Multi-Curve Bootstrapping and Implied Discounting Curves in Illiquid Markets

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MPhil in Mathematical Finance,
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Declaration

I declare that this dissertation is my own, unaided work. It is being submitted for the Degree of Master of Philosophy at the University of the Cape Town. It has not been submitted before for any degree or examination to any other University.

Nina Sender

June 8, 2017

Abstract

The credit and liquidity crisis of 2007 has triggered a number of inconsistencies in the interest rate market, questioning some of the standard methods and assumptions used to price and hedge interest rate derivatives. It has been shown that using a single risk-free curve (constructed from market instruments referencing underlying rates of varying tenors) to forecast and discount cash flows is not theoretically correct. Standard market practice has evolved to a multi-curve approach, using different curves to forecast and discount cash flows. The risk-free discount curve is proxied by the Overnight-Indexed Swap (OIS) curve. In South Africa there is no liquid market for OIS. In this dissertation a method is developed to estimate the ZAR OIS curve. A cointegration relationship between the SAFEX Overnight Rate, and the 3-month JIBAR rate is shown to exist. This relationship is used in a dual bootstrap algorithm, to simultaneously estimate the ZAR OIS curve and 3-month JIBAR tenor curve, while maintaining arbitrage relationships. The tractability of this method is shown, by pricing options written on ZAR OIS.

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I am very grateful to my parents for their continued support, love and guidance, without it I would not be where I am today.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 The Effect of the 2007 Financial Crisis on Interest Rate Markets

Before the credit and liquidity crisis in 2007, interest rate markets were characterized by textbook consistencies, Morini (2009). Practitioners assumed the existence of a single, well-defined term-structure of risk-free interest rates. Risk-free in this sense refers to the absence of credit and liquidity risk. Forward rate agreements (FRA's) could be replicated by going long and short the respective deposits. The difference between the deposit-implied FRA rate, and the market FRA rate was negligible. Interest rate swaps of the same maturity, but indexed to underlying rates with different tenors had similar rates. Deposit and Overnight Indexed Swap (OIS) rates tracked each other closely. With the onset of the crisis these rates that were so closely related, became segmented into different quantities carrying different credit and liquidity premia, Mercurio (2009). Large spreads developed between rates that were previously considered the same, invalidating many traditional fixed income pricing methodologies, Gallitschke *et al.* (2014). Non-zero spreads is not an entirely new phenomenon, but before 2007 they were small enough to be considered negligible.

Figure 1.1 shows the divergence between FRA rates implied by overnight deposits and the market quoted FRA rate. Figure 1.2 shows the divergence of swap rates of the same maturity, that reference rates with different tenors. Figure 1.3 shows the divergence between the 1-month EONIA[1] deposit rate and 1-month EURIBOR deposit rate. A 1-month EONIA deposit is an overnight deposit which is rolled daily over the period of 1-month. In Figure 1.3 the 1-month EURIBOR deposit rate lies above the 1-month EONIA deposit rate, as it carries higher credit and liquidity risk.

[1] Euro Overnight Index Average, a weighted average of all overnight rates corresponding to all unsecured lending transaction in the Euro-zone interbank market.

Fig. 1.1: EUR 3x6 EONIA rates versus 3x6 FRA rates

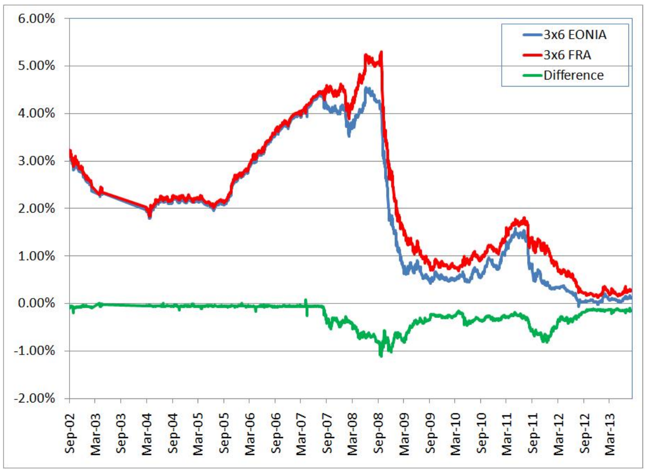


Fig. 1.2: 2-year swap rates (3-month versus 6-month)

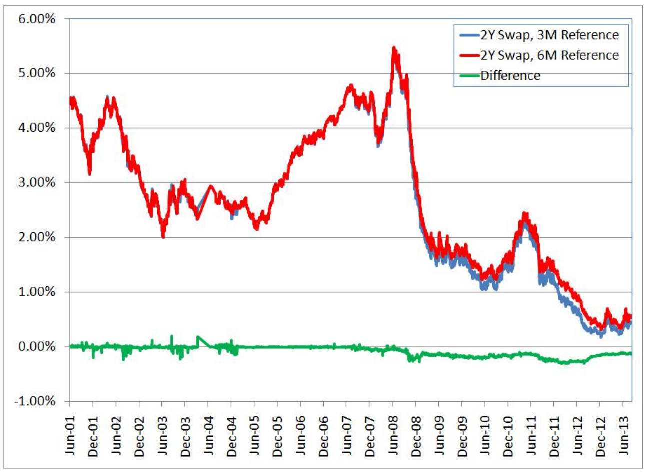
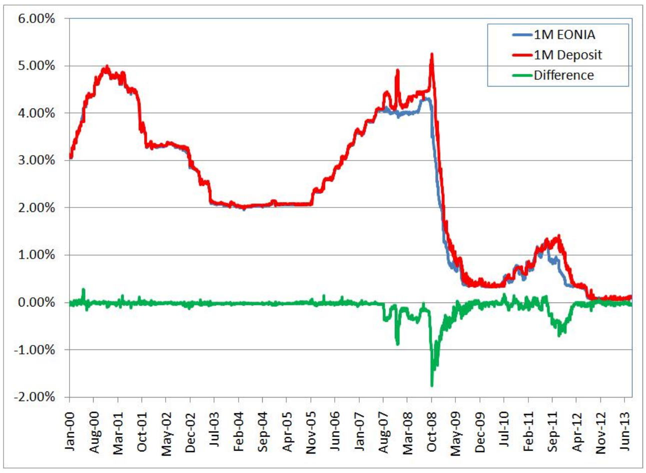


Fig. 1.3: EUR 1-month EONIA rates versus 1-month deposit rates



Figures 1.1, 1.2 and 1.3 [2], illustrate the divergence of previously similar rates. This has led to a segmentation in the interest rate market in the sense that underlying rates of different tenors, typically 1-month, 3-month, 6-month and 12-month, carry different levels of credit and liquidity risk, Bianchetti (2008). Each underlying tenor is characterised by a different credit and liquidity risk premium with shorter tenors carrying less credit and liquidity risk, and in this way the interest rate market is segmented. It is important to note that the divergence in values does not create arbitrage opportunities when credit and liquidity risk are taken into account. This is shown in, Morini (2009), where a theoretical framework is developed, which justifies the divergence in value of similar rates, by looking at a stochastic default probability. Practitioners and academics have adopted an empirical approach referred to as the Multi-Curve framework, where a different curve is constructed for each different rate tenor, so that each curve has homogeneous credit and liquidity risks. In post-crisis fixed income pricing this method has rapidly become the predominant market standard, Gallitschke *et al.* (2014).

In the remainder of this section the traditional single-curve and the new multiple-curve framework are briefly outlined. The changes in the interest rate markets, which we have looked at in a developed economy are then looked at in the South African context.

1.2 Single Curve Framework

Constructing a single yield curve to price and hedge interest rate sensitive instruments was standard market practice before the financial crisis in 2007. Interbank credit and liquidity issues were assumed negligible and not considered for pricing. The following procedure from, Ametrano and Bianchetti (2009), summarises the pre-crisis standard market practice:

1. Select the curve inputs (these should be liquid). Typically money market instruments are used for the short-end, FRA's in the middle and interest rate swaps for the long end of the term structure.
2. Build **one yield curve** using the selected inputs. The yield curve is built using a pre-specified bootstrapping algorithm, which incorporates a chosen interpolation method.

[2] The data for these figures is taken from Bloomberg.

3. From the yield curve calculate forward rates, cash-flows and discount factors needed for derivatives pricing.
4. Calculate the price of the derivative using the relevant pricing formula.

The pre-crisis approach is no longer applicable to the current market environment in light of the following stylized facts that have emerged from the financial crisis, Bianchetti and Carlicchi (2011):

- The divergence between deposit (EURIBOR-based) rates and OIS rates;
- The divergence between FRA rates and the corresponding deposit-implied forward rates;
- The explosion of Basis Swap spreads;
- The shift from unsecured to secured market instruments;
- The shift towards Credit Support Annex (CSA) discounting for collateralised cash flows.

1.3 Modern Pricing Paradigm

Since the financial crisis of 2007, the Classical Pricing Framework outlined in Section 1.2 has been replaced by the modern pricing approach (Multiple-Curve Framework). This entails the construction of multiple zero-coupon curves, one for each tenor. For example in Europe the following curves are constructed:

- 1-month nominal swap curve, referencing the 1-month EURIBOR rate
- 3-month nominal swap curve, referencing the 3-month EURIBOR rate
- 6-month nominal swap curve, referencing the 6-month EURIBOR rate
- 12-month nominal swap curve, referencing the 12-month EURIBOR rate

Each curve is constructed using a set of instruments homogeneous in underlying rate tenor, which provides a consistent representation of credit and liquidity risk. In the multiple curve framework, there are forecasting or tenor curves, shown in the examples above and there is a discounting curve. The forecasting curves are yield curves used to compute forward rates, and the discounting curve is used to discount future cash flows. The forecasting curve will be selected according to the underlying rate tenor of the selected interest rate derivative. The choice of discounting curve is however less clear.

Derivatives are priced in a risk-neutral framework, using risk-free rates for discounting. Before the 2007 credit crunch EURIBOR and EURIBOR swap rates were used as proxies for risk-free rates. The validity of this approach was questioned during the credit crisis, as banks became reluctant to lend to each other because of credit concerns. EURIBOR rates carry credit and liquidity risk premia, which differ across tenors. The banking sector is no longer considered risk-free. The value of a derivative must be considered along with credit risk and collateral agreements. When derivative dealers trade in the non-centrally cleared over-the-counter (OTC) market credit has become a major concern, and this additional risk needs to be taken into account in pricing. The interest rate paid on cash collateral also affects pricing. The standard approach to factor in credit risk is to first calculate the default-free value of the derivative, and then make credit and other valuation adjustments. The most common valuation adjustments are:

- Credit Value Adjustment (CVA) - this is the expected cost to the dealer if the counterparty were to default;
- Debt Value Adjustment (DVA) - this is the expected value to the counterparty if the dealer were to default;
- Funding Value Adjustment (FVA) - this is the adjustment to the contract value when the bank's funding rate exceeds the risk-free rate;
- Liquidity Value Adjustment (LVA) - this the adjustment to the contract value related to the cost of carry of collateral on a collateralised deal. The cost of carry is when the collateral rate is different to the cost of funding/investment for the dealing party.

The next consideration is the choice of discounting curve to calculate the default-free value.

1.3.1 OIS as a proxy for the risk-free rate

The next consideration is the choice of discounting curve to calculate the default-free value, i.e the best proxy for the risk-free curve. It is argued in, Hull and White (2013), that the OIS curve is the best proxy for the risk-free curve. Banks can borrow money in the overnight market on both an unsecured and secured basis, however the overnight EURIBOR rate (EONIA) is unsecured and therefore not entirely risk-free.

The LIBOR/EURIBOR overnight rates are referenced by Overnight Index Swaps (OIS). An OIS is a fixed for floating interest rate swap, where the floating leg references an overnight rate. The fixed rate is exchanged for a geometric average of the floating overnight rate at the end of the term of the swap.

There are two sources of risk in an OIS, Hull and White (2013):

1. The credit risk in the EURIBOR overnight rate, which is arguably small;
2. The credit risk that arises from the possible default of one of the counterparties.

The probability of the counterparty defaulting will lead to an adjustment to the fixed rate of the swap. This adjustment can safely be assumed to be zero for collateralised transactions, Hull and White (2013). From this it can be concluded that the OIS swap rate is a reasonable proxy for the risk-free rate.

Multi-Curve Framework

The post-crisis, multiple curve framework is summarised below, following, Ametrano and Bianchetti (2013).

1. Select the relevant OIS instruments to build the OIS swap curve (the discounting curve).
2. Select the separate sets of vanilla interest rate inputs, which will be used to construct the tenor curves. Each set of instruments should be homogeneous in their underlying rate tenor. There is typically a 1-month, 3-month, 6-month and 12-month curve, depending on the market.
3. Build multiple separate forecasting curves and the discounting curve using the inputs selected in (1 and 2) along with the discounting curve and a set of bootstrapping rules.
4. The forecasting curves are then used to calculate the forward rates and corresponding cash flows.
5. The discount factors are calculated from the discount curve calculated in 1.
6. Calculate the default-free price of the derivative using the relevant pricing formula and curves, and make the required value adjustments.

Valuation of Derivatives in the modern Paradigm

This section outlines how the assumptions underlying derivative pricing have changed as a result of the 2007 financial crisis. The following rates are assumed for a portfolio of derivatives:

Source	**Rate**
Risk-free valuation of derivatives	r_t
Rate earned on collateral	c_t
Rate paid on funding/hedging of derivatives	$\tilde{r_t}$

If perfect world (Black-Scholes) assumptions hold in the market then,

- $r_t = c_t$;
- There is no credit risk or liquidity risk;
- There are no funding costs, so $\tilde{r_t} = 0$.

In this case, the adjusted value of the derivative is equal to value of the derivative at the risk-free rate r_t.

If the following rates apply to a portfolio of derivatives:

Source	**Rate**
Risk-free valuation of derivatives	r_t
Rate earned on collateral	c_t $(c_t \neq r_t)$
Rate paid on funding/hedging of derivatives	$\tilde{r_t}$ $(\tilde{r_t} \neq c_t)$

and some of the perfect market assumptions are relaxed then,

- there is credit risk in the transaction,
- there are imperfect funding/hedging costs,
- the rate earned on collateral is not equal to the risk-free rate,

Then the value of the derivatives portfolio is given by the following equations, for collateralized and uncollateralized portfolios:

$$\text{Collateralized: } V_{adjusted} = V_{risk-free} + \text{FVA} + \text{LVA}$$
$$\text{Uncollateralized: } V_{adjusted} = V_{risk-free} + \text{FVA-CVA+DVA}$$

These adjustments ensure that the price of the derivative correctly accounts for additional credit and liquidity risks. When the risk-free value of the derivative is calculated the cash flows are calculated using the LIBOR rates they are linked to, and discounted the a risk-free rate (arguably OIS). Discounting using LIBOR has been found to be theoretically incorrect in most cases, Hull and White (2013).

1.4 South African Context

The use of the OIS curve as the default-free discount curve is known as OIS discounting and has become accepted market practice and is universally accepted, Daniels *et al.* (2014). Derivative cash flows are estimated using forecasting curves described in section 1.3, and cash flows are discounted using the OIS curve to calculate the risk-free value of the derivative, which is adjusted for credit and liquidity risk appropriately. In developed markets such as Europe and the United States, there is a liquid OIS market meaning the OIS curve can be easily bootstrapped using the liquid market instruments. In South Africa there are no observable quotes that can be used to construct an OIS curve, making OIS discounting difficult to implement.

Market participants are aware of the need for an OIS market in South Africa, however there is no general consensus on how the initiation of an OIS market should be approached, Daniels *et al.* (2014). The financial crisis did not affect South Africa to the extent it did other markets, which means the spreads described above were not as exaggerated. The need for an OIS market in South Africa is not urgent, and may take time to implement. In light of this, we show that multi-curve bootstrapping and OIS discounting in South Africa can be implemented without a formal OIS market, by estimating the OIS curve. In South Africa we need a method to construct the best possible estimate of the ZAR OIS curve, a required input to the modern pricing framework, Daniels *et al.* (2014).

In South Africa there are two overnight rates that can be used, the South African Futures Exchange (SAFEX) Rate overnight rate, and the South African Benchmark overnight rate (SABOR). The SAFEX overnight rate is the average rate that the exchange receives on its deposits in the bank, weighted by the size of the investment placed at each bank, i.e it is the rate earned on ZAR collateral. It should be noted that SAFEX deposits form only a small portion of the total overnight funding in South Africa, and therefore may not be an entirely representative reflection on the weighted average call rates paid on Rand deposits at all banks, Daniels *et al.* (2014). In this dissertation we use the SAFEX overnight rate as it is currently the best proxy available for a tradable overnight rate, Jakarasi *et al.* (2015).

The SAFEX overnight rate and the 3-month JIBAR rate move together historically. We model this relationship using cointegration. In South Africa there are enough liquidly traded instruments referencing the 3-month JIBAR to build a 3-month tenor-homogeneous curve, and so we can use the tenor curve to estimate the OIS curve. This thesis develops a method of using this cointegration relationship in a simultaneous bootstrap to estimate the OIS curve. This builds on the research presented in, Jakarasi *et al.* (2015), by clearly outlining the simultaneous bootstrap methodology such that it can be easily replicated and implemented. It is important that the simultaneous bootstrap is clearly defined because it ensures that the instruments used to strip the relevant tenor-homogeneous curve will price back to par maintaining arbitrage relationships. This method can be used to construct an OIS curve estimate in any market with the following characteristics:

- OIS swaps are not liquidly traded, or not traded at all
- An overnight rate that is easily available
- Has a liquid interest rate swap market

In Chapter 2 we outline how the standard theory of pricing interest rates swaps and forward rate agreements has changed as a result of the financial crisis. In Chapter 3, we develop the bootstrap algorithms, both for the single and multiple curve framework, assuming a liquid OIS market. In Chapter 4 we address the multi-curve bootstrap in South Africa, and develop an algorithm to similtaneously estimate the tenor curve and the ZAR OIS curve.

Chapter 2

Notation and Theoretical Background

This chapter outlines the theory of pricing Forward Rate Agreements and Interest Rate Swaps and illustrates changes between pricing in a single-curve setting in comparison to a multiple-curve setting. These are the instruments used to strip the zero curves. The pricing formulae developed here form the basis of the bootstrap algorithms in Chapters 3 and 4.

2.1 Notation

Here we summarise some of the key notation used in the remaining sections, as well as providing some important definitions.

- $L(t,T)$: Floating reference rate of the contract (It is a LIBOR rate, and refers to any other equivalent such as EURIBOR or JIBAR).
- $r(0,t)$: is the continuously compounded zero-rate for the period $(0,t)$.
- τ_i: this is the day count fraction of the associated rate i.e. if we have $L(t_0,t_i)$ then $\tau = t_i - t_0$ or if we have $L(t_{i-1},t_i)$ then $\tau_i = t_i - t_{i-1}$, where t is a day count fraction.
- Q: is defined as the risk-neutral probability measure. The risk-neutral probability measure is a set of probabilities under which the asset price discounted at the risk-free rate is a martingale.
- $Z(t_0,t)$: this is the discount factor/ zero-coupon bond price for the period (t_0,t).
- Q_T: is the T-forward measure. Under the T-forward measure the discounted zero-coupon bond price $Z(t,T)$ is a martingale.

- $Q_{T_{OIS}}$: Under the T-forward measure the discounted zero-coupon bond price $Z(t,T)$ is a martingale, and OIS discounting is used.

In general the LIBOR spot rate is the rate of the return eared when one unit of a default-free zero-coupon bond is purchased at time t_0, and sold at maturity T. The LIBOR spot rate can therefore be written as follows:

$$L(t_0,T) = \frac{1}{\tau}\left(\frac{1}{Z(t_0,T)} - 1\right) \tag{2.1}$$

where $Z(t_0,T)$ is the default-free discount factor.

2.1.1 Defining the Forward Rate

The forward rate between t and T is:

$$f(t_0,t,T) = \frac{1}{\tau}\left(\frac{Z(t_0,t)}{Z(t_0,T)} - 1\right) \tag{2.2}$$

To show that this holds a no-arbitrage argument can be used:

Assume we are at time t_0 and consider the cash flow $NL(t,T)\tau$ at time t, the following strategy is implemented:

- At time t_0 enter into the following transactions:

Transaction	**Value = 0**
Borrow $NZ(t_0,t)$ for maturity at T at $L(t_0,T)$ (1)	$+NL(t_0,t)$
Deposit $NZ(t_0,t)$ for maturity at t at $L(t_0,t)$ (2)	$-NZ(t_0,t)$

- At time t enter into the following transactions:

Transaction	**Value = 0**
Deposit matures (2)	$+N$
Deposit N for maturity at T at $L(t,T)$ (3)	$-N$

- At time T we have the following:

- The net payoff at maturity = $[NL(t,T)\tau - Nf(t_0;t,T)]$

Transaction	Value
Settle loan (1)	$-N\frac{Z(t_0,t)}{Z(t_0,T)} = -N[1+f(t_0;t,T)\tau]$
Deposit matures (3)	$+N[1+L(t,T)\tau]$

- Given total cost of setting up this strategy is zero, for the absence of arbitrage opportunities it must hold that $[NL(t,T)\tau - Nf(t_0;t,T)] = 0$, so $L(t,T) = f(t_0;t,T)$
- Mathematically this implies that $f(t_0,t,T) = \mathbb{E}^{\mathbb{Q}_T}\left[L(t,T)|\mathcal{F}_{t_0}\right] = \frac{1}{\tau}\left(\frac{Z(t_0,t)}{Z(t_0,T)} - 1\right)$ as written in 2.2.

2.1.2 Defining the forward measures $\mathbb{Q}_T$ and $\mathbb{Q}_{T_{OIS}}$

The theory outlined in this section follows, Shreve (2004). A numéraire is a quantity whose values are used as units for expressing an asset price. A numéraire must correspond to the price of a tradable asset in the same market. The money market account is often used implicitly as a numéraire, however, using an alternative numéraire can simplify calculations significantly. Using the zero-coupon bond price as a numéraire when pricing interest rate derivatives is particularly helpful. In this section we outline the theory used to change risk-neutral measures, such that the asset price process will be a martingale process when expressed in units of the numéraire.

In this section the model is driven by the standard Brownian Motion $W_t, 0 \leq t \leq T$, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}_t, 0 \leq t \leq T$ is the filtration generated by the Brownian Motion. There is an adapted interest rate process $r(t), 0 \leq t \leq T$. We use this to create the money-market account, with its price at time t given by:

$$B_t = e^{\int_0^t r(u)\,du} \tag{2.3}$$

This gives the amount of capital that would be available at time t after investing one unit of currency in the money market account a time zero, when the grows continuously at the short term interest rate $r(t)$. The discount process is then given by:

$$B_t{}^{-1} = \frac{1}{B_t} = e^{-\int_0^t r(u)\,du} \tag{2.4}$$

We define X as the primary asset, and its price satisfies the following equation:

$$dX(t) = \alpha(t)X(t)\,dt + \sigma(t)X(t)\,dW(t) \tag{2.5}$$

We construct the risk-neutral measure using the Girsanov Theorem, so that under $\mathbb{Q}$, the Brownian motion is given by:

$$\tilde{W}(t) = W(t) + \int_0^t \Theta(u)du \tag{2.6}$$

where Θ is defined as the market price of risk and is given by:

$$\Theta(t) = \frac{\alpha(t) - r(t)}{\sigma(t)} \tag{2.7}$$

Under the risk-neutral measure $\mathbb{Q}$ the discounted asset price $B(t)^{-1}X(t)$ is a martingale, and the discounted value of every portfolio process is also a martingale, and therefore $\mathbb{Q}$ is said to be risk-neutral with the money market account as the numeraire. If we change the numeraire, so that the asset price is no longer denominated in units of the money market account, then it will no longer be a martingale under $\mathbb{Q}$. This implies that when the numeraire is changed, in order for the asset price to be a martingale when denominated in units of the new numeraire we need to change measures. Below we define the measures $\mathbb{Q}^T$ and $\mathbb{Q}^{T_{OIS}}$ associated with using the zero-coupon bond price as the numeraire.

Zero-Coupon Bonds

Consider a zero-coupon bond that pays 1 unit of domestic currency at maturity T. Using the risk-neutral pricing formula, the value of the zero-coupon bond at time $t\epsilon[0,T]$ is:

$$Z(t,T) = E_{\mathbb{Q}}\left[\frac{B(t)}{B(T)}|\mathcal{F}_t\right] \tag{2.8}$$

$$= E_{\mathbb{Q}}\left[e^{-\int_t^T r(u)du}|\mathcal{F}_t\right] \tag{2.9}$$

In particular $Z(T,T) = 1$. A zero-coupon bond is an asset and therefore the discounted bond price $B(t)^{-1}Z(t,T)$ is a martingale under $\mathbb{Q}$, with $E_{\mathbb{Q}}\left[B(T)^{-1}Z(T,T)|\mathcal{F}_{tt}\right] = Z(0,T)$.

The Stochastic Representation of Assets Theorem states that there is a volatility process $\sigma^*(t,T)$ for the bond such that:

$$d((B(t))^{-1}Z(t,T)) = -\sigma^*(t,T)B(t)^{-1}Z(t,T)\,d\tilde{W}(t) \tag{2.10}$$

Zero Coupon Bond as a Numeraire

We now change the numeraire to the zero-coupon bond using the Change of Risk-Neutral Measure Theorem. For a fixed maturity date T we define the T-forward measure $\mathbb{Q}_T$ by:

$$\mathbb{Q}^{(T)}(A) = \int_A \frac{B(T)^{-1}Z(T,T)}{Z(0,T)}(\omega)d\mathbb{Q}(\omega) \tag{2.11}$$

$$= \int_A \frac{B(0,T)^{-1}}{Z(0,T)}(\omega)d\mathbb{Q}(\omega) \tag{2.12}$$

The Radon-Nikodym derivative is $\frac{d\mathbb{Q}_T}{d\mathbb{Q}} = \frac{B(T)^{-1}}{Z(0,T)}$ and the Radon Nikodym derivative process is given by $P^{\mathbb{T}}(t) = \frac{Bt)^{-1}Z(t,T)}{Z(0,T)}$.

Under the measure $\mathbb{Q}_T$ there is a $\mathbb{Q}_T$ - Brownian Motion

$$\tilde{W}(t)^{\mathbb{T}} = \tilde{W}(t) + \int_0^t \sigma^*(u,T)du \tag{2.13}$$

Under the T-forward measure, all assets denominated in units of the zero-coupon bond maturing at time T are martingales, i.e. $\frac{X(t)}{Z(t,T)}$ is a martingale and therefore all T-forward prices are martingales under $\mathbb{Q}_T$. The T-forward measure is introduced to simplify the risk-neutral pricing formula. According to the risk-neutral pricing formula the value at time t of a contract that pays $V(T)$ at maturity, T, is:

$$V(t) = B(t)E^{\mathbb{Q}}\left[B(T)^{-1}V(T)|\mathcal{F}_t\right] \tag{2.14}$$

Evaluating this requires that the dependence between the discount factor $B(T)^{-1}$ and the payoff $V(T)$ of the derivative is known. If the derivative security is dependent on an interest rate, such as for FRA's and interest rate swaps, then this dependence is difficult to model. Under the T-Forward measure we have:

$$E^{\mathbb{Q}_T}\left[V(T)|\mathcal{F}_t\right] = \frac{1}{B(t)^{-1}Z(t,T)}E^{\mathbb{Q}}\left[B(T)^{-1}V(T)|\mathcal{F}_t\right] \tag{2.15}$$

$$= \frac{1}{Z(t,T)}V(t) \tag{2.16}$$

This gives the following simplified formula:

$$V(t) = Z(t,T)E^{\mathbb{Q}_T}\left[V(T)|\mathcal{F}_t\right] \tag{2.17}$$

The difference between $\mathbb{Q}_T$ and $\mathbb{Q}_{T_{OIS}}$

When defining the measure $\mathbb{Q}_T$ the rate used to calculate the zero-coupon bond price $Z(t,T)$ is calculated off a LIBOR discount curve. We can define $\mathbb{Q}_{T_{OIS}}$ in using exactly the same procedure as above, however, the zero-coupon bond price used as the numeraire is given as follows:

$$Z_{OIS}(t,T) = E_{\mathbb{Q}}\left[e^{-\int_t^T r_{OIS}(u)du}|\mathcal{F}_t\right] \tag{2.18}$$

where $r_{OIS}(t)$ is calculated from the OIS curve. A similar process happens to simplify the risk-neutral pricing formula, except it will only hold for assets denominated in units of $Z_{OIS}(t,T)$ and under the risk-neutral measure $\mathbb{Q}_{T_{OIS}}$.

2.2 Forward Rate Agreements

A FRA is an over-the-counter (OTC) agreement to exchange a fixed interest rate for a floating interest rate for a fixed period starting at a future date. Figure 2.1 below shows the floating and fixed cash flows of a FRA.

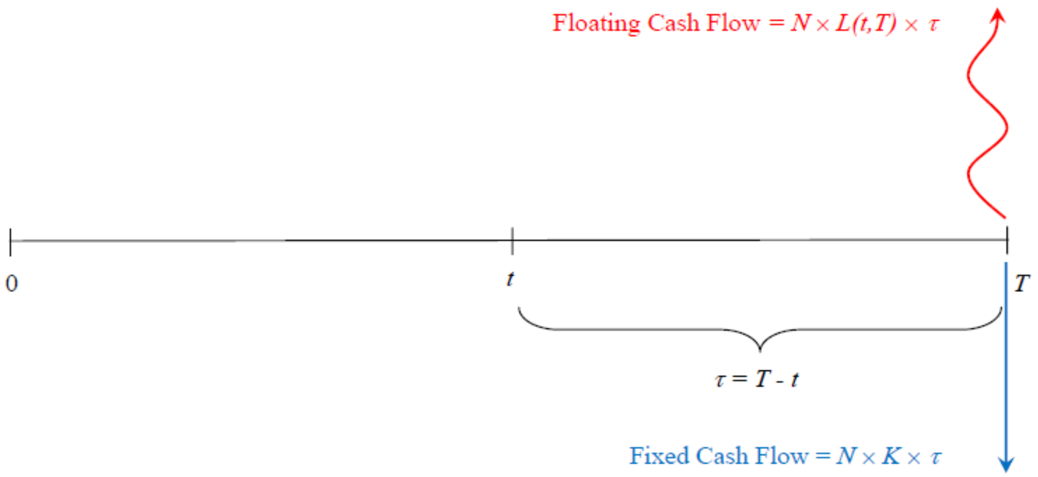


Fig. 2.1: Cash Flow Diagram of a Forward Rate Agreement

where N is the notional on the contract, $L(t,T)$ is the floating reference rate. K is the strike/fixed rate, t is the reset time, T is the maturity of the contract and $\tau = T - t$[1].
The payoff at maturity of a FRA referencing the x-month LIBOR rate $L(t,T)$ at strike rate K is given by:

$$V_{FRA}(T) = \alpha N \left[L(t,T) - K \right] \tau \qquad (2.19)$$

where $\alpha = 1$ for a long position and $\alpha = -1$ for a short position.

[1] Note: when we refer to a time t it is a year count fraction.

2.2.1 FRA value using the Single-Curve Approach

The value of a FRA at time s is given by:

$$V_{FRA}(s) = \alpha N \left[\mathbb{E}^{\mathbb{Q}_T}[L(t,T)|\mathcal{F}_t] - K\right] \tau Z(s,T) \tag{2.20}$$

$$= \alpha N \left[\frac{1}{\tau}\left(\frac{Z(s,t)}{Z(s,T)} - 1\right) - K\right] \tau Z(s,T) \tag{2.21}$$

where $Z(s,\cdot)$ is the discount factor calculated from the single default-free curve at time s.

The fair FRA rate is defined as the strike rate K that makes the value of the contract zero, therefore from (2.20) we have that the fair strike rate of a FRA is the forward rate:

$$K = \mathbb{E}^{\mathbb{Q}_T}\left[L(t,T)|\mathcal{F}_t\right] \tag{2.22}$$

2.2.2 FRA value using the Multiple-Curve Approach

In the multi-curve framework, the tenor of the underlying reference rate needs to be specified. A x-month tenor curve is used to estimate the forward rate, and the OIS curve is used for discounting. The value of a FRA at time t_0 is therefore given by:

$$V_{FRA}(t_0) = \alpha N \left[\mathbb{E}^{\mathbb{Q}_T^{OIS}}[L_x(t,T)|\mathcal{F}_t] - K\right] \tau Z_{OIS}(t_0,T)$$

$$= \alpha N \left[\frac{1}{\tau}\left(\frac{Z_x(t_0,t)}{Z_x(t_0,T)} - 1\right) - K\right] \tau Z_{OIS}(t_0,T)$$

where $Z_{OIS}(t_0,\cdot)$ is the discount factor calculated from the OIS curve, and $Z_x(t_0,\cdot)$ is the discount factor calculated from the x-month LIBOR curve at time t_0.

2.3 Interest Rate Swaps

Swaps are OTC agreements to exchange a floating interest rate for a fixed interest, which is known as the rate swap rate, over a specific period. It is the multi-period extension of a FRA. The cash flows of a vanilla interest rate swap are shown below in Figure 2.2, where the red lines are the floating cash flows and the blue lines are the fixed cash flows.

The payoff at maturity of a vanilla interest rate swap with the floating rate referenced the x-month LIBOR rate with strike rate K is given by:

$$V_{IRS}(t_n) = \sum_{i=1}^{n} \alpha N L(t_{i-1},t_i)\tau_i - \sum_{i=1}^{n} \alpha N K \tau_i \tag{2.23}$$

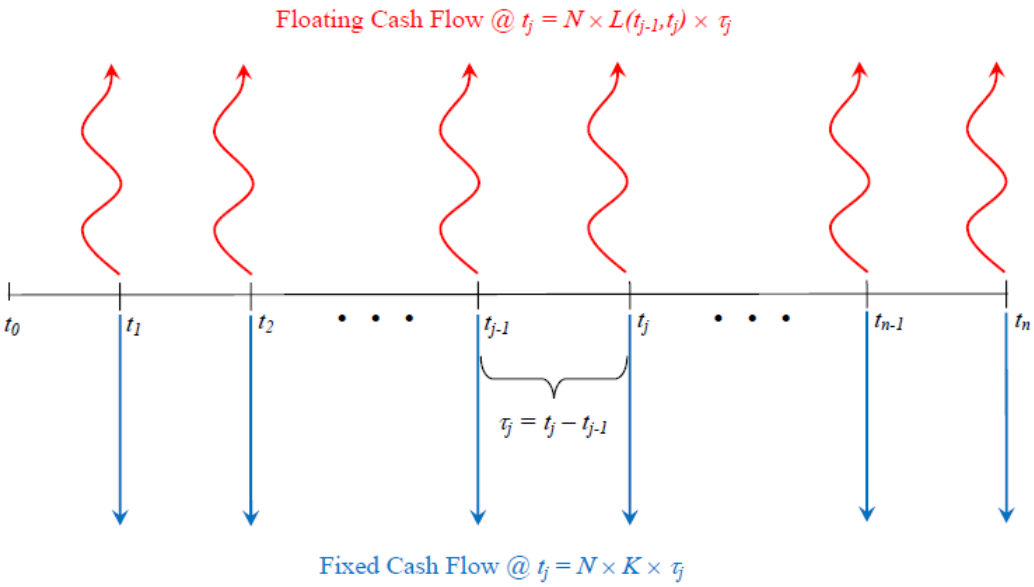


Fig. 2.2: Cash Flow Diagram of an Interest Rate Swap

where $\{t_0, ..., t_{i-1}\}$ denote the reset times of the floating rate, and $\{t_1, ..., t_n\}$ denote the payment times, $\alpha = 1$ for a long position and $\alpha = -1$ for a short position and N is the notional of the swap.

2.3.1 Value under the Single-Curve Approach

Assuming we have a swap zero curve at t_0, a term structure of zero-rates going out at least as far as t_n, then the value of an IRS at t_0 is given by:

$$V_{IRS}(t_0) = \sum_{i=1}^{n} \alpha N \, \mathbb{E}^{\mathbb{Q}^{t_i}} \left[L(t_{i-1}, t_i) | \mathcal{F}_{t_{i-1}} \right] \tau_i Z(t_0, t_i) - \sum_{i=1}^{n} \alpha N K \tau_i Z(t_0, t_i) \tag{2.24}$$

$$= \sum_{i=1}^{n} \alpha N \frac{1}{\tau_i} \left(\frac{Z(t_0, t_{i-1})}{Z(t_0, t_i}) - 1 \right) \tau_i Z(t_0, t_i) - \sum_{i=1}^{n} \alpha N K \tau_i Z(t_0, t_i) \tag{2.25}$$

$$= \alpha N \sum_{i=1}^{n} [Z(t_0, t_{i-1}) - Z(t_0, t_n)] - \alpha N K \sum_{i=1}^{n} \tau_i Z(t_0, t_) \tag{2.26}$$

$$= \alpha N \left[1 - Z(t_0, t_n) \right] - \sum_{i=1}^{n} \alpha N \tau_i Z(t_0, t_i) \tag{2.27}$$

The fair swap rate is the fixed rate that ensures $V_{IRS}(t_0) = 0$, therefore:

$$K = \frac{1 - Z(t_0, t_n)}{\sum_{i=1}^{n} \tau_i Z(t_0, t_i)} \tag{2.28}$$

2.3.2 Value under the Multi-Curve Approach

Assuming we have a swap zero curve at t_0, two term structures of zero-coupon rates derived from OIS and IRS referencing the x-month LIBOR rate going out at least as far as t_n, then the value of an IRS at t_0 is given by:

$$V_{IRS}(t_0) = \sum_{i=1}^{n} \alpha N \, \mathbb{E}^{\mathbb{Q}_{t_i}^{OIS}} \left[L_x(t_{i-1}, t_i) | \mathcal{F}_{t_{i-1}} \right] \tau_i Z(t_0, t_i) - \sum_{i=1}^{n} \alpha N K \tau_i Z(t_0, t_i) \tag{2.29}$$

$$= \sum_{i=1}^{n} \alpha N \frac{1}{\tau_i} \left(\frac{Z_x(t_0, t_{i-1})}{Z_x(t_0, t_i)} - 1 \right) \tau_i Z^{OIS}(t_0, t_i) - \sum_{i=1}^{n} \alpha N K \tau_i Z^{OIS}(t_0, t_i) \tag{2.30}$$

The fair swap rate is the fixed rate rate that ensures $V_{IRS}(t_0) = 0$, therefore:

$$K = \frac{\sum_{i=1}^{n} \frac{1}{\tau_i} Z_{OIS}(t_0, t_i) \left(\frac{Z_x(t_0, t_{i-1})}{Z_x(t_0, t_i)} - 1 \right)}{\sum_{i=1}^{n} \tau_i Z^{OIS}(t_0, t_i)} \tag{2.31}$$

2.4 Overnight-Indexed Swaps

The payoff of an OIS, referencing an overnight rate floating rate, at maturity is given by:

$$V_{OIS}(t_n) = \sum_{i=1}^{n} \alpha N \overline{L_i} \tau_i \tag{2.32}$$

where $\overline{L_i}$ denotes the compounded floating overnight rate for the ith period. $\overline{L_i}$ is the effective simple rate equivalent to rolling an overnight investment daily over the ith period (t_{i-1}, t_i), written as:

$$1 + \overline{L_i} \tau_i = \prod_{i=1}^{n} (1 + L(t_{i-1}, t_i) \tau_i) \tag{2.33}$$

The valuation of an OIS is treated the same as a vanilla interest rate swap.

2.4.1 Value under the Single-Curve Approach

The value of an OIS at t_0, at strike rate K is given by:

$$V_{OIS}(t_0) = \sum_{i=1}^{n} \alpha N \, \mathbb{E}^{\mathbb{Q}_{t_i}} \left[\overline{L_i} | \mathcal{F}_{t_{i-1}} \right] \tau_i Z(t_0, t_i) - \sum_{i=1}^{n} \alpha N K \tau_i Z(t_0, t_i) \tag{2.34}$$

$$= \alpha N \left[1 - Z(t_0, t_n) \right] - \sum_{i=1}^{n} \alpha N K \tau_i Z(t_0, t_i) \tag{2.35}$$

2.4.2 Value under the Multi-Curve Approach

The value of an OIS at t_0, at strike rate K is given by:

$$V_{OIS}(t_0) = \sum_{i=1}^{n} \alpha N \, \mathbb{E}^{\mathbb{Q}_{t_i}^{OIS}} \left[\overline{L_i} | \mathcal{F}_{t_{i-1}} \right] \tau_i Z_{OIS}(t_0, t_i) - \sum_{i=1}^{n} \alpha N K \tau_i Z_{OIS}(t_0, t_i) \quad (2.36)$$

$$= \alpha N \left[1 - Z_{OIS}(t_0, t_n) \right] - \sum_{i=1}^{n} \alpha N K \tau_i Z_{OIS}(t_0, t_i) \quad (2.37)$$

In the pricing formulae derived above we see the difference between pricing in a single-curve framework and a multiple curve framework. The theory in this section forms the basis of the bootstrap algorithms developed in Chapters 3 and 4.

Chapter 3

Bootstrapping Algorithm and Implementation in a Liquid Market

In this chapter we focus on the algorithms used to bootstrap the swap curve in the single and multiple curve setting. Before examining the algorithms, we consider the selection of instruments that should be used to build the curves and the interpolation methods, which play a crucial role in the bootstrapping process.

3.1 Instrument Selection

There are numerous combinations of bonds, futures and swaps that can be used to construct a yield curve. The choice of instruments would not matter if markets were arbitrage free and data were complete. This is not the case and the selection of instruments used to construct the yield curve can have a significant impact on the shape of the curve. The short end of the curve is typically constructed using deposits, forward rate agreements (FRA's) and interest rate futures, and the long-end using interest rate swaps (IRS's), which are sometimes liquidly traded out to 30 years, Alexander (2008).

Interest rate FRA's and futures typically cover the period up to 2 years. Both FRA's and futures enable the investor to secure a rate of return between two dates and thus provide information about forward EURIBOR rates. The main difference between the two is that the settlement of a FRA occurs at the maturity of the contract, while futures are marked to market daily, which results in a stream of cash flows between parties throughout the life of the contract. Futures tend to provide biased information about forward rates, and need an interest rate derivatives pricing model to value. FRA's however, are model free and provide unbiased information about forward rates, but they are typically not as liquid as futures and deposits. When quotes of an instrument are stale as a result of it not being very liquidly traded, it will not reflect changes in the yield curve. This is problematic for yield curve construction. Therefore neither FRA's nor futures are ideal to use in the bootstrapping process. In this dissertation we use FRA's over futures, as the main focus is the implementation in the South African market. The Johannesburg Stock Exchange recommend the use of FRA's in the construction of the swap curve, which we followed here JSE (2012).

3.2 Review of interpolation methods

When considering the bootstrap method it is very important to note that the bootstrap itself is intimately connected to the interpolation algorithm. In the bootstrap process, there is initially an incomplete set of information. This information is completed in a non-unique way using the chosen interpolation method. If a nonsensical interpolation scheme is chosen, results will not be correct or consistent. So before moving to a multiple curve framework it is important to decide on the interpolation method to get consistent single-curve results. There are a few key considerations when looking at and interpolation methods, highlighted in, Hagan and West (2008):

1. Structure of the forward rates: they should be positive and continuous. The positivity requirement is needed for there to be no arbitrage. Continuity of forward rates is important for pricing interest rate sensitive derivatives, which are sensitive to the stability of forward rates.

2. Locality of interpolation methods: If an input is changed, it is desirable for the function to only change nearby, with little spill-over. In this case the method is local. When an input is changed and there are material changes elsewhere, it is not a desirable interpolation method.

3. Stability of forward rates: As well as forward rates being positive and continuous the stability of forward rates is also important. The stability of the forward curve can be assessed by looking at the maximum basis point change in the forward curve given a basis point change in one of the inputs.

4. locality of hedges: If an interest rate derivative of a particular tenor is selected, a particular set of possible hedging instruments will be assigned. Testing the locality of the hedge involves assessing whether the delta risk is assigned to hedging instruments that have maturities close to the selected tenor, or if a significant amount filters into other areas of the curve.

Based on these criteria, Hagan and West (2008) compare a comprehensive list of interpolation methods, in a single curve framework. The best interpolation methods appear to be raw and monotone convex interpolation. Both of these methods have positive forward rates, which is critical. In this dissertation raw interpolation is used throughout, as it is shown to be one of the better interpolation methods as well as being easy to implement.

We now define the bootstrap algorithms used to strip the zero-curve in a single and multiple curve setting.

3.3 Bootstrapping Algorithms

The bootstrapping methodology will converge to a discrete set of yields. We interpolate these yields, using the same interpolation method as used in the bootstrap to derive the yield curve $\{r(t_0, t_i); t_i \epsilon [t_0, t_n]\}$, where $t = 0$ denotes the current time and t_n denotes the maturity of the input instrument with the longest tenor. The function $r(t_0, t_n)$ defines the set of continuously compounded market interest rates, which price all the respective input instruments back to par.

3.3.1 Single Curve Bootstrapping Algorithm

If we consider the i^{th} generic fair FRA, issued at time t, for the period $[t_{1,i}, t_{2,i}]$ with fair continuously compounded strike rate K, then it is known that

$$r(t, t_{t_{2,i}}) = \frac{K(t_{2,i} - t_{1,i}) + r(t_0, t_{1,i})t_{1,i}}{t_{2,i}} \tag{3.1}$$

Following a similar process for a generic fair IRS, $\{i\}$, contracted at time t_0 with payment times $\{t_{1,i}, ..., t_{n,i}\}$, we can derive the following[1]

$$r(t_0, t_{n,i}) = -\frac{1}{\tau_{n,i}} \ln \left[\frac{1 - K \sum_{j=1}^{n-1} (t_{j,i} - t_{j-1,i}) Z(t_0, t_{j,i})}{1 + K\tau_{n,i}} \right] \tag{3.2}$$

We use these instruments as inputs to the bootstrapping process, and the above equations form the basis of the iterative bootstrapping process. The bootstrapping algorithm proceeds as follows:

1. Select a set of input instruments, consisting of a set of money market instruments maturing at times $\{T_1, ..., t_m\}$, a set of FRA's maturing at times $\{t_{m+1}, ..., t_k\}$ and a set of IRS's maturing at times $\{t_{k+1}, ..., t_n\}$.

2. Deduce the values of $\{r(t_0, t_{k+1}), ..., r(t_0, t_n)\}$.

[1] the derivation can be found in Appendix A1

3. Select an appropriate interpolation method, and interpolate between $\{t_1, ..., t_n\}$ and $\{r(t_0, t_1), ..., r(t_0, t_n)\}$ to estimate the rates corresponding to each cash flow of the input instrument.

4. Put the rates obtained in the previous step into the above equations for each IRS to get new estimates of $\{r(t_0, t_{k+1}), ..., t(t_0, t_n)\}$.

5. Repeat steps 3 and 4 until convergence.

3.3.2 Dual Curve Bootstrapping Algorithm

In the multiple curve framework outlined in section 1.3, to price a given interest rate derivative we need a discount curve and a forecasting curve. The discount curve is the OIS curve. The OIS curve can be bootstrapped using the single curve bootstrap algorithm, as the discount and associated forecast rates are equivalent as in a single-curve setting. The OIS curve is used as an input to the dual curve bootstrap algorithm. Firstly we note that equation (3.2) changes in the case when we assume differential discounting (OIS discounting).

$$r(t_0, t_{j,i}) = \frac{1}{\tau_{n,i}} \ln \left[\frac{1}{Z_x(t_0, t_{n-1,i})} \left(1 + \tau_{n,i} \left(\frac{K \sum_{j=1}^{n} Z_{OIS}(t_0, t_{j,i})\tau_j - \sum_{j=1}^{n-1} Z_{OIS}(t_0, t_{j,i})\tau_j r_x(t_0, t_{j-1,i}, t_{j,i})}{\tau_{n,i} Z_{OIS}(t_0, t_{n,i})} \right) \right) \right]$$

The dual curve bootstrapping algorithm may be applied as follows:

1. Build the discounting curve using the procedure outlined in the single-curve framework above, with OIS.

2. Select a set of input instruments all homogeneous in their underlying rate tenor, consisting of a set of money market instruments maturing at times $\{t_{1,i}, ..., t_{m,i}\}$, a set of FRA's maturing at times $\{t_{m+1,i}, ..., t_{k,i}\}$ and a set of IRS's maturing at times $\{t_{k+1,i}, ..., t_{n,i}\}$.

3. Deduce the values of $\{r_{x,i}(t_0, t_{k+1,i}), ..., r_{x,i}(t_0, t_{n,i})\}$.

4. Select an appropriate interpolation method, and interpolate between $\{t_{1,i}, ..., t_{n,i}\}$ and $\{r_{x,i}(t_0, t_{1,i}), ..., r_{x,i}(t_0, t_{n,i})\}$ to estimate the rates corresponding to each cash flow of the input instrument.

5. Put the rates obtained in the previous step into the above equations for each IRS to get new estimates of $\{r_{x,i}(t_0, t_{k+1,i}), ..., r_{x,i}(t_0, t_{n,i})\}$.

6. Repeat steps 3 and 4 until convergence.

3.4 Implementation in the European Market

The European market is used as an example of the dual curve bootstrap and as a basis of comparison to test the methods that will be applied in the South African market. Figure (3.1) shows the EONIA curve, and the EURIBOR 1M, 3M, 6M and 12M tenor curves bootstrapped using the above dual curve bootstrap algorithm. The curves are bootstrapped using raw interpolation.

From these curves we can see that the longer the underlying rate tenor, the higher the curve. This illustrates the segmentation of the interest rate market and the difference in credit and liquidity risk between instruments based on different underlying rate tenors. The OIS curve is the lowest, which is expected because it is the closest proxy to a 'risk-free' curve. If we implemented the same process using a value date before the onset of the financial crisis we would see that the curves are much closer together. In the short end of the curve we also observe negative yields, which not conventional. In light of extreme market conditions and negative rates it is important to have a bootstrapping methodology which is robust enough to handle negative rates. From Figure 3.1, we can see that the bootstrapping algorithm implemented reflects the negative rates, as required.

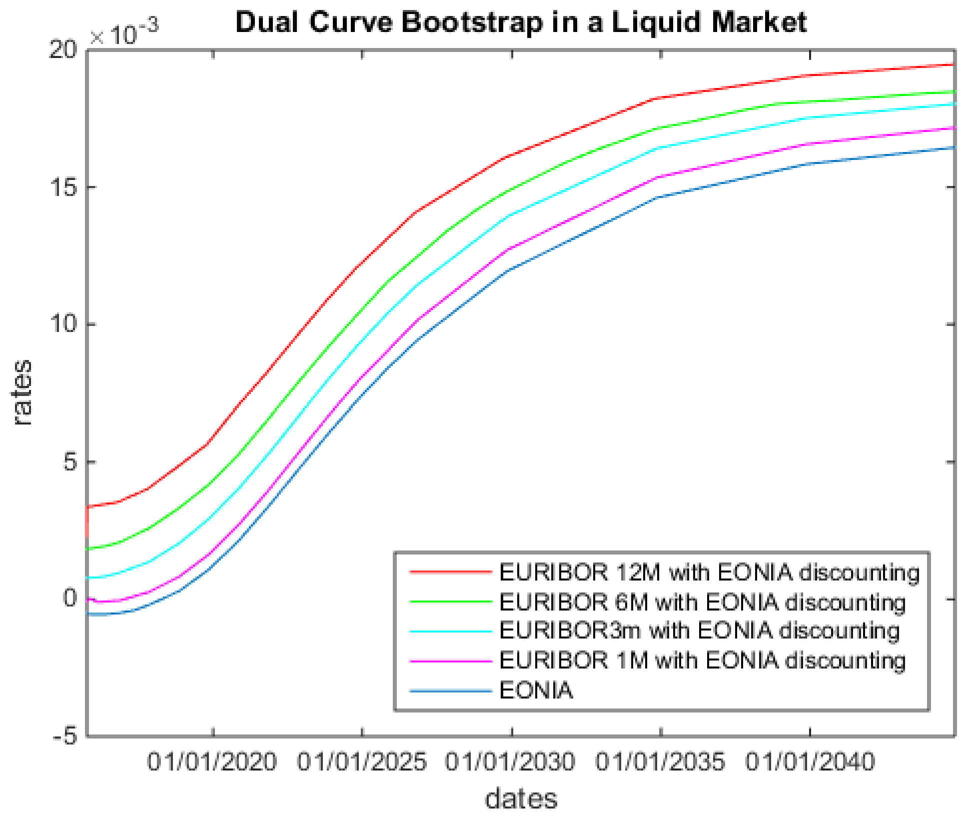


Fig. 3.1: Tenor curves for the Euro Zone market. The data used for each of these curves can be found in Appendix 2.

Chapter 4

Multi-curve Bootstrap in the Absence of an OIS Market

In Chapters 2 and 3 it has been assumed that there is a liquid OIS market. The OIS curve can be bootstrapped using a single curve algorithm. The dual curve bootstrap requires the discount curve as an input. Since OIS discounting has become standard market practice in developed markets, it is important to find a method of estimating the ZAR OIS curve.

Figure 4.1 shows the historical relationship between the SAFEX Overnight Rate, the 1-month JIBAR and 3-month JIBAR rates. These rates move together, and are very closely related. As expected the 1 month rate is closer to the overnight rate than the 3 month rate. We observe the same relationships between the relevant Euro zone rates (EONIA, 1-month EURIBOR and 3-month EURIBOR rates) shown in figure 4.2.

We aim to estimate the relationship between these two rates the overnight rate and either 1 month or 3 month EURIBOR/JIBAR rate. Along with the bootstrap algorithms this provides the framework enabling us to build an estimate of the ZAR OIS curve. The theory of cointegration is used to estimate the relationship. In most circumstances performing an Ordinary Least Squares (OLS) regression with time-series variables that are not stationary will not provide meaningful results, however, if the variables are cointegrated the OLS regression defines the long-run equilibrium relationship between the variables. In the next section we outline the basics of cointegration theory, which will be used to estimate the relationship between the overnight rate and 3-month JIBAR rate, and between the EONIA rate and 1-month and 3-month EURIBOR rates.

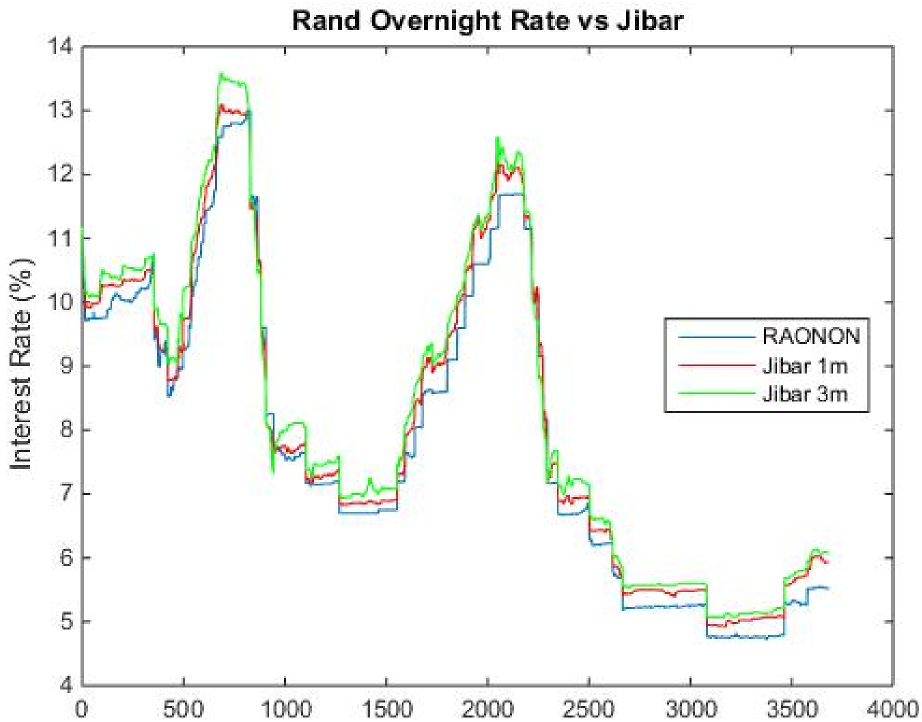


Fig. 4.1: Historical values of the Rand Overnight Rate, 1-month JIBAR and 3-month JIBAR rates from 01/05/2000 - 12/11/2014.

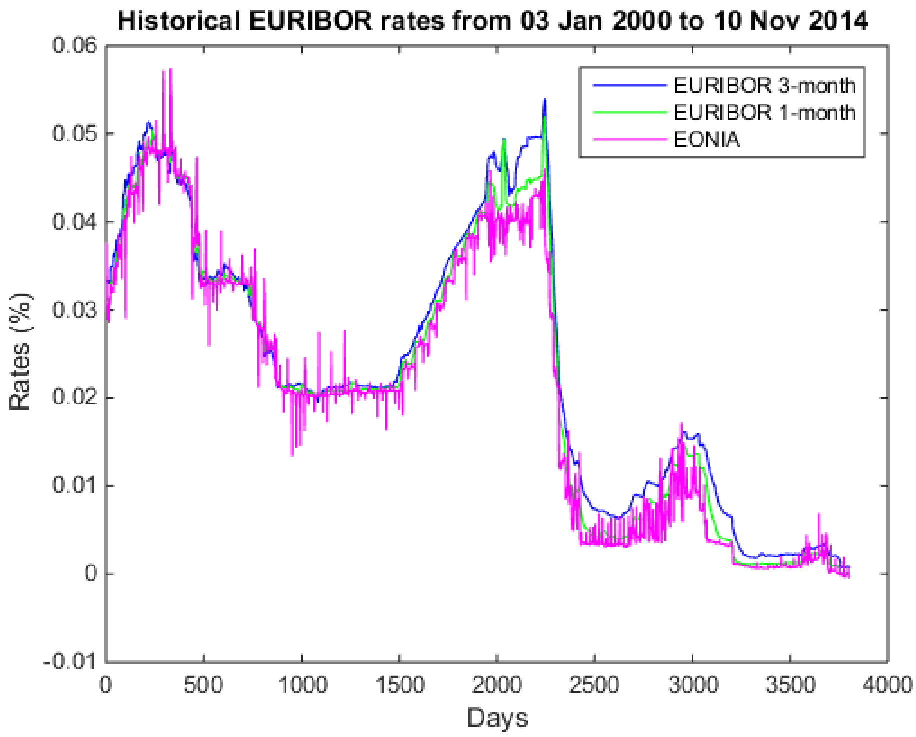


Fig. 4.2: Historical values of the EONIA Rate, 1-month EURIBOR and 3-month EURIBOR rates from 03/01/2000 - 10/11/2014.

4.1 Basic Time Series Models and Cointegration

In this section we outline the basic theory of cointegration, following Alexander (2008). Asset prices are said to be cointegrated when there is a long-run equilibrium relationship between them. The estimators obtained from an Ordinary Least Squares (OLS) regression are not consistent unless the residuals are stationary, Alexander (2008). As a result of this the standard requirement is that both the independent and dependent variables are stationary. The only exception to this is when two variables are cointegrated. Before discussing the cointegration methodology we introduce the concepts of stationary and integrated time series.

4.1.1 Stationary Processes

A stochastic process $\{X_t\}_{t=1}^T$ is stationary if the following conditions hold:

1. The expected value of X_t, $E(X_t)$ is a finite constant
2. The variance of X_t, $V(X_t)$ is a finite constant
3. The joint distribution of (X_s, X_t) is only dependent on $t - s$

If $\{X_t\}_{t=1}^T$ is stationary then we write $X_t \sim I(0)$, that is, X_t is integrated of order zero.

4.1.2 Integrated Processes

A time series is integrated of order 1 $I(1)$ if it is not stationary but its first difference is stationary, and integrated of order n if the n^{th} difference is stationary. We now summarize unit root tests, which are used to test for stationarity.

4.1.3 Unit Root Tests

A unit root test is a statistical hypothesis test for stationrity. The null hypothesis of non-stationarity is tested against the alternative hypothesis of stationarity. An autoregressive process is stationary if and only if the roots of its characteristic polynomial lie strictly inside the unit circle, Alexander (2008). So we have

$$H_0 : X_t \sim I(1) \tag{4.1}$$

$$H_1 : X_t \sim I(0) \tag{4.2}$$

If the test statistic lies inside the critical region at the required confidence level, then in can be concluded that the series is stationary. If we cannot reject the null hypothesis, it does not directly imply that it is true. When the test statistic lies outside the critical region, we perform another unit root test on the first difference of the series, with

$$H_0 : \triangle X_t \sim I(1) \tag{4.3}$$
$$H_1 : \triangle X_t \sim I(0) \tag{4.4}$$

where $\triangle$ denotes the difference operator. If the null hypothesis, H_0, of this test is rejected, then we can conclude that the time series is $I(1)$, and not a higher order of integration.

4.1.4 Dickey-Fuller Unit Root Test

The Dickey-Fuller unit root test, introduced in Dickey and Fuller (1979) is based on the Dickey-Fuller regression

$$\triangle X_t = \alpha + \beta X_{t-1} + \varepsilon_t \tag{4.5}$$

The t-ratio on the estimate $\widehat{\beta}$, is the test statistic. It is a one-side hypothesis test with

$$H_0 : \beta = 0 \tag{4.6}$$
$$H_1 : \beta < 0 \tag{4.7}$$

To observe how this applies to the null and alternative hypotheses, we assume that the series is generated by an $AR(1)$ process of the form

$$X_t = \alpha + \varrho X_{t-1} + \varepsilon_t \tag{4.8}$$

with $\varepsilon_t \sim I(0)$ then we require that $\beta = \varrho - 1$, then 4.3 and 4.4 are equivalent to

$$H_0 : \varrho = 1 \tag{4.9}$$
$$H_1 : \varrho < 1 \tag{4.10}$$

It is important to note that when variables in the regression are non-stationary, the regression R^2 will always be close to 1, and the t-ratios are high and severely biased, Alexander (2008). However, it was found in Dickey and Fuller (1981) that the correct critical values for the standard t-ratios in a regression with non-stationary variables, are increased by an amount proportional to the sample size. The major downfall of the original Dickey-Fuller regression is that even with the adjusted critical values, when there is autocorrelation in the residuals of the regression, the critical values are still biased. Autocorrelation in the residuals is often caused by missing predictor variables in the regression. To address this Dickey and Fuller (1981) recommended augmenting the original Dickey-Fuller regression to include as many lagged dependent variables as needed to remove the structure from the residuals. This is known as the Augmented Dickey-Fuller (ADF) test. The ADF test of order q is based on the following regression as presented in, Alexander (2008)

$$\triangle X_t = \alpha + \beta X_{t-1} + \gamma_1 \triangle X_{t-1} + ... + \gamma_q \triangle X_{t-q} + \varepsilon_t \tag{4.11}$$

The test is still implemented as the original Dickey-Fuller test discussed above, but the critical values for the t-ratio depend on the number of lags q included in the regression.

4.1.5 Engle and Granger Cointegration

A simple method of testing for cointegration was formulated in Engle and Granger (1987). Let x and y be two time series variables that are I(1). The basic method of analysing Engle and Granger cointegration as summarised in, West (2008) is outlined below:

1. Check that x_t and y_t are not I(0)
2. Check that x_t and y_t are I(1)
3. If x_t and y_t are I(1), estimate β by performing an OLS regression

$$y_t = \alpha + \beta x_t + \varepsilon_t \tag{4.12}$$

4. Verify that the residuals of this regression are I(0)

Testing for stationarity of time series variables is done using the Augmented Dickey-Fuller (ADF) test discussed in Section 4.1.4.

4.2 Cointegration Implementation

We estimate 3 cointegration relationships. We firstly look at relationship between EONIA and 1M EURIBOR, then EONIA and 3M EURIBOR. We then look at South Africa and the relationship between the SAFEX overnight rate and 3-month JIBAR rate. The Table 4.1 shows the results for step 1 and 2 above. The critical values for the ADF test at the 5% level is -1.9416, and -2.5684 at the 1% level, for the relevant sample sizes.

Table 4.1 confirms steps 1 and 2 of the Engle and Granger Cointegration. In the first line of the Table we note that the null hypothesis of non-stationarity cannot be rejected. The second line of the table shows the ADF test statistics for the

Tab. 4.1: Test Statistics for the ADF test

	EONIA	**SAFEX**	**1M EUR**	**3M EUR**	**3M JIBAR**
Data is I(0)	-0.8824	0.9418	1.1384	1.8395	1.6312
Data is I(1)	-73.9161	-60.3410	-46.4728	-39.4871	-52.5578

Note: the critical value at the 5% level is -1.9416, and -2.5684 at the 1% level. EONIA refers to the EONIA overnight rate, SAFEX refers to the SAFEX overnight rate, 1M EUR and 3M EUR refer to the 1-month and 3-month EURIBOR rates.

first difference of each series. For each series the null hypothesis is rejected, showing that the series are all I(1) as required. We now estimate the parameters of the cointegration equations using an OLS regression. The following regression equations are estimated:

$$EONIA_t = \alpha_1 + \beta_1 EURIBOR1m_t + \varepsilon_{1,t} \tag{4.13}$$

$$EONIA_t = \alpha_2 + \beta_2 EURIBOR3m_t + \varepsilon_{2,t} \tag{4.14}$$

$$RAONON_t = \alpha_3 + \beta_3 JIBAR3m_t + \varepsilon_{3,t} \tag{4.15}$$

The results from these regressions are in Table 4,2 , with the parameter values and R^2 of each equation. These equations provide the long-run equilibrium relationships between the variables, which will be used to derive the discount curve and simultaneously bootstrap the Tenor curve. The coefficients obtained here are of key importance in the next section.

Tab. 4.2: Parameters of Cointegration Equations

	EQ 4.13	**EQ 4.14**	**EQ 4.15**
α	-0.0009	-0.0022	-0.0014
β	0.9865	0.9844	0.9674
R^2	0.98	0.96	0.99

Lastly we ensure that the residuals from these regressions are I(0): Table 4.3 shows the ADF test statistics, checking the stationarity of the residuals of the regression equations. In each case the test statistic is significantly larger than the critical values, which are the same as those used above. The null hypothesis of non-stationarity is therefore rejected, so we may conclude that the residuals are sta-

Tab. 4.3: Test statistics of ADF test on Residuals

	EQ 4.13	**EQ 4.14**	**EQ 4.15**
Residuals are I(0)	-17.1689	-11.3596	-6.5985

Note: the critical value at the 5% level is -1.9416, and -2.5684 at the 1% level.

tionary. This is important to check before using the regression equations. Since the residuals of the required regression equations are stationary, we may use them to give a fairly reliable estimate of the long-run relationship between the variables.

4.3 Implying the discount curve

In a liquid market, we have an observable market rate $x(t)$ which is referenced by liquidly traded market instruments. No arbitrage relationships are used to bootstrap the term structure of interest rates $r_x(t)$. In South Africa the 3-month JIBAR rates are observable in the market and interest rate swaps referencing 3-month JIBAR are liquidly traded, so a 3-month ZAR swap curve can be constructed. In the previous section it was illustrated that there is a cointegration relationship between the 3-month JIBAR rate $x(t)$, and the SAFEX overnight rate $y(t)$, i.e $y(t) = \alpha + \beta x(t)$. In South Africa we have an observable overnight rate, the SAFEX overnight rate, but there are no liquidly traded market instruments that can be used to strip a term structure of overnight rates.

We propose deriving a set of forward interest rates from $r_x(t)$, the JIBAR curve. The cointregration relationship along with the relevant no arbitrage conditions are used to derive a term structure of interest rates referencing the SAFEX overnight rate. In a liquid market the multi-curve bootstrap is a two step procedure:

1. Bootstrap the OIS curve or overnight zero-curve using the relevant market instruments.

2. Bootstrap the respective tenor-homogeneous curves using the bootstrapped OIS zero curve for discounting.

In the cases where there is no liquid OIS market we modify this procedure. As mentioned above a cointegration relationship is derived between the 3-month JIBAR rate and the SAFEX overnight rate. The overnight zero-curve is dependent on the tenor curve, and so for no arbitrage relationships to be preserved the two curves are estimated simultaneously. In this section the methodology to bootstrap the ZAR OIS curve is outlined in detail so that it can be easily replicated and implemented. The ideas build off those presented in, Jakarasi *et al.* (2015) which illustrates the results but does not clearly define the details of the algorithm used to implement the simultaneous bootstrap.

4.3.1 Multi-curve bootstrap and OIS curve estimation algorithm

From Section 4.1 we have a relationship between the overnight rate and the 3-month rate for a specific day. We have sufficient liquid instruments to calculate forward 3 month rates, from which we can calculate daily forward rates. These daily forward rates can then be used to estimate the daily forward overnight rates. The daily forward overnight rates form the basis of the OIS curve. This is based on the following

$$Z_{OIS}(t,T_i) = \exp(-L(t,T_i)\tau_i) \tag{4.16}$$

$$= \exp\left(-\sum_{i=1}^{n} f_{OIS}(t,t_{i-1},t_i)\tau_i\right) \tag{4.17}$$

$$= \exp\left(-\sum_{i=1}^{n}(\alpha_i + \beta_i f_x(t,T_i-1,T_i))\tau_i\right) \tag{4.18}$$

The algorithm to estimate the OIS curve and tenor curve simultaneously is given below:

1. Estimate α and β using the Engle and Granger cointegration technique followed in section 4.1

2. Select a set of input instruments all homogeneous in their underlying rate tenor, consisting of a set of money market instruments maturing at times $\{T_{1,i},...,T_{m,i}\}$, a set of FRA's maturing at times $\{T_{m+1,i},...,T_{k,i}\}$ and a set of IRS's maturing at times $\{T_{k+1,i},...,T_{n,i}\}$.

3. Deduce the values of $\{L_{x,i}(t,T_{k+1,i}),...,L_{x,i}(t,t_{n,i})\}$.

4. Select an appropriate interpolation method, and interpolate between $\{T_{1,i},...,T_{n,i}\}$ and $\{L_{x,i}(t,t_{1,i}),...,L_{x,i}(t,T_{n,i})\}$ to estimate the rates corresponding to each cash flow of the input instrument.

5. Calculate the OIS Cointegration Estimate:

    - Use the same interpolation method to calculate daily zero rates

- Calculate the relevant capitalisation factors and daily forward rates
- Estimate daily forward overnight rates using the relevant cointegration equation
- Sum the daily OIS rates to form the OIS curve estimate

6. Put the OIS curve and the rates obtained in Step 4, into equation (3.3) to get new estimates of $\{L_{x,i}(t,t_{1,i}),...,L_{x,i}(t,T_{n,i})\}$
7. Repeat steps 3-6 until convergence is obtained.

The output of this algorithm is the estimate tenor-homogeneous curve and the ZAR OIS curve. The code is implemented in Matlab. Below the results of the implementation of this method in both the European market, where there is an actual OIS curve for comparison, and the South African market are analysed.

4.4 Discussion of Results

The results of the dual curve bootstrap and OIS estimation in the European market are shown in Figures 4.3 to 4.6. Figures 4.3 and 4.4 show the OIS curve estimated using the 1-month EURIBOR and using the 3-month EURIBOR rate, in comparison to the actual EONIA curve. Figures 4.5 and 4.6 show a comparison of the corresponding tenor-homogeneous curves bootstrapped simultaneously using the EONIA curve, and the curve estimate. Table 4.5 has the data corresponding to these graphs.

Figures 4.3 and 4.4 show OIS curve estimates that are very close to the actual OIS curve. The curves diverge slightly in the short and long end. The OIS curve estimated using the 1-month EURIBOR rate is a better estimate than the curve estimated using the 3-month EURIBOR rate, which is logical as the 1-month EURIBOR curve follows the shape of the OIS curve more closely. The tenor curves, in Figures 4.5 and 4.6, bootstrapped using the dual-curve algorithm, and OIS curve estimate compare very well to the actual 1-month and 3-month EURIBOR tenor curves. Table 4.5 and 4.6 display the information in Figures 4.5 and 4.6. Although this is not a formal test, it is enough to deduce whether this method produces tractable and usable results.

From observing how small the differences are between the actual and estimated curves, suggests that this method produces a good OIS curve estimate, and all the instruments on the curve will price back to par, which is an important feature to ensure the curve is being bootstrapped correctly.

The dual-curve bootstrap implementation in South Africa shown in Figure 4.11. The shape of the ZAR yield curves is not standard, and this is due to less liquidity in the market. We also observe the ZAR rates are significantly higher than the EURIBOR rates, which is typical for a developing economy. This OIS estimate will enable the pricing of interest rate derivatives using differential discounting, which is important in the post-crisis framework. It is also preferred for pricing methods to be consistent, and given that differential discounting has become standard market practice in developed economies, an adapted pricing framework is needed for economies with no liquid OIS market.

Table 4.4 below shows the prices of the interest rate swaps used to strip the 3-month JIBAR curve. They are priced using the estimated tenor and ZAR OIS curves. In method 1, the OIS curve is estimated separately using the ZAR swap curve. Method 2 estimates the 3-month tenor curve and the ZAR OIS curve simultaneously such that when the swaps used to strip the curve are priced, they price back to zero. Method 2 illustrates an integral feature of this method, as it ensures that no arbitrage relationships are still maintained, and ensures we build the best possible estimate of the ZAR OIS curve. If the curves are not stripped simultaneously there will be pricing differences that will be more noticeable for trades with a larger notional or a longer tenor. The IRS prices below are based on a notional of R1 000 000.

Tab. 4.4: Interest Rate Swap Prices

Instrument	Prices (Method 1)	Prices(Method 2)
SWP3Y	34.70	- 6.08
SWP4Y	70.00	- 1.08
SWP5Y	120.16	- 0.83
SWP6Y	188.12	1.31
SWP7Y	271.69	- 1.87
SWP8Y	376.46	6.90
SWP9Y	483.75	- 0.04
SWP10Y	628.16	0.38
SWP12Y	841.72	35.72
SWP15Y	1,405.49	181.03
SWP20Y	1,664.79	- 0.01
SWP25Y	1,369.06	- 82.59
SWP30Y	- 648.08	- 0.00

4.4.1 The effect of cointegration parameters α and β on the OIS curve estimate

The OIS curve estimate is defined by its cointegration parameter values α and β. In figures 4.7 and 4.8 we show how the estimates of α and β change over time. We use a minimum of 365 days of data, and incrementally add another day of data to the estimation, with a start date of 03/01/2000 and end date of 10/11/2014. It can be seen that the values of α and β vary significantly over time. It is important to capture these changes for the most accurate OIS curve estimate. In the figures 4.9 and 4.10 we illustrate the effect of different values of α and β on the OIS curve estimate. A sizeable impact is noted. It is therefore important that to get the best results from this method, the cointegration estimates be updated frequently.

4.5 Application: Pricing options on ZAR OIS

In West (2008) it is shown that options written on the South African prime rate can be valued using the prime curve estimate which is constructed in a similar way to the ZAR OIS curve estimate derived here. We show that similarly options written on ZAR OIS can be valued. We look at the pricing of a caplet/floorlet using the Black (1976) model. In South Africa there are existing option prices on the 3-month

JIBAR. We use the cointegration equation between the 3-month JIBAR rate, $x(t)$ and the SAFEX overnight rate, $y(t)$ developed above:

$$y(t) = \alpha + \beta x(t) \tag{4.19}$$

Using this we can convert the 3-month option prices into option prices on ZAR OIS. As in West (2008), under Black (1976) the required forward JIBAR rate, $f(t_0, t, T)$, has volatility measure σ. We price a caplet, with strike rate r_x, that pays off t maturity T if $y(T) > r_x$, where $y(T)$ is the OIS rate at maturity. From this we have:

$$y(T) > r_x \tag{4.20}$$

$$\alpha + \beta x(t) > r_x \tag{4.21}$$

$$x(t) > \frac{r_x - \alpha}{\beta} \tag{4.22}$$

which implies that the payoff of the option is:

$$\alpha - \beta x(t) - r_x \tag{4.23}$$

$$= \beta \left(x(t) - \frac{r_x - \alpha}{\beta} \right) \tag{4.24}$$

which is β caplets written on 3-month JIBAR. The appropriate volatility to use, from the volatility skew, in the option price is that corresponding to the JIBAR strike rate $\frac{r_x - \alpha}{\beta}$ as this is observable in the market. If we assume the option is written on the 3-month JIBAR rate for the period $[t, T]$, now assuming $t_0 < t < T$, where t_0 is the current time, we can use the Black (1976) model to calculate the value of the option:

$$V = \beta \gamma Z_{OIS}(t_0, t) \frac{T - t}{365} \left[f(t_0, t, T) N(\gamma d1) - \frac{r_x - \alpha}{\beta} N(\gamma d2) \right] \tag{4.25}$$

$$d1 = \frac{\ln \left(\frac{\beta f(t_0, t, T)}{r_x - \alpha} \right) + \frac{1}{2} \sigma^2 t}{\sigma \sqrt{t}} \tag{4.26}$$

$$d2 = d1 - \sigma \sqrt{t} \tag{4.27}$$

where $\gamma = 1$ for a caplet and $\gamma = -1$ for a floorlet, $N(\cdot)$ is the cumulative distribution function of the standard normal distribution, and $f(t_0, t, T)$ is the simple forward rate calculated off the 3-month JIBAR curve.

This section proves the usefulness of the ZAR OIS curve estimate we derived. The option could then be hedged in the FRA and swap market referencing 3-month JIBAR.

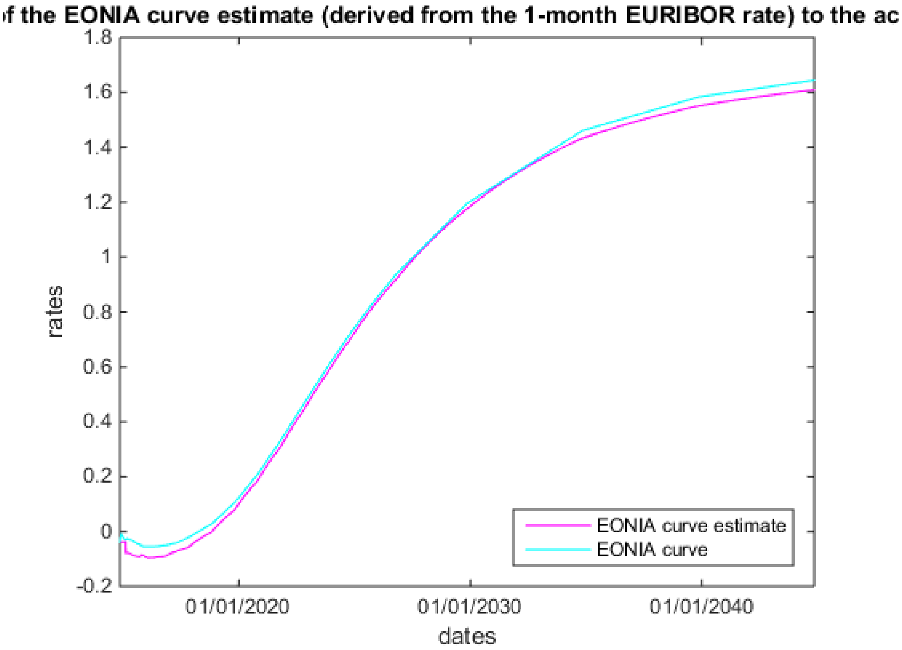


Fig. 4.3: Comparison of the EONIA curve estimate using the 1m EURIBOR rate to the actual EONIA curve

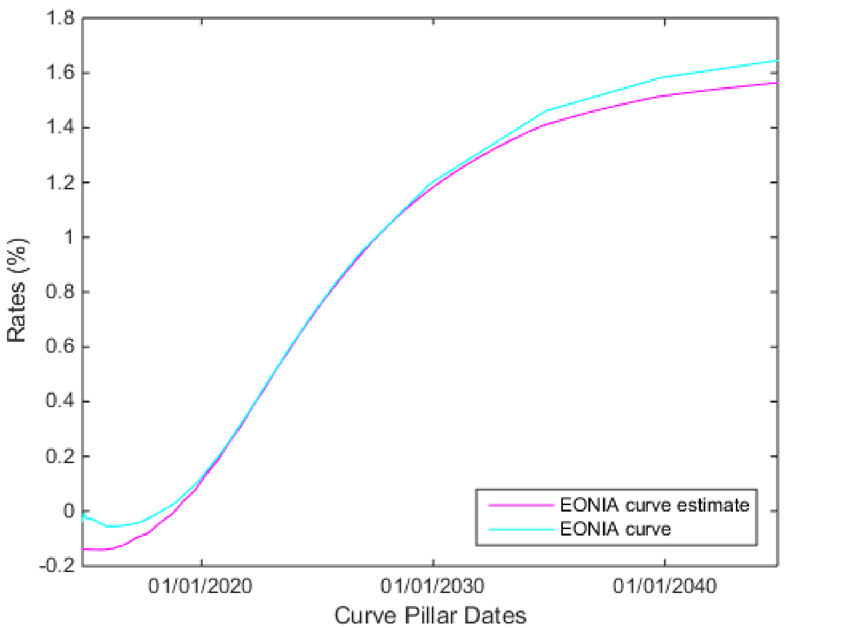


Fig. 4.4: Comparison of the EONIA curve estimate using the 3m EURIBOR rate to the actual EONIA curve

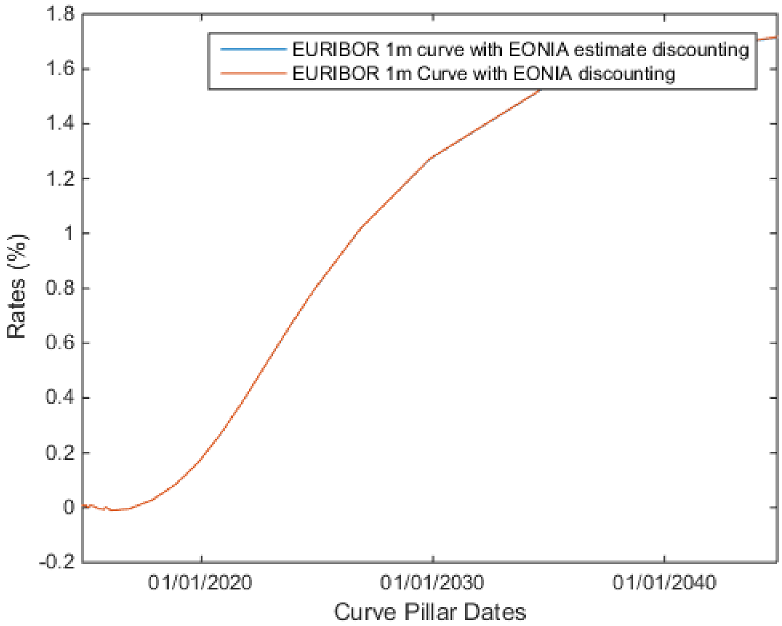


Fig. 4.5: Comparison of the 1-month EURIBOR curve estimate to the actual 1-month EURIBOR curve

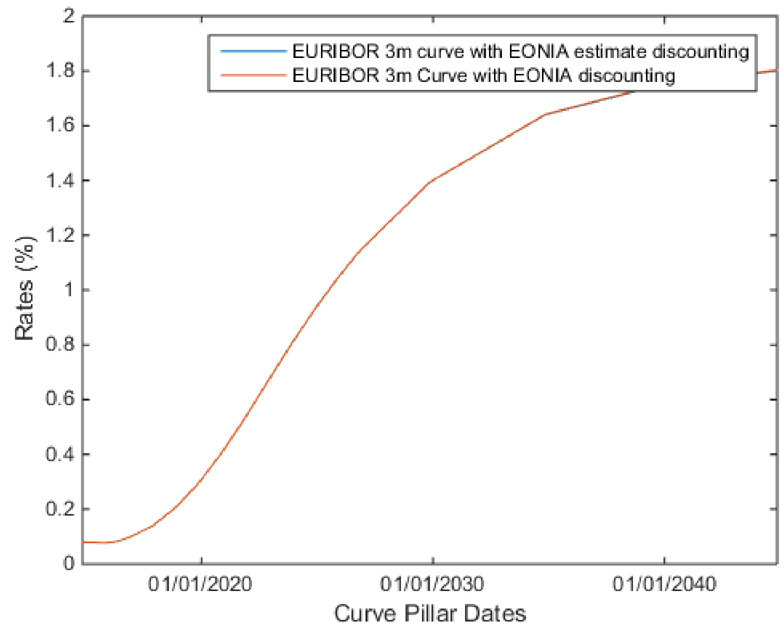


Fig. 4.6: Comparison of the 3-month EURIBOR curve estimate to the actual 3-month EURIBOR curve

Tab. 4.5: Comparison of the 1-month EURIBOR curve striped using the EONIA cuvre, to the 1-month EURIBOR curve stripped simultaneously with the EONIA curve estimate

Curve Pillars	EURIBOR 1m Curve	EURIBOR 1m Curve Estimate	% Difference
10/11/2014	0.00008	0.00008	0.0000%
10/12/2014	0.00008	0.00008	0.0000%
12/01/2015	0.00007	0.00007	0.0003%
10/02/2015	0.00002	0.00002	0.0089%
10/03/2015	0.00008	0.00008	-0.0029%
10/04/2015	0.00007	0.00007	-0.0016%
11/05/2015	0.00005	0.00005	0.0026%
10/06/2015	0.00001	0.00001	0.0612%
10/07/2015	-0.00002	-0.00002	-0.0465%
10/08/2015	-0.00004	-0.00004	-0.0271%
10/09/2015	-0.00005	-0.00005	-0.0227%
12/10/2015	-0.00007	-0.00007	-0.0180%
10/11/2015	0.00002	0.00002	0.0268%
10/02/2016	-0.00010	-0.00010	-0.0240%
10/05/2016	-0.00009	-0.00009	-0.0197%
10/08/2016	-0.00006	-0.00006	-0.0117%
10/11/2016	-0.00005	-0.00005	-0.0057%
10/11/2017	0.00027	0.00027	-0.0333%
12/11/2018	0.00083	0.00083	-0.0370%
11/11/2019	0.00164	0.00164	-0.0369%
10/11/2020	0.00272	0.00272	-0.0369%
10/11/2021	0.00398	0.00397	-0.0389%
10/11/2022	0.00533	0.00533	-0.0418%
10/11/2023	0.00667	0.00667	-0.0439%
11/11/2024	0.00794	0.00794	-0.0454%
10/11/2026	0.01016	0.01016	-0.0421%
12/11/2029	0.01273	0.01272	-0.0472%
10/11/2034	0.01535	0.01534	-0.0661%
10/11/2039	0.01656	0.01655	-0.0810%
10/11/2044	0.01717	0.01715	-0.0891%

Tab. 4.6: Comparison of the 3-month EURIBOR curve striped using the EONIA curve, to the 3-month EURIBOR curve stripped simultaneously with the EONIA curve estimate

Curve Pillars	EURIBOR 3m Curve	EURIBOR 3m Curve Estimate	% Difference
10/11/2014	0.00080	0.00080	0.0000%
11/11/2014	0.00080	0.00080	0.0000%
10/02/2015	0.00080	0.00080	0.0000%
11/05/2015	0.00079	0.00079	0.0000%
10/06/2015	0.00080	0.00080	0.0000%
10/07/2015	0.00078	0.00078	0.0000%
10/08/2015	0.00078	0.00078	0.0000%
10/09/2015	0.00078	0.00078	0.0000%
12/10/2015	0.00077	0.00077	0.0000%
10/11/2015	0.00078	0.00078	0.0000%
10/02/2016	0.00080	0.00080	0.0000%
10/05/2016	0.00083	0.00083	0.0000%
10/08/2016	0.00089	0.00089	0.0000%
10/11/2016	0.00097	0.00097	0.0000%
10/11/2017	0.00137	0.00137	-0.0127%
12/11/2018	0.00204	0.00204	-0.0141%
11/11/2019	0.00293	0.00293	-0.0060%
10/11/2020	0.00404	0.00404	0.0047%
10/11/2021	0.00530	0.00530	0.0120%
10/11/2022	0.00666	0.00666	0.0158%
10/11/2023	0.00800	0.00800	0.0171%
11/11/2024	0.00926	0.00926	0.0164%
10/11/2025	0.01041	0.01042	0.0140%
10/11/2026	0.01145	0.01145	0.0102%
12/11/2029	0.01394	0.01394	-0.0135%
10/11/2034	0.01642	0.01640	-0.0760%
10/11/2039	0.01752	0.01749	-0.1232%
10/11/2044	0.01804	0.01801	-0.1197%

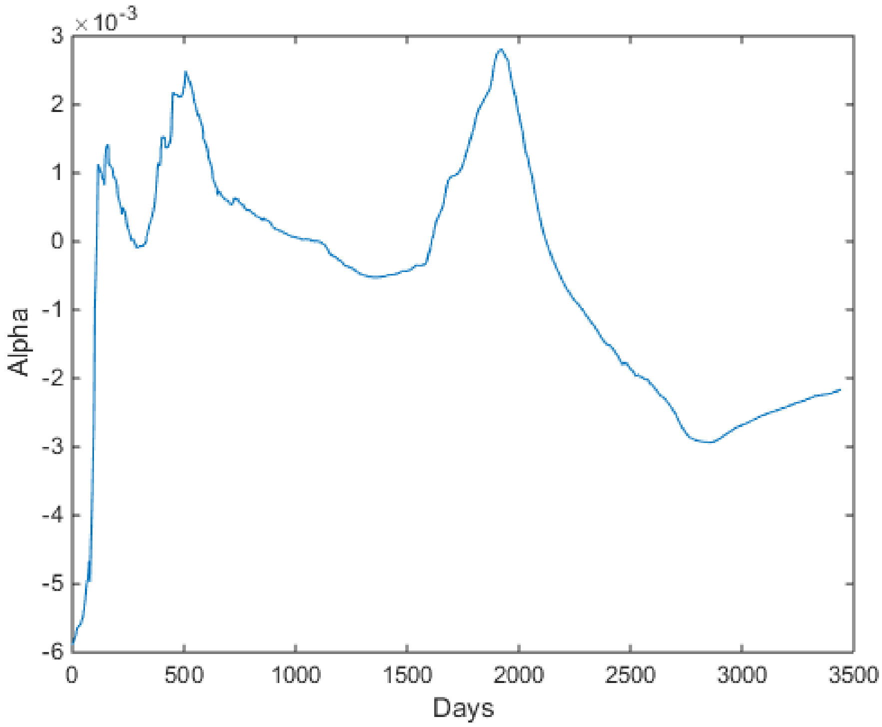


Fig. 4.7: How the cointegration parameter estimate α changes over time as more data is used

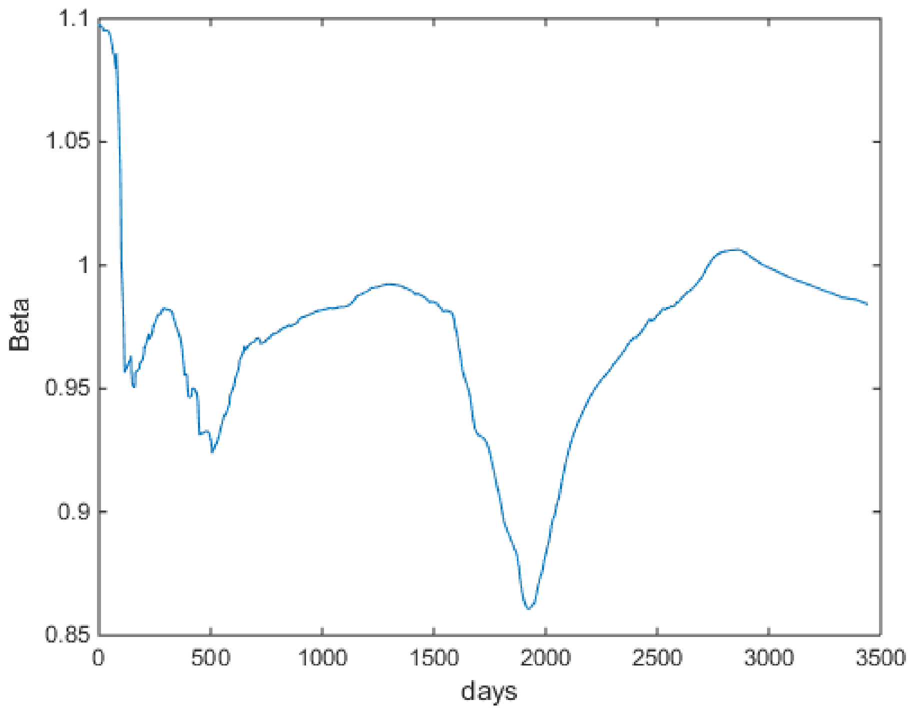


Fig. 4.8: How the cointegration parameter estimate β changes over time as more data is used

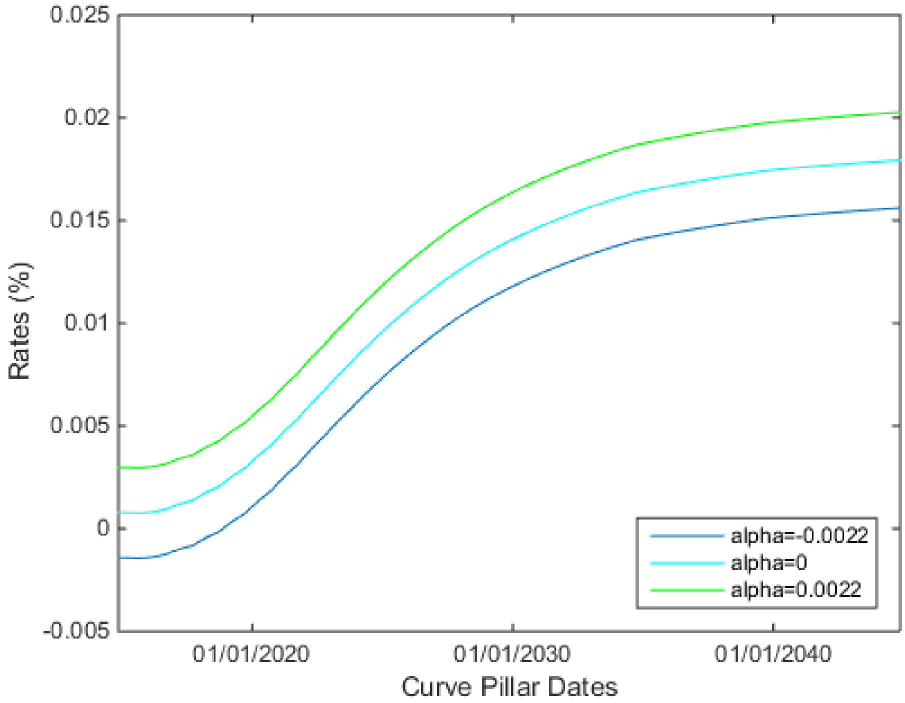


Fig. 4.9: How the cointegration parameter estimate α affects the OIS curve estimate

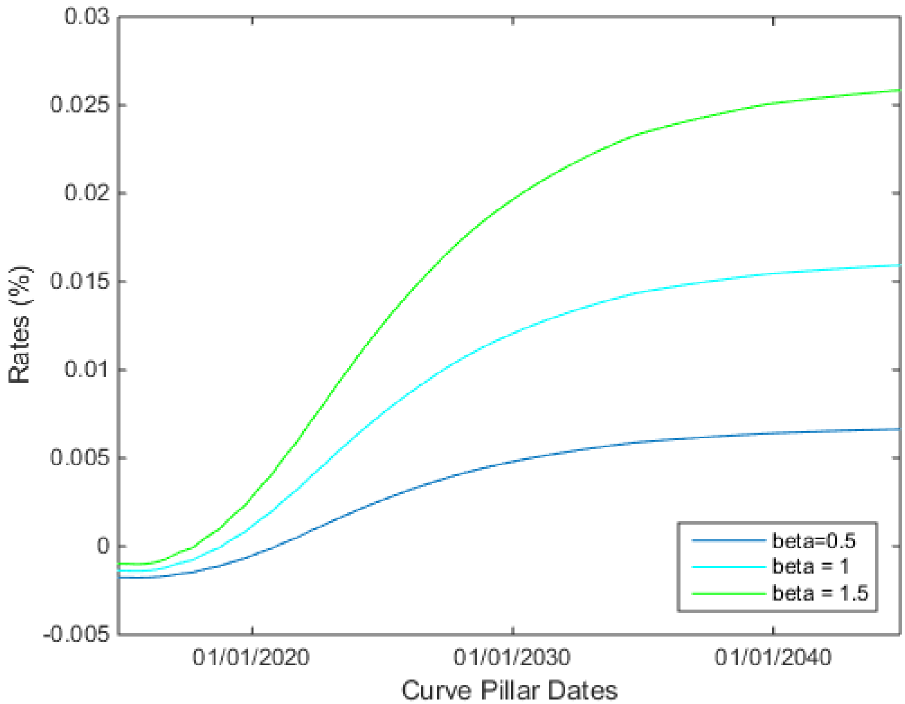


Fig. 4.10: How the cointegration parameter estimate β affects the OIS curve estimate

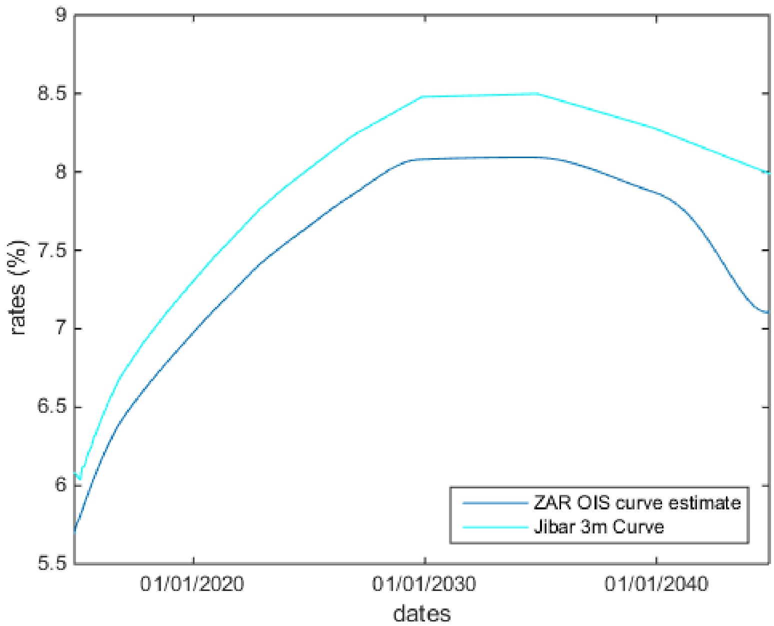


Fig. 4.11: South African OIS curve estimate and 3m JIBAR curve

Chapter 5

Conclusion

Since the financial crisis standard market practice to price interest rate derivatives has evolved, as it has been shown that derivative valuation should be spilt up in order to understand and express the different sources of risk in the transaction. It has been shown that LIBOR discounting is not theoretically correct in most cases. To calculate the risk-free value of the derivatives, cash flows of the derivative are generated using the rates to which they are linked (i.e using the appropriate tenor curve), and discounted using the default-free curve. The OIS curve is the best proxy for the default-free curve. In a market where there are no observable quotes, an OIS curve needs to be estimated or modelled, until such a market develops.

A method is successfully developed to estimate the OIS curve in the absence of a liquid OIS market. A cointegration relationship is established between the SAFEX overnight rate and the 3-month JIBAR rate. There are enough liquid instruments to construct the 3-month swap tenor curve in South Africa. This is exploited, along with a cointegration equation to estimate the ZAR OIS curve. After testing this method against the European market where the curve estimates can be compared to the actual curves, it is observed that it produces very good estimates. It must be noted that the cointegration parameters α and β do vary over time, so they need to be updated regularly to maintain the accuracy of the curve estimates. It is also shown that the cointegration relationship, along with the bootstrap algorithm, can be easily used to price and hedge interest rate derivatives written on ZAR OIS, with specific reference to caps and floors. In conclusion this dissertation presents a feasible solution to implement OIS discounting in South Africa, until a liquid OIS market is developed.

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Appendix A

First Appendix

A.1 Derivation of Single Curve Bootstrap Formula

We have that the fair swap rate is given by

$$K = \frac{1 - Z(t_0, t_n)}{\sum_{i=1}^{n} \tau_i Z(t_0, T_0)}$$

We invert this formula to derive the single curve bootstrap formula as follows

$$K \sum_{i=1}^{n} \tau_i Z(t_0, t_j) = 1 - Z(t_0, t_n) \tag{A.1}$$

$$K \sum_{i=1}^{n-1} \tau_i Z(t_0, t_i) + K\tau_n Z(t_0, t_n) = 1 - Z(t_0, t_n) \tag{A.2}$$

$$Z(t_0, t_n)(1 + K\tau_n) = 1 - K \sum_{i=1}^{n-1} \tau_i Z(t_0, t_j) \tag{A.3}$$

$$Z(t_0, t_n) = \frac{1 - K \sum_{i=1}^{n-1} \tau_i Z(t_0, t_i)}{(1 + K\tau_n)} \tag{A.4}$$

$$e^{-r(t_0, t_n)\tau_n} = \frac{1 - K \sum_{i=1}^{n-1} \tau_i Z(t_0, t_i)}{(1 + K\tau_n)} \tag{A.5}$$

$$r(t_0, t_n) = -\frac{1}{\tau_n} \ln \left(\frac{1 - K \sum_{i=1}^{n-1} \tau_i Z(t_0, t_i)}{(1 + K\tau_n)} \right) \tag{A.6}$$

A.2 Derivation of Dual Curve Bootstrap Formula

This derivation follows a similar process to the single curve framework. We have

$$K = \frac{\sum_{i=1}^{n} \tau_i Z_{OIS}(t_0, t_i) \left(\frac{Z_x(t_0, t_{i-1})}{Z_x(t_0, t_i)} - 1 \right)}{\sum_{i=1}^{n} \tau_i Z_{OIS}(t_0, t_i)}$$

$$K \sum_{i=1}^{n} \tau_i Z_{OIS}(t_0, t_i) = \sum_{i=1}^{n-1} \tau_i Z_{OIS}(t_0, t_i) L(t_0, t_{i-1}, T_i) + Z_{OIS}(t_0, t_n) L_x(t_0, t_{n-1}, t_n)\tau_n$$

Solving for $L_x(t, T_{N-1}, T_N)$, as above we have

$$L(t, t_{n-1}, t_n) = \frac{K \sum_{i=1}^{n} Z_{OIS}(t_0, t_i)\tau_i - \sum_{i=1}^{n-1} Z_{OIS}(t_0, t_i) L_x(t_0, t_{i-1}, t_i)}{Z_{OIS}(t_0, t_n)\tau_n}$$

Now we have that

$$L_x(t_0, t_{n-1}, t_n) = \frac{1}{\tau_n}\left(\frac{Z_x(t_0, t_{n-1})}{Z_x(t_0, t_n)} - 1\right)$$

Now we solve for $r_x(t_0, t_n)$

$$\frac{Z_x(t_0, T_{n-1})}{Z_x(t_0, t_n)} = 1 + \tau_n \left(\frac{K \sum_{i=1}^{n} Z_{OIS}(t_0, t_i)\tau_i - \sum_{i=1}^{n-1} Z_{OIS}(t_0, t_i) L_x(t_0, t_{i-1}, t_i)}{Z_{OIS}(t_0, t_n)\tau_n}\right)$$

$$e^{-r_x(t_0,t_n)\tau_n} = \frac{1}{Z_x(t_0, t_{n-1})}\left(1 + \tau_n \left(\frac{K \sum_{i=1}^{n} Z_{OIS}(t_0, t_j)\tau_i - \sum_{i=1}^{n-1} Z_{OIS}(t_0, t_i) L_x(t_0, t_{i-1}, t_i)}{Z_{OIS}(t_0, t_n)\tau_n}\right)\right)$$

$$r_x(t_0, t_n) = \frac{1}{\tau_n} \ln\left[\frac{1}{Z_x(t_0, t_{n-1})}\left(1 + \tau_n \left(\frac{L \sum_{i=1}^{n} Z_{OIS}(t_0, t_i)\tau_i - \sum_{i=1}^{n-1} Z_{OIS}(t_0, t_i) L_x(t_0, t_{i-1}, t_i)}{Z_{OIS}(t_0, t_n)\tau_n}\right)\right)\right]$$

Appendix B

Second Appendix

Tab. B.1: EONIA Curve Inputs Instruments

Instrument	**Rate**
OIS1D	-0.038
OIS7D	-0.0208
OIS14D	-0.0065
OIS1M	-0.0122
OIS2M	-0.025
OIS3M	-0.03
OIS4M	-0.025
OIS5M	-0.03
OIS6M	-0.0295
OIS7M	-0.035
OIS8M	-0.04
OIS9M	-0.045
OIS10M	-0.045
OIS11M	-0.05
OIS1Y	-0.055
OIS18M	-0.055
OIS2Y	-0.05
OIS30M	-0.04
OIS3Y	-0.02
OIS4Y	0.03
OIS5Y	0.109
OIS6Y	0.213
OIS7Y	0.337
OIS8Y	0.468
OIS9Y	0.596
OIS10Y	0.716
OIS11Y	0.825
OIS12Y	0.923
OIS15Y	1.159
OIS20Y	1.402
OIS25Y	1.5145
OIS30Y	1.573

Tab. B.2: EURIBOR 1M Curve Input Instruments

Instrument	Rates
DEP1M	0.008
SWP2M	0.007
SWP3M	0.002
SWP4M	0.008
SWP5M	0.007
SWP6M	0.005
SWP7M	0.001
SWP8M	-0.002
SWP9M	-0.004
SWP10M	-0.005
SWP11M	-0.007
SWP12M	0.002
SWP15M	-0.01
SWP18M	-0.009
SWP21M	-0.006
SWP2Y	-0.005
SWP3Y	0.027
SWP4Y	0.083
SWP5Y	0.164
SWP6Y	0.271
SWP7Y	0.395
SWP8Y	0.528
SWP9Y	0.658
SWP10Y	0.78
SWP12Y	0.991
SWP15Y	1.229
SWP20Y	1.467
SWP25Y	1.577
SWP30Y	1.634

Tab. B.3: EURIBOR 3M Curve Input Instruments

Instrument	Rate
DEP3M	0.08
FRA1x4	0.085
FRA2x5	0.081
FRA3x6	0.078
FRA4x7	0.08
FRA5x8	0.076
FRA6x9	0.077
FRA7x10	0.075
FRA8x11	0.076
FRA9x12	0.077
FRA12x15	0.086
FRA15x18	0.1015
FRA18x21	0.1265
FRA21x24	0.149
SWP3Y	0.137
SWP4Y	0.2045
SWP5Y	0.293
SWP6Y	0.403
SWP7Y	0.528
SWP8Y	0.661
SWP9Y	0.791
SWP10Y	0.913
SWP11Y	1.023
SWP12Y	1.121
SWP15Y	1.353
SWP20Y	1.578
SWP25Y	1.679
SWP30Y	1.72

Tab. B.4: JIBAR 3M Curve Input Instruments

Instrument	Rate
DEP1D	6.083
DEP3M	6.083
FRA1x4	6.18
FRA2x5	6.22
FRA3x6	6.32
FRA4x7	6.36
FRA5x8	6.46
FRA6x9	6.5
FRA7x10	6.6
FRA8x11	6.64
FRA9x12	6.74
FRA12x15	6.89
FRA15x18	7.04
FRA18x21	7.15
FRA21x24	7.29
SWP3Y	6.95
SWP4Y	7.13
SWP5Y	7.29
SWP6Y	7.44
SWP7Y	7.573
SWP8Y	7.7035
SWP9Y	7.807
SWP10Y	7.8985
SWP12Y	8.0635
SWP15Y	8.236
SWP20Y	8.2875
SWP25Y	8.231
SWP30Y	8.1415

Tab. B.5: EURIBOR 6M Curve Input Instruments

Instrument	Rates
DEP6M	0.181
SWP1Y	0.185
SWP18M	0.193
SWP2Y	0.208
SWP3Y	0.2593
SWP4Y	0.3336
SWP5Y	0.419
SWP6Y	0.5285
SWP7Y	0.656
SWP8Y	0.786
SWP9Y	0.911
SWP10Y	1.028
SWP11Y	1.1405
SWP12Y	1.2275
SWP13Y	1.3156
SWP14Y	1.388
SWP15Y	1.4503
SWP16Y	1.5
SWP17Y	1.548
SWP18Y	1.589
SWP19Y	1.624
SWP20Y	1.6583
SWP21Y	1.678
SWP22Y	1.7
SWP23Y	1.7215
SWP24Y	1.74
SWP25Y	1.7483
SWP26Y	1.755
SWP27Y	1.764
SWP28Y	1.772
SWP29Y	1.78
SWP30Y	1.786

Tab. B.6: EURIBOR 12M Curve Input Instruments

Instrument	Rates
DEP1D	-0.038
DEP1W	-0.016
DEP1M	0.008
DEP2M	0.044
DEP3M	0.080
DEP6M	0.181
DEP9M	0.255
DEP12M	0.336
SWP2Y	0.2165
SWP3Y	0.2619
SWP4Y	0.3358
SWP5Y	0.4672
SWP6Y	0.5485
SWP7Y	0.6735
SWP8Y	0.8035
SWP9Y	0.9295
SWP10Y	1.0455
SWP12Y	1.2415
SWP15Y	1.4778
SWP20Y	1.6848
SWP25Y	1.7728
SWP30Y	1.788