREM

Let S be a nonempty bounded convex set in a normed space X .

If $T: S \rightarrow S$ is nonexpansive then for every $\epsilon > 0$ there will be an ϵ -fixed point for T in S .

PROOF

Assume w.l.o.g. that S contains 0 .

(If not we translate the set.)

If $\delta(S) = 0$ then the result follows. Suppose that $\delta(S) > 0$.

Let $\epsilon > 0$.

Define $T' = \left[1 - \frac{\epsilon}{2\delta(S)}\right]T$.

Then $T' : S \rightarrow S$ since S is convex, invariant under T and $0 \in S$.

For $x, y \in S$, $\|T'x - T'y\| \leq \left[1 - \frac{\epsilon}{2\delta(S)}\right]\|x - y\|$.

Hence T' is a contraction mapping. By the above theorem T' has an $\frac{\epsilon}{2}$ -fixed point, say x_0 .

Thus $\|Tx_0 - x_0\| \leq \|Tx_0 - T'x_0\| + \|T'x_0 - x_0\| \leq \frac{\epsilon}{2\delta(S)}\|Tx_0\| + \frac{\epsilon}{2} \leq \epsilon$. □

§ 1.2 Existence of fixed points for nonexpansive mappings whose domains include sets repeatedly approached by all orbits of the mapping

The strong result of Schauder guarantees the fixed point property for any compact convex set in a normed space. Thus every nonexpansive mapping which maps a compact convex set into itself has a fixed point. If we delete compactness from the assumption then we may not obtain a fixed point for a nonexpansive mapping and hence, in general, for a continuous mapping. An example given in this chapter (1.4.4) suffices to show this.

However, the following theorems show that we can still obtain a fixed point by assuming, under some suitable conditions, the existence of a subset of the domain satisfying a property which we define as follows:

1.2.0 DEFINITION

Let C be a nonempty set in a normed space and $T: C \rightarrow C$ a continuous mapping.

Let S be a nonempty subset of C .

Then S will be said to be repeatedly approached by all orbits of T , if for every x in C , the sequence $\{T^n x: n \in \mathbb{N}\}$ has a closure point in S .

Remark:

Note that if x is a fixed point for T , then the orbit $\{T^n x: n \in \mathbb{N}\}$ is just x itself.

Clearly then any set S which is repeatedly approached by all orbits of T includes all the fixed points for T .

The following theorem shows that this set of fixed points is nonempty whenever such a set S is compact and the domain is bounded, closed and convex.

1.2.1 THEOREM [Göhde]

Let C be a bounded closed convex set in a normed space X . Let $S \subset C$ be compact and let $T: C \rightarrow C$ be nonexpansive such that S is repeatedly approached by all orbits of T . Then there exists at least one fixed point for T in S .

PROOF

W.l.o.g. assume that $0 \in C$.

For $0 < q < 1$ define $T_q x = q Tx$. Then we easily see that T_q is a contraction mapping of C into itself.

Thus by 1.1.2, for every $\epsilon > 0$, T_q has an ϵ -fixed point $x_q(\epsilon)$ in C .

Thus $\|T_q x_q(\epsilon) - x_q(\epsilon)\| \leq \epsilon$.

Now $\epsilon = 1 - q$ implies that $\|T_q x_q(\epsilon) - x_q(\epsilon)\| \leq 1 - q$.

$$\begin{aligned} \text{Hence} \quad & \|Tx_q(\epsilon) - x_q(\epsilon)\| \\ = & \|Tx_q(\epsilon) - q Tx_q(\epsilon) + q Tx_q(\epsilon) - x_q(\epsilon)\| \\ \leq & (1 - q) \|Tx_q(\epsilon)\| + \|T_q x_q(\epsilon) - x_q(\epsilon)\| \\ \leq & (1 - q) \|Tx_q(\epsilon)\| + (1 - q) \\ = & (1 - q) (1 + \|Tx_q(\epsilon)\|). \end{aligned}$$

Hence, since T is nonexpansive, by the above inequality we get

$$\|T^{n+1} x_q(\epsilon) - T^n x_q(\epsilon)\| \leq (1 - q) r \text{ for every } n, \text{ where } r = 1 + \|Tx_q(\epsilon)\|.$$

Let $S_{x_q(\epsilon)} = \{T^n x_q(\epsilon) : n \in \mathbb{N}\}$ and let $\delta > 0$.

By hypothesis, there exists $y_q(\epsilon) \in \overline{S_{x_q(\epsilon)}} \cap S$.

Thus $\|T^n x_q(\epsilon) - y_q(\epsilon)\| < \delta$ for some $n \in \mathbb{N}$.

$$\begin{aligned} \text{Now} \quad & \|Ty_q(\epsilon) - y_q(\epsilon)\| \\ \leq & \|Ty_q(\epsilon) - T^{n+1} x_q(\epsilon)\| + \|T^{n+1} x_q(\epsilon) - T^n x_q(\epsilon)\| + \|T^n x_q(\epsilon) - y_q(\epsilon)\| \\ < & \delta + (1 - q) r + \delta. \end{aligned}$$

Since this is true for every $\delta > 0$, $\|Ty_q(\epsilon) - y_q(\epsilon)\| \leq (1 - q) r$.

If we enumerate the rational values of q in $(0,1)$ as a sequence we see that by compactness of S , $\{y_{q_i}(\epsilon)\}$ has a convergent subsequence $y_{q_i}(\epsilon) \rightarrow y \in S$ when $q_i \rightarrow 1$.

Thus $\lim_i \|Ty_{q_i}(\epsilon) - y_{q_i}(\epsilon)\| \leq \lim_i (1 - q_i) r = 0$.

$$\Rightarrow \lim_i Ty_{q_i}(\epsilon) = \lim_i y_{q_i}(\epsilon)$$

But $Ty_{q_i}(\epsilon) \rightarrow Ty$ by continuity of T .

Hence $Ty = y$. □

In the next theorem we weaken the assumption on the set S ; assuming S is weakly compact rather than norm compact. However, in order to reach the same conclusions as previously, it is necessary to assume the normed space to be complete and the domain to have normal structure:

1.2.2 DEFINITION cf [BM]

A convex set S in a Banach space X is said to have **normal structure** if for each bounded convex subset H of S which contains more than one point, there exists a point $x \in H$ such that x is not a diametrical point of H .

i.e. There exists $x \in H$ such that $r_x(H) = \sup \{\|x - y\| : y \in H\} < \delta(H)$, where $\delta(H)$ denotes the diameter of H defined in 0.15.

It is clear that any convex subset of S will also have normal structure.

1.2.3 EXAMPLE

Let S be a nonempty compact convex set in a Banach space X . Then S has normal structure.

PROOF

Let C be a convex subset of S such that C has at least two elements. Now \bar{C} is compact

and convex.

We shall now prove the following claim which was given as a lemma in [DeMarr] for Banach spaces. However, it is valid in general for normed spaces.

Claim: Let S be a nonempty compact subset of a normed space X and let $K = \overline{\text{co}}(S)$. If $\delta(S) > 0$ then there exists $u \in K$ such that $\sup \{\|x - u\| : x \in S\} < \delta(S)$.

(Since S is nonempty and compact there exist $x_1, x_2 \in S$ such that $\|x_1 - x_2\| = \delta(S)$.)

Now there exists a maximal set $M \subset S$ such that $x_1, x_2 \in M$ and $\|x - y\| = \delta(S)$ for $x, y \in M$ and $x \neq y$.

M is finite: if not we can find a sequence $\{x_n\}$ of distinct points in M with $\|x_m - x_n\| = \delta(S) > 0$ for $m \neq n$. Then $x_n \in S$ and S being compact, x_n must have a convergent subsequence which is clearly not possible.

So let $M = \{x_1, \dots, x_n\}$. Define $u = \sum_{k=1}^n \frac{1}{n} x_k = \frac{1}{n} \sum_{k=1}^n x_k \in K$.

Again, since S is compact there exists $y \in S$ with $\|y - u\| = \sup \{\|x - u\| : x \in S\}$.

Now $\|y - u\| = \left\| y - \sum_{k=1}^n \frac{1}{n} x_k \right\| \leq \sum_{k=1}^n \frac{1}{n} \|y - x_k\| \leq \delta(S)$.

Suppose $\|y - u\| = \delta(S)$, then $\|y - x_k\| = \delta(S)$ for every $1 \leq k \leq n$.

This implies that $y \in M$ by the maximality of M . Then $y = x_k$ for some $1 \leq k \leq n$, which is a contradiction.

Hence $\sup \{\|x - u\| : x \in S\} = \|y - u\| < \delta(S)$.

Hence claim.

If we take a compact convex set S in a normed space such that $\delta(S) > 0$ then it follows from this claim that there exists $u \in S$ with $\sup \{\|x - u\| : x \in S\} < \delta(S)$.

Thus applying this claim to our case we have some $u \in \overline{C}$ such that

$$\sup \{\|u - x\| : x \in \overline{C}\} < \delta(\overline{C}).$$

But $\delta(\overline{C}) = \delta(C)$ and $\sup \{\|u - x\| : x \in C\} \leq \sup \{\|u - x\| : x \in \overline{C}\}$.

Choose $u_0 \in C$ such that $\|u - u_0\| < 1/2 (\delta(C) - \sup \{\|u - x\| : x \in C\})$.

Now $\|x - u_0\| \leq \|x - u\| + \|u - u_0\|$ for every $x \in C$.

Hence $\sup \{\|x - u_0\| : x \in C\} \leq \sup \{\|x - u\| : x \in C\} + \|u - u_0\|$.

Thus $\sup \{\|u_0 - x\| : x \in C\} < \delta(C)$.

Hence S has normal structure. □

Remarks:

It clearly follows that any nonempty bounded closed convex set in a finite dimensional space will have normal structure.

In 1.5.2 we will give an example of a space which lacks normal structure.

1.2.4 THEOREM [BellKirk]

Let C be a nonempty bounded closed convex set in a Banach space X . Suppose that C has normal structure and let S be a weakly compact subset of C .

Let $T: C \rightarrow C$ be nonexpansive for which all orbits of T repeatedly approach S .

Then there exists at least one fixed point for T in S .

PROOF

Let $\mathcal{S} = \{M' \subset C: M' \neq \emptyset, \text{ closed, convex, } T(M') \subset M' \text{ and } M' \cap S \neq \emptyset\}$. Clearly, \mathcal{S} is nonempty since $C \in \mathcal{S}$. Order \mathcal{S} by \supset . Let \mathcal{C} be a chain in \mathcal{S} . Each $M' \in \mathcal{S}$ is weakly closed since it is closed and convex. Since S is weakly compact, each $M' \cap S$ is weakly closed and so each $M' \cap S$ is weakly compact.

But $\{M' \cap S: M' \in \mathcal{C}\}$ satisfies the finite intersection property since \mathcal{S} is ordered by \supset .

Thus $\cap \{M' \cap S : M' \in \mathcal{E}\} \neq \emptyset$. Hence $\wedge \mathcal{E} = \cap \{M' : M' \in \mathcal{E}\}$ is nonempty, closed, convex and invariant under T . Hence $\wedge \mathcal{E} = \cap \{M' : M' \in \mathcal{E}\}$ is a lower bound for \mathcal{E} in \mathcal{S} . Therefore by Zorn's lemma there exists a minimal element M in \mathcal{S} .

Our aim is to show that M is a singleton.

Suppose that $\delta(M) > 0$.

By normal structure, if $\delta(M) > 0$ then there exists a point $x \in M$ such that

$$\sup \{\|x - z\| : z \in M\} = r < \delta(M).$$

Let $K = \overline{\text{co}}(T(M))$.

Now $K \subset M$ since $T(M) \subset M$ and M is closed and convex.

Hence $T(K) \subset T(M) \subset \overline{\text{co}}(T(M)) = K$. Also $\overline{T(M)} \subset K$. By hypothesis on T we have that $\overline{T(M)} \cap S \neq \emptyset$. Hence $K \cap S \neq \emptyset$.

By minimality of M we have that $K = M$.

Let $C_1 = \{z \in M : \|z - y\| \leq r \text{ for each } y \in M\}$. Note that C_1 is nonempty since $x \in C_1$.

Let $C_2 = \{z \in M : \|z - y\| \leq r \text{ for each } y \in T(M)\}$. Clearly, $C_1 \subset C_2$.

Next let $z \in C_2$.

Then, clearly, the closed ball $B(z, r)$ must contain $T(M)$.

Hence it must contain $M = K = \overline{\text{co}}(T(M))$ and so if $y \in M$ then $\|z - y\| \leq r$.

Thus $z \in C_1$.

Hence $C_2 \subset C_1$.

$$\Rightarrow C_1 = C_2$$

Let $z \in C_1$ and let $y \in T(M)$. Then $y = Tx$ for some $x \in M$.

Now $Tz \in M$ and $\|Tz - y\| = \|Tz - Tx\| \leq \|z - x\| \leq r$. Thus $Tz \in C_2 = C_1$.

Hence $T(C_1) \subset C_1$.

But C_1 is closed and convex, and so $\overline{T(C_1)} \subset C_1$.

By hypothesis on T we have that $\overline{T(C_1)} \cap S$ is nonempty. Hence $C_1 \cap S \neq \emptyset$.

Hence by minimality of M we have that $C_1 = M$.

However, for any $x, y \in C_1$ we have that $\|x - y\| \leq r$.

Thus $\delta(C_1) \leq r < \delta(M)$.

This yields a contradiction by which we have that $\delta(M) = 0$.

This implies that T leaves the point in M fixed.

But $M \cap S \neq \emptyset$.

This implies that the fixed point is in S . □

§ 1.3 Existence of fixed points for nonexpansive mappings in reflexive Banach spaces with domains having normal structure

As we have seen, weakening the assumption of compactness of the domain of a nonexpansive mapping will under suitable conditions yield a fixed point for the mapping. Another possibility is to weaken the type of compactness involved, for example, to consider weak compactness.

The theory is most satisfactory in reflexive spaces: the useful property we have in reflexive spaces is that the unit ball is weakly compact. Thus it follows that a weakly closed bounded set is weakly compact, which is an improvement on nonreflexive spaces. Moreover, normed closedness is equivalent to weak closedness provided that a given set is convex. Thus in a reflexive Banach space, all we need to deduce the weak compactness of a bounded convex set is either closedness or weak closedness.

Unfortunately, if we have a nonexpansive mapping which maps a nonempty weakly compact convex set into itself in a Banach space then we are not guaranteed a fixed point for the mapping, as shown in the last section of this chapter (§ 1.10).

It is our aim here to investigate some appropriate conditions which, in addition to weak compactness, will give a set the fixed point property for nonexpansive mappings.

The following theorem of Kirk commences this section where we assume the domain of a nonexpansive mapping has normal structure.

Let us first introduce some concepts.

1.3.1 DEFINITION cf [BellKirk1]

Let H and K be nonempty subsets of a metric space X , where H is bounded. We define :

- (i) $r(H,K) = \inf \{r_x(H) : x \in K\}$
- (ii) $\mathcal{C}(H,K) = \{x \in K : r_x(H) = r(H,K)\}$, where $r_x(H)$ is defined in 0.15.

If $K = X$ then, for short, we shall denote $r(H,K)$ and $\mathcal{C}(H,K)$ by $r(H)$ and $C(H)$ respectively.

In such an event, we shall refer to $r(H)$ as the radius of H and $C(H)$ as the center of H (in X).

Thus it follows that $r(H) = \inf \{r_x(H) : x \in X\}$ and $C(H) = \{x \in X : r_x(H) = r(H)\}$.

1.3.2 THEOREM [Kirk]

Let C be a nonempty bounded closed convex set in a reflexive Banach space X . Suppose that C has normal structure. If $T: C \rightarrow C$ is nonexpansive then T has a fixed point.

Before proving this we must establish the following results which we need for the proof.

1.3.3 LEMMA [BellKirk1]

Let X be a Banach space. Let K be weakly compact and convex in X . If H is a bounded set in X then $\mathcal{C}(H,K)$ is nonempty, closed and convex.

PROOF

For $x \in H$, let $F(x,n) = \{y \in K : \|x - y\| \leq r(H,K) + 1/n\}$.

Let $C_n = \bigcap_{x \in H} F(x,n)$.

We shall now show that each C_n is nonempty.

Let $n \in \mathbb{N}$. Then there exists $y_n \in K$ with $r_{y_n}(H) \leq r(H,K) + 1/n$.

$$\Rightarrow y_n \in F(x,n) \quad \forall x \in H$$

$$\Rightarrow y_n \in C_n$$

Hence each C_n is nonempty.

Thus for every x in H , $F(x,n)$ is nonempty. Moreover, for every x in H , $F(x,n)$ is closed and convex.

Thus each C_n is closed and convex since it is an intersection of nonempty closed convex sets.

Now $C_{n+1} \subset C_n$ since

$$y \in C_{n+1}$$

$$\Rightarrow y \in F(x,n+1) \quad \forall x \in H$$

$$\Rightarrow \|x - y\| \leq r(H,K) + \frac{1}{(n+1)} \leq r(H,K) + \frac{1}{n} \quad \forall x \in H$$

$$\Rightarrow y \in C_n$$

Hence $\{C_n\}$ forms a decreasing sequence of nonempty closed convex subsets of K . Each C_n is weakly closed and hence weakly compact since K is weakly compact. Also $\{C_n\}$ satisfies the finite intersection property. Hence $\bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$. In addition, $\bigcap_{n \in \mathbb{N}} C_n$ is closed and convex.

We now aim to show that $\mathcal{C}(H,K) = \bigcap_{n \in \mathbb{N}} C_n$ which would complete the proof.

Let $y \in \mathcal{C}(H,K)$.

$$\Rightarrow y \in K$$

Fix any $n \in \mathbb{N}$. Now $r_y(H) = r(H,K) \leq r(H,K) + 1/n$.

$$\Rightarrow \|x - y\| \leq r_y(H) \leq r(H,K) + 1/n \quad \forall x \in H$$

$$\Rightarrow y \in C_n$$

Since n is arbitrary, $y \in \bigcap_{n \in \mathbb{N}} C_n$.

Next let $y \in \bigcap_{n \in \mathbb{N}} C_n$.

$$\Rightarrow y \in C_n = \bigcap_{x \in H} F(x,n) \quad \forall n \in \mathbb{N}$$

$$\Rightarrow y \in K \text{ and } \|x - y\| \leq r(H,K) + 1/n \quad \forall x \in H, n \in \mathbb{N}$$

$$\Rightarrow r_y(H) \leq r(H,K) + 1/n \quad \forall n \in \mathbb{N}$$

$$\Rightarrow r_y(H) \leq r(H,K)$$

$$\Rightarrow r_y(H) = r(H,K)$$

$$\Rightarrow y \in \mathcal{C}(H,K)$$

$$\Rightarrow \bigcap_n C_n \subset \mathcal{C}(H,K)$$

Hence $\mathcal{C}(H,K) = \bigcap_{n \in \mathbb{N}} C_n$. □

1.3.4 LEMMA cf [Kirk]

Let F be a bounded closed convex set with at least two elements in a Banach space X . If F has normal structure then $\delta(\mathcal{C}(F,F)) < \delta(F)$.

PROOF

By definition of normal structure there exists $x \in F$ with $r_x(F) < \delta(F)$.

If $z, w \in \mathcal{C}(F,F)$ then $\|z - w\| \leq r_z(F) = r(F,F)$.

Hence $\delta(\mathcal{C}(F,F)) = \sup \{\|z - w\| : w, z \in \mathcal{C}(F,F)\} \leq r(F,F) \leq r_x(F) < \delta(F)$. □

We are now ready to prove 1.3.2.

PROOF (1.3.2)

Let $\mathcal{S} = \{K \subset C : K \neq \emptyset, \text{ closed, convex and } T(K) \subset K\}$. \mathcal{S} is nonempty since C belongs to \mathcal{S} . Order \mathcal{S} by \supset . Let \mathcal{F} be a chain in \mathcal{S} . Each element of \mathcal{S} is bounded (since C is bounded) and weakly closed. Therefore each element in \mathcal{S} is weakly compact since X is reflexive. Now \mathcal{F} satisfies the finite intersection property since \mathcal{S} is ordered by \supset .

Hence $\bigcap \mathcal{F} = \bigcap \{K : K \in \mathcal{F}\}$ is nonempty. Moreover, $\bigcap \mathcal{F}$ is closed, convex, and invariant under T since each element of \mathcal{F} is invariant under T .

Thus $\bigcap \mathcal{F}$ is a lower bound for \mathcal{F} in \mathcal{S} .

By Zorn's lemma, \mathcal{S} has a minimal element M .

If we show that M is a singleton then the proof is complete.

So assume the contrary.

Let us show that $\mathcal{C}(M,M) \in \mathcal{S}$.

Note that since M is weakly compact and convex, $\mathcal{C}(M,M)$ is nonempty, closed and convex by 1.3.3. If we show that $\mathcal{C}(M,M)$ is invariant under T then $\mathcal{C}(M,M) \in \mathcal{S}$.

So let $x \in \mathcal{C}(M,M)$.

Then $\|Tx - Ty\| \leq \|x - y\| \leq r(M,M) \quad \forall y \in M$. Hence $T(M) \subset B(Tx, r(M,M))$.

Let $B_x = B(Tx, r(M,M))$. Now $M \cap B_x \neq \emptyset$, closed, convex and

$T(M \cap B_x) \subset T(M) \subset M \cap B_x$. By minimality of M we have that $M \cap B_x = M$. If not, we will have that $M \cap B_x \subsetneq M$ contradicting the minimality of M .

Let $y \in M$. Then $\|Tx - y\| \leq r(M,M)$ since $M = B_x \cap M$.

$$\Rightarrow r_{T_x}(M) \leq r(M, M).$$

But $Tx \in M$ since $x \in \mathcal{S}(M, M) \subset M$.

$$\Rightarrow r(M, M) = r_{T_x}(M)$$

Therefore $Tx \in \mathcal{S}(M, M)$.

Thus $\mathcal{S}(M, M)$ is invariant under T .

Therefore $\mathcal{S}(M, M) \in \mathcal{S}$.

Since $\mathcal{S}(M, M) \subset M$ and M has normal structure (M is a convex subset of C , where has normal structure), by the above lemma (1.3.4) we have that $\mathcal{S}(M, M)$ is strictly contained in M . This contradicts the minimality of M .

Hence M is a singleton. □

Remark:

Although Kirk stated the following as a corollary, we shall state it as a theorem since it is an improvement on the above theorem (1.3.2) as it does not require C to be necessarily bounded.

1.3.5 THEOREM [Kirk]

Let C be a nonempty closed convex (not necessarily bounded) set with normal structure in a reflexive Banach space X and let $T: C \rightarrow C$ be nonexpansive. Let the sequence $\{T^n p\}$ be bounded for some $p \in C$. Then T has a fixed point.

PROOF

Let $S = \{T^n p\}$. Choose $r > 0$ such that $S \subset B(p, r)$. Let $B_n = B(T^n p, r)$ and let

$$C_n = B_n \cap C.$$

We shall now show that each C_n is nonempty.

Let $x \in S$ and let $n \in \mathbb{N}$.

$$\Rightarrow x = T^m p \text{ for some } m \in \mathbb{N}.$$

$$\Rightarrow T^m p \in B(p, r)$$

$$\text{Now } \|T^n p - T^n x\| = \|T^n p - T^{n+m} p\| \leq \|T^{n-1} p - T^{n+m-1} p\| \leq \|T^m p - p\| \leq r.$$

$$\Rightarrow T^n x \in B_n$$

But $T^n x \in C$.

Hence $T^n x \in C_n$. Therefore C_n is nonempty.

In addition, C_n also contains $T^n(S)$.

Next, let $W_k = \bigcap_{n=k}^{\infty} C_n$ and let $W = \bigcup_{k \in \mathbb{N}} W_k$.

Claim(1): W is nonempty

(Let us show that W_1 is nonempty, i.e. $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$, by which we shall affirm the claim.

Each C_n is closed and convex, and hence weakly closed. But each C_n is bounded.

Therefore, since X is reflexive, each C_n is weakly compact.

Hence $\bigcap_{n=1}^{\infty} C_n \neq \emptyset$ provided that the system $\{C_n : n \in \mathbb{N}\}$ satisfies the finite intersection property. So all we need to verify is that any finite number of elements in $\{C_n : n \in \mathbb{N}\}$ intersect.

To this end, consider a system $\mathcal{S} = \{C_i : i \in F \subset \mathbb{N}\}$ where F is finite. Fix any $i \in F$. Let $j \in F$ be arbitrary ($i \neq j$). Assume w.l.o.g. that $i \leq j$.

Now $\|T^j p - T^i p\| \leq \|T^{j-i} p - p\| \leq r$ since $S \subset B(p, r)$. Thus if we pick an any $i \in F$ then the centers of every other balls $B(T^j p, r)$, for which $B(T^j p, r) \cap C = C_j \in \mathcal{S}$, are contained in $B(T^i p, r)$. Thus, clearly, $\bigcap \mathcal{S} \neq \emptyset$.

Hence claim(1).

Claim(2): W is bounded, convex and invariant under T .

Let $x \in W$. Then $x \in W_k$ for some $k \in \mathbb{N}$. Thus $x \in C_n \quad \forall n \geq k$.

Now $\|x\| = \|x - T^n p + T^n p\| \leq r + \|T^n p\| \quad \forall n \geq k$. But $S = \{T^n p\}$ is bounded.

Hence W is bounded.

Next let $x, y \in W$ and let $\alpha \in [0,1]$. Then $x \in W_k$ and $y \in W_m$ for some $k, m \in \mathbb{N}$. Assume w.l.o.g. that $k \leq m$. Hence $\alpha x + (1 - \alpha) y \in C_n \quad \forall n \geq m$, since each C_n is convex.

Thus $\alpha x + (1 - \alpha) y \in W_m \subset W$. Hence W is convex.

Finally, let $x \in W$. Then as above, for some m , $x \in C_n \quad \forall n \geq m$.

Now $Tx \in C$ and $\|T^{n+1} p - Tx\| \leq \|T^n p - x\| \leq r \quad \forall n \geq m$. Thus $Tx \in C_n \quad \forall n \geq m + 1$.

Hence $Tx \in W_{m+1} \subset W$.)

Thus claim(2).

By continuity of T we have that $T(\overline{W}) \subset \overline{W}$.

Moreover, \overline{W} is convex and bounded.

Since C has normal structure so does \overline{W} .

By theorem 1.3.2, T has a fixed point in \overline{W} . □

§ 1.4 Existence of fixed points for nonexpansive mappings in
uniformly convex spaces

In this section we focus some attention on a narrower class of reflexive spaces called uniformly convex spaces:

1.4.1 DEFINITION [DG] § 2 (7.9)

A Banach space X is said to be **uniformly convex** if there exists a monotone increasing surjection $\varphi : [0,2] \rightarrow [0,1]$ continuous at 0, with $\varphi(0) = 0$, $\varphi(2) = 1$ such that

$\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$ implies that $\left\| \frac{x + y}{2} \right\| \leq (1 - \varphi(\epsilon))$ for $x, y \in X$.

Remark:

A uniformly convex space is strictly convex. To see this:

Suppose $x, y \in X$ such that $x \neq y$ and $\|x\|, \|y\| \leq 1$. Then $\|x - y\| \geq \epsilon$, for some $\epsilon > 0$.

Thus by definition of of uniform convexity it follows that $\|x + y\| \leq 2(1 - \varphi(\epsilon))$.

But $\varphi(\epsilon) > 0$. Hence $\|x + y\| < 2$.

1.4.2 EXAMPLE

Any Hilbert space is uniformly convex.

PROOF

Define $\varphi : [0,2] \rightarrow [0,1]$ by $\varphi(x) = 1 - \sqrt{1 - x^2/4}$ which is a monotone increasing surjection and it is continuous at 0 with $\varphi(0) = 0$ and $\varphi(2) = 1$.

Suppose that $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$.

By the parallelogram identity $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$, we have that

$$\|x + y\|^2 \leq 4 - \|x - y\|^2 \leq 4 - \epsilon^2.$$

$$\text{Thus } \left\| \frac{x+y}{2} \right\|^2 \leq 1 - \left(\frac{\epsilon}{2} \right)^2.$$

$$\Rightarrow \left\| \frac{x+y}{2} \right\| \leq \sqrt{1 - \left(\frac{\epsilon}{2} \right)^2}$$

$$\text{But } \varphi(\epsilon) = 1 - \sqrt{1 - \left(\frac{\epsilon}{2} \right)^2}.$$

$$\text{Thus } \left\| \frac{x+y}{2} \right\| \leq 1 - \varphi(\epsilon). \quad \square$$

In fact, B.J.Pettis showed in [Pettis] (which appeared in 1939) that every uniformly convex space is reflexive.

Dugundji and Granas credit Brodskii and Milman for the result that if K is a bounded closed convex set in a uniformly convex Banach space X , then K has normal structure ([DG] § 2 (7.12 (e))).

It then follows that the following theorem is a special case of 1.3.2.

It should be noted, however, that both these theorems were published in 1965. Moreover, neither Kirk nor Browder have made any reference to the result of the other except that Browder acknowledges having received an unpublished version of Kirk's paper, subsequent to him sending his note for publication. Thus it is evident that these results were discovered independently.

Therefore we find it interesting to study the proof and so we shall give a separate proof of this theorem.

1.4.3 THEOREM [Browder1]

Let C be a nonempty, bounded, closed and convex set in a uniformly convex Banach space X . If $T: C \rightarrow C$ is nonexpansive then T has a fixed point.

PROOF

Let $\mathcal{S} = \{S \subset C : S \neq \emptyset, T(S) \subset S, S \text{ is closed and convex}\}$. Now $\mathcal{S} \neq \emptyset$ since $C \in \mathcal{S}$. Order \mathcal{S} by \supset .

Now \mathcal{S} satisfies the finite intersection property and each element of \mathcal{S} is weakly compact since they are bounded and weakly closed in the reflexive space X (note that X is reflexive by the result of B.J Pettis).

Thus if \mathcal{E} is a chain in \mathcal{S} then $\bigwedge \mathcal{E} = \bigcap \{S : S \in \mathcal{E}\}$ is nonempty. Moreover, $\bigwedge \mathcal{E}$ is closed, convex and invariant under T . Thus $\bigwedge \mathcal{E}$ is a lower bound for \mathcal{E} in \mathcal{S} . Hence by Zorn's lemma \mathcal{S} contains a minimal element M .

Now $T(M) \subset M$ and hence $\overline{\text{co}}(T(M)) \subset M$. Thus $T(\overline{\text{co}}(T(M))) \subset T(M) \subset \overline{\text{co}}(T(M))$. Hence by minimality of M , $\overline{\text{co}}(T(M)) = M$.

We now show that M is a singleton.

Assume the contrary.

Then M has at least two elements. Let $d = \delta(M) > 0$. There exists $x_1, x_2 \in M$ such that

$\|x_1 - x_2\| \geq \frac{d}{2}$. Let $x = \frac{x_1 + x_2}{2}$. Then $x \in M$ by convexity of M .

For any $y \in M$, $x - y = \frac{(x_1 - y) + (x_2 - y)}{2}$. Now $\|x_1 - y\| \leq d$, $\|x_2 - y\| \leq d$ and

$\|(x_1 - y) - (x_2 - y)\| = \|x_1 - x_2\| \geq \frac{d}{2}$.

i.e. $\left\| \frac{x_1 - y}{d} \right\| \leq 1$, $\left\| \frac{x_2 - y}{d} \right\| \leq 1$ and $\left\| \frac{(x_1 - y)}{d} - \frac{(x_2 - y)}{d} \right\| \geq 1/2$

Hence by uniform convexity, $\left\| \frac{(x_1 - y) + (x_2 - y)}{2} \right\| \leq (1 - \varphi(1/2)) d$.

i.e. $\|x - y\| \leq (1 - \varphi(1/2)) d < d$ since $0 < \varphi(1/2) < 1$

Let $d_1 = [1 - \varphi(1/2)] d < d = \delta(M)$ and let $M' = \bigcap_{y \in M} \{ u \in M : \|u - y\| \leq d_1 \}$.

Then $M' \subset M$ and M' is nonempty since $x \in M'$.

Moreover, M' is a closed convex subset of M since it is an intersection of closed convex sets.

Claim: $T(M') \subset M'$

(Let $z \in M'$, and let $y \in M$. Since $\overline{\text{co}}(T(M)) = M$, for every $\epsilon > 0$ we can find a convex

combination $\sum_{i=1}^n \lambda_i T(x_i)$ (where $\{x_i\} \subset M$) such that $\left\| y - \sum_{i=1}^n \lambda_i T(x_i) \right\| < \epsilon$, where $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^n \lambda_i = 1$.

$$\begin{aligned} \text{Hence } & \|Tz - y\| \\ & \leq \left\| Tz - \sum_{i=1}^n \lambda_i T(x_i) \right\| + \left\| \sum_{i=1}^n \lambda_i T(x_i) - y \right\| \\ & < \left\| Tz - \sum_{i=1}^n \lambda_i T(x_i) \right\| + \epsilon \\ & = \left\| \sum_{i=1}^n \lambda_i (Tz - Tx_i) \right\| + \epsilon \\ & \leq \sum_{i=1}^n \lambda_i \|Tz - Tx_i\| + \epsilon. \end{aligned}$$

But $\|Tz - Tx_i\| \leq \|z - x_i\| \leq d_1$ since $z \in M'$ and $x_i \in M$ for $1 \leq i \leq n$.

Hence $\|Tz - y\| \leq \sum_{i=1}^n \lambda_i d_1 + \epsilon$. Since ϵ was arbitrary, $\|Tz - y\| \leq d_1 \quad \forall y \in M$.

Further, $Tz \in M$. Thus $Tz \in M'$.

Hence M' is invariant under T .

Thus claim.

However, this causes a contradiction to the minimality of M .

Hence M is a singleton by which we obtain the result. □

Remarks:

An interesting feature of this theorem is that it does not require C to be compact, so that it is a partial extension of Schauder's fixed point theorem.

On the other hand, if T is weakly continuous then $T(C)$ is weakly compact, since C is weakly compact (X is reflexive and C is weakly closed and bounded).

Thus $\text{cl}(T(C)) = T(C)$ is weakly compact. Thus this theorem becomes a special case of Tychonov's fixed point theorem.

Browder published an earlier version of this theorem for Hilbert spaces in [Browder]. We shall state the Hilbert space version in § 1.6 which deals with nonexpansive mappings in Hilbert spaces.

The following example (which Browder owes to Richard Beals) shows that the above theorem (1.4.3) is not extendable to the general class of Banach spaces.

1.4.4 Example [Browder1]

Let $X = c_0$, the space of sequences converging to zero, with the supremum norm, and let B be the unit ball in c_0 .

Define $T: c_0 \rightarrow c_0 : (x_1, x_2, x_3, \dots) \rightarrow (1, x_1, x_2, x_3, \dots)$

Then $\|T\underline{x} - T\underline{y}\|$

$$= \|(1 - 1, x_1 - y_1, x_2 - y_2, \dots)\|$$

$$= \|\underline{x} - \underline{y}\|.$$

Hence T is nonexpansive.

Clearly T maps B into B .

But T has no fixed points in B since c_0 consists of sequences converging to zero. □

Remark:

As mentioned at the beginning of § 1.2, this example shows that Schauder's fixed point theorem fails if we delete compactness from its hypotheses.

§ 1.5 Fixed point property of sets (for nonexpansive mappings in reflexive Banach spaces) which have asymptotic normal structure.

Baillon and Schöneberg referred to a Banach space X having the Browder–Göhde–Kirk property (abbreviation: B–G–K property) if every nonempty bounded closed convex subset C of X has the fixed point property for nonexpansive mappings.

The reason for this terminology is that both Browder and Göhde independently proved that every uniformly convex Banach space has the above-mentioned property, while Kirk established this property for the wider class of reflexive Banach spaces by assuming that C has normal structure.

All three of these results appeared in 1965: we have seen Browder's theorem in 1.4.3, Göhde's theorem can be found in [Göhde1], and Kirk's theorem was seen in 1.3.2.

The following theorem (1.5.3) of Baillon and Schöneberg extends 1.3.2 by establishing this property for reflexive Banach spaces, by assuming that the set C has a weaker structure than normal structure, called asymptotic normal structure:

1.5.1 DEFINITION cf [BS]

A convex set S in a Banach space X is said to have **asymptotic normal structure**, if every bounded closed convex subset C of S , where C consists of more than one point, satisfies the following condition:

if $\{x_n\}$ is any sequence in C satisfying $x_n - x_{n+1} \rightarrow 0$ ($n \rightarrow \infty$), then there exists a point $x \in C$ such that $\liminf_n \|x_n - x\| < \delta(C)$.

Remark:

Note that normal structure implies asymptotic normal structure.

To see this :

Let S have normal structure and let C be a bounded closed convex subset of S which

consists of more than one point. Suppose that $\{x_n\}$ is a sequence in C with $x_n - x_{n+1} \rightarrow 0$ as $n \rightarrow \infty$. By definition of normal structure, there exists $x \in C$ with $\sup \{\|x - y\| : y \in C\} < \delta(C)$. Hence $\|x_n - x\| \leq \sup \{\|x - y\| : y \in C\} < \delta(C) \quad \forall n$.

$$\Rightarrow \alpha_n = \inf_{r \geq n} \|x_r - x\| \leq \sup \{\|x - y\| : y \in C\} < \delta(C) \quad \forall n$$

$$\Rightarrow \lim_n \alpha_n \leq \sup \{\|x - y\| : y \in C\} < \delta(C)$$

$$\text{i.e. } \liminf_n \|x_n - x\| < \delta(C)$$

Hence S has asymptotic normal structure. □

Let us state the following theorem which will serve as an example to show that asymptotic normal structure does not imply normal structure.

1.5.2 THEOREM [BS]

Let $\beta \geq 1$ and let X_β be the ℓ^2 space renormed according to

$$\|x\|_\beta = \max \{\|x\|_2, \beta \|x\|_\infty\} \text{ where } \|x\|_\infty \text{ denotes the } \ell^\infty \text{ norm.}$$

Then

- (i) X_β has normal structure if and only if $\beta < \sqrt{2}$ and
- (ii) X_β has asymptotic normal structure if and only if $\beta < 2$.

From this theorem it is clear that asymptotic normal structure does not imply normal structure. Further, as mentioned in the remark following 1.2.3, this theorem gives examples of spaces which lack normal structure (for example, X_2).

Since for our purpose this theorem serves only to provide examples, we shall omit the proof.

1.5.3 THEOREM [BS]

Let X be a reflexive Banach space. Let C be a nonempty bounded closed convex set in X .

Suppose that C has asymptotic normal structure.

If $T: C \rightarrow C$ is a nonexpansive mapping then T has a fixed point in C .

We first establish two lemmas which will be needed for the proof of this theorem.

1.5.4 LEMMA [BS]

Let X be a reflexive Banach space, C be a nonempty bounded closed convex set in X and

let $T: C \rightarrow C$ be nonexpansive. Then there exists a sequence $\{x_n\} \subset C$ such that

$x_n - Tx_n \rightarrow 0$ and $x_n - x_{n+1} \rightarrow 0$ as $n \rightarrow \infty$.

PROOF

Fix any $z \in C$.

For each $n \in \mathbb{N}$, define $T_n: C \rightarrow C: x \rightarrow z/n + (1 - 1/n)Tx$. Then $T_n: C \rightarrow C$ by convexity of C . Further, each T_n is a contraction mapping. Since C is closed, we can apply Banach's fixed point theorem on C to get a unique fixed point, say x_n , for T_n .

Thus $x_n = z/n + (1 - 1/n)T(x_n)$.

Hence $x_n - Tx_n = \frac{z - T(x_n)}{n} \rightarrow 0$ ($n \rightarrow \infty$) (note that $\{Tx_n\}$ is bounded).

By nonexpansiveness of T ,

$$\begin{aligned} & \|x_n - x_{n+1}\| \\ = & \left\| \frac{1}{(n+1)n} (z - Tx_n) + \left(1 - \frac{1}{n+1}\right) (Tx_n - Tx_{n+1}) \right\| \\ \leq & \frac{1}{n(n+1)} \|z - Tx_n\| + \left(1 - \frac{1}{n+1}\right) \|x_n - x_{n+1}\|. \end{aligned}$$

Hence $\|x_n - x_{n+1}\| \leq \frac{\|z - Tx_n\|}{n} \rightarrow 0$ ($n \rightarrow \infty$). □

The hard work in the proof of 1.5.3 comes in the following lemma which says that ϵ -fixed points in a minimal closed convex set are nearly "ends of diameters".

1.5.5 LEMMA [Karlovitz]

Let S be a weakly compact convex set in a Banach space X and let $T: S \rightarrow S$ be nonexpansive.

Suppose that S is minimal in the sense that it contains no proper closed convex subset which is invariant under T . Let $\{x_n\}$ be a sequence in S with $\|Tx_n - x_n\| \rightarrow 0$.

Then $\lim_n \|x - x_n\| = \delta(S) \quad \forall x \in S$.

PROOF

Suppose $x \in S$.

By the weak compactness of S , $\{\|x_n - x\|\}$ is bounded.

Suppose $\{\|x_{n_k} - x\|\}$ is any convergent subsequence of $\{\|x_n - x\|\}$ with limit $s'(x)$. It will suffice to show that $s'(x) = \delta(S)$.

We first show that for any $z \in S$, $\{\|x_{n_k} - z\|\} \rightarrow s'(x)$.

Suppose $\{\|x_{n_{k_1}} - z\|\}$ is any convergent subsequence of $\{\|x_{n_k} - z\|\}$ with limit $s''(z)$. By the same reasoning as before, it will be sufficient to show that $s''(z) = s'(x)$.

Claim(1): $\|x_{n_{k_1}} - z\| \rightarrow s'(x)$

Let $E = \{y \in S: \limsup_1 \|x_{n_{k_1}} - y\| \leq \min \{s''(z), s'(x)\}\}$.

E is nonempty:

If $s''(z) \leq s'(x)$ then $z \in E$, and if $s'(x) \leq s''(z)$ then $x \in E$.

E is convex:

Suppose $y_1, y_2 \in E$ and $t \in [0,1]$.

$$\begin{aligned} & \text{Then } \limsup_1 \|x_{nk_1} - t y_1 - (1-t) y_2\| \\ & \leq t \limsup_1 \|x_{nk_1} - y_1\| + (1-t) \limsup_1 \|x_{nk_1} - y_2\| \\ & \leq \min \{s''(z), s'(x)\}. \end{aligned}$$

E is closed:

Suppose $\{y_i\} \subset E$ and $y_i \rightarrow y$.

$$\begin{aligned} & \text{Then } \limsup_1 \|x_{nk_1} - y\| \leq \limsup_1 (\|x_{nk_1} - y_i\| + \|y_i - y\|) \quad \forall i. \\ & \Rightarrow \limsup_1 \|x_{nk_1} - y\| \leq \min \{s''(z), s'(x)\} \text{ by taking limits in } i \end{aligned}$$

E is invariant under T:

Suppose $y \in E$.

$$\begin{aligned} & \Rightarrow \limsup_1 \|x_{nk_1} - Ty\| \\ & \leq \limsup_1 \|x_{nk_1} - Tx_{nk_1}\| + \limsup_1 \|Tx_{nk_1} - Ty\| \\ & = \limsup_1 \|Tx_{nk_1} - Ty\| \quad \text{since } \|Tx_n - x_n\| \rightarrow 0 \\ & \leq \limsup_1 \|x_{nk_1} - y\| \quad \text{since } T \text{ is nonexpansive} \\ & \leq \min \{s''(z), s'(x)\} \end{aligned}$$

So $Ty \in E$.

By the minimality condition on S it follows that $E = S$.

Hence for every $y \in S$ we have that $\limsup_1 \|x_{nk_1} - y\| \leq \min \{s''(z), s'(x)\}$.

But since $\|x_{nk_1} - z\| \rightarrow s''(z)$ it follows that $s''(z) \leq s'(x)$

and since $\|x_{nk} - x\| \rightarrow s'(x)$ it follows that $s'(x) \leq s''(z)$.

Hence $s''(z) = s'(x)$.

Thus claim(1).

Thus we have established that for any $z \in S$, $\|x_{n_k} - z\| \rightarrow s'(x)$.

We now complete the proof by showing that $s'(x) = \delta(S)$.

Claim(2): $s'(x) = \delta(S)$

(Clearly $s'(x) \leq \delta(S)$.)

Let $F = \{y \in S: \|y - z\| \leq s'(x) \quad \forall z \in S\}$.

F is nonempty:

By weak compactness of S , there exists a weakly convergent subsequence $\{x_{n_{k_r}}\}$ of $\{x_{n_k}\}$, with limit $u \in S$, say.

Then for every $z \in S$,

$$\begin{aligned} & \|u - z\| \\ = & \|\text{w-lim}_r(x_{n_{k_r}} - z)\| \\ \leq & \liminf_r \|x_{n_{k_r}} - z\| \quad (\text{by the weak lower semicontinuity of the norm}) \\ = & s'(x), \quad \text{as previously established.} \end{aligned}$$

Thus $u \in F$.

F is convex:

Suppose $y_1, y_2 \in F$ and $t \in [0,1]$.

For $z \in S$,

$$\begin{aligned} & \|t y_1 + (1-t) y_2 - z\| \\ \leq & t \|y_1 - z\| + (1-t) \|y_2 - z\| \\ \leq & s'(x). \end{aligned}$$

F is closed:

Suppose $\{y_i\} \subset F$ and $y_i \rightarrow y$. Let $z \in S$.

Then $\|y - z\| \leq \|y_i - z\| + \|y_i - y\| \leq s'(x) + \|y_i - y\|$.

$\Rightarrow \|y - z\| \leq s'(x)$ by taking limits in i

F is invariant under T:

Suppose $y \in F$ and $\epsilon > 0$ is given.

Note that by the minimality condition on S, $S = \overline{\text{co}}(T(S))$.

Hence for $z \in S$ we can find a convex combination $\sum_{i=1}^n \lambda_i T x_i$ (where $\{x_i\} \subset S$)

such that $\|z - \sum_{i=1}^n \lambda_i T x_i\| < \epsilon$.

Then $\|T y - z\|$

$$< \|T y - \sum_{i=1}^n \lambda_i T x_i\| + \epsilon$$

$$\leq \sum_{i=1}^n \lambda_i \|y - x_i\| + \epsilon$$

$$\leq s'(x) + \epsilon \quad \text{since } y \in F.$$

Thus $\|T y - z\| \leq s'(x)$, since ϵ was arbitrary.

Note that $T y \in S$.

Thus $T y \in F$.

Hence F is invariant under T.

By the minimality condition on S it follows that $F = S$.

Thus $s'(x) \geq \delta(S)$, and so $s'(x) = \delta(S)$.

Hence claim(2).

Thus any convergent subsequence of $\{\|x_n - x\|\}$ has limit $\delta(S)$, and this completes the proof. □

We shall now prove 1.5.3.

PROOF (1.5.3)

Let $\mathcal{S} = \{K \subset C : K \neq \emptyset, \text{ bounded, closed, convex and } T(K) \subset K\}$.

Then $\mathcal{S} \neq \emptyset$ since $C \in \mathcal{S}$.

Since X is reflexive, each element of \mathcal{S} is weakly compact. If we order \mathcal{S} by \supset then \mathcal{S} satisfies the finite intersection property. Thus if \mathcal{C} is a chain in \mathcal{S} then $\bigcap \mathcal{C}$ is nonempty. Moreover, $\bigcap \mathcal{C}$ is bounded, closed, convex and invariant under T . Hence $\bigcap \mathcal{C}$ is a lower bound for \mathcal{C} in \mathcal{S} . By Zorn's lemma \mathcal{S} has a minimal element M .

Claim: M is a singleton

(By 1.5.4 there exists a sequence $\{x_n\}$ in M with $x_n - Tx_n \rightarrow 0$ and

$x_n - x_{n+1} \rightarrow 0$ as $n \rightarrow \infty$.

Hence, since M is weakly compact, by 1.5.5 it follows that for every x in M

$\|x_n - x\| \rightarrow \delta(M)$ as $n \rightarrow \infty$.

Suppose that M has at least two elements.

By asymptotic normal structure of C there exists $y \in M$ with $\liminf_n \|x_n - y\| < \delta(M)$.

However, according to what we have said above by using 1.5.5, putting y for x we see that

$\delta(M) = \lim_n \|x_n - y\| = \liminf_n \|x_n - y\| < \delta(M)$.

This gives the required contradiction.)

Hence M is a singleton which is fixed under T . □

§ 1.6 Fixed point theorems for nonexpansive mappings in Hilbert spaces

Throughout this section, \mathcal{X} will always denote a Hilbert space.

First, as mentioned in § 1.4, we shall now give the following special case of 1.4.3 which was published earlier than 1.4.3.

Note that the theorems in § 1.3 and § 1.5 which require the space to be reflexive are valid for Hilbert spaces in particular.

1.6.1 THEOREM [Browder]

If $T: C \rightarrow C$ is nonexpansive, where C is bounded, closed and convex in \mathcal{X} , then T has a fixed point in C .

In 1.4.2 we showed that every Hilbert space is uniformly convex. Thus it is clear that this theorem is a special case of 1.4.3. Since we have already proven a general version of this theorem (1.4.3), we shall omit the proof here.

The following theorem serves as an example to show that the unit ball in an arbitrary Hilbert space need not have the fixed point property for maps which are "arbitrarily close to being nonexpansive".

1.6.2 THEOREM [Browder]

Let $\epsilon > 0$, \mathcal{X} of infinite dimension. Then there exists a mapping T of the unit ball C in \mathcal{X} into C such that T has no fixed points in C , while for every $u, v \in C$

$$\|Tu - Tv\| \leq (1 + \epsilon) \|u - v\|.$$

PROOF

We first establish the result for the Hilbert space ℓ^2 . Suppose $\epsilon > 0$.

Define $T: \ell^2 \rightarrow \ell^2: (x_1, x_2, \dots) \rightarrow (\epsilon(1 - \|\underline{x}\|), x_1, x_2, \dots)$

If $\underline{x} = \underline{0}$ then $T\underline{x} \neq \underline{0} = \underline{x}$.

If $\|\underline{x}\| = 1$ then $T\underline{x} = (0, x_1, x_2, \dots) \neq \underline{x}$.

If $0 \leq \|\underline{x}\| < 1$ then $\|T\underline{x}\| > \|\underline{x}\|$.

Hence T has no fixed points.

Also note that the unit ball C is invariant under T .

If $\underline{x}, \underline{y} \in C$, then

$$\begin{aligned} & \|T\underline{x} - T\underline{y}\| \\ = & \|\epsilon(1 - \|\underline{x}\|)e_1 + R\underline{x} - \epsilon(1 - \|\underline{y}\|)e_1 - R\underline{y}\| \\ = & \|\epsilon(\|\underline{y}\| - \|\underline{x}\|)e_1 + R(\underline{x} - \underline{y})\| \\ \leq & \epsilon\|\underline{y} - \underline{x}\| + \|R\|\|\underline{x} - \underline{y}\| \\ = & (1 + \epsilon)\|\underline{x} - \underline{y}\|. \end{aligned}$$

Hence $\|T\underline{x} - T\underline{y}\| \leq (1 + \epsilon)\|\underline{x} - \underline{y}\| \quad \forall \underline{x}, \underline{y} \in C$.

For any separable space \mathcal{X} we have that $\mathcal{X} \cong \ell^2$; so the result holds for separable Hilbert spaces.

If \mathcal{X} is not separable we form a subspace \mathcal{X}_s by forming a countable orthonormal basis.

Then $\mathcal{X}_s \cong \ell^2$. Let P be the orthogonal projection: $\mathcal{X} \rightarrow \mathcal{X}_s$ and let T_0 be a fixed point

free map: $\mathcal{X}_s \rightarrow \mathcal{X}_s$, with the required property: $\|T_0(\underline{u}) - T_0(\underline{v})\| \leq (1 + \epsilon)\|\underline{u} - \underline{v}\|$

for $\underline{u}, \underline{v}$ in the unit ball of \mathcal{X}_s . Define $T: \mathcal{X} \rightarrow \mathcal{X}_s$ by $T = T_0 \circ P$.

Let $\underline{x}, \underline{y} \in C$, where C is the unit ball in \mathcal{X} .

$$\begin{aligned} \text{Then } & \|T\underline{x} - T\underline{y}\| \\ = & \|T_0(P\underline{x}) - T_0(P\underline{y})\| \\ \leq & (1 + \epsilon)\|P\underline{x} - P\underline{y}\| \\ \leq & (1 + \epsilon)\|P\|\|\underline{x} - \underline{y}\| \\ = & (1 + \epsilon)\|\underline{x} - \underline{y}\| \quad \text{as } \|P\| = 1. \end{aligned}$$

1.6.3 THEOREM [DG] § 2 (1.5)

Let C be the closed ball of center 0 and radius δ in a Hilbert space \mathcal{X} and let $T: C \rightarrow \mathcal{X}$ be nonexpansive. Then either

- (i) T has a fixed point or
- (ii) there exists $x \in \partial C$ and a $\lambda \in (0,1)$ such that $x = \lambda Tx$.

We first establish the following lemma.

1.6.4 LEMMA [DG] § 2(1.4)

Let \mathcal{X} be a Hilbert space and let C be a closed ball with center 0 and radius δ .

Define a map $r: \mathcal{X} \rightarrow C$ by $rx = \begin{cases} x & \text{if } \|x\| \leq \delta \\ \frac{\delta x}{\|x\|} & \text{if } \|x\| \geq \delta \end{cases}$

Then r is nonexpansive.

PROOF

W.l.o.g. assume that \mathcal{X} is a real Hilbert space.

Claim: $\langle u - ru, rv - ru \rangle \leq 0$ if $u, v \neq 0$

(If $\|u\| \leq \delta$ then $ru = u$; hence claim.)

Next assume that $\|u\| \geq \delta$.

We have $\langle u - ru, rv - ru \rangle = \begin{cases} \left(1 - \frac{\delta}{\|u\|}\right) (\langle u, v \rangle - \delta \|u\|) & \text{if } \|v\| \leq \delta \\ \left(1 - \frac{\delta}{\|u\|}\right) \left(\frac{\delta}{\|v\|} \langle u, v \rangle - \delta \|u\|\right) & \text{if } \|v\| \geq \delta. \end{cases}$

Since $|\langle u, v \rangle| \leq \|u\| \|v\|$, we have $\langle u - ru, rv - ru \rangle \leq 0$ when $\|u\| \geq \delta$, since \mathcal{X} is real.)

Hence claim.

Now $x - y = rx - ry + x - rx + ry - y = rx - ry + a$ where $a = x - rx + ry - y$.

Hence $\|x - y\|^2 = \|rx - ry\|^2 + \|a\|^2 + 2 \langle a, rx - ry \rangle$.

Hence from our claim we have the following:

$$\begin{aligned} & \langle a, rx - ry \rangle \\ = & \langle x - rx + ry - y, rx - ry \rangle \\ = & - \langle x - rx, ry - rx \rangle - \langle y - ry, rx - ry \rangle \\ \geq & 0 \quad \text{if } x, y \neq 0. \end{aligned}$$

Hence $\|x - y\|^2 \geq \|rx - ry\|^2$.

Note that if $x = 0$ or $y = 0$ then the inequality holds.

Hence result. □

Let us return to the proof of the theorem.

PROOF (1.6.3)

By 1.6.4, since the map $r : X \rightarrow C$ is nonexpansive, so is $r \circ T : C \rightarrow C$. By 1.6.1

there exists $x \in C$ with $rTx = x$. If $Tx \in C$ then $x = rTx = Tx$ by which x is a fixed point for T .

If $Tx \notin C$, then $x = rTx = \frac{\delta}{\|Tx\|} Tx$.

Hence $x \in \partial C$.

Putting $\lambda = \frac{\delta}{\|Tx\|} < 1$ we get $x = \lambda Tx$. □

Remark:

Note that if T has no fixed points then the conclusion of this theorem is quite the opposite to the conclusion of Schaefer's theorem in 0.13.

1.6.5 COROLLARY [DG] § 2 (1.6)

Let X be real, C a closed ball with center at 0 and radius of δ and let $T: C \rightarrow X$ be nonexpansive.

Suppose that for each $x \in \partial C$ any one of the following conditions holds:

- (a) $\|Tx\| \leq \|x\|$
- (b) $\|Tx\| \leq \|x - Tx\|$
- (c) $\|Tx\|^2 \leq \|x\|^2 + \|x - Tx\|^2$
- (d) $\langle x, Tx \rangle \leq \|x\|^2$.

Then T has a fixed point.

PROOF

Assume the contrary. Then T has no fixed points.

By 1.6.3 there exists $x \in \partial C$ and a $\lambda > 1$ with $Tx = \lambda x$.

The following hold:

- (a') $\|Tx\| = \lambda \|x\| > \|x\|$
- (b') $\|x - Tx\| = \|Tx - x\| = \|\lambda x - x\| = (\lambda - 1) \|x\| < \lambda \|x\| = \|Tx\|$
- (d') $\langle x, Tx \rangle = \langle x, \lambda x \rangle = \lambda \langle x, x \rangle = \lambda \|x\|^2 > \|x\|^2$

Clearly (a'), (b') and (d') contradict (a), (b) and (d) respectively.

Hence at least (c) must hold. Now

- (c') for $\lambda' = \frac{1}{\lambda}$ we have that $0 < \lambda' < 1$ and $\lambda'Tx = x$.

Moreover, by (c), $\|Tx\|^2 \leq \|\lambda'Tx\|^2 + \|\lambda'Tx - Tx\|^2$.

Hence $1 \leq (\lambda')^2 + (\lambda' - 1)^2$ ($\|Tx\| \neq 0$ since $\lambda x \neq 0$).

But $(\lambda')^2 + (\lambda' - 1)^2 < \lambda' + (1 - \lambda') = 1$ contradicting the above inequality.

Hence T must have a fixed point. □

In the previous sections we have dealt with normed spaces in which either the whole space has a specific property (eg § 1.4, § 1.6) or the domain of a nonexpansive mapping has a specific property (eg § 1.3).

The typical spaces were Hilbert spaces or classes of spaces for which Hilbert spaces are standard examples, such as uniformly convex spaces or reflexive Banach spaces.

In this section we concentrate on a special class of metric spaces called hyperconvex spaces. Even when considering hyperconvex normed spaces, it transpires that this class is quite distinct from the class of Hilbert spaces: it will be seen that (real) ℓ^∞ is hyperconvex (but not Hilbert) and the real Hilbert space \mathbb{R}^2 is not hyperconvex.

We first define the notion of hyperconvexity and study some basic properties which are relevant to us, since this notion is not so well known in functional analysis.

However, a theorem of Nachbin and Kelly states that a real Banach space is hyperconvex if and only if it is a space $C(E)$ of continuous real functions on a stonian (extremally disconnected compact Hausdorff) space E ; i.e. the self adjoint part of a commutative von Neumann algebra. See [Baillon] and [Takesaki]. Thus (real) ℓ^∞ and L^∞ are hyperconvex.

Moreover, hyperconvex spaces have some significance in Category Theory since, due to an important result of Aronszajn and Panitchpakdi (which we shall give in 1.9.2) it follows that in the category of metric spaces \mathbf{Met} , with morphisms all nonexpansive mappings, an object (X,d) is injective if and only if it is hyperconvex.

An interesting aspect of this theory is that the existence of fixed points for nonexpansive mappings in hyperconvex spaces does not rely on the usual assumptions on the domain of the nonexpansive mapping, such as compactness or convexity. To begin with,

hyperconvexity is, in general, defined for metric spaces and thus convexity does not play any role. Further, we shall show that a bounded hyperconvex space has the fixed point property for nonexpansive mappings where the space need not be compact.

1.7.1 DEFINITIONS [Baillon]

A metric space (M, d) is called:

- (a) **metrically convex** if for any two distinct points, x and y , and for any $\alpha, \beta > 0$ such that $d(x, y) = \alpha + \beta$, implies that there exists $z \in M$ with $d(x, z) = \alpha$ and $d(y, z) = \beta$.
- (b) **hyperconvex** if for any indexed class of closed balls in M , $B(x_i, r_i)$, $i \in I$, satisfying the condition that $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$, we have that $\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$.

Remarks:

Any convex set S in a normed space is metrically convex:

Let $x, y \in S$ be such that $\|x - y\| = \alpha + \beta$ for $\alpha, \beta > 0$.

Then $\frac{\beta x}{\alpha + \beta} + \frac{\alpha y}{\alpha + \beta} \in S$. Let $z = \frac{\beta x}{\alpha + \beta} + \frac{\alpha y}{\alpha + \beta}$.

Now $\|x - z\|$

$$= \left\| x - \frac{\beta x}{\alpha + \beta} - \frac{\alpha y}{\alpha + \beta} \right\|$$

$$= \left\| \frac{\alpha x + \beta x - \beta x - \alpha y}{\alpha + \beta} \right\|$$

$$= \left\| \frac{\alpha x - \alpha y}{\alpha + \beta} \right\|$$

$$= \frac{\alpha}{\alpha + \beta} \|x - y\|$$

$$= \alpha$$

$$\text{Similarly, } \|z - y\| = \left\| \frac{\beta x + \alpha y - \alpha y - \beta y}{\alpha + \beta} \right\| = \frac{\beta}{\alpha + \beta} \|x - y\| = \beta.$$

Thus S is metrically convex.

Now if $B(x_i, r_i)$ and $B(x_j, r_j)$ are closed balls in a convex set such that $\|x_i - x_j\| \leq r_i + r_j$ then from this argument above, it is clear that there exists some z such that $\|x_i - z\| \leq r_i$ and $\|x_j - z\| \leq r_j$. Thus $B(x_i, r_i) \cap B(x_j, r_j)$ is nonempty.

Thus a convex set satisfies the definition of hyperconvexity, provided we restrict the indexed class of balls to having only two elements.

Next, hyperconvexity implies metric convexity:

let H be hyperconvex. Let $x, y \in H$ and let $d(x, y) = \alpha + \beta$, $\alpha, \beta > 0$. Then for $z \in B(x, \alpha) \cap B(y, \beta)$ it follows that $d(x, z) \leq \alpha$ and $d(y, z) \leq \beta$ which are in fact equalities since $d(x, z) + d(z, y) \geq \alpha + \beta$.

1.7.2 DEFINITION [Baillon]

Let H be a metric space. H is said to have the **binary intersection property** if for any set of closed balls $B(x_i, r_i)$, $i \in I$, such that every two balls intersect, then $\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$.

1.7.3 DEFINITION [Baillon]

A family $\{H_\alpha\}_{\alpha \in \Lambda}$ of hyperconvex spaces is said to have the **finite intersection property** if any finite intersection of elements in the family is nonempty and hyperconvex.

1.7.4 PROPOSITION [Baillon]

A metric space H is hyperconvex if and only if it is both metrically convex and has the binary intersection property.

PROOF

Assume hyperconvexity.

Suppose $B(x_i, r_i) \cap B(x_j, r_j) \neq \emptyset \quad \forall i, j \in I$ then $d(x_i, x_j) \leq r_i + r_j \quad \forall i, j \in I$.

Hence $\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$ by hyperconvexity. Hence the binary intersection property follows.

In addition, in our remark above, we have shown that hyperconvexity implies metric convexity.

Conversely, assume that H is metrically convex and H has the binary intersection property.

Take any indexed class of closed balls in H , $B(x_i, r_i)$, $i \in I$, where

$$d(x_i, x_j) \leq r_i + r_j \quad \forall i, j \in I.$$

W.l.o.g. we assume that $r_i > 0 \quad \forall i \in I$.

Let $i, j \in I$. If $d(x_i, x_j) = r_i + r_j$ then by metric convexity $B(x_i, r_i) \cap B(x_j, r_j) \neq \emptyset$.

If $d(x_i, x_j) < r_i + r_j$ then $d(x_i, x_j) = \frac{r_i}{r_i + r_j} d(x_i, x_j) + \frac{r_j}{r_i + r_j} d(x_i, x_j)$.

Let $\alpha_i = \frac{r_i}{r_i + r_j} d(x_i, x_j)$ and let $\alpha_j = \frac{r_j}{r_i + r_j} d(x_i, x_j)$. Then again by metric convexity

$$B(x_i, \alpha_i) \cap B(x_j, \alpha_j) \neq \emptyset.$$

$$\Rightarrow B(x_i, r_i) \cap B(x_j, r_j) \neq \emptyset \quad \forall i, j \in I.$$

Hence $\bigcap_{i \in I} B(x_i, r_i) \neq \emptyset$ by binary intersection property.

Hence hyperconvexity. □

1.7.5 EXAMPLES [Baillon]

(a) The real line \mathbb{R} with its usual metric is hyperconvex.

(b) The (real) space ℓ^∞ is hyperconvex.

PROOF

(a)

Suppose that we take a system of closed intervals $[a_i, b_i]$, $i \in I$ for some index set I such that every two of the intervals intersect. Then, by using induction and the fact that the intersection of any finite number of intervals, say $[a_1, b_1], \dots, [a_k, b_k]$ is nonempty if and only if $\max\{a_1, \dots, a_k\} \leq \min\{b_1, \dots, b_k\}$, it can be shown that the intersection of any

finite number of such intervals is nonempty.

Then the intersection of the whole system is nonempty by compactness of these intervals.

Thus \mathbb{R} has the binary intersection property. Clearly, \mathbb{R} is metrically convex.

Hence \mathbb{R} is hyperconvex.

(b)

We have shown in the remark following 1.7.1 that any convex set is metrically convex.

Thus it follows that any normed space is metrically convex.

So in order to show that ℓ^∞ is hyperconvex, we now show that it has the binary intersection property.

Let $\{B(\underline{x}_i, r_i) : i \in I\}$ be a system of closed balls in ℓ^∞ such that every two balls intersect.

Let $i, j \in I$.

$$\Rightarrow B(\underline{x}_i, r_i) \cap B(\underline{x}_j, r_j) \neq \emptyset$$

Then there exists \underline{y} such that $\|\underline{x}_i - \underline{y}\| \leq r_i$ and $\|\underline{x}_j - \underline{y}\| \leq r_j$.

$$\Rightarrow \sup \{ |x_{i,n} - y_n| : n \in \mathbb{N} \} \leq r_i \text{ and } \sup \{ |x_{j,n} - y_n| : n \in \mathbb{N} \} \leq r_j$$

$$\Rightarrow |x_{i,n} - y_n| \leq r_i \text{ and } |x_{j,n} - y_n| \leq r_j \quad \forall n \in \mathbb{N}$$

Thus $y_n \in B(x_{i,n}, r_i) \cap B(x_{j,n}, r_j) \quad \forall n \in \mathbb{N}$. Fix any $n \in \mathbb{N}$.

Now $\{B(x_{i,n}, r_i) : i \in I\}$ has the binary intersection property since they are all closed balls in \mathbb{R} , where \mathbb{R} is hyperconvex.

$$\Rightarrow \exists z_n \in \bigcap_{i \in I} B(x_{i,n}, r_i)$$

$$\text{i.e. } |x_{i,n} - z_n| \leq r_i \quad \forall i \in I$$

Since n is arbitrary, $\sup \{ |x_{i,n} - z_n| : n \in \mathbb{N} \} \leq r_i$ for every $i \in I$.

Thus $\|\underline{x}_i - \underline{z}\| \leq r_i$ for every $i \in I$. Clearly $\underline{z} \in \ell^\infty$.

$$\Rightarrow \underline{z} \in B(\underline{x}_i, r_i) \quad \forall i \in I \text{ (in } \ell^\infty)$$

$$\Rightarrow \bigcap_{i \in I} B(\underline{x}_i, r_i) \neq \emptyset$$

Hence ℓ^∞ has the binary intersection property

Combining the binary intersection property and metric convexity, it follows from 1.7.4 that ℓ^∞ is hyperconvex. □

1.7.6 PROPOSITION [Baillon]

Any hyperconvex space is complete.

PROOF

Let $\{x_n\}$ be Cauchy sequence and let $p_n = \sup_{m \geq n} d(x_m, x_n)$. Then for any $m, n \in \mathbb{N}$, $m \geq n$

$d(x_m, x_n) \leq p_n + p_m$ and similarly for $n \geq m$ $d(x_m, x_n) \leq p_n + p_m$.

By hyperconvexity there exists $x \in \bigcap_{n \in \mathbb{N}} B(x_n, p_n)$.

Hence $d(x_n, x) \leq p_n \rightarrow 0$ ($n \rightarrow \infty$) which implies that $x_n \rightarrow x$ ($n \rightarrow \infty$). □

1.7.7 EXAMPLES

Note that \mathbb{R}^2 is not hyperconvex. Indeed, a simple diagrammatic argument suffices to show this. In particular, any circle with radius strictly larger than zero is not hyperconvex. Thus neither completeness nor closedness (nor, indeed, compactness) imply hyperconvexity.

However, note that if a set S is hyperconvex in a metric space H then it is closed in H since S is complete by the above proposition.

However as already noted, \mathbb{R} is hyperconvex in itself, so a hyperconvex set need not be compact.

1.7.8 DEFINITION [Baillon]

For a bounded subset A of a metric space X , define

$\mathcal{B}(A) = \bigcap \{B : B \supset A \text{ and } B \text{ a closed ball}\}.$

Let A be a bounded subset of a hyperconvex space H .

Then the following hold:

- (i) $\mathcal{B}(A) = \cap \{B(x, r_x(A)) : x \in H\}$
- (ii) $\delta(A) = 2 r(A)$
- (iii) $r_x(\mathcal{B}(A)) = r_x(A)$ for any $x \in H$

PROOF

(i)

Let $S = \cap \{B(x, r_x(A)) : x \in H\}$.

Suppose $y \in \mathcal{B}(A)$.

$\Rightarrow y \in B \quad \forall B \supset A, B$ a closed ball

$\Rightarrow y \in B(x, r_x(A)) \quad \forall x \in H$, since $B(x, r_x(A))$ contains A for every $x \in H$

$\Rightarrow y \in S$

Thus $\mathcal{B}(A) \subset S$

Conversely suppose $y \in S$.

If $z \in H$ such that $B(z, \delta) \supset A$ for $\delta > 0$ then $r_z(A) \leq \delta$.

But $y \in S$ implies that $y \in B(z, r_z(A))$. Hence $d(y, z) \leq \delta$.

$\Rightarrow y \in B(z, \delta)$

Since $B(z, \delta)$ is arbitrary, $y \in \mathcal{B}(A)$.

Thus $S \subset \mathcal{B}(A)$

(ii)

Let $a, b \in A$. Then $d(a, b) \leq \delta(A) = \delta$ (say) $= \delta/2 + \delta/2$.

By hyperconvexity there exists $x \in \cap_{a \in A} B(a, \delta/2)$.

$$\Rightarrow d(a,x) \leq \delta/2 \quad \forall a \in A$$

$$\Rightarrow r_x(A) \leq \delta/2$$

$$\Rightarrow r(A) \leq \delta/2$$

$$\Rightarrow 2 r(A) \leq \delta$$

Fix any $x \in A$. Let $y \in A$. Now $d(x,y) \leq d(x,z) + d(z,y) \quad \forall z \in H$.

$$\Rightarrow d(x,y) \leq r_z(A) + r_z(A) = 2 r_z(A) \quad \forall z \in H$$

$$\Rightarrow d(x,y) \leq 2 r(A)$$

$$\Rightarrow r_x(A) \leq 2 r(A)$$

$$\Rightarrow \delta = \delta(A) \leq 2 r(A) \quad (\text{since } x \text{ is arbitrary})$$

Hence $\delta(A) = 2 r(A)$.

(iii)

Let $x \in H$.

For any $y \in \mathcal{B}(A)$, $y \in B(x, r_x(A))$.

$$\Rightarrow d(x,y) \leq r_x(A)$$

Hence $r_x(\mathcal{B}(A)) = \sup \{d(x,y) : y \in \mathcal{B}(A)\} \leq r_x(A)$.

Conversely, for any $y \in A$, $y \in \mathcal{B}(A)$.

$$\Rightarrow d(x,y) \leq r_x(\mathcal{B}(A))$$

$$\Rightarrow r_x(A) \leq r_x(\mathcal{B}(A))$$

□

The following three propositions are extensions of some unproven observations included in [Baillon] Theorem 5.

1.7.10 PROPOSITION

Let $\{H_\alpha : \alpha \in \Lambda\}$ be a family of nonempty hyperconvex spaces contained in a metric space (M, d) .

Let $\{B(x_i, r_i) : i \in I\}$ be a system of closed balls in M such that $\{x_i\}_{i \in I} \subset H_\alpha \quad \forall \alpha \in \Lambda$
and $\bigcap_{i \in I} B(x_i, r_i) \cap H_\alpha \neq \emptyset \quad \forall \alpha \in \Lambda$.

Then $\bigcap_{i \in I} B(x_i, r_i) \cap H_\alpha$ is hyperconvex for every $\alpha \in \Lambda$.

PROOF

Let $\alpha \in \Lambda$ be arbitrary and let $B_\alpha = \bigcap_{i \in I} B(x_i, r_i) \cap H_\alpha$.

We need to show that B_α is hyperconvex.

Let $\{B(x_j, r_j) \cap B_\alpha : j \in J\}$ be a system of closed balls in B_α such that $d_\alpha(x_j, x_k) \leq r_j + r_k$ for every $j, k \in J$, where $d_\alpha(x_j, x_k) = d(x_j, x_k)|_{B_\alpha}$.

To show that B_α is hyperconvex, we need to show that $\bigcap_{j \in J} (B(x_j, r_j) \cap B_\alpha)$ is nonempty.

Let $K = I \cup J$.

Claim: $\bigcap_{j \in J} (B(x_j, r_j) \cap B_\alpha) \neq \emptyset$

(Let $j \in J$. Then $x_j \in B_\alpha$.)

$\Rightarrow d(x_i, x_j) \leq r_i \quad \forall i \in I$

Since $j \in J$ is arbitrary, $d(x_i, x_j) \leq r_i \leq r_i + r_j \quad \forall i \in I, j \in J$.

Further, $d(x_{i_1}, x_{i_2}) \leq r_{i_1} + r_{i_2}$ for i_1 and $i_2 \in I$, since $\bigcap_{i \in I} B(x_i, r_i)$ is nonempty.

Thus $d|_{H_\alpha}(x_i, x_j) \leq r_i + r_j \quad \forall i, j \in K$, since $x_i, x_j \in H_\alpha \quad \forall i, j \in K$.

By hyperconvexity of H_α , $\bigcap_{i \in K} (B(x_i, r_i) \cap H_\alpha) \neq \emptyset$ since $B(x_i, r_i) \cap H_\alpha$ is a closed ball in

H_α for every $i \in K$. Thus $\bigcap_{j \in J} (B(x_j, r_j) \cap B_\alpha) \neq \emptyset$.

Hence claim.

Hence B_α is hyperconvex.

Since $\alpha \in \Lambda$ is arbitrary, $\bigcap_{i \in I} B(x_i, r_i) \cap H_\alpha$ is hyperconvex for every $\alpha \in \Lambda$. □

1.7.11 PROPOSITION

Let A be a nonempty bounded set in a hyperconvex space H . Then $C(A) \neq \emptyset$ and

$$C(A) = \bigcap_{x \in A} B(x, r(A)).$$

PROOF

For each $x \in A$ and $n \in \mathbb{N}$, let $C(x, n) = \{y \in H: d(x, y) \leq r(A) + 1/n\} = B(x, r(A) + 1/n)$.

Let $C_n = \bigcap_{x \in A} B(x, r(A) + 1/n)$. Then each C_n is nonempty since for every $n \in \mathbb{N}$ there exists

$y \in H$ such that $r_y(A) \leq r(A) + 1/n$. Moreover, $\{C_n\}$ is a decreasing sequence.

Hence any two elements in the sequence intersect. But each C_n is an intersection of closed balls. Thus any two balls in the system $\{B(x, r(A) + 1/n) : x \in A, n \in \mathbb{N}\}$ intersect.

Thus since H has the binary intersection property (recall that hyperconvexity implies the binary intersection property), we have that

$$\bigcap_n C_n = \bigcap_{n \in \mathbb{N}} \left(\bigcap_{x \in A} B(x, r(A) + 1/n) \right) \neq \emptyset.$$

Claim: $C(A) = \bigcap_n C_n$

(Let $y \in C(A)$.)

$$\Rightarrow r(A) = r_y(A)$$

$$\Rightarrow d(x, y) \leq r(A) \leq r(A) + 1/n \quad \forall x \in A, n \in \mathbb{N}$$

$$\Rightarrow y \in \bigcap_n C_n$$

Conversely let $y \in \bigcap_n C_n$.

$$\Rightarrow d(x, y) \leq r(A) + 1/n \quad \forall x \in A, n \in \mathbb{N}$$

$$\Rightarrow r_y(A) \leq r(A) + 1/n \quad \forall n \in \mathbb{N}$$

$$\Rightarrow r_y(A) = r(A)$$

$$\Rightarrow y \in C(A)$$

Hence claim.

Thus $C(A)$ is nonempty.

Next $y \in C(A)$

$$\Leftrightarrow r_y(A) = r(A)$$

$$\Leftrightarrow d(x,y) \leq r(A) \quad \forall x \in A$$

$$\Leftrightarrow y \in \bigcap_{x \in A} B(x, r(A))$$

Therefore $C(A)$ is nonempty and $C(A) = \bigcap_{x \in A} B(x, r(A))$ □

1.7.12 PROPOSITION

Suppose that H is a hyperconvex space. Let A be a nonempty bounded subset of H .

Then $C(A) \cap \mathcal{B}(A) \neq \emptyset$, where $\mathcal{B}(A) = \bigcap_{x \in H} B(x, r_x(A))$ as in 1.7.9 (i).

PROOF

By 1.7.11, $C(A)$ is nonempty and equals $\bigcap_{x \in A} B(x, r(A))$.

Consider the following system of closed balls

$\mathcal{S} = \{ \{ B(x, r(A)) : x \in A \} \cup \{ B(x, r_x(A)) : x \in H \} \}$. Since H is hyperconvex, all we need to show is that the distance between the centers of any two closed balls in \mathcal{S} is at most equal to the sum of their corresponding radii, by which it follows that $\bigcap \mathcal{S} = C(A) \cap \mathcal{B}(A) \neq \emptyset$.

Thus we need to consider the following three cases in order to show this.

(i)

Let $x, y \in A$.

Consider the balls $B(x, r(A))$ and $B(y, r(A))$.

Then $d(x,y) \leq r_x(A) \leq \delta(A)$.

But $\delta(A) = 2r(A)$ by 1.7.9. Hence $d(x,y) \leq r(A) + r(A)$.

(ii)

Let $x \in A$ and let $y \in H$.

Consider the balls $B(x, r_x(A))$ and $B(y, r_y(A))$. Then $d(x,y) \leq r_y(A) \leq r_y(A) + r(A)$.

(iii)

Let $x, y \in H$.

Consider the balls $B(x, r_x(A))$ and $B(y, r_y(A))$.

Let $a \in A$. Then $d(x,y) \leq d(x,a) + d(a,y) \leq r_x(A) + r_y(A)$.

Thus under (i), (ii) and (iii), and using the hyperconvexity of H ,

we have that $C(A) \cap \mathcal{B}(A) = \wedge \mathcal{S} \neq \phi$. □

We now show that a nonempty bounded hyperconvex space has the fixed point property for nonexpansive mappings. Following this, we will show that any closed ball in (real) ℓ^∞ is hyperconvex, which will give a nontrivial example of a bounded hyperconvex space.

1.7.13 THEOREM [Baillon]

If H is a nonempty bounded hyperconvex space and $T: H \rightarrow H$ is a nonexpansive mapping then the set of fixed points for T is nonempty and hyperconvex.

PROOF

We prove this in three parts. The first part shows that $F(T) \neq \phi$, the second part shows that $F(T)$ is metrically convex and the third part shows that $F(T)$ has the binary intersection property. Combining the second and third parts we conclude that $F(T)$ is hyperconvex by 1.7.4.

(1) $\underline{F(T)} \neq \phi$:

Let $\mathcal{S} = \{A \subset H : A \neq \phi, A = \mathcal{B}(A) \text{ and } T(A) \subset A\}$. The system \mathcal{S} is nonempty since $H \in \mathcal{S}$.

Order by \supset . Let \mathcal{C} be a chain in \mathcal{S} and let $A_1, \dots, A_n \in \mathcal{C}$. Then $\bigcap_{i=1}^n \mathcal{B}(A_i) = \bigcap_{i=1}^n A_i \neq \phi$, since \mathcal{S} is ordered by \supset . Now each element of \mathcal{C} is an intersection of closed balls since $\mathcal{B}(A) = A$ for every A in \mathcal{S} . Hence if we take any finite collection of closed balls, say B_i , each containing some $A_i \in \mathcal{C}$, then they intersect since $\bigcap_{i=1}^n A_i = \bigcap_{i=1}^n \mathcal{B}(A_i) \neq \phi$.

By hyperconvexity of H (using the binary intersection property) we conclude that if we collect every closed ball in H which contains at least one element in \mathcal{C} and if we intersect all these balls then the intersection of all these balls are nonempty.

This is equivalent to $\bigwedge \mathcal{C} \neq \phi$.

We now show that $\bigwedge \mathcal{C} = \mathcal{B}(\bigwedge \mathcal{C})$.

$\mathcal{B}(\bigwedge \mathcal{C}) \supset \bigwedge \mathcal{C}$ is clear.

Next, if $y \in \mathcal{B}(\bigwedge \mathcal{C})$ then $y \in B(x, r_x(\bigwedge \mathcal{C})) \quad \forall x \in H$.

But $r_x(\bigwedge \mathcal{C}) \leq r_x(A) \quad \forall x \in H, A \in \mathcal{C}$

$\Rightarrow y \in B(x, r_x(A)) \quad \forall x \in H, A \in \mathcal{C}$

$\Rightarrow y \in \bigcap_{\substack{x \in H \\ A \in \mathcal{C}}} B(x, r_x(A))$

$\Rightarrow y \in \mathcal{B}(A) \quad \forall A \in \mathcal{C}$

$\Rightarrow y \in A \quad \forall A \in \mathcal{C}$

$\Rightarrow y \in \bigwedge \mathcal{C}$

Thus $\mathcal{B}(\bigwedge \mathcal{C}) = \bigwedge \mathcal{C}$.

Clearly, $T(\bigwedge \mathcal{C}) \subset \bigwedge \mathcal{C}$.

Hence $\bigwedge \mathcal{C}$ is a lower bound for \mathcal{C} in \mathcal{S} .

Thus by Zorn's lemma \mathcal{S} has a minimal element M .

If we show that M is a singleton then it follows that $F(T)$ is nonempty.

$$\text{Claim(1): } M = \bigcap_{x \in H} B(x, r_x(T(M)))$$

(Clearly, for any set $S \subset H$, $\mathcal{B}(S) = \mathcal{B}(\mathcal{B}(S))$.)

Since $T(M) \subset M$ it follows that $\mathcal{B}(T(M)) \subset \mathcal{B}(M)$.

Hence $\mathcal{B}(\mathcal{B}(T(M))) = \mathcal{B}(T(M))$ and $T(\mathcal{B}(T(M))) \subset T(\mathcal{B}(M)) = T(M) \subset \mathcal{B}(T(M))$.

Since $T(M)$ is nonempty, so is $\mathcal{B}(T(M))$.

Hence $\mathcal{B}(T(M)) \in \mathcal{S}$.

But $\mathcal{B}(T(M)) \subset \mathcal{B}(M) = M$. Hence $M = \mathcal{B}(T(M))$ by minimality.

$$\Rightarrow M = \bigcap_{x \in H} B(x, r_x(T(M))) \text{ by 1.7.9(i)}$$

Hence claim(1).

It follows that $r_x(M) = r_x(\mathcal{B}(T(M))) = r_x(T(M))$ (for any $x \in H$) by 1.7.9(iii).

We now aim to show that $C(M) \cap M \in \mathcal{S}$.

Now $C(M) \neq \emptyset$ and $C(M) = \bigcap_{x \in M} B(x, r(M))$ by 1.7.11.

Further, by 1.7.12 we have that $C(M) \cap M$ is nonempty since $M = \bigcap_{x \in H} B(x, r_x(M))$.

If $y \in C(M)$ then $r_{T_y}(M) = r_{T_y}(T(M)) \leq r_y(M) = r(M)$.

$$\Rightarrow r_{T_y}(M) = r(M) \quad (\text{by definition of } r(M))$$

Hence $T_y \in C(M)$.

$$\Rightarrow T(C(M)) \subset C(M)$$

Thus $T(M \cap C(M)) \subset M \cap C(M)$.

Claim(2): $M \cap C(M) = \mathcal{B}(M \cap C(M))$.

$(M \cap C(M) \subset \mathcal{B}(M \cap C(M)))$ is obvious.

Now $\mathcal{B}(M \cap C(M)) \subset \mathcal{B}(M) = M$ and

$\mathcal{B}(M \cap C(M)) \subset \mathcal{B}(C(M)) = \cap \{B : B \supset C(M), B \text{ closed ball}\}$.

But $C(M) \subset B(x, r(M)) \quad \forall x \in M$ by 1.7.11.

$\Rightarrow \mathcal{B}(C(M)) \subset \cap_{x \in M} B(x, r(M))$ by definition of $\mathcal{B}(C(M))$

By 1.7.11, $\cap_{x \in M} B(x, r(M)) = C(M)$.

$\Rightarrow \mathcal{B}(C(M)) \subset C(M)$

Thus $\mathcal{B}(C(M)) = C(M)$ which implies that $\mathcal{B}(M \cap C(M)) \subset \mathcal{B}(C(M)) = C(M)$.

$\Rightarrow \mathcal{B}(M \cap C(M)) \subset M \cap C(M)$

Hence claim(2).

Thus we have shown that $M \cap C(M)$ is nonempty, invariant under T and it is equal to $\mathcal{B}(M \cap C(M))$.

By minimality of M we have that $M = M \cap C(M)$.

$\Rightarrow \delta(M) = \delta(M \cap C(M))$

We now show that $\delta(M) = r(M)$.

Let $y \in C(M) \cap M$.

$\Rightarrow y \in C(M)$

$\Rightarrow r_y(M) = r(M)$

$\Rightarrow r_y(C(M) \cap M) = r(M)$ since $C(M) \cap M = M$

Since $y \in C(M) \cap M$ is arbitrary, $r(M) = r_y(M \cap C(M)) = r_y(M) \quad \forall y \in M \cap C(M)$.

$\Rightarrow r(M) = \delta(M \cap C(M))$

$\Rightarrow \delta(M) = r(M)$

But $\delta(M) = 2 r(M)$ by 1.7.9 (ii).

Thus $\delta(M) = 0$.

Hence M is a singleton. Thus T leaves the point in M fixed.

(2) $F(T)$ is metrically convex:

Let $x, y \in F(T)$, let $d(x,y) = \alpha + \beta$ for $\alpha, \beta > 0$ and let $S = B(x,\alpha) \cap B(y,\beta)$.

The set S is nonempty, since H is metrically convex, and bounded.

Moreover, if we take each $H_\alpha = H$, $M = H$ and $\{B(x_i, r_i) : i \in I\} = \{B(x,\alpha), B(y,\beta)\}$ in 1.7.10, then it follows that S is hyperconvex.

If $z \in S$ then

$$\begin{aligned} & d(Tz, x) \\ = & d(Tz, Tx) \quad (\text{since } x \in F(T)) \\ \leq & d(z, x) \quad (\text{since } T \text{ is nonexpansive}) \\ \leq & \alpha \end{aligned}$$

Similarly, $d(Tz, y) \leq \beta$.

Thus S is a nonempty bounded hyperconvex space, where $T: S \rightarrow S$.

Hence by (1), T has a fixed point z in S . Thus there exists $z \in F(T)$ with

$$d(x, z) \leq \alpha \text{ and } d(y, z) \leq \beta.$$

$$\text{But } \alpha + \beta = d(x, y) \leq d(x, z) + d(z, y) \leq \alpha + \beta$$

$$\Rightarrow d(x, z) = \alpha \text{ and } d(y, z) = \beta$$

Hence $F(T)$ is metrically convex.

(3) $F(T)$ has the binary intersection property:

Consider the indexed balls $B(x_i, r_i) \cap F(T)$, where $i \in I$, $B(x_i, r_i)$ is a closed ball in H for every $i \in I$, $x_i \in F(T)$ and $d(x_i, x_j) \leq r_i + r_j \quad \forall i, j \in I$.

(We are first assuming that every two of these balls in $F(T)$ intersect, by which we get the inequality above.)

Let $S' = \bigcap_{i \in I} B(x_i, r_i)$. Then S' is nonempty by the hyperconvexity of H and S' is bounded.

Further, by 1.7.10 (if we take $M = H$ and $H_\alpha = H \quad \forall \alpha \in \Lambda$) S' is seen to be hyperconvex.

Let $y \in S'$. Then $d(x_i, y) \leq r_i$ for every $i \in I$. Moreover, since $x_i \in F(T)$ for every $i \in I$ and T is nonexpansive we have that

$$\begin{aligned} & d(x_i, Ty) \\ = & d(Tx_i, Ty) \\ \leq & d(x_i, y) \\ \leq & r_i \quad \text{for every } i \in I. \end{aligned}$$

Thus $T(S') \subset S'$.

Hence S' is a nonempty bounded hyperconvex space, where $T: S' \rightarrow S'$.

Hence T has a fixed point in S' by (1).

Thus $S' \cap F(T) \neq \emptyset$.

i.e. $\bigcap_{i \in I} (B(x_i, r_i) \cap F(T)) \neq \emptyset$

Therefore $F(T)$ has the binary intersection property.

By (2), (3) and 1.7.4 we have that $F(T)$ is hyperconvex. □

Remark:

Note that boundedness is necessary in the hypothesis of this theorem to guarantee the existence of a fixed point for T , since, as an example, if we consider the equation

$Tx = x + 1$ in \mathbb{R} then we see that T is nonexpansive but has no fixed points.

1.7.14 COROLLARY

Any closed ball in (real) ℓ^∞ has the fixed point property for nonexpansive mappings.

PROOF

Let B be a closed ball in ℓ^∞ . We have seen that ℓ^∞ is hyperconvex.

Taking $M = \ell^\infty$, $H_\alpha = \ell^\infty$ for every $\alpha \in \Lambda$ and $\{B(x_i, r_i) : i \in I\} = \{B\}$ in 1.7.10 we have that $B \cap \ell^\infty = B$ is hyperconvex.

Since B is bounded, the result follows by the above theorem (1.7.13). □

Remark:

A more general version of this corollary is that any nonempty set which is an intersection of closed balls in a hyperconvex space has the fixed point property for nonexpansive mappings, which we show as follows:

Let S be a nonempty intersection of closed balls, say $B(x_i, r_i)$ for $i \in I$, in a hyperconvex space H .

If we take $M = H$ and each $H_\alpha = H$ for $\alpha \in \Lambda$, in 1.7.10 then it follows that $\bigcap_{i \in I} B(x_i, r_i) \cap H$

is hyperconvex. But this is precisely $\bigcap_{i \in I} B(x_i, r_i)$.

Moreover, $\bigcap_{i \in I} B(x_i, r_i)$ is bounded.

By 1.7.13 it follows that $\bigcap_{i \in I} B(x_i, r_i)$ has the fixed point property for nonexpansive

mappings.

However, the specific version above (1.7.14) is much more interesting.

It is well known that any closed convex set in a Hilbert space \mathcal{K} is a nonexpansive retract of \mathcal{K} . (See for instance, [Diemling] § 9 Proposition 9.2.)

If $T: C \rightarrow \mathcal{K}$ is a nonexpansive mapping, where C is closed and convex, then since \mathcal{K} is strictly convex $F(T)$ is convex by 0.12. Further, $F(T)$ is closed in \mathcal{K} . Thus $F(T)$ is a nonexpansive retract of \mathcal{K} .

However, the following example adapted from [Bruck1] shows that this need not be the case in the general class of Banach spaces.

1.8.0 EXAMPLE (of a map whose set of fixed points is not a nonexpansive retract)

Let $C = B(0,1)$ be the unit ball in $C[0,1]$ and let $f \in C[0,1]$, where

$$f(t) = \begin{cases} 2t & 0 \leq t \leq 1/2 \\ 1 & 1/2 \leq t \leq 1 \end{cases}$$

Define $T: C[0,1] \rightarrow C[0,1] : g(t) \rightarrow f(t)g(t)$

Clearly T maps C into C .

Further, T is nonexpansive since for $g, h \in C$ we have that

$$\begin{aligned} & \|Tg - Th\| \\ = & \sup \{ |f(t)g(t) - f(t)h(t)| : t \in [0,1] \} \\ \leq & \sup \{ |g(t) - h(t)| : t \in [0,1] \} \\ = & \|g - h\|. \end{aligned}$$

However, we will show that we cannot find a nonexpansive retraction of C onto $F(T)$.

To see this :

Assume the contrary.

Clearly, $F(T) = \{ g \in C : g(t) = 0 \text{ for } 0 \leq t \leq 1/2 \}$.

Now there exists a nonexpansive retraction $r : C \rightarrow F(T)$.

Let k denote the constant function $1/2$.

$\Rightarrow k \notin F(T)$

For any $g \in F(T)$ we have that

$$(*) \quad \|rk - rg\| \leq \|k - g\| = \|k - rg\|.$$

But $rk(1/2) = r(1/2)$ and since $rk \in F(T)$ we have that $rk(1/2) = 0$.

Thus $rk(1/2) = r(1/2) = 0$.

Then there exists some $t_0 \in (1/2, 1)$ with $rk(t_0) < 1/2$.

If not then we have $rk(t) \geq 1/2 \quad \forall t \in (1/2, 1)$ which breaks down the continuity of rk .

$$\text{Define } h(t) = \begin{cases} 0 & 0 \leq t \leq 1/2 \\ \frac{t - 1/2}{t_0 - 1/2} & 1/2 < t \leq t_0 \\ 1 & t_0 \leq t \leq 1 \end{cases}$$

$\Rightarrow h \in F(T)$

Now $\|k - h\| = \sup \{|1/2 - h(t)| : 0 \leq t \leq 1\} = 1/2$.

But $\|rk - rh\| = \|rk - h\| \geq |rk(t_0) - h(t_0)| > 1/2 = \|k - h\|$. This contradicts (*).

Hence there can be no nonexpansive retraction. □

However, we will establish in 1.8.7 that $F(T)$ is indeed a nonexpansive retract in a Banach space (not necessarily a Hilbert space) under suitable conditions.

First let us establish some definitions and notations.

1.8.1 DEFINITION [Bruck1]

Let C be a nonempty closed convex set in a Banach space X .

Then C is said to have the **conditional fixed point property** (for nonexpansive mappings) if every nonexpansive mapping $T: C \rightarrow C$ satisfies

(CFP): either T has no fixed points in C or T has a fixed point in every nonempty bounded closed convex T -invariant subset of C .

1.8.2 DEFINITION [Bruck1]

Let C be a nonempty closed convex set in a Banach space X .

Then C is said to have the **hereditary fixed point property** (for nonexpansive mappings) if every nonempty bounded closed convex subset of C has the fixed point property for nonexpansive mappings.

Remark:

As an example, any compact convex set C has the hereditary fixed point property for nonexpansive mappings, since if S is a nonempty closed convex subset of C then S is compact and hence it has the fixed point property by Schauder's fixed point theorem. Thus in particular, S has the fixed point property for nonexpansive mappings.

One sees quite clearly that the hereditary fixed point property implies the conditional fixed point property.

Notations:

Let \underline{X} denote the topological space where X is a normed space and it is equipped with the weak topology. Analogously \underline{C} will denote the subspace of \underline{X} induced by C , where $C \subset X$. We shall say a set C is locally weakly compact if every point in C has a weak neighbourhood (in \underline{C}) which is compact in \underline{C} .

1.8.3 DEFINITIONS [Bruck]

Suppose that F is a nonempty subset of a closed convex locally weakly compact set C in a Banach space X .

Define $N(F) = \{f: C \rightarrow C : f \text{ is nonexpansive and } fx = x \text{ for all } x \in F\}$, which clearly describes the set of nonexpansive mappings: $C \rightarrow C$ which fix every point of F . Note that $N(F)$ is nonempty for any nonempty F , since the identity mapping belongs to $N(F)$.

Fix $x_0 \in F$ and define $C_x = \{y \in C : \|y - x_0\| \leq \|x - x_0\|\}$, which clearly describes the set of points that are closer to x_0 than x is.

Define $P = \prod_{x \in C} C_x$.

Notation:

For $S \subset P = \prod_{x \in C} C_x$, $\pi\text{-cl}(S)$ will denote the closure of S in the topology of weak pointwise convergence.

1.8.4 LEMMA [Bruck]

Let C be nonempty, closed, convex and locally weakly compact in a Banach space X .

For a nonempty set $F \subset C$, $N(F)$ is compact in the topology of weak pointwise convergence.

PROOF

Fix $x_0 \in F$.

Then $C_x = B(x_0, \delta_x) \cap C$ where $\delta_x = \|x - x_0\|$.

Note that if $x \in C$ and $f \in N(F)$ are arbitrary, then $f(x) \in C$ and

$$\|f(x) - x_0\| = \|f(x) - f(x_0)\| \leq \|x - x_0\|.$$

Therefore $f(x) \in C_x$.

Hence $N(F) \subset \prod_{x \in C} C_x = P$.

Claim(1): For every $x_0 \in C$, there exists some neighbourhood $B(x_0, \delta)$ such that $B(x_0, \delta) \cap C$ is weakly compact (in X).

(Let $x_0 \in C$. Then x_0 has a weakly compact neighbourhood in \underline{C} . This implies that there exists some neighbourhood $U \in \mathcal{U}_{x_0}$ (where \mathcal{U}_{x_0} denotes the neighbourhood system of x_0 in \underline{X}) such that $U \cap C$ is compact in \underline{C} .)

Now there exists an open set say V in \underline{X} such that $x_0 \in V \subset U$. But V is open in the norm topology. Thus there exists some $\delta > 0$ such that $B(x_0, \delta) \subset V$ by which we have that $B(x_0, \delta) \cap C \subset V \cap C \subset U \cap C$.

But $B(x_0, \delta)$ and C are both weakly closed and so $B(x_0, \delta) \cap C$ is weakly closed.

i.e. $B(x_0, \delta) \cap C$ is closed in \underline{X}

Since $U \cap C$ is compact in \underline{C} , it is compact in \underline{X} .

This implies that $B(x_0, \delta) \cap C \subset U \cap C$ is compact in \underline{X} .

Thus claim(1).

Claim(2): Each C_x is convex and weakly compact.

(W.l.o.g. assume that $x_0 = 0$. Convexity of C_x is immediate from its definition.

For weak compactness we proceed as follows.

By the above claim, for $x_0 \in C$, there exists some spherical neighbourhood

$B(\delta) = B(x_0, \delta) \cap C$ which is weakly compact in X .

If $\delta_x < \delta$ then $C_x \subset B(\delta)$ implying that C_x is weakly compact (since C_x is convex and norm closed and hence weakly closed).

Next, if $\delta_x \geq \delta > 0$ then $C_x = (B(x_0, \delta_x) \cap C) \subset \frac{\delta_x}{\delta} B(\delta)$, since if $y \in C$ and $\|y - x_0\| \leq \delta_x$

then $\|\frac{\delta}{\delta_x} y - x_0\| = \frac{\delta}{\delta_x} \|y - 0\| \leq \delta$ and $\frac{\delta}{\delta_x} y \in C$ ($0 \in C$ and C is convex).

Since $\frac{\delta_x}{\delta} B(\delta)$ is weakly compact, so is C_x by the same argument in the previous case.)

Hence claim(2).

Hence if C_x is given the weak topology then P is compact in the corresponding product topology, by Tychonoff's theorem.

Claim(3) $N(F)$ is closed in P with P having the product topology

(Let $f \in \pi\text{-cl}(N(F))$. Then there exists some net $\{f_\lambda : \lambda \in \Lambda\}$ in $N(F)$ converging to f .

Thus for each $x \in C$, $f_\lambda(x) \xrightarrow{w} f(x)$. Thus since C is weakly closed $f(x) \in C \quad \forall x \in C$.

If $x \in F$ then $f_\lambda(x) = x$ for each $\lambda \in \Lambda$. Hence $f(x) = w\text{-}\lim_\lambda f_\lambda(x) = x$ for $x \in F$.

Moreover, $\|f(x) - f(y)\| = \|w\text{-}\lim_\lambda (f_\lambda x - f_\lambda y)\| \leq \lim_\lambda \inf \|f_\lambda(x) - f_\lambda(y)\| \leq \|x - y\|$

for every $x, y \in C$.

Hence f is nonexpansive.

Hence $f \in N(F)$.)

Hence claim(3).

Thus the result clearly follows from claim(3). □

1.8.5 LEMMA [Bruck]

Let F be a nonempty subset of a closed convex locally weakly compact set C in a Banach space X .

Then there exists $r \in N(F)$ such that

$$\|f(rx) - f(ry)\| = \|rx - ry\| \quad \text{for all } f \in N(F) \text{ and } x, y \in C.$$

PROOF

Define an order \prec on $N(F)$ as follows:

$f \prec g$ iff $\|f(x) - f(y)\| \leq \|g(x) - g(y)\| \quad \forall x, y \in C$. Clearly, \prec defines a partial order on $N(F)$.

For $f \in N(F)$ we define the initial segment of f , $Is(f)$, by $Is(f) = \{g \in N(F) : g \prec f\}$.

Claim(1): $Is(f)$ is closed in $N(F)$, where $N(F)$ is given the topology of weak pointwise convergence.

(Let $g \in \pi\text{-cl}(Is(f))$. Then there exists some net $\{g_\lambda : \lambda \in \Lambda\}$ in $Is(f)$ converging to g .

Thus for each $x \in C$, $g_\lambda(x) \xrightarrow{w} g(x)$. Thus since C is weakly closed, $g(x) \in C \quad \forall x \in C$.

If $x \in F$ then $g_\lambda(x) = x$ for each $\lambda \in \Lambda$. Hence $g(x) = w\text{-}\lim_\lambda g_\lambda(x) = x$ for $x \in F$.

Moreover,

$$\|g(x) - g(y)\| = \|w\text{-}\lim_\lambda (g_\lambda x - g_\lambda y)\| \leq \lim_\lambda \inf \|g_\lambda(x) - g_\lambda(y)\| \leq \|f(x) - f(y)\| \leq \|x - y\|$$

for every $x, y \in C$. Thus $g \prec f$ and g is nonexpansive.

Hence $g \in Is(f)$.

Hence claim(1).

Hence $Is(f)$ is compact in the topology of weak pointwise convergence, since $N(F)$ is compact in the same topology by the above lemma (1.8.4).

Claim(2): $(N(F), \prec)$ satisfies the hypothesis of Zorn's lemma

(Let $\{g_\lambda : \lambda \in \Lambda\}$ be a chain in $(N(F), \prec)$. Then the family $\{Is(g_\lambda) : \lambda \in \Lambda\}$ is linearly ordered by inclusion. Now the family $\{Is(g_\lambda) : \lambda \in \Lambda\}$ satisfies the finite intersection property. Since each $Is(g_\lambda)$ is compact in the topology of weak pointwise convergence and clearly nonempty, there exists $g \in \bigcap_\lambda Is(g_\lambda)$.

Hence there exists $g \in N(F)$ such that $g \prec g_\lambda$ for each $\lambda \in \Lambda$.)

Hence claim(2).

Thus by Zorn's lemma there exists a minimal element r in $(N(F), \prec)$.

If $f \in N(F)$ then $\|f(rx) - f(ry)\| \leq \|rx - ry\| \quad \forall x, y \in C$, since f is nonexpansive.

Hence $fr \prec r$. Clearly $fr \in N(F)$. If $\|f(rx) - f(ry)\| < \|rx - ry\|$ for some $x, y \in C$, then we have a contradiction to the minimality of r .

Hence equality holds. □

1.8.6 THEOREM [Bruck]

Let C be a nonempty closed convex locally weakly compact set in a Banach space X and let F be a nonempty subset of C . Suppose that for each $z \in C$ there exists $h \in N(F)$ such that $h(z) \in F$. Then F is a nonexpansive retract of C .

PROOF

By the previous lemma there exists $r \in N(F)$ such that

$$(*) \quad \|f(rx) - f(ry)\| = \|rx - ry\| \quad \forall f \in N(F) \text{ and } x, y \in C.$$

Since r fixes each point of F we need only to show that $r(p) \in F$ for each $p \in C$, in order to show that r is a retraction.

So let $p \in C$ be arbitrary.

By hypothesis, for $r(p) = z \in C$ there exists $h \in N(F)$ with $h(r(p)) \in F$.

Let $y = h(r(p))$ for $r(p) = z \in C$.

$$\text{By } (*), \|h(r(p)) - h(r(y))\| = \|r(p) - r(y)\|.$$

Since $y \in F$, $r(y) = y \in F$ and so $h(r(y)) = y$. By definition of y , $y = h(r(p))$.

Hence $0 = \|h(r(p)) - h(r(y))\| = \|r(p) - r(y)\|$ by which it follows that $r(p) = r(y)$.

Hence $r(p) = y \in F$. □

1.8.7 THEOREM [Bruck]

If C is a nonempty closed convex locally weakly compact set in a Banach space X and if $T: C \rightarrow C$ is nonexpansive satisfying CFP, then $F(T)$ is a nonexpansive retract of C .

PROOF

Assume w.l.o.g. that $F(T)$ is nonempty.

Fix any $z \in C$. Let $K_z = \{f(z) : f \in N(F(T))\}$.

Claim: K_z is weakly compact

(Now K_z is the image of $N(F(T))$ under the z^{th} coordinate projection map: $\underline{P} \rightarrow \underline{C}_z$, where

\underline{P} , \underline{C}_z and $N(F(T))$ (with F is replaced by $F(T)$) are defined in 1.8.3.

By 1.8.4, $N(F(T))$ is compact in the topology of weak pointwise convergence. Hence K_z is compact in \underline{C}_z since the projection: $\underline{P} \rightarrow \underline{C}_z$ is continuous.)

Hence claim.

Thus K_z is weakly closed and bounded. Also K_z is clearly nonempty.

Next, let $f(z), g(z) \in K_z$ and let $\lambda \in [0,1]$. Then, clearly, $\lambda f + (1-\lambda)g \in N(F(T))$ which implies that $\lambda f(z) + (1-\lambda)g(z) \in K_z$. Thus K_z is convex.

Hence convexity together with weak closedness of K_z would imply that K_z is closed.

Thus K_z is a nonempty bounded closed convex subset of C .

Further, $T \circ f \in N(F(T))$ if $f \in N(F(T))$. Thus $T(K_z) \subset K_z$.

Since T satisfies CFP and $F(T)$ is nonempty, T has a fixed point in K_z .

i.e. There exists $h \in N(F(T))$ with $h(z) \in F(T)$.

Since z is arbitrary, by the above theorem (1.8.6) $F(T)$ is a nonexpansive retract of C . \square

Remark:

In [Bruck1], Bruck proves that in 1.8.7 the assumption that C is locally weakly compact can be replaced with separability.

Our aim here is to prove the following result analogous to 1.8.7 under the hyperconvex space setting.

Although Baillon has proved the following result for families of commuting nonexpansive mappings (which we shall give in chapter 2), we shall treat the version for a single nonexpansive mapping separately, by giving this result here.

1.9.1 THEOREM

Let H be a nonempty bounded hyperconvex space and let $T: H \rightarrow H$ be nonexpansive. Then $F(T)$ is a nonempty nonexpansive retract of H .

We need to establish the following theorem and a corollary in order to prove this theorem. The following theorem gives the relationships between hyperconvex spaces and nonexpansive mappings. This theorem has an application, which we have mentioned in the introduction of § 1.7, in Category Theory.

1.9.2 THEOREM [AP]

Let H be a metric space.

Then H is hyperconvex

iff

(*) every nonexpansive mapping T from any metric space D into H has a nonexpansive extension T_1 from M into H where M is any metric space containing D metrically.

PROOF

(\Leftarrow)

Assume (*).

We show that H is metrically convex and has the binary intersection property. Then hyperconvexity of H follows from 1.7.4.

Let $x, y \in H$ and let $a, b > 0$ with $d(x, y) = a + b$. Let $D = \{x, y\}$ and let T be the identity mapping: $D \rightarrow H$. Put $E = D \cup \{\xi\}$ where $\xi \notin H$ and the distance d on E is extended as follows:

$$d(\xi, \xi) = 0, d(\xi, x) = d(x, \xi) = a \text{ and } d(\xi, y) = d(y, \xi) = b.$$

Then there exists a nonexpansive extension $T_1: E \rightarrow H$.

$$\text{Now } T_1\xi \in H \text{ and } d(Tx, Ty) \leq d(x, y) \leq d(x, T_1\xi) + d(y, T_1\xi).$$

$$\Rightarrow a + b \leq d(x, T_1\xi) + d(T_1\xi, y)$$

$$\text{But } d(x, T_1\xi) = d(T_1x, T_1\xi) \leq d(x, \xi) = a \text{ and } d(y, T_1\xi) = d(T_1y, T_1\xi) \leq d(y, \xi) = b.$$

$$\text{Hence } d(x, T_1\xi) = a \text{ and } d(y, T_1\xi) = b.$$

Hence H is metrically convex.

Next, take an indexed class of balls $B(x_i, r_i), i \in I$, in H in which each two of them intersect.

$$\text{i.e. } d(x_i, x_j) \leq r_i + r_j \quad \forall i, j \in I$$

Let $D = \bigcup_{i \in I} \{x_i\}$ and let T be the identity mapping on D . Let $E = D \cup \{\xi\}$ for $\xi \notin H$ as

before. The distance on D is given by the distance on H and the distance on E is extended by

$$d(\xi, \xi) = 0 \text{ and } d(\xi, x) = d(x, \xi) = \inf \{r : \text{there exists } i \in I, B(x, r) \supset B(x_i, r_i)\} \text{ with } x \in D.$$

Now there exists a nonexpansive extension T_1 from E into H with $T_1\xi \in H$ and

$$d(x_i, T_1\xi) = d(T_1x_i, T_1\xi) \leq d(x_i, \xi) \leq r_i \text{ for each } i. \text{ Hence } T_1\xi \in \bigcap_{i \in I} B(x_i, r_i).$$

Hence binary intersection property.

Therefore by 1.7.4 it follows that H is hyperconvex.

(\Rightarrow)

Assume that H is hyperconvex. We show (*).

Let D be any metric space, T a nonexpansive mapping: $D \rightarrow H$. Let $D \subset E$ metrically.

Consider the system

$\mathcal{S} = \{(T_F, F) : T_F \text{ is nonexpansive: } F \rightarrow H \text{ and } D \subset F \subset E \text{ metrically}\}$ and the restriction of T_F on D is T . Then \mathcal{S} is nonempty since $(T, D) \in \mathcal{S}$. Order \mathcal{S} by $(T_F, F) < (T_G, G)$ iff $F \subset G$ and $T_G|_F = T_F$. Let $\mathcal{C} = \{(T_{F_\beta}, F_\beta) : \beta \in \Lambda\}$ be a chain in \mathcal{S} .

Then $D \subset \bigvee \mathcal{C} \subset E$. If $x \in \bigcup_\beta F_\beta$ then $x \in F_\beta$ for some $\beta \in \Lambda$. Hence define

$T_{\bigvee \mathcal{C}} : \bigcup_\beta F_\beta \rightarrow H$ by $T_{\bigvee \mathcal{C}}(x) = T_{F_\beta}(x)$.

Then $T_{\bigvee \mathcal{C}}|_{F_\beta} = T_{F_\beta} \quad \forall F_\beta \in \mathcal{C}$ and $T_{\bigvee \mathcal{C}}|_D = T$ since $T_{F_\beta}|_D = T \quad \forall \beta \in \Lambda$.

Further, by our definition of $T_{\bigvee \mathcal{C}}$ (and using the fact that \mathcal{C} is an increasing chain) it is clear that $T_{\bigvee \mathcal{C}}$ is nonexpansive. Thus \mathcal{C} has an upper bound $\bigvee \mathcal{C}$. With this order \mathcal{S} satisfies the hypothesis of Zorn's lemma.

Hence there exists a maximal element (T_1, F_1) in \mathcal{S} .

Assume that there exists no nonexpansive mapping $T_E : E \rightarrow H$ with $T_E|_D = T$.

Then $E \neq F$ for each $(T_F, F) \in \mathcal{S}$. In particular $F_1 \neq E$. So there exists $z \in E$ such that $z \notin F_1$.

Let $F' = F_1 \cup \{z\}$.

We now define a nonexpansive extension of T_1 on F' :

Consider the balls $B(T_1x, d(x, z))$, $x \in F_1$.

Now $d(T_1x, T_1y) \leq d(x, y) \leq d(x, z) + d(y, z) \quad \forall x, y \in F_1$.

Since H is hyperconvex, these balls intersect.

So there exists $\xi \in \bigcap_{x \in F_1} B(T_1x, d(x, z))$.

Define $T' : F' \rightarrow H$ by $T'z = \xi$ and $T'x = T_1x$ for $x \in F_1$.

Now $d(T'z, T'x) = d(\xi, T_1x) \leq d(z, x) \quad \forall x \in F_1$.

Hence T' is nonexpansive.

Also $T'|_{F_1} = T_1$. Hence $(T', F') > (T_1, F_1)$.

A contradiction to the maximality of (T_1, F_1) .

Hence there exists a nonexpansive extension: $E \rightarrow H$.

Thus (*). □

1.9.3 COROLLARY [AP]

If H is hyperconvex and is contained metrically in a metric space M , then there exists a nonexpansive retraction of M onto H .

PROOF

The identity map $i : H \rightarrow H$ is nonexpansive. By the above theorem (1.9.2) it has an extension $i' : M \rightarrow H$, where i' is nonexpansive. □

We shall now prove 1.9.1.

PROOF (1.9.1)

By 1.7.13 $F(T)$ is nonempty and hyperconvex.

But $F(T)$ is metrically contained in H .

Therefore by the above corollary (1.9.3) it follows that $F(T)$ is a nonexpansive retract of H . □

§ 1.10 Alspach's solution

In § 1.3 we stated that a nonempty weakly compact convex set need not have the fixed point property for nonexpansive mappings. It is our aim here to prove this by giving an example given by Dale E. Alspach (in [Alspach]), which appeared in 1981. This solved a long standing question on whether various additional assumptions placed on K , in order for it to have the fixed point property for nonexpansive maps, were needed.

Thus it seems to us that when the results in § 1.3, § 1.4 and § 1.5 were published the question of a weakly compact convex set having the fixed point property for nonexpansive mappings remained open as they appeared much earlier than 1981.

Moreover, these results required additional assumptions such as normal structure, uniform convexity or asymptotic normal structure, and thus they were not necessarily applicable in a general situation, where the domain was weakly compact and convex in any Banach space.

It is worth noting that when Kirk proved his result (theorem 1.3.2) he raised the question of whether normal structure is essential in the hypothesis of the theorem. Thus it is clear that at that stage the question of a weakly compact convex set having the fixed point property for nonexpansive mappings was open.

1.10.1 DEFINITIONS [DG] § 2 (7.14)

Let $K = \{f \in L_1[0,1] : \int_I f d\mu = 1/2 \text{ with } 0 \leq f \leq 1 \text{ a.e}\}$ where $I = [0,1]$.

Let $T: I^2 \rightarrow I^2$ be the "Baker's transformation" given by

$$T(x,y) = \begin{cases} (x/2, 2y) & 0 \leq y \leq 1/2 \\ (x/2 + 1/2, 2y - 1) & 1/2 < y \leq 1 \end{cases}$$

which can be seen as first squeezing I^2 into the rectangle $\{(x,y) : 0 \leq x \leq 1/2, 0 \leq y \leq 2\}$ then cutting off the top and placing it next to the lower half. It is known and intuitively

obvious that T is measure preserving.

Next define $\hat{T} : K \rightarrow K$ by $\hat{T}f(x) = \begin{cases} \min [2f(2x), 1] & 0 \leq x \leq 1/2 \\ \max [2f(2x - 1) - 1, 0] & 1/2 < x \leq 1 \end{cases}$

The graph of $\hat{T}f$ is obtained from the graph of f , after the top half of the squeezed rectangle is placed next to the lower half.

Throughout this section, μ will denote the Lebesgue measure.

1.10.2 PROPOSITION [DG] § 2 (7.14)

\hat{T} is nonexpansive.

PROOF

First note that if A_f is the ordinal set $\{(x,y) \in I^2 : y \leq f(x)\}$ for $f \in L_1[0,1]$

then $T(A_f) = A_{\hat{T}f}$.

For $f, g \in L_1[0,1]$ we have $\|\hat{T}f - \hat{T}g\| = \mu(A_{\hat{T}f} \Delta A_{\hat{T}g})$ where $A_{\hat{T}f} \Delta A_{\hat{T}g}$ is the symmetric difference of the ordinal sets.

$$\begin{aligned} \text{Now } & \mu(A_{\hat{T}f} \Delta A_{\hat{T}g}) \\ = & \mu(T(A_f) \Delta T(A_g)) && (\text{since } A_{\hat{T}f} = T(A_f) \text{ for } f \in L_1[0,1]) \\ = & \mu(T(A_f \Delta A_g)) && (\text{since } T \text{ is one-to-one and onto}) \\ = & \mu(A_f \Delta A_g) && (\text{since } T \text{ is measure preserving and bijective}) \\ = & \|f - g\|. \end{aligned}$$

□

1.10.3 PROPOSITION [DG] § 2 (7.14)

K is weakly compact.

First, we need to establish some results on which our proof depends.

We make use of the following result, which was given as a corollary in [DS], to show that K is weakly sequentially compact.

1.10.4 LEMMA [DS] IV.8.11

If $\nu(S) < \infty$ and if \mathcal{M} is a bounded set in $L_1(S, \Sigma, \nu)$ with $\lim_E \int f d\nu = 0$ as $\nu(E) \rightarrow 0$ uniformly for f in \mathcal{M} , then \mathcal{M} is weakly sequentially compact.

1.10.5 LEMMA

K is weakly sequentially compact.

PROOF

In order to establish weak sequential compactness, all we need to check is that we have satisfied the hypothesis of the above lemma.

First note that $\mu(S) = \mu([0,1]) < \infty$.

Let $\epsilon > 0$. Choose $\delta = \epsilon$. If $E \in \Sigma$ (Borel σ -algebra) with $\mu(E) < \delta$ then for any $f \in K$, $\int_E f d\mu \leq \mu(E)$ since $0 \leq f \leq 1$.

Thus $\mu(E) < \delta = \epsilon$ implies that $\int_E f d\mu < \epsilon$.

Clearly the set K is bounded.

Thus by the above lemma (1.10.4) K is weakly sequentially compact. □

1.10.6 LEMMA cf [Alspach]

K is weakly closed.

PROOF

It suffices to show that K is strongly closed and convex.

Convexity of K is clear.

Next, let $f \in K$. Then there exists a sequence $\{f_n\} \in K$ with $f_n \rightarrow f$ strongly.

Hence $0 \leq f \leq 1$. Further, by the Lebesgue's dominated convergence theorem $\int_I f d\mu = 1/2$.

Hence K is strongly closed. □

PROOF (1.10.3)

Now $L_1[0,1]$ is a Banach space.

From 1.10.5 and 1.10.6 it follows that K is weakly sequentially compact and weakly closed respectively.

Therefore by the Eberlein–Smulian theorem it follows that K is weakly compact. □

1.10.7 PROPOSITION [DG] § 2 (7.14)

\hat{T} has no fixed points in K.

PROOF

Suppose that \hat{T} has a fixed point f , in K. Then $f = 0$ or $f = 1$ a.e.

This would imply that $\int_I f d\mu \neq 1/2$.

A contradiction. □

CHAPTER 2

FAMILIES OF NONEXPANSIVE MAPPINGS AND THEIR COMMON FIXED POINTS

Kakutani and Markov have shown that if we have a family of affine continuous commuting mappings in a normed space, which leave some nonempty compact convex set invariant, then the family has a common fixed point in that invariant set.

A natural question arises: if we weaken the assumptions, or remove some assumptions completely, do we still obtain a common fixed point. For example, can we just drop the word "affine"? No.

For instance, [Huneke] gives methods for constructing commuting pairs of continuous functions which map $[0,1]$ into $[0,1]$ with no common fixed point.

In this chapter we address this question to nonexpansive mappings in Banach spaces.

We establish some generalised results, by various authors, in which the assumptions are progressively weakened in most cases. Theorem 2.4.5 is to our knowledge the greatest weakening of the assumptions available in the published literature.

Sections 2.1 to 2.5 are primarily devoted to investigating the abovementioned question. In addition, § 2.4 and § 2.5 are extensions of what we had in chapter 1 regarding the set of fixed points being a nonexpansive retract.

In § 2.6 we give an example of a family $\{T_i\}_{i=1}^{\infty}$ of nonexpansive mappings for which any finite subfamily has a common fixed point, but there exists no common fixed point for the entire family.

In § 2.7 we briefly introduce the concept of demicompact mappings in metric spaces, where we show that a family of commuting nonexpansive mappings (in a Banach space) which

map a nonempty bounded closed convex set into itself, has a common fixed point, provided that at least one element in the family is demicompact.

§ 2.1 Existence of common fixed points for families of nonexpansive mappings in compact sets

We commence our investigation with a theorem of De Marr. Here we have a strong assumption on the domain of the family in question, namely that it should be compact.

2.1.1 THEOREM [DeMarr]

Let S be a nonempty compact convex subset of a Banach space X and let \mathcal{F} be a nonempty commutative family of nonexpansive mappings: $S \rightarrow S$.

Then \mathcal{F} has a common fixed point in S .

Let us establish the following lemmas which are needed to prove this theorem.

2.1.2 LEMMA

Let M be a nonempty compact set in a normed space X . Let \mathcal{F} be a family of continuous mappings: $M \rightarrow M$. Suppose that M is minimal in the sense that, for any T in \mathcal{F} , it contains no proper nonempty compact T -invariant set.

Then T maps M onto itself, for every T in \mathcal{F} .

PROOF

Let $T \in \mathcal{F}$.

Then $T(M)$ is compact. Also M is invariant under T .

Suppose that there exists some $T' \in \mathcal{F}$ with $T'(M) = N \neq M$. For any $y \in N$ there exists $x \in M$ such that $T'x = y$. Since elements of \mathcal{F} commute we have that

$Ty = T(T'x) = T'(Tx) \in N$ since $Tx \in M$.

Hence $T(N) \subset N \subset M$.

But this would contradict the minimality of M , since $N \subset M$, N is compact and N is nonempty. □

2.1.3 LEMMA [DeMarr]

Let S be a nonempty convex set in a Banach space X and let $T: S \rightarrow S$ be nonexpansive. Suppose that there exists a compact set $M \subset S$ such that M is mapped onto itself and M has at least two points.

Then there exists a nonempty closed convex set K_1 such that $K_1 \cap S$ is invariant under T and M intersects the complement of K_1 .

PROOF

Let $K = \overline{\text{co}}(M)$.

By our claim in the proof of 1.2.3, since M has at least two points, there exists $u \in K$ such that $\rho_1 = \sup \{\|x - u\| : x \in M\} < \delta(M)$.

For each $x \in M$, let $U(x) = \{y \in X : \|y - x\| \leq \rho_1\}$.

Put $K_1 = \bigcap_{x \in M} U(x)$. Then K_1 is nonempty since $u \in K_1$.

Clearly K_1 is closed and convex.

Let us now show that $K_1 \cap S$ is invariant under T .

Assume w.l.o.g. that $K_1 \cap S$ is nonempty.

Let $x \in K_1 \cap S$ and let $z \in M$. Then $x \in U(z)$ by definition of K_1 . Hence $\|x - z\| \leq \rho_1$.

Since $M = \{Ty : y \in M\}$ there exists $y \in M$ with $z = Ty$.

Now $\|Tx - z\| = \|Tx - Ty\| \leq \|x - y\| \leq \rho_1$ (Since $x \in U(y)$ by definition of K_1 .)

i.e. $Tx \in U(z)$

Since $z \in M$ was arbitrary, $Tx \in K_1$. Further, $Tx \in S$.

Thus $Tx \in K_1 \cap S$.

Since M is compact there exists $x_0, x_1 \in M$ with $\|x_0 - x_1\| = \delta(M) > \rho_1$.

Hence $x_1 \notin U(x_0) \supset K_1$.

i.e. $x_1 \in M \cap K_1^c$

Thus M intersects the complement of K_1 . □

Let us now prove theorem 2.1.1.

PROOF (2.1.1)

Let $\mathcal{S} = \{K \subset S : K \neq \emptyset, \text{compact, convex}, T(K) \subset K \quad \forall T \in \mathcal{F}\}$.

The system \mathcal{S} is nonempty since $S \in \mathcal{S}$. Order \mathcal{S} by \supset . Let \mathcal{C} be a chain in \mathcal{S} . Every element in \mathcal{C} is compact and hence closed. Now \mathcal{C} satisfies the finite intersection property.

Hence $\bigcap \mathcal{C} = \bigcap \{S : S \in \mathcal{C}\}$ is nonempty, closed, convex and invariant under each $T \in \mathcal{F}$.

Being a closed subset of every element in \mathcal{C} which are all compact, $\bigcap \mathcal{C}$ is also compact.

Hence $\bigcap \mathcal{C}$ is an element of \mathcal{S} and is clearly a lower bound for \mathcal{C} in \mathcal{S} .

Thus by Zorn's lemma, \mathcal{S} has a minimal element, say L .

If L is a singleton then we are done.

So assume that L has at least two points.

Again applying Zorn's lemma to the system of nonempty compact (not necessarily convex) subsets of L which are invariant under each T , we obtain a minimal element M in this system as before.

By 2.1.2 we have that any element of \mathcal{F} maps M onto M .

Let us again assume that M has at least two points. If not the theorem is proven.

Applying 2.1.3 to each $T \in \mathcal{F}$ we see that there exists a nonempty closed convex set K_1 such that $K_1 \cap L$ is invariant under each $T \in \mathcal{F}$.

(Note that although we apply 2.1.3 separately for each $T \in \mathcal{F}$, the set K_1 remains

universal for every element of \mathcal{S} since the construction of K_1 in 2.1.3 was not dependent on T in 2.1.3.)

Claim(1): $K_1 \cap L$ is nonempty

(Let us refer to the proof of 2.1.3 in accordance with the notations used in its proof.

The set L is closed and convex in our case. Hence $\overline{\text{co}}(M) = K \subset L$.

Now $u \in K_1$. But $u \in K \subset L$.)

Hence claim(1).

Since K_1 is closed and convex, $K_1 \cap L$ is a nonempty compact convex subset of L .

By 2.1.3, since $\phi \neq M \cap K_1^c \subset L \cap K_1^c$ we have that $L \cap K_1 \neq L$ which contradicts the minimality of L .

Hence L is a singleton. □

The following simple examples show that we cannot delete compactness or convexity from the hypothesis of the above theorem (2.1.1).

2.1.4 EXAMPLE

Define the family $\{T_n\}$ of mappings by $T_n : (0,1) \rightarrow (0,1) : x \rightarrow x/n$

The interval $(0,1)$ is convex but not compact in \mathbb{R} .

Now $|T_n x - T_n y| = 1/n |x - y| \leq |x - y|$. Hence each map is nonexpansive. Clearly, the family commute, but we see that they have no common fixed point. □

2.1.5 EXAMPLE

Let C be the unit sphere in a finite dimensional space. Then C is compact but not convex.

Define $T: C \rightarrow C : \underline{x} \rightarrow -\underline{x}$.

(the map which maps each point of the sphere to its diametrically opposite point). Let S be

the identity map on the sphere. Then S and T commute and they are both nonexpansive. However, T has no fixed point in C . □

As a simple application of the above theorem, we have the following corollary.

2.1.6 COROLLARY

Let S be a nonempty compact convex set in a Banach space X . Let $T: S \rightarrow S$ be a not necessarily continuous mapping.

Suppose that there exists $k \in \mathbb{N}$ for which T^n is nonexpansive for every $n \geq k$. Then T has a fixed point.

PROOF

The commuting family $\{T^n: n \geq k\}$ has a common fixed point, say x , by the above theorem, since each element of the family is nonexpansive.

$$\text{Now } T(T^n x) = T^{n+1} x = x = T^n x \quad \forall n \geq k.$$

Hence $T^k x$ is a fixed point for T . □

As we have shown, compactness and convexity are needed for 2.1.1. However, the following theorem shows that we can relax these conditions to some extent under an additional assumption, namely, that there exists a compact subset of the domain C which is repeatedly approached by all orbits of some map in the family; a hypothesis considered by Göhde in 1.2.1. We show this by proving the following theorem, which generalises 2.1.1 since it does not require S to be invariant under T for any $T \in \mathcal{F}$ or S to be convex or C to be compact. On the other hand, taking C to be equal to S we get 2.1.1.

In addition to the following theorem, we shall give another generalisation of 2.1.1 (to "demicompact" mappings) in § 2.7.

2.1.7 THEOREM [BellKirk]

Let X be a Banach space.

Let C be nonempty, bounded, closed and convex in X and let S be a compact subset of C .

Let \mathcal{F} be a commuting family of nonexpansive mappings which map C into C such that for some $T_1 \in \mathcal{F}$, all orbits of T_1 repeatedly approach S .

Then there exists a point x in S such that $Tx = x \quad \forall T \in \mathcal{F}$.

PROOF

Let $\mathcal{S} = \{H \subset C : H \neq \emptyset, H \text{ is closed convex, } H \cap S \neq \emptyset \text{ and } T(H) \subset H \quad \forall T \in \mathcal{F}\}$.

Order \mathcal{S} by \supset . The system $\mathcal{S} \neq \emptyset$, since $C \in \mathcal{S}$. Let \mathcal{C} be a chain in \mathcal{S} . Then \mathcal{C} has the finite intersection property.

Therefore the system $\{H \cap S : H \in \mathcal{C}\}$ also has the finite intersection property. Since S is closed, $H \cap S$ closed for each $H \in \mathcal{C}$. Hence since $H \cap S$ is compact for each $H \in \mathcal{C}$,
 $\cap \{H \cap S : H \in \mathcal{C}\} \neq \emptyset$.

$\Rightarrow \quad \cap \{H : H \in \mathcal{C}\} \neq \emptyset$

Hence $\cap \{H : H \in \mathcal{C}\}$ is a lower bound for \mathcal{C} in \mathcal{S} , since in addition to being nonempty, it is closed, convex and invariant under every element of \mathcal{F} .

Thus by Zorn's lemma we can get a minimal element $M \subset C$ in \mathcal{S} .

Let $S' = M \cap S$. Then $S' \neq \emptyset$. Further, S' is compact, since M is closed and S is compact.

Let $x \in M \subset C$. Then the sequence $\{T_1^n x\}$ has a closure point in S . This implies that the sequence $\{T_1^n x\}$ has a closure point in $M \cap S = S'$, since $\{T_1^n x\} \subset M$ and M is closed.

If we replace C by M , S by S' and T by T_1 in 1.2.1 we see that T_1 has a fixed point in S' .

Further, from the remark preceding 1.2.1 we see that the set $F_M(T_1)$, of all the fixed points for T_1 in M is contained in S' .

Now $F_M(T_1)$ is closed. Hence $F_M(T_1)$ is compact since S' is compact.

Since \mathcal{F} is a commuting family, $F(T_1)$ is mapped into itself by any $T \in \mathcal{F}$.

i.e. $T(F_M(T_1)) \subset F_M(T_1)$ for any $T \in \mathcal{F}$

Let $\mathcal{S}_1 = \{K \subset F_M(T_1) : K \neq \emptyset, \text{ closed and } T(K) \subset K \quad \forall T \in \mathcal{F}\}$.

Now $\mathcal{S}_1 \neq \emptyset$, since $F_M(T_1) \in \mathcal{S}_1$. Order \mathcal{S}_1 by \supseteq . Each element of \mathcal{S}_1 is compact since $F_M(T_1)$ is compact. Every chain \mathcal{C}_1 has the finite intersection property. Let \mathcal{C}_1 be a chain in \mathcal{S}_1 . Then $\bigcap \mathcal{C}_1$ is nonempty, closed and invariant under every T in \mathcal{F} .

Hence $\bigcap \{K : K \in \mathcal{C}_1\}$ is a lower bound for \mathcal{C}_1 in \mathcal{S}_1 . Therefore by Zorn's lemma, \mathcal{S}_1 has a minimal element H' . By 2.1.2, $T(H') = H'$ for every $T \in \mathcal{F}$.

Now S' is compact in the Banach space X . This implies that $\overline{\text{co}}(S')$ is compact.

Hence $\overline{\text{co}}(H') \subset \overline{\text{co}}(S')$ is compact.

We now aim for a contradiction by supposing that $\delta(\overline{\text{co}}(H')) > 0$.

If $\delta(\overline{\text{co}}(H')) > 0$ then by the claim in the proof of 1.2.3, there exists $x' \in \overline{\text{co}}(H')$ such that

$$(*) \quad \sup \{\|x' - z\| : z \in \overline{\text{co}}(H')\} = r < \delta(\overline{\text{co}}(H')).$$

Let $S_1 = \{w \in \overline{\text{co}}(H') : \|w - z\| \leq r \quad \forall z \in H'\}$ and let

$$S_2 = \{w \in M : \|w - z\| \leq r \quad \forall z \in H'\}.$$

Clearly, S_2 is nonempty since $x' \in \overline{\text{co}}(H') \subset \overline{\text{co}}(S') \subset M$.

$$\text{Clearly, } S_2 \cap \overline{\text{co}}(H') = \{w \in \overline{\text{co}}(H') : \|w - z\| \leq r \quad \forall z \in H'\} = S_1.$$

Let $T \in \mathcal{F}$.

$$\text{Claim: } T(S_2) \subset S_2$$

(Let $x \in S_2$ and let $z \in H'$.)

Now $T(H') = H'$. Thus $z = Ty$ for some $y \in H'$.

$$\text{Now } \|Tx - z\| = \|Tx - Ty\| \leq \|x - y\| \leq r.$$

Hence claim.

Further, note that if $x \in S_2$ then $\overline{\{T_1^n x: n \in \mathbb{N}\}} \subset S_2$, since S_2 is closed. Thus by hypothesis, $S \cap S_2 \neq \emptyset$. But S_2 is a convex subset of M .

By minimality of M we have that $S_2 = M$.

Hence $S_2 \cap \overline{\text{co}}(H') = S_1$ implies that $S_1 = \overline{\text{co}}(H')$.

Since $\delta(\overline{\text{co}}(H')) = \delta(H')$, there exists $x, y \in H'$ with $\|x - y\| > r$ ($r < \delta(\overline{\text{co}}(H'))$) in (*).

But $H' \subset \overline{\text{co}}(H') = S_1$.

Hence $\|x - y\| \leq r$.

This contradiction shows that $\delta(H') = 0$ and hence $H' \subset S'$ consists of a single point which is fixed for every $T \in \mathcal{F}$. □

§ 2.2 Extension of a fixed point theorem of De Marr, for families of nonexpansive mappings on a uniformly convex space.

Our aim here is to extend 2.1.1 for S compact, by removing the compactness restriction which is however replaced by a requirement of uniform convexity. The proof follows closely Markov's original proof of the Markov–Kakutani theorem. This is possible since Markov uses the "affine" requirement in his theorem only to show that each $F(T_\lambda)$ is convex, and 0.12 shows that this property is true for nonexpansive mappings in strictly convex spaces.

2.2.1 THEOREM [Browder1]

Let X be a uniformly convex Banach space and let $\mathcal{T} = \{T_\lambda\}$ be a commuting family of nonexpansive mappings: $C \rightarrow C$, where C is a nonempty bounded closed convex set in X . Then $\{T_\lambda\}$ has a common fixed point in C .

PROOF

First let us show that $F(T_\lambda)$ is convex for each T_λ :

Since X is uniformly convex, it is strictly convex by the remark following 1.4.1.

Therefore since C is convex so is $F(T_\lambda)$, for each T_λ , by 0.12.

For each T_λ we have that $F(T_\lambda)$ is nonempty, by 1.4.3, and clearly closed. Moreover, each $F(T_\lambda)$ is invariant under each element of the family $\{T_\lambda\}$ since the elements of the family commute.

Hence by 1.4.3 again, T_β has a fixed point in $F(T_\lambda)$ for any T_β and T_λ in the family.

But the set of fixed points for T_β in $F(T_\lambda)$ is precisely $F(T_\lambda) \cap F(T_\beta)$, where $F(T_\lambda)$ and $F(T_\beta)$ are the fixed point sets of T_λ and T_β respectively, in C .

This implies that $F(T_\beta) \cap F(T_\alpha)$ is nonempty bounded closed and convex. Proceeding inductively we see that the family $\{F(T_\lambda)\}$ has the finite intersection property.

Now each $F(T_\lambda)$, being bounded, closed and convex in the reflexive space X , is

weakly compact.

(Recall that X is reflexive by the result of B.J Pettis. See § 1.4.)

Hence $\cap \{F(T_\lambda) : T_\lambda \in \mathcal{S}\} \neq \emptyset$.

But this is precisely the set of common fixed points for the family. □

Remark:

In chapter 1, 1.4.3 gives the same theorem for a single nonexpansive mapping.

Following this, 1.4.4 gives an example to show that 1.4.3 cannot be extended to the general class of Banach spaces. Thus it clearly follows that this theorem (2.2.1) cannot be extended to the general class of Banach spaces.

§ 2.3 Existence of common fixed points for families of nonexpansive mappings
when their domain has normal structure

Kirk established the existence of a fixed point for a nonexpansive mapping in a reflexive Banach space which maps a nonempty bounded closed convex set with normal structure into itself (theorem 1.3.2).

The following theorem of Belluce and Kirk extends this to finite families of commuting nonexpansive mappings. Further, it generalises 1.3.2 since it does not require the space to be reflexive. Further, it also generalises 1.4.3 since we have established in § 1.4 that 1.4.3 is a special case of 1.3.2.

2.3.1 THEOREM [BellKirk]

Suppose that C is a nonempty weakly compact convex set with normal structure in a Banach space X .

Let \mathcal{F} be a finite family of commuting nonexpansive mappings of C into itself.

Then there exists $x \in C$ such that $Tx = x$ for all T in \mathcal{F} .

PROOF

Let $\mathcal{F} = \{T_1, T_2, \dots, T_n\}$.

Let $\mathcal{S} = \{M' \subset C : M' \neq \emptyset, \text{ closed, convex and } T(M') \subset M' \quad \forall T \in \mathcal{F}\}$. Since C is weakly closed and convex, it is closed. So we have that $C \in \mathcal{S}$. Order \mathcal{S} by \supset . Let \mathcal{C} be a chain in \mathcal{S} . Every element of \mathcal{S} is closed and convex and hence weakly closed. Thus since C is weakly compact, so are these elements of \mathcal{S} . Now \mathcal{C} satisfies the finite intersection property. Hence $\bigcap \mathcal{C}$ is nonempty. Moreover, it is closed, convex and invariant under every element of \mathcal{F} . Thus $\bigcap \mathcal{C}$ is a lower bound for \mathcal{C} in \mathcal{S} . Hence applying Zorn's lemma as usual, we obtain a minimal element M in \mathcal{S} .

Let $T = T_1 T_2 \dots T_n$.

The set M has normal structure, since it is a convex subset of C , where C has normal structure. Now $T: M \rightarrow M$. Ofcourse, T is nonexpansive.

Also $\{\overline{T^n x} : n = 1, 2, 3, \dots\} \subset M$ for every $x \in M$, since M is closed.

Now $M \in \mathcal{S}$ is weakly compact. If we replace C by M in 1.2.4 and if we then take $S = M$ in 1.2.4 then we see that the set of fixed points for T (in M), denoted by $F_M(T)$ is nonempty.

Claim(1): $T_i(F_M(T)) = F_M(T)$ for every $1 \leq i \leq n$

(Now $T_i(F_M(T)) \subset F_M(T)$ for each $1 \leq i \leq n$ since M is invariant under every element of \mathcal{S} , \mathcal{S} is finite and elements of \mathcal{S} commute.

Let $x \in F_M(T)$.

$$\Rightarrow x = T_1 T_2 \dots T_n x = T_i T_1 \dots T_{i-1} T_{i+1} \dots T_n x$$

Let $T'_i = T_1 \dots T_{i-1} T_{i+1} \dots T_n$. Now $T_n x \in F_M(T)$ since $T_i(F_M(T)) \subset F_M(T)$ for $1 \leq i \leq n$.

Hence proceeding inductively we see that $T'_i x \in F_M(T)$.

$$\Rightarrow x = T_i(T'_i x) \in T_i(F_M(T))$$

Hence $F_M(T) \subset T_i(F_M(T))$.

Thus claim(1).

Let $K = \overline{\text{co}}(F_M(T))$. By normal structure of C there exists $x' \in K$ with

$$\sup \{\|x' - z\| : z \in K\} = r < \delta(K), \text{ provided } \delta(K) > 0.$$

Suppose that $\delta(K) > 0$. Once we obtain a contradiction we are done.

Let $C_1 = \{x \in M : \|x - z\| \leq r \quad \forall z \in K\}$. Then $x' \in C_1 \neq \phi$, closed and convex.

Further, let $C_2 = \{x \in M : \|x - z\| \leq r \quad \forall z \in F_M(T)\}$.

Clearly $C_1 \subset C_2$. Next, if $x \in C_2$ then the closed ball $B(x, r)$ will contain $F_M(T)$.

Hence $B(x, r) \supset K$. Thus $x \in C_1$.

$$\Rightarrow C_1 = C_2$$

Claim(2): $T_i(C_2) \subset C_2$ for $1 \leq i \leq n$

(Let T_i be arbitrary for $1 \leq i \leq n$.)

Let $x \in C_2$ and let $z \in F_M(T)$.

Then there exists $y \in F_M(T)$ with $T_i y = z$ since $T_i(F_M(T)) = F_M(T)$. Now $T_i(x) \in M$ and $\|T_i x - z\| = \|T_i x - T_i y\| \leq \|x - y\| \leq r$. Hence $T_i x \in C_2$.)

Thus claim(2).

By minimality of M we have that $C_1 = C_2 = M$.

But $\delta(C_1 \cap K) \leq r < \delta(K)$.

This would imply that since $M \cap K = K$, $\delta(K) = \delta(M \cap K) = \delta(C_1 \cap K) < \delta(K)$ giving a contradiction.

Hence $F_M(T)$ is a singleton.

Since $T_i(F_M(T)) = F_M(T)$ for $1 \leq i \leq n$ by claim(1), we see that $F(T)$ contains the desired fixed point. □

Remark:

When Belluce and Kirk proved the above theorem they did not know whether the theorem remained valid in general for infinite families. However, they did know that if the norm on the space was strictly convex then the result holds for infinite families. (For if the norm is strictly convex, the fixed point set for each $T \in \mathcal{S}$ becomes convex in addition to being nonempty, bounded and closed. Hence these fixed point sets are weakly compact due to the weak compactness of C . Further, they satisfy the finite intersection property by our theorem above (2.3.1). Thus there exists a point common to all of them.)

In order to prove the theorem for arbitrary families of nonexpansive mappings (without

assuming strict convexity of the norm), they strengthened the notion of normal structure by defining complete normal structure:

2.3.2 DEFINITION [BellKirk1]

Let C be a bounded closed convex set in a Banach space X . Then C is said to have complete normal structure if every closed convex subset K of C which contains more than one point satisfies the following condition:

For every decreasing net $\{K_\alpha : \alpha \in \Lambda\}$ of subsets of K for which $r(K_\alpha, K) = r(K, K)$ for $\alpha \in \Lambda$; $\bigcup_{\alpha \in \Lambda} \overline{\mathcal{C}}(K_\alpha, K)$ is a proper nonempty subset of K . (cf 1.3.1)

In the remark which follows the proof of 2.3.3, we will mention some examples of sets which have complete normal structure.

2.3.3 THEOREM [BellKirk1]

Let C be nonempty, weakly compact and convex with complete normal structure in a Banach space X .

Let \mathcal{F} be a commutative family of nonexpansive mappings of C into C .

Then there exists $x \in C$ such that $Tx = x \quad \forall T \in \mathcal{F}$.

Belluce and Kirk gave a condensed proof. The proof below is lengthy since we have given the proof in every detail.

The outline of the proof is as follows:

We find a set $M \subset C$ which is minimal in the sense of being nonempty, closed, convex and invariant under every element of \mathcal{F} . Once we establish that M is a singleton, the result follows. To this end, we assume that this set consists of more than one point. We then construct a decreasing net $\{M_\sigma : \sigma \in \mathcal{F}_f\}$ of subsets of M for which $r(M_\sigma, M) = r(M, M)$

for $\sigma \in \mathcal{F}_f$. We then show that $\overline{\bigcup_{\sigma \in \mathcal{F}_f} \mathcal{E}(M_\sigma, M)}$ is closed, convex and invariant under every element of \mathcal{F} .

But by definition of complete normal structure it follows that since M consists of more than one point, $\overline{\bigcup_{\sigma \in \mathcal{F}_f} \mathcal{E}(M_\sigma, M)}$ is a proper nonempty subset of M . This would contradict the minimality of M . Hence M is a singleton.

Let us get down to the proof.

PROOF

Let $\mathcal{S} = \{S \subset C : S \neq \emptyset, \text{ closed, convex and invariant under each } T \in \mathcal{F}\}$.

Note that C is closed, since it is weakly compact and convex.

Then $\mathcal{S} \neq \emptyset$, since $C \in \mathcal{S}$. Order \mathcal{S} by \supset . Let \mathcal{C} be a chain in \mathcal{S} . Then \mathcal{C} satisfies the finite intersection property. Every element in \mathcal{S} is weakly compact, since they are weakly closed subsets of the weakly compact set C . Thus $\bigcap \mathcal{C} \neq \emptyset$. Also $\bigcap \mathcal{C}$ is closed, convex and invariant under each $T \in \mathcal{F}$. Hence $\bigcap \mathcal{C}$ is a lower bound for \mathcal{C} in \mathcal{S} .

Hence by Zorn's lemma we obtain a minimal element, say M , in \mathcal{S} .

Thus M is weakly compact.

If M is a singleton the proof is complete. So assume that $\delta(M) > 0$.

Let \mathcal{F}_f be the family of all nonempty finite subsets of \mathcal{F} .

By the preceding theorem, for each $\sigma \in \mathcal{F}_f$, $M_\sigma = \{x \in M : Tx = x \quad \forall T \in \sigma\} \neq \emptyset$.

Let σ_0 be arbitrary but fixed in \mathcal{F}_f and let $\delta = r(M_{\sigma_0}, M)$. Then

$$(a) \quad \delta \leq r(M, M) \text{ since } r_x(M_{\sigma_0}) \leq r_x(M) \quad \forall x \in M.$$

For $\sigma_0 \subset \sigma$, let $H_\sigma = \{x \in M : M_\sigma \subset B(x, \delta)\}$.

Claim(1): Each H_σ ($\sigma_0 \subset \sigma$) is nonempty

(Let $\sigma \supset \sigma_0$. Note that if $x \in M$ then $r_x(M_\sigma) \leq r_x(M_{\sigma_0})$ since $M_\sigma \subset M_{\sigma_0}$.

$$\Rightarrow r(M_\sigma, M) \leq r(M_{\sigma_0}, M)$$

Also $\mathcal{E}(M_{\sigma_0}, M)$ is nonempty by 1.3-3. Hence there exists $x \in M$ such that

$$r_x(M_\sigma) = r(M_\sigma, M) \leq r(M_{\sigma_0}, M) = \delta. \text{ This implies that for every } y \in M_\sigma, \|x - y\| \leq \delta.$$

$$\Rightarrow M_\sigma \subset B(x, \delta)$$

Thus claim(1).

Now each H_σ is convex, since for $\alpha \in [0, 1]$ and for $x, y \in H_\sigma$ we have that

$$\alpha x + (1 - \alpha) y \in M \text{ and } M_\sigma \subset B(\alpha x + (1 - \alpha) y, \delta). \text{ Thus convexity.}$$

Let us direct the family $\{\sigma : \sigma \in \mathcal{F}_f\}$ by \subset . Then the net $\{M_\sigma : \sigma \in \mathcal{F}_f\}$ is decreasing and the net $\{H_\sigma : \sigma \in \mathcal{F}_f\}$ is increasing.

Hence $\{H_\sigma : \sigma \in \mathcal{F}_f, \sigma_0 \subset \sigma\}$ is an increasing net.

Claim(2): $H_{\sigma_0} = \mathcal{E}(M_{\sigma_0}, M)$

(Let $x \in H_{\sigma_0}$. Then $x \in M$ and $M_{\sigma_0} \subset B(x, \delta)$. Hence $\|x - y\| \leq \delta \quad \forall y \in M_{\sigma_0}$.

$$\Rightarrow r_x(M_{\sigma_0}) \leq \delta$$

But by definition of δ , $r_x(M_{\sigma_0}) \geq \delta$. Thus $r_x(M_{\sigma_0}) = \delta$

Hence $x \in \mathcal{E}(M_{\sigma_0}, M)$.

Conversely, if $x \in \mathcal{E}(M_{\sigma_0}, M)$ then $x \in M$ and $r_x(M_{\sigma_0}) = \delta$.

$$\Rightarrow \|x - y\| \leq \delta \quad \forall y \in M_{\sigma_0}$$

$$\Rightarrow M_{\sigma_0} \subset B(x, \delta)$$

$$\Rightarrow x \in H_{\sigma_0}$$

Hence claim(2).

Now let $H = \{x \in M : x \in H_\sigma, \text{ for some } \sigma_0 \subset \sigma\}$. Clearly, H is nonempty. Moreover, H is convex since each H_σ is convex for $\sigma_0 \subset \sigma$ and the net $\{H_\sigma : \sigma_0 \subset \sigma\}$ is increasing.

Claim(3): H is invariant under each $T \in \mathcal{F}$

(Let $x \in H$ and let $T \in \mathcal{F}$. Since the family of H_σ 's ($\sigma_0 \subset \sigma$) is increasing there exists σ_1 such that $\sigma_0 \subset \sigma_1$, with $T \in \sigma_1$ and $x \in H_{\sigma_1}$.

For $y \in M_{\sigma_1}$, $\|Tx - y\| = \|Tx - Ty\| \leq \|x - y\| \leq \delta$. Further, $Tx \in M$. Hence $Tx \in H_{\sigma_1}$.)

Hence claim(3).

Since each $T \in \mathcal{F}$ is nonexpansive and $T(H) \subset H$, we have that $T(\bar{H}) \subset \bar{H}$. Since H is convex so is \bar{H} . Thus by minimality of M , $\bar{H} = M$.

Now let $\epsilon > 0$ and let $x \in M$.

Claim(4): $M_\sigma \subset B(x, \delta + \epsilon)$ for some $\sigma \supset \sigma_0$

(There exists $y \in H$ such that $\|x - y\| \leq \epsilon$. By definition of H there exists $\sigma \supset \sigma_0$ with $M_\sigma \subset B(y, \delta)$. Let $t \in M_\sigma$. Then $\|x - t\| \leq \|x - y\| + \|y - t\| \leq \epsilon + \delta$.)

Hence claim(4).

Thus $\overline{\text{co}}(M_\sigma) \subset B(x, \delta + \epsilon)$. This implies that for every $x \in M$ there exists $\sigma(x) \supset \sigma_0$ such that $\overline{\text{co}}(M_{\sigma(x)}) \subset B(x, \delta + \epsilon)$.

Each $\overline{\text{co}}(M_\sigma)$ is weakly compact since they all are subsets of C .

The system $\{\overline{\text{co}}(M_\sigma) : \sigma \in \mathcal{F}_f, \sigma \supset \sigma_0\}$ is decreasing and hence has the finite intersection property.

$$\Rightarrow \quad \emptyset \neq \bigcap_{\sigma \supset \sigma_0} \overline{\text{co}}(M_\sigma) \subset \bigcap_{\sigma(x) \supset \sigma_0} \overline{\text{co}}(M_{\sigma(x)}) \subset \bigcap_{x \in M} B(x, \delta + \epsilon)$$

$$\Rightarrow \quad \exists \quad z \in \bigcap_{\sigma \supset \sigma_0} \overline{\text{co}}(M_\sigma) \subset \bigcap_{x \in M} B(x, \delta + \epsilon)$$

$$\Rightarrow r_z(M) \leq \delta + \epsilon$$

Hence $r(M, M) \leq \delta + \epsilon$.

Since $\epsilon > 0$ was arbitrary, by (a) we have

$$(b) \quad \delta = r(M_{\sigma_0}, M) = r(M, M), \text{ for } \sigma_0 \text{ arbitrary.}$$

Since the sets $\{M_\sigma : \sigma \in \mathcal{F}_f\}$ form a decreasing net in M ,

by definition of complete normal structure, $\phi \neq \overline{\bigcup_{\sigma \in \mathcal{F}_f} \mathcal{E}(M_\sigma, M)} \subset M$.

Claim(5): $H_\sigma = \mathcal{E}(M_\sigma, M)$ for each $\sigma \in \mathcal{F}_f$.

(Let $x \in H_\sigma$.

$$\Rightarrow x \in M \text{ and } M_\sigma \subset B(x, \delta)$$

$$\Rightarrow r_x(M_\sigma) \leq \delta$$

But by (b) $\delta \leq r_x(M_\sigma)$.

$$\Rightarrow r_x(M_\sigma) = \delta$$

$$\Rightarrow x \in \mathcal{E}(M_\sigma, M) = \{x \in M : r_x(M_\sigma) = r(M_\sigma, M)\} = \{x \in M : r_x(M_\sigma) = \delta\} \text{ by (b).}$$

Conversely, let $x \in \mathcal{E}(M_\sigma, M)$.

$$\Rightarrow x \in M \text{ and } r_x(M_\sigma) = r(M_\sigma, M) = \delta$$

$$\Rightarrow \|x - y\| \leq \delta \quad \forall y \in M_\sigma$$

$$\Rightarrow M_\sigma \subset B(x, \delta)$$

$$\Rightarrow x \in H_\sigma$$

$$\Rightarrow \mathcal{E}(M_\sigma, M) \subset H_\sigma$$

Thus claim(5).

Now $\bigcup_{\sigma \in \mathcal{F}_f} \mathcal{E}(M_\sigma, M) = \bigcup_{\sigma \in \mathcal{F}_f} H_\sigma$ is convex since each H_σ is convex and family of H_σ 's are

increasing.

Claim(6): $\bigcup_{\sigma \in \mathcal{F}_f} \mathcal{C}(M_\sigma, M)$ is invariant under each $T \in \mathcal{F}_f$

(Let $x \in \bigcup_{\sigma \in \mathcal{F}_f} \mathcal{C}(M_\sigma, M)$.)

$\Rightarrow x \in H_{\sigma_1}$ for some $\sigma_1 \in \mathcal{F}_f$

Let $T \in \mathcal{F}$.

$\Rightarrow T \in \sigma_2$ for some $\sigma_2 \in \mathcal{F}_f$

$\Rightarrow \sigma = \sigma_1 \cup \sigma_2$ is finite

$\Rightarrow x \in H_\sigma$ and $T \in \sigma$

Again, since the family $\{H_\sigma : \sigma \in \mathcal{F}_f\}$ is increasing, we have that $x \in H_\sigma$ and $T \in \sigma$.

If $y \in M_\sigma$ then $\|Tx - y\| = \|Tx - Ty\| \leq \|x - y\| \leq \delta$.

$\Rightarrow Tx \in H_\sigma$

$\Rightarrow Tx \in \mathcal{C}(M_\sigma, M)$.

Thus claim(6).

Hence $\overline{\bigcup_{\sigma \in \mathcal{F}_f} \mathcal{C}(M_\sigma, M)}$ is closed, convex and invariant under each T and yet, a proper

nonempty subset of M . This contradicts the minimality of M .

Hence $\delta(M) = 0$ which concludes the proof. □

Remarks:

Let us now show how this theorem generalises 2.1.1 and 2.2.1.

Belluce and Kirk proved in their paper ([BellKirk1]) that a set C has complete normal structure if

either

(i) C is bounded, closed and convex in a uniformly convex Banach space

or

(ii) C is compact and convex in a Banach space.

Now the hypotheses of 2.1.1 require S to be compact and convex. Hence S is weakly compact, and by (ii) S has complete normal structure. Thus the above theorem (2.3.3) generalises 2.1.1.

Next, it also generalises 2.2.1. To see this:

2.2.1 requires X to be a uniformly convex Banach space and C to be nonempty, bounded, closed and convex. Hence by (i), we have that C has complete normal structure. Further, by the result of Pettis (in [Pettis]) which we have mentioned in § 1.4, we have that X is reflexive. Hence C is weakly compact.

Apparently, neither Belluce nor Kirk knew of an example of a weakly compact convex set which did not possess normal structure or complete normal structure. However, Alspach's paper ([Alspach]) indirectly solves this problem:

Indeed, Alspach gave an example of a nonempty weakly compact convex set K in a Banach space which lacks the fixed point property for nonexpansive mappings. This implies that for an arbitrary family of commuting nonexpansive mappings: $K \rightarrow K$ it does not follow that the family has a common fixed point. This would imply that according to our theorem above (2.3.3), K cannot have complete normal structure.

(To be precise, K cannot even have normal structure by 2.3.1.)

Since Belluce and Kirk proved the above theorem (2.3.3) in their paper which appeared in 1967, the question of the existence of a common fixed point for infinite families under the normal structure setting remained unsolved until Lim produced the following strong result which appeared in 1974.

2.3.4 THEOREM [Lim]

Let C be a nonempty weakly compact convex set with normal structure in a Banach space X . Let \mathcal{F} be an arbitrary family of commuting nonexpansive maps from C into itself. Then \mathcal{F} has a common fixed point.

Transfinite induction is used to establish this result. The proof is neatly done. However, it relies on some propositions which involve some complicated notions and these notions in turn, yield complicated arguments in their proofs. We therefore omit the proof.

Clearly 2.3.4 generalises 2.3.1 and 2.3.3.

Note that for our purpose this result relegates the notion of complete normal structure to being merely of historical significance.

§ 2.4

When is $\bigcap_{\alpha} F(T_{\alpha})$ a nonexpansive retract?

In chapter 1 (§ 1.8) we considered some conditions under which a set of fixed points for a single nonexpansive mapping is a nonexpansive retract. Here we extend this possibility to families of nonexpansive mappings.

In addition, we give a theorem of Bruck (2.4.5) as a way of generalising the main results in the preceding sections.

2.4.1 THEOREM [Bruck]

Let C be a nonempty compact convex set in a Banach space X . Let $\{T_{\alpha} : \alpha \in \Lambda\}$ be a family of commuting nonexpansive mappings with $T_{\alpha} : C \rightarrow C$ for each $\alpha \in \Lambda$.

Then $\bigcap_{\alpha} F(T_{\alpha})$ is a nonempty nonexpansive retract of C .

Remark:

Note that $\bigcap_{\alpha} F(T_{\alpha})$ being nonempty in the above theorem is nothing but 2.1.1. However, 2.1.1 makes no assertion of $\bigcap_{\alpha} F(T_{\alpha})$ being a nonexpansive retract.

Before proving this we need to establish some results.

2.4.2 LEMMA [Bruck]

Let C be nonempty, weakly compact and convex with the hereditary fixed point property in a Banach space X . Let R be a nonempty nonexpansive retract of C and let $T : C \rightarrow C$ be nonexpansive such that $T(R) \subset R$.

Then $F(T) \cap R$ is a nonempty nonexpansive retract of C .

PROOF

There exists a nonexpansive retraction $r : C \rightarrow R$. Hence $Tr : C \rightarrow R \subset C$ is nonexpansive. Since C is bounded and has the hereditary fixed point property, $F(Tr) \neq \emptyset$. Now C has the

conditional fixed point property. Hence Tr satisfies CFP defined in 1.8.1.

Since C is weakly compact, it is locally weakly compact. Hence by 1.8.7 $F(Tr)$ is a nonexpansive retract of C .

Once we show that $F(Tr) = F(T) \cap R$ we are done.

Claim: $F(Tr) = F(T) \cap R$

(Clearly, $F(T) \cap R \subset F(Tr)$ since every element of R is a fixed point for r .)

Next assume that $Trx = x$ for some $x \in C$. Now $rx \in R$ and $T(R) \subset R$.

$$\Rightarrow x \in R$$

$$\Rightarrow rx = x$$

But this implies that $Trx = x = Tx$.

Hence $x \in F(T) \cap R$.)

Hence $F(Tr) = F(T) \cap R$.

Thus the result follows. □

2.4.3 THEOREM [Bruck]

Let C be a nonempty weakly compact convex set with the hereditary fixed point property in a Banach space X . Let $\{T_i : 1 \leq i \leq n\}$ be a family of commuting nonexpansive mappings with $T_i : C \rightarrow C$ for every i .

Then $\bigcap_{i=1}^n F(T_i)$ is a nonempty nonexpansive retract of C .

PROOF

Proof is by induction on n .

Let $n = 1$. Since C has the hereditary fixed point property, it follows that C has the conditional fixed point property. By definition of conditional fixed point property we have

that every nonexpansive $T: C \rightarrow C$ satisfies CFP. Moreover, C is locally weakly compact since it is weakly compact.

Thus by 1.8.7 it follows that $F(T)$ is a nonexpansive retract of C for every nonexpansive $T: C \rightarrow C$. Also $F(T)$ is nonempty by definition of the hereditary fixed point property.

Hence for $n = 1$ the theorem holds.

Suppose that the theorem holds for any set of cardinality n and that $\{T_i : 1 \leq i \leq n + 1\}$ is a commuting family of cardinality $n + 1$.

Let $F = \bigcap_{i=2}^{n+1} F(T_i)$. By the induction hypothesis, F is a nonempty nonexpansive retract of C .

Claim: $T_1(F) \subset F$

(Let $x \in F$. Then $T_i x = x$ for $n + 1 \geq i \geq 2$.)

Hence $T_i T_1 x = T_1 T_i x = T_1 x$ for $n + 1 \geq i \geq 2$. Hence $T_1 x \in F(T_i)$ for $n + 1 \geq i \geq 2$.

$\Rightarrow T_1 x \in F$)

Thus claim.

By 2.4.2, $F(T_1) \cap F$ is a nonempty nonexpansive retract of C .

But $F(T_1) \cap F = \bigcap_{i=1}^{n+1} F(T_i)$. □

2.4.4 LEMMA [Bruck]

Let C be nonempty, closed, convex and locally weakly compact in a Banach space X .

Let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of weakly closed nonexpansive retracts of C .

If the family is directed by \supset , then $\bigcap_\alpha F_\alpha$ is a nonexpansive retract of C .

PROOF

Let $F = \bigcap_\alpha F_\alpha$. W.l.o.g. assume that $F \neq \emptyset$.

Let $z \in C$ and define $K_z = \{f(z) : f \in N(F)\}$, where $N(F)$ is defined in 1.8.3.

By 1.8.4, $N(F)$ is compact in the topology of weak pointwise convergence.

Thus K_z is weakly compact by the claim in the proof of 1.8.7.

Since F_α is weakly closed, each $K_z \cap F_\alpha$ is weakly compact.

Claim: $K_z \cap F_\alpha$ is nonempty for each $\alpha \in \Lambda$

(Let $\alpha \in \Lambda$ be arbitrary. There exists a nonexpansive retraction r_α of C onto F_α . Since $F \subset F_\alpha$ we have that $r_\alpha \in N(F)$. Hence $r_\alpha(z) \in K_z$. But $r_\alpha(z) \in F_\alpha$ also.

Hence $K_z \cap F_\alpha \neq \emptyset$)

Thus claim.

Since F_α 's are directed by \supset , the family $\{K_z \cap F_\alpha : \alpha \in \Lambda\}$ has the finite intersection property. Since each $K_z \cap F_\alpha$ is weakly compact, $\bigcap_\alpha K_z \cap F_\alpha \neq \emptyset$.

i.e. $K_z \cap F \neq \emptyset$

This implies that there exists $h \in N(F)$ such that $h(z) \in F$.

Since z is arbitrary, by 1.8.6, F is a nonexpansive retract of C . □

We shall now prove 2.4.1.

PROOF (2.4.1)

The set C has the hereditary fixed point property since it is compact and convex. Also C is weakly compact. So 2.4.3 applies.

Let \mathcal{F} be the family of sets of the form $\bigcap_\alpha \{F(T_\alpha) : \alpha \in \Lambda'\}$ where Λ' runs through all possible finite subsets of Λ . Each element in \mathcal{F} is a closed subset of the compact set C and hence compact. Hence each element of \mathcal{F} is weakly compact.

By 2.4.3, each of these elements are nonempty nonexpansive retracts of C .

Note that C is locally weakly compact.

Let us direct \mathcal{F} by \supset .

$$\text{i.e. } \bigcap_{\alpha \in \Lambda_1} F(T_\alpha) \supset \bigcap_{\beta \in \Lambda_2} F(T_\beta) \Leftrightarrow \Lambda_1 \subset \Lambda_2$$

Then by 2.4.4 we have that $\bigcap \{F : F \in \mathcal{F}\}$ is a nonexpansive retract of C which is then nonempty, since each of these elements are compact and finite intersections of these elements are nonempty.

Finally, $\bigcap_{\alpha \in \Lambda} F(T_\alpha) = \bigcap \{F : F \in \mathcal{F}\}$.

Hence the result follows. □

Next, we state a strong result by Bruck (in [Bruck1]) which generalises 2.1.1, 2.2.1, 2.3.1, 2.3.3 and 2.3.4.

2.4.5 THEOREM [Bruck1]

Let C be a nonempty closed convex set in a Banach space X .

Suppose that C has both the fixed point property (for nonexpansive mappings) and the conditional fixed point property and C is either weakly compact or bounded and separable.
Then for any commuting family \mathcal{F} of nonexpansive mappings of C into C , the set $F(\mathcal{F})$ of common fixed points of \mathcal{F} is a nonempty nonexpansive retract of C .

The proof of this result is lengthy. We therefore omit the proof.

However, this theorem is of great importance. To see this:

As we have seen, 2.3.3 unifies 2.1.1 and 2.2.1. Then 2.3.4 unifies 2.3.1 and 2.3.3. Thus to affirm the importance of this theorem, we need only to show that it generalises 2.3.4.

In 2.3.4, C is a nonempty weakly compact convex set with normal structure. By 2.3.1, C has the fixed point property (for nonexpansive mappings).

Next, if S is any nonempty closed convex subset of C then S is weakly compact and has normal structure. Thus every nonexpansive mapping: $S \rightarrow S$ has a fixed point, again by

2.3.1. Thus C has the conditional fixed point property.

Moreover, the above theorem (2.4.5) does not require C to have normal structure as in

2.3.4.

Thus the theorem of Lim (2.3.4) is a special case of the above (2.4.5).

§ 2.5 When is $\bigcap_{\alpha} F(T_{\alpha})$ a nonexpansive retract in hyperconvex spaces?

In chapter 1 we showed that if H is a nonempty bounded hyperconvex space and $T: H \rightarrow H$ is nonexpansive then $F(T)$ is nonempty and hyperconvex and thus a nonexpansive retract of H .

It is our aim here to extend this to arbitrary families of commuting nonexpansive mappings. This is achieved in theorem 2.5.3.

The lengthy part of getting to this extension is the following theorem.

It is interesting to note that this theorem establishes a result for hyperconvex spaces that is quite similar to an analogous result for compact sets.

2.5.1 THEOREM cf [Baillon]

Let M be a hyperconvex space.

Let $\{H_{\alpha}\}_{\alpha \in \Lambda}$ be a decreasing family of nonempty bounded hyperconvex subsets of M indexed by Λ .

Then $\bigcap_{\alpha} H_{\alpha}$ is nonempty and hyperconvex.

This proof is lengthy (9 pages!), since we have given every detail.

PROOF

The proof is divided into 6 parts:

(1) Notation and definition

Denote the product space by $\mathcal{H} = \prod_{\alpha} H_{\alpha}$ with the projection π_{α} on \mathcal{H} onto H_{α} by $\pi_{\alpha}(x) = x(\alpha)$ if $(x(\alpha)) = x$.

Since the family is decreasing, Λ can be totally ordered by:

$$\alpha \leq \beta \quad \Leftrightarrow \quad H_{\alpha} \supset H_{\beta}$$

For any bounded subset E of M , define $B_\alpha(E)$ by $B_\alpha(E) = \bigcap_{x \in H_\alpha} B(x, r_x(E))$.

Note that if $E \subset H_\alpha$ then $B_\alpha(E) = \bigcap \{B \supset E : B \text{ a closed ball in } H_\alpha\}$ by 1.7.8 (e) and 1.7.9 (i). Further, $B_\alpha(B_\alpha(E)) = B_\alpha(E)$ by 1.7.9 (iii).

Consider $\mathcal{S} = \{A = \prod_\alpha A_\alpha : A \neq \emptyset, A_\alpha \subset H_\alpha, B_\alpha(A_\alpha) \cap H_\alpha = A_\alpha, A_\beta \subset A_\alpha \text{ if } \beta \geq \alpha\}$.

The system \mathcal{S} is nonempty since $\mathcal{H} \in \mathcal{S}$.

(2) Minimal element of \mathcal{S}

The inclusion is the natural order of \mathcal{S} .

Claim(A): \mathcal{S} satisfies the hypothesis of Zorn's lemma

(Let $\{A_i : i \in I\}$ be a totally ordered family of elements of \mathcal{S} with $A = \bigcap_{i \in I} A_i$. Then A is a

lower bound for $\{A_i : i \in I\}$. Moreover, we have:

(a) $A = \prod_\alpha \pi_\alpha(A)$ where $\pi_\alpha(A) = \bigcap_{i \in I} \pi_\alpha(A_i)$.

(b) for each $\alpha \in \Lambda$, H_α is hyperconvex and $\pi_\alpha(A_i)$ is an intersection of closed balls in H_α .

If we show that

(i) $A \neq \emptyset$ and $\pi_\alpha(A) \subset H_\alpha$,

(ii) $B_\alpha(\pi_\alpha(\bigcap_{i \in I} A_i)) \cap H_\alpha = \pi_\alpha(\bigcap_{i \in I} A_i)$ and

(iii) $\pi_\beta(A) \subset \pi_\alpha(A)$ if $\beta \geq \alpha$

then it follows that $A \in \mathcal{S}$. This will affirm the claim.

(i)

Fix any $\alpha \in \Lambda$.

First, each $A_i \neq \emptyset$. Next, $\pi_\alpha(A) = \pi_\alpha(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} \pi_\alpha(A_i)$.

For any finite $I' \subset I$ (in particular, when I' has only two elements), $\bigcap_{i \in I'} A_i \neq \emptyset$, since

inclusion is the order. Hence $\bigcap_{i \in I'} \pi_\alpha(A_i) \neq \emptyset$. But each $\pi_\alpha(A_i)$ is an intersection of closed

balls in H_α . Therefore by hyperconvexity of H_α , $\bigcap_{i \in I} \pi_\alpha(A_i)$ is nonempty.

$$\Rightarrow \pi_\alpha(A) = \pi_\alpha\left(\bigcap_{i \in I} A_i\right) \neq \emptyset \quad \text{for each } \alpha \in \Lambda$$

$$\Rightarrow A \neq \emptyset$$

Further, $\pi_\alpha(A) \subset \pi_\alpha(A_i)$ and $\pi_\alpha(A_i) \subset H_\alpha \quad \forall i \in I$.

$$\Rightarrow \pi_\alpha(A) \subset H_\alpha$$

Hence (i).

(ii)

Let $y \in B_\alpha(\pi_\alpha(\bigcap_{i \in I} A_i)) \cap H_\alpha$.

$$\Rightarrow y \in H_\alpha \text{ and } y \in B(x, r_x(\pi_\alpha(\bigcap_{i \in I} A_i))) \quad \forall x \in H_\alpha \quad (\text{by definition of } B_\alpha(E) \text{ in (1)})$$

But $\pi_\alpha(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} \pi_\alpha(A_i) \subset \pi_\alpha(A_i) \quad \forall i \in I$.

$$\Rightarrow r_x(\pi_\alpha(\bigcap_{i \in I} A_i)) \leq r_x(\pi_\alpha(A_i)) \quad \forall i \in I$$

$$\Rightarrow y \in B(x, r_x(\pi_\alpha(A_i))) \quad \forall x \in H_\alpha, i \in I$$

$$\Rightarrow y \in B_\alpha(\pi_\alpha(A_i)) \cap H_\alpha = \pi_\alpha(A_i) \quad \forall i \in I$$

$$\Rightarrow y \in \bigcap_{i \in I} \pi_\alpha(A_i) = \pi_\alpha(\bigcap_{i \in I} A_i)$$

$$\Rightarrow B_\alpha(\pi_\alpha(\bigcap_{i \in I} A_i)) \cap H_\alpha \subset \pi_\alpha(\bigcap_{i \in I} A_i)$$

Next, let $y \in \pi_\alpha(\bigcap_{i \in I} A_i)$.

$$\Rightarrow y \in \bigcap_{i \in I} \pi_\alpha(A_i)$$

$$\Rightarrow y \in B_\alpha(\pi_\alpha(A_i)) \cap H_\alpha \quad \forall i \in I$$

Since $y \in \bigcap_{i \in I} \pi_\alpha(A_i)$, $d(x, y) \leq r_x(\bigcap_{i \in I} \pi_\alpha(A_i)) \quad \forall x \in H_\alpha$

$$\Rightarrow y \in B(x, r_x(\bigcap_{i \in I} \pi_\alpha(A_i))) \quad \forall x \in H_\alpha$$

$$\Rightarrow y \in \bigcap_{x \in H_\alpha} B(x, r_x(\bigcap_{i \in I} \pi_\alpha(A_i)))$$

$$\Rightarrow y \in B_\alpha(\bigcap_{i \in I} \pi_\alpha(A_i))$$

$$\Rightarrow y \in B_\alpha(\pi_\alpha(\bigcap_{i \in I} A_i)) \cap H_\alpha$$

$$\Rightarrow \pi_{\alpha} \left(\bigcap_{i \in I} A_i \right) \subset B_{\alpha} \left(\pi_{\alpha} \left(\bigcap_{i \in I} A_i \right) \right) \cap H_{\alpha}$$

Hence equality.

(iii)

$$\pi_{\beta} \left(\bigcap_{i \in I} A_i \right) = \bigcap_{i \in I} \pi_{\beta}(A_i) \subset \pi_{\beta}(A_i) \subset \pi_{\alpha}(A_i) \quad \text{for every } i \in I \text{ if } \beta \geq \alpha \text{ by our definition of } \mathcal{S}.$$

$$\Rightarrow \pi_{\beta} \left(\bigcap_{i \in I} A_i \right) \subset \bigcap_{i \in I} \pi_{\alpha}(A_i) = \pi_{\alpha} \left(\bigcap_{i \in I} A_i \right) \quad \text{if } \beta \geq \alpha$$

$$\text{i.e. } \pi_{\beta}(A) \subset \pi_{\alpha}(A) \quad \text{if } \beta \geq \alpha$$

Hence it is verified that $A \in \mathcal{S}$.)

Thus claim(A) is true.

Hence there exists a minimal element A in \mathcal{S} .

(3) Operation B_{α} and minimal element A

From now on, A_{α} will denote $\pi_{\alpha}(A)$ where A is the minimal element in \mathcal{S} (which we have found in (2)).

For any fixed $\beta \in \Lambda$, define A' by the following way:

$$\pi_{\alpha}(A') = \begin{cases} B_{\beta}(A_{\beta}) \cap A_{\alpha} & \text{for } \alpha \leq \beta \\ A_{\alpha} & \text{for } \beta \leq \alpha \end{cases}$$

Note that $B_{\beta}(A_{\beta}) \cap A_{\beta} = A_{\beta}$ since $A \in \mathcal{S}$ (so that $\pi_{\alpha}(A')$ is well defined).

Claim(B): $A' \in \mathcal{S}$

(The only conditions worth checking are $A' \neq \emptyset$, $\pi_{\alpha_2}(A') \subset \pi_{\alpha_1}(A')$ for $\alpha_1 \leq \alpha_2$ and

$B_{\alpha}(\pi_{\alpha}(A')) \cap H_{\alpha} = \pi_{\alpha}(A')$ for $\alpha \in \Lambda$. The condition $\pi_{\alpha}(A') \subset H$ is clear.

$A' \neq \phi$:

(If $\beta \leq \alpha$ then $\phi \neq A_\alpha = \pi_\alpha(A')$. Next, if $\alpha \leq \beta$ then $A_\beta \subset A_\alpha$.

Hence $\phi \neq A_\beta = B_\beta(A_\beta) \cap A_\beta \subset B_\beta(A_\beta) \cap A_\alpha = \pi_\alpha(A')$.

Hence $A' \neq \phi$.)

$\pi_{\alpha_2}(A') \subset \pi_{\alpha_1}(A')$ for $\alpha_1 \leq \alpha_2$:

(We show this for each of the following three possibilities:

(i) $\beta \leq \alpha_1 \leq \alpha_2$

$$\pi_{\alpha_2}(A') = A_{\alpha_2} \subset A_{\alpha_1} = \pi_{\alpha_1}(A')$$

(ii) $\alpha_1 \leq \alpha_2 \leq \beta$

$$\pi_{\alpha_2}(A') = B_\beta(A_\beta) \cap A_{\alpha_2} \subset B_\beta(A_\beta) \cap A_{\alpha_1} = \pi_{\alpha_1}(A')$$

(iii) $\alpha_1 \leq \beta \leq \alpha_2$

$$\pi_{\alpha_2}(A') = A_{\alpha_2} \subset A_\beta = B_\beta(A_\beta) \cap A_\beta \subset B_\beta(A_\beta) \cap A_{\alpha_1} = \pi_{\alpha_1}(A')$$

Thus $\pi_{\alpha_2}(A') \subset \pi_{\alpha_1}(A')$ for $\alpha_1 \leq \alpha_2$.

Next, $B_\alpha(\pi_\alpha(A')) \cap H_\alpha = \pi_\alpha(A')$:

(If $\beta \leq \alpha$ then clearly the equality holds. So assume that $\alpha \leq \beta$.)

Let $y \in B_\alpha(\pi_\alpha(A')) \cap H_\alpha$.

$$\Rightarrow y \in B_\alpha(B_\beta(A_\beta) \cap A_\alpha) \cap H_\alpha$$

$$\Rightarrow y \in H_\alpha \text{ and } y \in \bigcap_{x \in H_\alpha} B(x, r_x(B_\beta(A_\beta) \cap A_\alpha))$$

$$\Rightarrow y \in B(x, r_x(A_\alpha)) \quad \forall x \in H_\alpha \text{ (since } B_\beta(A_\beta) \cap A_\alpha \subset A_\alpha)$$

$$\Rightarrow y \in B_\alpha(A_\alpha)$$

$$\Rightarrow y \in B_\alpha(A_\alpha) \cap H_\alpha = A_\alpha$$

Let us now show that $y \in B_\beta(A_\beta)$.

Since $\alpha \leq \beta$ we have that $H_\beta \subset H_\alpha$.

Now $y \in \bigcap_{x \in H_\alpha} B(x, r_x(B_\beta(A_\beta)))$ (since $B_\beta(A_\beta) \cap A_\alpha \subset B_\beta(A_\beta)$)

$$\Rightarrow y \in B(x, r_x(B_\beta(A_\beta))) \quad \forall x \in H_\beta$$

$$\Rightarrow y \in B(x, r_x(A_\beta)) \quad \forall x \in H_\beta \text{ (by 1.7.9 (iii))}$$

$$\Rightarrow y \in B_\beta(A_\beta)$$

$$\Rightarrow y \in B_\beta(A_\beta) \cap A_\alpha = \pi_\alpha(A')$$

$$\Rightarrow B_\alpha(\pi_\alpha(A')) \cap H_\alpha \subset \pi_\alpha(A')$$

Next let $y \in \pi_\alpha(A') = B_\beta(A_\beta) \cap A_\alpha \subset H_\alpha$.

$$\Rightarrow y \in H_\alpha \text{ and } y \in B_\alpha(B_\beta(A_\beta) \cap A_\alpha).$$

$$\Rightarrow y \in B_\alpha(\pi_\alpha(A')) \cap H_\alpha$$

Hence equality.

Then $A' \in \mathcal{S}$ with $A' \subset A$.

Thus claim(B) is true.

Hence $A' = A$ by minimality of A in \mathcal{S} .

Thus $B_\beta(A_\beta) \cap A_\alpha = A_\alpha$ for $\alpha \leq \beta$.

Now $A_\beta \subset A_\alpha \subset B_\beta(A_\beta)$ for $\alpha \leq \beta$. This implies that $A_\alpha \subset H_\beta$ for $\alpha \leq \beta$.

Thus $B_\beta(A_\beta) \subset B_\beta(A_\alpha) \subset B_\beta(B_\beta(A_\beta)) = B_\beta(A_\beta)$ for $\alpha \leq \beta$.

Hence $B_\beta(A_\beta) = B_\beta(A_\alpha)$ for $\alpha \leq \beta$.

(4) Radius of A_α

Now $r_x(A_\beta) = r_x(A_\alpha) \quad \forall x \in H_\beta$ and $\beta \geq \alpha$.

(We get this by noting that $B_\beta(A_\alpha) = B_\beta(A_\beta)$, $A_\alpha \subset H_\alpha$ and using 1.7.9 (iii))

Then $r(A_\beta)$

$$= \delta(A_\beta)/2$$

$$= \inf_{x \in H_\beta} r_x(A_\beta) \quad (\text{since } A_\beta \subset H_\beta \subset M)$$

$$\begin{aligned}
&= \inf_{x \in H_\beta} r_x(A_\alpha) \\
&\geq \inf_{x \in H_\alpha} r_x(A_\alpha) \\
&= \delta(A_\alpha)/2 \quad (\text{since } A_\alpha \subset H_\alpha \subset M) \\
&= r(A_\alpha) \quad \text{if } \beta \geq \alpha.
\end{aligned}$$

But $A_\beta \subset A_\alpha$ implies that $r(A_\beta) \leq r(A_\alpha)$.

Hence $r(A_\beta) = r(A_\alpha)$ for $\beta \geq \alpha$.

(5) A_α is a singleton

Define A'' as follows:

$$\pi_\alpha(A'') = C(A_\alpha) \cap A_\alpha \text{ where } C(A_\alpha) \text{ is the center of } A_\alpha \text{ (in } M) \text{ defined in 1.3.1.}$$

Claim(C): $A'' \in \mathcal{S}$

(First, $A'' \neq \phi$:

$$C(A_\alpha) \cap A_\alpha \neq \phi :$$

(Taking $H = H_\alpha$ and $B = B_\alpha(A_\alpha) \cap H_\alpha$ in 1.7.12 we see that $C(A_\alpha \cap H_\alpha) \cap A_\alpha$ is nonempty.

$$\Rightarrow C(A_\alpha) \cap A_\alpha = C(A_\alpha \cap H_\alpha) \cap A_\alpha \neq \phi)$$

$$\Rightarrow \pi_\alpha(A'') \neq \phi \quad \forall \alpha \in \Lambda$$

$$\text{Next, } B_\alpha(C(A_\alpha) \cap A_\alpha) \cap H_\alpha = C(A_\alpha) \cap A_\alpha :$$

(Clearly, $C(A_\alpha) \cap A_\alpha \subset B_\alpha(C(A_\alpha) \cap A_\alpha) \cap H_\alpha$.)

Next let $y \in B_\alpha(C(A_\alpha) \cap A_\alpha) \cap H_\alpha$.

$$\Rightarrow y \in A_\alpha \text{ since } B_\alpha(A_\alpha) \cap H_\alpha = A_\alpha.$$

$$\text{Now } y \in B(x, r_x(C(A_\alpha) \cap A_\alpha)) \quad \forall x \in H_\alpha$$

$$\Rightarrow y \in B(x, r_x(C(A_\alpha))) \quad \forall x \in H_\alpha$$

We shall now show that $r_x(C(A_\alpha)) \leq r(A_\alpha) \quad \forall x \in A_\alpha$.

Fix any $x \in A_\alpha$. Let $z \in C(A_\alpha)$.

$$\Rightarrow r_z(A_\alpha) = r(A_\alpha)$$

$$\Rightarrow d(x,z) \leq r(A_\alpha)$$

Since z is arbitrary, $r_x(C(A_\alpha)) \leq r(A_\alpha)$.

Thus $r_x(C(A_\alpha)) \leq r(A_\alpha)$ for every $x \in A_\alpha$ by which we have that $y \in B(x, r(A_\alpha))$ for every $x \in A_\alpha$.

Now $y \in B(x, r(A_\alpha))$ for every $x \in A_\alpha$ implies that $r_y(A_\alpha) \leq r(A_\alpha)$.

$$\Rightarrow r_y(A_\alpha) = r(A_\alpha)$$

$$\Rightarrow y \in C(A_\alpha)$$

Thus $B_\alpha(C(A_\alpha) \cap A_\alpha) \cap H_\alpha \subset C(A_\alpha) \cap A_\alpha$.

Thus $B_\alpha(C(A_\alpha) \cap A_\alpha) \cap H_\alpha = C(A_\alpha) \cap A_\alpha$ and also

$$C(A_\beta) \cap A_\beta = \{x \in A_\beta : r_x(A_\beta) = r(A_\beta)\} = \{x \in A_\beta : r_x(A_\alpha) = r(A_\alpha)\} \subset C(A_\alpha) \cap A_\alpha$$

if $\beta \geq \alpha$. Hence $\pi_\beta(A'') \subset \pi_\alpha(A'')$ if $\beta \geq \alpha$.

Clearly, $C(A_\alpha) \cap A_\alpha \subset H_\alpha$ for every $\alpha \in \Lambda$. Hence $A'' \in \mathcal{S}$.

Thus claim(C) is true.

But $A'' \subset A$.

Thus $C(A_\alpha) \cap A_\alpha = A_\alpha \quad \forall \alpha \in \Lambda$ by minimality of A . Further, by 1.7.11

$$C(A_\alpha) = \bigcap_{x \in A_\alpha} B(x, r(A_\alpha)).$$

Claim(D): $\delta(C(A_\alpha) \cap A_\alpha) = r(A_\alpha)$

(Let $x \in C(A_\alpha) \cap A_\alpha$.

$$\Rightarrow x \in B(y, r(A_\alpha)) \quad \forall y \in A_\alpha$$

$$\Rightarrow d(x,y) \leq r(A_\alpha) \quad \forall y \in A_\alpha$$

$$\Rightarrow r_x(A_\alpha) \leq r(A_\alpha)$$

But $r(A_\alpha) = \inf \{r_x(A_\alpha) : x \in M\}$

$$\Rightarrow r(A_\alpha) = r_x(A_\alpha)$$

Since $x \in C(A_\alpha) \cap A_\alpha$ is arbitrary, $r(A_\alpha) = r_x(A_\alpha) = r_x(C(A_\alpha) \cap A_\alpha)$

for every $x \in C(A_\alpha) \cap A_\alpha$.

$$\Rightarrow r(A_\alpha) = \delta(C(A_\alpha) \cap A_\alpha)$$

Thus claim(D) is true.

Thus $\delta(A_\alpha) = \delta(C(A_\alpha) \cap A_\alpha) = r(A_\alpha)$. Thus $\delta(A_\alpha) = 0$ since $2r(A_\alpha) = \delta(A_\alpha) = r(A_\alpha)$.

(6) Conclusions

$A_\alpha = \{x(\alpha)\}$ contains $A_\beta = \{x(\beta)\}$ if $\beta \geq \alpha$. Hence $x(\alpha)$ is constant for any $\alpha \in \Lambda$.

Thus $\bigcap_\alpha H_\alpha \neq \emptyset$.

We now show that $H = \bigcap_\alpha H_\alpha$ is hyperconvex:

Let $\{B_H(x_i, r_i) : i \in I\}$ be a system of closed balls in H such that $d_H(x_i, x_j) \leq r_i + r_j$ for every $i, j \in I$, where $B_H(x_i, r_i) = B(x_i, r_i) \cap H$ and $d_H(x_i, x_j) = d(x_i, x_j)|_H \quad \forall i, j \in I$ (where d is the metric on M).

Now $\{B(x_i, r_i) \cap H_\alpha : i \in I\}$ is a system of closed balls in each H_α and $\bigcap_{i \in I} B(x_i, r_i) \cap H_\alpha$ is nonempty for each $\alpha \in \Lambda$ (by hyperconvexity of each H_α).

Thus each $B_\alpha = \bigcap_{i \in I} B(x_i, r_i) \cap H_\alpha$ is hyperconvex by 1.7.10. Thus $\{B_\alpha\}_{\alpha \in \Lambda}$ is also a decreasing family of hyperconvex spaces, since $\{H_\alpha\}_{\alpha \in \Lambda}$ is a decreasing family.

Therefore by what we have shown above, $\bigcap_\alpha B_\alpha \neq \emptyset$.

i.e. $\bigcap_{i \in I} (B(x_i, r_i) \cap H_\alpha) \neq \emptyset$

$$\Rightarrow \bigcap_{i \in I} B(x_i, r_i) \cap H \neq \emptyset$$

$$\Rightarrow \bigcap_{i \in I} B_H(x_i, r_i) \neq \emptyset$$

Thus H is hyperconvex. □

2.5.2 COROLLARY of [Baillon]

Let M be a hyperconvex space.

Let $\{H_\alpha : \alpha \in \Lambda\}$ be a family of bounded hyperconvex subsets of M with the finite intersection property (as defined in 1.7.3).

Then $\bigcap_{\alpha \in \Lambda} H_\alpha$ is nonempty and hyperconvex.

PROOF

Let $\mathcal{F} = \{I \subset \Lambda : \text{for all nonempty finite } J \subset I, \bigcap_{\alpha \in J} H_\alpha \neq \emptyset \text{ and hyperconvex}\}$.

Order \mathcal{F} by inclusion. \mathcal{F} is nonempty since $\emptyset \in \mathcal{F}$.

Let \mathcal{C} be a chain in \mathcal{F} .

Claim: $\vee \mathcal{C}$ is an upper bound for \mathcal{C} in \mathcal{F}

(First $\vee \mathcal{C} \subset \Lambda$. Let J be finite and let $I' = \vee \mathcal{C} \cup J$. Now for each $I \in \mathcal{C}$, $\bigcap_{\alpha \in I \cup J} H_\alpha$ is

nonempty and hyperconvex by definition of \mathcal{F} . Since \mathcal{F} is ordered by inclusion,

$\left\{ \bigcap_{\alpha \in I \cup J} H_\alpha : I \in \mathcal{C} \right\}$ is a nonempty decreasing family of hyperconvex spaces. Thus by the

above theorem (2.5.1) it follows that $\bigcap_{I \in \mathcal{C}} \bigcap_{\alpha \in I \cup J} H_\alpha$ is nonempty and hyperconvex.

Hence it suffices to show that $\bigcap_{\alpha \in I'} H_\alpha = \bigcap_{I \in \mathcal{C}} \bigcap_{\alpha \in I \cup J} H_\alpha$ in order to prove the claim.

First $\bigcap_{\alpha \in \vee \mathcal{C}} H_\alpha \subset \bigcap_{\alpha \in I} H_\alpha \quad \forall I \in \mathcal{C}$ (since $\vee \mathcal{C} = \{\alpha \in \Lambda : \alpha \in I \text{ for some } I \in \mathcal{C}\}$).

$\Rightarrow \bigcap_{\alpha \in I'} H_\alpha \subset \bigcap_{\alpha \in I \cup J} H_\alpha \quad \forall I \in \mathcal{C}$.

$$\Rightarrow \bigcap_{\alpha \in I'} H_\alpha \subset \bigcap_{I \in \mathcal{I}} \bigcap_{\alpha \in I \cup J} H_\alpha$$

Next, let $y \in \bigcap_{I \in \mathcal{I}} \bigcap_{\alpha \in I \cup J} H_\alpha$.

$$\Rightarrow y \in \bigcap_{\alpha \in I \cup J} H_\alpha \quad \forall I \in \mathcal{I}.$$

$$\Rightarrow y \in \bigcap_{\alpha \in I'} H_\alpha$$

Thus equality.)

Thus the claim is proven.

Hence \mathcal{I} satisfies the hypothesis of Zorn's lemma.

Hence there exists a maximal element I in \mathcal{I} . For any $\alpha \in \Lambda$, $I \cup \{\alpha\} \in \mathcal{I}$.

By maximality of I , $\alpha \in I \quad \forall \alpha \in \Lambda$. □

2.5.3 THEOREM [Baillon]

Let $T_\alpha : H \rightarrow H$ be a commuting family of nonexpansive mappings from H into H , where H is nonempty, bounded and hyperconvex.

Then $\mathcal{F} = \bigcap_{\alpha \in \Lambda} F(T_\alpha)$ is nonempty and hyperconvex.

Moreover, \mathcal{F} is a nonexpansive retract of H .

PROOF

Let $\alpha, \beta \in \Lambda$.

$F(T_\alpha)$ is nonempty hyperconvex by 1.7.13. Since T_α and T_β commute,

$T_\beta(F(T_\alpha)) \subset F(T_\alpha)$. Thus by 1.7.13, there exists $x_\alpha \in F(T_\alpha)$ such that $x_\alpha \in F(T_\beta)$.

Hence $F(T_\alpha) \cap F(T_\beta)$ is nonempty.

Next we show that $F(T_\alpha) \cap F(T_\beta)$ is hyperconvex:

Let $T'_\beta = T_\beta|_{F(T_\alpha)}$. Then $T'_\beta : F(T_\alpha) \rightarrow F(T_\alpha)$. By 1.7.13, $F(T'_\beta)$ is nonempty and hyperconvex. But this is precisely $F(T_\alpha) \cap F(T_\beta)$.

Thus proceeding inductively in the same manner, we see that $\{F(T_\alpha) : \alpha \in \Lambda\}$ is a family of bounded hyperconvex spaces with the finite intersection property.

Thus by the above corollary (2.5.2), $\bigcap_\alpha F(T_\alpha)$ is nonempty and hyperconvex.

By 1.9.3, $\bigcap_\alpha F(T_\alpha)$ is a nonexpansive retract of H . □

In chapter 1 (§ 1.10) we had Alspach's example of a nonexpansive mapping with a weakly compact convex domain that had no fixed points. Here we present Schechtman's extension of this idea to families of nonexpansive mappings.

We construct an infinite family of commuting nonexpansive mappings with a weakly compact convex domain such that the family admits no common fixed points but any finite subfamily has a common fixed point.

2.6.1 DEFINITION OF THE OPERATOR T_τ [Sch]

Let (S, \mathcal{F}, μ) be a measure space and let $\tau^{-1} : S \rightarrow [0,1] \times S$ be a measure preserving transformation, where $\tau : [0,1] \times S \rightarrow S$ is bijective.

Define an operator T_τ on $\{f \in L_1(S) : 0 \leq f \leq 1\}$ by $T_\tau f(s) = \chi_{\tau^{-1}(\{t,s\} : 0 \leq t \leq f(s))}$.

2.6.2 PROPOSITION [Sch]

T_τ is an isometry on $\{f \in L_1(S) : 0 \leq f \leq 1\}$.

PROOF

Let $f \in \{f \in L_1(S) : 0 \leq f \leq 1\}$

$$\begin{aligned} &\Rightarrow \int T_\tau f \, d\mu \\ &= \mu(\tau^{-1}(\{t,s\} : 0 \leq t \leq f(s))). \end{aligned}$$

Now $(\lambda \times \mu) \{(t,s) : 0 \leq t \leq f(s)\}$

$$= \mu\{(\tau^{-1})^{-1}(\{t,s\} : 0 \leq t \leq f(s))\} \quad (\text{since } \tau^{-1} \text{ is measure preserving})$$

$$= \mu\{\tau(\{t,s\} : 0 \leq t \leq f(s))\} \quad (\text{since } \tau \text{ is bijective}).$$

$$\Rightarrow \mu(\tau^{-1}(\{t,s\} : 0 \leq t \leq f(s))).$$

$$\begin{aligned}
&= (\lambda \times \mu) \{(t,s) : 0 \leq t \leq f(s)\} \\
&= \int \chi_{\{(t,s) : 0 \leq t \leq f(s)\}} d(\lambda \times \mu) \\
&= \int_{s \in S} \int_0^1 \chi_{\{(t,s) : 0 \leq t \leq f(s)\}} d\lambda d\mu \quad (\text{by Fubini's theorem}) \\
&= \int_{s \in S} \int_0^{f(s)} 1 d\lambda d\mu \\
&= \int_{s \in S} f(s) d\mu \\
&= \int f d\mu
\end{aligned}$$

Hence $\int T_\tau f d\mu = \int f d\mu$.

Since $T_\tau f$ and f are positive functions, $\|T_\tau f\|_1 = \|f\|_1$. □

2.6.3 PROPOSITION [Sch]

T_τ is nonexpansive.

PROOF

Let $f, g \in \{f \in L_1(S) : 0 \leq f \leq 1\}$ and let $A = \{x \in S : f(x) \geq g(x)\}$.

$$\begin{aligned}
&\|T_\tau f - T_\tau g\|_1 \\
&= \int_A |T_\tau f - T_\tau g| d\mu + \int_{A^c} |T_\tau f - T_\tau g| d\mu \\
&= \int_A T_\tau f - T_\tau g d\mu - \int_{A^c} T_\tau f - T_\tau g d\mu \quad (\text{since } T_\tau f \geq T_\tau g \Leftrightarrow f \geq g) \\
&= \int_A f - g d\mu - \int_{A^c} f - g d\mu \quad (\text{by linearity of the integral and } \int T_\tau f d\mu = \int f d\mu)
\end{aligned}$$

$$= \int |f - g| d\mu$$

$$= \|f - g\|_1$$

Hence T_τ is nonexpansive. □

2.6.4 LEMMA [Sch]

T_τ and T_σ commute as operators on $\{f \in L_1(S) : 0 \leq f \leq 1\}$ if and only if

(*) $\tau([0,1] \times \sigma(B)) = \sigma([0,1] \times \tau(B))$ a.e for every $B \subset [0,1] \times S$, B measurable.

PROOF

Assume (*).

Then $T_\tau \circ T_\sigma f$

$$= \chi_{\tau([0,1] \times \sigma\{(t,s) : 0 \leq t \leq f(s)\})}$$

$$= \chi_{\sigma([0,1] \times \tau\{(t,s) : 0 \leq t \leq f(s)\})}$$

$$= T_\sigma \circ T_\tau f \quad \forall f \in \{f \in L_1(S) : 0 \leq f \leq 1\}.$$

Conversely, if T_τ and T_σ commute then taking $f = \alpha \chi_A$ for $A \in \mathcal{F}$ and for $\alpha \in [0,1]$

we have that

$$\chi_{\tau([0,1] \times \sigma\{[0,\alpha] \times A\})}$$

$$= \chi_{\tau([0,1] \times \sigma\{(t,s) : 0 \leq t \leq f(s)\})}$$

$$= T_\tau \circ T_\sigma f$$

$$= T_\sigma \circ T_\tau f$$

$$= \chi_{\sigma([0,1] \times \tau\{(t,s) : 0 \leq t \leq f(s)\})}$$

$$= \chi_{\sigma([0,1] \times \tau\{[0,\alpha] \times A\})}$$

Thus $\tau([0,1] \times \sigma\{[0,\alpha] \times A\}) = \sigma([0,1] \times \tau\{[0,\alpha] \times A\})$ a.e for every sets of form $[0,\alpha] \times A$.

Thus true for all measurable $B \subset [0,1] \times S$. □

2.6.5 DEFINITION OF $W_{\alpha,\beta}$ [Sch]

For $0 < \alpha \leq \beta < 1$ define $W_{\alpha,\beta} = \{f \in L_1(S) : 0 \leq f \leq 1, \alpha \leq \int f \leq \beta\}$.

2.6.6 PROPOSITION cf [Sch]

The set $W_{\alpha,\beta}$ is a T_τ -invariant convex subset of $L_1(S)$.

Moreover, if $\mu(S) < \infty$ then $W_{\alpha,\beta}$ is weakly compact.

PROOF

Note that by definition of T_τ and by what we have shown above (in proving that T_τ is an isometry), we have that $T_\tau(W_{\alpha,\beta}) \subset W_{\alpha,\beta}$.

Convexity follows from the linearity of the integral.

Suppose that $\mu(S) < \infty$.

We now aim to prove that $W_{\alpha,\beta}$ is weakly compact.

Claim(1): $W_{\alpha,\beta}$ is weakly sequentially compact.

(Let $\epsilon > 0$. Choose $\delta = \epsilon > 0$. Let $E \in \mathcal{F}$. If $\mu(E) < \delta$ then for any $f \in W_{\alpha,\beta}$

$\int_E f d\mu \leq \mu(E) < \epsilon$. Hence by 1.10.4, we justify our claim.)

E

Thus weakly sequentially compact.

Claim(2): $W_{\alpha,\beta}$ is weakly closed

(Let $f \in \overline{W_{\alpha,\beta}}$. Then there exists $\{f_n\} \in W_{\alpha,\beta}$ with $\|f_n - f\| \rightarrow 0$ ($n \rightarrow \infty$).

$\Rightarrow 0 \leq f \leq 1$

Also $\int f d\mu = \lim_n \int f_n d\mu$ by the Dominated convergence theorem.

$\Rightarrow \alpha \leq \int f \leq \beta$

Thus strongly closed.

But $W_{\alpha,\beta}$ is convex. Hence $W_{\alpha,\beta}$ is weakly closed.)

Thus claim(2) is true.

Hence $W_{\alpha,\beta}$ is weakly closed and weakly sequentially compact.

Now $L_1(S)$ is a Banach space.

Therefore by combining our claims and using the Eberlein–Smulian theorem, we see that

$W_{\alpha,\beta}$ is weakly compact, provided that $\mu(S) < \infty$. □

2.6.7 LEMMA [Sch]

Let $0 < \alpha \leq \beta < 1$.

T_τ has a fixed point χ_A in $W_{\alpha,\beta}$ if and only if there exists a set $A \in \mathcal{F}$ with $\alpha \leq \mu(A) \leq \beta$ and $\tau([0,1] \times A) = A$ a.e .

PROOF

First note that if $A \in \mathcal{F}$ then $\{(t,s) : 0 \leq t \leq \chi_A(s)\} = [0,1] \times A$ a.e.

Supposing there exists $A \in \mathcal{F}$ with $\alpha \leq \mu(A) \leq \beta$ and $\tau([0,1] \times A) = A$ a.e then

$$\begin{aligned} & T_\tau \chi_A \\ = & \chi_{\tau\{(t,s) : 0 \leq t \leq \chi_A(s)\}} \\ = & \chi_{\tau([0,1] \times A)} \\ = & \chi_A . \end{aligned}$$

Conversely, if χ_A is a fixed point then $A \in \mathcal{F}$ and it is apparent that $\tau([0,1] \times A) = A$ a.e by the equalities used above in proving that χ_A is a fixed point for T_τ .

We henceforth suppose that $S = [0,1]^{\mathbb{N}}$.

In the following definition we remind the reader of the canonical σ -algebra and the measure defined on this product space. (See [KT] for details.)

2.6.8 DEFINITIONS

Let $X_i = [0,1]$ for $i \in \mathbb{N}$, let \mathcal{F}_i be the Borel σ -algebra on X_i and let μ_i be the Lebesgue measure on X_i .

Let \mathcal{F} be the product σ -algebra of subsets of $[0,1]^{\mathbb{N}}$ which is generated by the cylinder sets of the form $E_1 \times E_2 \times \dots \times E_k \times \prod_{i=k+1}^{\infty} X_i$, where $E_i \in \mathcal{F}_i$ for $1 \leq i \leq k + 1$.

Let μ denote the unique measure on \mathcal{F} such that

$$\mu(E_1 \times E_2 \times \dots \times E_k \times \prod_{i=k+1}^{\infty} X_i) = \mu_1(E_1) \cdot \mu_2(E_2) \cdot \dots \cdot \mu_k(E_k).$$

Given a one-to-one measure preserving transformation $\rho : [0,1] \rightarrow [0,1]$, Schechtman defines $\tau(\rho) : [0,1] \times [0,1]^{\mathbb{N}} \rightarrow [0,1]^{\mathbb{N}} : (t, (s_1, s_2, \dots)) \rightarrow (\rho(s_1), t, s_2, s_3, \dots)$.

2.6.9 LEMMA [Sch]

If ρ_1 and ρ_2 commute then $T_{\tau(\rho_1)}$ and $T_{\tau(\rho_2)}$ commute.

PROOF

By 2.6.4, it suffices to show that $\tau(\rho_1)([0,1] \times \tau(\rho_2)(B)) = \tau(\rho_2)([0,1] \times \tau(\rho_1)(B))$ a.e for every $B \subset [0,1] \times S$, B measurable, where $\tau(\rho_1), \tau(\rho_2) : [0,1] \times S \rightarrow S$.

Let $y \in \tau(\rho_1)([0,1] \times \tau(\rho_2)(B))$ for any measurable $B \subset [0,1] \times S$.

Then $y = \tau(\rho_1)(\underline{x})$, where $\underline{x} = (t, (\rho_2(s_1), t', s_2, s_3, \dots))$ for $(t', (s_1, s_2, s_3, \dots)) \in B$.

$$\begin{aligned} \Rightarrow y &= \tau(\rho_1)(\underline{x}) \\ &= (\rho_1(\rho_2(s_1)), t, t', s_2, \dots) \\ &= (\rho_2(\rho_1(s_1)), t, t', s_2, \dots) \quad (\text{since } \rho_1 \text{ and } \rho_2 \text{ commute}) \\ &= \tau(\rho_2)(t, (\rho_1(s_1), t', s_2, \dots)) \end{aligned}$$

$$= \tau(\rho_2)(t, \tau(\rho_1)(t', (s_1, s_2, \dots))).$$

But $(t, \tau(\rho_1)(t', (s_1, s_2, \dots))) \in [0, 1] \times \tau(\rho_1)(B)$.

$$\Rightarrow y \in \tau(\rho_2)([0, 1] \times \tau(\rho_1)(B))$$

Hence $\tau(\rho_1)([0, 1] \times \tau(\rho_2)(B)) \subset \tau(\rho_2)([0, 1] \times \tau(\rho_1)(B))$.

Other inclusion is done by interchanging $\tau(\rho_1)$ and $\tau(\rho_2)$ in the above equalities which were used to show that $\tau(\rho_1)([0, 1] \times \tau(\rho_2)(B)) \subset \tau(\rho_2)([0, 1] \times \tau(\rho_1)(B))$. \square

2.6.10 THEOREM [Sch]

There exists a weakly compact, convex subset W of $L_1(S)$ and a sequence T_1, T_2, \dots of commuting nonexpansive operators of W into itself such that any finite number of them have a common fixed point but there exists no common fixed point for the entire family.

PROOF

Let $W = W_{1/2, 1/2}$ as defined in 2.6.5 and let $\{r_i\}$ be an enumeration of the rationals in $[0, 1]$.

For any given n , let m be a common denominator for r_1, \dots, r_n and let

$$A_0 = \bigcup_{k=0}^{m-1} (k/m, k/m + 1/2m).$$

Then $\mu(A_0) = 1/2$.

Define $\rho_i : [0, 1] \rightarrow [0, 1]$ by $\rho_i(t) = t + r_i \pmod{1}$ and let $T_i = T_{\tau(\rho_i)}$.

Then ρ_i is measure preserving and bijective. Hence 2.6.9 applies.

By 2.6.9, $\{T_i\}_{i=1}^{\infty}$ commute.

Claim(1): $\tau(\rho_i)^{-1} : [0, 1]^{\mathbb{N}} \rightarrow [0, 1] \times [0, 1]^{\mathbb{N}} : (s_1, s_2, s_3, \dots) \rightarrow (s_2, \rho_i^{-1}(s_1), s_3, \dots)$ is measure preserving

(First note that since ρ_i is bijective, so is $\tau(\rho_i)$.)

Let $A = A_1 \times A_2 \times \dots \times A_k \times \prod_{i=k+1}^{\infty} X_i$ be a cylinder, where $X_i = [0, 1]$.

We need to show that $\mu(\tau(\rho_i)^{-1}(A)) = \mu(A)$.

Thus it suffices to show that $\mu(\tau(\rho_i)(A)) = \mu(A)$. To this end,

$$\begin{aligned}
 (\tau(\rho_i)(A)) &= \rho_i(A_2) \times A_1 \times \dots \times A_k \times \prod_{i=k+1}^{\infty} X_i \\
 \Rightarrow \mu(\tau(\rho_i)(A)) &= \\
 &= \mu_2(\rho_i(A_2)) \cdot \mu_1(A_1) \cdot \dots \cdot \mu_k(A_k) \\
 &= \mu_2(A_2) \cdot \mu_1(A_1) \cdot \dots \cdot \mu_k(A_k) \quad (\text{since } \rho_i \text{ is measure preserving and bijective}) \\
 &= \mu(A)
 \end{aligned}$$

Hence claim(1) is true.

Thus 2.6.3 applies. Further, note that since $\mu(S) = 1 < \infty$, 2.6.6 applies.

By 2.6.3, each T_i is nonexpansive.

By 2.6.6, W is weakly compact in $L_1(S)$, convex and invariant under T_i .

Now $\rho_i(A_0) = A_0$ for $1 \leq i \leq n$. Moreover, for any i , $\tau(\rho_i)([0,1] \times A_0 \times S) = \rho_i(A_0) \times S$ and $\rho_i(A_0) \times S = A_0 \times S$ a.e if and only if $\rho_i(A_0) = A_0$ a.e.

Let $A = A_0 \times S$. By 2.6.7, χ_A is a fixed point for T_i where $1 \leq i \leq n$.

Suppose that χ_A is a fixed point for $\{T_i\}_{i=1}^{\infty}$.

Then $\tau(\rho_i)([0,1] \times A) = A \quad \forall i \in \mathbb{N}$.

Claim(2): $\tau(\rho_\alpha)([0,1] \times A) = A$ for any ρ_α of the form
 $\rho_\alpha(t) = t + \alpha \pmod{1}$ for $\alpha \in [0,1]$.

(Let $\alpha \in [0,1]$ and let $\epsilon > 0$.)

Choose r_i such that $|\alpha - r_i| < \epsilon/2$.

Now $\mu(\tau(\rho_i)([0,1] \times A) \Delta \tau(\rho_\alpha)([0,1] \times A)) < \epsilon$, where Δ is the symmetric difference.

But $\tau(\rho_i)([0,1] \times A) = A$.

Since ϵ is arbitrary, $A = \tau(\rho_\alpha)([0,1] \times A)$.

Hence claim(2) is true.

$$\Rightarrow A_0 \times S = \tau(\rho_\alpha)([0,1] \times A_0 \times S)$$

$$\Rightarrow \rho_\alpha(A_0) = A_0$$

But if α is irrational then p_α is ergodic.

$$\Rightarrow \mu(A_0) = 0 \text{ or } 1 \text{ giving a contradiction.} \quad \square$$

§ 2.7 Demicompact mappings

A method one sometimes adopts to show the existence of a fixed point for a continuous map is to construct a sequence $\{x_n\}$ of ϵ -fixed points (say in a metric space X) and to show that it converges to some x in X . Then by the continuity of the map, say f , $f(x_n) \rightarrow f(x)$. Then it follows that $f(x_n) \rightarrow x$ also, since the sequence contains ϵ -fixed points, by which x becomes a fixed point.

Obviously, in order to adopt this method, we rely heavily on the assumptions of the theorem. For instance, the proof of the Banach fixed point theorem uses this method, where we are given that X is a complete metric space and the map is contractive. This enables us to construct a convergent sequence of ϵ -fixed points.

The concept of demicompactness allows us to relax the assumptions on the space and even the mapping itself to some extent and still adopt this method.

2.7.1 DEFINITION cf [NSW]

Let X be a metric space. A mapping $T: X \rightarrow X$ is said to be **demicompact** if it is continuous and every bounded sequence $\{x_n\}$ for which $d(x_n, Tx_n) \rightarrow 0$ contains a convergent subsequence.

The essence of this definition is that if we have ϵ -fixed points for every $\epsilon > 0$ then we are guaranteed a fixed point, provided that the map is demicompact.

2.7.2 EXAMPLES

2.7.2.1

Let X be any normed space. Define $T: X \rightarrow X$ by $Tx = kx$, where k is a constant such that $k \neq 1$.

If $\{x_n\}$ is a bounded sequence for which $\|Tx_n - x_n\| \rightarrow 0$ ($n \rightarrow \infty$) then $(1 - k) \|x_n\| \rightarrow 0$ ($n \rightarrow \infty$). Thus $\{x_n\}$ is convergent. □

2.7.2.2

Define $T: \mathbb{R} \rightarrow \mathbb{R}$ by $Tx = x + \frac{1}{1 + |x|}$. If $|Tx_n - x_n| \rightarrow 0$ then $\{x_n\}$ is unbounded. Thus T is demicompact in a vacuous sense. □

As a trivial example, any continuous mapping whose domain is compact is demicompact.

At this stage, we are not sure of any application of this concept in terms of generalisations or extensions of results involving nonexpansive mappings except in the following theorem.

2.7.3 THEOREM cf [NSW]

Let X be a Banach space and let C be a nonempty bounded closed convex set in X . Let \mathcal{F} be a family of commuting nonexpansive mappings: $C \rightarrow C$, where $F(T) \neq \emptyset \quad \forall T \in \mathcal{F}$ and \mathcal{F} has at least one element which is demicompact.

Then \mathcal{F} has a common fixed point.

Let us establish the following lemmas as we need them to prove of this theorem.

2.7.4 LEMMA

Let S be a nonempty bounded convex set in a normed space X .

If $T: S \rightarrow S$ is demicompact and nonexpansive then T has a fixed point in S .

PROOF

By 1.1.2, for every $\epsilon > 0$, T has an ϵ -fixed point in S .

By definition of demicompactness, T has a fixed point. □

2.7.5 LEMMA [NSW]

Let X be any metric space and let S be a bounded closed subset of X . If $T: S \rightarrow S$ is a demicontact mapping then $F(T)$ is compact.

PROOF

Assume w.l.o.g. that $F(T) \neq \emptyset$.

Let $\{x_n\}$ be a sequence in $F(T)$. Then $d(x_n, Tx_n) = 0 \quad \forall n \in \mathbb{N}$.

By demicontactness of T there exists a subsequence x_{n_k} converging to some x in S since S is closed.

By continuity of T we have that $x = Tx$.

Thus $x \in F(T)$.

Compactness of $F(T)$ follows. □

Let us now prove 2.7.3.

PROOF (2.7.3)

By hypothesis, there exists a demicontact mapping $T' : C \rightarrow C$ in \mathcal{F} .

Let $\mathcal{S} = \{S \subset C : S \neq \emptyset, \text{ bounded, closed, convex and invariant under each } T \in \mathcal{F}\}$.

The system \mathcal{S} is nonempty, since $C \in \mathcal{S}$.

Claim(1): If $S \in \mathcal{S}$ then the set $\{x \in S : T'x = x\}$ is nonempty and compact.

 i.e. $F(T') \cap S$ is nonempty and compact.

(Let $S \in \mathcal{S}$. Then $T' : S \rightarrow S$. If $\{x_n\}$ is a bounded sequence in S such that

$\|x_n - T'x_n\| \rightarrow 0$ then $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$, by demicontactness of

$T' : C \rightarrow C$. But S is closed. Thus $\{x_{n_k}\}$ converges to an element of S . Hence T' is

demicontact on S also. Hence 2.7.4 and 2.7.5 apply.

By 2.7.4 we have that the set $\{x \in S : T'x = x\}$ is nonempty.

By 2.7.5 we have that $\{x \in S : T'x = x\}$ is compact.)

Thus claim(1).

Order \mathcal{S} by \supset . Let $\mathcal{C} = \{S_\alpha : \alpha \in \Lambda\}$ be a chain in \mathcal{S} .

Let $\alpha \in \Lambda$ and let $F_\alpha = \{x \in S_\alpha : T'x = x\}$. Then $\{F_\alpha : \alpha \in \Lambda\}$ satisfies the finite intersection property, since \mathcal{C} has the total order \supset . But each F_α is nonempty and compact by claim(1).

Hence $F = \bigcap \{F_\alpha : \alpha \in \Lambda\}$ is a nonempty subset of $\bigwedge \mathcal{C} = \bigcap \{S_\alpha : \alpha \in \Lambda\}$.

$\Rightarrow \bigwedge \mathcal{C} \neq \emptyset$

Further, $\bigwedge \mathcal{C} = \bigcap \{S_\alpha : \alpha \in \Lambda\}$ is bounded, closed, convex and invariant under each T .

Hence $\bigwedge \mathcal{C} \in \mathcal{S}$ is a lower bound for \mathcal{C} .

By Zorn's lemma, \mathcal{S} has a minimal element M .

Let $H = \{x \in M : T'x = x\}$. By claim(1), H is a nonempty compact subset of M .

For $x \in H$, $T'(Tx) = T(T'x) = Tx \in M \quad \forall T \in \mathcal{F}$.

Hence $T(H) \subset H \quad \forall T \in \mathcal{F}$.

Let $\mathcal{S}_1 = \{H_1 \subset H : H_1 \text{ nonempty, compact and } T(H_1) \subset H_1 \quad \forall T \in \mathcal{F}\}$.

Then $\mathcal{S}_1 \neq \emptyset$, since $H \in \mathcal{S}_1$. Order \mathcal{S}_1 by \supset . If \mathcal{C}_1 is a chain in \mathcal{S}_1 then \mathcal{C}_1 has the finite intersection property and hence $\bigwedge \mathcal{C}_1$ is nonempty since every element of \mathcal{C}_1 is compact.

Now $\bigwedge \mathcal{C}_1$ is a closed subset of every H_1 in \mathcal{C}_1 . Hence $\bigwedge \mathcal{C}_1$ is compact. Clearly, $\bigwedge \mathcal{C}_1$ is invariant under every element of \mathcal{F} . Thus $\bigwedge \mathcal{C}_1$ is a lower bound for \mathcal{C}_1 in \mathcal{S}_1 .

By Zorn's lemma, \mathcal{S}_1 has a minimal element M_1 .

By 2.1.2 we have that $T(M_1) = M_1 \quad \forall T \in \mathcal{F}$.

Suppose that M_1 consists of more than one element.

By our claim in the proof of 1.2.3, there exists $u \in \overline{\text{co}}(M_1)$ with

$p = \sup \{\|x - u\| : x \in M_1\} < \delta(M_1)$. Since $M_1 \subset M$ we have that $u \in M$. For each $x \in M_1$ we have that $u \in B(x, p)$.

Let $B = \bigcap_{x \in M_1} B(x, p)$ and let $B_M = B \cap M$. Then B_M is nonempty, closed and convex.

Claim(2): $T(B_M) \subset B_M \quad \forall T \in \mathcal{F}$

(Let $z \in B$ and let $T \in \mathcal{F}$. Then $\|x - z\| \leq p \quad \forall x \in M_1$.)

Since $T(M_1) = M_1$, for each $x \in M_1$ there exists $y \in M_1$ with $Ty = x$.

Now $\|Tz - x\| = \|Tz - Ty\| \leq \|z - y\| \leq p \quad \forall x \in M_1$.

Hence $Tz \in B(x, p) \quad \forall x \in M_1$.

$\Rightarrow T(B) \subset B \quad \forall T \in \mathcal{F}$

Also $T(M) \subset M$ for each $T \in \mathcal{F}$.

$\Rightarrow T(B_M) \subset B_M \quad \forall T \in \mathcal{F}$

Hence claim(2).

Hence $B_M \in \mathcal{S}$.

By minimality of M in \mathcal{S} we have that $B_M = M$.

Since M_1 is compact there exists $x_0, y_0 \in M_1$ such that $\|x_0 - y_0\| = \delta(M_1) > p$.

$\Rightarrow y_0 \notin B(x_0, p)$

$\Rightarrow y_0 \notin B$

But this would imply that $y_0 \notin B_M = M$ giving a contradiction, since $M_1 \subset M$.

Thus M_1 is a singleton.

Hence the point in M_1 is a common fixed point for the family since we have established that $T(M_1) = M_1$ for all T in \mathcal{F} . □

2.7.6 EXAMPLE

In the hypothesis of 2.7.3, we cannot delete the assumption that at least one mapping must be demicompact.

PROOF

Consider the set W in 2.6.10, which is nonempty, weakly compact and convex in the Banach space $L_1(S)$.

This implies that W is closed and bounded.

The situation in 2.6.10 is that we have a family $\{T_i\}$ of commuting nonexpansive mappings which map W into itself such that any finite number of mappings have a common fixed point. This implies that $F(T_i) \neq \emptyset$ for every $i \in \mathbb{N}$.

However, there exists no common fixed point for the entire family.

Hence none of the maps in the family can be demicompact. □

Remark:

Due to the assumption of at least one of the mappings being demicompact in 2.7.3, we cannot claim this theorem (2.7.3) to be a generalisation of the previous theorems involving families of nonexpansive mappings, unless the domain of the family of mappings is compact. The reason for this is that if the domain of a continuous mapping is compact, then by definition of demicompactness the map is demicompact, as mentioned in one of our examples.

2.7.7 THEOREM

Let X be a Banach space and let C be a nonempty bounded closed convex set in X . Let \mathcal{F} be a family of commuting nonexpansive mappings: $C \rightarrow C$, where $F(T) \neq \emptyset \quad \forall T \in \mathcal{F}$ and \mathcal{F} has at least one element T' , for which $T'(C)$ is compact.

Then \mathcal{F} has a common fixed point.

PROOF

Let the hypothesis hold.

Then $T'(C)$ is compact for some $T' \in \mathcal{F}$. Let $\{x_n\}$ be a sequence in C such that

$\|T'x_n - x_n\| \rightarrow 0$. By compactness of $T'(C)$, $\{T'x_n\}$ has a subsequence $\{T'x_{n_k}\}$ converging to some $T'x$.

Now $0 \leq \|x_{n_k} - T'x\| \leq \|x_{n_k} - T'x_{n_k}\| + \|T'x_{n_k} - T'x\| \rightarrow 0$ as $k \rightarrow \infty$.

Thus T' is demicompact.

By the above theorem (2.7.3) the result is immediate. □

Remarks:

As mentioned in § 2.1, clearly, this theorem (2.7.7) generalises theorem 2.1.1, since the domain in 2.1.1 is compact.

It is worth noting that 2.7.7 does not place strong assumptions on set C when compared to 2.4.5, which we have seen to be the the most general theorem compared to its predecessors.

In this way, 2.7.7 is a short and a neat generalisation of 2.1.1 when compared to 2.4.5.

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