

Temperature Anisotropies:

Covariant CMB anisotropies and nonlinear corrections

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"One must be sure that one has enabled science to make a great advance if one is to burden it with many new terms and require that readers follow research that offers them so much that is strange" A. L. Cauchy.

"I think therefore I am, is a statement of an intellectual who underrates toothaches." - Milan Kundera

"The CMB is the cleanest cosmological observable ... The anisotropy in the CMB is the Rosetta stone for models of structure formation. Any mechanism that produces gravitationally bounded objects, like clusters of galaxies, will produce fluctuations in the temperature of the CMB ..." - Lyman Page.

PROLOGUE

The questions I ask myself are generally all along the lines of "so where did all this structure come from?". I hoped that work in the CMB and its cosmological implications would give me insight into this. It is an adventure that is still young.

I began my PhD with an investigation of some formal aspects of Ehlers-Ellis Relativistic Kinetic Theory in mind – the implications of the truncation conditions found in the exact theory. I ended up trying to calculate CMB anisotropies as an application of this beautiful and somewhat purist formalism. The Ehler-Ellis (1+3) Lagrangian approach to General Relativity (GR) and Relativistic Kinetic Theory (RKT) are apparently not well known nor well used and have only recently begun to show advantages over the more usual ADM and Bardeen perturbative approaches to astrophysical cosmology when combined with the Ellis-Bruni perturbation theory.

The CMB, for me, represents an opening move in bettering our understanding of the restrictions on our knowledge of the geometry of our universe, the development of structure in it, and the assumptions required in order to make theoretical statements workable. A feel for what we really know about the universe, about the implications that general relativity and kinetic theory have for explaining the way it is. The large scale CBR imprint may tell us more about the very early universe and spacetime topology, while the small and intermediate scales hold the imprint of local physics and structure as well as the perturbations seeded in the early universe.

This piece of work focuses on the reasons for being able to use linear-FRW models in the context of CMB studies, and how to get them from the exact theory. Behind it are four assumptions : Gaussianity of the perturbations, the Copernican principle (cosmological principle), and a trust in General Relativity and Relativistic Kinetic Theory.

I have been asked many times "So, what is new?": I find (I) new non-linear (non-perturbative) corrections to the canonical linear-FRW treatment of temperature

formalism, using timelike integrations, for the study of temperature anisotropies, as developed here, not only recovers the canonical linear treatment, but highlights where the canonical treatment could go wrong (for example, the need to locate null-geodesics) and how one could address these shortcomings.

One of the most obvious missing components in this work, in the almost-FLRW part of the thesis, has been the omission of (A) a discussion of tensor perturbations [78]. This is for two reasons: (i) formally a shear-free threading does not admit gravitational waves [180], (ii) the focus here is on the CMB imprint due to structure and structure formation. This is the essence of the Newtonian-like description. (B) That of vector perturbations, once again there is an ambiguity in their definition unless the u^a -frame is totally fixed – these do not seem to have a manifestly gauge invariant meaning.

Although in the exact equations everything that is admitted by GR is accounted for, in reducing the exact theory to the canonical treatment, focusing on the Newtonian like threading, it was expedient to avoid these two issues. The canonical approach investigates the scalar induced temperature anisotropies within the Newtonian frame, the matter kinematics and dynamics in the total frame, and the gravitational wave effects in the total frame. In order to facilitate the jump between these two frames one needs knowledge of the potentials (Φ_A and Φ_H) and the peculiar velocities (v_a between the total and Newtonian frames). In the exact theory the total frame coincides with the preferred Lagrangian threading while the Newtonian frame ($\sigma_{ab} \approx 0 \approx \omega^a$) is an Eulerian choice that is only appropriate at linear order.

The thesis itself is based on original work written up in six papers that I have had the privilege to work on during the 3 year research period of my PhD related to temperature anisotropies and the calculation thereof in the 1+3 Lagrangian threading formalism : [66, 120, 167, 67, 40, 68]. Each chapter is based on the key results of the respective papers.

The thesis is arranged into six chapters, these are put into a form that is as free-standing as possible, the idea being that the individual chapters can be read without the requirement of detailed knowledge of the contents of the prior chapters. Each chapter is introduced by a brief summary with an indication of the key results pertaining to that chapter. These *italicized summaries* should not be viewed as formal abstracts; they were included to give the reader an idea of what terms and equations to expect, as well as outlining the key results of the respective chapters. Formal summaries are provided at the end of the thesis (see chapter 7).

On preconceptions

In order to place my preconceptions in context I quote Feyerabend as [abridged] by myself:

“[I do not claim] to possess special knowledge about what is good and what is bad in science and that I want to impose this knowledge upon my readers. Everyone can read the terms in his own way and in accordance with the tradition to which he belongs ... [It must be remembered that] some of the most important formal properties of a theory are found by contrast, and not by analysis ... [For the cosmologist needing to maximize the empirical content] of the views he holds and who wants to understand them as clearly as he possibly

can must therefore introduce other views ... each single theory, each fairy tale, each myth that is part of the collection forcing the others into greater articulation and all of them contributing, via this process of competition ... "

This brings across the point clearly. I view the mathematics and the models encoded therein, and these models along with the explanations, the narratives and stories: The Theories, as nothing more than representations of reality, not realities in themselves. Such a world is pluralistic, where different ways of doing things and the contrasts between these bring out the interesting explanations, the interesting features and better our understanding.

"We need dreamworlds in order to discover the features of the realworld we think we inhabit",

in this regard

"prejudices are found by contrast, not analysis".

However, at the end of the day one should consult nature.

It is then important to understand the problem of where to hide what I do not know. To assign a probability to a property of the universe is only possible if we are dealing with an ensemble; this allows the existence of relative frequencies of different outcomes. Since only one universe is known, arguments about the prior probabilities are inevitable. If this is neglected entirely then in the Bayesian case, where probabilities are like odds, the probability of an outcome given the data becomes proportional to the probability of getting the data given the particular outcome. One needs to be careful with such maximum likelihood assumptions.

"It is actually quite difficult to construct a theory where everything we see is all there is " - A. Albrecht

The notion of ensembles is slightly more problematic, it is a concept used everytime we apply a probability to an event - we imagine an infinite sequence of repeated trails. Here the actual field found in a given member of the ensemble is a realization of the statistical process. It is common sense to treat the volume average over realizations in all parts of space as equivalent to the ensemble average. It is interesting to note that we are only able to observe one realization - so how then is the variance measured ? and we have no evidence supporting the existence of the ensemble itself.

"Unamity of opinion may be fitting for a church, for the frightened or greedy [acolytes] of some (ancient or modern) myth, of for the weak and willing followers of some tyrant. Variety of opinion is necessary for objective knowledge. And a method that encourages variety is also the only method that is compatible with a humanitarian outlook. " - P. Feyerabend

Acknowledgements

In this endeavour I owe most to my mentor, George Ellis, who has taught me about the beauty of GR, about science, about cosmology and about humanism – about finding hidden structure in all things and for the need to maintain a deep sense of fun and lightness in a very complex world. I owe much to Roy Maartens, for teaching me what is and what is not important, not only with respect to research, and for first introducing me to GR. Both contributed to the development of this work – in detail and support.

They have shown me where simplicity in clarity is to be found. In many ways cosmology is unique: one needs to honestly face the metaphysics in which the subject is embedded. To better understand one's own prejudices. Treating the subject as merely an exercise in modeling, as in observational cosmology, is unacceptable as nothing new will be learnt beyond getting better numbers for the free parameters. In this regard I have also been strongly influenced by Bill Stoeger, whose insight and views have greatly influenced my own. Bruce Bassett has the "belief" that science is a religion. His assertive upfrontedness, in light of this ideal, has been fun. Astrophysical cosmology without alternative theories, either from relativistic cosmology or other more exotic sources, will ultimately be sterile.

In addition the criticism of Peter Dunsby (as well as comments pertaining to the thesis itself) and enjoyable discussions with Henk van Elst, I hope, have ultimately lead to a better understanding of the mathematics and the physics. I must also thank George Ellis for proof reading the thesis; all remaining errors, of either a typographical or mathematical nature, are mine. The staff of the Mathematics and Applied Mathematics department, in particular Di Loureiro, have made life easier by providing a pleasant work environment. Most of my thanks must go to my friends for their support and company.

Tim Gebbie, Cape Town, South Africa, Africa, August 1999

"Siyawela laphesheya lulezontaba ezimnyama ... Siyayilanda"

I would like to thank the external examiners: John Barrow, Roy Maartens and William Stoeger, for their comments, criticism and encouragement, for taking the time to look through my work. The external reports were received by UCT during the middle of November 1999. I was notified by the Doctorial Degrees Board in early December 1999 to finally received the examiners reports in January 2000. This draft was then submitted to the Library. The entire process, from submission to graduation, as overseen by the Doctorial Degrees Board took 10 months.

NOTATION and CONVENTIONS

I follow the notation and conventions of Ellis [50, 58, 62] with the improvements of Maartens [113]. The units are geometrical : $c = 8\pi G = k_B = +1$ (unless otherwise stated). The signature is $(-1, +1, +1, +1)$. Spacetime indices are early Latin ($abc\dots$), spatial indices are early Greek ($\alpha\beta\dots$), and are indicative of either tetrad or coordinate basis. The Coordinate basis indices will be late Latin for the spacetime ($ijk\dots$).

The curvature tensor is $R^a{}_{bcd} = -e_d\Gamma^a{}_{bc} + \dots$, the Ricci identity is $2\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d$, the Ricci tensor $R_{ab} = R^c{}_{acb}$ and $R = R^a{}_a$. The connection in terms of a tetrad e_a is $\Gamma_{abc} = e_a \cdot \nabla_{e_b} e_c$ (different from [51]) and the vorticity differs in sign from the convention in [51]. A_ℓ denotes the index string $a_1 a_2 \dots a_\ell$. Indices on tensors that are PSTF are denoted by $\langle A_\ell \rangle$, symmetric by (A_ℓ) , antisymmetric indices by $[A_\ell]$ and will denote screenspace PSTF $\{A_\ell\}$.

We will normalize the scale factor here and now : $a_0 = +1$ (so that we can use that $\rho_C = 3H_0^2\Omega_0 a^{-3}$). We will use η_* to denote the conformal time at the surface of last scattering, and η_0 , that here and now. We will then use $\Delta\eta_* = \eta_0 - \eta_*$ and $\Delta\eta = \eta_0 - \eta$ (following [82]). Furthermore, we will also introduce $\delta\eta = \eta - \eta'$, $\delta\eta_* = \eta - \eta_*$ and $\delta\eta_0 = \eta - \eta_0$.

Within the text: I have used *italics* to emphasize what I consider important statements or observations, **boldface** to highlight the introduction or definition of important terminology, and have Capitalized (C) terms where I introduce acronyms.

Acronym	Comments
CBR	Cosmic Background Radiation
CDM	Cold Dark Matter
CMB	Cosmic Microwave Background (CMBR)
EFE	Einstein Field Equations
FL	Friedmann-Lemaître dynamics
GR	General Relativity
GW	Gravity Waves, typically $H_{ab} \neq 0$ and excluding tidal gravitational forces.
GRF	Gaussian Random Fields
IBE	energy Integrated Boltzmann Equation
MDE	Multipole Divergence Equations
PSTF	Projected, Symmetric & Trace Free tensors (<i>e.g.</i> $F_{\langle A_\ell \rangle}$).
RKT	Relativistic Kinetic Theory (Israel-Stewart)
RW	Robertson-Walker geometry
linear FRW	The linearization about a Friedmann-Robertson-Walker metric in the context of Bardeen's Gauge Invariant variables (Bardeen 1980); typically expressed in terms of gauge invariant potentials.
almost FLRW	A linearization of the exact field equations to those linear about FL-dynamics and consistent with RW-geometries in the context of the 1+3 Covariant approach of Ellis (Ellis & Bruni 1989); using geometric quantities.
GI	3+1 Gauge Invariant perturbative foliation of the spacetime about FRW surfaces of homogeneity (<i>al la</i> Bardeen 1980).
CGI	1+3 Covariant and Gauge Invariant perturbative threading of the spacetime (<i>al la</i> Ellis & Bruni 1989).
EGS	Ehlers Geren and Sachs theorem
almost EGS	The CGI generalization of the Ehlers Geren and Sachs Theorem.

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0.1 An overview of the CMB

I attempt to outline, first, the history of the development of our understanding of temperature anisotropies, second, the standard view of the thermal history of the CMB, and third, where the relativistic cosmology approach to the CMB fits into the canonical treatment.

0.1.1 The CMB, a social history

The first theoretical estimates of the radiation temperature were based on a theory of element synthesis in the early universe worked out by George Gamow in the 1940's [64]. The first observational detection leading to the calculation of the background temperature seems to have been made by Andrew McKellar in 1940 at the Dominion Observatory¹, British Columbia [134]. It is unlikely that he had any idea of the cosmological implications of what he had uncovered, he was however able to quote an average bolometric temperature of $T_{R0} = 2.3^\circ \text{K}$ – based on the study of interstellar absorption lines. He was prompted to try and find the average temperature of the interstellar medium given the then recent spectral work of Adams from the Mount Wilson Observatory. He computed “a temperature for molecules in interstellar space”, the “rotational” or “effective” temperature that governed the population of the lowest states of the molecules giving rise to the “interstellar molecular lines” (see for instance Weinberg [185] pg 514). This was nevertheless a remarkable and sophisticated achievement.

In the 1950's a somewhat more detailed theoretical analysis of present radiation temperature as an artifact of some early hot era were undertaken by Ralph Alpher and Robert Herman [5]. The idea was that the individual photons arising from an era with temperatures of the order of 10^9K would have been absorbed long before today. It was realized that because the photon entropy per baryon is very large the matter temperature would relax as a^{-1} , leading one to the idea that the photons emitted as the universe was becoming transparent would have the same value of $T_R a$ as found during the element synthesis; a consequence of an expanding FRW cosmology. The remarkable prediction of a 5K black-body radiation was attained. These results were allowed to sink into obscurity.

It was only in the mid sixties that the problem of determining the radiation temperature was once again taken up.

It seems that two Russians; Andrei Doroshkevich and Igor Novikov, consulted the Bell Laboratories Technical Journal in 1963, to see if there were any interesting microwave background measurements. They found a paper by Ed Ohm in the Bell Systems Technical Journal, dating back to 1961, which reported an excess of noise at the level of 3°K from measurements using Bell Laboratories 20-foot antenna. The same instrument later used by Penzias and Wilson. Not only was Ohm unable to rule out static due to instrument noise [159], there was some confusion by the Russians over what Ohm's sky temperature meant [159, 186]. Ya. B. Zeldovich had carried out the correct calculation in 1964, but thought that the measurement by Ohm gave a temperature less than a degree Kelvin. This in tandem with low estimates of cosmic element abundance

¹Copies are available from the DAO library, I am thankful for their helpfulness.

led the Russians to abandon the idea of a hot big bang. Doroshkevich and Novikov published their work in 1964.

In 1965 the argument of Robert Dicke and Jim Peebles was that the early universe was hotter than 10^{10} ° K because it either expanded from a singularity with $a = 0$, or there were cyclic oscillations between finite values of a ; it would get hot enough to dissociate the heavy elements left over from previous cycles. They suggested that the energy density of the CMB would be such that T_{R0} is somewhere less than or close to 40 K; the predictions of the previous decade had been significantly better. At last the CMB was again being taken seriously. An experiment by Roll and Wilkinson was prepared to measure the radiation temperature. In order to detect the temperature a radiometer designed by Dicke in the mid forties was to be used – the Dicke switching radiometer which jumped between two receivers a hundred times per second; one pointed at the sky, the other at a liquid Helium bath. Before Roll and Wilkinson could complete a measurement of the radiation temperature they learned that Penzias and Wilson had already made the observations.

Penzias and Wilson observed a weak background signal from a Horn antenna at Holmdel, New Jersey. Though McKellar actually had the additional *a priori* intent of calculating this temperature from observations he did not have anyone to tell him about its cosmological source. For Penzias and Wilson it was a truly serendipitous, well placed and beautifully timed discovery; a temperature detection of $T_{R0} \approx 3.5 \pm 1$ °K resulting from an antenna intended to track the Echo satellite. It was at one frequency only, so made very little impact with regards to the expectation of a blackbody spectrum. The impact on cosmology and the public perception of cosmology was immense.

This observation, published in 1965 [146], along with the work of Dicke, Peebles, Roll and Wilkinson [35] was to hail the beginning of modern CMB physics in cosmology. The key point was that at wavelengths in the range of centimeters to millimeters the extraterrestrial electromagnetic radiation is dominated by a nearly isotropic component, the Cosmic Microwave Background (CMB). The closeness to isotropy suggests that the CMB uniformly fills space, meaning that an observer in another galaxy would see almost the same intensity of radiation – this is consistent with the Copernican principle. The spectrum is close to black-body, in fact the best example of a black body known. It has a thermal Planckian form at a temperature near 3°K. This suggests that the radiation has almost completely relaxed to thermodynamic equilibrium. This could not have happened recently as the universe is currently optically thin to radiation – we can see distant galaxies and stars. The CMB can move across the present universe on scale of the Hubble length with little change beyond that caused by expansion.

The interpretation is that the CMB is left over from an earlier time when the expanding universe was so dense and hot that interaction rates between particles were rapid enough to have allowed a relaxation to thermal equilibrium, thus filling space with a sea of black-body radiation. Furthermore, when interaction are negligible, cooling is due to expansion, preserving the thermal spectrum. When the radiation interacts with the matter, because the heat capacity of the radiation is very much larger than that of the matter, the spectrum will still tend to remain close to blackbody. A nearly thermal spectrum of blackbody radiation is thus an expected signature of an expanding universe in which the radiation is that left over from a early hot dense era.

There is however structure in the universe, we see galaxies and super-cluster of galaxies, stars and other interesting objects and phenomena, including the Earth. If the CMB was perfectly isotropic one would have expected there to have been no deviations from isotropy and homogeneity in the early universe, where then would the structure come from?

In the Big Bang Model, complex structures arise from primordial perturbations, the perturbations grow by gravitational instability as modified by the expansion. Even though the CMB was expected to be an artifact of an earlier less structurally complex phase, the fluctuations should be between the 10^{-6} to the 10^{-5} level in order to be consistent with the simplest gravitational instability models.

The detection of anisotropies in the CMB, by the COBE team, in 1992, was thus a most auspicious moment in the history of cosmology. It brought into play an era of precision cosmology, both on the theoretical and observational fronts. It vindicated the idea that there should be small fluctuations in the early universe that would seed the formation of structure and promised to provide a testing ground for the physics describing the nature of the primordial fluctuations and hence large scale structure of the observable universe.

Launched on November 18, 1989, the COBE satellite carried three experiments: the Far Infrared Absolute Spectrophotometer (FIRAS) to compare the spectrum of the CMB with a precise blackbody, a Differential Microwave Radiometer (DMR) to create an all-sky map of the cosmic radiation, and a Diffuse Infrared Background Experiment (DIRBE) to search for the Cosmic Infrared Background Radiation. The COBE satellite revealed the CMB to have an almost perfect thermal spectrum (from the FIRAS experiment) of a temperature $T_{R0} = 2.726 \pm 0.010\text{K}$ (95%CL) with a maximum deviation of 3×10^{-4} [131] and noise weighted deviations of its peak intensity of under 5×10^{-5} . It was also to show that the CMB had temperature anisotropies (from the DMR experiment) near to one part in 10^5 [160] (after subtraction of the 10^{-3} dipole).

0.1.2 The CMB, a thermal history

The general idea of the thermal history of the CMB is worthwhile keeping in mind – the canonical picture is as follows [139]. Spectral decoupling, when the last new photons comprising the CMB were produced, ended near a redshift $z \sim 3 \times 10^6$ (or about after 2 months of Einstein de Sitter (EdS) bang-time). Departures from the Planckian form of the CMB are a result of energy transfer to the CMB.

- Before $z \sim 3 \times 10^6$ double Compton scattering and free-free emission easily re-establish the thermal equilibrium of the CMB with its surroundings by the creation of photons. This then means that inputs of energy into the CMB merely relax into a hotter CMB – the black-body spectrum is maintained.
- Between $10^5 < z < 3 \times 10^6$ single Compton scattering dominates the interaction of the photons and its surroundings. Single Compton scattering conserves the number of photons. The CMB is thus in statistical equilibrium with its surroundings and has distortions which can be characterized by a chemical potential μ , here with a *photon flux*: $S_\nu(T, \mu) \sim 2h\nu^3 / (\exp[(h\nu/kT_{CMB}) + \mu] - 1)$.
- Hot electrons begin to interact more readily with the CMB from $z < 10^5$. The hot electrons are in neither thermal nor statistical equilibrium with the CMB. They can inverse Compton scatter the CMB photons to induce Compton- y distortions: $y \sim 1/m_e c^2 \int (k(T_e - T_{CMB})) d\kappa_e$. Here κ_e is the optical depth and T_e the electron temperature. This distortion can be thought of as the fractional energy change per scattering event times the number of scatterings, when there are a small number of events.
- For relatively recent times, $z < 10^3$, that is since the CMB has decoupled, if the universe is still ionized, there may be enough free-free emission from the plasma to increase the photon occupation numbers at low frequencies : thus increasing the CMB temperature there. This distortion is parametrized by $Y_{ff} = (h\nu/kT)^2 [T_p(\nu) - T_{CMB}] / T_{CMB}$. Here T_p is the plasma temperature.

The FIRAS limits on these distortions give :

$$|y| < 1.5 \times 10^{-5}, \quad \text{and} \quad |\mu| < 9 \times 10^{-5}, \quad (1)$$

while low frequency fits find [139]

$$Y_{ff} < 1.9 \times 10^{-5}.$$

These lend the Big-Bang story much credibility. It also supports the adiabatic models of structure formation; in which the spectrum of primordial fluctuations in the gravitational potential is described well by $\mathcal{P}(k) = Ak^{n+1}$. The idea is then that oscillations in the primordial photon-baryon fluid, in response to these fluctuations, are damped; this leads to energy being transferred to the CMB. This results in distortions – larger n implies larger distortions in $|\mu|$, this makes $n > 1.5$ seem infeasible.

It is known from the Gunn-Peterson test [75] that the nearby universe is well ionized; when or how the universe became ionized is not clear. The history of the ionized electrons can be constrained by knowledge of Y_{ff} and y .

0.1.3 Developments in CMB studies

It is because of these developments that the careful calculation of temperature anisotropies as arising from primordial perturbations has become an important aspect of modern cosmology. It is only since April 1992 that CMB anisotropies have taken up their place with spectral distortions, BBN element abundances and large scale structure constraints. The DMR experiment made a map of the sky with a resolution set by a 7° FWHM beam which meant that only low-order multipoles were accessible to the experiment ($\ell \sim 1 \dots 20$); DMR probes the pure Sachs-Wolfe effect. This makes it straight-forward to relate the DMR detection limits to those of the power spectrum [45]. It also provides sufficient data (the dipole, quadrupole and octopole limits) in order to place observational limits directly on the geometry using the almost-EGS theorem [166, 117, 118], the so called COBE-Copernican limits [167].

The future of the subject is anticipated to be rich given that the American Microwave Anisotropy Probe (MAP)² promises to obtain precision all-sky small-scale and polarization maps of the CMB by 2002 (it should be launched in Fall 2000), and the European PLANCK satellite will give even higher precision maps by the end of the next decade. In this regard the two outstanding issues in astrophysical cosmology are those of the foregrounds and nonlinearity, while the issues of astrophysical cosmology, those pertaining to the nature and formation of the primordial anisotropies, are still open. I would recommend the wonderful PhD thesis of Wayne Hu for details of the canonical treatment of theoretical CMB anisotropies [80].

The canonical calculation of temperature anisotropies in universes with small deviations from perfect isotropy and homogeneity can be divided into the study of, first : primary sources (projected) and secondary sources (integrated), and second : the development of gauge-invariant perturbation theory.

The **primary sources** arise due to effects at the time of recombination, there are three important contributions. First, the Sachs-Wolfe effect, where photons climb out of potential wells which induce a redshift (Sachs-Wolfe 1967 [152]). Second, the intrinsic photon temperature at last scattering due to acoustic oscillations, where the photons have been glued to the baryons by Thompson scattering off electrons such that the coupled radiation-baryon fluid can compress the radiation leading to a higher temperature. Below the horizon but above the photon diffusion scale the radiation-baryon fluid evolves adiabatically leading to acoustic oscillations as the photon pressure resists gravitational compression (Peebles-Yu 1970 [143]). Third, the plasma has a non-zero velocity at recombination which leads to Doppler shifts in the frequency and hence becomes brighter (Sunyaev-Zel'dovich 1970 [170]).

The **secondary sources** are generated by scattering along the line of sight. These are characterized by differential gravitational redshifts the Integrated Sachs-Wolfe effects (Sachs-Wolfe 1967 [152]). Additionally it was realized that diffusion damps out the intrinsic temperature fluctuations, meaning that the acoustic oscillation will be damped in strongly re-ionized models as well as during an era of slow decoupling (Kaiser 1984 [95]). In the re-ionized case other sources arising from fluctuations in the matter are important. These sources can yield strong contributions if last scattering occurs after

²See appendix C for the http addresses for some important CMB experiments

the matter has been released from Compton drag (Vishniac 1987 [183]). A good code, CMBFAST³ (Seljak-Zaldariagga 1996 [155]), is available for the calculation of linear-FRW temperature anisotropies. Software, HEALPix⁴, is available with which to create, display and statistically analyze discrete full sky-maps on a sphere, at a high angular resolution.

The **gauge-invariant perturbation** theory, in the canonical approach, uses the developments of Bardeen 1980 [6, 7] and Kodama-Sasaki 1984 [98] (building on the work of Lifshitz [104]).

The canonical approach to the CMB, the **Astrophysical cosmology** programme, is focussed on the sophisticated predictions arising from the linear-FRW models and the comparison of these to observations. Although these papers provide an almost exhaustive treatment of many of the issues involved in CMB physics it is limited to a perturbative analysis – see [194, 193, 110, 82, 83, 155, 198, 42, 93]. They also have the tendency to draw heavily from Newtonian analogues, place the emphasis on the use of coordinates and mode functions.

The approach of Sachs-Wolfe was that of integrating the photons down the null-cone. This was appropriate when the photons were either free-streaming or tightly-coupled to the matter. In the situation of interest, particularly for the primary sources, the photons are not perfectly coupled to the matter, nor is decoupling instantaneous. A kinetic theory description of the physics of temperature anisotropies, being more fundamental than the fluid approach, became necessary, particularly when dealing with re-ionization and other foreground physics. The line-of-sight code, CMBFAST, is a compromise between these two approaches and is only relevant to the linear regime. The Sachs-Wolfe approach forms a literature of its own⁵. It is however the kinetic theory approach that is of interest here – being physically more interesting.

The **Relativistic cosmology** approach to the CMB draws on the (i) covariant Lagrangian dynamics of Ehlers [48] and Ellis [50] (ii) the perturbation theory of Hawking [78] and Ellis-Bruni [52] (iii) The Relativistic Kinetic Theory of Ellis-Matras-Treciokas [58, 62] and Thorne [172, 173] and; (iv) COBE-Copernican approach of Maartens-Ellis-Stoeger [166, 117, 118]. By its very nature the 1+3 Lagrangian threading approach to CMB studies incorporates nonlinearity – it is exact, while its reduction to the linear theory, the 1+3 covariant and gauge invariant formalism, is complementary to the canonical one, particularly when reduced to a linear theory about a FRW background, the almost-FLRW theory. Here the emphasis is on manifest covariance – the equations are developed for any tetrad or coordinate choice.

This thesis is concerned primarily with the second tradition and subsequently with the contrasting of this with the prior. The idea is that they both have much to offer one another, particularly in better understanding the consistency of various approximations schemes used in astrophysical cosmology and ensuring that key non-perturbative effects have not been dropped in an ad-hoc fashion.

Lastly, the nature of the primordial perturbations is an issue of **Astrofunda-**

³To be found at <http://www.sns.ias.edu/matiasz/CMBFAST/cmbfast.html>

⁴From <http://www.tac.dk/healpix/>

⁵One important application of the test-field null-cone approach within relativistic cosmology has been that for Bianchi cosmologies, see for example [137]

mental physics, although there are of course overlaps between this and relativistic cosmology. I do not deal with those issues at all. However, much of the canonical linear-FRW programme (and hence the precision cosmology programme) is deeply tied into the inflationary paradigm, a central theme of Astrofundamental physics. This is essential inasmuch as any research into cosmology must at some time deal with initial conditions, first causes in some sense. None of these issues are dealt with here beyond the use of a primordial power spectrum for the perturbations; in fact we emphasize the use of the matter power spectrum now. We would like to emphasize the integration of the temperature anisotropies down the matter worldlines to last scattering and the integration of the matter variables forward to now; with the intention of comparing both today without *a priori* primordial assumptions. The primordial spectrum must come from some process, which must have some explanation, and that explanation must be sufficiently rich, robust and constrained to be able to generate the rich structures we see today. Physics is more a collection of calculations, explanations and observable histories woven together so as to create the illusion of a whole, than a formally complete and self-consistent mathematical description of reality that can be developed bottom up in a seamless manner. I think it is naive, though not impossible, to expect one to be able to find a formally complete description of the universe about us, when there are no examples of complete theories. But that should not stop us from asking questions about the nature of things under the assumption that such a description is possible.

“ ... if you really feel, Look, its too hard to deal with real problems, there are lots of ways to avoid doing so. One of them is to go off on wild goose chases that don't matter. Another is to get involved in academic cults that are very divorced from any reality and that provide a defense against dealing with the world as it really is. There is plenty of that going around ... ”

– Noam Chomsky 1994

Chapter 1

Algebraic relations

Algebraic relations for mode and multipole representations : *The approach represents radiation anisotropies by projected symmetric and trace-free tensors, $\tau(x, e) = \sum_{\ell > 1} \tau_{A_\ell} O^{A_\ell}$. Given that we use the Gaussian assumption, the angular correlation functions for the mode coefficients are found in terms of multipole coefficients, following the Wilson-Silk approach for the mode representation, but derived and dealt with in 1+3 covariant and gauge invariant (CGI) form :*

$$C_\ell = \frac{2}{\pi} \frac{\beta_\ell^2}{(2\ell + 1)^2} \int_0^\infty \frac{dk}{k} k^3 |\tau_\ell(k, \eta)|^2 \quad \text{and} \quad \langle \tau_{A_\ell} \tau^{A_\ell} \rangle = (2\ell + 1) \Delta_\ell^{-1} C_\ell.$$

The covariant multipole and mode-expanded angular correlation functions are related to the usual treatments in the literature. The CGI mode expansion, $\tau(x, e) \approx \sum_{\ell > 1, k_a} \tau_\ell(x) G_\ell[Q]$ with $G_\ell[Q] = O^{A_\ell} Q_{A_\ell}$ (where $\tau_{A_\ell}(x) = \sum_{k^a} \tau_\ell(k, t) Q_{A_\ell}(x, k)$) is related to the coordinate approach by linking the Legendre functions to the projected symmetric trace-free representation, using a covariant addition theorem, $O_{A_\ell} O'^{A_\ell} = \beta_\ell P(e^a e'_a)$, for the tensors to generate the Legendre Polynomial recursion relation.

1.1 Introduction

Ellis, Treciokas and Matravers (ETM) [58, 62] introduced a covariant kinetic theory formalism in which an irreducible representation of the rotation group based on Projected Symmetric and Trace-Free (PSTF) tensors, orthogonal to a physically defined 4-velocity u^a , gives a covariant representation of the Cosmic Background Radiation (CBR) anisotropies, which is gauge-invariant when the geometry is an almost-Robertson Walker (RW) geometry. This Covariant and Gauge-Invariant (CGI) formalism has been used in a series of papers [166, 117, 118, 167] to look at the local generation of CBR anisotropies by matter and spacetime inhomogeneities and anisotropies in an almost-Friedmann Lemaitre (FL) universe model ¹. By contrast, the present work ² uses this formalism to investigate

¹Here 'Robertson-Walker' refers to the geometry, whatever the field equations; 'Friedman-Lemaitre' assumes that the Einstein gravitational field equations with a perfect fluid matter source are imposed on such a geometry.

²The series of papers developed by Gebbie, Ellis, Dunsby and Maartens [66, 120, 167, 67, 40, 68]; a series by Challinor and Lasenby [27, 28] is similar in method at the linear level, but different in detail and focus.

CBR anisotropies in the non-local context of emission of radiation near the surface of last scattering in the early universe and its reception here and now (the Sachs-Wolfe (SW) effect and its further developments).

There is of course a vast literature investigating these anisotropies both from a photon viewpoint, developing further the methods of the original Sachs-Wolfe paper [152], and from a kinetic theory viewpoint, so it is useful to comment on why the covariant and gauge invariant philosophy and programme [78, 52] make the present work worthwhile. Rather than beginning with a background described in particular coordinates and perturbing away from this background, this approach centres on covariantly defined geometric quantities, and develops exact nonlinear equations for their evolution. These equations are then systematically linearized about a Friedmann-Lemaître (FL) background universe with a Robertson-Walker (RW) geometry resulting in description by gauge-invariant variables and equations [52]. Because the definitions and equations used are coordinate-independent, one can adopt any suitable coordinate or tetrad system to specialize the tensor equations to specific circumstances when carrying out detailed calculations; a harmonic or mode analysis can be carried out at that stage, if desired.

This approach is arguably (see [27, 28]) geometrically transparent because of the CGI variable definitions used. In contrast to the various gauge dependent approaches to perturbations in cosmology (see for example [6, 7, 19, 163]), the differential equations are of just the right order needed to describe the true physical degrees of freedom; no non-physical gauge modes occur. In many cases the variables used are gauge-dependent, so non-physical gauge modes result from differential equations that are of higher order than is needed to describe the true physical degrees of freedom. Some coordinate-dependent gauge-invariant approaches have been used to analyze density contrasts and the CBR anisotropy, notably Bardeen's non-local GI approach [6]; the present variables provide a description that is equivalent when a harmonic decomposition is introduced in the linear case [20], but have a more transparent meaning than those used in that approach, and do not imply linearization of the equations from the outset, as occurs in that formalism.

Thus the advantage of the present formalism is precisely its 1+3 covariant and gauge invariant nature, together with the fact that we are able to write down the exact non-linear equations governing the growth of structure and the propagation of the radiation, and then linearize them in a transparent way in an almost-RW situation – the formal basis of both the recovery of linear-FRW equations (based on a 3+1 foliation as opposed to the almost-FLRW theory) and the Newtonian like equations (using the Newtonian like u^a -frame threading). This means it can be extended to non-linear analyses in a straightforward way, which will be essential in developing the theory of finer CBR anisotropy structure as reliable small-angle observations become available. Some of the successes of the covariant and gauge invariant approach with respect to the CMB are the almost-EGS Theorem [166], related model-independent limits on inhomogeneity and isotropy [117, 118, 119, 167], and derivation of exact anisotropic solutions of the Liouville equation in a RW geometry ([132], see also [59]).

This chapter deals with algebraic issues, developing further the formalism of ETM: namely an irreducible representation of radiation anisotropies based on PSTF tensors [172, 147]. It considers this irreducible representation and its relation to observable quantities, both generically and in the context of models linearized about RW geome-

tries [166]. In section 1.2 and 1.3, the underlying 1+3 decomposition is outlined and the basic CGI harmonic formalism for anisotropies developed. The angular correlation functions are constructed from 1 + 3 Lagrangian threading variables which make the results relevant both to the exact variables (described in chapter 2) and the CGI variables (described in detail in chapter 5), assuming that the multipole coefficients are generated by superpositions of homogeneous and isotropic Gaussian random fields. The multipole expansion is discussed in detail, extending the results of ETM, giving the construction of the multipole coefficient mean-square and developing its link to the angular correlation function. In section 1.4, the mode coefficients are found following the Wilson-Silk approach, but derived and dealt with in the CGI form; the covariant and gauge invariant multipole and mode expanded angular correlation functions are related to the usual treatments used in the literature [194, 70, 82, 83, 45]. In this discussion, the covariant and gauge invariant mode expansion is related to the coordinate approach by linking the Legendre Tensors to the PSTF representation, using a covariant addition theorem to generate the Legendre Polynomial recursion relation. The key result is the construction of the angular correlation functions in the covariant and gauge invariant variables, and their link to the (non-local) GI Mode functions [70].

The following chapters look at the Boltzmann equation and multipole divergence relations, solution of the resulting mode equations, and relation of the kinetic theory approach to the photon based formalism of the original Sachs-Wolfe paper. Exact non-linear equations are obtained and then linearized, allowing a transparent linearization process from the non-linear equations that is free from ambiguities and gauge modes.

1.2 Temperature anisotropies

A radiation temperature measurement is associated with an antenna temperature, $T(x, e)$, measured by an observer moving with 4-velocity u^a at position x^i in a direction e^a on the unit sphere ($e^a e_a = 1$, $e^a u_a = 0$). We assume u^a can be uniquely define in the cosmological situation, corresponding to the motion of ‘fundamental observers’ in cosmology [50]³. The direction e^a can be given in terms of an orthonormal tetrad frame⁴, for example by :

$$e^a(\theta, \phi) = (0, \sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta). \quad (1.1)$$

The temperature $T(x, e)$ can be unambiguously decomposed into the all-sky average bolometric temperature⁵ $T(x)$ at position x^i , given by

$$T(x) = \frac{1}{4\pi} \int_{4\pi} T(x, e) d\Omega. \quad (1.2)$$

³In the early universe, when matter and radiation average velocities differ, there may be several competing possibilities for covariant definition of u^a ; however once a choice has been made between these possibilities, this vector field is uniquely defined.

⁴The description of such a tetrad frame is briefly discussed in appendix, A.4

⁵Note that this is *not* the same as the background temperature, for that quantity varies only with cosmic time t , whereas the true isotropic component of the temperature varies with spatial position as well as time.

where Ω is the solid angle on the sky, and the anisotropic **temperature perturbation** $\delta T(x, e)$ (the difference from the average over the unit sphere surrounding x^i [117]), can be defined :

$$T(x, e) = T(x) + \delta T(x, e). \quad (1.3)$$

From the Stefan-Boltzmann law it follows – if the radiation is almost black-body, which we assume – that the radiation energy density is given in terms of the average bolometric temperature by $\rho_R(x) = k_B T^4(x)$. (k_B is Stefan-Boltzmann constant). Both the quantities $T(x)$ and $\delta T(x, e)$ are covariant and gauge invariant, for $T(x)$ is defined in a physically unique frame in the real universe (because u^a is assumed to be uniquely defined), and $\delta T(x, e)$ vanishes in any background without temperature anisotropies.

We can define the fractional temperature variation $\tau(x, e)$ by [166, 117]

$$\tau(x, e) := \frac{\delta T(x, e)}{T(x)}, \quad (1.4)$$

and take a covariant (angular) harmonic expansion of this,

$$\tau(x, e) = \sum_{\ell=1}^{\infty} \tau_{a_1 a_2 a_3 \dots a_\ell}(x) e^{a_1} e^{a_2} e^{a_3} \dots e^{a_{\ell-1}} e^{a_\ell} \equiv \sum_{\ell \geq 1} \tau_{A_\ell} \hat{e}^{A_\ell}. \quad (1.5)$$

We introduce the shorthand notation using the compound index $A_\ell = a_1 a_2 \dots a_\ell$. Here $\tau_{a_1 a_2 a_3 \dots a_\ell}(x)$ are trace-free symmetric tensors orthogonal to u^a :

$$\tau_{A_\ell} = \tau_{(A_\ell)}, \quad \tau_{A_\ell ab} h^{ab} = 0, \quad \tau_{A_\ell a} u^a. \quad (1.6)$$

Round brackets “(.)” denote the symmetric part of a set of indices, angle brackets “⟨.⟩” the (orthogonally-) Projected Symmetric Trace-Free (PSTF) part of the indices : $\tau_{A_\ell} = \tau_{(A_\ell)}$.

Because of (1.1), this expansion is entirely equivalent to a more usual expansion in terms of spherical harmonics:

$$\tau(x, e) = \sum_{l=1}^{\infty} A_l^m(x) Y_l^m(\theta, \phi) \quad (1.7)$$

(see [58] for details), but is more closely related to a tensor description, and so results in more transparent relations to physical quantities.

We wish to measure the temperature in two different directions to find the temperature difference associated with the directions e^a and e'^a such that (using 1.2) :

$$\Delta T(x; e, e') = T(x, e) - T(x, e') \Rightarrow \Delta T(x; e, e') = \delta T(x, e) - \delta T(x, e'). \quad (1.8)$$

It follows from (1.8), (1.4) and (1.5) that $\Delta T(x; e, e') = T(x) \sum_{\ell} \tau_{A_\ell} (e^{A_\ell} - e'^{A_\ell})$, where $\Delta T/T$ represents the *real fractional temperature difference* on the current sky. Due to the covariant and gauge invariant nature of $T(x)$ we may relate this directly to the *real temperature perturbations* (no background model is involved in these definitions).

The relation between the two directions e^a and e'^a at x^i is characterized by

$$e^a e'_a = \cos(\beta) =: X. \quad (1.9)$$

i.e. they are an angular distance β apart ⁶. If analogous to (1.1) we write $e^a(\theta', \phi') \equiv e'^a = (0, \sin \theta' \sin \phi', \sin \theta' \cos \phi', \cos \theta')$, then it can be shown from (1.9) that

$$\cos \beta = \sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta'. \quad (1.10)$$

In later applications, it is important to relate the different terms of the harmonic expansion to angular scales on the observers sky. A useful approximation is $l \approx \frac{1}{\theta}$, where θ is in radians.

1.2.1 The angular correlation function

The two-point correlations are an indication of the fraction of temperature measurements, $T(x, e)$, that are the same for a given angular separation. This corresponds to the correlation between $\delta T(x, e)$ and $\delta T(x, e')$ or equivalently between $\tau(x, e)$ and $\tau(x, e')$, given by the **angular position correlation function**

$$C(e, e') = \langle \tau(x, e), \tau(x, e') \rangle, \quad (1.11)$$

where the angular brackets representing an angular average over the complete sky. Note this is a function in the sky. If we write $\tau(x, e)$ and $\tau(x, e')$ in terms of the angular harmonic expansion (1.5), we can also define correlation functions C_ℓ for the anisotropy coefficients $\tau_{A_\ell}(x, e), \tau_{A_\ell}(x, e')$ by

$$C_\ell(e, e') = (2\ell + 1)^{-1} \Delta_\ell \langle \tau_{A_\ell}(x, e) \tau^{A_\ell}(x, e') \rangle. \quad (1.12)$$

Here the right-hand side term in brackets is the all-sky mean-square value of the ℓ -th temperature coefficient $\tau_{A_\ell}(x, e)$, and the coefficient Δ_ℓ is defined in (A.19). The numerical factor $(2\ell + 1)^{-1} \Delta_\ell$ is included in order to agree with definitions normally used in the literature (see later). This can be thought of as the momentum space version of (1.11), as we have taken an angular Fourier series of the quantities in that equation; it says, for each choice of e^a, e'^a , how much power there is in that expression for that angular separation as contributed by a particular ℓ -th valued multipole moment on average.

1.2.2 The Central-Limit Theorem

We consider an ensemble of temperature anisotropies, where a sequence of repeated trials is replaced by a complete ensemble of outcomes. The temperature anisotropy, τ_{A_ℓ} ⁷, found in a given member of the ensemble is a realization of the statistical process represented by the ensemble.

The physically measure anisotropy is taken to be one such realization. The variance of the ensemble, for example $\langle \tau_{A_\ell} \tau^{A_\ell} \rangle$, is in principle found by averaging over a sufficiently large number of experiments, where we assume the results will approach the true ensemble variance – this is the assumption of ergodicity.

⁶It should be pointed out that e'^a is distinct from e^a , the first denotes a direction vector different from e^a in a given tetrad frame, while the second means the same direction vector in a different tetrad frame.

⁷We could use Π_{A_ℓ} here, i.e. the generalized anisotropy (see chapter 2). However, we have in mind the situation where the anisotropies are small “here-and-now”.

On Fourier transforming, we make the assumption that to a good approximation the phases of the various multipole moments are uncorrelated and random. This corresponds to treating the anisotropies as a form of random noise. The random phase assumption has a useful consequence: the sum of a large number of independent random variables will tend to be normally distributed. By the central-limit theorem (Kendall-Stuart), this is true for all quantities that are derived from linear sums over waves. The end result is that one ends up with a Gaussian Random Field (GRF) which is characterized by a power spectrum. The central limit theorem holds as long as there exists a finite second moment, *i.e.* a finite variance.

We will assume that the angular variance $\langle \tau_{A_\ell} \tau^{A_\ell} \rangle$ is independent of position; this is the assumption of statistical homogeneity. Its plausibility lies in the underlying use of the weak Copernican assumption. Additionally (in the context of almost-FLRW models) it is convenient to assume that the power spectrum will have no directional dependence, that it will be isotropic: $P(k^a) = P(|k^a|)$. Together these imply the statistical distribution respects the symmetries of the RW background geometry.

One needs to be careful in using the central limit theorem to motivate Gaussian Random Fields, particularly in the presence of nonlinearity, which could result in the elements of the ensemble no longer being independent. While the assumption of primordial homogeneous and isotropic GRF's is plausible, because the perturbations are made up of a sufficiently large number of independent random variables, the key point is to realize that these are assumptions that should be tested if possible [72, 125, 156, 109, 63].

The simplest possible test of weak non-Gaussianity is looking for a three-point angular or spatial correlation, for the Gaussian assumption ensures that all the odd higher moments are zero and the even ones can be expressed in terms of the variance alone. If the primordial perturbations are made up of GRF's, then non-Gaussianity of the CBR anisotropy spectrum should arise primarily from foreground contamination due to local physical processes. If the non-Gaussian effects due to these later physical processes or evolutionary effects are small enough, one can attempt to determine a cosmological primordial signature.

1.2.3 Gaussian perturbations

A general *Gaussian perturbation* [109, 4], $\tau(x, e)$, will be a superposition of functions, τ_{A_ℓ} , *i.e.* (1.5) is satisfied, where the probability, P , of finding a particular valued temperature coefficient is given by $(\sigma_\ell^2 = \langle \tau_{A_\ell} \tau^{A_\ell} \rangle)$:

$$P(\tau_{A_\ell}) = \frac{1}{\sqrt{2\pi\sigma_\ell^2}} \exp \left\{ \frac{-\tau_{A_\ell} \tau^{A_\ell}}{2\sigma_\ell^2} \right\}. \quad (1.13)$$

Note that τ_{A_ℓ} is both the amplitude of the ℓ -th component, and determines the probability of that amplitude. The probability of a temperature perturbation, τ , is given by the sum of the Gaussian probability distributions (1.13) weighting the various angular scales, given by ℓ , of the general perturbation (1.5).

Considering isotropic (and homogeneous in the context of linear-FRW models) Gaussian random fields, the angular position correlation function $C(e, e')$ is a function

only of the angular separation β of the two temperature measurements. We then write (1.11) as

$$\langle \tau \cdot \tau' \rangle = C(\beta) = W(X), \quad (1.14)$$

where the expression on the left is shorthand for $\langle \tau(x, e), \tau(x, e') \rangle_\beta$, the **2-point angular correlation function** for a given angular separation β between the on-sky temperature measurements, and $X = e^a e'_a = \cos \beta$. This expression is now independent of position in the sky. Gaussian Fields are completely specified by the angular power spectrum coefficients C_ℓ (1.12), which are now just constants, because ℓ is uniquely related to β , so the power spectrum is a function of the modulus of the wavenumber only. One thus expects the temperature perturbations in this case to be fully specified by the **mean squares**, $\langle \tau_{A_\ell} \tau^{A_\ell} \rangle$, when (1.5) is substituted in (1.14). Equivalently they are uniquely determined by the angular Fourier transform of the 2-point angular correlation function.

1.3 Multipole expansions

In this section we examine the anisotropy properties of radiation described in terms of the covariant multipole formalism (1.5), which is equivalent to the usual angular harmonic formalism but much more directly related to space-time tensors. Note that the relations in this section hold at any *point* in the space-time, and in particular at the event x_0 (“here-and-now”) where observations take place. Here we consider the PSTF part of, e^{A_ℓ} ; some useful properties of e^{A_ℓ} are listed in appendix A.2.

1.3.1 The PSTF part of e^{A_ℓ}

Because the coefficients in (1.5) are symmetric and trace-free, the important directional quantities defined by directions e^a at a position x^i are the PSTF quantities

$$O^{A_\ell} = e^{A_\ell} = e^{a_1 a_2 a_3 \dots a_{\ell-1} a_\ell}, \quad (1.15)$$

for clearly

$$\tau(x, e) = \sum_{\ell=1}^{\infty} \tau_{A_\ell}(x) e^{A_\ell}(\theta, \phi) = \sum_{\ell=1}^{\infty} \tau_{A_\ell}(x) O^{A_\ell}(\theta, \phi). \quad (1.16)$$

Indeed the standard spherical harmonic properties are contained in these quantities.

Now the symmetric trace-free (STF) part of a 3-tensor is given in general by Pirani [147],

$$F_{\langle A_\ell \rangle} = \sum_{n=0}^{[\ell/2]} B^{\ell n} h_{(a_1 a_2 \dots a_{2n} a_{2n+1} \dots a_\ell)} F_{a_{2n+1} \dots a_\ell} \quad \text{with} \quad B_{\ell n} = \frac{(-1)^n \ell! (2\ell - 2n - 1)!!}{(\ell - 2n)! (2\ell - 1)!! (2n)!!} \quad (1.17)$$

Here $[\ell/2]$ means the largest integer part less than or equal to $\ell/2$. The following definitions have also been used : $\ell! = \ell(\ell - 1)(\ell - 2)(\ell - 3)\dots(1)$, and $\ell!! = \ell(\ell - 2)(\ell - 4)(\ell - 6)\dots(2 \text{ or } 1)$.

We take the PSTF part of $e^{(A_\ell)}$ [147, 172, 173, 58] to find

$$O^{A_\ell} = e^{(A_\ell)} = \sum_{k=0}^{[\ell/2]} B_{\ell k} h^{(A_{2k} e^{A_{\ell-2k}})}, \quad (1.18)$$

where $h^{(A_{2k} e^{A_{\ell-2k}})} \equiv h^{(a_1 a_2 \dots h^{a_{2k-1} a_{2k}} e^{a_{2k+1} e^{a_{2k+2}} \dots e^{a_\ell})}$ and $B_{\ell k}$ are given by (1.17).

From [58] we can now construct recursion relations that play a key role later on. First,

$$O^{(A_\ell e^{a_{\ell+1}})} = O^{(A_\ell e^{a_{\ell+1}})} - \frac{\ell}{2\ell+1} e^a h_{ad} O^{d(A_{\ell-1} h^{a_\ell a_{\ell+1}})}. \quad (1.19)$$

From (1.19), (1.15), and using

$$e_{a_1} O^{A_\ell} = \frac{\ell}{(2\ell-1)} O^{A_{\ell-1}}, \quad (1.20)$$

it can then be shown that

$$O^{A_{\ell+1}} = e^{(a_{\ell+1} O^{A_\ell})} - \frac{\ell^2}{(2\ell+1)(2\ell-1)} h^{(a_{\ell+1} a_\ell O^{A_{\ell-1}})} \quad (1.21)$$

relates the $(\ell+1)$ -th term to the ℓ -th term and the $(\ell-1)$ -th term.

The orthogonality, addition theorem and double integral relations of O^{A_ℓ} are listed in appendix A.2. Using the orthogonality relations we obtain the inversion of the harmonic expansion :

$$\tau(x, e) = \sum_{\ell=0}^{\infty} \tau_{A_\ell}(x) O^{A_\ell} \Leftrightarrow \tau_{A_\ell}(x) = \Delta_\ell^{-1} \int_{4\pi} d\Omega O_{A_\ell} \tau(x, e). \quad (1.22)$$

The polynomial $L_\ell \equiv O^{A_\ell} O'_{A_\ell} = \sum_{m=0}^{[\ell/2]} B_{\ell m} X^{\ell-2m}$, (see (A.24)) is the natural polynomial that arises in the PSTF tensor approach (equivalent to the Legendre polynomials, see below), where the coefficients $B_{\ell m}$ are defined by (1.17). It follows from this that

$$L_\ell(1) = O^{A_\ell} O_{A_\ell} = \sum_{m=0}^{[\ell/2]} B_{\ell m} =: \beta_\ell \quad \text{with} \quad \beta_\ell = \left(\frac{(\ell!)^2 2^\ell}{(2\ell)!} \right) = \frac{\ell!}{(2\ell-1)!!}. \quad (1.23)$$

The β_ℓ 's satisfy the recursive relations

$$\beta_\ell = \frac{(2\ell+1)}{(\ell+1)} \beta_{\ell+1}, \quad \beta_\ell = \frac{\ell}{(2\ell-1)} \beta_{\ell-1}, \quad \frac{\beta_{\ell+1}}{\beta_{\ell-1}} = \frac{\ell(\ell+1)}{(2\ell+1)(2\ell-1)}. \quad (1.24)$$

Any function $W(X)$ can be expanded in terms of the polynomials $L_\ell(X)$ ⁸ and then upon combining (A.24) and expansion in terms of $L_\ell(X)$ one finds the expansion in terms of O^{A_ℓ} :

$$W(X) = \sum_{\ell=1}^{\infty} \hat{C}_\ell L_\ell(X) \quad \text{and} \quad W(X) = \sum_{n=1}^{\infty} \hat{C}_n O^{A_n} O'_{A_n}. \quad (1.25)$$

⁸We include the dipole, $\ell=1$, however the monopole is dropped as we will only be considering the expansion of the covariant and gauge invariant perturbations (1.5) where the isotropic part has been factored out according to (1.4). We must beware that this does not cause problems later by omitting the spatial gradients of the isotropic term.

When $W(X)$ is the angular correlation function, the \hat{C}_ℓ are the corresponding angular power spectrum coefficients (see below).

Relationship to Legendre polynomials

A Legendre polynomial $P_\ell(X)$ is given by re-normalizing the polynomials $L_\ell(X)$ defined in (A.24) so that $P_\ell(1) = 1$. By (1.23), this implies

$$P_\ell(X) = (\beta_\ell)^{-1} L_\ell(X) \Rightarrow P_\ell(1) = 1; \quad (1.26)$$

consequently from (A.24),

$$O^{A_\ell} O'_{A_\ell} = \beta_\ell P_\ell(X), \quad (1.27)$$

where β_ℓ are given by (1.23). It follows from (1.27) that

$$P_\ell(X) = \sum_{m=0}^{[\ell/2]} A_{\ell m} X^{\ell-2m}, \quad \text{with } A_{\ell k} = \frac{(-1)^k (2\ell - 2k)!}{2^\ell k! (\ell - k)! (\ell - 2k)!} \quad (1.28)$$

are related to the $B_{\ell k}$ in (A.23) by

$$B_{\ell k} = \left(\frac{(\ell!)^2 2^\ell}{(2\ell)!} \right) A_{\ell k} \quad (1.29)$$

Any function $W(X)$ can be expanded in terms of both sets of polynomials - see (1.25) and the corresponding expression

$$W(X) = \sum_{\ell=1}^{\infty} \frac{2\ell + 1}{4\pi} C_\ell P_\ell(X). \quad (1.30)$$

These two expansions can then be related as follows: equating (1.25) and (1.30), and using (1.28) and (1.18), gives

$$\sum_{\ell=1}^{\infty} \sum_{m=0}^{[\ell/2]} \hat{C}_\ell B_{\ell m} X^{\ell-2m} = \sum_{\ell=1}^{\infty} \sum_{m=0}^{[\ell/2]} \frac{(2\ell + 1)}{4\pi} C_\ell A_{\ell m} X^{\ell-2m} \Rightarrow \hat{C}_\ell B_{\ell m} = \frac{(2\ell + 1)}{4\pi} C_\ell A_{\ell m}, \quad (1.31)$$

from (1.29) this gives the relation between the expansion coefficients :

$$\hat{C}_\ell = \left[\frac{(2\ell + 1)(2\ell)!}{4\pi 2^\ell (\ell!)^2} \right] C_\ell = \Delta_\ell^{-1} C_\ell. \quad (1.32)$$

1.3.2 The mean square of PSTF coefficients : $\langle F_{A_\ell} F^{A_\ell} \rangle$

It is known as before, from evaluating $\int d\Omega f^2$ and constructing the orthogonality conditions on O_{A_ℓ} , that inversion

$$F_{A_\ell} = \Delta_\ell^{-1} \int_{4\pi} d\Omega O_{A_\ell} f(x, e) \Leftrightarrow f(x, e) = \sum_{\ell=0}^{\infty} F_{A_\ell} O^{A_\ell}, \quad (1.33)$$

can be constructed. From this we can build

$$F_{A_t} F^{B_m} = \Delta_\ell^{-1} \Delta_m^{-1} \int_{4\pi} d\Omega O_{A_t} f(x, e) \int_{4\pi} d\Omega' O'^{B_m} f(x, e'), \quad (1.34)$$

to find

$$F_{A_t} F^{B_m} = \Delta_\ell^{-1} \Delta_m^{-1} \int \int d\Omega d\Omega' O_{A_t} O'^{B_m} f(x, e) f(x, e'). \quad (1.35)$$

Taking the ensemble average [109, 193] then gives

$$\langle F_{A_t} F^{B_m} \rangle = \Delta_\ell^{-1} \Delta_m^{-1} \int \int d\Omega d\Omega' O_{A_t} O'^{B_m} \langle f \cdot f' \rangle \quad (1.36)$$

where $\langle \dots \rangle$ indicates an ensemble average over sufficiently many realizations of the angular correlation function.

In order to evaluate this further we assume that the correlations between the function $f(x, e)$, *i.e.*, $\langle f(x, e) f(x, e') \rangle$ are a function of the angular separation between the two directions only,

$$\langle f \cdot f' \rangle = W(e^a e'_a) = W(X). \quad (1.37)$$

This is a consequence of the Gaussian assumption (1.14), which allows one to evaluate the angular correlation functions (1.11) in a straightforward way⁹. With this assumption (1.36) becomes

$$\langle F_{A_t} F^{B_m} \rangle = \Delta_\ell^{-1} \Delta_m^{-1} \int \int d\Omega d\Omega' O_{A_t} O'^{B_m} W(X). \quad (1.38)$$

Substituting (1.25) into (1.38), we find

$$\langle F_{A_t} F^{B_m} \rangle = \Delta_\ell^{-1} \Delta_m^{-1} \int \int d\Omega d\Omega' O_{A_t} O'^{B_m} \sum_{n=1}^{\infty} \hat{C}_n O^{A_n} O'^{A_n}. \quad (1.39)$$

Rearranging terms,

$$\langle F_{A_t} F^{B_m} \rangle = \Delta_\ell^{-1} \Delta_m^{-1} \sum_{n=1}^{\infty} \hat{C}_n \left\{ \int_{4\pi} d\Omega O_{A_t} O^{A_n} \right\} \left\{ \int_{4\pi} d\Omega' O'^{B_m} O'^{A_n} \right\}, \quad (1.40)$$

where the integrals can be evaluated using the orthogonality conditions on the O_{A_t} 's, (A.19),

$$\begin{aligned} \langle F_{A_t} F^{B_m} \rangle &= \Delta_\ell^{-1} \Delta_m^{-1} \sum_{n=1}^{\infty} \hat{C}_n \left\{ \delta^{\ell n} \Delta_\ell h_{\langle A_t \rangle}^{\langle A_n \rangle} \right\} \left\{ \delta^{mn} \Delta_m^{-1} h_{\langle A_n \rangle}^{\langle B_m \rangle} \right\} \\ &= \sum_{n=1}^{\infty} \hat{C}_n \delta^{\ell n} \delta^{mn} h_{\langle A_t \rangle}^{\langle A_n \rangle} h_{\langle A_n \rangle}^{\langle B_m \rangle}. \end{aligned} \quad (1.41)$$

⁹Ideally one would prefer to evaluate the angular correlation function without using the Gaussian assumption, as this is one of the features one should test rather than assume.

Thus

$$\langle F_{A_\ell} F^{B_m} \rangle = \hat{C}_\ell \delta^{\ell m} h_{\langle A_\ell \rangle}^{(B_m)} \quad \text{to find} \quad \langle F_{A_\ell} F^{A_\ell} \rangle = \hat{C}_\ell h_{\langle A_\ell \rangle}^{(A_\ell)}, \quad (1.42)$$

so using $h_{\langle A_\ell \rangle}^{(A_\ell)}$ (A.21) the mean-square is found to be

$$\langle F_{A_\ell} F^{A_\ell} \rangle = \hat{C}_\ell (2\ell + 1) \quad (1.43)$$

giving the angular power spectrum coefficients \hat{C}_ℓ in terms of the ensemble-averages of the harmonic coefficients¹⁰. If we use the Legendre expansion (1.30) instead of the covariant expansion coefficients (1.25), then from (1.43) and (1.32) the relation is

$$\langle F_{A_\ell} F^{A_\ell} \rangle = (2\ell + 1) \Delta_\ell^{-1} C_\ell, \quad (1.44)$$

where C_ℓ are the usual Legendre angular power spectrum coefficients.

1.3.3 The covariant and gauge invariant angular correlation function

We can now gather the results above in terms of the application we have in mind, namely anisotropy of the CBR. Consider on-sky perturbations made of Gaussian Random Fields: The **angular correlation function** $C(\beta) = W(X)$ is given by (1.14); the **angular power spectrum coefficients** C_ℓ are given by the mean-square of the ℓ -th temperature coefficient through (1.44):

$$\langle \tau_{A_\ell} \tau^{A_\ell} \rangle = (2\ell + 1) \Delta_\ell^{-1} C_\ell \quad (1.45)$$

where the constants Δ_ℓ are given by (A.19). These quantities are related by (1.30):

$$W(X) = \sum_{m=1}^{\infty} C_m \frac{(2m+1)}{4\pi} P_m(X) \quad (1.46)$$

where the $P_m(X)$ are given by (1.26) from (1.30) and (1.23).

Cosmic variance

The observations are in fact of a_ℓ^2 (which is $\sum_{m=-\ell}^{+\ell} |a_{\ell m}|^2 / 4\pi$ in the usual notation). This is what is effectively found from experiments, such as the COBE-DMR experiment. This is a *single* realization of the angular power spectrum C_ℓ . The finite sampling of events generated by random processes (in this case GRF's) leads to an intrinsic uncertainty in the variance even in exactly perfect experiments - this is **sample variance**, or in the cosmological setting, **cosmic variance**. We are measuring a single realization of a process that is assumed to be random; there is an error associated with how we fit the single realization to the *averaged* angular power spectrum.

The quantities $\langle \tau_{A_\ell} \tau^{A_\ell} \rangle$ represent the averaged (over the entire ensemble of possible C_ℓ 's) angular power spectrum, this is what one is in fact dealing with in the theory as the reductions are done in terms of Gaussian Random Fields, where the entire ensemble is considered, not a single experimental realization. The a_ℓ^2 are a sum of the $2\ell + 1$

¹⁰This corrects an error in [167], removing a spurious factor of $3^\ell = h_{A_\ell}^{A_\ell}$ which follows from the orthogonality conditions in [58] which are corrected here.

Gaussian Random Variables $a_{\ell m}$, this is taken to be χ^2 distributed with $2\ell + 1$ degrees of freedom. Each multipole has $2\ell + 1$ samples¹¹.

The key point here: cosmic variance is proportional to $\ell^{-1/2}$ and is then less significant for smaller angular scales than larger scales. Although cosmic variance is not an issue on small scales, systematic error could be underestimated [93]. Physical process deviations and instrument noise are expected to dominate the small scales rather than non-Gaussian effects in the primordial perturbations (small deviations from Gaussianity), but on large scales the uncertainty due to cosmic variance would swamp out a non-Gaussian signature. It then seems plausible that on large and small scales the assumption of Gaussian Perturbations is acceptable; however on intermediate scales this is not the case, on these scales the effects of cosmic variance would be small enough to allow a non-Gaussian signature to be apparent.

1.4 Mode expansions

We now consider spatial harmonic analysis of the angular coefficients discussed in the previous section. Note that the relations in this section hold in space-like *surfaces*, namely the background space-like surfaces in an almost-FL model. The application in the following sections will be to the projection into these spacelike surfaces of null cone coordinates associated with the propagation of the CBR down the null cone.

Following the Wilson-Silk approach [193, 70, 82] we consider the following CGI expansions. Eigenfunctions $Q(x)$ are chosen to satisfy the Helmholtz equation

$$D^a D_a Q = -k_{\text{phys}}^2 Q \quad (1.47)$$

in the (background) space sections of the given space-time of interest, where the Q 's are time-independent scalar functions with the physical wavenumber $k_{\text{phys}}(t) = k/a(t)$, the wave number k being independent of time¹². These define tensors $Q_{A_\ell}(k^\nu, x)$ that are Projected, Symmetric, and Trace-Free, and in the case of scalar perturbations are chosen to be given by PSTF covariant derivatives of the eigenfunctions Q :

$$Q_{A_\ell} = (-k_{\text{phys}})^{-\ell} D_{(A_\ell)} Q \quad (1.48)$$

Using these we define functions of direction and position:

$$G_\ell[Q](x^\nu, e) = O^{A_\ell} Q_{A_\ell}. \quad (1.49)$$

with the O^{A_ℓ} defined by (1.15). Here the G_ℓ are called **mode operators** and the objects $G_\ell[Q]$ are called **mode functions**¹³. It follows that

$$G_\ell[Q] = (-k_{\text{phys}})^{-\ell} O^{A_\ell} D_{(A_\ell)} Q \quad (1.50)$$

¹¹The uncertainty in C_ℓ as $\frac{\Delta C_\ell}{C_\ell} = \sqrt{\frac{2}{2\ell+1}}$

¹²The function Q will be associated with a direction vector $e_{(k)}^a$ and wave vector $k_a = k e_a^{(k)}$ normal to the surfaces $Q = \text{const}$, see the following subsection.

¹³Note these are functions in phase space, not on M .

and we can expand a given function $f(x, e)$ in terms of these functions. In our case this serves as a way of harmonically analyzing the coefficients $\tau_{A_\ell}(x)$ in (1.5) and (1.16): expanding the temperature anisotropy in terms of the mode functions,

$$\tau(x, e) = \sum_{\ell} \sum_k \tau_{\ell}(t, k) G_{\ell}[Q] \quad (1.51)$$

where the ℓ -summation is the angular harmonic expansion and the k -summation the spatial harmonic expansion (in fact k will be a 3-vector because space is 3-dimensional, see below). Using the expansion (1.5) on the left and (1.50) on the right,

$$\sum_{\ell} \tau_{A_\ell}(x) O^{A_\ell} = \sum_{\ell} \sum_k \tau_{\ell}(t, k) (-k_{\text{phys}})^{-\ell} O^{A_\ell} D_{(A_\ell)} Q \quad (1.52)$$

and so

$$\tau_{A_\ell}(x) = \sum_k \tau_{\ell}(t, k) (-k_{\text{phys}})^{-\ell} D_{(A_\ell)} Q(x^\nu) \quad (1.53)$$

which is the spatial harmonic expansion of the radiation anisotropy coefficients in terms of the symmetric, trace-free spatial derivatives of the harmonic function Q . The quantities $\tau_{\ell}(t, k)$ are the corresponding mode coefficients. Note that we have not as yet restricted the geometry of the Q 's: they could be either spherical or plane-wave harmonics, for example.

By successively applying the background 3-space Ricci identity,

$$D_{abA_\ell} Q - D_{baA_\ell} Q = + \sum_{n=1}^{\ell} \frac{K}{a^2} \left(\delta_b^{b_n} h_{a_n a} - \delta_a^{b_n} h_{a_n b} \right) D_{\bar{A}_\ell} Q, \quad (1.54)$$

where $\bar{A}_\ell = a_1 \dots a_{n-1} b_n a_{n+1} \dots a_\ell$, *i.e.*, the sequence of ℓ indices with the n -th one replaced with a contraction. First, the **curvature-modified Helmholtz equation** is found ¹⁴ (using A.36, A.37, A.35):

$$D^a D_a Q_{(A_\ell)} = -\bar{k}_\ell^2 Q_{(A_\ell)} \quad \text{with} \quad -\bar{k}_\ell^2 = \frac{1}{a^2} \left(K\ell(\ell+2) - k^2 \right). \quad (1.55)$$

Second, we are able to construct the **mode recursion relation** (using A.33, A.34 in 1.21) :

$$e^a D_a [G_\ell[Q]] = +k_{\text{phys}} \left[\frac{\ell^2}{(2\ell+1)(2\ell-1)} \left(1 - \frac{K}{k^2} (\ell^2 - 1) \right) G_{\ell-1}[Q] - G_{\ell+1}[Q] \right]. \quad (1.56)$$

The latter is the basis of the standard derivation of the linear-FRW mode hierarchy for scalar modes. The derivation of these are given in appendix A.3 (we consider only scalar eigenfunctions). It will be seen, in chapter 5, that this relation can be used in place of the general divergence relations which allow the construction of generic multipole divergence equations [58] if one restricts oneself to constant curvature space-times.

Given the recursion relation one can immediately make the connection with the usual Legendre tensor treatment (A.29) this is shown in appendix A (A.29-A.31).

¹⁴This will be necessary in order to switch from the almost-FLRW multipole divergence equations in chapter 5 to the mode representation where we use (1.20) to construct $D^a D_{(a} Q_{A_\ell)}$.

1.4.1 The mean-square, $|\tau_\ell|^2$, in almost-FLRW universes

We now relate the multipole mean-squares $\langle \tau_{A_\ell} \tau^{A_\ell} \rangle$ of the ensemble average over the multipole moments with that of the mode coefficient mean-squares $|\tau_\ell(k)|^2$.

In order to carry this out we relate two separate spatial harmonic expansions (1.53) for the same function: the first is one associated with plane wave harmonics (Q^k), naturally used in describing structure existing at any time t , and the second, one associated with radial and multipole harmonics ($O_{(X)}^{A_\ell} R_{A_\ell}$), *i.e.*, a spherical expansion based at the point of observation, naturally arises when we project the null cone angular harmonics into a surface of constant time. These are both related to the Mode Function formulation which becomes useful in the non-flat constant curvature cases.

Plane-waves and Mode functions

Considering flat FRW universes, each set of eigenfunctions satisfy (1.47). The temperature anisotropy (1.4) can be expressed in terms of its plane-wave spatial Fourier transform:

$$\tau(x, e) = \sum_{k_\nu} \tau(k, t, e^{(k)}, e) Q_{\text{FLAT}}. \quad (1.57)$$

(For a more detailed treatment see appendix A.3.4). It can be shown that for the flat case, $K = 0$, that is (A.61) holds in (A.57) and (A.58), we find from (A.60) :

$$D_{(A_\ell)} Q_{\text{FLAT}} = (-ik_{\text{phys}})^\ell O_{A_\ell}^{(k)} Q_{\text{FLAT}} \quad (1.58)$$

where $O_{A_\ell}^{(k)}$ are the PSTF tensors associated with the direction $e_a^{(k)}$. Then from (1.48) we find that :

$$Q_{\text{FLAT} A_\ell} = (-1)^\ell O_{A_\ell}^{(k)} Q_{\text{FLAT}}, \quad (1.59)$$

holds along with (1.53), to get the temperature multipole:

$$\tau_{A_\ell} = (-1)^\ell \sum_k \tau_\ell(t, k) Q_{\text{FLAT}} O_{A_\ell}^{(k)}. \quad (1.60)$$

Radial expansions and Mode functions

We use the flat, $K = 0$, spherical eigenfunctions centered on a point x_0^i , with associated radial direction vector $e_a^{(x)}$. The latter is the same as the (spherically symmetric) projection into the constant time surfaces of the tangent vector e^a of the radial null geodesics, so we need not distinguish it from that vector. In this case the ℓ -th harmonic is

$$Q_\ell(x^\alpha, e_{(X)})|_X = R_{A_\ell}(r) O_X^{A_\ell}, \quad D_a r = e_a, \quad D^a e_a = \frac{2}{r} \quad (1.61)$$

where $e^a = dx^a/dr$ is the unit radial vector. Cartesian coordinates in space are given by r and e^a through $x^i = r e^i$. Defining the projection tensor

$$p_{ab} = h_{ab} - e_a e_b \Rightarrow p_{ab} u^a = 0 = p_{ab} e^a, \quad p_a^a = 2, \quad p^a_b p^b_c = p^a_c, \quad (1.62)$$

then (e^a is shear and curl free)

$$D_a e_b = \frac{1}{r} p_{ab} \Rightarrow D_a p_{bc} = -\frac{1}{r} (p_{ab} e_c + p_{ac} e_b), \quad D^a p^b{}_a = -\frac{2}{r} e^b \quad (1.63)$$

so from (1.47)

$$D_a D^a (R_{A_\ell} O^{A_\ell}) = -k_{\text{phys}}^2 (R_{A_\ell} O^{A_\ell}). \quad (1.64)$$

To work out the *l.h.s.*, we must first calculate $D_a Q_\ell$ (A.64) and $D_a D^a Q_\ell$ (A.65). These are then used to calculate: $D_a R_{A_\ell}$ (A.66), $D^a D_a R_{A_\ell}$ (A.67), $D_a R^{A_\ell}$ (A.68) and $D_a D^a O^{A_\ell}$ (A.69) respectively.

Putting these in (A.65) we find (A.71) from which we get :

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R_{A_\ell}}{\partial r} \right) + R_{A_\ell} \left(-\frac{\ell(\ell+1)}{r^2} \right) = -k_{\text{phys}}^2 R_{A_\ell}, \quad (1.65)$$

the spherical Bessel equation. This provides the PSTF derivation of the spherical Bessel equation in terms of the irreducible representation.

Now consider that we can choose any basis we like for the tensor basis here, independent of the spatial coordinates used. It is convenient to use the plane wave decomposition to get a parallel vector basis. We do this by writing

$$R_{A_\ell}(r) = \sum_{k^a} R_{A_\ell}^{(k)} = \sum_{k^a} R_\ell(k, r) O_{A_\ell}^{(k)}, \quad (1.66)$$

this expresses the tensor eigenfunctions in terms of the monopole eigenfunctions. When this is substituted into (A.71) we obtain the *radial equation* which has solutions that are *spherical Bessel functions* in the flat case :

$$\frac{\partial}{\partial r} \left(r^2 \frac{\partial R_\ell}{\partial r} \right) + \left[k_{\text{phys}}^2 r^2 - \ell(\ell+1) \right] R_\ell = 0, \quad \text{to find } R_\ell(r) = \alpha_\ell j_\ell(k_{\text{phys}} r) \quad (1.67)$$

where $k_{\text{phys}}^2 = k^2/a^2$ and α_ℓ are integration constants (the second set of constants for this second order equation vanish because we choose $R_\ell(0)$ to be finite; the Neumann functions are not finite at $r = 0$).

From (1.61), (1.66), and (1.67) we have found that the solutions to the Helmholtz equations give the eigenfunctions

$$Q_\ell(x^\nu) = \alpha_\ell \sum_{k^\nu} j_\ell(k_{\text{phys}} r) O_{A_\ell}^{A_\ell} O_{A_\ell}^{(X)} = \alpha_\ell \sum_{k^\nu} j_\ell(k_{\text{phys}}(k, r)r) L_\ell(X). \quad (1.68)$$

and we can set $\alpha_\ell = (\Delta_\ell)^{-1}$ so that $\int Q_\ell d\Omega = 1$.

It is important to notice that the functions $L_\ell(X)$ depend both on k^a and on e^a , and so for each k^a is a function of (θ, ϕ) , thus Q is indeed a function of all spatial coordinates. We can pick any direction k^a to find the particular eigenfunctions

$$Q_\ell^{(k)} = (\Delta_\ell)^{-1} j_\ell(k_{\text{phys}} r) O_{A_\ell}^{A_\ell} O_{A_\ell}^{(X)} \quad (1.69)$$

associated with that direction. The general ℓ -th eigenfunction is a sum of such eigenfunctions over a basis of directions k^a ¹⁵:

$$Q_\ell(x^\nu) = \sum_{k^a} Q_\ell^{(k)}. \quad (1.70)$$

¹⁵Many treatments choose a particular direction for k : $k^a = \delta_3^a$ or similar and omit the summation.

Now we find $O^{A_\ell} D_{A_\ell} Q$ (A.76) in terms of R_ℓ (as show in the appendix using PSTF techniques, A.72-A.76) :

$$O_{(x)}^{A_\ell} D_{(A_\ell)} Q = (-k_{\text{phys}})^\ell O_{(x)}^{A_\ell} O_{A_\ell}^{(k)} R_\ell \Rightarrow O^{A_\ell} Q_{A_\ell} = (-k_{\text{phys}})^\ell Q_\ell. \quad (1.71)$$

Putting this in the expansion (1.53)

$$\tau_{A_\ell}(x) = \sum_{k^\nu} \tau_\ell(t, k) (-k_{\text{phys}})^{-\ell} D_{(A_\ell)} Q(x^\nu) = \sum_{k^\nu} \sum_n \tau_\ell(t, k) (-k_{\text{phys}})^{-\ell} D_{(A_\ell)} Q_n \quad (1.72)$$

gives the present version of (1.51)

$$\tau(x, e_{(x)}) = \sum_\ell \tau_{A_\ell} O^{A_\ell} = \sum_\ell \tau_\ell(k, t) G_\ell[Q] \quad (1.73)$$

directly analyzing the coefficients τ_{A_ℓ} in terms of these functions Q - given that we have 3-dimensions worth of variability so as to represent arbitrary spatial functions - with purely time-dependent coefficients (parametrized by a vector k^a).

Radial expansions and plane-waves

Now consider the inversion¹⁶

$$e^{(+ik_\alpha x^\alpha)} = \sum_\ell \tilde{R}_{A_\ell}(r) O_{(x)}^{A_\ell}, \quad \Longleftrightarrow \quad \tilde{R}_{A_\ell} = \Delta_\ell^{-1} \int d\Omega_{(x)} e^{+ik_\alpha x^\alpha} O_{A_\ell}^{(k)} \quad (1.74)$$

by taking a Taylor expansion using $k_\alpha x^\alpha = k_{\text{phys}} r e_b^{(k)} e_b^{(x)}$ we find

$$\begin{aligned} \tilde{R}_{A_\ell} &= \Delta_\ell^{-1} \sum_{n=0}^{\infty} \frac{(+i)^n (k_{\text{phys}} r)^n}{n!} e_{B_n}^{(k)} \int d\Omega O_{A_\ell}^{(x)} e_{(x)}^{B_n} \\ &= \Delta_\ell^{-1} \sum_{n=0}^{\infty} \frac{(+i)^n (k_{\text{phys}} r)^n}{n!} \frac{\delta_n^{\ell+2m} n! (n-\ell+1)!!}{(n-\ell+1)! (n+\ell+1)!!} 4\pi e_{B_n}^{(k)} h_{(A_\ell)}^{(B_\ell)} h^{B_n - \ell} \end{aligned} \quad (1.75)$$

on using (A.27). Putting (A.20) into (1.75) we find

$$\begin{aligned} \tilde{R}_{A_\ell} &= (+i)^\ell \Delta_\ell^{-1} 4\pi \sum_{m=0}^{\infty} \frac{(-1)^m (k_{\text{phys}} r)^{\ell+2m} (2m+1)!!}{(2m+1)! (2m+2\ell+1)!!} O_{A_\ell}^{(k)} \\ &= 4\pi (+i)^\ell \Delta_\ell^{-1} \sum_{m=0}^{\infty} \frac{(-1)^m (k_{\text{phys}} r)^{\ell+2m}}{2^m m! (2m+2\ell+1)!!} O_{A_\ell}^{(k)}. \end{aligned} \quad (1.76)$$

This can be re-expressed as

$$\tilde{R}_{A_\ell} = 4\pi (+i)^\ell \Delta_\ell^{-1} j_\ell(k_{\text{phys}} r) O_{A_\ell}^{(k)} = (+i)^\ell (2\ell+1) \beta_\ell^{-1} j_\ell(k_{\text{phys}} r) O_{A_\ell}^{(k)}. \quad (1.77)$$

¹⁶Contrast with (1.61): there the l.h.s is the spherical eigenfunction; here it is the plane one, expressed in terms of spherical ones.

Hence

$$Q_{\text{FLAT}}(x, e^{(k)}) = e^{+ik^a x_a} = 4\pi \sum_{n, k^\nu}^{+\infty} (+i)^n j_n(k_{\text{phys}} r) O_{(k)}^{A_n} O_{A_n}^{(\chi)} \Delta_n^{-1} \quad (1.78)$$

links the plane-waves to the spherical expansion. This recovers, from $L_\ell(e^a e'_a) = \beta_\ell P_\ell(e^a e'_a)$ the more usual relation:

$$e^{+ik_{\text{phys}} r X} = \sum_{\ell} \tilde{R}_{A_\ell} O^{A_\ell}(\chi) = \sum_{\ell} (+i)^\ell (2\ell + 1) j_\ell(k_{\text{phys}} r) P_\ell(X). \quad (1.79)$$

$|\tau_\ell|^2$ for $K = 0$ (flat) almost-FLRW models

We now return to the relationship between the τ_{A_ℓ} and τ_ℓ . Now from (A.60), (A.63), (1.78) and (1.72) :

$$\tau_{A_\ell} = (+i)^\ell \int \frac{k^2 dk}{(2\pi)^3} \int d\Omega_k \tau_\ell(k, t) O_{A_\ell}^{(k)} \sum_{n=0}^{+\infty} (+i)^n j_n(\lambda r) O_{(k)}^{B_n} O_{B_n}^{(\chi)} \beta_n^{-1} (2n + 1). \quad (1.80)$$

This can be reduced further (A.77-A.82), where upon using (A.78) one finds that

$$\tau_{A_\ell} = 4\pi O_{A_\ell}^{(\chi)} \int \frac{k'^2 dk'}{(2\pi)^3} \tau_\ell(k', t) R_\ell(k', \chi), \quad (1.81)$$

where $\chi = r/a$ and by using $R_\ell(k, \chi) = j_\ell(\lambda r)$ we can identify this with (A.77), the equivalent of the multipole moments found using the explicit form of the plane-waves. The radial eigenfunctions will be normalized to unity under the ensemble average, this will introduce an additional factor of $(4\pi)^{-1}$ in order to correct for the solid angle part of the average ¹⁷.

In order to proceed further, from (1.81) we construct the ensemble average

$$\begin{aligned} \langle \tau_{A_\ell} \tau^{A_\ell} \rangle &= (4\pi)^2 O_{A_\ell}^{(\chi)} O_{(X)}^{A_\ell} \left\langle \int \frac{k'^2 dk'}{(2\pi)^3} \tau_\ell(k', t) R_\ell(k', r) \int \frac{k^2 dk}{(2\pi)^3} \tau_\ell(k, t) R_\ell(k, r) \right\rangle, \\ &= \frac{4\pi}{(2\pi)^3} \beta_\ell \int k^2 dk |\tau_\ell(k, t)|^2. \end{aligned} \quad (1.82)$$

What has happened here is that we imagine an ensemble of universes and we use an ensemble average rather than the space average; we have to do this because τ_ℓ is not square-integrable, *i.e.*, we cannot use the r.m.s value as we cannot integrate the square over the space in general; this operation is not well defined. In order to deal with the ensemble average over the mode coefficients, *i.e.* $\langle \tau_\ell^k \tau_m^{k'} \rangle$, we assume the perturbations

¹⁷Remembering that

$$\langle j_\ell(k' r) j_\ell(k r) \rangle_{z_a} = 4\pi \int_0^\infty r^2 dr j_\ell(k' r) j_\ell(k r) = 4\pi \left(\frac{\pi}{2} \right) \frac{1}{k^2} \delta(k - k').$$

this is multiplied through by the additional 4π in order to compensate for the solid angle integration. This then gives a factor of $2\pi^2$.

to be fairly homogeneously spread throughout the space and not confined in a particular region, and assume that there are no correlations between perturbations with different wavenumbers. Here the Gaussian assumption is useful, we have that

$$\langle \tau_\ell(k, t) \tau_\ell(k', t) \rangle = (2\pi)^3 \delta(k' - k) |\tau_\ell|^2$$

which fixes the normalization, and cancels the additional factor of $(2\pi)^{-3}$.

$|\tau_\ell|^2$ for almost-FLRW models

It is useful to notice that an alternative although equivalent avenue of approach is also possible proceeding directly from the mode expansion in $G_\ell[Q] = O_{(x)}^{A_\ell} Q_{A_\ell}$; this can be used for any K .

Notice that as before at the observer at x_0^i ,

$$\tau(x_0, e) = \sum_\ell \tau_{A_\ell} O^{A_\ell}, \quad \Longleftrightarrow \quad \tau_{A_\ell}(x_0) = \Delta_\ell^{-1} \int d\Omega O_{A_\ell} \tau(x_0, e). \quad (1.83)$$

Now in the spatial section in general we can write

$$\tau(x, e_{(k)}) = \sum_\ell \tau_{A_\ell} O_{(k)}^{A_\ell}, \quad \Longleftrightarrow \quad \tau_{A_\ell}(x) = \Delta_\ell^{-1} \int d\Omega^{(k)} O_{A_\ell}^{(k)} \tau(x, e_{(k)}). \quad (1.84)$$

Now the point x_0^ν is chosen at some earlier time in a spatial section, with radial direction vector $e_{(x)}^\alpha$; for FRW models we can consider $e_{(x)}^\alpha$ and e^α to be equivalent and write

$$\begin{aligned} \tau(x_0, e) &= \tau(x_0, e_{(x)}) = \sum_\ell \tau_{A_\ell}(x_0) O_{(x)}^{A_\ell} \equiv \sum_\ell \tau_{A_\ell}(x_0) O^{A_\ell} \\ &\Longleftrightarrow \tau_{A_\ell}(x_0) = \Delta_\ell^{-1} \int d\Omega^{(x)} O_{A_\ell}^{(x)} \tau(x_0, e_{(x)}). \end{aligned} \quad (1.85)$$

Remember that in some spatial section (where the integral relations (A.83,A.84,A.85) are useful)

$$\tau(x, e) = \sum_{\ell, k^\nu} \tau_\ell(k, t) O_{(x)}^{A_\ell} Q_{A_\ell}. \quad (1.86)$$

Now from

$$\tau_{A_\ell}(x_0) = \Delta_\ell^{-1} \int d\Omega \sum_{m, k^\alpha} \tau_m O_{A_\ell} G_\ell[Q], \quad (1.87)$$

we identify $O_{(x)}^{A_\ell}$ with O^{A_ℓ} , to find

$$\tau_{A_\ell} = \sum_{k^\alpha} \tau_\ell(k, t) Q_{A_\ell}(k, x), \quad (1.88)$$

which means that for $K = 0$ ($R_\ell \propto j_\ell$) (1.82).

$$\langle \tau_{A_\ell} \tau^{A_\ell} \rangle = \sum_{k^\nu} |\tau_\ell|^2 \langle Q_{A_\ell} Q^{A_\ell} \rangle = (4\pi) \beta_\ell \sum_k |\tau_\ell|^2. \quad (1.89)$$

What is particularly useful about the last method of calculation is that for constant K surfaces it can be shown on using relation (A.89) recursively that

$$\int d\Omega_k \langle |G_\ell[Q]G_m[Q]| \rangle = 4\pi \Xi_\ell^2 \delta_\ell^m, \quad \text{with } \Xi_\ell^2 = \prod_{n=1}^{\ell} (\alpha_n)^2 \quad (1.90)$$

and $\langle | \dots | \rangle$ denotes the ensemble average of the eigenfunctions. What is useful here is to notice that (using the normalization for $K = 0$),

$$\Xi_\ell^2|_{K=0} = \prod_{n=1}^{\ell} \frac{n^2}{(2n+1)(2n-1)} = \frac{\beta_\ell^2}{2\ell+1} = (4\pi)^{-2} (2\ell+1) \Delta_\ell^2. \quad (1.91)$$

This means that for $K = 0$

$$\int d\Omega_{(k)} \langle |G_\ell^k G_m^{k'}| \rangle = \delta_\ell^m \frac{2\ell+1}{4\pi} \Delta_\ell^2. \quad (1.92)$$

Now we can extend the above to FL models with $K \neq 0$. The mean-square for constant K is found by modifying the normalization after identifying the $K = 0$ normalizations in (A.85) and (1.92) :

$$\langle \tau_{A_\ell} \tau^{A_\ell} \rangle = \sum_{k^\nu} |\tau_\ell|^2 \langle Q_{A_\ell} Q^{A_\ell} \rangle = (4\pi) \beta_\ell \sum_k |\tau_\ell|^2 \Xi_\ell^2, \quad (1.93)$$

or alternatively, keeping the form of the mean-square in (1.89), by redefining the mode function expansion [194] by a wavelength-dependent coefficient; then

$$\tau(x, e) = \sum_{\ell, k^\nu} \tilde{\tau}_\ell(k, t) M_\ell[Q], \quad \text{and } M_\ell[Q] = \Xi_\ell(k, K) G_\ell[Q], \quad (1.94)$$

defines the new coefficients $\tilde{\tau}_\ell(k, t)$ so that

$$|\tilde{\tau}_\ell|^2 = |\tau_\ell|^2 \Xi_\ell^2. \quad (1.95)$$

Because we have redefined the mode functions in (1.94), the form of the equations for $K \neq 0$ is the same as in the case $K = 0$. However the coefficients are different because they are from a different expansion. Using the results from the $K = 0$ case, either (1.81) or (A.77), we have that

$$\langle \tau_{A_\ell} \tau^{A_\ell} \rangle = \left(\frac{1}{2\pi^2} \right) \beta_\ell \int \frac{dk}{k} k^3 |\tau_\ell(k, t)|^2 \quad (1.96)$$

Hence, on using $\hat{C}_\ell = \Delta^{-1} C_\ell$ and $\langle \tau_{A_\ell} \tau^{A_\ell} \rangle = \hat{C}_\ell (2\ell+1)$, we now have

$$\langle \tau(x_0, e) \tau(x_0, e') \rangle = \sum_{\ell=0}^{\infty} (2\ell+1)^{-1} \langle \tau_{A_\ell} \tau^{A_\ell} \rangle O_{B_\ell} O'^{B_\ell} \quad (1.97)$$

which can now be written from (1.96) and (1.95) as

$$\begin{aligned} \langle \tau(x_0, e) \tau(x_0, e') \rangle &= \frac{1}{2\pi^2} \sum_{\ell=0}^{\infty} (2\ell+1)^{-1} \beta_\ell \int_0^\infty \frac{dk}{k} k^3 |\tilde{\tau}_\ell(k, t)|^2 O_{A_\ell} O'^{A_\ell} \\ &= \frac{1}{2\pi^2} \sum_{\ell=0}^{\infty} \beta_\ell^2 (2\ell+1)^{-1} \int_0^\infty \frac{dk}{k} k^2 |\tilde{\tau}_\ell(k, t)|^2 P_\ell(e^a e'_a) \end{aligned} \quad (1.98)$$

In this way we reobtain the results of White [191] and Wilson [193] :

$$C_\ell = \frac{2}{\pi} \frac{\beta_\ell^2}{(2\ell + 1)^2} \int_0^\infty \frac{dk}{k} k^3 |\bar{\tau}_\ell(k, t)|^2. \quad (1.99)$$

The additional factors of $(2\ell + 1)^2$ will cancel out with those that will be recovered from the integral solutions in a latter chapter. Notice that for $\ell > 2$ we have that $\Theta_\ell = \beta_\ell \tau_\ell$, however things are more complex when attempting to relate the dipole (Θ_1) and monopole (Θ_0) to the CGI temperature perturbation and dipole, in the CGI theory there is no monopole anisotropy as it is pure gauge. These issues are dealt with in later chapters (see chapters 5 and 6). The Θ_ℓ is only meaningful within the linear-FRW theory – its interpretation and gauge invariance are questionable in the nonlinear extension that the 1+3 Lagrangian approach accommodates.

This relates the amount of power in a given wavenumber, $|\tau_\ell(k, t_0)|^2$, at the observer x_0^i , on an angular scale ℓ to given the angular correlations found on the angular scale $X = e^a e'_a$, *i.e.*, X is the separation between measurements.

The relationship between the mode wavenumbers k , and hence the angular scale ℓ at the observer, to actual scale at the surface of last scattering requires knowledge of the nature of the past null-cone of the observer, x_0^i , and its intersection with the real last-scattering surface. This requires the integration of geodesic deviation equations in general, so as to construct the area-distance relations.

The relationship between the power on a given scale now, $|\tau_\ell(k, t_0)|^2$, and that at the last scattering surface, $|\tau_\ell(k, t_*)|^2$, as evolved down a given worldline, requires the calculation of the radiation transfer functions. This is dealt with in later chapters.

We have given here a comprehensive survey of the covariant and gauge invariant representation of CMB anisotropies, both in general and in almost-FLRW universes, and related this formalism to the other major formalisms in use for this purpose at the present time.

This chapter has been concerned with algebraic relations, specifically the Multipole, (*e.g.* for $\tau_{A\ell}$), and Mode, (*e.g.* for τ_ℓ), formalisms and the relationship between them.

Chapter 2

Temperature Anisotropies

Temperature anisotropies in inhomogeneous and anisotropic spacetimes : A new approach to local nonlinear effects in CMB anisotropies is developed. Two new effects are shown : a coupling between the radiation multipole and the baryonic velocity via nonlinear Thompson scattering and a gravitational coupling between the radiation multipoles and the kinematic quantities. The qualitative features of these effects are discussed. The covariant evolution and constraint equations are given. The nonlinear generalization of the radiation multipole hierarchy, the *Multipole Divergence Equations (MDE)*'s are given from the exact *IBE* :

$$-[\dot{\Pi}(x, e) + e^a D_a \Pi(x, e)] = 4\mathcal{D}(x, e)\Pi(x, e) + K(x, e).$$

where $\Pi(x, e)$ is the generalized temperature anisotropy, $\mathcal{D}(x, e)$ the gravitational source term and $K(x, e)$ the scattering correction, i.e. , we find the evolution equations for $\Pi_{A_l}(x)$. The main application is to higher order effects about FLRW universes. A secondary application - using the tetrad formulation of the resulting equations, is to Bianchi cosmologies. The equations are covariantly and gauge invariantly linearized about a Friedmann background recovering the almost-FLRW MDE's.

2.1 Multipole Divergence Relations

In order to place this approach in correct context, one should have in mind the conceptual basis of formulation : the existence of the divergence relations [58]. The Ellis-Matravas-Treciokas integration technique can be used to construct a general (exact) multipole hierarchy for the moments of the Integrated Boltzmann Equations in terms of a PSTF representation. The key is the **divergence identity**¹ for the moments $\int p^{A_l} f d\Omega$ of the distribution function $f(x, p)$ (see a later chapter for a more systematic treatment) and the Liouville operator L : namely,

$$\int P^{A_l} C[f] d\Omega = \int P^{A_l} L(f) E^q dE d\Omega = \nabla_a \left(\int P^{A_l a} f E^q dE d\Omega \right) \quad (2.1)$$

¹This follows from the identity $L(m^2) = g_{ab;c} p^a p^b p^c = 0$ and $L(f) = C[f]$ [58].

for ($q > 0$) where $p^a = Eu^a + \lambda e^a$ (where on mass-shell $\lambda^2 - E^2 = -m^2$: relates the mass, m , energy E , and 3-momentum λe^a) and $P^{A_t} = p^{(A_t)}$. This is a set of exact space-time relations (*i.e.* they are valid on the manifold M) resulting from an integral over energy and directions of the Boltzmann equation (2.47), which holds in the on-mass shell tangent bundle $T_m(M)$ over M . These will be called the Moment or Multipole Divergence Equations (MDE's) because on using the 1+3 splitting, in the Lagrangian threading formulation, this is a set of divergence relations for the moments $\int \int E^{m-\ell} \lambda^\ell e^{A_t} f E dE d\Omega$ (here $\ell \geq m \geq 0$), which are space-time tensors (the energy E and the direction vectors e^a having been integrated out). The important point is that an exact moment hierarchy has thus been constructed for both massive and massless particles as describe by a single particle distribution function.

These can be expressed in terms of the harmonic expansion (1.33, 2.41 and 2.42) to find the multipole formulation. Equivalently, the PSTF parts of the MDE are relations for the PSTF moments of the distribution function integrated over the energy. In order to relate this to local physics and thus the local observables, closeness to equilibrium is assumed. This then allows the construction of the closely related temperature multipole divergence equations.

So there are three procedures for the derivation of the multipole divergence equations; the first two make use of the divergence relations, the first, actively, the second passively, while the third procedure make use of the orthonormal tetrad – these are all equivalent for massless particles.

1. The first, uses the **divergence relations** for any symmetric tensor K_{A_t} , effectively in the form:

$$L(K_{A_t} P^{A_t}) = P^{A_t a} \nabla_a K_{A_t}, \quad (2.2)$$

the resulting equations can then be separated out into the multipole PSTF parts by factoring off the coefficients of O_{A_t} at each order in ℓ (as is done in the original papers [58, 62]).

2. The second procedure utilizes the additional conditions that photons obey, *i.e.* $p^a = E(u^a + e^a)$; they are massless (2.40). This then allows one to directly evaluate $d/dv[K_{A_t} e^{A_t}]$ using only the exact form of dE/dv (see appendix (D.1) and section (2.4)) and then taking the PSTF part of the resulting equations to find the multipole hierarchy (see section (2.4) and equation (2.73) – this is the approach used in [120] as it is the least technical in the case of zero rest mass particles).
3. The third makes use of the tetrad by transforming from the tuple (x^i, p^a) in $T(M)$, to the tuple (x^i, E, e^a, m) on the mass shell part $T_m(M)$, and then to the reduced bundle $B(M)$ with tuple (x^i, e^a) , to directly find equations in terms of the generalized temperature anisotropy $\Pi(x, e)$ (see chapter 3 and section 3.2). This is mentioned in the original EMT papers but not carried out; it is done here both for completeness and to provide the link between the relativistic kinetic theory approach and the Sachs-Wolfe one. The generalized temperature anisotropy can then be reduced to moment form using a PSTF harmonic expansion.

The key difference here is that the first two methods go directly to the exact multipole equations and do not need the explicit use of the orthonormal tetrad, while the third method allows one to construct equations in terms of the generalized temperature anisotropy – not explicitly in terms of its multipole moments. We will in fact follow the second procedure in this chapter, because it has not been done explicitly this way before, and because in the case of massless particles it is straightforward to directly construct the multipole hierarchy from the energy Integrated Boltzmann Equations in a transparent manner. We do this rather than either using the full power of the divergence relation machinery developed by Ellis-Matras-Treciokas (that is necessary for generic on-mass-shell particles) or the full power of the orthonormal tetrad approach (the tetrad approach will of course be crucial for the study of Bianchi cosmologies).

These equations, which include the hydrodynamic energy and momentum equations and their higher order moment analogues, generically contain timelike derivatives. They are typically integrated along timelike curves in spacetime (not along null geodesics), even when looking at the evolution of photon distributions – as in our present context. This timelike integration is related to the nullcone testfield integrations by making homogeneity assumptions about the distribution of matter in (spacelike) surfaces of constant time (see chapter 3). The relationship between spatial scales on an early hypersurface and angular scales in the multipole hierarchy are not contained in the MDE's.

In order to facilitate the derivation of the MDE's we will need (i) the covariant Lagrangian threading formalism as applied to general relativity, (ii) the covariant Lagrangian kinetic theory (which are developed next), and (iii) knowledge of the irreducible representations of the rotation group with respect to the u^a -frame as developed in chapter 1.

2.2 1+3 Covariant Lagrangian threading formalism

The Ehlers-Ellis 1+3 formalism [46, 50, 101] is a covariant Lagrangian threading approach², *i.e.* every quantity has a natural interpretation in terms of observers comoving with the fundamental 4-velocity u^a (where $u^a u_a = -1$), in the so called u^a -frame. Provided this is defined uniquely in an invariant manner, all related quantities have a direct physical or geometric meaning, and may in principle be measured in the instantaneous rest space of the comoving fundamental observers. Any coordinate system or tetrad can be used when specific calculations are made. These features are a crucial part of the strengths of the formalism and of the perturbation theory that is derived from it.

We will follow the development of the formalism given by Maartens [113], which makes explicit use of irreducible quantities and derivatives; it develops the identities which these quantities and derivatives obey (see also [122, 115, 180, 51]) which are compatible and most easily interfaced with the canonical linear-FRW approach, relativistic kinetic theory and the techniques of relativistic cosmology.

²We are contrasting the 1+3 threading approach to the 3+1 foliation approach; the prior threads the spacetime with respect to a preferred frame given by a four velocity u^a while the latter foliates the spacetime with respect to spatial hypersurfaces with normals n^a . Additionally we use a Lagrangian choice initially for the formulation with respect to the four velocity u^a and then boost to recover the exact Eulerian formulation – this is where the formalism becomes useful, when it is applied to nonperturbative effects.

The basic algebraic tensors are: (a) the spatial projector $h_{ab} = g_{ab} + u_a u_b$, where g_{ab} is the spacetime metric, which projects into the instantaneous rest space of comoving observers; and (b) the projected alternating tensor $\varepsilon_{abc} = \eta_{abcd} u^d$, where $\eta_{abcd} = -\sqrt{|g|} \delta^0_{[a} \delta^1_b \delta^2_c \delta^3_{d]}$ is the spacetime alternating tensor. Thus

$$\eta_{abcd} = 2u_{[a} \varepsilon_{b]cd} - 2\varepsilon_{ab[c} u_{d]}, \quad \varepsilon_{abc} \varepsilon^{def} = 3! h_{[a}{}^d h_b{}^e h_c{}^f].$$

The projected symmetric tracefree (PSTF) parts of vectors and rank-2 tensors are

$$V_{(a)} = h_a{}^b V_b, \quad S_{(ab)} = \left\{ h_{(a}{}^c h_{b)}{}^d - \frac{1}{3} h^{cd} h_{ab} \right\} S_{cd},$$

with higher rank formulas given in [66]. The skew part of a projected rank-2 tensor is spatially dual to the projected vector $S_a = \frac{1}{2} \varepsilon_{abc} S^{[bc]}$, and then any projected rank-2 tensor has the irreducible covariant decomposition

$$S_{ab} = \frac{1}{3} S h_{ab} + \varepsilon_{abc} S^c + S_{(ab)},$$

where $S = S_{cd} h^{cd}$ is the spatial trace. In the 1+3 covariant formalism, all quantities are either scalars, projected vectors, or PSTF rank-2 tensors. The equations governing these quantities involve a covariant vector product and its generalization to PSTF rank-2 tensors:

$$[V, W]_a = \varepsilon_{abc} V^b W^c, \quad [S, Q]_a = \varepsilon_{abc} S^b{}_d Q^{cd}.$$

The covariant derivative ∇_a defines 1+3 covariant time and spatial derivatives

$$j^{A_t}{}_{B_m} = u^c \nabla_c j^{A_t}{}_{B_m}, \quad D_c j^{A_t}{}_{B_m} = h^{A_t}{}_{C_t} h^{D_m}{}_{B_m} h_c{}^d \nabla_d j^{C_t}{}_{D_m}. \quad (2.3)$$

Note that $D_c h_{ab} = 0 = D_d \varepsilon_{abc}$, while $\dot{h}_{ab} = 2u_{(a} \dot{u}_{b)}$ and $\dot{\varepsilon}_{abc} = 3u_{[a} \varepsilon_{bc]d} \dot{u}^d$. The projected derivative D_a further splits irreducibly into a 1+3 covariant spatial divergence and curl [113]

$$\text{div } V = D^a V_a, \quad (\text{div } S)_a = D^b S_{ab}, \quad (2.4)$$

$$\text{curl } V_a = \varepsilon_{abc} D^b V^c, \quad \text{curl } S_{ab} = \varepsilon_{cd(a} D^c S_{b)}{}^d, \quad (2.5)$$

and a 1+3 covariant spatial distortion [116]

$$D_{(a} V_{b)} = D_{(a} V_{b)} - \frac{1}{3} (\text{div } V) h_{ab},$$

$$D_{(a} S_{bc)} = D_{(a} S_{bc)} - \frac{2}{5} h_{(ab} (\text{div } S)_{c)}.$$

Note that div curl is not in general zero, for vectors or rank-2 tensors (see [113, 122, 180, 115] for the relevant formulae). The covariant irreducible decompositions of the derivatives of scalars, vectors and rank-2 tensors are given in exact (nonlinear) form by [118]

$$\nabla_a \psi = -\dot{\psi} u_a + D_a \psi, \quad (2.6)$$

$$\begin{aligned} \nabla_b V_a &= -u_b \left\{ \dot{V}_{(a)} + A_c V^c u_a \right\} + u_a \left\{ \frac{1}{3} \Theta V_b + \sigma_{bc} V^c + [\omega, V]_b \right\} \\ &\quad + \frac{1}{3} (\text{div } V) h_{ab} - \frac{1}{2} \varepsilon_{abc} \text{curl } V^c + D_{(a} V_{b)}, \end{aligned} \quad (2.7)$$

$$\begin{aligned} \nabla_c S_{ab} &= -u_c \left\{ \dot{S}_{(ab)} + 2u_{(a} S_{b)d} A^d \right\} + 2u_{(a} \left\{ \frac{1}{3} \Theta S_{b)c} + S_b{}^d (\sigma_{cd} - \varepsilon_{cde} \omega^e) \right\} \\ &\quad + \frac{3}{5} (\text{div } S)_{(a} h_{b)c} - \frac{2}{3} \varepsilon_{dc(a} \text{curl } S_{b)}{}^d + D_{(a} S_{bc)}. \end{aligned} \quad (2.8)$$

The algebraic correction terms in equations (2.7) and (2.8) arise from the relative motion of comoving observers, as encoded in the kinematic quantities: the expansion $\Theta = D^a u_a$, the 4-acceleration $A_a \equiv \dot{u}_a = A_{(a)}$, the vorticity³ $\omega_a = -\frac{1}{2}\text{curl} u_a$, and the shear $\sigma_{ab} = D_{(a} u_{b)}$.

The irreducible parts of the Ricci identities produce commutation identities for the irreducible derivative operators. In the simplest case of scalars:

$$\text{curl} D_a \psi \equiv \varepsilon_{abc} D^{[b} D^{c]} \psi = -2\dot{\psi} \omega_a, \quad (2.9)$$

$$D_a \dot{\psi} - h_a{}^b (D_b \psi)' = -\dot{\psi} A_a + \frac{1}{3} \Theta D_a \psi + \sigma_a{}^b D_b \psi + [\omega, D\psi]_a. \quad (2.10)$$

Identity (2.9) reflects the relation of vorticity to non-integrability; non-zero ω_a implies that there are no constant-time 3-surfaces everywhere orthogonal to u^a , since the instantaneous rest spaces cannot be patched together smoothly.⁴ Identity (2.10) is the key to deriving evolution equations for spatial gradients, which covariantly characterize inhomogeneity [52]. Further identities are given in [55, 113, 180, 118, 122].

The **kinematic quantities** govern the relative motion⁵ of neighboring fundamental worldlines, and describe the universal expansion and its local anisotropies (by Eq. (2.7)):

$$\nabla_b u_a = -A_a u_b + \frac{1}{3} \Theta h_{ab} + \varepsilon_{abc} \omega^c + \sigma_{ab}. \quad (2.11)$$

The **dynamic quantities** describe the sources of the gravitational field, and directly determine the Ricci curvature locally via Einstein's field equations. They are the (total) energy density $\rho = T_{ab} u^a u^b$, isotropic pressure $p = \frac{1}{3} h_{ab} T^{ab}$, energy flux $q_a = -T_{(a)b} u^b$, and anisotropic stress $\pi_{ab} = T_{(ab)}$, where T_{ab} is the total energy-momentum tensor

$$T^{ab} = \rho u^a u^b + 2q^{(a} u^{b)} + (ph^{ab} + \pi^{ab}) = \sum_I T_I^{ab} \quad (2.12)$$

given by the sum of all the individual contributions⁶, labeled by I . The total energy momentum tensor is conserved⁷: $\nabla_b T^{ab} = 0$.

The **locally free gravitational field**, i.e. the part of the spacetime curvature not directly determined locally by dynamic sources, is given by the Weyl tensor C_{abcd} . This

³Here the irreducible vector ω_a is used, the vorticity tensor is $\omega_{ab} = \varepsilon_{abc} \omega^c$ ($D^b \omega_{ab} = \text{curl} \omega_a$). The sign conventions, following [46, 50], are such that in the Newtonian limit, $\vec{\omega} = -\frac{1}{2} \vec{\nabla} \times \vec{v}$.

⁴In this case, which has no Newtonian counterpart, the D_a operator is not intrinsic to a 3-surface, but it is still a well-defined spatial projection of ∇_a in each instantaneous rest space.

⁵The convention used here arises historically from the use of the semi colon: $u_{a;b} \equiv \nabla_b u_a$ following [50, 184]: where a sign problem in the vorticity will arise when comparing to the convention in [51].

⁶In addition to the usual fluid (as in 2.12) and gas formulations (2.44) there are two other useful ones: the electromagnetic one: $T_{ab}^H = F_a{}^c F_{cb} - \frac{1}{4} F_{cd} F^{cd} g_{ab}$ and $F_{ab}^H = 2u_{[a} E_{b]} + \varepsilon_{abc} H^c$, and that for a scalar field: $T_{ab}^\phi = \nabla_a \phi \nabla_b \phi - g_{ab} (\frac{1}{2} \nabla_c \phi \nabla^c \phi - V(\phi))$ (a cosmological constant is then the case with $\phi = \Lambda = \text{constant}$ and $V(\phi) = \phi$). [51]

⁷The partial energy-momentum tensors have current contributions: $\nabla_b T_I^{ab} = J_I^a$ where $J_I^a = U_I u^a + M_I^a$ and (in the total component frame) U_I is the rate of energy density transfer and M_I^a the rate of momentum density transfer. During Late tight-coupling the CDM and neutrino components are decoupled: $J_C^a = 0 = J_N^a$. The baryon and radiation components are coupled via Thompson scattering (via the electron plasma) $J_R^a = -J_B^a = U_T u^a + M_T^a$ [120]

splits irreducibly into the gravito-electric and gravito-magnetic fields

$$E_{ab} = C_{abcd}u^c u^d = E_{\langle ab \rangle}, \quad H_{ab} = \frac{1}{2}\varepsilon_{acd}C^{cd}{}_{be}u^e = H_{\langle ab \rangle},$$

which provide a covariant Lagrangian description of tidal forces and gravitational radiation.

The Ricci identity for u^a and the Bianchi identities $\nabla^d C_{abcd} = \nabla_{[a}(-R_{b]c} + \frac{1}{6}Rg_{b]c})$ produce the fundamental evolution and constraint equations governing the above covariant quantities [46, 50]. Einstein's equations are incorporated⁸ via the algebraic replacement of the Ricci tensor R_{ab} by $T_{ab} - \frac{1}{2}T_c{}^c g_{ab}$. These equations, in exact (nonlinear) form and for a general source of the gravitational field, are [118, 51, 120, 11]:

Evolution:

$$\dot{\rho} + (\rho + p)\Theta + \text{div } q = -2A^a q_a - \sigma^{ab}\pi_{ab}, \quad (2.13)$$

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + \frac{1}{2}(\rho + 3p) - \text{div } A = -\sigma_{ab}\sigma^{ab} + 2\omega_a\omega^a + A_a A^a, \quad (2.14)$$

$$\dot{q}_{\langle a \rangle} + \frac{4}{3}\Theta q_a + (\rho + p)A_a + D_a p + (\text{div } \pi)_a = -\sigma_{ab}q^b + [\omega, q]_a - A^b \pi_{ab}, \quad (2.15)$$

$$\dot{\omega}_{\langle a \rangle} + \frac{2}{3}\Theta\omega_a + \frac{1}{2}\text{curl } A_a = \sigma_{ab}\omega^b, \quad (2.16)$$

$$\dot{\sigma}_{\langle ab \rangle} + \frac{2}{3}\Theta\sigma_{ab} + E_{ab} - \frac{1}{2}\pi_{ab} - D_{\langle a} A_{b \rangle} = -\sigma_{c\langle a}\sigma_{b \rangle}{}^c - \omega_{\langle a}\omega_{b \rangle} + A_{\langle a} A_{b \rangle}, \quad (2.17)$$

$$\dot{E}_{\langle ab \rangle} + \Theta E_{ab} - \text{curl } H_{ab} + \frac{1}{2}(\rho + p)\sigma_{ab} + \frac{1}{2}\dot{\pi}_{\langle ab \rangle} + \frac{1}{2}D_{\langle a} q_{b \rangle} + \frac{1}{6}\Theta\pi_{ab} \quad (2.18)$$

$$= -A_{\langle a} q_{b \rangle} + 2A^c \varepsilon_{cd\langle a} H_{b \rangle}{}^d + 3\sigma_{c\langle a} E_{b \rangle}{}^c - \omega^c \varepsilon_{cd\langle a} E_{b \rangle}{}^d - \frac{1}{2}\sigma^c{}_{\langle a} \pi_{b \rangle}{}^c - \frac{1}{2}\omega^c \varepsilon_{cd\langle a} \pi_{b \rangle}{}^d,$$

$$\dot{H}_{\langle ab \rangle} + \Theta H_{ab} + \text{curl } E_{ab} - \frac{1}{2}\text{curl } \pi_{ab} = 3\sigma_{c\langle a} H_{b \rangle}{}^c - \omega^c \varepsilon_{cd\langle a} H_{b \rangle}{}^d \quad (2.19)$$

$$- 2A^c \varepsilon_{cd\langle a} E_{b \rangle}{}^d - \frac{3}{2}\omega_{\langle a} q_{b \rangle} + \frac{1}{2}\sigma^c{}_{\langle a} \varepsilon_{b \rangle}{}^c{}_{cd} q^d.$$

Constraint:

$$\text{div } \omega = A^a \omega_a, \quad (2.20)$$

$$(\text{div } \sigma)_a - \text{curl } \omega_a - \frac{2}{3}D_a \Theta + q_a = -2[\omega, A]_a, \quad (2.21)$$

$$\text{curl } \sigma_{ab} + D_{\langle a} \omega_{b \rangle} - H_{ab} = -2A_{\langle a} \omega_{b \rangle}, \quad (2.22)$$

$$(\text{div } E)_a + \frac{1}{2}(\text{div } \pi)_a - \frac{1}{3}D_a \rho + \frac{1}{3}\Theta q_a = [\sigma, H]_a - 3H_{ab}\omega^b + \frac{1}{2}\sigma_{ab}q^b - \frac{3}{2}[\omega, q]_a, \quad (2.23)$$

$$(\text{div } H)_a + \frac{1}{2}\text{curl } q_a - (\rho + p)\omega_a = -[\sigma, E]_a - \frac{1}{2}[\sigma, \pi]_a + 3E_{ab}\omega^b - \frac{1}{2}\pi_{ab}\omega^b \quad (2.24)$$

Here we notice that there is no evolution equation for the acceleration (for \dot{A}_a) nor for anisotropic pressure (for $\dot{\pi}_{ab}$). Eq. (2.13) and Eq. (2.15) are just the conservation of energy-momentum. The generalized Gauss relation arises from the trace of 3-Ricci tensor

⁸Note that one constraint Einstein equation is not explicitly contained in this set – see [50, 112].

[51], when one includes the energy momentum tensor as substituted for the Ricci tensor it is also known as the generalized Friedmann equation:

$${}^3R = 2\rho - \frac{2}{3}\Theta^2 + 2\sigma^2 - 2\omega^2. \quad (2.25)$$

The Gauss equation is the case when $\omega^a = 0$, which then fully determines the 3-space average curvature. The Codacci equation is included in the constraint equations for the shear. In the Hamiltonian formulation of gravity the Gauss equation is often referred to as the Hamiltonian constraint, while the Codacci equation, as the momentum constraint.

A FLRW universe

A FLRW (background) universe, with its unique preferred 4-velocity u^a , is covariantly characterized as follows:

$$\text{kinematics: } D_a\Theta = 0, A_a = 0 = \omega_a, \sigma_{ab} = 0, \quad (2.26)$$

$$\text{dynamics: } D_a\rho = 0 = D_a p, q_a = 0, \pi_{ab} = 0, E_a = 0 = H_a, \quad (2.27)$$

$$\text{gravito-electric/magnetic field: } E_{ab} = 0 = H_{ab}. \quad (2.28)$$

The Hubble rate is $H = \frac{1}{3}\Theta = \dot{a}/a$, where $a(t)$ is the scale factor and t is cosmic proper time. In spatially homogeneous but anisotropic universes (Bianchi and Kantowski-Sachs models), the quantities $q_a, \pi_{ab}, \sigma_{ab}, E_{ab}$ and H_{ab} in the preceding list may be non-zero. There is no electromagnetic field (characterized by E_a and H_a , the electric and magnetic fields) in the background FLRW universe, there can however be a background cosmic magnetic-field, B_0 [174]. The FLRW field equations are:

$$\rho = 3H^2 + \frac{3K}{a^2}, \quad (2.29)$$

$$\dot{H} = -H^2 - \frac{1}{6}(\rho + 3p), \quad (2.30)$$

$$\dot{\rho} = -3H(\rho + p). \quad (2.31)$$

The basic parameters that characterize a FLRW universe are $H_0, \Omega_0, \Omega_\Lambda, q_0$ and $\mathcal{K}_0 = K/a_0^2$ [51]. The existence of the big-bang in these models follows from the Raychaudhuri equation (2.14,2.30) with the energy assumption $\rho + 3p > 0$ and that the universe is expanding today: $\Theta_0 > 0$. The simplest FLRW models [51] are

Einstein static :	$a(t) = \text{constant}, K = +1, \Lambda = \frac{1}{2}(\rho + 3p),$ (everything is constant)
de Sitter:	$a(t) = \exp(Ht), H = \text{constant}, K = 0,$ (it is empty) $\rho + p = 0.$
Milne:	$a(t) = t, K = -1,$ (it is empty), $\rho = 0,$ and $p = 0.$
Einstein-de Sitter:	$a(t) = \alpha t^{2/3}, K = 0 = \Lambda, \alpha = \text{constant}$ if $p = 0.$

An almost FLRW universe

If the universe is close to an FLRW model, then quantities that vanish in the FLRW limit are $\mathcal{O}(\epsilon)$, where ϵ is a dimensionless smallness parameter, and the quantities are suitably normalized (e.g. $\sqrt{\sigma_{ab}\sigma^{ab}}/H < \epsilon$, etc.). The Einstein Field Equations (2.13-2.24) are covariantly and gauge-invariantly linearized [52] by dropping all terms $\mathcal{O}(\epsilon^2)$ (all those on the right-hand side), and by replacing scalar coefficients of $\mathcal{O}(\epsilon)$ terms by their background values. In addition there are three 1+3 covariant and gauge invariant perturbation equations which arise from taking the spatial gradients of the density evolution equations (2.13), the expansion evolution equations (2.14) and the three curvature (2.25) and linearizing as described above; this then gives the density perturbation evolution equation, the expansion perturbation evolution equation and the curvature perturbation respectively.

To understand why this approach is gauge invariant (see [52, 163, 164]) we need:

Theorem 1 (Stewart-Walker Lemma)⁹ : *If a quantity $T^{A_t}{}_{B_m}$ vanishes in the background, then it is gauge invariant (to all orders).*

The resulting linear equations (see appendix D for some useful almost-FL relations) as well as their implementation and implications for almost-FLRW temperature anisotropies are discussed in chapter (5).

2.2.1 1+3 Covariant Eulerian threading formulation

The 1+3 Covariant Eulerian threading formulation of the theory can be recovered by boosting the exact Lagrangian equations to another frame. The initial choice of u^a is replaced by a new choice \tilde{u}^a :

$$\tilde{u}^a = \gamma(u^a + v^a) \quad \text{where} \quad v_a u^a = 0, \quad \gamma = (1 - v_a v^a)^{-1/2}, \quad (2.33)$$

where v^a is the (covariant) velocity of the new frame relative to the original frame. The exact transformations of all relevant quantities are given in the appendix B, these along with a more complete description of the multi-fluid case can be found in [120].

The formalism described above applies for *any* covariant choice of u^a . Typically one chooses \tilde{u}_a to be physically defined and timelike. If the physics picks out only one u^a , then that becomes the natural and obvious 4-velocity to use. In a complex multi-fluid situation, however, there are various possible choices (see [120, 39] for a more complete discussion of the multi-fluid case). The different particle species in cosmology will each have distinct 4-velocities (u_I^a); we could choose any of these as the fundamental frame, and other choices such as the centre of mass frame are also possible. Six typical species included as fluids (with unique four velocities, except for the cosmological constant) in

⁹The proof is straightforward, for \bar{T} the background part of T :

$$\text{If } \bar{T}^{\dots} = 0, \text{ then } \delta T^{\dots} = T^{\dots} - \bar{T}^{\dots} = T^{\dots}, \quad (2.32)$$

making the variables manifestly independent of the point mappings from the background \bar{S} to the lumpy universe S [51, 52, 164].

the field equations are (as measured in the respective rest frames):

$$\text{baryonic matter : } p_B = w_B \rho_B, \quad q_B^a = 0 = \pi_B^{ab}, \quad (2.34)$$

$$\text{cold dark matter: } p_C = 0 = q_C^a = \pi_C^{ab}, \quad (2.35)$$

$$\text{radiation : } p_R = \frac{1}{3} \rho_R, \quad (2.36)$$

$$\text{neutrinos(light): } p_N = \frac{1}{3} \rho_N, \quad (2.37)$$

$$\text{cosmological constant: } p_\Lambda = -\rho_\Lambda, \quad q_\Lambda^a = 0 = \pi_\Lambda^{ab} = v_\Lambda^a, \quad (2.38)$$

$$\text{electromagnetic fields: } \rho_H = \frac{1}{2}(E^2 + H^2) = \frac{1}{3} p_H. \quad (2.39)$$

The electromagnetic choice above is possible only if the *electro-magnetic*-fields are short wavelength and statistically homogeneous¹⁰.

This allows a variety of covariant choices of 4-velocities, each leading to a slightly different 1+3 covariant description. One particularly important choice is the total fluid or *overall* u^a -frame¹¹, defining the exact relative velocity of a particular component with respect to this frame, v_i^{*a} ; in the exact case (as shown in [120]) much care needs to be taken, however, in the linear case there is no difference between the dynamic quantities measured in the individual u_i^a -frames and the fundamental frame except for a simple correction to the heat flux – making a variety of additional gauge tricks available in the linear theory [80, 7]. One can regard a choice between these different possibilities as a partial gauge-fixing (but determined in a covariant and physical way). Any differences between such 4-velocities will be $\mathcal{O}(\epsilon)$ in the almost-FLRW case and will disappear in the FLRW limit,¹² as is required in a consistent 1+3 covariant and gauge-invariant linearization about an FLRW model (see chapter 5 and [52, 27] for further discussion). In addition the invariance of the linear order Weyl tensor under linear frame boosts ensures the existence of invariant potentials – such as Φ_A and Φ_H within the Bardeen Newtonian gauge formulation.

2.3 1+3 Covariant Lagrangian Kinetic Theory

Relativistic kinetic theory (see e.g. [106, 48, 162, 32, 15, 157]) provides a self-consistent microscopically based treatment where there is a natural unifying framework in which to deal with a gas of particles in circumstances ranging from hydrodynamic to free-streaming behavior. The photon gas undergoes a transition from hydrodynamic tight coupling with matter, through the process of decoupling from matter, to non-hydrodynamic free streaming. This transition is characterized by the evolution of the photon mean free path from (effectively) zero to infinity. The range of behavior can appropriately be described by kinetic theory with Thomson scattering [188, 178], and the baryonic matter

¹⁰Generically homogeneous fields do not behave like this particularly given that the pressure anisotropy then becomes important [10]

¹¹Usually denoted u^a , the Lagrangian frame choice; it is denoted u^{*a} in the linear theory for convenience; this choice underlies the multifluid formulation of the Lagrangian approach [113, 120].

¹²A similar situation occurs in relativistic thermodynamics, where suitable 4-velocities are close to the equilibrium 4-velocity, and hence to each other [89].

with which radiation interacts can reasonably be described hydrodynamically during these times. (The basic physics of radiation and matter and density perturbations in cosmology was developed in the works of Sachs and Wolfe [152], Silk [158], Peebles and Yu [143], Weinberg [187], Kaiser [95] and others, see [80]).

In the covariant Lagrangian approach of [58] (see also [47, 175]), the photon 4-momentum p^a (where $p^a p_a = 0$) is split as

$$p^a = E(u^a + e^a), \quad e^a e_a = 1, \quad e^a u_a = 0, \quad (2.40)$$

where $E = -u_a p^a$ is the energy and $e^a = p^{(a)}/E$ is the direction, as measured by a comoving (fundamental) observer. Then the photon distribution function (see 3.18) is decomposed into covariant harmonics via the expansion [58, 173]

$$f(x, p) = f(x, E, e) = F + F_a e^a + F_{ab} e^a e^b + \dots = \sum_{\ell \geq 0} F_{A_\ell} e^{(A_\ell)}, \quad (2.41)$$

where $e^{A_\ell} \equiv e^{a_1} e^{a_2} \dots e^{a_\ell}$, and $e^{(A_\ell)} \equiv O^{A_\ell}$ provides a representation of the rotation group [66] (as in 1). The covariant multipoles are irreducible since they are PSTF, i.e.

$$F_{A_\ell} = F_{\langle A_\ell \rangle} \Leftrightarrow F_{A_\ell} = F_{\langle A_\ell \rangle}, \quad F_{A_\ell} u^{a_\ell} = 0 = F_{A_\ell} h^{a_\ell - 1 a_\ell}.$$

They encode the anisotropy structure of the distribution in the same way as the usual spherical harmonic expansion

$$f = \sum_{\ell \geq 0} \sum_{m=-\ell}^{+\ell} f_\ell^m(x, E) Y_\ell^m(\vec{e}),$$

but here (a) the F_{A_ℓ} are covariant, and thus independent of any choice of coordinates in momentum space, unlike the f_ℓ^m ; (b) F_{A_ℓ} is a rank- ℓ tensor field on spacetime for each fixed E , and directly determines the ℓ -multipole of radiation anisotropy after integration over E . The multipoles can be recovered from the distribution function via [58, 66]

$$F_{A_\ell} = \Delta_\ell^{-1} \int f(x, E, e) e_{\langle A_\ell \rangle} d\Omega, \quad \text{with } \Delta_\ell = 4\pi \frac{(\ell!)^2 2^\ell}{(2\ell + 1)!}, \quad (2.42)$$

where $d\Omega = d^2 e$ is a solid angle in momentum space. A further useful identity is [58] (see chapter 1 for a more detailed account)

$$\int e^{A_\ell} d\Omega = \frac{4\pi}{\ell + 1} \begin{cases} 0 & \ell \text{ odd,} \\ h^{(a_1 a_2} h^{a_3 a_4} \dots h^{a_{\ell-1} a_\ell)} & \ell \text{ even.} \end{cases} \quad (2.43)$$

The first 3 multipoles arise from the radiation energy-momentum tensor, which is

$$T_R^{ab}(x) = \int p^a p^b f(x, p) d^3 p = \rho_R u^a u^b + \frac{1}{3} \rho_R h^{ab} + 2q_R^{(a} u^{b)} + \pi_R^{ab}, \quad (2.44)$$

where $d^3 p = E dE d\Omega$ is the covariant volume element on the future null cone at event x . It follows that the dynamic quantities of the radiation (in the u^a -frame) are:

$$\rho_R = 4\pi \int_0^\infty E^3 F dE, \quad q_R^a = \frac{4\pi}{3} \int_0^\infty E^3 F^a dE, \quad \pi_R^{ab} = \frac{8\pi}{15} \int_0^\infty E^3 F^{ab} dE. \quad (2.45)$$

The radiation dynamic quantities are relative to the fundamental frame; we do not need to relate them to their values in the radiation frame.

We extend these dynamic quantities to all multipole orders by defining¹³ [58]

$$\text{brightness anisotropy: } \Pi_{A_\ell} = \int E^3 F_{A_\ell} dE, \quad (2.46)$$

so that $\Pi = \rho_R/4\pi$, $\Pi^a = 3q_R^a/4\pi$ and $\Pi^{ab} = 15\pi^{ab}/8\pi$.

The Boltzmann equation is

$$\frac{df}{dv} \equiv p^a \frac{\partial f}{\partial x^a} - \Gamma^a_{bc} p^b p^c \frac{\partial f}{\partial p^a} = C[f], \quad (2.47)$$

where $p^a = dx^a/dv$ and $C[f]$ is the collision term, which determines the rate of change of f due to emission, absorption and scattering processes. This term is also decomposed into covariant harmonics:

$$C[f] = \sum_{\ell \geq 0} b_{A_\ell}(x, E) e^{A_\ell}, \quad (2.48)$$

where the multipoles $b_{A_\ell} = b_{\langle A_\ell \rangle}$ encode covariant irreducible properties of the particle interactions. Then the Boltzmann equation is equivalent to an infinite hierarchy of covariant multipole equations

$$L_{A_\ell}(x, E) = b_{A_\ell}[F_{A_m}](x, E),$$

where L_{A_ℓ} are the multipoles of df/dv , and will be given in the next section. These multipole equations are tensor field equations on spacetime for each value of the photon energy E – as discussed in the introduction (but note that energy changes along each photon path). Given the solutions $F_{A_\ell}(x, E)$ of the equations, the relation (2.41) then determines the full photon distribution $f(x, E, e)$ as a scalar field over phase space.

Over the period of importance for CMB anisotropies, i.e. considerably after electron-positron annihilation (spectral decoupling), the average photon energy is much less than the electron rest mass and the electron thermal energy may be neglected, so that the Compton interaction between photons and electrons (the dominant interaction between radiation and matter) may reasonably be described in the Thomson limit. (See [148] for refinements.) We will also neglect the effects of polarization (see e.g. [198, 29]). For Thomson scattering

$$C[f] = \sigma_T n_B E_B [\bar{f}(x, p) - f(x, p)], \quad (2.49)$$

where $E_B = -p_a u_B^a$ is the photon energy relative to the baryonic (i.e. baryon-electron) frame u_B^a , and $\bar{f}(x, p)$ determines the number of photons scattered into the phase space volume element at (x, p) . The differential Thomson cross-section is proportional to $1 + \cos^2 \alpha$, where α is the angle between initial and final photon directions in the baryonic frame; so $\cos \alpha = e_B^a e'_{Ba}$ where e'_{Ba} is the initial and e_B^a is the final direction, so that

$$p'^a = E_B (u_B^a + e'^a_B), \quad p^a = E_B (u_B^a + e^a_B). \quad (2.50)$$

¹³Because photons are massless, we do not need the complexity of the moment definitions used in [58]. In [28], $J_{A_\ell}^{(\ell)}$ is used, where $J_{A_\ell}^{(\ell)} = \Delta_\ell \Pi_{A_\ell}$. From now on, all energy integrals will be understood to be over the range $0 \leq E \leq \infty$.

Here we have used $E'_B = E_B$, which follows since the scattering is elastic. Here u_B^a is given by Eq. (2.33 with $\tilde{u}^a = u_B^a$), where v_B^a is the velocity of the baryonic frame relative to the fundamental frame u^a , with $v_B^a u_a = 0$. Then \bar{f} is given by [28, 148]:

$$\bar{f}(x, p) = \frac{3}{16\pi} \int f(x, p') \left[1 + \left(e_B^a e'_{Ba} \right)^2 \right] d\Omega'_B. \quad (2.51)$$

The exact forms of the photon energy and direction in the baryonic frame follow on using the boost equations (2.33 to find $p^a u_a^B$) and (B.2 to find $h_B^{ab} p_b$):

$$E_B = E \gamma_B \left(1 - v_B^a e_a \right), \quad (2.52)$$

$$e_B^a = \frac{1}{\gamma_B (1 - v_B^c e_c)} \left[e^a + \gamma_B^2 \left(v_B^b e_b - v_B^2 \right) u^a + \gamma_B^2 \left(v_B^b e_b - 1 \right) v_B^a \right]. \quad (2.53)$$

Anisotropic scattering will source polarization, and small errors are introduced by assuming that the radiation remains unpolarized [86]. A fully consistent and general treatment requires the incorporation of polarization. However, for simplicity, and in line with many previous treatments, we will neglect polarization effects here – in fact, we will follow the canonical approach, to include the corrections in the almost-FLRW theory via corrections to the damping scale.

2.3.1 Truncation conditions

The Ellis-Matruers-Treciokas treatment [62] makes various general statements about exact relativistic kinetic theory in the free streaming case. Of these the most relevant are :

1. **high- ℓ truncation** : If *any* four consecutive harmonics vanish, say those with $\ell = L + 1, L + 2, L + 3, L + 4$, but those for $\ell = L$ are non-zero, then

$$F_{\langle A_\ell \sigma_{a_{L+1} a_{L+2}} \rangle} = 0. \quad (2.54)$$

This means, as F_{A_ℓ} is non-zero, that the shear must vanish exactly, $\sigma_{ab} = 0$. The results arises from the regularity conditions: $\lim_{E \rightarrow 0} E^n F_{A_\ell} = 0$ and $\lim_{E \rightarrow \infty} E^n F_{A_\ell} = 0$.

2. **low- ℓ truncation** : It can also be shown that if the first 3 multipole harmonics are zero, *i.e.* , $\ell = 1, 2, 3$, once again the shear is necessarily zero, $\sigma_{ab} = 0$:

$$\sigma_{ab} \int_m^\infty E^5 \frac{\partial F}{\partial E} dE = 0. \quad (2.55)$$

Thus in both cases the resulting space-times are highly restricted, and do not include generic perturbations.

2.4 Multipole Divergence Equations

The full Boltzmann equation in photon phase space contains more information than necessary to analyze radiation anisotropies in an inhomogeneous universe. For that purpose, when the radiation is close to black-body we do not require the full spectral behaviour of the distribution multipoles, but only the energy-integrated multipoles. The monopole leads to the average temperature, while the higher order multipoles determine the temperature fluctuations. The 1+3 covariant and gauge-invariant definition of the average temperature T is given by [117]

$$\text{Stefan-Boltzmann: } \rho_r(x) = 4\pi \int E^3 F(x, E) dE = rT(x)^4, \quad (2.56)$$

where r is the radiation constant. If f is close to a Planck distribution, then T is the thermal black-body average temperature to identify r with k_B . But note that no notion of *background* temperature is involved in this definition. There is an all-sky average implied in (2.56). Fluctuations across the sky are measured by integrating the higher multipoles (a precise definition is given below), i.e. the fluctuations are determined by the Π_{A_ℓ} ($\ell \geq 1$) defined in Eq. (2.46).

The form of $C[f]$ shows that covariant equations for the temperature fluctuations arise from decomposing the **energy-integrated Boltzmann equation**

$$\int E^2 \frac{df}{dv} dE = \int E^2 C[f] dE \quad (2.57)$$

into 1+3 covariant multipoles. We begin with the right hand side, which requires the covariant form of the Thomson scattering term (2.51). Since the baryonic frame will move nonrelativistically relative to the fundamental frame in all cases of physical interest, it is sufficient to linearize only in v_B , and not in the other quantities. Thus we drop terms in $\mathcal{O}(\epsilon v_B^2, v_B^3)$ but do not neglect terms that are $\mathcal{O}(v_B^2, \epsilon v_B)$ or $\mathcal{O}(\epsilon^2)$ relative to the FLRW limiting background [120]. We make no restrictions on the geometric and physical quantities that covariantly characterize the spacetime, apart from assuming a nonrelativistic relative average velocity for matter. The resulting expression will in particular be applicable for covariant second-order effects in FLRW backgrounds (recognising that polarization effects should be included for a complete treatment), or for first-order effects in Bianchi backgrounds.

The order-of-magnitude notation

$$\mathcal{O}[3] \equiv \mathcal{O}(\epsilon v_B^2, v_B^3), \quad (2.58)$$

will be used, noting that this does *not* imply any second-order restriction on the dynamic, kinematic and gravito-electric/magnetic quantities. It follows from equations (2.43) and (2.51) that

$$4\pi \int \bar{f} E_B^3 dE_B = (\rho_R)_B + \frac{3}{4} (\pi_R^{ab})_B e_{Ba} e_{Bb}, \quad (2.59)$$

where the dynamic radiation quantities are evaluated in the *baryonic* frame. This approach, introduced by Maartens [120] relies on the frame-transformations given in appendix B; it allows us to evaluate the Thomson scattering integral directly. In the

process, we are also generalizing to include nonlinear effects. We use equations (B.8) and (B.11) (linearized in velocity) to transform back to the fundamental frame:¹⁴

$$\begin{aligned} (\rho_R)_B &= \rho_R \left[1 + \frac{4}{3} v_B^2 \right] - 2q_R^a v_{Ba} + \mathcal{O}[3], \\ (\pi_R^{ab})_B &= \pi_R^{ab} + 2v_{Bc} \pi_R^{c(a} u^{b)} - 2q_R^{(a} v_B^{b)} + \frac{4}{3} \rho_R v_B^{(a} v_B^{b)} + \mathcal{O}[3]. \end{aligned}$$

Now

$$\begin{aligned} \int E^2 C[f] dE &= n_E \sigma_T \left[1 + 3v_B^c e_c + (v_B^c e_c)^2 - \frac{3}{2} v_B^2 \right] \int E_B^3 \bar{f} dE_B \\ &\quad - n_E \sigma_T \left[1 - v_B^c e_c + \frac{1}{2} v_B^2 \right] \int f E^3 dE + \mathcal{O}[3]. \end{aligned} \quad (2.60)$$

In addition, we need the following identity, valid for any projected vector v^a :

$$v^a e_a f = \sum_{\ell \geq 0} \left[F_{\langle A_{\ell-1} v_{a\ell} \rangle} + \left(\frac{\ell+1}{2\ell+3} \right) F_{A_{\ell a} v^a} \right] e^{\langle A_{\ell} \rangle}. \quad (2.61)$$

(we use the convention that $F_{A_\ell} = 0$ for $\ell < 0$.) This identity may be proved using Eq. (2.43) and the identity (2.62) (see [58], p. 470):

$$\text{EMT Lemma 2:} \quad V_{\langle b} S_{A_\ell \rangle} = V_{(b} S_{A_\ell)} - \left(\frac{\ell}{2\ell+1} \right) V^c S_{c(A_{\ell-1} h_{a\ell b})} \quad (2.62)$$

$$\begin{aligned} \text{EMT Lemma 3:} \quad W_{\langle ab} S_{A_\ell \rangle} &= W_{(ab} S_{A_\ell)} - \frac{1}{2\ell+3} W_{cd} h^{cd} S_{(A_\ell h_{ab})} - \frac{2\ell}{2\ell+3} W_{d(a\ell} h_{ab} S_{A_{\ell-1})c} h^{cd} \\ &\quad + \frac{\ell(\ell-1)}{(2\ell+1)(2\ell+3)} h^{eg} h^f h^h W_{gh} S_{ef(A_{\ell-2} h_{a\ell-1} a_\ell h_{ab})}, \end{aligned} \quad (2.63)$$

where $S_{A_\ell} = S_{\langle A_\ell \rangle}$, $V_a = V_{(a)}$, $W_{ab} = W_{\langle ab \rangle}$ and $\ell \geq 0$. Using the above equations, we find that¹⁵

$$\begin{aligned} 4\pi \int E^2 C[f] dE &= n_E \sigma_T \left[\frac{4}{3} \rho_R v_B^2 - q_R^a v_{Ba} \right] \\ &\quad - n_E \sigma_T \left[3q_R^a - 4\rho_R v_B^a - 3\pi_R^{ab} v_{Bb} \right] e_a \\ &\quad - n_E \sigma_T \left[\frac{27}{4} \pi_R^{ab} - \frac{3}{2} q_R^{(a} v_B^{b)} - \frac{12}{7} \pi \Pi^{abc} v_{Bc} - 3\rho_R v_B^{(a} v_B^{b)} \right] e_{\langle a} e_{b \rangle} \\ &\quad - n_E \sigma_T \left[4\pi \Pi^{abc} - \frac{45}{4} \pi_R^{(ab} v_B^{c)} - \frac{16}{9} \pi \Pi^{abcd} v_{Bd} \right] e_{\langle a} e_b e_{c \rangle} + \dots + \mathcal{O}[3]. \end{aligned} \quad (2.64)$$

Now it is clear from equations (2.59) and (2.60) that the first four multipoles are affected by Thomson scattering differently than the higher multipoles. This is confirmed by the form of equation (2.64). Defining the energy-integrated scattering multipoles

$$K_{A_\ell} = \int E^2 b_{A_\ell} dE,$$

¹⁴As noted in Section III, we retain the $\mathcal{O}(v_B^2)$ term in $(\rho_R)_B$ since ρ_R is zero-order.

¹⁵A. Challinor has independently derived the same result [26].

we find from Eq. (2.64) that

$$K = n_E \sigma_T \left[\frac{4}{3} \Pi v_B^2 - \frac{1}{3} \Pi^a v_{Ba} \right] + \mathcal{O}[3], \quad (2.65)$$

$$K^a = -n_E \sigma_T \left[\Pi^a - 4 \Pi v_B^a - \frac{2}{3} \Pi^{ab} v_{Bb} \right] + \mathcal{O}[3], \quad (2.66)$$

$$K^{ab} = -n_E \sigma_T \left[\frac{9}{10} \Pi^{ab} - \frac{1}{2} \Pi^{(a} v_B^{b)} - \frac{3}{7} \Pi^{abc} v_{Bc} - 3 \Pi v_B^{(a} v_B^{b)} \right] + \mathcal{O}[3], \quad (2.67)$$

$$K^{abc} = -n_E \sigma_T \left[\Pi^{abc} - \frac{3}{2} \Pi^{(ab} v_B^{c)} - \frac{4}{9} \Pi^{abcd} v_{Bd} \right] + \mathcal{O}[3], \quad (2.68)$$

and, for $\ell > 3$:

$$K^{A\ell} = -n_E \sigma_T \left[\Pi^{A\ell} - \Pi^{(A\ell-1} v_B^{a\ell)} - \left(\frac{\ell+1}{2\ell+3} \right) \Pi^{A\ell a} v_{Ba} \right] + \mathcal{O}[3]. \quad (2.69)$$

Equations (2.65)–(2.69) are a nonlinear generalization of the linear results [28, 82]. They show the new *coupling of baryonic bulk velocity to the radiation multipoles, arising from local nonlinear effects in Thomson scattering*. If we linearize fully, i.e. neglect all terms containing v_B except the $\rho_R v_B^a$ term in the dipole K^a , which is first-order, then our equations reduce to those in [28]. The generalized nonlinear equations apply to the analysis of second-order effects on an FLRW background, to first-order effects on a spatially homogeneous but anisotropic background, and more generally, to any situation where the baryonic frame is non-relativistic relative to the fundamental u^a -frame.

Next we require the multipoles of df/dv . These can be read directly (with a little algebra) from the general expressions first derived in [58], which are exact, 1+3 covariant and also include the case of massive particles. For clarity (as pointed out in the introduction), we outline an alternative, 1+3 covariant derivation [120] (the derivation in [58] uses the divergence relations in an ONT-tetrad). We require the identity [47, 175]

$$\frac{dE}{dv} = -E^2 \left[\frac{1}{3} \Theta + A_a e^a + \sigma_{ab} e^a e^b \right], \quad (2.70)$$

which follows directly from (energy) $E = -p^a u_a$, (geodesic equation) $p^b \nabla_b p^a = 0$ and (kinematics) $\nabla_b u_a = -A_a u_b + D_b u_a$. Then¹⁶

$$\begin{aligned} \frac{d}{dv} \left[F_{A\ell}(x, E) e^{A\ell} \right] &= \frac{d}{dv} \left[E^{-\ell} F_{A\ell}(x, E) P^{A\ell} \right] \\ &= E \left\{ \left[\frac{1}{3} \Theta + A_b e^b + \sigma_{bc} e^b e^c \right] (\ell F_{A\ell} - E \partial_E [F_{A\ell}]) e^{A\ell} \right. \\ &\quad \left. + (u^{a_1} + e^{a_1}) \dots (u^{a_\ell} + e^{a_\ell}) \left[\dot{F}_{A\ell} + e^b \nabla_b F_{A\ell} \right] \right\}, \end{aligned}$$

¹⁶The relationship to the divergence relations can be seen from :

$$\frac{d}{dv} (F_{A\ell} P^{A\ell}) = \frac{d}{dv} (F_{A\ell} E^\ell (u^{A\ell} + e^{A\ell})) = \frac{d}{dv} (F_{A\ell} E^\ell \mathcal{O}^{A\ell})$$

and

$$\frac{d}{dv} (F_{A\ell} P^{A\ell}) = P^{A_{\ell+1}} \nabla_{\alpha_{\ell+1}} F_{A\ell} + \frac{dE}{dv} P^{A_{\ell+1}} \frac{\partial F_{A\ell}}{\partial E}.$$

where a “ ∂_E ” denotes $\partial/\partial E$. The first term is readily put into irreducible PSTF form using the identity (2.62) with $V_a = A_a$, and its extension to the case when V_a is replaced by a rank-2 PSTF tensor W_{ab} (2.63), with $W_{ab} = \sigma_{ab}$. In the second term, when the round brackets are expanded, only those terms with at most one u^{ar} survive, and

$$u^{a\ell} \dot{F}_{A_\ell} = -A^{a\ell} F_{A_\ell}, \quad u^{a\ell} \nabla^a F_{A_\ell} = -\left(\frac{1}{3}\Theta h^{aa\ell} + \sigma^{aa\ell} - \epsilon^{aa\ell c} \omega_c\right) F_{A_\ell}. \quad (2.71)$$

Thus the covariant multipoles b_{A_ℓ} of df/dv are

$$\begin{aligned} \frac{b_{A_\ell}}{E} &= \dot{F}_{(A_\ell)} - \frac{1}{3}\Theta E \partial_E [F_{A_\ell}] + D_{\langle a\ell} F_{A_{\ell-1}\rangle} + \frac{(\ell+1)}{(2\ell+3)} D^a F_{aA_\ell} \\ &- \frac{(\ell+1)}{(2\ell+3)} E^{-(\ell+1)} \partial_E \left[E^{\ell+2} F_{aA_\ell} \right] A^a - E^\ell \partial_E \left[E^{1-\ell} F_{(A_{\ell-1})} \right] A_{a\ell} \\ &- \ell \omega^b \varepsilon_{bc(a\ell} F_{A_{\ell-1})}{}^c - \frac{(\ell+1)(\ell+2)}{(2\ell+3)(2\ell+5)} E^{-(\ell+2)} \partial_E \left[E^{\ell+3} F_{abA_\ell} \right] \sigma^{ab} \\ &- \frac{2\ell}{(2\ell+3)} E^{-1/2} \partial_E \left[E^{3/2} F_{b(A_{\ell-1})} \right] \sigma_{a\ell}{}^b - E^{\ell-1} \partial_E \left[E^{2-\ell} F_{(A_{\ell-2})} \right] \sigma_{a_{\ell-1}a_\ell}. \end{aligned} \quad (2.72)$$

This regains the result of [58] [equation (4.12)] in the massless case, with minor corrections. The form given here benefits from the streamlined 1+3 covariant formalism of Maartens [113]. We reiterate that this result is exact and holds for any photon or (massless) neutrino distribution in any spacetime. We now multiply Eq. (2.72) by E^3 and integrate over all energies, using integration by parts and the fact that $E^n F_{A_\ell} \rightarrow 0$ as $E \rightarrow \infty$ for any positive n . We obtain the **multipole divergence equations** that determine the brightness multipoles Π_{A_ℓ} :

$$\begin{aligned} K_{A_\ell} &= \dot{\Pi}_{(A_\ell)} + \frac{4}{3}\Theta \Pi_{A_\ell} + D_{\langle a\ell} \Pi_{A_{\ell-1}\rangle} + \frac{(\ell+1)}{(2\ell+3)} D^b \Pi_{bA_\ell} \\ &- \frac{(\ell+1)(\ell-2)}{(2\ell+3)} A^b \Pi_{bA_\ell} + (\ell+3) A_{\langle a\ell} \Pi_{A_{\ell-1}\rangle} - \ell \omega^b \varepsilon_{bc(a\ell} \Pi_{A_{\ell-1})}{}^c \\ &- \frac{(\ell-1)(\ell+1)(\ell+2)}{(2\ell+3)(2\ell+5)} \sigma^{bc} \Pi_{bcA_\ell} + \frac{5\ell}{(2\ell+3)} \sigma^b{}_{\langle a\ell} \Pi_{A_{\ell-1}\rangle}{}^b - (\ell+2) \sigma_{\langle a_\ell a_{\ell-1}} \Pi_{A_{\ell-2}\rangle}. \end{aligned} \quad (2.73)$$

For photons undergoing Thomson scattering, the left hand side of Eq. (2.73) is given by Eq. (2.69), which is exact in the kinematic and dynamic quantities, but first order in the relative baryonic velocity. The equations (2.69) and (2.73) thus constitute a nonlinear generalization of the FLRW-linearized case [28, 82].

These equations describe evolution along the timelike world-lines of fundamental observers, not along the lightlike geodesics of photon motion. The timelike integration is related to light cone integrations by making homogeneity assumptions about the distribution of matter in (spacelike) surfaces of constant time, as is discussed in chapter 3 and [40].

The monopole and dipole of equation (2.73) give the evolution equations of energy and momentum density:

$$K = \dot{\Pi} + \frac{4}{3}\Theta \Pi + \frac{1}{3} D^a \Pi_a + \frac{2}{3} A^a \Pi_a + \frac{2}{15} \sigma^{ab} \Pi_{ab}, \quad (2.74)$$

$$\begin{aligned}
 K^a &= \dot{\Pi}^{(a)} + \frac{4}{3}\Theta\Pi^a + D^a\Pi + \frac{2}{3}D_b\Pi^{ab} \\
 &\quad + \frac{2}{3}A_b\Pi^{ab} + 4\Pi A^a - [\omega, \Pi]^a + \sigma^{ab}\Pi_b.
 \end{aligned} \tag{2.75}$$

For photons, K and K^a are given by equations (2.65) and (2.66) and determine the Thomson rates of transfer between the baryons and radiation ¹⁷.

Finally, we return to the definition of temperature anisotropies. As noted above, these are determined by the $\Pi_{A\ell}$. Generalizing the linearized 1+3 covariant approach in [166, 117], we define the **temperature fluctuation** $\tau(x, e)$ via the directional bolometric brightness:

$$T(x) [1 + \tau(x, e)] = \left[\frac{4\pi}{r} \int E^3 f(x, E, e) dE \right]^{1/4}. \tag{2.76}$$

This is a 1+3 covariant and gauge-invariant definition which is also exact. We can rewrite it explicitly in terms of the $\Pi_{A\ell}$:

$$\tau(x, e) = \left[1 + \left(\frac{4\pi}{\rho_R} \right) \sum_{\ell \geq 1} \Pi_{A\ell} e^{A\ell} \right]^{1/4} - 1. \tag{2.77}$$

In principle, we can extract the irreducible PSTF **temperature fluctuation multipoles** by using the inversion in Eq. (2.42):

$$\tau_{A\ell}(x) = \Delta_\ell^{-1} \int \tau(x, e) e_{(A\ell)} d\Omega. \tag{2.78}$$

In the almost-FLRW case, when τ is $O(\epsilon)$, we regain from Eq. (2.77) the linearized definition given in [166, 117]:

$$\tau_{A\ell} \approx \left(\frac{\pi}{\rho_R} \right) \Pi_{A\ell}, \tag{2.79}$$

where $\ell \geq 1$. In particular, the dipole and quadrupole are related to the hydrodynamic variables:

$$\tau^a \approx \frac{3q_R^a}{4\rho_R} \quad \text{and} \quad \tau^{ab} \approx \frac{15\pi_R^{ab}}{2\rho_R}. \tag{2.80}$$

2.5 Temperature Multipole Divergence Equations

We can normalize the radiation dynamic multipoles $\Pi^{A\ell}$ to define the dimensionless **generalized temperature anisotropies** ($\ell \geq 1$)

$$\mathcal{T}^{A\ell} = \left(\frac{\pi}{rT^4} \right) \Pi^{A\ell} \approx \tau^{A\ell}.$$

Thus the multipoles $\mathcal{T}^{A\ell}$ are equal to the temperature anisotropy multipoles plus non-linear corrections. In terms of these quantities, the hierarchy of radiation multipoles

¹⁷Total energy-momentum conservation has that $J_R^a = -J_B^a = U_T u^a + M_T^a$ where the energy transfer is $U_T = 4\pi K$ and the momentum transfer is $M_T^a = \frac{4\pi}{3} K^a$

becomes:

$$\begin{aligned} \frac{\dot{T}}{T} &= -\frac{1}{3}\Theta - \frac{1}{3}D_a \mathcal{T}^a \\ &\quad - \frac{4}{3}\mathcal{T}^a \frac{D_a T}{T} - \frac{2}{3}A_a \mathcal{T}^a - \frac{2}{15}\sigma_{ab}\mathcal{T}^{ab} + \frac{1}{3}n_E \sigma_T v_{Ba} (v_B^a - \mathcal{T}^a) + \mathcal{O}[3], \end{aligned} \quad (2.81)$$

$$\begin{aligned} \dot{\mathcal{T}}^a &= -4 \left(\frac{\dot{T}}{T} + \frac{1}{3}\Theta \right) \mathcal{T}^a - \frac{D^a T}{T} - A^a - \frac{2}{5}D_b \mathcal{T}^{ab} + n_E \sigma_T (v_B^a - \mathcal{T}^a) \\ &\quad + \frac{2}{5}n_E \sigma_T \mathcal{T}^{ab} v_{Bb} - \sigma^a_b \mathcal{T}^b - \frac{2}{5}A_b \mathcal{T}^{ab} + [\omega, \mathcal{T}]^a - \frac{8}{5}\mathcal{T}^{ab} \frac{D_b T}{T} + \mathcal{O}[3], \end{aligned} \quad (2.82)$$

$$\begin{aligned} \dot{\mathcal{T}}^{ab} &= -4 \left(\frac{\dot{T}}{T} + \frac{1}{3}\Theta \right) \mathcal{T}^{ab} - \sigma^{ab} - D^{(a} \mathcal{T}^{b)} - \frac{3}{7}D_c \mathcal{T}^{abc} - \frac{9}{10}n_E \sigma_T \mathcal{T}^{ab} \\ &\quad + n_E \sigma_T \left(\frac{1}{2}\mathcal{T}^{(a} v_B^{b)} + \frac{3}{7}\mathcal{T}^{abc} v_{Bc} + \frac{3}{4}v_B^{(a} v_B^{b)} \right) - 5A^{(a} \mathcal{T}^{b)} - \frac{4}{21}\sigma_{cd} \mathcal{T}^{abcd} \\ &\quad + 2\omega^c \varepsilon_{cd} {}^{(a} \mathcal{T}^{b)d} - \frac{10}{7}\sigma_c {}^{(a} \mathcal{T}^{b)c} - \frac{12}{7}\mathcal{T}^{abc} \frac{D_c T}{T} + \mathcal{O}[3], \end{aligned} \quad (2.83)$$

and, for $\ell > 3$:

$$\begin{aligned} \dot{\mathcal{T}}^{A\ell} &= -4 \left(\frac{\dot{T}}{T} + \frac{1}{3}\Theta \right) \mathcal{T}^{A\ell} - D^{(A\ell} \mathcal{T}^{A\ell-1)} - \frac{(\ell+1)}{(2\ell+3)} D_b \mathcal{T}^{bA\ell} - n_E \sigma_T \mathcal{T}^{A\ell} \\ &\quad + n_E \sigma_T \left[\mathcal{T}^{(A\ell-1} v_B^{a\ell)} + \left(\frac{\ell+1}{2\ell+3} \right) \mathcal{T}^{A\ell b} v_{Bb} \right] + \frac{(\ell+1)(\ell-2)}{(2\ell+3)} A_b \mathcal{T}^{bA\ell} \\ &\quad - (\ell+3) A^{(A\ell} \mathcal{T}^{A\ell-1)} + \ell \omega^b \varepsilon_{bc} {}^{(A\ell} \mathcal{T}^{A\ell-1)c} + (\ell+2) \sigma^{(A\ell} v_B^{a\ell-1} \mathcal{T}^{A\ell-2)} \\ &\quad + \frac{(\ell-1)(\ell+1)(\ell+2)}{(2\ell+3)(2\ell+5)} \sigma_{bc} \mathcal{T}^{bcA\ell} - \frac{5\ell}{(2\ell+3)} \sigma_b {}^{(A\ell} \mathcal{T}^{A\ell-1)b} - 4 \frac{(\ell+1)}{(2\ell+3)} \mathcal{T}^{A\ell b} \frac{D_b T}{T} + \mathcal{O}[3]. \end{aligned} \quad (2.84)$$

For $\ell = 3$, the Thomson term $\mathcal{T}^{(A\ell-1} v_B^{a\ell)}$ must be multiplied by $\frac{3}{2}$.

The nonlinear multipole equations given in this form show more clearly the evolution of temperature anisotropies (including the monopole, i.e. the average temperature T). Although the $\mathcal{T}^{A\ell}$ only determine the actual temperature fluctuations $\tau^{A\ell}$ to linear order, they are a useful dimensionless measure of anisotropy. Furthermore, equations (2.81)–(2.84) apply as the evolution equations for temperature anisotropies when the radiation anisotropy is small (i.e. $\mathcal{T}^{A\ell} = \tau^{A\ell}$), but the spacetime inhomogeneity and anisotropy are not restricted. This includes the particular case of small CMB anisotropies in general Bianchi universes (in which the ONT-tetrad formulation is of particular interest), or in perturbed Bianchi universes.

2.5.1 Almost-FLRW temperature MDE

FLRW-linearization, i.e. the case when only first order effects relative to the FLRW limit are considered, reduces the above equations to:

$$\frac{\dot{T}}{T} \approx -\frac{1}{3}\Theta - \frac{1}{3}D_a \tau^a, \quad (2.85)$$

$$\dot{\tau}^a \approx -\frac{D^a T}{T} - A^a - \frac{2}{5} D_b \tau^{ab} + n_E \sigma_T (v_B^a - \tau^a), \quad (2.86)$$

$$\dot{\tau}^{ab} \approx -\sigma^{ab} - D^{(a} \tau^{b)} - \frac{3}{7} D_c \tau^{abc} - n_E \sigma_T \tau^{ab}, \quad (2.87)$$

and, for $\ell \geq 3$:

$$\dot{\tau}^{A\ell} \approx -D^{(a\ell} \tau^{A\ell-1)} - \frac{(\ell+1)}{(2\ell+3)} D_b \tau^{bA\ell} - n_E \sigma_T \tau^{A\ell}. \quad (2.88)$$

These are the 1+3 covariant and gauge-invariant multipole generalizations of the Fourier mode formulation of the integrated Boltzmann equations used in the standard literature (see e.g. [82] and the references therein). Equations (2.85)–(2.87) were given in [117] in the free-streaming case $n_E = 0$. The set of equations here (2.85)–(2.87) (which describes the gravitational sources and scattering sources coupling to the anisotropies) and (2.88) (which gives the recursion relations that will be used to find the radiation transfer function) will be the focus of chapter 5, where explicit solutions of these equations will be constructed in a manifestly gauge invariant way. In practice we will drop the factor $\frac{1}{10}$ correction due to anisotropic scattering (as we now have in Eq. (2.87)) – and include it as a correction to the damping scale (see chapter 5).

As noted before, there is still a gauge freedom here associated with the choice of 4-velocity u^a . Given any physical choice for this 4-velocity which tends to the preferred 4-velocity in the FLRW limit, the $\ell \geq 1$ equations are gauge-invariant.

Truncation

A simplistic approach to linearization will suggest that the truncation relations in section 2.3.1 and their implications can be ignored in (linear-FRW) almost-FL universes, because the equations leading to these results are second-order relations and so can be dropped when linearizing. However that argument is not correct; both statements (2.54 and 2.55) will hold in almost-FL universes too. This can be seen as follows: although both $F_{A\ell}$ and σ_{ab} are at most $\mathcal{O}[1]$ (or $\mathcal{O}(\epsilon, v)$) in the almost-EGS sense [166], there are no zero or first order terms in the relevant equations leading to the above results, to explicitly linearize with respect to. Thus they cannot be dropped relative to larger (first order) terms in these equations, as there are no such terms; the first non-zero terms are second-order, and hence these equations with these terms must be obeyed even if we carrying out a (first-order) linearization. Thus they are both at most $\mathcal{O}(\epsilon^2)$ equations, but are both still valid in the almost-FL universes¹⁸. What this means is that one must be very careful about any form of *ad-hoc* truncation in the multipole hierarchy, even in the almost-FL universes - this includes not only the Free-Streaming case but the case with a generalized Krook equation for the scattering term [62].

In the matter dominated almost-FL models with scalar perturbations, any truncation leading to zero shear would suppress the perturbations, reducing the dynamics to that of *exactly* FL. This doesn't mean that one cannot consistently damp the higher

¹⁸Although $\mathcal{O}(\epsilon^2) \ll \mathcal{O}(\epsilon)$, this doesn't mean that $\mathcal{O}(\epsilon^2) = 0$ on its own, even though formally the notation $\mathcal{O}(\epsilon^2) \simeq 0$ is often adopted. These are not equivalent.

moments out, it just means that they cannot be formally truncated – when higher moments are ignored the consistency condition in the exact theory should still be taken into account.

Where this does seem to make a difference, as one example, is on the intermediate scale where one is tempted to drop everything with $\ell \geq 3$, *i.e.*, when one is close to tight-coupling. However, this is, strictly speaking a truncation and hence problematic. This is why a perturbative analysis in the Thompson scattering time is important; one can in this way consistently, without truncation, build up the entire Multipole Divergence Equation hierarchy perturbatively - provided one has a meaningful sense of smallness, in this case the relaxation time (we do this for small scales in chapter 5). There one is decoupling a subset of the hierarchy from the full set of Multipole Divergence Equations, as in the Gauge Invariant formulation of Hu-Sugiyama [82, 83]. This suggests that one needs to be extremely careful when approximating relativistic gas effects in terms of truncated theories in generic geometries or even almost-FL. Such truncations may not always be consistent with gravity to the order of the calculation. The use of the Newtonian frame, at linear order, is one way of trying to get around this problem – but is in itself highly restrictive with application limited to anisotropies due to gravitational tidal forces only.

2.6 Tetrad Multipole Divergence Equations

In order to contrast the 1+3 Lagrangian threading formulation using a covariant choice of u^a -frame we outline the tetrad formulation of the MDE's. Why?, because nonperturbative temperature anisotropies can be investigated within the context of the tetrad formulation of the Multipole Divergence Equations following the approach of [58] given that the angular correlation functions can be constructed for generic temperature anisotropies [66] (see E.25 and chapter 1). The 1+3 Lagrangian formulation is valid for any tetrad or coordinate system as it is formulated in a general basis.

The ONT tetrad formalism used in cosmology is as outlined in [51], however, the conventions of [58, 62, 50] are used, second, the MDE's are put explicitly into the tetrad.

2.6.1 Tetrad formalism in cosmology

A **tetrad** is a set of four orthogonal unit basis vector fields $\{\mathbf{e}_a\}$ which can be written in terms of a local coordinate basis using the **tetrad components** : $E_a^i(x^i) = \{u_a(x^i), e_a^i(x^i)\}$ as in [58] (See appendix A.4)

$$\mathbf{e}_a = E_a^i(x^j) \frac{\partial}{\partial x^i} \Rightarrow \mathbf{e}_a(x^j) = E_a^i(x^j). \quad (2.89)$$

Here the notation is as : $\mathbf{e}_a(f) = E_a^i(x^j) \partial f / \partial x^i$. The metric tensor components in this tetrad are as :

$$g_{ab} = g_{ij} E_a^i E_b^j = \mathbf{e}_a \cdot \mathbf{e}_b = \eta_{ab} \quad (2.90)$$

where $\eta_{ab} = \text{diag}(-1, +1, +1, +1)$ is the Minkowski metric. The coordinate form of the metric are constructed from the tetrad as : $g_{ij}(x^k) = \eta_{ab} E_a^i(x^k) E_b^j(x^k)$. The commutators defined by a tetrad give the **structure functions** $\gamma^a_{bc}(x^i)$:

$$[\mathbf{e}_a, \mathbf{e}_b] = \gamma^c_{ab}(x^i) \mathbf{e}_c. \quad (2.91)$$

This means that the structure functions are antisymmetric in their lower indices a and b . The connection components Γ^a_{bc} for the tetrad¹⁹ are

$$\Gamma_{abc} = \mathbf{e}_a \cdot \nabla_{\mathbf{e}_b} \mathbf{e}_c = E_{a_i} E_b^j \nabla_j E_c^i.$$

Now all the covariant derivatives in the 1+3 Lagrangian threading approach can be written out in tetrad components. The useful point here is that for $g_{ab} = \eta_{ab}$ we have that $\mathbf{e}_a(g_{bc}) = 0$ giving us torsion free connections such that

$$\gamma^a_{bc} = +(\Gamma^a_{bc} - \Gamma^a_{cb}).$$

Finally, the tetrad must satisfy the **integrability conditions** (from the Jacobi identities) : $\mathbf{e}_{[a}(\gamma^d_{bc])} = -\gamma^e_{[ab}\gamma^d_{c]e}$.

For cosmology we choose \mathbf{e}_0 to be the unit tangent of the matter flow, u^a . We will thread the spacetime with respect to \mathbf{e}_0 . This reduces the freedom in the tetrad down to three-parameter rotations of the spatial frame \mathbf{e}_α with triad components e^a_i . As in [58] we then have

$$\Gamma^0_{0\nu} = A_\nu, \quad (2.92)$$

$$\Gamma_{\nu\mu 0} = \sigma_{\nu\mu} + \frac{1}{3}\Theta\delta_{\nu\mu} + \epsilon_{\nu\mu\sigma}\omega^\sigma, \quad (2.93)$$

$$\Gamma_{\mu 0\nu} = \epsilon_{\mu\nu\sigma}\Omega^\sigma, \quad (2.94)$$

$$\Gamma_{\kappa\mu\nu} = \frac{1}{2}(\epsilon_{\mu\nu\tau}n_{\kappa}{}^\tau + \epsilon_{\kappa\mu\tau}n_{\nu}{}^\tau - \epsilon_{\nu\kappa\tau}n_{\mu}{}^\tau) + \delta_{\nu\mu}a_\kappa - \delta_{\mu\kappa}a_\nu. \quad (2.95)$$

Here the rotation coefficients are in terms of the 3-dimensional quantities : A_ν , $\sigma_{\mu\nu}$, ω_ν , Θ , Ω^ν , $n_{\nu\mu}$ and a_ν . These are the acceleration, shear, vorticity, expansion, the rotation of the basis and the symmetric and antisymmetric parts of $\gamma^\mu_{\kappa\sigma}\epsilon^{\nu\kappa\sigma}$ which give the space-like connection.

2.6.2 Temperature anisotropies in a tetrad

One can then construct the conversion from the 1+3 Lagrangian threading formalism (l.h.s. is on the manifold M) to that in terms of the tetrad (r.h.s. is in the tangent bundle $T(M)$) [58, 62] (2.73), where Π_{A_ℓ} is PSTF with respect to a u^a -frame:

$$D^c \Pi_{\langle A_\ell \rangle c} = h^{cd} \mathbf{e}_d(\Pi_{\langle A_\ell \rangle c}) + 2\ell \Pi_{bc\langle A_{\ell-1} \rangle \epsilon_{a_\ell}{}^{bd} n^c{}_d - 2(\ell+1)a^b \Pi_{b\langle A_\ell \rangle}, \quad (2.96)$$

$$\dot{\Pi}_{\langle A_\ell \rangle} = \mathbf{e}_0(\Pi_{\langle A_\ell \rangle}) - \ell \Pi_{d\langle A_{\ell-1} \rangle \epsilon^d{}_{a_\ell} \Omega^c}, \quad (2.97)$$

$$\begin{aligned} D_{\langle a_\ell \rangle \Pi_{A_{\ell-1} \rangle} &= \mathbf{e}_{\langle a_\ell \rangle}(\Pi_{A_{\ell-1} \rangle}) - (\ell-1) \Pi_{\langle A_{\ell-2} \rangle n_{a_{\ell-1} \rangle}{}^c \epsilon^d{}_{a_\ell} \rangle c \\ &\quad - (\ell-1) a^d \Pi_{d\langle A_{\ell-2} \rangle \delta_{a_{\ell-1} \rangle a_\ell} \rangle + (\ell-1) \Pi_{\langle A_{\ell-1} \rangle a_{a_\ell} \rangle}. \end{aligned} \quad (2.98)$$

Using these in the 1+3 MDE Eq. (2.73) we find the MDE in the tetrad :

$$\mathbf{e}_0(\Pi_{\langle a_\ell \rangle}) + \mathbf{e}_{\langle a_\ell \rangle}(\Pi_{A_{\ell-1} \rangle}) + \mathbf{e}_\alpha(\Pi^\alpha_{\langle A_\ell \rangle}) + \frac{4}{3}\Theta \Pi_{A_\ell}$$

¹⁹The convention in [51] is : $\Gamma_{abc} = \mathbf{e}_a \cdot \nabla_{\mathbf{e}_c} \mathbf{e}_b$.

$$\begin{aligned}
& -\ell \Pi_{d(A_{\ell-1} \epsilon^d_{a\ell} c) \Omega^c} - (\ell-1) \Pi_{(A_{\ell-2} n_{a\ell-1} c \epsilon^d_{a\ell} c)} - (\ell-1) a^d \Pi_{d(A_{\ell-2} \delta_{a\ell-1} a_\ell)} \\
& + (\ell-1) \Pi_{(A_{\ell-1} a_{a\ell})} + 2\ell \Pi_{bc(A_{\ell-1} \epsilon_{a\ell})}{}^{bd} n^c{}_d - 2(\ell+1) a^b \Pi_{b(A_\ell)} \\
& = K_{A_\ell} + \frac{(\ell+1)(\ell-2)}{(2\ell+3)} A^b \Pi_{bA_\ell} - (\ell+3) A_{(a_\ell} \Pi_{A_{\ell-1})} + \ell \omega^b \epsilon_{bc(a_\ell} \Pi_{A_{\ell-1})}{}^c \quad (2.99)
\end{aligned}$$

$$+ \frac{(\ell-1)(\ell+1)(\ell+2)}{(2\ell+3)(2\ell+5)} \sigma^{bc} \Pi_{bcA_\ell} - \frac{5\ell}{(2\ell+3)} \sigma^b{}_{(a_\ell} \Pi_{A_{\ell-1})b} + (\ell+2) \sigma_{(a_\ell a_{\ell-1}} \Pi_{A_{\ell-2})}. \quad (2.100)$$

This is not pursued further here, these are however the equations that would need to be solved in order to recover the anisotropies in the context of Bianchi models.

2.7 Qualitative analysis of Nonlinearity

The generalized equations are given first, and can form the basis for investigating the implications of nonlinear dynamical effects in general. The linearized equations are then briefly given in hydrodynamic form – together these can be used to investigate second-order effects against a FLRW background. More quantitative and detailed investigations along these lines are taken up in chapter 6. Here we will confine ourselves to a qualitative analysis: first, effects on the kinematics, gravitational and dynamic quantities, and second, nonlinear effects on the radiation multipoles.

2.7.1 generalized equations

For photons, the equations follow from the previous Section (2.2) as:

$$\dot{\rho}_R + \frac{4}{3} \Theta \rho_R + D_a q_R^a + 2A_a q_R^a + \sigma_{ab} \pi_R^{ab} = n_E \sigma_T \left(\frac{4}{3} \rho_R v_B^2 - q_R^a v_{Ba} \right) + \mathcal{O}[3], \quad (2.101)$$

$$\dot{q}_R^{(a)} + \frac{4}{3} \Theta q_R^a + \frac{4}{3} \rho_R A^a + \frac{1}{3} D^a \rho_R + D_b \pi_R^{ab} \quad (2.102)$$

$$+ \sigma^a{}_b q_R^b - [\omega, q_R]^a + A_b \pi_R^{ab} = n_E \sigma_T \left(\frac{4}{3} \rho_R v_B^a - q_R^a + \pi_R^{ab} v_{Bb} \right) + \mathcal{O}[3].$$

The nonlinear dynamical equations are completed by the integrated Boltzmann multipole equations – (see 2.73).

The quadrupole evolution equation is

$$\begin{aligned}
& \dot{\pi}_R^{(ab)} + \frac{4}{3} \Theta \pi_R^{ab} + \frac{8}{15} \rho_R \sigma^{ab} + \frac{2}{5} D^{(a} q_R^{b)} + \frac{8\pi}{35} D_b \Pi^{ab} \\
& + 2A^{(a} q_R^{b)} - 2\omega^c \epsilon_{cd}{}^{(a} \pi_R^{b)d} + \frac{2}{7} \sigma_c{}^{(a} \pi_R^{b)c} - \frac{32\pi}{315} \sigma_{cd} \Pi^{abcd} \\
& = -n_E \sigma_T \left[\frac{9}{10} \pi_R^{ab} - \frac{1}{5} q_R^{(a} v_B^{b)} - \frac{8\pi}{35} \Pi^{abc} v_{Bc} - \frac{2}{5} \rho_R v_B^{(a} v_B^{b)} \right] + \mathcal{O}[3]. \quad (2.103)
\end{aligned}$$

In the free-streaming case $n_E = 0$, equation (2.103) reduces to the result first given in [166]. This quadrupole evolution equation is central to the proof that almost-isotropy of the CMB after last scattering implies almost-homogeneity of the universe [166] (see chapter 4).

The higher multipoles ($\ell > 3$) evolve according to

$$\begin{aligned}
 \dot{\Pi}^{(A_\ell)} + \frac{4}{3}\Theta\Pi^{A_\ell} + D^{(a_\ell}\Pi^{A_{\ell-1})} + \frac{(\ell+1)}{(2\ell+3)}D_b\Pi^{bA_\ell} \\
 - \frac{(\ell+1)(\ell-2)}{(2\ell+3)}A_b\Pi^{bA_\ell} + (\ell+3)A^{(a_\ell}\Pi^{A_{\ell-1})} - \ell\omega^b\varepsilon_{bc}^{(a_\ell}\Pi^{A_{\ell-1})c} \\
 - \frac{(\ell-1)(\ell+1)(\ell+2)}{(2\ell+3)(2\ell+5)}\sigma_{bc}\Pi^{bcA_\ell} + \frac{5\ell}{(2\ell+3)}\sigma_b^{(a_\ell}\Pi^{A_{\ell-1})b} - (\ell+2)\sigma^{(a_\ell a_{\ell-1}}\Pi^{A_{\ell-2})} \\
 = -n_E\sigma_T \left[\Pi^{A_\ell} - \Pi^{(A_{\ell-1}v_B^{a_\ell})} - \left(\frac{\ell+1}{2\ell+3} \right) \Pi^{A_\ell a} v_{Ba} \right] + \mathcal{O}[3]. \tag{2.104}
 \end{aligned}$$

For $\ell = 3$, the second term in square brackets on the right of Eq. (2.104) must be multiplied by $\frac{3}{2}$. The temperature fluctuation multipoles τ_{A_ℓ} are determined in principle from the radiation dynamic multipoles Π_{A_ℓ} via equations (2.77) and (2.78).

These equations show precisely which physical effects are directly responsible for the evolution of CMB anisotropies in an inhomogeneous universe. They show how the matter content of the universe generates anisotropies. This happens directly through the interaction of matter with the radiation, as encoded in the Thompson scattering terms on the right of equations (2.101), (2.102), (2.103) and (2.104). It happens indirectly, as matter generates inhomogeneities in the gravitational field via the field equations (2.13)–(2.24) and the evolution equation (2.13) and for the baryons in the overall frame [120], for the baryonic²⁰ velocity v_B^a . This in turn feeds back into the multipole equations via the kinematic quantities, the baryonic velocity v_B^a , and the spatial gradient $D_a\rho_R$ in the dipole equation (2.102). The coupling of the multipole equations themselves provides an up and down cascade of effects, shown in general by equation (2.104).

Power is transmitted to the ℓ -multipole by lower multipoles through the dominant (linear) *distortion term*, $D^{(a_\ell}\Pi^{A_{\ell-1})}$, as well as through nonlinear terms coupled to the *4-acceleration* ($A^{(a_\ell}\Pi^{A_{\ell-1})}$ – to generate couplings to non-geodesic effects), *baryonic velocity* ($v_B^{(a_\ell}\Pi^{A_{\ell-1})}$ – to generate couplings to the peculiar velocities and nonlinear kinetic-SZ effects), and *shear* ($\sigma^{(a_\ell a_{\ell-1}}\Pi^{A_{\ell-2})}$ – to generate, not only, nonlinear coupling to gravitational tidal forces by also to gravitational radiation). Power cascades down from higher multipoles through the linear *divergence term*, $(\text{div } \Pi)^{A_\ell}$ (which is how the lower multipoles see the spatial curvature in the almost-FLRW linearization), and the nonlinear terms coupled to A^a , v_B^a and σ^{ab} . The vorticity coupling does not transmit across multipole levels.

2.7.2 Linearized hydrodynamic equations

The equations for the radiation (and neutrino) multipoles generalize the equations in the literature [28, 82], to which they reduce when we remove all terms $O(\epsilon v_B)$ and $O(\epsilon^2)$. In this case, *i.e.* the FLRW-linearization, there is major simplification of the equations (in the hydrodynamics variables) :

$$\dot{\rho}_R + \frac{4}{3}\Theta\rho_R + \text{div } q_R \approx 0, \tag{2.105}$$

²⁰it is important to realize that the $c_s^2 = \dot{p}_B/\dot{\rho}_B$ only holds to linear order

$$\dot{q}_R^a + 4Hq_R^a + \frac{4}{3}\rho_R A^a + \frac{1}{3}D^a \rho_R + (\text{div } \pi_R)^a \approx n_E \sigma_T \left(\frac{4}{3}\rho_R v_B^a - q_R^a \right) \quad (2.106)$$

$$\dot{\pi}_R^{ab} + 4H\pi_R^{ab} + \frac{8}{15}\rho_R \sigma^{ab} + \frac{2}{5}D^{(a} q_R^{b)} + \frac{8\pi}{35} (\text{div } \Pi)^{ab} \approx -\frac{9}{16}n_E \sigma_T \pi_R^{ab}, \quad (2.107)$$

and for $\ell \geq 3$ (where the hydrodynamic multipole variable is $J_{A_\ell} = \Delta_\ell \Pi_{A_\ell}$)

$$\dot{\Pi}^{A_\ell} + 4H\Pi^{A_\ell} + D^{(A_\ell} \Pi^{A_{\ell-1})} + \frac{(\ell+1)}{(2\ell+3)} (\text{div } \Pi)^{A_\ell} \approx -n_E \sigma_T \Pi^{A_\ell}. \quad (2.108)$$

These linearized equations, together with the linearized equations governing the kinematic and free gravitational quantities, given by equations (2.13)–(2.24) with zero right hand sides, may be covariantly split into scalar, vector and tensor modes, as described in [20, 39, 6, 81]. The modes can then be expanded in covariant eigentensors of the comoving Laplacian (to give a Helmholtz equation), and the Fourier coefficients obey ordinary differential equations, facilitating numerical integration (using the CMBFAST code) [155] or [28] (using a modified CMBFAST code in the CDM frame). The canonical analytic treatment can be found from [82, 83] with the 1+3 covariant and gauge invariant formulation in chapter 5 (and [67]).

However, in the nonlinear case, it is no longer possible to split into scalar, vector and tensor modes [150, 133, 21]. An example of this arises in dust spacetimes – a simplified model after last scattering in which we neglect the dynamical effects of baryons, radiation and other components. If one attempts to carry over the linearized scalar-mode conditions [20, 28], $\omega_a = 0 = H_{ab}$ (no vorticity and no gravitational waves), into the nonlinear regime, it turns out that a non-terminating chain of integrability conditions must be satisfied, so that the models are in general inconsistent unless they have high symmetry [182, 121] or additional physical restrictions (such as the use of perturbative assumptions). In particular, gravitational radiation, with $\text{curl } H_{ab} \neq 0$ (see [38, 79, 116]), must in general be present. Even in this simplest case it is not possible to locate exact scalar modes. In the case of the linearized perturbative treatment ($\omega_a \approx 0 \approx H_{ab}$) – the almost-FLRW Newtonian frame formalism, we show in chapter 6 that even with additional weak (or mildly nonlinear) assumptions, that it is not possible on small scales to expect non-trivial nonlinearity in the multipole divergence equations for the temperature anisotropies.

2.7.3 Nonlinear effects on kinematic, gravitational and dynamic quantities

Evolution of the expansion of the universe Θ , given by equation (2.14), is retarded by the nonlinear shear term $-\sigma_{ab}\sigma^{ab}$, and accelerated by the nonlinear vector terms $+A_a A^a$ and $+2\omega_a \omega^a$ (see also [50]). The vorticity evolution equation (2.16) has a nonlinear coupling $\sigma_{ab}\omega^b$ of vorticity to shear, whose effect will depend on the alignment of vorticity relative to the shear eigendirections. The shear evolution equation (2.17) has tensor-tensor and vector-vector type couplings, which are the tensor counterpart of similar terms in the expansion evolution. Relative velocity effects enter via the total anisotropic stress term [120]. Baryonic and cold dark matter contributions of the form $\rho v^{(a} v^{b)}$ to the shear evolution arise at the nonlinear level. The constraint equations (2.20) and (2.21) show

that acceleration and vorticity provide scalar ($A^a\omega_a$) and vector ($[\omega, A]_a$) nonlinear source terms for respectively the vorticity and shear.

The free gravitational fields, which 1+3 covariantly describe tidal forces and gravitational radiation (see [50, 78, 38, 79, 116]), and therefore in particular control the tensor contribution to CMB anisotropies, are governed by the Maxwell-like equations (2.18), (2.19), (2.23) and (2.24). This is the foundation for the electromagnetic analogy. The role of nonlinear coupling terms in these equations is more complicated – see [115] for a full discussion. Here we note that nonlinear couplings of the shear and vorticity to the energy flux and gravito-magnetic field act as source terms for the gravito-electric field – see Eq. (2.23), while nonlinear couplings of the shear and vorticity to the anisotropic stress and gravito-electric field act as source terms for the gravito-magnetic field – see Eq. (2.24). This leads one to suspect that low- ℓ nonlinear pathologies in the multipole hierarchy are influenced by nonlinear gravitational couplings between gravitational waves and the gravitational tidal forces.

The linear parts of the baryonic (perfect fluid) and cold dark matter (dust) energy and relative velocity equations have nonlinear relative velocity terms acting as sources. While the 4-acceleration A_a is involved in correction terms in all these equations, the vorticity ω_a and shear σ_{ab} only enter nonlinear corrections of the velocity equations, and not the energy density equations. This reflects the fact that vorticity and shear are volume-preserving.

The kinematic corrections to the evolution of matter relative velocity are of the form $A_a v^a$, $[\omega, v]_a$ and $\sigma_{ab} v^b$. For the massless species the same form of corrections arises in the energy flux evolution [120], since energy flux is of the form $\frac{4}{3}\rho v^a$ when the photon and neutrino frames are chosen as the energy frame. Vorticity also does not affect energy density, but shear does, owing to the intrinsic anisotropic stress of photons and neutrinos, which couples with the shear.

Baryonic and radiation conservation equations are both affected by nonlinear Thomson correction terms, which involve a coupling of the baryonic relative velocity v_B^a to the radiation energy density, momentum density and anisotropic stress. In particular, we note that there is a *nonzero energy density transfer* due to Thomson scattering at second-order – that is a pathology of corrections due to the nonlinear kinetic-SZ effects.

2.7.4 Nonlinear effects on radiation multipoles

Nonlinear Thomson scattering corrections also affect the evolution of the radiation quadrupole π_R^{ab} , as shown by equation (2.103). In this case, the baryonic relative velocity couples to the radiation dipole q_R^a and octopole Π^{abc} :

The general evolution equation (2.104) for the radiation dynamic multipoles $\Pi^{A\ell}$ shows that *five successive multipoles*, i.e. for $\ell - 2, \dots, \ell + 2$, are linked together in the nonlinear case. Furthermore, the 4-acceleration A_a couples to the $\ell \pm 1$ multipoles, the vorticity ω_a couples to the ℓ multipole, and the shear σ_{ab} couples to the $\ell \pm 2$ and ℓ multipoles. All of these couplings are nonlinear, except for $\ell = 1$ in the case of A_a , and $\ell = 2$ in the case of σ_{ab} . These latter couplings that survive linearization are shown in the dipole equation (2.102) (i.e. $\rho_R A^a$) and the quadrupole equation (2.103) (i.e. $\rho_R \sigma^{ab}$). The latter term drives diffusion damping during the decoupling process, for example, it determines the anisotropic pressure within the tight-coupling approximation (5.135) –

nonlinear matter shear breaks the tight-coupling approximation (see [123]). Nonlinear corrections introduce additional acceleration and shear terms. *Vorticity corrections are purely nonlinear*, i.e. vorticity has no effect at the linear level, and a linear approach could produce the false impression that vorticity has no direct effect at all on the evolution of CMB anisotropies. However, for very high ℓ , i.e. on very small angular scales, the nonlinear vorticity term could in principle be non-negligible.

The disappearance of most of the kinematic terms upon linearization is further reflected in the fact that the linearized equations link only *three* successive moments, i.e. ℓ , $\ell \pm 1$. This is clearly seen in equation (2.108).

In addition to A_a and ω_a , there is a further vector coupling at the nonlinear level, i.e. the coupling of the baryonic velocity v_B^a to the $\ell \pm 1$ multipoles in the Thomson scattering source term of the evolution equation (2.104). In fact these nonlinear velocity corrections are of precisely the same tensorial form as the acceleration corrections on the left hand side, only with different weighting factors. Linearization, by removing these terms, also has the effect of removing the nonlinear contribution of the radiation multipoles $\Pi^{A_{\ell \pm 1}}$ to the collision multipole $K^{A_{\ell}}$.

High- ℓ nonlinear effects

One notable feature of the nonlinear terms is that some of them scale like ℓ for large ℓ , as already noted in the case of vorticity. There are no purely linear terms with this property, which has an important consequence, i.e. that *for very high ℓ multipoles (corresponding to very small angular scales in CMB observations), certain nonlinear terms can reach the same order of magnitude as the linear contributions.* (Note that the same effect applies to the neutrino background.) The relevant nonlinear terms in Eq. (2.104) are (for $\ell \gg 1$):

$$-\ell \left(\frac{1}{4} \sigma_{bc} \Pi^{bcA_{\ell}} + \sigma^{(a_{\ell} a_{\ell-1}} \Pi^{A_{\ell-2}} \right) - A^{(a_{\ell}} \Pi^{A_{\ell-1}} + \frac{1}{2} A_a \Pi^{bA_{\ell}} + \omega^b \varepsilon_{bc}^{(a_{\ell}} \Pi^{A_{\ell-1}c)} \right).$$

The observable imprint of this effect will be made after last scattering. In the free-streaming era, it is reasonable to neglect the vorticity relative to the shear. We can remove the acceleration term by choosing u^a as the dynamically dominant cold dark matter frame (i.e. choosing $v_C^a = 0$), as in [28]. It follows from equations (2.79) and (2.104) that the nonlinear correction to the rate of change of the linearized temperature fluctuation multipoles is

$$(\delta \tau_{NL}^{A_{\ell}}) \sim \ell \left(\frac{1}{4} \sigma_{bc} \tau^{bcA_{\ell}} + \sigma^{(a_{\ell} a_{\ell-1}} \tau^{A_{\ell-2}} \right) \quad \text{for } \ell \gg 1. \quad (2.109)$$

We write $\tau_{A_{\ell}}$ instead of $\Pi_{A_{\ell}}$ as the nonlinear feedback is implicitly excluded – we are interested in the weak (or mildly) nonlinear effect. The linear solutions for $\tau_{A_{\ell}}$ and σ_{ab} can be used on the right hand side to estimate the correction. This correction is essentially non-perturbative and so should not be confused with a second or higher order perturbative approach. Nonlinear effects due to structure formation and evolution will generate mildly nonlinear corrections to the shear which are investigated in chapter 6. The point here is that nonperturbative high- ℓ effects can be investigated in the context of weakly nonlinear corrections to an almost-FLRW model.

The new nonlinear effect on observed anisotropies will be estimated by integrating $\delta(\dot{\tau}^{A_t})$ from last scattering to now. (See chapter 1 [66] for the relation between the τ_{A_t} and the angular correlations). In the case of scalar perturbations, these solutions are given in the manifestly gauge invariant form and reduced to the Newtonian frame²¹ in chapter 5 [67].

Finally, we note that the well-known Vishniac (or higher order kinetic-SZ effects) and Rees-Sciama second-order effects also become significant at high ℓ , and can eventually dominate the linear contributions to CMB anisotropies on small enough angular scales (typically $\ell > 10^3$ or more) [82]. These will provide weak (or mildly nonlinear) source terms to either the shear or scattering correction (in the total frame) which will then couple to the anisotropies. These too can be investigated within a similar weakly nonlinear almost-FLRW framework.

Low- ℓ nonlinear corrections

The important exclusion in this thesis is that of the low- ℓ nonlinear corrections (typically $\ell = 1 \dots 30$). These can be seen, from Eq. (2.104), as being due to shear, vorticity and acceleration couplings to anisotropies. This would be a pathology of large scale anisotropies with respect to a background that has intrinsic weak gravitational nonlinearity. For this reason low- ℓ nonlinear effects are beyond the scope of this thesis – it requires the explicit application of the 1+3 covariant and gauge invariant perturbation theory to backgrounds that have at least non-vanishing shear. The simplest such case is that of almost-Bianchi I. This would result in the inclusion of weak couplings between gravitational waves and the matter and radiation dynamics in the CDM models. This is discussed further in chapter 7; in the context of important loose ends.

²¹This has $H_{ab} \approx 0$ and E_{ab} as being sourced by gravitational tidal forces as a result of the choice of frame as : $u^a = n^a$ for $D_{(a}n_{b)} = 0$ and $\text{curl } n_a = 0$. Which means that the frame has no shear nor vorticity; this is Newtonian-like because gravitational waves are not admitted and well define surfaces of constant cosmic time can be found. Such a frame is only consistent for small relative velocities between the total and Newtonian frame [180].

The EFE then provide the energy density conservation equations for the radiation, this can be put into the form

$$\Theta = -\frac{3}{4} \frac{1}{\rho_R} \left(\dot{\rho}_R + D_a q_R^a + 2q_R^a A_a + \pi^{ab} \sigma_{ab} \right). \quad (3.4)$$

Here we are using the hydrodynamical variables.

A key point here is that the mean energy $\langle E \rangle$ should not be confused with the energy $E(v) = -u_a p^a$. One needs to evoke the Stefan-Boltzmann law relating the energy density to the temperature, $\rho_R(x) \propto T(x)^4$; such that the number density $n_R(x) \propto T(x)^3$ to find that the **mean energy** over a photon bundle is

$$\langle E(x) \rangle = \rho_R(x)/n_R(x) \propto T(x). \quad (3.5)$$

The use of such an averaging argument essentially excludes most lensing effects beyond the most basic weak corrections (a limited smoothing of the angular correlation functions for example). It is quite difficult to convince oneself that de^a/dv can be anything but vanishly small. This means that (3.3) can now be written, after switching the parameterization from v to t ($dt = E dv$), substituting for the expansion from (3.4) and integrating from an initial emission point x_* (or x_E) to the point of reception x_0 (or x_R), as (see [37] for example)

$$\int_{E_*}^{E_0} \frac{dE}{E} = + \int_{t_*}^{t_0} \left[\frac{1}{4\rho_R} \left\{ D_a q_R^a + 2q_R^a A_a + \pi^{ab} \sigma_{ab} \right\} - \sigma_{ab} e^a e^b - A_a e^a \right] dt. \quad (3.6)$$

Upon using the definition

$$\mathcal{D}(x, e) = \frac{1}{4} e^a D_a (\ln \rho_R) - \frac{1}{4} \frac{D_a q_R^a}{\rho_R} - \frac{\pi_{ab} \sigma^{ab}}{4\rho_R} - \frac{q_a A^a}{2\rho_R} + \sigma_{ab} e^a e^b + A_a e^a, \quad (3.7)$$

this construction of the mean energy allows the Sachs-Wolfe equation for temperature to be obtained on first noticing that $\dot{\rho}_R \equiv u^a \nabla_a \rho_R = \frac{d\rho_R}{dt} - e^a D_a \rho_R$ and using (3.5):

$$\ln \left(\frac{T(x_0, e)}{T(x_*)} \right) - \frac{1}{4} \ln \left(\frac{\rho_R(x_0)}{\rho_R(x_*)} \right) = - \int_{t_*}^{t_0} \mathcal{D}(x, e) dt. \quad (3.8)$$

Then following the covariant definition of temperature anisotropies

$$T(x, e) = T(x) + \delta T(x, e) = T(x)(1 + \tau(x, e)) \quad (3.9)$$

one is able to show that

$$\ln \left(\frac{T(x_0, e)}{T(x_*)} \right) - \frac{1}{4} \ln \left(\frac{\rho_R(x_0)}{\rho_R(x_*)} \right) = \ln(1 + \tau(x_0, e)). \quad (3.10)$$

The general **Sachs-Wolfe equation** can then be written out in covariant integral form:

$$\ln(1 + \tau(x_0, e)) = - \int_{t_*}^{t_0} \mathcal{D}(x(t), e(t)) dt. \quad (3.11)$$

This is then inverted into differential form, either directly in terms of the null derivative *i.e.* $p^a \nabla_a$, or in terms of the 1+3 threading, which is what we do here so as to make the identification the with IBE easier in the next section :

$$-(\dot{\tau} + e^a D_a \tau) = (1 + \tau) \mathcal{D}(x, e). \quad (3.12)$$

The important point here is the we are on the spacetime manifold M , near event x^i , where the direction vector e^a is essentially a function of the coordinate: $e^a(x)$.

3.1.1 Almost-FLRW Sachs-Wolfe equation

The almost-FLRW Sachs-Wolfe equation is a null integration of the form

$$\tau(x_0, e) \approx + \int_{s_0}^{s^*} \left[\frac{1}{4} e^a D_a \ln \rho_R - \frac{1}{3} D_a \tau^a + \sigma_{ab} e^a e^b + A_a e^a \right] E(x) \frac{dv}{ds} ds. \quad (3.13)$$

(from [152, 1, 51, 124] and using that $dx^a/dv = p^a$, $k^a = p^a/E$). This can be written in terms of the Weyl sourced effect (found by using the constraint equations) and matter sourced effect when the radiation is tightly coupled to the matter (or the radiation is a test field over the matter geometry – $A_a \approx 0$, as in a CDM universe). The CDM almost-FLRW Sachs-Wolfe equation is:

$$\begin{aligned} \tau(x_0, e) \approx & + \frac{1}{3} \int_{v_0}^{v^*} \left[\frac{3}{4} e^a D_a \ln \rho_R - D^a \tau_a \right] E(v) dv \\ & - 2 \int_{v_0}^{v^*} \left[\left(\frac{E_{ab}}{\rho_B} \right) - \text{curl} \left(\frac{H_{ab}}{\rho_B} \right) \right] e^a e^b E(v) dv. \end{aligned} \quad (3.14)$$

For adiabatic scalar perturbations this can be further simplified to find an **adiabatic CDM scalar almost-FLRW Sachs-Wolfe equation**:

$$\tau(x_0, e) \approx + \frac{1}{3} \int_{v_0}^{v^*} [e^a D_a \ln \rho_B - D^a \tau_a] E dv - 2 \int_{v_0}^{v^*} \left[\frac{E_{ab}}{\rho_B} \right] e^a e^b E dv. \quad (3.15)$$

Using the constraint equation for the divergence of the electric part of the Weyl tensor 2.23, in this situation, one finds

$$\tau(x_0, e) \approx + \frac{1}{3} \int_{v_0}^{v^*} \left[3\rho_B^{-1} e^a D^b E_{ab} - D^a \tau_a \right] E dv - 2 \int_{v_0}^{v^*} \left[\frac{E_{ab}}{\rho_B} \right] e^a e^b E dv. \quad (3.16)$$

The interesting point is that we need only know the gravitational tidal forces and the radiation dipole to fully determine the temperature anisotropies. If decoupling was instantaneous, which is necessary if one is using a consistent multi-fluid approach – one jumps from an exactly tight-coupled situation to a free-streaming one. Then one can argue that $D^a \tau_a \approx 0$ before and after last scattering and write the temperature anisotropy directly in terms of the gravitational tidal forces alone. This is of course a gross approximation but is illustrative of how one can reduce, in the Sachs-Wolfe situation, to equations that are quite simple¹:

$$\tau(x_0, e) \approx 3H_0^2 \Omega_0^{-1} \int_{v_0}^{v^*} (E(v) dv) \left[a^3 e^a D^b D_{(a} D_{b)} \Phi_E - 2e^a e^b (a^3 D_{(a} D_{b)} \Phi_E) \right]. \quad (3.17)$$

The physically more realistic almost-FLRW temperature anisotropies are given by the more sophisticated kinetic theory treatment in chapter 5 (see Eq. 5.185).

¹using $\rho_B = a^{-3} \rho_0 = a^{-3} (3H_0^2 \Omega_0)$

3.2 The phase space : The Boltzmann equation

Suppose dN is the number of photons seen by the observer in the rest space volume dV about event x^i , with four momentum in the range dP around p^a . Then we can define the single particle distribution function $f(x, p)$, with respect to the future light cone, as giving dN by :

$$dN = f(x, p)(-u_a p^a) dV dP. \quad (3.18)$$

We begin with the Boltzmann equation in the frame bundle $T(M)$ over M with tuple (x^i, p^a) (see section 2.6 and appendix A.4):

$$L(f) = p^a e_a f - \Gamma^a_{bc} p^b p^c \frac{\partial f}{\partial p^a} = C[f], \quad (3.19)$$

where we consider free-streaming only, such that $C[f] = 0$ (the Einstein-Liouville equation). This contains sufficient information to deal with the photon anisotropies once we put this on the reduced bundle $B(M)$ over M with local co-ordinates (x^i, e^a) which is obtained from the zero-rest mass bundle $T_0(M)$ by integrating out the energy :

$$\int_0^\infty E^2 dE \left\{ p^a e_a f - \Gamma^a_{bc} p^b p^c \frac{\partial f}{\partial p^a} \right\} = 0, \quad (3.20)$$

This is done as follows.

First, consider the the change of variables from $f(x, p) \rightarrow f(x, E, e, m)$ where we are now in the *on mass-shell* bundle $T_m(M)$ (having used $p^a p_a = -m^2$). This is then put in $T_0(M)$ the *zero-mass* bundle with the tuple (x^i, e^β, E) (having used $m = 0 \iff \lambda = E$); notice that we are still using $-u^a p_a = E$.

Second, we energy integrate the Boltzmann equation over the forward light cone ($u_a p^a < 0$) to find the (energy)-Integrated Boltzmann Equation now explicitly in terms of the tuple (x^i, e^a, E) . Using the orthonormal tetrad basis [58, 120] we find

$$\int_0^\infty E^3 dE \left\{ e_0 f - e_a f - \mathcal{D}(x, e) E \frac{\partial f}{\partial E} + C^\alpha(x, e) \frac{\partial f}{\partial e^\alpha} \right\} = 0, \quad (3.21)$$

where the following definitions have been used (arising from the use of the orthonormal tetrad (ONT) – see appendix A.4) :

$$C^\alpha(x, e) = A_\beta e^\alpha e^\beta - A^\alpha - (\sigma^\alpha_\beta + \epsilon^\alpha_{\beta\delta} \omega^\delta + \epsilon^\alpha_{\beta\gamma} \Omega^\gamma) e^\beta + (\sigma_{\beta\delta} e^\alpha - \Gamma^\alpha_{\beta\delta}) e^\beta e^\delta \quad (3.22)$$

$$\Gamma^\alpha_{\beta\delta} e^\beta e^\delta = \epsilon^\alpha_{\beta\delta} n^\delta e^\beta e^\delta + a^\alpha - e^\beta a_\beta e^\alpha, \quad (3.23)$$

$$\mathcal{D}(x, e) = \frac{1}{3} \Theta + \sigma_{\alpha\beta} e^\alpha e^\beta + A_\alpha e^\alpha. \quad (3.24)$$

Here the relationship between the time-like partial derivative, $e_0 = u^a E_a^i(x) \partial / \partial x^i$ and the space-like partial derivatives ∂_α in $B(M)$ (on the righthand side) and the partial derivatives on M (on the lefthand side) are given by

$$\dot{f} = e_0 f - \left[\epsilon^\nu_{\alpha\beta} \Omega^\beta e^\alpha - A^\nu + A_\beta e^\beta e^\nu \right] \frac{\partial f}{\partial e^\nu}, \quad (3.25)$$

$$e^a(x) D_a f = e^a e_a f - \left[\Gamma^\nu_{\alpha\beta} e^\alpha e^\beta + (\sigma_{\nu\beta} + \epsilon^\nu_{\beta\delta}) e^\beta \omega^\delta - \sigma_{\alpha\beta} e^\alpha e^\beta e^\nu \right] \frac{\partial}{\partial e^\nu}. \quad (3.26)$$

For massless particle the energy integration only affects the third term in the IBE (3.21), so we could effectively carry everything out without needing the ONT explicitly [120]. It is however instructive to carry the calculation out in its full generality particularly if one is interested in extending the treatment to include massive relativistic particles.

Third, in the relativistic kinetic theory one can still arrive at the IBE *without* needing to explicitly invoke a temperature or the almost blackbody assumption which is necessary for the Sachs-Wolfe equation, one need only use the variable $\Pi(x, e)$:

$$\Pi(x, e) = \int_0^\infty E^3 dE f(x, e, E). \quad (3.27)$$

Once again we use the EFE to replace the expansion, *i.e.* use (3.4), carry out the integration by parts and use the regularity conditions to pick up the term $4E^3 dE f$ replacing $E^4(\partial F/\partial E)$ and then use (3.31) to obtain an equation in terms of the temperature anisotropy. This can then be written covariantly on the manifold by changing from the tuple (x^i, e^a) in $B(M)$ (on the lefthand side) to $(x^i, e^a(x))$ on M (on the righthand side) :

$$4\mathcal{D}\Pi + e_0\Pi + e^a e_a \Pi + C^a \frac{\partial \Pi}{\partial e^a} = 0 \iff -(\dot{\Pi} + e^a D_a \Pi) = 4\mathcal{D}(x, e)\Pi. \quad (3.28)$$

Fourth, with the aim of pulling out a temperature, at this point it is useful to be reminded that the energy momentum tensor due to the radiation is now (as in 2.44):

$$T_{Rab}(x) = \int_{T_x} p^a p^b f(x, e, E) d^3p \quad (3.29)$$

where² $d^3p = E dE d\Omega$ is the ($p^a p_a = 0$, $u^a p_a < 0$) tangent space volume element. The energy density of the radiation is then

$$\rho_R(x) = T_R^{ab} u_a u_b = \int_0^\infty E^2 f(x, e, E) d^3p = 4\pi \int E^3 dE f_0(x, E) \quad (3.30)$$

where the isotropic temperature $T_0(x)$ arise from the isotropic part $f_0(x, E)$ of $f(x, e, E)$. This allows a covariant definition of the fractional temperature anisotropy $\tau(x, e)$ for almost black body radiation in the direction e^a in terms of $T(x, e)$, the directionally dependent temperature measurement in direction e^a around event x^i :

$$T(x, e) = T_0(x)(1 + \tau(x, e)) = \left[\frac{4\pi}{r} \int_0^\infty E^3 f(x, E, e) dE \right]^{\frac{1}{4}} = \left[\frac{4\pi}{r} \Pi(x, e) \right]^{\frac{1}{4}} \quad (3.31)$$

Hence we make the identification

$$4\pi \Pi(x, e) = r T^4(x, e), \quad (3.32)$$

to then find from (3.28)

$$-(\dot{\tau} + e^a D_a \tau) = \mathcal{D}(x, e)(1 + \tau). \quad (3.33)$$

This in itself is not particularly surprising, given the assumptions we have made along the way. What may be surprising is that upon taking a harmonic expansion we can without

²In the notation of [58], the volume element, π , is as $\pi \equiv d^3p$

loss of generality turn these into equations that are in general timelike integrations, not nullcone ones. As a passing comment, although the procedure above, the energy integration, is straightforward for mass-less particles, it is problematic for massive particles, however the massive particles can be energy integrated with ease, on-mass shell, using the divergence relations [58].

On linearizing $\mathcal{D}(x, e)$ with respect to the FLRW background using the 1+3 covariant and gauge invariant perturbative approach we then have:

$$\mathcal{D}(x, e) \approx \mathcal{B}(x, e) = -\frac{1}{3}D^a\tau_a + (D_a \ln T + A_a)e^a + \sigma_{ab}e^ae^b. \quad (3.34)$$

Here we have demonstrated the essential equivalence of the Sachs-Wolfe and Integrated Boltzmann equations in the covariant theory, given that one has the *a priori* definition of the brightness temperature, a situation where a test-field approach along with some sort of averaging procedure over the photon-bundles is realistic. What we will notice in the multipole formulation is that the resulting equations will be on the spacetime manifold M , *i.e.*, there will be no need to replace the phase space derivatives in the tangent bundle with the covariant ones on M as the resulting moment equations will be space-time equations on M . There are two key points to be aware of when considering the issue of the relationship between the timelike and nullcone approaches.

First, one needs to explicitly deal with the null-integrations in order to extract information about physical scales, as one is dealing with the intersection of the past light-cone of “here-and-now” with the covariantly defined last scattering surface (the latter is defined by its ionization fraction). This is in fact precisely the information that is lost when one uses the timelike integration approach arising naturally from relativistic kinetic theory – there is no scale information.

Second, one does not need any prior knowledge of the null-geodesics in order to extract temperatures – the kinetic theory integrates volume averaged information down time-like worldlines, this is possible because of the divergence relations that follow from using the on-mass shell part of the tangent bundle. There is however a computational advantage of using the fact that in the FRW background the relationship between the data on the intersection of the past light-cone and the last scattering surface is a simple projection into spherical coordinates; the conformal time-like integration parameter can be easily related to the radial distances, turning the time-like integrations into null ones³. This is possible because *a priori* assumptions about the nature of the global structure of the space-time have been made using the almost-Copernican principle (see chapter 4) thus putting direct limits on the statistical nature of the temperature anisotropies as averaged over the intersection of the past null-cone and the surface of last scattering.

One may integrate the temperature, $T(x, e)$, down a null-geodesic, and then decompose it “here-and-now” into its moments ΔT_ℓ (by taking a spherical decomposition). This is the essence of the Sachs-Wolfe approach. In order to correctly reconstruct the observations “here-and-now” one needs to integrate $T(x, e)$ up at least ℓ distinct null-geodesics when considering the ℓ -th order moment, ΔT_ℓ (as found from $\tau_{A_\ell}(x_0)$), while in the timelike approach, one is integrating up one worldline. However, one is then integrating, in general, ℓ distinct moments $\tau_{A_\ell}(x)$.

³ $d\chi = \eta_0 - \eta$ can be used once the null cone has been fixed by choosing the space-time point of reception x_0^i

Chapter 4

COBE-Copernican limits

COBE-Copernican limits : *Upon assuming that the weak Copernican principle holds (the cosmological principle) a detailed scheme linking anisotropies in the CMB with anisotropies and inhomogeneities in the large scale structure of the universe has been constructed by Maartens-Ellis-Stoeger. The MES scheme is briefly discussed, showing the connection between the multipole moments of the temperature anisotropies and the quadrupole and octopole limits from COBE using the Gaussian assumption.*

All anisotropy measures are shown to be less than 10^{-4} when unaffected by the expansion rate. Those influenced by the expansion rate are shown to be less than 10^{-6} Mpc^{-1} . This demonstrates that the observable universe is indeed close to FLRW on scales probed by COBE, the largest scales accessible to observations, given that the universe has been dominated by pressure-free dust since last scattering.

4.1 Almost-EGS Theorem

It has been shown that after last scattering in a CDM universe, if all comoving observers see an almost isotropic CMB, then the universe is almost-FLRW [117]. This result has become known as the almost-EGS Theorem. Starting with a generically lumpy universe Stoeger, Maartens and Ellis were able to show that the surfaces of constant matter density are almost homogeneous; the three-curvature is that of an almost-FLRW universe (Eq. 49 [166])

$${}^3R \approx 6K + \mathcal{O}(\epsilon) \approx \frac{6K}{\langle a(t) \rangle^2} + \mathcal{O}(\epsilon). \quad (4.1)$$

The average scale factor is given by the expansion of the normals to the surface. The Ellis-Bruni (1+3) Lagrangian perturbation theory provides a precise meaning for the almost-FLRW linearization [117, 120, 67] (see chapter 2). The point is that one cannot change the limiting background curvatures since only small corrections can be made to the dominant term – this is stable, the corrections vanish in the limit. Although the theorem makes no assumption about the universes' geometry to start with, it does require the input of matter-energy that is present. It is important to realize that the almost-EGS result means that one cannot mistake a curvature varying FLRW universe for a linearly perturbed one, given the assumption of the dominant pressure free dust.

Theorem 4.1.1 (almost-EGS Theorem) *If the Einstein-Liouville equations are satisfied in an expanding universe, where there is present pressure-free matter with four velocity u^a , and that freely propagating background radiation is everywhere almost-isotropic relative to u^a in some domain, then, the spacetime is almost-FLRW in that region [166, 51].*

This theorem provides the basis for putting observational limits on the inhomogeneity and anisotropy of the universe, by using the following two conditions:

- (A1) All fundamental observers measure the CMB to be almost isotropic in some domain (almost Copernican assumption).
- (A2) The space-time geometry is almost-FLRW.

We know that (A2) follows from (A1), from the almost-EGS theorem. The result is a local one. It has no implications for the global topology – it says nothing about small universes [61, 102]. The size of the region in which one deduces that the integrated deviation from FLRW is small depends on the details of the estimation procedure of the spatial homogeneity gradients in the local region. That almost isotropy implies almost homogeneity seems to have been first discussed in the 1983 Cargèse lectures of Barrow [9]. One should note a technical point: in practice one needs to assume that the derivatives of the temperature multipoles are small. There are two ways that one can do this, and they are both elaborated in the discussion that follows; it is these two approaches that make the almost-EGS approach of Maartens-Ellis-Stoeger novel.

These assumptions underly the usual Sachs-Wolfe analysis and provides the *starting point* needed to set up the Sachs-Wolfe equations; in their usual form this requires special homogeneity and isotropy conditions in order to correctly (i) do the null-integrations (ii) and match these to the surface of last scattering, *i.e.*, to match the past-null cone to the surface of last scattering in an unambiguous way. Using the almost-EGS theorem one can demonstrate the self-consistency of the use of almost-FLRW models given the known data and assumption (A1).

The Ehler-Ellis-Bruni (1+3) Lagrangian formalism has been used in this way to place limits on the geometric, dynamical and kinematic deviations from FLRW in terms of the CMB anisotropy multipoles [118]. These give the Maartens-Ellis-Stoeger (MES) equations so setting limits on the anisotropy and inhomogeneity. These arise in two flavours, the first uses the assumption that :

$$\text{scheme 1: } |\dot{\sigma}_{ab}| < \alpha\Theta|\sigma_{ab}|, |D_{ab}\mu| < \beta\Theta|D_a\mu|, \text{ and that } |D_a E_{bc}| < \gamma\Theta|E_{ab}| \quad (4.2)$$

for α , β and γ constants less than one [117]; these are essentially *non-observational assumptions*, the second, (and improved scheme) uses *in principle observational assumptions alone* [118]:

$$\text{scheme 2: } |D_a\tau_{A\ell}| \sim |\dot{\tau}_{A\ell}| < \alpha|\tau_{A\ell}|\Theta. \quad (4.3)$$

These results have become known as the COBE-Copernican Theorem (it is outlined in more depth below). This has been contrasted with assumption of *a priori* homogeneity related now with the emphasis on the weak Copernican principle – all observers see

almost isotropic CMB in some statistical sense [119]. The weak Copernican assumption can in principle be tested using the SZ -effect as discussed by Goodman [69] and in the guise of cosmic variance, by Kamionkowski-Loeb [96].

4.2 MES equations

Essentially the Maartens-Ellis-Stoeger (MES) equations provide the framework to find limits on inhomogeneity and anisotropy, and give the following theorem.

Theorem 4.2.1 (COBE-Copernican) *If we accept the strong form of the almost-Copernican assumption (A1) then the COBE limits on the CMB temperature anisotropy (of at least 10^{-5}) are consistent with the use of an almost-FLRW universe where pressure free dust dominates the background. [118, 51].*

The COBE-Copernican theorem¹ embodies the weak-Copernican (or almost-Copernican) principle : that all fundamental observers in the relevant spacetime domain measure at most the same level of CMB anisotropy (the strong-Copernican principle asserts that all observations are precisely the same everywhere).

The basis of the argument is that we observe the limits on the temperature anisotropy multipoles, $\tau_{A_\ell}(x_0)$, “here-and-now”, x_0^i :

$$\text{here-and-now: } |\tau_{A_\ell}(x_0)| < \epsilon_\ell. \quad (4.4)$$

By the almost-Copernican assumption (A1) we can extend this to all observers, first, to all observers at the same comoving time-difference along their respective worldlines since last scattering (weak almost-Copernican assumption):

$$\text{weak almost-Copernican: } |\tau_{A_\ell}(t_0, x)| < \epsilon_\ell. \quad (4.5)$$

If the anisotropies are of the order ϵ since last scattering we can then extend this generically (strong almost-Copernican assumption):

$$\text{strong almost-Copernican: } |\tau_{A_\ell}(x)| < \epsilon_\ell. \quad (4.6)$$

In addition we need to make the following three assumptions (these are in principle observational and are implications of the strong almost-Copernican assumption): we assume that there exist smallness parameters, ϵ_ℓ^* of at least order ϵ such that:

$$\text{(B1)} \quad \begin{cases} |\dot{\tau}_{A_\ell}(x^a)| < \epsilon_\ell^* \Theta(x^a), \\ |\ddot{\tau}_{A_\ell}(x^a)| < \epsilon_\ell^{**} \Theta(x^a) \\ |\tau_{A_\ell}''(x^a)| < \epsilon_\ell^{***} \Theta(x^a) \end{cases} \quad (4.7)$$

$$\text{(B2)} \quad \begin{cases} |D_a \tau_{A_\ell}(x^a)| < \epsilon_\ell^\dagger \Theta(x^a), \\ |D_{ab} \tau_{A_\ell}(x^a)| < \epsilon_\ell^{\dagger\dagger} \Theta(x^a) \\ |D_{abc} \tau_{A_\ell}(x^a)| < \epsilon_\ell^{\dagger\dagger\dagger} \Theta(x^a) \end{cases} \quad (4.8)$$

¹Although “COBE-Copernican” theorem is perhaps not a good name; as the idea could just as well be applied to ground based experiments, where “CMB-Copernican” theorem could then be more appropriate. The application and use of the theorem here is explicitly in the context of the COBE data.

$$(\mathbf{B3}) \begin{cases} |(D_a \tau_{A\ell})^\cdot(x^a)| < \epsilon_\ell^\dagger \Theta(x^a), \\ |(D_{ab} \tau_{A\ell})^\cdot(x^a)| < \epsilon_\ell^{\dagger**} \Theta(x^a) \\ |(D_{abc} \tau_{A\ell})^\cdot(x^a)| < \epsilon_\ell^{\dagger\dagger*} \Theta(x^a) \end{cases} \quad (4.9)$$

Here one realizes that ϵ_ℓ^* , ϵ_ℓ^\dagger , $\epsilon_\ell^{\dagger\dagger}$ and ϵ_ℓ^{**} are, in practice, not known from observations. Hence they are estimated [118], the idea being that the spatial gradients of the temperature multipoles are not greater than their time derivatives:

$$(\mathbf{C1}) \begin{cases} \epsilon_\ell^\dagger \leq \epsilon_\ell^*, \\ \epsilon_\ell^{\dagger\dagger}, \epsilon_\ell^{\dagger*} \leq \epsilon_\ell^{**}, \\ \epsilon_\ell^{\dagger\dagger\dagger}, \epsilon_\ell^{\dagger**}, \epsilon_\ell^{\dagger\dagger*} \leq \epsilon_\ell^{***}. \end{cases} \quad (4.10)$$

The bounds on the time derivatives are then estimated in terms of the characteristic CMB length and time scales : $d_R = T/|D_a T|$ and $t_R = T/|\dot{T}|$ where $d_R \gg t_R$ (from the radiation conservation equations) [117, 118]:

$$(\mathbf{C2}) \begin{cases} \Theta \epsilon_\ell^* \simeq \epsilon_\ell / t_R, \\ \Theta^2 \epsilon_\ell^{**} \simeq \epsilon_\ell / t_R^2, \\ \Theta^3 \epsilon_\ell^{***} \simeq \epsilon_\ell / t_R^3. \end{cases} \quad (4.11)$$

This, finally, allows one to write the bounds on the time-derivatives of the anisotropies in terms of those on the anisotropies themselves:

$$\epsilon_\ell^* \simeq \frac{1}{3} \epsilon_\ell, \quad \epsilon_\ell^{**} \simeq \frac{1}{9} \epsilon_\ell, \quad \epsilon_\ell^{***} \simeq \frac{1}{27} \epsilon_\ell. \quad (4.12)$$

The upper bounds are then put into the Einstein Field Equations to find that the geometric measures of anisotropy and inhomogeneity can be written in terms of ϵ_1 , ϵ_2 and ϵ_3 . Using the strong observational assumptions (assumptions **B1-B2** and **C1 - C2**) on the spatial gradients and time-derivatives of the temperature harmonics, the observational limit equations on various kinematic, dynamic and geometric indicators of anisotropy and inhomogeneity are found [118] (The MES equations):

$$\frac{|D_a \rho_R|}{\rho_R} = 4 \frac{|D_a T|}{T} < H(12\epsilon_1 + \frac{24}{5}\epsilon_2), \quad (4.13)$$

$$\frac{|\sigma_{ab}|}{\Theta} < \frac{5}{3}\epsilon_1 + 3\epsilon_2 + \frac{3}{7}\epsilon_3, \quad (4.14)$$

$$\frac{|\omega_{ab}|}{\Theta} < \frac{10}{3}\epsilon_1 + \frac{2}{15}\epsilon_2, \quad (4.15)$$

$$\frac{|D_a \rho_M|}{\Theta} < \frac{9}{2}H\epsilon_2 + (\frac{H}{\Omega_M})[60\epsilon_1 + 134\epsilon_2 + \epsilon_3] + (\frac{\Omega_R}{\Omega_M})H[16\epsilon_1 + \frac{61}{3}\epsilon_2], \quad (4.16)$$

$$\frac{|D_a \Theta|}{\Theta} < H(\frac{205}{3}\epsilon_1 + 8\epsilon_2) + 4(2\Omega_R + \Omega_M)H\epsilon_1, \quad (4.17)$$

$$\frac{|E_{ab}|}{\Theta} < H(\frac{55}{3}\epsilon_1 + \frac{103}{3}\epsilon_2 + \frac{23}{7}\epsilon_3) + \frac{4}{45}(11\Omega_R + 15\Omega_M)H\epsilon_2, \quad (4.18)$$

$$\frac{|H_{ab}|}{\Theta} < H(\frac{16}{3}\epsilon_1 + \frac{52}{15}\epsilon_2 + \frac{1}{21}\epsilon_3). \quad (4.19)$$

Here ρ_R and ρ_M are the radiation and matter densities, respectively, and Ω_R and Ω_M are the ratios of the radiation energy density and the matter densities, respectively, to the critical density of the universe. Here ϵ_1 gives the limits on the dipole components, ϵ_2 on the quadrupole components, ϵ_3 on the octopole moment.

One can then show that given an anisotropy measure of ϵ and taking:

$$\epsilon = \max(\epsilon_1, \epsilon_2, \epsilon_3), \quad \text{and,} \quad (\Omega_R)_0 \ll 1, \quad (4.20)$$

the COBE-Copernican Theorem can be used to support claims that the universe is nearly homogeneous on large enough scales given that anisotropies are at least $\epsilon \sim 10^{-5}$ once the dipole anisotropy has been removed by the appropriate velocity choice. The assumption being that the dipole anisotropy $\bar{\epsilon}_1 \sim 10^{-3}$ is due primarily to local peculiar motions.

By using the COBE-Copernican Theorem along with the COBE quadrupole and octopole limits directly one can compute more precise limits – this is the essence of this chapter, this is carried out here using the actual data rather than an upper bound. These more precise limits support the use of the almost-FLRW universe under the assumptions: the (1) *universe is dominated by cold dark matter*, (2) *the weak almost-Copernican principle holds* and (3) *the Gaussian assumption is good enough*. The latter can be motivated by a combination of arguments that the perturbations are linearly independent and sufficiently dense in the region of spacetime of interest in order to make use of the central limit theorem – which implies that the perturbations in that region can then be well described by Gaussian random fields at linear order. This is needed for the construction and interpretation of the angular correlation function, the crucial link to the COBE-DMR observations.

We can determine ϵ_1 , ϵ_2 and ϵ_3 from CMB measurements, and have observational estimates of H , Ω_R and Ω_M from other observations. So we can determine limits on all these anisotropy and inhomogeneity indicators. However, first, we need to discuss the relationship between the $|\tau_{A_\ell}|$ given in the equations and the multipole moment results determined by the COBE Differential Microwave Radiometers (DMR) and other CMB anisotropy detectors.

4.3 CMB multipole anisotropy data

We have mentioned above that the harmonic decomposition represented by the $|\tau_{A_\ell}|$ is equivalent to that in terms of spherical harmonics. The multipole results recovered from COBE-DMR data cannot be simply substituted into the MES-equations equations. The multipole results presented in the COBE papers can be used as measurements of the mean squares of the moments $|\tau_{A_\ell}|$, as used in our limit equations, as long as we use the real rms dipole, quadrupole, octopole moments that they obtain. We cannot use those associated with obtaining the power spectrum, which are often the focus of their reported results. Finding the first three multipoles is then the essence of this section.

The numerical factors relating the dipole, quadrupole and octopole moments, defined in terms of Legendre polynomials, to the those in the PSTF representation $|\tau_{a_1 \dots a_\ell}|$ [66] of chapter 1 are given below (4.23). In addition, it should be realized that the moments are usually given as the squareroot of the sum of the squares of the $(2\ell + 1)$

components of the ℓ -pole, the rms ℓ -pole – not as values of each separate component of the multipole in question.

Furthermore, in the COBE data the multipoles are quoted for δT , instead of $\frac{\delta T}{T}$ (or τ), for which our $|\tau_{a_1 \dots a_\ell}|$ are the multipoles. The COBE rms values published are all in units of μK , therefore. To translate these values into what we need we must thus divide them by T , the average CMB background temperature over the sky.

From four years of data the COBE workers give us a best fit quadrupole value of [160, 167]

$$\text{COBE-quadrupole: } Q_{rms} = 10.7 \pm 7 \mu K \quad [95\% \text{CI}]. \quad (4.21)$$

We can modify this result for use in the MES-equations (4.13-4.19). This quantity is to be carefully distinguished in the COBE results from Q_{rms-PS} which is often referred to and which is *not* the true best fit value of the quadrupole, but rather the value of the quadrupole derived from a power-spectrum fit (when a power law is assumed) of the other higher-order multipole moments.

The dipole moment can be neglected, assuming that it is all due to our peculiar motion with respect to the rest frame of the microwave background. This is in fact what is done in the COBE anisotropy analysis [13]. We should be aware that there could in principle be a small non-Doppler contribution to the CMB dipole (see [118, 119]). In our calculations below we shall set $\epsilon_1 = 0$.

The COBE workers have informed us that the rms octopole results from COBE are (see [167] too):

$$\text{COBE-octopole: } O_{rms} = 16 \pm 8 \mu K. \quad (4.22)$$

This is consistent with Wright *et al* [196].

There are several important observational and data-reduction issues, and one theoretical issue, which we should be briefly mention here, to provide the background against which we can understand and appreciate these COBE multipole results : (1) the COBE-DMR experiment does not directly measure the dipole, quadrupole nor octopole moments, but rather the two -point correlation function of the temperature anisotropies, (2) there is a great deal of contamination by experimental systematic errors, galactic emission, and the entire region containing the galaxy is removed from the data sets and (3) the theoretical-observational issue of cosmic variance. Points (1) and (2) need to be kept in mind, point (3) will not effect our conclusions here (for a more detailed discussion of these see the appendix [167] C).

Our concern here, however, is not to compare the observed power spectrum of CMB anisotropies with the theoretical spectrum of density perturbations generated by an inflationary scenario. It is merely to use the best values of the CMB multipole moments we have available – however they are generated and whatever their spectrum – to set definite limits on the large scale anisotropy and inhomogeneity of the observable universe itself. Thus, cosmic variance falls outside those issues which we need to consider in arriving at those limits.

4.4 COBE-Copernican limits on homogeneity and isotropy

We now use the values for the rms dipole, quadrupole and octupole the COBE team have so far obtained to determine the anisotropy and inhomogeneity of the universe on very large scales. We set the dipole equal to zero – equation (12). Thus, we shall set $\epsilon_1 = 0$ in the MES equations (4.13-4.19). In order to transform quadrupole and octupole equations, (4.21) and (4.22) into values of $|\tau_{ab}|$ and $|\tau_{abc}|$, respectively, we need to divide the results by $T = 2.73K$, since our multipoles are for $\delta T/T$. We do not need to divide them by $(2\ell+1)^{1/2}$, since our multipole quantities are given as the absolute values, which we *defined* as the squareroot of the sum of the squares of the components.

Finally, we relate the ΔT_ℓ^2 , the coefficients in the usual Legendre polynomial expansion, to the $|\tau_{a_1\dots a_\ell}|$. The numerical relationship between the multipole moments and the angular correlations in the temperature anisotropies (see chapter 1 section 1.3.2)

$$\langle \tau_{A_\ell} \tau^{A_\ell} \rangle = (2\ell + 1) \beta_\ell^{-1} \Delta T_\ell^2. \quad (4.23)$$

Assuming Gaussian fluctuations, and dropping the factor $(4\pi)^{-1}$ (as they have been included in the measurement definitions), the explicit value of the normalization is $\beta_\ell^{-1} = (2\ell)!/2^\ell(\ell!)^2$. This gives the quadrupole and octupole mean squares respectively :

$$|\tau_{ab}|^2 = 7.5 \bar{Q}_{rms}^2, \quad |\tau_{abc}|^2 = 17.5 \bar{O}_{rms}^2. \quad (4.24)$$

Here, $Q_{rms} = 10.7 \pm 7\mu K$ and $O_{rms} = 16 \pm 8\mu K$, the values of the quadrupole and octupole, are divided through by the average bolometric temperature, $T = 2.73 K$, to find \bar{Q}_{rms} and \bar{O}_{rms} , the temperature normalized quadrupole and octupole respectively. We then obtain that

$$\langle \epsilon_2 \rangle = 1.1(\pm 0.8) \times 10^{-5}, \quad \langle \epsilon_3 \rangle = 2.5(\pm 1.3) \times 10^{-5}. \quad (4.25)$$

The dipole is set to zero; $\epsilon_1 = 0$. Using $H = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$ and $0.4 < h < 1.0$ where we are neglecting terms containing the factor Ω_R since this is presently so small ($\Omega_R = 4.11h^{-2} \times 10^{-5}$) we obtain (after having inserted c^{-1} where H are found in order to get the correct units and order of magnitude):

$$\text{kinematics:} \quad \left\{ \begin{array}{l} \left\langle \frac{|\sigma_{ab}|}{\Theta} \right\rangle \leq 4.4 \times 10^{-5}, \\ \left\langle \frac{|\omega_{ab}|}{\Theta} \right\rangle \leq 1.5 \times 10^{-6}, \\ \left\langle \frac{|D_a \Theta|}{\Theta} \right\rangle \leq 3.0h \times 10^{-8} \text{Mpc}^{-1}, \\ |A_a| = 0 \quad \text{dominant CDM assumption} \end{array} \right. \quad (4.26)$$

$$\text{dynamics:} \quad \left\{ \begin{array}{l} \left\langle \frac{|D_a \rho_R|}{\rho_R} \right\rangle \leq 1.8 \times 10^{-8} \text{Mpc}^{-1}, \\ |\tau_a| = 0 \quad \text{dipole assumption}, \\ |\tau_{ab}| = 1.1(\pm 0.8) \times 10^{-5}, \\ |\tau_{abc}| = 2.5(\pm 1.3) \times 10^{-5}, \\ \left\langle \frac{|D_a \rho_M|}{\Theta} \right\rangle \leq (0.02 + 0.54\Omega_M^{-1})h \times 10^{-6} \text{Mpc}^{-1}, \\ q_M^a = 0 = \pi_M^{ab} \quad \text{dominant CDM assumption} \end{array} \right. \quad (4.27)$$

$$\text{gravito-electro/magnetic:} \quad \left\{ \begin{array}{l} \left\langle \frac{|E_{ab}|}{\Theta} \right\rangle \leq (1.5 + 48 \Omega_M)h \times 10^{-7} \text{Mpc}^{-1}, \\ \left\langle \frac{|H_{ab}|}{\Theta} \right\rangle \leq 1.6h \times 10^{-8} \text{Mpc}^{-1} \end{array} \right. \quad (4.28)$$

The above are found by substituting the r.m.s values of ϵ_1 , ϵ_2 and ϵ_3 into (4.13-4.19) [167, 118].

What is important about this scheme is that we have not assumed exact homogeneity, or even that the inhomogeneities and anisotropies are small. They are small because the CMB anisotropies are small "here-and-now". The principle assumption is (A1) as used in its strong form, the sense of the COBE-Copernican Theorem – or to put it another equivalent way the strong form of the weak-Copernican principle. Other groups have placed limits on these quantities – they have assumed exact spatial homogeneity in order to do so (See [167] for additional references). We do not. This approach provides limits on the full range of possible measures of inhomogeneity and anisotropy including the electric and magnetic parts of the Weyl tensor. The Weyl tensor components measure those parts of the gravitational curvature that are not determined by local mass-energy distributions. Specifically, the magnetic part of the Weyl tensor measures the amount of gravitational radiation in the spacetime, while the electric part measures tidal and shear inducing forces of the global gravitational field : it introduces a shearing in the timelike (and null) congruences. A final point is that one may expect a significant cosmological constant : $p_\Lambda = -\rho_\Lambda$ and $q_\Lambda^a = \pi_\Lambda^{ab} = v_\Lambda^a = 0$. The only contribution to the matter geodesics would be via the Raychaudhuri equations (a term like -3Λ). The point is that there is no such contribution in the null Raychaudhuri equations – so one does not expect the almost-EGS result to be affected, although one could expect marginally different figures for the COBE-Copernican limits. This has been discussed to a limited degree in the exact situation [31].

Two final caveats. First, on the issue of *averaging* [200, 25]: (i) these limits only provide plausibility of an almost-FLRW universe on the scales probed by COBE and (ii) we assume that the effective theory of gravity on cosmological scales is general relativity. Second, on the issue of *the dipole*, if the dipole is not due primarily to local peculiar motions these limits are questionable (the MES equations show the dependence of the results on our interpretation of the cosmological dipole, ϵ_1 , being small). In the almost FLRW CDM models cosmological vorticity is at least second order making it unlikely to contribute to the dipole. While the shear is first order, it could not provide cosmological relative velocity sources of the order of magnitude of the measured dipole without violating the smallness limits imposed by the almost-FLRW assumptions. It is thus plausible, and self-consistent, to assume that the dipole is mostly due to local peculiar motions that are astrophysical rather than cosmological.

In order to get the above limits from the weak-Copernican assumption we have made explicit use of : (1) the almost-EGS result, used to give (2) the COBE-Copernican result and (3) the Gaussian assumption. The Gaussian assumption may be regarded as more questionable than the first two; given that it is included here by construction. At a pragmatic level, the plausibility of using the Gaussian assumption, at least at the level of the primordial signature, is mainly because most models in the inflationary family seem to predict it.

An obvious question should be asked:

"When is the COBE-Copernican result inappropriate?"

Two cases of interest have arisen, as regarding the validity of the COBE-Copernican result. Neither function as counter examples, but they do question the nature of the geo-

metric and kinematic assumptions underlying the theory: whether the weak Copernican principle in either its strong form, Eq. (4.6), or its weaker form, Eq. (4.5), are viable, in a physical sense.

The first, that of Wainwright and co-workers [137], is a Bianchi example where the quadrupole anisotropy is small but there is large Weyl curvature. These are anisotropic models with no inhomogeneities. They have not checked that the higher multipoles are small – in fact these will most likely be large given that the large Weyl effects will lead to large shear oscillations. Their models will probably at most work at a given time instant – the point is that under evolution the oscillating shear will produce a big quadrupole. Their example is very special. Furthermore, these authors also ignore the important contributions made by collisionless particles [11]. Shear oscillations in Bianchi type VII were first investigated by Barrow and Sonoda [12]. This is a class of models that should be better understood in the relativistic kinetic theory setting which models the large scale implications of gravitational waves that are not limited by the almost-FLRW restrictions.

The second, that of Clarkson-Barrett [31], gives a counter example based on Stephani universes – these conformally flat models reduce to the FRW case when the acceleration vanishes. These are inhomogeneous models that are isotropic at the observer. They exploit the almost-EGS assumption that pressure gradients are not permitted in the background – essentially they are challenging the assumption of a CDM dominated universe after last-scattering. This can be seen as unreasonable given that exotic forms of matter are required – an unrealistic equation of state is needed. However, if Quintessence does become fashionable, this could be an important model. In that instance one needs to question the philosophical formulation and empirical evidence supporting an almost-FLRW universe, as well as most of the canonical paradigm of anisotropy formation.

A more significant question then arises:

“When does the almost-FLRW model become inapplicable when considering realistic scenarios?”,

where we take realistic scenarios as models that have both inhomogeneity and anisotropies. Towards answering that question, we need a good understanding of the nature of anisotropy formation within the canonical linear-FRW paradigm; the simplest model with generic linear order inhomogeneity and anisotropy.

The obvious omission in the COBE-Copernican limits is that of the important higher multipoles : $|\tau_{A_\ell}|^2$ for $\ell > 2$. We can now proceed to flesh out the finer details of implications of the COBE-Copernican theorem on anisotropy formation and development in the preferred model, the almost-FLRW universe: that, as we have argued, given our assumptions, best explains and is consistent with the known CMB observations.

The next chapter provides the explicit 1+3 Lagrangian threading formulation of temperature anisotropies, τ_{A_ℓ} , as reduced using the 1+3 covariant and gauge invariant perturbation theory to an almost-FLRW universe.

Chapter 5

Scalar almost-FLRW universes

Temperature anisotropies in almost-FLRW universe : *The covariant and gauge invariant treatment is extended using kinetic theory to examine the CMB temperature anisotropies arising from inhomogeneities in the early universe. The Mode form of the Integrated Boltzmann Equations are derived, giving a covariant version of the standard derivation using the mode recursion relations. This then links the mode IBE to the MDE. Analytic methods of solving the resulting equations are discussed. A general manifestly gauge invariant integral form of solution is obtained for the equations with Thomson scattering. The covariant FL multipole form of the transport equations are found near tight-coupling using the CGI generalization of the Peebles-Yu expansion in Thomson scattering time. The dispersion relations and damping scale are then found from the covariant approach. The equations are integrated to give the covariant and gauge invariant equivalent of the standard (non-local) treatment of the expected primary-source anisotropies in the $K = 0$ (flat background) case. A simple treatment of the matter dominated free-streaming projection, slow decoupling, and tight-coupling are carried out. The covariant and gauge invariant temperature anisotropy multipole coefficients are then recovered in the Newtonian frame for the flat case:*

$$\begin{aligned} \tau_{A_\ell}(x_0) \approx & \sum_{k_a} (2\ell + 1) \beta_\ell^{-1} Q_{A_\ell}(x_0, k^a) \left\{ [\delta\tilde{T} + \Phi_A](k, \eta_*) j_\ell(k\Delta\eta_*) \right. \\ & + \int_{\eta_*}^{\eta_0} d\eta \mathcal{V} e^{-(k/k_D)^2} \left[\frac{1}{3} \tau_1(k, \eta) + [\kappa' \tilde{v}'_B + \kappa'' \tilde{v}_B](k, \eta) \right] j_\ell(k\Delta\eta) \\ & \left. + \int_{\eta_*}^{\eta_0} \mathcal{V} e^{-(k/k_D)^2} \left[[\Phi'_A - \Phi'_H](k, \eta) + aH[\delta\tilde{T} + 3\Phi_A](k, \eta) \right] j_\ell(k\Delta\eta) \right\} \quad (5.1) \end{aligned}$$

The aim is both providing a unified transparent derivation of this range of results and clarifying the connection between the more usual approaches (for example that of Hu-Sugiyama) and the CGI treatment for scalar perturbations.

5.1 Introduction

The previous three chapters have done the following: First, chapter 1) established the link between the angular correlation functions in the mode representation, for almost-

FLRW models, and the angular correlation functions for generalized (exact) temperature anisotropies in the multipole representation, using GRF's. Second, chapter 2, established the exact form of the multipole divergence equations describing temperature anisotropies and their reduction to almost-FLRW temperature multipole divergence equations using the 1+3 covariant and gauge invariant approach to cosmological perturbations. Third, chapter 4) has established the nature of the assumptions necessary to interpret the current low- ℓ temperature anisotropy data (such as that from the COBE-DMR experiment) as supporting the use of an almost-FLRW universe on large scales.

Having established the link between exact relativistic kinetic theory and exact general relativity with the linear-FRW models and the current large scale CMB observations (low- ℓ); we now proceed to recover the detailed implications of almost-FLRW universes on measurements of temperature anisotropies from the 1+3 covariant and gauge invariant perspective (for arbitrary- ℓ), to so recover the canonical treatment of primary and secondary sourced anisotropies using the adiabatic CDM linear-FRW paradigm (see section 0.1.3 for an overview). We will need this to understand the implications of the small-scale (high- ℓ) non-perturbative corrections, that we uncovered in chapter 2 as an extension to the linear-FRW paradigm.

Once again we recall that the ETM papers introduced a covariant kinetic theory formalism based on a 1+3 covariant representation of temperature anisotropies in terms of Projected Symmetric Trace-Free (PSTF) tensors orthogonal to a preferred time-like vector field u^a [172, 147]. The benefits of the approach have been briefly summarized in chapter 1. In essence, it provides :

1. Clarity of definition of variables used (chapter 1, 2 and 4).
2. 1+3 Covariant and Gauge-Invariant (CGI) variables and equations (chapter 2).
3. A sound basis from which to proceed to non-linear calculations (chapter 2 and 6).
4. The possibility of using any desired coordinate and tetrad system for evaluating the variables and solving the equations in specific cases (section 2.6 (the ONT tetrad example) and section 5.2.4 and 5.7.1 (the conformal Newtonian gauge example)).

This chapter will draw strongly on the algebraic relations given in chapter 1, which dealt with the covariant and gauge invariant irreducible representation of CMB anisotropies by PSTF tensors, and their relation to observable quantities (specifically, the angular correlation functions). Particularly it dealt with the reduction from the general framework to that specialized to almost-FL universe models ¹ [166, 117], as well as dealing with multipole and mode expansions and the relation to the usual formalisms in the literature [194, 70, 82, 83, 45].

In this chapter we use the CGI formalism to study CMB anisotropies in almost-FL models in an analytic manner by timelike integration of the almost-FL differential relations. Our emphasis is on the canonical linearized model for CMB anisotropies [152, 193, 1, 161, 70, 82, 83, 190], systematically developing the CGI approach (also considered

¹In order to be clear on the use of these acronyms, RW refers to the Robertson-Walker *geometry* whatever the dynamics, while FL refers to such a geometry which additionally obeys the Friedmann-Lemître *dynamics* implied by imposing the Einstein Field Equations (EFE).

by Challinor and Lasenby [27, 28]) and providing a comprehensive and transparent link to the alternative analytic GI treatments based on Bardeen's GI variables. We develop these results both in mode and multipole form from within the CGI approach, emphasizing the different physical processes and assumptions and demonstrating how these are dealt with in the CGI context. This requires the covariant mode form of the IBE, based on the recursion relations for almost-FL mode functions (see chapter 1 [66]), enabling a direct mirroring of standard treatments based on Wilson's seminal paper [193, 70, 82, 83, 190], but carried out in a CGI fashion, thus forming a sound basis for extension to non-linear effects.

One of the advantages of the CGI approach, in the context of the generic multipole divergence equations, is the ability to include non-linear corrections to the almost-FL treatment (this is explicitly done in chapter 6). Towards this end, the relationship between the covariant mode formulations and the almost-FL covariant multipole treatment are given (see also the linear-FRW CDM frame formulation of Challinor and Lasenby [28]), based on the results in chapter 1. These will be necessary for a non-linear extension of the almost-FL treatment given here in the context of the possible non-linear effects outlined in chapter 2 [120]. We emphasize here that in our treatment, $\langle \tau_{A\ell} \tau^{A\ell} \rangle$ (the multipole form of the angular correlation function) is given for small temperature anisotropies irrespective of the form of the geometry [120], making it the natural representation for the inclusion of non-linear dynamic effects, while the analysis for $|\tau_\ell|^2$ (the mode coefficient mean square) is specifically for almost-FL models [66]. The non-linear extension of the almost-FL treatment given here will be based on the multipole-to-mode relations, leaving the use of mode expansion to the latest possible stage.

Our focus is on the era following **spectral decoupling** (near 500 eV) – as discussed briefly in section 0.1. A complex series of interactions take place at the various epochs of the expansion of the universe. The kinetic equations developed in chapter 2 can represent almost any such interactions provided we use appropriate interaction ('collision') terms; the issue is how to obtain simplified models that are reasonably accurate in the various epochs. We will consider only two kinds of representation here: namely

1. **Thomson scattering**, valid at late times when particle and photon numbers are conserved and the matter is non-relativistic (during decoupling an alternative approach is to use a visibility function);
2. an **effective two-fluid** description, obtained by truncation of the hierarchy and valid at earlier times when strong interactions take place establishing equilibrium or close to equilibrium conditions between the components, i.e. when the interaction time-scale is much less than the expansion time-scale; an alternative description is to use a single imperfect fluid [122]. These descriptions can be used even when the matter is relativistic. The detailed form of interactions does not need to be represented in this case, because the state of the matter depends only on its equilibrium condition, characterized by its equation of state.

At some times either form is valid and they can then be related to each other. We do not attempt here to give a description of earlier non-equilibrium eras when processes such as pair production, nucleosynthesis, etc, occur, nor do we consider issues such as inflation and reheating after inflation, and the differences between the inflation sourced

perturbations as opposed to those based on other phase transitions. Thus our models will be valid after the end of any period of inflation that may have occurred and after strong non-equilibrium processes have ceased. During this era the processes occurring determine the final CBR anisotropy that occurs as a result of initial fluctuations left over from the earlier non-equilibrium epochs.

Specifically, we deal with four eras of interest. Going backwards in time from the present, they are, firstly, free streaming from last scattering to here and now, in the matter dominated almost-FL context; secondly, slow decoupling during a matter dominated era, during which the CMB spectrum freezes out; thirdly, the late tight-coupling era after matter-radiation equality, during which structure formation begins; and fourthly, the main tight coupling era after any inflationary epoch and before matter-radiation equality, during which acoustic modes occur in the tightly-coupled fluid, the initial matter perturbations having been seeded by earlier conditions and processes (for example, by inflation). We then show how to put these CGI results together to determine the major features of the expected anisotropy spectrum. We develop sound models of the dominant effects in each of the eras we consider, but there will always be a need for refinement of these models by taking into account further effects (in particular polarization).

In more detail:

- **Free-Streaming** : We find the CGI integral solutions to the almost-FL MDE with no collision term, and use them to project the initial data from decoupling to the current sky. We explicitly do this for instant decoupling. Neither the Vishniac, Rees-Sciama, thermal Sunyaev-Zel'dovich nor lensing effects are considered here – we are emphasizing the CGI model of the dominant processes, and these further effects will introduce detailed modifications. However a comprehensive understanding of such secondary higher order effects relies on a derivation of the anisotropy effects given here.
- **Slow-Decoupling** : Here we consider modification of the previous results when slow decoupling of the interactions due to Thomson scattering is taken into account. We consider the damped integral solution for slow recombination, and as an alternative description modify the integral solutions appropriately with a visibility function carrying the functional dependence of the varying electron fraction, in a matter dominated context. Effectively, recombination is complete before the radiation decouples. This means that the surface of last scattering is found a little after the end of recombination ². It is during this era that photon diffusion damping scale effects become important – the damping scale is affected by the duration of this era. This will be investigated in the context of almost-FL universes after matter radiation equality.

²A note on nomenclature: By **decoupling** we have in mind the situation when the interaction rate per particle becomes less than the expansion rate. By **last scattering surface** we mean the surface upon which the diffusion scale is equal to the horizon scale, after which it is larger than that scale and the free-streaming approximation is sufficient. The photons will decouple from the thermal plasma near 0.2 eV, and from the matter after recombination has effectively ended, near 0.3 eV. Free-streaming is considered to be a good approximation from about 0.26 eV.

- **Tight-coupling:** This is the key to the entire treatment. We give the CGI version of the tight-coupling approximation of Hu *et al* [82, 83, 85]. In the almost-FL treatment, the slow decoupling and free-streaming era's will only "damp" and "project" the spectrum formed at the end of tight-coupling onto the current sky ³.

We consider two different tight-coupling regimes.

1. The **late tight coupling** era is separated, conceptually, from the early tight coupling era by matter-radiation equality, after which time the matter perturbations have effectively decoupled and (CDM-based) structure formation begins. In this era, strong interactions have ceased and a Thomson scattering description can be used. We first carry out the near tight-coupling treatment of this era based on Peebles & Yu [143, 94, 95], and then reduce these equations to the covariant tight-coupling equations equivalent to those of Hu *et al* [82, 83, 85]. This provides the basis for understanding the acoustic signatures in the temperature anisotropies within the CGI approach.
2. The **early tight coupling** era occurs between the matter-radiation equality and spectral decoupling (after strong interactions have ceased), when the Thomson scattering description will also be sufficient. This era is characterized by acoustic oscillations in a tightly coupled fluid; for calculation convenience this can be represented as a single dissipative fluid [122], or for a slightly more sophisticated treatment by tightly coupled two-fluid models [39, 37]. We give a CGI derivation of the harmonic oscillator equation providing the primary source terms in the standard model of acoustic and Doppler peak formation by acoustic oscillations.

We put these results together in sections 7 and 8, where the equations are integrated to give the CGI treatment in the $K = 0$ (flat background) case in terms of an integral solution. The primary sources of the temperature anisotropies (the acoustic and Doppler contributions near last scattering resulting in "Acoustic Peaks" today) are demonstrated ⁴. This recovers the Sachs-Wolfe family of effects for flat background RW geometries, but derived from a CGI kinetic theory viewpoint as opposed to the photon-propagation description used in the original Sachs-Wolfe paper.

The form of the angular correlation function is determined for the primary effects (although not given explicitly in terms of the matter power spectrum). The normalization to standard CDM (adiabatic CDM) is presented in terms of CGI variables. This demonstrates the basic effects in the CGI formalism, and links our approach to the standard literature, see for example [190] and references therein, where further details of this era are given.

It is important to note that the integrations considered here are carried out along timelike curves, even though the CBR radiation reaches us along null curves. These are alternative approaches that are equivalent in the context of linearization about FL models; differences will however occur in the context of non-linear corrections. Briefly,

³This useful feature of the almost-FL models is due to the homogeneity and isotropy conditions in the background FL universe and is not generic [120].

⁴following the analytic treatment of Hu & Sugiyama [82, 83]

This then allows us to write the shear and acceleration in terms of the scalar potentials and perturbation variables:

$$\frac{1}{2}(\rho + p)\sigma_{ab} \approx (D_{(a}D_{b)}\Phi_H) + 3HD_{(a}D_{b)}\Phi_H - HD_{(a}D_{b)}(\Phi_H + \Phi_A) + \left\{\frac{1}{2}D_{(a}q_{b)}\right\} \quad (5.32)$$

$$(\rho + p)A_a \approx -D_a p - D^b D_{(a}D_{b)}(\Phi_H + \Phi_A) - \{q_a + 4Hq_a\} . \quad (5.33)$$

5.2.4 Frame transformations and Gauge fixing

There is freedom associated with the choice of reference velocity u^a , which we call a **frame choice**¹⁵. This is to be distinguished from the choice of coordinates in the realistic universe model, which can be done independently of the choice of u^a . It is equivalent to the choice of timelike world lines mapped into each other by the perturbation gauge chosen, but is independent of the choice of time surfaces in that mapping. Given a particular covariantly defined choice for this velocity, the frame choice is physically specified and the equations are covariantly determined and gauge invariant under the remaining gauge freedom. Gauge fixing requires in addition a specification of correspondence of time surfaces in the realistic and background models (effectively specified by determining the choice of surfaces of constant time in the realistic universe model) and of points in initial spacelike surfaces (i.e. the point-to-point mappings between the background model and the realistic lumpy universe model, see for example Ellis and Bruni [52, 21]). In simple situations this choice will be unique, however in more complex situations several choices of this velocity are possible, each leading to a somewhat different CGI description – essentially specified by fixing the relative velocity between the 1+3 Lagrangian choice (total frame typically) and the various 1+3 Eulerian frames – as discussed generally in chapter 2.

When we restrict ourselves to a particular frame in order to simplify calculations, we can straightforwardly make the appropriate simplifications in the general equations to see what the implications are (for example setting $q^a = 0$ for the *energy frame*, the quantities in the braces in equations (5.32,5.33) above vanish). However it is also useful to explicitly transform from one frame to another and examine the resulting effect on dynamic and kinematic quantities.

Under a frame transformation $\tilde{u}^a \approx u^a + v^a$, $|v^a| \ll 1$ (restricting our analysis to non-relativistic relative velocities¹⁶), the following relations [120] hold:

$$\tilde{\sigma}_{ab} \approx \sigma_{ab} + D_{(a}v_{b)}, \quad (5.34)$$

$$\tilde{A}_a \approx A_a + \dot{v}_a + H v_a, \quad (5.35)$$

$$\tilde{\Theta} \approx \Theta + \text{div } v, \quad (5.36)$$

$$\tilde{q}_a \approx q_a - (\rho + p)v_a, \quad (5.37)$$

$$\tilde{\omega}_a \approx \omega_a - \frac{1}{2}\text{curl } v_a. \quad (5.38)$$

¹⁵In the general basis we have in mind a preferred family of fluid flow aligned frames; see appendix A.4 for the ONT formulation.

¹⁶It is important to recall that gauge invariance is only guaranteed if the choice of velocity v_a coincides *exactly* with its value in the background spacetime. This is not difficult in practice, as appropriate physically defined frames \tilde{u}_a will necessarily obey this condition because of the RW symmetries.

The quantities ρ , p , π_{ab} , E_{ab} and H_{ab} , remain unchanged to linear order in almost FL universes (e.g. $\tilde{\pi}_{ab} \approx \pi_{ab}$ and $\tilde{E}_{ab} \approx E_{ab}$), and the temperature anisotropies ($\tau_{A\ell}$) for $\ell > 1$ are similarly invariant for the small velocity transformations considered. The baryon and radiation (dipole) relative velocities change according to:

$$\tilde{v}_a^B \approx v_a^B - v_a, \quad (5.39)$$

$$\tilde{\tau}_a \approx \tau_a - v_a. \quad (5.40)$$

Also, the projection tensor h_{ab} changes if we boost to the frame \tilde{u}_a giving \tilde{h}_{ab} , hence any spatial gradients need to be modified and \tilde{D}_a will be the totally projected derivative in that frame. The consequence is that the perturbation variables change accordingly: Thus for any species I , (using the transformation equations – appendix B [120])

$$\tilde{D}_a \ln \rho_I \approx D_a \ln \rho_I - 3Hv_a(\rho^I + p^I)/\rho^I. \quad (5.41)$$

For example $I = R$ and $I = B$ give the equations for radiation and baryons respectively, implying:

$$\tilde{D}_a \ln T \approx D_a \ln T - v_a H, \quad \tilde{D}_a \ln \rho_B \approx D_a \ln \rho_B - 3Hv_a. \quad (5.42)$$

These equations allow us to determine the required transformation to obtain desired properties of a particular choice \tilde{u}^a . The almost-FL MDE (5.19-5.18) are valid in any frame; in particular, if a frame transformation is performed as above, they can be given in terms of the resulting variables in the new frame, \tilde{u}_a , with whatever restrictions result.

While various choices of \tilde{u}^a are available in a multi-fluid description of the early universe [120], there are seven particularly useful choices.

1. The **zero acceleration frame** (or CDM frame): $\tilde{u}_a = u_a^C \approx u_a + v_a^C$. From the CDM velocity equation [28, 120] and (5.35) we then find: $\dot{v}_a^C + Hv_a^C + A_a \approx 0 \Rightarrow \tilde{A}_a \approx 0$ [110, 28, 120]. This choice is particularly useful in multi-species situations, as this frame will be geodesic right through tight-coupling, slow decoupling and into the free-streaming era.
2. The **Newtonian frame**: $\tilde{u}_a = n_a$ in which the vorticity and shear of the reference frame vanishes: $\tilde{\sigma}_{ab} \approx D_{\langle a} n_{b \rangle} = 0$, when such a frame can be found. This frame is only consistent in restricted cases [180], but is particularly useful in making comparisons with much of the analytic literature [82, 83, 70] and in making connections with the local physics in terms of Newtonian analogues via Eulerian coordinates, for example, the matter shear can be then thought of in terms of distortion due to the relative velocities (5.34): $\sigma_{ab} \approx -D_{\langle a} v_{b \rangle}^N$. This frame is the most convenient one in which to consider the photon physics – to calculate scalar almost-FLRW temperature anisotropies, and the evolution equations for the temperature perturbations and dipole near tight-coupling.
3. The **constant expansion frame**: $\tilde{D}_a \tilde{\Theta} = 0$. This choice is sometimes useful when discussing perturbations on small scales.

4. The **CMB-frame**: $\tau_a = 0$, is useful when dealing with the temperature anisotropies in a manner which excludes the need to consider sources of peculiar velocity-like effects.
5. The **total frame** (also know as the overall frame, or fundamental frame) is the most convenient for the nonlinear extension [120] because the conservation equations for individual components are best given in the overall u^a -frame (denote by u^{*a} in the perturbative setting). This is simply because this frame can be defined irrespective of the types and number of different particle species included [120] and which is easily given in terms of the individual relative velocities v_I^a ; as discussed in depth in chapter 2. The total energy momentum tensor is defined in terms of overall u^a -frame while the individual contributions, T_I^{ab} , are with respect to the individual u_I^a -frame velocities ($\tilde{u}_a = u_I^a$) for the individual frame ($u^{*a} = \gamma_I(u_I^a + v_I^{*a})$). This frame is that in which it is very convenient to treat the almost-FLRW matter kinematics, matter dynamics and gravitational wave effects – it coincides with the energy frame in the case of a CDM dominated model.
6. The **energy frame** (or Landau-Lifshitz): $\tilde{q}_a = 0$ is preferred when dealing with two coupled particle species, as in the two fluid scenario [37]. This is useful as the EFE are simplified to a form which takes on a similar structure to the matter dominated equations, and even in the strong interaction case may be expected to be unaffected by collisions because of energy-momentum conservation during such collisions (this choice is dealt with in more detail below in the context of scalar perturbations).
7. The **particle frame** (or Eckart): $\tilde{j}_a = 0$, where the number current is $N^a = nu^a + j^a$ ($j^a u_a = 0$). This is in general distinct from the energy frame in which the energy flux vanishes. For a single fluid in equilibrium the energy and particle frames will coincide¹⁷.

These various choices will simplify the equations in significant ways, and enable us to recover many of the standard results. It should be noted however, that the covariant equations we have given above are general and do not require either gauge or coordinate restrictions to be meaningful, and the covariant quantities have in them a natural invariant geometric meaning. We will retain the covariant form and not restricted ourselves to any particular frame nor gauge choice for most of this thesis; we retain the freedom to make such a choice however when useful. If and when we do pick a particular frame this will be explicitly stated along with the reason for doing so.

The Newtonian Frame link to the Bardeen variables

Here we demonstrate the direct link between our variables in the scalar case, and those used in the Newtonian gauge, in terms of the Bardeen variables. From (5.34) and (5.35)

¹⁷The energy momentum tensor and particle 4-current are respectively : $T^{ab} = \int p^a p^b f dP$ and $N^a = \int p^a f dP$, for f a single particle distribution function and p^a the four momentum with respect to the u^a -frame.

we find easily that for $\tilde{u}^a = n^a$ where $D_{\langle a} n_{b\rangle} = 0$, $n_a = u_a + v_a^N$ (the consistency of this choice is discussed in [180]):

$$0 \approx \sigma_{ab} - D_{\langle a} v_{b\rangle}^N, \quad (5.43)$$

$$\tilde{A}_a \approx A_a + \dot{v}_a^N + H v_a^N, \quad (5.44)$$

$$\tilde{D}_a \ln T \approx D_a \ln T - H v_a^N, \quad (5.45)$$

$$\tilde{\tau}_a \approx \tau_a - v_a^N, \quad (5.46)$$

$$\tilde{\Theta} \approx \Theta + \text{div } v^N, \quad (5.47)$$

$$\tilde{q}_a \approx q_a - (\rho + p) v_a^N. \quad (5.48)$$

The effect of this frame transformation is to modify the $\ell = 1$ and 2 MDE (5.18-5.20):

$$-\dot{\tilde{\tau}}_a \approx \tilde{D}_a \ln T + \tilde{A}_a + \frac{2}{5} D^c \tau_{ab} - \sigma_T n_e (v_a^B - \tau_a), \quad (5.49)$$

$$-\dot{\tilde{\tau}}_{ab} \approx D_{\langle a} \tilde{\tau}_{b\rangle} + \frac{3}{7} D^c \tau_{abc} + \sigma_T n_e \tau_{ab}. \quad (5.50)$$

The $\ell > 2$ equations (5.21) remain unchanged, however the field equations as well the perturbation equations need to be modified, if necessary using the transformation relations (5.35 -5.40). For example, (5.18) becomes

$$(\tilde{D}_a \ln T) + H(\tilde{D}_a \ln T + \tilde{A}_a) \approx -\frac{1}{3} \tilde{D}_a \tilde{\Theta} - \frac{1}{3} D_a (D^c \tau_c), \quad (5.51)$$

which can easily be checked to be invariant under the frame transformations $\tilde{u}_a \approx u_a + v_a$. From the shear evolution equation (2.17) :

$$D_{\langle a} \tilde{A}_{b\rangle} \approx E_{ab} - \frac{1}{2} \pi_{ab}, \quad \Rightarrow \quad \tilde{A}_a \approx D_a \Phi_A, \quad (5.52)$$

From the momentum constraint (2.21) and the propagation equation for the electric part of the Weyl tensor (2.18) one finds respectively that:

$$\frac{1}{3} D_{\langle a} \tilde{D}_{b\rangle} \tilde{\Theta} \approx -\frac{1}{2} D_{\langle a} \tilde{q}_{b\rangle}, \quad \frac{1}{3} D_{\langle a} \tilde{D}_{b\rangle} \tilde{\Theta} \approx D_{\langle a} D_{b\rangle} \dot{\Phi}_H - H D_{\langle a} D_{b\rangle} \Phi_A. \quad (5.53)$$

This then gives one the scalar monopole equation for the temperature perturbation:

$$D_{\langle a} D_{b\rangle} (\ln T) \approx -D_{\langle a} D_{b\rangle} \dot{\Phi}_H - \frac{1}{3} D_{\langle a} D_{b\rangle} (D^c \tau_c). \quad (5.54)$$

The important point to note here is that in the shear-free frame we can interpret the acceleration directly in terms of the Φ_A potential; that is in terms of its Newtonian analog, while Φ_H can be interpreted as a curvature perturbation. In terms of the potentials used by [82, 41] one can identify $\Psi = \Phi_A$ and $\Phi = \Phi_H$.

The above formulation is useful in linking the covariant work to the usual GI treatments. When we take the mode expansion of the potentials, one finds on dropping the k -index on the right,

$$A_\alpha \approx (\Phi_{A|\alpha} + V'_{S|\alpha} + H V_{S|\alpha}) = (V_S^{(0)} + \frac{a'}{a} V_S^{(0)} - k \Phi_A) Y_\alpha^{(0)}, \quad (5.55)$$

$$\sigma_{\alpha\beta} \approx a (D_{\langle \alpha} D_{\beta\rangle} V_S) = -a k V_S^{(0)} Y_{\alpha\beta}^{(0)}, \quad (5.56)$$

where the prime (') denotes the time-derivative with respect to the conformal time, Y are the eigenfunctions of $Y|_{\alpha} = -k^2 Y$ and following Kodama and Sasaki [98, 20], the bar ($|_{\alpha}$) denotes spatial derivatives with respect to surfaces of constant curvature in the background. Furthermore, one can identify V_S as a relative velocity.

The Energy Frame

In order to be clear on the consequences and subtleties involved in fixing the frame, here we give the source terms in the *energy frame* ($\tilde{u}^a = u_a^E$ such that $\tilde{q}_a = 0$) for scalar perturbations. The important point is that this is a physical frame, uniquely defined by the local physics. The equations (5.32) and (5.33) then take on the form :

$$(\rho + p)\tilde{\sigma}_{ab} \approx 2(D_{(a}D_{b)}\Phi_H)' + 4HD_{(a}D_{b)}\Phi_H - 2HD_{(a}D_{b)}\Phi_A + D_{(a}v_{b)}^E, \quad (5.57)$$

$$(\rho + p)\tilde{A}_a \approx -D_a p - D^b D_{(a}D_{b)}(\Phi_H + \Phi_A) + \dot{v}_a^E + H v_a^E. \quad (5.58)$$

Many CGI treatments use this frame [37], and have the advantage that the equations take on a form which is similar to that for the matter dominated case, but can still be used near to matter-radiation equality.

Matter Domination

During matter domination (we have in mind the CDM dominated case) there is a unique physically relevant frame defined by the matter 4-velocity, hence without restricting the almost-FL universe further there is natural frame, u^a , in which the variables will be gauge invariant and the above relations hold – in matter domination the energy frame, total frame and matter frame coincide. The equations in the matter dominated scenario are thus considered to be total-frame equations.

In this frame, equations (5.32) and (5.33) become

$$a^3 \rho_M \sigma_{ab} \approx -2(a^3 D_{(a}D_{b)}\Phi_H)', \text{ and } A_a \approx 0. \quad (5.59)$$

(where $\Phi_H(x) \approx -\Phi_E(x)$). Here $\rho \approx \rho_M$ is now the density of the matter content only. The key point is that to retain a consistent linearization scheme as well as retaining gauge invariance, we now have three smallness parameters: v (non-relativistic relative velocities), η (radiation-baryon ratio is at least 10^{-2}), ϵ (the universe is almost FL given that ϵ is at least 10^{-5}) [117, 120]. It follows that ρ_R (the radiation energy density) is now $O(\eta)$ and we can drop all terms at least $O(\eta\epsilon)$, $O(\epsilon^2)$ and $O(\eta^2)$ such as, for example, $p\sigma_{ab} \approx 0$ or $\frac{4}{3}\rho_R\tau_{ab} \simeq \pi_{ab} \approx 0$. This is how the anisotropic pressure is eliminated to the order of the calculation in this scheme.

The link to the matter distribution in the space-time comes through the mode coefficients $\sigma(k, t)$ (of the shear), $A(k, t)$ (of the acceleration), and $\delta T(k, t)$ (of the temperature perturbation). A mode analysis leads to a particular solution of the linearized EFE due to scalar modes as in (D.41 - D.44):

$$E_{ab} \approx \bar{\Phi} Q_{ab} = \frac{k^2}{a^2} \Phi_H(t, k) Q_{ab}, \quad (5.60)$$

$$\sigma_{ab} \approx -\frac{2}{3}(H_0^2 \Omega_0)^{-1} k^2 (a \Phi_H(k, t))' Q_{ab}, \quad (5.61)$$

$$D_a \ln \rho_M \approx \frac{2}{3} k (H_0^2 \Omega_0)^{-1} \Phi_H(k, t) [k^2 - 3K] Q_a. \quad (5.62)$$

For a matter dominated open model ($K \neq 0$) where $a_0 = +1$ we have $H_0^2 \approx K/(\Omega_0 - 1)$. Notice that the factor $2/3$ in the shear equation arises from the factor $(3H_0^2\Omega_0)$ while that for the matter density gradients from a term $D^b Q_{ab}$.

If we add the **adiabatic assumption** (see append D.5.1) we find

$$D_a \ln T \approx +\frac{1}{3} D_a \ln \rho_B, \quad (5.63)$$

where we have used that $\rho_B \approx 3H_0^2\Omega_0 a^{-3}$ in the background. One can then put the mode coefficients, in the **matter dominated scalar adiabatic almost-FLRW models** into the form :

$$\delta T(k, t) \approx \frac{1}{3} (H_0^2\Omega_0)^{-1} ({}_a\Phi_H) \left[\frac{2}{3} (k^2 - 3K) \right], \quad (5.64)$$

$$A(k, t) \approx 0 \approx v_B(k, t), \quad (5.65)$$

$$\sigma(k, t) \approx -\frac{2}{3} (H_0^2\Omega_0)^{-1} ({}_a\Phi_H) \left[k^2 \right]. \quad (5.66)$$

The first expression gives the direct effect of the gravitational potential on the CBR anisotropies (Sachs-Wolfe effect), and the third the effect of the time variation of the potential on these anisotropies. These are investigated in detail in later sections. The matter dominated EFE are at least $O(\epsilon\eta)$ and fix the form of the shear, the acceleration and the temperature perturbations $D_a \ln T$ as they enter the IBE (which is how the geometry enters into these equations). The hierarchy itself is $O(\epsilon)$ and although the radiation variables do not enter the almost-FL (matter) EFE, they remain non-zero, and therefore there are still temperature anisotropies, τ_{A_i} . This is an important but subtle point – matter domination implies the radiation moves as a test field over the geometry.

5.2.5 Some Newtonian like equations

Within the matter dominated almost-FLRW models there are two useful relations (i) the relationship between the gravitational tidal forces and the matter perturbations, and (ii) the relationship between the shearing of the comoving volumes and gravitational effects; from (2.17) and (2.23) (respectively) we construct the matter dominated almost-FLRW equations:

$$D^b E_{ab} \approx +\frac{1}{3} D_a \rho_M - \left\{ \frac{1}{3} H q_a \right\}, \quad (5.67)$$

$$-\frac{1}{2} (\rho_M + p_M) \sigma_{ab} \approx \dot{E}_{ab} + 3H E_{ab} - \text{curl } H_{ab} + \left\{ \frac{1}{2} D_{(a} q_{b)} \right\}. \quad (5.68)$$

Here, in the total (u^{*a}) and matter frames (u_M^a), $q_b = 0$; this is not the case in general u^a -frame equations. This is important when we boost to the Newtonian frame from the total (or matter frame); we will pick up an additional flux term as pointed out in (5.37), (5.48) and chapter 2.

In the (implicitly scalar) Newtonian-like formulation the two useful relations (5.67) and (5.68) will be reduced to (i) the relationship between *the gravitational tidal forces and the matter perturbations*, and (ii) the relationship between *the relative velocity difference between the Newtonian frame and the total frame, and the gravitational tidal forces* :

$$\text{Newtonian frame: } \begin{cases} D^b E_{ab} \approx +\frac{1}{3} \tilde{D}_a \rho_M + \frac{1}{3} H (\rho_M + p_M) v_a, \\ \dot{E}_{ab} + 3H E_{ab} \approx -(\rho_M + p_M) D_{(a} v_{b)}. \end{cases} \quad (5.69)$$

Here $E_{ab} = D_{(a}D_{b)}\Phi_A$ and $\tilde{A}_a = D_a\Phi_A$. We also have that in the total frame u^{*a} with respect to the Newtonian frame (5.43) $D_{(a}v_{b)} = -\sigma_{ab}$. The Newtonian frame equations give one a physical interpretation of the relative velocities and the potentials. The point of the exercise is that in the total frame u^{*a} (and matter frame) we have in fact that:

$$\text{total frame: } \begin{cases} D^b E_{ab} \approx +\frac{1}{3}\tilde{D}_a\rho_M, \\ \dot{E}_{ab} + 3HE_{ab} \approx +\frac{1}{2}(\rho_M + p_M)D_{(a}v_{b)}. \end{cases} \quad (5.70)$$

Although $E_{ab} \approx D_{(a}D_{b)}\Phi_A$, we cannot now interpret Φ_A as an acceleration potential. This last set of equations are then the almost-FLRW equivalent of the usual equations describing the link between the gravitational tidal forces and the matter over densities and the relative velocities. To get the usual constants in the CDM case, use $a_0 = 1$ so that $\rho_C = 3H_0^2\Omega_0 a^{-3}$ and $p_C = 0$. To understand where this has taken us, we need the evolution equation for the relative velocities; consider (5.44), where in the total frame u^{*a} , $A^a \approx 0$:

$$\tilde{A}_a \approx D_a\Phi_A \approx \dot{v}_a + H v_a \quad (5.71)$$

which can also be obtained from (2.14) in the almost-FLRW situation – in the generic case it is found from (B.5). In the Newtonian frame we think of the relative velocity, v^a , as that of the matter: *the peculiar velocity field associated with a mass density distribution*¹⁸. Notationally whenever v^a or $v(k, t)$ are encountered we have in mind this peculiar velocity (as opposed to v_B^a the baryon relative velocity with respect to the u^a -frame). In the matter dominated situation the baryon relative velocity that appears in the Newtonian frame temperature anisotropy equations can be identified with the peculiar velocity.

We have recovered the canonical Newtonian-like equations from the 1+3 Lagrangian perspective. By *Lagrangian* we mean the total frame equations [120] (see chapter 2) – in which the energy momentum tensor takes on its most general form; and by *Eulerian*, we mean any other physically defined frames – typically the Newtonian choice in the almost-FLRW theory. For explicit solutions to these equations for the CDM dominated flat almost-FLRW case, see appendix D.6.1. The important point here – the temperature anisotropies, the temperature perturbations and dipole near tight-coupling – will be investigated in the Newtonian frame, while the matter dynamics and kinematics will be investigated in the total frame.

5.2.6 Linearizations, approximations and scales

In this section we discuss the various linearization and approximations (already mentioned in the last section), that will be used in this chapter.

Almost-FLRW linearization

Here we drop all terms that are at least $\mathcal{O}(\epsilon^2)$, the implication of this is that one can only consider small velocity boosts, large ones would break the linearity about the FLRW

¹⁸One may then speculate upon being able to recover the Zel'dovich approximation from (B.5), Eq. (5.71); upon setting $\omega^a = 0 = A^a$, linearizing in terms of v^a in (B.5), and substituting back into the exact field equations. To then recover the difference in the distortions in the energy density between the Eulerian and Lagrangian frames at linear order.

background. Hence we include $v^2 = |v^a v_a| \ll 1$ as an almost-FLRW limit, dropping the additional terms that are at least $\mathcal{O}(v\epsilon, v^2)$.

The important subtlety here is that ϵ_ℓ is in fact the temperature anisotropy smallness parameter as related to the temperature moment mean squares $|\tau_{A_\ell}| \propto \epsilon_\ell$. The almost-FLRW limits on the geometry, ϵ , (which define σ_{ab} , A_a and $D_a\Theta$, for example, as $\mathcal{O}(\epsilon)$, in appropriate dimensionless units [166, 167]) is related to ϵ_ℓ via the almost-EGS theorem (see chapter 4). In other words, limits on the temperature anisotropies, ϵ_ℓ , put bounds on the size of the smallness parameter ϵ , given that a weak Copernican principle holds. Furthermore, the limits on ϵ in turn place consistency limits on the size of the v/c boosts that are applicable (here in units of $c = 1$). Thus almost-FLRW means keeping terms that are at most :

$$\text{almost-FLRW} \approx \mathcal{O}(\epsilon, v). \quad (5.72)$$

Matter dominated linearization

This is based on the radiation-baryon ratio, $\eta \propto \frac{\rho_R}{\rho_M}$. We keep every term that is $\mathcal{O}(\eta)$ but in the almost-FLRW case of matter domination we then drop everything that is at least $\mathcal{O}(\eta v, \eta\epsilon, \epsilon v, \eta^2, \epsilon^2, v^2)$. We then have that **matter dominated almost-FLRW** (almost-FL) means keeping terms that are at most :

$$\text{almost-FL} \approx \mathcal{O}(\epsilon, v, \eta). \quad (5.73)$$

Expansion in Thompson scattering time

We will introduce a perturbative scheme in the Thompson scattering time, $t_c = (\sigma_T n_e)^{-1}$, and will consider terms up to $\mathcal{O}(t_c^3)$ during the tight-coupling calculation – such an expansion will be used to generate equations near to tight-coupling, the limiting case being when $t_c = 0$. Additionally an equivalent scheme can be constructed in terms of the differential optical depth κ' . This scheme is useful in the slow-decoupling era, *i.e.* in expansions where κ'' and $(\kappa')^2$ are sufficiently small to be ignored when compared to terms of order κ . This approximation allows one to additionally consider the case when $\kappa' e^{-\kappa} \ll e^{-\kappa}$.

Small and large scales

We will find it convenient to introduce the notion of small and large scales. We will do this in two heuristically equivalent ways. The first scheme is based on the parameter ϵ_H , where the Hubble expansion is of order ϵ_H , and is used when considering situations outside the Hubble flow; thus in the almost-FLRW small-scale case one would ignore all terms at least $\mathcal{O}(\epsilon_H^2, \epsilon^2, v^2, v\epsilon_H, v\epsilon, \epsilon\epsilon_H)$. This scheme is useful since it can be used without a mode expansion. It is ideal for making qualitative statements without the details which arise when introducing mode functions; specifically avoiding the complexity of mode-mode coupling in the small scale non-linear situation. By small-scale almost-FL we have in mind keeping only terms that are at most:

$$\text{small-scale almost-FLRW} \approx \mathcal{O}(\epsilon_H, \epsilon, v). \quad (5.74)$$

A second and more precise scheme is based on the Hubble scale $\lambda_H = a_*/k_H$ defined near the time of last scatter (say), allowing one to use $k/k_H > 1$ and $k/k_H < 1$ as characterizing large and small scales respectively.

5.3 1+3 covariant integral solutions

This section has three aims: (i) Reproducing the integral solution of the free-streaming mode equations and modifying them in order to take into account Thomson scattering, using the CGI variables [83, 155]. We carry out a timelike integration, instead of a null-cone integration corresponding to the original Sachs-Wolfe paper [152], restricting ourselves to scalar perturbations with adiabatic modes only and assuming for the most part a $K = 0$ almost-FL background universe. (ii) Showing in the CGI formulation how fluctuations at last scattering time result in measurable CBR anisotropies. (iii) Demonstrating how the solution can be related to standard formalisms by choosing specific frames; in particular we consider the Newtonian-like frame based on a shear free congruence.

The basic equation we are concerned with in this section is the IBE (5.5). In covariant form it is given by

$$\dot{\tau}(x, e) + e^a D_a \tau(x, e) + B(x, e) \simeq C[x, e], \quad (5.75)$$

where the **gravitational source term**, B , and **Thompson source term**, C , for damping by Thompson scattering are respectively represented by:

$$B = -\frac{1}{3} D^a \tau_a + (D_a \ln T + A_a) e^a + \sigma_{ab} e^a e^b, \quad C[x, e] \approx \dot{\kappa}(e^a v_a^B - \tau), \quad (5.76)$$

and in the almost-FL situation in mind¹⁹, the EFE give:

$$3\rho_M^{-1} (\text{div } E)_a \approx D_a \ln \rho_M, \quad \sigma_{ab} \approx -2[(\rho_M^{-1} E_{ab}) - \text{curl}(\rho_M^{-1} H_{ab})]. \quad (5.77)$$

The mode expanded form of this equation for the flat case ($K = 0$) can be written in the compact form as follows:

$$\tau'_\ell + k \left[\frac{(\ell + 1)^2}{(2\ell + 1)(2\ell + 3)} \tau_{\ell+1} - \tau_{\ell-1} \right] + \kappa' \tau_\ell \simeq S_B, \quad (5.78)$$

where

$$S_B = - [aB_0 \delta_{\ell 0} + (aB_1 + \kappa' v_B) \delta_{\ell 1} + aB_2 \delta_{\ell 2}]. \quad (5.79)$$

This combines (5.14)-(5.16) in a single equation for $\tau_\ell(\eta, k)$ (see [66, 120]), where $\tau = \sum_\ell \tau_\ell G_\ell[Q]$ is written in terms of conformal time η ($dt = a d\eta$), rather than proper time t . It is valid for all $\ell \geq 0$, with $\tau_0 = 0$ a solution as required, consistent with the definitions we introduced above (B_0 cancels the dipole term on the right in this case).

¹⁹One can compare this to the formulation of Durrer [41, 43] (eqns 3.5 and 18). To see how this is linked to the Bardeen potentials Ψ and Φ , we can use $E_{ab} = \frac{1}{2} D_{(a} D_{b)} (\Psi - \Phi)$ [43]. Notice that Durrer's integral solutions take on the Sachs-Wolfe form and can be compared with the treatments in [37, 124], while ours follow the form of [193, 155].

In what follows, we will deal with the integral solutions to (5.78) given the source terms – as in (5.79). The paradigm is that of matching an (almost-FL) era of free-streaming to one of tight-coupling. We will construct the homogeneous solution (that without inhomogeneous gravity or scattering) first, then include the gravitational effects to construct the free-streaming solution (*i.e.* after decoupling to the present day) and finally include the Thompson damping to find the integral solution including scattering (which can be used during slow decoupling or to include effects of reionization). Diffusion damping is included in this full solution with Thompson scattering, which in general has to be solved numerically. However it is helpful to introduce various analytical approximations for the different stages described by the solution; this will be done later, where the visibility function approximation is used and the damping scale derived. Additional effects, such as the anisotropic correction and polarization correction, have to be dealt with separately.

5.3.1 Integral solutions (flat almost-FL case)

Here we wish to find the general solution to (5.78) without collision terms, *i.e.* with $\kappa' = 0$, integrating along timelike curves using conformal variables. In order to do this we find, first, the solution to the homogeneous version of the equations (*i.e.* for no inhomogeneous gravity and no scattering), second, an integral solution of the inhomogeneous equation with inhomogeneous gravity taken into account from the homogeneous solution, and third, the general solution for free-streaming. In the next sections we consider the effect of Thompson scattering ($\kappa \neq 0$), and the transition from tight coupling to free-streaming.

The approach here is similar to the Seljak-Zaldariagga treatment [155, 198], however, they have taken the Sachs-Wolfe like formulation of the IBE, which is an integration down the null cone, and integrated out the angular dependence over Legendre polynomials (angular averaging) in order to construct the mode coefficients; then the conformal radial distance, χ , is written in terms of the conformal time η , leading to an integral solution dependent only on the conformal time. Thus, formally they have carried out a null-integration. By contrast, what is carried out here is in effect an integration of the IBE down the matter world-lines, thus this is a time-like integration, onto an initial surface ('last scattering'); the corresponding initial data near our past world line on that surface can then be related to that on the intersection of the past null-cone and the surface by means of a suitable homogeneity assumption. The timelike nature of the integration is often not made particularly clear in the literature, but solutions of the generic multipole divergence equations (from which the almost-FL mode hierarchy of temperature anisotropies are derived) are usually based on timelike integrations in the relativistic kinetic theory [120].

Finding the homogeneous solution

The $\ell = 0, 1$ and 2 MDE and hence mode form of the IBE are exceptional, given that $\tau_0 = 0$. The point of the integral solutions is to cast the exceptional equations $\ell = 0, 1, 2$ into a form that allows analytic investigation. We now find the covariant homogeneous solutions.

Consider the homogeneous equation (valid for $\ell \geq 1$) :

$$\tau_\ell^{(0)'} + k \left[\frac{(\ell+1)^2}{(2\ell+1)(2\ell+3)} \tau_{\ell+1}^{(0)} - \tau_{\ell-1}^{(0)} \right] = 0, \quad (5.80)$$

for the background $K = 0$ case²⁰ and without damping. The functions

$$\tau_\ell^{(0)}(k, \eta) = (2\ell+1)\beta_\ell^{-1} j_\ell(k\eta) \quad (5.81)$$

are solutions of (5.94), if the coefficients β_ℓ obey the recursion relations [66]:

$$(\ell+1)\beta_\ell = (2\ell+1)\beta_{\ell+1}, \quad \beta_\ell(2\ell-1) = \ell\beta_{\ell-1}. \quad (5.82)$$

This can be shown by multiplying (5.80) through by $\alpha_\ell = \beta_\ell(2\ell+1)^{-1}$ and comparing with the recursion relation for the spherical Bessel function. If the function $j_0(k\eta)$ satisfies the equation (5.80) for $\ell = 0$ (there are of course no terms with $\ell < 1$) then the rest of the equations ($\ell \geq 1$) will be satisfied because of the recursion relations :

$$-(2\ell+1)(\alpha_\ell \tau_\ell)' \simeq k[(\ell+1)(\alpha_{\ell+1} \tau_{\ell+1}) - \ell(\alpha_{\ell-1} \tau_{\ell-1})]. \quad (5.83)$$

The freedom in $\beta_\ell(k)$ is in $\beta_0(k)$ and $\beta_1(k)$. Given that the Bessel function is finite at the origin: $j_\ell(0) = \delta_{\ell 0}$, these can be chosen to satisfy $\beta_0 = \beta_1 = +1$, the rest are generated through the recursion relations on β_ℓ and then determine the solution $\tau_\ell^{(0)}(k, \eta)$. The arbitrarily specifiable initial data are later fixed by introducing an integral solution ((5.87) below) containing arbitrary functions $C_A(\eta)$ (see (5.88)) which are determined by the EFE through $\mathcal{B}_I(\eta)$.

The corresponding mode functions are

$$\tau^{(0)}(x, e) = \sum_{\ell=1}^{\infty} \beta_\ell^{-1} (2\ell+1) j_\ell(k\eta) O^{A_\ell} Q_{A_\ell} \Big|_{\text{FLAT}}, \quad (5.84)$$

(cf. (5.8)) and the corresponding multipole coefficients can then be found:

$$\tau_{A_\ell} \Big|_{\text{FLAT}} \simeq \beta_\ell^{-1} (2\ell+1) j_\ell(k\eta) Q_{A_\ell} \Big|_{\text{FLAT}}. \quad (5.85)$$

Notice that this differs by a factor of $i^{-\ell}$ from Wilson [193] since we are using plain mode functions instead of plane waves, although these can be easily related. Note there is no explicit mode mixing in this approximation, but such mixing is implicitly determined by the recursion relations (5.82). This shows that we should be careful with any truncation procedure we propose (see chapter 2). This procedure can be easily extended to the open case using the recursion relations for the open mode functions (see appendix D.9).

Construction of the integral solution.

Given that we have the solution $\tau_\ell^{(0)}(k, \eta)$ to the homogeneous equation of the form (5.94), now consider the equation with given gravitational source terms, but still without damping:

$$\tau_\ell' + k \left[\frac{(\ell+1)^2}{(2\ell+1)(2\ell+3)} \tau_{\ell+1} - \tau_{\ell-1} \right] = -[a\mathcal{B}_1 \delta_{\ell 1} + a\mathcal{B}_2 \delta_{\ell 2}]. \quad (5.86)$$

²⁰Or for the open or closed cases by following [82] [70].

What is important to notice here, is that this equation is valid for $\ell \geq 1$, not $\ell \geq 0$ as in (5.78); indeed $\tau_0 = 0$. We need to find a particular solution to this equation.

We proceed as follows. Consider the ansatz in terms of A_ℓ using that $\delta\eta = \eta - \eta'$, along with the Liebnitz rule for differentiation of integrals:

$$\tau_\ell^P(\eta) = \int_0^\eta d\eta' A_\ell(\eta, \eta') \Rightarrow \frac{\partial \tau_\ell^P}{\partial \eta}(\eta) = \int_0^\eta d\eta' \frac{\partial}{\partial \eta} A_\ell(\eta, \eta') + A_\ell(\eta, \eta). \quad (5.87)$$

Now we define the kernel, A_ℓ , as in [193, 198]:

$$A_\ell(\eta, \eta') = C_0(\eta) \tau_\ell^{(0)}(\delta\eta) + C_1(\eta) \frac{\partial}{\partial \eta'} \tau_\ell^{(0)}(\delta\eta) + C_2(\eta) \frac{\partial^2}{\partial \eta'^2} \tau_\ell^{(0)}(\delta\eta). \quad (5.88)$$

It can then be shown from (5.87) and (5.88) that :

$$\tau_\ell^P + k \left[\frac{(\ell+1)^2}{(2\ell+1)(2\ell+3)} \tau_{\ell+1}^P - \tau_{\ell-1}^P \right] = \mathcal{S}_C, \quad (5.89)$$

where

$$\mathcal{S}_C = C_0(\eta) \tau_\ell^{(0)}(0) + C_1(\eta) \tau_\ell^{(0)'}(0) + C_2(\eta) \tau_\ell^{(0)''}(0), \quad (5.90)$$

given $\tau_\ell^P(\eta)$ as in (5.87-5.88); thus (5.86) is satisfied by our ansatz provided the coefficients $C_0(\eta)$, $C_1(\eta)$, and $C_2(\eta)$ in the integral solution are found in terms of the CGI variables $\mathcal{B}_0(\eta)$, $\mathcal{B}_1(\eta)$, and $\mathcal{B}_2(\eta)$ determined by the EFE. This will be ensured next, when we put the parts of the solution together to obtain (5.98).

Inclusion of damping.

Here we extend the previous solution (5.86), where the relationship between the coefficients in the IBE (D.29) can be read off from (5.90), including damping through κ' . We notice that if $\tau_\ell^{(0)}$ is a solution to (5.94), then

$$\tau_\ell^*(\eta) = e^{-\kappa(\eta)} \tau_\ell^{(0)}(\eta) \quad (5.91)$$

will be a solution to

$$\tau_\ell^{*'} + k \left[\frac{(\ell+1)^2}{(2\ell+1)(2\ell+3)} \tau_{\ell+1}^* - \tau_{\ell-1}^* \right] + \kappa' \tau_\ell^* \simeq 0. \quad (5.92)$$

Similarly we find that for the integral solution (5.87), of (5.86), the expressions

$$\tau_\ell^{*P}(\eta) = \int_0^\eta d\eta' e^{-\kappa(\eta')} A_\ell(\eta, \eta') \quad \text{or} \quad \tau_\ell^{*P}(\eta) = e^{-\kappa(\eta)} \int_0^\eta d\eta' A_\ell(\eta, \eta'), \quad (5.93)$$

will be particular solutions to (5.78), given the correct choice of C_0 , C_1 and C_2 . Hence we can modify the solutions of the previous section to include Thompson scattering by simply including the damping terms as in these equations. The extended equations thus obviously include the free streaming case too, as the special case when for some interval

of time $\kappa = 0$; thus they can extend all the way from late tight coupling to the present day, if we include a suitably time-dependent scattering coefficient κ .

In more detail: we have that $\tau_\ell^{(0)}(\eta_*) = 0$. We assume that free-streaming begins at some η_d . As before we have (now using $\delta\eta^* = \eta - \eta_*$)

$$\tau_\ell^{(0)}(\eta) = (2\ell + 1)\beta_\ell^{-1}j_\ell(k\delta\eta^*) \Rightarrow \tau_\ell^*(\eta) \simeq \int_0^{\delta\eta^*} d\eta' A_\ell(\eta, \eta'), \quad (5.94)$$

where once again the initial data is now $C_I(\eta' - \eta_*)$; notice that we do not introduce an additional η_* as we will be using the solution of $\tau_\ell^{(0)}$ already including the initial conditions ²¹. Once again we have the integral solution $\tau_\ell^P(\eta)$ integrated from 0 to $\delta\eta^*$ such that the anisotropies are now determined by $\tau_\ell^P(\eta_0)$ – setting the integral from 0 to $\Delta\eta_*$ (for $\Delta\eta = \eta_0 - \eta'$), we find

$$\begin{aligned} \tau_\ell(\eta_0) = & \int_{\eta_*}^{\eta_0} d\eta' e^{-\kappa(\eta' - \eta_*)} \left[C_0(\eta' - \eta_*)\tau_\ell^{(0)}(\Delta\eta) \right. \\ & \left. + C_1(\eta' - \eta_*)\tau_\ell^{(0)'}(\Delta\eta) + C_2(\eta' - \eta_*)\tau_\ell^{(0)''}(\Delta\eta) \right], \end{aligned} \quad (5.95)$$

where the damping term now enters explicitly.

The complete solution.

We can now construct the general solution to (5.78) with $\kappa' \neq 0$ by putting the previous results together. The homogeneous seed solution $\tau_\ell^{(0)}(\eta)$ is given by (5.81). The particular integral solution $\tau_\ell^P(\eta)$ is given, in terms of $\delta\eta = \eta - \eta'$ by (5.93). The general solution is then given by

$$\tau_\ell(\eta) = e^{-\kappa(\eta)}\tau_\ell^{(0)}(\eta) + \tau_\ell^{*P}(\eta). \quad (5.96)$$

Substituting this into the general equation (5.78) and using the radial eigenfunctions evaluated at zero (in particular $j_\ell(0) = \delta_{\ell 0}$ along with the recursion relation) gives

$$C_0\delta_{\ell 0} + C_1\beta_1^{-1}k\delta_{\ell 1} + C_2k^25\beta_2^{-1}\left(\frac{2}{15}\delta_{\ell 2} + \frac{1}{3}\delta_{\ell 0}\right) = -[(a\mathcal{B}_1 + \kappa'v_B)\delta_{\ell 1} + a\mathcal{B}_2\delta_{\ell 2}], \quad (5.97)$$

relating the functions determining the solution to the time-dependent coefficients in the equation. From (5.97) the functions in the integral solution are found in terms of the dynamical CGI variables:

$$-C_0(\eta) \simeq +\frac{5}{2}a\mathcal{B}_2, \quad -C_1(\eta) \simeq +\frac{1}{k}(a\mathcal{B}_1 + \kappa'v_B), \quad -C_2(\eta) \simeq +\frac{1}{k^2}a\mathcal{B}_2. \quad (5.98)$$

For scalar perturbations, the term C_2 is effectively the coefficient of the shear; the term C_1 will be the coefficient of the gradient of the temperature and the acceleration; the coefficient C_0 the shear contribution; see (D.40) noting that these are CGI with respect to the u^a -frame.

These functions are all evaluated at a time η , which takes all values from η_d , the time of decoupling, to η_0 , the time of observation. In the nullcone formulation of the

²¹We could have used $\tau_\ell^{(0)}(\delta\eta) \rightarrow \tau_\ell^{(0)}(\delta\eta + \eta_*)$ along with the original homogeneous solution unchanged.

problem this input of new information corresponds to the way the null geodesics from the point of emission to the observer keep crossing new matter and hence encounter new information. Because we are integrating on a timelike curve, this information is represented here as varying with time along that curve; and in some simple circumstances, the values at later times are determined fully by the values at earlier times (as happens, for example, in the original Sachs-Wolfe case: $K = 0$, $p = 0$, and only decaying scalar modes).

5.3.2 Integration by parts

In order to deal easily with the initial data it is now useful to write the general solution for $K = 0$ in terms of the present time, η_0 , and the initial time, η_* , by integrating with respect to conformal time and defining $\Delta\eta_* = \eta_0 - \eta_*$ ²². Notice that from $d\tau^{(0)}/dt = 0$, we have $\tau^{(0)}(x(\eta), e(\eta)) = \tau^{(0)}(x(\eta_*), e(\eta_*))$, and $j_\ell(0) = \delta_{\ell 0}$ (we have chosen the solution to be finite at origin). We choose the initial conditions $\tau_\ell(\eta_*) = 0$ [193] and $\tau_0(\eta) = 0$ and using the parameter freedom in the homogeneous solution, set $\beta_0(k) = +1$ and $\beta_1(k) = +1$. The homogeneous solution is now fixed as in (5.94), for $\delta\eta^* = \eta - \eta_*$ and this in turn sets the integral solution to (5.93) [193]:

$$\tau_\ell^{*P}(\eta) = \int_0^{\delta\eta^*} e^{-\kappa(\eta')} A_\ell(\eta, \eta') d\eta', \quad (5.99)$$

where we still have the freedom of setting the initial data for the integral solution from the $C_I(\eta)$'s which are fixed by the EFE. Putting this all together it is then found that

$$\tau_\ell(\eta_0) = \tau_\ell^{(0)}(\eta_*) + \tau_\ell^{*P}(\eta_0) = \tau_\ell^{*P}(\eta_0) = \int_0^{\Delta\eta_*} e^{-\kappa} A_\ell(\eta_0, \eta') d\eta'. \quad (5.100)$$

We can change the integration limits from $(0, \Delta\eta_*)$ to (η_*, η_0) i.e. $\eta' \rightarrow \eta' - \eta_*$, and setting $C_I(\eta') \rightarrow C_I(\eta' - \eta_*)$.

On changing the integration to from η_* to η_0 in (5.95), integrating by parts, and using the initial conditions (once again $\tau_\ell^{(0)}(\eta_*) = 0$) we find :

$$\begin{aligned} \tau_\ell(\eta_0) &\simeq [C_2'(\eta_*) - C_1(\eta_*)] e^{-\kappa} \tau_\ell^{(0)}(\eta_0) - C_2(\eta_*) e^{-\kappa} \tau_\ell^{(0)'}(\eta_0), \\ &+ \int_{\eta_*}^{\eta_0} d\eta' e^{-\kappa} [C_0(\eta') - C_1'(\eta') + C_2''(\eta')] \tau_\ell^{(0)}(\eta_0 + \eta_* - \eta'), \\ &+ \int_{\eta_*}^{\eta_0} d\eta' (\kappa' e^{-\kappa}) [C_1(\eta') - 2C_2(\eta')] \tau_\ell^{(0)}(\eta_0 + \eta_* - \eta'), \dots \\ &+ \int_{\eta_*}^{\eta_0} d\eta' ((\kappa')^2 - \kappa'') e^{-\kappa} C_2(\eta') \tau_\ell^{(0)}(\eta_0 + \eta_* - \eta'). \end{aligned} \quad (5.101)$$

The initial data for the solution $\tau_\ell^{(0)}(k, \eta)$ is the set of constants $C_I(\eta_*)$ which are determined by $\mathcal{B}_A(k) = \{\mathcal{B}_0(k), \mathcal{B}_1(k), \mathcal{B}_2(k)\}$; these must be matched to the initial distribution function on an appropriate initial surface Σ (for example, the 'surface of last

²²The relationship between the conformal time η and the radial distance χ is $d\chi = -d\eta$ so $\chi = \eta_R - \eta$ which follows for the homogeneity and isotropy in the background.

scattering' which can be covariantly and gauge invariantly defined). This then determines the solution up to the present day (and after). We are free to choose any Q 's as long as they solve the Helmholtz equation in the relevant background. The choice of Q basis then explicitly determines $G_\ell[Q]$; we are free to choose Q in terms of the spherical or planewave basis for example. In practice we naturally use two sets of mode functions $G_\ell[Q]$, matching those for the null-cone (given in a spherical basis) to those in some initial surface (given in terms of a plane-wave basis). The matching of these two sets of harmonics is then given by the relations usually written into the construction of the mode coefficients (see (5.84)). This matching is based on mode functions $G_I[Q]$ in the RW background, this is acceptable because of the homogeneity assumption. By using $G_\ell[Q]$ we do not actually need the explicit form of the Q 's.

Equation (5.101) shows (r.h.s. of the first line) how major parts of the CBR anisotropy are determined directly from the initial conditions set (e.g. at last scattering, for the freely propagating radiation). The integrated effect arises through the coefficients $C_I(\eta)$ as integrated down timelike geodesics in the remaining terms on the right hand side. In general there is a non-linear coupling through the field equations between the matter, the radiation and the acceleration and shear terms that arise in the integrated part. The situation is much simpler when this back-reaction can be neglected; for this reason it is convenient to consider the case of matter domination, the radiation considered as a test-field propagating on a background determined by the matter content. However we can also consider the general set of linearized field equations (2.13 -2.25) and the coupling to the radiation via the source terms, first the gravitational source, \mathcal{B} , and second, the scattering source, $C[\tau]$, (5.76), in (5.75). In the following sections we look at the various approximations that can be applied at different epochs.

5.4 Free-streaming

Using the integral solution (5.101) we construct the almost-FL free-streaming projection of the initial conditions near last scattering to here and now (the determination of these initial conditions is demonstrated in latter sections) and the integrated secondary contributions arising during the period after last-scattering until now (we have dropped the baryon relative velocity effect using the instantaneous decoupling assumption):

$$\begin{aligned} \frac{\tau_\ell(\eta_0)\beta_\ell}{(2\ell+1)} &\simeq \left[\frac{1}{k}[a\mathcal{B}_1](\eta_*) - \frac{5}{3}\frac{1}{k^2}[a\mathcal{B}_2]'(\eta_*) - \frac{1}{k}[a\mathcal{B}_2](\eta_*)\frac{\partial}{\partial\eta_0} \right] j_\ell(k\Delta\eta_*) \\ &- \int_{\eta_*}^{\eta_0} d\eta \left\{ \frac{5}{6}a\mathcal{B}_2 - \frac{1}{k}(a\mathcal{B}_1)' + \frac{5}{3}\frac{1}{k^2}(a\mathcal{B}_2)'' \right\} j_\ell(k\Delta\eta). \end{aligned} \quad (5.102)$$

Here we used as final conditions :

$$[a\mathcal{B}_1]'(\eta_0) = [a\mathcal{B}_2](\eta_0) = 0. \quad (5.103)$$

The first term on the right, the \mathcal{B}_1 term, will generate the acoustic primary effect on the anisotropies, the second term is the Doppler contribution due to the radiation dipole (the baryon velocity contribution which would arise through C_1 (5.98), the third and fourth

terms give the effect of any shear, near last scattering (through the initial conditions of \mathcal{B}_2). The remaining terms represent the integrated Sachs-Wolfe effect.

The above equation will be modified in the following section to include slow decoupling, but first we demonstrate how to recover the basic Sachs-Wolfe effect.

5.4.1 The almost-FL Sachs-Wolfe effect

We will now find the solutions for matter dominated, free-streaming era, with adiabatic modes only. Using the EFE (see *e.g.* in [120], appendix B and 5.62), it follows that in the matter-dominated case (at instant last scattering and in the free-streaming era):

$$\text{Source terms: } \begin{cases} a\mathcal{B}_0 = -\frac{1}{3}k\tau_1, \\ a\mathcal{B}_1 = \frac{2}{3}k^{-1}(\bar{\Phi}\rho_M^{-1})(k^2 - 3K) \approx \frac{k}{a}\delta T, \\ a\mathcal{B}_2 = -2(\bar{\Phi}\rho_M^{-1})', \end{cases} \quad (5.104)$$

where we have used that $E_{ab} = \bar{\Phi}Q_{ab}$ ²³.

In the total frame (or matter) the dipole is vanishingly small – we ignore it. The shear contribution is small on large scales – hence we ignore it too.

On substituting these into the flat almost-FL integral solution (5.101) with $K = 0$ in (5.104) we can find the free-streaming almost-FL solutions for the temperature anisotropies (5.102). Using $(\bar{\Phi}\rho_M^{-1})' \sim 0$ ²⁴ we find the CGI kinetic theory equivalent of the Sachs-Wolfe formula for CBR anisotropies in terms of matter inhomogeneities at last scattering at various wavelengths, together with an integral term. In various cases, in particular matter-dominated spatially flat solutions with only growing scalar modes, the integral terms vanishes and we then obtain (in the total matter frame):

$$\text{Sachs-Wolfe effect: } \frac{\tau_\ell^{SW}(\eta_0, k)\beta_\ell}{(2\ell + 1)} \approx \frac{2}{3}[\bar{\Phi}\rho_M^{-1}](\eta_*)j_\ell(k\Delta\eta_*). \quad (5.105)$$

This gives the (approximate) projection of the large scale potential inhomogeneities at last scattering onto the sky today in the CGI invariant framework; the mean-squares $|\tau_\ell|^2$ can then be constructed, using the results from chapter 1 [66]. The effect arises from the terms $D_a \ln T \approx \frac{1}{3}D_a \ln \rho \approx \rho_M^{-1}D^b E_{ab}$, having used the adiabatic assumption in the total frame. This recovers the standard Sachs-Wolfe result [152, 37] at large scales, where the potential fluctuations are just due to primordial initial inhomogeneities that are unchanged by intervening physics (they are at super-horizon scales). Additionally it gives the anisotropy at large scales where acoustic oscillations have led to periodic potential fluctuations at last scattering.

We discuss the origin of these fluctuations in later sections – they are given by solving the equations before decoupling, which for example implies the existence of acoustic oscillations. These potential fluctuations are what seed structure formation through the production of matter perturbations undergoing gravitational collapse beneath the Jeans scale. The matter perturbations effectively decouple near matter-radiation equality, making the large scale temperature anisotropies the key link between the radiation

²³This differs from the definition $E_{ab} = D_{(a}D_{b)}\Phi_E$ used to link to Φ_H and Φ_A in [7] by k^2/a^2 .

²⁴In fact one has that for an EdS background: $[\bar{\Phi}\rho_M^{-1}](\eta_*) = -(3H_0^2\Omega_0)^{-1}D_*[k^2\Phi_A(k, 0)]$ and $(\Omega_0 D_*/a_*)^2 \approx \Omega_0^{1.54}$

anisotropies now and the potential fluctuations then (near radiation decoupling), and so to the matter power spectrum both on large and small scales today.

Notice that these equations will hold for any choice of 4-velocity that is close to the matter 4-velocity, i.e. there is still a frame freedom here associated with that freedom of choice. As has been remarked various times, there are several possible physical choices for this 4-velocity (which will all agree at late times); the interpretation of the physical meaning of the CBR anisotropy sources will change depending on our choice. The important difference between the derivation of the Sachs-Wolfe effect here and other treatments (i) is that this result *is found in the total u^a -frame* (5.102), (ii) the integration *is explicitly timelike*.

In the Lagrangian approach this corresponds in a natural way to the total frame results – given that we have assumed a CDM dominated almost-FLRW model this also corresponds to the matter and energy frames. If we do not assume matter domination we could still carry the calculation out in the generic u^a -frame too. However, it then becomes convenient to evaluate the temperature anisotropies in the Newtonian threading.

We have obtained this result by a timelike integration, so it is not treated mathematically as a projection along null rays but rather as the evolution of the anisotropies of radiation in a small comoving box, as explained in the introduction. Thus the initial data here is not at the intersection of the past light cone and the last scattering surface, but rather at the intersection of the world line of the observer and the last scattering surface.

This analysis can be compared to the derivation of the primary anisotropy source term of the gauge-invariant treatment used [83]. The subtle difference between the Bardeen variable gauge-invariant approach and the CIG approach used here is that the Doppler source, which in their case arises through \mathcal{B}_0 , now enters through the integral terms only; there is no direct Doppler contribution at last scattering from the first term in the integrated solution.

5.5 Slow decoupling

To deal with slow-decoupling, we return to the general damped solution, (5.101), and introduce the slow decoupling approximation

$$\kappa'' \ll 1, \quad (\kappa')^2 \ll 1, \quad (5.106)$$

to find, on substituting in from the coefficient relations (5.98):

$$\begin{aligned} \frac{\tau_\ell(\eta_0)\beta_\ell}{(2\ell+2)} &\simeq e^{-\kappa} \left[+\frac{1}{k}[a\mathcal{B}_1](\eta_*) + \frac{1}{k^2}\frac{5}{3}[a\mathcal{B}_2]'(\eta_*) - \frac{1}{k}[a\mathcal{B}_2](\eta_*)\frac{\partial}{\partial\eta_0} \right] j_\ell(k\Delta\eta_*) \\ &\quad - \int_{\eta_*}^{\eta_0} d\eta' e^{-\kappa} \left\{ \frac{5}{6}a\mathcal{B}_2 - \frac{1}{k}(a\mathcal{B}_1)' + \frac{5}{3}\frac{1}{k^2}(a\mathcal{B}_2)'' \right\} j_\ell(k\Delta\eta) \\ &\quad + \frac{1}{k}(\kappa' e^{-\kappa})v_B(\eta_*)j_\ell(k\Delta\eta_*) + \int_{\eta_*}^{\eta_0} d\eta' (v_B'\kappa' + \kappa''v_B)e^{-\kappa} \frac{1}{k}j_\ell(k\Delta\eta) \\ &\quad + \int_{\eta_*}^{\eta_0} d\eta' (\kappa' e^{-\kappa}) \left[-\frac{1}{k}(a\mathcal{B}_1) + 2\frac{1}{k^2}(a\mathcal{B}_2) \right] j_\ell(k\Delta\eta). \end{aligned} \quad (5.107)$$

We see that damping effects are controlled by $e^{-\kappa}$ and $\kappa'e^{-\kappa}$. A further approximation would be to take $e^{-\kappa} \gg \kappa'e^{-\kappa}$, so that we need only consider the free-streaming like solutions, which we then convolve with the damping factor, defined as a combination of the visibility function and the diffusion damping envelope – this is done later using the damping envelope as derived from the dispersion relations in section (5.5.2). Later we will explicitly recover these equations in the Newtonian-like frame of section (5.2.4).

5.5.1 Silk Damping

Diffusion damping will occur and introduce a damping scale, the Silk scale, giving a cut-off in the matter perturbations, and there will be a corresponding diffusion damping effect in the photons. This is naturally included in our general damped solutions in terms of the exponential envelope implied by the equations, which can be demonstrated heuristically.

The cut-off arising through photon diffusion occurs when the term in κ' in (5.78) dominates the other terms; that is, when for any ℓ , k is large enough that

$$k \left[\frac{1}{4}\tau_{\ell+1} - \tau_{\ell-1} \right] \approx \frac{3}{4}k\tau_{\ell} \ll \kappa'\tau_{\ell}, \quad (5.108)$$

the approximation assuming that the damped modes are roughly of the same magnitude (independent of ℓ when this condition is satisfied). This then implies an exponential decay in the relevant modes:

$$k \ll \frac{4}{3}\kappa' \Rightarrow \tau'_{\ell} \approx -\kappa'\tau_{\ell} \Rightarrow \tau_{\ell}(\eta) \approx \exp(-\kappa'\eta)\tau_{\ell}(0). \quad (5.109)$$

Thus small scales will be heavily damped by this process and long wavelengths unaffected, leading to a wavelength-dependent damping envelope. The resulting cut-off in perturbation amplitude at a critical wavelength at last scattering will result in a corresponding cut-off in CBR anisotropy amplitudes observed at a critical angular scale.

A more detailed examination undertaken later will show the explicit wavelength dependence of this cut-off effect.

5.5.2 The visibility function

An alternative approach to the slow decoupling solution (5.107), is to argue that the dominant contribution during slow-decoupling arises from the visibility function defined by $\mathcal{V}(k, \eta) \approx \kappa'e^{-\kappa}$ as convolved with the free-streaming integral solution. The visibility function gives the probability of a photon last scattering during a small time interval $d\eta$. From Hu & Sugiyama [82] it is useful to define the damping factor (now including diffusion damping, which will be derived from the coupled baryon-photon equations in (5.178)):

$$\mathcal{D}(\eta_0, k) = \int_{\eta_*}^{\eta_0} d\eta \mathcal{V}(\eta, k) e^{-(k/k_D)^2} \approx e^{-(k/k_D)^2}. \quad (5.110)$$

The visibility function will model the changing ionization fraction, this does not include the diffusion damping, which is added in by hand above, through the damping scale²⁵.

Now, we can modify the free-streaming projection by including the damping factor \mathcal{D} and the baryon velocity effect (which must be put in from (5.101)). In this approximation, we can effectively drop the last two lines of (5.101) except for the initial baryon velocity contribution, to obtain:

$$\begin{aligned} \frac{\tau_\ell(\eta_0)\beta_\ell}{(2\ell+1)} &\simeq [C'_2(\eta_*) - C_1(\eta_*)] \mathcal{D}(\eta_0, k) j_\ell(k\Delta\eta_*) \\ &+ kC_2(\eta_*)\mathcal{D}(\eta_0, k) \left[\frac{\ell}{(2\ell+1)} j_{\ell-1}(k\Delta\eta_*) - \frac{(\ell+1)}{(2\ell+1)} j_{\ell+1}(k\Delta\eta_*) \right] \\ &+ \int_{\eta_*}^{\eta_0} d\eta \mathcal{V} e^{-(k/k_D)^2} \{C_0(\eta) - C'_1(\eta) + C''_2(\eta)\} j_\ell(k\Delta\eta), \end{aligned} \quad (5.111)$$

where the coefficients C_1 , C_2 and C_3 are given by (5.98). This has the effect of taking the previous more general solution (5.101) and specializing it to the most important regime as far as decoupling is concerned, thus giving a major improvement on the sharp decoupling approximation, while avoiding the complications of the complete integral solution given above. A similar correction is made, using the visibility function, in the integrated part of the solution in order to best deal with a changing ionization fraction, given that we will once again only be using almost-FL solutions that are either first order or zero-th order in the scattering times (discussed below).

5.5.3 Slow decoupling in the conformal Newtonian frame

Here we cast the above derived solutions (5.102, 5.107, 5.111) based on the integral solution (5.101) into the CGI Newtonian frame (the shear-free frame described in section 5.2.4) for the case of scalar perturbations, in terms of the Bardeen like scalar potentials Φ_H and Φ_A .

The vanishing shear condition $\bar{\sigma}_{ab} \approx 0$ implies that $\tilde{C}_0(\eta) \approx 0$ and $\tilde{C}_2(\eta) \approx 0$, hence we can use these conditions directly to find the slow-decoupling anisotropy solution for an almost-FL model in the Newtonian frame:

$$\frac{\tau_\ell(\eta_0)\beta_\ell}{(2\ell+1)} \simeq -\tilde{C}_1(\eta_*)\mathcal{D}(\eta_0, k)j_\ell(k\Delta\eta_*) - \int_{\eta_*}^{\eta_0} d\eta \mathcal{V} e^{-(k/k_D)^2} \tilde{C}'_1(\eta)j_\ell(k\Delta\eta) \quad (5.112)$$

Using the results from section (5.2.4) and the almost-FL relations (given in the appendix), it can be shown that the key quantity of interest: $\tilde{B}_a \approx \tilde{D}_a \ln T + D_a \Phi_A$, in the case of scalar perturbations, obeys the following relation in the Newtonian frame:

$$D_{(a}\dot{\tilde{B}}_{b)} \approx D_{(a}D_{b)}(\dot{\Phi}_A - \dot{\Phi}_H) - 2HD_{(a}D_{b)}\Phi_A - \frac{1}{3}D_{(a}D_{b)}(D^c\tilde{\tau}_c) \quad (5.113)$$

²⁵It should be realized here is that the diffusion damping is naturally included in the original almost-FL integral solution (5.101), however, given that we use solutions that are first-order in the scattering time in our tight-coupling approximation, the diffusion damping factors are not included as they are second order in scattering time, which is why we have to put this effect in by hand.

Now we need to find the mode coefficients for \mathcal{B} in terms of the quantities defined in the Newtonian frame. Writing

$$\tilde{D}_a \ln T \approx \frac{k}{a} \delta \tilde{T} Q_a, \quad D_a \Phi_A \approx \frac{k}{a} \Phi_A Q_a, \quad \text{and} \quad D_a \Phi_H \approx \frac{k}{a} \Phi_H Q_a, \quad (5.114)$$

we find that

$$\tilde{B}_a \approx \frac{k}{a} (\delta \tilde{T} + \Phi_A) Q_a \Rightarrow \tilde{B}_1 \approx \frac{k}{a} (\delta \tilde{T} + \Phi_A). \quad (5.115)$$

On mode expanding (5.113)²⁶ and transforming to the conformal time derivative we obtain:

$$(a\tilde{B}_1)' \approx -a^2 H \tilde{B}_1 + k(\Phi'_A - \Phi'_H) - 2Hak\Phi_A + \frac{1}{3}k^2\tilde{\tau}_1. \quad (5.116)$$

Now using (5.98) we find that:

$$-\tilde{C}'_1(\eta) - (\kappa' \tilde{v}_B)' \approx +\frac{1}{k}(a\tilde{B}_1)'. \quad (5.117)$$

We can now put this all together, first, from (5.98) and (5.115) to find:

$$-\tilde{C}_1(\eta) \approx \kappa' \tilde{v}_B + (\delta \tilde{T} + \Phi_A), \quad (5.118)$$

and second from (5.117), (5.118) and (5.116) to find:

$$-\tilde{C}'_1(\eta) \approx (\kappa' \tilde{v}_B)' + \left((\Phi'_A - \Phi'_H) - aH(\delta \tilde{T} + \Phi_A) - 2aH\Phi_A + \frac{1}{3}k\tilde{\tau}_1 \right). \quad (5.119)$$

Substituting these results into the integral solution (5.112) we obtain:

$$\begin{aligned} \frac{\tau_\ell(\eta_0)\beta_\ell}{(2\ell+1)} &\approx \left[(\delta \tilde{T} + \Phi_A) + \kappa' \tilde{v}_B \right] (\eta_*, k) \mathcal{D}(\eta_0, k) j_\ell(k\Delta\eta_*) \\ &+ \int_{\eta_*}^{\eta_0} d\eta \mathcal{V} e^{-(k/k_D)^2} \left\{ (\kappa' \tilde{v}'_B + \kappa'' \tilde{v}_B) + \frac{1}{3}k\tilde{\tau}_1 \right\} j_\ell(k\Delta\eta) \\ &+ \int_{\eta_*}^{\eta_0} d\eta \mathcal{V} e^{-(k/k_D)^2} \left\{ (\Phi'_A - \Phi'_H) - aH(\delta \tilde{T} + 3\Phi_A) \right\} j_\ell(k\Delta\eta). \end{aligned} \quad (5.120)$$

Here the second order terms (both in terms of the scattering time and in the almost-FL sense) have been dropped. We can then pull out the canonical solution when we ignore the Doppler contribution, the initial baryon relative velocity at last scattering (it is tightly coupled to the radiation velocity and is thus small already). We also ignore the intermediate scale integrated effect which contributes to the early-ISW effect, the result is:

$$\frac{\tau_\ell(\eta_0)\beta_\ell}{(2\ell+1)} \approx \left[\delta \tilde{T} + \Phi_A \right] \mathcal{D}(\eta_0, k) j_\ell(k\Delta\eta_*) + \int_{\eta_*}^{\eta_0} d\eta \mathcal{V} e^{-(k/k_D)^2} \left[\Phi'_A - \Phi'_H \right] j_\ell(k\Delta\eta). \quad (5.121)$$

This completes the recovery of the standard integral solution results using the 1+3 CGI approach at linear order, where we have notationally suppressed the k -dependence of the temperature anisotropy ($\tau_\ell(\eta_0) \equiv \tau_\ell(\eta_0, k)$). It corroborates the standard anisotropy derivations based on the 3+1 hypersurface foliation approach which uses the Bardeen formalism in the conformal Newtonian gauge.

²⁶We will use $D_{(a}D_{b)}\dot{\Phi}_A \approx -\dot{\Phi}_A D_{(a}D_{b)}Q$, $D^a\tau_c \approx +\frac{k}{a}\tau_1 Q$ and $D_{(a}\dot{B}_{b)} \approx -\frac{c}{k}(\dot{B}_1 + 2HB_1)D_{(a}D_{b)}Q$.

5.6 Late tight-coupling

Here we extend the Thomson scattering analysis of the previous sections to include a simple model of late-tight coupling and hence of fast decoupling.

We aim to reproduce in covariant form the Peebles and Yu *near-tight* coupling [143] and Hu and Sugiyama *tight-coupling approximation* [83] treatments, valid for the period of late tight-coupling, up to and including decoupling²⁷. Remember that we are ignoring the anisotropic and polarization effects as these can be corrected at the level of the damping scale.

5.6.1 Integrated Boltzmann Equation near-tight coupling

Here the almost-FL IBE is used to construct a set of MDEs that describe the radiation near tight-coupling. These are the intermediate scale equations, valid in the tight-coupling era. This is done by carrying out a CGI version of perturbation theory in terms of the scattering time.

The scattering-strength expansion

The solutions we have considered so far are linearized through a small-parameter expansion in terms of the anisotropy function τ . The basic idea now, following the method of Peebles and Yu [143], is that additionally a second expansion is constructed in terms of the collision parameter $t_c = (\sigma_T n_e)^{-1}$, without truncating the Boltzmann hierarchy at the order of the calculation, and thus avoiding the problems inherent in exact truncation [58] (see chapter 2). We thus find the evolution equations for the energy density, momentum flux, and the anisotropic flux of the radiation close to tight coupling.

Consider the almost-FL IBE (5.5) and (5.75) for isotropic (in the baryon frame) Thompson scattering (5.76) [120]; this is inverted to find :

$$\tau(x, e) = v_B^\alpha e_\alpha - t_c [\mathcal{B} + \dot{\tau} + e^a D_a \tau] . \quad (5.122)$$

We now systematically approximate (5.122) in terms of the smallness parameter t_c . The right-hand side (the scattering term) is used to find the zero-th order collision-time correction to the total bolometric temperature, with corresponding temperature anisotropy given by

$$\tau_{(0)}(x, e) \approx v_B^\alpha e_\alpha . \quad (5.123)$$

The equation is now perturbed about the zero order velocity perturbation and one can then recover the first and second order corrections in t_c to the zero-th order temperature

²⁷Here we are explicitly making a distinction between the treatment [83] (what we call *tight-coupling approximation*) and that in [143] (what we call *near-tight* coupling). By tight-coupling approximation we mean that κ^{-1} is sufficiently small that it can be ignored (inducing a contribution of the order of magnitude (say) of at least 10^{-6} when multiplying quantities of linear order such as the shear). The near-tight coupling includes the radiation quadrupole in the case of isotropic Thompson scattering (in the matter frame).

anisotropies, to find, $\tau_{(1)}$ and $\tau_{(2)}$ respectively, where the n -th. order correction is denoted by $\tau_{(n)}$. We obtain an almost-FL perturbative expansion in t_c :

$$\tau_{(n)} \approx v_B^a e_a - t_c \left[\mathcal{B} + \dot{\tau}_{(n-1)} + e^a D_a \tau_{(n-1)} \right]. \quad (5.124)$$

The tight-coupling limit is recovered when $t_c = 0$. This treatment is then a consistent (in the sense of the truncation conditions described in chapter 2) near-to-tight-coupling treatment in almost-FL universes. (The first three temperature anisotropies for $n = 1, 2, 3$, are given in appendix D.2).

Solid angle integration

Now the temperature anisotropy is integrated over the solid sphere to ensure the condition that there is no contribution to the bolometric average $T_{(b)}(x^i)$,

$$\int_{4\pi} \tau(x, e) d\Omega = 0. \quad (5.125)$$

It should be clear why the second order correction to the temperature anisotropy is needed even though we intend to keep the expansion only to first order in t_c ; the integrations over term the $e^a v_a$ will vanish. Now by integrating $\tau_{(2)}$ (D.23) over the solid angle and using (5.125) and orthogonality of $O^{A\ell}$ the gradient of the radiation flux is found (some additional details are given the appendices, see for example D.2),

$$D_a \tau^a \simeq D_a v_B^a - \alpha_c \left[(D_a v_B^a)' - (D_a \tau^a)' + \frac{1}{4} D^2 (\ln \rho_{-R}) + (D_a \dot{v}_B^a) + (D_a A^a) \right]. \quad (5.126)$$

By taking spatial gradients of the radiation flux (5.130) we find on comparing with (5.126), that in order for there to be no contributions to scalars,

$$(\text{div } v)_B \approx (\text{div } \tau). \quad (5.127)$$

The above equation (5.126) then becomes

$$(\text{div } \tau) \simeq (\text{div } v)_B - t_c \left[\frac{1}{4} D^2 \ln \rho_R + (D_a \dot{v}_B^a) + (\text{div } A) \right]. \quad (5.128)$$

The transport equations

Finally the individual PSTF multipoles are recovered at a given order:

$$\tau_{A\ell} = \Delta_\ell^{-1} \int_{4\pi} O_{A\ell} \tau_{(n)}(x, e) d\Omega. \quad (5.129)$$

Here Δ_ℓ is defined as before in chapter 1. The second order temperature multipoles are now found from (5.129) integrating $\tau_{(2)}(x, e)$ (D.22) after the combination of direction vectors has been replaced by PSTF tensors (one can use D.24-D.26) :

$$\tau^b \approx v_B^b - t_c \left[D^b \ln T + A^b + \dot{v}_B^b \right] + t_c^2 \left[(D^b \ln T + A^b)' + \dot{v}_B^b - \frac{1}{3} D^b D_c \tau^c \right], \quad (5.130)$$

$$\tau^{ab} \approx -t_c \left[\sigma^{ab} + D^{(a} v_B^{b)} \right] + t_c^2 \left[(D^{(a} v_B^{b)})' + D^{(a} (D^b \ln T + A^b)) + D^{(a} \dot{v}_B^{b)} + \dot{\sigma}^{ab} \right] \quad (5.131)$$

$$\tau^{abc} \approx +t_c^2 \left[D^{(ab} v_B^{c)} \right], \quad (5.132)$$

$$\tau^{A\ell} \approx 0 \quad \forall \ell > 3, \quad (5.133)$$

where we have dropped terms of $\mathcal{O}(t_c^3)$. These are the key results of this section. They are the appropriate **transport equations** for the late-tight coupling era, *i.e.* up to last scattering, and are essentially equivalent to the **causal transport equations** given by causal relativistic thermodynamics [122].

What we have shown here is that if we are interested in the behaviour of the photon-baryon systems to first order in the scattering time, a dissipative fluid approximation is sufficient to describe the radiation (cf. the papers by Israel and Stewart [89]), and will not result in an explicit truncation of the Boltzmann multipole hierarchy, rather it gives a systematic approximation scheme where we can, to the appropriate accuracy, ignore the third order and higher terms. This is significant; one cannot merely drop the higher order moments and truncate to a fluid description (see section 2.3.1), as the kinetic theory treatment fixes the transport equations. Here we have consistently decoupled the $\ell < 3$ multipole equations from the rest of the hierarchy and the consistency of this decoupling is maintained through (5.126) and (5.130 - 5.133).

The solutions to these equations, which lead to acoustic oscillations during this period (5.163), will then affect the CBR anisotropies by setting initial conditions for the free-streaming solution discussed in the previous section (5.120). We give a derivation of these results in the following section.

5.6.2 Late tight-coupling and the oscillator equation

Here we derive the CGI equivalent of the analytic tight-coupling approximation used by Hu & Sugiyama [82, 83]. This approach uses the tight-coupling limit in order to get rid of the radiation quadrupole during late tight-coupling²⁸ and covariantly reproduces Hu and Sugiyama's conclusions about the CBR anisotropy due to inhomogeneities, acoustic and Doppler sources (what they call "primary sources"). This gives the "Sachs-Wolfe effect" due to the Newtonian potential near last scattering, but not the "integrated Sachs-Wolfe effect" due to changing potentials after tight coupling (resulting from more complex matter models and/or spatial curvature).

Near tight-coupling equations

We start with the **near-tight coupling equations** (5.130, 5.131) and (5.133) rewritten to first order and in terms of the optical depth so as to be closer to the notation of the better known treatments [83]:

$$\tau_a \simeq v_a^B - \kappa^{-1} \left[D_a(\ln T) + A_a + \dot{v}_a^B \right], \quad (5.134)$$

$$\tau_{ab} \simeq -\kappa^{-1} \left[\sigma_{ab} + D_{\langle a} v_{b \rangle}^B \right], \quad (5.135)$$

$$\tau^{A\ell} \simeq 0 \quad \forall \ell > 2. \quad (5.136)$$

Here we have assumed that the collisions are dominated by Thompson scattering and is therefore isotropic in the matter frame.

²⁸If there are any anisotropic contributions such as anisotropic scattering (in the matter frame) or large shear (from gravitational waves) at that time this sort of approximation should be considered with care - such phenomena would break tight-coupling.

The relative velocity of the matter w.r.t. the preferred reference frame is $v_B^a \approx u_B^a - u^a$. Rewriting (5.135) in terms of the shear of the baryon frame, we have $\tau_{ab} \approx -\dot{\kappa}^{-1} D_{(a} v_{b)}^B$, so the quadrupole is given entirely by the shear of the matter. It is for this reason that one expects the quadrupole to include a gravitational wave contribution.

Notice that $\pi_{ab} = \rho_R \tau_{ab} \approx 0$, as the case of matter domination. This condition is not sufficient to ensure that τ_{ab} can be ignored in equations when it appears on its own, even though the quadrupole is small. The key point, which was discussed in section 2.5, is that there are four principal linearizations: the almost-FL one: at least $O(\epsilon^2)$, the almost-FL radiation isotropy one $O(\epsilon\eta)$, $O(\eta^2)$, (implying the previous by the almost-EGS theorem), the non-relativistic assumption $O(v\eta)$, $O(v^2)$, $O(v\epsilon)$, and the linearization scheme based on the differential optical depth. Hence care must be taken when approximations are made to the equations.

Tight coupling: Momentum equations

The tight-coupling calculation is now carried out, assuming (5.134-5.136) hold. Consider once again the radiation energy & momentum conservation equations ($\ell = 0$ and $\ell = 1$ MDE):

$$(\ln T)' + \frac{1}{3}\Theta \simeq -\frac{1}{3}\text{div } \tau, \quad (5.137)$$

$$-\dot{\tau}_a \simeq A_a + D_a(\ln T) - \dot{\kappa}(v_a^B - \tau_a) + \frac{2}{5}D^c \tau_{ac}, \quad (5.138)$$

and the baryon energy & momentum conservation equations ²⁹

$$(\ln \rho)' + \Theta \simeq -D^a v_a^B, \quad (5.139)$$

$$-\dot{v}_a^B \simeq +Hv_a^B + A_a + R^{-1}\dot{\kappa}(v_a^B - \tau_a). \quad (5.140)$$

Here $\dot{\kappa}$ is the optical depth, and the radiation-baryon ratio in the real universe is given by (using the enthalpy $h = \rho + p$) by

$$R(x^i) = \frac{h_B(x)}{h_R(x)} = \frac{\rho_B + p_B}{\rho_R + p_R} \approx \frac{3}{4} \frac{\rho_B(x^i)}{\rho_R(x^i)} \Rightarrow \dot{R} \simeq HR. \quad (5.141)$$

This is related to the speed of sound ³⁰ in the background via $c_s^2 = (1/3(R+1))$. The matter momentum equations, (5.140), give

$$v_a^B \simeq \tau_a - \frac{R}{\dot{\kappa}} [\dot{v}_a^B + Hv_a^B + A_a]. \quad (5.142)$$

Substituting (5.134) into (5.142) and retaining all terms up to linear order (in the relaxation time) we obtain

$$v_a^B \simeq \tau_a - R\dot{\kappa}^{-1} [\dot{\tau}_a + A_a + H\tau_a] + \mathcal{O}(\dot{\kappa}^{-2}). \quad (5.143)$$

²⁹This is found to $O[1]$ by substituting the matter energy conservation equation (dust part of 2.13) into matter momentum equation (dust part of 2.14) all to $O[1]$.

³⁰By speed of sound we mean adiabatic sound speed: $c_s^2 = \frac{\dot{p}}{\dot{\rho}_M + \dot{\rho}_R} = \frac{1}{3} \frac{\dot{\rho}_R}{(\dot{\rho}_M + \dot{\rho}_R)} = \frac{1}{3} \frac{1}{(\dot{\rho}_M/\dot{\rho}_R) + 1} = \frac{1}{3(3\rho_M/4\rho_R) + 1}$, for matter domination $c_s \approx 0$ unless there are pressure gradients.

This is then substituted into (5.138) in order to remove the velocity terms, and with a little algebra, we find

$$-\dot{\tau}_a \simeq \dot{u}_a + \frac{1}{1+R} D_a \ln T + \frac{\dot{R}}{(1+R)} \tau_a - \frac{2}{5} \frac{\kappa^{-1}}{(1+R)} D^b D_{(a} u_{b)}^M, \quad (5.144)$$

where the last term has been written in terms of the matter shear. We can now consider the situation where κ^{-1} becomes sufficiently small (but non-zero) so that the last term can be ignored. This is possible as the matter shear is already at least first order. We then find (cf [83] eqn (B2 b)):

$$\dot{\tau}_a + \frac{\dot{R}}{(1+R)} \tau_a + \frac{1}{(1+R)} D_a (\ln T) \simeq -A_a. \quad (5.145)$$

This is the momentum flux equation for the radiation and is a key result. It can be rewritten as

$$[(1+R)\tau_a]' + D_a (\ln T) \simeq -(1+R)A_a, \quad (5.146)$$

or on taking its divergence as

$$[a(1+R)(D^a \tau_a)]' + a(D^2 \ln T) \simeq -(1+R)a(D^a A_a). \quad (5.147)$$

Spatial gradients and the oscillator equation on small scales

The basis of this derivation is the “small-scale” assumption which effectively means that on small enough scales we can ignore the expansion (see section 2.5). This is just the statement that the scale of inhomogeneity is much less than the Hubble scale (5.74).

What we have in mind here is the situation below or near to the Jeans scale near the relevant times so we drop all terms of $O(\epsilon\epsilon_H)$ (see section 2.5.4).

Our aim is to recover the standard oscillator equation (the source equation for the acoustic oscillations) using the 1+3 CGI formalism with $u^a = \delta_0^a$. Note however that we still have the freedom to set the relative velocity in the boost equations (which we will do in the next section).

Taking the spatial gradient³¹ of the radiation energy conservation equation, (5.137), we find

$$-\frac{1}{3} D_a (D_c \tau^c) \simeq (D_a \ln T)' + \frac{1}{3} D_a \Theta + H(D_a \ln T + A_a), \quad (5.148)$$

and taking the divergence of the resulting equation above gives

$$-\frac{1}{3} (D^2 (D_c \tau^c)) \simeq (D^2 \ln T)' + 2H(D^2 \ln T) + \frac{1}{3} D^2 \Theta + H(\text{div } A), \quad (5.149)$$

and this can be written as

$$-\frac{1}{3} (a^2 D^2 (D_c \tau^c)) \simeq (a^2 D^2 \ln T)' + \frac{1}{3} (a^2 D^2 \Theta) + H(a^2 \text{div } A). \quad (5.150)$$

³¹Using the identity $(D_a f)' \simeq D_a \dot{f} - H D_a f + A_a \dot{f}$ we find that $(D_a \ln T)' \simeq D_a \ln T' - H(D_a \ln T + A_a)$ from the almost-FL $\ell = 0$ MDE and $H = \frac{\dot{a}}{a}$ [120].

This is analogous to equation (B3) in [83]. Then using (5.147) we find

$$[(1+R)aD_c(D_a\tau^a)] + H(1+R)aD_c(D^a\tau_a) + aD_c(D^2\ln T) \simeq -(1+R)aD_c(\text{div } A). \quad (5.151)$$

Substituting (5.148) into (5.151) and using the fact that $HD^a A_a \approx \mathcal{O}(\epsilon_H\epsilon)$, we obtain

$$\begin{aligned} & -3[a(1+R)(aD_c\ln T)] + a^2D_c(D^2\ln T) \\ & \simeq [a(1+R)(aD_c\Theta)] - a(1+R)D_c(\text{div } A). \end{aligned} \quad (5.152)$$

Finally, transforming to conformal time, $dt = a d\eta$ gives

$$\begin{aligned} & 3[(1+R)(aD_c\ln T)'] - a^2D_c(D^a(aD_a\ln T)) \\ & \simeq -[a(1+R)(aD_c\Theta)]' + a^2(1+R)D_c(\text{div } A). \end{aligned} \quad (5.153)$$

On using the small-scale linearization scheme described in section (5.74) we find, on dividing through by $3(1+R)$, the **oscillator equation**:

$$(aD_c\ln T)'' \approx +\frac{a^3}{3(1+R)}D_c(D^2\ln T) - a^2(D_c\Theta)' + \frac{a^2}{3}D_c(D^a A_a). \quad (5.154)$$

This is the covariant harmonic-oscillator equation which describes the acoustic modes³². We can compare it to the usual gauge invariant result found in the Newtonian gauge by transforming to the shear-free frame $\tilde{u}_a = n^a$ where $D_{\langle a}n_{b\rangle} = 0$.

We will investigate this equation in more detail in the next section and relate our results to those in the standard literature (which are expressed in the Newtonian gauge).

The Newtonian frame oscillator equation

In this section we recover the harmonic oscillator equation of Hu and Sugiyama in the Newtonian gauge [82] from the CGI formalism. The difference between this and the previous section is that here we apply the small scale approximation at the very end of the calculation.

Using the Newtonian frame choice $n^a \approx u^a + v_N^a$, $D_{\langle a}n_{b\rangle} = 0$, equations (5.145) and (5.18) become

$$\text{dipole:} \quad \dot{\tilde{\tau}}_a + \frac{\dot{R}}{1+R}\tilde{\tau}_a + \frac{1}{1+R}\tilde{D}_a(\ln T) \approx -\tilde{A}_a \approx -D_a\Phi_A, \quad (5.155)$$

$$\text{monopole:} \quad (\tilde{D}_a\ln T) + H(\tilde{D}_a\ln T + \tilde{A}_a) + \frac{1}{3}\tilde{D}_a\tilde{\Theta} \approx -\frac{1}{3}D_a(D_c\tilde{\tau}^c). \quad (5.156)$$

These can be re-written and put into the following form by using the almost-FL EFE, together with the transformation relations given in the appendix D:

$$(D_a\tilde{\tau}^a) + H(D_a\tilde{\tau}^a) \approx -\frac{\dot{R}}{1+R}(D_a\tilde{\tau}^a) - \frac{1}{1+R}(\tilde{D}^2\ln T) - (D^2\Phi_A), \quad (5.157)$$

$$D_{\langle a}\tilde{D}_{b\rangle}(\ln T) \approx -D_{\langle a}D_{b\rangle}\dot{\Phi}_H - \frac{1}{3}D_{\langle a}D_{b\rangle}(D_c\tilde{\tau}^c). \quad (5.158)$$

³²This can also be obtainable from a two-fluid CGI description [37], as well as from the imperfect-fluid description [122] – the point here is that we have derived it from a self consistent kinetic theory approach, listing along the way the necessary physical approximations required to reduce it to the standard acoustic oscillator form.

Taking the time derivative of (5.158) and substitution into equation (5.157) after first taking PSTF derivatives we obtain the full equation in $D_{\langle a}D_{b\rangle} \ln T$:

$$\begin{aligned} (D_{\langle a}D_{b\rangle} \ln T)^{\cdot} &+ \dot{H}(D_{\langle a}D_{b\rangle} \Phi_A + 2D_{\langle a}D_{b\rangle} \ln T) + H \left[(D_{\langle a}D_{b\rangle} \Phi_A)^{\cdot} + 2(D_{\langle a}D_{b\rangle} \ln T)^{\cdot} \right] \\ &\approx -(D_{\langle a}D_{b\rangle} \Phi_H)^{\cdot} - 2\dot{H}D_{\langle a}D_{b\rangle} \Phi_H - 2H(D_{\langle a}D_{b\rangle} \Phi)^{\cdot} + \frac{1}{1+R}D_{\langle a}D_{b\rangle}D^2 \ln T \\ &- D_{\langle a}D_{b\rangle}D^2 \Phi_A + \left[\frac{\dot{R}}{(1+R)} - 3H \right] \left(D_{\langle a}D_{b\rangle}(\ln T)^{\cdot} + D_{\langle a}D_{b\rangle} \dot{\Phi}_H \right) \end{aligned} \quad (5.159)$$

On dropping all terms $O(\epsilon\epsilon_H)$, we once again obtain the 1+3 covariant form of the small scale Newtonian frame oscillator equation (without using a mode expansion):

$$\begin{aligned} (D_{\langle a}D_{b\rangle} \ln T)^{\cdot} &+ \frac{\dot{R}}{(1+R)}D_{\langle a}D_{b\rangle}(\ln T)^{\cdot} + \frac{1}{1+R}D_{\langle a}D_{b\rangle}D^2 \ln T \\ &\approx -(D_{\langle a}D_{b\rangle} \Phi_H)^{\cdot} + \frac{\dot{R}}{(1+R)}D_{\langle a}D_{b\rangle} \dot{\Phi}_H - D_{\langle a}D_{b\rangle}D^2 \Phi_A. \end{aligned} \quad (5.160)$$

The techniques used to derive the above equation become useful later when dealing with the non-linear terms as they avoid the complication of mode-mode couplings when understanding the qualitative features of various effects [120]. These equations are also valid for scalar, vector and tensor modes. However the mode form is necessary for calculation of transfer functions; in this case we need to explicitly deal with mode-mode couplings.

Upon using the mode decomposition definitions for the temperature perturbations, radiation dipole and the scalar Newtonian and curvature perturbations respectively³³, we can write the mode decomposition of (5.157) and (5.158) as:

$$\text{mode dipole:} \quad \dot{\tilde{\tau}}_1 \approx -\frac{\dot{R}}{1+R}\tilde{\tau}_1 - \frac{1}{1+R}\frac{k}{a}\delta\tilde{T} - \frac{k}{a}\Phi_A, \quad (5.161)$$

$$\text{mode monopole:} \quad \delta\dot{\tilde{T}} \approx -H\Phi_A - \dot{\Phi}_H + \frac{k}{a}\frac{1}{3}\tilde{\tau}_1. \quad (5.162)$$

Upon ignoring the expansion coupled term (i.e the small scale approximation) and including the curvature fluctuations, since this makes the resulting equations applicable up to the Jeans length (above which the matter would not be gravitationally bound), and substituting the second equation (5.162) into (5.161) we obtain the well known equation describing the acoustic oscillations in the radiation [82]:

$$\text{acoustic:} \quad \delta\tilde{T}'' + \frac{R'}{1+R}\delta\tilde{T}' + k^2c_S^2\delta\tilde{T} \approx -\Phi_H'' - \frac{R'}{1+R}\Phi_H' - \frac{k^2}{3}\Phi_A. \quad (5.163)$$

Here we have used conformal time (since we are now working in the conformal Newtonian frame).

³³We use, as before, $\tilde{D}_a \ln T \approx \frac{k}{a}\delta\tilde{T}Q_a$, $\tilde{\tau}_a \approx \tilde{\tau}_1 Q_a$, $D_a \Phi_A = \frac{k}{a}\Phi_A Q_a$ and $D_a \Phi_H \approx \frac{k}{a}\Phi_H Q_a$.

The dispersion relations and photon damping scale

In this section we derive the dispersion relations for small scale anisotropies, where the focus is once again on developing generic covariant equations in parallel to the usual gauge invariant treatments [94, 99, 81]. In this regard we begin by iterating the baryon velocity equation in much the same manner as we iterated the IBE for the radiation.

We begin with the the baryon relative velocity equation (5.140) which is once again inverted in order for it to take the form in (5.142). This is then turned in to the basis of an iteration scheme in terms of the scattering time:

$$v_{B(n)}^a \approx \tau_a - \frac{R}{\dot{\kappa}} \left[\dot{v}_{B(n-1)}^a + H v_{B(n-1)}^a + A^a \right]. \quad (5.164)$$

Using this equation and the zero-th order tight-coupling approximation, $v_{B(0)}^a \approx \tau_a$, the covariant second order baryon velocity equation is found:

$$v_a^B \approx \tau_a - \frac{R}{\dot{\kappa}} \left[\dot{v}_a^B + H v_a^B + A_a \right] + \frac{R^2}{\dot{\kappa}^2} \left[\ddot{v}_a^B + (\dot{H} + H^2) v_a^B + \dot{A}_a + H A_a + 2H \dot{v}_a^B \right] \quad (5.165)$$

where we have dropped terms of $\mathcal{O}(\dot{\kappa}^{-3})$. We now consider the small scale version of this equation, by ignoring terms scaled by the Hubble parameter H and effects due to the gravitational potentials (see 5.74):

$$v_a^B \approx \tau_a - \frac{R}{\dot{\kappa}} [\dot{\tau}_a + A_a] + \frac{R^2}{\dot{\kappa}^2} \ddot{\tau}_a. \quad (5.166)$$

Expanding the transport equation for the second order radiation quadrupole (5.131) to first order and ignoring the shear contribution which is negligible in the almost-FL small scale limit, we obtain

$$\tau^{ab} \approx -t_c D^{(a} v_B^{b)} \approx -\dot{\kappa}^{-1} D^{(a} \tau^{b)}, \quad (5.167)$$

where again we retain only first order terms. We now substitute (5.166) and (5.167) into the radiation dipole evolution equation (5.138) to find:

$$-\dot{\tau}_a = (A_a + D_a \ln T) - R(\dot{\tau}_a + A_a) + R^2 \dot{\kappa}^{-1} \ddot{\tau}_a - \frac{2}{5} \dot{\kappa}^{-1} D^c D_{(a} \tau_{c)}, \quad (5.168)$$

which can be compared to the mode equation (A-11) in Hu & Sugiyama [81]. This is the key equation from which we will now proceed to recover the dispersion relations and hence standard damping scale results. The key-point here is that the diffusion damping is second order in the scattering time, while the acoustic oscillations are first order.

We now take a covariant mode expansion of the necessary quantities: $\tau_a = \tau_1 Q_a$, $A_a = A_k Q_a$ and $D_a \ln T = (\frac{k}{a} \delta T) Q_a$, and use the well known result (see chapter 1 for the general case): $D^b Q_{ab} = -\frac{2}{3} (ak)^{-1} (-k^2 + 3K) Q_a$. We consider only the flat case here, so we set $K = 0$. Putting these all into the second order radiation dipole equation (5.168) we obtain:

$$-\dot{\tau}_1 \approx (A + (\frac{k}{a} \delta T)) + R(\dot{\tau}_1 + A) + R^2 \dot{\kappa}^{-1} \ddot{\tau}_1 + \frac{4}{15} \dot{\kappa}^{-1} \frac{k^2}{a^2} \tau_1. \quad (5.169)$$

Now we use the WKB approximation:

$$\tau_1 \propto \exp i \int (\omega/a) dt, \quad (5.170)$$

and drop terms scaled by H and use $a \approx R$. This gives

$$-i(1+R)\frac{\omega}{a}\tau_1 \approx \left[\left(\frac{k}{a}\delta T\right) + (1+R)A\right] - R^2\dot{\kappa}^{-1}\frac{\omega^2}{a^2}\tau_1 + \frac{4}{15}\dot{\kappa}^{-1}\frac{k^2}{a^2}\tau_1. \quad (5.171)$$

Now in order to deal with the terms arising from A_a and $D_a \ln T$ we consider the covariant radiation monopole perturbation equation (5.18):

$$(D_a \ln T) + \frac{1}{3}D_a \Theta + H(D_a \ln T + A_a) \simeq -\frac{1}{3}D_a(D_c \tau^c). \quad (5.172)$$

We find, on taking a mode expansion and applying the WKB approximation again, dropping terms scaled by H and the expansion gradient (the small scale approximation described in section 2.5), that

$$\left(\frac{k}{a}\delta T\right) \approx \frac{1}{3}\frac{k^2}{a}(-i\omega)\tau_1. \quad (5.173)$$

Upon substituting (5.173) into (5.171), factoring out the dipole coefficient τ_1 , writing the differential optical in terms of conformal time: $\dot{\kappa}^{-1} = (a\kappa')^{-1}$ and multiplying through by $i\omega a$ we get:

$$\omega^2(1+R) + R^2\kappa'^{-1}\omega^2(i\omega) - \frac{4}{15}\kappa'^{-1}k^2(i\omega) \approx \frac{1}{3}k^2. \quad (5.174)$$

On re-arranging terms we finally obtain

$$\omega^2 \approx \frac{k^2}{3(1+R)} + k^2(i\omega\kappa'^{-1}) \left(\frac{R^2}{3(1+R)} + \frac{4}{15} \right). \quad (5.175)$$

Splitting ω up into its natural frequency ω_0 and the diffusion damping term γ and then solving the quadratic for the frequency ω we obtain

$$\omega \approx \pm\omega_0 + i\gamma, \quad (5.176)$$

where

$$\omega_0 \approx \frac{k}{\sqrt{3(1+R)}} \approx c_s k, \quad \text{and} \quad \gamma \approx k^2 \left(\frac{\kappa^{-1'}}{6} \right) \left[\frac{R^2 + \frac{4}{3}(1+R)}{(1+R)^2} \right] \approx k^2/k_D^2. \quad (5.177)$$

This then defines the **diffusion damping scale** k_D , where c_s is the baryotropic speed of sound in the matter. We get oscillations when $k < k_D$ (see the next section), and diffusion damping when $k > k_D$, leading to a damping envelope.

The above covariant result is equivalent to the analytic small scale approximation scheme of [82, 83, 81], who have demonstrated its robustness and closeness to more precise numerical studies.

This derivation can be easily corrected to include the effect of anisotropic scattering (which breaks the tight-coupling approximation) and polarization. This is done by correcting the scattering terms in the calculation of the damping scale. We follow the approach of Kaiser [94, 82, 83, 81] and include the correction f_2 to find the modified damping factor:

$$\text{damping factor: } \gamma^* \approx k^2 \left(\frac{\kappa^{-1'}}{6} \right) \left[\frac{R^2 + \frac{4}{3}f_2^{-1}(1+R)}{(1+R)^2} \right] \approx k^2/k_D^{*2}, \quad (5.178)$$

where, first, for the anisotropic effect, $f_2 = \frac{9}{10}$, and second, to compensate for the polarization, $f_2 = \frac{3}{4}$. Equation (5.178) defines the modified damping scale k_D^* . Hence we get the diffusion damping envelope for τ_1 when $k > k_D$.

The temperature oscillations

To examine the solution when $k < k_D$, we take a mode expansion, using the Q_{A_i} 's defined in chapter 1:

$$a(D_c \ln T)^k = k(\delta T)Q_c \Rightarrow aD_c(D^a D_a \ln T) = \left(-\frac{k^3}{a}\right)(\delta T)Q_c, \quad (5.179)$$

where the driving term is written in terms of a generic potential Φ_F defined by

$$-\frac{a^3}{3}D_c(D^2\Phi_F) \approx -a^2(D_c\Theta)' + \frac{a^2}{3}D_c(D^a A_a) \quad \text{and} \quad a^3D_c(D^2\Phi_F) \approx +\frac{k^3}{3}\Phi_F Q_c. \quad (5.180)$$

From (5.154) we find (once again working in the Newtonian frame *cf* [83] eqn (B3) and our equation (5.163)), the oscillator equation.

The solution at first order must be convolved with the damping envelope, found from the dispersion relations, in order to include the damping cut-off. We have waited until the form of this damping is derived, so that it can be easily included by simply replacing the natural oscillator frequency, ω_0 , with $\omega = \omega_0 + i\gamma$, which now includes the damping. We use equation (5.154) in a manifestly gauge invariant form, to find upon mode expanding

$$(\delta T)'' + k^2 c_s^2 (\delta T) \approx -\frac{k^2}{3}\Phi_F, \quad (5.181)$$

where the *sound speed* c_s is given by $[3(1+R)]^{-\frac{1}{2}}$. This gives the well known solution (see for example [82, 85])

$$(\delta T)(\eta) \approx [(\delta T)(0) + (1+R)\Phi_F] \cos(kr_s) - (1+R)\Phi_F, \quad (5.182)$$

where the *sound horizon scale* is given by $r_s = \int c_s d\eta \approx c_s \eta$. This describes the source term for the acoustic oscillations with the isocurvature term dropped³⁴.

By putting this back into the momentum flux equation (5.145) in mode form we obtain

$$\tau_1'(\eta, k) \simeq -\frac{1}{1+R}(k\delta T) - ak\Phi_F, \quad (5.183)$$

and using the temperature anisotropy mode expansion that results when (5.135) holds, the Doppler contribution to the temperature anisotropies at decoupling can be found. These are given by

$$\tau_1(\eta, k) \approx 3[\delta T(0) + (1+R)\Phi_F]c_s \sin(kr_s). \quad (5.184)$$

If we restrict ourselves to the Newtonian frame from the onset (which we have not) then this can also be found from the direct substitution of (5.182) (with $\Phi_F \approx \Phi_A$) into the monopole evolution equation (5.162); *i.e.* solve (5.163) to find $\delta T(\eta, k)$ and substitute it into (5.162) as in [83].

³⁴The other part of this solution comes from $\sin(kr_s)$; this is the isocurvature part ($\delta T'(0) \neq 0$), giving the full solution:

$$(\delta T)(\eta) = [(kc_s)^{-1}(\delta T)'(0)] \sin(kr_s) + [(\delta T)(0) + (1+R)\Phi_F] \cos(kr_s) - (1+R)\Phi_F.$$

In this work we consider adiabatic perturbations only so $(\delta T)'(0) = 0$.

5.7 Temperature anisotropies from integral solutions

In this section we derive the explicit form of the temperature anisotropies, using the general u^a -frame integral solutions and the tight-coupling approximation solutions.

Before we do this, there are two important points that need to be discussed. Firstly, where does the high- ℓ cut-off come from? The natural frequency of the system is ω_0 and is set by the sound speed in the tight-coupled system to first order in Thompson scattering time – the oscillator equation. The diffusion damping is a second order effect whose correction is found by deriving the dispersion relations at second order in the Thompson expansion for the baryon and photon momentum equations and using the WKB approximation, finding the new oscillator frequency to be : $\omega \approx \omega_0 + i\gamma$. The assumption that the anisotropies are sourced by these oscillations in the radiation is the key to the physics; the initial conditions before free-streaming are $\tau_{A\ell} \approx 0$ for $\ell > 2$ (where the anisotropic correction at $\ell = 2$ is included as a correction to the damping scale). The free-streaming radiation transfer function (the spherical Bessel function in k -space) is then convolved with the initial power sourced by the oscillations in the average temperature and the dipole (essentially a cosine and sine function in k -space respectively). As free-streaming continues power is shifted up into the higher ℓ 's, as the radiation transfer functions' maximum is near $\ell \propto k\Delta\eta_*$ (where $\Delta\eta_*$ is the elapsed time since last scattering). This maximum moves into higher ℓ for longer times to give one a sense for how the power is distributed through the $\ell \pm 1$ moment couplings in the almost-FLRW IBE. Obviously at high- ℓ (which corresponds to high- k) the amount of power surviving the $(k/k_D)^2$ damping will be very small, and hence as time progresses the peak in the transfer functions drops off for higher- k , thus giving the high- ℓ cut-off.

Given that in free-streaming there is no diffusion damping to cut-off the high- k power, the truncation of the hierarchy is only consistent and meaningful if the significant anisotropy signal is sourced from the initial conditions at low- ℓ near to tight-coupling.

Secondly, why do we find angular variations in the temperature anisotropies that are entirely due to oscillations in the dipole and monopole of the radiation – particularly given that in relativistic kinetic theory one has timelike integrations ³⁵?

Let us first consider the issue of timelike integrations. In the almost-FLRW case, can one think of the little box being integrated down a single timelike world line as being equivalent to the average of all the little boxes being integrated down the null-cone from its intersection with the surface of last scattering without some ensemble averaging procedure, based on a more restricted form of the weak Copernican principle? If so one could then treat a little box being integrated down a null-geodesic as equivalent to one being integrated down a timelike world line. It then follows, since relativistic kinetic theory is formulated with respect to timelike worldlines, that one can think of the initial data as being projected from the intersection of the past null-cone and the surface of last scattering onto the unit sphere here and now.

This is important for non-linear extensions of the almost-FLRW model [120], since care should be taken when treating ensemble averaged quantities integrated down the null-cone as being generally equivalent to those integrated down a single timelike world-

³⁵Only in the almost-FLRW subcase can one think of the averaged relationship between the null-projection

line. The issue will become even more complex once the Gaussian averaging necessary for the construction of the angular correlation functions is relaxed. It is this, along with the weak Copernican principle that makes possible the extension of the data here and now, as integrated in a little box down a timelike worldline to last scattering, to global statements.

Now we return to the issue of sourcing the temperature anisotropies from oscillations in the dipole and monopole. The integral solutions give the full almost-FLRW solution to the radiation anisotropies given the appropriate initial data. Generically it represents the transfer of power from low- ℓ initial data up into high- ℓ , with the accompanying reverse transfer of power; the $\ell \pm 2$, $\ell \pm 1$ transfer of power where there is a wall at $\ell = 0$ but none at high- ℓ . There is strictly no ℓ_{max} unless the geometry is exactly FRW [62]. The tight-coupling approximation gives the monopole and dipole at the $\ell = 0$ initial wall. Diffusion damping in the initial power near last scattering as well as additional damping through slow decoupling cuts off the transfer of power. This power is sourced by those initial conditions and therefore gives the temperature anisotropies in terms of the dipole and monopole alone. Additional integrated effects will only modify the primary projection, effectively leaving the cut-off unchanged.

5.7.1 Temperature anisotropies

The integral solution gives the projection of these conditions at last scattering onto the current sky. These results are covariant.

The diffusion damping can be found from the dispersion relations arising from the coupled baryon-photon equations using the WKB method applicable to tight-coupling, which was described in section 5.6.2. We can now construct the primary source temperature anisotropies, by first considering the acoustic and Doppler contributions in the free-streaming projection (5.102). Our solutions for B_0 and B_1 are first order in scattering time expansion hence to include the second order diffusion cut off, we multiply the solution through by $\exp \gamma$ (or $\exp \gamma^*$ if we want to include the corrections for the anisotropic and polarization effects).

We can re-write γ or γ^* in terms of the damping scale, k_D (cf [83] eqn (A7)-(A8)).

Sources of temperature anisotropies

In the manifestly gauge invariant integral solution of the temperature anisotropies in almost-FLRW spacetime, using the slow decoupling approximation with scalar perturbations (5.107,5.111), there is additional complexity of having to deal with shear terms. We therefore choose the Newtonian frame in which the shear contribution does not appear³⁶. We are then able to recover the canonical scalar treatment [193, 161, 70, 82, 83]. This can be written out in terms of the scalar perturbation potentials Φ_A and Φ_H , the temperature perturbation, δT , the radiation dipole τ_1 (through slow decoupling) and the baryon relative velocity, v_B , all within the Ehler-Ellis-Bruni approach (including Doppler con-

³⁶Incidentally this would also remove any problems we may have with the introduction of high- ℓ truncation as discussed in chapter 2

tributions (5.120)):

$$\frac{\beta_\ell \tau_\ell(\eta_0)}{(2\ell + 1)} \approx \underbrace{\mathbf{S}_P(k, \eta_*) j_\ell(k\Delta\eta_*)}_{\text{primary sources}} + \underbrace{\int_{\eta_*}^{\eta_0} d\eta (\mathbf{S}_{DISW} + \mathbf{S}_{ISW}) j_\ell(k\Delta\eta)}_{\text{secondary sources}}. \quad (5.185)$$

Here the source terms are given by:

$$\mathbf{S}_P(\eta, k) = \mathcal{D}(\eta_0, k) \left[(\tilde{\delta}T + \Phi_A) + \kappa' v_B \right], \quad (5.186)$$

$$\mathbf{S}_{DISW}(\eta, k) = \mathcal{V} e^{-(k/k_D)^2} \left[\frac{1}{3} k \tilde{\tau}_1 + (\kappa' \tilde{v}'_B + \kappa'' \tilde{v}_B) \right], \quad (5.187)$$

$$\mathbf{S}_{ISW}(\eta, k) = \mathcal{V} e^{-(k/k_D)^2} \left[[\Phi'_A - \Phi'_H](k, \eta) - aH[\tilde{\delta}T + 3\Phi_A](k, \eta) \right]. \quad (5.188)$$

In the first line, on the RHS, we have the SW and acoustic sourced projection effects which are the primary sources. The next term describe secondary Doppler effects during slow decoupling. The last term models the integrated Sachs-Wolfe (ISW) effects; the late-ISW and early-ISW respectively. We have used $\Delta\eta_* = \eta_0 - \eta_*$ (the conformal time difference between the surface of last scattering and the time of reception, here and now) and $\Delta\eta = \eta_0 - \eta$.

In this way we have recovered the solution from the exact equations in a systematic manner using the 1+3 CGI formalism, rather than recovering the solution from a perturbation theory about a foliation of FRW surfaces of homogeneity. Note that there is no monopole temperature anisotropy in the CGI approach, while there is one in the canonical treatment.

The angular correlation functions, C_ℓ , for the general small temperature anisotropy case have been derived in terms of a superposition of homogeneous and isotropic Gaussian random fields with respect to the temperature anisotropy multipoles to find the multipole mean-squares, $\langle \tau_{A_\ell} \tau^{A_\ell} \rangle$, in terms of the an ensemble average (see chapter 1). The relationship between the multipole mean-squares (in the almost-FLRW case) and the mode coefficient mean-squares $|\tau_\ell|^2$ in terms of the RW mode functions $G_\ell[Q]$ are also given in chapter 1. We do not discuss these further here. It must be emphasized that the multipole mean-squares are given for generic geometries, while the mode means squares are only for almost-FL. This is why the application of the non-linear extension is possible; where the corrections to the almost-FL standard model are calculated using the multipole formalism of chapter 2.

Sachs-Wolfe effect and the acoustic source

The Newtonian frame result differs from the total frame result. In the Newtonian frame (5.186) we find that the Sachs-Wolfe effect arises from the combination $\bar{D}_a \ln T + \bar{A}_a \approx \frac{k}{a} (\delta\tilde{T} + \Phi_A) Q_a$. From the solution to the oscillator equation in the Newtonian frame (Eq. (5.163) or take Eq. 5.182 and let $\Phi_F \approx \Phi_A$) along with $\tau_s \sim c_s \eta$ and ignoring the time evolution of the potential (for the equivalent canonical version see [82]) we obtain

$$\delta\tilde{T}(\eta, k) + \Phi_A(\eta_*, k) \approx [\delta\tilde{T}(0, k) + (1 + R)\Phi_A(0, k)] \cos(kr_s) - R\Phi_A(\eta_*, k). \quad (5.189)$$

Upon taking the matter dominated limit ($R \approx \mathcal{O}(\epsilon^2)$) and then using the adiabatic assumption in the tight-coupling approximation $\delta T(0, k) \sim \frac{1}{3}\Delta(0, k) \sim -\frac{2}{3}\Phi_A(0, k)$ (adiabatic flat CDM model) and finally taking $r_s^* \sim 0$ (by using that $k\eta_* \ll 0$) we recover the usual results as in [82, 84]:

$$[\delta\tilde{T} + \Phi_A](\eta_*, k) \approx +\frac{1}{3}\Phi_A(0, k)(1 + 3R_*) \cos(kr_s^*) + R_*\Phi_A(0, k) \sim \frac{1}{3}\Phi_A(0, k). \quad (5.190)$$

Here we have used that $\Phi_A(\eta_*, k) \approx \Phi_A(0, k)$, from the EdS result that the potential is constant when we drop the decaying mode (see the appendix). The physics of the Sachs-Wolfe effect as opposed to the acoustic oscillations, is then quite clear. There are no oscillations just an imprint due to an acceleration potential. Although the adiabatic assumption is invariant to order $\mathcal{O}(\epsilon)$, the relationship between the Electric part of the Weyl tensor and the perturbations are not – so we will always use the matter perturbation in the total frame, where it can be easily related to the Newtonian potential, and the temperature perturbation in the Newtonian frame, where the oscillator equation takes on a useful form. In a similar manner we find the explicit form of the radiation dipole and acoustic oscillations in the slow decoupling era:

$$\begin{aligned} \text{acoustic source: } \delta\tilde{T}(k, \eta) &\approx \frac{1}{3}\Phi_A(0, k)(1 + 3R) \cos(kc_s\eta) + (1 + R)\Phi_A(0, k), \\ \text{Doppler source: } \tilde{\tau}_1(k, \eta) &\approx \Phi_A(0, k)(1 + 3R)c_s \sin(kc_s\eta). \end{aligned} \quad (5.191)$$

Weak-coupling for the small scale solution

Here we briefly consider the ISW effect in connection with the weak-coupling approximation. The idea is that the anisotropies fall with ℓ more rapidly than a simple projection would imply. What one has in mind is the situation where the anisotropy contributes across many wavelengths of the fluctuation allowing cancellations on small scales; the secondary sources, in particular the term $(\Phi'_A - \Phi'_H)$, varies slowly on small scales [84]. Specifically we consider the situation where

$$\int_{\eta_*}^{\eta_0} d\eta \mathcal{V} e^{-(k/k_D)^2} \{(\Phi'_A - \Phi'_H)\} j_\ell(k\Delta\eta) \approx \sqrt{\frac{\pi}{2\ell}} \frac{1}{k} (\Phi'_A - \Phi'_H)(\Delta\eta_*) \mathcal{D}(k, \eta_0), \quad (5.192)$$

where we used $\int_0^\infty j_\ell(k\Delta\eta) d\eta = [\sqrt{\pi}/2k][\Gamma((\ell+1)/2)/\Gamma((\ell+2)/2)]$. The weak-coupling solution is implied by the assumption that:

$$(\ddot{\Phi}_H - \ddot{\Phi}_A) \ll k(\dot{\Phi}_H - \dot{\Phi}_A). \quad (5.193)$$

This is nothing more than a useful approximation allowing the direct construction of analytic solutions. Some care should be taken when using the weak-coupling approximation and trying to estimate the early-ISW (the free-streaming analogue of the acoustic driving effect) and late-ISW (due to non-vanishing curvature or cosmological constant, which will dominate the expansion rate at latter time) effects. Ideally one should evaluate the slowly-varying function, which has been taken out of the integral, at the ℓ -th peak; $\eta_\ell = \eta_0 - (\ell + \frac{1}{2})/k$, rather than at $\Delta\eta_* = \eta_0 - \eta_*$.

In the case of the early-ISW effect, since it satisfies neither the tight-coupling nor weak-coupling criteria, our approximations schemes here are not entirely appropriate; the

decay time and wavelength are comparable [82, 83, 84]. However we can use the weak-coupling approximation in the case of the late-ISW effect because cancellation effects lead to damping on small scales. The temperature perturbation decays, and hence the potential decays on the order of the expansion time near the end of the matter dominated era. The photons will free-stream across many wavelengths of the perturbations below the Hubble horizon, leading to cancellation and damping effects.

The important point about the late-ISW effect is that there will be an imprint due to the exit from the matter dominated era into a Λ - or curvature-dominated one. In the context of the equations here, we can consider Λ -dominated effects by investigating the evolution equations for the potentials in a Λ -dominated case.

In this case the effective expansion changes, given here for late-times (well after matter-radiation equality) by:

$$H^2 = a^{-3}\Omega_0 H_0^2 + \frac{1}{3}\Lambda \quad (5.194)$$

for the $\Omega_0 + \Omega_\Lambda = +1$ model. This has been dealt with in depth in the literature (see for example [84, 83]).

The mode coefficients

The angular correlation functions, measured here and now (x_0^i), can then be computed using the the mode coefficients from chapter 1 [66]:

$$C_\ell = \frac{2}{\pi} \frac{\beta_\ell^2}{(2\ell + 1)^2} \int_0^\infty \frac{dk}{k} k^3 |\tau_\ell(k, \eta_0)|^2. \quad (5.195)$$

The final step is to construct the temperature anisotropy solutions in terms of the matter power spectra. We discuss this in the next section. At linear order, there is no important difference in our solutions from those found in the canonical treatment [82, 83], however the advantage is that we can easily relate our formulation of the temperature anisotropies to the mean-squares of the multipoles (τ_{A_ℓ}) and hence to the mean-square of generalized temperature anisotropies, (Π_{A_ℓ}) [120]. This was not attainable in the canonical treatment.

The multipole coefficients

The mean-square of the multipole moments are related to the almost-FL mode coefficients by (from chapter 1)

$$\langle \tau_{A_\ell} \tau^{A_\ell} \rangle \approx \frac{1}{2\pi^2} \beta_\ell \int k^2 dk |\tau_\ell(k, \eta_0)|^2, \quad (5.196)$$

giving the **angular correlation function** (see chapter 1 [66]):

$$C_\ell = \Delta_\ell (2\ell + 1)^{-1} \langle \tau_{A_\ell} \tau^{A_\ell} \rangle. \quad (5.197)$$

This relates the matter fluctuation amplitude at last scattering to the present day via the Newtonian like potential, which in turn may be related to the matter power spectrum

directly. A plethora of numerical studies of Doppler peak features exist in the gauge invariant literature (see for example [87, 80] and [168]).

We will now summarize the standard picture of acoustic peak characteristics before discussing the matter power spectrum.

5.7.2 Some “Acoustic peak” characteristics

The key features of the standard model of CBR primary sourced Doppler peaks are listed below [82, 83, 84, 85, 190, 93]. The standard model of acoustic peak formation has been given in its analytic form above.

1. The j – th **peak positions** (as given in the flat adiabatic case) is given by

$$\ell_j \approx k_j |r_\theta(\eta)| = \left| \frac{r_\theta}{r_s} \right| j\pi, \quad (5.198)$$

where r_θ is the comoving angular diameter and r_s is the sound-horizon near decoupling. Notice that $j_\ell(k\Delta\eta)$ peaks near $\ell \sim k\Delta\eta$, where free-streaming projects this physical scale onto the angular scale $\theta\Delta\eta$ on the current sky. The first peak is dependent on the driving force, which is model independent. It provides a way of fixing the angular size distance, when using the linear-FRW assumptions. The adiabatic CDM model makes the prediction that the position of the primary peak is at $\ell \sim 1/(220\sqrt{\Omega_0})$ [82, 83].

2. The **acoustic relative peak spacings** are given by

$$k_{j+1} - k_j = k_A \approx \frac{\pi}{r_s} \iff \Delta\ell \sim \ell_A = k_A r_\theta. \quad (5.199)$$

The peak spacing is fixed by the natural frequency of the oscillator: $\omega = kc_s$, which is independent of the driving force. The factor c_s is the photon-baryon sound speed. The adiabatic models make the prediction that the ratio of peaks spacing is as 1:2:3:4, while the isocurvature models predict the ratios to be as 1:3:5:7 [82, 83].

3. The peak ratios arise from the angular power spectrum ratio $C_{jk} = C_j/C_k$ found from C_ℓ in terms of $|\tau_\ell|^2$ or $\langle \tau_{A\ell} \tau^{A\ell} \rangle$.
4. The **damping tail** provides yet another angular size distance test of curvature [84] via the damping scale k_D . The peak spacings $\Delta\ell$ and the damping tail location ℓ_D depend only on the background quantities, they are robust to model changes, assuming that secondary effects do not overwhelm the signal. The diffusion scale near decoupling is the angular scale of the wavenumber $k_D \sim \sqrt{\kappa/\Delta\eta} \sim \sqrt{a/\Delta\eta\bar{a}} \sim \Delta\eta^{-1}$. Given that the damping tail of the acoustic oscillations takes on the form $e^{-(k/k_D)^2}$, k_D can be found from (as derived previously):

$$k_D^{-2} \approx \frac{1}{6} \int_{\eta_*}^{\eta_d} d\eta \frac{1}{\kappa'} \frac{R^2 + \frac{4}{5}f_2^{-1}(1+R)}{(1+R)^2}, \quad (5.200)$$

for no anisotropic stress [84, 94]. This can be used to find that the damping tail location is $\ell_D = k_D r_\theta$.

5. The ratio of the damping tail position to the peak scale, ℓ_D/ℓ_A , is a good measure of the number of peaks and gives an indication of the delay in recombination independently of the area distance.

5.8 The matter power spectrum

In this section we relate the temperature anisotropies in the case of the Sachs-Wolfe effect to the matter power spectrum, using the 1+3 CGI formalism. We now deal with basics of the normalization of the CBR angular power spectrum to the matter power spectrum.

Up to this point everything we have derived has been found covariantly from general relativity and from relativistic kinetic theory, however on small scales there is no covariant analytic derivation of the transfer function, although it is well known for large scales, so this is the only gap in our treatment ³⁷.

The aim here is to try to predict the matter distribution today from the CBR spectrum. From the CBR spectrum one finds the matter power spectrum (as fixed by the CBR data today) from which the current distribution of matter is found (in terms of σ_8).

Briefly, in the literature there have been two methods for normalizing the CBR data in relation to COBE: *firstly* the $\sigma(10^0)$ normalization where the r.m.s temperature fluctuation was averaged over a 10^0 FWHM beam, and *secondly* the Q_{rms-PS} ³⁸ normalization which uses the best fitting amplitude for a $n = 1$ Harrison-Zel'dovich (HZ) spectrum quoted for the quadrupole, (Q). The first method had the advantage of being a model independent way of fitting the data – it is observationally determined, however, because of this it does not provide the most accurate normalization for a specific model and care must be taken in order to properly deal with cosmic variance. The second approach is model dependent and works well only for the HZ spectrum [45, 24].

Improved, more general normalization schemes have now been developed and there currently seem to be two favoured ones, both having variations of the HZ spectrum in mind. It is these two schemes we now briefly consider here. We emphasise the so-called *CDM-models* with $\Omega_0 = 1$. These models have a baryon fraction Ω_B , with the rest of the matter being made up of massive Cold Dark Matter (CDM). Of these the most easily dealt with is the so called *standard-CDM* model, where initial fluctuations are assumed to have a Gaussian distribution and have adiabatic, scalar density fluctuations with a HZ spectrum on large scales:

$$P(k) \propto k^{n-1}, \quad n = 1, \quad H_0 = 50 \text{km}^{-1} \text{Mpc}^{-1}, \quad \Omega_B = 0.05. \quad (5.201)$$

There are three other popular models: the *Λ CDM model* of Peebles, which is a version of *standard-CDM*; the *ICDM* model (which is an isocurvature version of the *Λ CDM model*); and lastly the *Λ CDM model* which is the large cosmological constant version of *standard-CDM*. More recently Hu has introduced a Generalized Dark Matter model (GDM) [88].

³⁷There is still some hope that the almost FL Quasi-Newtonian models, will be useful in this regard [180] providing us a covariant derivation of the Harrison-Zel'dovich spectrum.

³⁸ $Q_{rms-PS} \approx \frac{5}{4\pi} C_2$ and $C_l^{SW} \sim \frac{6}{l(l+1)} C_2$.

5.8.1 The Power Spectrum

The Power Spectrum $\mathcal{P}(k)$ is fully specified by a **shape** and **normalization** (see appendix D.8 for additional details and references). The normalization is fixed by the amplitude of the temperature fluctuations in the CBR³⁹.

There are two normalization scales, large and small (see appendix D.8). On small scales the normalization is expressed in terms of σ_8 , the ratio of the r.m.s mass fluctuations to the galaxy number fluctuations - both averaged over randomly located spheres of radius $8h^{-1}\text{Mpc}$; this is the variance of the density field in these spheres.

On large scales the power spectrum is found by fixing the shape function [44] which is a measure of the horizon scale at matter radiation equality. We look at these now for the case of a flat matter dominated almost-FL model (the open case is considered in appendix D.9).

In the 1+3 covariant and gauge invariant theory the emphasis is on the quantity $aD_a \ln \rho_M \approx a\delta_M Q_a$. Here $\delta_M = a\delta_M(0, k)$ and $\langle \delta_M(k, 0)\delta_M(k', 0) \rangle = (2\pi)^3 \mathcal{P}(k)\delta(k-k')/k^2$ for $|\delta_M(k, 0)|^2 = \mathcal{P}(k)$. It becomes preferable in the context here to use the variable Δ : $\frac{k}{a}\Delta(k, t) = \delta_M(k, t)$ ⁴⁰. We will, however, be using the power spectrum for the gauge invariant acceleration potential because the application in mind is the matter dominated adiabatic one. In the general situation it is best to use $\bar{\Phi}$; in the matter dominated situation we can use that $\Phi_A = -\Phi_H$ and write everything in terms of either Φ_H or Φ_A .

The relationship between the power spectrum of the acceleration potential and the power spectrum of the matter perturbation is

$$|\Phi_A(k, 0)|^2 = P_{\Phi_A}(k), \quad \text{and} \quad |\Delta(k, 0)|^2 = P(k), \quad (5.202)$$

where these are then related to each other in the total frame via the constraints for the electric part of the Weyl tensors (see the appendix) for the flat case :

$$P_{\Phi_A}(k, 0) = \left(\frac{3}{2}H_0^2\Omega_0\frac{D}{a}\right)^2 k^{-4}P(k). \quad (5.203)$$

Here $\bar{\Phi} = -\frac{k^2}{a^2}\Phi_A$ and as noted before (see appendix) :

$$(\bar{\Phi}\rho_M^{-1})(k, t) \approx -(3H_0^2\Omega_0)^{-1}D(t)[k^2\Phi_A(k, 0)]. \quad (5.204)$$

and $D \approx a \approx \eta^2$ for a flat almost-FLRW dust model. We are also going to use that $a_0 = +1$.

5.8.2 Relating the power spectrum to temperature anisotropies

We now combine the covariant and gauge invariant Sachs-Wolfe effect (due to potential fluctuations) and the matter power spectrum to express the anisotropies in terms of the matter power spectrum near decoupling.

³⁹It seems that models normalized to COBE which are fixed in both the amplitude at small scales and by the shape predicted by *standard-CMD*, are inconsistent with observations [97, 24].

⁴⁰ $\Delta(k)^2 \approx \frac{a^2}{k^2}|\delta_M(k)|^2 = \frac{a^2}{k^2}Ak^{n+1}$

Small scales

For small scales one can use the σ_8 normalization via (5.207) and $P(k) = BT^2(k)k^n$ (D.92) and the parametrized transfer function (D.94). This gives

$$C_\ell \simeq \frac{1}{2\pi} H_0^4 \Omega_0^{1.54} \int \frac{dk}{k^2} B k^n T^2(k) j_\ell^2(k\Delta\eta_*) . \quad (5.211)$$

For example using the *standard-CDM model* with $n = 1$, $h = 0.5$, $\Omega_B = 0.05$ and $\Gamma = 0.48$ we can relate B and A to the scale of fluctuations near horizon crossing and the quadrupole:

$$B = 2\pi^2 A = (6\pi^{2/5}) \langle Q \rangle / T_0^2 . \quad (5.212)$$

To normalize to σ_8 we can choose $\sigma_8 \simeq 1.3$. This then allows us, in principle, to invert the angular correlation function to find the matter power spectrum once the initial spectrum is known.

We are however more concerned with normalizing the radiation angular correlation function to CDM on horizon scales near decoupling. In this regard we once again begin with the matter power spectrum (we set $T(k) = 1$) and on using (5.208) for $m = 1$ we find that:

$$C_\ell^{(CDM)} = \left[\frac{1}{2\pi} H_0^4 \Omega_0^{1.54} B \frac{\pi^{1/2}}{8} \right] \frac{1}{\ell(\ell+1)} . \quad (5.213)$$

C_ℓ normalized to Λ CDM

Finally we normalize the angular correlation function C_ℓ (found from 5.111) to the potential fluctuations normalized to Λ CDM (5.213) above:

$$D_\ell = [C_\ell / C_\ell^{(CDM)}] = \left[\frac{1}{2\pi} H_0^4 \Omega_0^{1.54} B \frac{\pi^{1/2}}{8} \right]^{-1} \ell(\ell+1) C_\ell . \quad (5.214)$$

The angular correlation function C_ℓ can be found from the primary (5.186) and secondary (the integrated part of 5.111) source that make up the total: $C_\ell = C_\ell^{(P)} + C_\ell^{(S)}$. This then allows one to remove that part of the angular correlation function arising from the standard Sachs-Wolfe potential fluctuations leaving a signal which is dominated by the photon primary and secondary sourced physics. The convention is to use D_ℓ rather than C_ℓ [93], so we have from (5.207 and 5.213) for the flat case:

$$D_\ell = \ell(\ell+1) C_\ell = \int \frac{dk}{k^2} \mathcal{P}^*(k) |j_\ell(k\chi)|^2 \quad \text{using} \quad \mathcal{P}^*(k) = \left(\frac{k}{k_s} \right)^{n_s + \alpha \ln(k/k_s)} , \quad (5.215)$$

where k_s is the normalization scale, α gives the deviation from the power law, and n_s give the scalar power law index. The angular power per $\ln \ell$ is $(\ell(\ell+1)/4\pi) C_\ell$.

Chapter 6

Scalar Nonlinear corrections

The weakly nonlinear imprint of structure on the CMB: *The prospects of realistic non-perturbative relativistic effects being important in the study of temperature anisotropies in the context of relativistic cosmology is outlined. Maartens-Gebbie-Ellis have found, using qualitative arguments, that nonlinear terms dominate the small-scales temperature anisotropies. These terms are not included in the canonical treatment as they arise from the exact equations. The two new small-scale non-perturbative corrections are (from chapter 2):*

$$\begin{aligned}\delta\dot{\tau}_{NL} &\approx -\ell O_{A_\ell} \left[\frac{1}{4} \sigma_{bc} \tau^{bcA_\ell} + \sigma^{(a_\ell a_{\ell-1} \tau^{A_\ell-2})} - (A^{(a_\ell \tau^{A_\ell-1})} - \frac{1}{2} A_b \tau^{bA_\ell}) \right], \quad (6.1) \\ \delta\dot{C}_{NL} &\approx +\sigma_T n_e O_{A_\ell} \left[\tau^{(A_\ell-1} v_B^{a_\ell)} + \frac{1}{2} \tau^{A_\ell a} v_a^B \right].\end{aligned}$$

Here a weak nonlinear investigation of these effects are carried out with an emphasis on Rees-Sciama like effects – the imprint of structure formation on the CMB. It is shown that gravitational nonlinearity in the weak nonlinear extension of almost-FLRW temperature anisotropies leads to cancellation on small-scales when threading in the Newtonian frame. Generically this cancellation does not occur, it is unique to the shear free threading. In the context of a flat almost-FLRW CDM model we provide a heuristic argument for a non-perturbative small scale correction to the Rees-Sciama effect of not more than $\Delta T/T \sim 10^{-6} - 10^{-5}$ near $\ell \sim 100 - 300$. Weak nonlinearity cannot be expected to be generically ignorable.

6.1 Introduction

In the context of classical cosmology and CMB studies : Do we need to worry about nonlinear effects ? Popular wisdom tells us, No. Is the almost-FLRW geometry really good enough on all observable cosmological scales ? apparently, Yes. However the evidence supporting both these statements, although being plausible and self-consistent, is not only slim, but circumstantial; essentially it is an extension of the COBE-Copernican theorem (See chapter 4).

The emphasis here is not on trying to overthrow the use of almost-FLRW models but on trying to extend the usual treatment of temperature anisotropies away from

analysis of self-consistent linear-FRW cosmologies (and their associated free parameters) towards comparative studies between realistic alternative cosmologies and sources of anisotropies. This would go a long way towards providing more convincing support, using relativistic cosmology, for the high-precision linear-FRW programme that much of the community is pursuing. In fact, we find a weakly nonlinear signature that *could* discriminate between the use of almost-FLRW and more generic nonlinear geometries.

Using a non-perturbative scheme [120] to construct small-scale corrections to temperature anisotropy calculations one could be able to (i) calculate angular correlation functions for a much broader range of relativistic cosmologies with ease, (ii) verify the stability of the almost-FLRW models, (iii) provide a scheme within which to investigate the implications of nonlinear gravitational phenomena and the implications for the CMB.

It should be noted that when doing higher order perturbation theory about a particular background, the very existence of the background and its use have secured two issues: (i) the existence of the necessary averaging procedure, (ii) the convergence of the perturbation theory itself. Without these latter two points, talking about a background, or even closeness to a background, is meaningless. It should then not be unexpected to find the second order corrections to a particular background to be small. Furthermore, one may conjecture that for any convergent perturbative analysis about a high symmetry background the back-reaction will of course be small as will higher order corrections – as the analysis itself has the a-priori assumption of (i) and (ii). At best this means that one can only test the self-consistency of perturbations about the background – discriminate between small (consistent) and large (inconsistent) backreactions (say) as opposed to realistic nonlinear corrections.

However, in many situations where fluctuations in the system are large or nonlinear, where fluctuations are distinguished as spontaneous deviations in a system due to the large number of interacting entities, one cannot find a unique average background. One may end up, by construction, with an average that is self consistent with the systems observables; so giving quantitative meaning to the importance of fluctuations relative to the average, the variance, and the degree of correlation between various spatial regions, the covariance. However, the uniqueness and physical relevance of such a construction can be questionable, multiple means exist in many physical situations, and the situation where the variance is of the same order of magnitude as a given mean is also possible in nonlinear systems. One way of ensuring the validity of the use of linear models, at least in the cosmological setting we have in mind here, is to look for and isolate specific dominant nonlinear effects on specific scales, to add these on as corrections to the linear theory, and to then look for these effects in the observations. This would then give a clearer view of how and why the linear models seem to work so well and anticipate and correct for nonlinear effects that are ubiquitous to general relativity, while still retaining the advantages and physical sophistication possible within the linear theory.

Thus we are attempting to isolate feasible relativistic cosmological sources of anisotropy that are due to the nonlinear nature of general relativity, features that can be embedded within the self-consistent linear-FRW subclass – or alternatively carry out direct calculations of exact anisotropies.

The key issue that are dealt with here is that the 1+3 Lagrangian threading formalism gives generic equations for general temperature anisotropies. It has been

found, using qualitative arguments, that nonlinear terms dominate on small-scales. These terms are not included in the canonical treatment as they arise from the exact equations. In addition the 1+3 Lagrangian approach, in as much as one reduces the exact theory down to recover the perturbative one, is not plagued by the necessity of constructing higher order gauge invariant variables when using the Stewart-Walker lemma.

An outline of the results in this chapter is now given: in the main section we calculate the gravitational corrections induced via the weakly nonlinear corrections using the almost-FLRW formulation of the Rees-Sciama (RS) effect over an Einstein-de Sitter background (EdS). Section 6.5 is divided into four subsections. The first introduces the gravitational corrections in terms of the 1+3 mode functions defined in [66], and explicitly constructs the Fourier coefficients for the situation of aligned wavenumbers – this simplifies the mode-mode coupling considerably. In this section the correction is written in terms of a conformal time integration from last scattering until now. It is constructed in a manifestly covariant and gauge invariant form for scalar perturbations. In section 6.5.1, we calculate the effect in the Newtonian frame (see [120, 67])¹ – the effect is shown to be suppressed. The third subsection, section 6.5.2, gives the calculation in the total frame (see [120, 67]). The effect is shown to be non-trivial in the total frame. The fourth subsection, section 6.5.3, then attempts to approximate the resulting angular correlation function for the total frame effect. This is done by reducing it to a form that is similar to the Rees-Sciama calculation given in section 6.4.1. This then allows me to compare the nonlinear correction to the Rees-Sciama one. A brief outline of the specific effects and the reasoning behind them is now given in the form of a set of back-of-the-envelope arguments.

From an order of magnitude argument using the COBE-Copernican limits [167] in the exact anisotropy equations [120] or Eq. (6.6) one can approximate the weakly nonlinear correction by finding the order of magnitude of the induced temperature anisotropy multipole, τ_{A_ℓ} : $\mathcal{O}(\tau_{A_\ell}) \sim \ell \mathcal{O}((\sigma_{ab}/\Theta)\Pi_{A_\ell})$. Here we have ignored the acceleration by assuming CDM domination, and have used the usual assumption that $|\dot{\tau}_{A_\ell}| < \Theta|\tau_{A_\ell}|$; these are consistent with the almost-EGS approach [117] (see chapter 4). Three results follow.

First, using $\tau_{A_\ell} \sim 10^{-5}$, for almost-FLRW sources (using $(\sigma_{ab}/\Theta) \sim 10^{-4}$) [167] and that the lower bound on Π_{A_ℓ} is that on τ_{A_ℓ} , we find that almost-FLRW sourced nonlinear corrections would only dominate the anisotropies at $\ell \sim \mathcal{O}(|\sigma_{ab}/\Theta|^{-1}) \sim 10^4$, long since diffusion damping and system noise have dominated the signal.

Second, if there are nonlinear sources of shear (due to structure evolution or formation), even at only 10^{-3} (of the order of the local dipole), one can expect an effect near $\ell \sim 1000$. If the nonlinear effects are an order of magnitude larger, then one expects a noticeable effect around $\ell \sim 100$.

An important point here is then that we have entirely ignored feedback of the anisotropy source into the correction terms, there could be an accumulation effect.

Third, we attempt to take the feedback into account by modeling the situation as N repeated sources along the line of sight, an accumulated effect: $\mathcal{O}(\Pi_{A_\ell}) \sim \sqrt{N}\mathcal{O}(\tau_{A_\ell})$. One then finds that $\ell \sim (1/\sqrt{N}) \times 10^4$.

Although this line of argument leads to an unreasonably low value of ℓ , even for

¹The 1+3 covariant and gauge invariant equivalent of the Newtonian Gauge

a moderate number of iterations (or sources along line of sight), one gets a feeling for the situation – the universe needs to be very special for there to be no nonlinear effect. We have excluded the effect of dissipation via mode-mode coupling, which would smooth the effect and could also push ℓ up to higher values, as will any inherent cancellation effect – as found in the Newtonian frame. However, in general, it seems that one should still expect an observable effect. It is for this reason that I investigate the gravitational effects in more depth in this section.

Perhaps the point is best seen as two fold: first, nonlinear sources of shear (or Φ_H) would result in a moderately amplified correction apparently not accounted for in the canonical treatment because of its, by construction, linearity, second, nonlinear feedback (or, if that does not do the trick, accumulated sources) could have an even more noticeable effect.

The general attitude towards the importance of nonlinearity in both observational cosmology and CMB studies can be understood in the context of popular lore surrounding the Rees-Sciama effect (see for example [127, 128]).

The two points that are usually made with regards to the Rees-Sciama effect are: first, that numerical simulations (where nonlinear effects dominate the linear effects only above $\ell > 5000$, so that the effects are a problem near to a sensitivity of 10^{-7}) corroborate the scaling arguments and second, that the scaling arguments indicate that the effects are ignorable: that the effect suffers cancellation as $(k\delta\eta)^{-1}$ for $\delta\eta$ being the time-scale of change in the potential [80]. The idea is that the Poisson equation relates potentials to densities as $(k\eta)^{-4}$ while the volume from the mode coupling introduces terms like k^3 meaning that the effect scales as $k^2 P(k)$ – which is ignorable, particularly when compared with the dominant Vishniac corrections (which cannot be ignored).

The novelty of using the Rees-Sciama effect as the source of nonlinear gravitational effects is that the peak in the nonlinear correction seems to be near to the peak in the radiation transfer function, $\ell \sim k(\eta_0 - \eta_*)$, somewhere near $\ell = 100 - 300$ [154]. The additional ℓ scaling from the nonlinear correction then introduces an additional k^2 scaling in the angular correlation function. It is for this reason that the scalar gravitational correction will be investigated here in substantially more depth, using the Rees-Sciama corrections as the source term. It does not seem unreasonable to suspect that it could dominate the acoustic effect.

The additional linear scaling in ℓ ([120] and Eq. (6.6)) may cause the scaling argument to fail, nonlinear effects could become important well before $\ell > 5000$. It is the latter point that needs to be properly understood, especially given that this scaling does not seem to arise in the perturbative analysis; it appears as a non-perturbative feature of the exact small scale treatment.

In addition, many established treatments carry out the analysis in the linear gauges which set the shear to zero : such as the longitudinal, Newtonian or conformal Newtonian choices [150, 127], which have no nonlinear corrections of the form discussed here – this will be shown here in detail.

A second-order EdS extension of Pine-Carrol [150] was carried out by Mollerach and Matarrese [135] using the Poisson gauge, which gives the appropriate Newtonian extension at higher order; it requires that $D^a \tilde{\sigma}_{ab} \approx 0$, somewhat less restrictive than $\tilde{\sigma}_{ab} \approx 0$, it is however a perturbative analysis and so will not uncover the corrections

discussed here. The main problem with a strict perturbation theory approach is the need to maintain gauge invariance at each order [21, 130]. In addition the strict perturbative approach is a computational *tour de force*, even in the case of investigating “mildly nonlinear effects”.

The points that are being made here are: (1) that when calculating the influence of nonlinearity on temperature anisotropies using a Newtonian gauge choice within the linear theory context one should view the results of such calculations with suspicion. This is because: (i) the gauge is inconsistent at higher order [180], and (ii), suppresses the mode-mode coupling artificially – the cancellation should not be a surprise. (2) When using a higher order perturbation theory, one should not expect to find these effects either; the effects discussed here are not of a perturbative nature.

Although these arguments may seem esoteric, the point at hand is understanding the stability and validity of the use of the almost-FLRW models on all observationally interesting scales to see whether or not specific features in the observations can be explained using non-almost-FLRW corrections and addressing the question of whether non-almost-FLRW corrections need to be considered when interpreting the anisotropies. From the perspective put forward here, this is still an open question, given the nature of the construction of the canonical linear-FRW programme.

The situation may become even more complex if one has to worry about a significant gravitational wave (GW) background (or other large scale effects).

A similar set of arguments can be made with regards to lensing effects. The idea that the only lensing effects that need to be considered are those due to weak lensing, which will smooth the acoustic peaks at a few percent, is widely accepted [80]. Not only should one be more careful about the effects of almost-FLRW caustic and cusp formation leading to the possibility of strong lensing effects particularly if there are non almost-FLRW deviations on small scales, but the weak lensing argument is used in a self-referential manner based on the idea that nonlinearity and deviations from linear-FRW are ignorable.

There are essentially five independent but related issues that we wish to make headway on:

- Is the almost-FLRW geometry ($|E_{ab}| < \epsilon\Theta$, $|H_{ab}| < \epsilon\Theta$), applicable on all observationally meaningful scales?
- Is the almost-FLRW kinematics ($|\nabla_b u_a - H| < \epsilon\Theta$ for ϵ small) applicable on all meaningful scales?
- Can the observed temperature anisotropies be explained by non-almost-FLRW sources? Should one be worried about nonlinear gravitational effects? How can one realistically attack the problem of excluding non-almost-FLRW sources when most of the studies exclude them by construction?
- Is weak nonlinearity important to almost-FLRW based studies of the implications of temperature anisotropies?
- If the universes geometry and kinematics can be well modeled by the almost-FLRW cosmologies – at least in the context of cosmology and CMB studies, what keeps it

that way? The conditions that must be satisfied for it to remain close to almost-FLRW are those given by the set of almost-EGS assumptions – when do these fail, can they fail in a reasonable cosmology, are there any conditions or constraints missing?

These issues are discussed below and some methods of attack are outlined.

The scattering correction is not dealt with here, beyond being posed. One may expect cancellation of the aligned effect. One would then expect the dominant correction to arise as a nonlinear Ostriker-Vishniac-like effect. That is, one would need to consider the transverse effect within the coupling between the anisotropies and the baryon relative velocities – this is not carried out here. The sources are cast into the canonical formalism in the final appendix of the thesis.

In brief, we attempt to show the weak nonlinearity in the temperature anisotropies cancels out in the Newtonian frame : the frame is stable to mild nonlinearity. It is shown that there is a small but non-negligible effect in the generic threading frame which peaks near to the peak in the Rees-Sciama effect – which just happens to be near the acoustic peak.

What needs to be emphasized here is that, although the main result of this thesis is the construction of the exact MDE's (see chapter 2), the aim has been to demonstrate that small scale nonlinear effects due to the imprint of structure evolution and dynamics seem to have been excluded, by construction, in the canonical linear-FRW approach. In this regards, this chapter provides the first tentative attempts at understanding the quantitative implications of these corrections in the language of the canonical treatments – as opposed to the qualitative demonstration of the chapter 2.

6.2 Stability to inhomogeneity

Why should the time derivatives and spatial gradients of the dynamical and kinematic quantities be small? Within this is the additional problem that one can in principle have nonlinear matter evolution (dynamics) without breaking the linearity of the geometry. The real issue is instability against inhomogeneity within a generic relativistic cosmology.

The coupling between inhomogeneity and anisotropies is subtle. For example in the Bianchi VII_0 models one could think of the solution as a frozen or standing gravity wave, a slight inhomogeneous fluctuation will unfreeze it – leading to rapid growth of propagating GW inhomogeneity. Inhomogeneity is generic to our universe; even if it is small, it is still there, and so needs to be taken into account. The almost-EGS assumptions (hence the COBE-Copernican result) seem to be the minimal set that has both sufficient complexity to be an interesting cosmology as well as sufficiently constrained to fit the data sets.

It is unreasonably artificial to exclude the feed-back from inhomogeneity, into the EFE – this is, by far, stricter than the assumptions underlying the COBE-Copernican approach, which allows for both inhomogeneity and anisotropy. Although these must be small, one has the additional freedom of using nonlinear solutions from other exact geometries as ansatz solutions and checking stability as well as self-consistency. Excluding inhomogeneity *a priori* is very unnatural and special – and at best, such models, may provide interesting alternative backgrounds and provide the basis for more realistic

cosmologies, *e.g.*, almost-Bianchi I or almost-FLRW. It is for this reason that I regard the use of Bianchi VII, as discussed in chapter 4, as very artificial – they are simply not realistic models on cosmological scales.

6.3 Weakly nonlinear almost-FLRW

We now define the weakly nonlinear formulation of the equations describing temperature anisotropies.

The exact equations for the total brightness temperature anisotropies were found in chapter 2 (2.73) by Maartens-Gebbie-Ellis [120, 51, 58] along with their reduction to the almost-FLRW case (2.85-2.88) about the FLRW background (see section 2.2). The almost-FLRW model is expressed in terms of the 1+3 covariant and gauge invariant perturbation theory. The temperature anisotropies are defined in terms of the covariant total direction brightness temperature and the bolometric average in the almost-FLRW universe: $T(x, e) = T(x)(1 + \tau(x, e))$ [120, 67]. The almost-FLRW temperature anisotropies for generic- ℓ (in particular $\ell > 2$) (see Eq. 5.185) were the focus of chapter 5 while the implication of the observational constraints on the dipole, quadrupole and octopole were the focus of chapter 4 (see section 4.4 – Eq. 4.26 - 4.28) – where the theoretical foundations of the use of the post-decoupling CDM dominated almost-FLRW models was established, and hence the plausibility of the use of almost-FLRW models on large scales.

The notation used here follows [50, 120] as in the previous chapter. In summary the temperature anisotropy $\tau(x, e)$ has been expanded in terms of a multipole expansion, $\tau = \sum_{\ell} \tau_{A\ell} O^{A\ell}$ with $O^{A\ell} = e^{(A\ell)}$ forming a PSTF basis [66], while the equations are expressed in terms of the 1+3 threading variables derived with respect to a preferred u^a -frame [50, 51]: $\nabla_b u_a = +\frac{1}{3}h_{ab}\Theta + \sigma_{ab} + \epsilon_{abc}\omega^c - A_a \dot{u}_b$, defining the expansion: $D^a u_a = \Theta$, the shear: $D_{(a} u_{b)} = \sigma_{ab}$, the vorticity vector: $\omega_a = -\frac{1}{2}\epsilon_{abc}D^b u^c$ and the acceleration: $A_a = \dot{u}_a = u^b \nabla_b u_a$. The dot is indicative of comoving time with respect to the u^a frame, while a prime, ', will be used to denote conformal time. The spatially projected covariant derivative is denoted: $h^{ab}\nabla_b = D^a$.

Here $u^a u_a = -1$, and the direction vectors e^a are defined such that $e^a e_a = +1$ and $e^a u_a = 0$: the momentum is given by $p^a = E u^a + \lambda e^a$ such that $E^2 = m^2 + \lambda^2$. We refer the reader to references [78, 50, 120, 51] and chapter 2 for further details on the formalism.

The weakly non-linear equation for the brightness temperature anisotropy is:

$$\underbrace{\dot{\tau} + e^a D_a \tau}_{\text{almost-FLRW}} \approx \mathcal{B} + C[\tau] + \underbrace{\left\{ (\delta\dot{\tau})_{OV} + (\delta\dot{\tau})_{NL} + \delta\dot{C}_{NL} \right\}}_{\text{weakly nonlinear terms}}. \quad (6.2)$$

This defines the weakly nonlinear theory. The respective terms used in equation (6.2) are: the linear order coupling between the anisotropies and the field equations, the linear order Thompson scattering correction, the Ostriker-Vishniac (OV) correction, and the dominant non-linear (NL) corrections :

$$\mathcal{B}(x, e) \approx -\frac{1}{3}D^a \tau_a + (D_a \ln T + A_a)e^a + \sigma_{ab}e^a e^b, \quad (6.3)$$

$$C[\tau](x, e) \approx -\sigma_T n_e (\bar{f}_1 e^a v_a^B + \bar{f}_2 e^a e^b \tau_{ab} - \tau), \quad (6.4)$$

$$\delta\dot{\tau}_{OV} \approx -\sigma_T n_e p^{ab} D_a \rho v_b, \quad (6.5)$$

$$\begin{aligned} \delta\dot{\tau}_{NL} \approx & -\ell O_{A_\ell} \left[\frac{1}{4} \sigma_{bc} \tau^{bc A_\ell} + \sigma^{(a_\ell a_{\ell-1} \tau^{A_\ell-2})} \right. \\ & \left. - (A^{(a_\ell \tau^{A_\ell-1})} - \frac{1}{2} A_b \tau^{b A_\ell}) + \omega^b \epsilon_{bc}^{(a_\ell \tau^{A_\ell-1} c)} \right], \end{aligned} \quad (6.6)$$

$$\delta\dot{C}_{NL} \approx +\sigma_T n_e O_{A_\ell} \left[\tau^{(A_\ell-1} v_B^{a_\ell)} + \frac{1}{2} \tau^{A_\ell a} v_a^B \right]. \quad (6.7)$$

The important point here is that, as was shown in chapter 2, the dominant non-linear contribution to the temperature anisotropies at high- ℓ will arise from term (6.6) above; if gravitational non-linearity between decoupling and now significantly alters the small scale angular correlations the dominant contribution will arise from this term. By considering this term in isolation, that is, outside of the context of the exact equations and within the context of corrections to the almost-FLRW theory, we are trying to put bounds on the possible effects that this term may have on the small scale angular correlation functions and hence on the acoustic peak. One notices that for almost-constant coupling one may expect to find an effect that increases linearly in ℓ . Furthermore, these equations are valid for the so-called scalar, vector and tensor perturbation split – at least at linear order where such a split is meaningful in the almost-FLRW theory.

The scattering term includes two weighting factors which are used to deal with the anisotropic scattering correction and the polarization correction, this has been discussed in detail elsewhere [82, 83, 94, 67].

The third and fifth terms are respectively the OV correction and an additional non-linear scattering correction – these are probably the most important outstanding issues from the perspective of the covariant and gauge invariant theory. The OV contribution² represents additional variations in the baryon streaming velocities, induced by gradients in the matter energy flux that generate flows perpendicular to the line of sight e^a . These will not be investigated any further in this thesis.

A last comment on other outstanding effects is with regards to the kinetic-SZ effect, this arises through $e_a v_B^a$ in re-ionized re-scattering while the thermal-SZ effect arises from including a electron-baryon pressure term. The kinetic effect is accumulative, and is due to along-line-of-sight streaming velocity differences – this tends to cancel in much the same manner as the baryon infall effect does during slow decoupling. These non-linear scattering effects are important in order to fully understand the effects of patchy re-ionization within the almost-FLRW models.

Here we are primarily interested in bettering the understanding of the gravitational non-linear effects on the temperature anisotropies. We aim to attain two immediate goals:

²The energy flux in the Newtonian frame with respect to the total frame is : $D_b \bar{q}_a \approx D_b q_a - (D_b \rho + D_b p) v_a$. In the situation with dominate pressure-free dust one then has an additional scattering contribution as :

$$\delta\tau_{OV}(\eta_0) \approx - \int_{\eta_1}^{\eta_2} (\kappa' e^{-\tau}) \{ D^a \rho_B v_a \} d\eta.$$

Here, the divergence of the energy flux will vanish along the line of sight by cancellation due to falling in and out of potential wells on small scales – however the transverse components of the quantities in the brackets will not vanish generically.

first, to demonstrate how this effect would arise as a correction to the standard gauge invariant treatment following the work of Hu et al [82, 83] in the formalism of Bardeen [6] and Wilson [194, 193], and second, to demonstrate how to implicitly calculate such corrections analytically by finding the corrections to the mode coefficients of the temperature anisotropies. Such a programme is necessary in order to calculate the non-linear scattering corrections as the next step. We will investigate the scalar effect in the conformal Newtonian frame using weak-coupling [84] and high- ℓ approximation techniques, all within the covariant and gauge invariant theory. These calculations represent new effects that do not arise in the standard treatments.

An additional caveat here is that in the Newtonian frame, which is equivalent to the Newtonian gauge choice used in the bulk of the literature following the Bardeen formalism, any extension to second order must be treated with extreme care. Such a threading is inconsistent beyond linear order [51]. We are relying on weak coupling and sufficiently small velocity corrections in order to argue validity and hence consistency of our treatment.

Ideally such a calculation should be carried out in a manifestly gauge invariant manner (using the generic Lagrangian threading) or in frames that are known to be consistent at higher order, such as the constant expansion frame or CDM frame: we do this for a nonlinear Rees-Sciama imprint on the temperature anisotropies. With such frame choices one then has the additional freedom to move away from weak non-linearity.

Before we can calculate the weakly nonlinear correction (with no nonlinear feedback) using the weak coupling approximation, that is the effect of terms of the form $\dot{\Phi}\Phi$, $\dot{\Phi}\delta T$ and $\dot{\Phi}_H^2$, we must review the almost-FLRW results.

6.4 The almost-FLRW anisotropy sources

The calculation is intended to reproduce the results of Martinez-Gonzalez *et al* [128] and Seljak [154], where they use an *ad-hoc* nonlinear correction at second order in a perturbation theory about Einstein-de Sitter (EdS). We now use the temperature anisotropy results we have computed in chapter 5 (see 5.185).

As before the almost-FLRW primary source (5.186) (using matter domination to get $\Phi_A \approx -\Phi_H$, η_0 is the conformal time now, and η^* is that near last scattering):

$$\frac{\beta_\ell \tau_\ell^P(k, \eta_0)}{(2\ell + 1)} \approx [\delta T - \Phi_H](k, \eta^*) j_\ell(k(\eta_0 - \eta_*)) \quad (6.8)$$

The almost-FLRW secondary source (5.188) is:

$$\frac{\beta_\ell \tau_\ell^S(k, \eta_0)}{(2\ell + 1)} \approx -2 \int_{\eta^*}^{\eta_0} d\eta' e^{-\kappa} \Phi'_H(k, \eta') j_\ell(k(\eta_0 - \eta')). \quad (6.9)$$

Using weak-coupling and not specifying the end time ($\eta_0 \rightarrow \eta$) the latter becomes

$$\tau_\ell^{Swc}(k, \eta) \approx -2 \frac{(2\ell + 1)}{\beta_\ell} \sqrt{\frac{\pi}{2\ell}} \frac{1}{k} \Phi'_H(k, \eta) e^{-\kappa}, \quad \text{and} \quad \eta_\ell \approx \eta - (2\ell + 1)/2k. \quad (6.10)$$

To summarize the power spectrum definitions³, we recall that: $P(k) = |\Delta(k, \eta_0)|^2$, for $\Delta = D(\eta)\Delta(k, \eta_0)$ and $D(\eta) \approx (\eta^2/\eta_0^2)$, in addition⁴ one has

$$\frac{3}{2}H_0^2\Omega_0\Delta(\eta, k) \approx -k^2(a\Phi_A(k, 0)). \quad (6.11)$$

We are then able to write the primary and secondary sources as:

$$\tau_\ell^P(k, \eta) \approx \frac{(2\ell + 1)}{\beta_\ell} \left[\delta T_0 - \frac{3}{2}H_0^2\Omega_0k^{-2} \left(\frac{D_\star}{a_\star} \right) \Delta(k, \eta_0) \right] j_\ell(k(\eta - \eta_\star)), \quad (6.12)$$

$$\tau_\ell^{S-wc}(k, \eta) \approx -\frac{(2\ell + 1)}{\beta_\ell} 3H_0^2\Omega_0 \sqrt{\frac{\pi}{2\ell}} \frac{1}{k^3} e^{-\kappa} \left(\frac{D(\eta_\ell)}{a(\eta_\ell)} \right)' \Delta(k, \eta_0). \quad (6.13)$$

We assume the standard Einstein-de-Sitter (EdS) results : $D(\eta) = a(t)/a_0$ from $a/a_0 = \eta^2$ and that $P(k, t) = P(k, 0)D(t)$ along with the useful relation $(\Omega_0 D_\star/a_\star)^2 \approx \Omega^{1.54}$. The intention here is to use the EdS growth factors but a nonlinear correction to Δ . As pointed out before, we are modeling the effect of small-scale nonlinearity in the matter density so as to recover the Rees-Sciama like corrections for the exact theory. In addition one could include damping as κ'/k to include cancellation and diffusion (see [84]).

We assume here that the dominate contribution will arise from the ISW terms via a Rees-Sciama like correction, this will dominate the early-ISW, Sachs-Wolfe and acoustic sourced primary effects, given that there was no nonlinearity near last scattering. Hence we need only consider the secondary sourced effect.

6.4.1 The Rees-Sciama effect (RS)

In order to understand the nature of the non-perturbative corrections we need to understand the Rees-Sciama effect. Here we reproduce the well known result. The Rees-Sciama effect arises when the gravitational potential changes with time due to nonlinear evolution, typically of the matter; in models that are linear perturbations about Einstein de Sitter, the potential does not change with conformal time, so making the Rees-Sciama effect in these universes a nonlinear effect. When the time rate of change of the potential is due to Ω being far from unity, this would best be termed the ISW effect, being distinct from the Rees-Sciama one. The Rees-Sciama effect is specific to the effect of nonlinear matter evolution, via gravitational tidal forces, on photons. From the almost-FLRW temperature anisotropies, using that $\Phi_H(k, \eta)' \approx D'(k, \eta)\Phi(k, \eta_0)$ ⁵ and following [154] but using the covariant and gauge invariant notation (6.10 and 6.9):

$$\tau_\ell^{RS}(k, \eta_0) \approx -2 \frac{(2\ell + 1)}{\beta_\ell} \sqrt{\frac{\pi}{2\ell}} \frac{1}{k} e^{-\kappa} D'(k, \eta_\ell) \Phi_H(k, \eta_0), \quad \text{for } k\eta_0 \geq \ell. \quad (6.14)$$

³Note that the definition using δ_k differs from that using Δ by k^3 :

$$\Delta^2 = \frac{d\langle \delta \rangle^2}{d \ln k} \propto k^3 \mathcal{P}(k) \quad \text{for, } \mathcal{P}(k) \equiv \langle |\delta_k \delta_{k'}|^2 \rangle.$$

We will be using the dimensionless form; the variance per $\ln k$.

⁴ $\frac{1}{3}D_a \ln \rho_B \approx \rho_M^{-1} D^b E_{ab} \iff D_a \ln \rho_B \approx +\frac{k}{a} \Delta(k, \eta) Q_a$ and $E_{ab} \approx \frac{1}{2} D_{(a} D_{b)} (\Phi_A(x) - \Phi_H(x)) \approx +D_{(a} D_{b)} \Phi_A(x) \Rightarrow \frac{3}{2} H_0^2 \Omega_0 \Delta(k, \eta) \approx -k^2 \Phi_A(k, 0)$.

⁵This is nonlinear for the CDM dominate flat almost-FLRW models – where $\Phi_H(k, \eta)' \approx \Phi_H(k, 0)' \approx 0$. To understand the notation used here. The Rees-Sciama correction arises from (i) the term $\Phi_H(k, \eta) = D(k, \eta)\Phi_H(k, \eta_0)$, while the (ii) generic ISW effect arises from $\Phi_H(k, 0)'$ which is written as $(D(\eta)\Phi_H(k, 0)/a(\eta))'$.

This term is zero for $k\eta_0 < \ell$. Now using the definition of the angular correlation function:

$$C_\ell^{\text{RS}} = \frac{2}{\pi} \frac{\beta_\ell^2}{(2\ell + 1)^2} \int_0^\infty k^2 dk |\tau_\ell^{\text{RS}}(k, \eta_0)|^2, \quad (6.15)$$

we find the angular correlation function or the Rees-Sciama effect in terms of the power spectrum of the potential:

$$C_\ell^{\text{RS}} \approx \frac{4}{\ell} (4\pi)^2 \int_0^\infty k^2 dk (D'(k, \eta_\ell))^2 P_{\Phi_H}(k) \frac{1}{k^2}. \quad (6.16)$$

The power spectrum of the time changing potential is defined as

$$P_{\Phi'_H}(k, \eta) = (D'(k, \eta))^2 P_{\Phi_H}(k) \quad (6.17)$$

[154] so we find that (here we are using $\eta_\ell = \eta_0 - \ell/k$ such that $k = \ell/(\eta_0 - \eta)$ for all $k\eta_0 \geq \ell$):

$$\text{Rees-Sciama: } C_\ell^{\text{RS}} \approx \frac{4}{\ell} (4\pi)^2 \int_0^\infty dk P_{\Phi'_H}(k, \eta_0 - \frac{\ell}{k}). \quad (6.18)$$

We can re-scale the integration variable from k to η , $dk \approx [\ell/(\eta_0 - \eta)^2](d\eta)$, and change the integration limits (using that the integral is non-vanishing for $k \geq \ell/\eta_0$) to write the angular correlation function in terms of the area distance $r \approx (\eta_0 - \eta)$:

$$C_\ell^{\text{RS}} \approx 4(4\pi)^2 \int_0^{\eta_0} \frac{d\eta}{r^2} P_{\Phi'_H}(\ell/r, \eta). \quad (6.19)$$

This recovers the result of [154]. From [154] we have that

$$P_{\Phi'_H} = \frac{9}{4} \frac{H_0^2}{k^4} (a')^2 P_{(2)}(k). \quad (6.20)$$

This then give the linear Rees-Sciama effect. The nonlinear power spectrum is approximated from [128, 154] : $P_{\Phi'_A}(k) \propto k^{-4} P_{(2)}(k)$. The relationship between the second order power spectrum and the linear one can be found from:

$$P_{(2)}(k) \propto \int_0^\infty q^2 dq P(k) P(|k - q|) D_{(2)}^2(k, |k - q|) \quad (6.21)$$

where $D_{(2)}$ is the second order growth factor, and $P(k) = A(k/k_0^5 + k^4)^{-1} e^{-ka}$ for $k_0 = h/30 \text{ Mpc}$, $a = 8h^{-1} \text{ Mpc}$ such that for $k < k_0$ one has $P(k) \propto k^4$ and for $a^{-1} > k > k_0$ one finds $P(k) \propto 1/k$. In this latter region one finds then that $P_{(2)}(k) \propto k^3 P(k)$. With the additional cancellation as $(k\delta\eta)^{-1}$ one finds the usual result that the source scales as $k^2 P(k)$ in this region of interest – as pointed out in the introduction. Hence one then finds, $P'_{\Phi_A}(k) \propto k^{-2} P(k)$, as the power spectrum of the time rate of change of the potential in the region $a^{-1} > k > k_0$: leading to the usual scaling argument.

We can now investigate the nonlinear Rees-Sciama effect (NLRS) arising from the non-perturbative small-scale corrections in the exact MDE equations (here nonlinear in the MDE's rather than in the sense of the second order theory breaking down for the matter dynamics). In order to clarify our terminology, we will be using mildly nonlinear

matter evolution (6.17), as in the construction (6.14) of the Rees-Sciama effect (6.18), and the weakly nonlinear formulation of the temperature anisotropies (6.2) using the non-perturbative small scale corrections (6.6) due to gravitational tidal forces; together they will give a nonlinear Rees-Sciama effect. The key worry in this calculation is the possibility of cancellation between the primary source feedback and the secondary source feed back. This will be explicitly demonstrated in the Newtonian frame calculation.

6.5 Weakly nonlinear gravitational corrections

We use the usual scalar mode decomposition but now include a mode-mode coupling between the wavenumber (assuming that the directions are all aligned along the line of sight)⁶.

The non-linear correction multipole coefficient and the temperature anisotropy multipole coefficient can be put into mode coefficient form:

$$\delta\tau_{\text{NL}}^{A_\ell} = \sum_{k^a} (\delta\tau)_\ell^k Q_k^{A_\ell}, \quad \text{and} \quad \tau^{A_\ell} = \sum_{k^a} \tau_\ell^k Q_k^{A_\ell}. \quad (6.22)$$

Using the Fourier convolution theorem, with a little algebra and integrating over the solid angle, these, along with the mode form of the scalar potential [67], using the PSTF relations $e_{a\ell+1} O^{A_{\ell+1}}$ and an identity for the delta function, $\delta(2k) = \frac{1}{2}\delta(k)$, allow us to write (6.6) as a mode coefficient for the correction in terms of those for the temperature anisotropy and scalar potential sourced term.

We have four different cases arise in Eq. (6.6), using in addition that $\ell \gg 1$ and that $e_a O^{aA_\ell} = (\ell + 1)/(2\ell + 1) O^{A_\ell}$:

$$A_{\langle a_\ell \tau_{A_{\ell-1}} \rangle} \sim (4\pi) Q_{A_\ell}(x^a, k_*^a) \int_0^\infty \frac{k^2 dk}{(2\pi)^3} A(k) \tau_{\ell-1}(|k^* - k|), \quad (6.23)$$

$$A^a \tau_{aA_\ell} \sim (4\pi) \frac{\ell + 1}{2\ell + 1} Q_{A_\ell}(x^a, k_*^a) \int_0^\infty \frac{k^2 dk}{(2\pi)^3} A(k) \tau_{\ell+1}(|k^* - k|), \quad (6.24)$$

$$\sigma_{\langle a_\ell a_{\ell-1} \tau_{A_{\ell-2}} \rangle} \sim (4\pi) Q_{A_\ell}(x^a, k_*^a) \int_0^\infty \frac{k^2 dk}{(2\pi)^3} \sigma(k) \tau_{\ell-2}(|k^* - k|), \quad (6.25)$$

$$\sigma^{ab} \tau_{abA_\ell} \sim (4\pi) Q_{A_\ell}(x^a, k_*^a) \frac{(\ell + 1)(\ell + 2)}{(2\ell + 1)(2\ell + 3)} \int_0^\infty \frac{k^2 dk}{(2\pi)^3} \sigma(k) \tau_{\ell+2}(|k^* - k|). \quad (6.26)$$

Our approximation scheme only holds for high- ℓ , this then gives us:

$$A_{\langle a_\ell \tau_{A_{\ell-1}} \rangle} \sim (4\pi) Q_{A_\ell}(x^a, k_*^a) \int_{k_0}^\infty \frac{k^2 dk}{(2\pi)^3} A(k) \tau_{\ell-1}(|k^* - k|), \quad (6.27)$$

$$A^a \tau_{aA_\ell} \sim (4\pi) \frac{1}{2} Q_{A_\ell}(x^a, k_*^a) \int_{k_0}^\infty \frac{k^2 dk}{(2\pi)^3} A(k) \tau_{\ell+1}(|k^* - k|), \quad (6.28)$$

⁶We do not need to worry about the direction couplings then (which is of course essential for the Vishniac effect and the nonlinear scattering effect which would generate a non linear Vishniac effect). We therefore do not have terms as : $\frac{1}{2} \sum_{k^a} (A(k^a)M(k^{*a} - k^a) + A(k^{*a} - k^a)M(k^a))$.

$$\sigma_{\langle a_\ell a_{\ell-1} \tau_{A_{\ell-2}} \rangle} \sim (4\pi) Q_{A_\ell}(x^\alpha, k_*^\alpha) \int_{k_0}^{\infty} \frac{k^2 dk}{(2\pi)^3} \sigma(k) \tau_{\ell-2}(|k^* - k|), \quad (6.29)$$

$$\sigma^{ab} \tau_{ab A_\ell} \sim (4\pi) Q_{A_\ell}(x^\alpha, k_*^\alpha) \frac{1}{4} \int_{k_0}^{\infty} \frac{k^2 dk}{(2\pi)^3} \sigma(k) \tau_{\ell+2}(|k^* - k|). \quad (6.30)$$

We are dropping the vorticity from Eq. (6.6) as we are assuming matter domination for which it becomes natural to treat vorticity effects as small. We then have the weakly nonlinear correction on small scales as [120] (from Eq. 6.6 using Eqns. 6.27 – 6.30):

$$\begin{aligned} \delta\tau_{A_\ell}^{\text{NL}} &\approx -(4\pi)\ell Q_{A_\ell}(x^\alpha, k_*^\alpha) \int_{k_0}^{\infty} \frac{k^2 dk}{(2\pi)^3} \\ &\times \left[\sigma(k, t) \tau_{\ell-2}(|k^* - k|, t) + \frac{1}{16} \sigma(k, t) \tau_{\ell+2}(|k^* - k|, t) \right. \\ &\left. - A(k, t) \tau_{\ell-1}(|k^* - k|, t) + \frac{1}{4} A(k, t) \tau_{\ell+1}(|k^* - k|, t) \right]. \quad (6.31) \end{aligned}$$

The question is how does this compare to the linear primary and secondary sources from the canonical treatment. We are interested in finding the effect on scales : $k > k_0$. The above equations is valid for any u^a -frame : we have not yet frame-fixed the theory. We drop the mode functions:

$$\begin{aligned} \frac{d}{dt}(\delta\tau)_\ell^{\text{NL}}(k^*, t) &\approx -(4\pi)\ell \int_{k_0}^{\infty} \frac{k^2 dk}{(2\pi)^3} \left[\sigma(k, t) \tau_{\ell-2}(|k^* - k|, t) + \frac{1}{16} \sigma(k, t) \tau_{\ell+2}(|k^* - k|, t) \right. \\ &\left. - A(k, t) \tau_{\ell-1}(|k^* - k|, t) + \frac{1}{4} A(k, t) \tau_{\ell+1}(|k^* - k|, t) \right]. \quad (6.32) \end{aligned}$$

Finally, we can write this out in terms of the conformal time through : $dt = a d\eta$, and write the effect today in terms of a timelike integration (under the assumption that the effects are secondary – as opposed to primary):

$$\begin{aligned} \delta\tau_\ell^{\text{NL}}(k^*, \eta_0) &\approx -(4\pi)\ell \int_{\eta_*}^{\eta_0} a(\eta) d\eta \int_{k_0}^{\infty} \frac{k^2 dk}{(2\pi)^3} \left[\sigma(k, \eta) \tau_{\ell-2}(|k^* - k|, \eta) \right. \\ &\left. + \frac{1}{16} \sigma(k, \eta) \tau_{\ell+2}(|k^* - k|, \eta) - A(k, \eta) \tau_{\ell-1}(|k^* - k|, \eta) + \frac{1}{4} A(k, \eta) \tau_{\ell+1}(|k^* - k|, \eta) \right] \quad (6.33) \end{aligned}$$

There are two distinct effects that we wish to highlight, the first, is the result of mode-mode coupling between the linear primary and linear secondary anisotropies (6.13) and the linear shear and linear acceleration sources, this will most likely lead to a small smoothing effect at high- ℓ , and second, the effect of local nonlinear matter dynamics coupling through into the anisotropies. It is this latter effect that we will emphasize here, because it promises to be a dominant source of nonlinearity. The manner in which we are going to try and find an upper bound on the effect of local nonlinearity in the matter on the temperature anisotropies due to nonlinear couplings is to compute the correction for the Rees-Sciama effect (6.18).

6.5.1 Newtonian-frame correction

Given that much of the canonical literature is formulated in the Newtonian gauge, we first carry out our treatment in the 1+3 Lagrangian threading equivalent of this for the almost-FLRW universe; the Newtonian frame. We will show that on small-scales (at high- ℓ), in this restrictive frame, the effect cancels; one should not expect smoothing nor any feedback effects in the small scale Newtonian frame. However, in the generic frame no such cancellation can be expected. This latter point will be demonstrated by deriving the aligned scalar-sourced shear correction in the total frame in section 6.5.2.

We now investigate the nonlinear Rees-Sciama effects (NLRS). Once again the high- ℓ correction (from chapter 2), from (6.6) is:

$$(\delta\dot{\tau})^{A\ell} \approx -\ell \left(\frac{1}{4}\sigma_{bc}\tau^{bcA\ell} + \sigma^{(a\ell}a_{\ell-1}\tau^{A\ell-2)} - A^{(a\ell}\tau^{A\ell-1)} + \frac{1}{2}A_b\tau^{bA\ell} + \omega^b\epsilon_{bc}^{(a\ell}\tau^{A\ell-1)c)} \right) \quad (6.34)$$

in the Newtonian frame (see chapter 5 where $\tilde{u}^a = n^a \approx u^a + v^a$ where $D_{\langle a}n_{b\rangle} = \tilde{\sigma}_{ab} = 0$ which if dominated by pressure free dust leads one to expect $\omega^a \approx 0$). First, equation (6.34) can be written in terms of the acceleration alone [67], it can then be written, for scalar perturbations, in terms of the Newtonian potential [67] :

$$\delta\dot{\tau}^{A\ell} \approx \ell \left(\bar{A}^{(a\ell}\tau^{A\ell-1)} - \frac{1}{2}\tilde{A}_b\tau^{bA\ell} \right) \approx \ell \left((D^{(a}\Phi_A)\tau^{A\ell-1)} - \frac{1}{2}(D_b\Phi_A)\tau^{bA\ell} \right). \quad (6.35)$$

Here we have used that : $A(k) = -(|k^a|/a)\Phi_A(k, \eta)$ for $\tilde{A}_a \approx D_a\Phi_A$.

Expanding out the correction

We have changed from comoving time, (t), to conformal time, (η), in (6.35). Invert the resulting equation into an integral equation and now work in the conformal Newtonian frame using the conformal time variable : $dt = ad\eta$. The integral inversion will only work for weak nonlinearity as we have excluded the general feedback from the anisotropy into the small-scale correction and back into the anisotropy:

$$(\delta\dot{\tau})_\ell^{NL}(k^*, \eta) \approx -(4\pi)\ell \int_{\eta_*}^{\eta_0} \int_{k_0}^{\infty} \frac{k'^2 dk' k'}{(2\pi)^2 a} \Phi_A(k', \eta) \left[\tau_{\ell-1}(k-k', \eta) - \frac{1}{4}\tau_{\ell+1}(k-k', \eta) \right] \quad (6.36)$$

It becomes convenient to rewrite this in terms of :

$$T_\ell(k^*, \eta) = T_\ell^P(k^*, \eta) + T_\ell^S(k^*, \eta) = \frac{\beta_\ell}{(2\ell+1)} \left[\tau_{\ell-1}(k^*, \eta) - \frac{1}{4}\tau_{\ell+1}(k^*, \eta) \right], \quad (6.37)$$

where S and P denote the secondary and primary induced corrections. Essentially the argument is that the time integration over $T_\ell(k, \eta)$ cancels when using weak coupling, that is for high- ℓ (small scales).

The Primary Sourced correction

Using the almost-FLRW integral solution [67] for the temperature anisotropies in the mode coefficient formulation of the non-linear correction (6.36) and using (6.37):

$$\frac{\beta_\ell(\delta\tau)_\ell^{NLP}}{(2\ell+1)}(\eta_0, k^*) \approx +2\ell \int_{\eta_*}^{\eta_0} d\eta \int_{k_0}^{\infty} \frac{k^2 dk}{(2\pi)^2} k\Phi_A(k, \eta) T_\ell^P(k^*, \eta) \quad (6.38)$$

$$T_\ell^P(k^*, \eta) \approx [\delta T + \Phi_A](k^* - k, \eta_*) \left[\frac{(2\ell - 1)j_{\ell-1}(|k^* - k|)}{\beta_{\ell-1}} - \frac{1}{4} \frac{(2\ell + 3)j_{\ell+1}(|k^* - k|)}{\beta_{\ell+1}} \right] \quad (6.39)$$

Notationally what is important to realize is that the Bessel functions have conformal time arguments as $\eta_0 - \eta$ inside the time integration and the co-efficients inside the square brackets are evaluated at η_* while the ones outside at η . The best way to think of this is that the integral solution is integrated up to some arbitrary time parameter η , this is then convolved as in the integral above and integrated from η_* until now, η_0 .

Now, to get any further we will again use the weak-coupling approximation [84] arguing that the variations in the temperature anisotropies are significantly more rapid than those in potentials, this will allow us to reduce the potential term and remove the conformal time integration down the worldline:

$$\begin{aligned} \frac{\beta_\ell}{(2\ell + 1)} \int_{\eta_*}^{\eta_0} d\eta k \Phi_A(k, \eta) T_\ell^P(k^*, \eta) &\approx + \sqrt{\frac{\pi}{2}} \frac{k}{|k^* - k|} [\delta T_0 + \Phi_A](|k^* - k|, \eta_*) \\ &\times \left[\Phi_A(k, \eta_{\ell-1}^{k^*-k}) \frac{(2\ell - 1)\beta_\ell}{\beta_{\ell-1}\sqrt{(\ell - 1)}} - \frac{1}{4} \Phi_A(k, \eta_{\ell+1}^{k^*-k}) \frac{(2\ell + 3)\beta_\ell}{\beta_{\ell+1}\sqrt{(\ell + 1)}} \right]. \end{aligned} \quad (6.40)$$

We expect some cancellations to occur between the k^* and $-k^*$ terms but before we can deal with that we use that $\beta_\ell/\beta_{\ell-1} = \ell/(2\ell - 1)$ and $\beta_\ell/\beta_{\ell+1} = (2\ell + 1)/(\ell + 1)$ [66]. This allows the following reduction :

$$\frac{(2\ell + 3)\beta_\ell}{\beta_{\ell+1}\sqrt{(\ell + 1)}} = \frac{(2\ell + 3)(2\ell + 1)}{(\ell + 1)^{\frac{3}{2}}}, \quad \text{and} \quad \frac{(2\ell - 1)\beta_\ell}{\beta_{\ell-1}\sqrt{(\ell - 1)}} = \frac{\ell}{\sqrt{(\ell - 1)}}. \quad (6.41)$$

We now take the high- ℓ approximation (large- k) again:

$$\ell(2\ell + 3)/(\ell + 1)^{\frac{3}{2}} \approx 2\sqrt{\ell}, \quad \text{and} \quad \ell^2/(2\ell + 1)\sqrt{(\ell - 1)} \approx \frac{1}{2}\sqrt{\ell}, \quad (6.42)$$

to find that putting (6.41) into (6.40) and then using (6.42), upon collecting terms to cancel factors of $1/4$, we recover :

$$\begin{aligned} \frac{\beta_\ell}{(2\ell + 1)} \int_{\eta_*}^{\eta_0} d\eta k \Phi_A(k, \eta) T_\ell^P(k^*, \eta) &\approx \frac{1}{2} \sqrt{\frac{\pi}{2\ell}} \frac{k}{|k^* - k|} \\ &\times [\delta T_0 + \Phi_A] \left(\Phi_A(k, \eta_{\ell-1}^{k^*-k}) - \Phi_A(k, \eta_{\ell+1}^{k^*-k}) \right). \end{aligned} \quad (6.43)$$

Putting this back together (6.38 and 6.43) we then find (the additional ℓ in front of (6.39) leads to the $\sqrt{\ell}$ scaling below and the factor 2 cancels):

$$\begin{aligned} \frac{\beta_\ell(\delta\tau)_\ell^{\text{NL-P}}}{(2\ell + 1)}(\eta_0, k^*) &\approx \sqrt{\frac{\pi\ell}{2}} \int_{k_0}^{\infty} \frac{k^2 dk}{(2\pi)^2} \frac{k}{|k^* - k|} \\ &\times [\delta T_0 + \Phi_A] \left(\Phi_A(k, \eta_{\ell-1}^{k^*-k}) - \Phi_A(k, \eta_{\ell+1}^{k^*-k}) \right) \end{aligned} \quad (6.44)$$

First, for sufficiently high- ℓ (that is small scales) we could argue that the peaks of the Bessel functions are sufficiently close that $\eta_\ell \approx \eta_{\ell\pm 1}$; one immediately notices that all the relevant terms will cancel out in (6.44), we then find

$$\frac{\beta_\ell}{(2\ell + 1)}(\delta\tau)_\ell^{\text{NL-P}}(\eta_0, k) \approx 0. \quad (6.45)$$

So it would appear that in the weakly non-linear case there is no effect. The cancellation has nothing to do with the damping factor, $\mathcal{D}(\eta_R, k)$, which we can safely ignore here. We have shown that that primary sourced effect cancels. We now show that the secondary sourced effect cancels too. This is more important as this is how the Rees-Sciama effect would arise in the Newtonian frame.

Secondary source correction

From (6.36) the term of interest is:

$$T_\ell^S(k^*, \eta) = \frac{\beta_\ell}{(2\ell + 1)} \left[\tau_{\ell-1}^{k^*} - \frac{1}{4} \tau_{\ell+1}^{k^*} \right]. \quad (6.46)$$

Now from the ISW source anisotropy we have (6.13 and 6.10)

$$\tau_\ell^{\text{RS}} \approx \left[\frac{-2\ell + 1}{\beta_\ell} \sqrt{\frac{\pi}{2\ell}} \right] \left[\frac{\Delta(k^*, \eta_0)}{k^{*3}} \right] \mathbf{S}_{\text{ISW}}(\eta_\ell^{k^*}), \quad (6.47)$$

$$\mathbf{S}_{\text{ISW}}(\eta) \approx 3H_0^2 \Omega_0 \frac{D}{a} \left(\frac{D'}{D} - \frac{a'}{a} \right). \quad (6.48)$$

Putting (6.48) into (6.46) we find on using that $\beta_\ell/\beta_{\ell-1} \approx \frac{1}{2}$ and that $\beta_\ell/\beta_{\ell+1} \approx 2$ (for $\ell \gg 1$):

$$\begin{aligned} T_\ell^S(k^*, \eta) &\approx - \left(\frac{2\ell}{\beta_{\ell-1}} \frac{\beta_\ell}{2\ell} \right) \sqrt{\frac{\pi}{2\ell}} \left(\frac{\Delta(k^*, \eta_0)}{k^{*3}} \right) \mathbf{S}_{\text{ISW}}(\eta_{\ell-1}^{k^*}) \\ &+ \frac{1}{4} \left(\frac{2\ell}{\beta_{\ell+1}} \frac{\beta_\ell}{2\ell} \right) \sqrt{\frac{\pi}{2\ell}} \left(\frac{\Delta(k^*, \eta_0)}{k^{*3}} \right) \mathbf{S}_{\text{ISW}}(\eta_{\ell+1}^{k^*}) \end{aligned} \quad (6.49)$$

$$\approx \frac{1}{2} \frac{\Delta(k^*, \eta_0)}{k^{*3}} \sqrt{\frac{\pi}{2\ell}} \left(\mathbf{S}_{\text{ISW}}(\eta_{\ell+1}^{k^*}) - \mathbf{S}_{\text{ISW}}(\eta_{\ell-1}^{k^*}) \right). \quad (6.50)$$

Now, using that $\eta_{\ell-1} \approx \eta_{\ell+1} \approx \eta_\ell \approx \eta - \ell/k$, we then find that:

$$T_\ell^S(k^*, \eta) \approx 0, \quad \Rightarrow (\delta\tau)_\ell^{NL} \approx 0. \quad (6.51)$$

Hence we have the result that acceleration sourced nonlinearity is at best small and in the limit vanishes. This means that in the Newtonian frame there may be mild nonlinearity (in the sense of second order perturbation theory) but there are no weakly nonlinear effects due to the acceleration potential in the Newtonian frame – as arising from non-perturbative small scale effects, which are the dominant gravitational effects on small scales [120].

The important point is that this provides some indications that the effects of weak nonlinearity are suppressed in the Newtonian frame; in some sense the Newtonian frame calculation is stable to weakly nonlinear contamination making it an ideal first approximation when trying to include additional non-perturbative source terms. To contrast this consider the shear sourced effect (once again treating the case where the wave-vectors are aligned along line of sight for simplicity).

In the Newtonian frame one needs only include the linear Rees-Sciama effect (6.18) which is known to be small [154]. In the CDM-frame (or any generic frame) one would need to be more careful with the nonlinear Rees-Sciama corrections (NLRS).

6.5.2 Total-frame correction

In the CDM dominated almost-EdS universe we have that $A_a \approx 0$, one then needs to only consider the nonlinear corrections due to the shear (given that the vorticity is small) – this is a nonlinear shear-sourced correction. Here we consider the nonlinear corrections (6.34) due to the shear in the generic u^a -frame (6.33):

$$(\delta\tau)_\ell^{NL}(k^*, \eta_0) \approx 4\pi\ell \int_{\eta_*}^{\eta_0} d\eta' \int_{k_0}^{\infty} \frac{k^2 dk}{(2\pi)^3} \sigma(k, \eta) \left[\tau_{\ell-2}(|k^* - k|) + \frac{1}{16} \tau_{\ell+2}(|k^* - k|) \right] \quad (6.52)$$

This will not generically vanish on small-scales: however in the case of sufficient matter domination one would expect the effect to be small at linear level. In the context of an EdS background the ISW sourced effect will very small, because the effect of $(\Phi_H(k, \eta)/a)' \approx (D/a)(D'/D - a'/a)\Phi_H(k, 0)$ is vanishingly small – which sources the ISW terms in the temperature anisotropy. We need to consider the nonlinear-ISW effect – the Rees-Sciama effect.

The coupling between the primary sourced effect and the linear-FLRW shear could be expected to, at best, lead to a smoothing of the acoustic peaks; this needs to be checked in more depth. If we interpret the shear as sourced by nonlinear dynamics (which leads to the Rees-Sciama effect – the imprint of nonlinear dynamics on the temperature anisotropies) we can then isolate three possible effects :

1. The **primary correction** about the flat CDM background : due to the coupling between the almost-FLRW shear (a term like $(a'/a)\Phi_A(k, 0)$) to the primary sources of anisotropies - the acoustic and Sachs-Wolfe effects. This may be important on intermediate scales.
2. The **primary nonlinear correction** about a flat CDM background : due to the coupling between the primary sources of temperature anisotropies and nonlinear shear, induced by nonlinear small scale matter dynamics. This would be important on small scales.
3. The **secondary nonlinear correction** (or *Nonlinear Rees-Sciama effect*) about the flat CDM background : due to the coupling between the nonlinear-ISW effect (Rees-Sciama effect) in the temperature anisotropies and the nonlinear corrections to the shear (a term like $(D'/a)\Phi_A(k, 0)$). This effect is expected to be the dominant small scale correction – and is due to small scale nonlinearity in the matter dynamics generating anisotropies which couple to the nonlinear shear.

We focus on this latter possibility with the idea of getting a feel for the form of the angular correlation function due to this effect. The interest in the secondary correction – a Nonlinear Rees-Sciama effect, lies in the realization that the peak in the Rees-Sciama effect is between $\ell = 200 - 300$; where one expects to find the peak in the angular correlation function (apparently due to the projection of Doppler and acoustic oscillations) [93]. We provide a simple argument demonstrating this contribution. Once again this calculation should be viewed as the departure point towards more sophisticated treatments – it does nothing more than demonstrate that further work is required.

The almost-FLRW shear source

The shear, in the energy-frame (which coincides with the total-frame in the CDM dominated case considered here), is given by [67] (see chapter 5 and appendix D.6.1):

$$\frac{1}{2}(\rho + p)\sigma_{ab} \approx (D_{(a}D_{b)}\Phi_H) + 3HD_{(a}D_{b)}\Phi_H - HD_{(a}D_{b)}(\Phi_H + \Phi_A). \quad (6.53)$$

In addition, for sufficient matter domination [67] ($A_a \approx 0$ and hence $\Phi_H \approx -\Phi_A$):

$$\rho(\eta_0)\sigma_{ab} \approx -2(a^3D_{(a}D_{b)}\Phi_H). \quad (6.54)$$

Now $a^2\rho(\eta_0) \approx 3K\Omega_{0m}/\Omega_{0m} - 1 \approx 3a^2H_0^2\Omega_0$ and $\rho(\eta) = a^{-3}\rho(\eta_0)$, which for scalar perturbations [66, 67] gives the mode coefficient for the shear⁷ using a flat CDM dominated model:

$$\frac{3}{2}\frac{H_0^2\Omega_0}{a^3}\sigma(k, \eta)Q_{ab} \approx \left(-\frac{k^2}{a^2}\Phi_H Q_{ab}\right) + 3H\left(-\frac{k^2}{a^2}\Phi_H Q_{ab}\right) \approx -\frac{k^2}{a^2}H\Phi_H(k, 0)Q_{ab} \quad (6.55)$$

we then find, on using that $\Phi_H(k, \eta) = D(\eta, k)\Phi_H(k, \eta_0)$ (Eq. D.67) and $\Phi_H(k, \eta_0) = \Phi_H(k, 0)$ in the dust case, and changing to conformal time ($dt = a d\eta$);

$$\frac{3}{2}\frac{H_0^2\Omega_0}{a^3}\sigma(k, \eta) \approx -\frac{k^2}{a^2}\frac{D'}{a^2}\Phi_H(k, \eta_0). \quad (6.56)$$

We have in mind $\dot{\Phi}_H \gg H\Phi_H$ for $\Phi_H(k, \eta) = D(\eta)\Phi_H(k, 0)$: (small scale nonlinear situation), where in the linear case $D = a$. this gives

$$\frac{3}{2}H_0^2\Omega_0\sigma(k, \eta) \approx -k^2\frac{D'}{a}\Phi_H(k, 0). \quad (6.57)$$

Equation (6.57) gives the almost-FLRW shear in terms of the scalar potentials $\Phi_A(k, 0) \approx -\Phi_H(k, 0)$. However, more importantly it also shows how the nonlinear dynamics will affect the shear in the 1+3 covariant and gauge invariant approach via the growth factor $D(k, \eta)$. Nonlinear source of $\dot{\Phi}_H + H\Phi_H$ will give a weakly nonlinear shear effect.

An important point here is that we have used the shear in its *total-frame* formulation (that natural to the Lagrangian threading), while the temperature anisotropies have been found in the *Newtonian frame* (that given by an Eulerian threading in the exact theory) – to understand why we can use these two together we recall (i) that E_{ab} is invariant under linear order frame boosts, (ii) $\tau_\ell(k\eta)$ is invariant to linear order for $\ell > 1$ (we have already got the source terms for the temperature perturbation, $\delta T(k, \eta)$, and the dipole $\tau_1(k, \eta)$ from chapter 5 in the Newtonian frame). We can use the scalar sourced anisotropies as found in the Newtonian frame (5.185) – we use the shear in the total frame along, with the evolution equations for the other various dynamic quantities (the EFE are solved in the total frame, for scalar (chapter 5 and tensor [78]) and solved these in terms of the scalar potentials ($\Phi_A(k, \eta)$, $\Phi_H(k, \eta)$) and peculiar velocity ($v(k, \eta)$) – which can then be substituted into the Newtonian frame solution. Of course

⁷Using $D_a\Phi_H = +\frac{k}{a}\Phi_H(k, \eta)Q_a$ and $D_{(a}D_{b)}\Phi_H = -\frac{k^2}{a^2}\Phi_H(k, \eta)Q_{ab}$

the dipole in the almost-FLRW Newtonian frame solution will not be that found in the almost-FLRW total-frame solution – while it will be zero in the CMB-frame.

Notice that from the div- E equation, $k^2\Phi_H(k, 0) \approx +\frac{3}{2}(D/a)H_0^2\Omega_0\Delta(k, \eta_0)$ (appendix D.6.1), now we have that $\Delta(k, \eta) \sim a\delta + a^2\delta_2$, where in EdS only the second order part contributes to the time changing potential, hence the growth factor $D \sim D_+(\eta) \sim a^2(\eta)$ so

$$\sigma(k, \eta)^{NL} \approx -\frac{D'(\eta)}{a(\eta)}\Delta_{NL}(k, \eta_0), \quad (6.58)$$

which is for sufficient matter domination on small scales.

Nonlinear Rees-Sciama effect

The nonlinear Rees-Sciama effect will take the form

$$\begin{aligned} \frac{\beta_\ell(\delta\tau)_\ell^{\text{NLRS}}}{(2\ell+1)}(k^*, \eta_0) &\approx \frac{\beta_\ell}{(2\ell+1)}(4\pi\ell) \int_{\eta_*}^{\eta_0} d\eta \int_{k_0}^{\infty} \frac{k^2 dk}{(2\pi)^3} \mathbf{S}_\ell^{\text{NLRS}}, \\ \mathbf{S}_\ell^{\text{NLRS}} &= \sigma(k, \eta) \left[\tau_{\ell-2}^{\text{RS}}(|k^* - k|, \eta) + \frac{1}{16}\tau_{\ell+2}^{\text{RS}}(|k^* - k|, \eta) \right], \end{aligned} \quad (6.59)$$

with the linear Rees-Sciama source terms

$$\sigma(k, \eta) \approx -\frac{D'}{a}\Delta(k, \eta_0) \approx -\frac{2}{3}\frac{1}{H_0^2\Omega_0}k^2\Phi_H(k, 0)\frac{D'}{a}, \quad (6.60)$$

$$\tau_\ell^{\text{RS}}(k, \eta) \approx -2\frac{(2\ell+1)}{\beta_\ell}\sqrt{\frac{\pi}{2\ell}}\frac{e^{-\kappa}}{k}D'(\eta_\ell)\Phi_H(k, 0), \quad \forall k\eta \geq \ell, \quad (6.61)$$

where $\eta_\ell \approx \eta - (\ell + \frac{1}{2})/k$, as before. Substituting (6.60) and (6.61) into (6.59) we are able to find:

$$\frac{\beta_\ell(\delta\tau)_\ell^{\text{NLRS}}}{(2\ell+1)}(k^*, \eta_0) \approx \frac{8\pi}{3H_0^2\Omega_0}\sqrt{\frac{\pi\ell}{2}}(e^{-\kappa}) \quad (6.62)$$

$$\times \int_{k_0}^{\infty} \frac{k^2 dk}{(2\pi)^3} \left[\frac{k^2}{|k^* - k|}\Phi_H(k)\Phi_H(|k^* - k|) \right] \mathbf{I}_\ell(|k^* - k|),$$

$$\mathbf{I}_\ell(|k^* - k|) \approx \int_{\eta_*}^{\eta_0} d\eta \frac{D'(\eta)}{a(\eta)} D'(\eta - \frac{\ell}{|k^* - k|}) \quad (6.63)$$

where we have used the high- ℓ assumption, which gives one the ratios of the β_ℓ coefficients:

$$\frac{\beta_\ell}{\beta_{\ell-2}} \sim \frac{1}{4}, \quad \text{and,} \quad \frac{\beta_\ell}{\beta_{\ell+2}} \sim 4, \quad \forall \ell \gg 1, \quad (6.64)$$

and using that $\eta_{\ell-2} \sim \eta_{\ell+2}$, we can show that :

$$\mathbf{I}_\ell(|k^* - k|) \sim 16 \left[\frac{1}{5} - \frac{1}{2}\frac{\ell}{k} + \frac{1}{3}\frac{\ell^2}{k^2} \right] \sim 16 \left[\frac{1}{5} - \frac{1}{2}\frac{\ell}{|k^* - k|} \right]. \quad (6.65)$$

We also used that the second order growth parameter is $D(a) \sim a^2(\eta)$ and for $\ell \sim k$ the higher order terms will drop off faster under the k -space integration.

6.5.3 Approximating the angular correlation function

In order to find the angular correlation function we first find the mean-square of the correction to the temperature anisotropy.

We use that⁸

$$\begin{aligned} \langle \Phi_H(k^* - k) \Phi_H(k) \Phi_H(k'' - k') \Phi_H(k') \rangle &= (2\pi)^6 P_{\Phi_H}(k^* - k) P_{\Phi_H}(k) \\ &\times \left[\frac{\delta(k^* - k'')}{k^{*2}} \frac{\delta(k - k')}{k^2} + \frac{\delta(k^* - k'')}{k^{*2}} \frac{\delta(k^* - k - k')}{(k^* - k)^2} \right], \end{aligned} \quad (6.66)$$

to then find the mean-square of the correction to the temperature anisotropy due to the nonlinear Rees-Sciama coupling between the local matter nonlinearities and the temperature anisotropies. The basis of the argument used to construct the maximum mean square contribution is that it will be near the peak in the radiation transfer function : $k \sim |k^* - k| \sim \ell(\eta_0 - \eta)$. This along with $(D')^2 P_{\Phi} \approx P_{\Phi'}$ shows

$$\begin{aligned} \frac{\beta_\ell^2}{(2\ell + 1)^2} |\delta\tau_\ell^{\text{NLRS}}(k^*, \eta)|^2 &= (4\pi)^2 \left[\frac{8\pi}{3H_0^2 \Omega_0} e^{-\kappa} \sqrt{\frac{\pi\ell}{2}} \right]^2 \\ &\times \int_{k_0}^{\infty} k^2 dk P_{\Phi'_H}(k, \eta_0) P_{\Phi'_H}(k^* - k, \eta_\ell) \frac{k^4}{k^{*2} |k^* - k|^2}. \end{aligned} \quad (6.67)$$

The correction to the angular correlation function due to the nonlinear Rees-Sciama terms (6.67) is constructed from the definition of the angular correlation function [66, 67]:

$$C_\ell^{\text{NLRS}} \approx \frac{2}{\pi} \frac{\beta_\ell^2}{(2\ell + 1)^2} \int_0^\infty k^{*2} dk^* |\delta\tau_\ell^{\text{NLRS}}(k^*, \eta_0)|^2. \quad (6.68)$$

It is then found that the approximate maximum correction to the angular correlation function is:

$$C_\ell^{\text{NLRS}} \sim (16\pi^2) \left[\frac{8\pi}{3H_0^2 \Omega_0} e^{-\kappa} \right]^2 \ell \int_0^\infty dk^* \int_{k_0}^\infty k^2 dk P_{\Phi'_H}(k, \eta_0) P_{\Phi'_H}(k^* - k, \eta_\ell) \left[k^2 \frac{\delta(k^* - \ell\eta_0)}{(\ell\eta_0)^2} \right]. \quad (6.69)$$

We have approximated the time integral by using the assumption that the effect is most prevalent near $\ell \sim k(\eta_0 - \eta)$. We use that $P_{\Phi'_H}(k, \eta) \approx D'(k, \eta) P_{\Phi_H}(k)$. We have that :

$$C_\ell^{\text{NLRS}} \sim (16\pi^2) \left[\frac{8\pi}{3H_0^2 \Omega_0} e^{-\kappa} \right]^2 \ell \int k^2 dk P_{\Phi'_H}(k, \eta_0) P_{\Phi'_H}(k, \eta_0 - \ell/k). \quad (6.70)$$

Now we argue that we can approximate the result by arguing that $\ell \sim k\eta_0$ in the first power spectrum and using that $P_{\Phi'_H}(k) \propto k^{-1} P(k)$ (which excludes the cancellation of $(k\delta\eta)^{-1}$) to give on small scales $P_{\Phi'_H} \propto k^{-2}$. Then one finds that:

$$C_\ell^{\text{NLRS}} \sim \alpha_0 (32\pi^2) e^{-2\kappa} \ell \int_{k_0}^\infty dk P_{\Phi'_H}(k, \eta_0 - \ell/k). \quad (6.71)$$

⁸This arises from $\delta(k^a - k'^a) = \frac{1}{k^{\frac{a-1}{2}}} \delta(k - k') \delta(e^a - e'^a)$. Recall that $\langle |\Phi(k, 0) \Phi(k', 0)| \rangle = (2\pi)^3 P_\Phi(k) \delta(k^a - k'^a)$

Here $\alpha_0 = 2A [8\pi/3H_0^2\Omega_0]^2$. We once again use that $k \approx \ell(\eta_0 - \eta)$ and that $r \sim (\eta_0 - \eta)$ to then find that

$$C_\ell^{\text{NLRS}} \sim \ell^2 (32\pi^2) \int_0^{\eta_0} \frac{d\eta}{r^2} P_{\Phi_H'}\left(\frac{\ell}{r}, \eta\right). \quad (6.72)$$

This can be readily compared to the Rees-Sciama calculation (where we ignore the damping):

$$C_\ell^{\text{NLRS}} \sim \ell^2 C_\ell^{\text{RS}}. \quad (6.73)$$

This is of course a gross approximation, however, it does go some distance to demonstrate that if C_ℓ^{RS} dominates the angular correlation functions near $\ell \sim 5000$ one can expect the nonlinear Rees-Sciama effect to dominate on scales much larger than this.

1. If C_ℓ^{RS} gives $\Delta T_{\text{RS}}/T \sim 10^{-7} - 10^{-6}$ between $\ell \sim 100 - 300$ (where the Rees-Sciama effect peaks [154]), one can naively expect, from (6.73) to find that $\Delta T_{\text{NLRS}}/T \sim 10^{-5} - 10^{-4}$ between $\ell \sim 100 - 300$.
2. If we include damping and cancellation as κ/k we then find : $C_\ell^{\text{NLRS}} \sim \ell C_\ell^{\text{RS}}$. This then gives $\Delta T_{\text{NLRS}}/T \sim 10^{-6} - 10^{-5}$ between $\ell \sim 100 - 300$. The latter is probably more realistic. The keypoint is that it is not a negligible effect when using the above approximation scheme.

The key point is that the primary peak height could be lifted by more than an order of magnitude by effects that do not arise in the linear theory. This is one possible explanation for the high peak values that current observations have found (see the appendix C). A similar result can be found from using an approximation scheme based on the approach of Futamase [92] – this is not given here in as much as a similar conclusion is found by much tedious algebra with little additional insight.

A weakness in the above approach is that the almost-FLRW anisotropies for *high*- ℓ are found in the Newtonian frame, then boosted to the total frame. As τ_{A_ℓ} is invariant under small frame boost the solutions are the same as those found in the Newtonian frame. The relationship between the shear and the acceleration potentials, the potentials and the matter perturbations and the peculiar velocities and the matter perturbations are all found in the total frame. Such tricks are only consistent for small relative velocities – at best this treatment is valid for mild nonlinearity.

Chapter 7

Summary and Loose Ends

Higher order formalisms and other nonlinear problems: *The conclusions of various chapters are outlined. The use of the almost-FLRW models as a good description of the imprint of structure and structure formation on the CMB is emphasised – both in the linear and weakly nonlinear context. A brief overview of the literature for the higher-order perturbative formalism is given – the nonlinear extension of the Bardeen theory. The future direction of the use of the 1+3 Lagrangian threading formalism and the 1+3 covariant and gauge invariant perturbative reduction is given in the context of both the tetrad-MDE and almost-Bianchi I approaches towards understanding the gravitational wave effects.*

7.1 Summary : chapter outlines

A broad outline of this work is given.

CHAPTER 1 Algebraic relations :

We have provided the 1+3 Lagrangian threading formalism with the multipole angular correlation function. Its definition is independent of geometric assumptions – it does not require the use or existence of unique mode functions. The multipole angular correlation function for small temperature anisotropies was reduced to the mode angular correlation functions applicable to almost-FLRW models (both in the open and flat cases). This provides the link between the theoretical developments in chapter 2 and the observations (see chapters 4 and 5). The Gaussian assumption underlies the development of the multipole angular correlation function.

CHAPTER 2 Temperature Anisotropies :

We have used a covariant Lagrangian approach, in which all the relevant physical and geometric quantities occur directly and transparently, as PSTF tensors measured in the comoving rest space. There is no restriction on the deviation of geometric and physical quantities from FLRW limiting values, so that arbitrary nonlinear behavior may in principle be treated. We have derived the corresponding equations governing the generation and evolution of inhomogeneities and CMB anisotropies in nonlinear generality, without *a priori* restrictions on spacetime geometry or specific assumptions about early-

universe particle physics, structure formation history, etc. Thus we have developed a useful approach to the analysis of local nonlinear effects in CMB anisotropies, with the clarity and transparency arising from 1+3 covariance. The equations are readily linearized in a gauge-invariant way, and then the methods of [66] may be used to expand in scalar modes and regain well-known first-order results as in chapter 5 [67].

This approach allowed us to identify and qualitatively describe some of the key local nonlinear effects. We calculated the nonlinear form of Thomson scattering multipoles (given the initial simplifying assumption of no polarization), revealing the new effect of coupling between the baryonic bulk velocity and radiation brightness multipoles of order $\ell \pm 1$. We also found the nonlinear effects of relative velocities of particle species on the dynamic quantities that source the gravitational field. These effects also operate on the conservation equations, including evolution equations for the relative velocities of baryonic and cold dark matter.

Nonlinear effects come together in the hierarchy of evolution equations for the radiation dynamic (brightness) multipoles, which determine the CMB temperature anisotropies. In addition to the nonlinear Thomson contribution, we identified nonlinear couplings of the kinematic quantities to the multipoles of order $\ell \pm 2$, $\ell \pm 1$, and ℓ . These quantities themselves are governed by nonlinear evolution equations, which provide part of the link between CMB anisotropies and inhomogeneities in the gravitational field and sources. The link is also carried by the spatial gradient of radiation energy density (equivalently, average radiation all-sky temperature), and the baryonic relative velocity. Furthermore, there is internal up- and down-transmission of power within the multipole hierarchy, supported by the kinematic couplings as well as by distortion and divergence derivatives of the multipoles.

We have provided the tetrad equations that will form the basis of any realistic kinetic theory investigation of temperature anisotropies in exact relativistic cosmologies.

We used our analysis of the radiation multipoles to identify new effects that operate at high ℓ . In particular, we showed that there is a nonlinear shear correction effect on small angular scales, whose impact on the angular power spectrum was qualitatively described. The quantitative analysis of this and other nonlinear effects will involve a choice of coordinates in order to tackle the solution of partial differential equations. This is a subject of further research (the scalar case was dealt with in chapter 6). We also anticipate nonlinear shear effects due to couplings between gravitational tidal force induced matter dynamics and gravitational waves at low- ℓ .

CHAPTER 3 Sachs-Wolfe and Kinetic Theory:

In order to contrast the Sachs-Wolfe and Relativistic Kinetic theory both have been used to derive equations for the temperature anisotropies, allowing a better understanding of the distinction between the null- and time-like integrations – crucial to the formulation of the nonlinear (and exact) investigation of temperature anisotropies. The missing component is the *ab initio* derivation of the area-distance from the null-geodesic deviation equations – to provide the generic link between angular scales “here-and-now” and spatial scales near decoupling.

A key feature of the Sachs-Wolfe approach in almost-FLRW models is the need to only integrate up a single null geodesic. This gives the Sachs-Wolfe approach a computa-

tional advantage over the timelike relativistic kinetic theory approach which integrates, typically, ℓ -poles up a single matter worldline.

CHAPTER 4 COBE-Copernican limits :

The formal basis for the use of the CDM dominated almost-FLRW model is developed. The COBE-DMR limits on the quadrupole and octopole are used to motivate the use of the almost-FLRW models. This motivation – essentially arising from the use of the weak Copernican principle, is not only a check on the stability and self consistency of the almost-FLRW universe, but provides the crucial CMB-based observational basis for the use and development of the almost-FLRW models in the later chapters.

CHAPTER 5 Scalar almost-FLRW universes:

The Newtonian frame almost-FLRW temperature anisotropies are developed. They describe the imprint of structure formation on the CMB – a scalar model describing gravitational tidal effects and the effect of this on the intermediate and small scale dynamics and kinematics of the matter and radiation within a CDM dominated cosmology. This is needed so as to provide the basis for the weakly nonlinear corrections developed in chapter 6. The total frame formulation of the matter dynamics and kinematics is developed. The Newtonian frame formulation of the photon physics along with the total frame formulation of the matter dynamics completely recovers the canonical formulation (see [80]). The formulation of these results, in the 1+3 Lagrangian threading formalism, is a critical component of the nonlinear extension (as used in chapters 2 and 6).

In this chapter we have carried out a covariant analytic timelike integration reproducing the well known primary effects generating the “Acoustic Peaks” measured here and now for CDM almost-FL universes with adiabatic scalar perturbations. We have also demonstrated how, in the CGI formalism, the angular correlation functions are constructed in terms of the matter power spectrum and normalized on large and small scales for *standard-CDM*, given appropriate approximations for the transfer functions.

As pointed out initially, the aim of this chapter was to clarify the link between the standard 3+1 gauge-invariant and 1+3 CGI treatments of CBR anisotropies, and provide a strong basis from which to tackle non-linear and gravitational wave effects using CGI methods.

We have followed a strict time-like integration philosophy throughout chapters 1 and 2; so as to separate out the use of hidden homogeneity assumptions that make the Sachs-Wolfe procedure seem plausible only for (i) test-field applications or (ii) in the context of almost-FLRW universes. This has meant that we have not explicitly needed to construct the angular diameter and area distance relations nor match the intersection of the past null-cone with the surface of last scattering. Nor are we in principle restricted by statistical assumptions about the perturbations; unless we wish to construct an angular correlation function.

Some of the key outstanding issues are:

1. How does one deal with anisotropic scattering and anisotropic stresses before and during decoupling within the covariant approach, specifically in such a manner that consistency is maintained with general relativity, its covariant linearizations and relativistic kinetic theory.

2. The small anisotropy equations developed in chapter 2 with the application to space-times with arbitrary anisotropy and inhomogeneity have yet to be properly investigated; these become applicable when one ignores the Copernican principle that underlies the almost-EGS theorem, which in turn provides the theoretical basis for using almost-FL space-time dynamics. An investigation of their consequences on the CBR may provide an alternative method of testing the Copernican principle other than the Sunyaev-Zel'dovich effect or via the use of polarization maps.
3. There is a need to find a working non-Gaussian treatment from which one can construct a generic characterization of observables here and now (other than the angular power spectrum alone (see chapter 1) and, second, finding an *ab initio* covariant analysis of transfer functions extending the post-Newtonian treatments which use periodic boundary conditions.
4. An exact 1+3 Lagrangian formulation of the area distance and its reduction to the almost-FLRW case has yet to reach maturity. This is the key to correctly recovering the relationship between angular scales now and physical scales near last scattering – beyond the almost-FLRW models which use FRW area distances with weak lensing corrections (see [80, 81] and references therein).

CHAPTER 6 Scalar Nonlinear corrections:

Our results on nonlinearity are by no means conclusive – they are a tentative departure point, warning against having to. Much faith in the linear theory. First, we have evidence that the Newtonian threading suppresses nonperturbative nonlinearity on small scales. This along with the well known result that the Newtonian threading is inconsistent beyond linear order are a strong indication that such treatments are generically inadequate for the study of relativistic cosmology unless an *a priori* assumption is made; that the universe is close to almost-FLRW on all observationally relevant scales. Although self-consistency of this assumption has been shown, such an approach is not generic. The implication is that if one tries to calculate the weakly nonlinear imprint of structure formation in the temperature anisotropies using the Newtonian frame form of the anisotropies, no effect will be found beyond the Rees-Sciama effect (and in the case of scattering – the Vishniac correction).

If this assumption is relaxed one discovers that there could be additional small-scale effects that have been excluded, by definition, from the canonical treatments of temperature anisotropies. One such small-scale effect, is the Rees-Sciama effect (we call this the nonlinear Rees-Sciama effect to indicate that this requires a nonperturbative small scale correction – here found in chapter 2). When included in the framework of the nonperturbative small-scale corrections one is lead to the conclusion that the contributions are of the same order of magnitude near the peak in the radiation transfer functions as the anisotropies themselves; or at best an order of magnitude less. This conclusion is not consistent with the canonical treatment - from which it is excluded by construction. One is then lead to the question of whether or not the primary peak then could be dominated by the imprint of nonlinear structure on small-scales – as opposed to an acoustic sourced primary peak. The answer to this is unclear at this time. It is certainly an interesting area for future work.

Second, we have provided some evidence demonstrating that nonlinearity seems to become problematic before the standard $\ell = 5000$ limit, in the case of no feedback between the temperature anisotropies and the nonlinearities, using an extension of the effects of nonlinearity through a Rees-Sciama like correction – the scaling is different from the canonical treatment inasmuch as an additional k scaling is introduced from the exact treatment. It can be expected, when the feedback is included, that the effect will be even more noticeable. There is probably a deep connection between the calculations carried out in order to model active perturbations, and those needed to understand the nonlinear effects due to gravity, as outlined here, which are also active. The connection between the pathologies of the effects discussed in the active perturbation literature [126] and those here need to be better understood.

7.2 Some comments on higher order formalisms

The generation of gravity waves by density perturbations at second order was considered in Matarrese-Terranova 1996 [129], using a Lagrangian approach. Second-order perturbations applied to the CMB – using a Bardeen-like generalization of the longitudinal and synchronous gauges, rather than a fluid-flow approach, is considered in Mollerach-Matarrese [135]. The theory of second-order perturbations is given in Bruni-Matarrese-Mollerach-Sonego [21]. This goes into the mathematical subtleties of gauge invariance, and the use of Knight diffeomorphisms, which are needed in order to extend the Bardeen gauge invariant metric approach. The Lagrangian approach does not have a higher order gauge problem; gauge invariance here is based on the vanishing of higher order quantities in the background, in a manifestly covariant fashion, the so called "second order" gauge problem is only a pathology of the perturbed metric approach and not the 1+3 Lagrangian theory. The backreaction of super-Hubble Gravity Waves produced in inflation has been considered by Abramo [3] following earlier work with Mukhanov-Brandenberger [136].

7.3 Loose Ends : the gravitational-wave imprint

The 1+3 covariant and gauge invariant gravitational wave imprint in the CMB using the Ehler-Ellis-Bruni approach as carried out by Maartens-Gebbie-Ellis needs to be completed. The almost-FLRW gravitational wave equations have been developed, see Hawking 1966 [78] in the vacuum case. Within the almost-FLRW theory, because there is no mixing at linear order between the gravitational wave (trace-free transverse tensor) part of the theory and the scalar part, and given that the electric and magnetic parts of the Weyl tensor are invariant under small frame transformations ($E_{ab} \approx \bar{E}_{ab}$ and $H_{ab} \approx \bar{H}_{ab}$), we can use the Newtonian frame formulation for the scalar almost-FLRW part of the theory and retain the total frame formulation of the linear gravitational wave part of the theory as given by Hawking 1966. In addition one needs to use the m -th order mode functions $Q_{A_\ell}^m = (-k/a)^{m-\ell} D_{(A_{m-\ell}} Q_{A_m)}^m$ for $D^{A_n} Q_{A_m}^m = 0 \forall n < m$ and $D^2 Q_{(A_m)} = -(k^2/a^2) Q_{(A_m)}$: $m = 2$ for tensor perturbations. The almost-FLRW version is well understood [80]. A version of this has also been formulated by Challinor using the

original work of Hawking [78] and the total angular momentum techniques of the canonical approach [82, 81, 93]– it is a recovery of some canonical results which emphasizes the astrophysical cosmology application to parameter fitting the almost-FLRW model.

However, the issue here is very different from those pursued by the canonical and precision cosmology programmes. The almost-FLRW models are useful for the understanding of the *effect of local structures and structure formation on temperature anisotropies* – even in the weakly nonlinear situation (as shown in chapter 6). These are essential local, or small scale (high- ℓ), effects of nonlinear dynamics – the paradigm being that of:

- *weakly nonlinear dynamics and kinematics over a globally linear perturbed geometry.*

One finds that the essential physics is that of understanding the scalar perturbations – an ideal problem for the Newtonian frame ($H_{ab} \approx 0$ and E_{ab} is sourced by tidal effects only). The global nonlinear effects cannot be tackled usefully in this approach – it is small-scale by construction. However, given that the scalar-sourced anisotropies in the CMB arise from local physics during and since decoupling, most likely after matter-radiation equality, this approach is suitable for those times, particularly the Newtonian and total-frame formulations of the linear and weakly nonlinear almost-FLRW universes. This is not necessarily the case for gravitational waves.

There is little reason to suspect that gravitational wave phenomena (encoded in E_{ab} and H_{ab}) need to be well described by almost-FLRW models – beyond the constraints imposed via the COBE-Copernican limits of chapter 4. Thus an important test of the use of almost-FLRW models (and the COBE-Copernican limits) is to understand the effect of a globally nonlinear geometry on the local kinematics and dynamics. The imprint of this is most likely to be important at low- ℓ ; one may expect to find an imprint of the coupling between gravitational waves and tidal induced scalar perturbations in the form of set of resonance peaks in the large scale angular power spectrum. The key move towards a better theoretical understanding of this, and thus the question of the gravitational wave imprint, is to understand the dust almost-Bianchi I cosmologies and their CMB imprint – the paradigm being that of:

- *weakly nonlinear geometry and kinematics but linearly perturbed dynamics.*

Dunsby's (unpublished) work on the dynamics of dust almost-Bianchi I is the key to this endeavour. The important point is that there is coupling between the tidal gravitational effects and the gravitational wave phenomena which is nontrivial – a feature not found in almost-FLRW studies. As pointed out, almost-FLRW models are more suited to studying structure formation, the evolution of structure and the imprint of this on the CMB, on cosmological scales – loosely below the Jeans-scale at matter-radiation equality and above the scale of gravitationally compact objects. Although Bianchi I models are not generic, it would be a strong move forward in better understanding the gravitational wave imprint.

The gravitational wave imprint near decoupling may become a key probe to very early physics, given that it is sufficiently well understood theoretically – in particular a direct probe of the era in which the gravity waves (which free-stream through the

tight-coupled era) form to leave an imprint in the CMB (at only linear order in the quadrupole) during the slow decoupling era, before radiation decoupling is complete. The linear behaviour is well known [78]. The important point here is that in the linear-FRW theory the gravitational radiation decays as at least a^{-1} (the energy density decays as radiation: a^{-4}). Its imprint on the CMB would be very small. This would arise due to an induced quadrupole via the shear of the matter congruences. The importance in understanding the perturbed Bianchi models is precisely for this reason. The almost-FLRW models may be ineffective at modeling the global geometry – which may (or may not) be linear perturbed FRW. In addition one could expect non-almost-FLRW large scale effects to mimic the pathologies of active perturbations (for a discussion of active perturbations using the Hu-Sugiyama formalism see [126]).

The problem is twofold : (I) understanding weakly nonlinear gravitational wave effects in order to have a suitable interesting and sophisticated cosmology that can be interfaced with the knowledge of CMB effects due to structure formation (that with an almost-FLRW geometry using the Newtonian frame formulation). This may be facilitated by the study of **almost-Bianchi** models, such as those with either flat type I or VII backgrounds (the background models need to be homogeneous to trivially ensure gauge invariance). (II) Understanding exact cosmologies that are inherently nonlinear and have interesting pathologies – the **ONT-tetrad-MDE approach** is the key to this. In tandem (I) and (II) offer the possibility of better understanding the correct context for the almost-FLRW models.

The view put forward here is as follows: first, treat the physics of the high- ℓ temperature anisotropies as well described by a mildly nonlinear almost-FLRW model, second, investigate the relevance of almost-FLRW models on large scales by modeling the low- ℓ anisotropy contributions by comparing it with a perturbation theory about homogeneous, but anisotropic, background models and exact homogeneous but anisotropic models (as undertaken by Wainwright and coworkers). This approach, given by (I) and (II) should be the key future areas of work in the 1+3 covariant and gauge invariant approach to cosmological perturbations towards better understanding the implications of the observed temperature anisotropies.

– F I N –

“To a theoretical physicist, there is no greater joy than to see this curious activity we call calculation – the depositing of ink on paper, followed by throwing away the paper and depositing new ink on more paper – actually tell us something about reality. ” – Alan Guth

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Appendix A

Algebraic relations

A.1 Spherical Harmonics

A.1.1 Basic relations

A Spherical Harmonic (SH) $Y_{\ell,m}(\theta, \phi)$ is related to an Associated Legendre Polynomial (ALP) [172, 147],

$$Y^{\ell m} = C^{\ell m} e^{im\phi} P^{\ell m}(\cos \theta), \quad (\text{A.1})$$

$$= C^{\ell m} (e^{i\phi} \sin \theta)^m \sum_{j=0}^{[(\ell-m)/2]} A^{\ell m j} (\cos \theta)^{\ell-m-2j} \quad \forall m \geq 0. \quad (\text{A.2})$$

Here,

$$C^{\ell m} = (-1)^m \frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}, \quad \text{and} \quad A^{\ell m j} = \frac{(-1)^j (2\ell-2j)!}{2^\ell j! (\ell-j)! (\ell-m-2j)!}, \quad (\text{A.3})$$

along with

$$Y^{\ell m} = (-1)^m Y^{\ell |m|*} \quad \forall m \leq 0. \quad (\text{A.4})$$

Now we can relate the SH, $Y^{\ell m}$, to the direction vector product e^{A_ℓ} ,

$$\mathcal{Y}_{A_\ell}^{\ell m} = \mathcal{Y}_{A_\ell}^{\ell m} e^{A_\ell}(\theta, \phi), \quad (\text{A.5})$$

where following [147] (from making the substitution $e^1 + ie^2 = e^{i\phi} \sin \theta$ and $e^3 = \cos \theta$ into the above relation)

$$\begin{aligned} \mathcal{Y}_{A_\ell}^{\ell m} &= C^{\ell m} \sum_{j=0}^{[(\ell-m)/2]} A^{\ell m j} \prod_{k=0}^m \left(h^1_{(a_k)} + i h^2_{(a_k)} \right) \prod_{p=m+1}^{\ell-2j} h^3_{a_p} \\ &\quad \times \prod_{q=1}^j \left(h^\alpha_{a_{2q-1+t-2j}} h_{\alpha a_{2q+t-2j}} \right). \end{aligned} \quad (\text{A.6})$$

Furthermore, it can then be shown that

$$f = \sum_{\ell} F_{A_{\ell}} e^{A_{\ell}}, \quad \text{and} \quad F_{A_{\ell}} = \sum_{m=-\ell}^{m=+\ell} a^{\ell m} y_{A_{\ell}}^{\ell m}. \quad (\text{A.7})$$

This is not unexpected.

A.1.2 Consequences

Closure

$$\sum_{\ell=0}^{\infty} \sum_{\ell'=-m}^{\ell'=+m} Y_{\ell m}^*(\Omega') Y_{\ell m}(\Omega) = \delta(\Omega - \Omega'). \quad (\text{A.8})$$

Addition

$$\sum_{\ell=-m}^{\ell=+m} Y_{\ell m}(\Omega') Y_{\ell m}(\Omega) = \frac{(2\ell+1)}{4\pi} P_{\ell} = \Delta_{\ell}^{-1} O_{A_{\ell}} O^{A_{\ell}}. \quad (\text{A.9})$$

Orthonormality

$$\int_{4\pi} d\Omega Y_{\ell' m'}(\Omega') Y_{\ell m}(\Omega) = \delta_{\ell \ell'} \delta_{m m'}. \quad (\text{A.10})$$

Matching plane waves to spherical harmonics

$$e^{(ik\chi)} = 4\pi \sum_{\ell} i^{\ell} j_{\ell}(k\chi) Y_{\ell m} Y'_{\ell, -m} = 4\pi \sum_{\ell} i^{\ell} j_{\ell} \Delta_{\ell}^{-1} O^{A_{\ell}} O'_{A_{\ell}}, \quad (\text{A.11})$$

$$e^{(ik\chi)} = \sum_{\ell} i^{\ell} (2\ell+1) j_{\ell} P_{\ell} = \sum_{\ell} i^{\ell} (2\ell+1) j_{\ell} \beta_{\ell}^{-1} O^{A_{\ell}} O'_{A_{\ell}}. \quad (\text{A.12})$$

A.2 Multipole relations

A.2.1 Properties of $e^{A_{\ell}}$

Normalization

The normalization for $e^{A_{\ell}}$ is found from [58], for *odd* and *even* ℓ respectively :

$$\frac{1}{4\pi} \int_{4\pi} e^{A_{2\ell+1}} d\Omega = 0, \quad \text{and} \quad \frac{1}{4\pi} \int_{4\pi} e^{A_{2\ell}} d\Omega = \frac{1}{2\ell+1} h^{(A_{2\ell})}. \quad (\text{A.13})$$

From this (contracting with $h_{(A_{2\ell})}$) it can be shown that

$$h_{(A_{2\ell})} h^{(A_{2\ell})} = (2\ell+1), \quad (\text{A.14})$$

this can also be shown algebraically [32].

Orthogonality

From [58] we also have that

$$\int_{4\pi} e^{A_\ell} e^{B_m} d\Omega = \frac{4\pi}{\ell + m + 1} h^{(A_\ell B_m)}. \quad (\text{A.15})$$

if $\ell+m$ is even, and is zero otherwise (this follows from the above because $e^{A_\ell} e^{B_m} = e^{A_\ell+m}$ on relabeling indices: $b_1 \dots b_m \rightarrow a_{\ell+1} \dots a_{\ell+m}$.)

Addition Theorem

From (1.9) it follows that

$$e^{A_\ell} e'_{A_\ell} = (X)^\ell \Rightarrow \sum_{\ell=0}^{\infty} e^{A_\ell} e'_{A_\ell} = \sum_{\ell} (X)^\ell = \frac{\cos \beta}{1 - \cos \beta}, \quad (\text{A.16})$$

where $X = \cos \beta$. It may be useful to compare these to the relations for standard spherical harmonics, which are given in the appendix A. Note that

$$\int_{4\pi} d\Omega e^{A_n} e'_{A_n} = \int_{4\pi} d\Omega (e^a e'_a)^n = 2\pi \int_{-1}^{+1} dX X^n = \begin{cases} 0 & \forall n \text{ odd} \\ \frac{4\pi}{n+1} & \forall n \text{ even} \end{cases}, \quad (\text{A.17})$$

where the integral is taken over e^a with e'^a fixed.

Orthogonality of O^{A_ℓ}

The orthogonality conditions can be found from

$$\sum_{m,\ell} \int d\Omega (F_{A_\ell} O^{A_\ell}) (F_{B_m} O^{B_m}), \quad (\text{A.18})$$

see [58]. Here F_{A_ℓ} are arbitrary PSTF harmonic components of some $f(e^a, x^i)$. Using (A.15), (1.18), and (1.17) we find

$$\int d\Omega O^{A_\ell} O_{B_m} = \delta_m^{\ell} \Delta_\ell h^{(A_\ell)}_{(B_\ell)} \quad \text{with} \quad \Delta_\ell := \frac{4\pi}{(2\ell+1)} \frac{2^\ell (\ell!)^2}{(2\ell)!}. \quad (\text{A.19})$$

where $h^{(A_\ell)}_{(B_\ell)} = h^{(a_1}_{(b_1} \dots h^{a_\ell)}_{b_\ell)}$. From this it follows that

$$e^{B_n} h^{(A_\ell)}_{(B_\ell)} h_{B_{n-\ell}} = e^{(A_\ell)} (+1)^{n-\ell} = O^{A_\ell}. \quad (\text{A.20})$$

It should also be noticed that from (A.19),

$$h^{(A_\ell)}_{(A_\ell)} = (2\ell+1), \quad (\text{A.21})$$

can be also shown algebraically [32]¹.

Using these relations we obtain the inversion of the harmonic expansion (1.5):

$$\tau(x^i, e^a) = \sum_{\ell=0}^{\infty} \tau_{A_\ell}(x^i) O^{A_\ell} \Leftrightarrow \tau_{A_\ell}(x^i) = \Delta_\ell^{-1} \int_{4\pi} d\Omega O_{A_\ell} \tau(x^i, e^a). \quad (\text{A.22})$$

¹Note the contrast with (A.14).

Addition of O^{A_ℓ}

The addition theorem for O_{A_ℓ} can be found from

$$O^{A_\ell} O'_{A_\ell} = \sum_{k=0}^{[\ell/2]} \sum_{k'=0}^{[\ell/2]} B_{\ell k} B_{\ell k'} h^{(A_{2k} h_{(A_{2k'})} e^{A_\ell})} e'_{A_\ell}. \quad (\text{A.23})$$

The resulting polynomial

$$L_\ell(X) \equiv O^{A_\ell} O'_{A_\ell} = \sum_{m=0}^{[\ell/2]} B_{\ell m} X^{\ell-2m}, \quad (\text{A.24})$$

is the natural polynomial that arises in the PSTF tensor approach.

Double Integrals

First, note that

$$\begin{aligned} O'_{A_\ell} e'_{B_n} \int d\Omega O^{A_\ell} e^{B_n} &= \int d\Omega O'_{A_\ell} O^{A_\ell} e'_{B_n} e^{B_n} \\ &= \beta_\ell \frac{4\pi}{n+\ell+1} \frac{n!(n-\ell+1)!!}{(n-\ell+1)!(n+\ell-1)!!} \end{aligned} \quad (\text{A.25})$$

on integrating a Legendre polynomial, where $n \geq \ell$ and $(n-\ell)$ is even, so we can write $n-\ell=2m$ for m an integer². Consequently, on remembering

$$h^{(A_\ell)}_{(B_n h_{B_{n-\ell}})} O'_{A_\ell} e'^{B_n} = O'_{A_\ell} e'^{A_\ell} = \beta_\ell, \quad (e'^a e'_a)^{(n-\ell)} = +1, \quad (\text{A.26})$$

we find

$$\int d\Omega O^{A_\ell} e^{B_n} = 4\pi \delta_n^{\ell+2m} \frac{n!(n-\ell+1)!!}{(n-\ell+1)!(n+\ell+1)!!} h^{(A_\ell)}_{(B_\ell h_{B_{n-\ell}})}. \quad (\text{A.27})$$

Here m are positive integers. Also we will need

$$\begin{aligned} \int d\Omega_k O^{(k)}_{A_\ell} \int d\Omega_{k'} O^{B_m}_{(k')} &= \int d\Omega_k \int d\Omega_{k'} O^{(k')}_{A_\ell} O^{B_m}_{(k)} = 4\pi \int d\Omega_k O^{(k)}_{A_\ell} O^{B_m}_{(k')} \\ &= 4\pi \delta_\ell^m \Delta_\ell h^{(B_\ell)}_{(A_\ell)} \delta(e_k^a - e_{k'}^a). \end{aligned} \quad (\text{A.28})$$

A.2.2 Legendre tensors and the PSTF tensors

One can immediately make the connection between this formulation and the one usually used in terms of Legendre tensors, and see that the Legendre tensors used in Wilson [193] in the coordinate basis (indicated by late romans) can be related to irreducible representation O^{A_ℓ} in terms of its associated tetrad frame $E_i^a = \{u^a, e_\mu^a\}$. The direction

²This can also be seen from using $F_{A_\ell} h^{(A_\ell B_n)} = (n!/(n-\ell)!(n+\ell-1)!!) F_{A_\ell} h^{A_\ell(B_\ell h_{B_{n-\ell}})}$, the definition of O^{A_ℓ} in terms of e^{A_ℓ} and evaluating $\int d\Omega F_{A_\ell} O^{A_\ell} e^{B_n}$ for F_{A_ℓ} PSTF.

vectors e^a in the triad with components e_μ^a are related to γ^i , the direction cosines used in the Wilson-Silk coordinate basis treatment :

$$P_{(\ell)}^{i_1 i_2 \dots i_\ell}(\gamma^i) = (\beta_\ell)^{-1} e_{a_1}^{i_1} e_{a_2}^{i_2} \dots e_{a_\ell}^{i_\ell} O^{A_\ell}(e^a). \quad (\text{A.29})$$

This connection to the Legendre polynomials can be seen by using the relation between spherical harmonics and the PSTF tensor along with the addition theorem for the PSTF tensors :

$$O^{A_\ell} = y_{\ell m}^{A_\ell} Y^{\ell m}(\Omega), \quad \Leftrightarrow \quad \beta_\ell P_\ell(e^a e'_a) = O^{A_\ell} O'_{A_\ell}. \quad (\text{A.30})$$

Multiply (1.21) by $\beta_{\ell+1}^{-1}$ and use (A.29) and (1.24) to find the recursion relation for the Wilson-Silk [194, 193] Legendre polynomials:

$$P_{(\ell+1)}^{i_1 \dots i_{\ell+1}} = \frac{(2\ell+1)}{(\ell+1)} \gamma^{(i_1} P_{(\ell)}^{i_2 \dots i_{\ell+1})} - \frac{\ell}{(\ell+1)} \gamma^{(i_1 i_2} P_{(\ell-1)}^{i_3 \dots i_{\ell+1})}. \quad (\text{A.31})$$

It is seen that the recursion relations (1.21) for the irreducible representation O^{A_ℓ} can be reduced to that of the Legendre polynomials. This links the CGI-PSTF approach to the usual GI-Legendre tensor approach.

A.3 Mode relations

A.3.1 The curvature modified Helmholtz equation and the mode recursion relation

By successively applying the background 3-space Ricci identity,

$$D_{abA_\ell} Q - D_{baA_\ell} Q = + \sum_{n=1}^{\ell} \frac{K}{a^2} \left(\delta_b^{b_n} h_{a_n a} - \delta_a^{b_n} h_{a_n b} \right) D_{\bar{A}_\ell} Q, \quad (\text{A.32})$$

where $\bar{A}_\ell = a_1 \dots a_{n-1} b_n a_{n+1} \dots a_\ell$, *i.e.*, the sequence of ℓ indices with the n -th one replaced with a contraction, the following useful relations are found:

$$e^{(a_1} O^{A_\ell)} D_{A_{\ell+1}} Q = e^{a_1} O^{A_\ell} D_{A_\ell} Q - \frac{1}{3} \frac{K}{a^2} \frac{\ell^2(\ell-1)}{(2\ell-1)} O^{A_{\ell-1}} D_{A_{\ell-1}} Q, \quad (\text{A.33})$$

$$h^{(a_1 a_2} O^{A_{\ell-1}}) D_{A_{\ell+1}} Q = \left(-k_{\text{phys}}^2 + \frac{1}{3} \frac{K}{a^2} \frac{(\ell+2)}{(\ell-1)} \right) O^{A_{\ell-1}} D_{A_{\ell-1}} Q, \quad (\text{A.34})$$

$$O^{A_\ell} D_{cA_\ell}^c Q = \left(-k_{\text{phys}}^2 + \frac{1}{2} \frac{K}{a^2} \ell(\ell+3) \right) O^{A_\ell} D_{A_\ell} Q, \quad (\text{A.35})$$

$$O^{A_\ell} D_{cA_\ell} Q = O^{A_\ell} D_{A_\ell c} Q + \frac{1}{2} \frac{K}{a^2} \ell(\ell-1) O_c^{A_{\ell-1}} D_{A_{\ell-1}} Q. \quad (\text{A.36})$$

Now consider $O^{A_\ell} D_{cA_\ell}^c Q$. From (A.36) we find

$$O^{A_\ell} D_{cA_\ell}^c Q = O^{A_\ell} D_{A_\ell c}^c Q + \frac{1}{2} \frac{K}{a^2} \ell(\ell-1) O_c^{A_{\ell-1}} D_{A_{\ell-1}}^c Q \quad (\text{A.37})$$

where the first term on the left of the equality above, (A.37), can be reduced to one in terms of $O^{A\ell}D_{A\ell}Q$ using (A.35) while the last term can also be rewritten in terms of $O^{A\ell}D_{A\ell}Q$. Now on dropping the $O^{A\ell}$ and making the identification of $Q_{A\ell} = (-k_{\text{phys}})^{-\ell}D_{(A\ell)}Q$ it is then shown that the $Q_{A\ell}$ satisfy the *curvature-modified Helmholtz equation*

$$D^a D_a Q_{(A\ell)} = (-k_{\text{phys}}^2 + \frac{K}{a^2} \ell(\ell+2)) Q_{(A\ell)}, \quad (\text{A.38})$$

i.e. , the Helmholtz equation with modified wavelength using as before $k_{\text{phys}} = k/a$:

$$-\tilde{k}_\ell^2 = \frac{1}{a^2} (K\ell(\ell+2) - k^2). \quad (\text{A.39})$$

On using (A.38) and taking the PSTF part of the lower indices, $D^a D_{(a} Q_{A\ell)}$, and using the PSTF tensor relation (1.20) we find:

$$D^a D_{(a} Q_{A\ell)} = \frac{(\ell+1)}{(2\ell+1)} \left(-\frac{k^2}{a^2} \right) \left[1 - \frac{K}{k^2} \ell(\ell+2) \right] Q_{(A\ell)}. \quad (\text{A.40})$$

On substituting the first two relations (A.33, A.34) into the recursion relation for the PSTF tensors (1.21), we find

$$e^a D_a [G_\ell[Q]] = +k_{\text{phys}} \left[\frac{\ell^2}{(2\ell+1)(2\ell-1)} \left(1 - \frac{K}{k^2} (\ell^2 - 1) \right) G_{\ell-1}[Q] - G_{\ell+1}[Q] \right], \quad (\text{A.41})$$

A.3.2 Fourier transform conventions

In the flat space case we have followed the following conventions with respect to the mode analysis :

$$f(x^a) = \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3 k_a e^{+ik_a x^a} f(k^a), \quad (\text{A.42})$$

$$f(k^a) = (+1) \int_{-\infty}^{+\infty} d^3 x_a e^{-ik_a x^a} f(x^a). \quad (\text{A.43})$$

We have that the delta function is :

$$\int_{-\infty}^{+\infty} d^3 x_a e^{+i(k_a - k'_a)x^a} = (2\pi)^3 \delta(k^a - k'^a). \quad (\text{A.44})$$

The reader is also reminded that because we are only considering Gaussian random fields, all even moments can be written in terms of the variance (over the ensemble) and all odd moments are zero :

$$\langle \Delta(k^a) \Delta^*(k'^a) \rangle = (2\pi)^3 \delta(k^a - k'^a) P(k^a). \quad (\text{A.45})$$

We will further restrict ourselves (as previously pointed out) to homogeneous and isotropic random perturbations. The reality conditions hold : $\Delta^*(k_a) = \Delta(-k_a)$. We will also, often , jump between a sum convention and an integral one :

$$\sum_{k_a} \rightarrow \int_{-\infty}^{+\infty} d^3 k_a \rightarrow \int_0^\infty k^2 dk \int_{4\pi} d\Omega_k \rightarrow \int_0^{+\infty} k^2 dk \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta, \quad (\text{A.46})$$

these are all equivalent, we will not be considering closed universes hence the loose nature of the notation.

From the convolution theorem we have that :

$$W(x^a) = N(x^a)M(x^a), \quad (\text{A.47})$$

such that

$$W(k^a) = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{d^3 k_a}{(2\pi)^3} (N(k_a - k'_a)M(k'_a) + N(k'_a)M(k_a - k'_a)). \quad (\text{A.48})$$

It becomes convenient when $N(k^a) = N(|k^a|)$, $M(k^a) = M(|k^a|)$ and the wavenumbers are aligned in direction e^a to use :

$$W(k) = 4\pi \int_0^\infty \frac{k^2 dk}{(2\pi)^3} (N(k')M(|k - k'|)). \quad (\text{A.49})$$

A.3.3 Evaluating the mode-mode coupling terms

In the almost-flat-FLRW case with scalar perturbations we have

$$Q_{A_\ell}(x^a) = (-1)^\ell O_{A_\ell} Q(k^a, x_a), \quad (\text{A.50})$$

$$Q(k'^a, x^a)Q(k^a, x_a) = Q(k'^a + k^a, x^a), \quad (\text{A.51})$$

$$(\text{A.52})$$

and

$$Q(k'^a, x^a)Q^*(k^a, x_a) = Q(k'^a - k^a, x_a). \quad (\text{A.53})$$

In this instance we find that for

$$W_{A_\ell B_m}(x^a) = N_{A_\ell}(x^a)M_{B_m}(x^a) \quad (\text{A.54})$$

and

$$W_{A_\ell B_m}(x^a) = 4\pi(-1)^\ell O_{A_\ell} \int_0^\infty W_{\ell+m}(k) \quad (\text{A.55})$$

the mode decomposition is just :

$$W_{A_\ell B_m}(k^*) = (4\pi)O_{A_\ell}O_{B_m} \int_0^\infty \frac{k^2 dk}{(2\pi)^3} N_\ell(k)M_m(|k^* - k|). \quad (\text{A.56})$$

Clearly one should worry about the direction integration - we do not do that here.

A.3.4 Plane waves, spherical waves and mode functions

Plane wave and mode function relations

We consider only flat, $K = 0$, universes at present. Each set of harmonic functions $Q_{(k)}(x^a)$ satisfying (1.47) has associated with it $k_{\text{phys}} = k/a$, the physical wavenumber, a

variation vector field, q^a , and a direction e^a ($e^a e_a = 1$, $e^a u_a = 0$) determined by ³

$$D_a Q = Q q_a, \quad q_a = q e_a, \quad q^2 = q^a q_a \quad (\text{A.57})$$

the first equality defining $q_a(x^i)$ (but not necessarily so as to factor out Q) and the second splitting it into its magnitude and direction. It follows that

$$D^a D_a Q = Q q^2 + Q D^a q_a = Q(q^2 + e^a D_a q + q D^a e_a) \quad (\text{A.58})$$

so that (1.47) becomes

$$q^2 + e^a D_a q + q D^a e_a = -k_{\text{phys}}^2, \quad \iff D^a q_a = -q^2 - k_{\text{phys}}^2. \quad (\text{A.59})$$

Using the $K = 0$ plane wave eigenfunctions with associated direction vector $e_a^{(k)}$:

$$Q(x^i, e_{(k)}^a)|_{flat} = \exp\{-ik_{\text{phys}} e_a^{(k)} x^a\}, \quad (\text{A.60})$$

where $k_{\text{phys}}(k, t) = k/a$, expresses the temperature anisotropy (1.4) in terms of its plane wave spatial Fourier Transform (1.57). In this case

$$q_a = -ik_{\text{phys}} e_a^{(k)}, \quad D_a q_b = 0 = D_a k_{\text{phys}} = D_a e_b^{(k)}, \quad q^2 = -\frac{k^2}{a^2} = -k_{\text{phys}}^2 \quad (\text{A.61})$$

holds in equation (A.57) and (A.58) respectively. We find (from (A.60))

$$D_{(A_\ell)} Q(x^i, e_{(k)}^a)|_{flat} = (-ik_{\text{phys}})^\ell O_{A_\ell}^{(k)} Q(x^i, e_{(k)}^a)|_{flat} \quad (\text{A.62})$$

where the $O_{A_\ell}^{(k)}$ are the PSTF tensors associated with the direction $e_{(k)}^a$ in the tangent spaces on the spatial section. Thus from (1.48)

$$(Q(x^i, e_{(k)}^a)|_{flat})_{(A_\ell)} = (-1)^\ell O_{A_\ell}^{(k)} Q(x^i, e_{(k)}^a)|_{flat} \quad (\text{A.63})$$

and from (1.53) we find (1.60).

Radial expansion and mode function relations

$$D_a Q_\ell = D_a (R_{A_\ell} O^{A_\ell}) = O^{A_\ell} D_a R_{A_\ell} + R_{A_\ell} D_a O^{A_\ell}, \quad (\text{A.64})$$

which implies that

$$D_a D^a Q_\ell = (D^a D_a R_{A_\ell}) O^{A_\ell} + 2(D_a R_{A_\ell} D^a O^{A_\ell}) + R_{A_\ell} (D_a D^a O^{A_\ell}). \quad (\text{A.65})$$

Now we need to work out $(D_a R_{A_\ell})$, $D^a D_a R_{A_\ell}$, $D_a O^{A_\ell}$ and $D_a D^a O^{A_\ell}$, say (a), (b), (c), (d) respectively. Calculating (a) :

$$D_a R_{A_\ell}(\tau) = \frac{\partial R_{A_\ell}}{\partial \tau} D_a \tau = \frac{\partial R_{A_\ell}}{\partial \tau} e_a, \quad (\text{A.66})$$

³The vector e^a defined here is in general different from that associated with the angular harmonic expansion in (1.5). When ambiguity can arise, we explicitly put in the k -dependence : $q_{(k)}^a$, to signify both this dependence and the definition of e^a from (A.57) : thus strictly we should write, for example, $D_a Q_{(k)} = Q_{(k)} q_{(k)}^a = Q_{(k)} q_{(k)}^a e_a^{(k)}$. We will suppress the k when this causes no ambiguity.

hence (b) follows:

$$D^a D_a R_{A_\ell}(r) = \frac{\partial^2 R_{A_\ell}}{\partial r^2} + \frac{2}{r} \frac{\partial R_{A_\ell}}{\partial r}. \quad (\text{A.67})$$

Next, (c) is:

$$D_a O^{A_\ell} = \frac{\ell}{r} p_a^{(a_\ell)} O^{A_{\ell-1}} \Rightarrow e^a D_a O^{A_\ell} = 0 \quad (\text{A.68})$$

which gives (d) :

$$\begin{aligned} D^a D_a O^{A_\ell} &= \frac{\ell}{r} (D^a p_a^{(a_\ell)} O^{A_{\ell-1}}) + \frac{\ell}{r} p_a^{a(a_\ell)} (D_a O^{A_{\ell-1}}) - \frac{\ell}{r^2} D^a(r) p_a^{(a_\ell)} O^{A_{\ell-1}}, \\ &= \frac{-2\ell}{r^2} e^{(a_\ell)} O^{A_{\ell-1}} + \frac{\ell(\ell-1)}{r^2} p^{(a_\ell a_{\ell-1})} O^{A_{\ell-2}} - \frac{\ell}{r^2} e^a p_a^{(a_\ell)} O^{A_{\ell-1}}, \\ &= -\frac{\ell(\ell+1)}{r^2} O^{A_\ell}. \end{aligned} \quad (\text{A.69})$$

Now put these in (A.65) to find,

$$\begin{aligned} D_a D^a Q_\ell &= \left[\frac{\partial^2 R_{A_\ell}}{\partial r^2} + \frac{2}{r} \frac{\partial R_{A_\ell}}{\partial r} \right] O^{A_\ell} + 2 \left[\frac{\partial R_{A_\ell}}{\partial r} e_a \right] \left[\frac{\ell}{r} p_a^{(a_\ell)} O^{A_{\ell-1}} \right] \\ &+ R_{A_\ell} \left[-\frac{\ell(\ell+1)}{r^2} O^{A_\ell} \right] = -k_{\text{phys}}^2 R_{A_\ell} O^{A_\ell}, \end{aligned} \quad (\text{A.70})$$

which simplifies to

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial R_{A_\ell}}{\partial r} \right) O^{A_\ell} + R_{A_\ell} \left[-\frac{\ell(\ell+1)}{r^2} O^{A_\ell} \right] = -k_{\text{phys}}^2 R_{A_\ell} O^{A_\ell}. \quad (\text{A.71})$$

Now we need to find $O_{(X)}^{A_\ell} D_{(A_\ell)} Q$ in terms of R_0 , then relate the resulting Rodrigues formulae of R_ℓ to get R_ℓ in terms of R_0 and hence Q^{A_ℓ} in terms of R_ℓ . In this regard, now in (1.66), $D_a e_b^{(k)} = 0$ (see (A.59)) which implies $D_a O_{A_\ell}^{(k)} = 0$. Thus

$$D_a R_{A_\ell}(r) = \sum_{k^a} \frac{\partial R_\ell(k, r)}{\partial r} e_a O_{A_\ell}^{(k)} = \frac{\partial R_0(k, r)}{\partial r} e_a. \quad (\text{A.72})$$

We can find $Q_{(A_\ell)}$ from (1.48) obtaining

$$\begin{aligned} D_{(A_\ell)} Q &= \sum_{m, k^\nu} D_{(A_\ell)} R_{B_\ell} O_{(X)}^{B_\ell} = \sum_{m, k^\nu} D_{(A_\ell)} R_m(r, k) O_{B_m}^{(k)} O_{(X)}^{(B_m)} = D_{(A_\ell)} R_0(k, r), \\ &= D_{(A_{\ell-1})} \frac{\partial R_0}{\partial r} e_{a_\ell} = D_{(A_{\ell-2})} \left(\frac{\partial^2 R_0}{\partial r^2} e_{a_{\ell-1}} e_{a_\ell} + \frac{1}{r} p_{a_{\ell-1} a_\ell} \frac{\partial R_0}{\partial r} \right), \\ &= D_{(A_{\ell-3})} \left(\frac{\partial^3 R_0}{\partial r^3} e_{a_{\ell-2}} e_{a_{\ell-1}} e_{a_\ell} + \frac{3}{r} \frac{\partial^2 R_0}{\partial r^2} p_{a_{\ell-2} a_{\ell-1}} e_{a_\ell} \right. \\ &\quad \left. - \frac{2}{r} \frac{\partial R_0}{\partial r} p_{a_{\ell-2} a_{\ell-1}} e_{a_\ell} - \frac{1}{r} \frac{\partial R_0}{\partial r} (D_{a_{\ell-2}} r) p_{a_{\ell-1} a_\ell} \right). \end{aligned} \quad (\text{A.73})$$

Now we note that

$$\begin{aligned} O_{(X)}^{A_\ell} D_{(A_\ell)} Q &= O_{(X)}^{A_\ell} D_{(A_\ell)} R_0(k, r) = O_{(X)}^{A_\ell} D_{(A_{\ell-2})} \left(\frac{\partial^2 R_0}{\partial r^2} - \frac{1}{r} \frac{\partial R_0}{\partial r} \right) e_{a_{\ell-1}} e_{a_\ell} \\ &= O_{(X)}^{A_\ell} (-r)^\ell \left(-\frac{1}{r} \frac{\partial}{\partial r} \right)^\ell R_0 e_{(A_\ell)} \end{aligned} \quad (\text{A.74})$$

which follows from $p_{ab} e'^{(a} e'^{b)} = h_{ab} O'^{ab} - e_a e_b O'^{ab} = -O_{ab} O'^{ab}$ and that

$$R_\ell = \frac{r^\ell}{k_{\text{phys}}^\ell} \left(-\frac{1}{r} \frac{\partial}{\partial r} \right)^\ell R_0. \quad (\text{A.75})$$

Used in (A.74) this gives that

$$O_{(X)}^{A_\ell} D_{(A_\ell)} Q = (-k_{\text{phys}})^\ell O_{A_\ell}^{(k)} R_\ell. \quad (\text{A.76})$$

A.3.5 Mode mean square relations

Flat relations

$$\begin{aligned} \tau_{A_\ell} &= (+i)^\ell \int \frac{k^2 dk}{(2\pi)^3} \int d\Omega_k \tau_\ell(k, t) O_{A_\ell}^{(k)} \sum_{n=0}^{\infty} (+i)^n j_n(\lambda r) O_{(k)}^{B_n} O_{B_n}^{(X)} \beta_n^{-1} (2n+1), \\ &= \frac{(-1)^\ell}{2\pi^2} O_{A_\ell}^{(X)} \int k^2 dk \tau_\ell(k, t) j_\ell(k\chi). \end{aligned} \quad (\text{A.77})$$

Equivalently from

$$\tau(x^i, e^a) = \sum_m \tau_m Q_m, \quad Q_m = \sum_{k^a} R_{C_m D_m} O_{(k)}^{D_m} O_{(X)}^{C_m}, \quad \text{and} \quad R_{C_m D_m} = R_\ell(k, r) h_{C_m D_m}. \quad (\text{A.78})$$

Invert the multipole expansion

$$\tau_{A_\ell} = \Delta_\ell^{-1} \int d\Omega_k O_{A_\ell}^{(k)} \left\{ \sum_{m, k^a} \tau_m R_{C_m B_m} O_{(k)}^{B_m} O_{(X)}^{C_m} \right\}, \quad (\text{A.79})$$

to find, on using (A.28), that

$$\tau_{A_\ell} = \frac{1}{2\pi^2} \Delta_\ell^{-1} \sum_m \int k'^2 dk' \tau_m(k', t) R_{C_m B_m} O_{(X)}^{C_m} \left[\delta_\ell^m \Delta_\ell h_{(A_\ell)}^{(B_\ell)} \right], \quad (\text{A.80})$$

and hence that

$$\tau_{A_\ell} = \frac{1}{2\pi^2} \int k'^2 dk' \tau_\ell(k', t) R_{A_\ell C_\ell} O_{(X)}^{C_\ell}. \quad (\text{A.81})$$

Using (A.78) this becomes

$$\tau_{A_\ell} = \frac{1}{2\pi^2} O_{A_\ell}^{(X)} \int k'^2 dk' \tau_\ell(k', t) R_\ell(k', \chi), \quad (\text{A.82})$$

Constant curvature relations

We now have that

$$\int d\Omega_{(x)} O_{(x)}^{A_\ell} G_\ell[Q] = \delta_m^\ell \Delta_\ell Q^{A_\ell}, \quad (\text{A.83})$$

$$\int d\Omega_{(x)} G_\ell[Q] G_m[Q] = \delta_\ell^m Q^{A_\ell} Q_{A_\ell} \Delta_\ell, \quad (\text{A.84})$$

$$\begin{aligned} \int d\Omega_{(k)} G_\ell[Q] G'_m[Q] &= \int d\Omega_{(k)} O_{(x)}^{A_\ell} R_\ell O_{A_\ell}^{(k)} O_{(x)}^{B_m} R_m O_{B_m}^{(k)} = \delta_m^\ell \Delta_\ell R_\ell^2 O_{A_\ell}^{(x)} O'^{A_\ell}_{(x)} \\ &\Rightarrow \int d\Omega_{(k)} G_\ell G_m = \delta_m^\ell \Delta_\ell^2 \frac{(2\ell+1)}{4\pi} R_\ell^2. \end{aligned} \quad (\text{A.85})$$

Furthermore, we have from the recursion relations (1.56)

$$e_{(x)}^a D_a[G_\ell] = +k_{\text{phys}} [\alpha_\ell^2 G_{\ell-1} - G_{\ell+1}] \quad (\text{A.86})$$

where

$$(\alpha_\ell)^2 = \frac{\ell^2}{(2\ell+1)(2\ell-1)} \left(1 - \frac{K}{k^2} (\ell^2 - 1) \right) \quad (\text{A.87})$$

and using [194]⁴,

$$\int d\Omega_k \int dx^\nu e_{(x)}^a D_a[G_\ell G_{\ell-1}] = 0 \quad (\text{A.88})$$

we find

$$\int d\Omega^{(k)} \langle |G_n|^2 \rangle = (\alpha_n)^2 \int d\Omega_{(k)} \langle |G_{n-1}|^2 \rangle. \quad (\text{A.89})$$

Here we have defined the mean square to pick out the power spectrum which is a function only of the absolute value of the wavelength for a Gaussian distribution (there is no directional dependence, the modulus is only dependent on the wave number).

A.4 1+3 Orthonormal Tetrad relations

We use an orthonormal tetrad approach (cf. [58]). Consider an orthonormal tetrad basis e_a with components $E_a^i(x^j)$ relative to a coordinate basis; here indices a,b,c... , that is early letters, are used for the tetrad basis, while late letters i,j,k,... are used for the coordinate basis. The differential operators $e_a \equiv \partial_a = E_a^i \partial / \partial x^i$ are defined by the inverse basis components⁵

$$E_a^i E_b^j = \delta_i^j \Leftrightarrow E_a^i E_b^j = \delta_b^a. \quad (\text{A.90})$$

The tetrad components of a vector X^i are $X^a = E_a^i X^i$, and similarly for any tensor. Tetrad indices are raised and lowered using the tetrad components of the metric

$$g_{ab} = g_{ij} E_a^i E_b^j = \text{diag}(-1, +1, +1, +1), \quad g^{ab} g_{bc} = \delta_c^a, \quad (\text{A.91})$$

⁴The idea is to use this to fix the normalization of $\int d\Omega_{(k)} G_\ell[Q] = D_\ell \int d\Omega_{(k)} Q(x^i, e_{(k)}^a) O_{(k)}^{A_\ell} O_{A_\ell}^{(x)}$.

⁵In the context of RKT we prefer to use the the operators ∂_a and $\partial / \partial p^a$ (this is purely a notational choice) in T(M)

the form of these components being the necessary and sufficient condition that the tetrad basis vectors used are orthonormal, which we will always assume.

For an observer with 4-velocity u^a , there is a preferred family of orthonormal tetrads associated with u^a *i.e.* a frame for which the timelike tetrad basis E_0 is parallel to the velocity u^a . In such a tetrad basis

$$u^a = \delta_0^a \tag{A.92}$$

$$h_{ab} = \text{diag}(0, +1, +1, +1) \tag{A.93}$$

All our work is based on such a tetrad, which leads to a preferred set of associated rotation coefficients. The issue for the present is that we have a preferred family of local orthonormal frames at each point (usually matter flow aligned), and carry out our algebraic analysis of observational quantities relative to that orthonormal frame.

Appendix B

The 1+3 Eulerian threading relations

The exact relative velocity equations are now given from [120, 113]. From a change in 4-velocity :

$$\tilde{u}_a = \gamma(u_a + v_a) \quad \text{where} \quad \gamma = (1 - v^2)^{-1/2}, \quad v_a u^a = 0. \quad (\text{B.1})$$

Change in fundamental algebraic tensors:

$$\tilde{h}_{ab} = h_{ab} + \gamma^2 \left[v^2 u_a u_b + 2u_{(a} v_{b)} + v_a v_b \right], \quad (\text{B.2})$$

$$\tilde{\epsilon}_{abc} = \gamma \epsilon_{abc} + \gamma \left\{ 2u_{[a} \epsilon_{b]cd} + u_c \epsilon_{abd} \right\} v^d. \quad (\text{B.3})$$

Transformed kinematic quantities are defined by

$$\nabla_b \tilde{u}_a = \frac{1}{3} \tilde{\Theta} \tilde{h}_{ab} + \tilde{\sigma}_{ab} + \tilde{\epsilon}_{abc} \tilde{\omega}^c - \tilde{A}_a \tilde{u}_b,$$

which implies, using $\nabla_a \gamma = \gamma^3 v^b \nabla_a v_b$, and Eq. (2.7), the following kinematic transformations [114]:

$$\tilde{\Theta} = \gamma \Theta + \gamma (\text{div } v + A^a v_a) + \gamma^3 W, \quad (\text{B.4})$$

$$\begin{aligned} \tilde{A}_a &= \gamma^2 A_a + \gamma^2 \left\{ \dot{v}_{(a)} + \frac{1}{3} \Theta v_a + \sigma_{ab} v^b - [\omega, v]_a + \left(\frac{1}{3} \Theta v^2 + A^b v_b + \sigma_{bc} v^b v^c \right) u_a \right. \\ &\quad \left. + \frac{1}{3} (\text{div } v) v_a + \frac{1}{2} [v, \text{curl } v]_a + v^b D_{(b} v_{a)} \right\} + \gamma^4 W (u_a + v_a), \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} \tilde{\omega}_a &= \gamma^2 \left\{ \left(1 - \frac{1}{2} v^2 \right) \omega_a - \frac{1}{2} \text{curl } v_a + \frac{1}{2} v_b \left(2\omega^b - \text{curl } v^b \right) u_a + \frac{1}{2} v_b \omega^b v_a \right. \\ &\quad \left. + \frac{1}{2} [A, v]_a + \frac{1}{2} [\dot{v}, v]_a + \frac{1}{2} \epsilon_{abc} \sigma^b{}_d v^c v^d \right\}, \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} \tilde{\sigma}_{ab} &= \gamma \sigma_{ab} + \gamma (1 + \gamma^2) u_{(a} \sigma_{b)c} v^c + \gamma^2 A_{(a} [v_{b)} + v^2 u_{b)}] \\ &\quad + \gamma D_{(a} v_{b)} - \frac{1}{3} h_{ab} \left[A_c v^c + \gamma^2 (W - \dot{v}_c v^c) \right] \\ &\quad + \gamma^3 u_a u_b \left[\sigma_{cd} v^c v^d + \frac{2}{3} v^2 A_c v^c - v^c v^d D_{(c} v_{d)} + \left(\gamma^4 - \frac{1}{3} v^2 \gamma^2 - 1 \right) W \right] \end{aligned}$$

$$\begin{aligned}
& + \gamma^3 u_{(a} v_b) \left[A_c v^c + \sigma_{cd} v^c v^d - \dot{v}_c v^c + 2\gamma^2 \left(\gamma^2 - \frac{1}{3} \right) W \right] \\
& + \frac{1}{3} \gamma^3 v_a v_b \left[\text{div } v - A_c v^c + \gamma^2 \left(3\gamma^2 - 1 \right) W \right] + \gamma^3 v_{(a} \dot{v}_{b)} + v^2 \gamma^3 u_{(a} \dot{v}_{b)} \\
& + \gamma^3 v_{(a} \sigma_{b)c} v^c - \gamma^3 [\omega, v]_{(a} \left\{ v_b \right\} + v^2 u_b \left. \right\} + 2\gamma^3 v^c D_{(c} v_{a)} \left\{ v_b \right\} + u_b \left. \right\} , \quad (\text{B.7})
\end{aligned}$$

where

$$W \equiv \dot{v}_c v^c + \frac{1}{3} v^2 \text{div } v + v^c v^d D_{(c} v_{d)} .$$

Transformed dynamic quantities are [114]:

$$\tilde{\rho} = \rho + \gamma^2 \left[v^2 (\rho + p) - 2q_a v^a + \pi_{ab} v^a v^b \right] , \quad (\text{B.8})$$

$$\tilde{p} = p + \frac{1}{3} \gamma^2 \left[v^2 (\rho + p) - 2q_a v^a + \pi_{ab} v^a v^b \right] , \quad (\text{B.9})$$

$$\begin{aligned}
\tilde{q}_a &= \gamma q_a - \gamma \pi_{ab} v^b - \gamma^3 \left[(\rho + p) - 2q_b v^b + \pi_{bc} v^b v^c \right] v_a \\
&\quad - \gamma^3 \left[v^2 (\rho + p) - (1 + v^2) q_b v^b + \pi_{bc} v^b v^c \right] u_a , \quad (\text{B.10})
\end{aligned}$$

$$\begin{aligned}
\tilde{\pi}_{ab} &= \pi_{ab} + 2\gamma^2 v^c \pi_{c(a} \left\{ u_b \right\} + v_b \left. \right\} - 2v^2 \gamma^2 q_{(a} u_b) - 2\gamma^2 q_{(a} v_b) \\
&\quad - \frac{1}{3} \gamma^2 \left[v^2 (\rho + p) + \pi_{cd} v^c v^d \right] h_{ab} \\
&\quad + \frac{1}{3} \gamma^4 \left[2v^4 (\rho + p) - 4v^2 q_c v^c + (3 - v^2) \pi_{cd} v^c v^d \right] u_a u_b \\
&\quad + \frac{2}{3} \gamma^4 \left[2v^2 (\rho + p) - (1 + 3v^2) q_c v^c + 2\pi_{cd} v^c v^d \right] u_{(a} v_b) \\
&\quad + \frac{1}{3} \gamma^4 \left[(3 - v^2) (\rho + p) - 4q_c v^c + 2\pi_{cd} v^c v^d \right] v_a v_b . \quad (\text{B.11})
\end{aligned}$$

For the gravito-electric/magnetic field: using [113]

$$\begin{aligned}
C_{ab}{}^{cd} &= 4 \left\{ u_{[a} u^{[c} + h_{[a}{}^{[c} \right\} E_{b]}{}^{d]} + 2\varepsilon_{abe} u^{[c} H^{d]e} + 2u_{[a} H_{b]e} \varepsilon^{cde} \\
&= 4 \left\{ \tilde{u}_{[a} \tilde{u}^{[c} + \tilde{h}_{[a}{}^{[c} \right\} \tilde{E}_{b]}{}^{d]} + 2\tilde{\varepsilon}_{abe} \tilde{u}^{[c} \tilde{H}^{d]e} + 2\tilde{u}_{[a} \tilde{H}_{b]e} \tilde{\varepsilon}^{cde} ,
\end{aligned}$$

we find the transformation [114]:

$$\begin{aligned}
\tilde{E}_{ab} &= \gamma^2 \left\{ (1 + v^2) E_{ab} + v^c \left[2\varepsilon_{cd(a} H_{b)}{}^d + 2E_{c(a} u_b) \right. \right. \\
&\quad \left. \left. + (u_a u_b + h_{ab}) E_{cd} v^d - 2E_{c(a} v_b) + 2u_{(a} \varepsilon_{b)cd} H^{de} v_e \right] \right\} , \quad (\text{B.12})
\end{aligned}$$

$$\begin{aligned}
\tilde{H}_{ab} &= \gamma^2 \left\{ (1 + v^2) H_{ab} + v^c \left[-2\varepsilon_{cd(a} E_{b)}{}^d + 2H_{c(a} u_b) \right. \right. \\
&\quad \left. \left. + (u_a u_b + h_{ab}) H_{cd} v^d - 2H_{c(a} v_b) - 2u_{(a} \varepsilon_{b)cd} E^{de} v_e \right] \right\} . \quad (\text{B.13})
\end{aligned}$$

This may be compared with the electromagnetic transformation

$$\begin{aligned}
\tilde{E}_a &= \gamma \left\{ E_a + [v, H]_a + v^b E_b u_a \right\} , \\
\tilde{H}_a &= \gamma \left\{ H_a - [v, E]_a + v^b H_b u_a \right\} ,
\end{aligned}$$

where

$$F_{ab}^H = 2u_{[a}E_{b]} + \varepsilon_{abc}H^c = 2\tilde{u}_{[a}\tilde{E}_{b]} + \tilde{\varepsilon}_{abc}\tilde{H}^c.$$

All the transformations above are given explicitly in terms of irreducible quantities (i.e. irreducible in the original u^a -frame).

The second order (or n -th order – in the relative velocities) theory can be easily constructed using that :

$$\gamma \approx_2 \left(1 + \frac{1}{2}v^2\right), \quad \gamma^2 \approx_2 (1 - v^2), \quad \gamma^3 \approx_2 \left(1 - \frac{1}{2}v^2\right), \quad \gamma^4 \approx_2 (1 - 2v^2). \quad (\text{B.14})$$

These, when used in the exact relative velocity equations, will allow us to recover a class of second order frames that can be used to do second order covariant and gauge invariant perturbation theory. Once such extension of interest would be carrying out the nonlinear extension in this manner along with the condition that $D_a^3 R \approx 0$ (to the order of the extension). This would recover higher order equations that would could well model nonlinear effects on small scales when the curvature perturbations are small – as in structure formation models.

Appendix C

Some observational CMB issues

C.1 Key observational COBE-DMR issues

There are three pivotal issue pertaining to the COBE-DMR experiment. (as in [167]).

First, that the COBE DMR experiment does not directly measure the dipole, quadrupole, octopole moments of the temperature anisotropy, but rather the two-point correlation function of the temperature anisotropy [160, 140] (see chapter 1)

$$C(\alpha) = \langle S(\mathbf{e}_1)S(\mathbf{e}_2) \rangle = \sum_{\ell=1}^{\infty} \Delta T_{\ell}^2 W(\ell)^2 P_{\ell}[\cos \alpha], \quad (\text{C.1})$$

where α is the angle between the two points, \mathbf{e}_1 and \mathbf{e}_2 are unit vectors denoting the two different directions, so that $\cos \alpha = \mathbf{e}_1 \cdot \mathbf{e}_2$, and the averaging brackets signify the average over all pairs of points on the sky with separation angle α . Furthermore, $S(\mathbf{e}_1) \equiv \delta T(\mathbf{e}_1)$, the temperature anisotropy in a given direction \mathbf{e}_1 on the plane of the sky. The ΔT_{ℓ}^2 (with $\ell \equiv L$) are the squares of the rotationally invariant rms multipole moments (thus, $\Delta T_2^2 = Q_{rms}^2$ - see [13])

$$\Delta T_{\ell}^2 = \frac{1}{4\pi} \sum_m |a_{\ell m}|^2, \quad (\text{C.2})$$

where the $a_{\ell m}$ are the coefficients of the expansion of the temperature anisotropy in spherical harmonics, i. e.

$$S(\mathbf{e}_1) \equiv S(\theta, \phi) = \sum_{\ell, m} a_{\ell m} Y_{\ell m}(\theta, \phi), \quad (\text{C.3})$$

θ and ϕ being, of course, the angular coordinates of the direction at which the temperature anisotropy is being measured. In equation (C.1), finally, $P_{\ell}[\cos \alpha]$ is just the Legendre polynomial of degree ℓ given as a function of $\cos \alpha$, and $W(\ell)$ is the window function, which describes the smoothing properties of the instruments beam. For detectors measuring large-angle CMB anisotropies it essentially weights the multipole moments in such a way that the higher multipoles are smoothed over - that is, the instrument is insensitive to anisotropies on angular scales less than a certain ℓ -pole, and $W(\ell)$ describes that insensitivity and resulting transfer of power in the measurement from higher multipoles to lower ones. Thus, when the actual quadrupole or octopole is

determined from the data, ΔT_ℓ must be de-convolved from $W(\ell)$. When the rms dipole, quadrupole, octopole results are given by the COBE researchers, this de-convolution has already been performed. This is one of the procedures which must be effected to give us the real rms multipole moments.

Second, a major observational and data-reduction problem is that in the COBE measurements there is a great deal of contamination by experimental systematic errors, including Galactic emission [13, 160]. The lower multipole moments are the most susceptible to these distortions. Furthermore, when the data is processed, the entire region containing the Galaxy is removed from the data set. This “Galactic cut” destroys the orthogonality of the spherical harmonics and leads to further aliasing of higher order multipole power onto the lower multipoles (dipole, quadrupole, octopole, etc.). Corrections for this are estimated on the basis of Monte Carlo simulations [13] and included in the published values for the rms multipoles. There are a number of other complex issues which it has been necessary to resolve in arriving at these values – see [13, 160, 196].

Finally, there is the theoretical-observational issue of “cosmic variance”. Cosmic variance does not affect our concerns with the COBE-Copernican result. It is important to realize why it does not. It may explain why the multipoles we measure have the values they have relative to theoretical models of the perturbation spectrum, but it does not lead to observational errors, which would have to be corrected for. It *does* affect the comparison of the measured multipole power spectrum with the theoretical power spectrum predicted from, say, inflationary models [2]. If the primordial perturbation spectrum originated due to fluctuations in the inflaton field then the values of the temperature anisotropy multipoles they induce will be random variables with a certain distribution, probably Gaussian with zero mean, and thus with a certain variance. Our observable universe is only one realization of that ensemble of universes represented by the probability distribution. Therefore, the value for each multipole we obtain from our observations will give us just one point of the distribution, which, in general, will not reflect the ensemble averaged value. It will deviate from it by a certain amount, which can be theoretically estimated by the variance [103]. This variance goes as $2/(2\ell + 1)$ and so will be more significant for the lower multipoles [2, 160, 93].

C.2 Some other CMB Experiments

Although the thesis has emphasized the theoretical aspects of the high- ℓ temperature anisotropies and the COBE-DMR data there are variety of complete and ongoing CMB experiments. The more significant ones can be found at the following www sites (a more complete table summarizing the anisotropy data can be found in [8] using the compilations of Tegmark (<http://www.sns.ias.edu/max/cmb/experiments.ps>)). Some CMB experiments :

ARGO: <http://oberon.roma1.infn.it/argo.htm>
 BAM: <http://cmbr.physics.ubc.ca/experimental.html>
 CAT: <http://www.mrao.cam.ac.uk/telescopes/cat/index.html>
 COBE: http://www.gsfc.nasa.gov/astro/cobe/cobe_home.html
 FIRS: <http://pupgg.princeton.edu/cmb/rs.html>
 (upcoming) MAP: <http://map.gsfc.nasa.gov>
 MAT: <http://imogen.princeton.edu/page/matdir/www/index.html>
 MSAM: <http://topweb.gsfc.nasa.gov/>
 OVRO: http://www.cco.caltech.edu/emleitch/ovro/ovro_cmb.html
 (upcoming) Planck: <http://astro.estec.esa.nl/Planck>
 PYTHON: <http://cmbr.phys.cmu.edu/pyth.html>
 QMAP: <http://pupgg.princeton.edu/cmb/qmap/qmap.html>
 Saskatoon: http://pupgg.princeton.edu/cmb/skintro/sask_intro.html
 SPort: http://tonno.tesre.bo.cnr.it/stefano/sp_draft.html
 Tenerife: <http://clarin.ll.iac.es/>

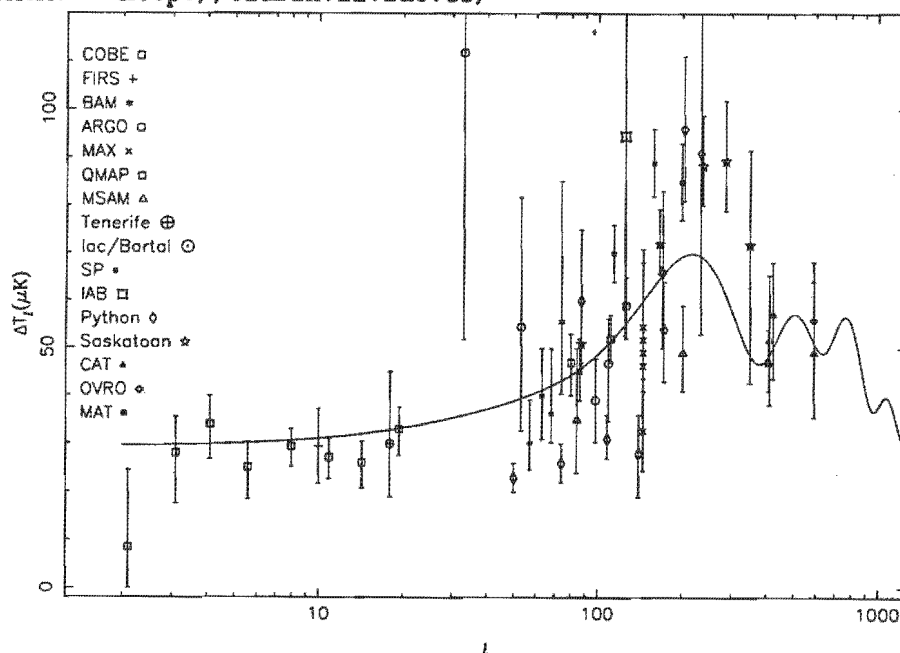


Figure C.1: Anisotropy detections in the CBR with the error bars showing the 95% confidence level. The solid line corresponds to predicted CMB anisotropy angular power spectrum for a flat standard CDM linear-FRW model.

Appendix D

Almost-FL relations

Some useful covariant linearized differential identities are given below [122]:

$$\text{curl } D_a \psi = -2\dot{\psi} \omega_a, \quad (\text{D.1})$$

$$D^2 (D_a \psi) \approx D_a (D^2 \psi) + \frac{2}{3} (\rho - 3H^2) D_a \psi + 2\dot{\psi} \text{curl } \omega_a, \quad (\text{D.2})$$

$$(D_a \psi)' \approx D_a \dot{\psi} - H D_a \psi + \dot{\psi} A_a, \quad (\text{D.3})$$

$$({}_a D_a J^{Bm}{}_{A\ell})' \approx {}_a D_a \dot{J}^{Bm}{}_{A\ell}, \quad (\text{D.4})$$

$$(D^2 \psi)' \approx D^2 \dot{\psi} - 2H D^2 \psi + \dot{\psi} D^a A_a, \quad (\text{D.5})$$

$$D_{[a} D_{b]} V_c \approx \frac{1}{3} (3H^2 - \rho) V_{[a} h_{b]c}, \quad (\text{D.6})$$

$$D_{[a} D_{b]} S^{cd} \approx \frac{2}{3} (3H^2 - \rho) S_{[a} {}^{(c} h_{b]}{}^{d)}, \quad (\text{D.7})$$

$$D^a \text{curl } V_a \approx 0, \quad (\text{D.8})$$

$$D^b \text{curl } S_{ab} \approx \frac{1}{2} \text{curl} (D^b S_{ab}), \quad (\text{D.9})$$

$$\text{curl curl } V_a \approx D_a (D^b V_b) - D^2 V_a + \frac{2}{3} (\rho - 3H^2) V_a, \quad (\text{D.10})$$

$$\text{curl curl } S_{ab} \approx \frac{3}{2} D_{(a} D^c S_{b)c} - D^2 S_{ab} + (\rho - 3H^2) S_{ab}, \quad (\text{D.11})$$

$$D^a D^b D_{(a} V_{b)} \approx \frac{2}{3} D^2 \text{div } V + (\rho_0 - 3H^2) \text{div } V, \quad (\text{D.12})$$

where the vectors and tensors are $O(\epsilon)$ and $S_{ab} = S_{(ab)}$.

D.1 Integrated Boltzmann Equation (IBE) relations

Here we repeat some useful results from [120]. The integrated Boltzmann equation is :

$$\int_0^\infty L(f) E^2 dE = \int_0^\infty E^2 dE \left[p^a \partial_a f - \Gamma^a{}_{bc} p^b p^c \frac{\partial f}{\partial p^a} \right] = \int_0^\infty C[f] E^2 dE. \quad (\text{D.13})$$

Here f is the single particle distribution function, $C[f]$ the scattering correction. The Liouville operator for the photons, $L(f) = \frac{df}{dv}$ and $d/dv = p^a \nabla_a$ is the null derivative. Hence we find

$$L(f(E, x^i, e^a)) = \frac{df}{dv} = p^a \nabla_a f + \frac{dE}{dv} \frac{\partial f}{\partial E}, \quad (\text{D.14})$$

as in [120], so that given the identity (see [120])

$$\frac{dE}{dv} = -E^2 \left[\frac{1}{3} \Theta + A_a e^a + \sigma_{ab} e^a e^b \right], \text{ and } p^a = E(u^a + e^a), \quad (\text{D.15})$$

we have covariant derivation of the almost-FL IBE. The IBE is covariant on M after we have integrated out the energy dependence for the photons.

In order to consider the effect of electromagnetic fields in the almost-FLRW universes a test-field procedure is more appropriate (as opposed to the usual weak-field approach which puts the electromagnetic fields into the background (see for example [174]) – which leaves one wondering about the gauge invariance and consistency). That is one would need to introduce another dimensionless smallness parameter, say η_H (where E_a and H_a are now of this order):

$$\mathcal{O}(E_a H^a) = \mathcal{O}(\eta_H) = \mathcal{O}(E^2) = \mathcal{O}(H^2), \text{ and } F_{ab} = \mathcal{O}(\eta_H). \quad (\text{D.16})$$

This allows one to compare the electromagnetic effects to the gravitational ones (once the appropriate units have been chosen). An additional correction to the scattering term in the Einstein-Boltzmann equation is required :

$$C_{em}[f](x^i, e^a) \approx \frac{e}{m_e} \int F_H^{ab} p_a \frac{\partial f}{\partial p^b} E^2 dE. \quad (\text{D.17})$$

We then choose to set $\eta_H^2 \approx \epsilon$: to construct an almost-FLRW linearization scheme dropping everything of the order ϵ^2 , $\epsilon \eta_H^2$, η_H^3 , ϵv , v^2 and $v \eta_H^2$. The advantage of this scheme over the usual weak field schemes is that it is gauge invariant to all orders and easily extended to allow for nonlinear gravito-magnetic couplings. In addition it is important to differentiate between cosmological magnetic fields and those due to local physics, associated with photons [174].

D.2 Scattering strength expansion

The almost-FL IBE for Thompson scattering is

$$\mathcal{B} + \dot{\tau} + e^a D_a \tau \simeq t_c^{-1} [v^a e_a - \tau]. \quad (\text{D.18})$$

This enables us to find

$$\tau(x, e) = v^a e_a - t_c [\mathcal{B} + \dot{\tau} + e^a D_a \tau]. \quad (\text{D.19})$$

We now systematically approximate (D.19) in terms of the smallness parameter t_c :

$$\tau_{(n)} \approx v^a e_a - t_c \left[\mathcal{B} + \dot{\tau}_{(n-1)} + e^a D_a \tau_{(n-1)} \right]. \quad (\text{D.20})$$

Up to second order the following anisotropies are recovered:

$$\tau_{(0)}(x, e) \approx v^a e_a, \quad (\text{D.21})$$

$$\tau_{(1)}(x, e) \approx v^a e_a - t_c \left[\mathcal{B} + e^a \dot{v}_a + e^a e^b D_a v_b \right], \quad (\text{D.22})$$

$$\begin{aligned} \tau_{(2)}(x, e) \approx & v^a e_a - t_c \left[\mathcal{B} + e^a \dot{v}_a + e^a e^b D_a v_b \right] + t_c^2 \left[\dot{\mathcal{B}} + e^a \ddot{v}_a \right. \\ & \left. + e^a e^b (D_a v_b) + e^c D_c \mathcal{B} + e^c e^d D_c \dot{v}_b + e^c e^a e^b D_c D_a v_b \right]. \end{aligned} \quad (\text{D.23})$$

In order to carry out the solid angle integration over the sky (5.125), the following results will be useful. From the normalization of the direction vectors, e^a , along with the recursive definition of O^{A_i} , it can be demonstrated that

$$e^a e^b D_a v_b = O^{ab} D_a v_b + \frac{1}{3} h^{ab} D_a v_b = O^{ab} D_a v_b + \frac{1}{3} D_a v^a, \quad (\text{D.24})$$

$$e^a e^b D_a \dot{v}_b = O^{ab} D_a \dot{v}_b + \frac{1}{3} D_a \dot{v}^a, \quad (\text{D.25})$$

$$e^a e^b e^c D_a D_b v_c = O^{abc} D_a D_b v_c - \frac{1}{5} \left(O^a D_{(a} D_b) v^b + O^b D_a D^a v_b \right). \quad (\text{D.26})$$

Using the orthogonality conditions we find, first, from (5.76):

$$\int_{4\pi} d\Omega O^{A_i} \mathcal{B} \simeq \delta_0^\ell \left[-\frac{4\pi}{3} D_a \tau^a \right] + \delta_1^\ell \left[\frac{4\pi}{3} \left(\frac{1}{4} D^{a_1} \ln \rho_R + A^{a_1} \right) \right] + \delta_2^\ell \left[\frac{8\pi}{15} \sigma^{a_1 a_2} \right], \quad (\text{D.27})$$

and second,

$$\int_{4\pi} e^a D_a \mathcal{B} d\Omega = +\frac{4\pi}{3} \left(\frac{1}{4} D^a D_a \ln \rho_R + D_a A^a \right). \quad (\text{D.28})$$

The integration over the solid angle can now be carried out finding the equation for the gradient of the radiation flux.

D.3 Integral solutions

Beginning with an integral ansatz of the form (here $\eta^* = \eta - \eta'$):

$$\tau_\ell^P(\eta) = \int_0^\eta d\eta' \left[C_0(\eta') \tau_\ell^{(0)}(\eta^*) + C_1(\eta') \frac{\partial}{\partial \eta'} \tau_\ell^{(0)}(\eta^*) + C_2(\eta') \frac{\partial^2}{\partial \eta'^2} \tau_\ell^{(0)}(\eta^*) \right], \quad (\text{D.29})$$

where $\eta^* = \eta - \eta'$. Using the Leibnitz rule for differentiation of integrals we obtain:

$$\begin{aligned} \frac{\partial \tau_\ell^P}{\partial \eta}(\eta) = & \int_0^\eta d\eta' \frac{\partial}{\partial \eta} \left[C_0(\eta') \tau_\ell^{(0)}(\eta^*) + C_1(\eta') \tau_\ell^{(0)'}(\eta^*) + C_2(\eta') \tau_\ell^{(0)''}(\eta^*) \right] \\ & + \left[C_0(\eta) \tau_\ell^{(0)}(0) + C_1(\eta) \tau_\ell^{(0)'}(0) + C_2(\eta) \tau_\ell^{(0)''}(0) \right]. \end{aligned} \quad (\text{D.30})$$

If we hold η' constant in the partial derivatives, it follows from (5.94) and (D.29) that

$$k \left[\frac{(\ell+1)^2}{(2\ell+1)(2\ell+3)} \tau_{\ell+1}^P(\eta^*) - \tau_{\ell-1}^P(\eta^*) \right] = -\frac{\partial}{\partial \eta} \tau_\ell^P(\eta^*). \quad (\text{D.31})$$

This then gives us

$$\begin{aligned} & k \left[\frac{(\ell+1)^2}{(2\ell+1)(2\ell+3)} \tau_{\ell+1}^P(\eta^*) - \tau_{\ell-1}^P(\eta^*) \right] \\ &= - \int_0^\eta d\eta' \left[C_0(\eta') \frac{\partial}{\partial \eta} \tau_\ell^{(0)}(\eta^*) + C_1(\eta') \frac{\partial}{\partial \eta} \tau_\ell^{(0)'}(\eta^*) + C_2(\eta') \frac{\partial}{\partial \eta} \tau_\ell^{(0)''}(\eta^*) \right] \end{aligned} \quad (\text{D.32})$$

Putting (D.29), (D.30) and (D.32) together, we find :

$$\tau_\ell'^P + k \left[\frac{(\ell+1)^2}{(2\ell+1)(2\ell+3)} \tau_{\ell+1}^P - \tau_{\ell-1}^P \right] = C_0(\eta) \tau_\ell^{(0)}(0) + C_1(\eta) \tau_\ell^{(0)'}(0) + C_2(\eta) \tau_\ell^{(0)''}(0) \quad (\text{D.33})$$

D.4 Linking a number of different expansions

The gauge invariant and covariant mode expansion (5.7) and (5.8) [66, 28] in the almost-flat-FL case ($K = 0$) give the mode coefficient recursion relations for $\ell \geq 2$:

$$-\dot{\tau}_\ell \simeq \frac{k}{a} \left[\frac{(\ell+1)^2}{(2\ell+3)(2\ell+1)} \tau_{\ell+1} - \tau_{\ell-1} \right]. \quad (\text{D.34})$$

Multiplying through by β_ℓ , one finds

$$-(\beta_\ell \tau_\ell)' \simeq \frac{k}{a} \left[\frac{(\ell+1)}{(2\ell+3)} (\beta_{\ell+1} \tau_{\ell+1}) - \frac{\ell}{(2\ell-1)} (\beta_{\ell-1} \tau_{\ell-1}) \right]. \quad (\text{D.35})$$

Changing from the proper time derivative in (D.35) to the conformal time derivative ' using $dt = a d\eta$ we find

$$-(\beta_\ell \tau_\ell)' \simeq k \left[\frac{(\ell+1)}{(2\ell+3)} (\beta_{\ell+1} \tau_{\ell+1}) - \frac{\ell}{(2\ell-1)} (\beta_{\ell-1} \tau_{\ell-1}) \right]. \quad (\text{D.36})$$

This can be immediately seen to be the same mode equation for $\ell > 2$ as in Hu & Sugiyama [82] and Wilson & Silk [193] (eqn 7).

If now β_ℓ is replaced with $\alpha_\ell^{-1}(2\ell+1) = \beta_\ell$, we find that (D.36) can be rewritten (on first multiplying through by $(2\ell+1)^{-1}$) as

$$-(2\ell+1)(\alpha_\ell^{-1} \tau_\ell)' \simeq k \left[(\ell+1)(\alpha_{\ell+1}^{-1} \tau_{\ell+1}) - \ell(\alpha_{\ell-1}^{-1} \tau_{\ell-1}) \right]. \quad (\text{D.37})$$

This can be recognized as the form of the $\ell > 2$ free-streaming Integrated Boltzmann Equation of Ma & Bertschinger [110] (*cf.* ℓ -th mode equation, 49 or 50) or Seljak & Zaldarriaga [155] equation 3d. Once again, it may be useful to remind the reader of our nomenclature, τ_ℓ are the mode coefficients (from the mode expansion) while τ_{A_ℓ} are the multipole coefficients (from the multipole expansion). This distinction is not made in the Bardeen-variable Gauge Invariant treatments based on the FL mode expansion.

The solution to the mode coefficients with respect to the timelike integration in the flat ($K = 0$) almost-FL case, are spherical Bessel functions. This should not be surprising given that the recursion relation in the linear FL case is merely a projection from an initial section onto a sphere around here and now, modified as a result of RW expansion.

D.5 Almost-FL source terms $\mathcal{B}(x)$

D.5.1 The adiabatic condition

The entropy perturbation S_a , for a radiation-dust almost-FL universe, is given by

$$S_a \simeq \frac{1}{4} D_a \ln \rho_R - \frac{1}{3} D_a \ln \rho_M, \quad (\text{D.38})$$

and this gives the relation used in [37]:

$$D^b E_{ba} \simeq \left(\frac{1}{4} D_a \ln \rho_R - S_a \right) \rho_M. \quad (\text{D.39})$$

The adiabatic condition is then characterized by $S_a = 0$ ¹. This can either be written as $D_a \rho_R = R D_a \rho_M$, $\frac{k}{a} \delta T D^a Q_a = +\frac{1}{3} \delta^M D^a Q_a$ and that $\delta T(k, t) = \frac{1}{3} \Delta(k, t)$.

D.5.2 Source terms

From [120] we have that the source terms in the almost-FLRW MDE are (see chapter 2 and chapter 5):

$$\text{IBE source terms: } \begin{cases} \mathcal{B}_0(x) \simeq -\frac{1}{3} D^a \tau_a(x), \\ \mathcal{B}_a(x) \simeq D_a \ln T(x) + A_a(x), \\ \mathcal{B}_{ab}(x) \simeq \sigma_{ab}(x), \end{cases} \quad (\text{D.40})$$

for the case of scalar perturbations in a matter-dominated almost-FL universe [120]. Using the mode expansion [66], the electric part of the Weyl tensor E_{ab} (gravitational tidal effects), the acceleration A_a and anisotropic pressure π_{ab} can be related to their corresponding potentials:

$$E_{ab} = \bar{\Phi}(k, t) Q_{ab}, \quad A_a = \Phi_u(k, t) Q_a, \quad \pi_{ab} = \Phi_\pi(k, t) Q_{ab}. \quad (\text{D.41})$$

Using the Mode functions Q , Q_a and $Q_{(ab)}$, taking care of the additional wavenumber factors² now introduced [66], the following mode decomposition will be used:

$$(D^b E_{ab})^k = \Phi(k, t) D^b Q_{ab}, \quad (D^b \pi_{ab})^k = \Phi_\pi D^b Q_{ab}, \quad (\text{D.42})$$

$$(D_a \ln \rho_M)^k = \delta^M(k, t) Q_a = +\frac{k}{a} \Delta(k, t) Q_a, \quad (\text{D.43})$$

$$(D_a \ln \rho_R)^k = 4\frac{k}{a} \delta T(k, t) Q_a, \quad (\text{D.44})$$

It then follows that³

¹By adiabatic we mean that the comoving entropy density is constant.

²Once again one should not confuse the Fourier coefficients of the potentials here with those defined from $E_{ab} = D_{(ab)} \Phi(x^i)$, $A_a = D_a \Phi_u(x^i)$ and $\pi_{ab} = D_{(ab)} \Phi_\pi(x^i)$.

³To see where these factors come from notice that $a^2 D^c D_{ac} Q = (-k^2 + 2K) D_a Q \Rightarrow a^2 D^c D_{(ac)} Q = \frac{2}{3} (-k^2 + 3K) D_a Q$ (after removing the trace) for the general l -th order relations see [66]. $(-\lambda)^{-1} D_{(A_i)} Q = Q_{A_i}$, to find that $D^b Q_{ab} = -\frac{2}{3} (ak)^{-1} (-k^2 + 3K) Q_a$. Where as before $D^a D_a Q = -\lambda^2 Q$ and $\lambda = \frac{k}{a}$ [20, 66].

$$\text{mode IBE source terms: } \begin{cases} \mathcal{B}_0(k, t) \approx -\frac{1}{3}\frac{k}{a}\tau_1(k, t), \\ \mathcal{B}_1(k, t) \approx \frac{2}{3}(\rho^{-1}(t)\bar{\Phi}(k, t))\frac{k}{a}\left(\frac{k^2-3K}{k^2}\right), \\ \mathcal{B}_2(k, t) \approx -2[\rho^{-1}(t)\bar{\Phi}(k, t)]^*. \end{cases} \quad (\text{D.45})$$

In the case of matter domination (without using the adiabatic assumption - it makes no difference whether or not it is used here) we then find the relationship between the Newtonian potential and the matter density gradients:

$$2\frac{a}{k}(k^2 - 3K)\bar{\Phi}(k, t) \simeq (a^2\rho_M)\delta^M \approx \frac{k}{a}(a^2\rho_M)\Delta(k, t). \quad (\text{D.46})$$

Replacing $a^2\rho_M$ in terms of the density parameter and the curvature constant from the Friedmann equation (See Appendix A) for both the $K = 0$ ($\Omega_0 = +1$) and $K < 0$ ($\Omega_0 < 1$) matter dominated cases respectively, we obtain

$$\begin{aligned} \text{flat: } \bar{\Phi}(k, t) &\approx \frac{3}{2}(H_0^2\Omega_0)D(\eta)\Delta(k, \eta_0) \\ &\approx \frac{3}{2}\frac{a}{k}H_0^2\delta^M, \end{aligned} \quad (\text{D.47})$$

$$\begin{aligned} \text{open: } \bar{\Phi}(k, t) &\approx \frac{3}{2}\left[\frac{1}{(k^2 - 3K)}\frac{K\Omega_0}{(\Omega_0 - 1)}\right]D(\eta)\Delta(k, \eta_0) \\ &\approx \frac{3}{2}\frac{k}{a}\frac{1}{(k^2 - 3K)}\frac{K\Omega_0}{(\Omega_0 - 1)}\delta^M. \end{aligned} \quad (\text{D.48})$$

These are then the equations that will be used through the slow decoupling and free-streaming eras.

D.5.3 The temperature monopole

Using the energy conservation equations ($\ell = 0$ MDE), ⁴

$$(D_a \ln T)' + H(D_a \ln T + A_a) + \frac{1}{3}D_a\Theta \simeq +\frac{1}{3}D_a(D^c\tau_c), \quad (\text{D.49})$$

and taking another spatial covariant derivative, we obtain one of key equations in the primary source calculation:

$$(D^2 \ln T)' + 2H(D^2 \ln T) - \frac{1}{3}D^2(D^c\tau_c) \simeq -\frac{1}{2}(D_a D_b \sigma^{ab} - D_a q^a) - H(D^a A_a). \quad (\text{D.50})$$

D.6 Almost-FL scalar perturbations

In the case of scalar perturbations, the almost-FL EFE for matter domination reduce to the following CGI perturbation equations:

$$\dot{\sigma}_{ab} + 2H\sigma_{ab} + E_{ab} \approx 0 \iff (a^2\sigma_{ab})' \approx -a^2 E_{ab} \quad (\text{D.51})$$

$$\dot{E}_{ab} + 3HE_{ab} + \frac{1}{2}\rho_M\sigma_{ab} \approx 0 \iff (\rho_M^{-1}E_{ab})' \approx -\frac{1}{2}\sigma_{ab}. \quad (\text{D.52})$$

⁴Notice that $D_a(\ln T)' \simeq (D_a \ln T)' + H(D_a(\ln T) + A_a)$ where $D_a = h_{ab}\nabla^b$ and $-\dot{u}_a(\ln T)' \simeq H\dot{u}_a$. It is also well known that $D^a(D_a \ln T)' \simeq (D^2 \ln T)' + H(D^2 \ln T)$

Taking the time derivative of the above equations, we obtain

$$(a^2\sigma_{ab})'' + H(a^2\sigma_{ab})' \approx +\frac{1}{2}a^2\rho_M\sigma_{ab}, \quad (\text{D.53})$$

$$(\rho_M^{-1}E_{ab})' + 2H(\rho_M^{-1}E_{ab}) \approx \frac{1}{2}E_{ab}. \quad (\text{D.54})$$

To see how these relate to evolution equations for the density gradient, we take the spatial divergence of (D.54) [20].

A useful consequence of the above relations is that for matter domination⁵, the monopole equations from the MDE take on a simple form:

$$(D_a \ln T)' + H(D_a \ln T) + \frac{1}{3}D_a(D^c \tau_c) \approx -\frac{1}{2}D^b \sigma_{ab}. \quad (\text{D.55})$$

To summarize, the matter dominated limit leads to the following simple relationships between the dynamical variables and the electric part of the Weyl tensor:

$$A_a \approx 0, \quad (\text{D.56})$$

$$\sigma_{ab} \approx -\frac{2}{3}(H_0^2\Omega_0)^{-1}u^c\nabla_c(a^3E_{ab}), \quad (\text{D.57})$$

$$D_a \ln \rho \approx (H_0^2\Omega_0)D^b(a^3E_{ab}), \quad (\text{D.58})$$

$$\frac{2}{3}D^a\Theta \approx -\frac{2}{3}(H_0^2\Omega_0)^{-1}\left[(a^3D_bE^{ab})' + H(a^3D_bE^{ab})\right], \quad (\text{D.59})$$

using (for the electric part of the Weyl tensor)

$$E_{ab} = \sum_k \bar{\Phi} Q_{ab} \equiv D_{(a}D_{b)}\Phi_E(x) \quad \text{and} \quad D_a\Phi_E(x) = \sum_k \frac{k_a}{a}\Phi_E(k,t)Q_a \quad (\text{D.60})$$

$$\Rightarrow \bar{\Phi}(k,t) \approx -\frac{k^2}{a^2}\Phi_E(k,t). \quad (\text{D.61})$$

Note that in [37], $E_{ab} = \sum_k (k^2/a^2)\Phi_k Q_{ab}$ which is based on the notation of Kodama and Sasaki [98]. The relationship between Φ_k and the potential used here is $\bar{\Phi} = (k^2/a^2)\Phi_k$.

The evolution equation for the Newtonian-like potential follows from (D.54) [54, 55]:

$$(\rho_M^{-1}\bar{\Phi})'' + 2H(\rho_M^{-1}\bar{\Phi})' \approx \frac{1}{2}\bar{\Phi}, \quad (\text{D.62})$$

In terms of the conformal time derivative ($dt = a d\eta$) which we denote by a prime “ \prime ” we have:

$$(\rho_M^{-1}\bar{\Phi})'' + H(\rho_M^{-1}\bar{\Phi})' \approx \frac{1}{2}a\bar{\Phi}. \quad (\text{D.63})$$

On rearranging (D.62) using (D.61) and (2.29-2.31) we find the evolution equation for the potential

$$\left[\ddot{\Phi}_E + 4H\dot{\Phi}_E\right] \approx \left[\frac{1}{2}\rho_M - \dot{H} - 3H^2\right]\Phi_E \approx \left(\frac{2K}{a^2}\right)\Phi_E, \quad (\text{D.64})$$

⁵A further consequence of $(a^3E_{ab})' \approx -\frac{1}{2}a^3\rho_0\sigma_{ab}$ and $\rho = 3H_0^2\Omega_0 a^{-3}$ for $a_0 = +1$; is that $(a^3E_{ab})' \approx -\frac{3}{2}H_0^2\Omega_0^2\sigma_{ab}$.

which is just

$$(a\Phi_E)''(t, k) + 2H(a\Phi_E)'(t, k) \approx \frac{3}{2}H_0^2\Omega_0 a^{-2}\Phi_E(t, k). \quad (\text{D.65})$$

Notice that for $K = 0$ one recovers the well known $\ddot{\Phi}_E \approx -4H\dot{\Phi}_E$ for dust used in [37]; for dust this means that $(a^4\Phi_E(k, t))' = 0$ (which gives the well known $\Phi_E(k, t) \approx \Phi_E^+(k, 0) + \Phi_E^-(k, t)$)⁶. Considering the growing mode only, we then have :

$$(\bar{\Phi}_{\rho_M}^{-1}) \approx (3H_0^2\Omega_0)^{-1}a[k^2\Phi_E(k, 0)] \approx (3H_0^2\Omega_0)^{-1}a[k^2\Phi_A(k, 0)]. \quad (\text{D.66})$$

We have that $(\bar{\Phi}_{\rho_M}^{-1})' \sim k^2\dot{a}$ and that $\Phi_A' \approx -\Phi_H' \approx 0$. Here we are recovering the standard result⁷ of that the ‘‘potential fluctuations’’ Φ_E is time independent for a $K = 0$ matter dominated dust scenario.

We can write $a\Phi_A(k, 0) = D/a(a\Phi_A(k, 0)) = (D/a)\Phi_A(k, t)$; for the linear growth factor D , where the flat dust case is recovered using $D = a$:

$$(\bar{\Phi}_{\rho_M}^{-1}) \approx (3H_0^2\Omega_0)^{-1}\frac{D}{a}[k^2\Phi_A(k, t)]. \quad (\text{D.67})$$

D.6.1 Newtonian like equations

From the *total frame equations* (5.70) for the evolution and constraint for the electric part of the Weyl tensor in a CDM dominated flat almost-FLRW model with respect to the Newtonian frame we have the following definitions:

$$\text{peculiar velocity:} \quad \dot{E}_{ab} + 3HE_{ab} \approx a^{-3}(a^3E_{ab})' \approx +\frac{1}{2}\rho_M D_{(a}v_{b)}, \quad (\text{D.68})$$

$$\text{matter over-density:} \quad +\frac{1}{3}D_a \ln \rho_M = \rho_M^{-1}D^a E_{ab}. \quad (\text{D.69})$$

We now use that with $v_a = vQ_a$,

$$D_a\Phi_A = \frac{k}{a}\Phi_A(k, 0)Q_a, \quad D_{(a}v_{b)} \approx -\frac{k}{a}v(k, \eta)Q_{ab} \quad D_a \ln \rho_M \approx \frac{k}{a}\Delta(k, \eta)Q_a, \quad (\text{D.70})$$

to find, first, from (D.68) and $v \approx -v_B$ (here v is the relative velocity of the Newtonian frame with respect to the total frame, and v_B , is the relative velocity of the matter with respect to the Newtonian frame):

$$-\frac{k}{a}v_B(k, t) \approx \frac{k}{a}v(k, \eta) \approx \dot{a}(\eta) \left[\frac{2}{3}(H_0^2\Omega_0)^{-1}[k^2\Phi_A(k, 0)] \right], \quad (\text{D.71})$$

and, second, from (D.69):

$$\Delta(k, \eta) \approx D(\eta)\Delta(k, \eta_0) \approx -\frac{2}{3}(H_0^2\Omega_0)^{-1}a(\eta)[k^2\Phi_A(k, 0)]. \quad (\text{D.72})$$

⁶For dust $\rho_M \sim a^{-3}$ and $a \sim t^{2/3}$ to then find that $\Phi_E^- \sim t^{-5/3}$ and $\Phi_E^+ \sim \text{constant}$

⁷In the notation of [20] $\ddot{\Delta} + 2H\dot{\Delta} - \frac{1}{2}\rho_0\Delta = 0$ for $\Delta = a^2D^2 \ln \rho$ with two solutions (growing mode) $\Delta \sim t^{2/3}$ and (decaying mode) $\Delta \sim t^{-1}$ for dust, and from the constraint equations $\frac{1}{3}\Delta = (a^2\rho_0^{-1})D^a D^b E_{ab}$. To get this in the form of that used in [54] in the variable \mathcal{X}_k use $\rho_0 = 3H^2 + \frac{3k^2}{a^2}$ from the Hamiltonian constraint and notice that $\mathcal{X}_a = aD_a \ln \rho$, such that $\Delta = aD^a \mathcal{X}_a$. It is in order to avoid such notational problems we have adopted the middle ground of using the variables that the general covariant EFE (2.13-2.24) are expressed in following historical conventions and notation [50, 48, 78, 113, 58, 184], these seem the most straight forward to deal with in the context of the Kinetic Theory equations.

Hence we are able to recover the canonical results from (Eq.D.72 – noting that $\Phi_A(k, \eta) = D(\eta)\Phi_A(k, 0) \approx a(\eta)\Phi(k, 0)$) and (Eq. D.71 – using that $\dot{a} = (a'/a)$ and that $\Delta(k, \eta) = D(\eta)\Delta(k, \eta_0)$):

$$\text{matter peculiar velocity: } k v_B(k, \eta) \approx a' \Delta(k, 0) \approx \Delta'(k, \eta). \quad (\text{D.73})$$

$$\text{matter over-density: } k^2 \Phi_A(k, \eta) \approx -\frac{3}{2} H_0^2 \Omega_0 \Delta(k, \eta), \quad (\text{D.74})$$

The shear in the total frame can then be found from the peculiar velocity,

$$\text{shear: } \sigma_{ab} \approx -D_{(a} v_{b)} \quad (\text{D.75})$$

giving (5.43) on using $\sigma_{ab} = \sigma(k, t) Q_{ab}$ – this is an equivalent but alternative derivation to that in section 6.5.2 which starts with Eq. (6.53) :

$$\sigma(k, t) \approx \frac{k}{a} v(k, t) \approx -\dot{\Delta}(k, t) \approx -\frac{D'}{a} \Delta(k, \eta_0), \quad (\text{D.76})$$

such that

$$\sigma(k, \eta) \approx +\frac{2}{3} (H_0^2 \Omega_0)^{-1} \frac{D'(\eta)}{a(\eta)} [k^2 \Phi_A(k, 0)]. \quad (\text{D.77})$$

D.6.2 On relating $a^2 \rho$ to the curvature and Ω

The Friedmann equation in a FL universe is

$$H^2 + \frac{K}{a^2} \simeq \frac{1}{3} \rho \Rightarrow a^2 H^2 + K = \frac{1}{3} a^2 \rho. \quad (\text{D.78})$$

Using the definition of the density parameters:

$$\Omega = \frac{\rho}{3H^2}, \text{ and } \Omega_r = \frac{\rho_r}{3H^2}. \quad (\text{D.79})$$

we can show that

$$a^2 \rho \simeq 3a^2 H^2 \Omega, \text{ and } a^2 \rho_r \simeq 3a^2 H^2 \Omega_r. \quad (\text{D.80})$$

Eq. (D.78) and Eq.(D.80) can be used to deduce that

$$a^2 H^2 + K = a^2 H^2 \Omega, \Rightarrow a^2 H^2 = \frac{K}{(\Omega - 1)}. \quad (\text{D.81})$$

Using this and (D.80) we obtain

$$a^2 \rho \simeq \frac{3K\Omega}{(\Omega - 1)}. \quad (\text{D.82})$$

D.6.3 Relating $D^b\sigma_{ab}$ to $D_a\Theta$

Furthermore, by considering the almost-FL momentum constraint equation (2.21) for matter domination $q_a \approx q_a^{(m)} = 0$ in the physical frames (either matter, particle or energy) we find

$$D_b\sigma^{ab} \approx +\frac{2}{3}D^a\theta + \text{curl}\omega^a. \quad (\text{D.83})$$

Taking the curl of the constraint equation for the magnetic part of the Weyl tensor, we find that for scalar perturbations ($H_{ab} \approx 0$) and matter domination in the physical frames :

$$-\rho \left(D_b\sigma^{ab} - \frac{2}{3}D^a\theta \right) \approx 0. \quad (\text{D.84})$$

Here we have used (D.83) and that $D^a q_a \approx D^2 q_a \approx q_a \approx 0$ which follows from matter domination $q_a \approx q_a^{(m)}$, and carried out the calculation in the physical frames $q_a \approx 0$ (matter, particle or energy). The identity $\text{curl}\text{curl}q_a \simeq D_a(D^b q_b) - D^2 q_a + 2(\frac{1}{3}\rho - H^2)q_a$ has been used too.

In passing it is worthwhile to notice that in a general frame, from the constraint equations, $\text{curl}(D_b\sigma^{ab}) \simeq \text{curl}(\frac{2}{3}D^a\Theta) + \text{curl}\text{curl}\omega^a$, $D^a\omega_a \simeq 0$ and $D^b(\text{curl}\sigma_{ab}) \simeq \frac{1}{2}\text{curl}(D^b\sigma_{ab})$ and then from the propagation equation for the vorticity vector it can be respectively shown that :

$$D^b D_{(a}\omega_{b)} \approx +\frac{1}{3}\text{curl}(D_a\theta) - \frac{1}{2}D^2\omega_a + (\frac{1}{3}\rho_0 - H^2)\omega_a, \quad \dot{\omega}_a \approx -2H\omega_a. \quad (\text{D.85})$$

What is important about (D.84) is that the expansion perturbation is determined in terms of the gravitational tidal forces for the scalar perturbation matter dominated almost-FL universe.

D.7 The Correlation Function

If $\langle(\delta N/N)^2\rangle$ is the square of the variance in the number of objects in a volume V , then the correlation function, the excess probability over a random variable of finding an object within a distance χ of a given object [144], is given by

$$\xi(\chi) = \frac{d}{dV} \left[V(\chi) \left\langle \left(\frac{\delta N}{N} \right) \right\rangle_{V(\chi)} \right], \quad (\text{D.86})$$

where $V(\chi)$ is the volume enclosed with a radius χ for flat universe ($K = 0$). The power spectrum $\mathcal{P}(k)$ is related ⁸ to the correlation function for a given distribution [97]:

$$\xi(\chi) = \frac{1}{2\pi^2} \int k^2 dk \mathcal{P}(k) \frac{\text{sink}\chi}{k\chi} \iff \mathcal{P}(k) = 4\pi \int \chi^2 d\chi \xi(\chi) \frac{\text{sink}\chi}{k\chi}. \quad (\text{D.87})$$

In an open universe the definition for the correlation function can still be retained. To see how, one first notices that in a flat universe $\langle(\delta N/N)^2\rangle_{V(\chi)} \sim \langle N \rangle_V^{-1} \sim V^{-\alpha}$, where $\langle N \rangle_V$ is the average number of objects in a volume V . In a space of constant negative

⁸Care should be taken with the normalization convention.

curvature the volume enclosed by a sphere of radius χ is $V(\chi) = \pi(\sinh(2\chi) - 2\chi)$ so that

$$\xi(\chi) \sim V^{-\alpha} \sim (\sinh(2\chi) - 2\chi)^{-\alpha}. \quad (\text{D.88})$$

The power spectrum for a power law correlation function in the volume in an open universe is then

$$\mathcal{P}(k) = \frac{1}{2\pi^2} B \int \sinh^2 \chi d\chi \frac{\sin k\chi}{k \sinh \chi} \frac{1}{(\sinh(2\chi) - 2\chi)^\alpha}, \quad (\text{D.89})$$

where B is the normalization constant. This diverges for small χ so a small-scale cut-off is necessary [97]. To relate this to the power spectrum today on scales measured by galaxy surveys, the power spectrum is multiplied by the square of a transfer function $T^2(k)$. The power spectrum can then be normalized to σ_8 .

D.8 Power Spectrum Normalization

There are two possibly normalization schemes to follow.

Large Scales

On considering a power spectrum of the form

$$P(k) = A(k\eta_0)^{n-1} \quad (\text{D.90})$$

where $\eta_0 \simeq 3t_0 \simeq 2H_0^{-1}$ for $\Omega_0 = 1$ gives the conformal time today, the scale factor can be normalized and A is one way of expressing the amplitude of scalar perturbations since it is related to the dimensionless scale of matter fluctuations at horizon crossing, δ_H , by [142, 2, 17, 192]:

$$\delta_H^2 = \frac{4}{\pi} A. \quad (\text{D.91})$$

Here $\Delta^2(k) \approx \delta_H^2(k/H_0)^{n+3}$ (using $a_0 = c = +1$). As before, $|\delta(k)|^2 \propto k^{n+1}$.

Small Scales

The alternative scheme is to consider a power spectrum of the form

$$P(k) = Bk^n T^2(k), \quad (\text{D.92})$$

where the transfer function $T(k) \sim +1$ on large scales. This means that if the fluctuations arise purely from the SW effect caused by potential fluctuations near decoupling, the B and A can be related at $n = +1$ [192]:

$$P(k) = 2\pi^2 \eta_0^4 A k T^2(k). \quad (\text{D.93})$$

For *standard CMD* it is the convention to use the parametrized transfer function [18]:

$$T(k) = \left[1 + \left(ak + (bk)^{\frac{3}{2}} + (ck)^2 \right)^\nu \right]^{\frac{1}{\nu}}, \quad (\text{D.94})$$

where

$$a = 6.4\Gamma^{-1}h^{-1}\text{Mpc}, \quad b = 3.0\Gamma^{-1}h^{-1}\text{Mpc}, \quad c = 1.7\Gamma^{-1}h^{-1}\text{Mpc}, \quad \nu = 1.13. \quad (\text{D.95})$$

Now the **shape function** Γ can be given as approximately $\Gamma \simeq \Omega_0 h$ (by choosing $h = 0.5$ and $\Omega_B = 0.05$ the shape function is given as $\Gamma = 0.48$).

Large scale flows provide a measure of the power spectrum in as much as the variance of the velocity field sphere of radius x_f , $v_{rms}^2(x_f)$ can be expressed as an integral over the power spectrum. On small scales (clusters of galaxies) the power spectrum is normalize to σ_8 , the variance of the galaxy distribution on scales of $x_f = 8h^{-1}\text{Mpc}$ [34]:

$$\sigma_8^2 = \frac{1}{b_\rho^2} = \frac{1}{2\pi^2} \int k^2 dk \mathcal{P}(k) T^2(k) W^2(kx_f), \quad (\text{D.96})$$

where b_ρ is the x_f scale ‘‘bias’’ such that $\sigma_8^m = \sigma_8/b_\rho$ and the appropriate variance and $W(x) = 3(\sin x - x \cos x)/x^3$ is the top-hat function. It should also be pointed out that it is more convenient to use the form

$$\sigma_8^2 = \int_0^\infty \frac{dk}{k} A(k\eta_0)^{n+3} T^2(k) \left(\frac{3j_1(kx_f)}{kx_f} \right). \quad (\text{D.97})$$

The variance of galaxies possibly biased to the matter ($\delta_{gal} = b\delta_\rho$) is roughly unity on the scale of $8h^{-1}\text{Mpc}$. Interestingly enough the *standard-CDM* normalization from COBE seems to give $\sigma_8 \simeq 1.3$ which seems to imply that it is not correct to assume a pure SW-HZ power spectrum or even a $n = 1.15$ SW one. An appropriate table of *standard-CDM* normalizations is provided in [24].

D.9 The open almost-FLRW case

D.9.1 Extending the integral solution to the open case

In the main body of this paper we keep to the almost-flat-FL case for clarity; the extension to the open case is straight forward. The point is that we have shown that the standard results can be recovered from the CGI approach without anything particularly new beyond some important differences in the formalism. Given that the generic linear FRW is well treated in the standard literature, we provide only an outline for the open case. The essence of the open solution is given via:

$$\begin{aligned} \tau'_\ell + k \left[\frac{(\ell+1)^2}{(2\ell+3)(2\ell+1)} \left(1 - \frac{K}{k^2} ((\ell+1)^2 - 1) \right) \tau_{\ell+1} - \tau_{\ell-1} \right] \\ \approx -\kappa' \tau_\ell - [a\mathcal{B}_0 \delta_{\ell 0} + (a\mathcal{B}_1 + \kappa' v_B) \delta_{\ell 1} + a\mathcal{B}_2 \delta_{\ell 2}]. \end{aligned} \quad (\text{D.98})$$

The left hand side, the homogeneous case, is solved in very much the same way as for the flat case.

The flat radial eigenfunctions are found from (D.100):

$$\frac{d}{d\chi} j_\ell(k\chi) = \frac{\ell}{(2\ell+1)} k j^{\ell-1}(k\chi) - \frac{\ell+1}{(2\ell+1)} k j^{\ell+1}(k\chi), \quad (\text{D.99})$$

where $Q_{lm} = j_\ell(k\chi)\mathcal{Y}_{\ell m}^{A_\ell} O_{A_\ell}$. For the general almost-FL (linearized FL) models [82, 70, 198]⁹:

$$\frac{d}{d\chi} X_\nu^\ell = \frac{\ell}{(2\ell+1)} k X_\nu^{\ell-1} - \frac{(\ell+1)}{(2\ell+1)} k \left[1 - \ell(\ell+2) \frac{K}{k^2} \right] X_\nu^{\ell+1}. \quad (\text{D.100})$$

Thus, in order to construct the open solution, we just replace the homogeneous solution for $\nu^2 + 1 = k^2/(-K)$ [70, 193] after reading off the solution using the recursion relation:

$$\tau_\ell^{(0)}(k, \eta) = (2\ell+1)\beta_\ell^{-1} X_\nu^\ell(\eta). \quad (\text{D.101})$$

Of course one needs to be careful to redefine the wavenumber. The integral solution (5.95) is then used in (D.98), *i.e.*, the solutions (D.101) are substituted into (5.101) to recover the open integral solutions.

One carries out the same treatment as for $K = 0$ but using (D.100) and (5.14-5.16). Alternatively one can re-define the mode expansion $M_\ell[Q]$ [66].

All that remains is to solve the evolution equations for the scalar perturbations, the coefficients $C_I(\eta, k)$ now include curvature terms when written out in terms of the \mathcal{B} terms (one uses the open recursion relation instead of the flat). In turn, these terms, \mathcal{B} , will also pick up curvature terms when written out in terms of the perturbation variables. When the curvature starts to dominate the evolution, one gets an additional ISW contribution (which will be similar to the late ISW effect for a Λ dominated flat model).

D.9.2 Extending the power spectrum to the open matter dominated case

We consider the primary anisotropy term:

$$\frac{\tau_\ell^{SW}(\eta_0)\beta_\ell}{(2\ell+1)} \simeq \mathbf{S}_{DEC} X_\ell^\nu(\Delta\eta_*), \quad (\text{D.102})$$

$$\mathbf{S}_{DEC} \simeq +\frac{a}{k} \mathcal{B}_1(\eta_*), \quad (\text{D.103})$$

$$\mathcal{B}_1(\eta_E) \simeq +\frac{2}{3}(ak)^{-1} \left(\bar{\Phi}(k, \eta_*) \rho^{-1} \right) (k^2 - 3K). \quad (\text{D.104})$$

This gives the SW effect in the open case.

For spaces of constant negative curvature in an almost-FL model:

$$\langle \bar{\Phi}(k^a) \bar{\Phi}(k'^a) \rangle = \left[(2\pi)^3 \frac{3}{2} \frac{K}{3K - k^2} \frac{\Omega_0}{(\Omega_0 - 1)} \right] \mathcal{P}(k). \quad (\text{D.105})$$

Using (D.102-D.104) with $K < 0$ we find

$$\tau_\ell(\eta_0) \frac{\beta_\ell}{(2\ell+1)} \simeq -\frac{2}{3} \bar{\Phi}(k, \eta_*) \rho_*^{-1} \frac{1}{k^2} (3K - k^2) X_\ell^\nu(\Delta\eta). \quad (\text{D.106})$$

⁹Notice that $(\nu^2 + (\ell+1)^2)/(\nu^2 + 1) = (1 - \ell(\ell+2)(K/k^2))$.

It follows that (noticing that $\Delta\eta_* = \chi$)

$$\frac{|\tau_\ell|^2 \beta_\ell^2}{(2\ell + 1)^2} \approx (2\pi)^3 \left[\frac{\rho_0^{-1} K}{k^2} \frac{\Omega_0}{(\Omega_0 - 1)} \right]^2 \mathcal{P}(k). \quad (\text{D.107})$$

Then we can use

$$\langle \tau_{A_\ell} \tau^{A_\ell} \rangle = \frac{2}{\pi} \beta_\ell \int k^2 dk \Xi_n^2 |\tau_\ell|^2 j_\ell^2(k\chi), \quad (\text{D.108})$$

for

$$\Xi_\ell^2 = \prod_{n=1}^{\ell} \frac{n^2}{(2n+1)(2n-1)} \left[1 - \frac{K}{k^2} (n^2 - 1) \right], \quad (\text{D.109})$$

and (5.207) to find the angular correlation function. It is more straight forward to redefine the mode function $G_\ell[Q]$ in the mode function expansion, that is we use $M_\ell[Q] = (\Xi_\ell)^{-1} G_\ell[Q]$ instead of $G_\ell[Q]$. This then allows us to retain the flat-like form for the angular correlation function (5.207), however we must retain the flat mode eigenfunction normalization:

$$C_\ell = 16\pi^2 \left[\frac{K}{\rho_0^{-1}} \frac{\Omega_m}{(\Omega_m - 1)} \right]^2 \int \frac{d\nu}{\nu^2} \mathcal{P}(\nu) j_\ell^2(\nu\chi), \quad (\text{D.110})$$

where $\nu^2 = k^2 - 1$.

The angular power spectrum can then be normalized to some structure formation theory such as *standard-CDM*. It would seem that the favoured model (by current observational limits) is the Λ CDM [107].

Also, in the open case from (D.110 and 5.213):

$$D_\ell = \left(\frac{K \Omega_m}{a^2 H^2 (\Omega_m - 1)} \right)^2 \int \frac{d\nu}{\nu^2} \mathcal{P}^*(\nu) j_\ell^2(\nu\chi). \quad (\text{D.111})$$

Here \mathcal{P}^* (5.215) is defined as before, while the mode expansion has been carried out in terms of $M_\ell[Q]$ rather than $G_\ell[Q]$ [66].

D.10 Additional comments on approximations

Given an integral of the form

$$S_\ell(\eta, k) = \int_{\eta_*}^{\eta} \mathbf{S}(k, \eta') j_\ell(k(\eta_* - \eta')) d\eta', \quad (\text{D.112})$$

we can find the following approximations: First, when \mathbf{S} is slowly varying when compared with the transfer function $\mathbf{S}' \ll k\mathbf{S}$:

$$\int_{\eta_*}^{\eta_0} d\eta S(k, \eta) j_\ell \sim S(\eta_\ell) \int_0^\infty d\eta j_\ell. \quad (\text{D.113})$$

Second, when the transfer function is slowly varying with respect to \mathbf{S} ($\mathbf{S}' \gg k\mathbf{S}$):

$$\int_{\eta_*}^{\eta_0} d\eta \mathbf{S} j_\ell \sim j_\ell(k(\eta_0 - \eta_*)) \int_0^\infty d\eta \mathbf{S}(k, \eta). \quad (\text{D.114})$$

We are then, in addition, able to construct the Early-Times (ET) and Late-Time (LT) approximations for the integral. Third, if the contribution of \mathbf{S} is mostly at early times :

$$\int_{\eta_*}^{\eta} \mathbf{S} j_\ell d\eta \simeq j_\ell(k\eta_0) \int_{\eta_*}^{\eta_0} \mathbf{S}(k, \eta) d\eta. \quad (\text{D.115})$$

Fourth, and last, if the contribution is mostly at late times :

$$\int \mathbf{S} j_\ell d\eta \simeq j_\ell(k(\eta_0 - \eta_*)) \int_0^\infty \mathbf{S}(k, \eta) d\eta. \quad (\text{D.116})$$

An example of where these are used, is in the estimation of the ISW effect, which can be approximated by using :

$$- \int_{\eta_*}^{\eta_0} (\Phi' - \Psi') j_\ell(k(\eta_0 - \eta)) d\eta \approx (\Psi - \Phi)|_{\eta_*}^{\eta_0} j_\ell(k\eta_0). \quad (\text{D.117})$$

Appendix E

Nonlinear extension

E.1 The scattering correction in the canonical formalism

Here I give the weakly nonlinear scattering correction (6.7) in terms of the canonical formalism for flat almost-FLRW models. The primary and secondary sourced corrections are given using the standard ansatz for primary and secondary anisotropies. These corrections do not arise in the canonical treatment.

E.1.1 Summary of the canonical adiabatic results

Ignoring the isocurvature and Doppler effects from Hu-Sugiyama PRD 1995 :

$$\Theta_\ell(\eta_0) \approx (\Theta_0 + \Psi)(\eta_*) j_\ell(k\Delta\eta_*)(2\ell + 1) + (2\ell + 1) \int_{\eta_*}^{\eta_0} (\Psi' - \Phi') j_\ell(k\Delta\eta) d\eta, \quad (\text{E.1})$$

for which we have used : $\Delta\eta_* = \eta_0 - \eta_*$ and $\Delta\eta = \eta_0 - \eta$. Now using the WKB method, from HS95-PRD, we include the photon diffusion damping by making the replacement :

$$(\Theta_0 + \Psi)(\eta_*) \approx (\hat{\Theta}_0 + \Psi)(\eta_*) \mathcal{D}(k), \quad \text{for, } \mathcal{D}(k) = \int_0^{\eta_0} d\eta \kappa' e^{-\kappa} e^{-(k/k_D)^2}. \quad (\text{E.2})$$

Here k_D is the diffusion wavelength, and κ the optical depth. We now consider the case for adiabatic perturbations only, this allows the following initial conditions to be used in the solutions to the tight-coupling (approximation) equations, the explanation being that photons are overdense in the potential wells :

$$\hat{\Theta}_0(0) \approx -\frac{1}{2}\Psi(0) \quad (\text{E.3})$$

where the solution close to tight-coupling are:

$$(\hat{\Theta}_0 + \Psi)(\eta_*) \approx (\hat{\Theta}_0(0) + (1 + R)\Psi(0)) \cos(kr_s^*) - R\Psi(k, \eta^*). \quad (\text{E.4})$$

This then describes the acoustic oscillations where, $r_s^* \approx c_s \eta_*$, and $c_s^2 = \frac{1}{3}(1/1 + R)$ while the normalized scale factor ($\frac{3}{4}$ at photon-baryon equality) is just, $R = (3\rho_b/4\rho_R)$.

The manner in which we intend to deal with the non-linearity, in the intermediate scale, will once again rely upon the use of weak-coupling :

$$\int_0^{\eta_0} \Psi(k, \eta) [\Theta_0 + \Psi](\bar{k}, \eta_*) j_\ell(\bar{k}(\eta_0 - \eta)) \simeq \sqrt{\frac{\pi}{2\ell}} \frac{1}{k} \Psi(k, \eta_\ell) [\hat{\Theta}_0 + \Psi](\bar{k}, \eta_*) D(k), \quad (\text{E.5})$$

where : $\eta_\ell = \eta_0 - \ell + \frac{1}{2}/\bar{k}$. This is how we will deal with the SW and acoustic oscillation contributions : the primary contributions.

E.1.2 Canonical primary and secondary effects

The primary effect will take the form

$$\frac{\Theta_\ell^P(k, \eta_0)}{(2\ell + 1)} \approx \left[\Theta_0 - \frac{3}{2} \frac{1}{k^2} H_0^2 \Omega_0 \Delta_B(k, \eta_*) \right] j_\ell(k(\eta_0 - \eta_*)). \quad (\text{E.6})$$

It is convenient to then write the secondary effects following [84] (we now also include the Doppler effect along with the ISW effects) where it is written out in terms of the baryon perturbations using the adiabatic matter dominated CDM model :

$$\begin{aligned} \frac{\Theta_\ell^S(k, \eta_0)}{(2\ell + 1)} &\approx \frac{1}{k} \int_{\eta_d}^{\eta_0} \left[(D''\kappa' + \kappa''D') \Delta_B(k, \eta_0) + (D'\kappa') \Delta'_B - 3k \frac{D}{a} H_0^2 \Omega_0 \Delta_B \right] \\ &\times e^{-\kappa} j_\ell(k(\eta_0 - \eta)) d\eta. \end{aligned} \quad (\text{E.7})$$

where I have used that ($\hat{k}_a \equiv e_a^k$) (see D.6.1 for an alternative derivation using 1+3 covariant techniques)

$$k^2 \Psi(k, \eta) \approx -\frac{3}{2} \frac{D}{a} H_0^2 \Omega_0 \Delta_b(k, \eta_0), \quad (\text{E.8})$$

$$k_a v_B(k, \eta) \approx \dot{\Delta}_B \hat{k}_a. \quad (\text{E.9})$$

Furthermore in the Newtonian frame we can identify $\Phi_A(k, \eta)$ and $\Psi(k, \eta)$ as being the same potentials, we also have in mind the situation of matter domination such that $\Phi_H \approx -\Phi_A$ (hence the factor of 2). We can then substitute (E.7) into (6.7) using (E.10), where the sum is replaced by an integral.

The idea of introducing the Δ_B here is that this will be replaced by the quasi-linear and non-linear perturbations, Δ_{NL} following the usual HKLM procedure (see Peacocks book for a nice summary [141]). The relationship between the 1 + 3 covariant and gauge invariant theory is given by :

$$\tau_{A_\ell}^m \approx \sum_{\ell > 1} \Theta_\ell^m \beta_\ell Q_{A_\ell}^m, \quad (\text{E.10})$$

where $m = 0, 1, 2$ for scalar, vector and tensor mode respectively. In the instance of scalar perturbations we then have that :

$$\beta_\ell \tau_\ell(\eta, k) \approx \Theta_\ell(k, \eta) \quad \forall \ell \geq 2. \quad (\text{E.11})$$

using this we can the construct the nonlinear corrections to the canonical theory.

E.1.3 The scattering correction in canonical notation

From equation (6.7) we have that the source term takes the form

$$\begin{aligned} \mathbf{S}_{NLS} \approx & \left(v_B(k^a - k'^a) \beta_{\ell-1}^{-1} \Theta_{\ell-1}(k') + v_B(k'^a) \beta_{\ell-1}^{-1} \Theta_{\ell-1}(|k^a - k'^a|) \right) \\ & + \frac{1}{2} \left(v_B(k^a - k'^a) \beta_{\ell+1}^{-1} \Theta_{\ell+1}(k'^a) + v_B(k'^a) \beta_{\ell+1}^{-1} \Theta_{\ell+1}(|k^a - k'^a|) \right). \end{aligned} \quad (\text{E.12})$$

Here the nonlinear correction takes the form

$$(\delta C)_\ell^{NLS}(k, \eta_R) \approx \int_{\eta_*}^{\eta_0} d\eta' \mathcal{V} e^{-(k/k_D)^2} \sum_{k^a} \mathbf{S}_{NLS}(k^a, k'^a, \eta_0, \eta'). \quad (\text{E.13})$$

The canonical source term can be simplified by multiplying through by β_ℓ and using the high- ℓ approximations : $\beta_\ell/\beta_{\ell-1} = \ell/2\ell - 1 \sim \frac{1}{2}$ and $\beta_\ell/\beta_{\ell+1} = 2\ell + 1/\ell = 1 \sim 2$. Doing this we find that from (E.12) we are able to get :

$$\begin{aligned} \mathbf{S}_{NLS} \approx & v_B(k^a - k'^a) \left[\frac{1}{2} \Theta_{\ell-1}(|k'^a|) + \Theta_{\ell+1}(|k'^a|) \right] \\ & + v_B(k'^a) \left[\frac{1}{2} \Theta_{\ell-1}(|k^a - k'^a|) + \Theta_{\ell+1}(|k^a - k'^a|) \right]. \end{aligned} \quad (\text{E.14})$$

We can further split the source term into the primary and secondary source anisotropy terms : $\mathbf{S}_{NLS} \approx \mathbf{S}_{NLS}^P + \mathbf{S}_{NLS}^S$. We do this below by first calculating the secondary sourced corrections and the primary sourced corrections.

Primary sourced corrections

The contribution arising from the Primary effect, using (E.14) and (E.6) is

$$\begin{aligned} \mathbf{S}_{NLS}^P(k^a, k'^a, \eta_*, \eta_0) \approx & \frac{\Delta'(|k^a - k'^a|, \eta')}{(|k^a - k'^a|)} \left[\left(\Theta_0 - \frac{3}{2} \frac{1}{k'^2} H_0^2 \Omega_0 \Delta_B(k', \eta') \right) \right. \\ & \times \left. \left(\frac{1}{2} j_{\ell-1}(k'(\eta_0 - \eta')) + j_{\ell+1}(k'(\eta_0 - \eta')) \right) \right] \\ & + \frac{\Delta'(k', \eta')}{(k')} \left[\left(\Theta_0 - \frac{3}{2} \frac{1}{(k^a - k'^a)^2} H_0^2 \Omega_0 \Delta_B(|k^a - k'^a|, \eta') \right) \right. \\ & \times \left. \left(\frac{1}{2} j_{\ell-1}((|k^a - k'^a|)(\eta_0 - \eta')) + j_{\ell+1}((|k^a - k'^a|)(\eta_0 - \eta')) \right) \right]. \end{aligned} \quad (\text{E.15})$$

Secondary sourced corrections

Now we substitute in the almost-FLRW adiabatic temperature anisotropy ansatz (E.7) for the secondary effects :

$$\mathbf{S}_{NLS}^S(k^a, k'^a, \eta_0, \eta_d, \eta_*) \approx \frac{3}{2} \frac{\Delta'_B(|k^a - k'^a|, \eta')}{|k^a - k'^a| k'} \int_{\eta_d}^{\eta'} d\eta \left[(D'' \kappa' + \kappa'' D') \Delta_B(k', \eta') \right]$$

$$\begin{aligned}
& + (D'\kappa')\Delta'_B(k', \eta') - 3k' \frac{D}{a} H_0^2 \Omega_0 \Delta_B(k', \eta')] e^{-\kappa} \left(\frac{1}{2} j_{\ell-1}(k'(\eta' - \eta)) + j_{\ell+1}(k'(\eta' - \eta)) \right) \\
& + \frac{3}{2} \frac{\Delta'_B(k', \eta')}{|k^a - k'^a| k'} \int_{\eta_d}^{\eta'} d\eta [(D''\kappa' + \kappa''D')\Delta_B(|k^a - k'^a|, \eta')] \\
& + (D'\kappa')\Delta'_B(|k^a - k'^a|, \eta') - 3(|k^a - k'^a|) \frac{D}{a} H_0^2 \Omega_0 \Delta_B(|k^a - k'^a|, \eta')] \\
& \times e^{-\kappa} \left(\frac{1}{2} j_{\ell-1}(|k^a - k'^a|(\eta' - \eta)) + j_{\ell+1}(|k^a - k'^a|(\eta' - \eta)) \right). \tag{E.16}
\end{aligned}$$

We can then substitute this into the integral solution for the correction (E.13) to find the correction in terms of the density contrasts due to the secondary effects as integrated from the last scattering surface until now. From this one can then construct the mean-squares to write the effect in terms of the matter power spectrum.

Correction to angular correlation functions

We use the weakly nonlinear ansatz – which excludes integrated feedback to source the calculation of the correction to the angular correlation function:

$$\delta C_\ell^{NLS} = \frac{2}{\pi} \int_0^\infty \frac{dk}{k} k^3 |\delta \Theta_\ell^{NLS}(k, \eta)|^2. \tag{E.17}$$

($\eta_0 \equiv \eta_R$ and $\eta_E = \eta_*$).

In order to understand why what we have done is so convenient, recalling that $\Delta_B(k, \eta) = D(\eta)\Delta_B(k, \eta_0)$, where in the linear theory $D(\eta) = (\eta/\eta_0)^2$ from $D \propto H \int da (aH)^{-3}$. The power spectrum is defined by $\mathcal{P}(|k|) = |\Delta_B(k, \eta_0)|^2$. In order to include nonlinear matter evolution one would modify the power spectrum by using a nonlinear extension of usual matter perturbation variable $\Delta_B \rightarrow \Delta_{NL}$ perhaps using the HKLM procedure.

E.2 Towards a nonlinear theory

Moving towards a nonlinear theory of small-scale non-perturbative correction, one would like a systematic algorithm for the inclusion of non-perturbative sources within a perturbed cosmology. Ideally one would like to have a less restrictive situation that includes mode-mode coupling. In this regard one possible recipe is outlined – but not developed further.

The next level of complexity would be to carry such a calculation out, either in the CDM frame – where the shear and vorticity would then become the interesting sources of dynamical nonlinearity or in the manifestly gauge invariant (the relative velocity free or frame unfixed) version of the theory.

That is, given a nonlinear source term, its small scale effects on the CMB would be approximated by substituting the corrections both into the general almost-FLRW solutions (the ungauged or frame fixed version as in [67]) along with the additional nonlinear shear effect so as to include the mode-mode coupling. The source term would be of the form of a mode coefficient, with a time and wavenumber dependence. The

calculation of these would in themselves be quite tricky – unless one has flat surfaces of homogeneity or similar symmetries that would allow the use of known mode functions.

The scheme is as follows, one begins with the linear ansatz :

$$\tau_{A_\ell}^{(0)} \approx \tau_{A_\ell}^P + \tau_{A_\ell}^S. \quad (\text{E.18})$$

The small scale non-perturbative correction is then constructed:

$$\tau_{A_\ell}^{(1)} \approx \tau_{A_\ell}^{(0)} + \delta\tau_{A_\ell}[\tau_{A_\ell}^{(0)}, \sigma_{ab}, A_a, \omega_a], \quad (\text{E.19})$$

(as we have done for the Rees-Sciama in the CDM almost-Eds case : with a nonlinear σ_{ab}^{RS} and $A_a \approx 0 \approx \omega_a$). To find the weakly nonlinear small-scale corrections, one can extend this scheme to test stability, we extend it iteratively as a small scale scheme for general temperature anisotropies Π_{A_ℓ} :

$$\Pi_{A_\ell}^{(n)} \approx \Pi_{A_\ell}^{(0)} + \int_{t_1}^{t_0} \dots \int_{t_0}^{t_n} dt_1 \dots dt_n \Pi_{A_\ell}^{(n-1)}, \quad (\text{E.20})$$

$$\Pi_{A_\ell}^{(0)} \approx \tau_{A_\ell}^{(0)} + \delta\tau_{A_\ell}[\tau_{A_\ell}^{(0)}, \sigma_{ab}^{NL}, A_a^{NL}, \omega_a^{NL}]. \quad (\text{E.21})$$

The subtle point here is that if one is picking *ad hoc* sources of nonlinearity, the issue of correctly modeling their backreaction through the Einstein Field Equations becomes a problem. One simplistic remedy to this problem is to use the initial almost-FLRW ansatz as restricted to the Newtonian frame (N denotes Newtonian frame almost-FLRW anisotropies):

$$\tau_{A_\ell}^{N(0)} \approx \tau_{A_\ell}^{N(1)}, \quad (\text{E.22})$$

using (E.19) and (E.18); as we have shown for the Newtonian threading. Hence we modify (E.21) such that :

$$\Pi_{A_\ell}^{(n)} \approx \Pi_{A_\ell}^{(0)} + \int_{t_1}^{t_0} \dots \int_{t_0}^{t_n} dt_1 \dots dt_n \Pi_{A_\ell}^{(n-1)}, \quad (\text{E.23})$$

$$\Pi_{A_\ell}^{(0)} \approx \tau_{A_\ell}^{N(0)} + \delta\tau_{A_\ell}[\tau_{A_\ell}^{N(0)}, \sigma_{ab}^{NL}, A_a^{NL}, \omega_a^{NL}]. \quad (\text{E.24})$$

Here σ_{ab}^{NL} , A_a^{NL} and ω_a^{NL} are found as in a fixed u^a -frame, in the exact theory, but with the linear part removed in terms of the linear relative velocity difference between the general u^a -frame and the linear Newtonian frame : $\sigma_{ab}^{NL} \approx D_{\langle a} v_{b \rangle} + \sigma_{ab}$, $A_a^{NL} \approx \dot{v}_a + H v_a + A_a$ and $\omega_a^{NL} \approx \omega_a - \frac{1}{2} \text{curl} v_a$. In the almost-FLRW case $\sigma_{AB}^{NL} \approx 0 \approx \omega_a^{NL}$ and A_a^{NL} will cancel on small scales so as to recover the canonical Newtonian-like results (E.22).

The angular correlation function can be generically constructed using the PSTF basis and the assumption of Π_{A_ℓ} being an element of an ensemble made up of Gaussian Random Fields [66] :

$$C_\ell = \frac{\Delta_\ell}{(2\ell + 1)} \langle \Pi_{A_\ell} \Pi^{A_\ell} \rangle. \quad (\text{E.25})$$

This makes the use of exact relativistic kinetic theory plausible in the context of comparison with the observed angular correlation functions. One can in principle use the exact Multipole Divergence Equations (MDE's) in a tetrad to compute Π_{A_ℓ} : the generalized temperature anisotropies.

E.3 Related Issues

In addition to the key question about the nature of the global geometry in the context of the gravitational wave imprint (as outlined in chapter (7)) there are other loose ends that could be interesting to understand better.

- **Mode-Mode couplings** : A good understanding of generic mode-mode coupling pathologies has yet to be obtained. The issue of the dissipative and dispersive effects that such couplings have on coherent behaviour found in the linearized systems is a key point. Not only is there a generic breaking of the linear independence between different modes, but resonance effects as well as the washing out of preceding structure needs to be understood within the context of GR. The key here is to understand the effects of generic smoothing due to mode-mode couplings.
- **Nonlinear Foregrounds and Integrated effects**: This still needs much work. The issue of the relevance and approximation of nonlinear astrophysical foreground effects, nonlinear general relativistic foreground effects, and the integrated nature of these effects has yet to reach the mature clarity that would be necessary in order to claim that these effects are irrelevant to modern cosmology. More importantly, if one is to take the precision cosmology programme seriously these need to be very carefully scrutinized. A key-point towards resolution of this is the understanding of generic area distance and diameter distance relations.
- **Gaussian Assumption** : A better formulation of the statistics of near linear systems in a way so as to provide a better foundations for the use of the Gaussian Assumption (See Ferreira-Mageujio [63]). Perhaps a better understanding of the process of primordial perturbation formation would provide reasons for the statistics being so close to superpositions of GRF's – leaving non-Gaussianity as a foreground issue.
- **Weak Copernican Principle** : Explicit tests of the weak Copernican principle should be carried out, not only as an exercise of getting around cosmic variance but also as an attempt to falsify the COBE-Copernican Theorem. This could be carried out using a combination of SZ, polarization and X-ray maps (see [96, 69, 119]).
- **Area-distance and Angular diameter-distance**: The exact formulation of the area distance and angular diameter distance relations from the geodesic deviation equations have not been sufficiently well developed within the 1+3 Lagrangian approach – the almost-FLRW case is well understood.
- **Nonlinear Polarization**: The Ehlers-Ellis-Theory does not yet have a non-perturbative formulation of polarization. This linear formulation is well described in the canonical literature (see Kosowsky [99] for a description of the linear theory) and has been put into 1+3 CGI almost-FLRW form by Challinor [29].
- **Spatial boundary conditions**: If the universe were only almost-FLRW in a small region of space-time, rather than the entire space-time, the surface terms on the boundary need to be understood and taken into account. One way of

doing this would be to treat the background as globally FRW but a subregion¹ of this as almost-FLRW (a small subset). Additionally one could attempt to match the perturbations across the boundary of this region in order to investigate the implication of nontrivial topologies within the the threading approach [61, 102]. That is, to construct a (more sophisticated) realistic Swiss-Cheese approach.

- **Passive and Active perturbations** : Is there a link between non-perturbative relativistic effects and active perturbations [126] – can nonlinear gravitational corrections from relativistic cosmology be well modeled by techniques from linear response theory [126] or other tricks utilized in the study of active/passive and coherent/incoherent perturbations ?

¹One could consider normals to the boundary of some region U , n^a , they will be curl free and timelike. Using the projection tensor $p^a_b = \delta^a_b - n^a n_b$ for all $\delta^a_b = -u^a u_b + h^a_b$. The boundary equations are then linearized the along with the idea that $n_a u^a \approx 0$ (close to FRW) at the boundary:

$$0 = (G_{ab} - T_{ab})n^a n^b, \quad 0 = (G_{bc} - T_{bc})n^b p^c_a, \quad 0 = (\nabla_b T^{ab})u_a, \quad 0 = (\nabla_c T^{bc})p^a_b.$$

The point is then that one can have a space-time region that is almost-FLRW as opposed to the space-time being globally almost-FLRW and impose the boundary conditions by specifying the symmetries in $n^a = \nabla_a F(x)$. Henk van Elst humoured my discussion of this and help clarify its practicality.